2.5 Corollary *The monotone clone of an unbounded semilattice is a subclone of a central maximal clone.*

3 On Order Varieties

As we can notice in the literature, the study of near unanimity functions in monotone clones lead to a classification of some ordered sets. One method is via the notion of an order variety. A class $\mathcal{V}$ of ordered sets is an *order variety* if it is closed under isomorphisms, products and retracts (a *retraction* on an ordered set **P** is an order-preserving map $f : \mathbf{P} \to \mathbf{P}$ such that $f(f(x)) \leq f(x)$ for all x in P; $\mathcal{V}$ is closed under retracts if whenever **P** is in $\mathcal{V}$ and f is a retract on **P**, then $f(\mathbf{P})$ is in $\mathcal{V}$). Unfortunately, the classes $\mathcal{K}$ and $\mathcal{C}$ of all finite unbounded connected ordered sets whose monotone clones are subclones of the maximal clones of all operations preserving relations of either Class(5) or Class(6); respectively, do not form order varieties. For example: $\mathbf{2}+\mathbf{2}$ is a crown whose monotone clone is a subclone of the Słupecki clone [7]; hence, it is in class $\mathcal{K}$ but, by Theorem 2.4, its retracts $\mathbf{1}+\mathbf{2}$ and $\mathbf{2}+\mathbf{1}$ are only in class $\mathcal{C}$. Now, by using the binary central relation ρ defined as in Theorem 2.4, one can easily show that the monotone clone of $\mathbf{2}+\mathbf{1}+\mathbf{2}$ is a subclone of $Pol(\rho)$; hence, $\mathbf{2}+\mathbf{1}+\mathbf{2}$ is in class $\mathcal{C}$ but its retract $\mathbf{1}+\mathbf{1}$ is a bounded ordered set whose clone is maximal.

For products of ordered sets in class $\mathcal{K}$ or class $\mathcal{C}$, we have the followings.

3.1 Theorem *Let P be an ordered set whose monotone clone is contained in a k-regularly generated maximal clone for some $3 \leq k \leq |P|$. Then the monotone clone of the product $\mathbf{P}^N$ for $N \geq 1$ is contained in a k-regularly generated maximal clone.*

Proof : Let a maximal clone of **P** preserve a k-regularly generated relation associated with equivalence relations $\theta_1, \ldots, \theta_m$ on P for some $m \geq 1$ and $3 \leq k \leq |P|$ and let N be a positive integer. For each $1 \leq i \leq N$ and $1 \leq j \leq m$, we define $\theta_i^j = \{(\bar{x}, \bar{y}) \in P^N \times P^N | \pi_i(\bar{x})\theta_j\pi_i(\bar{y})\}$. Then θ_i^j is an equivalence relation on P^N for each $1 \leq j \leq m$ and $1 \leq i \leq N$. Clearly, for each $1 \leq i \leq N$ and $1 \leq j \leq m$ the map $\psi_i^j : [\bar{x}]\, \theta_i^j \to [\pi_i(\bar{x})]\, \theta_j$ is a bijection between the set ξ_i^j of all equivalence classes of θ_i^j and the set ξ_j of all equivalence classes of θ_j; hence each $\theta_i^j, 1 \leq i \leq N; 1 \leq j \leq m$ has k equivalence classes.

Let ρ be the k-regularly generated relation on the set P^N associated with θ_i^j for $1 \leq j \leq m$ and $1 \leq i \leq N$. If $(\bar{x}^1, \ldots, \bar{x}^k) \in \rho \subseteq (P^N)^k$, then for each

$1 \leq j \leq m$ and $1 \leq i \leq N$ there exists $r < s < k$ such that $\bar{x}^r \theta_i^j \bar{x}^s$ which means that $(x_i^r, x_i^s) = (\pi_i(\bar{x}^r), \pi(\bar{x}^s)) \in \theta_j$ where $\bar{x}^t = (x_1^t, \ldots, x_N^t)$ for all $1 \leq t \leq k$; hence, for each $1 \leq i \leq N$ we have $|\{x_i^1, \ldots, x_i^k\}| < |\xi_i| = k$.

Let F be an n-ary monotone operation on $\mathbf{P}^N$ for $n \geq 1$. To show that F preserves ρ, let $(\bar{a}^{c_1}, \ldots, \bar{a}^{c_k}) \in \rho$ with $\bar{a}^{c_t} = (a_1^{c_t}, \ldots, a_N^{c_t})$ for all $1 \leq t \leq k$ and $1 \leq c \leq n$. Then, by the above remark, for each $1 \leq c \leq n$ and $1 \leq i \leq N$ we have $|\{a_i^{c_1}, \ldots, a_i^{c_k}\}| < k$.

Suppose that $(F(\bar{a}^1), F(\bar{a}^2), \ldots, F(\bar{a}^k)) \notin \rho$ where $\bar{a}^t = (\bar{a}^{1_t}, \bar{a}^{2_t}, \ldots, \bar{a}^{n_t})$ for $1 \leq t \leq k$. This means that there are $1 \leq i \leq N$ and $1 \leq j \leq m$ such that $\pi_i(F(\bar{a}^r)) \neq \pi_i(F(\bar{a}^s))$ for all $r < s \leq k$ which implies (for this fixed i) that $|\pi_i(F(\bar{a}^1)), \ldots, \pi_i(F(\bar{a}^k))| = |\xi_i| = k$. Therefore, the map $\pi_i \circ F$ is monotone from P^{Nn} to P and its range intersects every θ_j-classes. Since the monotone clone of $\mathbf{P}$ is a subclone of the k-regularly generated maximal clone which is associated with $\theta_1, \ldots, \theta_m$ and by Theorem 1.1, there exists $1 \leq u \leq Nn$ and $1 \leq v \leq m$ and a function $f_j : \xi_v \to \xi_j$ such that $[F(\bar{x})]\theta_j = f_j([x_u]\theta_v)$ for all $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \in P^{Nn}$ where $\bar{x}^c = (x_1, \ldots, x_N) \in P^N$; or equivalently, $\pi_i \circ F = f_j \circ \bar{\pi}_u$ where $\bar{\pi}_u : P^{Nn} \to P$ is the projection map to the u^{th}-component. Since $u = (c-1)N + d$ for some $1 \leq c \leq n$ and $1 \leq d \leq N$, we have $\bar{\pi}_u(\bar{a}^t) = a_d^{c_t}$ for all $1 \leq t \leq k$. Hence, the set $\{\pi_i(F(\bar{a}^1)), \ldots, \pi_i(F(\bar{a}^k))\}$ is $\{(f_j \circ \bar{\pi}_u)(\bar{a}^1), \ldots, (f_j \circ \bar{\pi}_u)(\bar{a}^k)\}$ which is $\{f_j(a_d^{c_1}), \ldots, f_j(a_d^{c_k})\}$. Now, the cardinality of $\{f_j(a_d^{c_1}), \ldots, f_j(a_d^{c_k})\}$ is k while the cardinality of $\{a_d^{c_1}, \ldots, a_d^{c_k}\}$ is less than k which implies that f_j cannot be a function; which is a contradiction. Therefore, $(F(\bar{a}^1), \ldots, F(\bar{a}^k)) \in \rho$. Thus, F preserves ρ. □

3.2 Corollary *Let* **k** *be an ordered set of size $k \geq 3$ whose monotone clone is a Slupecki clone. Then the clone of the product* $\mathbf{k}^m$ *for $m \geq 1$ is contained in a k-regularly generated maximal clone.*

3.3 Theorem *Let* **P** *be an ordered set whose monotone clone is a subclone of a central maximal clone. Then the monotone clone of the product* $\mathbf{P}^N$ *for $N \geq 1$ is also a subclone of a central maximal clone.*

Proof: Let ρ be a k-ary central relation on P for some $2 \leq k < |P|$ such that $Pol(\leq) \subseteq Pol(\rho)$. Let $\bar{\rho} \subseteq (P^N)^k$ be the set all $(\bar{x}^1, \ldots, \bar{x}^k)$ where $\bar{x}^t = (x_1^t, \ldots, x_N^t)$ such that $(\pi_i(\bar{x}^1), \ldots, \bar{\pi}_i(\bar{x}^k)) \in \rho$ for all $1 \leq i \leq N$. It is clear that $\bar{\rho}$ is a central relation on P^N where its center is the N^{th} product of the center of ρ.

Let F be an n-ary monotone operation on P^N for $n \geq 1$. To show that F preserves $\bar{\rho}$, let $(\bar{a}^{c_1}, \ldots, \bar{a}^{c_k}) \in \bar{\rho}$ with $\bar{a}^{c_t} = (a_1^{c_t}, \ldots, a_N^{c_t})$ for all $1 \leq t \leq k$ and $1 \leq c \leq n$. Then $(a_i^{c_1}, \ldots, a_i^{c_k}) \in \rho$ for all $1 \leq c \leq n$ and $1 \leq i \leq N$.

Since $\pi_i \circ F$ is monotone from P^{N_n} to P for each $1 \leq i \leq N$, we have $((\pi_i \circ F)(\bar{a}^1), \ldots, (\pi_i \circ F)(\bar{a}^k)) \in \rho$ where $\bar{a}^t = (a^{1_t}, \ldots, a^{n_t})$; hence, the definition of $\bar{\rho}$ implies that $(F(\bar{a}^1), \ldots, F(\bar{a}^k)) \in \bar{\rho}$. □

References

[1] K.A. Baker and A.F. Pixley, (1975), *Polynomial interpolation and the Chinese Remainder Theorem for Algebraic Systems*, Math. Z. 14(165-174).

[2] B.A. Davey, R.W. Quackenbuch and D. Schweigert, (1990), *Monotone Clones and the Varieties They Determine*, Order 7(145-167).

[3] B.A. Davey, J.B. Nation, R.N. McKenzie and P.P. Palfy, (1994), *Braids and their Monotone Clones*, Algebra Universalis, 32(153-176).

[4] J. Demetrovics, L. Hannak and L. Ronyai, (1984), *Near Unanimity Functions of Partial Ordering*, Proc. 14 internat. Sympos. Muliple-valued Logic, Winnipeg, Man., May 1984, IEEE,(52-56).

[5] J. Demetrovics, L. Hannak and L. Ronyai, (1986), *On Algebraic Properties of Monotone Clones*, Order 3 (219-225).

[6] J. Demetrovics, L. Hannak and L. Ronyai, (1987), *On Monotone Clones*, MTA SZTAKI Tanulmanyok(39-62).

[7] J. Demetrovics and L. Ronyai, (1989), *Algebraic Properties of Crowns and Fences*, Order 6 (91-99).

[8] D. Lau, (1978), *Bestimmung der Ordnung maximaler Klassen von Funktionen der k-wertigen Logik*, Z. Math. Logik u.Grundlagen d. Math. 24(79-96).

[9] R.W. Quackenbush, I. Rival and I.G. Rosenberg, (1990), *Clones, Order Varieties, Near Unanimity Functions and Holes*, Order 7 (239-247)

[10] I.G. Rosenberg, (1990), *Über die funktionale Vollständigkeit in den mehrwertigen Logiken, Rozpravy Ceskoslovenské Akademie Ved*, 80(3-93).

[11] G. Tardos,(1986). *A Maximal clone of Monotone Operations which is not Finitely Generated*, Order 3(211-218).

[12] L. Zádori, (1993), *Series Parallel Posets with Nonfinitely Generated Clones*, Order 10(305-316).

[13] D. Duffus and I. Rival, (1981), *A Structure Theory of Order Sets*, Discrete Mathematics, 35(53-118).

[14] R.N. McKenzie, G.F. McNulty, W.F. Taylor, (1987), *Algebras, Lattices, Varieties, Vol. 2*, manuscript.

Department of Mathematics, Faculty of Science, Silpakorn University, Thailand. email:ratach@su.ac.th

ภาคผนวก (ข)

ALL MAXIMAL CLONES CONTAINING A CROWN

C. RATANAPRASERT

ABSTRACT. I.G. Rosenberg has classified all maximal clones over a finite set A by finding six classes of relations such that maximal clones are just the clones of operations on A preserving one of these relations. In [7], we studied the clone of all operations preserving a partial order on a finite set and called the monotone clone on the ordered set; we have shown that the monotone clone on an unbounded connected ordered set is a subclone of a maximal clone preserving only central relations or k-regularly generated relations. In the paper, we investigate some properties of monotone operations of all crowns by using the comparabilities of their elements. This enables us present explicit all arities of central relations and all k-regularly generated relations which admit the monotone clone of a crown. From the results, one can give examples of unbounded order sets whose monotone clone contained in maximal clone preserving central relations and k-regularly generated relations of all arities by using crowns.

1. INTRODUCTION

A set of finitary operations on a finite set A is called a **clone** over A and denoted by Clo(A) if it contains all projections and is closed under superpositions([4], [6], [11]). A clone is finitely generated if it is the smallest clone containing some of its finite subsets. A set of all clones over a finite set is an ordered set with respect to inclusion; in fact, it is a complete lattice with the dual atoms being the maximal clones. It is known that every proper subclone is contained in a maximal one. I.G. Rosenberg[10] has classified the maiximal clones by finding six classes of relations such that maximal clones are just the clones of operations preserving one of these relations. Two classes of these relations are all central relations and all k-regularly generated relations for all integers $3 \leq k \leq |A|$.

A k-ary relation $\rho \subseteq A^k$ $(k \geq 1)$ is **totally symmetric** if for any permutation α on $\{1, \ldots, k\}$ we have $(a_1, \ldots, a_k) \in \rho$ if and only if $(a_{\alpha(1)}, a_{\alpha(2)}, \ldots, a_{\alpha(k)}) \in \rho$; and is **totally reflexive** if $\{(a_1 \ldots, a_k) | a_i = a_j \text{ for some } i \neq j\} \subseteq \rho$. The **center** of ρ is the set of all $a \in A$ such that $(a, a_2, \ldots, a_k) \in \rho$. for all $a_2, \ldots, a_k \in A$. We say that ρ is **central** if it is totally symmetric, totally reflexive, and has a center which is a non-empty, proper subset of A. Note that these conditions imply that $k < |A|$.

For $3 \leq k \leq |A|$, a set T $= \{\theta_1, \theta_2, \ldots, \theta_m\}$ $(m \geq 1)$ of equivalence relations on A is **k-regularly** if each θ_i, $1 \leq i \leq m$, has exactly k equivalence classes and the intersection $\bigcap_{i=1}^{m} \epsilon_i$ of arbitrary equivalence classes ϵ_i of θ_i is nonempty. A k-ary relation $\rho = \{(a_1, \ldots, a_k) | a_i \in A \text{ for all } i = 1, \ldots, k\}$ is **k-regularly generated** by T if for each $1 \leq i \leq m$, at least two of the elements $a_1, \ldots, a_k$ are equivalence modulo θ_i.

The following theorem can be found in [5].

Research Supported by the Thailand Research Fund.

