

รายงานวิจัยฉบับสมบูรณ์

โครงการ

ตัวดำเนินการบางตัวที่เกี่ยวข้องกับตัวดำเนินการไดมอนด์

On Some Operators Related to Diamond Operators

โดย ศาสตราจารย์ อำนวย ขนนไทย และ คณะ

พฤษภาคม 2545

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

กิตติกรรมประกาศ

คณะผู้วิจัยขอขอบคุณ สำนักงานกองทุนสนับสนุนการวิจัย โดยเฉพาะอย่างยิ่ง ฝ่ายวิชาการ ที่ให้การสนับสนุนในด้านงบประมาณทั้งหมดให้แก่โครงการวิจัยนี้ จนกระทั่งโครงการวิจัยได้บรรลุวัตถุประสงค์ตามเป้าหมายทุกประการ

พร้อมกันนี้คณะผู้วิจัยขอขอบคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้เอื้ออำนวยสถานที่ และ อุปกรณ์การวิจัย ทำให้งานวิจัยลุล่วงไปด้วยดี

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หัวหน้าโครงการวิจัย

บทคัดย่อ

ตัวดำเนินการบางอย่างที่เกี่ยวข้องกับตัวดำเนินการไดมอนด์

(On Some Operators related to Diamond Operator)

ในงานวิจัยนี้ เราได้ศึกษาหาคำตอบขั้นพื้นฐาน การหาคอนวอลูชัน และการหาการแปลงฟูเรียร์ของตัวดำเนินการ

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

และ $\odot^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k$ ซึ่งเกี่ยวข้องกับตัวดำเนินการไดมอนด์

$$\diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k$$

เมื่อ $p+q=n$ และ $x \in R^n$ ซึ่ง R^n เป็นปริภูมิยูคลิดีียนมิติ n และ k เป็นจำนวนเต็มที่ไม่เป็นลบ

นอกจากนี้เรายังพบว่า เมื่อ $p=1$ และ $q=n-1$ และ $k=1$ แล้วตัวดำเนินการดังกล่าวจะกลายเป็นตัวดำเนินการคลื่น

$$\square = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \dots - \frac{\partial^2}{\partial x_n^2}$$

ซึ่งเป็นตัวดำเนินการที่สำคัญในวิชาฟิสิกส์และเป็นที่รู้จักกันดีมาเป็นเวลานานแล้ว

On Some Operators related to Diamond Operator

Abstact

In this research, we studied the elementary solution and the convolution and the Fourier transform of the operators

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

and $\odot^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k$ related to the Diamond operator

$$\diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k$$

where $p+q=n$ is the dimension of the space R^n of Euclidean, $x \in R^n$ and k is nonnegative integer.

Moreover, if $p=1$, $q=n-1$ and $k=1$ such the operators reduce to the well known wave operator

$$\square = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \dots - \frac{\partial^2}{\partial x_n^2}$$

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**แบบเสนอโครงการวิจัย
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1. ชื่อโครงการ (ภาษาไทย) ตัวดำเนินการบางตัวที่เกี่ยวข้องกับตัวดำเนินการ
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3. ความสำคัญและที่มาของปัญหา

นักคณิตศาสตร์ได้ศึกษาค้นคว้าตัวดำเนินการ (operators) ที่เกี่ยวข้องกับสมการเชิงอนุพันธ์ย่อย (Partial differential equations) เป็นเวลานาน ทั้งนี้เพราะตัวดำเนินการช่วยให้เกิดความสะดวกในการแยกแยะชนิดของสมการเชิงอนุพันธ์ย่อย และมีบทบาทสำคัญในการหาคำตอบ (solutions) ของสมการ ตัวดำเนินการที่นักคณิตศาสตร์คุ้นเคยกันดีในอดีต ได้แก่

1. ตัวดำเนินการลาปลาซ (Laplace operator)

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad \text{เกี่ยวข้องกับสมการเชิงวงรี}$$

(Elliptic equation) ซึ่งมีบทบาทสำคัญในทฤษฎีศักย์ (Potential theory)

2. ตัวดำเนินการของความร้อน (Heat operator)

$\frac{\partial}{\partial t} - \Delta$ เกี่ยวข้องกับสมการเชิงพาราโบลา (Parabolic equation) ซึ่งเป็นตัวแทนสมการของความร้อน

3. ตัวดำเนินการของคลื่น (Wave equation)

$\frac{\partial^2}{\partial t^2} - \Delta$ เกี่ยวข้องกับสมการเชิงไฮเพอโบลิก (Hyperbolic equation) มีบทบาทสำคัญในสมการคลื่นชนิดต่าง ๆ

ปัจจุบันการค้นคว้าวิจัยตัวดำเนินการ ก็ยังกระทำกันอย่างต่อเนื่อง จนเกิดตัวดำเนินการใหม่ ๆ ขึ้น เช่น S.E.Trone [14] ได้ค้นพบคำตอบขั้นพื้นฐาน (Fundamental solution) ของตัวดำเนินการอัลตราไฮเพอโบลิก (Ultra - hyperbolic)

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k$$

เมื่อ $p + q = n$ (ขนาดของปริภูมิ R^n) และ k เป็นจำนวนเต็มที่ไม่เป็นเลขลบ ในเวลาใกล้เคียงกัน M. Aguirre Tellez [13] ได้แสดงให้เห็นว่า การเกิดคำตอบของตัวดำเนินการ อัลตราไฮเพอโบลิกนั้น

ขึ้นอยู่กับค่าของ p, q และ n ว่าเป็นคู่หรือคี่ ในการค้นพบคำตอบของตัวดำเนินการ อัลตราไฮเพอโบลิกนี้ เป็นการขยายความรู้ของสมการคลื่นให้กว้างขวางขึ้น

ต่อมา A. Kananthai [6] ซึ่งเป็นหัวหน้าโครงการนี้ ได้ศึกษาหาคำตอบขั้นพื้นฐานของตัวดำเนินการอัลตราไฮเพอโบลิก ในรูป

$$\square^k u(x) = \sum_{i=1}^m c_i \square^k \delta, \quad c_i \text{ เป็นค่าคงที่และ } \delta \text{ คือไดเรคเดลต้า-}$$

ดิสทริบิวชัน ซึ่งผลงานนี้ได้รับการตีพิมพ์ใน Proceedings of International conference in Functional Analysis and Global Analysis ได้จัดขึ้นที่ เมืองมนิลา ประเทศฟิลิปปินส์ October 20 - 26, 1996 (Page 157 - 161)

หลังจากนั้น A. Kananthai [2] ได้ศึกษาการเกิดคำตอบขั้นพื้นฐานของสมการที่อยู่ในรูป

$$e^{\alpha t} \square^k u(t) = \delta$$

และได้รับการตีพิมพ์ในวารสารคือ Journal of Computational and Applied Mathematics 84 (1997) 101 - 106. Elsevier Science B.V. Amsterdam.

ในระยะเวลาใกล้เคียงกัน A. Kananthai [1] ได้ค้นพบตัวดำเนินการตัวใหม่และได้ตั้งชื่อว่า ตัวดำเนินการไดมอน

$$\Diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k$$

ซึ่งตัวดำเนินการ \Diamond สามารถเขียนอยู่ในรูปคอมโพสิตของตัวดำเนินการ \square และ Δ ดังนี้ $\Diamond = \square \Delta$ นอกจากนี้ A. Kananthai ได้ค้นพบคำตอบพื้นฐานของสมการ

$$\Diamond^k u(x) = \sum_{i=1}^n c_i \Diamond^k \phi_i \text{ และได้ค้นพบว่าการเกิดคำตอบของสมการข้างบนนี้ขึ้น}$$

อยู่กับความสัมพันธ์ระหว่าง k และ n ผลงานชิ้นนี้ได้รับการตีพิมพ์วารสารระดับนานาชาติ Applied Mathematics and Computation 88 : 27 - 37 (1997) Elsevier Science B.V New York NY10010.

ในงานวิจัยที่เสนอในโครงการนี้จะศึกษาตัวดำเนินการ

$$\odot^k \equiv \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k \quad \text{และ}$$

$$\oplus^k \equiv \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

ซึ่งเป็นตัวดำเนินการดังกล่าวยังไม่มีผู้ใดศึกษามาก่อน ในการวิจัยที่สามารถแยกออกได้เป็น 3 ขั้นตอน คือ

1. ศึกษาคุณสมบัติที่สำคัญทั้งหมดของตัวดำเนินการดังกล่าว

2. ศึกษาหาคำตอบขั้นพื้นฐานของสมการที่เกี่ยวข้องกับตัวดำเนินการนี้

3. ศึกษาหาคำตอบเชิงตัวเลข (Numerical solutions) ของสมการที่เกี่ยวข้องกับตัวดำเนินการนี้

4. วัตถุประสงค์ของงานวิจัย

1. เพื่อศึกษาค้นคว้าหาคุณสมบัติต่าง ๆ ของตัวดำเนินการใหม่ที่สร้างขึ้นในเชิงทฤษฎี

2. เพื่อหาคำตอบของสมการของตัวดำเนินการที่สร้างขึ้น

3. เพื่อศึกษาคอนโวลูชัน (convolution) ของตัวดำเนินการที่สร้างขึ้น

4. เพื่อศึกษาการแปลงฟูริเยร์ (Fourier's transform) ของตัวดำเนินการที่สร้างขึ้น

5. เพื่อหาคำตอบเชิงตัวเลข (numerical solutions) ของสมการที่เกี่ยวข้องกับตัวดำเนินการดังกล่าว

5. ระเบียบวิธีวิจัย

1. ศึกษาความรู้พื้นฐานเกี่ยวกับตัวดำเนินการ (operators) ต่าง ๆ ที่เกี่ยวข้องเช่น ตัวดำเนินการลาปลาซ (Laplace operator), ตัวดำเนินการอัลตรา ไฮเพอโบลิค (Ultra Hyperbolic operator) จากเอกสารและตำราที่มีอยู่

2. ศึกษาความรู้พื้นฐานที่เกี่ยวข้องกับตัวดำเนินการไดมอนด์จากเอกสารที่เกี่ยวข้อง

3. ศึกษาวิธีการหาคำตอบของสมการที่เกี่ยวข้องกับตัวดำเนินการต่าง ๆ ในข้อ

1. และ 2.

