Appendix 8: The exchange property of modules with the finite exchange property.
Revised

THE EXCHANGE PROPERTY OF MODULES WITH THE FINITE EXCHANGE PROPERTY

Sompong Dhompongsa*

Department of Mathematics, Chiang Mai University,

Chiang Mai 50200, Thailand

e-mail: sompongd@chiangmai.ac.th

Somyot Plubtieng and Hansuk Tansee†

Department of Mathematics, Naresuan University,

Phitsanulok 65000, Thailand

e-mail: somyotp@nu.ac.th

e-mail: g4165001@cm.edu

Abstract

The question, posed in Crawley and Jónsson [2], whether the finite exchange property implies the unrestricted exchange property for any modules, is still open. In this note we obtain the equivalence of the finite exchange property and the unrestricted exchange property for a class of modules. The result includes duo modules, modules whose principal right ideals of their endomorphism rings are two-sided, modules whose endomorphism rings are semiprime, modules whose idempotent endomorphisms are central, and non-singular square free modules (and hence non-singular quasi-continuous modules by [9]).

1 Introduction

Given a cardinal $\aleph$, an R-module M is said to have the $\aleph$-*exchange property* if for any module X and decompositions $X = M' \oplus Y = \oplus_{i \in I} N_i$ where $M' \simeq M$ and $|I| \leq \aleph$, there exist

*Supported by The Thailand Research Fund

†Supported by The Royal Golden Jubilee Project

submodules $N_i' \subset N_i$ such that $X = M' \oplus (\oplus_{i \in I} N_i')$. Clearly, by the modular law, $N_i' \subset^{\oplus} N_i$ for each $i \in I$. A module M has the *exchange property* if it has the $\aleph$-exchange property for every cardinal $\aleph$. A module M has the *finite exchange property* if it has the $\aleph$-exchange property for every finite cardinal $\aleph$. It is known that the finite exchange property and the 2-exchange property are equivalent (cf. [3, Lemma 2.5]). For a finitely generated module, the exchange property is the same as the finite exchange property. Injective modules [12], quasi-injective modules [4], pure-injective modules [15], continuous modules [7], projective modules over perfect rings ([5], [13]), projective modules over a certain Boolean ring [6] and projective modules over von Neumann regular rings [11] have the exchange property. It is not known whether the exchange property and the finite exchange property are equivalent for arbitrary modules. They are equivalent for indecomposable modules [2], modules having decompositions $\oplus_{\alpha \in \Lambda} M_\alpha$ where each M_α has a local endomorphism ring [15], and quasi-continuous modules [10, Theorem 3.2]. The case of non-singular quasi-continuous modules had been shown earlier by Mohamed and Müller [9, Theorem 6]. In [10], it is noted that discrete modules have the exchange property and, for quasi-discrete modules, the finite exchange property implies the exchange property.

In this note we extend the result to include modules such as duo modules, modules whose principal right ideals of their endomorphism rings are two sided, modules whose endomorphism rings are semiprime, modules whose idempotent endomorphisms are central (and hence distributive modules since every idempotent endomorphism of a distributive module is central by [14, Proposition 2.3], and modules whose endomorphism rings are reduced by [8, page 40]), and non-singular square free modules. For convenience, we let $\mathcal{A}$ be the class of all these modules. A module M is said to be a *duo module* if $s(N) \subset N$ for any endomorphism s of M and all submodules N of M. Finally, M is called *square free* if it does not contain a nonzero square, that is, there are no nonzero submodules X and Y with $X \simeq Y^2$. Our main result is proved that

For $M \in \mathcal{A}$, the finite exchange property and the exchange property are equivalent.

By [8, Theorem 2.37], every quasi-continuous module is a direct sum of a quasi-injective module and a square free module, and the finite exchange property of modules whose idempotent endomorphisms are central implies the countable exchange property [14, Theorem 1.1]. Thus our result covers non-singular quasi-continuous modules [9], and modules whose idempotent endomorphisms are central [14]. The proof we provided here is not only a simpler one but it may be applied to wider classes of modules.

2 The Results

Throughout this paper we consider associative rings with identity and unitary right modules. Given two modules U and V, a family $(f_i)_{i\in I}$ of homomorphisms from U into V is called *summable* if for each $u \in U$, $f_i(u) = 0$ for almost all $i \in I$. Thus Σf_i is a well-defined homomorphism from U into V. We write Π^U_V to denote the canonical projection Π^U_V : $U \oplus V \longrightarrow V$ for modules U and V. If $X \oplus Z = X \oplus Y_1 \oplus Y_2$ and $\bar{Z}_2 = \Pi^X_Z(Y_2)$, then $X \oplus Z = X \oplus Y_1 \oplus \bar{Z}_2$. This can be verified by showing that $\Pi^{X\oplus Y_1}_{Y_2}|_{\bar{Z}_2} : \bar{Z}_2 \longrightarrow Y_2$ is an isomorphism, and the assertion follows from [1, Proposition 5.5]. We verify this fact as follow. If $y_2 \in Y_2$, then $y_2 = x + z_2$ for some $x \in X$ and $z_2 \in \bar{Z}_2$, i.e. $\bar{z}_2 = -x + 0 + y_2 \in X \oplus Y_1 \oplus Y_2$ and consequently $\Pi^{X\oplus Y_1}_{Y_2}|_{\bar{Z}_2}(\bar{z}_2) = y_2$. Thus, $\Pi^{X\oplus Y_1}_{Y_2}|_{\bar{Z}_2}$ is onto. To prove that $\Pi^{X\oplus Y_1}_{Y_2}|_{\bar{Z}_2}$ is 1-1, we let $\Pi^{X\oplus Y_1}_{Y_2}|_{\bar{Z}_2}(\bar{z}_2) = 0$. Write $\bar{z}_2 = x + y_1$ for some $x \in X$, $y_1 \in Y_1$, and $y_2 = x' + \bar{z}_2$ for some $x' \in X$ and $y_2 \in Y_2$. Now $y_2 - x' = x + y_1$ implies $y_2 = 0$ and hence $\bar{z}_2 = 0$ as desired.

Lemma 1. [15, Proposition 3.] *The following are equivalent for a module M :*

(1) *M has the $\aleph$-exchange property;*

(2) *For any $|I| \leq \aleph$ and for each summable family $(f_i)_{i\in I}$ in $S = End_R(M)$ with $\Sigma f_i = 1$, there exist orthogonal idempotents $e_i \in Sf_i$ such that $\Sigma e_i = 1$.*

The following is a key lemma of this paper.

Lemma 2. *Let M have the finite exchange property, e_1, f_2, f_3 be elements in $S = End_R(M)$ with e_1 an idempotent, $f_2 + f_3 = 1 - e_1 = h_1$. Then there exist orthogonal idempotents e_1^*, e_2, h_2 such that $e_1^* = e_1 + h_1 e_1^* \in Se_1$, $e_2 \in Sf_2$, $h_2 \in Sf_3$, and $e_1^* + e_2 + h_2 = 1$. Moreover, $e_1^* = e_1^* e_1$, $e_1^* : e_1 M \longrightarrow e_1^* M$ is an isomorphism, $e_1^* M \oplus h_1 M = M$, $h_1 = e_2 h_1 + h_2 h_1$, $e_2 = h_1(1 - e_1^*)e_2$, and $h_2 = h_1(1 - e^*)h_2$.*

Proof. We apply the idea in the proof of [8, Proposition 3.22]. Put $A_i = M$, $A = \oplus^3_{i=1} A_i$, $f : M \longrightarrow A$ $(m \longmapsto (e_1 m, f_2 m, f_3 m))$, $g : A \longrightarrow M((a_1, a_2, a_3) \longrightarrow a_1 + a_2 + a_3)$.

Thus $gf = 1_M$, and

$$A = fM \oplus \ker g. \tag{2.1}$$

As $M \simeq f(M)$ and M has the finite exchange property, we can write

$$A = fM \oplus C_1 \oplus C_2 \oplus C_3, \text{ for some submodules } C_i \text{ of } A_i\ (i = 1, 2, 3). \tag{2.2}$$

(2.1) and (2.2) imply $\Pi^{fM}_{\ker g}|_{C_1 \oplus C_2 \oplus C_3} : C_1 \oplus C_2 \oplus C_3 \longrightarrow \ker g$ is an isomorphism. Put $\Psi = \Pi^{fM}_{\ker g}|_{C_1 \oplus C_2 \oplus C_3}$. We first find the images in $\ker g$, under the map Ψ of elements in each C_i.

C_1 : Let $c_1 \in C_1$ and $(c_1, 0, 0) = (e_1 m, f_2 m, f_3 m) + (x, y, z)$ for some $m \in M$ and $(x, y, z) \in \ker g$. Now $x = c_1 - e_1 m$, $y = -f_2 m$, $z = -f_3 m$, together with $x + y + z = 0$ imply $c_1 = m$. Thus

$$(c_1, 0, 0) \overset{\Psi}{\longmapsto} (h_1 c_1, -f_2 c_1, -f_3 c_1) \quad (c_1 \in C_1). \tag{2.3}$$

C_2 : Let $c_2 \in C_2$ and put $(0, c_2, 0) = (e_1 m, f_2 m, f_3 m) + (x, y, z)$ for some $m \in M$ and $(x, y, z) \in \ker g$. Again $x = -e_1 m$, $y = c_2 - f_2 m$, $z = -f_3 m$, and $x + y + z = 0$ imply $c_2 = m$. Thus

$$(0, c_2, 0) \overset{\Psi}{\longmapsto} (-e_1 c_2, c_2 - f_2 c_2, -f_3 c_2) \quad (c_2 \in C_2). \tag{2.4}$$

C_3 : As for C_2 we have

$$(0, 0, c_3) \overset{\Psi}{\longmapsto} (-e_1 c_3, -f_2 c_3, c_3 - f_3 c_3) \quad (c_3 \in C_3). \tag{2.5}$$

From (2.3), (2.4) and (2.5), we have

$$(c_1, c_2, c_3) \overset{\Psi}{\longmapsto} (h_1 c_1 - e_1(c_2 + c_3), c_2 - f_2(c_1 + c_2 + c_3), c_3 - f_3(c_1 + c_2 + c_3)). \tag{2.6}$$

We observe the following.

For each $m \in M$, $(h_1 m, -f_2 m, -f_3 m) \in \ker g$, then there must be a unique $(c_1, c_2, c_3) \in C_1 \oplus C_2 \oplus C_3$ such that $\Psi(c_1, c_2, c_3) = (h_1 m, -f_2 m, -f_3 m)$. Thus , by (2.6), $h_1 c_1 - e_1(c_2 + c_3) = h_1 m$. Therefore $h_1 m = h_1 c_1$, i.e. $h_1 m \in h_1 C_1$ and hence

$$h_1 M = h_1 C_1. \tag{2.7}$$

Put $C_1' = e_1M \cap C_1$, and let

$$C_1'' = \{c_1'' \in C_1 : \Psi(c_1'', c_2, c_3) = (h_1c_1'', -f_2h_1c_1'', -f_3h_1c_1'') \text{ for some } c_i \in C_i,\ i = 2, 3\}.$$

We claim that $C_1 = C_1' \oplus C_1''$: If $c_1'' \in C_1' \cap C_1''$, then $\Psi(c_1'', c_2, c_3) = (0, 0, 0)$ for some $(c_2, c_3) \in C_2 \oplus C_3$, and $c_1'' = 0$ as Ψ is 1-1. That is the sum $C_1' + C_1''$ is direct. Next let $c_1 \in C_1$. By (2.3), we have

$$\begin{aligned}\Psi(c_1, 0, 0) &= (h_1c_1, -f_2h_1c_1, -f_3h_1c_1) + (0, -f_2e_1c_1, -f_3e_1c_1)\\ &= \Psi(c_1'', c_2, c_3) + \Psi(c_1', -c_2, -c_3)\end{aligned}$$

for some $c_1', c_1'' \in C_1$ and $c_i \in C_i$ for $i = 2, 3$. $c_1'' \in C_1''$ by the definition of C_1''. Indeed, by (2.6), $h_1c_1 = h_1c_1'' - e_1(c_2 + c_3)$ which implies $h_1c_1 = h_1c_1''$ and thus $\Psi(c_1'', c_2, c_3) = (h_1c_1'', -f_2h_1c_1'', -f_3h_1c_1'')$. For c_1', we conclude, by (2.6), that $h_1c_1' + e_1(c_2 + c_3) = 0$, i.e. $c_1' \in C_1'$. Since Ψ is 1-1 we have $c_1 = c_1' + c_1''$, and hence we conclude that $C_1 = C_1' \oplus C_1''$ as claimed.

Put $\bar{C}_1 = \Psi(C_1')$. Thus, by (2.3), $\bar{C}_1 = \{(0, f_2c_1', f_3c_1') : c_1' \in C_1'\} \subset A_2 \oplus A_3$. If $\Psi(c_1', 0, 0) = (0, c_2, c_3)$ for some $c_1' \in C_1'$, $c_2 \in C_2$, and $c_3 \in C_3$, then $(0, c_2, c_3) \in \ker g$, and $\Psi(c_1', 0, 0) = (0, c_2, c_3) = \Psi(0, c_2, c_3)$. Therefore $c_1' = c_2 = c_3 = 0$. This shows that $\bar{C}_1 \cap (C_2 \oplus C_3) = 0$.

From the remark preceding Lemma 1 applying to $A = fM \oplus \ker g$ and $A = fM \oplus C_1' \oplus C_1'' \oplus C_2 \oplus C_3$, we obtain

$$A = fM \oplus \bar{C}_1 \oplus (C_1'' \oplus C_2 \oplus C_3).$$

Here we let $X = fM$, $Z = \ker g$, $Y_1 = C_1'' \oplus C_2 \oplus C_3$, and $Y_2 = C_1'$.

Write $A = fM \oplus C_1'' \oplus D$, where $D = \bar{C}_1 \oplus C_2 \oplus C_3$.
We show now that

$$e_1M \oplus C_1'' = M. \tag{2.8}$$

To prove this, let $m \in M$. By (2.7), $h_1m = h_1c_1$ for some $c_1 \in C_1$. Write $c_1 = c_1' + c_1''$, for some $c_1' \in C_1'$ and $c_1'' \in C_1''$. Now $c_1' + c_1'' = c_1 = e_1c_1 + h_1c_1$. Thus $h_1m = h_1c_1 = (c_1' - e_1c_1) + c_1'' \in e_1M + C_1''$, and hence $M = e_1M + h_1M \subset e_1M + C_1''$. As $e_1M \cap C_1'' = 0$ is obvious, the sum $e_1M + C_1''$ is direct and this proves (2.8).

From $A = fM \oplus C_1'' \oplus D$, $C_1'' \subset A_1$, $D \subset A_2 \oplus A_3$, we can find (see the proof, for example, of [3, Lemma 2.5]) direct summands $C_2'' \subset^{\oplus} A_2$ and $C_3'' \subset^{\oplus} A_3$ such that

$$A = fM \oplus C_1'' \oplus C_2'' \oplus C_3''. \tag{2.9}$$

By (2.8) we have

$$A = (e_1M \oplus B_2 \oplus B_3) \oplus (C_1'' \oplus C_2'' \oplus C_3'')(2.10)$$

for some direct summands B_2, B_3 such that $A_2 = B_2 \oplus C_2''$, $A_3 = B_3 \oplus C_3''$. (2.9) and (2.10) imply $\Pi^{C_1''\oplus C_2''\oplus C_3''}_{e_1M\oplus B_2\oplus B_3}|_{fM} : fM \longrightarrow e_1M \oplus B_2 \oplus B_3$ is an isomorphism. Put $p = \Pi^{C_1''\oplus C_2''\oplus C_3''}_{e_1M\oplus B_2\oplus B_3}|_{fM}$ and $p^{-1} : e_1M \oplus B_2 \oplus B_3 \longrightarrow fM$, the inverse of p.

Let $\rho_1 : e_1M \oplus C_1'' \longrightarrow e_1M$, $\rho_2 : B_2 \oplus C_2'' \longrightarrow B_2$ and $\rho_3 : B_3 \oplus C_3'' \longrightarrow B_3$ be the canonical projections. Finally, define

$$e_1^* = gp^{-1}\rho_1 e_1, \quad e_2 = gp^{-1}\rho_2 f_2, \quad h_2 = gp^{-1}\rho_3 f_3.$$

Thus, as in the proof of Proposition 3.22 ((2)⇒(3)) in [8], $\{e_1^*, e_2, h_2\}$ is a family of orthogonal idempotents such that $e_1^* + e_2 + h_2 = 1$. It is seen that, $e_1^* \in Se_1, e_2 \in Sf_2$, and $h_2 \in Sf_3$. For each $m \in M$, let $p^{-1}(e_1m) = f(n_m)\Big(= (e_1n_m, f_2n_m, f_3n_m)\Big)$ for a unique $n_m \in M$. Thus $e_1^*(m) = gp^{-1}\rho_1e_1(m) = gf(n_m) = n_m$ and

$$(e_1m, 0, 0) = (e_1n_m, f_2n_m, f_3n_m) + (c_1'', c_2'', c_3'')$$

for some $c_i'' \in C_i''$ $(i = 1, 2, 3)$. From (2.8) we have $c_1'' = 0$, $e_1(n_m) = e_1(m)$ and so $e_1^*(m) = e_1(n_m) + h_1(n_m) = e_1(m) + h_1e_1^*(m)$. Consequently $e_1^* = e_1 + h_1e_1^*$ and $e_1e_1^* = e_1$. Since $e_1^* \in Se_1$, it is clear that $e_1^*e_1 = e_1^*$. Also the mapping $e_1m \longmapsto e_1m + h_1n_m$ is an isomorphism from e_1M onto e_1^*M. Now, if $h_1m = e_1^*(m') = e_1(m') + h_1e_1^*(m')$ for some $m, m' \in M$, then $e_1m' = 0$ and $e_1^*(m') = e_1^*e_1(m') = 0$. Therefore the sum $e_1^*M + h_1M$ $(= M)$ is direct. Finally, from $e_1^*h_1 = e_1^*e_1h_1 = 0$, and $e_1^* + e_2 + h_2 = 1$, we see that $h_1 = e_2h_1 + h_2h_1$. Also, writing $1 = (e_1 + h_1e_1^*) + (h_1 - h_1e_1^*) = e_1^* + e_2 + h_2$ we obtain $h_1(1 - e^*) = e_2 + h_2$. Consequently, $e_2 = h_1(1 - e_1^*)e_2$ and $h_2 = h_1(1 - e_1^*)h_2$. □

Corollary 3. *Under condition in Lemma 2 if, in addition, $M \in \mathcal{A}$, then $e_1^* = e_1$.*

Proof. We modify the proof of Lemma 2.

(Case : M is a duo module or every principal right ideal of S is two-sided) If M is duo, then $e_1^*M = e_1^*e_1M \subset e_1M$. Thus $h_1e_1^* = 0$ and therefore $e_1^* = e_1 + h_1e_1^* = e_1$. If S

has above assumption, then e_1S is a left ideal. Thus $e_1^*e_1 \in e_1S$ and $e_1^*e_1M \subset e_1M$ and the proof can be continued as before.

(Case : Every idempotent element in S is central) It is clear.

(Case : M is a non-singular square free module) By [8, Lemma 3.4], $\ker(e_1^*e_1 - e_1e_1^*)$ is essential in M. Since M is non-singular, $e_1^*e_1 - e_1e_1^* = 0$, and consequently, $e_1^* = e_1^*e_1 = e_1e_1^* = e_1$ as desired.

(Case : S is semiprime) Since $(h_1e_1^*)^2 = 0$, we conclude that $h_1e_1^* = 0$ because S is semiprime. □

Theorem 4. *For $M \in \mathcal{A}$, the finite exchange property and the exchange property are equivalent.*

Proof. In the course of the proof we shall apply repeatedly the following fact: If $\{\alpha_i\}_{i\in\Lambda}$ is a summable family of orthogonal idempotents in $S = End_R(M)$, and if α is anothor idempotent in S which is orthogonal to $\Sigma_\Lambda\alpha_i$, then α and α_i are orthogonal for all $i \in \Lambda$. To prove this we first consider αm for $m \in M$ and see that $\Sigma_\Lambda\alpha_i\alpha m \in \oplus_\Lambda\alpha_i M$. From $(\Sigma_\Lambda\alpha_i)\alpha = 0$ we have $\alpha_i\alpha m = 0$ for all $i \in \Lambda$. Secondly, we observe that $\alpha_{i_0} = (\Sigma_\Lambda\alpha_i)\alpha_{i_0}$ for each $i_0 \in \Lambda$. Thus $\alpha\alpha_{i_0} = \alpha(\Sigma_\Lambda\alpha_i)\alpha_{i_0} = 0$, and the assertion is proved. Anothor fact that we often need in the upcoming argument is that $\Sigma_\Lambda\alpha_i$ is also an idempotent whenever $\{\alpha_i : i \in \Lambda\}$ is a summable family of orthogonal idempotents in S.

Now to apply Lemma 1, we are given a summable family $(f_i)_{i\in I}$ of elements in S with $\Sigma f_i = 1$. Assume I is a well ordered set of ordinals: $I = \{0, 1, 2, ..., \omega, \omega+1, ...\}$. We may enlarge I so that it does not contain its supremum. Thus for each $m \in M$, we can find an $i_0 \in I$ such that $f_k(m) = 0$ for $i_0 \leq k$. For each $i < j$ in I, put $F_i^j = \Sigma_{i\leq k<j} f_k$ and $F_i^\infty = \Sigma_{k\geq i} f_k$. For each $i \in I$, let $J_i = \{j \in I : j < i\}$ and i^+ (if exists) be the smallest limit ordinal in I with $i^+ > i$.

We are going to construct a family of idempotents $\{e_i,\ E_{i^+},\ H_i : i \in I\}$ in S such that $e_i \in Sf_i$, $H_i \in SF_i^{i^+}$, $E_{i^+} \in SF_{i^+}^\infty$ for each $i \in I$, and

(3.1) For each $i \in I$, the idempotents H_i, E_{i^+}, and e_j $(j \in J_i)$ are orthogonal and

$$H_i + E_{i^+} + \Sigma_{J_i} e_j = 1.$$

When i^+ in I does not exist, put $E_{i^+} = 0$ and $F_i^{i^+} = F_i^\infty$.

Step 1. Apply Lemma 1 for $\aleph = 2$ to the endomorphisms F_0^ω and F_ω^∞ to obtain orthogonal idempotents H_0, E_ω such that $H_0 \in SF_0^\omega$, $E_\omega \in SF_\omega^\infty$ and $H_0 + E_\omega = 1$. That is, (3.1) holds for $i = 0$. Write $H_0 = s_0 F_0^\omega$ for some $s_0 \in S$, and put $f_{01} = H_0 s_0 f_0$ and $f_{02} = H_0 s_0 F_1^\omega$. Apply Corollary 3 to the endomorphisms E_ω, f_{01}, f_{02} to obtain orthogonal idempotents e_0, H_1 such that $e_0 \in Sf_{01} \subset Sf_0$, $H_1 \in Sf_{02} \subset SF_1^\omega$, and $e_0 + H_1 + E_\omega = 1$. Thus (3.1) holds for $i = 1$. Next let $\alpha_1 = e_0 + E_\omega$, $H_1 = s_1 F_1^\omega$ for some $s_1 \in S$, $f_{11} = H_1 s_1 f_1$, and $f_{12} = H_1 s_1 F_2^\omega$. Note that α_1 is an idempotent and $f_{11} + f_{12} = H_1 = 1 - \alpha_1$. Apply Corollary 3 to the endomorphisms α_1, f_{11}, f_{12} and obtain orthogonal idempotents e_1, H_2 such that $e_1 \in Sf_{11} \subset Sf_1$, $H_2 \in Sf_{12} \subset SF_2^\omega$ and $\alpha_1 + e_1 + H_2 = 1$. Thus, from above observation, (3.1) holds for $i = 2$. Suppose for some $i \in J_\omega$ there are idempotents such that (3.1) holds. Let $\alpha_i = E_\omega + \Sigma_{J_i} e_j$, $H_i = s_i F_i^\omega$ for some $s_i \in S$, $f_{i1} = H_i s_i f_i$ and $f_{i2} = H_i s_i F_{i+1}^\omega$. Clearly, α_i is an idempotent. Apply Corollary 3 to α_i, f_{i1}, f_{i2} and obtain orthogonal idempotents e_i, H_{i+1} such that $e_i \in Sf_{i1} \subset Sf_i$, $H_{i+1} \in Sf_{i2} \subset SF_{i+1}^\omega$ and $\alpha_i + e_i + H_{i+1} = 1$. We now have idempotents $e_j \quad (j \in J_i \cup i)$ and H_{i+1} such that (3.1) holds for $i+1$. By induction we obtain idempotents $E_\omega, e_i, H_i \quad (i \in J_\omega)$ such that (3.1) holds for all $i \in J_\omega$. We show now that

$$E_\omega + \Sigma_{J_\omega} e_j = 1. \tag{3.2}$$

Let $m \in M$. By summability of $(f_i)_{i \in I}$ we can find some $i_0 \in J_\omega$ such that $f_k(m) = 0$ for all $i_0 \leq k < \omega$. Thus $H_{i_0}(m) = 0$ and $e_k(m) = 0$ for all $i_0 \leq k < \omega$. By (3.1) for $i = i_0$ we have $m = E_\omega(m) + \Sigma_{J_\omega} e_j(m)$, proving (3.2).

