โครงการสมการวิวัฒนาการแบบอิมพัลส์ไม่เชิงเส้นอย่างเข้มและการควบคุมเหมาะที่สุด

โดยรองศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม และคณะ

เมษายน 2548

สัญญาเลขที่ BRG4780011

รายงานวิจัยฉบับสมบูรณ์

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย

สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

งานวิจัยนี้ทำการศึกษาสมการวิวัฒนาการแบบอิมพัลส์ไม่เชิงเส้นอย่างเข้ม โดยได้มีการพิสูจน์ว่าสมการดังกล่าวมีผลเฉลยและได้มีการศึกษาคุณสมบัติที่สำคัญของผลเฉลยด้วย

ต่อจากนั้นได้มีการประยุกต์ผลลัพธ์ข้างต้นเข้ากับปัญหาการควบคุมเหมาะที่สุดแบบลากรองจ์ และได้พิสูจน์ว่าการควบคุมเหมาะที่สุดดังกล่าวมีอยู่จริงด้วย เพื่อเป็นการแสดงให้เห็นจริงได้มีการพิจารณาตัวอย่างสมการคล้ายเชิงเส้นพาราโบลิกในปริภูมิยูคลิเดียน และได้มีการประยุกต์ใช้ระบบที่ได้พิสูจน์ไว้แล้วในตอนแรก เพื่อยืนยันถึงการมีการควบคุมเหมาะที่สุดของระบบตัวอย่าง

คำสำคัญ สมการวิวัฒนาการแบบอิมพัลส์ไม่เชิงเส้นอย่างเข้ม ตัวดำเนินการโมโนโทนไม่เชิงเส้น วิวัฒนาการสามชั้น

Abstract

Strongly nonlinear impulsive evolution equations are investigated. Existence of solutions of strongly nonlinear impulsive equations is proved and some properties of the solutions are discussed.

These results are applied to Lagrange problem of optimal control and we proved existence results. For illustration, an example of a quasi-linear impulsive parabolic differential equation and the corresponding optimal control is also presented.
Keywords: Nonlinear impulsive evolution equations; Nonlinear monotone operator; evolution triple.

Executive Summary

โครงการ "สมการวิวัฒนาการแบบอิมพัลส์ไม่เชิงเส้นอย่างเข้มและการควบคุมเหมาะที่สุด

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งบประมาณ	:	417,500.00 บาท

ปัญหาที่ทำการวิจัย

Consider the following impulsive controlled system.

$$\dot{x}(t) + Ax(t) = g(x(t)) + B(t)u(t), \quad t > 0, \; t \neq t_i, \tag{1a}$$

$$x(0) = x_0, \tag{1b}$$

$$\Delta x(t_i) = F_i\left(x(t_i)\right), \tag{1c}$$

where $0 < t_1 < t_2 < \ldots < t_n < T$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $i = 1, 2, \ldots, n$. A is nonlinear monotone operator, g is a nonmonotone perturbation, $B(t)$, and F_i's are some operators. The impulsive condition (1c) represents the jump in the state x at time t_i; with F_i determining the size of jump at t_i.

The **first research problem** is on proving that the impulsive controlled system (1) has a solution. Moreover, we define a cost function as follows:

$$J(x,u) = \int_0^T L(t, x(t), u(t))\,dt \tag{2}$$

where L is measurable in the variable t, semicontinuous function in the variable x, and a convex function in the variable u. The **second research problem** is to prove that there is a control function u and a trajectory x which satisfies equation (1) and minimize the cost function $J(x,u)$.

This is the optimal control problem.

ความสำคัญของปัญหา

Impulsive control is important in the case of the plant has at least one "impulsively" changeable state variable or when the plant has impulse effects.

We give some examples of plants whose state variables, can be changed instantaneously.

Example 1. In a financial system, we suppose that one state variable is the amount of money in a market and the other state-variables are saving rates of a central bank. As it always occurs, the former can be controlled to a desired value by changing the latter instantaneously.

Example 2. In a nanoscale electronic circuit consisting impulsive capacitor called single-electron tunnel junctions (SETJ). In the SETJ device, electron can tunnel through the barrier in a time period of order of 10^{-15} s and causes the voltage of the junction capacitor a jump in the order of 10^{-3} V. Since the junction capacitor voltage is a state variable of the SETJ model, it is practical to model SETJ using impulsive differential equations. Since SETJ has been used in many nanoelectronic circuit models, the impulsive control theory becomes a very important tool for designing and programming nanodevices which will be the building black for the next generation of computers.

The above two examples emphasize the important of impulsive control system which is the main problem for investigation in this project.

วัตถุประสงค์

The purpose of this project is to study the existence of the classical solutions of the impulsive evolution equations 1(a) – 1(c) and optimal control where the operator A is **nonlinear monotone operator**. We will solve this problem by using monotone operator approach.

ระเบียบวิธีวิจัย

1. For the construction of a classical solution, we propose to use the Schauder fixed point theorem on a suitable Banach space
2. For the construction of an optimal control pair, we propose to use Balder's result about strong-weak lower semicontinuity of integral functionals.

แผนการดำเนินงานวิจัย

1) Study the existence of a classical solution of the impulsive evolution equation.
2) Study the existence of an optimal pair of the control of the impulsive system.
3) Give some examples for an illustration.

Operation Plan

Activities	Duration / Month			
	1^{st}- 4^{th} month	5^{th}– 8^{th} month	9^{th}– 10^{th} month	11^{th}– 12^{th} month
1. Try to prove existence of solution of the impulsive evolution equations.	⟷			
2. Try to prove the existence of an optimal pair.		⟷		
3. Give some examples.			⟷	
4. Writing a paper				⟷

ผลของการทดลอง

We can proved the following two main Theorems

Theorem 1. Under some suitable conditions on the operators A, g, F_i's and B, the controlled system (1) has a solution (See, Theorem C, in the Appendix for the detail of the proof).

Theorem 2 (Existence of an optimal control pair)

Under some suitable condition on the operator A, g, F_i's, B, and L, the optimal control problem (1) together with (2) has a solution (See, Theorem D, in the Appendix for the detail of the proof).

Finally, we apply our model to a quasi-linear parabolic differential equation in the Euclidean space R^n. By utilizing Theorem 2, we can prove that such a quasi-linear problem has a solution (see Theorem E, in the Appendix for the detail of the proof).

ข้อสรุป:

By using our impulsive model, we can continue to investigate relaxation of the impulsive system and periodic impulsive system which are also important models in finance and nanoelectronic.

End of Executive Summary

เนื้อหางานวิจัย

We consider the following strongly nonlinear impulsive evolution

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = g(\mathrm{t}, x(t)) , \\ x(0) = x_0 \in H, \\ \Delta x(t_i) = F_i(x(t_i)), i = 1,2,\ldots,n. \end{cases} \tag{2}$$

By using the technique of evolution triple, monotone operator, and Schauder's fixed point theorem, we proved that the (2) has a solution.

Next, we consider the following impulsive controlled system

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = g(\mathrm{t}, x(t)) + B(t)u(t) , \\ x(0) = x_0 \in H, \\ \Delta x(t_i) = F_i(x(t_i)), i = 1,2,\ldots,n. \end{cases} \tag{3}$$

$(0 < t_1 < t_2 < \ldots < t_n < T)$ where the control function $u(t)$ is an element of space U_{ad}. We proved that, for each u, one can find a trajectory x such that the admissible pair (x,u) is a solution of (3).