4. ศึกษาคอนโวลูชัน (convolution) และ การแปลงฟูรีเยร์ (Fourier's transforms) ของตัวดำเนินการที่สร้างขึ้น

5. ค้นคว้าหาความรู้เพิ่มเติมจากเอกสารหรือจากการพบปะเพื่อแลกเปลี่ยนความคิดเห็นกับนักคณิตศาสตร์ที่ทำงานเกี่ยวข้องทั้งภายในและต่างประเทศ

6. เมื่อได้ศึกษาค้นคว้าตามข้อ 1 - 5 แล้ว ขั้นตอนมาก็คือการทบทวนค้นคว้าเพื่อให้ได้องค์ความรู้ใหม่ที่สอดคล้องกับวัตถุประสงค์ของโครงการ

6. แผนการดำเนินงานตลอดโครงการวิจัยและผล (output) ที่จะได้
(ระบุผลงานคาดว่าจะตีพิมพ์ได้ด้วย)

กิจกรรมและขั้นตอนดำเนินงาน	2542 (6 เดือนแรก)						2542 (6 เดือนหลัง)					
	1	2	3	4	5	6	7	8	9	10	11	12
1. ศึกษาความรู้พื้นฐานเกี่ยวกับ Ultra - hyperbolic จากเอกสาร [13] และ [14]	↔											
2. ศึกษาความรู้เกี่ยวกับสมการ Ultra - hyperbolic และ Elliptic จากเอกสาร [2] , [3], [4], [6], [10]	↔											
3. ศึกษาพื้นฐานและคุณสมบัติต่าง ๆ เกี่ยวกับตัวดำเนินการโดมอนต์จากเอกสาร [1], [5], [7]			↔									
4. ศึกษาคำตอบวิคของตัวดำเนินการโดมอนต์จากเอกสาร [9]				↔								
5. ศึกษาการแปลงฟูรีเยร์ของตัวดำเนินการโดมอนต์จากเอกสาร [5]					↔							
6. เดินทางไปหาเอกสารที่เกี่ยวข้อง						↔						
7. รายงานความคืบหน้า 6 เดือน							↔					

กิจกรรมและขั้นตอนดำเนินงานวิจัย	2544 (6 เดือนแรก)						2544 (6 เดือนหลัง)					
	1	2	3	4	5	6	1	2	3	4	5	6
17. ศึกษาคุณสมบัติต่าง ๆ ที่เกี่ยวข้องกับ การแปลงฟูรีเยร์ที่เกี่ยวกับตัวดำเนินการที่กำลังศึกษา												
18. ค้นคว้าวิจัยเพื่อหาทฤษฎีใหม่ ๆ ของการแปลงฟูรีเยร์และสมการที่เกี่ยวข้องกับการแปลงฟูรีเยร์ของตัวดำเนินการที่กำลังศึกษา												
19. รายงานความคืบหน้าของโครงการ 6 เดือนแรกของปี 2544												
20. เดินทางไปหาเอกสารที่เกี่ยวข้อง												
21. ศึกษาและค้นคว้าเพื่อหาคำตอบเชิงตัวเลขของสมการที่เกี่ยวข้องกับตัวดำเนินการที่กำลังศึกษาอยู่												
22. ส่งผลงานตีพิมพ์ในวารสารนานาชาติ												
23. รายงานความคืบหน้าของโครงการ 6 เดือนหลังของ 2544												
24. รายงานฉบับสมบูรณ์												

7. ประโยชน์ที่คาดว่าจะได้รับ

1. ได้องค์ความรู้ใหม่ในตัวดำเนินการที่เกี่ยวข้องกับตัวดำเนินการไดมอนด์
2. ได้องค์ความรู้ใหม่ที่จะเป็นเครื่องมือในการศึกษาคำตอบของสมการเชิงอนุพันธ์ย่อย ที่เกี่ยวข้องกับตัวดำเนินการไดมอนด์
3. ได้หัวข้อวิจัยสำหรับนักศึกษาระดับปริญญาโทหรือเอก ที่จะป็นหัวข้อวิทยานิพนธ์
4. ได้ผลงานตีพิมพ์ในระดับนานาชาติ เพื่อเป็นการเผยแพร่ผลงานทางวิชาการของนักคณิตศาสตร์ไทย
5. ทำให้เกิดความร่วมมือระหว่างนักคณิตศาสตร์ไทย และ นักคณิตศาสตร์ระดับนานาชาติ อันจะนำมาซึ่งการพัฒนาคณิตศาสตร์ของประเทศไทยให้เจริญรุ่งเรือง

8. ผลงานที่จะตีพิมพ์ในวารสารวิชาการระดับนานาชาติในแต่ละปี

ปีที่ 1. เรื่อง "On the weak solution of the operator related to the Diamond operator"

วารสาร " Applied Mathematics and Computation, New York.

ปีที่ 2. เรื่อง "On the convolution of the operator related to the Diamond operator"

วารสาร "Computational and Applied Mathematics, Amsterdam.

ปีที่ 3. เรื่อง "On the Fourier transform of the operator related to the Diamond operator"

วารสาร "Applied Mathematics Letters , United Kingdom.

9. เอกสารอ้างอิง

1. A. Kananthai , *On the solution of the n - dimensional Diamond operator*, Applied Mathematics and computation 88 : 27 - 37 (1997), Elsevier Science Inc, New York NY 10010.
2. A. Kananthai , *On the Distribution related to the ultra - hyperbolic Equations* Journal of Computational and Applied Mathematics 84 (1997) 101 - 106. Elsevier Science B.V, Amsterdam, Netherlands.
3. A. Kananthai , *On the tempered distribution related to hyperbolic Equations*, Fasciculi Mathematici (1997) , Poland.
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5. A. Kananthai , *On the Fourier transform of the Diamond Kernel of Marcel Riesz* Applied Mathematics and Compeetation . Elsevier Science Inc. New York NY 10010 , to appear.
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9. A. Kananthai *On the weak solution of the compound Diamond operators* Proceedings of International Conference on Computational Mathematics, Chulalongkorn University 1997.

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 11. A. Kananthai *The Distribution solutions of ordinary differential Equation with polynomial Coefficients*. J. Sci. Fac. CMU. (1996) 23 (2) 52 - 58.
 12. A. Kananthai *On the $e^{at}\delta^{(k)}$ and it's Applications*, Annual Conference in Mathematics, King Mongkut's Institute of Technology, Ladkrabang, Thailand 1996.
 13. Ageeirre Tellez, *The distributional Itankel transform of Mareel Riesz's ultra - hyperbolic Kernel*, Studies in Applied Mathematics 93 : 133 - 162 (1994). M.I.T, Elsevier Sc Inc.
 14. S.E. Trione *On Mareel Riesz's ultra - hyperbolic Kernel*, trabajos de Matematica, 116 Preprint, 1987.
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เนื้อหางานวิจัยเรื่อง

ตัวดำเนินการบางตัวที่เกี่ยวข้องกับตัวดำเนินการไดมอนด์ (On Some Operators related to Diamond Operator)

ในงานวิจัยนี้เป็นการศึกษาตัวดำเนินการ

$$\odot^k \equiv \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k \quad \text{และ}$$

$$\oplus^k \equiv \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

ซึ่งเป็นตัวดำเนินการอนุพันธ์ย่อย ที่ยังไม่มีผู้ใดศึกษามาก่อน งานวิจัยนี้สามารถแยกออกได้เป็น 3 ขั้นตอน คือ

1. ศึกษาคุณสมบัติที่สำคัญทั้งหมดของตัวดำเนินการดังกล่าว
2. ศึกษาหาคำตอบชั้นพื้นฐานของสมการที่เกี่ยวข้องกับตัวดำเนินการนี้
3. ศึกษาหาคำตอบเชิงตัวเลข (Numerical solutions) ของสมการที่เกี่ยวข้องกับตัวดำเนินการนี้

วัตถุประสงค์ของงานวิจัย

1. เพื่อศึกษาค้นคว้าหาคุณสมบัติต่าง ๆ ของตัวดำเนินการใหม่ที่สร้างขึ้นในเชิงทฤษฎี
2. เพื่อหาคำตอบของสมการของตัวดำเนินการที่สร้างขึ้น
3. เพื่อศึกษาคอนโวลูชัน (convolution) ของตัวดำเนินการที่สร้างขึ้น
4. เพื่อศึกษาการแปลงฟูริเยร์ (Fourier's transform) ของตัวดำเนินการที่สร้างขึ้น

ขึ้น

5. เพื่อหาคำตอบเชิงตัวเลข (numerical solutions) ของสมการที่เกี่ยวข้องกับตัวดำเนินการดังกล่าว

เนื้อหางานวิจัยแบ่งเป็น 4 บท คือ บทที่ 1 กล่าวถึงความสำคัญและที่มาของปัญหา บทที่ 2 เป็นการศึกษาคำตอบขั้นพื้นฐานของตัวดำเนินการ \odot^k และ \oplus^k บทที่ 3 ศึกษาเรื่องคอนวอลูชันของตัวดำเนินการ \oplus^k และในบทที่ 4 เป็นการศึกษาการแปลงฟูริเยร์ของตัวดำเนินการ \oplus^k

