รายงานการวิจัยเรื่อง

กราฟและกราฟทิศทาง ที่สอดคล้องกับคุณสมบัติที่กำหนด

วัชรพงษ์ อนันต์ชื่น

สาขาวิชาศิลปศาสตร์
มหาวิทยาลัยสุโขทัยธรรมาธิราช
นนทบุรี

พ.ศ. 2542

Research Report

On graphs and digraphs with prescribed properties

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School of Liberal Arts
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Nonthaburi

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กิตติกรรมประกาศ

บางส่วนของงานวิจัยชิ้นนี้ได้ทำขณะที่ผู้วิจัยไปศึกษาค้นคว้าทางวิชาการ ณ. Department of Electrical and Electronics Engineering, Ehime University, Matsuyama, Japan โดยการสนับสนุนของ The Matsumae International Foundation, Tokyo, Japan

ผู้วิจัยขอขอบพระคุณ Prof. Dr. Louis Caccetta, School of Mathematics and Statistics, Curtin University of Technology, Perth, Western Australia ที่ได้อนุเคราะห์ให้ข้อเสนอแนะและความคิดเห็นที่เป็นประโยชน์อย่างยิ่งต่อการพัฒนาและปรับปรุงงานวิจัยชิ้นนี้

งานวิจัยครั้งนี้สำเร็จลุล่วงได้ด้วยดี โดยการสนับสนุนของ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ภายใต้โครงการทุนวิจัยองค์ความรู้ใหม่ที่เป็นพื้นฐานต่อการพัฒนา รหัส BRG/07/2541 ผู้วิจัยขอขอบพระคุณเป็นอย่างสูงมา ณ. โอกาสนี้

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ชื่อโครงการวิจัย กราฟและกราฟทิศทางที่สอดคล้องกับคุณสมบัติที่กำหนด

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ให้ m และ n เป็นจำนวนเต็มบวกหรือศูนย์ และ k เป็นจำนวนเต็มบวกใด ๆ เรากล่าวว่ากราฟ G มีคุณสมบัติ P(m,n,k) ก็ต่อเมื่อ สำหรับทุก ๆ เซต A และ B ที่เป็นเซตต่างสมาชิกกันของจุดของ G โดยที่ | A | = m และ | B | = n จะมีอีกอย่างน้อย k จุด ซึ่งแต่ละจุดต่างประชิดกับจุดทุกจุดใน A แต่ไม่ประชิดกับจุดใด ๆ ใน B เลย สำหรับกรณี m, n $\geq$ 2 ปัญหาการสร้างกลุ่มของกราฟที่มีคุณสมบัติ P(m,n,k) เป็นปัญหาที่ค่อนข้างยาก นับจนถึงปัจจุบัน กลุ่มของกราฟที่มีคุณสมบัติดังกล่าวที่เรารู้จักมีเพียงกลุ่มเดียวคือกลุ่มของกราฟพาเลย์ G_q ซึ่งนิยามดังนี้ ให้ q เป็นจำนวนเฉพาะกำลังซึ่ง $q \equiv 1 \pmod 4$ จุดของกราฟ G_q คือสมาชิกของสนามจำกัด F_q จุด a และ b ใด ๆ ประชิดกันก็ต่อเมื่อ ผลต่างของ a และ b เป็นส่วนตกค้างกำลังสอง โดยการใช้ส่วนตกค้างกำลังสูงกว่า เราสามารถสร้างกลุ่มของกราฟกลุ่มใหม่ ซึ่งจะเรียกว่าการวางนัยทั่วไปของกราฟพาเลย์ ในงานวิจัยครั้งนี้เราแสดงว่า สำหรับทุก ๆ จำนวนเต็ม m, n และ k กราฟที่สร้างโดยการใช้ส่วนตกค้างกำลังสามและกำลังสี่ ที่มีจำนวนจุดมากพอ มีคุณสมบัติ P(m,n,k)

กราฟทิศทาง D มีคุณสมบัติ Q(n,k) ก็ต่อเมื่อ สำหรับเซต A ใด ๆ ของจุดของ D ที่มีจำนวน n จุด จะมีอีกอย่างน้อย k จุด ซึ่งแต่ละจุดต่างครอบครองจุดทุกจุดใน A สำหรับ $q \equiv 5 \pmod 8$ ที่เป็นจำนวนเฉพาะกำลังจะนิยามกราฟทิศทางพาเลย์กำลังสี่ $D_q^{(4)}$ ดังนี้จุดของกราฟทิศทาง $D_q^{(4)}$ คือสมาชิกของสนามจำกัด F_q จุด u ครอบครองจุด v ก็ต่อเมื่อ ผลต่างของ u และ v เป็นส่วนตกค้างกำลังสี่ ในรายงานฉบับนี้เราแสดงว่าสำหรับ q ที่ใหญ่มากพอ $D_q^{(4)}$ มีคุณสมบัติ Q(n,k)

Abstract

Research title	On graphs and digraphs with prescribed properties.
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Let m and n be non-negative integers and k a positive integer. A graph G is said to have property P(m,n,k) if for any m + n distinct vertices of G there are at least k other vertices, each of which is adjacent to the first m vertices but not adjacent to any of the latter n vertices. We know that almost all graphs have property P(m,n,k). However, for the case m, n $\geq$ 2, almost no graphs have been constructed, with the only known examples being Paley graphs which defined as follows. For q $\equiv$ 1 (mod 4) a prime power, the Paley graph G_q of order q is the graph whose vertices are elements of the finite field F_q; two vertices a and b are adjacent if and only if their difference is a quadratic residue. By using higher order residues on finite fields we can generate other classes of graphs which we refer to as generalized Paley graphs. For any m, n and k, we show that all sufficiently large (order) graphs obtained by taking cubic and quadruple residues satisfy property P(m,n,k).

A digraph D is said to has property Q(n,k) if for every subset of n vertices of D is dominated by at least k other vertices. Let q $\equiv$ 5(mod 8) be a prime power. Define a quadruple Paley digraph $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite field F_q. Vertex u joins to vertex v by an arc if and only if $u - v = x^4$ for some $x \in F_q$. In this report, we show for sufficiently large q, $D_q^{(4)}$ has property Q(n,k).

Contents

1. Introduction

A **graph** G consists of a non-empty set of elements, called **vertices**, and a list of unordered pair of these elements, called **edges**. The set of vertices of the graph G is called **vertex set** of G, and the list of edges is called **edge set** of G. If a and b are vertices of a graph G, then an edge of the form ab or ba is said to **join** a and b. We also say that a and b are **adjacent**. A **loop** is an edge of a graph joining a vertex to itself. Two or more edges joining the same pair of vertices are called **multiple edges**. All graphs considered in this paper are finite, loopless and have no multiple edges. For the most part, our notation and terminology follows that of Bondy and Murty [12]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges.