From $E_\omega \in SF_\omega^\infty$, write $E_\omega = t_\omega F_\omega^\infty = E_\omega t_\omega F_\omega^{2\omega} + E_\omega t_\omega F_{2\omega}^\infty$ for some $t_\omega \in S$. Observe that $\Sigma_{J_\omega} e_j$ is an idempotent. Apply Corollary 3 to endomorphisms $\Sigma_{J_\omega} e_j, E_\omega t_\omega F_\omega^{2\omega}, E_\omega t_\omega F_{2\omega}^\infty$ and obtain orthogonal idempotents H_ω and $E_{2\omega}$ such that $H_\omega \in SF_\omega^{2\omega}$, $E_{2\omega} \in SF_{2\omega}^\infty$, and $\Sigma_{J_\omega} e_j + H_\omega + E_{2\omega} = 1$. That is (3.1) holds for $i = \omega$.

Step 2. Suppose for some $i \in I$ orthogonal idempotents $e_j \quad (j < i), H_i$ and E_{i^+} in S have been constructed so that $e_j \in Sf_j \quad (j < i)$, $H_i \in SF_i^{i^+}$, $E_{i^+} \in SF_{i^+}^\infty$, and $\Sigma_{J_i} e_j + H_i + E_{i^+} = 1$. Put $\alpha_i = E_{i^+} + \Sigma_{J_i} e_j$, $H_i = s_i F_i^{i^+}$ for some $s_i \in S$, $f_{i1} = H_i s_i f_i$ and $f_{i2} = H_i s_i F_{i+1}^{i^+}$. From Step 1, it is seen that we can consider the case of being a non-limit ordinal or a limit ordinal of i simultaneously (see for example the case when $i = \omega$). Apply Corollary 3 to α_i, f_{i1}, f_{i2} and obtain orthogonal idempotents e_i, H_{i+1} such that $e_i \in Sf_{i1} \subset Sf_i, H_{i+1} \in Sf_{i2} \subset SF_{i+1}^{i^+}$ and $\alpha_i + e_i + H_{i+1} = 1$. We now have idempotents

$e_j \quad (j \in J_i \cup i), H_{i+1}$ and E_{i+} such that (3.1) holds. By induction we conclude that idempotents $\{e_i, E_{i+}, H_i \ : \ i \in I\}$ satisfying (3.1) have been constructed. We show that $\Sigma_I e_i = 1$. Let $m \in M$. By the choice of I, we take an $i_0 \in I$ such that $f_k(m) = 0$ for $i_0 \leq k$. By (3.1) for $i = i_0$ we have, as before, $m = H_i(m) + E_{i+}(m) + \Sigma_{J_i} e_j(m) = 0 + 0 + \Sigma_{J_i} e_j(m)$, completing the proof of Theorem. □

Remark. As we can see in the proof of the main Theorem, we do not use all what we state in Lemma 2. However, we put them there in case it might help to apply them to a wider class of modules.

ACKNOWLEDGEMENT

We would like to thank Professor N. V. Dung for calling our attention to the study of the topic and for many useful discussions during his visit to Naresuan University, Phitsanulok, Thailand.

References

[1] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," Graduate Texts in Math.No.13, Springer-Verlag, New York, 1992.

[2] P. Crawley and B. Jónsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math., **14**(1964), 797–855.

[3] A. Faccini "Module Theory," Birkhäuser Verlag, Basel, 1998.

[4] L. Fuchs, *On quasi-injective modules,* Ann. Scuola Norm. Sup. Pisa, **23**(1969), 541–546.

[5] M. Harada and T. Ishii, *On perfect rings and the exchange property,* Osaka J. Math. *12*(1975), 483–491.

[6] M. Kutami and K. Oshiro, *An example of a ring whose projective modules have the exchange property,* Osaka J. Math., **17**(1980), 415–420.

[7] S. Mohamed and B. J. Müller, *Continuous modules have the exchange property,* Proc. Perth Conf. Abelian Groups, Contemporary Math., 1989.

[8] S. H. Mohamed and B. J. Müller, "Continuous and Discrete Modules," Cambridge Univ. Press, 1990.

[9] S. H. Mohamed and B. J. Müller, *The exchange property for quasi-continuous modules*, Ring Theory (Ohio State-Denison), 1992, 242–247.

[10] K. Oshiro and S. T. Rizvi, *The exchange property of quasi-continuous modules with the finite exchange property,* Osaka J. Math., **33**(1996), 217–234.

[11] J. Stock, *On rings whose projective modules have the exchange property*, J. Algebra, **103**(1986), 437–453.

[12] R. B. Warfield, *Decompositions of injective modules*, Pacific J. Math., **31**(1969), 263–276.

[13] K. Yamagata, *On projective modules with the exchange property*, Sci. Rep. Tokyo Kyoiku Daigaku Sec. A, 1974, 149–158.

[14] H.P. Yu, *On modules for which the finite exchange property implies the countable exchange property*, Comm. Algebra, **22**(1994), 3887–3901.

[15] B. Zimmermann-Huisgen and W. Zimmermann, *Classes of modules with the exchange property*, J. Algebra, **88**(1984), 416–434.

Appendix 9: The Faith's conjecture on Quasi-Frobenius rings. Being submitted

Note

Since the following article is being considered for submission, we would like to keep the manuscript unseen.

The Faith's conjecture on quasi-Frobenius rings

S. Dhompongsa and H. Tansee

Appendix 10: On the endomorphism ring of a semi-projective module.
In Preparation

On The Endomorphism Ring of A Semi-Projective Module

S. Chotchaisthit
Depart. of Mathematics, Khon Kaen University,
Khon Kaen, 40002, Thailand,

S. Dhompongsa*
Depart. of Mathematics, Chiang Mai University,
Chiang Mai 50200, Thailand

H. Tansee
Depart. of Mathematics, Naresuan University,
Phitsanulok 65000, Thailand

S. Wongwai
Rajamangala Institute of Technology,
Khon Kaen Campus, Khon Kaen, 40000, Thailand

Abstract

Several properties of the endomorphism ring of a semi-projective module are obtained from the knowledge on M, and vice versa. As for examples, we show: (1) The endomorphism ring of a semi-projective module is regular if and only if the image of every endomorphism is a direct summand. (2) The kernel of every endomorphism of a semi-projective module is a direct summand if and only if the endomorphism ring S is a PP-ring and $\operatorname{Ker} F(s) \supset \operatorname{Ker} s$ for all $F \in \operatorname{Hom}_S(sS, S)$ and all $s \in S$. (3) If the endomorphism ring of a semi-projective module is principally injective (mininjective), then the module itself is quasi-principally injective (respectively, quasi-mininjective). The converse is also considered. Finally, we investigate when the principal right ideals of the endomorphism ring of a semi-projective module are both projective and injective.

1991 Mathematics Subject Classification: 16D50,16D70,16D80.
Keywords: Semi-projective modules, Endomorphism rings.

*Supported by Thailand Research Fund.

1 Introduction

Let R be a ring. A right R-module M is called principally injective if every R-homomorphism from a principal right ideal of R into M can be extended to R. This notion was introduced by Camillo for a commutative ring (1989). In 1995, Nicholson and Yousif [7] studied intensively the structure of principally injective rings and gave some applications. Since then, they continued to study rings with some other kinds of injectivity, namely, mininjective rings [8] and simple-injective rings [9]. Recently, Nicholson, Park, and Yousif [10] extended the notion of principally injective rings to the one for modules. In 1999 [11], Sanh and others introduced a quasi-principally injective module, an idea parallel to the notion of a principally quasi-injective module given in [10]. Dually, a right R-module N is called M-principally projective if every R-homomorphism from N to an M-cyclic submodule of M can be lifted to an R-homomorphism from N to M. Motivated by these and following Wisbauer [14] we consider quasi-principally (or semi-) projective modules, especially, their endomorphism rings.

Throughout this paper R will be an associative ring with unity and all modules are unitary. For a module M over a ring R we write M_R $({}_RM)$ to indicate that M is a right (left) R-module. In the following we consider the ring $S = \mathrm{End}_R(M_R)$ of all R-endomorphisms on a module M_R. A submodule N of M is said to be an *M-cyclic submodule* of M if it is the image of some element of S. By the notation $N \subset^{\oplus} M$ $(N \ll M)$ we mean that N is a direct summand (superfluous submodule) of M. We denote the Jacobson radical and the singular submodule of a module M by $J(M)$ and $Z(M)$, respectively. All modules considered here are right modules, except state otherwise.

2 Principal Projectivity

Definition 2.1 Let M and N be modules. A module N is called *M-principally projective* if for any M-cyclic submodule $s(M)$ of M, any R-homomorphism ϕ from N to $s(M)$ can be factored through to a homomorphism from N to M and s.

Lemma 2.2 *Let M and N be modules. Then N is M-principally projective if and only if for each $s \in S$, $\mathrm{Hom}_R(N, s(M)) = s\mathrm{Hom}_R(N, M)$.*

Example 2.3 (1) $\mathbb{Z}/4n\mathbb{Z}$ is $\mathbb{Z}$-principally projective but not $\mathbb{Z}$-projective.

(2) For a module M which is M-principally projective (later called *semi-projective*) and $s \in S$, if $\mathrm{Ker}\, s \subset^{\oplus} M$, then $s(M)$ is M-principally projective. (See Proposition 3.1).

(3) If $K \simeq N$ and N is M-principally projective, then K is M-principally projective.

(4) If $M \simeq M'$ and N is M-principally projective, then N is M'-principally projective.

Clearly, every K-cyclic submodule of K is an M-cyclic submodule of M for every M-cyclic submodule K of M. Thus we have the following

Proposition 2.4 *N is M-principally projective if and only if N is K-principally projective for every M-cyclic submodule K of M. In particular, if K is a direct summand of M and N is M-principally projective, then N is both K-principally projective and M/K-principally projective.*

For a direct sum we have

Proposition 2.5 *$\bigoplus_i N_i$ is M-principally projective if and only if N_i is M-principally projective for all i.*

Corollary 2.6 *If N is M-principally projective, then so is every direct summand.*

Lemma 2.7 *Let K be an M-cyclic submodule of M.If K is M-principally projective,then $K \simeq D$ for some direct summand D of M.*

Proof. Let $K = s(M)$ where $s \in S$. Then $s(M)$ is M-principally projective, so that s splits. Hence $M = \text{Ker } s \oplus D$ for some submodule D of M. Therefore $D \simeq M/\text{Ker } s \simeq s(M) = K$. ☐

In general, we can not have K be a direct summand as the following example shows.

Example 2.8 $n\mathbb{Z}$ is a $\mathbb{Z}$-cyclic submodule of $\mathbb{Z}$ which is $\mathbb{Z}$-principally projective and $n\mathbb{Z} \simeq \mathbb{Z} \subset^{\oplus} \mathbb{Z}$. But $n\mathbb{Z}$ is not a direct summand of $\mathbb{Z}$ if $n > 1$.

Proposition 2.9 *Let M and N be R-modules. Then M is N-injective and every submodule of N is M-principally projective if and only if N is M-principally projective and every M-cyclic submodule of M is N-injective.*

Proof. ($\Leftarrow$) Assume that N is M-principally projective and every M-cyclic submodule of M is N-injective. Let N' be a submodule of N, $s \in S$, and $f : N' \longrightarrow s(M)$ a homomorphism . Since $s(M)$ is N-injective, there exists a homomorphism $g : N \longrightarrow s(M)$ such that $g\iota = f$ where $\iota : N' \longrightarrow N$ is the inclusion mapping. Also, being M-principally projective of N, there exists a homomorphism $h : N \longrightarrow M$ with $sh = g$. Now $h\iota : N' \longrightarrow M$ and $sh\iota = g\iota = f$. Hence N' is M-principally projective.

($\Rightarrow$) Assume that M is N-injective and every submodule of N is M-principally projective. Let $s \in S$ and let $f : N' \longrightarrow s(M)$ be a homomorphism from a submodule N' of N to $s(M)$. By assumption, there exists a homomorphism $h : N' \longrightarrow M$ such that $sh = f$. Since M is N-injective, there exists a homomorphism $k : N \longrightarrow M$ which is an extension of h to N. Set $\alpha = sk$. Then $\alpha : N \longrightarrow s(M)$ and it is an extension of f to N. Hence $s(M)$ is N-injective. □

Corollary 2.10 *Let M be injective. Then every submodule of M-principally projective module is M-principally projective if and only if every M-cyclic submodule of M is injective.*

Proof. This follows from the fact that the module R_R is projective and a module is R-injective if and only if it is N-injective for all modules N. □

Corollary 2.11 *If R is injective, then R is regular if and only if every submodule of R-principally projective module is R-principally projective.*

Definition 2.12 A module M is called *semi- projective* if it is M-principally projective.

In general, we have

projective $\Longrightarrow$ quasi-projective $\Longrightarrow$ semi- projective $\Longrightarrow$ direct-projective.

Recall that an R-module M is *direct-projective* if given any $P \subset^{\oplus} M$, every $M \longrightarrow P \longrightarrow 0$ splits. Direct-injective modules are defined dually. A ring R is (von Neumann) regular if $a \in aRa$ for each $a \in R$.

Remark 2.13 (1) Any direct summand of a semi-projective module is again semi-projective. This follows from Proposition 2.4.

(2) A submodule N of M is called a *fully invariant submodule* of M if $s(N) \subset N$ for every $s \in S$.

If M is quasi-projective, $K \subset M$ is a fully invariant submodule, then M/K is semi-projective.

(3) It is easy to see that a module is direct projective if and only if it satisfies the property (D_2) where

(D_2): If $M/A \simeq B$, and B is a direct summand of M, then A is a direct summand of M.

It is well-known that (D_2) implies

(D_3): If A and B are direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

Thus, every semi-projective module satisfies the property (D_2) and hence (D_3).

Lemma 2.14 *Let M be an R-module. Then the following conditions are equivalent:*

(1) *M is semi- projective.*

(2) [14, p. 260] *For each $s \in S$, $sS = \mathrm{Hom}_R(M, s(M))$.*

(3) *For each s and t in S, if $s(M) \subset t(M)$, then $sS \subset tS$.*

(4) *For each s and t in S, $\{u \in S : tu(M) \subset ts(M)\} = sS + \{v \in S : v(M) \subset \operatorname{Ker} t\}$.*

Condition (2) in Lemma 2.14 is quite useful. As an example we can easily see that, for every semi-projective module M, $\mathrm{Hom}_R(M, s(M))$ is simple as a right S-module for every $s \in S$ with $s(M)$ simple (see Lemma 4.3). However, to what follows, we will employ condition (3) throughout the study.

3 The endomorphism ring and its Jacobson radical

Proposition 3.1 *Let M be a semi-projective module and $s \in S$. Thus* $\operatorname{Ker} s$ *is a direct summand of M if and only if $s(M)$ is M-principally projective.*

Proof. Assume that $\operatorname{Ker} s$ is a direct summand of M. Then $M/\operatorname{Ker} s$ is isomorphic to a direct summand of M. Hence $M/\operatorname{Ker} s$ is M-principally projective. Conversely, let $g : M/\operatorname{Ker} s \to M$ be a homomorphism, $p : M \to M/\operatorname{Ker} s$ the natural projection, and i the identity mapping on $M/\operatorname{Ker} s$. Since $M/\operatorname{Ker} s$ is M-principally projective, there exists a homomorphism $g : M/\operatorname{Ker} s \to M$ such that $pg = i$. Then $\operatorname{Ker} s = \operatorname{Ker} p$ is a direct summand of M. □

Proposition 3.2 *Let M be a semi-projective module and $s, t \in S$.*

(1) *If $s(M)$ embeds into $t(M)$, then sS can be embedded into tS.*

(2) *If $t(M)$ is an image of $s(M)$, then tS is an image of sS.*

(3) *If $s(M) \simeq t(M)$, then $sS \simeq tS$.*

Proof. (1) Let $f : s(M) \to t(M)$ be a monomorphism. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $fs = tg$. Define $\sigma : sS \to tS$ by $su \mapsto tgu$ for all $u \in S$. Clearly, σ is a well-defined homomorphism. If $tgu = 0$, then $su = 0$, since f is monic. Hence σ is a monomorphism.

(2) Let f, g and σ be as above, only this time, f is epic. Since M is semi-projective, if $th \in tS$, there exists $u \in S$ such that $th = tgu$. Hence σ is epic.

(3) Follows from the proofs of (1) and (2). □

Write

$$\nabla = \{s \in S : s(M) \ll M\}, \quad \text{and} \quad \Diamond = \{s \in S : (1 + ts)(M) = M \text{ for all } t \in S\}.$$

It is known that ∇ is an ideal of S [5]. Note also that $\nabla \subset \Diamond$ and

$$(1 + s)(M) = M \iff s(M) \subset (1 + s)s(M).$$

It is well-known that, for a quasi-discrete module M, M is discrete if and only if S/∇ is regular and $J(S) = \nabla$ [5, Theorem 5.4]. We now investigate when $J(S) = \nabla$.

Recall that an R-module is called *π-projective* if each of its submodules lies above a direct summand [14, 41.18]. A ring R is called *semiregular* if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. Equivalently, R is semiregular if and only if for each element $a \in R$, there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$.

A module is *hollow* if each of its proper submodule is superfluous.

Proposition 3.3 *Let M be semi-projective.*

(1) $J(S) = \Diamond$.

(2) *If S is local, then $J(S) = \{s \in S : s(M) \neq M\}$.*

(3) *If S/∇ is regular, then $J(S) = \nabla$.*

(4) *If $S/J(S)$ is regular (e.g. S is semilocal or S is semiregular), then S/∇ is regular if and only if $J(S) = \nabla$.*

(5) *If* $\operatorname{Ker} s \ll M$ *where $s \in S$, then any epimorphism $t : M_R \to s(M)$ can be lifted to an epimorphism in S.*

(6) *If M is hollow, then S is a local ring and $J(S) = \nabla$.*

(7) *For $s \in S$, if M is hollow and s is right invertible, then s is invertible.*

(8) *If M is finitely generated, then $J(S) = \nabla$ if and only if $J(S)(M) = \nabla(M)$. Here $J(S)(M) = \sum_{s \in J(S)} s(M)$ and $\nabla(M) = \sum_{s \in \nabla} s(M)$.*

(9) *M is hollow if and only if S is local and M is π-projective.*

(10) *If M is π-projective, then M is hollow if and only if S is local.*

(11) *If M is hollow, then $Z({}_SS) \subset J(S)$.*

Proof. (1) For $s \in J(S)$ and $t \in S$, $(1+ts)g = 1$ for some $g \in S$. Thus $M \supset (1+ts)(M) \supset (1+ts)g(M) = 1_M(M) = M$, and hence $J(S) \subset \Diamond$. On the other hand, if $(1+ts)(M) = M$, then $S = (1+ts)S$ which implies $1 = (1+ts)g$ for some g. Since this is true for all $t \in S$, we have $s \in J(S)$.

(2) Since S is local, $J(S) = \{s \in S : sS \neq S\}$ (see [1, 15.15]). But for a semi-projective module,

$$sS \neq S \Leftrightarrow s(M) \neq M,$$

by Lemma 2.14(3). Thus the assertion follows.

(3) If $s \in J(S)$, then $s(1-\alpha s) = s - s\alpha s \in \nabla$ for some $\alpha \in S$. Thus $s \in \nabla$ since $1 - \alpha s$ has a right inverse. The other inclusion is obvious by (1).

(4) If R/∇ is regular, then $J(S) = \nabla$ by (3). The other direction is obvious.

(5) Since M is semi-projective, there exists $h \in S$ such that $sh = t$. Hence $h(M) + \operatorname{Ker} s = M$. Since $\operatorname{Ker} s \ll M$, $h(M) = M$.

(6) Since M is direct projective, S is local provided that M is hollow [14, 41.19]. Now $J(S) = \nabla$ is clear.

(7) Follows from (6) and (2).

(8) ($\Leftarrow$) $s \in J(S)$ implies $s(M) \subset \nabla(M)$. Since M is finitely generated, there exist $t_1, \ldots, t_n \in \nabla$ such that $s(M) \subset t_1(M) + \ldots + t_n(M) \ll M$. Thus $s \in \nabla$ and therefore $J(S) \subset \nabla$. Now $J(S) = \nabla$ is obvious.

(9) ($\Rightarrow$) Follows from (6).

($\Leftarrow$) Let $U, V \subset M$ be such that $U + V = M$. As M is π-projective, we can choose $s \in S$ so that $s(M) \subset U$ and $(1-s)(M) \subset V$. Note that either s or $1-s$ belong to $J(S)$. If $s \in J(S)$, then $M = (1-s)M \subset V$. Otherwise, $M = U$.

(10) This is a direct consequence of (9).

(11) If $\alpha \in Z({}_SS)$ and $t \in S\backslash 0$, then there exists $s \in S$ such that $st \neq 0$ but $st\alpha = 0$. Thus $s(1+t\alpha) = s$ and $s(1+t\alpha)(M) = s(M)$. Since $\operatorname{Ker}(s) \neq M$, $(1+t\alpha)(M) = M$. That is $\alpha \in \Diamond = J(S)$. □

Note that (9) and (10) in Proposition 3.4 also hold for a direct projective module M. The following Proposition is modified from [1, 17.12].

Proposition 3.4 *If $J(M_R) \ll M_R$, then*

(1) $\nabla = \mathrm{Hom}(M, J(M))$, *and*

(2) S/∇ *is embedded in* $\mathrm{End}_R(M/J(M))$ *as a subring.*

Proof. (1) If $s \in \nabla$, then $s(M) \ll M$. So $s(M) \subset J(M)$. If, on the other hand, $s(M) \subset J(M)$, then $s(M) \ll M$ and $s \in \nabla$.

(2) For each $s \in S$, let $\theta(s)$ be a map from $M/J(M)$ into itself defined by $(\theta(s))(m + J(M)) = s(m) + J(M)$. Clearly, $\theta(s) \in \mathrm{End}_R(M/J(M))$ and $\theta : S \to \mathrm{End}_R(M/J(M))$ is a ring homomorphism. Observe that $\theta(s) = 0$ if and only if $s(M) \subset J(M)$ if and only if $s \in \nabla$. □

Corollary 3.5 *If M_R is semi-projective and local and if every R-homomorphism in $\mathrm{End}_R(M/J(M_R))$ lifts to an R-homomorphism in S, then $J(S) = \nabla$.*

Proof. Since M is local, we have $J(M)$ is a maximal submodule of M and $J(M) \ll M$. So $M/J(M))$ is simple and $\mathrm{End}_R(M/J(M))$ is a division ring by Schur's Lemma. By the mapping $s \mapsto \theta(s)$ in the proof of Proposition 3.4(2) and by the hypothesis, we see that S/∇ is indeed a division (sub)ring of $\mathrm{End}_R(M/J(M))$. Thus S/∇ is regular and so $J(S) = \nabla$ by Proposition 3.3(3). □

A module M is said [13] to have the summand intersection property (SIP) if the intersection of two direct summands is again a direct summand. We call a module M a *duo module* if every submodule of M is fully invariant. In [10, Proposition 3.3], it is shown that every duo and PQ-injective module has the SIP. Here a module M is said to be *principally quasi-injective* (PQ-injective) [10] if for each element $m \in M$, every R-homomorphism $mR \to M$ can be extended to an endomorphism in S.