บทที่ 1

ความสำคัญและที่มาของปัญหา

นักคณิตศาสตร์ได้ศึกษาค้นคว้าตัวดำเนินการ (operators) ที่เกี่ยวข้องกับสมการเชิงอนุพันธ์ย่อย (Partial differential equations) เป็นเวลานาน ทั้งนี้เพราะตัวดำเนินการช่วยให้เกิดความสะดวกในการแยกแยะชนิดของสมการเชิงอนุพันธ์ย่อย และมีบทบาทสำคัญในการหาคำตอบ (solutions) ของสมการ ตัวดำเนินการที่นักคณิตศาสตร์คุ้นเคยกันดีในอดีต ได้แก่

1. ตัวดำเนินการลาปลาซ (Laplace operator)

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad \text{เกี่ยวข้องกับสมการเชิงวงรี}$$

(Elliptic equation) ซึ่งมีบทบาทสำคัญในทฤษฎีศักย์ (Potential theory)

2. ตัวดำเนินการของความร้อน (Heat operator)

$$\frac{\partial}{\partial t} - \Delta \quad \text{เกี่ยวข้องกับสมการเชิงพาราโบลา (Parabolic}$$

equation) ซึ่งเป็นตัวแทนสมการของความร้อน

3. ตัวดำเนินการของคลื่น (Wave equation)

$$\frac{\partial^2}{\partial t^2} - \Delta \quad \text{เกี่ยวข้องกับสมการเชิงไฮเพอโบลา (Hyperbolic}$$

equation) มีบทบาทสำคัญในสมการคลื่นชนิด

ต่าง ๆ

ปัจจุบันการค้นคว้าวิจัยตัวดำเนินการ ก็ยังกระทำกันอย่างต่อเนื่อง จนเกิดตัวดำเนินการใหม่ ๆ ขึ้น เช่น S.E.Trone [14] ได้ค้นพบคำตอบขั้นพื้นฐาน (Fundamental solution) ของตัวดำเนินการอัลตราไฮเพอโบลิค (Ultra - hyperbolic)

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k$$

เมื่อ $p + q = n$ (ขนาดของปริภูมิ R^n) และ k เป็นจำนวนเต็มที่ไม่เป็นเลขลบ ในเวลาใกล้เคียงกัน M. Aguirre Tellez [13] ได้แสดงให้เห็นว่า การเกิดคำตอบของตัวดำเนินการ อัลตราไฮเปอโบลิกนั้น

ขึ้นอยู่กับค่าของ p, q และ n ว่าเป็นคู่หรือคี่ ในการค้นพบคำตอบของตัวดำเนินการ อัลตราไฮเปอโบลิกนี้ เป็นการขยายความรู้ของสมการคลื่นให้กว้างขวางขึ้น

ต่อมา A. Kananthai [6] ซึ่งเป็นหัวหน้าโครงการนี้ ได้ศึกษาหาคำตอบขั้นพื้นฐานของตัวดำเนินการอัลตราไฮเปอโบลิก ในรูป

$$\square^k u(x) = \sum_{i=1}^m c_i \square^k \delta, \quad c_i \text{ เป็นค่าคงที่และ } \delta \text{ คือไดเรคเดลต้า-}$$

ดิสทริบิวชัน ซึ่งผลงานนี้ได้รับการตีพิมพ์ใน Proceedings of

International conference in Functional Analysis and Global Analysis ได้จัดขึ้นที่ เมืองมินลา ประเทศฟิลิปปินส์ October 20 - 26, 1996 (Page 157 - 161)

หลังจากนั้น A. Kananthai [2] ได้ศึกษาการเกิดคำตอบขั้นพื้นฐานของสมการที่อยู่ในรูป

$$e^{\alpha t} \square^k u(t) = \delta$$

และได้รับการตีพิมพ์ในวารสารคือ Journal of Computational and Applied Mathematics 84 (1997) 101 - 106. Elsevier Science B.V. Amsterdam.

ในระยะเวลาใกล้เคียงกัน A. Kananthai [1] ได้ค้นพบตัวดำเนินการตัวใหม่และได้ตั้งชื่อว่า ตัวดำเนินการไดมอน

$$\Diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k$$

ซึ่งตัวดำเนินการ \Diamond สามารถเขียนอยู่ในรูปคอมโพสิตของตัวดำเนินการ \square และ Δ ดัง

นี้ $\Diamond = \square \Delta$ นอกจากนี้ A. Kananthai ได้ค้นพบคำตอบพื้นฐานของสมการ

$$\diamond^k u(x) = \sum_{i=1}^n c_i \diamond^k \delta \text{ และได้ค้นพบว่าการเกิดคำตอบของสมการข้างบนนี้ขึ้น}$$

อยู่กับความสัมพันธ์ระหว่าง k และ n ผลงานชิ้นนี้ได้รับการตีพิมพ์วารสารระดับนานาชาติ

ชาติ Applied Mathematics and Computation 88 : 27 - 37 (1997) Elsevier

Science B.V New York NY10010.

ในงานวิจัยที่เสนอในโครงการนี้จะศึกษาตัวดำเนินการ

$$\odot^k \equiv \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k \quad \text{และ}$$

$$\oplus^k \equiv \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

ซึ่งเป็นตัวดำเนินการดังกล่าวยังไม่มีผู้ใดศึกษามาก่อน ในการวิจัยนี้ได้แบ่งการวิจัยเป็น 3 ขั้นตอน คือ

1. ศึกษาคุณสมบัติที่สำคัญทั้งหมดของตัวดำเนินการดังกล่าว
2. ศึกษาหาคำตอบชั้นพื้นฐานของสมการที่เกี่ยวข้องกับตัวดำเนินการนี้
3. ศึกษาหาคำตอบเชิงตัวเลข (Numerical solutions) ของสมการที่เกี่ยวข้องกับตัวดำเนินการนี้

วัตถุประสงค์ของงานวิจัย

1. เพื่อศึกษาค้นคว้าหาคุณสมบัติต่าง ๆ ของตัวดำเนินการใหม่ที่สร้างขึ้นในเชิงทฤษฎี
2. เพื่อหาคำตอบของสมการของตัวดำเนินการที่สร้างขึ้น
3. เพื่อศึกษาคอนโวลูชัน (convolution) ของตัวดำเนินการที่สร้างขึ้น

4. เพื่อศึกษาการแปลงฟูริเยร์ (Fourier's transform) ของตัวดำเนินการที่สร้างขึ้น
5. เพื่อหาคำตอบเชิงตัวเลข (numerical solutions) ของสมการที่เกี่ยวข้องกับตัวดำเนินการดังกล่าว

บทที่ 2

คำตอบชั้นพื้นฐานของตัวดำเนินการ \odot^k และ \oplus^k

ในบทนี้เราจะกล่าวถึงสมบัติพื้นฐานเกี่ยวกับตัวดำเนินการเชิงเส้น ตัวดำเนินการเชิงอนุพันธ์ ตัวดำเนินการเชิงอนุพันธ์ย่อย ตัวดำเนินการเชิงรี (Elliptic operator) ตัวดำเนินการอัลตราไฮเพอร์โบลิค (Ultra-hyperbolic operator) ตัวดำเนินการไดมอนด์ (Diamond operator) และ กล่าวถึงพื้นฐานของการแปลงฟูรีเยร์ของตัวดำเนินการไดมอนด์

เนื้อหาหลักของบทที่ 2 นี้เป็นการค้นพบทฤษฎีต่างๆที่สำคัญเกี่ยวกับการหาคำตอบวีคของตัวดำเนินการ \oplus^k และ เกี่ยวกับ Distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$ นอกจากนี้ยังศึกษา Green function ของตัวดำเนินการ \oplus^k และ \odot^k

คุณสมบัติพื้นฐานที่เกี่ยวกับตัวดำเนินการไดมอนด์
BASIC PROPERTIES OF OPERATORS RELATED TO
DIAMOND OPERATOR

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BASIC PROPERTIES OF OPERATORS RELATED TO DIAMOND OPERATOR

1. Linear Operators

In this section we study the operators which preserve the two algebraic operations of vector space. These operators are called linear operators. There are two important types of linear operators: bounded linear operators and unbounded linear operators. These two types of linear operators play important roles in the theory of operators. Especially, the unbounded linear operators occur in many applications, notably in connection with differential equations and in quantum mechanics. Their theory is more complicated than of bounded operators.

The aim of this section is to give some basic knowledge concerning bounded and unbounded linear operators in order to understand the later sections better.

1.1 Bounded Linear Operators

Definition 1.1.1 A linear operator T is an operator such that

(i) the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field

(ii) for all $x, y \in D(T)$ and scalar α ,

$$T(x + y) = Tx + Ty$$

$$T(\alpha x) = \alpha Tx$$

The null space of T is the set

$$N(T) = \{x \in D(T) \mid Tx = 0\}.$$

It is a subspace of $D(T)$.

Definition 1.1.2 Let X and Y be normed spaces and $T : X \rightarrow Y$ a linear operator. The operator T is said to be bounded if there is a real number c such that for all $x \in X$,

$$\|Tx\| \leq c \|x\| \quad (1)$$

By (1) we have for $x \in X, x \neq 0$

$$\frac{\|Tx\|}{\|x\|} \leq c$$

Hence $\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \leq c$. We define the norm of T by

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

We can easily show that

$$\|T\| = \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\|$$

We shall now consider important general properties of bounded linear operators.

Theorem 1.1.3 (Continuity and Boundedness)

Let $T : X \rightarrow Y$ be a linear operator, where X and Y are normed spaces.