Let m and n be non–negative integers and k a positive integer. A graph G is said to have property P(m,n,k) if for any disjoint sets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property P(m,n,k) is denoted by $\mathcal{G}(m,n,k)$. The cycle C_ν of length ν is a member of $\mathcal{G}(1,1,1)$ for every $\nu \geq 5$. The well–known Petersen graph is a member of $\mathcal{G}(1,2,1)$ and also of $\mathcal{G}(1,1,2)$. The class $\mathcal{G}(m,n,k)$ has been studied by Ananchuen and Caccetta [2, 3, 4, 7, 8], Blass et. al. [9], Blass and Harary [10], Exoo [15], Exoo and Harary [16, 17]. In addition, some variations of the above adjacency property have been studied by Alspach et. al. [1], Ananchuen and Caccetta [6], Bollobás [11], Caccetta et. al. [13, 14] and Heinrich [19].

In 1979, Blass and Harary [10] established, using probabilistic methods, that almost all graphs have property P(n,n,1). From this it is not too difficult to show that almost all graphs have property P(m,n,k). Despite this result, few graphs have been constructed which exhibit the property P(m,n,k); some constructions for the class $\mathcal{G}(1,n,k)$ were given in [7].

An important graph in the study of the class $\mathcal{G}(m,n,k)$ is the so–called **Paley graph** G_q defined as follows. Let $q \equiv 1 \pmod 4$ be a prime power. The vertices of G_q are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbf{F}_q$.

In [4, 8] we proved that for a prime power $q \equiv 1 \pmod 4$:

$G_q \in \mathcal{G}(1,n,k)$ for every $q > \{(n-2)2^n + 2\}\sqrt{q} + (n+2k-1)2^n - 2n - 1$;

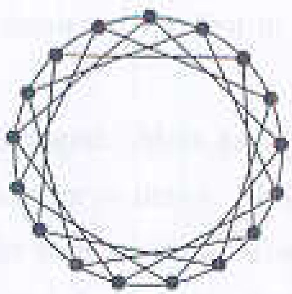
$G_q \in \mathcal{G}(n,n,k)$ for every $q > \{(2n-3)2^{2n-1} + 2\}\sqrt{q} + (n+2k-1)2^{2n-1} - 2n^2 - 1$;

and $G_q \in \mathcal{G}(m,n,k)$ for every $q > \{(t-3)2^{t-1} + 2\}\sqrt{q} + (t+2k-1)2^{t-1} - 1$, where $t \geq m + n$.

By using higher order residues on finite fields we can generate other classes of graphs. More specifically, for $q \equiv 1 \pmod 3$ a prime power we define the **cubic Paley graph** $G_q^{(3)}$ as follows. The vertices of $G_q^{(3)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^3$ for some $y \in \mathbf{F}_q$. Since $q \equiv 1 \pmod 3$ is a prime power, -1 is a cubic in $\mathbf{F}_q$. The condition -1 is a cubic in $\mathbf{F}_q$ is needed to ensure that ab is defined to be an edge when ba is defined to be an edge. Consequently, $G_q^{(3)}$ is well-defined. Figure 1(a) gives an example.

For $q \equiv 1 \pmod 8$ a prime power, define the **quadruple Paley graph** $G_q^{(4)}$ as follows. The vertices of $G_q^{(4)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^4$ for some $y \in \mathbf{F}_q$. Since $q \equiv 1 \pmod 8$ is a prime power, -1 is a quadruple in $\mathbf{F}_q$. The condition -1 is a quadruple in $\mathbf{F}_q$ is needed to ensure that ab is defined to be an edge when ba is defined to be an edge. Figure 1(b) gives an example.

(a) $G_{13}^{(3)}$



(b) $G_{17}^{(4)}$

Figure 1. Graphs $G_{13}^{(3)}$ and $G_{17}^{(4)}$.

In this report the adjacency properties of the classes $G_q^{(3)}$ and $G_q^{(4)}$ are studied. More specifically, we prove that:

- $G_q^{(3)} \in \mathcal{G}(2,2,k)$ for every $q > [\frac{1}{4}(79 + 3\sqrt{36k + 701})]^2$;
- $G_q^{(3)} \in \mathcal{G}(m,n,k)$ for every $q > (t2^{t-1} - 2^t + 1)2^m\sqrt{q} + (m + 2n + 3k - 3)2^{-n}3^{t-1}$, where $t \geq m + n$; and
- $G_q^{(4)} \in \mathcal{G}(m,n,k)$ for every $q > (t2^{t-1} - 2^t + 1)3^m\sqrt{q} + (m + 3n + 4k - 4)3^{-n}4^{t-1}$, where $t \geq m + n$.

If we think of the edge between two vertices as an ordered pair, a natural direction from the first vertex to the second vertex can be associated with the edge. Such an edge will be called an **arc**, and a graph which each edge has such a direction will be called a **directed graph or digraph**. A digraph is **strict** if it has no (directed) loops and no two arcs with the same ends have the same orientation. A **tournament** is a digraph with no (directed) loops in which any two distinct vertices are joined by exactly one arc. For our purpose, all digraphs are finite and strict.

If there exists an arc directed away from a vertex u to a vertex v, we say that u **dominates** v and that v is dominated by u. A set of vertices U dominates a set of vertices V if every vertex of U dominates every vertex of V. A digraph D is said to have property Q(n,k) if every subset of n vertices of D is dominated by at least k other vertices. Further, a digraph D is said to have property Q(m,n,k) if for any set of m + n distinct vertices of D there exist at least k other vertices each of which dominates the first m vertices and is dominated by the latter n vertices.

A special digraph arises in round robin tournaments. More precisely, consider a tournament T_q with q players 1, 2, ..., q in which there are no draws. This gives rise to a digraphs in which either (u, v) or (v, u) is an arc for each pair u, v. Tournaments with property Q(n, k) have been studied by Ananchuen and Caccetta [5] Bollobás [11] and Graham and Spencer [18].

Graham and Spencer [18] defined the following tournament. Let p ≡ 3(mod 4) be a prime. The vertices of digraph D_p are {0, 1, ... , p − 1} and D_p contains the arc (u, v) if and only if $u - v$ is a quadratic residue modulo p. The digraph D_p is sometimes referred to as the **Paley tournament**. Graham and Spencer [18] proved that D_p has property $Q(n, 1)$ whenever $p > n^2 2^{2n-2}$. Bollobás [11] extended these results to prime powers. More specifically, if $q \equiv 3 \pmod 4$ is a prime power, the Paley tournament D_q is defined as follows. The vertex set of D_q are the elements of the finite field $\mathbf{F}_q$. Vertex u joins to vertex v by an arc if and only if $u - v$ is a quadratic residue in $\mathbf{F}_q$. Bollobás [11] noted that D_q has property $Q(n,1)$ whenever

$$q > \{(n-2)2^{n-1} + 1\}\sqrt{q} + n2^{n-1}.$$

In [5], Ananchuen and Caccetta proved that D_q has property $Q(n,k)$ whenever

$$q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + k2^n - 1.$$

Ananchuen and Caccetta [5] proved that D_q has property $Q(m,n,k)$ for every

$$q > \{(t-3)2^{t-1} + 2\}\sqrt{q} + (t + 2k - 1)2^{t-1} - 1,$$

where $t = m + n$.