Now, let us define a Lagrange cost function

$$J(x,u) = \int_0^T L(t,x(t),u(t))dt$$

where L is a convex function in the variable u. By using Balder's result, we can prove that there is an admissible pair (x_0,u_0) such that

$$J(x_0,u_0) = \inf_{(x,u)\in A_{ad}} J(x,u)$$

This prove the existence of an optimal control pair.

Finally, we apply our model to a quasi-linear partial differential equation in R^n and proved that such a model in R^n has an optimal pair.

Out put ที่ได้จากโครงการ

We proved the following four main theorems

<u>Theorem B</u> Under assumptions (A), (G) and (F), system (2) has a solution (see Appendix page 1012).

Theorem C Assume that hypotheses (A), (G), (B) and (U) hold. Then the admissible set $A_{ad} \neq \varnothing$ and X_{ad} is bounded in $PW_{pq}(I) \cap PC(\overline{I}, H)$ (see Appendix page 1014).

Theorem D Assume that hypotheses (A), (G), (U), (B) and (L) hold. There exists an admissible control pair (x_*, u_*) such that $J(x_*, u_*) = m$ (see Appendix page 1015).

Theorem E Assume that hypotheses (G') and (F') hold and $x_0(\cdot) \in L_2(\Omega), u(\cdot,\cdot) \in L_2(I \times \Omega)$, then the quasi linear parabolic problem has a solution $x \in L_p(I, PW_0^{1,p}(\Omega)) \cap PC(I, L_2(\Omega))$ such that $\partial x / \partial t \in L_q(I, W^{-1,q}(\Omega))$. (see Appendix page 1019).

ภาคผนวก

(Appendix)

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SCIENCE DIRECT®
Nonlinear Analysis
Nonlinear Analysis 57 (2004) 1005 – 1020
www.elsevier.com/locate/na

Strongly nonlinear impulsive evolution equations and optimal control ☆

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Received 26 June 2003; accepted 10 March 2004

Abstract

Strongly nonlinear impulsive evolution equations are investigated. Existence of solutions of strongly nonlinear impulsive equations is proved and some properties of the solutions are discussed.

These results are applied to Lagrange problems of optimal control and we proved existence results. For illustration, an example of a quasi-linear impulsive parabolic differential equation and the corresponding optimal control is also presented.

Keywords: Nonlinear impulsive evolution equations; Nonlinear monotone operator; Evolution triple

1. Introduction

Let $I =: (0,T)$ be a bounded open interval of the real line and let the set $D =: \{t_1, t_2, \dots, t_n\}$ be a partition on $(0,T)$ such that $0 < t_1 < t_2 < \cdots < t_n < T$. A strongly nonlinear impulsive system can be described by the following evolution equation:

$$\dot{x}(t) + A(t, x(t)) = g(t, x(t)), \quad t \in I \backslash D, \tag{1a}$$

$$x(0) = x_0, \tag{1b}$$

$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1, 2, \dots, n, \tag{1c}$$

☆ This work was supported by Thailand Research Fund Grant BRG 47 2004.
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0362-546X/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2004.03.025

where A is a nonlinear monotone operator, g is a nonlinear nonmonotone perturbation in Banach spaces, $\Delta x(t_i) \equiv x(t_i^+) - x(t_i^-) \equiv x(t_i^+) - x(t_i)$, $i = 1, 2, \ldots, n$, and F_i's are some operators. The impulsive condition (1c) represents the jump in the state x at time t_i; with F_i determining the size of the jump at time t_i (for definition of the operators A, g, and F_i will be given in Section 2). Interesting examples of impulsive systems are found in the dynamic of populations subject to abrupt changes caused by diseases or harvesting [7].

For impulsive evolution equations with an unbounded linear operator A of the form

$$\dot{x}(t) + A(t, x(t)) = g(t, x(t)), \quad t > 0, t \neq t_i$$

$$x(0) = x_0$$

$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1, 2, \ldots, n$$

have been considered in several papers by Ahmed [1], Liu [6], and Rogovchenko [7]. The questions of existence and regularity of solutions have been discussed. Ahmed applied these results to study Bolza and Lagrange problem of optimal control. However, these questions are still open when the operator A is nonlinear.

The purpose of this paper is to study the existence of classical solutions of the strongly nonlinear impulsive evolution equations (1a)–(1c) on $(0, T)$ and we will apply these results to study Lagrange optimal control problem.

2. System description

The mathematical setting of our problem is the following. Let H be a real separable Hilbert space, V be a dense subspace of H having structure of a reflexive Banach space, with the continuous embedding $V \hookrightarrow H \hookrightarrow V^*$, where V^* is the topological dual space of V. The system model considered here is based on this evolution triple. Let the embedding $V \hookrightarrow H$ be compact.

Let $\langle x, y \rangle$ denote the paring of an element $x \in V^*$ and an element $y \in V$. If $x, y \in H$, then $\langle x, y \rangle = (x, y)$, where (x, y) is the scalar product on H. The norm in any Banach space X will be denoted by $\| \cdot \|_X$.

Let $0 \leqslant s < T < +\infty, I_s \equiv (s, T), I_0 \equiv I \equiv (0, T)$, and let $p, q \geqslant 1$, be such that $1/p + 1/q = 1$ where $2 \leqslant p < +\infty$. For p, q satisfying the preceding conditions, it follows from reflexivity of V that both $L_p(I, V)$ and $L_q(I, V^*)$ are reflexive Banach spaces and the paring between $L_p(I, V)$ and $L_q(I, V^*)$ denoted by $\langle\!\langle , \rangle\!\rangle$.

Define

$$W_{pq}(I_s) = W_{pq}(s, T) = \{x : x \in L_p(I_s, V), \quad \dot{x} \in L_q(I_s, V^*)\},$$

$$\|x\|_{W_{pq}(I_s)} = \|x\|_{L_p(I_s, V)} + \|\dot{x}\|_{L_q(I_s, V^*)}$$

and

$$W_{pq}(s, u) = \{x : x \in L_p((s, u), V), \quad \dot{x} \in L_q((s, u), V^*)\}, \quad 0 \leqslant s < t < u < T,$$

where $\dot{x}$ denotes the derivative of x in the generalized sense. Furnished with the norm $\|\cdot\|_{W_{pq}(I_s)}$, the space $(W_{pq}(I_s), \|\cdot\|_{W_{pq}(I_s)})$ becomes a Banach space which is clearly reflexive and separable. Moreover, the embedding $W_{pq}(I_s) \hookrightarrow C(\bar{I}_s, H)$ is continuous. If the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq}(I_s) \hookrightarrow L_p(I_s, H)$ is also compact (see Problem 23.13(b) of [9]). Consider the following impulsive evolution equation:

$$\dot{x}(t) + A(t, x(t)) = g(t, x(t)), \quad t \in I \backslash D, \tag{2a}$$

$$x(0) = x_0 \in H, \tag{2b}$$

$$\Delta x(t_i) = F_i(x(t_i)), \ i = 1, 2, \ldots, n \quad \text{and} \quad 0 < t_1 < t_2 < \cdots < t_n < T, \tag{2c}$$

where the operators $A : I \times V \to V^*$, $g : I \times H \to V^*$ and $F_i : H \to H$. For a partition $0 < t_1 < t_2 < \cdots < t_n < T$ on $(0, T)$, we define the set $PW_{pq}(0, T) = \{x \in W_{pq}(t_i, t_{i+1})$, $i=0, 1, 2, \ldots, n$ where $t_0=0$, $t_{n+1}=T\}$. For each $x \in PW_{pq}(0, T)$, we define $\|x\|_{PW_{pq}(0,T)} =: \sum_{i=0}^{n} \|x\|_{W_{pq}(t_i, t_{i+1})}$. As a result, the space $(PW_{pq}(0, T), \|\cdot\|_{PW_{pq}(0,T)})$ becomes a Banach space. Let $PC([0, T], H) = \{x : x$ is a map from $[0, T]$ into H such that x is continuous at every point $t \neq t_i$, left continuous at $t = t_i$, and possesses right-hand limit $x(t_i^+)$ for $i = 1, 2, \ldots, n\}$. Equipped with the supremum norm topology, it is a Banach space.