We prove here the corresponding result for a duo and semi-projective module. Note that every direct summand of M is of the form $s(M)$ for some $s \in S$.

Proposition 3.6 *Every duo and semi-projective module has the* SIP.

Proof. Let M_R be a duo and semi-projective module. Let $s(M)$ and $t(M)$ be summands of M where $s,\ t \in S$. Write $M = s(M) \oplus K$ and $M = t(M) \oplus L$. Now $s(M) = s(t(M) \oplus L) \subset st(M) + s(L) \subset (s(M) \cap t(M)) + (s(M) \cap L) = (s(M) \cap t(M)) \oplus (s(M) \cap L) \subset s(M)$. Thus $s(M) \cap t(M) \subset^{\oplus} M$. □

Remark 3.7 In fact, from the proof above, we can show more that $K = \mathrm{Ker}\, s := K_s$: From $M = s(M) \oplus K$ it follows that $s(K) \subset s(M) \cap K = 0$. Thus $K \subset K_s$, and $M = s(M) + K_s$. By the condition (D_2) K_s is a direct summand of M, thus, by the condition (D_3), we have $s(M) \cap K_s \subset^{\oplus} M$. Write $M = (s(M) \cap K_s) \oplus P$. $s(M) = s(P) \subset P \cap s(M)$. Therefore $P \supset s(M)$, and $s(M) \cap K_s = 0$. Now $M = s(M) \oplus K_s$, and $K = K_s$.

Recall that a module M has the property (C_2) if any of its submodules which is isomorphic to a direct summand is itself a direct summand. Every direct-injective module has the property (C_2). (C_2) implies (C_3) where we call a module M to have the property (C_3), if whenever N and K are direct summands with $N \cap K = 0$, then $N \oplus K$ is also a direct summand.

If we assume that M is quasi-principally injective, then $\operatorname{Ker} s \subset^{\oplus} M$ implies $s(M) \subset^{\oplus} M$. (A module M_R is said to be *quasi-principally injective* [11] if every element in $\operatorname{Hom}_R(s(M), M)$ can be extended to an element in S. A ring R is called *principally injective* if the module R_R is quasi-principally injective.) This is obvious since every quasi-principally injective module satisfies the property (C_2) (see [11, Lemma 2.5]).

The module is said [3] to have the summand sum property (SSP) if the sum of any two summands is again a summand.

Proposition 3.8 *Every duo and semi-projective module with the property (C_3) has the* SSP.

Proof. The proof follows the idea in [10]. Let M_R be a duo and semi-projective module with the property (C_3). Let $s(M)$ and $t(M)$ be summands of M where $s,\ t \in S$. By Proposition 3.6, we can write $M = (s(M) \cap t(M)) \oplus N$ for some $N \subset M$. Since M is duo, we see that $s(M) + t(M) = s(M) + (t(M) \cap ((s(M) \cap t(M)) \oplus N)) = s(M) + (s(M) \cap t(M)) + (t(M) \cap N) = s(M) + (t(M) \cap N) = s(M) \oplus (t(M) \cap N)$. To apply the property (C_3), we shall show that $t(M) \cap N$ is a summand of M. But this follows from the SIP of M (Proposition 3.6). □

Proposition 3.9 *Suppose M is a semi-projective and π-projective module. If S is semiperfect, then $M = \bigoplus_{i=1}^{n} H_i$, where H_i is hollow and semi-projective for each i.*

Proof. Since S is semiperfect and M is semi-projective, $M = H_1 \oplus \ldots \oplus H_n$, where each $\operatorname{End}_R(H_i)$ is local. Note that H_i is semi-projective. Each H_i is π-projective [14, 41.14], thus by Proposition 3.3(9) we see that H_i is hollow. □

Let

$$D(M) = \{s \in S : s(M) \subset^{\oplus} M\} \quad \text{and} \quad K(M) = \{s \in S : \operatorname{Ker} s \subset^{\oplus} M\}.$$

We know that $D(M) \subset K(M)$ when M is semi-projective, and the reverse inclusion holds for a principally quasi-injective module M.

To say that a module M has the property $(*)$ if

$(*)$: For each $s \in S$, there exists an idempotent $\alpha \in S$ such that $s(M) = \alpha(M)$.

It is known that S is regular if and only if $D(M) = S = K(M)$ [14, 37.3]. Thus S is regular if M has the property (C_2) and $K(M) = S$.

Also observe that the property $(*)$ implies $D(M) = S$.

Theorem 3.10 *For a semi-projective module M, S is regular if and only if M has the property $(*)$.*

Proof. $(\Rightarrow)$ Let $s \in S$. Thus $sS = \alpha S$ for some idempotent $\alpha \in S$. Hence M has the property $(*)$.

$(\Leftarrow)$ Let $s \in S$. Thus $s(M) = \alpha(M)$ for some $\alpha = \alpha^2 \in S$. By semi-projectivity we have $sS = \alpha S$. Since α is an idempotent, αS and hence sS is a direct summand of S. □

A module M_R is called *a principal selfgenerator* if for each $m \in M$, mR is M-cyclic. Clearly $M = R$ has this property as does every module in which each principal submodule is a summand (for example regular modules [15]). Here M is *regular* if, for any $m \in M$, $m = m\alpha(m)$ for some $\alpha \in \mathrm{Hom}_R(M, R)$. It is worth to include the following

Theorem 3.11 *For a finitely generated module M, M is regular if and only if mR is projective for all $m \in M$, M is a principal selfgenerator, and $D(M) = S$.*

Proof.$(\Rightarrow)$ It is known [15, Theorem 2.2] that every regular module M is a principal selfgenerator and for each $m \in M$, mR is projective. Finally, let $s \in S$. Thus $s(M) = s(m_1)R + \cdots + s(m_n)R$ for some $m_1, \ldots, m_n \in M$, and this sum is a direct summand of M by [15, Theorem 2.2].

$(\Leftarrow)$ Let $m \in M$. As M is a principal selfgenerator, $mR = s(M)$ for some $s \in S$. Now $D(M) = S$ implies $mR \subset^{\oplus} M_R$. Then apply [15, Theorem 2.2]. □

Theorem 3.12 *For a semi-projective module M,*
$D(M) = S$ if and only if S is regular
if and only if M has the property $(*)$

Proof. From the Remark after Proposition 3.6 we know that $D(M) \subset K(M)$. Thus the assertion follows from Theorem 3.10. □

It follows immediately that for a finitely generated semi-projective module M, if M is regular, then $\mathrm{End}_R(M)$ is regular.

Corollary 3.13 *Let R be direct-injective. Then R is a regular ring if and only if every principal right ideal of R is projective.*

Corollary 3.14 [4, Corollary 3.2(ii)] *If M is quasi-projective, then S is regular if and only if $D(M) = S$.*

Theorem 3.15 *For a semi-projective module M, if S is semiregular, then $(*')$ holds, where $(*')$ is the condition*

$(*')$: *For every $s \in S\backslash J(S)$, there exists $0 \neq \alpha^2 = \alpha \in S$ such that $\alpha(M) \subset s(M)$ and $(1-\alpha)s(M) \neq M$.*

If, in addition, S is local, then the converse is true.

Proof. Let $s \in S\backslash J(S)$. Take $\alpha^2 = \alpha \in sS$ such that $(1-\alpha)s \in J(S)$. Thus $\alpha \neq 0$. Now $\alpha S \subset sS$ implies $\alpha(M) \subset s(M)$. Write $\alpha' = 1 - \alpha$. If $\alpha' s(M) = M$, then $\alpha' sq = 1_M$ for some $q \in S$. Applying α on both sides, we have $\alpha = 0$, a contradiction.

The converse follows from Proposition 3.3(2). □

4 Injectivity

We now consider an interplay between M_R and it endomorphism ring S in terms of injectivity. Let M be a semi-projective module. Observe that the relation

$$s(M) \longleftrightarrow sS$$

establishes a bijection between the set of M-cyclic submodules of M and the set of principal right ideals of S.

For each $s \in S$, consider the mapping $\theta : \mathrm{Hom}_R(s(M), M) \to \mathrm{Hom}_S(sS, S)$ defined by $\theta(f) = F$, for each $f : s(M) \to M$ where $F : sS \to S$ is defined by $F(st) = fst$. It is clear that $\mathrm{Ker}\,\theta(f)(s) = \mathrm{Ker}\,F(s) \supset \mathrm{Ker}\,s$. On the other hand, if $F \in \mathrm{Hom}_S(sS, S)$ with $\mathrm{Ker}\,F(s) \supset \mathrm{Ker}\,s$, we define $f(sm) = F(s)(m)$ for $m \in M$. We see that $f \in \mathrm{Hom}_R(s(M), M)$, and $\theta(f) = F$.

The relation

$$f \longleftrightarrow F \tag{4.1}$$

establishes a bijection between $f \in \mathrm{Hom}_R(s(M), M)$ and $F \in \mathrm{Hom}_S(sS, S)$ with $\mathrm{Ker}\,F(s) \supset \mathrm{Ker}\,s$ in such a way that

$$F(st) = fst \text{ for all } s,\ t \in S, \tag{4.2}$$

and

$$f(s(m)) = F(s)(m) \text{ for all } s \in S \text{ and } m \in M.$$

Lemma 4.1 *If M is semi-projective and $s \in K(M)$, then* $\operatorname{Ker} F(s) \supset \operatorname{Ker} s$.

Proof. Write $M = \operatorname{Ker} s \oplus K$ for some submodule K of M. Let $t = 1_{\operatorname{Ker} s} \oplus 0$ associate to this decomposition. Thus $st = 0$. If $m \in \operatorname{Ker} s$, we have $F(s)(m) = F(s)(t(m)) = 0$. Hence $\operatorname{Ker} s \subset \operatorname{Ker} F(s)$. □

Theorem 4.2 *Let M be a semi-projective module.*

(1) *If S is principally injective, then M is quasi-principally injective.*

(2) *The converse in* (1) *is true if we assume, in addition, that $K(M) = S$.*

Proof. (1) To show that M is quasi-principally injective we let $s \in S$ and $f \in \operatorname{Hom}_R(s(M), M)$. Define $F \in \operatorname{Hom}_S(sS, S)$ corresponding to f by (4.2). Extend F to $\psi \in \operatorname{Hom}_S(S, S)$ and define $\varphi : M \to M$ corresponding to ψ by (4.2). Clearly, $\varphi \in S$. If $m \in M$, we have $\varphi(s(m)) = \psi(s)(m) = F(s)(m) = (fs)(m)$. That is, φ is an extension of f.

(2) If $F \in \operatorname{Hom}_S(sS, S)$, then as $f \in \operatorname{Hom}_R(s(M), M)$ by (4.2) and Lemma 4.1, there exists $\varphi \in S$ such that $\varphi|_{s(M)} = f$. Let $\psi \in \operatorname{Hom}_S(S, S)$ be the one corresponds to $\varphi \in S$ according to (4.2). We can see as above that ψ is an extension of F. □

A module M is called *quasi-mininjective* if for each $s \in S$ with $s(M)$ simple, every R-homomorphism in $\operatorname{Hom}_R(s(M), M)$ can be extended to an endomorphism in S. A ring R is called *mininjective* if the module R_R is quasi-mininjective.

Lemma 4.3 *For a semi-projective module M and $s \in S$, if $s(M)$ is simple, then sS is simple. The converse is true if, in addition, M is a principal self generator.*

Proof. Suppose $s(M)$ is simple. If $0 \neq stS \neq sS$ for some $t \in S$, then $st(M)$ is a nonzero proper submodule of $s(M)$, a contradiction. On the other hand, suppose M is a principal selfgenerator and sS is simple. If $0 \neq s(mR) \neq s(M)$, then for some $t \in S$, $st(M) = s(mR) \neq s(M)$. This implies $0 \neq stS \neq sS$, a contradiction. □

Theorem 4.4 *Let M be a semi-projective module.*

(1) *If S is mininjective, then M is quasi-mininjective.*

(1) *The converse in (1) is also true if we assume, in addition, that M is a principal selfgenerator and $s \in K(M)$ for all $s \in S$ with sS simple.*

Proof. With the help of Lemma 4.3 and 4.1, the proof follows the same lines as above. □

5 PP-endomorphism rings

A ring R is called a right *PP-ring* if each of its principal right ideal is projective. In [14, 39.10(4)] it is shown that

If a module M_R is a self-generator or ${}_SM$ is flat, and if S is a right PP-ring, then $K(M) = S$.

We consider the result for a semi-projective module.

Theorem 5.1 *For a semi-projective module M, $K(M) = S$ if and only if S is a right PP-ring and* $\operatorname{Ker} F(s) \supset \operatorname{Ker} s$ *for all* $F \in \operatorname{Hom}_S(sS, S)$ *and all* $s \in S$.

Proof. ($\Rightarrow$) The first part is [14, 39.10(1)]. The second part is Lemma 4.1.

($\Leftarrow$) By Proposition 3.1, it is enough to show that $s(M)$ is M-principally projective for all $s \in S$. Let $s,\ t \in S$ and $f : s(M) \longrightarrow t(M)$ be an R-homomorphism. Define $F \in \operatorname{Hom}_S(sS, S)$ as in (4.2). By projectivity of sS we get $\psi \in \operatorname{Hom}_S(sS, S)$ such that $t\psi = F$. That is, $t(\psi(sq)) = F(sq)$ for all $q \in S$. We now get a $\varphi \in \operatorname{Hom}_R(s(M), M)$ via (4.2) for ψ. For each $m \in M$,

$$(t\varphi)(s(m)) = (t\varphi(s))(m) = t\psi(s)(1_M(m)) = F(s1_M)(m) = (fs)(m) = f(s(m)).$$

This shows that $t\varphi = f$. □

A module M is said to be $(*)$-*hereditary* if every submodule of M is M-principally projective.

Lemma 5.2 *Let M be a semi-projective module. If s, t_1 and t_2 are endomorphisms on M such that $t_1(M) \cap t_2(M)$ and* $\operatorname{Ker} s$ *are direct summands of M and $s(M) = t_1(M) + t_2(M)$, then $s \in t_1S + t_2S$.*

Proof. Write $K_2 = t_1(M) \cap t_2(M)$, $t_1(M) = K_1 \oplus K_2$, $t_2(M) = K_2 \oplus K_3$. Put $M_i = s^{-1}(K_i)$ $(i = 1, 2, 3)$, and $K_s = \operatorname{Ker} s$. Thus $M_1 \cap M_2 \cap M_3 = K_s = M_i \cap M_j$ for $i \neq j$. Write $M_i = M'_i \oplus K_s$. From the fact that $K_i \cap t_j(M) = 0$ for $2 \neq i \neq j$ we have the sum $M'_1 + M'_2 + M'_3 + K_s$ is direct. For, if $m'_i \in M'_i$ for $i = 1, 2, 3$ and $k \in K_s$ such that $m'_1 + m'_2 + m'_3 + k = 0$, then, for example, $s(m'_1) = -s(m'_2) - s(m'_3) \in K_1 \cap t_2(M) = 0$. This implies $m'_1 \in M'_1 \cap K_s = 0$. Therefore $m'_1 = m'_3 = 0$ which implies $m'_2 + k = 0$ as well. Now, as $M'_2 \cap K_s = 0$, we see that $m'_2 = k = 0$.

Claim $M = M'_1 \oplus M'_2 \oplus M'_3 \oplus K_s$: Let $m \in M$. Since $s(M) = t_1(M) + t_2(M)$, there exist $x \in t_1(M)$ and $y \in t_2(M)$ such that $s(m) = x + y$. Write $x = a + b$ and $y = c + d$ for some $a \in K_1$, $d \in K_3$ and b and c are in K_2. As $t_1(M) \subset s(M)$ and by the definition of M_i, we can find $m'_i \in M'_i$ for $i = 1$ and 3, and h_1, $h_2 \in M'_2$ such that $s(m'_1) = a$, $s(m'_3) = d$, $s(h_1) = b$, and $s(h_2) = c$. Put $m' = m'_1 + (h_1 + h_2) + m'_3$.

Thus $m' \in M'_1 \oplus M'_2 \oplus M'_3$ and $s(m') = s(m'_1+h_1)+s(h_2+m'_3) = (a+b)+(c+d) = x+y = s(m)$. But then $m-m' \in K_s$, and so $m \in M'_1 \oplus M'_2 \oplus M'_3 \oplus K_s$. This proves the claim.

With respect to this decomposition of M, decompose $s = s_1 \oplus s_2 \oplus s_3 \oplus 0$ so that $s_i(M) = s_i(M'_i) = K_i$. Since $(s_1 \oplus s_2)(M) \subset t_1(M)$ and $s_3(M) \subset t_2(M)$, we have by semi-projectivity of M that $s_1 \oplus s_2 \in t_1S$ and $s_3 \in t_2S$. Therefore, $s \in t_1S + t_2S$. □

Theorem 5.3 *Let M be semi-projective. Consider the following conditions.*

(i) *M is $(*)$-hereditary.*

(ii) *$s(M)$ is M-injective for all $s \in S$.*

(iii) *sS is projective and injective for all $s \in S$.*

(1) *If M is quasi-injective, then* (i) *and* (ii) *are equivalent.*

(2) *If M has both the properties* (SIP) *and* (SSP) *(eg. M satisfies conditions in Proposition 3.8), then* (ii) *implies* (iii).

(3) *If M is a principal selfgenerator, then* (iii) *implies* (ii).

Proof. (1) This is Proposition 2.9 with $N = M$.

(2) From (ii) we have $D(M) = K(M) = S$, i.e. S is regular. Thus sS is projective for each $s \in S$. To show sS is injective we let $F \in \mathrm{Hom}_S(T, sS)$ where T is a right ideal of S. Put $T(M) = \sum_T t(M)$ and define a map $f : T(M) \to s(M)$ by $f(t(m)) = F(t)(m)$ for $m \in M$. Since $K(M) = S$, if $t(m) = 0$ where $t \in T$ and $m \in M$, then by Lemma 4.1 we have $F(t)(m) = 0$.

Claim $f \in \mathrm{Hom}_R(T(M), s(M))$: If $\sum_{i=1}^n t_i(m_i) = 0$ for some $t_i \in T$, and $m_i \in M$. Write $\sum_{i=1}^{n-1} t_i(M) = \alpha(M)$ for some $\alpha\ (= \alpha^2) \in S$ and $\alpha(M) + t_n(M) = \beta(M)$ for some $\beta\ (= \beta^2) \in S$. By Lemma 5.2 and by induction we have

$$\beta \in \alpha S + t_nS \subset \sum_{i=1}^{n} t_iS \subset T.$$

Now $0 = \sum_{i=1}^n t_i(m_i) = \beta(m)$ for some $m \in M$. Finally, $f(\sum_{i=1}^n t_i(m_i)) = f(\beta(m)) = F(\beta)(m) = 0$ as shown above. This completes the proof of the claim.

By injectivity of $s(M)$, we choose an extension $\varphi \in \mathrm{Hom}_R(M, s(M))$ of f. Since M is semi-projective, $\varphi = sq$ for some $q \in S$. We define a map $\psi : S \to sS$ via $\psi(t) = sqt$. ψ is an extension of F since, if $t \in T$ and $m \in M$, then $\psi(t)(m) = (sqt)(m) = f(t(m)) = f(t)(m)$.

(3) Let $f \in \mathrm{Hom}_R(N, s(M))$ where N is a submodule of M and $s \in S$. Define $T = \{t \in S : t(M) \subset N\}$. Thus T is a right ideal of S, and by the assumption,

$T(M) = N$. Define a map $F : T \to sS$ by $F(t) = ft$. Note that $ft(M) \subset s(M)$. Thus $ft \in sS$ and so $F \in \mathrm{Hom}_S(T, sS)$. By (iii), F is left multiplicative, that is, $F = sq \cdot -$ for some $q \in S$. Now define $\varphi \in \mathrm{Hom}_R(M, s(M))$ by $\varphi(m) = (sq)(m)$. We verify that φ is an extension of f. For this, let $n \in N$. Take $m \in N$ and $t \in T$ such that $t(m) = n$. Now $\varphi(n) = \varphi(t(m)) = (sqt)(m) = F(t)(m) = f(n)$. □

References

[1] F. W. Anderson and K. R. Fuller," Rings and Categories of Modules," Graduate Texts in Math.No.13 ,Springer-verlag, New York, 1992.

[2] V. Camillo, *Commutative rings whose principal ideals are annihilators,* Portugal Math., 46(1989), 33–37.

[3] J. L. Garcia, *Properties of direct summands of modules,* Comm. Algebra, **17**(1989), 73–92.

[4] S. M. Khuri, *Modules with regular, perfect, Noetherian or Artinian endomorphism rings,* Non-Commutative Ring Theory, Proc. Conf. Athens/OH(USA)1989, Lecture Notes in Math. 1448(1990), 7–18.

[5] S. H. Mohamed and B. J. Müller, "Continuous and Discrete Modules," London Math. Soc. Lecture Note Series 14, Cambridge Univ. Press, 1990.

[6] W. K. Nicholson, *Semiregular modules and rings,* Canad. J. Math., **28**(1976), 1105–1120.

[7] W. K. Nicholson and M. F. Yousif, *Principally injective rings,* J. Algebra, **174**(1995), 77–93.

[8] W. K. Nicholson and M. F. Yousif, *Mininjective rings,* J. Algebra, **187**(1997), 548–578.

[9] W. K. Nicholson and M. F. Yousif, *On perfect simple injective rings,* Proc. A.M.S., **125**(1997), 979–985.

[10] W. K. Nicholson, J. K. Park and M. F. Yousif, *Principally quasi-injective modules,* Comm. Algebra, **27**:4(1999), 1683–1693.

[11] N. V. Sanh; K. P. Shum ; S. Dhompongsa and S. Wongwai, *On quasi-principally injective modules,* Algebra Coll., **6**: 3(1999), 269–276.

[12] R. Ware, *Endomorphism rings of projective modules,* Trans. A.M.S., **155**(1971), 233–256.

[13] G. V. Wilson, *Modules with the summand intersection property,* Comm. Algebra, 14(1986), 21–38.

[14] R. Wisbauer, "Foundations of Module and Ring Theory," Gordon and Breach London, Tokyo e.a. 1991.

[15] J. M. Zelmanowitz, *Regular modules,* Trans. A.M.S., **163**(1972), 341–355.

Appendix 11:On V-rings and pV-modules.
In Preparation

ON V-RINGS AND pV-MODULES*

Jintana Sanwong

Abstract

A right R-module M is called a pV-module if every simple right R-module is p-M-injective. In this note it is shown that R is a right V-ring if and only if every cyclic right R-module is a pV-module.

2000 Mathematics Subject Classification. 16D30, 16D50, 16D60, 16D90

1 Introduction

Let M and N be right R-modules. Call M a *principally-N-injective* (*p-N-injective*) module if every R-homomorphism from a cyclic submodule of N to M extends to N. If M is principally R-injective, we call M a *principally injective* (*p-injective*) module.

Following [?], a ring R is a right V-ring if every simple right R-module is injective. A module M is a V-module or co-semisimple module if every simple right R-module is M-injective ([9]). It is known that R is a right V-ring if and only if every right R-module is a V-module. Tominaga in [7] call the ring R a right pV-ring if every simple right R-module is p-injective. Generalize this notion, we study pV-modules and prove that R is a right V-ring if and only if every cyclic right R-module is a pV-module.

2 Characterizations of pV-modules

Throughout this paper all rings R are associative with unity, and Mod-R is the category of unital right R-modules. Following [?], $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are all R-modules subgenerated by M. For a right R-module M

*Supported by Thailand Research Fund

with $X \subseteq M$ and $A \subseteq R$, the right annihilator of X in R is $r_R(X) = \{r \in R \mid Xr = 0\}$ and the left annihilator of A in M is $\ell_M(A) = \{x \in M \mid xA = 0\}$. For singletons $\{x\}$ and $\{a\}$, we abbreviate to $r_R(x)$ and $\ell_M(a)$. The notations $N \subset M$ and $N \subset^{max} M$ will mean N is a submodule and a maximal submodule of a module M respectively. As usual we denote the Jacobson radical of M by $Rad(M)$.