Theorem 1. [5] *Let $3 \leq k \leq |A|$ and ρ be a k-regularly generated relation on a finite set A associated with $\theta_1, \theta_2, \ldots, \theta_m$ for some positive integers $m \geq 1$. Then the maximal clone preserving ρ is the set of all n-ary operations F for all positive integers $n \geq 1$ which satisfy for each $j < m$ either (i): the range of F intersects fewer than k θ_j − classes; or (ii): there exists $u < n$ and $v < m$ and a function $f_j : A/\theta_v \to A/\theta_j$ such that $F(\bar{x})/\theta_j = f_j(x_u/\theta_v)$ for all $\bar{x} = (x_1, \ldots, x_n)$ in A^n.*

We note that for $k = |A|$, the condition (ii) of Theorem 1 implies that F is essentially unary (defined below) and Clo(A) preserving the $|A|$-regularly generated relation is known as **Slupecki clone.** So, Theorem 1 can be restated [1] that "Clo(A) is the Slupecki clone if Clo(A) consists of all nonsurjective operations and all essentially unary operations on A"

A nonempty subset $\rho \subseteq A^k$ is a k-ary relation over A. We call that ρ **admits** Clo(A) if ρ is a subalgebra of the algebra $\langle A; f\rangle^n$ for all n-ary operations f on A. If ρ is an order relation $\leq$ over A admitting Clo(A), we will call it, the **monotone clone** of A $= \langle A; \leq\rangle$. Even there are finitely many maximal clones on a finite set A, it is in general not easy to decide which maximal clones on A contain the monotone clone of A. In [7], we showed that the classes of central relations and k-regularly generated relations of Rosenberg's six classes are the only classes of relations admitting the monotone clones of all unbounded connected ordered sets. In [2], they showed a finite generating set of the monotone clones of fences and crowns while, in [1], they studied the class of ordered set known as braids which is a natural extention of the class of crowns and showed that the monotone clones of some braids are Slupecki clones. It was shown in [12] that the clone of the linear sum of two finite antichains, each of which having at least two elements, is contained in the Slupecki clone on the underlying set.

For $n \geq 2$, an **n-crown** is an ordered set $\mathbf{C}_n = \langle A; \leq\rangle$ with 2n elements $u_1, u_2, \ldots, u_n, d_1, d_2, \ldots, d_n$ such that $d_1 < u_1 > d_2 < \cdots > d_n < u_n > d_1$ with no other comparabilities. Denote the set of all minimal elements $d_1, d_2, \ldots, d_n$ of $\mathbf{C}_n$ by D and the set of all maximal elements $u_1, u_2, \ldots, u_n$ by U ; so, A $= U \cup D$. And please note whenever we write u_m (or d_m) for a positive integer m, we mean u_i (or d_i) for some $1 \leq i \leq n$ where i and m are congruence modulo n. We call elements x and y of an n-crown $\mathbf{C}_n$, **successors or successive elements** if $x, y \in \{d_i, d_j\}$ or $x, y \in \{u_i, u_j\}$ where i and j are successive integers or $\{i, j\} = \{1, n\}$. A map $f : A^n \to A$ is called **essentially unary** if there exist i and a map $\sigma : A \to A$ such that $f(\bar{x}) = \sigma(x_i)$ for all $\bar{x} \in A^n$.

Crowns arise frequently in the theory of finite ordered sets since they play an importance role in studies of symmetries of ordered sets and of the fixed point problem [8]. Their structures and algebraic properties are considered in [3].

In the paper, we study a k-ary relations that will admit the monotone clone Clo($\mathbf{C}_n$) of a crown $\mathbf{C}_n$ for all positive integers $n \geq 2$ and $2 \leq k \leq 2n$. We investigate some properties of monotone operations of all crowns by using the comparabilities of their elements. This enables us see the only possible arities of central relations and all k-regularly generated relations which admit the monotone clone Clo($\mathbf{C}_n$). From the results, crowns are examples of unbounded ordered sets whose clones contain in a maximal clone preserving central relations and regularly generated relations of all arities.

2. Some algebraic properties of operations on a crown

To get instances of crowns possessing clones contained in maximal clones preserving some certain relaions, we need some algebraic properties of their operations. But in [2], they showed that the monotone clone of a crown is finitely generated by its unary and binary operations. In this section, we will study mostly unary and binary operations; and then we get a result that an operation on a crown is onto only if it is onto on either its all maximal or its all minimal elements.

We first state the results which follows from the fact that every element of a crown dominates a maximal element or a minimal element.

Lemma 1. *If the range of a monotone operation on an n-crown C_n for $n \geq 2$ contains all maximal(minimal) elements, then the restriction of the operation to the set of all maximal(minimal) elements also contains all maximal(minimal) elements.*

Proof. Let F be a monotone operation on C_n for $n \geq 2$. Assume that $F(C_n)$ contains U and $u \in U$. Then, there is an $a \in C_n$ such that $u = F(a)$. By the comparabilities of elements in C_n, we have $u_i \in U$ such that $a \leq u_i$; hence, $u = F(a) \leq F(u_i)$. So, maximality of u implies that $u = F(u_i) \in F(U)$. Therefore, the restriction of F to U contains U. □

Lemma 2. *Let $n \geq 3$ and $g : C_n \to C_n$ be a monotone operation satisfying the assumption of Lemma 1. Then, g is a bijection with $g(U) = U$ and $g(D) = D$.*

Proof. By dually and Lemma 1, we may assume that $g(U) \supseteq U$. Then $|g(U)| \geq |U|$. Since U is finite, $|g(U)| \leq |U|$ which implies $g(U) = U$. Hence, the restriction of g to U is bijective on U. Therefore, $g(D) \subseteq D$. Let $d_i, d_j \in D$ with $g(d_i) = g(d_j) \in D$ for $1 < i, j \leq n$. The comparabilities of elements in a crown imply that $u_{i-1} \geq d_i \leq u_i$ and $u_{j-1} \geq d_j \leq u_j$ for $u_i, u_{i-1}, u_j, u_{j-1} \in U$. Now, $\{g(u_{i-1}), g(u_i)\} = \{g(u_{j-1}), g(u_j)\}$ since each minimal element dominates only one pair of maximal successors. Since g is injective on U, we have $\{u_{i-1}, u_i\} = \{u_{j-1}, u_j\}$. But $n \geq 3$ and $d_i \neq d_j$ imply that at least three of $u_i, u_{i-1}, u_j, u_{j-1}$ are distinct which implies $\{u_{i-1}, u_i\} \neq \{u_{j-1}, u_j\}$. So, $d_i = d_j$. Therefore, g is also injective on D; thus, $g(D) = D$. □

In the following Lemma 3 to Lemma 6, we consider only those binary monotone operations on a crown C_n for $n \geq 3$; and we denote ab, the product of elements a and b under a binary monotone operation on C_n.

Lemma 3. *(i) if u_iu_j and u_iu_{j+1} (or $u_{i+1}u_j$) are distinct, then either u_iu_j and u_iu_{j+1}(or $u_{i+1}u_j$) are successive maximal or $\{u_iu_j, u_iu_{j+1}$ (or $u_{i+1}u_j)\} = \{u, d\}$ where d is a minimal element dominated by the maximal element u.*

(ii) if $u_{i+1}u_{j+1}$ is minimal then $u_iu_{j+1}, u_{i+1}u_j, u_{i+1}u_{j+2}$ and $u_{i+2}u_{j+1}$ must be maximal elements dominated by $u_{i+1}u_{j+1}$

(iii) the elements u_iu_j, u_iu_{j+1} and $u_{i+1}u_j$ cannot be all distinct.

Proof. (i) Assume that u_iu_j and u_iu_{j+1} (or $u_{i+1}u_j$) are distinct. Since $u_j \geq d_{j+1} \leq u_{j+1}$ (or $u_i \geq d_{i+1} \leq u_{i+1}$) , we have $u_iu_j \geq u_id_{j+1} \leq u_iu_{j+1}$ (or $u_iu_j \geq d_{i+1}u_j \leq u_{i+1}u_j$) which implies that u_iu_j and u_iu_{j+1}(or $u_{i+1}u_j$) cannot be both minimal. If u_iu_j and u_iu_{j+1}(or $u_{i+1}u_j$) are both maximal, then u_id_{j+1} (or $d_{i+1}u_j$) is the minimal element dominated by u_iu_j and u_iu_{j+1}; hence, u_iu_j and u_iu_{j+1} (or $u_{i+1}u_j$) are successsors. But, if one of u_iu_j and u_iu_{j+1}(or $u_{i+1}u_j$) is a maximal element u and the other is a minimal element d then $d = u_i d_{j+1}$ (or $d_{i+1}u_j) \leq u \in \{u_i u_j, u_i u_{j+1}(\text{ or } u_{i+1}u_j)\}$ which completes the proof.

(ii) Assume that $u_{i+1}u_{j+1}$ is minimal. The results in (i) imply that the four elements $u_i u_{j+1}, u_{i+1}u_j, u_{i+1}u_{j+2}$ and $u_{i+2}u_{j+1}$ cannot be minimal; hence, they are the two maximal elements dominated by $u_{i+1}u_{j+1}$.

(iii) By the comparabilities of elements in a crown, we have $u_{i+1}u_j \geq d_{i+1}d_{j+1} \leq u_i u_j \geq d_{i+1}d_{j+1} \leq u_i u_{j+1}$. If one of the $u_i u_j, u_i u_{j+1}$ and $u_{i+1}u_j$ is minimal, then by(i) either $\{u_i u_j, u_i u_{j+1}, u_{i+1}u_j\}$ is a singleton set of d or the set of two elements u and d where d is a minimal element dominated by the maximal element u; so, the lemma follows. We may assume that $u_i u_j, u_i u_{j+1}$ and $u_{i+1}u_j$ are all maximal with $u_i u_j \neq u_{i+1}u_j$. Then, $d_{i+1}d_{j+1}$ will be the minimal element dominated by $u_i u_j$ and $u_{i+1}u_j$. Since each minimal element of a crown is dominated by only one pair of maximal elements, we have $u_i u_{j+1} = u_{i+1}u_j$ or $u_i u_{j+1} = u_i u_j$. □

One can easily prove that the dual of all results in Lemma 3 are also true.

Lemma 4. *If the range of a binary operation on a crown C_n with $n \geq 3$ contains all the maximal elements, then there exists an integer $1 \leq i \leq n$ such that $U^i = \{u_j u_i | j = 1, \ldots, n\} = U$ or $U_i = \{u_i u_j | j = 1, \ldots, n\} = U$.*

Proof. We notice from the fact that $|U_i| \leq |U| = n$ and $|U^i| \leq |U| = n$; so, if the set of all maximal elements U of a crown C_n is a subset of U_i (or U^i) for some integers $1 \leq i \leq n$ then $U_i = U$ (or $U^i = U$). We will prove by supposing on the contrary that U is not a subset of U_i and U^i for all integers $1 \leq i \leq n$. Since U is a subset of the range of the binary operation and by Lemma 1, there is an integer $1 \leq i \leq n$ such that $U_i \cap U \neq \phi$. We may assume by the cyclical nature that a proper subset $U' = \{u_1, u_2, \ldots, u_k\}$ of U is labelled in the places of U_i for some integers $k < n$. By Lemma 3, one can easily seen that k is at most $\frac{n+1}{2}$ (if n is an odd integer) or $\frac{n}{2} + 1$ (if n is an even integer). Since U_i has n places to put $u_i u_j$ and by Lemma 3, we must have $n \geq 2(k-2) + 2 + a$ where a is the number of j with $u_i u_j = u_i u_{j+1} \in U'$ or $\{u_i u_j, u_i u_{j+1}\} = \{u, d\}$ where $u \in U'$ and d is a minimal element dominated by u . But the restriction of the operation to the set of all maximal elements contains U, we may let $u_{k+b} \in U_{i+j_b}$ where b and j_b are integers such that $k + b$ and $i + j_b$ are integers congruence modulo n to some integers in $\{1, \ldots, n\}$.

It is obvious from Lemma 3 that we cannot put u_c in the set U_{j+1} if the set U_j does not contain u_c or the successive maximal elements of u_c; and so, for a positive integer r, we must have at least r+1 times of u_c in U_j if we have r times of u_{c+1} in U_{j+1} and U_j does not contain u_{c+1} and u_{c+2}. Therefore, we have at least n-k+1 times of u_k in U_i if $U_{i+j_{n-k}}$ contains an u_n. Thus, $n \geq 2(k-2) + 2 + (n-k+1)$ which implies $k \leq 1$. Now, let u_n be in the place $u_{i+j_{n-k}}u_{j'}$. Since $U^{j'} \neq U$, we have $n \geq 2(n-k-1) + 2$ which impies $n \leq 2k$. Thus, $n \leq 2$ which contradicts to the assumption of the lemma that $n \geq 3$. □

Corollary 1. *If the range of a binary operation on a crown C_n with $n \geq 3$ contains all the maximal elements, then $\{u_i u_j | i = 1, \ldots, n; j = 1, \ldots, n\} = U$.*

Proof. Suppose that $\{u_i u_j | i = 1, \ldots, n; j = 1, \ldots, n\}$ is not a subset of U; that is, there exist a minimal element d and $1 \leq i \leq n$ such that $d \in U_i$. If $U_i \subseteq D$, then U is not a subset of U^j for all $j = 1, \ldots, n$; so, by using a symmetry proof to Lemma 4, one can get the same contradiction. If U_i is not a subset of D and $d \in U_i$ for some $d \in D$, there is a proper subset U' of U such that $U' \subseteq U_i$; hence, by using the proof of Lemma 4, one can also get the same contradiction. Now, Lemma 1 implies the equality. □

In the following Lemma 5 and Lemma 6 we refer the sets U_i and U^i as defined in Lemma 4.

Lemma 5. *Let the range of a binary operation on a crown C_n with $n \geq 3$ contain U. Then either*

(i) $U_i = U$ for all $i = 1, \ldots, n$ and U^j is a singleton set of a maximal element for each $j = 1, \ldots, n$; or

(ii) $U^i = U$ for all $i = 1, \ldots, n$ and U_j is a singleton set of a maximal element for each $j = 1, \ldots, n$.

Proof. By symmetry and Lemma 4, we may consider the case that there is an $1 \leq i \leq n$ such that $U_i = U$. By Corollary 1 and Lemma 3(i), we may assume that $u_i u_{j+c} = u_{k+c}$ for all integers c with i, j+c and k+c being integers in the set $\{1, \ldots, n\}$. Hence, we will have $u_{i+1} u_{j+c} \in \{u_{k+c}, u_{k+c-1}, u_{k+c+1}\}$. But the results of Lemma 3(iii) implies that $u_{i+1} u_{j+c} = u_{k+c}$. By induction, $U^{j+c} = \{u_{k+c}\}$, a singleton set of a maximal element. But c is an arbitary integers, the lemma follows. □

Lemma 6. *If the range of a binary operation on a crown C_n with $n \geq 3$ contain U, then the binary operation is onto.*

Proof. By symmetry and Lemma 4, we assume that there is an $1 \leq i \leq n$ such that $U_i =$ U. By Lemma 5, we may assume without loss of generality that $u_i u_j = u_{k+j-1}$ for each $j = 1, \ldots, n$. Then, $u_i u_j \geq u_i d_{j+1} \leq u_i u_{j+1}$ and $u_i u_j \neq u_i u_{j+1}$ for all integers j imply that $\{u_i d_j | j = 1, \ldots, n\}$ contains all the n minimal elements each of which is dominated by a pair of maximal successors. Hence, the binary operation is onto. □

The dual results as in Lemma 4 to lemma 6 and their corollaries are also true for the set D^i and D_i which are defined dually to the set U^i and U_i; respectively.