By a (classical) solution x of problem (2), we mean a function $x \in PW_{pq}(0, T) \cap PC([0, T], H)$ such that $x(0) = x_0$ and $\Delta x(t_i) = F_i(x(t_i))$ for $i = 1, 2, \ldots, n$ which satisfies

$$\langle \dot{x}(t), v \rangle + \langle A(t, x), v \rangle = \langle g(t, x), v \rangle$$

for all $v \in V$ and μ-a.e. on I , where μ is the Lebesgue measure on I.

We need the following hypothesis on the data of problem (2).

(A) $A : I \times V \to V^*$ is an operator such that
  (1) $t \mapsto A(t, x)$ is weakly measurable, i.e., the functions $t \mapsto \langle A(t, x), v \rangle$ is μ-measurable on I, for all $x, v \in V$.
  (2) For each $t \in I$, the operator $A(t) : V \to V^*$ is uniformly monotone and hemicontinuous, that is, there is a constant $c_1 \geqslant 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geqslant c_1 \|x_1 - x_2\|_V^p$$

  for all $x_1, x_2 \in V$, and the map $s \mapsto \langle A(t, x + sz), y \rangle$ is continuous on $[0, 1]$ for all $x, y, z \in V$.
  (3) *Growth condition*: There exists a constant $c_2 > 0$ and a nonnegative function $a_1(\cdot) \in L_q(I)$ such that

$$\|A(t, x)\|_{V^*} \leqslant a_1(t) + c_2 \|x\|_V^{p-1}$$

  for all $x \in V$, for all $t \in I$.

(4) *Coerciveness*: There exists a constant $c_3 > 0$ and $c_4 \geqslant 0$ such that

$$\langle A(t,x),x\rangle \geqslant c_3\|x\|_V^p - c_4 \quad \text{for all } x\in V, \text{ for all } t\in I.$$

Without loss of generality, we can assume that $A(t,0)=0$ for all $t\in \bar{I}$.

(G) $g: I\times H \to V^*$ is an operator such that

(1) $t\mapsto g(t,x)$ is weakly measurable.

(2) $g(t,x)$ is Hölder continuous with respect to x with exponent $0<\alpha\leqslant 1$ in H and uniformly in t. That is, there is a constant L such that

$$\|g(t,x_1)-g(t,x_2)\|_{V^*} \leqslant L\|x_1-x_2\|_H^\alpha$$

for all $x_1,x_2\in H$ and for all $t\in I$. This assumption implies the map $x\mapsto g(t,x)$ is continuous.

(3) There exists a nonnegative function $h_1(\cdot)\in L_q(I)$ and a constant $c_5>0$ such that

$$\|g(t,x)\|_{V^*} \leqslant h_1(t) + c_5\|x\|_H^{k-1}$$

for all $x\in V$, $t\in I$, where $1\leqslant k<p$ is constant.

(F) $F_i: H\to H$ is locally Lipschitz continuous on H, i.e., for any $\rho>0$, there exists a constant $L_i(\rho)$ such that

$$\|F_i(x_1)-F_i(x_2)\|_H \leqslant L_i(\rho)\|x_1-x_2\|_H$$

for all $\|x_1\|_H, \|x_2\|_H < \rho$ $(i=1,2,\ldots,n)$.

It is sometimes convenient to rewrite system (2) into an operator equation. To do this, we set $X=L_p(I,V)$ and hence $X^*=L_q(I,V^*)$. Moreover, we set

$$\begin{cases} A(x)(t) = A(t,x(t)), \\ G(x)(t) = g(t,x(t)) \end{cases} \tag{3}$$

for all $x\in X$ and for all $t\in(0,T)$. Then the original problem (2) is equivalent to the following operator equation (see [9, Theorem 30.A]):

$$\begin{cases} \dot{x} + Ax = G(x), \\ x(0) = x_0 \in H, \\ \Delta x(t_i) = F_i(x(t_i)), \quad i=1,2,\ldots,n \quad \text{and} \quad 0<t_1<t_2<\cdots<t_n<T. \end{cases} \tag{4}$$

Remark. It follows from Theorem 30.A of Zeidler [9] that Eq. (3) defines an operator $A: X\to X^*$ such that A is uniformly monotone, hemicontinuous, coercive, and bounded. Moreover, by using hypothesis (G)(3) and using the same technique as in Theorem 30.A, one can show that the operator $G: L_p(I,H)\to X^*$ is also bounded and satisfies

$$\|G(u)\|_{X^*} \leqslant M_1 + M_2\|u\|_{L_p(I,H)}^{k-1}$$

for all $u\in L_p(I,H)$.

3. Preliminaries

In order to get a solution of Eq. (2) in the space $PW_{pq}(I)$, we firstly show that the following Cauchy problem

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = g(t,x(t)), & 0 \leqslant s < t < T, \\ x(s) = x_s \in H \end{cases} \tag{5}$$

has a solution in the space $W_{pq}(s,T)$. To prove this we need some lemmas.

Lemma 1. *Under assumption* (G), *the operator* $G : L_p(I,H) \to L_q(I,V^*)$ *is Hölder continuous with exponent* $\alpha, 0 < \alpha \leqslant 1$, *and* $G(x_n) \to G(x)$ *in* $L_q(I,V^*)$ *whenever* $x_n \xrightarrow{w} x$ *in* $W_{pq}(I)$.

Proof. The proof is the same as in Lemma 1 of [8, p .101]. □

Lemma 2. *Let* X_s *be the set of solution of Eq.* (5) *where* $0 \leqslant s < T$. *Then* X_s *is bounded in* $W_{pq}(I)$, *i.e.*, $\|x\|_{W_{pq}(I)} \leqslant M$ *and, moreover*, $\|x\|_{C([\bar{I},H])} \leqslant M$, $\forall x \in X_s$.