Then :

- (1) T is continuous if and only if T is bounded
- (2) If T is continuous at a single point, then it is continuous.

Proof (1) For $T = 0$ the statement is trivial.

Let $T \neq 0$. Then $\|T\| \neq 0$. We assume T to be bounded and consider any $x_0 \in X$. Let any $\varepsilon > 0$ be given. Then, since T is linear, for every $x \in X$ such that

$$\|x - x_0\| < \delta \quad \text{where} \quad \delta = \frac{\varepsilon}{\|T\|}.$$

We obtain

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\| \|x - x_0\| < \|T\| \delta = \varepsilon$$

Since $x_0 \in X$ was arbitrary, this shows that T is continuous.

Conversely, assume that T is continuous at an arbitrary $x_0 \in X$. Then, given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$(*) \quad \|Tx - Tx_0\| \leq \varepsilon \quad \text{for all } x \in X \text{ satisfying } \|x - x_0\| \leq \delta.$$

We now take any $y \neq 0$ in X and let $x = x_0 + \frac{\delta}{\|y\|} y$. Then $x - x_0 = \frac{\delta}{\|y\|} y$. Hence $\|x - x_0\| = \delta$, so that we may use (*).

Since T is linear, we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \|T(\frac{\delta}{\|y\|} y)\| = \frac{\delta}{\|y\|} \|Ty\|$$

and (*) implies

$$\frac{\delta}{\|y\|} \|Ty\| \leq \varepsilon. \quad \text{Thus } \|Ty\| \leq \frac{\varepsilon}{\delta} \|y\|$$

or

$$\|Ty\| \leq c \|y\|,$$

where $c = \frac{\varepsilon}{\delta}$. Therefore T is bounded.

(2) Continuity of T at a point implies boundedness of T by the second part of the proof of (1), and this implies continuity of T by (1).

Corollary 1.1.5 Let T be a bounded linear operator. Then :

- (1) $x_n \rightarrow x$ implies $Tx_n \rightarrow Tx$.
- (2) The null space $N(T)$ is closed.

Linear Functionals

A functional is an operator whose range lies on the real line R or in the complex plane C . A linear functional f on a vector space X is a linear operator with range in the scalar field K of X ; thus

$$f : X \rightarrow K$$

where $K = R$ if X is real and $K = C$ if X is complex.

Definition 1.1.6 A bounded linear functional f on a normed space X is a linear functional on X such that there exists $c > 0$ with

$$|f(x)| \leq c \|x\| \quad \text{for all } x \in X.$$

Furthermore, the norm of f is

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$$

or

$$\|f\| = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|$$

The following theorem is a special case of Theorem 1.1.4

Theorem 1.1.7 A linear functional f on a normed space X is continuous if and only if f is bounded.

Examples 1.1.8

(1) **Identity operator.** The identity operator $I : X \rightarrow X$ on a normed space $X \neq \{0\}$ is bounded and has norm $\|I\| = 1$.

(2) **Zero operator.** The zero operator $0 : X \rightarrow Y$ on a normed space X is bounded and has norm $\|0\| = 0$.

(3) **Integral operator.** We can define an integral operator $T : C[0, 1] \rightarrow C[0, 1]$ by

$$y = Tx \text{ Where } y(t) = \int_0^1 k(t, \tau) x(\tau) d\tau.$$

Here k is a given function, which is called the kernel of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau$ - plane, where $J = [0, 1]$.

This operator is linear.

We can prove that T is bounded.

To prove this, we first note that the continuity of k on the closed square implies that k is bounded, say, $|k(t, \tau)| \leq k_0$ for all $(t, \tau) \in G$, where k_0 is a real number. Furthermore,

$$|x(t)| \leq \max_{t \in \tau} |x(t)| = \|x\|$$

Hence

$$\begin{aligned} \|y\| = \|Tx\| &= \max_{t \in \tau} \left| \int_0^1 k(t, \tau) x(\tau) d\tau \right| \\ &\leq \max_{t \in \tau} \int_0^1 |k(t, \tau)| |x(\tau)| d\tau \\ &\leq k_0 \|x\| \end{aligned}$$

Hence T is bounded.

(4) A real matrix $A = (a_{ik})$ with r rows and n columns defines an operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^r$ by means of $y = Ax$ where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_r)$,

$$y_i = \sum_{k=1}^n a_{ik} x_k \quad (i \leq r)$$

T is linear because matrix multiplication is a linear operation.

We can prove that T is bounded.

To prove this, use the definition of norm on \mathbb{R}^n

$$\|x\| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}};$$

By Cauchy - Schwarz inequality we thus obtain

$$\begin{aligned} \|Tx\|^2 &= \sum_{i=1}^r y_i^2 = \sum_{i=1}^r \left(\sum_{k=1}^n a_{ik} x_k \right)^2 \\ &\leq \sum_{i=1}^r \left[\left(\sum_{k=1}^n a_{ik}^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^n x_m^2 \right)^{\frac{1}{2}} \right]^2 \\ &= \|x\|^2 \sum_{i=1}^r \sum_{k=1}^n a_{ik}^2 \end{aligned}$$

This implies that T is bounded.

(5) **Differentiation operator.**

Let X be the normed space of all polynomials on $I = [0, 1]$ with norm $\|x\| = \max |x(t)|, t \in I$.

A differentiation operator T is defined on X by

$$Tx(t) = x'(t)$$

This operator is linear but not bounded. Indeed, let $x_n(t) = t^n$, where $n \in \mathbb{N}$, then $\|x_n\| = 1$ and

$$Tx_n(t) = x'_n(t) = nt^{n-1}$$

So that $\|Tx_n\| = n$ and $\frac{\|Tx_n\|}{\|x_n\|} = n$.

This shows that T is not bounded.

Since differentiation is an important operation, our result seems to indicate that unbounded operators are also of practical importance.

(6) Let $C^\infty(\mathbb{R}^n)$ be the space of all infinitely differentiable functions on \mathbb{R}^n . Let $D : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ be a partial differential operator defined by

$$D(f(x_1, \dots, x_n)) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_n)$$

The same as in example (5) we have that D is not bounded.

There are many important partial differential operators such as the elliptic Operators, the ultra-hyperbolic operators and the diamond operators. We will discuss in more details about these operators in the section 2.

2. The Partial Differential operator

In the first part we have studied the properties of Linear operators which are the important basic concepts and needed for our research, particularly in the unbounded operators. The partial differential operator is an example of the unbounded operators. In the second part we shall study the partial differential operators. The following operators are concerned with our research.

2.1 The elliptic operator or the Laplacian

We denote by Δ the Laplacian and is defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad \dots\dots (2.1.1)$$

This operator is well known, particularly in the potential theory.

Consider the equation $\Delta u(x) = f(x)$ (2.1.2)

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the n – dimensional Euclidian space $u(x)$ is an unknown function and $f(x)$ is a given function.

If $f(x) = 0$ in (2.1.2), then $u(x)$ is called a harmonic function in an open subset Ω of \mathbb{R}^n and if $f(x) = \delta$ - The Dirac – delta distribution, then $u(x)$ is called the elementary solution or a weak solution of (2.1.2)

Now we are finding the elementary solution of (2.1.2).

First, consider the equation $\Delta u = 0$ (2.1.3)

This equations is preserved under rotations about a point ξ , that is under orthogonal linear substitution for $x - \xi$. Thus there is a solution $v(x)$ of (2.1.3) that are invariant under rotation about ξ . That is have the same value at all point x at the some distance from ξ . Such solution would be of the form

$$v = \psi(r) \text{ where } r = |x - \xi| = \sqrt{\sum_{i=1}^n (x_i - \xi_i)^2}.$$

Thus, by (2.1.3) and the chain rule of differentiations

$$\text{we have } \Delta v = \psi''(r) + \frac{(n-1)}{r} \psi'(r) = 0$$

and obtaining the solution, $\psi'(r) = Cr^{1-n}$

$$\text{and } \psi(r) = \begin{cases} \frac{Cr^{2-n}}{2-n} & \text{for } n > 2 \\ c \log r & \text{for } n = 2 \end{cases} \quad \dots\dots (2.1.4)$$

where C is a constant. We see that the function $v(x) = \psi(x)$ satisfies (2.1.3) for $r > 0$, that is for $x \neq \xi$. Now v is singular at $x = \xi$, let $B(\xi, \rho)$ be a ball contained in Ω with center ξ , radius ρ and let $S(\xi, \rho)$ be the boundary of $B(\xi, \rho)$. Let $\Omega_\rho = \Omega - B(\xi, \rho)$.

Since $\Delta v = 0$ in Ω_ρ , thus by applying the Green identity we have

$$\int_{\Omega_\rho} (v \Delta u - u \Delta v) dx = \int_{\partial \Omega} (v \frac{\partial v}{\partial x} - u \frac{\partial v}{\partial x}) ds$$

$$\text{Thus } \int_{\Omega_\rho} v \Delta u dx = \int_{\partial \Omega} (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) ds + \int_{S(\xi, \rho)} (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) ds$$

Now we have $v = \psi(\rho)$

$$\begin{aligned} \frac{\partial v}{\partial n} &= \frac{\partial \psi(\rho)}{\partial \rho} \cdot \frac{\partial \rho}{\partial n} = \psi'(\rho) (-1) \left[\frac{\partial \rho}{\partial n} = \nabla \cdot n \rho = \sum_{i=1}^n n_{x_i} \frac{\partial \rho}{\partial x_i} \right] \\ &= -\psi'(\rho) \end{aligned}$$

$$\begin{aligned} \text{consider } \int_{S(\xi, \rho)} v \frac{\partial v}{\partial n} ds &= \psi(\rho) \int_{S(\xi, \rho)} \frac{\partial u}{\partial n} ds \\ &= -\psi(\rho) \int_{B(\xi, \rho)} \Delta u dx \end{aligned}$$

Since u and Δu are continuous at ξ .