Definition 2.1 A module M is called a *pV-module* if every simple right R-module is p-M-injective. A ring R is called a *right pV-ring* if R_R is a pV-module.

Example 2.2 (1). Every V-module is a pV-module.

(2). If every cyclic submodule of M is a direct summand, then M is a pV-module. In particular, every regular module (see [?]) is a pV-module. Since there is a regular module which is not a V-module, not every pV-module is a V-module.

Before we present some characterizations of pV-modules, the following lemma about principal injectivity is needed.

Lemma 2.3 *Let $\{M_i\}_{i\in I}$ be an indexed set of p-N-injective modules.*

(1) *$\Pi_{i\in I}M_i$ is p-N-injective.*

(2) *$\oplus_{i\in I}M_i$ is p-N-injective.*

Proof. (1) We prove that $\Pi_{i\in I}M_i$ is p-N-injective. Let $n \in N$ and $\alpha : nR \longrightarrow \Pi_{i\in I}M_i$ be an R-homomorphism. Let $\pi_j : \Pi_{i\in I}M_i \longrightarrow M_j$ be the natural projection. Since M_j is p-N-injective, there exists an R-homomorphism $\beta_j : N \longrightarrow M_j$ with $\beta_j\imath = \pi_j\alpha$ where $\imath : nR \longrightarrow N$ is the inclusion map. Now, let $\gamma : N \longrightarrow \Pi_{i\in I}M_i$ be such that $\pi_i\gamma = \beta_i$ for each $i \in I$. Then $\pi_i\gamma(nr) = \beta_i(nr) = \beta_i\imath(nr) = \pi_i\alpha(nr)$ for all $r \in R$, all $i \in I$. Therefore $\gamma\imath = \alpha$.

(2) We show that $\oplus_{i\in I}M_i$ is p-N-injective. Let $n \in N$, $\alpha : nR \longrightarrow \oplus_{i\in I}M_i$ an R-homomorphism and $\imath : nR \longrightarrow N$ be the inclusion map. Since $\alpha(n) \in \oplus_{i\in I}M_i$, $\alpha(n) = (m_i)_{i\in I}$ where $m_i = 0$ for almost all i. Set $F = \{i \in I \ / \ m_i \neq 0\}$ and $K = \{(x_i) \in \oplus_{i\in I}M_i \ / \ x_i = 0 \text{ for all } i \notin F\}$. Then F is a finite subset of I and K is a submodule of $\oplus_{i\in I}M_i$. Thus it follows that $\alpha(nR) = \alpha(n)R = (m_i)_{i\in I}R$ is a submodule of K which is isomorphic to $\oplus_{i\in F}M_i$. Let $\imath' : \alpha(nR) \longrightarrow K$ be the inclusion map and let $\beta : K \longrightarrow \oplus_{i\in F}M_i$ be the obvious isomorphism. Because M_i is p-N-injective for all $i \in F$ and $\oplus_{i\in F}M_i = \Pi_{i\in F}M_i$, so by (1) we get $\oplus_{i\in F}M_i$ is p-N-injective. Hence there exists $\gamma : N \longrightarrow \oplus_{i\in F}M_i$ with $\gamma\imath = \beta\imath'\alpha$. Put $\bar{\alpha} = \beta^{-1}\gamma$. Then $\bar{\alpha} : N \longrightarrow K \subset \oplus_{i\in I}M_i$ is an R-homhmorphism which extends α. □

Proposition 2.4 *The following statements are equivalent for a module M:*

(1) *M is a pV-module;*

(2) *Every semisimple module in $\sigma[M]$ is p-M-injective;*

(3) *Every simple module in $\sigma[M]$ is p-M-injective;*

(4) *For any cyclic submodule C of M, any maximal submodule K of C, there exists a maximal submodule L of M such that $K = L \cap C$.*

Proof. (1) $\Longrightarrow$ (2) Let S be semisimple in $\sigma[M]$. Then $S = \oplus_{i \in I} S_i$ where S_i is simple for each $i \in I$. Since M is a pV-module, each S_i is p-M-injective, and thus $S = \oplus_{i \in I} S_i$ is p-M-injective by Lemma ??.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) follows from the fact that every simple module outside $\sigma[M]$ is p-M-injective.

(1) $\Longrightarrow$ (4) Let C be a cyclic submodule of M, and let K be a maximal submodule of C. Thus C/K is simple. Since M is a pV-module, there is an $h : M \longrightarrow C/K$ with $h\imath = \eta$ where $\imath : C \longrightarrow M$ is the inclusion map and $\eta : C \longrightarrow C/K$ is the canonical map. From the fact that $M/\ker h \simeq C/K$ which is simple we get that $L := \ker h$ is a maximal submodule of M, and $K = ker\eta = kerh \cap C = L \cap C$.

(4) $\Longrightarrow$ (1) Let S be a simple right R-module, C a cyclic submodule of M and $0 \neq \alpha \in \mathrm{Hom}(C, S)$. Then $K := \ker \alpha$ is a maximal submodule of C, since $C/K \simeq S$ is simple. Thus by (4), there is a maximal submodule L of M such that $K = L \cap C$. This implies $C \not\subseteq L$ and hence $M = C + L$. It follows that $M/K = C/K \oplus L/K$. Therefore α can be extended to M. □

If M, N are any right R-modules and $t \in M$, write

$$_tN = \{n \in N \mid r_R(t) \subseteq r_R(n)\}.$$

Then $_tN$ is a left S-module where $S = End(N_R)$.

For a cyclic right R-module M we have a characterization of M to be pV. At first we need the following lemma which is motivated by the work of Nicholson and Yousif in [6].

Lemma 2.5 *Let N be a right R-module, and let $M = tR$ be a cyclic right R-module. Then the following conditions are equivalent:*

(1) *N is p-M-injective;*

(2) *For each $a \in R$ and $\alpha \in \mathrm{Hom}(taR, N)$, $\alpha(ta) \in {}_tNa$;*

(3) *For each* $a \in R$, $\ell_N(r_R(ta)) = {}_tNa$.

Proof. (1) $\Longrightarrow$ (2) Let $a \in R$ and $\alpha \in \mathrm{Hom}(taR, N)$. Then there is an R-homomorphism $\beta : M \longrightarrow N$ such that $\beta|_{taR} = \alpha$. So $\alpha(ta) = \beta(t)a$ where $\beta(t) \in {}_tN$.

(2) $\Longrightarrow$ (3) Let $a \in R$ and $n \in \ell_N(r_R(ta))$. Then $nu = 0$ for all $u \in r_R(ta)$. Consider the mapping $\alpha : taR \longrightarrow N$ defined by $\alpha(tar) = nr$, we see that α is an R-homomorphism. Thus, by assumption, $n = n.1 = \alpha(ta.1) = \alpha(ta) \in {}_tNa$. The other inclusion always holds.

(3) $\Longrightarrow$ (1) Let $\alpha : taR \longrightarrow N$, $ta \in M$ be an R-homomorphism. Suppose $\alpha(ta) = n$, then $r_R(ta) \subseteq r_R(n)$. So $n \in \ell_N(r_R(n)) \subseteq \ell_N(r_R(ta)) = {}_tNa$ by (3), that is $n = ka$, $k \in {}_tN$. Thus the mapping $\beta : M \longrightarrow N$ defined by $\beta(tr) = kr$ is an R-homomorphism and $\beta(tar) = \beta(ta)r = (ka)r = nr = \alpha(ta)r = \alpha(tar)$. □

We can now characterize cyclic pV-modules.

Proposition 2.6 *Let $M = tR$ be a cyclic module. Then the following conditions are equivalent:*

(1) *M is a pV-module;*

(2) *For each $K \subset^{max} R_R$ and each $a \in R$,*

$$\ell_{R/K}(r_R(ta)) = {}_t(R/K)a.$$

Proof. (1) $\Longrightarrow$ (2) Let K be a maximal right ideal of R and $a \in R$. Then R/K is simple. Thus by (1), R/K is p-M-injective. So by Lemma ?? we get

$$\ell_{R/K}(r_R(ta)) = {}_t(R/K)a.$$

(2) $\Longrightarrow$ (1) Let S be simple. We prove that S is p-M-injective by using Lemma ??. Since $S \simeq R/K$ for some maximal right ideal K of R, by assumption we have $\ell_{R/K}(r_R(ta)) = {}_t(R/K)a$ for all $a \in R$. Thus R/K is p-M-injective, this infers that S is also p-M-injective. □

Proposition 2.7 *The following conditions are equivalent for a ring R:*

(1) *R is a right pV-ring;*

(2) *Every semisimple right R-module is p-M-injective;*

(3) *For any principal right ideal P of R, any maximal right ideal K of P, there exists a maximal right ideal L of R such that $K = L \cap P$;*

(4) *For each $K \subset^{max} R_R$ and each $a \in R$, $\ell_{R/K}(r_R(a)) = (R/K)a$;*

(5) *Every factor ring of R is a right pV-ring.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) follow from Proposition ??.

(1) $\Longleftrightarrow$ (4) Put $M = R = 1R$ in Proposition ??, then ${}_1(R/K) = \{\bar{x} \in R/K \mid r_R(1) \subseteq r_R(\bar{x})\} = R/K$. Thus we get the equivalence of (1) and (4).

(5) $\Longrightarrow$ (1) is trivial.

(1) $\Longrightarrow$ (5) Assume that R is a right pV-ring. Let I be an ideal of R. For convenience we set $M = R/I$ and thus $M = tR$ where $r_R(t) = I$. We prove that M is a right pV-ring. Let S be simple, $m = ta \in M$ for some $a \in R$ and $\alpha \in \mathrm{Hom}(mR, S)$. If $\alpha = 0$, then it is clear that $\alpha(m) = 0 \in {}_tSa$. Now, we consider the case $\alpha \neq 0$, and define $\beta : aR \longrightarrow mR$ by $\beta(ar) = mr$. Thus β is an R-homomorphism. Since R is a right pV-ring, there is $\delta \in \mathrm{Hom}(R, S)$ such that δ extends $\alpha\beta$, this gives $\delta(1)a = \delta(a) = \alpha\beta(a) = \alpha(\beta(a)) = \alpha(m) \neq 0$. From the fact that $r_R(t) = I$ is a two-sided ideal, $r_R(t) \subseteq r_R(\alpha(m))$. This infers that $0 \neq \alpha(m) \in {}_tS$. Because S is simple, so $\delta(1) = \alpha(m)b$ for some $b \in R$. Thus $r_R(t) \subseteq r_R(\alpha(m)b) = r_R(\delta(1))$ and that $\delta(1) \in {}_tS$. Hence $\alpha(m) = \delta(1)a \in {}_tSa$. Therefore by Lemma ??, S is p-M-injective and thus $M = R/I$ is a right pV-ring as required. □

Remark 2.8 The equivalence of (1) and (3) is Theorem 1 in Yue Chi Ming [?].

Recall that a ring R is a *right duo (quasi-duo) ring* if every (maximal) right ideal of R is a left ideal. Thus for a right quasi-duo ring we obtain:

Corollary 2.9 *If R is a right quasi-duo right pV-ring, then $r_R(a) \neq 0$ for all $a \in R$ with $aR \neq R$.*

Proof. Since R is a right pV-ring, $\ell_{R/M}(r_R(a)) = (R/M)a$ for all $M \subset^{max} R_R$ and all $a \in R$. Suppose there is an $a \in R$ with $aR \neq R$, but $r_R(a) = 0$. Let K be a maximal right ideal of R with $aR \subseteq K \subseteq R$. Then $R/K = \ell_{R/K}(r_R(a)) = (R/K)a$. Because R is right quasi-duo and $a \in K$, so $(R/K)a = \{K\}$. This means $R/K = \{K\}$ which contradicts to the maximality of K. □

3 V-rings and pV-modules

The ring R is a right V-ring if every simple right R-module is injective, equivalently every right R-module is a V-module. In this section the characterizations of right V-rings in terms of injectivity and V-modules will be weakened.

Lemma 3.1 *Let E be p-N-injective for all cyclic right R-modules N. If* $\mathrm{Hom}(S, E) \neq 0$ *for all simple right R-modules S, then E cogenerates every cyclic right R-module.*

Proof. Let N be a cyclic right R-module. We show that E cogenerates N. Let $0 \neq n \in N$. Since nR is cyclic, nR contains a maximal submodule, say L. So by hypothesis there is $0 \neq h \in \mathrm{Hom}(nR/L, E)$. Let $p : nR \longrightarrow nR/L$ be the canonical map. Then there is $\varphi \in \mathrm{Hom}(N, E)$ with $\varphi|_{nR} = hp$ since E is p-N-injective. Thus $\varphi(n) \neq 0$ and that $Rej_N(E) = \bigcap\{\ker h \mid h \in \mathrm{Hom}(N, E)\} = 0$. Therefore E cogenerates N as required. $\square$

If K, I are any right ideals of the ring R, we set

$$_I(R/K) = \{\bar{x} \in R/K \mid xI \subseteq K\}.$$

Thus it follows that $_I(R/K) = \{\bar{x} \in R/K \mid r_R(1 + I) \subseteq r_R(\bar{x})\} = {}_{1+I}(R/K)$ where $1 + I \in R/I$. So we have the following theorem.

Theorem 3.2 *For any ring R, the following are equivalent:*

(1) *R is a right V-ring;*

(2) *Every simple right R-module is p-M-injective for all right R-modules M;*

(3) *Every simple right R-module is p-N-injective for all cyclic right R-modules N;*

(4) *Every right R-module is a pV-module;*

(5) *Every cyclic right R-module is a pV-module;*

(6) *For all $K \subset^{max} R_R$ and $I \subset R_R$,*

$$\ell_{R/K}(r_R(a + I)) = {}_I(R/K)a$$

for all $a \in R$.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (5) and (2) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are obvious implications.

(5) $\Longrightarrow$ (1) We prove that R is a right V-ring by proving that $Rad(N) = 0$ for all cyclic right R-modules N.

Let $\{S_i \mid i \in I\}$ be a set of representatives of the distinct isomorphism classes of simple right R-modules. Then S_i is p-N-injective for all cyclic right R-modules N. Thus by Lemma 2.3, $E = \Pi_{i \in I} S_i$ is p-N-injective for all cyclic right R-modules N. From the fact that E cogenerates every simple right R-modules we get $\mathrm{Hom}(S, E) \neq 0$ for all simple right R-modules S. Hence by Lemma 3.1, E cogenerates every cyclic right R-module. So for each cyclic right R-module N, there is an indexed set A with $N \longrightarrow E^A$ a monomorphism. This infers that N is cogenerated by the class of simple modules. Hence $Rad(N) = 0$ for all cyclic right R-modules N.

(5) $\Longrightarrow$ (6) Let $K \subset^{max} R_R$, $I \subset R_R$ and $a \in R$. Then R/I is cyclic. Thus by (5), $R/I = (1 + I)R$ is a pV-module. So by Proposition ?? we get

$$\ell_{R/K}(r_R(a + I)) = {}_{1+I}(R/K)a = {}_I(R/K)a.$$

(6) $\Longrightarrow$ (5) Let N be a cyclic right R-module. Then $N \simeq R/I$ for some $I \subset R_R$. We prove that $R/I = (1 + I)R$ is a pV-module. Let K be a maximal right ideal of R and $a \in R$. Then by (6) we get $\ell_{R/K}(r_R(a + I)) = {}_I(R/K)a = {}_{1+I}(R/K)a$. Thus Proposition ?? shows that R/I is a pV-module. Hence N is a pV-module as required. $\square$

Corollary 3.3 *The following conditions are equivalent for a right duo ring R :*

(1) *R is a right V-ring;*

(2) *R is a right pV-ring;*

(3) *R is a von Neumann regular ring.*

Proof. (1) $\Longrightarrow$ (2) is evident. (1) $\Longleftrightarrow$ (3) is shown in Brown [?].

(2) $\Longrightarrow$ (1) Since R is a right pV-ring, by Proposition ?? we get that every factor ring of R is a right pV-ring. Because R is right duo, so every cyclic module is a pV-module. Thus R is a right V-ring by Theorem ??. $\square$

References

[1] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules", Graduate Texts in Math.No.13, Springer-Verlag, New York, 1992.

[2] S. H. Brown, *Rings over which every simple module is rationally complete*, Canad. J. Math., **25**(1973), 693–701.

[3] J. Cozzens and C. Faith, "Simple Noetherian Rings", Cambridge Univ. Press, 1975.

[4] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, "Extending Modules", Research Notes in Mathematics Series **313**, Pitman, London, 1994.

[5] S. H. Mohamed and B. J. Müller, "Continuous and Discrete Modules", Cambridge Univ. Press, 1990.

[6] W. K. Nicholson and M. F. Yousif, *Principally injective rings*, J. Algebra, **174**(1995), 77–93.

[7] H. Tominaga, *On S-unital rings*, Math. J. Okayama Univ., **18**(1976), 117–133.

[8] H. Tominaga, *On s-unital rings, II*, Math. J. Okayama Univ., **19**(1977), 171–182.

[9] R. Wisbauer, "Foundations of Modules and Ring Theory", Gordon and Breach, Reading 1991.

[10] R. Yue Chi Ming, *On simple p-injective modules*, Math. Japan., **19**(1974), 173–176.

[11] R. Yue Chi Ming, *On V-rings and prime rings*, J. Algebra, **62**(1980), 13–20.

[12] R. Yue Chi Ming, *On V-rings, P-V-rings and injectivity*, Kyungpook Math. J., **32**(1992), 219–227.

[13] J. Zelmanowitz, *Regular modules*, Trans. Amer. Math. Soc., **163**(1972), 341–355.

Department of Mathematics
Chiang Mai University
Chiang Mai 50200
Thailand

Reprints

On Generalized Q.F.D. Modules and Rings

Nguyen Van Sanh[1], Sompong Dhompongsa[2], and Jintana Sanwong[2]

[1] Department of Mathematics, Hue University, Vietnam
[2] Department of Mathematics, Chiang Mai University, Thailand

Abstract

Let R be a ring. A right R-module M is called a generalized q.f.d module if every M-singular quotient has finitely generated socle. In this paper we introduce and characterize this class of modules by means of the weak injectivity.

1. Introduction

In [8] Kurshan defined a class of q.f.d rings, that are rings over which every cyclic right module has finitely generated socle. Let R be a ring. A right R-module M is called a q.f.d module if every quotient has a finitely generated socle and generalized q.f.d module if every M-singular quotient has a finitely generated socle. A ring R is called a *right q.f.d ring (resp. a generalized q.f.d ring)* if R_R is a q.f.d module (resp. a generalized q.f.d module). The class of q.f.d modules was introduced and investigated by Camillo in [4] and some characterisations of q.f.d rings were given by Al-Huzali, Jain and López-Permouth in [1]. We now study some properties of generalized q.f.d modules and rings.

2. Results

Throughout the paper R is an associative ring with identity and Mod-R the category of unitary right R-modules. Let M be a right R-module. A module N is said to be *M-generated* if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite, then N is called *finitely M-generated*. Especially, N is called M-cyclic if it is isomorphic to M/L for some submodule $L \subset M$. The socle of the

Keywords: generalized q.f.d modules. wealky injective, M-singular module
1991 Mathematics Subject Classification: 16D50, 16D70, 16D80
This research is partially supported by Thailand Research Fund

module M is denoted by Soc(M). For $M \in$ Mod-R we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [11]).

Let M and N be right R-modules. N is called singular in $\sigma[M]$ or M-singular if there exists a module L in $\sigma[M]$ containing an essential submodule K such that $N \simeq L/K$ (see [11]). By definition, every M-singular module belongs to $\sigma[M]$. For $M = R_R$, the notion of being R-singular is identical to the usual definition of singular right R-module (see [6]). The class of all M-singular modules is closed under submodules, homomorphic images and direct sums (e.g. [11, 17.3 and 17.4]). Hence every module $N \in \sigma[M]$ contains a largest M-singular submodule which we denote by $Z_M(N)$.

Definition. A right module M is called a *generalized q.f.d module* if every M-singular quotient has a finitely generated socle. A ring R is called a right generalized q.f.d ring if it is a generalized q.f.d module as a right R-module.

At first we need:

Lemma 1. *If $0 \to A \to M \to B \to 0$ is an exact sequence of modules such that A and B are finite dimensional, then M is also finite dimensional and* $\dim M \leq \dim A + \dim B$.

Proof. See [8].

Theorem 2. *Let R be a ring and M be a right R-module. Then the following conditions are equivalent:*

(1) M is a generalized q.f.d module;

(2) Every M-cyclic M-singular module is finite dimensional;

(3) Every finitely M-generated M-singular module is finite dimensional;

(4) For every strictly increasing chain $A_1 \subset A_2 \subset \ldots$ of submodules of an M-cyclic M-singular right R-module N, there exists an integer n such that $A_k \subset^e A_{k+1}$ for any $k \geq n$;

(5) Every submodule N of an M-cyclic M-singular module contains a finitely generated submodule T such that N/T has no maximal submodules.

Proof. We use the same argument as that given in [10].

(1) $\Rightarrow$ (2). Suppose M is a generalized q.f.d module and N is an M-cyclic M-singular module. If N is not finite dimensional, then there is an infinite direct sum of nonzero submodules $T = \oplus_{i\in I} T_i \subset N$. Without loss of generality, we can assume that each T_i is cyclic. Then there exists submodules U_i of T_i which is maximal in T_i for each i. Put $U = \oplus_{i\in I} U_i$. Clearly N/U is M-cyclic and M-singular and its socle contains the infinite direct sum $T/U = \oplus_{i\in I} T_i/U_i$, which is contrary to the hypothesis.

(2) $\Rightarrow$ (3). Let $\varphi : M^n \longrightarrow N$ be an epimorphism and N is M-singular. We prove by induction on n. If $n = 1$, then N is M-cyclic and M singular and

therefore it is finite dimensional by (2). Now assume that the assertion is true for $n-1$, and set $M^n = M \oplus M^{n-1}$, $\varphi(M) = \{\varphi(m, 0, \ldots, 0) \in N | m \in M\}$ and $g = \varphi|_M$. Consider the diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota} & M \oplus M^{n-1} & \xrightarrow{\pi} & M^{n-1} & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow \varphi & & \downarrow f & & \\ 0 & \longrightarrow & \varphi(M) & \longrightarrow & N & \longrightarrow & N/\varphi(M) & \longrightarrow & 0 \end{array}$$

where $f : M^{n-1} \longrightarrow N/\varphi(M)$ is defined by $f(x_2, \ldots, x_n) = \varphi(0, x_2, \ldots, x_n) + \varphi(M)$. It is easy to see that g and f are epimorphisms. Clearly, $\varphi(M)$ and $N/\varphi(M)$ are M-singular. By hypothesis, $\varphi(M)$ and $N/\varphi(M)$ are finite dimensional. It follows from Lemma 1 that N is finite dimensional.

(3) $\Rightarrow$ (4) Suppose on the contrary that (4) fails. Let N be an M-cyclic M-singular module containing a strictly increasing chain of submodules $A_1 \subset A_2 \subset \ldots$ such that A_j is not essential in $\sum A_i$. Then there exists a subchain $A_{i_1} \subset A_{i_2} \subset \ldots$ such that A_{i_k} is not essential in $A_{i_{k+1}}$ for $k = 1, 2, \ldots$. Hence for every $k = 1, 2, \ldots$, we can find a nonzero submodule U_k such that $U_k \subset A_{i_{k+1}}$ and $U_k \cap A_{i_k} = 0$. It follows that the sum $\sum U_k$ is direct and hence N is not finite dimensional.

(4) $\Rightarrow$ (1) If $A = \oplus S_i$ is a semisimple submodule of an M-cyclic M-singular module N, then by (4) there is an integer n such that $S_1 \oplus S_2 \oplus \ldots \oplus S_n \subset^e A$, in this case $A = S_1 \oplus \ldots \oplus S_n$.