Theorem 2. *A monotone operation on a crown C_n with $n \geq 3$ is onto if and only if its range contains all maximal elements or all minimal elements.*

Proof. The necessary condition is clear. Let $n \geq 3$ and by dually, let f be a monotone operation on a crown $\mathbf{C}_n$ whose range contains all maximal elements. Since Clo($\mathbf{C}_n$) is finitely generated by all its unary and binary operations [3], there are finite numbers of unary or binary operations such that f is their superpositions. Let f_1 be the last map in the superposition of f. Then, clearly, the range of f_1 contains all maximal elements; hence, Lemma 2 or Lemma 6 imply that f_1 is onto.

If f_1 is binary, there exists $i \in \{1, 2\}$ and monotone operation $\sigma : \mathbf{C}_n \rightarrow \mathbf{C}_n$ such that $f_1(\bar{x}) = \sigma(x_i)$ for all $\bar{x} \in \mathbf{C}_n^2$ since $\mathbf{C}_n$ is Slupecki for all $n \geq 2$. Hence, f $= \sigma(g)$ where g is the superposition on i^{th}-component of f_1. And also, $U \subseteq \sigma(U)$. Now, let $u \in U$. Then, by Lemma 2, $\sigma(u) \in U$. Hence, by the assumption of the theorem, we have $f(\bar{x}) = \sigma(u)$ for some $\bar{x}$ in the domain of f. Thus, $f(\bar{x}) = \sigma(g(\bar{y}))$ for some $\bar{y}$ in the domain of g. Now, the injectivity of σ implies that g is onto. Again, if f_2 is the last map in the superposition of g then f_2 is also onto. If f_1 is unary, one can follow the proof to get the same conclusion.

By induction, f is a superposition of finite numbers of unary or binary operations all of which are onto. □

Corollary 2. *If the range of a monotone operation contains all maximal elements or all minimal elements, then it is a composition of finite surjective unary operations.*

3. The central relations admit the monotone clone of a crown

In this section, we will prove that a central relation on a set of 2n elements which admits $\mathrm{Clo}(C_n)$ has arity n and then we define the only such a certain relation.

From now on, let n and k be integers with $n \geq 2$ and $2 \leq k \leq 2n$, A be a set containing 2n elements and $R \subseteq A^k$ be a totally reflexive and totally symmetric relation which admits $\mathrm{Clo}(C_n)$ and let

$$C = \{a \in A | (a, x_1, \ldots, x_{k-1}) \in R \quad \text{for all } x_1, \ldots, x_{k-1} \in A\}$$

be the set of center elements with respect to R.

Lemma 7. *If C contains a minimal (or maximal) element of a crown C_n, then C contains all the minimal (or a maximal) elements of C_n.*

Proof. It is enough to prove that $d_i \in C$ implies that $d_{i+1} \in C$ for all integers i. We define a binary operation on C_n by

$d_i y = d_{i+1}$ and $u_i y = u_{i+1}$ for all $y \in A$ and all integers i.

It is obvious that this operation is monotone on C_n. Now, let $x_1, \ldots, x_{k-1} \in A$ and for each $1 \leq j \leq k$, we will choose

$$y_j = \begin{cases} d_{c-1}, & \text{if } x_j = d_c \text{ for some } c \in \{1, \ldots, n\} \\ u_{c-1}, & \text{if } x_j = u_c \text{ for some } c \in \{1, \ldots, n\} \end{cases}$$

Then, $y_j y_j = x_j$ for all $1 \leq j \leq k$. If $d_i \in C$, we have $(d_i, y_1, \ldots, y_{k-1}) \in R$ which together with $(y_1, y_1, \ldots, y_{k-1}) \in R$ implies that $(d_{i+1}, x_1, \ldots, x_{k-1}) \in R$; hence, $d_{i+1} \in C$. □

Lemma 8. *If $k \neq n$, then $C = \phi$ or $R = A^k$.*

Proof. Assume that $k \neq n$ and $C \neq \phi$. By Lemma 7, we may assume that $\{d_1, d_2, \ldots, d_n\} \subseteq C$. To show that $R = A^k$, let $\bar{x} = (x_1, \ldots, x_k) \in A^k$. By totally reflexivity of R and $\{d_1, d_2, \ldots, d_n\} \subseteq C$, we can consider only the case when all x_i, $1 \leq i \leq k$ are maximal. But, if $k > n$, then $\{x_1, \ldots, x_k\}$ cannot be the set of distinct maximal elements; hence, $\bar{x} \in R$. If $k < n$, then there is a maximal element $u \notin \{x_1, \ldots, x_k\}$. Let d and $\bar{d}$ be minimal elements and $\bar{u}$ be maximal element with $d \leq u \geq \bar{d} \leq \bar{u}$. If $x_j = \bar{u}$ for some $j = 1, \ldots, k$, we will define a binary monotone operation on C_n by

$$xy = \begin{cases} y, & \text{if } x \in A \text{ and } y \notin \{\bar{u}, u, d\} \\ \bar{u}, & \text{if } x \in A \text{ and } y \in \{\bar{u}, u, d\} \end{cases}$$

but if $x_j \neq \bar{u}$ for all $1 \leq j \leq k$ we will define binary monotone operation on C_n by

$$xy = \begin{cases} y, & \text{if } x \in A \text{ and } y \notin \{\bar{u}, u, d\} \\ x_j, & \text{if } x \in A \text{ and } y \in \{\bar{u}, u, d\} \end{cases}$$

In any cases, $(x_1, x_2, \ldots, d_j, \ldots, x_k) \in R$ where $x_i \notin \{\bar{u}, u, d\}$ for all $i \neq j$ and $(d_j, \ldots, d_j) \in R$ which imply that $(d_j x_1, \ldots, d_j d_j, \ldots, d_j x_k) = (x_1, x_2, \ldots, x_k) \in R$. □

Theorem 3. *Let ρ be an n-ary relation defined on a crown C_n with $n \geq 2$ by*

$$(x_1, x_2, \ldots, x_n) \in \rho \longleftrightarrow \{x_1, x_2, \ldots, x_n\} \neq U \quad (3.1)$$

or by

$$(x_1, x_2, \ldots, x_n) \in \rho \longleftrightarrow \{x_1, x_2, \ldots, x_n\} \neq D \quad (3.2)$$

Then, ρ is the only central relation admitting Clo(C_n).

Proof. It is easy to see that ρ is central. By dually, we will show that ρ, as defined in (3.1), admits Clo(C_n). Let . be a binary operation in Clo(C_n). If $(x_1, \ldots, x_n) and (y_1, \ldots, y_n)$ are n-tuples with $\{x_1y_1, \ldots, x_ny_n\} = U$, then surjectivity of the binary operation follows by Theorem 2; hence, the Słupecki property of a crown implies that $(x_1, \ldots, x_n) =$ U or $(y_1, \ldots, y_n) =$ U.

Conversely, let R be a central relation admitting Clo(C_n) with C being its set of center elements. Then, Lemma 8 implies that the arity of R is n. By Lemma 7, we may assume without loss of generality that C = U; that is , R is not a relation defined as in (3.1). Let ρ be a relation defined as in (3.2). If $(x_1, \ldots, x_n) \notin R$, then $\{x_1, \ldots, x_n\} \cap U = \phi$ and $x_1, \ldots, x_n$ are all distinct which imply that $\{x_1, \ldots, x_n\} = D$; hence, $(x_1, \ldots, x_n) \notin \rho$. Now, assume that $\{x_1, \ldots, x_n\} = D$ and $(x_1, \ldots, x_n) \in R$. Let $y_2, \ldots, y_n \in C_n$. If $y_i \in U$ for some $2 \leq i \leq n$ or $y_2, \ldots, y_n$ are not all distinct, we have $(x_1, y_2, \ldots, y_n) \in R$; but if $\{y_2, \ldots, y_n\} \subseteq D$ and $y_2, \ldots, y_n$ are all distinct, then $x_1 \in \{y_2, \ldots, y_n\}$ implies that $(x_1, y_2, \ldots, y_n) \in R$ and $x_1 \notin \{y_2, \ldots, y_n\}$ implies that $\{y_2, \ldots, y_n\} = \{x_2, \ldots, x_n\}$ which again implies that $(x_1, y_2, \ldots, y_n) = (x_1, \ldots, x_n) \in R$. These show that $x_1 \in C$; thus, $U \cap D \neq \phi$ which is a contradiction. Hence, $\{x_1, \ldots, x_n\} = D$ implies that $(x_1, \ldots, x_n) \notin R$.

Therefore, the relations defined as in (3.1) or (3.2) are the only central relations admitting Clo(C_n). □

4. THE REGULARLY GENERATED RELATIONS ADMIT THE MONOTONE CLONE OF A CROWN

Our aim of this section is to show explicit k-regularly generated relations admitting the monotone clone of a crown for all possibility $3 \leq k \leq |A|$. First, we introduce a special unary operation in Clo(C_n). Let g: $C_n \rightarrow C_n$ be defined by $g(u_i) = u_{i+1}$ and $g(d_i) = d_{i+1}$ for all integers i. It is readily seen that this operation is monotone and bijective; and we will call it, the **successors operation.**

Theorem 4. *Let C_n be an n-crown for $n \geq 2$ and ρ be a k-regularly generated relation on C_n for $3 \leq k \leq 2n$. If $3 \leq k \leq n$ or $n + 3 \leq k < 2n$, then ρ does not admit Clo(C_n).*

Proof. Let ρ be a k-regularly generated relation on C_n associated with $\theta_0, \ldots, \theta_{m-1}$ for some integers $m \geq 1$. Then, ρ is a k-ary relation and each $\theta_z, z = 0, \ldots, m-1$ has exactly k equivalence classes. Let $0 \leq z < m$. If $3 \leq k \leq n$ or $n+3 \leq k < 2n$, one of the following two cases will occur:

Case 1 : $(u_a, u_b) \in \theta_z$ and $(d_{a'}, d_{b'}) \in \theta_z$ for some integers $a < b, a' < b' \in \{1, \ldots, n\}$. We notice that if $(u_a, u_b) \in \theta_z$ and $(d_{a'}, d_{b'}) \in \theta_z$ for all integers $1 \leq a, b, a', b' \leq n$, then $k \leq 2$. So, there are $u_c \in U$ or $d_{c'} \in D$ such that $(u_b, u_c) \notin \theta_z$ or $(d_{b'}, d_{c'}) \in \theta_z$. Without loss of generality, we may assume that $(u_b, u_c) \notin \theta_z$ and we can cyclical label elements in U so that $a < b < c$. Since θ_z has exactly k equivalence classes, we can choose elements $x_{i_1}, \ldots, x_{i_k} \in C_n$ satisfying the following conditions:-

(i) if $j \neq j'$, then $(x_{i_j}, x_{i_{j'}}) \notin \theta_z$,

(ii) $(x_{i_j}, x_{i_j+1}) \notin \theta_z$ for all $j = 1, \ldots, k$ where x_{i_j+1} denote the successor of x_{i_j}, and

(iii) if $x_{i_j} = u_{i_j}$ and $(u_t, u_{i_j}) \in \theta_z$ (or dually, $x_{i_j} = d_{i_j}$ and $(d_t, d_{i_j}) \in \theta_z$), then $t < i_j$.

Case 1.1: $(x_{i_j-1}, x_{i_j}) \in \theta_z$ for some integers $1 \leq j \leq k$. In this case, we will have a k-tuple $\bar{x} \in \rho$ of the form

$$\bar{x} = (x_{i_1-1}, \ldots, x_{i_j-1}, x_{i_{j+1}-1}, \ldots, x_{i_{t-1}-1}, x_{i_j}, x_{i_{t+1}-1}, \ldots, x_{i_k-1}).$$

Please note that we skip x_{i_t} where $(x_{i_t}, x_{i_j+1}) \in \theta_z$. Hence, the successors operation g implies a k-tuple $g(\bar{x}) \notin \rho$ of the form

$$g(\bar{x}) = (x_{i_1}, \ldots, x_{i_j}, x_{i_{j+1}}, \ldots, x_{i_{t-1}}, x_{i_j+1} \equiv x_{i_t}, x_{i_{t+1}}, \ldots, x_{i_k}).$$

case 1.2 : $(x_{i_j-1}, x_{i_j}) \notin \theta_z$ for all $1 \leq j \leq k$ and we have elements x_{i_j}, and x_{i_t} for some $1 \leq t < j \leq k$ which are related by

$$(x_{i_j}, x_{i_t-1}) \in \theta_z \text{ and } x_{i_j} = x_{i_{j+1}-1} \tag{4.1}$$

Since $(x_{i_t}, x_{i_{j+1}}) \notin \theta_z$, for all possible elements x_{i_j} and x_{i_t} related as in (4.1), w e have a k-tuple $\bar{x} \in \rho$ of the form

$$\bar{x} = (x_{i_1-1}, \ldots, x_{i_t-1}, \ldots, x_{i_{j+1}-1} = x_{i_j}, \ldots, x_{i_k-1}).$$

So, the successors operation g implies a k-tuple $g(\bar{x}) \notin \rho$ of the form

$$g(\bar{x}) = (x_{i_1}, \ldots, x_{i_t}, \ldots, x_{i_j}, x_{i_{j+1}}, \ldots, x_{i_k}).$$

Case 1.3 : If $(x_{i_j-1}, x_{i_j}) \notin \theta_z$ for all $1 \leq j \leq k$ but $(x_{i_j}, x_{i_t-1}) \in \theta_z$ implies that $x_{i_j} \neq x_{i_{j+1}-1}$ for all $1 \leq t < j \leq k$, then $n \geq 4$ and k must be even. Therefore, $k \geq 4$.

If there are x_{i_j} and x_{i_t} such that $(x_{i_j}, x_{i_t-1}) \in \theta_z$, $x_{i_j} \neq x_{i_{j+1}-1}$ and $(x_{i_j+1}, x_{i_{j+1}}) \in \theta_z$. So, we have a k-tuple $\bar{x} \in \rho$ of the form

$$\bar{x} = (x_{i_1-1}, \ldots, x_{i_t-1}, \ldots, x_{i_j-1}, x_{i_j}, \ldots, x_{i_k-1}).$$

Again, the successor operation g implies a k-tuple $g(\bar{x}) \notin \rho$ of the form

$$g(\bar{x}) = (x_{i_1}, \ldots, x_{i_t}, \ldots, x_{i_j}, x_{i_j+1}, \equiv x_{i_{j+1}}, \ldots x_{i_k}).$$

For the case, $(x_{i_j}, x_{i_t-1}) \in \theta_z$ implies that $(x_{i_j-1}, x_{i_{j+1}}) \notin \theta_z$, for all possible $x_{i_j}, x_{i_t} \in \mathbf{C}_n$, we will have $k = 4$ and $(u_i, d_j) \notin \theta_z$ for all $1 \leq i, j \leq n$. Therefore, $|U/\theta_z| = |D/\theta_z| = 2$; so, $n \geq 4$ implies that $(d_2, d_4) \in \theta_z$. We define $g_1 : \mathbf{C}_n \to \mathbf{C}_n$ by

$$g_1(x) = \begin{cases} u_n, & x = u_2 \\ u_{n-1}, & x \in U - \{u_2\} \\ d_n, & x \in \{d_1, d_2, d_3\} \\ d_{n-1}, & x \in D - \{d_1, d_2, d_3\} \end{cases}$$

Then, clearly, g_1 is monotone. Now, we have $(u_1, u_2, d_2, d_4) \in \rho$ but

$$(g_1(u_1), g_1(u_2), g_1(d_2), g_1(d_4)) = (u_{n-1}, u_n, d_n, d_{n-1}) \notin \rho$$

Case 2 : $(u_a, u_b) \notin \theta_z$ or $(d_a, d_b) \notin \theta_z$ for all $1 \leq a < b \leq n$. By dually, we may assume that $(u_a, u_b) \notin \theta_z$ for all $1 \leq a < b \leq n$. Then, $k \geq n$.