Proof. Let $x \in X_s$, then x can be considered as an element in $W_{pq}(I)$ by defining $x(t) \equiv 0$ on $(0,s)$. Let $X = L_p(I,V)$ and $X^* = L_q(I,V^*)$, it follows from Eq. (5) that

$$\langle\langle \dot{x}, x\rangle\rangle + \langle\langle A(x), x\rangle\rangle = \langle\langle G(x), x\rangle\rangle.$$

Since A is coercive (hypothesis (A)) then

$$c_3\|x\|_X^p - c_4 \leqslant \langle\langle G(x), x\rangle\rangle - \langle\langle \dot{x}, x\rangle\rangle.$$

By using integration by part, Hölder inequality, and hypothesis (G), we get

$$\begin{aligned} c_3\|x\|_X^p &\leqslant c_4 + \langle\langle G(x), x\rangle\rangle - \frac{1}{2}[\|x(T)\|_H^2 - \|x(0)\|_H^2] \\ &\leqslant c_4 + \left(\int_0^T \|g(t,x)\|_{V^*}^q \,\mathrm{d}t\right)^{1/q} \left(\int_0^T \|x(t)\|_V^p\right)^{1/p} \\ &\quad - \frac{1}{2}[\|x(T)\|_H^2 - \alpha_1] \\ &\leqslant c_4 + \left(\int_0^T (h_1(t) + c_5\|x(t)\|_H^{k-1})^q \,\mathrm{d}t\right)^{1/q} (\|x\|_X) + \frac{\alpha_1}{2} \end{aligned}$$

for some constants $\alpha_1 \geqslant 0$. After, some simplification, we finally get

$$c_3\|x\|_X^p \leqslant \alpha + \beta\|x\|_X + \gamma\|x\|_X^k \tag{6}$$

for some constants $\alpha, \beta, \gamma > 0$. Multiply both sides of (6) by $\|x\|_X^{1-p}$ and using the fact $1 \leqslant k < p$ and $p \geqslant 2$, we can easily see that

$$\|x\|_X \leqslant M_1 \tag{7}$$

for some constant $M_1 > 0$ and for all $x \in X_s$. Next, we shall show that

$$\|\dot{x}\|_{X^*} \leqslant M_2 \quad \text{for all } x \in X_s.$$

Let $x \in X_s$ and $\phi \in X$ then it follows from Eq. (5) that

$$\langle\langle \dot{x}, \phi \rangle\rangle + \langle\langle A(x), \phi \rangle\rangle = \langle\langle G(x), \phi \rangle\rangle.$$

Applying Hölder inequality, we get

$$|\dot{x}(\phi)| \leqslant \|A(x)\|_{X^*}\|\phi\|_X + \|G(x)\|_{X^*}\|\phi\|_X.$$

Referring to the Remark at the end of Section 2, we know that the operators A and G are bounded. Thus,

$$|\dot{x}(\phi)| \leqslant (\alpha + \beta\|x\|_X^{p-1} + \gamma + \delta\|x\|_{L_p(I,H)}^{k-1})\|\phi\|_X \tag{8}$$

for some positive constants α, β, γ, and δ. Since the embedding $L_p(I,V) \hookrightarrow L_p(I,H)$ is continuous then Eqs. (7) and (8) imply

$$\|\dot{x}\|_{X^*} \leqslant M_2 \tag{9}$$

for some positive constant M_2.

Hence, by Eqs. (7) and (9), we get

$$\|x\|_{W_{pq}(I)} = \|x\|_X + \|\dot{x}\|_{X^*} \leqslant M_1 + M_2 = M_3.$$

Hence X_s is bounded in $W_{pq}(I)$.

Finally, we note that the embedding $W_{pq}(I) \hookrightarrow C[\bar{I}, H]$ is continuous; then

$$\|x\|_{C[\bar{I},H]} \leqslant \eta\|x\|_{W_{pq}(I)}$$

and hence

$$\|x\|_{C[\bar{I},H]} \leqslant M_4$$

for some positive constants η, M_4 and for all $x \in X_s$. Choosing $M = \max\{M_3, M_4\}$ the assertion follows. □

Theorem A. *Under assumptions* (A) *and* (G), *the Cauchy problem* (5) *has a solution* $x \in W_{pq}(s,T)$.

Proof. Let $I_s = (s,T)$. Define a mapping $H : L_p(I_s,H) \times [0,1] \to L_p(I_s,H)$ by $H(u,\sigma) = w$ where w is the solution of the following problem:

$$\begin{cases} \dot{x} + A(x) = \sigma G(u), & 0 \leqslant s < t < T, \\ x(s) = \sigma x_s \in H. \end{cases} \tag{10}$$

Here the operators $A : L_p(I_s,V) \to L_q(I_s,V^*)$ and $G : L_p(I_s,H) \to L_q(I_s,V^*)$ are assumed to be satisfied hypotheses (A) and (G) on the interval (s,T), respectively. It follows from Theorem 30.A of Zeidler [9], for each $u \in L_p(I_s,H)$, problem (10) has a unique solution $w \in W_{pq}(I_s)$. Hence H is well defined. Similar to the proof of

Theorem 3 in [8], one can show the map $H : L_p(I_s,H)\times[0,1] \to L_p(I_s,H)$ is continuous and compact.

We try to use Leray–Schauder fixed point theorem. Hence, firstly, we must show that the set

$$\{u \in L_p(I_s,H) : u = H(u,\sigma) \text{ for some } 0 \leqslant \sigma \leqslant 1\}$$

is bounded in $L_p(I_s,H)$. Let $u \in L_p(I_s,H)$ and $u = H(u,\sigma)$, for some $\sigma \in [0,1]$. Then $u \in W_{pq}(I_s)$ and satisfies the problem

$$\begin{cases} \dot{u} + A(u) = \sigma G(u), \\ u(s) = \sigma x_s. \end{cases} \tag{11}$$

By Lemma 2, we get $\|u\|_{W_{pq}(I)} \leqslant M$. Moreover, since the embedding $W_{pq}(I) \hookrightarrow L_p(I,H)$ is compact, then

$$\|u\|_{L_p(I,H)} \leqslant B \text{ and hence } \|u\|_{L_p(I_s,H)} \leqslant B$$

for some positive constant B.

Secondly, we shall show that

$$H(u,0) = 0 \quad \text{for all } u \in L_p(I_s,H).$$

For any $u \in L_p(I_s,H)$, set $H(u,0) = w$ where w satisfies

$$\begin{cases} \dot{w} + A(w) = 0, \\ w(s) = 0 \in H. \end{cases} \tag{12}$$

By uniqueness of the solution of Eq. (12), we get from $A(0) = 0$ (see hypothesis (A)(4)) that

$$w = 0 \text{ in } W_{pq}(I_s) \subset W_{pq}(I).$$

Since the embedding $W_{pq}(I) \hookrightarrow L_p(I,H)$ is continuous, we get

$$w = 0 \text{ in } L_p(I,H) \quad \text{and hence} \quad w = 0 \text{ in } L_p(I_s,H).$$

That is $H(u,0) = 0$ for all $u \in L_p(I_s,H)$.

Finally, we can invoke the Leray–Schauder fixed point theorem (see [4, p. 222]) in the space $L_p(I_s,H)$, there is one fixed point $x \in L_p(I_s,H)$ such that

$$x = H(x,1)$$

and $x \in W_{pq}(I_s) \cap L_p(I_s,H)$. That is x is a solution of problem (10). Since problem (10) is equivalent to problem (5), hence there exists a solution for nonlinear evolution equation (5). □

4. Impulsive evolution equation

In this section, we would like to investigate the classical solutions of Eq. (2). By virtue of Theorem A, we have the following theorem.

Theorem B. *Under assumptions* (A), (G) *and* (F), *system* (2) *has a solution.*

Proof. Let $0 < t_1 < t_2 < \cdots < t_n < T$ be a partition of $(0, T)$.

Case 1: Find a solution of Eq. (2) on the interval $(0, t_1)$. By Theorem A, Eqs. (2a) and (2b) have a solution $x \in W_{pq}(0, T)$. Let x_1 be the restriction of x on the interval $(0, t_1)$. It is obvious that $x_1 \in W_{pq}(0, t_1)$ and $x_1(0) = x_0$. Hence, x_1 is a solution of Eq. (2) on the interval $(0, t_1)$.