By using higher order residues on finite fields we can generate other classes of digraphs. Let $q \equiv 5 \pmod 8$ be a prime power. Define the **quadruple Paley digraph** $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex u joins to vertex v by an arc if and only if $u - v$ is a quadruple in $\mathbf{F}_q$; that is $u - v = y^4$ for some $y \in \mathbf{F}_q$. Since $q \equiv 5 \pmod 8$ is a prime power, -1 is not a quadruple in $\mathbf{F}_q$. The condition -1 is not a quadruple in $\mathbf{F}_q$ is needed to ensure that (v, u) is not defined to be an arc when (u, v) is defined to be an arc. Consequently, $D_q^{(4)}$ is well-defined. However, $D_q^{(4)}$ is not a tournament. Figure below displays the digraph $D_{13}^{(4)}$.

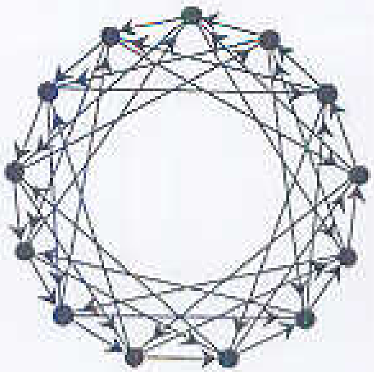


Figure 2. Paley digraph $D_{13}^{(4)}$.

In this report, we will show that $D_q^{(4)}$ has property Q(n,k) whenever

$$q > [1 + (3n-4)4^{n-1}]\sqrt{q} + (4k-3)4^{n-1},$$

and has property Q(m,n,k) whenever

$$q > (t2^{t-1} - 2^t + 1)3^m\sqrt{q} + (t + 4k - 4)3^m4^{t-1},$$

where t = m + n.

In the next section we present some results on finite fields which we mark use of in the proof of our main results.

2. Finite Fields

In this section, we present some results on finite fields that we make use of in establishing our main theorems. We begin with some basic notation and terminology.

Let $\mathbf{F}_q$ be a finite field of order q where q is a prime power and let $\mathbf{F}_q[x]$ be a polynomial ring over $\mathbf{F}_q$.

A **character** χ of $\mathbf{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbf{F}_q$, is a map from $\mathbf{F}_q^*$ to the multiplicative group of complex numbers with $|\chi(x)| = 1$ for all $x \in \mathbf{F}_q^*$ and with $\chi(xy) = \chi(x)\chi(y)$ for any $x, y \in \mathbf{F}_q^*$.

Among the character of $\mathbf{F}_q^*$, we have the **trivial** character χ_o defined by $\chi_o(x) = 1$

for all $x \in \mathbf{F}_q^*$; all other character of $\mathbf{F}_q^*$ are called **nontrivial**. With each character χ of $\mathbf{F}_q^*$, there is associated the **conjugate** character $\overline{\chi}$ defined by $\overline{\chi}(x) = \overline{\chi(x)}$ for all $x \in \mathbf{F}_q^*$. A character χ is of **order** d if $\chi^d = \chi_0$ and d is the smallest positive integer with this property.

It will be convenient to extent the definition of nontrivial character χ to the whole $\mathbf{F}_q$ by defining $\chi(0) = 0$. For χ_0 we define $\chi_0(0) = 1$.

Observe that

$$\chi^t(a) = \chi(a^t) \tag{2.1}$$

for any $a \in \mathbf{F}_q$ and t a positive integer.

If χ is a nontrivial character of $\mathbf{F}_q$, we known that (see [20]), for $a, b \in \mathbf{F}_q$ with $a \neq b$

$$\sum_{x \in F_q} \chi(x-a)\overline{\chi}(x-b) = -1. \tag{2.2}$$

The following lemma, due to Schmidt [21], is very useful to our work.

Lemma 2.1. Let χ be a nontrivial character of order d of $\mathbf{F}_q$. Suppose $f(x) \in \mathbf{F}_q[x]$ has precisely s distinct zero and it is not a d^{th} power; that is f(x) is not the form $c\{g(x)\}^d$, where $c \in \mathbf{F}_q$ and $g(x) \in \mathbf{F}_q[x]$. Then

$$\left| \sum_{x \in F_q} \chi(f(x)) \right| \leq (s-1)\sqrt{q}\,. \qquad \square$$

The next lemma is a generalization of Lemma 3.2 proved in [3].

Lemma 2.2. Let χ be a nontrivial character of order d of $\mathbf{F}_q$. If $a_1, a_2, \ldots, a_s$ are distinct elements of $\mathbf{F}_q$ and $s \equiv 0 \pmod d$, then there exist $c \in \mathbf{F}_q^*$ such that

$$\sum_{x \in F_q} \chi\{(x-a_1)(x-a_2)\ldots(x-a_s)\} = -1 + \sum_{x \in F_q} \chi\{c(x-b_1)(x-b_2)\ldots(x-b_{s-1})\}$$

for some distinct elements $b_1, b_2, \ldots, b_{s-1}$ of $\mathbf{F}_q$.

Proof: We write

$$\sum_{x\in F_q} \chi\{(x-a_1)(x-a_2)\ldots(x-a_s)\}$$

$$= \sum_{x\in F_q} \chi\{x(x+a_1-a_2)(x+a_1-a_3)\ldots(x+a_1-a_s)\}. \qquad (2.3)$$

Note the latter equality is valid, since x and $x + a_1$ assume all values in $\mathbf{F}_q$. Now, since a_1, a_2, … , a_s are distinct, then $c_i = a_1 - a_{i+1} \neq 0$ for $1 \leq i \leq s-1$.

If $x \neq 0$, then there exists an x^{-1} such that $xx^{-1} = 1$. Furthermore, $\chi(x^{-1})^s = 1$, since $s \equiv 0 (\text{mod } d)$ and χ is a character of order d. If x = 0, then $\chi(x) = 0$. Thus, we can write (2.3) as

$$\sum_{x\in F_q^*} \chi\{x(x+c_1)(x+c_2)\ldots(x+c_{s-1})\}$$

$$= \sum_{x\in F_q^*} \chi(x^{-1})^s\chi\{x(x+c_1)(x+c_2)\ldots(x+c_{s-1})\}$$

$$= \sum_{x\in F_q^*} \chi\{xx^{-1}(xx^{-1}+c_1x^{-1})(xx^{-1}+c_2x^{-1})\ldots(xx^{-1}+c_{s-1}x^{-1})\}$$

$$= \sum_{x\in F_q^*} \chi\{(1+c_1x^{-1})(1+c_2x^{-1})\ldots(1+c_{s-1}x^{-1})\}.$$

Since, for each i, $c_i \neq 0$, then c_i^{-1} exists. Further, $\chi(c_1c_1^{-1}c_2c_2^{-1}\ldots c_{s-1}c_{s-1}^{-1}) = 1$. Now using the same idea as above we can write

$$\sum_{x\in F_q^*} \chi\{(1+c_1x^{-1})(1+c_2x^{-1})\ldots(1+c_{s-1}x^{-1})\}$$

$$= \sum_{x\in F_q^*} \chi(c_1c_2\ldots c_{s-1})\chi\{(c_1^{-1}+x^{-1})(c_2^{-1}+x^{-1})\ldots(c_{s-1}^{-1}+x^{-1})\}. \qquad (2.4)$$

Let $c = c_1c_2 \ldots c_{s-1}$. Since $c_i \neq 0$, for each i, we have $c \neq 0$. As x assumes all values in $\mathbf{F}_q^*$, so does x^{-1}. Hence, we can write (2.4) as

$$\sum_{x\in F_q^*} \chi(c)\chi\{(x+c_1^{-1})(x+c_2^{-1})\ldots(x+c_{s-1}^{-1})\}$$

$$= \sum_{x\in F_q} \chi(c)\chi\{(x+c_1^{-1})(x+c_2^{-1})\ldots(x+c_{s-1}^{-1})\} - \chi(c)\chi(c^{-1})$$

$$= \sum_{x\in F_q} \chi\{c(x+c_1^{-1})(x+c_2^{-1})\ldots(x+c_{s-1}^{-1})\} - 1.$$

This completes the proof of the lemma. □

Using Lemma 2.1, we have the following corollary to Lemma 2.2.