If k = n and θ_x has n classes, there exists $1 \leq j < i \leq n$ such that $(u_i, d_j) \in \theta_x$. Without loss of generality, we may assume that $(u_i, d_1) \in \theta_x$. Define $\bar{g} : \mathbf{C}_n \to \mathbf{C}_n$ by $\bar{g}(u_j) = u_{n-j+1}$ and $\bar{g}(d_j) = d_{n-j+2}$ for all integers $1 \leq j \leq n$ where n-j+1 and n-j+2 are congruence modulo n to t and s for some $1 \leq t, s \leq n$, respectively. Then, clearly, $\bar{g} \in Clo(\mathbf{C}_n)$ and the range of $\bar{g}$ intersects every n θ_x - classes. Now, $\bar{x} = (u_n, u_{n-1}, \ldots, u_{n-i+2}, d_1, u_{n-i}, \ldots, u_i, \ldots, u_1) \in \rho$ but $\bar{g}(\bar{x}) = (u_1, u_2, \ldots, u_{i-1}, d_1 \equiv u_i, \ldots, u_{n-i+1}, \ldots, u_n) \notin \rho$

If $n + 3 \leq k < 2n$, there are integers $1 \leq a < b < c \leq n$ such that $(d_a, d_b) \in \theta_x$ but $(d_b, d_c) \notin \theta_x$. We will follow the proof of case 1 by substituting $x_{i_1}, \ldots, x_{i_k}$ by $d_{i_1}, \ldots, d_{i_{k-n}}$. Now, as in case 1.1, we will have a k-tuple $\bar{x} \in \rho$ of the form

$$\bar{x} = (u_1, \ldots, u_n, d_{i_1-1}, \ldots, d_{i_j-1}, d_{i_j}, d_{i_{j+1}-1}, \ldots, d_{i_{j'+1}-1}, \ldots, d_{i_{k-n}-1})$$

but the successor operation g implies that $g(\bar{x}) \notin \rho$ of the form

$$g(\bar{x}) = (u_2, \ldots, u_n, u_1, d_{i_1}, \ldots, d_{i_j}, d_{i_{j'}}, d_{i_{j+1}}, \ldots, d_{i_{j'+1}}, \ldots, d_{i_{k-n}})$$

And also, as in case 1.2, we will have $\bar{x} \in \rho$ of the form

$$(u_1, \ldots, u_n, d_{i_1-1}, \ldots, d_{i_{j_1-1}-1}, d_j, d_{i_{j_1+1}-1}, \ldots, d_{i_{j_2-1}-1}, d_{i_{j'}}, d_{i_{j_2+1}-1}, \ldots, d_{i_{k-n}-1})$$

but by applying the successor operation g, we have

$$g(\bar{x}) = (u_2, \ldots, u_n, u_1, d_{i_1}, \ldots, d_{i_{k-n}}) \notin \rho$$

which completes the proof. □

Corollary 3. *If n is odd, there is no (n+2)-regularly generated relation on A which admits Clo(C_n).*

Lemma 9. *Every 3-regularly generated relation on a 4-element set does not admit Clo(C_2).*

Proof. Let $\langle A; \leq \rangle$ be a crown $\mathbf{C}_2$ and ρ be a 3-regularly generated relation on A associated with 3 equivalence relations each of which has 3 classes. We first suppose that both maximal elements a and a' (or dually, both minimal elements) are in the same class. Then, each minimal elements b and b' is the only element in each of the other two classes; respectively. Consider the binary operations defined on $\mathbf{C}_2$ either by

$$xy = \begin{cases} a, & x \in A \text{ and } y \in \{a, a'\}; \text{ or x = a and } y \in A \\ b, & x \in \{a', b, b'\} \text{ and } y = b \\ b', & x \in \{a', b, b'\} \text{ and } y = b' \end{cases} \tag{4.2}$$

or

$$xy = \begin{cases} a, & x \in \{a, a'\} \text{ and y = a; or } x = a' \text{ and } y \in \{a, a'\} \\ a', & \text{x = a and } y = a' \\ b', & \text{otherwise} \end{cases} \tag{4.3}$$

It is readily seen that these operations are monotone. Now, by the operation defined by (4.2), we have $\bar{x} = (a, a', b) \in \rho$ and $\bar{y} = (b, b', b) \in \rho$; but $\bar{x}\bar{y} = (a, b', b) \notin \rho$.

If one of maximal elements, call it a, and one of minimal elements, call it b, are in the same class, then the operation defined by (4.3) will provide that $\bar{x} = (a, b, a) \in \rho$ and $(a, a', a') \in \rho$; but $(a, b', a') \notin \rho$. Hence, by symmetry, any 3-regularly generated relations on A will not admit Clo($\mathbf{C}_2$). □

Theorem 5. *Let C_n be an n-crown.*

(i) If $n \geq 2$, then $Clo(C_n)$ is Slupecki,

(ii) If $n \geq 3$, there is an (n+1)-regularly generated relation on A which admit $Clo(C_n)$,

(iii) If n is even, there is an (n+2)-regularly generated relation which admit $Clo(C_n)$.

Proof. (i) follows by [3]. Let $C_n = \langle A; \leq \rangle$ be an n-crown with $|A| = 2n$ and B be the n-element set $\{1, 2, \ldots, n\}$ with $1', 2' \notin B$. To prove (ii), we define $\phi : A \to B \cup \{1'\}$ by

$$\phi(x) = \begin{cases} i, & x = u_i \in U \\ 1', & x = d_i \in D \end{cases}$$

Then, clearly, ϕ is an onto mapping. Let $\rho \subseteq A^{n+1}$ be a relation on A defined by

$$(x_0, \ldots, x_n) \in \rho \longleftrightarrow \phi(x_i) = \phi(x_j) \quad \text{for some } 0 \leq i < j \leq n$$

Then, ρ is (n+1)-regularly generated associated with ϕ and $\theta = ker\phi$ and $|A/\theta|$ = n+1. So, Clo(ρ) is maximal. Now, let $F \in Clo(C_n)$ be a k-ary ($k \geq 1$) operation such that R(F), the range of F, intersects every $\theta - class$. By the definition of ϕ and θ, a θ-class of each maximal element is singleton which implies that the set of all maximal elements U is a subset of R(F); hence, by Theorem 2, F is onto. Since C_n is Slupecki, there are $1 \leq i \leq k$ and an onto monotone operation $\sigma : C_n \to C_n$ such that $F(\bar{x}) = \sigma(x_i)$ for all $\bar{x} \in C_n^k$. Recall from Lemma 3 that $\sigma(U) = U$ and $\sigma(D) = D$. We define $\bar{\sigma} : A/\theta \to A/\theta$ by $\bar{\sigma}(x/\theta) = \sigma(x)/\theta$ for all $x \in A$. Notice that $x/\theta = y/\theta$ if and only if $\phi(x) = \phi(y)$; if and only if $x, y \in U$ or $x, y \in D$. Let $x/\theta = y/\theta$. If $x, y \in U$, then x = y; hence, $\sigma(x) = \sigma(y)$ which implies $\sigma(x)/\theta = \sigma(y)/\theta$. And if $x, y \in D$, then $\sigma(x), \sigma(y) \in D$ since $\sigma(D) = D$; so, $|D/\theta| = 1$, implies $\sigma(x)/\theta = \sigma(y)/\theta$. Therefore, $\bar{\sigma}$ is well-defined. Now, for each $\bar{x} \in C_n^k$, we have $F(\bar{x})/\theta = \sigma(x_i)/\theta = \bar{\sigma}(x_i/\theta)$ which implies $F \in Clo(\rho)$. Therefore, ρ admits Clo(C_n).

To prove (iii), we define $\phi : A \to B \cup \{1', 2'\}$ by

$$\phi(x) = \begin{cases} i, & \text{if } x = u_i \in U \\ 1', & \text{if } x = d_i \in D \text{ and i is odd} \\ 2', & \text{if } x = d_i \in D \text{ and i is even} \end{cases}$$

It is obvious that ϕ is onto and $\phi(x) = \phi(y)$ if and only if x and y are not successors for all $x, y \in D$. Let $\rho \subseteq A^{n+2}$ be a relation on A by

$$(x_0, \ldots, x_{n+1}) \in \rho \longleftrightarrow \phi(x_i) = \phi(x_j) \quad \text{for some } 0 \leq i < j \leq n.$$

Then, ρ is (n+2)-regularly generated associated with ϕ and $\theta = \ker\phi$ and $|A/\theta| = n + 2$, Now, we will follow the proof of (ii) to conclude that ρ admits Clo(C_n). It remains to show that $\bar{\sigma}$ is well-defined. Let $x/\theta = y/\theta \in D/\theta$. Then $x, y \in D$ and $\phi(x) = \phi(y)$; so, x and y are not successors. The bijectivity of σ implies that $\sigma(x) = \sigma(y)$ or $\sigma(x)$ is not successive to $\sigma(y)$ in D. Hence, $\phi(\sigma(x)) = \phi(\sigma(x))$ shows $\sigma(x)/\theta = \sigma(y)/\theta$. □

Corollary 4. *The relation ρ defined in the proof of Theorem 4 and the Slupecki relation are the only k-regularly generated relations with $k \in \{n + 1, n + 2, 2n\}$ admitting the monotone clones of a crown C_n for all $n \geq 3$.*

REFERENCES

[1] B.A. Davey, J.B. Nation, R.N. McKenzie and P.P. Palfry, (1994), *Braids and their Monotone Clones*, Algbra Universallis, 32(153-176).
[2] J. Demetrovics and L. Ronyai, (1989), *Algebraic Properties of Crowns and Fences*, Order 6 (91-99).
[3] D.Duffus and I. Rival, (1981), *A Structure Theory of Ordered Sets*, Discrete Math. 35(53-118).
[4] S.V. Jablonskŭ, (1958), *Functional constructions in a k-valued logic(Russian)*, Trudy Mat, Inst. Steklov 51(5-142).
[5] R.N. McKenzie, G.F. McNulty and W.F. Taylor, (1987), *Algebras, Lattices, Varieties*, Vol. 2, manuscript.
[6] R. Pöschel and L.A. Kalužnin, (1979), *Funktionen und Relationenalgebren*, Math. Monograph. Bd 15 VEB Deutscher Verlag der Wissenschaften, Berlin, Math. Reihe, Bd. 67, Birkhäuser Verlag Basel und Stuttart.
[7] C. Ratanaprasert, *On Monotone Clones of Connected Ordered Sets*, General Algebra and Discrete Mathematics, $59^{\underline{th}}$ Arbeitstagung Allgemeine Algebra, Potsdam, Febuary 2000.
[8] I. Rival, (1976), *A fixed point theorem for finite partially ordered sets*, J. Comïn. Theory A21(309-318).
[9] I.G. Rosenberg, (1985), *Near unanimity orders*, Preprint CRM Université de Montréal.
[10] I.G. Rosenberg, (1990), Über die funktionale Vollständigkeit in den mehrwertigen Logiken, *Struktur der Funktionen von mehreren Veränderlichen auf endlichen Mengen*, Rozpravy Čs. Czechosloven Akademie Věd, Ser. Math. Nat. Sci. 80(3-93).
[11] A. Szendrei, (1986), *Clones in Universal Algebra* , Les Presses de l' Université de Montréal.
[12] L. Zadori, (1993), *Series Parallel Posets with Nonfinitely Generated Clones*, Order 10(305-316).

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE, SILPAKORN UNIVERSITY THAILAND.
E-mail address: ratach@su.ac.th

ภาคผนวก (ค)

ALL ORDERED SETS HAVING AMENABLE LATTICE ORDERS

C. RATANAPRASERT

ABSTRACT. Kolibair, Rosenberg and Schweigert proved that all compatible orders $\leq$ of a lattice $\mathbf{L} = (L; \leq^*)$ stem from 2-factor subdirect representations of **L**. We denote this by **P**#**L** and called **L**, *amenable lattice order* of an ordered set $\mathbf{P} = (L; \leq)$. We first give necessary and sufficient conditions for an order to be compatible with a lattice. We show that if we want to characterize all amenable lattice orders of an ordered set it is enough to characterize all amenable lattice orders of its connected components, and then we describe all connected ordered sets which have amenable lattice orders.

1. INTRODUCTION

Let **L** be a (semi)lattice and we consider an order $\leq$ on the underlying set P of **L** such that $\leq$ is a sublattice of $\mathbf{L}^2$. Then we say that $\leq$ is a *compatible ordering* of **L**. On the other hand, if **P** is a fixed ordered set and we consider a (semi)lattice order $\leq^*$ on the underlying set P of **P** such that $\leq$ is a sublattice of $\mathbf{L}^2$, then we say that $\leq^*$ is an *amenable* (semi)lattice order of **P** or that $\leq^*$ is a (semi)lattice order amenable with $\leq$.

In the paper, we denote the join (and the meet) operations of a (semi)lattice **L** by $\vee$ (resp. $\wedge$); and denote a compatible ordering and the corresponding order relation of **L** (on the same underlying set) by $\leq$ and $\leq^*$ respectively.

Kolibliar[5] and Rosenberg and Schweigert[8] have shown that there is a one-to-one correspondence between the set of all compatible orders which stem from 2-factor subdirect representations of **L** and the set C of all compatible orders of **L** such that $\leq \in$ C if and only if whenever $a \leq^* b \leq^* c$ the following conditions hold:

(i) $a \leq c \Rightarrow a \leq b \leq c$, and (ii) $c \leq a \Rightarrow c \leq b \leq a$

They proved that all compatible orders of a lattice **L** satisfy Conditions(i) and (ii); that is, all compatible orders of a lattice stem from 2-factor subdirect representations of **L**.

Bounded compatible orderings and compatible quasiorderings were described in [2] and [9] respectively. Cźedli, Huhn and Szabó [2] and Rosenberg and Schweigert[8] showed that compatible lattice orderings in a lattice are in one-to-one correspondence with the set of all direct decomposition of the lattice. In [7], we find necessary and sufficient conditions for a pair of semilattices S and S_1 such that S is a compatible order of S_1 and vice versa. In particular, S and S_1 will have isomorphic graphs.