Case 2: Find a solution of Eq. (2) on the interval $(0, t_2)$. Since $x_1 \in W_{pq}(0, t_1)$ and $W_{pq}(0, t_1) \hookrightarrow C([0, t_1], H)$. Then the left-hand limit $x_1(t_1^-)$ exists in H and we define $x_1(t_1) = x_1(t_1^-) \in H$. Moreover, define

$$x_1(t_1^+) = x_1(t_1) + F_1(x_1(t_1)).$$

By Hypothesis (F), we see that $x_1(t_1^+) \in H$. Now, consider the following equation:

$$\begin{cases} \dot{y}(t) + A(t, y(t)) = g(t, y(t)), & t \in (t_1, T), \\ y(t_1) = x_1(t_1^+). \end{cases} \tag{13}$$

Again, Theorem A implies that system (13) has a solution $y \in W_{pq}(t_1, T)$. Let x_2 be the restriction of y onto the interval (t_1, t_2) then $x_2 \in W_{pq}(t_1, t_2)$ and $x_2(t_1) = y(t_1) = x_1(t_1) + F_1(x_1(t_1))$. Hence, x_2 is the solution of Eq. (2) on the interval (t_1, t_2).

Now define a function x on $(0, t_2)$ as follows:

$$x(t) = \begin{cases} x_1(t); t \in (0, t_1], \\ x_2(t); t \in (t_1, t_2). \end{cases}$$

We see that $x \in PW_{pq}(0, t_2) \cap PC([0, t_2], H)$ and x satisfies Eq. (2a). Moreover, since $x(0) = x_1(0) = x_0$ and $\Delta x(t_1) \equiv x(t_1^+) - x(t_1^-) = x_1(t_1) + F_1(x_1(t_1)) - x_1(t_1) = F_1(x_1(t_1)) = F_1(x(t_1))$. Thus, x is the solution of Eq. (2) on the interval $(0, t_2)$. Continue this process through the interval $(0, T)$. We get that system (2) has a solution $x \in PW_{pq}(0, T) \cap PC([0, T], H)$. $\square$

5. Admissible trajectories and optimal control

In this section, we study the existence of optimal solutions for a Langrange optimal control problem which is governed by a class of impulsive strongly nonlinear evolution equation.

We model the control space by a separable reflexive Banach space E. By $P_f(E)$ ($P_{fc}(E)$) we denote a class of nonempty closed (closed and convex) subsets of E, respectively. Let $I = (0, T)$. Recall (see, for example, [5]) that a multifunction $\Gamma : I \to P_f(E)$ is said to be graph measurable if

$$G_r\Gamma =: \{(t, v) \in I \times E : v \in \Gamma(t)\} \in B(I) \times B(E),$$

where $B(I)$ and $B(E)$ are the Borel σ-field of I and E, respectively. For $2 \leqslant q < \infty$, we define the admissible space U_{ad} to be the set of all $L_q(I, E)$-selections of $\Gamma(\cdot)$, i.e.,

$$U_{\text{ad}} = \{u \in L_q(I, E) : u(t) \in \Gamma(t)\mu\text{-a.e. on } I\},$$

where μ is the Lebesgue measure on I. Note that the admissible space $U_{\mathrm{ad}} \neq \phi$ if $\Gamma : I \to P_f(E)$ is graph measurable and the map $t \mapsto |\Gamma(t)| =: \sup\{\|v\|_E : v \in \Gamma(t)\} \in L_q(I)$ (see [5, Lemma 3.2, p. 175]).

The Lagrange optimal control problem (P) under consideration is the following:

$$\inf J(x,u) = \int_0^T L(t,x(t),u(t))\,\mathrm{d}t = m, \tag{14a}$$

$$\dot{x}(t) + A(t,x(t)) = g(t,x(t)) + B(t)u(t), \tag{14b}$$

$$x(0) = x_0 \in H, \tag{14c}$$

$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1,2,\ldots,n \quad (0 < t_1 < t_2 < \cdots < t_n < T). \tag{14d}$$

Here, we require the operators A, g and F_i's of Eq. (14) satisfy hypotheses (A), (G) and (F), respectively, as in Section 2. We now give some new hypotheses for the remaining data.

(U) $\Gamma : I \to P_{fc}(E)$ is a measurable multifunction such that the map

$$t \mapsto |\Gamma(t)| = \sup\{\|v\|_E : v \in \Gamma(t)\}$$

belongs to $L_q(I)$.

(B) $B \in L_\infty(I, \mathscr{L}(E,H))$, where $\mathscr{L}(E,H)$ is the space of all bounded linear operators from E into H.

(L) $L : I \times V \times E \to \mathbb{R} \cup \{+\infty\}$ is an integrand such that

(1) $(t,x,u) \mapsto L(t,x,u)$ is measurable;

(2) $(x,u) \mapsto L(t,x,u)$ is sequentially lower semicontinuous;

(3) $u \mapsto L(t,x,u)$ is convex;

(4) there exists a nonnegative bounded measurable function $\phi(\cdot) \in L_1(0,T)$ and a nonnegative constant c_6 such that

$$L(t,x,u) \geqslant \phi(t) - c_6(\|x\|_V + \|u\|_E)$$

for all most $t \in I$, all $x \in V$, and all $u \in E$.

By using the same notation as in Eq. (4), we can rewrite the control system (14b)–(14d) into an equivalent operator equation as follows:

$$\dot{x} + A(x) = G(x) + B(u), \quad 0 < t < T, \tag{15a}$$

$$x(0) = x_0 \in H, \tag{15b}$$

$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1,2,\ldots,n \ (0 < t_1 < t_2 < \cdots < t_n < T), \tag{15c}$$

where the operators A, G, and $F_i (i = 1,2,\ldots,n)$ are the same as in Eq. (4). We set $B(u)(t) = B(t)u(t)$. This relation defines an operator $B : L_q(I,E) \to L_q(I,H)$ which is linear and continuous.

It follows immediately from hypothesis (U) that the admissible space $U_{\mathrm{ad}} \neq \phi$ and U_{ad} is a bounded closed convex subset of $L_q(I,E)$. Any solution x of Eqs. (15a)–(15c) is referred to as a state trajectory of the evolution system corresponding to $u \in U_{\mathrm{ad}}$ and the pair (x,u) is called an admissible pair. Let

$$A_{\mathrm{ad}} = \{(x,u) \in PW_{pq}(I) \times U_{\mathrm{ad}} : (x,u) \text{ is an admissible pair}\},$$

$$X_{\mathrm{ad}} = \{x \in PW_{pq}(I) : \exists u \in U_{\mathrm{ad}} \text{ such that } (x,u) \in A_{\mathrm{ad}}\}.$$

By using the preceding notation, our optimal control problem (14a)–(14d) can be restated as follows.

Problem (P). Find $(x_*,u_*) \in A_{\mathrm{ad}}$ such that

$$J(x_*,u_*) = \min_{(x,u)\in A_{\mathrm{ad}}} J(x,u) = m.$$

If such a pair (x_*,u_*) exists, then (x_*,u_*) is called an optimal control pair.

Theorem C. *Assume that hypotheses* (A), (G), (B) *and* (U) *hold. Then the admissible set* $A_{\mathrm{ad}} \neq \phi$ *and* X_{ad} *is bounded in* $PW_{pq}(I) \cap PC(\bar{I},H)$.