Corollary. Let χ be a nontrivial character of order d of $\mathbf{F}_q$. If $a_1, a_2, \ldots, a_s$ are distinct elements of $\mathbf{F}_q$ and $s \equiv 0 \pmod d$ then

$$\left| \sum_{x \in \mathbf{F}_q} \chi\{(x-a_1)(x-a_2)\ldots(x-a_s)\} \right| \le 1 + (s-2)\sqrt{q}. \qquad \square$$

Let g be a fixed primitive element of the finite field $\mathbf{F}_q$; that is g is a generator of the cyclic group $\mathbf{F}_q^*$. Define a function α by

$$\alpha(g^t) = e^{\frac{2\pi i t}{3}},$$

where $i^2 = -1$. Therefore, α is a cubic character, character of order 3, of $\mathbf{F}_q$. The values of α are the elements of the set $\{1, \omega, \omega^2\}$ where $\omega = e^{\frac{2\pi i}{3}}$. Note that α^2 is also a cubic character and $\overline{\alpha} = \alpha^2$. Moreover, if a is not a cubic of an element of $\mathbf{F}_q^*$, then $\alpha(a) + \alpha^2(a) = -1$.

Further, define a function β by

$$\beta(g^t) = i^t.$$

Therefore, β is the quadruple character, character of order 4, of $\mathbf{F}_q$. The values of β are in the set $\{1, -1, i, -i\}$. Observe that β^3 is also a quadruple character and $\overline{\beta} = \beta^3$ while β^2 is a quadratic character. Moreover, if a is not a quadruple of an element of $\mathbf{F}_q^*$, then $\beta(a) + \beta^2(a) + \beta^3(a) = -1$.

Lemma 2.3. Let α be a cubic character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$. Put

$$g = \sum_{x \in \mathbf{F}_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \ge 2^n q - (t2^{t-1} - 2^t + 1)2^t \sqrt{q},$$

where $|A| = m$, $|B| = n$ and $t = m + n$.

Proof: Let $A \cup B = \{c_1, c_2, \ldots, c_t\}$. Expanding g and noting that $\sum_{x \in F_q} 2^n = 2^n q$, we can write

$$
\begin{aligned}
|g - 2^n q| \le & \left| \sum_{x \in F_q} \sum_{\chi \in \{\alpha, \alpha^2\}} \sum_{i=1}^{t} 2^{t-1} \chi(x - c_i) \right| + \\
& \left| \sum_{x \in F_q} \sum_{\chi_j \in \{\alpha, \alpha^2\}} \sum_{i_1 < i_2} \{2^{t-2} \chi_1(x - c_{i_1}) \chi_2(x - c_{i_2})\} \right| + \ldots + \\
& \left| \sum_{x \in F_q} \sum_{\chi_j \in \{\alpha, \alpha^2\}} \sum_{i_1 < i_2 < \ldots < i_s} \{2^{t-s} \chi_1(x - c_{i_1}) \chi_2(x - c_{i_2}) \ldots \chi_s(x - c_{i_s})\} \right| + \ldots + \\
& \left| \sum_{x \in F_q} \sum_{\chi_i \in \{\alpha, \alpha^2\}} \{\chi_1(x - c_1) \chi_2(x - c_2) \ldots \chi_t(x - c_t)\} \right| .
\end{aligned}
$$

Now, by (2.1) and Lemma 2.1 we obtain

$$
\begin{aligned}
|g - 2^n q| &\le \sum_{s=1}^{t} 2^s 2^{t-s} \binom{t}{s} (s-1) \sqrt{q} \\
&= (t 2^{t-1} - 2^t + 1) 2^t \sqrt{q} .
\end{aligned}
$$

Therefore, $g \ge 2^n q - (t2^{t-1} - 2^t + 1)2^t \sqrt{q}$ as required. □

Lemma 2.4. Let α be a cubic character of $\mathbf{F}_q$ and A be a subset of m vertices of $\mathbf{F}_q$. Put

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \alpha(x - a) + \alpha^2(x - a)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \ge q - [1 - m^2 + m + (2m - 3)3^{m-1}] \sqrt{q} - (m^2 - m).$$

Proof: Let $A = \{a_1, a_2, \ldots, a_m\}$. We can write

$$
\begin{aligned}
g = & \sum_{x \in F_q} 1 + \sum_{x \in F_q} \sum_{\chi \in \{\alpha, \alpha^2\}} \sum_{i=1}^{m} \chi(x - a_i) + \\
& \sum_{x \in F_q} \sum_{\chi_j \in \{\alpha, \alpha^2\}} \sum_{i_1 < i_2} \{\chi_1(x - a_{i_1}) \chi_2(x - a_{i_2})\} + \ldots + \\
& \sum_{x \in F_q} \sum_{\chi_j \in \{\alpha, \alpha^2\}} \sum_{i_1 < i_2 < \ldots < i_s} \{\chi_1(x - a_{i_1}) \chi_2(x - a_{i_2}) \ldots \chi_s(x - a_{i_s})\} + \ldots +
\end{aligned}
$$

$$\sum_{x\in F_q} \sum_{\chi_j\in\{\alpha,\alpha^2\}} \{\chi_1(x-a_1)\chi_2(x-a_2)\ldots\chi_m(x-a_m)\}. \tag{2.5}$$

Consider

$$h = \sum_{x\in F_q} \sum_{\chi_j\in\{\alpha,\alpha^2\}} \{\chi_1(x-a_{i_1})\chi_2(x-a_{i_2})\}$$

for some a_{i_1}, a_{i_2} with $i_1 < i_2$. Then by using (2.2) we have

$$\begin{aligned} h &= \sum_{x\in F_q} \{\alpha(x-a_{i_1})\alpha(x-a_{i_2}) + \alpha(x-a_{i_1})\alpha^2(x-a_{i_2}) + \alpha^2(x-a_{i_1})\alpha(x-a_{i_2}) + \\ &\qquad \alpha^2(x-a_{i_1})\alpha^2(x-a_{i_2})\} \\ &= -2 + \sum_{x\in F_q} \{\alpha(x-a_{i_1})\alpha(x-a_{i_2}) + \alpha^2(x-a_{i_1})\alpha^2(x-a_{i_2})\}. \end{aligned}$$

Using the same idea as above together with (2.1), (2.2) and Lemma 2.1 we get from (2.5)

$$\begin{aligned} \left| g - [q-(m^2-m)] \right| &\le \sum_{s=3}^{m} 2^s \binom{m}{s} (s-1)\sqrt{q} + (m^2-m)\sqrt{q} \\ &= [1 + (2m-3)3^{m-1} - (m^2-m)]\sqrt{q}. \end{aligned}$$

Therefore, $g \ge q - [1 - m^2 + m + (2m-3)3^{m-1})]\sqrt{q} - (m^2-m)$ as required. ❑

Lemma 2.5. Let β be a quadruple character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$. Put

$$g = \sum_{x\in F_q} \prod_{a\in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b\in B} \{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \ge 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q},$$

where $|A| = m$, $|B| = n$ and $t = m + n$.