(2) $\Rightarrow$ (5). Suppose there exists a submodule N of an M-cyclic M-singular module L such that for every finitely generated submodule T of N, N/T has a maximal submodule. Let P_1 be a maximal submodule of N. For $x_2 \in N \setminus P_1$, let P_2 be a maximal submodule of N containing $x_2 R$. For $x_3 \in N \setminus P_2$ let $P_3 \subset N$ be a maximal submodule of N containing $x_2 R + x_3 R$. Continuing this process we obtain a sequence $x_1 = 0, x_2, x_3, \ldots,$ of elements of N and a sequence $P_1, P_2, P_3, \ldots,$ of maximal submodules of N such that

$$x_1 R + x_2 R + \ldots + x_k R \subset P_k \text{ and } x_{k+1} \notin P_k \text{ for each } k \geq 1.$$

Put $P = \cap_{i \in \mathbb{N}} P_i$ and $\overline{N} = N/P$. By induction,

$$\overline{N} = \left(\cap_{i=1}^{k} \overline{P}_i\right) \oplus \left(\cap_{i=k+1}^{\infty} \overline{P}_i\right).$$

Thus $0 \subset (\cap_{i=2}^{\infty} \overline{P}_i) \subset (\cap_{i=3}^{\infty} \overline{P}_i) \subset (\cap_{i=4}^{\infty} \overline{P}_i) \subset \ldots$ is a strictly ascending chain of direct summands of $\overline{N}$ and hence $\overline{N} = N/P$ is not finite dimensional. Therefore $\overline{L} = L/P$ is not finite dimensional. This contradicts to (2) because $\overline{L}$ is M-cyclic and M-singular.

(5) $\Rightarrow$ (1) Let N be an M-cyclic M-singular module. Then by assumption, Soc(N) contains a finitely generated submodule T such that Soc(N)/T has no maximal submodule. But Soc(N)/T is a homomorphic image of a semisimple module, thus it is semisimple and always has maximal submodules if it is not zero. Hence it must be zero and therefore Soc(N) $= T$. It follows that Soc(N) is finitely generated, as desired. $\square$

Let $M = R_R$, we obtain the following result:

Corollary 3. *For a ring R the following conditions are equivalent:*

(1) R is a generalized right q.f.d ring;

(2) Every cyclic singular right R-module is finite dimensional;

(3) Every finitely generated singular right R-module is finite dimensional;

(4) For every strictly increasing chain $A_1 \subset A_2 \subset \ldots$ of submodules of a cyclic singular right R-module N, there exists an integer n such that $A_k \subset^e A_{k+1}$ for any $k \geq n$;

(5) Every submodule N of a cyclic singular module contains a finitely generated submodule T such that N/T has no maximal submodules.

We now characterize generalized q.f.d modules and rings by means of weak injectivity. Let M be a right R-module. A module N is said to be weakly M-injective if for every homomorphism $f : M \longrightarrow E(N)$ there exists a monomorphism $\sigma : N \longrightarrow E(N)$ and a homomorphism $\widehat{f} : M \longrightarrow N$ such that $f = \sigma\widehat{f}$ (see [7]). We note that for any module $N \in \sigma[M]$ there exists an M-injective hull $\widehat{N}$ of N which is $\sum_{f:M\to E(N)} \operatorname{Im}(f)$, the trace of M in the injective hull $E(N)$ of N. Therefore, a module $N \in \sigma[M]$ is weakly M-injective if and only if for every homomorphism $f : M \longrightarrow \widehat{N}$ there exists a monomorphism $\sigma : N \longrightarrow \widehat{N}$ and a homomorphism $\widehat{f} : M \longrightarrow N$ such that $f = \sigma\widehat{f}$. If $N \in \sigma[M]$ is weakly X-injective for any module $X \in \sigma[M]$, then N is M-injective.

Following [3], a right R-module $N \in \sigma[M]$ is called *weakly injective in $\sigma[M]$* if for any finitely generated submodule X of $\widehat{N}$, the M-injective hull of N, there exists a submodule Y of $\widehat{N}$ which is isomorphic to N such that $X \subset Y$. Since every finitely generated submodule of $\widehat{N}$ is contained in a finitely M-generated submodule of $\widehat{N}$, then every module $N \in \sigma[M]$ which is weakly M^n-injective for all natural number n is weakly injective in $\sigma[M]$. It is easy to show that for a finitely generated module M, a module $N \in \sigma[M]$ is weakly M^n-injective for all natural number n if and only if it is weakly injective in $\sigma[M]$.

Theorem 4. *For a finitely generated right R-module M with $Z_M(M) = 0$ the following conditions are equivalent:*

(1) M is a generalized q.f.d module;

(2) Every direct sum of M-injective M-singular right R-modules is weakly injective in $\sigma[M]$;

(3) Every direct sum of M-singular right R-modules which are injective in $\sigma[M]$ is weakly injective in $\sigma[M]$;

(4) Every direct sum of M-singular right R-modules which are weakly injective in $\sigma[M]$ is weakly M-injective;

(5) Every direct sum of indecomposable M-injective M-singular right R-modules is weakly M-injective.

Proof. We use the same technique as that given in [1].

(1) $\Rightarrow$ (2). For convenience we denote $\widehat{X}$, the M-injective hull of X in $\sigma[M]$. Consider $X = \oplus_{i \in I} E_i$, where E_i is an M-injective M-singular module for each $i \in I$. Let N be a finitely M-generated submodule of $\widehat{X}$. Since M is non-M-singular, then $\widehat{X}$ is M-singular by [5, 4.1] and hence N is M-singular. By Theorem 2, N contains as an essential submodule a direct sum of uniform submodules $\oplus_{i=1}^{k} U_i$. Since X is essential in $\widehat{X}$, there exists $0 \neq q_i \in U_i \cap X$ for each $i = 1, 2, \ldots, k$. Therefore $\oplus_{i=1}^{k} q_i R$ is contained in a finite subsum $E_{i_1} \oplus E_{i_2} \oplus \ldots \oplus E_{i_t}$ of X. It follows that $E_{i_1} \oplus E_{i_2} \oplus \ldots \oplus E_{i_t}$ contains an M-injective hull E of $\oplus_{i=1}^{k} q_i R$. Since E is M-injective and contained in X, we may write $X = E \oplus K$, for some submodule K of X. On the other hand let $\widehat{N}$ be the M-injective hull of N inside $\widehat{X}$. Then $\widehat{N} = \oplus_{i=1}^{k} \widehat{U}_i = \oplus_{i=1}^{k} \widehat{q_i R} \simeq E$. Since $\oplus_{i=1}^{k} q_i R$ is essential in $\widehat{N}$, it follows that $\widehat{N} \cap K = 0$. So let $Y = \widehat{N} \oplus K \simeq E \oplus K = X$. Then $N \subset Y \simeq X$, proving our claim.

(2) $\Rightarrow$ (3). Let $X = \oplus_{i \in \Lambda} X_i$ be a direct sum of M-singular modules X_i, $i \in \Lambda$ which are weakly injective in $\sigma[M]$ and N a finitely M-generated submodule of $\widehat{X}$, the M-injective hull of X. By (2) the direct sum $\oplus_{i \in \Lambda} \widehat{X}_i$ of M-injective M-singular modules is weakly injective weakly injective in $\sigma[M]$ and

$$X \subset^{e} \oplus_{i \in \Lambda} \widehat{X}_i \subset^{e} \widehat{X}.$$

Therefore by (2) there exists a submodule $Y \subset \widehat{X}$ such that $N \subset Y$ and $Y \simeq \oplus_{i \in \Lambda} \widehat{X}_i$. We write $Y = \oplus_{i \in \Lambda} Y_i$, where $Y_i \simeq \widehat{X}_i, i \in \Lambda$. Since N is also finitely generated, hence there exists a finite subset $\Gamma \subset \Lambda$ such that $N \subset \oplus_{i \in \Gamma} \widehat{Y}_i = \widehat{\oplus_{i \in \Gamma} Y_i}$. Since Y_i's are weakly injective in $\sigma[M]$, the finite direct sum $\oplus_{i \in \Gamma} Y_i$ is again weakly injective in $\sigma[M]$ and hence there exists $Z_1 \simeq \oplus_{i \in \Gamma} Y_i \simeq \oplus_{i \in \Gamma} X_i$ such that $N \subset Z_1 \subset \widehat{\oplus_{i \in \Gamma} Y_i}$. But then $N \subset Z_1 \oplus (\oplus_{i \notin \Gamma} Y_i) \simeq X$, proving our assertion.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are trivial.

Now we prove (5) $\Rightarrow$ (1). Let M/L be an M-cyclic M-singular module. If $\mathrm{Soc}(M/L) = 0$, then we are done. Suppose $S = \mathrm{Soc}(M/L) \neq 0$.

Write $S = \oplus_{i \in \Lambda} S_i$ as a direct sum of simple modules S_i. We show that S is finitely generated. Clearly $\widehat{S} = \widehat{\oplus_{i \in \Lambda} S_i} = \oplus_{i \in \Lambda} \widehat{S}_i$. By hypothesis $\oplus_{i \in \Lambda} \widehat{S}_i$ is weakly M-injective, hence it is weakly M/L-injective. Consider the diagram

$$\begin{array}{ccc} S = \oplus_{i \in \Lambda} S_i & \xrightarrow{\lambda} & M/L \\ \downarrow{\scriptstyle \varphi} & & \\ \widehat{S} = (\widehat{\oplus_{i \in \Lambda} \widehat{S}_i}) & & \end{array}$$

where φ and λ are inclusion R-homomorphisms. By the M-injectivity of $\widehat{S}$, there exists $\widehat{\varphi} : M/L \to \widehat{S}$ such that $\widehat{\varphi}\lambda = \varphi$. Further, since $\oplus_{i \in \Lambda} \widehat{S}_i$ is weakly M/L-injective and since M is finitely generated by $\{m_1, m_2, \ldots, m_k\}$, say, there exists $X \subset \widehat{S}$ such that $\widehat{\varphi}(m_1 + L), \ldots, \widehat{\varphi}(m_k + L) \in X \simeq \oplus_{i \in \Lambda} \widehat{S}_i$. Hence there exists a finite subset Γ of Λ and an independent family of submodules $\{X_i\}_{i \in \Gamma}$

such that $\widehat{\varphi}(m_1 + L), \ldots, \widehat{\varphi}(m_k + L) \in \oplus_{i \in \Gamma} X_i$, and $X_i \simeq \widehat{S}_i$ for all $i \in \Gamma$. On the other hand, $S = \varphi(S) \subset \widehat{\varphi}(M/L) = \sum_{1 \leq j \leq k} \widehat{\varphi}(m_j + L)R \subset \oplus_{i \in \Gamma} X_i$. Since each X_i is uniform, S has finite dimension and is therefore finitely generated. The proof of the theorem is now complete. □

By taking $M = R_R$ we have the following result:

Corollary 5. *For a non-singular ring R the following conditions are equivalent:*

(1) R is a right generalized q.f.d ring;
(2) Every direct sum of injective singular right R-modules is weakly injective;
(3) Every direct sum of weakly injective singular right R-modules is weakly injective;
(4) Every direct sum of weakly injective singular right R-modules is weakly R-injective;
(5) Every direct sum of indecomposable injective singular right R-modules is weakly R-injective.

References

[1] Al-Huzali, A.H., Jain, S.K. and López-Permouth, S.R., *Rings whose cyclics have finite Goldie dimension*, J. Algebra, **153** No. 1 (1992), 37-40.

[2] Anderson, F.W., and Fuller, K.R., "Rings and Categories of Modules", Graduate Texts in Math. No.13, Springer-Verlag, New York, Heidelberg, Berlin, 1974.

[3] Brodskii, G.M., Saleh, M., Thuyet, L.V. and Wisbauer, R., *On weakly injectivity of direct sums of modules*, Vietnam J. Math., to appear.

[4] V. P., Camillo, V. P., *Modules whose quotients have finite Goldie dimension*, Pacific Journal of Mathematics, **69** No. 2 (1977), 387-388.

[5] Dung, N.V., Huynh, D.V., Smith, P.F. and Wisbauer, R. "Extending modules" Pitman, London, 1994.

[6] Goodearl, K.R., "Singular torsion and the splitting properties", Mem. Amer. Math. Soc. **No 124**, 1972.

[7] Jain, S.K., López-Permouth, S.R. and Singh, S., *On a class of QI-rings*, Glasgow Math. J. **34** (1992), 75-81.

[8] Kurshan, R.P., *Rings whose cyclic modules have finitely generated socle*, J. Algebra, **15** (1970), 376-386.

[9] López-Permouth, S.R., *Rings characterized by their weakly-injective modules*, Glasgow Math. J. **34** (1992), 349-353.

[10] Sanh, N.V. and Sanwong, J., *On modules whose quotients have finitely generated socles*, preprint.

[11] Wisbauer, R., "Foundations of Module and Ring Theory", Gordon and Breach, London, Tokyo e. a. 1991.

Algebras and Combinatorics. An International Congress. ICAC'97, Hong Kong.

On weak CS-modules*

N. v. Sanh, S. Dhompongsa, P. Jantagan

Abstract

A right R-module M is CS if every its complement is a direct summand. We now study a weaker form of this kind of modules. A right R-module M is called a weak CS-module if every semisimple submodule is essential in a direct summand of M. In his paper, P. Smith asked the question whether any direct summand of a weak CS-module is again weak CS. This question now is still open. In this note we prove that if M is a weak CS-module such that every direct summand is again weak CS and $M/Soc(M)$ satisfies ACC or DCC on direct summands, then $M = K \oplus L \oplus N$, where K is a modules with finite Goldie dimension, L is semisimple and N is a module with zero socle.

Key words: weak CS-module, chain conditions
1991 AMS Mathematics Subject Classification: 16D50, 16D70, 16D80

1 The Theorem

Throughout this paper R is an associative ring with identity and Mod-R is the category of unitary right R-modules. M will denote a right R-module. A submodule N of M is called essential in M if it has a non-zero intersection with any non-zero submodule of M. A family of submodules $\{N_i;\ i \in I\}$ is called independent if the sum $\sum_{i\in I} N_i$ is direct and all N_i are non-zero. For a module M, the socle (resp. radical) of M is denoted by $Soc(M)$ (resp. rad(M)). If $M = Soc(M)$, M is called a semisimple module. A complement submodule of a module M is a submodule N of M for which there is a submodule L of M such that N is maximal with respect to $L \cap N = 0$. A module is called a CS-module if every complement submodule is a direct summand or equivalently, if every submodule is essential in a direct summand. We use the terminologies ACC and DCC for the ascending and descending chain conditions.

Following [12], a right R-module M is called *weak CS*-module if every semisimple submodule is essential in a direct summand of M. The question in [12, 1.1] whether any direct summand of a weak CS module is again weak CS is now still open. Below we will call roughly such a module *weakly CS* module, i.e., M is a weakly CS-module if every direct

*The research of the first and the second author was partly supported by a grant of the Thailand Research Fund.

summand of M is weak CS. Since every module with zero socle is weakly CS, it is clear that the class of CS-modules and weakly CS-modules are different.

In [4], N.V. Dung showed that if M is a CS-module satisfying ACC on direct summands of $M/Soc(M)$, then M is a direct sum of a semisimple module and a module with finite Goldie dimension. In this note we get a similar result for weakly CS-modules as follows:

Theorem. *Let M be a weakly CS-module. If $M/Soc(M)$ has ACC or DCC on direct summands, then M is the sum $K \oplus L \oplus N$, where K is a module with finite Goldie dimension, L is semisimple and N is a module with zero socle.*

It was shown in [6] and [2] that $M/Soc(M)$ is Noetherian (resp. Artinian) if and only if it satisfies ACC (resp. DCC) on essential submodules. As an application of this fact and the above theorem we have the following corollary.

Corollary. *If M is weakly CS and satisfies ACC (resp. DCC) on essential submodules, then $M = K \oplus L$, where K is Noetherian (resp. Artinian) and L is semisimple.*

2 The Proof

We consider the following property (P) :

A right R-module is said to have property (P) if for any independent family $\{N_i \mid i \in I\!N\}$ of semisimple submodules of M, there exists a descending chain $\{M_i \mid i \in I\!N\}$ of direct summands of M such that $\bigoplus_{i \geq j} N_i$ is essential in M_j for every $j \in I\!N$.

Lemma 1. *Let M be a right R-modules. If M is a weakly CS-module, then M has property* (P).

Proof. Suppose that M is weakly CS. Let $\{N_i \mid i \in I\!N\}$ be an independent family of semisimple submodules of M. Let M_0 be a direct summand of M that contains essentially $\bigoplus_{i=1}^{\infty} N_i$. By definition, M_0 is weakly CS. Take M_1, a direct summand of M_0 that contains essentially $\bigoplus_{i=2}^{\infty} N_i$. Clearly M_1 is also a direct summand of M and $M_1 \subset M_0$. By induction, let M_k be a direct summand of M_{k-1} that contains essentially $\bigoplus_{i=k}^{\infty} N_i$. Then it is clear that M satisfies (P). ∎

Lemma 2. *Let M be a right R-module. If M satisfies ACC on direct summands, then it satisfies DCC on direct summands. Conversely, if M is weakly CS with essential socle and satisfies DCC on direct summands, then M satisfies ACC on direct summands.*

Proof. Suppose M satisfies ACC on direct summands and $\{M_i, i \in I\!N\}$ is a descending chain of direct summands of M

$$M_1 \supset M_2 \supset M_3 \supset \ldots \supset \cap_{i=1}^{n} M_i = M_n \supset \ldots . \qquad (*)$$

Since each M_i is a direct summand of M, then M_i is a direct summand of M_{i-1} by the modular law. Therefore $M_{i-1} = N_{i-1} \oplus M_i$ (for all i). Put $U_k = \bigoplus_{i \leq k} N_i$ then by our

construction. U_k is a direct summand of M for all k and they establish an ascending chain of direct summands of M. By hypothesis, this chain must be stationary, i. e., there exists n_0 such that $N_k = N_{n_0}$ for all $k \geq n_0$. This shows that the descending chain (*) is stationary. i. e., M satisfies DCC on direct summands.
Conversely, suppose that M is weakly CS with essential socle. Let $\{A_i, i \in \mathbb{N}\}$ be an ascending chain of direct summands of M and $A = \cup_{i \in \mathbb{N}} A_i$. Then $Soc(M) = Soc(A) \oplus U$ for some semisimple submodule U of M. Put $Soc(A) = \bigoplus_{i \in \Lambda} S_i$ where each S_i is simple. We will show that Λ is finite. We see that $Soc(A)$ is essential in a direct summand M_1 of M. Since M_1 is again weakly CS with essential socle, then for $i_1 \in \Lambda$, $\bigoplus_{i \in \Lambda, i \neq i_1} S_i$ is essential in a direct summand M_2 of M_1. Now we take $i_2 \in \Lambda \setminus \{i_1\}$. Then $\bigoplus_{i \in \Lambda \setminus \{i_1, i_2\}} S_i$ will be essential in a direct summand M_3 of M_2 (M_3 is a direct summand of M, since M_2 is). Continuing this process we have a descending chain $\{M_i, i \in \mathbb{N}\}$ of direct summands of M. By DCC condition, this chain must be stationary and therefore Λ is finite. This shows that M has ACC on direct summands. ■

Corollary 3. *Let M be a weakly CS-module with essential socle. Then M satisfies ACC or DCC on direct summands if and only if every independent family of semisimple submodules of M is finite.*

Proof. ($\Rightarrow$). By Lemma 2, it suffices to give the proof for the case of DCC. Let $\{N_i \mid i \in I\}$ be an independent family of semisimple submodules of M. Without loss of generality, we can assume that I is countable. Since M is weakly CS, then by Lemma 1, there exists a descending chain $\{M_i \mid i \in I\}$ of direct summands of M such that $\sum_{i \geq j} N_i$ is essential in M_j for every $j \in I$. Since M has DCC on direct summands, then the family $\{M_i \mid i \in I\}$ is finite. It follows that the family $\{N_i \mid i \in I\}$ is finite.
($\Leftarrow$). Let $M_1 \supset M_2 \supset \ldots \supset M_n \supset \ldots$ be a descending chain of direct summands of M. Since the socle of M is finitely generated and essential in M, this chain must be stationary. This shows that M has DCC on direct summands. ■

The following lemma is given by N.V. Dung in [4].

Lemma 4. *Let M be a module and $S = Soc(M)$. Then:*

(1) If A and B are submodules of M with $A \cap B = 0$, then

$$((A + S)/S) \cap ((B + S)/S) = 0;$$

(2) If A is a direct summand of M, then $(A + S)/S$ is a direct summand of M/S;

(3) If $\oplus_{i \in I} A_i$ is a direct sum of submodules of M, then $\oplus_{i \in I}((A_i + S)/S)$ is also a direct sum of submodules in M/S.

Lemma 5. ([8, Folgerung 9.1.5]). *If S is the socle of a direct sum $\oplus_{\alpha \in \kappa} N_\alpha$, then $S = \oplus_{\alpha \in \kappa} Soc(N_\alpha)$. Hence*

$$(\oplus_{\alpha \in \kappa} N_\alpha)/S \sim eq \oplus_{\alpha \in \kappa} (N_\alpha / Soc(N_\alpha)).$$

It is easy to see that:

Lemma 6. *Let K be a direct summand of a module M. If M has ACC (resp. DCC) on direct summands, then so does K.*

Now we are ready to give the proof of the theorem.

Proof of the Theorem. By definition, without loss of generality we can assume that M is weakly CS with essential socle. By Lemma 2, it is enough to give the proof for the DCC condition.

Let $\mathcal{F} = \{N_i \mid i \in I\}$ be a maximal independent family of direct summands of M such that for every $i \in I$, $Soc(N_i) \neq N_i$ and $Soc(N_i)$ is simple. Let K be a direct summand of M essentially containing $\oplus_{i\in I} Soc(N_i)$ and we write $M = K \oplus L$. Here we have some remarks.

Remark 1. For any direct summand N of K, we have $Soc(N) \neq N$. In fact, by the choice of $\mathcal{F}$, every simple submodule of K which is not a direct summand of M must be contained in rad(K), so is $Soc(K)$. Therefore any simple submodule of K is not a direct summand of K and the remark follows.

Remark 2. From the choice of $\mathcal{F}$, if N is direct summand of M such that $Soc(N)$ is finitely generated, $Soc(N) \neq N$ and there is no simple submodules of N which is its direct summand, then N must be contained in K. In fact, if there is such a direct summand $N \not\subset K$, then there exists $0 \neq E \subset N$, E is simple and $E \not\subset K$, otherwise every simple submodule of N is in K, then $N \subset K$, a contradiction. Take such a simple submodule E. Then E is essential in a direct summand E' of M with $E \neq E'$. This contradicts to the maximality of $\mathcal{F}$.

We may assume that I is countable. Since $\{Soc(N_i), i \in I\}$ is an independent family of simple submodules of K which is weak CS, then by Lemma 1, there exists a descending chain $\{M_n \mid n \in I\}$ of direct summands of K and hence of M such that $\bigoplus_{i\geq n} Soc(N_i)$ is essential in M_n for all $n \in I$. Put $S = Soc(K) = \oplus_{i\in I} Soc(N_i)$. By Lemma 4, $\{\overline{M}_n \mid n \in I\}$ is a descending chain of direct summands of M/S, where $\overline{M}_n = (M_n+S)/S$. By hypothesis, the descending chain $\{\overline{M}_n \mid n \in I\}$ is stationary. Thus there exists $k \in I$ such that

$$\overline{M}_k = \overline{M}_{k+j}$$

for all $j \in \mathbb{N}$. We show that $M_k = M_{k+j}$ for all $j \geq 1$. Since $k + j > k$, then $M_{k+j} \subset M_k$ and we have $M_k = M_{k+j} \oplus U, U \neq 0$. Therefore, $(M_k + S)/S = ((M_{k+j} + S)/S) \oplus ((U + S)/S) = \overline{M}_{k+j} \oplus ((U + S)/S) = \overline{M}_k \oplus ((U + S)/S)$. It follows that $(U + S)/S = 0$. Since U is a direct summand of K, then by Remark 1, $Soc(U) \neq U$ and hence $U \not\subset S$, therefore $U = 0$. Thus $M_k = M_{k+j}$. It follows that the descending chain $\{M_n \mid n \in I\}$ must be stationary, i.e., I is finite. Then $Soc(K) = \oplus_{i\in I} Soc(N_i)$ is finitely generated. Moreover $Soc(K)$ is semisimple and essential in K, then K has finite Goldie dimension.