In section 2, we first prove the properties of compatible orders of a (semi)lattice and then we give necessary and sufficient conditions for an order to be compatible with a lattice. In section 3, we show that if we want to characterize all amenable lattice orders of an orders set it is enough to characterize all amenable lattice orders

Research supported by the Thailand Reasearch Fund.

of its connected components, then we characterize all connected compatible orders of a lattice.

2. Amenable Lattice Orders and Subdirect Representations

Let P be a set and $\leq$ and $\leq^*$ be orders defined on P. If $a, b \in P$ with $a \leq b$, we define $[a, b]$ be the set of elements in P between a and b; that is,

$$[a,b] = \{x \in P | a \leq x \leq b\}.$$

Similarly, we define $[a,b]^* = \{x \in P | a \leq^* x \leq^* b\}$. We have the followings.

Lemma 1. *Let $\leq$ be a compatible ordering of a lattice $L = (P; \wedge, \vee, \leq^*)$. Then*
(i) $a \leq b$ implies that $a \wedge b$ and $a \vee b$ belong to $[a, b]$,
(ii) $a \leq b$ and $a \leq^* b$ imply that $[a,b] = [a,b]^*$, and
(iii) $a \leq b$ and $b \leq^* a$ imply that $[a,b] = [b,a]^*$.

Proof. (i) follows immediately from the definition of compatible ordering $\leq$. To prove (ii), let $a \leq b$ and $a \leq^* x \leq^* b$. Then $a = a \wedge x \leq b \wedge x = x = a \vee x \leq b \vee x = b$ since $\leq$ is compatible with $\vee$ and $\wedge$. Conversely, let $a \leq^* b$ and $a \leq x \leq b$. Then $a \leq x \leq b$ implies $a \leq a \wedge x$ and $b \wedge x \leq b$, which together with $a \wedge b = a$ yields $a \wedge x = a \wedge b \wedge x = a \wedge b = a$. Therefore, $a \leq^* x$. A similar argument with $a \vee b = b$ gives $x \leq^* b$. By duality, we get (iii). □

Given $a, b \in P$ we write $a \prec b$ (resp $a \prec^* b$) if $a < b$ (resp. $a <^* b$) and the interval [a, b] (resp. $[a,b]^*$) is a two element set.

Corollary 1. *Let $\leq^*$ be an amenable lattice order of $P = (P; \leq)$. If $a \prec b$, then $a \prec^* b$ or $b \prec^* a$.*

Proof. Assume that $a \prec b$. Then $a \leq b$ implies $a \vee b \in [a,b] = \{a, b\}$; hence, $a \vee b = a$ or $a \vee b = b$; that is, $a \leq^* b$ or $b \leq^* a$. By Lemma 1, we have $[a,b]^* = [a,b] = \{a,b\}$ or $[b,a]^* = [a,b] = \{a,b\}$ which shows that $a \prec^* b$ or $b \prec^* a$. □

For a (semi)lattice L, we denote the lattice of congruences by Con L with smallest element ω; the identity relation. The dual of an ordered set $\mathbf{P} = (P; \leq)$ is denoted by $\mathbf{P}^\partial = (P; \leq^\partial)$. The set of all equivalence classes of an equivalence relation θ on an order set $\mathbf{P}$ and the equivalence class containing an $a \in P$ are denoted by $\frac{P}{\theta}$ and $[a]\theta$; respectively.

Let $\mathbf{L} = (P; \leq^*)$ be a (semi)lattice. If θ_1 and θ_2 are congruence relations of $\mathbf{L}$ with $\theta_1 \cap \theta_2 = \omega$, then there exists an injective map $a \to (a_1, a_2)$ from P into $\frac{P}{\theta_1} \times \frac{P}{\theta_2}$. We define a binary relation $\leq$ on P by

$$\begin{aligned} a \leq b \quad &\Leftrightarrow \quad a_1 \geq^* b_1 (\text{ in } \frac{P}{\theta_1}) \text{ and } a_2 \leq^* b_2 (\text{ in } \frac{P}{\theta_2}); \text{ or} \\ &\Leftrightarrow \quad \text{the image of a is smaller than the image of b in} \\ &\qquad \text{the direct product } (\frac{P}{\theta_1})^\partial \times \frac{P}{\theta_2}. \end{aligned}$$

One can see that $a \leq b$ implies $[a]\theta_1 \geq^* [b]\theta_1$ and $[a]\theta_2 \leq^* [b]\theta_2$ which also implies $[a \vee c]\theta_1 \geq^* [b \vee c]\theta_1$, $[a \vee c]\theta_2 \leq^* [b \vee c]\theta_2$, $[a \wedge c]\theta_1 \geq^* [b \wedge c]\theta_1$ and $[a \wedge c]\theta_2 \leq^* [b \wedge c]\theta_2$; that is, $a \leq b$ implies $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$. Hence, $\leq$ is compatible with $\wedge$ and $\vee$.

Defnition 1. We say that $\leq$ stems from the 2-factor subdirect representation (θ_1, θ_2) of $\mathbf{L}$ and we will write $\mathbf{P}\#\mathbf{L}$ where $\mathbf{P} = (P; \leq)$.

Now let $\leq$ be a compatible ordering of a lattice $\mathbf{L} = (P; \wedge, \vee, \leq^*)$. Define relations θ_1 and θ_2 on P as follows:

$$\left.\begin{array}{lcl} a\theta_1 b & \Leftrightarrow & a \leq^* u \geq^* b \text{ and } a \leq u \geq b, \\ a\theta_2 b & \Leftrightarrow & a \leq^* v \geq^* b \text{ and } a \geq v \leq b \end{array}\right\} \qquad (2.1)$$

for some $u, v \in P$; or

$$\left.\begin{array}{lcl} a\theta_1 b & \Leftrightarrow & a \geq^* u \leq^* b \text{ and } a \geq u \leq b, \\ a\theta_2 b & \Leftrightarrow & a \geq^* v \leq^* b \text{ and } a \leq v \geq b \end{array}\right\} \qquad (2.2)$$

for some $u, v \in P$.

Kolibiar[5], Rosenberg and Schweigert[8] proved that θ_1 and θ_2, as defined either by (2.1) or (2.2), are congruence relations of the semilattices $(P; \vee)$ or $(P; \wedge)$, respectively. We prove the followings.

Lemma 2. *If $L = (P; \leq^*)$ is a semilattice, then the congruence relations θ_1 and θ_2 as defined in (2.1) or (2.2) are the transitive closure of R_1 and R_2 where R_1 is the set of all pairs $(a, b) \in P^2$ such that either*

$$(a \leq^* b \text{ and } a \leq b) \text{ or } (a \geq^* b \text{ and } a \geq b)$$

and R_2 is the set of all pairs $(a, b) \in P^2$ such that either

$$(a \leq^* b \text{ and } a \geq b) \text{ or } (a \geq^* b \text{ and } a \leq b).$$

Moreover, if L is a lattice, then the congruence relations θ_1 and θ_2 defined in (2.1) are the same congruence relations defined in (2.2).

Proof. It is enough to show that $R_i \subseteq \theta_i \subseteq \bar{R}_i$ (i =1, 2). Let $(a, b) \in R_1$. Then either $(a \leq^* b$ and $a \leq b)$ or $(b \leq^* a$ and $b \leq a)$. Hence, either $a \leq^* b \geq^* b$ and $a \leq b \geq b)$ or $(b \leq^* a \geq^* a$ and $b \leq a \geq a)$ proves $a\theta_1 b$.

Now let $a\theta_1 b$. Then there is $u \in P$ such that $a \leq^* u \geq^* b$ and $a \leq u \geq b$ which shows that (a, u) and (u, b) are elements of R_1; hence $(a, b) \in \bar{R}_1$.

We can prove $R_2 \subseteq \theta_2 \subseteq \bar{R}_2$ analogously. □

The following useful result is proved by Kolibiar[5], Rosenberg a and Schweigert[8].

Theorem 1. *([5], [8]) The followings are equivalent for a compatible order $\leq$ of a semilattice $L = (P; \leq^*)$ and the corresponding congruence relations θ_1 and θ_2,*

(i) $\theta_1 \cap \theta_2 = \omega$ and $\leq$ stems from the subdirect representation given by θ_1 and θ_2,

(ii) each interval $\{x \in P | a \leq x \leq b\}$ is a convex subset of L, and

(iii) if $a \leq^* b \leq^* c$ then $a \leq c$ implies $a \leq b \leq c$, and $c \leq a$ implies $c \leq b \leq a$.

In the proof of theorem 1, we can see that $\leq := (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. If $a \leq^* b$ and $a\theta_1 b$, then $a\theta_1 b\theta_2 b$ and $a \leq^* b \geq^* b$; hence $a \leq b$. Analogougsly, if $a \leq^* b$ and $a\theta_2 b$, then $a \geq b$.

Corollary 2. *For $a, b \in P$,*

(i) if $a \leq^* b$ and $a\theta_1 b$ then $a \leq b$, and

(ii) if $a \leq^* b$ and $a\theta_2 b$ then $b \leq a$.

Now, if **L** is a lattice, Lemma 1 shows that condition(iii) of Theorem 1 always holds. In [5] and [8], they showed that the map $\leq \rightarrow (\theta_1, \theta_2)$ induced a bijection between the set of compatible orders of a lattice and the set of orders stemming from 2-factor subdirect representation of the lattice. We have the following as its corollary.

Corollary 3. *Let L be a lattice. Then every congruence θ on L gives rise to compatible orders $\leq$ and $\leq^{\partial}$ where $\leq$ is given by*

$$a \leq b \iff a \leq^{*} b \text{ and } a\theta b.$$

Morover, if L is subdirectly irreducible then every compatible ordering of L arises in this way.

Proof. If **L** is subdirectly irreducible, then $\theta_1 \cap \theta_2 = \omega$ implies $\theta_1 = \omega$ or $\theta_2 = \omega$. □

If an order $\leq$ is compatible with $\vee$ and $\wedge$ of a lattice $\mathbf{L} = (P; \leq^{*})$ and θ_1, θ_2 are defined as in (2.1), then it follows from Theorem 1 that θ_1 and θ_2 are congruence relations of **L** with $\theta_1 \cap \theta_2 = \omega$ and $\leq = (\theta_1 \cap \leq^{*}) \circ (\theta_2 \cap \geq^{*})$. Therefore, $a\theta_1 b$ implies $(a \wedge b)\theta_1 a\theta_2 a\theta_1 (a \vee b)\theta_2 (a \vee b)$; that is, $a \wedge b \leq a \leq a \vee b$. Similarly, $a\theta_2 b$ implies $a \vee b \leq a \leq a \wedge b$.

Moreover, if $a \leq b$ then $a\theta_1 u\theta_2 b$ and $a \leq^{*} u \geq^{*} b$ for some $u \in P$. Therefore $a \leq^{*} a \vee b \leq^{*} u$ and $b \leq^{*} a \vee b \leq^{*} u$ yield $(a \vee b, u) \in \theta_1 \cap \theta_2$; that is $u = a \vee b$.

Corollary 4. *Let $\leq$ be a compatible order of a lattice $L = (P; \leq^{*})$ and let θ_1 and θ_2 be defined as in (2.1). Then for $a, b \in P$,*

(i) $a\theta_1 b$ implies $a \wedge b \leq a, b \leq a \vee b$,

(ii) $a\theta_2 b$ implies $a \vee b \leq a, b \leq a \wedge b$,

(iii) $a \leq b$ implies $a\theta_1(a \vee b)\theta_2 b$ and $a\theta_2(a \wedge b)\theta_1 b$, and

(iv) $a \prec b$ implies $a\theta_1 b$ or $a\theta_2 b$.

Theorem 2. *Let $P = (P; \leq)$ be an ordered set and $L = (P; \leq^{*})$ be a lattice. Then $P\#L$ if and only if there are lattice L_1 and L_2 (with underlying sets L_1 and L_2 and a map $\psi : P \rightarrow L_1 \times L_2$ such that*

(i) ψ is a lattice embedding of L into $L_1 \times L_2$, and

(ii) ψ is an order embedding of P into $L_1^{\partial} \times L_2$.

Proof. The forward implication follows by Theorem 1 since θ_1 and θ_2 defined as in (2.1) are congruence relations of L with $\theta_1 \cap \theta_2 = \omega$. Conversely, let θ_1 and θ_2 be congruence relations of **L** corresponding to $\mathbf{L}_1$ and $\mathbf{L}_2$ and identify $\mathbf{L}_1$ with $\frac{\mathbf{L}}{\theta_1}$ and $\mathbf{L}_2$ with $\frac{\mathbf{L}}{\theta_2}$ respectively. Then $\theta_1 \cap \theta_2 = \omega$. It remains to show that $\leq = (\theta_1 \cap \leq^{*}) \circ (\theta_2 \cap \geq^{*})$. Let $a \leq b$. Then $\psi(a) \leq \psi(b)$ where $\leq$ denotes the order relation of $\mathbf{L}_1^{\partial} \times \mathbf{L}_2$. So $[a]\theta_1 \geq^{*} [b]\theta_1$ and $[a]\theta_2 \leq^{*} [b]\theta_2$ yield $a\theta_1(a \vee b)\theta_2 b$; that is, $(a, b) \in (\theta_1 \cap \leq^{*}) \circ (\theta_2 \cap \geq^{*})$. If $(a, b) \in (\theta_1 \cap \leq^{*}) \circ (\theta_2 \cap \geq^{*})$, it follows from the argument above Corollary 4 that $a\theta_1(a \vee b)\theta_2 b$ which yields $([a]\theta_1, [b]\theta_2) = \psi(a \vee b) \in Im\psi$. Therefore $\psi(a) \leq \psi(a \vee b)$. Since ψ is an order-embedding of **P** into $\mathbf{L}_1^{\partial} \times \mathbf{L}_2$, we have $a \leq a \vee b$. Analogously $a \vee b \leq b$. Hence $a \leq b$. □

3. Connected Ordered Sets Having Amenable Lattice Orders

Let $\leq$ be an order on a set P and let $\leq^{c}$ denote the equivalence closure of $\leq$; that is, the smallest equivalence relation on P containing $\leq$. Then for $a_1, a_2, \ldots, a_n \in P$, $a_1 \leq a_2 \geq a_3 \leq \cdots \leq a_n$ implies $a_1 \leq^{c} a_n$. Hence, if θ is the set of all pairs $(a, b) \in P^2$ such that a and b are in the same component, then θ is a subset of $\leq^c$. But, in fact, θ is an equivalence relation on P containing $\leq$. Therefore, $\theta = \leq^c$. Moreover, if $\leq$ is a compatible ordering of a lattice $L = (P; \leq^*)$, then θ is a congruence relation of L. Let $a \in P$ and $x, y, z \in [a]\theta$ with $x \leq y$. Since $\leq$ and θ are compatible with $\wedge$ and $\vee$ of L, we have $x \wedge z \leq y \wedge z$, $x \vee z \leq y \vee z$, $(x \wedge z)\theta a\theta(y \wedge z)$ and $(x \vee z)\theta a\theta(y \vee z)$. This shows that each block of θ is amenable with the corresponding order-component.