Proof: Let $A \cup B = \{c_1, c_2, \ldots, c_t\}$. Expanding g and noting that $\sum_{x\in F_q} 3^n = 3^n q$, we can write

$$\begin{aligned} \left| g - 3^n q \right| &\le \left| \sum_{x\in F_q} \sum_{\chi\in\{\beta,\beta^2,\beta^3\}} \sum_{i=1}^{t} 3^{t-1}\chi(x-c_i) \right| + \\ &\qquad \left| \sum_{x\in F_q} \sum_{\chi_j\in\{\beta,\beta^2,\beta^3\}} \sum_{i_1<i_2} \{3^{t-2}\chi_1(x-c_{i_1})\chi_2(x-c_{i_2})\} \right| + \ldots + \end{aligned}$$

$$\left| \sum_{x \in F_q} \sum_{\chi_j \in \{\beta, \beta^2, \beta^3\}} \sum_{i_1 < i_2 < \ldots < i_s} \{3^{t-s} \chi_1(x - c_{i_1}) \chi_2(x - c_{i_2}) \ldots \chi_s(x - c_{i_s})\} \right| + \ldots +$$

$$\left| \sum_{x \in F_q} \sum_{\chi_j \in \{\beta, \beta^2, \beta^3\}} \{\chi_1(x - c_1) \chi_2(x - c_2) \ldots \chi_t(x - c_t)\} \right| .$$

Now, by (2.1) and Lemma 2.1 we have

$$| g - 3^n q | \le \sum_{s=1}^{t} 3^s 3^{t-s} \binom{t}{s} (s-1) \sqrt{q}$$

$$= (t 2^{t-1} - 2^t + 1) 3^t \sqrt{q} .$$

Therefore, $g \ge 3^n q - (t 2^{t-1} - 2^t + 1) 3^t \sqrt{q}$ as required. ❑

Lemma 2.6. Let β be a quadruple character of $\mathbf{F}_q$ and let A be a subset of m vertices of $\mathbf{F}_q$. Put

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \beta(x - a) + \beta^2(x - a) + \beta^3(x - a)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \ge q - [1 + (3n - 4) 4^{m-1}] \sqrt{q} .$$

Proof: Let $A = \{a_1, a_2, \ldots, a_m\}$. We can write

$$g = \sum_{x \in F_q} 1 + \sum_{x \in F_q} \sum_{\chi \in \{\beta, \beta^2, \beta^3\}} \sum_{i=1}^{m} \chi(x - a_i) +$$

$$\sum_{x \in F_q} \sum_{\chi_j \in \{\beta, \beta^2, \beta^3\}} \sum_{i_1 < i_2} \{\chi_1(x - a_{i_1}) \chi_2(x - a_{i_2})\} + \ldots +$$

$$\sum_{x \in F_q} \sum_{\chi_j \in \{\beta, \beta^2, \beta^3\}} \sum_{i_1 < i_2 < \ldots < i_s} \{\chi_1(x - a_{i_1}) \chi_2(x - a_{i_2}) \ldots \chi_s(x - a_{i_s})\} + \ldots +$$

$$\sum_{x \in F_q} \sum_{\chi_j \in \{\beta, \beta^2, \beta^3\}} \{\chi_1(x - a_1) \chi_2(x - a_2) \ldots \chi_m(x - a_m)\}.$$

Then, by (2.1) and Lemma 2.1 we have

$$| g - q | \le \sum_{s=1}^{m} 3^s \binom{m}{s} (s-1) \sqrt{q}$$

$$= [1 + (3m - 4) 4^{m-1}] \sqrt{q} .$$

Therefore, $g \ge q - [1 + (3m - 4) 4^{m-1}] \sqrt{q}$ as required. ❑

3. Preliminaries

For $q \equiv 1 \pmod 3$ a prime power, there exists a cubic character α of $\mathbf{F}_q$ and $\alpha(-a) = \alpha(a)$ for all $a \in \mathbf{F}_q$. Further, for $q \equiv 1 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and $\beta(-a) = \beta(a)$ for all $a \in \mathbf{F}_q$.

Observe that if a and b are any vertices of $G_q^{(3)}$, then for $t = 1$ and 2

$$\alpha^t(a-b) = \begin{cases} 1, & \text{if a is adjacent to b,} \\ 0, & \text{if } a = b, \\ \omega \text{ or } \omega^2, & \text{otherwise.} \end{cases}$$

Also, if a and b are any vertices of $G_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(a-b) = \begin{cases} 1, & \text{if a is adjacent to b,} \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character; that is

$$\beta^2(a-b) = \begin{cases} 1, & \text{if } a-b \text{ is a quadratic ressidue,} \\ 0, & \text{if } a = b, \\ -1, & \text{otherwise.} \end{cases}$$

We conclude this section by noting that for $q \equiv 5 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and $\beta(-u) = -\beta(u)$ for all $u \in \mathbf{F}_q$. Furthermore, if u and v are any vertices of $G_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(u-v) = \begin{cases} 1, & \text{if u dominates v,} \\ 0, & \text{if } u = v, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character; that is

$$\beta^2(u-v) = \begin{cases} 1, & \text{if } u-v \text{ is a qudratic ressidue,} \\ 0, & \text{if } u = v, \\ -1, & \text{otherwise.} \end{cases}$$

Before stating our results, we need the following notation. For disjoint subsets A and B of V(G), we denote by n(A/B) the number of vertices of G not in $A \cup B$ that are adjacent to each vertex of A but not adjacent to any vertex of B. When $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$, we sometimes write for convenience $n(A/B) = n(a_1, a_2, \dots, a_m / b_1, b_2, \dots, b_n)$.

4. Results On Generalized Paley Graphs

Recall that for $q \equiv 1 \pmod 3$ a prime power we define the **cubic Paley graph** $G_q^{(3)}$ as follows. The vertices of $G_q^{(3)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^3$ for some $y \in \mathbf{F}_q$. For $q \equiv 1 \pmod 8$ a prime power, define the **quadruple Paley graph** $G_q^{(4)}$ as follows. The vertices of $G_q^{(4)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^4$ for some $y \in \mathbf{F}_q$.

Our first result concerns cubic Paley graphs having property P(2,2,k).

Theorem 4.1. Let $q \equiv 1 \pmod 3$ be a prime power and k a positive integer. If

$$q > [\frac{1}{4}(79 + 3\sqrt{36k + 701})]^2,$$

then $G_q^{(3)} \in \mathcal{G}(2,2,k)$.

Proof: Let S = {a, b, c, d} be any set of distinct vertices of $G_q^{(3)}$. Then $n(a, b/c, d) \geq k$ if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin S^4}} \{[1 + \alpha(x-a) + \alpha^2(x-a)][1 + \alpha(x-b) + \alpha^2(x-b)]$$

$$[2 - \alpha(x-c) - \alpha^2(x-c)][2 - \alpha(x-d) - \alpha^2(x-d)]\}$$

$$\geq k3^4.$$

To show that $f \geq k3^4$, it is clearly sufficient to establish that $f > (k-1)3^4$.