Thus by (2.1.4) $\psi(\rho) \rightarrow 0$ as $\rho \rightarrow 0$

$$\text{Thus } \int_{S(\xi, \rho)} v \frac{\partial u}{\partial n} ds \rightarrow 0 \text{ as } \rho \rightarrow 0$$

$$\begin{aligned} \text{consider } \int_{S(\xi, \rho)} v \frac{\partial v}{\partial n} ds &= -\psi'(\rho) \int_{S(\xi, \rho)} u ds \\ &= -C\rho^{1-n} \int_{S(\xi, \rho)} u ds \end{aligned}$$

By Gauss's law of the arithmetic mean.

$$\begin{aligned} \text{we have } u(\xi) &= \frac{1}{\omega_n \rho^{n-1}} \int_{S(\xi, \rho)} u(\alpha) d\alpha \\ \text{where } |x - \xi| &= \rho \end{aligned}$$

$$\left[-C\rho^{1-n} \int_{S(\xi, \rho)} u ds = -C\rho^{1-n} u(\xi) \int ds = -C\rho^{1-n} u(\xi) \omega_n \right]$$

where $\alpha \omega_n \rho^{n-1}$ is the surface area of the sphere $|x - \xi| = \rho$.

$$\text{Thus } \int_{S(\xi, \rho)} u \frac{\partial v}{\partial n} ds = -C \omega_n u(\xi)$$

where $\omega_n = 2\pi^{n/2} / \Gamma(\frac{n}{2})$ is the surface area of unit sphere.

$$\text{Thus } \int_{\Omega} v \Delta u dx = \int_{\partial\Omega} (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) ds + C \omega_n u(\xi)$$

choose $C = \frac{1}{\omega_n}$, then we obtain

$$\psi(r) = \begin{cases} \frac{r^{2-n}}{(2-n)\omega_n} & \text{for } n > 2 \\ \frac{\log r}{2\pi} & \text{for } n = 2 \end{cases}$$

If we write $v = K(x, \xi) = \psi(r) = \psi(|x - \xi|)$.

$$\text{Then } u(\xi) = \int_{\Omega} K(x, \xi) \Delta u dx = \int_{\partial\Omega} (K(x, \xi) \frac{\partial u}{\partial n_x} - u \frac{\partial K(x, \xi)}{\partial n_x}) ds_x$$

for $\xi \in \Omega$. where the subscript x in s_x and ∂n_x indicates the variable of integration. respectively differentiation.

$$\text{Now } \psi(r) = \frac{r^{2-n}}{(2-n)\omega_n} \text{ for } n > 2.$$

If we let $\xi = 0$, we have $r = |x| = (\sum_{i=1}^n x_i^2)^{1/2}$

$$\text{Let } E(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}, \quad n > 2 \quad \dots\dots\dots(2.1.5)$$

Then $E(x)$ is call the elementary solution of (2.1.2) that is $\Delta E(x) = \delta$

Now in our work we study the operator

$$\Delta^k = (\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2})^k \text{ that is the Laplace operator iterated } k \text{ times } (k = 1, 2, 3, \dots)$$

We define the function $S_\alpha(x) = 2^{-\alpha} \pi^{-n/2} \Gamma\left(\frac{n-\alpha}{2}\right) \frac{|x|^{\alpha-n}}{\Gamma(\frac{\alpha}{2})}$ (2.1.6)

and α is complex parameter.

From (2.1.5) we have $E(x) = -S_2(x)$

Consider the equation $\Delta^k u(x) = \delta$ (2.1.7)

We can obtain $u(x) = (-1)^k S_{2k}(x)$ as an elementary solution of (2.1.7) or the

Green's function for the operator Δ^k . Since $\Delta^k u(x) = \Delta(\Delta^{k-1}u(x)) = \delta$

Thes $\Delta^{k-1}u(x) = E(x) = -S_2(x)$.

Convoluing both sides of the above equation by $-S_2(x)$. The we obtain

$$(-S_2(x) * \Delta^{k-1}u(x) = -S_2(x) * (-S_2(x))$$

$$\text{or } \Delta(-S_2(x)) * \Delta^{k-1}u(x) = -S_2(x) * (-S_2(x))$$

$$\delta * \Delta^{k-2}u(x) = -S_2(x) * (-S_2(x))$$

$$\Delta^{k-2}u(x) = (-S_2(x) * (-S_2(x)))$$

Convoluing both side of the equation again by $S_2(x)$ up until $k-2$ times, we then obtain

$$u(x) = \underbrace{(-S_2(x)) * (-S_2(x)) * \dots * (-S_2(x))}_{k \text{ times}}$$

$$= (-1)^k S_{2k}(x), \text{ since } S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x)$$

$$\text{Thus } u(x) = (-1)^k S_{2k}(x) \dots\dots\dots(2.1.8)$$

is an elementary solution of (2.1.7)

2.2 The Ultra – hyperbolic Operator.

This operator is denoted by \square and is defined by

$$\square \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \quad \dots\dots\dots(2.2.1)$$

where $p+q = n$ is the dimension of the Euclidean space $|R^n$ and $(x_1, x_2, \dots, x_n) \in |R^n$.

$$\text{Consider the equation } \square^k u(x) = f(x) \quad \dots\dots\dots(2.2.2)$$

where \square^k is an operator given by (2.2.1) with iterated k – times $(k = 1, 2, \dots, n)$. S.E. Trione [2] has shown that $u(x) = R_{2k}(x)$ is an elementary solution of (2.2.2), (2.2.23)

that is $\square^k R_{2k}(x) = \delta$ where

$$R_\alpha(x) = \frac{v^{\alpha-n}}{K_n(\alpha)}, \quad \alpha \text{ is an complex parameter in this case}$$

$$\alpha = 2k, v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$$

$$p+q = n \text{ and } k_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2-\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \quad \dots\dots\dots(2.2.4)$$

we have the properties of $R_\alpha(x)$ related to the operator \square^k

$$(i) \square R_{\alpha+2}(x) = R_\alpha(x)$$

$$(ii) R_{-2k}(x) = \square^k \delta$$

$$(iii) R_0(x) = \delta$$

$$(iv) \square^k R_{2k}(x) = \delta$$

$$\text{Proof (1) Since } R_{\alpha+2}(x) = \frac{v^{\frac{\alpha+2-n}{2}}}{K(\alpha+2)}, \text{ by (2.3)}$$

$$\text{Now } \frac{\partial}{\partial x_i} \left[v^{\frac{\alpha+2-n}{2}} \right] = (\alpha+2-n) v^{\frac{\alpha-n}{2}} \cdot x_i, \quad i = 1, 2, \dots, p$$

$$\text{and } \frac{\partial}{\partial x_j} \left[v^{\frac{\alpha+2-n}{2}} \right] = -(\alpha+2-n) v^{\frac{\alpha-n}{2}} \cdot x_j, \quad j = p+1, p+2, \dots, p+q.$$

also, we have

$$\frac{\partial^2}{\partial x_i^2} \left[v^{\frac{\alpha+2-n}{2}} \right] = (\alpha+2-n) \left[v^{\frac{\alpha-n}{2}} + (\alpha-n) v^{\frac{\alpha-2-n}{2}} \cdot x_i^2 \right]$$

$$\text{and } \frac{\partial^2}{\partial x_j^2} [v^{\frac{\alpha+2-n}{2}}] = -(\alpha+2-n) [v^{\frac{\alpha-n}{2}} - (\alpha-n) v^{\frac{\alpha-2-n}{2}} x_j^2]$$

$$\begin{aligned} \text{Thus } \square v^{\frac{\alpha+2-n}{2}} &= \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} v^{\frac{\alpha+2-n}{2}} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} v^{\frac{\alpha+2-n}{2}} \\ &= \alpha(\alpha+2-n) v^{\frac{\alpha-n}{2}} \dots \end{aligned} \quad (2.2.5)$$

We next show that $K_n(\alpha+2) = \alpha(\alpha+2-n) K_n(x)$
we have, from (2.2.4) that

$$K_n(\alpha+2) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+(\alpha+2)-n}{2}) \Gamma(\frac{1-(\alpha+2)}{2}) \Gamma(\alpha+2)}{\Gamma(\frac{2+(\alpha+2)-p}{2}) \Gamma(\frac{p-(\alpha+2)}{2})}$$

$$\text{we know that } \Gamma(\frac{2+\alpha+2-n}{2}) = \frac{(\alpha+2-n)}{2} \Gamma(\frac{\alpha+2-n}{2})$$

$$\Gamma(\alpha+2) = \alpha(\alpha+1) \Gamma(\alpha)$$

$$\Gamma(1+\frac{\alpha+2-p}{2}) \Gamma(\frac{p-\alpha-2}{2}) = \pi \operatorname{cosec} \pi(\frac{p-\alpha-2}{2})$$

$$= -\pi \operatorname{cosec} \pi(\frac{p-\alpha}{2})$$

$$= -\Gamma(1+\frac{\alpha+2-p}{2}) \Gamma(\frac{p-\alpha}{2})$$

$$\text{and } \Gamma(\frac{1-\alpha}{2}-1) = \frac{\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{1-\alpha}{2}-1)} = \frac{-2}{1+\alpha} \Gamma(\frac{1-\alpha}{2})$$