Lemma 3. *Let L be an amenable lattice order of an ordered set P. Then each order-component of P has an amenable lattice order which is a convex sublattice of L.*

Conversely, let $P = (\cup_{i \in I} P_i; \leq)$ be an ordered set where $P_i \cap P_j = \phi$ if $i \neq j$, let $L_i = (P_i; \leq_i^*)$ be an amenable lattice order of $P_i = (P_i; \leq)$ for all $i \in I$ and let $<$ be a strict total order of I. Define a binary relation $\leq^*$ on $P = \cup_{i \in I} P_i$ as follow:

(i) $a \leq^* b \Leftrightarrow a \leq_i^* b$ whenever $a, b \in P_i$ for some $i \in I$, or

(ii) $a \leq^* b \Leftrightarrow a \in P_i$, $b \in P_j$ for $i < j$.

Then, clearly, $\leq^*$ is a lattice order on P such that $\leq$ is a compatible ordering of $(P; \leq_i^*)$.

Theorem 3. *An ordered set has an amenable lattice order just if each its order components has.*

We shall now prove that the compatible orders of a lattice arising from complementary pairs of congruences are precisely the connected compatible orders. Moreover, connected compatible orders of a lattice satisfy the upper and lower bound properties(LBP and UBP) definded below.

Lemma 4. *Let $\leq$ be a connected compatible order of a lattice $L = (P; \wedge, \vee, \leq^*)$ and let θ_1 and θ_2 be as in (2.1). Then θ_1 is the complement of θ_2 in Con L.*

Proof. It remains to show that $\theta_1 \vee \theta_2 = P \times P$. Let $a, b \in P$. Since $\leq$ is connected, there are elements $a = a_0, a_1, \ldots, a_n = b$ such that $a_i \leq a_{i+1}$ or $a_{i+1} \leq a_i$ for all $i = 0, 1, \ldots, n-1$ which yields from Corollary 4(iii) that either $a_i\theta_1(a_i \vee a_{i+1})\theta_2 a_{i+1}$ or $a_{i+1}\theta_1(a_i \vee a_{i+1})\theta_2 a_i$. In either cases, we have $(a_i, a_{i+1}) \in \theta_1 \vee \theta_2$ for all $i = 0, 1, \ldots, n-1$. Hence, by the transitivity of $\theta_1 \vee \theta_2$, we have $(a, b) \in \theta_1 \vee \theta_2$. □

Remark: Let L be a lattice and let θ_1 and θ_2 be congruence relations of L. It is known[3] that $(a, b) \in \theta_1 \vee \theta_2$ if and only if there is a sequence $a \wedge b = z_0 \leq^* z_1 \leq^* \cdots \leq^* z_n = a \vee b$ such that $z_0\theta_1 z_1\theta_2 z_2 \ldots \theta_2 z_n$.

Lemma 5. *Let $L = (P; \wedge, \vee, \leq^*)$ be a lattice and let θ_1 and θ_2 be a complementary pair of congruences of L. Then the compatible order $\leq = (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$ is connected.*

Proof. Let $a, b \in P$. Then $(a, b) \in \theta_1 \vee \theta_2$ and hence there is a sequence $a \wedge b = z_0 \leq^* z_1 \leq^* \cdots \leq^* z_n = a \vee b$ such that for each $i = 0, 1, \ldots, n-1$ we have either $z_i\theta_1 z_{i+1}$ or $z_i\theta_2 z_{i+1}$. It follows from Corollary 3 that $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for all $i = 0, 1, \ldots, n-1$.

By using $a = a \vee (a \wedge b)$ and either $(a \vee z_{i+1})\theta_1(a \vee z_i)$ or $(a \vee z_{i+1})\theta_2(a \vee z_i)$ we have either $a \vee z_i \leq a \vee z_{i+1}$ or $a \vee z_{i+1} \leq a \vee z_i$. By a symmetric proof we obtain either $b \wedge z_i \leq b \wedge z_{i+1}$ or $b \wedge z_{i+1} \leq b \wedge z_i$ for all $i = 0, 1, \ldots, n-1$. Therefore, we have a sequence $a = c_0 = a \vee z_0$, $c_1 = a \vee z_1, \ldots, c_n = a \vee z_n = a \vee b = z_n$, $c_{n+1} = z_{n-1}, \ldots, c_{2n} = z_0 = a \wedge b = z_0 \wedge b, c_{2n+1} = z_1 \wedge b, \ldots, c_{3n} = z_n \vee b = b$ such that either $c_i \leq c_{i+1}$ or $c_{i+1} \leq c_i$ for all $i = 0, 1, \ldots, 3n$. Hence, $\leq$ is connected. □

The following corollaries follows directly from Corollary 4, Lemma 4 and Lemma 5.

Corollary 5. *The map $\leq\rightarrow(\theta_1,\theta_2)$ induced a bijection between the connected compatible orders of a lattice and the pairs of complementary congruence relations on the lattice.*

Corollary 6. *If L is a subdirectly irreducible lattice, then $\leq$ and $\geq$ are the only connected compatible order of L*

We say that an ordered set $\mathbf{P}$ satisfies the *lower bound property*(LBP) if any pairs of elements of $\mathbf{P}$ which have a lower bound have a greatest lower bound. Dually, $\mathbf{P}$ satisfies the *upper bound property*(UBP) if any pairs of elements of $\mathbf{P}$ which have an upper bound have a least upper bound.

We shall now show that a connected compatible order of a lattice satisfies the lower bound property and the upper bound property.

Lemma 6. *Let P be a connected ordered set having an amenable lattice order. Then P satified LBP and UBP.*

Proof. Let $\mathbf{P}=(P;\leq)$ be a connected ordered set and let $\mathbf{L}=(P;\wedge,\vee,\leq^*)$ be an amenable lattice order of $\mathbf{P}$. Let $\mu(x,y,z)$ denote a ternary function $(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$ in $\mathbf{L}$. Since $\leq$ is compatible with $\wedge$ and $\vee$, the function μ is monotone with respect to both $\leq$ and $\leq^*$. Let $a,b,u,l\in P$ with $a\leq u, b\leq u, l\leq a$ and $l\leq b$. It is easily seen that the upper bound $\mu(a,b,u)$ and the lower bound $\mu(a,b,l)$ of a and b are minimal and maximal respectively. By theorem 2, since the order relation of $\mathbf{L}_1^\partial\times\mathbf{L}_2$ is compatible with the operation of $\mathbf{L}_1\times\mathbf{L}_2$, a minimal upper bound $\mu(a,b,u)$ and a maximal lower bound $\mu(a,b,l)$ are unique. □

Let $\mathbf{C}=<C;\leq^*>$ be an infinite chain, $\mathrm{P}=C\cup\{a,b,c\}$ where $a,b,c\notin C$. Define an order relation $\leq$ on P as follow:

(i) $x\leq y\Leftrightarrow x\leq^* y$ for all $x,y\in C$,

(ii) $x\leq y$ for all $x\in C$ and $y\in\{a,b,c\}$, and

(iii) $a\leq c\geq b$

Then, $\mathbf{P}:=(P;\leq)$ is an example of ordered sets which does not satisfy the lower bound property and hence it has no amenable lattice order.

Let $\mathbf{P}=<P;\leq>$ be an ordered set and θ be an equivalence relation on P. Define a binary relation $\leq_\theta$ on $\frac{P}{\theta}$ by

$$[a]\theta\leq_\theta[b]\theta\ \Leftrightarrow\ a\theta c\leq d\theta b \text{ for some } c,d\in P.$$

Then $\leq_\theta$ need not be transitive. Let $\leq_\theta^t$ denote the transitive closure of $\leq_\theta$. It was proved in [2] that if $\leq$ is a compatible ordering of a lattice $\mathbf{L}$ and θ is also a congruence relation of $\mathbf{L}$ then $\leq_\theta^t$ is an order on $\frac{P}{\theta}$.

Lemma 7. *([2]) Let $L=(P;\wedge,\vee,\leq^*)$ be a lattice and let θ be a congruence relation of L. If $\leq$ is a compatible ordering of a lattice L, then $\leq_\theta^t$ is an order on $\frac{P}{\theta}$. Moreover, $\leq_\theta^t$ is a compatible ordering of $(\frac{P}{\theta};\leq^*)$.*

We shall now characterize all ordered sets which have an amenable lattice order.

Theorem 4. *Let $P=<P;\leq>$ be a connected ordered set, $\leq^*$ be an amenable lattice order of P and let θ_1 and θ_2 be defined as in (2.1). Then*

(i) $(\frac{P}{\theta_1};\leq_{\theta_1}^t)$ and $(\frac{P}{\theta_2};\leq_{\theta_2}^t)$ are lattices,

(ii) denote the join and meet on $(\frac{P}{\theta_1}; \leq^t_{\theta_1})$ *and on* $(\frac{P}{\theta_2}; \leq^t_{\theta_2})$ *by* $+, \cdot$ *and* $\cup, \cap$ *respectively; then for* $a, b \in P$ *there are unique* $c, d \in P$ *such that* $c \in [a]\theta_1 \cdot [b]\theta_1$, $c \in [a]\theta_2 \cup [b]\theta_2, d \in [a]\theta_1 + [b]\theta_1$ *and* $d \in [a]\theta_2 \cap [b]\theta_2$, *and*

(iii) if a and b are noncomparable, then $((a, c) \in \theta_1$ *implies* $(b, c) \notin \theta_2)$ *and* $((b, d) \in \theta_2$ *implies* $(a, d) \notin \theta_1)$.

Proof. Let $\vee$ and $\wedge$ denote the join and meet operations of the lattice $\mathbf{L} = (P; \leq^*)$. By the assumption and an application of Theorem 1, we have $\theta_1 \cap \theta_2 = \omega$, $\leq = (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$, and $(\frac{P}{\theta_1}; \wedge, \vee, \leq^*)$ and $(\frac{P}{\theta_2}; \wedge, \vee, \leq^*)$ are lattices. Hence the natural map $\psi : P \to \frac{P}{\theta_1} \times \frac{P}{\theta_2}$ is a lattice embedding of $\mathbf{L}$ into $\frac{\mathbf{P}}{\theta_1} \times \frac{\mathbf{P}}{\theta_2}$ and an order embedding of $\mathbf{P}$ into $(\frac{\mathbf{P}}{\theta_1})^\partial \times \frac{\mathbf{P}}{\theta_2}$ respectively.

(i) It remains to show that $\leq^t_{\theta_1}$ and $\leq^t_{\theta_2}$ are the restrictions of $\geq^*$ to $\frac{P}{\theta_1}$ and of $\leq^*$ to $\frac{P}{\theta_2}$ respectively. It is clear that $a \leq b$ implies $[a]\theta_1 \geq^* [b]\theta_1$ and $[a]\theta_2 \leq^* [b]\theta_2$. This shows that $\leq_{\theta_1}$ is a subset of $\geq^*$ restricted to $\frac{P}{\theta_1}$ and $\leq_{\theta_2}$ is a subset of $\leq^*$ restricted to $\frac{P}{\theta_2}$; so are $\leq^t_{\theta_1}$ and $\leq^t_{\theta_2}$.

Now, let $a, b \in P$ with $[a]\theta_1 \geq^* [b]\theta_1$. Then $a\theta_1(a \vee b)$ and $b\theta_1(a \wedge b)$. According to Lemma 3 and the remark, we have a sequence $a \wedge b = z_0 \leq^* z_1 \leq^* \cdots \leq^* z_n = a \vee b$ such that $z_0\theta_2 z_1 \theta_1 z_2 \ldots \theta_2 z_n$. It follows from Corollary 2 with $z_{2m}\theta_2 z_{2m+1}$ and $z_{2m} \leq^* z_{2m+1}$ for $0 \leq m < n$ that $z_{2m+1} \leq z_{2m}$ for all $0 \leq m < n$. Therefore $[b]\theta_1 = [z_0]\theta_1 \geq [z_1]\theta_1 \geq \cdots \geq [z_n]\theta_1 = [a]\theta_1$; that is, $[a]\theta_1 \leq^t_{\theta_1} [b]\theta_1$. Hence the restriction of $\geq^*$ to $\frac{P}{\theta_1}$ is a subset of $\leq^t_{\theta_1}$. Analogously, the restriction of $\leq^*$ to $\frac{P}{\theta_2}$ is a subset of $\leq^t_{\theta_2}$.

Denote the join and meet on the lattices $\frac{\mathbf{P}}{\theta_1} = (\frac{P}{\theta_1}; \leq^t_{\theta_1})$ and $\frac{\mathbf{P}}{\theta_2} = (\frac{P}{\theta_2}; \leq^t_{\theta_2})$ by $+, \cdot$ and $\cup, \cap$; respectively.

(ii) Since $\psi(a \vee b) = ([a]\theta_1 \vee [b]\theta_1, [a]\theta_2 \vee [b]\theta_2) = ([a]\theta_1 \cdot [b]\theta_1, [a]\theta_2 \cup [b]\theta_2)$ and $\psi(a \wedge b) = ([a]\theta_1 \wedge [b]\theta_1, [a]\theta_2 \wedge [b]\theta_2) = ([a]\theta_1 + [b]\theta_1, [a]\theta_2 \cap [b]\theta_2)$, we have $a \vee b$ and $a \wedge b$ corresponding to c and d in Condition(ii).

Condition (iii) is obvious from (ii) since $\leq = (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. □

We shall now use Theorem 4 to give an example of ordered sets having no amenable lattice orders. Let $\mathrm{P} = \{0, 1, 2, 3, 4, 5, 6\}$ and $\leq$ be an order on P defined by $0 \leq 1 \leq 2$, $0 \leq 4 \leq 3 \leq 2$, $5 \leq 4$, $5 \leq 6 \leq 3$. Suppose that $\mathbf{L} = (P; \leq^*)$ is an amenable lattice order of $\mathbf{P} = (P; \leq)$. According to Corollary 1 , we have that $(A = \{0, 1, 2, 3, 4\}; \leq^*)$ is a cover-preserving sublattice of $\mathbf{L}$ isomorphic to the subdirectly irreducible lattice $\mathbf{N}_5$; hence, Corollary 6 implies that $(A; \leq^*)$ is either $(A; \leq)$ or $(A; \leq)^\partial$.

Let θ_1 and θ_2 be defined as in (2.1) and denote the restriction of θ_1 and θ_2 to A by $\theta_1|A$ and $\theta_2|A$ respectively. Then one of $\theta_1|A$ or $\theta_2|A$ is the identity relation ω and the other is the universal relation $\iota = A \times A$. We may assume that $\theta_1|A = \omega$ and $\theta_2|A = \iota$. One can show by using Theorem 1 and Corollary 4 that $3\theta_1 6, 4\theta_1 5$ and $6\theta_2 5$. Hence, $(\frac{P}{\theta_1}, \leq^t_{\theta_1})$ is $\mathbf{N}_5$ and $(\frac{P}{\theta_2}, \leq^t_{\theta_2})$ is a 2-element chain. For $1, 5 \in P$, we have $[1]\theta_1 + [5]\theta_1 = [2]\theta_1 = \{2\}$ and $[1]\theta_2 \cap [5]\theta_2 = [5]\theta_2 = \{5, 6\}$ which have an empty intersection which contradicts to Condition(ii) of Theorem 4. If we assume that $\theta_1|A = \iota$ and $\theta_2|A = \omega$, then we get a similar contradiction. Hence, $\mathbf{P}$ has no amenable lattice order.