We can write

$$g = \sum_{x \in F_q} \{[1 + \alpha(x-a) + \alpha^2(x-a)][1 + \alpha(x-b) + \alpha^2(x-b)]$$

$$[2 - \alpha(x-c) - \alpha^2(x-c)][2 - \alpha(x-d) - \alpha^2(x-d)]\}$$

$$= \sum_{x \in F_q} 4 + \sum_{x \in F_q} \sum_{\chi \in \{\alpha, \alpha^2\}} \{4\chi(x-a) + 4\chi(x-b) - 2\chi(x-c) - 2\chi(x-d)\} +$$

$$\sum_{x \in F_q} \sum_{\chi_i \in \{\alpha, \alpha^2\}} \{\chi_1(x-c)\chi_2(x-d) - 2\chi_1(x-a)\chi_2(x-c) -$$

$$2\chi_1(x-a)\chi_2(x-d) - 2\chi_1(x-b)\chi_2(x-c) -$$

$$2\chi_1(x-b)\chi_2(x-d) + 4\chi_1(x-a)\chi_2(x-b)\} +$$

$$\sum_{x \in F_q} \sum_{\chi_i \in \{\alpha, \alpha^2\}} \{\chi_1(x-a)\chi_2(x-c)\chi_3(x-d) + \chi_1(x-b)\chi_2(x-c)\chi_3(x-d) -$$

$$2\chi_1(x-a)\chi_2(x-b)\chi_3(x-c) - 2\chi_1(x-a)\chi_2(x-b)\chi_3(x-d)\} +$$

$$\sum_{x \in F_q} \sum_{\chi_i \in \{\alpha, \alpha^2\}} \{\chi_1(x-a)\chi_2(x-b)\chi_3(x-c)\chi_4(x-d)\}. \qquad (4.1)$$

Now, by (2.1) (2.2) and Lemma 2.1 we get from (4.1)

$$g = 4q + 0 + [\sum_{x \in F_q} \alpha(x-c)\alpha(x-d) + \sum_{x \in F_q} \alpha^2(x-c)\alpha^2(x-d) - 2] -$$

$$2[\sum_{x \in F_q} \alpha(x-a)\alpha(x-c) + \sum_{x \in F_q} \alpha^2(x-a)\alpha^2(x-c) - 2] -$$

$$2[\sum_{x \in F_q} \alpha(x-a)\alpha(x-d) + \sum_{x \in F_q} \alpha^2(x-a)\alpha^2(x-d) - 2] -$$

$$2[\sum_{x \in F_q} \alpha(x-b)\alpha(x-c) + \sum_{x \in F_q} \alpha^2(x-b)\alpha^2(x-c) - 2] -$$

$$2[\sum_{x \in F_q} \alpha(x-b)\alpha(x-d) + \sum_{x \in F_q} \alpha^2(x-b)\alpha^2(x-d) - 2] +$$

$$4[\sum_{x \in F_q} \alpha(x-a)\alpha(x-b) + \sum_{x \in F_q} \alpha^2(x-a)\alpha^2(x-b) - 2] +$$

$$\sum_{x\in F_q}\sum_{\chi_i\in\{\alpha,\alpha^2\}}\{\chi_1(x-a)\chi_2(x-c)\chi_3(x-d)+\chi_1(x-b)\chi_2(x-c)\chi_3(x-d)-$$

$$2\chi_1(x-a)\chi_2(x-b)\chi_3(x-c)-2\chi_1(x-a)\chi_2(x-b)\chi_3(x-d)\}+$$

$$\sum_{x\in F_q}\sum_{\chi_i\in\{\alpha,\alpha^2\}}\{\chi_1(x-a)\chi_2(x-b)\chi_3(x-c)\chi_4(x-d)\}.$$

By first applying (2.1) and Lemma 2.2 and then applying Lemma 2.1 we obtain

$$|g-4q-10|\le 2\sqrt{q}+4\sqrt{q}+4\sqrt{q}+4\sqrt{q}+4\sqrt{q}+8\sqrt{q}+$$

$$[6((3-1)\sqrt{q})+2(3-2)\sqrt{q}]+[6((3-1)\sqrt{q})+2(3-2)\sqrt{q}]+$$

$$2[6((3-1)\sqrt{q})+2(3-2)\sqrt{q}]+2[6((3-1)\sqrt{q})+2(3-2)\sqrt{q}]+16(3\sqrt{q})$$

$$=158\sqrt{q}.$$

Therefore,

$$g\ge 4q+10-158\sqrt{q}.$$

Consider

$$\begin{aligned}g-f=&\{1+\alpha(a-b)+\alpha^2(a-b)\}\{2-\alpha(a-c)-\alpha^2(a-c)\}\{2-\alpha(a-d)-\alpha^2(a-d)\}+\{1\\&+\alpha(b-a)+\alpha^2(b-a)\}\{2-\alpha(b-c)-\alpha^2(b-c)\}\{2-\alpha(b-d)-\alpha^2(b-d)\}+2\{1\\&+\alpha(c-a)+\alpha^2(c-a)\}\{1+\alpha(c-b)+\alpha^2(c-b)\}\{2-\alpha(c-d)-\alpha^2(c-d)\}+2\{1\\&+\alpha(d-a)+\alpha^2(d-a)\}\{1+\alpha(d-b)+\alpha^2(d-b)\}\{2-\alpha(d-c)-\alpha^2(d-c)\}\\\le& 108,\end{aligned}$$

since $g-f$ achieves its maximum value when $ab, cd \notin E(G)$ and $ac, ad, bc, bd \in E(G)$. Consequently,

$$\begin{aligned}f&\ge g-108\\&\ge 4q+10-158\sqrt{q}-108.\end{aligned}$$

Hence, $f>(k-1)3^4$ for $q>[\frac{1}{4}(79+3\sqrt{36k+701})]^2$. As S is arbitrary, this completes the proof. ❑

Remark 1. When $k=1$, Theorem 4.1 above asserts that $G_q^{(3)}\in\mathcal{G}(2,2,1)$ for all prime powers ≥ 1609. We have verified, using the computer, that $G_q^{(3)}\in\mathcal{G}(2,2,1)$ only if q is a prime power of order 151, 157 or at least 223. Table I gives the maximum k for which $G_q^{(3)} \in \mathcal{G}(2,2,k)$; we give only some of the computational results.

Table I. Maximum k for which $G_q^{(3)} \in \mathcal{G}(2,2,k)$.

Maximum k	Order q	Maximum k	Order q
0	≤ 139 and 163	14	601, 613, 619, 631, 634
1	151, 157, 223	15	661
2	169, 181, 193, 199, 229	16	673, 625
3	211, 241, 271, 361	17	691, 709, 769
4	256, 277, 289, 313	18	727, 733, 757
5	283, 307, 331	19	751
6	337, 343, 349, 373, 379	20	739, 787, 811, 829
7	367, 397, 409	22	823
8	433, 439, 463, 523	23	859, 883
9	421, 457, 487, 529	24	853, 877, 907
11	499	25	919, 937
12	547, 571, 577	27	967, 991
13	541, 607	28	997, 1009

For the class $\mathcal{G}(m,n,k)$, we have the following result.