Substitute into (2.2.5) we obtain

$$k_n(\alpha+2) = \alpha(\alpha+2-n) k_n(\alpha)$$

$$\text{Thus } \square R_{\alpha+2}(x) = \square \frac{v^{\frac{\alpha+2-n}{2}}}{K_n(\alpha+2)}$$

$$= \frac{\alpha(\alpha+2-n)v^{\frac{\alpha-n}{2}}}{\alpha(\alpha+2-n)K_n(\alpha)} = \frac{v^{\frac{\alpha-n}{2}}}{K_n(\alpha)}$$

It follows that $\square R_{\alpha+2}(x) = R_{\alpha}(x)$

(ii) To show that $R_{-2k}(x) = \square^k \delta$

$$\text{Now } R_{-2k}(a) = \lim_{\alpha \rightarrow -2k} R_{\alpha}(x) = \lim_{\alpha \rightarrow -2k} \frac{v^{\frac{\alpha-n}{2}}}{K_n(\alpha)}$$

By using Legendre's duplication formula of $\Gamma(z)$

$$\text{That is } \Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2})$$

we obtain

$$R_{-2k}(x) = \lim_{\alpha \rightarrow -2k} \frac{v^{\frac{\alpha-n}{2}}}{2^{\alpha-1} \pi^{\frac{n-2}{2}} \Gamma(\frac{\alpha}{2})} \cdot \lim_{\alpha \rightarrow -2k} \left\{ \frac{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}{\Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\frac{\alpha+1}{2})} \right\}$$

By using the properties of Gamma function $\Gamma(z)$ and I.M. Gelfand and G.E. Shilov [], we obtain

$$R_{-2k}(x) = \square^k \delta$$

(iii) From (ii), by putting $k=0$, we obtain

$$R_0(x) = \square^0 \delta = \delta.$$

(iv) From (i), we have $\square R_{\alpha}(x) = R_{\alpha-2}(x)$

$$\text{and } \square^2 R_{\alpha}(x) = \square R_{\alpha-2}(x) = R_{\alpha-4}(x)$$

$$\text{Thus } \square^k R_{\alpha}(x) = R_{\alpha-2k}(x)$$

By putting $\alpha = 2k$ and by (iii) we obtain

$$\square^k R_{2\alpha}(x) = R_{2k-2k}(x) = R_0(x) = \delta$$

that is $R_{2\alpha}(x)$ is an elementary solution of (2.2.2)

In particular, if $p=1$ and $k=1$ then (2.2.1) becomes the wave operator, that is

$$\square \equiv \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2}$$

and if $\square u(x) = f(x)$ then $u(x) = \frac{w^{\frac{2-n}{2}}}{(2-n)\Gamma(\frac{2-n}{2})}$ is an elementary solution

for the operator \square where $w = x_1^2 - x_2^2 - \dots - x_n^2$, $x_1 > 0$

2.3 The Diamond Operator

This operator is denoted by \diamond and defined by

$$\diamond \equiv \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2, \quad p+q=n \quad \dots (2.3.1)$$

or $\diamond = \square \Delta = \Delta \square$ where \square is defined by (2.2.1)

and Δ is an elliptic operator or Laplacian defined by (2.1.1)

Consider the equation

$$\diamond^k u(x) = f(x) \quad \dots (2.3.2)$$

where \diamond^k is defined by (2.3.1) with iterated k times.

A. Kananthai [] has shown that the convolution $R_{2k}(x) * (-1)^k S_{2k}(x)$ is an elementary solution of (2.3.2).

That is

$$\diamond^k (R_{2k}(x) * (-1)^k S_{2k}(x)) = \delta$$

where $R_{2k}(x)$ is defined by (2.2.3) with $\alpha = 2k$ and $S_{2k}(x)$ is defined by (2.1.6) and $\alpha = 2k$

Now define the family $K_{\alpha,\beta} = R_{\alpha} * S_{\beta}$ where R_{α} and S_{β} are defined by (2.2.3) and (2.1.6) respectively.

We have the properties of $K_{\alpha,\beta}$ related to the Diamond operator \diamond^k

$$(i) \quad \diamond^k (K_{\alpha,\beta}(x)) = (-1)^k K_{\alpha-2k,\beta-2k}$$

$$(ii) \quad \diamond^k \delta(x) = K_{-2k,-2k}$$

$$(iii) \quad K_{0,0} = \delta$$

$$(iv) \quad \diamond^k ((-1)^k K_{2k,2k}) = \delta$$

$$\begin{aligned} \text{Proof (1) We have } \diamond^k (K_{\alpha,\beta}(x)) &= \square^k \Delta^k (R_{\alpha} * S_{\beta}) \\ &= \square^k (R_{\alpha}(x) * \Delta^k S_{\beta}) \end{aligned}$$

Since $\square^k R_{\alpha}(x) = R_{\alpha-2k}$ (by the previous proof) and also we can show that $\Delta^k S_{\beta} = (-1)^k S_{\beta-2k}$.

$$\begin{aligned} \text{Thus } \diamond^k (K_{\alpha,\beta}(x)) &= R_{\alpha-2k}(x) * (-1)^k S_{\beta-2k}(x) \\ &= (-1)^k R_{\alpha-2k}(x) * S_{\beta-2k}(x) \\ &= (-1)^k K_{\alpha-2k,\beta-2k}(x) \end{aligned}$$

$$(iii) \quad \text{Since } K_{\alpha,\beta} = R_{\alpha} * S_{\beta}$$

By putting $\alpha = \beta = 0$, we have $K_{0,0} = R_0 * S_0 = \delta * \delta = \delta$

(ii) putting $\alpha = \beta = 0$ in (1), we obtain

$$\phi^k (K_{0,0}(x)) = K_{-2k,-2k}(x)$$

Thus, by (iii), $\phi^k \delta = K_{-2k,-2k}(x)$

(iv) From (1), put $\alpha = \beta = 2k$, we have

$$\phi^k (K_{2k,2k}(x)) = (-1)^k (K_{0,0}(x)) = (-1)^k \delta$$

or $\phi^k ((-1)^k K_{2k,2k}(x)) = \delta$

Thus $(-1)^k K_{2k,2k}(x) = R_{2k}(x) * (-1)^k R_{2k}(x)$ is the elementary solution or weak solution for (2.3.2)

2.4 The Fourier Transform of the Diamond Operator

The Fourier transform plays an important role in Applied Mathematics. In this section we study the Fourier transform of the diamond operator from the paper of A. Kananthai [4]. We start with the definition of Fourier transform in n – dimensional Euclidean space.

Definition 2.4.1 Let $f(x)$ be continuous function and be an absolutely integrable for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then the Fourier transform of $f(x)$, denoted by

$\hat{F}f(x)$ or $\hat{f}(\xi)$ and defined by

$$Ff(x) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi x} f(x) dx \quad \dots (2.4.1)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$
and $\xi x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$.

The inverse Fourier transform of $\hat{f}(\xi)$ is defined by

$$F^{-1}\hat{f}(\xi) = f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi x} \hat{f}(\xi) d\xi \quad \dots (2.4.2)$$

By (2.4.1), we obtain the following formulas

$$(1) F \frac{\partial f}{\partial x_r} = (i\xi_r) \hat{f}(\xi)$$

$$(2) F D^\alpha f = (i\xi)^\alpha \hat{f}(\xi)$$

$$\text{where } D^\alpha f = \frac{\partial^{\alpha_1 + \alpha_2 + \dots + \alpha_n} f(x)}{\partial x^{\alpha_1} \partial x^{\alpha_2} \dots \partial x^{\alpha_n}} \text{ and } \xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}$$

$$(3) D^\alpha \hat{f}(\xi) = ((-ix)^\alpha f(x))^\wedge$$

$$(4) (f * g)^\wedge(\xi) = (2\pi)^{n/2} \hat{f}(\xi) \hat{g}(\xi)$$

$$(5) F \delta(x_1, x_2, \dots, x_n) = 1$$

$$(6) F D^\alpha \delta = (i\xi)^\alpha$$

$$(7) F D^\alpha (\delta(x-r)) = (i\xi)^\alpha e^{-i\xi r}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $r = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n$
and $\xi r = \xi_1 r_1 + \xi_2 r_2 + \dots + \xi_n r_n$

Lemma (The Fourier transform of $(\phi^k \delta)$)

$$F(\phi^k \delta) = \frac{1}{(2\pi)^{n/2}} [(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2]^k$$

where F is the Fourier transform defined by (2.4.1) and if the norm of ξ is given by

$$\|\xi\| = \left(\sum_{i=1}^n \xi_i^2 \right)^{1/2} \text{ then } |F(\phi^k \delta)| \leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{4k}$$

That is $F(\phi^k \delta)$ is bounded and F continuous on the space S' of tempered distribution. Moreover by (2.4.2) we obtain

$$\phi^k \delta = F^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right]^k$$

Proof we have $F(\phi^k \delta) = \frac{1}{(2\pi)^{n/2}} \langle \phi^k \delta, e^{-i\xi x} \rangle$

since $\phi^k \delta$ has a compact support.

$$\text{Then } \frac{1}{(2\pi)^{n/2}} \langle \phi^k \delta, e^{-i\xi x} \rangle = \frac{1}{(2\pi)^{n/2}} \langle \delta, \square^k \Delta^k e^{-i\xi x} \rangle$$

$$= \frac{1}{(2\pi)^{n/2}} (-1)^{2k} [(\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \dots - \xi_{p+q}^2)]$$

$$= \frac{1}{(2\pi)^{n/2}} [(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2]^k$$

$$\text{Thus } F(\phi^k \delta) = \frac{1}{(2\pi)^{n/2}} [(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2]^k$$

$$\text{Now } |F(\phi^k \delta)| \leq \frac{1}{(2\pi)^{n/2}} (|\xi_1^2 + \xi_2^2 + \dots + \xi_n^2|)^k = \frac{1}{(2\pi)^{n/2}} \|\xi\|^{4k}$$

Thus $F(\phi^k \delta)$ is bounded and it follows that F is continuous on the space S' and by (2.4.1) we obtain

$$\phi^k \delta = F^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right]^k$$

Theorem (The Fourier transform of $(-1)^k K_{2k,2k}(x)$)

$$F((-1)^k K_{2k,2k}(x)) = \frac{1}{(2\pi)^{n/2} [(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2) - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2]}$$

and $|F((-1)^k K_{2k,2k}(x))| \leq \frac{1}{(2\pi)^{n/2}} M$ for a large $\xi_i \in \mathbb{R}$.

where M is a constant. More over F is bounded and continuous on the space S' of the tempered distribution.