Now, it remains to show that L is semisimple. Note that $Soc(L)$ is essential in L. Assume on the contrary that L is not semisimple. Then there exists a finitely generated submodule E of L which is not semisimple. Let H be a direct summand of M essentially containing E. Then $Soc(H) = Soc(E) \neq H$. If $Soc(E)$ is finitely generated, then H contains a non-zero direct summand U which $Soc(U) \neq U$, $Soc(U)$ is finitely generated and no simple submodule of U is a direct summand of U. Then by Remark 1 above, $U \subset K$, a contradiction. Hence $Soc(E)$ is not finitely generated, say $Soc(E) = \oplus_{i\in I} E_i$. We may assume that I is countable, say $I = \mathbb{N}$, and we will show that $Soc(E)$ must contain a submodule which is not finitely generated and is a direct summand of H.

Since $\mathbb{N}$ can be written in the form $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, $N_i \cap (\bigcup_{j\in \mathbb{N}\setminus\{i\}} N_j) = \emptyset$, N_i is infinite. then there exists an infinite independent family $\{T_n \mid n \in \mathbb{N}\}$ of semisimple submodules of E such that for each $n \in \mathbb{N}$, T_n is not finitely generated. Since H is a weakly CS-module. there exists a descending chain $\{C_m \mid m \in \mathbb{N}\}$ of direct summands of H such that $\oplus_{n\geq m} T_n$ is essential in C_m for every $m \in \mathbb{N}$ (see Lemma 1). Hence, $\overline{C}_m = (C_m + Soc(H))/Soc(H)$ is a direct summand of $H/Soc(H)$ for every $m \in \mathbb{N}$ (Lemma 4) and therefore $\{\overline{C}_m \mid m \in \mathbb{N}\}$ is a descending chain of direct summands of $H/Soc(H)$.

Since $H/Soc(H)$, being isomorphic to direct summand of $L/Soc(L)$, also has DCC on direct summands (Lemma 6) and $\mathbb{N}$ is infinite, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\overline{C}_k = \overline{C}_{k+1}. \tag{1}$$

Since $Soc(H) = (Soc(H) \cap C_k) \oplus U$ for some $U \subseteq Soc(H)$, we have $C_k + Soc(H) = C_k \oplus U$. Since $Soc(H) = Soc(C_k) \oplus U$ and $C_{k+1} \oplus T_k$ is essential in C_k by our construction, we have

$$C_{k+1} + Soc(H) = C_{k+1} \oplus T_k \oplus U. \tag{2}$$

From this and (1) we have

$$(C_{k+1} \oplus T_k \oplus U)/Soc(H) = (C_k \oplus U)/Soc(H). \tag{3}$$

Since $C_{k+1} \oplus T_k \subseteq C_k$, it follows that $C_{k+1} \oplus T_k = C_k$. This shows that T_k is a direct summand of H and hence of E. But this is impossible because E is finitely generated and T_k is not finitely generated by our assumption. This contradiction shows that every finitely generated submodule of L must be semisimple. Therefore L is semisimple and the proof is complete. ■

Remark. Following the proof, we can replace the condition that M is weakly CS by assuming that M is weak CS and non-singular. With this condition, every direct summand of M is weak CS and the lemmas 1 and 2 are satisfied by an adaptation of the proofs in [12].

References

[1] F.W. Anderson and K.R. Fuller,"Rings and Categories of Modules", Graduate Texts in Math. No.13, Springer - Verlag, New York, Heidelberg, Berlin, 1974.

[2] E.P. Armendariz, *Rings with dcc on essential left ideals*, Comm. Algebra, **13** (1980) 229-308.

[3] V.P. Camillo and M.F. Yousif, *CS-modules with acc or dcc on essential submodules*, Comm. Algebra, **19**(2) (1991), 655-662.

[4] N.V. Dung, *Generalized injectivity and chain conditions*, Glasgow Math. J., **34** (1992), 319-326.

[5] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, "Extending modules", Longman, Harlow, 1994.

[6] D.V. Huynh, N. V. Dung and R. Wisbauer, *Quasi-injective modules with acc or dcc on essential submodules*, Arch.Math. **53** (1989), 252-255.

[7] C. Faith, "Algebra 2: Rings, Modules and Categories", Springer - Verlag, Berlin-Heidelberg - New York, 1981.

[8] F. Kasch, "Moduln und Ringe", Stuttgart, 1977.

[9] S.H. Mohamed and B.J. Müller, "Continuous and discrete modules", London Math. Soc. Lecture Note Series 14, Cambridge Univ. Press, 1990.

[10] B.L. Osofsky, *Chain conditions on essential submodules*, Proc. Amer. Math. Soc. **114**(1) (1992), 11-19.

[11] N.V. Sanh, *Chain condition on direct summands*, Studia Sci. Math. Hungarica, **32** (1996), 227-233.

[12] P.F. Smith, *CS-modules and weak CS-modules*, Noncommutative Ring Theory, Springer LNM 1448 (1990), 99-115.

[13] P.F. Smith, *Modules for which every submodule has a unique closure*, Ring Theory, Eds. S.K. Jain and S. Tariq Rizvi, World Sci. Singapore-New Jersey-London-Hong Kong (1993), 302-313.

[14] R. Wisbauer, "Foundations of Module and Ring Theory", Gordon and Breach, London, Tokyo , e.a., 1991.

author's addresses
Nguyen Van Sanh
Dept. of Mathematics, Hue University,
Vietnam and
Khon Kaen University,
Thailand
e-mail: vansanh@kku.ac.th

Sompong Dhompongsa and Preecha Jantagan
Dept. of Mathematics, Chiang Mai University,
Thailand
e-mail:sompongd@cmu.chiangmai.ac.th

Algebra Colloquium 6:3 (1999) 269–276

Algebra
Colloquium

On Quasi-principally Injective Modules

Nguyen Van Sanh*
Department of Mathematics, Hue University, Vietnam

Kar Ping Shum**
Department of Mathematics, The Chinese University of Hong Kong
Shatin, N. T., Hong Kong
E-mail: kpshum@euler.math.cuhk.edu.hk

Sompong Dhompongsa† **Sarun Wongwai**
Department of Mathematics, Chiang Mai University, Thailand
E-mail: sompongd@cmu.chiangmai.ac.th

Received 11 May 1998
Revised 30 November 1998

Communicated by R. Wiegandt

Abstract. Let R be a ring. A right R-module M is called principally injective if every homomorphism from a principal right ideal of R to M can be extended to R. We extend this notion to modules. A module N is called M-principally injective if every homomorphism from an M-cyclic submodule of M to N can be extended to M. In this paper, we give some characterizations and properties of quasi-principally injective modules which generalize results of Nicholson and Yousif.

1991 Mathematics Subject Classification: 16D50, 16D70, 16D80

Keywords: M-principally injective modules, quasi-principally injective modules, right self-principally injective rings, self-generator

1 Introduction

Let R be a ring. Call a right R-module M *principally injective* if any R-homomorphism from a principal right ideal of R to M can be extended

---

*Current address: Department of Mathematics, Khon Kaen University, Khon Kaen 40200, Thailand (E-mail: vansanh@kku.ac.th).

**Partially supported by CUHK grant #2010126.
†Supported by Thailand Research Fund.

to an R-homomorphism from R to M. This notion was introduced by Camillo [2] for commutative rings. In [7], Nicholson and Yousif studied the structure of self-principally injective rings and gave some applications. The nice structure of self-principally injective rings draws our attention to bring this notion to modules. We see that every principal right ideal I of a ring R can be considered as a homomorphic image of R and vice versa. We use this fact to generalize the notion of principal injectivity to M-principal injectivity for a given right R-module M. Throughout this paper, R is an associative ring with identity. Let M be a right R-module. A module N is called *M-generated* if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called *finitely M-generated*. In particular, N is called *M-cyclic* if it is isomorphic to M/L for some submodule $L \subseteq M$. As usual, the socle of the module M is denoted by $\mathrm{Soc}(M)$.

2 Principal Injectivity

Definition 2.1. Let M be a right R-module. A right R-module N is called *M-principally injective* if every homomorphism from an M-cyclic submodule of M to N can be extended to a homomorphism from M to N. Equivalently, for any endomorphism ε of M, every homomorphism from $\varepsilon(M)$ to N can be extended to a homomorphism from M to N. N is called *principally injective* if it is R-principally injective.

Lemma 2.2. *Let X_i $(1 \leq i \leq n)$ be M-principally injective modules. Then $\oplus_{i=1}^{n} X_i$ is M-principally injective.*

Proof. It is enough to prove the result for $n = 2$. Let $\varphi : C \to X \oplus Y$ be a homomorphism, where C is an M-cyclic submodule of M. Since X and Y are M-principally injective, there exist $\varphi_1 : M \to X$ and $\varphi_2 : M \to Y$ such that $\varphi_1 \iota = \pi_1 \varphi$ and $\varphi_2 \iota = \pi_2 \varphi$, where π_1 and π_2 are the natural epimorphisms from $X \oplus Y$ to X and Y, respectively, and $\iota : C \to M$ is the embedding. Put $\bar{\varphi} = \iota_1 \varphi_1 + \iota_2 \varphi_2 : M \to X \oplus Y$. Then it is clear to see that $\bar{\varphi}$ extends φ. □

Lemma 2.3. *Let C be an M-cyclic submodule of M. If C is M-principally injective, then it is a direct summand of M.*

Proof. Since the embedding $\iota : C \to M$ has a left inverse, it clearly splits and so C is a direct summand of M. □

Call a module M *quasi-principally injective* if it is M-principally injective. A ring R is *right self-principally injective* if R_R is R-principally injective. The following lemma is straightforward. By this lemma, we can produce many quasi-principally injective modules from a right self-principally injective ring.

Lemma 2.4. *Any direct summand of a quasi-principally injective module* *is again quasi-principally injective.*

The next lemma shows that the conditions (C_2) and (C_3) (see [6]) also hold in a quasi-principally injective module.

Lemma 2.5. *Let M be a quasi-principally injective module and A, B its submodules.*

(1) *If A is a direct summand of M and $B \simeq A$, then B is a direct summand of M.*
(2) *If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.*

Proof. (1) Since A is a direct summand of M, it is M-principally injective, and so is B. Therefore, B is a direct summand of M by Lemma 2.3.

(2) Let $M = A \oplus A'$ and $\pi : A \oplus A' \to A'$ be the projection. Then $\pi|_B : B \to A'$ is a monomorphism, and therefore, $B \simeq \pi(B)$. By (1), $\pi(B)$ is a direct summand of M, and hence, a direct summand of A' by the modular law. Thus, $A \oplus \pi(B)$ is a direct summand of M. Since $A \oplus B \simeq A \oplus \pi(B)$, it follows from (1) that $A \oplus B$ is again a direct summand of M. □

Corollary 2.6. *Let M be a quasi-principally injective, quasi-projective module and $s : M \to M$ an endomorphism. Then the following conditions are equivalent:*

(1) $\mathrm{Im}(s)$ *is a direct summand of M.*
(2) $\mathrm{Im}(s)$ *is M-principally injective.*
(3) $\mathrm{Im}(s)$ *is M-projective.*

Proof. (1)⇒(3). It follows from the projectivity of M.

(3)⇒(2). Since the sequence $0 \to \mathrm{Ker}(s) \to M \to \mathrm{Im}(s) \to 0$ splits, $\mathrm{Im}(s)$ is isomorphic to a direct summand of M. Therefore, it is a direct summand of M by Lemma 2.5. Hence, it is M-principally injective.

(2)⇒(1). It follows from Lemma 2.3. □

Following [3], a right R-module M is said to be direct-projective if, for any direct summand X of M, every epimorphism $f : M \to X$ splits (i.e., $\mathrm{Ker}(f)$ is a direct summand of M). Combining Lemma 2.3 and Theorem 37.7 in [9], we can state the following theorem.

Theorem 2.7. *Let $S = \mathrm{End}(M_R)$ be the endomorphism ring of a module M. If S is von Neumann regular, then every M-cyclic submodule of M is M-principally injective. Conversely, if M is direct-projective and every M-cyclic submodule is M-principally injective, then $S = \mathrm{End}(M_R)$ is von Neumann regular.*

We now consider the endomorphism ring of a quasi-principally injective module.

Theorem 2.8. *Let M be a quasi-principally injective module and $s, t \in S = \mathrm{End}(M_R)$.*

(1) *If $s(M)$ embeds into $t(M)$, then Ss is an image of St.*
(2) *If $t(M)$ is an image of $s(M)$, then St can be embedded into Ss.*
(3) *If $s(M) \simeq t(M)$, then $Ss \simeq St$.*

Proof. (1) Let $f : s(M) \to t(M)$ be a monomorphism. Let $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ be embeddings. Then there is a map $s' : M \to s(M)$ induced by $s : M \to M$ (i.e., $\iota_1 s' = s$). Since M is quasi-principally injective, the homomorphism $f : s(M) \to t(M)$ can be extended to $\bar{f} : M \to M$ such that $\bar{f}\iota_1 = \iota_2 f$. Let $\sigma : St \to Ss$ be defined by $\sigma(ut) = u\bar{f}s$ for every $u \in S$. Since $\mathrm{Im}(\bar{f}s) \subseteq t(M) = \mathrm{Im}(t)$, σ is well-defined. Moreover, σ is an S-homomorphism. For any $v \in S$, $v\iota_1 : s(M) \to M$ can be extended to an R-homomorphism $\varphi : M \to M$ such that $\varphi\iota_2 f = v\iota_1$. Consequently, we have $\sigma(\varphi s) = \varphi\bar{f}s = \varphi\bar{f}\iota_1 s' = \varphi\iota_2 f s' = v\iota_1 s' = vs$. This shows that σ is an epimorphism, proving our claim.

(2) By the same notation as in (1), let $f : s(M) \to t(M)$ be an epimorphism. Since M is quasi-principally injective, f can be extended to $\bar{f} : M \to M$ such that $\bar{f}\iota_1 = \iota_2 f$. Define $\sigma : St \to Ss$ by $\sigma(ut) = u\bar{f}s$ for any $ut \in St$. Then the map σ is well-defined since $\mathrm{Im}(t) = \mathrm{Im}(\bar{f}s)$. It is clear that σ is an S-monomorphism.

(3) This part follows immediately from (1) and (2). □

Corollary 2.9. [7, Theorem 1.1] *Let R be right self-principally injective and $a, b \in R$. Then the following statements hold:*
(1) *If bR embeds in aR, then Rb is an image of Ra.*
(2) *If aR is an image of bR, then Ra can be embedded in Rb.*
(3) *If $bR \simeq aR$, then $Ra \simeq Rb$.*

Proof. The above statements are direct consequences of Theorem 2.8. □

From now on, we use the symbols ℓ and r to indicate the left and right annihilators, respectively.

Theorem 2.10. *Let M be a right R-module and $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:*
(1) *M is quasi-principally injective.*
(2) *$\ell_S(\mathrm{Ker}(s)) = Ss$ for all s in S.*
(3) *$\mathrm{Ker}(t) \subseteq \mathrm{Ker}(s)$ implies $Ss \subseteq St$ for any $s, t \in S$.*
(4) *$\ell_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s)) = \ell_S(\mathrm{Im}(t)) + Ss$ for all $s, t \in S$.*

Proof. (1)⇒(2). For any $t \in \ell_S(\mathrm{Ker}(s))$, we have $t(\mathrm{Ker}(s)) = 0$. This leads to $\mathrm{Ker}(s) \subseteq \mathrm{Ker}(t)$. Let $s' : M \to s(M)$ and $t' : M \to t(M)$ be R-homomorphisms induced by s and t, respectively, and $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ the embeddings. Since s' is an epimorphism, there is an R-homomorphism $\varphi : s(M) \to t(M)$ such that $\varphi s' = t'$. Since M is quasi-principally injective, there exists an R-homomorphism $u : M \to M$ such that $u\iota_1 = \iota_2\varphi$. Hence, $t = us$ and $t \in Ss$. On the other hand, since $s \in \ell_S(\mathrm{Ker}(s))$, we see $Ss \subseteq \ell_S(\mathrm{Ker}(s))$.

(2)$\Rightarrow$(3). From $\mathrm{Ker}(t) \subseteq \mathrm{Ker}(s)$, we have $\ell_S(\mathrm{Ker}(s)) \subseteq \ell_S(\mathrm{Ker}(t))$. Therefore, we have $Ss \subseteq St$ by (2).

(3)$\Rightarrow$(1). Let $s' : M \to s(M)$ be an R-homomorphism induced by $s : M \to M$ and $\iota_1 : s(M) \to M$. Let $\varphi : s(M) \to M$. Then $\varphi s'$ is clearly an R-endomorphism of M and $\mathrm{Ker}(s) \subseteq \mathrm{Ker}(\varphi s')$. By (3), we have $S\varphi s' \subseteq Ss$, and therefore, $\varphi s' = us$ for some $u \in S$. This shows that M is quasi-principally injective.

(3)$\Rightarrow$(4). Let $u \in \ell_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s))$. Then $u(\mathrm{Im}(t) \cap \mathrm{Ker}(s)) = 0$. This implies $\mathrm{Ker}(st) \subseteq \mathrm{Ker}(ut)$. By (3), $ut = vst$ for some $v \in S$. It follows that $(u - vs)t = 0$, and therefore, $u - vs \in \ell_S(\mathrm{Im}(t))$, i.e., $u \in \ell_S(\mathrm{Im}(t)) + Ss$. This shows $\ell_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s)) \subseteq \ell_S(\mathrm{Im}(t)) + Ss$. Conversely, any $x \in \ell_S(\mathrm{Im}(t)) + Ss$ can be written in the form $x = u + v$, where $u(\mathrm{Im}(t)) = 0$ and $v(\mathrm{Ker}(s)) = 0$. It follows that $x \in \ell_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s))$.

(4)$\Rightarrow$(2). This part is clear by taking $t = 1_M$, the identity map of M. □

Corollary 2.11. [7, Lemma 1.1] *The following conditions are equivalent for a ring R:*

(1) *R is right self-principally injective.*
(2) *$\ell r(a) = Ra$ for all a in R.*
(3) *$r(b) \subseteq r(a)$ for $a, b \in R$ implies $Ra \subseteq Rb$.*
(4) *$\ell(bR \cap r(a)) = \ell(b) + Ra$ for all a, b in R.*

The following corollary is a generalization of Corollary 1.1 in [7].

Corollary 2.12. *If $K = s(M)$ is a simple submodule of a quasi-principally injective module M where $s \in S = \mathrm{End}(M_R)$, then $SK = \sum_{s \in S} s(K)$ is the homogeneous component of $\mathrm{Soc}(M)$ containing K.*

Proof. It is clear that, for any isomorphism $u : K \to H$ where $H \subseteq M$, we have $\mathrm{Ker}(u) = \mathrm{Ker}(us)$. By Theorem 2.10, $Ss = S(us)$, and hence, $H = us(M) \subseteq SK$. This shows that the K-component is in SK. The other inclusion always holds. □

Following [9], a module M is called a *self-generator* if it generates all its submodules. Since every homomorphism from M to its submodule can be considered as an endomorphism of M, for every $m \in M$, we have $mR = \sum_{s \in I} s(M)$ for some $I \subseteq S$ if M is a self-generator. The following theorem is a generalization of Theorem 2.1 in [7].

Theorem 2.13. *Let M be a right R-module, $S = \mathrm{End}(M_R)$, Δ the set of $s \in S$ such that $\mathrm{Ker}(s)$ is essential in M, and $J(S)$ the Jacobson radical of S. If M is a quasi-principally injective module which is a self-generator, then $J(S) = \Delta$.*

Proof. Since $\mathrm{Ker}(s) \cap \mathrm{Ker}(1 - s) = 0$, we have $\mathrm{Ker}(1 - s) = 0$ for any $s \in \Delta$. Hence, $S = \ell_S(\mathrm{Ker}(1 - s)) = S(1 - s)$ by Theorem 2.10(2). This shows that $J(S) \supseteq \Delta$. For the converse, we show that, for every $s \in J(S)$, if $t(M) \cap \mathrm{Ker}(s) = 0$ for $t \in S$, then $t = 0$. In fact, by Theorem 2.10(4), we have

$$\ell_S(\mathrm{Im}(t)) + Ss = \ell_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s)) = S,$$

so $\ell_S(\mathrm{Im}(t)) = S$, i.e., $t = 0$. Since M is a self-generator, for any $m \in M$, we have $mR = \sum_{t \in I} t(M)$ for some $I \subseteq S$. If $\mathrm{Ker}(s) \cap mR = 0$, then $\mathrm{Ker}(s) \cap t(M) = 0$ for all $t \in I$, and hence, $mR = 0$. This shows that $\mathrm{Ker}(s)$ is essential in M and the proof is complete. □

Corollary 2.14. [7, Theorem 2.1, Corollary 2.1] *If R is right self-principally injective, then $J(R) = Z(R_R)$. Consequently, if R is right and left self-principally injective, then $Z(R_R) = Z({}_RR)$.*

Theorem 2.15. *Let M be a quasi-principally injective module and $s_1, \cdots, s_n \in S = \mathrm{End}(M)$ such that the sum $\sum_{i=1}^n Ss_i$ is direct. Then any homomorphism $\alpha : \sum_{i=1}^n s_i(M) \to M$ can be extended to a homomorphism $\varphi : M \to M$.*

Proof. Each $s_i(M)$ is M-cyclic. By the quasi-principal injectivity of M, there exists a homomorphism $\varphi_i : M \to M$ such that $\varphi_i s_i(m) = \alpha s_i(m)$ for all $m \in M$. It follows that $\sum_{i=1}^n \varphi_i s_i = \sum_{i=1}^n \alpha s_i$. Since $(\sum_{i=1}^n s_i)(M) \subseteq \sum_{i=1}^n s_i(M)$, α can be extended to $\varphi : M \to M$ such that, for any $m \in M$,

$$\varphi\Big(\sum_{i=1}^{n} s_i\Big)(m) = \alpha\Big(\sum_{1 \le i \le n} s_i\Big)(m),$$

i.e., $\sum_{i=1}^n \varphi s_i = \sum_{i=1}^n \alpha s_i$. It follows that $\sum_{i=1}^n \varphi s_i = \sum_{i=1}^n \varphi_i s_i$. The direct sum $\oplus_{i=1}^n Ss_i$ implies $\varphi s_i = \varphi_i s_i$ for all $1 \le i \le n$. Therefore, for any $x \in \sum_{i=1}^n s_i(M)$, we have $\alpha(x) = \varphi(x)$, proving our theorem. □

Corollary 2.16. [7, Lemma 3.1] *Let R be right self-principally injective and assume $Rb_1 \oplus Rb_2 \oplus \cdots \oplus Rb_n$ is a direct sum where $b_i \in R$. Then any linear map*

$$\alpha : b_1R + b_2R + \cdots + b_nR \to R$$

can be extended to $\alpha : R \to R$.

Theorem 2.17. *Let M be a quasi-principally injective module and $s_1, \cdots, s_n \in S = \mathrm{End}(M)$ such that the sum $\sum_{i=1}^n Ss_i$ is direct. Write $A = s_1(M) + \cdots + s_k(M)$ and $B = s_{k+1}(M) + \cdots + s_n(M)$ where $1 \le k \le n$. Then*

$$\ell_S(A \cap B) = \ell_S(A) + \ell_S(B).$$

Proof. Clearly, $\ell_S(A) + \ell_S(B) \subseteq \ell_S(A \cap B)$. Let $u \in \ell_S(A \cap B)$. Consider the map $\alpha : A + B \to M$ given by $\alpha(a + b) = u(a)$. Since $u(x) = 0$ for any $x \in A \cap B$, it can be checked that the map α is well-defined and is a homomorphism. By Theorem 2.15, there exists a homomorphism $\varphi : M \to$

M that extends α. Clearly, $\varphi(b) = 0$ for all $b \in B$, and hence, $\varphi \in \ell_S(B)$ and $u - \varphi \in \ell_S(A)$. This shows $u = (u - \varphi) + \varphi \in \ell_S(A) + \ell_S(B)$, proving our claim. $\square$

Corollary 2.18. [7, Theorem 3.1] *If R is right self-principally injective and $Rb_1 \oplus \cdots \oplus Rb_n$ is direct where $b_i \in R$. Write $S = b_1R + \cdots + b_kR$ and $T = b_{k+1}R + \cdots + b_nR$ where $1 \leq k < n$. Then $\ell(S \cap T) = \ell(S) + \ell(T)$.*

3 Principal Injectivity and Weak Injectivity

We now consider the relationship between principal injectivity and weak injectivity. Let M be a right R-module. Recall that a module N is said to be *weakly M-injective* if, for every homomorphism f from M to the injective hull $E(N)$ of N, there exists a monomorphism $\sigma : N \to E(N)$ and a homomorphism $\widehat{f} : M \to N$ such that $f = \sigma\widehat{f}$. It is easy to see that N is weakly M-injective if and only if, for any M-cyclic submodule X of $E(N)$, there exists a submodule Z of $E(N)$ such that $X \subseteq Z \simeq N$ (see [4, 5]).