Theorem 5. *Let* $P = \langle P; \leq \rangle$ *be an ordered set and let* θ_1 *and* θ_1 *be equivalence relations on P satisfying Condition (i), (ii), and (iii) of Theorem 4. Then there is a lattice* L *such that* $P\#L$.

Proof. Define a binary relation $\leq^*$ on P as follow:

$$a \leq^* b \Leftrightarrow b \in [a]\theta_1 \cdot [b]\theta_1 \text{ and } b \in [a]\theta_2 \cup [b]\theta_2.$$

Let $a, b \in P$. Then there is an element $c \in P$ such that $[c]\theta_1 = [a]\theta_1 \cdot [b]\theta_1$ and $[c]\theta_2 = [a]\theta_2 \cup [b]\theta_2$. Hence, $[c]\theta_1 \leq^t_{\theta_1} [a]\theta_1$ and $[a]\theta_2 \leq^t_{\theta_2} [c]\theta_2$; which show that $[a]\theta_1 \cdot [c]\theta_1 = [c]\theta_1$ and $[a]\theta_2 \cup [c]\theta_2 = [c]\theta_2$; or equivalently, $c \in [a]\theta_1 \cdot [c]\theta_1$ and $c \in [a]\theta_2 \cup [c]\theta_2$. Thus $a \leq^* c$. By analogy, we have $b \leq^* c$. Now let $u \in P$ be such that $a \leq^* u$ and $b \leq^* u$. Then $[u]\theta_1 = [a]\theta_1 \cdot [b]\theta_1 \cdot [u]\theta_1 = [c]\theta_1 \cdot [u]\theta_1$ and $[u]\theta_2 = [a]\theta_2 \cup [b]\theta_2 \cup [u]\theta_2 = [c]\theta_2 \cup [u]\theta_2$; that is, $c \leq^* u$. Therefore, c is the least upper bound of a and b with respect to $\leq^*$; and we can prove analogously that every pair of elements in P has the greatest lower bound with respect to $\leq^*$. Hence, we have that $\leq^*$ is a lattice order on P. Let $\vee$ and $\wedge$ denote the join and meet operations of the lattice $\mathbf{L} = (P; \leq^*)$. To show that θ_1 and θ_2 are congruence relations of L, let $a, b, c \in P$ with $a\theta_1 b$. Then $a \vee c \in [a]\theta_1 \cdot [c]\theta_1 = [b]\theta_1 \cdot [c]\theta_1$ and $b \vee c \in [b]\theta_1 \cdot [c]\theta_1$ imply $(a \vee c)\theta_1(b \vee c)$. Analogously, we have $(a \wedge c)\theta_1(b \wedge c)$. A similar argument yields $(a \vee c)\theta_2(b \vee c)$ and $(a \wedge c)\theta_2(b \wedge c)$.

Since Condition(ii) implies $\theta_1 \cap \theta_2 = \omega$, the natural map $\psi : P \rightarrow \frac{P}{\theta_1} \times \frac{P}{\theta_2}$ is a lattice embedding of L into $\frac{\mathbf{L}}{\theta_1} \times \frac{\mathbf{L}}{\theta_2}$. Now $[a]\theta_1 \leq^t_{\theta_1} [b]\theta_1$ if and only if $[a \vee b]\theta_1 = [a]\theta_1 \cdot [b]\theta_1 = [a]\theta_1$ if and only if $[a]\theta_1 = [a \vee b]\theta_1 \geq^* [b]\theta_1$. Thus $(\frac{P}{\theta_1}; \leq^t_{\theta_1}) \cong (\frac{P}{\theta_1}; \geq^*) \cong (\frac{\mathbf{L}}{\theta_1})^\partial$. Similarly, $(\frac{P}{\theta_2}; \leq^t_{\theta_2}) \cong (\frac{P}{\theta_2}; \leq^*) \cong \frac{\mathbf{L}}{\theta_2}$. Finally, we will show that ψ is an order embedding of P into $(\frac{\mathbf{L}}{\theta_1})^\partial \times \frac{\mathbf{L}}{\theta_2}$; this is equivalent to prove that $\leq = (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. If $a \leq b$, then $[a]\theta_1 \leq^t_{\theta_1} [b]\theta_1$ and $[a]\theta_2 \leq^t_{\theta_2} [b]\theta_2$ imply that $[a]\theta_1 = [a \vee b]\theta_1$ and $[a \vee b]\theta_2 = [b]\theta_2$; that is, $a\theta_1(a \vee b)\theta_2 b$ which together with $a \leq^* a \vee b \geq^* b$ yields $(a, b) \in (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. Now let $(a, b) \in (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. Then $a\theta_1 u\theta_2 b$ and $a \leq^* u \geq^* b$ for some $u \in P$. Hence $[u]\theta_1 \geq^* [b]\theta_1$ and $[a]\theta_2 \leq^* [u]\theta_2$; or equivalently, $[u]\theta_1 \leq^t_{\theta_1} [b]\theta_1$ and $[a]\theta_2 \leq^t_{\theta_2} [u]\theta_2$. Thus $a \vee b \in [a]\theta_1 . [b]\theta_1 = [u]\theta_1 . [b]\theta_1 = [u]\theta_1$ and $a \vee b \in [a]\theta_2 \cup [b]\theta_2 = [a]\theta_2 \cup [u]\theta_2 = [u]\theta_2$; that is, $(a \vee b, u) \in \theta_1 \cap \theta_2$. So $a \vee b = u$. By Condition(iii), since $a\theta_1(a \vee b)\theta_2 b$, we have $a \leq b$ or $b \leq a$. But, $b \leq a$ implies $a = b$, we conclude that $a \leq b$.

It follows from Theorem 2 that $\leq^*$ is an amenble lattice order of P. □

References

[1] G. Birkhoff, (1967), *Lattice Theory*, 3^{rd} Ed., Amer. Math. Soc., Providence.

[2] G. Czédli, A. P. Huhn and L. Szabó(1980), *On compatible ordering of lattices*, Contributions to lattice theory (Szeged, 1980), Colloq. Math. Soc. J. Bolyai 33, North-Holland, Amsterdam-New York, (87-99).

[3] G. Grätzer(1978), *General Lattice Theory*, Akademic-Verlag Berlin.

[4] J. Jakubik and M. Kolibiar(1954), *On some properties of pairs of lattices(Russian)*, Czec. Math. J. 4, (1-27).

[5] M. Kolibiar(1983), *Compatible ordering in semilattices*, Contributions to genereal algebra 2, (Klagenfurt, 1982), Holder-Pichler-Tempsky, Vienna(215-220).

[6] M. Kolibiar(1982), *Interval, convex sublattices and subdirect representations of lattices*, Universal Algebra and Applications Banach Center Publ., 9 (Warsaw), (335-339).

[7] C. Ratanaprasert (1995), *Compatible Orders of Semilattices*, Tatra Mountains Math. Publ. 5 (177-187).

[8] I.G. Rosenberg and D. Schweigert (1984), *Compatible ordering and tolerances of lattices*, Orders description and roles (L' Arresle. 1982), North-Holland Math. Stud. 99, North-Holland, Amsterdam-New York, (119-150).

[9] L. Szabó(1985), *Characterization of compatible quasiorderings of lattice ordered algebras*, preprint. Bolyai Inst. Szeged.

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE, SILPAKORN UNIVERSITY THAILAND
E-mail address: kanate@ratach.ac.th

ภาคผนวก (ง)

GRAPH ISOMORPHISM OF ORDERED SETS

C. RATANAPRASERT

ABSTRACT. G. Birkhoff([1], problem 8) proposed the question of finding necessary and sufficient conditions on a lattice **L**, in order that every lattice **M** whose unoriented graph is isomophic to the graph of **L** be lattice isomorphic to **L**. J. Jakubík solved the problem in the class of modular lattices; he has also shown in [3] that if **L** and **M** are graphically isomorphic and all sublattices of certain types of **L** and **M** are preserved or reversed under the graph isomorphism then the condition which is equivalent to **L** and **M** being compatible to each other holds. In the paper, we show that all connected compatible orders of a lattice have graphs isomorphic to the graph of the lattice; and then describe all compatible orders of a lattice in term of subgraphs of the lattice. It turns out that consideration of the types of sublattices of a lattice **L** which are mentioned in [3] and [4] leads to necessary and sufficient conditions for all connected ordered sets whose graphs are isomorphic to **L** to be compatible order of **L**. The results shown in [3] and [4] become a special case when those compatible orders are compatible lattice orders.

An ordered set is called *discrete* if all its bounded chains are finite. All ordered sets which are dealt with in this paper are assumed to be discrte.

Let $\mathbf{P} := < P; \leq >$ be an ordered set. For $a, b \in P$ with $a \leq b$, the interval $[a, b]$ is the set $\{x \in P | a \leq x \leq b\}$; for the case when $[a, b] = \{a, b\}$ and $a \neq b$ we will write $a \prec b$ or $b \succ a$ and say, a is covered b or b covers a, respectively.

A subset X of an ordered set $\mathbf{P} = < P; \leq >$ is called a *c-subset* if, whenever $a, b \in X$ and a covers b in $(X; \leq)$ then a covers b in **P**. The definition of *c-sublattice* is analogous. Let $u, v, x_1, \ldots, x_m, y_1, \ldots, y_n$ be distinct elements in **P** such that

(i) $u \prec x_1 \prec \cdots \prec x_m \prec v, u \prec y_1 \prec \cdots \prec y_n \prec v$, and

(ii) either v is the least upper bound of x_1 and y_1 (denoted by $v = x_1 \vee y_1$) or u is the greatest lower bound of x_m and y_n (denoted by $v = x_m \wedge y_n$).

Then the set $C = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is said to be a *cell* of **P**. If $x_1 \vee y_1 = v$, we call C a *cell of type* $\vee(m, n)$. Dually, if $x_m \wedge y_m = u$, we call C a *cell of type* $\wedge(m, n)$. If $x_1 \vee y_1 = v$ and $x_m \wedge y_n = u$, we call C a *cell of type* $\Diamond(m, n)$. A cell C is called *proper* if $m > 1$ or $n > 1$.

By the graph G(**P**), we mean the (undirected) graph whose vertex set is P and whose edges are those pairs $\{a, b\}$ which satify either $a \prec b$ or $b \prec a$. Let **P** and **Q** be ordered sets. It is said that G(**P**) is isomorphic to G(**Q**) if there is a bijection $\psi : P \to Q$ such that for all $a, b \in P$, $\{a, b\}$ is an edge of G(**P**) if and only if $\{\psi(a), \psi(b)\}$ is an edge of G(**Q**). Without loss of generality, throughout this paper we may assume that P = Q and that ψ is the identity map if G(**P**) is isomorphic to G(**Q**), whence G(**P**) = G(**Q**); in this case, ψ is called a graph isomorphism of **P** onto **Q**.

Research support by the Thailand Research Fund.

Let ψ be a graph isomorphism of P onto Q and let $X \subseteq P$. We say that X is *preserved (reversed)* under ψ if , whenever $x, y \in X$ and $x \prec y$, then $\psi(x) \prec \psi(y)$ (or $\psi(x) \succ \psi(y)$, respectively).

In [3], Jakubík has shown that if L and M are lattices with G(L) = G(M) and all proper cell of L and M are preserved or reversed then the following Condition (a) holds.

(a) There are lattices L_1 and L_2 and a direct product representation via which L is isomorphic to $L_1 \times L_2$ and M is isomorphic to $L_1^\partial \times L_2$.

Jakubík proved in [4] that for discrete lattices (with no assumption of modularlity), Condition (a) is equivalent to Condition (b).

(b) L and M have isomorphic graphs and all proper cells of M are either preserved or reversed.

In [6], Kolibiar proved that for discrete semimodular semilattices S and S_1 on the same underlying set S, the graphs G(S) and G(S_1) are isomorphic if and only if the following Conditions (c) holds.

(c) there exist a lattice $A = (A; +, .)$, a semilattice $B = (B; \vee)$ and a map $\psi : S \rightarrow A \times B$ via which ψ is a subdirect embedding of S into $A \times B$ and S_1 into $A^\partial \times B$.

In [8], we gave a new characterization of Condition (c) by proving that Condition (c) holds if and only if $G(S) = G(S_1)$ and the graph isomophism preserves the order on some special types of cells and proper cells.

An order $\leq_1$ is said to be a *compatible order* of a (semi)lattice $L = < L; \leq >$ if $\leq_1$ is a sub(semi)lattice of L^2. If a compatible order $\leq_1$ of a (semi)lattice L is also a (semi)lattice order, we call $\leq_1$ a compatible (semi)lattice order of L.

In [9], we characterized all compatible orders of a lattice. In this paper, we will show that all connected compatible orders of a lattice L have graph isomorphic to G(L); and then, we describe all compatible orders of a lattice in terms of subgraphs of the lattice. It turns out that consideration of the types of sublattices of a lattice which are mentioned in [3] and [4] leads to necessary and sufficient conditions for all connected ordered sets whose graphs are isomophic to G(L) to be compatible orders of L. The results shown in [3] and [4] become a special case when those orders are compatible lattice orders.

A 4-element subset $\{a, b, c, d\}$ of P is said to be a *quadrilateral* if $a \prec b \prec d$ and $a \prec c \prec d$; and it is called a *crisscross* if $a, b \prec c, d$. We will denote these by $< a, b, c, d >$ and $< ab; cd >$ respectively. If G(P) = G(Q), then a quadrilateral of P can either be preserved, be reversed, be rotated through 90°, or be bent into a crisscross in Q. We have the following lemma.

Lemma 1. *Let P and Q be ordered sets with G(P) = G(Q) and let $< a, b, c, d >$ be a quadrilateral of P. If Q contains no crisscross, then the set $\{a, b\}$ is preserved(reverse) if and only if the set $\{c, d\}$ is preserved(reversed).*

Corollary 1. *([3], [4], [5]) Let P and Q be lattices with G(P) = G(Q). If $< a, b, c, d >$ is a quadrilateral in P, then the set $\{a, b\}$ is preserved(reversed) if and only if the set $\{c, d\}$ is preserved(reversed).*

Corollary 2. *Let P and Q be ordered sets with G(P) = G(Q). If Q contains no crisscross, then every c-subset of P which is isomorphic to M_n (the ordered set shown in Figure 1) is preserved or reversed in Q.*

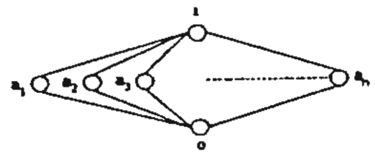

Figure 1.

Proof. It is enough to prove that the subset $\{0, 1, a_1, a_2, a_3\}$ of Figure 1 is preserved or reversed. We may assume that $\{0, a_1\}$ is preserved. It follows from Lemma 1 that $\{a_2, 1\}$ and $\{0, a_3\}$ are preserved since $< 0, a_1, a_2, 1 >$ and $< 0, a_2, a_3, 1 >$ are quadrilaterals. Now consider the quadrilateral $< 0, a_1, a_3, 1 >$. The preservation of $\{0, a_1\}$ and $\{0, a_3\}$ implies the preservation of $\{a_1, 1\}$ and $\{a_3, 1\}$. Hence, in the quadrilateral $< 0, a_1, a_2, 1 >$, the preservation of $\{a_1, 1\}$ implies the preservation of $\{0, a_2\}$. So $\{0, 1, a_1, a_2, a_3\}$ is preserved. □

An ordered set **P** is said to be *upper semimodular* if **P** satisfies the following *Upper Covering Condition(UCC)*:

(UCC) : If a *and* b *cover* c *with* $a \neq b$ *and a least upper bound of* a *and* b *(denoted by* $a \vee b$*) exists in* **P**, *then both* a *and* b *are covered by* $a \vee b$.