Theorem 4.2. Let $q \equiv 1 \pmod 3$ be a prime power and k a positive integer. If

$$q > (t2^{t-1} - 2^t + 1)2^m\sqrt{q} + (m + 2n + 3k - 3)2^{-m}3^{t-1}, \tag{4.2}$$

then $G_q^{(3)} \in \mathcal{G}(m,n,k)$ for all m, n with $m + n \leq t$.

Proof: It clearly suffices to establish the result for $m + n = t$. Let A and B be disjoint subsets of $V(G_q^{(3)})$ with $|A| = m$ and $|B| = n$. Then, $n(A/B) \geq k$ if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}$$

$$> (k-1)3^t.$$

Let

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Now, by Lemma 2.3 we have

$$g \geq 2^n q - (t2^{t-1} - 2^t + 1)2^t\sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Since, in the product $\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\}$ each factor is at most 3 and one factor is 1 and in the product $\prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}$ each factor is at most 3 and one factor is 2 we have

$$g - f \leq 3^{t-1}m + 3^{t-1}2n$$
$$= (m+2n)3^{t-1}.$$

Consequently,

$$f \geq 2^n q - (t2^{t-1} - 2^t + 1)2^t\sqrt{q} - (m+2n)3^{t-1}.$$

Now, if inequality (4.2) holds, then $f > (k-1)3^t$ as required. Since A and B are arbitrary, this completes the proof of the theorem. □

For the case $n = 0$, we have the following sharper result.

Theorem 4.3. Let $q \equiv 1 \pmod 3$ be a prime power and k a positive integer. If

$$q > [1 - m^2 + m + (2m-3)3^{m-1}]\sqrt{q} + (m^2 - m) + (3k-2)3^{m-1}, \qquad (4.3)$$

then $G_q^{(3)} \in \mathcal{G}(m,0,k)$.

Proof: Let A be any subset of m vertices of $G_q^{(3)}$. Then there are at least k other vertices, each of which is adjacent to every vertex of A if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A}} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} > (k-1)3^m.$$

Let

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\}.$$

Then, by Lemma 2.4 we have

$$g \geq q - [1 - m^2 + m + (2m-3)3^{m-1}]\sqrt{q} - (m^2 - m).$$

Consider

$$g - f = \sum_{x \in A} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\}$$
$$\leq 3^{m-1},$$

since, each factor is at most 3 and one factor is 1.

Therefore,

$$f \geq q - [1 - m^2 + m + (2m-3)3^{m-1}]\sqrt{q} - (m^2 - m) - 3^{m-1}.$$

Now, if inequality (4.3) holds, then $f > (k-1)3^m$ as required. As A is arbitrary, this completes the proof of the theorem. ❑

We now turn our attention to the adjacent properties of the quadruple Paley graph $G_q^{(4)}$.

Theorem 4.4. Let $q \equiv 1 \pmod 8$ be a prime power and k a positive integer. If

$$q > \left[\frac{1}{6}(291 + \sqrt{1024k + 85193})\right]^2, \tag{4.4}$$

then $G_q^{(4)} \in \mathcal{G}(2,2,k)$.

Proof: Let $S = \{a, b, c, d\}$ be any set of distinct vertices of $G_q^{(4)}$. Then $n(a, b/c, d) \geq k$ if and only if

$$f = \sum_{\substack{x \in Z_q \\ x \notin S}} \{[1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)][1 + \beta(x-b) + \beta^2(x-b) + \beta^3(x-b)]$$
$$[3 - \beta(x-c) - \beta^2(x-c) - \beta^3(x-c)][3 - \beta(x-d) - \beta^2(x-d) - \beta^3(x-d)]\}$$

$$> (k-1)4^4.$$

We can write

$$g = \sum_{x \in F_q} \{[1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)][1 + \beta(x-b) + \beta^2(x-b) + \beta^3(x-b)]$$

$$[3 - \beta(x-c) - \beta^2(x-c) - \beta^3(x-c)][3 - \beta(x-d) - \beta^2(x-d) - \beta^3(x-d)]\}.$$

Now using an argument similar to that used in the proof of Theorem 4.1 (except here we do not use (2.2)) we obtain:

$$|g - 9q| \leq 9(9\sqrt{q}) + 12(9\sqrt{q}) + 9\sqrt{q} + 54(2\sqrt{q}) + 162(2\sqrt{q}) + 81(3\sqrt{q})$$

$$= 873\sqrt{q}.$$

Observe that

$$g - f \leq 384,$$

since $g - f$ achieves its maximum value when $ab, cd \notin E(G)$ and $ac, ad, bc, bd \in E(G)$. Consequently,

$$f \geq 9q - 873\sqrt{q} - 384.$$

Hence, $f > (k-1)4^4$ when (4.4) holds. As S is arbitrary, this completes the proof. ❑

For the class $\mathcal{G}(m,n,k)$, we have the following result.

Theorem 4.5. Let $q \equiv 1 \pmod 8$ be a prime power and k a positive integer. If

$$q > (t2^{t-1} - 2^t + 1)3^m\sqrt{q} + (m + 3n + 4k - 4)3^{-n}4^{t-1}, \tag{4.5}$$

then $G_q^{(4)} \in \mathcal{G}(m,n,k)$ for all m, n with $m + n \leq t$.

Proof: It clearly suffices to establish the result for $m + n = t$. Let A and B be disjoint subsets of $V(G_q^{(4)})$ with $|A| = m$ and $|B| = n$. Then, $n(A/B) \geq k$ if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A \cup B}} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

$$> (k-1)4^t.$$

Let

$$g = \sum_{x \in F_q} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\},$$

Now, by Lemma 2.5, we have

$$g \geq 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q}.$$

Consider

$$g - f = \sum_{x \notin A \cup B} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}\prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Since, in the product $\prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}$ each factor is at most 4 and one factor is 1 and in the product $\prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$ each factor is at most 4 and one factor is 3 we have

$$\begin{aligned} g - f &\leq 4^{t-1}m + 4^{t-1}3n \\ &= (m+3n)4^{t-1}. \end{aligned}$$

Consequently,

$$f \geq 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q} - (m+3n)4^{t-1}.$$

Now, if inequality (4.5) holds, then $f > (k-1)4^t$ as required. Since A and B are arbitrary, this completes the proof of the theorem. ❑

For the case $n = 0$, we have the following result.

Theorem 4.6. Let $q \equiv 1 \pmod 8$ be a prime power and k a positive integer. If

$$q > [1 + (3m-4)4^{m-1}]\sqrt{q} + (4k-3)4^{m-1}, \tag{4.6}$$

then $G_q^{(4)} \in \mathcal{G}(m,0,k)$.

Proof: Let A be any subset of m vertices of $G_q^{(4)}$. Then there are at least k other vertices, each of which is adjacent to every vertex of A if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A}} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} > (k-1)4^m.$$

Now using the method of proof of Theorem 4.3 together with Lemma 2.6, we get $f > (k-1)4^m$ when (4.6) holds. Hence, the result. ❑

5. Results on Quadruple Paley Digraphs

Recall that for $q \equiv 5 \pmod 8$ a prime power we define the **quadruple Paley digraph** $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex u joins to vertex v by an arc if and only if $u - v$ is a quadruple in $\mathbf{F}_q$; that is $u - v = y^4$ for some $y \in \mathbf{F}_q$.