Proof. Let $u \in U_{\mathrm{ad}}$, define

$$g_u(t,x) = g(t,x) + B(t)u(t).$$

Since $B \in L_\infty(I,\mathscr{L}(E,H))$, then one can see that $g_u : I \times H \to V^*$ satisfies hypothesis (G). Hence, by virtue of Theorem B, Eq. (15) has a solution. Next, we shall show that X_{ad} is bounded in $PW_{pq}(I)$ by considering in each case separately. Let $x \in X_{\mathrm{ad}}$.

Case 1: $t \in (0,t_1)$. By Lemma 2, $\|x\|$ is bounded in $W_{pq}(0,t_1)$. Hence,

$$\|x\|_{W_{pq}(0,t_1)} \leqslant M_1 \quad \text{and} \quad \|x\|_{C([0,t_1],H)} \leqslant M_1.$$

Case 2: $t \in (t_1,t_2)$. Since $\|x(0)\|_H$ and $\|x(t_1)\|_H \leqslant M_1$ then, by hypothesis (F), we have

$$\begin{aligned}\|x(t_1^+)\|_H &\leqslant \|x(t_1)\|_H + \|F_1(x(t_1))\|_H \\ &\leqslant M_1[1 + 2L_1(M_1)] + \|F_1(x(0))\|_H,\end{aligned}$$

where $L(M_1)$ is a real constant depending on M_1. Hence, $\|x(t_1^+)\|_H$ is bounded.

Using Lemma 2 again, we have

$$\|x\|_{W_{pq}(t_1,t_2)} \leqslant M_2, \quad \text{and} \quad \|x\|_{C([t_1,t_2],H)} \leqslant M_2.$$

After a finite step, there exists $M > 0$ such that

$$\|x\|_{PW_{pq}(0,T)} \leqslant M \quad \text{and} \quad \|x\|_{C(\bar{I},H)} \leqslant M.$$

Hence, X_{ad} is bounded in $PW_{pq}(0,T) \cap PC(\bar{I},H)$. $\square$

6. Existence of optimal controls

Theorem D. *Assume hypotheses* (A), (G), (F), (U), (B), *and* (L) *hold. There exists an admissible control pair* (x_*, u_*) *such that* $J(x_*, u_*) = m$.

Proof. By Theorem C, we get $A_{\rm ad} \neq \phi$. If $m = +\infty$, then every control is admissible. Now suppose that $m < +\infty$. Choose a minimizing sequence $\{(x_k, u_k)\} \subset A_{\rm ad}$ such that

$$\lim_{k \to +\infty} J(x_k, u_k) = m.$$

Since, for each k, $(x_k, u_k) \in A_{\rm ad}$ then (x_k, u_k) must satisfy the operator equation

$$\dot{x}_k + A(x_k) = G(x_k) + B(u_k), \quad 0 < t < T, \tag{16a}$$

$$x_k(0) = x_0 \in H, \tag{16b}$$

$$\Delta x_k(t_i) = F_i(x_k(t_i)), \quad i = 1, 2, \ldots, n \quad (0 < t_1 < t_2 < \cdots < t_n < T) \tag{16c}$$

$(k = 1, 2, 3, \ldots)$. Since $U_{\rm ad}$ is bounded, the sequence $\{u_k\}$ is bounded in the reflexive Banach space $L_q(I, E)$. By passing to a subsequence if necessary, we may assume that

$$u_k \overset{w}{\to} u_* \quad \text{in } L_q(I, E) \quad \text{as } k \to \infty. \tag{17}$$

Moreover, since $U_{\rm ad}$ is a closed convex subset of $L_q(I, E)$. So, by Mazur's theorem (see [2, p. 7]), $U_{\rm ad}$ is weakly closed and hence $u_* \in U_{\rm ad}$.

Next, we shall find $x_* \in X_{\rm ad}$ such that $(x_*, u_*) \in A_{\rm ad}$. We shall do this by considering in each case separately.

Case 1: Find x_* on the interval $(0, t_1)$.

For notational convenience, we let $I_1 = (0, t_1)$, $X_1 = L_p(I_1, V)$, and $X_1^* = L_q(I_1, V^*)$. We note that $X_1 = L_p(I_1, V)$ can be considered as a closed subspace of $X = L_p(I, V)$. Let x_k^1 and u_k^1 be the restriction of the functions x_k and u_k on the interval I_1, respectively $(k = 1, 2, 3, \ldots)$. Since $\{x_k^1\}$ is the sequence of solution of Eq. (16) on the interval $(0, t_1)$, then by Theorem C, $\{x_k^1\}$ is bounded in $W_{pq}(I_1)$. By reflexivity of $W_{pq}(I_1)$, there is a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$x_k^1 \overset{w}{\to} x^1 \quad \text{in } W_{pq}(I_1) \quad \text{as } k \to \infty. \tag{18}$$

Since the embedding $W_{pq}(I_1) \hookrightarrow X_1$ is continuous, the embedding $W_{pq}(I_1) \hookrightarrow L_p(I_1, H)$ is compact, and the operator $A : X_1 \to X_1^*$ maps bounded sets to bounded sets, it follows from (18) that there is a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$x_k^1 \overset{w}{\to} x^1 \quad \text{in } X_1, \quad \dot{x}_k^1 \overset{w}{\to} \dot{x}^1 \quad \text{in } X_1^*,$$

$$x_k^1 \overset{s}{\to} x^1 \quad \text{in } L_p(I_1, H), \quad \text{and} \quad Ax_k^1 \overset{w}{\to} z \quad \text{in } X_1^*$$

as $k \to \infty$. It follows from Lemma 1 that

$$G(x_k^1) \to G(x^1) \quad \text{in } X_1^*.$$

Hence,

$$\langle\langle G(x_k^1),x_k^1\rangle\rangle_{X_1} \to \langle\langle G(x^1),x^1\rangle\rangle_{X_1} \quad \text{as } k\to\infty. \tag{19}$$

Moreover, since $B:L_q(I_1,E)\to L_q(I_1,H)$ is linear and continuous. Hence, we get from Eq. (17) that

$$Bu_k^1 \xrightarrow{w} Bu_*^1 \quad \text{in } L_q(I_1,H) \quad \text{as } k\to\infty.$$

Since $x_k^1 \xrightarrow{s} x^1$ in $L_p(I_1,H)$ (here, we identify $H=H^*$). Then

$$\langle\langle Bu_k^1,x_k^1\rangle\rangle_{X_1} \to \langle\langle Bu_*^1,x^1\rangle\rangle_{X_1} \quad \text{as } k\to\infty. \tag{20}$$

We note from Eq. (16a) that

$$\begin{aligned}\langle\langle A(x_k^1),x_k^1\rangle\rangle_{X_1} &= \langle\langle A(x_k^1),x^1\rangle\rangle_{X_1} - \langle\langle \dot{x}_k^1,x_k^1-x^1\rangle\rangle_{X_1}\\ &\quad + \langle\langle G(x_k^1),x_k^1-x^1\rangle\rangle_{X_1} + \langle\langle Bu_k^1,x_k^1-x^1\rangle\rangle_{X_1}.\end{aligned} \tag{21}$$

From the integration by part formula, we have

$$\begin{aligned}\langle\langle \dot{x}_k^1,x_k^1-x^1\rangle\rangle_{X_1} &= \langle\langle \dot{x}^1,x_k^1-x^1\rangle\rangle_{X_1}\\ &\quad + \frac{1}{2}(\|x_k^1(t_1)-x^1(t_1)\|_H^2 - \|x_k^1(0)-x^1(0)\|_H^2).\end{aligned} \tag{22}$$