Following [4], a right R-module is *weakly injective* if it is weakly R^n-injective for each natural number n. It was shown in [4] that a cyclic right R-module is weakly injective if and only if it is weakly R^2-injective. The following proposition generalizes this fact.

Proposition 3.1. *Let M be a right R-module. An M-cyclic module N is weakly M^n-injective for all $n \in \mathbb{Z}^+$ if and only if it is weakly M^2-injective.*

Proof. The necessity is trivial. We now prove the converse by using induction on n. Let N be an M-cyclic module. Since $M \simeq M^2/M$, it is true for $n = 1$. Let U be an M^n-cyclic submodule of $E(N)$. Then U can be written in the form of a sum $U = A + B$, where A is M^{n-1}-cyclic and B is M-cyclic. By the induction hypothesis, there is a submodule X of $E(N)$ such that $A \subseteq X \simeq N$. Since X and B are M-cyclic, $X - B$ must be M^2-cyclic. By the weak M^2-injectivity, there exists a submodule Y of $E(N)$ such that $X + B \subseteq Y \simeq N$, proving our proposition. $\square$

We now give a description for the quasi-injective modules.

Theorem 3.2. *A module M is quasi-injective if and only if it is weakly M^2-injective and quasi-principally injective.*

Proof. The necessity is trivial. For the converse part, suppose M is M-principally injective and weakly M^2-injective. It suffices to show $f(M) \subseteq M$ for any $f : M \to E(M)$. Assume $f(M) \not\subseteq M$ and consider $\mathrm{id}+f : M \oplus M \to E(M)$. Then $M \subsetneq \mathrm{Im}(\mathrm{id}+f)$ is an essential submodule of some $M' \subseteq E(M)$, $M' \simeq M$. But M' is quasi-principally injective, and hence, M is a direct summand of M', which is a contradiction. $\square$

As an application of Theorem 3.2, we re-obtain the following characterization for right self-injective rings.

Corollary 3.3. [7, Theorem 1.3] *A ring R is right self-injective if and only if it is right weakly injective and right self-principally injective.*

Acknowledgements. The first author is grateful to the Department of Mathematics, The Chinese University of Hong Kong, for their support of his visit in January 1998. The authors would like to thank Professor R. Wisbauer for a short proof of Theorem 3.2 without referring to Proposition 3.1.

References

[1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York-Heidelberg-Berlin, 1974.
[2] V.P. Camillo, Commutative rings whose principal ideals are annihilators, *Portugal Math.* **46** (1) (1989) 33–37.
[3] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman, London, 1994.
[4] S.K. Jain, S.R. López-Permouth, S. Singh, On a class of QI-rings, *Glasgow Math. J.* **34** (1992) 75–81.
[5] S.R. López-Permouth, Rings characterized by their weakly-injective modules, *Glasgow Math. J.* **34** (1992) 349–353.
[6] S.H. Mohamed, B.J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, Cambridge, 1990.
[7] W.K. Nicholson, M.F. Yousif, Principally injective rings, *J. Algebra* **174** (1995) 77–93.
[8] E.A. Rutter, Jr., Ring with the principal extension property, *Comm. Algebra* **3** (3) (1975) 203–212.
[9] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, London-Tokyo, 1991.

Southeast Asian Bulletin of Mathematics (2000) 24: 559–564

Southeast Asian Bulletin of Mathematics

On Finite Injectivity*

Prakit Jampachon and Jirasook Ittharat

Department of Mathematics, Khon Kaen University, Thailand
E-mail: prajam@kku1.kku.ac.th

Nguyen Van Sanh†

Department of Mathematics, Hue University, Vietnam

AMS Subject Classification (2000): 16D50, 16D70, 16D80

Abstract. Let R be a ring. A right R-module M is called finitely injective (briefly, f-injective) if every homomorphism from a finitely generated right ideal of R to M can be extended to R. We now extend this notion to modules. A module N is called M-finitely injective (or M-f-injective) if every homomorphism from a finitely M-generated submodule of M to N can be extended to M. In this note we give some characterizations and some properties of quasi f-injective modules.

Keywords: quasi-f-injective, quasi-p-injective, self generator, annihilator

1. Introduction

Throughout, R is an associative ring with identity and Mod-R the category of unitary right R-modules. Let M be a right R-module. A module N is called *M-generated* if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called *finitely M-generated*. In particular, N is called M-cyclic if it is isomorphic to M/L for some submodule $L \subset M$.

Let R be a ring. Call a right R-module M *finitely injective* (resp. *p-injective*) if any R-homomorphism from a finitely generated (principal) right ideal of R to M can be extended to an R-homomorphism from R to M. Right self p-injective rings were studied by Nicholson and Yousif in [8] and some properties of finitely injectivity were given there. Sanh et al. [10] generalized this notion to M-p-injectivity for a given right R-module M. A right R-module N is M-p-injective (*M-f-injective*) if every homomorphism from an M-cyclic (finitely M-generated) submodule of M to N can be extended to a homomorphism from M to N.

*Supported in part by the Thailand Research Fund.

†Corresponding author. Current address: Department of Mathematics, Khon Kaen University, Thailand

In this note, we give a characterization of quasi-f-injective modules which generalizes the Ikeda–Nakayama Lemma (see [4]).

2. Finitely Injectivity

Definition. *Let M be a right R-module. A right R-module N is called* M-finitely injective *(briefly,* M-f-injective*) if every homomorphism from a finitely M-generated submodule of M to N can be extended to a homomorphism from M to N. Equivalently, for any endomorphisms $s_1, s_2, \ldots, s_n$ of M, every homomorphism from $s_1(M)+s_2(M)+\cdots+s_n(M)$ to N can be extended to a homomorphism from M to N. N is called* f-injective *if it is R-finitely injective.*

Lemma 2.1. *Let X_i, $i \in I$ be M-f-injective modules. Then $\Pi_{i\in I} X_i$ is M-f-injective.*

Proof. Let U be a finitely M-generated submodule of M, $\iota : U \to M$ the embedding, and $\varphi : U \to \Pi_{i\in I} X_i$. For each i, let $\pi_i : \Pi_{i\in I} X_i \to X_i$ be the ith projection. Since each X_i is M-f-injective, there is $\varphi_i : M \to X$ such that $\varphi_i \iota = \pi_i \varphi$. By the definition of products, there is a homomorphism $\overline{\varphi} : M \to \Pi_{i\in I} X_i$ such that $\pi_i \overline{\varphi} = \varphi_i$ for all $i \in I$. It is clear that $\overline{\varphi}\iota = \varphi$, proving that $\Pi_{i\in I} X_i$ is M-f-injective. □

Lemma 2.2. *Let C be a finitely M-generated submodule of M. If C is M-f-injective, then it is a direct summand of M.*

Proof. Since the embedding $\iota : C \to M$ has a left inverse, it clearly splits and so C is a direct summand of M. □

Proposition 2.3. *Let $\{M_i, i \in I\}$ be any family of M-f-injective modules. If M is finitely generated, then $\oplus_{i\in I} M_i$ is M-f-injective.*

Proof. Since M is finitely generated, it follows that every finitely M-generated submodule U of M is again finitely generated. Hence, for any homomorphism $\varphi : U \to \oplus_{i\in I} M_i$, $\varphi(U)$ is contained in a finite direct sum $\oplus_{i\in I_0} M_i$ for some finite subset I_0 of I. By Lemma 2.1, $\oplus_{i\in I_0} M_i$ is M-f-injective, hence, φ can be extended to a homomorphism $\bar{\varphi}$ from M to $\oplus_{i\in I} M_i$, proving our proposition. □

Corollary 2.4. *Any direct sum of f-injective modules is again f-injective.*

It is clear that over a right Noetherian ring, every f-injective module is injective. By Proposition 2.3, we see that if every f-injective right R-module is injective, then R is right Noetherian.

Following [10], a right R-module N is M-p-injective if every homomorphism from an M-cyclic submodule of M to N can be extended to a homomorphism from M to N. M is *quasi-p-injective* if it is M-p-injective. According to Wisbauer [11], a module M is *direct projective* if, for any direct summand X of M, every epimorphism $f : M \to X$ splits (i.e., $\ker(f)$ is a direct summand of M). Clearly, every quasi-projective module is direct projective. The following theorem gives a description of this kind of modules.

Theorem 2.5. *Let M be a direct-projective module and $S = \mathrm{End}(M_R)$ its endomorphism ring. The following conditions are equivalent:*

(1) *S is von Neumann regular;*
(2) *every M-cyclic submodule of M is M-p-injective;*
(3) *every finitely M-generated submodule of M is M-p-injective;*
(4) *every finitely M-generated submodule of M is M-f-injective;*
(5) *every right R-module is M-f-injective.*

Proof. Clearly, (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2), and (2) $\Rightarrow$ (1) by Theorem 2.7 in [10]. We now show that (1) $\Rightarrow$ (5). Let $s_1, s_2 \in S$. Since S is von Neumann, then $s_1(M) = e(M)$ for some idempotent e of S. Hence, $s_1(M) + s_2(M) = e(M) \oplus (1-e)s_2(M)$. Again, $(1-e)s_2(M) = f(M)$ for some idempotent $f \in S$ with $ef = 0$. Let $h = e + f - fe$. Then h is an idempotent of S and we have $h(M) = s_1(M) + s_2(M)$. This shows that $s_1(M) + s_2(M)$ is a direct summand of M. By induction, we see that every finitely M-generated submodule of M is a direct summand of M, and therefore, any homomorphism from a finitely M-generated submodule of M to a module N can be extended to M. □

Proposition 2.6. *Let M be a right R-module and U a finitely M-generated submodule of M. If a module N is M-f-injective, then it is U-f-injective. Moreover, if M is quasi-projective, then N is M/U-f-injective.*

Proof. The proof for the first part is routine. We observe that for a quasi-projective module M, if U is a finitely M-generated submodule of M and $\overline{X} = X/U$, $X \subset M$, is a finitely $\overline{M}$-generated submodule of $\overline{M} = M/U$, then X is finitely M-generated. In fact, let $\nu_X : X \to \overline{X} = X/U$ be induced by the natural epimorphism $\nu : M \to M/U$. Since $\overline{X} = X/U$ is finitely $\overline{M}$-generated, there is an epimorphism $\eta : \overline{M}^n \to \overline{X} = X/U$ for some n. Let $k = \oplus\nu : \oplus M = M^n \to (M/U)^n$ be defined by components. Since M is quasi-projective, M^n is M-projective and is therefore X-projective. Hence, there exists a homomorphism $\xi : M^n \to X$ such that $\nu_X\xi = \eta k$. Clearly, for any $x \in X$, we can find $t \in M^n$ such that $\xi(t) - x \in U$, i.e., $x \in \xi(M^n) + U$. It follows that $X = \xi(M^n) + U$, and hence X is finitely M-generated.

Let $\iota : X \to M$ and $\bar{\iota} : \overline{X} \to \overline{M}$ be the embedding. Since N is M-f-injective, there exists $\varphi_1 : M \to N$ such that $\varphi_1\iota = \varphi\nu_X$. Then $U \subset \nu_X^{-1}(\mathrm{Ker}\varphi) = \mathrm{Ker}(\varphi\nu_X) = \mathrm{Ker}(\varphi_1\iota) = \iota^{-1}(\mathrm{Ker}\varphi_1) = X \cap \mathrm{Ker}\varphi_1$. This shows that $\ker\nu \subset \ker\varphi_1$. Hence, there is $\overline{\varphi} : \overline{M} \to N$ such that $\overline{\varphi}\nu = \varphi_1$. Clearly, $\overline{\varphi}|_{\overline{X}} = \varphi$. Hence, N is $\overline{M}$-f-injective, proving our proposition. □

Corollary 2.7. *Let I be a finitely generated right ideal of R. If N is f-injective, then it is I-f-injective and R/I-f-injective.*

3. Quasi-Finite Injectivity

Call a module M *quasi-f-injective* if it is M-f-injective. A ring R is right self-f-injective if R_R is R-f-injective. The following lemma is straightforward but, by this lemma, we can produce many quasi-f-injective modules from a right self-f-injective ring.

Lemma 3.1. *Any direct summand of a quasi-f-injective module is again quasi-f-injective.* □

Theorem 3.2. *Let M_R be a right R-module and $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:*

(1) *M is a quasi-f-injective module;*
(2) *for any finitely M-generated submodules A, B of M and for any $c \in S$, we have*
 (a) $\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)$.
 (b) $\ell_S r_M(c) = Sc$.
(3) *M is quasi-p-injective and $\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)$ for any finitely M-generated submodules A, B of M.*

Proof. (1) $\Rightarrow$ (2). First we prove (a). Clearly, $\ell_S(A) + \ell_S(B) \subset \ell_S(A \cap B)$. Conversely, let $s \in \ell_S(A \cap B)$. Then the map

$$\begin{aligned} \varphi : A + B &\to M \\ a + b \qquad &\varphi(a+b) = s(b) \end{aligned}$$

is well defined, and moreover, it is an R-homomorphism.

Since M is quasi-f-injective and $A + B$ is finitely M-generated, there exists $\overline{\varphi} : M \to M$ such that $\overline{\varphi}|_{A+B} = \varphi$. In particular, $\overline{\varphi}(a) = \varphi(a) = 0$ for all $a \in A$. Hence, $\overline{\varphi} \in \ell_S(A)$. Moreover, for all $b \in B$, we have $\overline{\varphi}(b) = \varphi(b)$, i.e., $(\overline{\varphi} - s)(b) = 0$ or $\overline{\varphi} - s \in \ell_S(B)$. It follows that $s = \overline{\varphi} + t \in \ell_S(A) + \ell_S(B)$, proving (a).

Recall that, for any $c \in S$, $r_M(c) = \{m \in M | c(m) = 0\} = \ker(c)$. We now prove that

$$\ell_S r_M(c) = Sc \qquad \text{for all } c \in S.$$

Clearly, $Sc \subset \ell_S r_M(c) = \ell_S\big(\ker(c)\big)$. Let $b \in \ell_S r_M(c)$. Then $r_M(c) \subset r_M(b)$ and hence, the map

$$\begin{aligned} c(M) &\to M \\ c(m) &\mapsto b(m) \end{aligned}$$

is well defined and is a homomorphism. Since M is quasi-f-injective, there exists $s : M \to M$ such that $sc(m) = b(m)$ for all $m \in M$. This shows that $b = sc \in Sc$, as required.

(2) $\Rightarrow$ (1). We now assume that M satisfies (a) and (b) in (1). Let $U = \Sigma_{i=1}^{n} s_i(M)$. We prove by induction on n. For $n = 1$, let $\varphi : s(M) \to M$ be a homomorphism. Since $s(m) = 0$ implies that $\varphi s(m) = 0$, we infer that $r_M(s) \subset r_M(\varphi s)$. It follows that $r_M(Ss) \subset r_M(S\varphi s)$. By (b), we have

$$S\varphi s = \ell_S r_M(S\varphi s) \subset \ell_S r_M(Ss) = Ss.$$

Therefore, there exists $\overline{\varphi} \in S$ such that $\varphi s = \overline{\varphi} s$, and hence, $\overline{\varphi}|_{s(M)} = \varphi$ as required.

We now transfer from n to $n+1$. Let

$$\varphi : \sum_{i=1}^{n+1} s_i(M) \to M$$

be a homomorphism. By induction, there exist $\varphi_1, \varphi_2 \in S$ such that, for any $\Sigma_{i=1}^{n+1} s_i(m_i) \in \Sigma_{i=1}^{n+1} s_i(M)$, we have

$$\varphi\Big(\sum_{i=1}^{n} s_i(m_i)\Big) = \varphi_1\Big(\sum_{i=1}^{n} s_i(m_i)\Big),$$
$$\varphi(s_{n+1}(m_{n+1})) = \varphi_2(s_{n+1}(m_{n+1})).$$

By (b), we have

$$\varphi_1 - \varphi_2 \in \ell_S\left(\Big(\sum_{i=1}^{n} s_i(M)\Big) \cap s_{n+1}(M)\right) = \ell_S\Big(\sum_{i=1}^{n} s_i(M)\Big) + \ell_S(s_{n+1}(M)),$$

i.e., there exist $s \in \ell_S\left(\Sigma_{i=1}^{n} s_i(M)\right)$, $t \in \ell_S(s_{n+1}(M))$ such that $\varphi_1 - \varphi_2 = s - t$. Put $\overline{\psi} = \varphi_1 - s = \varphi_2 - t$. Then for any $x = \Sigma_{i=1}^{n+1} s_i(m_i) \in \Sigma_{i=1}^{n+1} s_i(M)$,

$$\begin{aligned}
\varphi\Big(\sum_{i=1}^{n+1} s_i(m_i)\Big) &= \varphi\Big(\sum_{i=1}^{n} s_i(m_i)\Big) + \varphi s_{n+1}(m_{n+1}) \\
&= (\varphi_1 - s)\Big(\sum_{i=1}^{n} s_i(m_i)\Big) + (\varphi_2 - t)(s_{n+1}(m_{n+1})) \\
&= \overline{\psi}\Big(\sum_{i=1}^{n+1} s_i(m_i)\Big).
\end{aligned}$$

This shows that M is quasi f-injective, proving (1).

(2) $\Leftrightarrow$ (3) The condition (b) in (2) is a characterization of quasi-p-injective module (see [10, Theorem 2.10]). □

As an application, by putting $M_R = R_R$ we have immediately the Ikeda–Nakayama Lemma (see [4]).

Corollary 3.3. *Let R be a ring. The following conditions are equivalent.*

(1) *R is right self-f-injective;*
(2) *for any finitely generated right ideals A and B of R and any $c \in R$, we have*
  (a) $\ell_R(A \cap B) = \ell_R(A) + \ell_R(B)$
  (b) $\ell_R r_R(Rc) = Rc$;
(3) *R is right self p-injective and $\ell_R(A \cap B) = \ell_R(A) + \ell_R(B)$ for any finitely generated right ideals A and B of R.*

We now consider the relation between quasi-injectivity and quasi-f-injectivity in a special case.

Corollary 3.4. *Let M be a Noetherian right R-module which is a self-generator. Then M is quasi-injective if and only if it is quasi-f-injective.*

Proof. Since M is Noetherian, every submodule of M is finitely generated, and hence, it is finitely M-generated, because M is a self-generator. This shows that if M is quasi-f-injective, then it is quasi-injective. The converse is always true. □

It is well known that a ring R is QF (quasi-Frobenius) if it is right Noetherian and right self-injective. As an application, we have

Corollary 3.5. *If R is right Noetherian and satisfies $\ell(A+B)=\ell(A)+\ell(B)$ and $\ell r(c)=Rc$ for any finitely generated right ideals A, B of R and $c\in R$, then R is a QF-ring.*

References

1. Anderson, F.W., Fuller, K.R.: *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York, Heidelberg, Berlin, 1974.
2. Camillo, V.P.: Commutative rings whose principal ideals are annihilators, *Portugal. Math.* 46(1), 33–37 (1989).
3. Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R.: *Extending Modules*, Pitman, London, 1994.
4. Ikeda, M., Nakayama, T.: On some characteristic properties of quasi-Frobenius and regular rings, *Proc. A.M.S.* 5, 15–19 (1954).
5. Kasch, F.: *Moduln and Ringe*, Teubner, Stuttgart, 1977.
6. Lambek, J.: *Lecture on Rings and Modules*, Blaisedell Publishing Co., Waltham, 1966.
7. Mohamed, S.H., Müller, B.J.: *Continuous and Discrete Modules*, Cambridge University Press, 1990.
8. Nicholson, W.K., Yousif, M.F.: Principally injective rings, *J. Algebra* **174**, 77–93 (1995).
9. Rutter, E.A., Jr.: Ring with the principal extension property, *Comm. in Algebra* 3(3), 203–212 (1975).
10. Sanh, N.V., Shum, K.P., Dhompongsa, S., Wongwai, S.: On quasi-principally injective modules, *Algebra Colloq.* 6(3), 269–276 (1999).
11. Wisbauer, R.: *Foundations of Module and Ring Theory*, Gordon and Breach, London, Tokyo, 1991.

Algebra Colloquium 8:2 (2001) 227–236

Algebra Colloquium

On Modules Whose Singular Subgenerated Modules Are Weakly Injective*

S. Dhompongsa **J. Sanwong**
Department of Mathematics, Faculty of Science, Chiang Mai University
Chiang Mai 50200, Thailand
E-mail: sompongd@chiangmai.ac.th scmti004@chiangmai.ac.th

S. Plubtieng **H. Tansee**
Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
E-mail: somyotp@nu.ac.th g4165001@cm.edu

Received 21 October 1999
Revised 17 October 2000

Communicated by R. Wiegandt

Abstract. Rings over which every singular right module is injective (briefly, right SI-rings) were introduced and investigated by Goodearl. Weakly injective modules, as a generalization of injective modules, were introduced by Jain and López-Permouth. In this paper, we study the class of rings whose singular right modules are weakly injective, which we call SwI-rings. This concept is extended to SwI-modules, i.e., modules whose singular subgenerated modules are weakly injective. Several characterizations and properties of SwI-rings and SwI-modules are obtained which generalize some earlier known results on SI-rings and weakly semisimple rings.

2000 Mathematics Subject Classification: 16D50, 16D80

Keywords: weakly injective modules, singular modules, SI-rings, SI-modules

1 Introduction

Rings over which every singular right module is injective (briefly, right SI-rings) were introduced and investigated by Goodearl [2]. Using the category

*The work was supported by the Thailand Research Fund. The fourth author was also supported by the Royal Golden Jubilee Program.

Sompong Dhompongsa
March 21, 2001

$\sigma[M]$, the concept of SI-modules was defined similarly, and a structure theorem for finitely generated quasi-projective SI-modules was obtained (see [4]). Weakly injective modules, as a generalization of injective modules, were introduced by Jain and López-Permouth [5] and have been extensively studied [6, 8, 13]. In this paper, we introduce and investigate the class of rings whose singular right modules are weakly injective. In fact, most of our results are proved in a more general setting, namely for modules M whose singular subgenerated modules are M-weakly injective. We call the former right SwI-rings and the later right SwI-modules. These concepts may be regarded as generalizations of right SI-rings and weakly semisimple rings. Various characterizations of SwI-rings and SwI-modules are obtained and, under an additional condition for modules, right SwI-rings and right SwI-modules can also be characterized in terms of semiprimitive modules. For a right continuous (and hence, self-injective) ring R, if R is right SwI, then it is left SI. Finally, we obtain a necessary and sufficient condition for an SwI-ring (resp., SwI-module) to be an SI-ring (resp., SI-module).