Dually, **P** is said to be *lower semimodular* if **P** satisfies the dual of UCC which is called the *Lower Covering Condition(LCC)*. If **P** satisfies both UCC and LCC, then **P** is said to be *modular*.

We proved in [9] that for a connected compatible order $\leq$ of a lattice $\mathbf{L} = < L; \leq^* >$ there corresponds a pair (θ_1, θ_2) of complementary congruence relations of **L**. Thus, if $a \leq^* b$ in **L**, then there are elements $a = a_1 \leq^* a_2 \leq^* \cdots \leq^* a_n = b$ in **L** such that either $a_i \theta_1 a_{i+1}$ or $a_i \theta_2 a_{i+1}$ for all $0 \leq i < n$. Hence, if $a \prec^* b$, then $a\theta_1 b$ or $a\theta_2 b$ which together with Corollary 3 and Lemma 1 in [9] yield $[a, b]^* = [a, b]$ or $[a, b]^* = [b, a]$; thus $a \prec b$ or $b \prec a$. We have the following Condition (A):

(A) G(P) = G(L).

Although Condition (A) is necessary, it is not sufficient for **P** to be a compatible ordered set of **L** even when **P** itself is a lattice.

Let $\mathbf{C} = \{u \prec^* x \prec^* v \succ^* y_n \succ^* \cdots \succ^* y_1 \succ^* u\}$ be a cell of **L** of type $\Diamond(1, n)\}$ and let $x \geq u \geq y_1$. Then $v \leq x$ and $y_1 \leq v$, so $[y_1, v]^* = [y_1, v]$; that is, $y_1 \leq y_2 \leq \cdots \leq y_n$. Since $v \leq x$ implies $y_n \leq u$, we have $[u, y_n]^* = [y_n, u]$; that is, $y_n \leq y_{n-1} \leq \cdots \leq y_1$. This shows that $y_1 = y_2 = \cdots = y_n$ which contradicts $n > 1$. We shall get a similar contradiction if $x \leq u \leq y_1$. This means that a cell of **L** of type $\Diamond(1, n)$ cannot be "bent" in **P**. That is,

(B) all proper cells of L are preserved or reversed in P.

In [9] , we proved that if **P** is a compatible ordered set of a lattice **L**, then **P** satisfied both LBP and UBP; and hence, the following Condition (C) holds:

(C) P contains no crisscross as a c-subset.

We shall now prove that Conditions (A), (B) and (C) altogether are equivalent to the following Condition (D):

(D) P is a connected compatible ordered set of L.

And for a pair of discrete lattices **L** and $\mathbf{L}^*$, Condition (B) is equivalent to the following Condition (B'):

(B') *all proper cells of L and all proper cells of* L^* *are preserved or reversed.*

Thus, we answer a question raised by Jakubík.

Lemma 2. *Let $P = (P; \leq)$ be a connected compatible order of a lattice $L = (P; \leq^*)$. Then $G(P) = G(L)$ and all proper cells of P and all proper cells of L are preserved or reversed.*

Proof. By [9], P satisfies LBP and UBP. Let $a \wedge b$ and $a \vee b$ denote the greatest lower bound and the least upper bound of any a and b in P if they are bounded below or bounded above, respectively.

Let $C = \{u \prec x_1 \prec \dots x_m \prec v \succ y_n \succ \dots \succ y_1 \succ u\}$ be a proper cell of P; that is, ($m > 1$ or $n > 1$) and ($x_1 \vee y_1 = v$ or $x_m \wedge y_n = u$). We may assume that $x_1 \vee y_1 = v$ (if $x_m \wedge y_n = u$ we can argue analogously). Let $w = x_m \wedge y_n$. Since $u \leq x_1 \wedge w \leq x_1, u \prec x_1$ and $x_1 \vee y_1 = v \neq y_n$, we have $x_1 \wedge w = u$. Similarly, $y_1 \wedge w = u$. Hence, $A = (A = \{u, v, x_1, x_m, y_1, y_n, w\}; \vee, \wedge, \leq)$ is a lattice and $\leq^*$ is a compatible order of A.

Suppose $x_1 \geq^* u \geq^* y_1$. Then $y_1 \leq^* v \leq^* x_1$. Since P is a compatible order of L, we have $[x_1, v] = [v, x_1]^*$ and $[y_1, v] = [y_1, v]^*$; hence, $y_1 \leq^* y_n \leq^* v$ and $v \leq^* x_m \leq^* x_1$. Since $\leq^*$ is a compatible order of A, we have ($x_m \leq^* x_1$ implies $w = x_m \wedge w \leq^* x_1 \wedge w = u$) and ($y_1 \leq^* y_n$ implies $u = w \wedge y_1 \leq^* w \wedge y_n = w$), which yield w = u. Now, ($v \leq^* x_m$ implies $y_n \leq^* u$) yields $[u, y_n] = [y_n, u]^*$; that is , $y_1 = y_2 = \dots = y_n$. Similary, we have $x_1 = x_2 = \dots = x_m$. Thus, m = 1 and n =1 which is a contradiction. We will get a similar contradiction if $x_1 \leq^* u \leq^* y_1$. Hence, $x_1 \geq^* u \leq^* y_1$ or $x_1 \leq^* u \geq^* y_1$. In either cases, C is preserved or reversed.

We can prove that all proper cell of L are preserved or reversed analogously. □

In the following 3 lemmata, we assume that an ordered set P satisfies Condition(C).

Lemma 3. *Let $L = (P; \vee, \wedge, \leq^*)$ be a discrete lattice and $P = (P; \leq)$ be a discrete connected ordered set with $G(P) = G(L)$. Assume that all proper cells of L are preserved or reversed in P.*

(i) If $a \succ^* c \prec^* b$, then ($c \prec a$ implies $b \leq a \vee b$) and ($a \prec c$ implies $a \vee b \leq b$), and

(ii) If $a \prec^* c \succ^* b$, then ($c \prec a$ implies $b \leq a \wedge b$) and ($a \prec c$ implies $a \wedge b \leq b$).

Proof. (i) If $a \prec^* a \vee b \succ^* b$, then the lemma follows by Lemma 1. We may assume that $a = x_1 \prec^* x_2 \prec^* \dots \prec^* x_m \prec^* a \vee b$ and $b = y_1 \prec^* y_2 \prec^* \dots \prec^* y_m \prec^* a \vee b$ for some $x_2, \dots, x_m, y_2, \dots, y_n \in P$. Then the set $\{c, a \vee b, x_1, \dots, x_m, y_1, \dots, y_n\}$ is a proper cell of L. Hence, the interval $[b, a \vee b]$ is preserved(reversed) if the interval [c, a] is preserved(reversed).

We can prove(ii) analogously. □

Lemma 4. *Let $L = (P; \vee, \wedge, \leq^*)$ be a discrete lattice and $P = (P; \leq)$ be a discrete connected ordered set with $G(P) = G(L)$. Assume that all proper cells of L are preserved or reversed in P. If $a \prec^* b$, then for all $c \in P$*

(i) $a \prec b$ implies $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$ and

(ii) $b \prec a$ implies $b \vee c \leq a \vee c$ and $b \wedge c \leq a \wedge c$.

Moreover, if $a, b, c \in P$ then $a \leq b$ implies $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$.

Proof. (i) and (ii) follow directly from Lemma 5. Let $a \leq b$ and $c \in P$. We may assume $a = a_0 \leq a_1 \leq \dots \leq a_n = b$ for some $a_1, a_2, \dots, a_{n-1} \in P$. Since G(P) = G(L), we have $a_i \prec^* a_{i+1}$ or $a_{i+1} \prec^* a_i$ for all $0 \leq i < n$. It follows from (i) and (ii) that $a_i \vee c \leq a_{i+1} \vee c$ and $a_i \wedge c \leq a_{i+1} \wedge c$ for all $0 \leq i < n$. Hence, by induction, $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$. □

Theorem 1. *Let L be a discrete lattice and P be a discrete connected ordered set having no crisscross as a c-subset. Then the followings are equivalent:*

(i) P is a compatible ordered set of L

(ii) $G(P) = G(L)$ and all proper cells of L and all proper cell of P are preserved or reversed.

(iii) $G(P) = G(L)$ and all proper cells of L are preserved or reversed in P.

In [9] we proved that if **P** is also a lattice then Condition(i) of Theorem 1 is equivalent to Condition (a). We obtain the following corollary which answer in the afffirmative a question posed by Jakubík[3].

Corollary 3. *Let L and L_1 be discrete lattices. Then the followings are equivalent:*

(i) $G(L) = G(L_1)$ and all proper cells of L and all proper cells of L_1 are preserved or reversed.

(ii) $G(L) = G(L_1)$ and all proper cells of L are preserved or reversed in L_1.

If **P** is a compatible ordered set of both lattices **L** and $\mathbf{L}_1$, then G(L) = G(P) = G($\mathbf{L}_1$) and all proper cells of **L** are preserved or reversed in **P**; hence, by the equivalence of conditions(i) and (ii) of Theorem 1, they are preserved or reversed in $\mathbf{L}_1$. Therefore, **L** is a compatible lattice order of $\mathbf{L}_1$ and the converse also holds (see[7]).

Theorem 2. *Let L and L_1 be discrete lattices and P be a discrete connected ordered set. If P is a compatible order of L, then P is a compatible order of L_1 if and only if L is a compatible lattice order of L_1.*

Let **L** be discrete modular lattice; then **L** contains no proper cells. Hence, if **P** is a discrete connected ordered set having the same graph as **L**, then Conditions(iii) of Theorem 1 holds. We obtain the following corollaries.

Corollary 4. *Let L be a discrete modular lattice and P be a discrete connected ordered set satisfying Condition(C). Then $G(P) = G(L)$ if and only if P is a compatible ordered set of L.*

Corollary 5. *Let L be a discrete lattice and $P = (P; \leq)$ be a discrete connected ordered set having the same graph as L and satisfying Condition(C). If L is modular, then so is P.*

Proof. It follows from [9] that **P** is a compatible ordered set of **L**. Suppose that **P** is not modular. Then **P** fails either UCC or DCC; that is , there exist a, b, c with $a \neq b$ such that either

(i) $a \prec c \succ b$ but $a \not\succ a \wedge b$ or $b \not\succ a \wedge b$, or

(ii) $a \succ c \prec b$ but $a \not\prec a \vee b$ or $b \not\prec a \vee b$,

where $a \wedge b$ and $a \vee b$ denote the greatest lower bound and the least upper bound of a and b in **P** respectively. Hence, **P** contains a proper cell $C = \{a \wedge b \prec x_1 \prec \cdots \prec x_m = a \prec c \succ b = y_n \succ \ldots y_1 \succ a \wedge b\}$ or $D = \{c \prec a = x_1 \prec \cdots \prec x_m \prec a \vee b \succ y_n \succ \cdots y_1 = b \succ c\}$ for some $x_1, \ldots, x_m, y_1, \ldots, y_n \in P$. So C, C^{∂}, D or D^{∂} is a proper cell in **L** which is a contradiction. □

Corollary 6. *([3]) Let L and L_1 be discrete lattices whose graphs are isomorphic. If L is modular (distributive), then so is L_1.*

As Jakubík has observed in [3], the modularity condition in Corollary 4 cannot be replaced by semimodularity. In fact, we have examples(see Figure 2 (a) and (b)) of semimodular lattices whose graphs are isomorphic; but one is not a compatible order of the other.

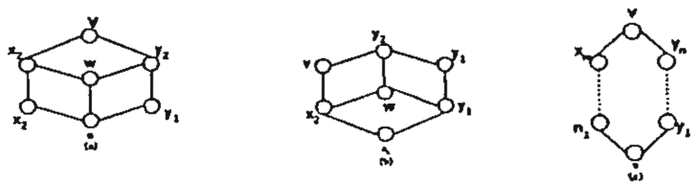

Figure 2

In [4], Jakubík has shown that a discrete lattice **L** is modular if and only if **L** does not contain a c-sublattice isomorphic to one of the lattices in Figure 2. In fact, all c-sublattices of a lattice **L** which are isomophic to one of the lattice in Figure 2 are proper cells of **L**. It is interesting to ask whether for a discrete lattice **L** and a discrete connected ordered set **P** have the same graphs and the isomorphism preserves the order on all c-sublattices of **L** which are isomorphic to one of the lattices in Figure 2. Unfortunately, Figure 3 and Figure 4 show that this is not the story.

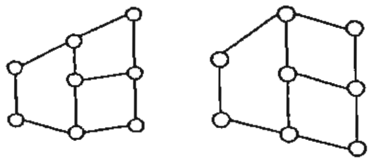

Figure 3

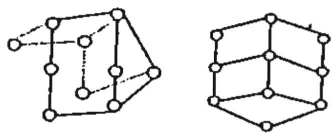

Figure 4

REFERENCES

[1] G. Birkhoff, (1967), *Lattice Theory*, 3rd Ed., Amer. Math. Soc., Providence.

[2] G. Czedli, A. P. Huhn and L. Szabo(1980), On compatible ordering of lattices, *Contributions to lattice theory* (Szeged, 1980), Colloq. Math. Soc. J Bolyai 33, North-Holland, Amsterdam-New York, (87-99).

[3] J. Jakubík(1985), Graph Isomorphisms of semimodular lattices, *Math. Slovaca 35*, (229-232).

[4] , J. Jakubík(1985), On isomorphism of graphs of lattice, *Czech. Math. J. 35*, (188-200).

[5] J. Jakubík(1954), On graph isomorphism of semimodular lattices(Slovak), *Math-Fyz. Casopis. Sloven. Akad. 4*, (162-177).

[6] M. Kolibiar(1983), Compatible ordering in semilattices, *Contributions to genereal algebra 2*, (Klagenfurt, 1982), Holder-Pichler-Tempsky, Vienna(215-220).

[7] C. Ratanaprasert and B. A. Davey(1987), Semimodular Lattices with Isomorphic Graphs, *Order 4*, by D. Reidel Publishing Company(1-13).

[8] C. Ratanaprasert (1995), Compatible Orders of Semilattices, *Tatra Mountains Math. Publ. 5* (177-187).

[9] C. Ratanaprasert, All Ordered Set having Amenable Lattice Orders, *East-West Journal of Mathematics*, Thailand. (Submitted)

[10] I.G. Rosenberg and D. Schweigert (1984), Compatible ordering and tolerances of lattices, *Orders description and roles*(L' Arresle. 1982), North-Holland Math. Stud. 99, North-Holland, Amsterdam-New York, (119-150).

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE, SILPAKORN UNIVERSITY THAILAND
E-mail address: kanate@ratach.edu.th