Proof Since $\phi^k((-1)^k K_{2k,2k}(x)) = \delta$

or $\phi^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta$ or $(\phi^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta$

By taking the Fourier transform both sides we obtain

$$(2\pi)^{n/2} \cdot F(\phi^k \delta) \cdot F[(-1)^k S_{2k}(x) * R_{2k}(x)] = F \delta$$

$$= \frac{1}{(2\pi)^{n/2}}$$

$$\frac{(2\pi)^{n/2}}{(2\pi)^{n/2}} \left[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2 \right]^k F((-1)^k K_{2k,2k}(x))$$

$$= \frac{1}{(2\pi)^{n/2}}$$

Thus $F((-1)^k K_{2k,2k}(x)) =$

$$\frac{1}{(2\pi)^{n/2} [(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2]} \dots \quad (2.4.3)$$

For $(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \dots - \xi_{p+q}^2) > 0$ and for a large ξ_i and a large k . Thus the right hand side of (2.4.3) converge to zero. It follows that it is bounded by a positive constant M say, that is

$$|F((-1)^k K_{2k,2k}(x))| \leq \frac{1}{(2\pi)^{n/2}} M.$$

Then F is continuous on the space S' .

That complete all required topics of studying for the first 6 months.

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ON THE WEAK SOLUTIONS OF THE EQUATION RELATED TO THE DIAMOND OPERATOR*

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Рассматривается функция Грина оператора \oplus^k , определенного следующим образом:

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k,$$

где $p + q = n$ — размерность пространства C^n векторов $x = (x_1, x_2, \dots, x_n)$ с n комплексными компонентами x_j , k — целое неотрицательное число. Выполнено исследование функции Грина, которая затем применяется для построения слабого решения уравнения $K(x)$, такого что

$$\oplus^k K(x) = f(x),$$

где f — обобщенная функция.

1. Introduction

The operator \oplus^k can be factorized in the following form

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \left[\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \left[\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \quad (1.1)$$

where $i = \sqrt{-1}$ and $p + q = n$. The operator $\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2$ has first been introduced by A. Kananthai [4] and is named the Diamond operator which is denoted by

$$\diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k. \quad (1.2)$$

Let us denote the operators

$$L_1^k = \left[\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \quad (1.3)$$

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and

$$L_2^k = \left[\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k. \quad (1.4)$$

Thus the operator \oplus^k , iterated k -times defined by (1.1) can be written in the form

$$\oplus^k = \diamond^k L_1^k L_2^k. \quad (1.5)$$

In this work, we obtain the Green function of the operator \oplus^k , i. e. $\oplus^k G(x) = \delta$ where δ is the Dirac-delta distribution and $G(x)$ is the Green function and $x \in R^n$.

Moreover, we find the weak solution of the equation

$$\oplus^k K(x) = f(x) \quad (1.6)$$

where f is a given generalized function and $K(x)$ is an unknown and $x \in R^n$.

2. Preliminary

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n) \in R^n$

Let us denote by

$$u = \sum_{i=1}^p x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \quad (2.1)$$

the nondegenerated quadratic form, whereas $p + q = n$ is the dimension of R^n .

Let $\Gamma_+ = \{x \in R^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma_+}$ denotes its closure.

For any complex number α , we define the function

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{(\alpha-n)}{2}}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \quad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{(n-1)}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}.$$

The function R_α^H is called The Ultra-Hyperbolic Kernel of Marcel Riesz and was introduced by Y. Nozaki (see [3], p. 72).

It is well known that R_α^H is an ordinary function if $\text{Re}(\alpha) \geq n$ and is a distribution of α if $\text{Re}(\alpha) < n$. Let us supp $R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$. Assume $R_\alpha^H(u) \subset \overline{\Gamma_+}$.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the Euclidean space R^n and

$$v = \sum_{i=1}^n x_i^2. \quad (2.3)$$

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Define the function

$$R_{\alpha}^e(v) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \quad (2.4)$$

where α is any complex number and the constant $H_n(\alpha)$ is given by the formula

$$H_n(\alpha) = \frac{\pi^{\frac{1}{2}} 2^{\alpha} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \quad (2.5)$$

Now the function $R_{\alpha}^e(v)$ is called the Elliptic Kernel of Marcel Riesz.

Definition 2.3. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the C^n and let

$$w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2), \quad (2.6)$$

where $i = \sqrt{-1}$ and $p + q = n$ is the dimension of R^n .

Define the function

$$S_{\alpha}(w) = \frac{w^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \quad (2.7)$$

where α is any complex number and $H_n(\alpha)$ is defined as the formula (2.5).

Definition 2.4. Define the function

$$T_{\alpha}(z) = \frac{z^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \quad (2.8)$$

where

$$z = x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2) \quad (2.9)$$

and $i = \sqrt{-1}$, $p + q = n$ and $H_n(\alpha)$ is defined as (2.5).

Lemma 2.1. The convolution $R_{2k}^H(u) * (-1)^k R_{2k}^e(v)$ is an elementary solution of the operator remove off \diamond^k where \diamond^k is defined by (1.2) and $R_{2k}^H(u)$ and $R_{2k}^e(v)$ are defined by (2.2) and (2.4) respectively with $\alpha = 2k$.

Proof. The elementary solution of \diamond^k is the solution of the equation $\diamond^k K(x) = \delta$ where δ is the Dirac-delta distribution, $K(x)$ is an unknown and $x \in R^n$. Now we need to prove that

$$K(x) = R_{2k}^H(u) * (-1)^k R_{2k}^e(v).$$

To prove this, see ([4], p. 33).

Lemma 2.2. (i) The function $K(x) = S_2(w)$ is the solution of the equation $L_1 K(x) = 0$ where L_1 is defined by (1.3) and $S_2(w)$ is defined by (2.7) with $\alpha = 2$.

(ii) The function $K(x) = (-1)^k (-i)^{\frac{q}{2}} S_{2k}(w)$ is an elementary solution of the operator L_1^k , where L_1^k is the operator iterated k times defined by (1.3) and $S_{2k}(w)$ is defined by (2.7) with $\alpha = 2k$.

Proof. (i) Now $L_1 = \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$.

We need to show that $L_1 S_2(w) = 0$. Now if α is real, we have for $1 \leq r \leq p$

$$\frac{\partial}{\partial x_r} S_{\alpha}(w) = \frac{\partial}{\partial x_r} \left(\frac{w^{\frac{\alpha-n}{2}}}{H_n(\alpha)} \right) = \frac{(\alpha-n)}{2} \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)} 2x_r = (\alpha-n)x_r \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)}.$$

$$\frac{\partial^2}{\partial x_r^2} S_\alpha(w) = (\alpha - n) \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)} + \frac{\alpha - n}{H_n(\alpha)} (\alpha - n - 2) w^{\frac{\alpha-n-4}{2}} x_r^2.$$

Thus

$$\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} S_\alpha(w) = p \frac{\alpha - n}{H_n(\alpha)} w^{\frac{\alpha-n-2}{2}} + \frac{\alpha - n}{H_n(\alpha)} (\alpha - n - 2) w^{\frac{\alpha-n-4}{2}} \sum_{r=1}^p x_r^2.$$

Similarly

$$i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} S_\alpha(w) = \frac{q(\alpha - n)}{H_n(\alpha)} w^{\frac{\alpha-n-2}{2}} - i \frac{\alpha - n}{H_n(\alpha)} (\alpha - n - 2) w^{\frac{\alpha-n-4}{2}} \sum_{j=p+1}^{p+q} x_j^2.$$

Thus

$$\begin{aligned} L_1 S_\alpha(w) &= \frac{(p+q)}{H_n(\alpha)} (\alpha - n) w^{\frac{\alpha-n-2}{2}} + \frac{(\alpha - n)(\alpha - n - 2)}{H_n(\alpha)} w^{\frac{\alpha-n-4}{2}} \left(\sum_{i=1}^p x_i^2 - i \sum_{j=p+1}^{q+p} x_j^2 \right) = \\ &= \frac{n(\alpha - n)}{H_n(\alpha)} w^{\frac{\alpha-n-2}{2}} + \frac{(\alpha - n)(\alpha - n - 2)}{H_n(\alpha)} w^{\frac{\alpha-n-2}{2}} = (\alpha - 2)(\alpha - n) \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)}. \end{aligned} \quad (2.10)$$

For $\alpha = 2$, we have $L_1 S_2 = 0$. That is $K(x) = S_2(w)$ is a solution of the homogeneous equation $L_1 K(x) = 0$.

(ii) To show that $K(x) = (-1)^k (-i)^{\frac{q}{2}} S_{2k}(w)$ is an elementary solution of L_1^k , that is $L_1^k (-1)^k (-i)^{\frac{q}{2}} S_{2k}(w) = \delta$. At first we need to show that $L_1^k (-1)^k S_\alpha(w) = S_{\alpha-2k}(w)$ and $S_{-2k}(w) = (-1)^k (i)^{\frac{q}{2}} L_1^k \delta$.