Substituting (22) into (21) and note that the second term on the right-hand side of (22) is always nonnegative, then we get

$$\begin{aligned}\langle\langle A(x_k^1),x_k^1\rangle\rangle_{X_1} &\leqslant \langle\langle A(x_k^1),x^1\rangle\rangle_{X1} - \langle\langle \dot{x}^1,x_k^1-x^1\rangle\rangle_{X_1} + \|x_k^1(0)-x^1(0)\|_H^2\\ &\quad + \langle\langle G(x_k^1),x_k^1-x^1\rangle\rangle_{X_1} + \langle\langle Bu_k^1,x_k^1-x^1\rangle\rangle_{X_1}.\end{aligned}$$

By Eq. (18), $x_k^1 \xrightarrow{w} x^1$ in $W_{pq}(0,t_1)$ and hence $x_k^1 \xrightarrow{w} x^1$ in $C([0,t_1],H)$. This means that $x_k^1(0) \xrightarrow{w} x^1(0)$ in H. Referring to the initial condition (16a), we have $x_k^1(0)=x_0\in H$ $(k=1,2,3,\ldots)$. Hence, by the uniqueness of weakly limit, we get $x_k^1(0)=x^1(0)=x_0$ for all k. Therefore

$$\overline{\lim_{k\to\infty}}\langle\langle A(x_k^1),x_k^1\rangle\rangle_{X_1} \leqslant \langle\langle z,x^1\rangle\rangle_{X_1}. \tag{23}$$

Since $A:X_1\to X_1^*$ is monotone and hemicontinuous on the reflexive Banach space $X_1=L_p(I_1,V)$ then by Example 27.2(a) [9, p. 584], we have

$$z=Ax^1.$$

That is

$$A(x_k^1) \xrightarrow{w} A(x^1) \quad \text{in } X_1^*.$$

For any $\phi\in X_1$, we have

$$\langle\langle \dot{x}_k^1,\phi\rangle\rangle_{X_1} + \langle\langle A(x_k^1),\phi\rangle\rangle_{X_1} = \langle\langle G(x_k^1),\phi\rangle\rangle_{X_1} + \langle\langle B(u_k^1),\phi\rangle\rangle_{X_1}.$$

Letting $k\to\infty$, we have

$$\langle\langle \dot{x}^1,\phi\rangle\rangle_{X_1} + \langle\langle A(x^1),\phi\rangle\rangle_{X_1} = \langle\langle G(x^1),\phi\rangle\rangle_{X_1} + \langle\langle B(u^1),\phi\rangle\rangle_{X_1}.$$

Hence, x^1 is the solution of the following system:

$$\dot{x}^1 + A(x^1) = G(x^1) + B(u^1), \quad 0 < t < t_1,$$

$$x^1(0) = x_0. \tag{24}$$

Moreover, one can show that $x_k^1(t_1) \to x^1(t_1)$ in H as $k \to \infty$. To see this we note that

$$\langle\langle \dot{x}_k^1 - \dot{x}^1, x_k^1 - x^1 \rangle\rangle_{X_1} = -\langle\langle A(x_k^1 - x^1), x_k^1 - x^1 \rangle\rangle_{X_1}$$
$$+ \langle\langle G(x_k^1) - G(x^1), x_k^1 - x^1 \rangle\rangle_{X_1} + \langle\langle B(u_k^1) - B(u^1), x_k^1 - x^1 \rangle\rangle_{X_1}.$$

By using integration by parts and noting that the operator A is monotone, we have

$$\frac{1}{2}(\|x_k^1(t_1) - x^1(t_1)\|_H^2 - \|x_k^1(0) - x^1(0)\|_H^2) \leqslant \langle\langle G(x_k^1) - G(x^1), x_k^1 - x^1 \rangle\rangle_{X_1}$$
$$+ \langle\langle B(u_k^1) - B(u^1), x_k^1 - x^1 \rangle\rangle_{X_1}.$$

Since $x_k^1 \xrightarrow{w} x^1$ in X_1, then the right-hand side the above inequality tend to 0 as $k \to \infty$. Thus, we have just proved

$$x_k^1(t_1) \to x^1(t_1) \text{ in } H \text{ as } k \to \infty. \tag{25}$$

This proves that x^1 satisfies Eqs. (15a)–(15c) on the interval $(0, t_1)$ and x^1 is the required x_* on $(0, t_1)$.

Case 2: Find x_* on the interval (t_1, t_2).

The proof is similar to case 1. Here, let $I_2 = (t_1, t_2), X_2 = L_p(I_2, V)$ and $X_2^* = L_q(I_2, V^*)$. Let x_k^2 and u_k^2 be the restriction of the functions x_k and u_k on the interval I_2, respectively ($k = 1,, 2, 3, \ldots$). It follows from Eq. (16) that the sequence (x_k^2, u_k^2) satisfies the operator equation

$$\dot{x}_k^2 - A(x_k^2) = G(x_k^2) + B(u_k^2), \quad t_t < t < t_2, \tag{26a}$$

$$x_k^2(t_1^+) = x_k^2(t_1^-) + F_1(x_k^2(t_1)), \tag{26b}$$

where $x_k^2(t_1^-) = x_k^2(t_1) = x_k^1(t_1)$ $(k = 1, 2, 3, \ldots)$. By using the same proof as in case 1, we get that

$$x_k^2 \xrightarrow{w} x^2 \quad \text{in } W_{pq}(t_1, t_2) \quad \text{and} \quad x_k^2 \xrightarrow{w} x^2 \quad \text{in } C([t_1, t_2], H),$$

which implies that $x_k^2(t_1^+) \to x^2(t_1^+)$ in H as $k \to \infty$ and moreover, x^2 satisfies the operator equation

$$\dot{x}^2 + A(x^2) = G(x^2) + B(u^2), \quad t_t < t < t_2.$$

We are left to verify the initial condition at t_1. To see this, we note that the expression on the right-hand side of Eq. (26b) converges to $x^1(t_1) + F_1(x^1(t_1))$ as $k \to \infty$ (see Eq. (25) and hypothesis (F)). On the other hand, the left-hand side $x_k^2(t_1^+) \to x^2(t_1^+)$ in H as $k \to \infty$. Hence, $x^2(t_1^+) = x^1(t_1) + F_1(x^1(t_1)) \equiv x^2(t_1^-) + F_1(x^2(t_2))$. This proves that x^2 satisfies Eqs. (15a)–(15c) on the interval (t_1, t_2) and x^2 is the required x_* on (t_1, t_2).

Continue this process, we can find x_* satisfies (15a)–(15c) on the interval $(0,T)$. This proves that $(x_*,u_*)\in A_{\mathrm{ad}}$.

Finally, we shall show that (x_*,u_*) is an optimal pair. Let (x_k,u_k) be the minimizing sequence as above, i.e.,

$$x_k^j \xrightarrow{w} x_*^j \quad \text{in } W_{pq}(I_j) \quad \text{and} \quad u_k \xrightarrow{w} u_* \quad \text{in } L_q(I,E),$$

where x_k^j and x_*^j are the restriction functions of x_k and x_* onto the interval $I_j =: (t_{j-1},t_j)$ $(j=1,2,\ldots,n)$, respectively, and $\lim_{k\to\infty} J(x_k,u_k)=m$. Since the embedding $W_{pq}(I_j)\hookrightarrow L_p(I_j,H)$ is compact then, by passing to a subsequence if necessary, $x_k^j \xrightarrow{s} x_*^j$ in $L_p(I_j,H)$ as $k\to\infty$. By piecing them together from $j=1$ to n and taking into account the impact of jumps, one can conclude that

$$x_k \xrightarrow{s} x_* \quad \text{in } L_p(I,H) \quad \text{as } k\to\infty.$$

Since the embedding $L_p(I,H)\hookrightarrow L_1(I,H)$ and $L_q(I,E)\hookrightarrow L_1(I,E)$ are continuous, then

$$x_k \xrightarrow{s} x_* \quad \text{in } L_1(I,H) \quad \text{and} \quad u_k \xrightarrow{w} u \quad \text{in } L_1(I,E)$$

as $k\to\infty$.