2 Definitions and Preliminaries

Throughout this paper, R is an associative ring with identity and we use Mod-R to denote the category of unitary right R-modules. Let M be a right R-module. An R-module N is said to be *M-generated* if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called a *finitely M-generated* module. Especially, N is called an *M-cyclic* module if it is isomorphic to M/L for some submodule L of M. N is said to be *M-subgenerated* if it is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are all M-subgenerated modules (see [12]). M is said to be a *QI-module* if every quasi-injective module in $\sigma[M]$ is M-injective. Let $\mathrm{Soc}(M)$ and $E(M)$ denote the socle and injective hull of M, respectively. Following Goodearl [2], M is said to be *singular* (resp., *non-singular*) if $Z(M) = M$ (resp., $Z(M) = 0$). Here, $Z(M)$ is the *singular* submodule $\{m \in M \mid mI = 0$ for some essential right ideal I of $R\}$. A ring R is right *singular* (resp, *non-singular*) if the module R_R is singular (resp., non-singular). N is called *singular* in $\sigma[M]$ or *M-singular* if there exists a module L in $\sigma[M]$ containing an essential submodule K such that $N \simeq L/K$. By definition, every M-singular module belongs to $\sigma[M]$. For $M = R_R$, the notion "R-singular" is identical to the usual definition of singular right R-modules defined above (see [1, 2, 12]). The class of all M-singular modules is closed under submodules, homomorphic images, and direct sums. Hence, every module $N \in \sigma[M]$ contains a largest M-singular submodule, which we denote by $Z_M(N)$. In our notation, $Z(N) = Z_R(N)$ is just the largest singular submodule of N and $Z_M(N) \subset Z(N)$. If $Z_M(N) = 0$, N is called *M-non-singular*. In this paper, the term "singular subgenerated modules" means all modules in $\sigma[M]$ which are M-singular.

Recall that for any module $N \in \sigma[M]$, there exists an M-injective hull $\widehat{N}$ of N which is $\sum_{f: M \to E(N)} \mathrm{Im}(f)$, the trace of M in $E(N)$. $\widehat{N}$ is the largest M-generated submodule of $E(N)$ (cf. [12]). A module N is said to be *weakly M-injective* if, for every homomorphism $f : M \to E(N)$, there exists a monomorphism $\sigma : N \to E(N)$ and a homomorphism $\widehat{f} : M \to N$ such that $f = \sigma\widehat{f}$ (see [6] and the references therein). Note that this concept generalizes M-injective modules, in which case we require σ to be the inclusion map. It is clear that a module $N \in \sigma[M]$ is weakly M-injective if and only if, for every homomorphism $f : M \to \widehat{N}$, there exists a monomorphism $\sigma : N \to \widehat{N}$ and a homomorphism $\widehat{f} : M \to N$ such that $f = \sigma\widehat{f}$. Note that, if $N \in \sigma[M]$ is weakly X-injective for every $X \in \sigma[M]$, then N is M-injective. From this fact, it leads to the following definition (see [11]).

Definition 2.1. For a right R-module M, a module $N \in \sigma[M]$ is called *M-weakly injective* if N is weakly M^n-injective for each natural number n. A right R-module is *weakly injective* if it is R-weakly injective.

From this definition and the observation stated above, we have the following.

Proposition 2.2. *A module $N \in \sigma[M]$ is M-weakly injective if and only if any finitely M-generated submodule X of $\widehat{N}$ is contained in a submodule Y of $\widehat{N}$ such that $Y \simeq N$.*

Lemma 2.3. [11, Proposition 18] *Let M be a right R-module. An M-cyclic module N is M-weakly injective if and only if it is weakly M^2-injective.*

Definition 2.4. A right R-module M is called *weakly semisimple* if every module in $\sigma[M]$ is M-weakly injective, and it is called an *SwI-module* if every M-singular module in $\sigma[M]$ is M-weakly injective. A ring R is *right weakly semisimple* if every right R-module is weakly injective, and it is called a *right SwI-ring* if every singular right R-module is weakly injective.

We begin with a module version of Lemma 1.8 in [6].

Lemma 2.5. *Let U be Q-injective, where $Q \in \sigma[M]$ is weakly M-injective. Then U is $\widehat{Q}$-injective.*

Proof. Suppose U is not $\widehat{Q}$-injective. By Zorn's Lemma, we have a submodule A of $\widehat{Q}$ and a homomorphism $f : A \to U$ which cannot be extended to any proper extension of A. Since A is essential in $\widehat{Q}$ and $M^{(\Lambda)} \to \widehat{Q}$ for some index set Λ, we have an epimorphism $M \to B$ for some submodule B of $\widehat{Q}$ and $B \not\subset A$. Since Q is weakly M-injective, B is embeddable in Q. So U is B-injective. Note that $A \cap B \neq 0$. We extend $f|_{A\cap B}$ to $g : B \to U$ and define $f' : A+B \to U$ via $a + b \mapsto f(a) + g(b)$, which is well defined. Thus, f' is a proper extension of f, a contradiction. □

Observe that, when $U = Q$, we have that Q is $\widehat{Q}$-injective. Thus, Q is a direct summand of $\widehat{Q}$, and hence, $Q = \widehat{Q}$ since Q is essential in $\widehat{Q}$. Therefore, Q is M-injective.

Lemma 2.6. *Every finitely generated weakly semisimple module is noetherian.*

Proof. Let M be a weakly semisimple module. If $Q \in \sigma[M]$, then Q is M-weakly injective. If Q is also quasi-injective, then Q is M-injective by the observation above. This shows that M is a QI-module. Let S_λ $(\lambda \in \Lambda)$ be simple modules in $\sigma[M]$. Since every semisimple module is quasi-injective and $\oplus_{\lambda\in\Lambda}S_\lambda \in \sigma[M]$, we see that $\oplus_{\lambda\in\Lambda}S_\lambda$ is M-injective. Therefore, $\widehat{\oplus_{\lambda\in\Lambda}S_\lambda} = \oplus_{\lambda\in\Lambda}S_\lambda = \oplus_{\lambda\in\Lambda}\widehat{S_\lambda}$. Hence, if M is also finitely generated, then M is noetherian. □

Lemma 2.7. *If M_R is finitely generated and $M/\mathrm{Soc}(M)$ is noetherian, then every finitely M-generated M-singular module has finite Goldie dimension.*

Proof. Let X be a finitely M-generated M-singular module. Clearly, X is finitely generated and $X \in \sigma[M/K]$ for some essential submodule K of M. Now $X \in \sigma[M/\mathrm{Soc}(M)]$ since $\mathrm{Soc}(M) \subset K$. Since X is finitely generated and $M/\mathrm{Soc}(M)$ is noetherian, X is noetherian. Thus, X has finite Goldie dimension. □

Lemma 2.8. *Let $A \in \sigma[M]$. Let X be a finitely M-generated submodule of $\widehat{A}$, N a submodule of $\widehat{X} \cap A$, and $\widehat{X} = \widehat{N}$. If N is M-weakly injective, then $A \simeq A'$ for some A' such that $X \subset A' \subset \widehat{A}$.*

Proof. The proof is based on the idea of Lemma 2.4 in [6]. Write $\widehat{A} = \widehat{X} \oplus L$ for some submodule L of $\widehat{A}$. From $X \subset \widehat{N}$, there exists a monomorphism $\sigma : \widehat{N} \to \widehat{N}$ such that $\sigma(N) \supset X$. Define a monomorphism $\eta = \sigma \oplus \iota_d : \widehat{A} = \widehat{N} \oplus L \to \widehat{A}$. Then $X \subset \sigma(N) = \eta(N) \subset \eta(A) \subset \widehat{A}$. □

Lemma 2.9. *Suppose every uniform M-cyclic module is M-weakly injective. Then every uniform module in $\sigma[M]$ is M-weakly injective.*

Proof. Let U be a uniform module in $\sigma[M]$ and X a finitely M-generated submodule of $\widehat{U}$. There exists a non-zero submodule Y of X and an epimorphism from M to Y. Now Y is M-weakly injective since it is uniform M-cyclic. Thus, X is M-weakly injective since Y is essential in X. By Lemma 2.8 (putting $A = U$ and $N = X$), U is M-weakly injective. □

3 Results

A module M is said to be *compressible* if it is embeddable in each of its non-zero submodules. The following theorem is the module-theoretic version of

Theorem 2.5 in [6].

Theorem 3.1. *For a finitely generated right R-module M, the following conditions are equivalent:*

(1) *M is weakly semisimple.*

(2) *Every finitely M-generated module is M-weakly injective and M is noetherian.*

(3) *Every finitely M-generated uniform module is compressible and M is noetherian.*

(4) *Every finitely M-generated module is compressible.*

Proof. By Lemma 2.6, we have (1)$\Rightarrow$(2).

(2)$\Rightarrow$(3). Let U be a uniform M^n-cyclic module. Then there exists a non-zero uniform M-cyclic submodule U_1 of U. By (2), U_1 is M-weakly injective, which implies that U is M-weakly injective. Let U' be a non-zero submodule of U. Then $U' \cap U_1$ is M-weakly injective by Lemma 2.9, U' is M-weakly injective, and $U \subset \widehat{U} = \widehat{U_1}$. Therefore, U is embeddable in U'.

(3)$\Rightarrow$(1). Let U be a uniform cyclic module in $\sigma[M]$. We show that U is M-weakly injective. Let K be a finitely M-generated submodule of $\widehat{U}$. By (3), K is compressible, i.e., there exists an isomorphism $K \to K' \subset U \cap K$ for some K'. Let $\sigma : K' \to K$ be an isomorphism. Since $\widehat{U}$ is M-injective, there is an extension $\widehat{\sigma} : U \to \widehat{U}$ of σ. We see that $\widehat{\sigma}$ is monic since K' is essential in U and σ is monic. Thus, $U \simeq \widehat{\sigma}(U) \supset \widehat{\sigma}(K') = K$. Hence, U is M-weakly injective. Finally, let $A \in \sigma[M]$. To show that A is M-weakly injective, let X be a finitely M-generated submodule of $\widehat{A}$. By Lemma 2.6, there exists a direct sum N of uniform cyclics such that N is essential in $\widehat{X} \cap A$ (which is essential in $\widehat{X}$). Thus, N and hence $\widehat{N}$ are M-weakly injective. Then Lemma 2.8 and Proposition 2.2 imply that A is M-weakly injective.

(3)$\Rightarrow$(4). Let X be an M^n-cyclic module and let N be essential in X. Since M is noetherian, X has finite Goldie dimension. Then N has an essential submodule $\oplus_{i=1}^n U_i$ which is a direct sum of uniform cyclic submodules U_i. Therefore, $\widehat{X} = \oplus_{i=1}^n \widehat{U_i}$. Put $X_i = \pi_i(X)$, where $\pi_i : \widehat{X} \to \widehat{U_i}$ is the natural projection. Since X is finitely M-generated, each X_i is also finitely M-generated and $X \subset \oplus_{i=1}^n X_i$. Noting that $U_i \cap X_i$ is essential in X_i and X_i is a uniform module, by (3), we have a monomorphism $g_i : X_i \to U_i$. Therefore, X is embedded into N by $\oplus_{i=1}^n g_i$, showing that X is compressible.

(4)$\Rightarrow$(3). Let S be a simple right R-module. If $S \notin \sigma[M]$, then S is M-injective. Now suppose $S \in \sigma[M]$. If $S \neq \widehat{S}$, since $\widehat{S}$ is M-generated, there exists an M-cyclic submodule $N \subset \widehat{S}$ such that $N \not\subset S$. But since $\widehat{S}$ is uniform, S is essential in N and then N is embeddable in S. Thus, $S = N$, a contradiction. Hence, $S = \widehat{S}$ and S is M-injective, i.e., M is a V-module.

Next, we show that every M-cyclic module K has finitely generated socle. Let X be a complement of $\mathrm{Soc}(K)$ in K. Hence, $\mathrm{Soc}(K) \oplus X$ is essential in K. This implies that $\mathrm{Soc}(K)$ is essential in K/X. Since K/X is compressible, we have an embedding $K/X \hookrightarrow \mathrm{Soc}(K)$, implying that K/X is a semisimple module. This shows that $\mathrm{Soc}(K)$ is isomorphic to K/X, and hence, is finitely generated. Now Lemma 2 in [13] says that M is noetherian. □

When $M = R$, the equivalence of (1)-(3) in the above theorem can also be obtained by Theorem 3.4 in [14].

Theorem 3.2. *For a finitely generated M-non-singular right R-module M, the following conditions are equivalent:*

(1) *M is an SwI-module.*
(2) *Every finitely M-generated M-singular module is weakly semisimple.*
(3) *Every finitely M-generated uniform M-singular module is compressible and $M/\mathrm{Soc}(M)$ is noetherian.*
(4) *Every finitely M-generated M-singular module is compressible.*

Proof. (1)⇒(2). Let A be finitely M-generated and M-singular, and $B \in \sigma[A]$. Then B is M-singular, and so B is M-weakly injective. Since A is finitely M-generated, B is A-weakly injective.

(2)⇒(1). Let A be a finitely generated M-singular module. Then $A \in \sigma[M/K_1]$ for some essential submodule K_1 of M. Let X be a finitely M-generated submodule of $\widehat{A} \in \sigma[M]$. Thus, X is a finitely generated M-singular module and $X \in \sigma[M/K_2]$ for some essential submodule K_2 of M. Let $K = K_1 \cap K_2$. Then K is essential in M, and $A, X \in \sigma[M/K]$. We now see that A is an (M/K)-weakly injective module. Note that $E(A) \supset \widehat{A} := \sum_{f: M \to E(A)} f(M) \supset \sum_{f: M/K \to E(A)} f(M/K) := \widehat{\widehat{A}} \in \sigma[M/K]$. We show $X \subset \widehat{\widehat{A}}$. From $X \in \sigma[M/K]$, we have an epimorphism $(M/K)^{(\Lambda)} \to Y \supset X$ for some Y. Since X is finitely generated, we have an epimorphism $B \to X$, where $B \subset (M/K)^k$ for some k. Since $\widehat{A}$ is M-injective, we also have some submodule C of $\widehat{A}$ containing X, which is an image of $(M/K)^k$. Therefore,

$$\widehat{\widehat{A}} = \sum_{f: M/K \to E(A)} f(M/K) = \sum_{f: M/K \to \widehat{A}} f(M/K) \supset C \supset X,$$

as desired. Now A is (M/K)-weakly injective and C is a finitely (M/K)-generated submodule of $\widehat{\widehat{A}}$. Thus, by (2), we have a submodule D of $\widehat{\widehat{A}}$ such that $D \supset C$ and $D \simeq A$. From $D, C \in \sigma[M]$, $D \subset \widehat{\widehat{A}} \subset \widehat{A}$, $X \subset D$, and $D \simeq A$, we see that A is M-weakly injective.

For the general case, let A be an M-singular module. We show that A is M-weakly injective. Again, let X be a finitely M-generated submodule of $\widehat{A} \in \sigma[M]$. Write $\widehat{A} = \widehat{X} \oplus L$ for some $L \subset \widehat{A}$. Note that $\widehat{X} \cap A$ is essential in $\widehat{X}$ since A is essential in $\widehat{A}$ and $\widehat{X} \subset \widehat{A}$. By Lemma 2.7, $\widehat{X} \cap A$ contains some essential submodule N, which is a finite direct sum of (uniform) cyclic submodules. Note also that $\widehat{X} \subset \widehat{N} = \widehat{X \cap A} = \widehat{X}$. Thus, $\widehat{X} = \widehat{N}$. Since N is a finitely generated M-singular module, it is M-weakly injective. By Lemma 2.8, A is M-weakly injective.

(1)$\Leftrightarrow$(3). This follows by applying (1)$\Leftrightarrow$(3) of Theorem 3.1 to the category of M-singular modules.

(3)$\Rightarrow$(4). Let X be an M^n-cyclic module. Since $M/\mathrm{Soc}(M)$ is noetherian, X has finite Goldie dimension by Lemma 2.7. Hence, X is compressible as in the proof (3)$\Rightarrow$(4) of Theorem 3.1.

(4)$\Rightarrow$(3). Let K be essential in M and $\overline{M} = M/K$. Note that any finitely $\overline{M}$-generated module is finitely M-generated and M-singular. Then every finitely $\overline{M}$-generated module is compressible, and hence, by (4)$\Rightarrow$(3) of Theorem 3.1, M/K is noetherian. Therefore, $M/\mathrm{Soc}(M)$ is noetherian. □

Applying Lemmas 2.8 and 2.9, and Theorem 5.5 in [1], we can furnish more equivalent conditions for Theorem 3.2:

(5) M/K is weakly semisimple for every essential submodule K of M.

(6) Every finitely M-generated M-singular module is M-weakly injective.

(7) Every M-cyclic M-singular module is weakly M^2-injective and $M/\mathrm{Soc}(M)$ is noetherian.

(8) Every uniform M-cyclic M-singular module is weakly M^2-injective and $M/\mathrm{Soc}(M)$ is noetherian.

Using (5) and Theorem 3.1, one can easily prove the following.

Corollary 3.3. *For a finitely generated M-non-singular right R-module M, the following conditions are equivalent:*

(1) *$M/\mathrm{Soc}(M)$ is weakly semisimple.*

(2) *M is an SwI-module and every finitely $(M/\mathrm{Soc}(M))$-generated uniform module is compressible.*

(3) *M is an SwI-module, and for any finitely $(M/\mathrm{Soc}(M))$-generated uniform module U, $\mathrm{Hom}(U, U_1) \neq 0$ for every non-zero submodule U_1 of U.*

A ring R is *right continuous* (cf. [10]) if it satisfies the following conditions:

(C_1) Every right ideal of R is essential in a direct summand of R.

(C_2) Every right ideal isomorphic to a direct summand of R is itself a direct summand of R.

Corollary 3.4. *If R is a right continuous right SwI-ring, then R is a left SI-ring.*

Proof. We first show that R is right non-singular, i.e., $Z(R) = 0$. We write $E(R) = E(Z(R)) \oplus N$ for some non-singular R-module N, and $1 = a + b$, where $a \in E(Z(R))$ and $b \in N$. Since $Z(R)$ is weakly R-injective, aR is embeddable in $Z(R)$ and so there exists an essential right ideal I such that

$aI = 0$. But then $bI = (1-a)I = I$, and hence, $I = bI \subseteq bR \subseteq N$. Since $Z(I) = I \cap Z(R) = 0$, we obtain $Z(R) = 0$ because I is essential in R_R. Next, we show that R is von Neumann regular. Let $0 \neq y \in R$ and let $A = r_R(y)$ be its right annihilator in R. It follows by Lemma 7.51 in [7] that A is closed in R_R. By (C_1), $R = A \oplus N$ for some right ideal N of R. By (C_2), $yR \simeq R/A \simeq N$ is a direct summand of R_R. Hence, R is von Neumann regular. The ring $R/\mathrm{Soc}(R)$ is von Neumann regular, as well as right noetherian by Theorem 3.2. By Proposition 1.2 in [2], we see that $R/\mathrm{Soc}(R)$ is a semisimple artinian ring. Hence, by Corollary 3.7 in [2], R is a left SI-ring. □

Consider the following condition on a module M:

$$\text{Every simple module in } \sigma[M] \text{ is } M\text{-cyclic.} \qquad (*)$$

In this case, we can deduce that M is weakly semisimple or M is SwI from a smaller subclass of modules in $\sigma[M]$, namely the subclass of all semiprimitive modules. Recall that a module is said to be *semiprimitive* if its Jacobson radical is zero. When R is a V-ring, i.e., every simple module is injective, every module is semiprimitive (see [9]). A ring R is semisimple artinian if and only if every cyclic semiprimitive R-module is injective. If every cyclic semiprimitive singular R-module is injective, then R is an SI-ring (see [3]).

Theorem 3.5. *For a finitely generated right R-module M satisfying $(*)$, the following conditions are equivalent:*

(1) *M is weakly semisimple.*
(2) *Every semiprimitive module in $\sigma[M]$ is M-weakly injective.*
(3) *Every M-cyclic semiprimitive module is weakly M^2-injective and M is noetherian.*
(4) *Every uniform M-cyclic semiprimitive module is weakly M^2-injective and M is noetherian.*

Proof. (1)⇒(2). It is clear.

(2)⇒(3). This can be proved similarly to the proof of Lemma 2.6 using the remark after Lemma 2.5 and noting that every semisimple module is semiprimitive.

(3)⇒(4). It is clear.

(4)⇒(1). By $(*)$, every simple module in $\sigma[M]$ is weakly M^2-injective. Let X be a uniform M-cyclic module. We first show that X is semiprimitive. If X is simple, then we are done. If X is not simple, we apply the same proof as for Lemma 2 in [3] to conclude that X is semiprimitive. Now by (4), X is weakly M^2-injective. Therefore, by Theorem 3.1, M is weakly semisimple. □

Corollary 3.6. *The following are equivalent for a hereditary noetherian ring R :*

(1) *R is a right QI-ring.*
(2) *Every semiprimitive right R-module is weakly injective.*
(3) *Every semiprimitive left R-module is weakly injective.*

Proof. This is an immediate consequence of Theorem 3.5 since a ring is right weakly semisimple if and only if it is left weakly semisimple (see [6, Theorem 3.1]). □

Theorem 3.7. *For a finitely generated M-non-singular right R-module M satisfying $(*)$, the following conditions are equivalent:*
(1) *M is an SwI-module.*
(2) *Every finitely M-generated semiprimitive M-singular module is M-weakly injective and $M/\mathrm{Soc}(M)$ is noetherian.*
(3) *Every M-cyclic semiprimitive M-singular module is weakly M^2-injective and $M/\mathrm{Soc}(M)$ is noetherian.*
(4) *Every uniform M-cyclic semiprimitive M-singular module is weakly M^2-injective and $M/\mathrm{Soc}(M)$ is noetherian.*

Proof. We only need to prove (4)⇒(1). For this, let X be a uniform M-cyclic M-singular module. Then X is semiprimitive and weakly M^2-injective by (4). Now apply Theorem 3.2 to see that M is an SwI-module. □

As all rings satisfy the condition $(*)$, Theorems 3.2 and 3.7 can apply immediately to all non-singular rings.

An R-module M is called a *GV-module* if every singular simple R-module is M-injective, and it is called a *GCO-module* if every singular simple R-module is M-injective or M-projective. Thus, every GV-module is a GCO-module (see [1]). The following is a necessary and sufficient condition for an SwI-module to be an SI-module. Recall that a module M is called an *SI-module* if every M-singular module is M-injective.

Theorem 3.8. *A finitely generated self-projective M-non-singular right R-module M is an SI-module if and only if it is an SwI-module and* $\mathrm{Soc}(M/K)$ $\neq 0$ *for every proper essential submodule K of M.*

Proof. We only need to prove the sufficiency. Let S be a singular simple R-module. Note that every simple R-module is M-singular or M-projective. If S is M-singular, then S is weakly M-injective and so M-injective since it is quasi-injective. This implies that M is a GV-module, and hence, a GCO-module. We note by Theorem 3.2 that $M/\mathrm{Soc}(M)$ is noetherian. Hence, by [1, 17.3], M is an SI-module. □

Since every right SwI-ring is right non-singular (Corollary 3.4), we immediately have the following.

Corollary 3.9. *A right SwI-ring R is an SI-ring if and only if* $\mathrm{Soc}(R/K)$ $\neq 0$ *for every proper essential right ideal K of R.*

Acknowledgement. The authors would like to thank the referees for their comments.

References

[1] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman, London, 1994

[2] K.R. Goodearl, *Singular Torsion and the Splitting Properties*, Mem. Amer. Math. Soc., No. 124, Providence, R.I., 1972.

[3] D.V. Huynh, H.K. Kim, J.K. Park, Some results on SI-rings, *J. Algebra* **174** (1995) 39–52.

[4] D.V. Huynh, R. Wisbauer, A structure theorem for SI-modules, *Glasgow Math. J.* **34** (1992) 83–86.

[5] S.K. Jain, S.R. López-Permouth, Rings whose cyclics are essentially embeddable in projectives, *J. Algebra* **128** (1990) 257–269.

[6] S.K. Jain, S.R. López-Permouth, S. Singh, On a class of QI-rings, *Glasgow Math. J.* **34** (1992) 75–81.

[7] T.Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York, 1999.

[8] S.R. López-Permouth, Rings characterized by their weakly injective modules, *Glasgow Math. J.* **34** (1992) 349–353.

[9] G.O. Michler, O.E. Villamayor, On rings whose simple modules are injective, *J. Algebra* **25** (1973) 185–201.

[10] S. Mohamed, B.J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, Cambridge, 1990.

[11] N.V. Sanh, K.P. Shum, S. Dhompongsa, S. Wongwai, On quasi-principally injective modules, *Algebra Colloq.* **6** (1999) 269–276.

[12] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, London, 1991.

[13] Y. Zhou, Notes on weakly-semisimple rings, *Bull. Austral. Math. Soc.* **53** (1996) 517–525.

[14] Y. Zhou, Weak injectivity and module classes, *Comm. Algebra* **25** (1997) 2395–2407.