Our first result in this section concerns quadruple Paley digraphs having property Q(n,k).

Theorem 5.1. Let $q \equiv 5 \pmod 8$ be a prime power and k a positive integer. If

$$q > [1 + (3n-4)4^{n-1}]\sqrt{q} + (4k-3)4^{n-1}, \tag{5.1}$$

then $D_q^{(4)}$ has property Q(n,k).

Proof: Let U be subsets of n vertices of $D_q^{(4)}$. Then, there are at least k other vertices each of which dominates U if and only if

$$h = \sum_{\substack{x \in F_q \\ x \notin U}} \prod_{u \in U} \{1 + \beta(x-u) + \beta^2(x-u) + \beta^3(x-u)\} \geq k4^n.$$

To show that $h \geq k4^n$, it is clearly sufficient to establish that $h > (k-1)4^n$.

Let

$$g = \sum_{x \in F_q} \prod_{u \in U} \{1 + \beta(x-u) + \beta^2(x-u) + \beta^3(x-u)\}.$$

Then, by Lemma 2.6, we have

$$g \geq q - [1 + (3n-4)4^{n-1}]\sqrt{q}.$$

Consider

$$g - h = \sum_{x \in U} \prod_{i=1}^{n} \{1 + \beta(x-u) + \beta^2(x-u) + \beta^3(x-u)\}.$$

If $g - h \neq 0$, then for some u_k the product

$$\prod_{i=1}^{n} \{1 + \beta(u_k - u_i) + \beta^2(u_k - u_i) + \beta^2(u_k - u_i)\} \neq 0. \tag{5.2}$$

For (5.2) to hold we must have $\beta(u_k - u_i) + \beta^2(u_k - u_i) + \beta^3(u_k - u_i) \neq -1$ for all i. This means that for $i \neq k$, $\beta(u_k - u_i) + \beta^2(u_k - u_i) + \beta^3(u_k - u_i) = 3$. Hence u_k dominates all other vertices in U. Therefore u_k is unique and $g - h = 4^{n-1}$. Then, since $g - h$ could be 0 we conclude that $g - h \leq 4^{n-1}$ and so

$$\begin{aligned} h &\geq g - 4^{n-1} \\ &\geq q - [1 + (3n-4)4^{n-1}]\sqrt{q} - 4^{n-1}. \end{aligned}$$

Now, if inequality (5.1) holds, then $h > (k-1)4^n$ as required. As U is arbitrary, this completes the proof.

❑

For the property Q(m,n,k), we have the following result.

Theorem 5.2. Let $q \equiv 5 \pmod 8$ be a prime power and k a positive integer. If

$$q > (t2^{t-1} - 2^t + 1)3^m\sqrt{q} + (1 + 4k - 4)3^{-n}4^{t-1}, \tag{5.3}$$

then $D_q^{(4)}$ has property Q(m,n,k) for all m, n with $t = m + n$.

Proof: Let U and V be disjoint subsets of vertices of $D_q^{(4)}$ with $|U| = m$ and $|V| = n$. Then, there are at least k vertices, each of which is dominates every vertex of U but is dominated by every vertex of V if and only if

$$h = \sum_{\substack{x \in \Gamma_q \\ x \notin U \cup V}} \prod_{u \in U} \{1 + \beta(x-u) + \beta^2(x-u) + \beta^3(x-u)\} \prod_{v \in V} \{3 - \beta(x-v) - \beta^2(x-v) - \beta^3(x-v)\}$$

$$> (k-1)4^t.$$

Let

$$g = \sum_{x \in \Gamma_q} \prod_{u \in U} \{1 + \beta(x-u) + \beta^2(x-u) + \beta^3(x-u)\} \prod_{v \in V} \{3 - \beta(x-v) - \beta^2(x-v) - \beta^3(x-v)\}.$$

Using Lemma 2.5 we have

$$g \geq 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q}.$$

Consider

$$g - h = \sum_{x \in U \cup V} \prod_{u \in U} \{1 + \beta(x-u) + \beta^2(x-u) + \beta^3(x-u)\} \prod_{v \in V} \{3 - \beta(x-v) - \beta^2(x-v) - \beta^3(x-v)\}.$$

Since, in each product, each factor is at most 4 and one factor is 1, so each of these terms is at most 4^{t-1} we have

$$g - h \leq t4^{t-1}.$$

Consequently,

$$h \geq 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q} - t4^{t-1}.$$

Now, if inequality (5.3) holds, then $h > (k-1)4^t$ as required. Since U and V are arbitrary, this completes the proof of the theorem. ❑

6. Some Open Problems

We conclude this report by highlighting some problems that have not yet been resolved. Almost all of our work in this report has been related to Paley constructions. It would be interesting to find other classes of graphs and digraphs with the properties P(m,n,k) and Q(n,k), respectively.

In this report, cubic and quadruple residues played an important role in constructing graphs and digraphs with prescribed properties. In general, we can choose q and d such that q is a prime power and

$$d > 1 \text{ odd or } \frac{q-1}{d} \text{ even.} \tag{5.1}$$

Define the "generalized" Paley graph $G_q^{(d)}$ as follows. The vertices of $G_q^{(d)}$ are the elements of the finite fields $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^d$ for some $y \in \mathbf{F}_q$. Since q is a prime power and $d > 1$ is odd or $\frac{q-1}{d}$ is even, $-1 = y^d$ for some $y \in \mathbf{F}_q$. Consequently, $G_q^{(d)}$ is well-defined.

We conjecture that for any m, n and k, all sufficiently large "generalized" Paley graphs satisfy property P(m,n,k).

For q a prime power and

$$d > 1 \text{ even and } \frac{q-1}{d} \text{ odd,} \tag{5.2}$$

we define the Paley digraphs $D_q^{(d)}$ as follows. The vertices of $D_q^{(d)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex u joins to a vertex v by an arc if and only if $u - v = y^d$ for some

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8. OUTPUT

จากผลการวิจัยภายใต้โครงการ "กราฟและกราฟทิศทางที่สอดคล้องกับคุณสมบัติที่กำหนด" เราสามารถนำมาเรียบเรียงเขียนเป็นบทความทางวิชาการได้ 2 บทความ โดยแยกเป็นบทความที่เกี่ยวกับคุณสมบัติการประชิดของจุดของกราฟ 1 บทความ และบทความที่เกี่ยวกับคุณสมบัติการครอบครองของจุดของกราฟทิศทางอีก 1 บทความคือ

1. W. Ananchuen, **On the adjacency properties of generalized Paley graphs,** The Australasian Journal of Combinatorics, (In press).
2. W. Ananchuen, **A note on constructing digraphs with prescribed properties,** (Submitted).

<u>หมายเหตุ</u> Prof. Dr. Louis Caccetta ได้แนะนำให้เปลี่ยนชื่อบทความบทที่ 2 จาก "Quadruple Paley digraphs with prescribed properties" เป็น "A note on constructing digraphs with prescribed properties"