It follows from hypothesis (L) and Theorem 2.1 of Balder [3] that

$$J(x_*,u_*)=\int_0^T L(t,x_*(t),u_*(t))\,\mathrm{d}t \leqslant \int_0^T \lim_{k\to\infty} L(t,x_k(t),u_k(t)) \leqslant m.$$

Hence (x_*,u_*) is an optimal control pair. □

7. Example

Let $I=(0,T)$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^1 boundary $\partial\Omega$. For $p\geqslant 2$ and $\theta\geqslant 0$, we consider the following optimal control problem:
(P′)

$$J(x,u)=\frac{1}{2}\int_0^T\int_\Omega |x(t,z)-y_0(z)|^2\,\mathrm{d}z\,\mathrm{d}t+\frac{\theta}{2}\int_0^T\int_\Omega |u(t,z)|^2\,\mathrm{d}z\,\mathrm{d}t \to \inf = m,$$

such that

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,z)-\displaystyle\sum_{i=1}^N D_i(|D_i x(t,z)|^{p-2}D_i x(t,z)), \\ \quad =\displaystyle\sum_{i=1}^N D_i f_i(t,z,x(t,z))+f_0(t,z,x(t,z))+b(t)u(t,z) \quad \text{a.e. on } I\times\Omega, \\ x|_{I\times\partial\Omega}=0, \quad x(0,z)=x_0(z), \quad |u(t,z)|\leqslant r(t,z) \quad \text{a.e. on } \Omega, \\ \Delta x(t_i,z)=F_i(x(t_i,z)), \quad i=1,2,\ldots,n, \\ \quad \text{where } (0<t_i<t_i<\cdots<t_n<T). \end{cases} \tag{27}$$

Here the operator $D_i = \partial/\partial x_i$ $(i = 1, 2, \dots, N)$. We need the following hypotheses on the data of (27).

(G′) $f_i : I \times \Omega \times \mathbb{R} \to \mathbb{R}$ $(i = 0, 1, \dots, N)$ are functions such that

(1) for every $x \in \mathbb{R}$, $(t, z) \mapsto f_i(t, z, x)$ is measurable;

(2) for all $(t, z) \in I \times \Omega$ and for all $x_1, x_2 \in \mathbb{R}$, we have $f_i(t, z, x)$ is Hölder continuous with respect to x and exponent $0 < \alpha \leqslant 1$; that is, there is a constant $L_i > 0$ such that

$$|f_i(t, z, x_1) - f_i(t, z, x_2)| \leqslant L_i |x_1 - x_2|^{\alpha},$$

(3) for almost all $(t, z, x) \in I \times \Omega \times \mathbb{R}$, we have

$$|f_i(t, z, x)| \leqslant a(t, z) + \gamma_1 |x|^{k-1}$$

with $1 \leqslant k < p$, $a(\cdot, \cdot) \in L_q(I \times \Omega)$, and $\gamma_1 > 0$.

(F′) $F_i : L_2(\Omega) \to L_2(\Omega)$ $(i = 1, 2, \dots, n)$ are operators such that for any $\rho > 0$ there exists a constant $L_i(\rho) > 0$ such that

$$\|F_i(x_1) - F_i(x_2)\|_{L_2(\Omega)} \leqslant L_i(\rho) \|x_1 - x_2\|_{L_2(\Omega)}$$

for all $\|x_1\|_{L_2(\Omega)}, \|x_2\|_{L_2(\Omega)} < \rho$ $(i = 1, 2, \dots, n)$.

(B′) $b(\cdot) \in L_\infty(I)$.

(R′) $r(\cdot, \cdot) \in L_q(I \times \Omega)$.

In order to study the existence for optimal control problem (P′), we firstly consider the existence of solutions for the impulsive quasi-linear control systems.

Theorem E. *If hypotheses* (G′) *and* (F′) *hold and* $x_0(\cdot) \in L_2(\Omega)$, $u(\cdot, \cdot) \in L_2(I \times \Omega)$, *then problem* (27) *has a solution* $x \in L_p(I, PW_0^{1,p}(\Omega)) \cap PC(I, L_2(\Omega))$ *such that* $\partial x/\partial t \in L_q(I, W^{-1,q}(\Omega))$.

Proof. In this problem, the evolution triple is $V = W_0^{1,p}(\Omega)$, $H = L_2(\Omega)$, and $V^* = W^{-1,q}(\Omega)$. All embedding are compact (Sobolev embedding theorem). Define an operator $A : I \times V \to V^*$ by

$$\langle A(t, x), y \rangle_V = \int_\Omega \sum_{i=1}^{N} |D_i x|^{p-2} (D_i x)(D_i y) \, \mathrm{d}z. \tag{28}$$

One can easily check that $A(t, x)$ satisfies hypothesis (A) in Section 2. The uniform monotonicity of $A(t, \cdot)$ is a consequence of the result of Zeidler [9, p. 783].

Next, by using the time-varying Dirichlet form $f : I \times H \times V \to \mathbb{R}$ by

$$f(t, x, y) = \int_\Omega \sum_{i=1}^{N} f_i(t, z, x) D_i y \, \mathrm{d}z + \int_\Omega f_0(t, z, x) y \, \mathrm{d}z.$$

Then, for each $t \in I$ and $x \in H$, the map $y \mapsto f(t, x, y)$ is a continuous linear form on V. Hence, there exists an operator $g : I \times H \to V^*$ such that

$$f(t, x, y) = \langle g(t, x), y \rangle_V. \tag{29}$$

By using hypothesis (G′), we obtain that g satisfies hypothesis (G) of Section 2. Using the operator A and g as defined in Eqs. (28) and (29), one can rewrite Eqs. (27) in an abstract form as in Eq. (15). So apply Theorem C, problem (27) has a solution.

Finally, consider the optimal control problem (P′). Let $E = L_q(\Omega)$, $V = W_0^{1,p}$ and $L : I \times V \times E \to \mathbb{R}$ with

$$L(t,x,u) = \frac{1}{2}\int_\Omega |x(t,z) - y_0(t,z)|^2 \, dz + \frac{\theta}{2}\int_\Omega |u(t,z)|^2 \, dz,$$

where $u \in L_q(I, L_q(\Omega))$, $r : I \times \Omega \to \mathbb{R}^+$ with $r \in L_q(I \times \Omega)$, and $y_0(\cdot) \in L_2(\Omega)$. Let $\Gamma : I \to P_{fc}(E)$ be defined by

$$\Gamma(t) = \{v \in L_q(\Omega) : \|v\|_{L_q(\Omega)} \leqslant \|r(t,\cdot)\|_{L_q(\Omega)}\}.$$

Then, it is easy to see that, with these definitions, problem (P′) satisfies all the hypothesis of Theorem D. Hence (P′) has at least one optimal pair. □

Acknowledgements

I would like to thank Prof. X.L. Xiang and the referees for their valuable suggestions and comments.

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