Now, from (2.10) and (2.5)

$$L_1 S_\alpha(w) = (\alpha - 2)(\alpha - n) \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)} = \frac{(\alpha - 2)(\alpha - n) w^{\frac{\alpha-n-2}{2}}}{\pi^{\frac{n}{2}} \frac{2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}}.$$

By direct calculation with the property of Gamma function we obtain

$$L_1 S_\alpha(w) = - \frac{w^{\frac{\alpha-n-2}{2}}}{\pi^{\frac{n}{2}} \cdot 2^{\alpha-2} \frac{\Gamma(\frac{\alpha-2}{2})}{\Gamma(\frac{n-(\alpha-2)}{2})}} = - \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha-2)} = -S_{\alpha-2}(w).$$

By keeping on operating the operator L_1 k -times to the function $S_\alpha(w)$, we obtain

$$L_1^k S_\alpha(w) = (-1)^k S_{\alpha-2k}(w)$$

or

$$L_1^k (-1)^k S_\alpha(w) = S_{\alpha-2k}(w). \quad (2.11)$$

Then we show that $S_{-2k} = (-1)^k (i)^{\frac{q}{2}} \delta$.

Now

$$S_{-2k}(w) = \lim_{\alpha \rightarrow -2k} S_\alpha(w) = \lim_{\alpha \rightarrow -2k} \left[\frac{w^{\frac{\alpha-n}{2}}}{H_n(\alpha)} \right] = \frac{\lim_{\alpha \rightarrow -2k} \left[w^{\frac{\alpha-n}{2}} \right]}{\lim_{\alpha \rightarrow -2k} \left[\Gamma(\frac{\alpha}{2}) \right]} \cdot \pi^{\frac{n}{2}} \cdot \lim_{\alpha \rightarrow -2k} 2^{-\alpha} \Gamma\left(\frac{n-\alpha}{2}\right). \quad (2.12)$$

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Now consider $\lim_{\alpha \rightarrow -2k} [w^{\frac{\alpha-n}{2}}]$. We have $w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$. By changing the variables, let $x_1 = y_1, x_2 = y_2, \dots, x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \dots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$.

Thus $w = y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 + \dots + y_{p+q}^2$, where $y_i (i = 1, 2, \dots, n)$ is real and $p + q = n$.

Let $r^2 = w = y_1^2 + y_2^2 + \dots + y_n^2$ and consider the distribution w^λ , where λ is a complex parameter. Since $\langle w^\lambda, Q \rangle = \int_Q w^\lambda Q(x) dx$, where $Q(x)$ is the element of the space D of the infinitely differentiable functions with compact supports and $x \in R^n, dx = dx_1 dx_2 \dots dx_n$. Thus

$$\begin{aligned} \langle w^\lambda, Q \rangle &= \int_{R^n} r^{2\lambda} \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} \cdot Q dy_1 dy_2 \dots dy_n = \\ &= \frac{1}{(-i)^{\frac{q}{2}}} \int_{R^n} r^{2\lambda} Q dy_1 dy_2 \dots dy_n = \frac{1}{(-i)^{\frac{q}{2}}} \langle r^{2\lambda}, Q \rangle. \end{aligned}$$

Now, by Gelfand and Shilov (see [1], p. 271), $\langle w^\lambda, Q \rangle$ has simple poles at $\lambda = \frac{-n}{2} - k$ and for $k = 0$ the residue of $r^{2\lambda}$ at $\lambda = \frac{-n}{2}$ is given by $\text{res}_{\lambda=\frac{-n}{2}} r^{2\lambda} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta(x)$.

Thus

$$\text{res}_{\lambda=\frac{-n}{2}} \langle w^\lambda, Q \rangle = \frac{1}{(-i)^{\frac{q}{2}}} \text{res}_{\lambda=\frac{-n}{2}} \langle r^{2\lambda}, Q \rangle = \frac{1}{(-i)^{\frac{q}{2}}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \langle \delta(x), Q \rangle$$

or

$$\text{res}_{\lambda=\frac{-n}{2}} w^\lambda = \frac{1}{(-i)^{\frac{q}{2}}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta(x). \quad (2.13)$$

Now we find $\text{res}_{\lambda=\frac{-n}{2}-k} w^\lambda$ for k is nonnegative integer by, Gelfand and Shilov (see [1], p. 272) we have

$$w^\lambda = \frac{1}{4^k (\lambda + 1)(\lambda + 2) \dots (\lambda + k)(\lambda + \frac{n}{2})(\lambda + \frac{n}{2} + 1) \dots (\lambda + \frac{n}{2} + k - 1)} L_1^k w^{\lambda+k}.$$

Thus

$$\text{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \text{res}_{\lambda=\frac{-n}{2}} L_1^k w^\lambda \cdot \frac{1}{4^k (\lambda + 1) \dots (\lambda + k)(\lambda + \frac{n}{2}) \dots (\lambda + \frac{n}{2} + k - 1)} \Big|_{\lambda=\frac{-n}{2}-k}$$

by (2.12) we have

$$\text{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \frac{1}{(-i)^{\frac{q}{2}}} \frac{2\pi^{\frac{n}{2}}}{4^k k! \Gamma(\frac{n}{2} + k)} L_1^k \delta(x). \quad (2.14)$$

Thus

$$\lim_{\alpha \rightarrow -2k} [w^{\frac{\alpha-n}{2}}] = \lim_{\lambda \rightarrow \frac{-n}{2}-k} w^\lambda.$$

Now from (2.12), we have

$$S_{-2k}(w) = \frac{\lim_{\alpha \rightarrow -2k} (\alpha + 2k) w^{\frac{\alpha-n}{2}}}{\lim_{\alpha \rightarrow -2k} (\alpha + 2k) \Gamma(\frac{\alpha}{2})} \pi^{-\frac{n}{2}} 2^{2k} \Gamma\left(\frac{n}{2} + k\right) = \frac{\text{res}_{\alpha=-2k} w^{\frac{\alpha-n}{2}}}{\text{res}_{\alpha=-2k} \Gamma(\frac{\alpha}{2})} \pi^{-\frac{n}{2}} 4^k \Gamma\left(\frac{n}{2} + k\right).$$

Now

$$\operatorname{res}_{\alpha=-2k} \Gamma\left(\frac{\alpha}{2}\right) = \frac{2(-1)^k}{k!}.$$

Thus by (2.14), we obtain

$$\begin{aligned} S_{-2k}(w) &= \frac{(-1)^k}{(-i)^{\frac{\alpha}{2}}} \frac{2\pi^{\frac{n}{2}} \pi^{-\frac{n}{2}} k! 4^k \Gamma\left(\frac{n}{2} + k\right)}{2 \cdot 4^k k! \Gamma\left(\frac{n}{2} + k\right)} L_1^k \delta(x) = \\ &= \frac{(-1)^k}{(-i)^{\frac{\alpha}{2}}} L_1^k \delta(x) = (-1)^k (i)^{\frac{\alpha}{2}} L_1^k \delta(x). \end{aligned}$$

Thus

$$S_0(w) = (i)^{\frac{\alpha}{2}} \delta(x). \quad (2.15)$$

From (2.11) and (2.15), we obtain

$$L_1^k (-1)^k S_{2k}(w) = S_{2k-2k}(w) = S_0(w) = (i)^{\frac{\alpha}{2}} \delta(x)$$

or

$$L_1^k (-1)^k (-i)^{\frac{\alpha}{2}} S_{2k}(w) = \delta.$$

It follows that $K(x) = (-1)^k (-i)^{\frac{\alpha}{2}} S_{2k}(w)$ is an elementary solution of the operator L_1^k . Similary $K(x) = (-1)^k (i)^{\frac{\alpha}{2}} T_{2k}(z)$ is an elementary solution of the operator L_2^k where z is defined by (2.9) and T_{2k} is defined by (2.8) with $\alpha = 2k$.

3. Main results

Theorem 3.1. *Given the equation*

$$\oplus^k K(x) = \delta \quad (3.1)$$

where \oplus^k is the operator iterated k -times defined by (1.1), δ is the Dirac-delta distribution, $x = (x_1, x_2, \dots, x_n) \in R^n$ and k is a nonnegative integer. Then the convolution

$$K(x) = R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{\alpha}{2}} S_{2k}(w) * (-1)^k (i)^{\frac{\alpha}{2}} T_{2k}(z) \quad (3.2)$$

is an elementary solution or the Green function of the equation (3.1) where $R_{2k}^H(u)$, $R_{2k}^e(v)$, $S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.2), (2.4), (2.7) and (2.8) respectively with $\alpha = 2k$.

Proof. By (1.5) the equation (3.1) can be written as

$$\oplus^k K(x) = \diamond^k L_1^k L_2^k K(x) = \delta. \quad (3.3)$$

Since the function $R_{2k}^H(u)$, $R_{2k}^e(v)$, $S_{2k}(w)$ and $T_{2k}(z)$ are tempered distributions (see [5], p. 34, Lemma 2.1) and the convolution of functions in (3.2) exists and is a tempered distribution (see [5], p. 35, Lemma 2.2 and [2], pp. 156-159). Now convolving both sides of (3.3) by $R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{\alpha}{2}} S_{2k}(w) * (-1)^k (i)^{\frac{\alpha}{2}} T_{2k}(z)$ we obtain

$$\begin{aligned} \diamond^k [R_{2k}^H(u) * (-1)^k R_{2k}^e(v)] * L_1^k [(-1)^k (-i)^{\frac{\alpha}{2}} S_{2k}(w)] * L_2^k [(-1)^k (i)^{\frac{\alpha}{2}} T_{2k}(z)] * K(x) = \\ = [R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{\alpha}{2}} S_{2k}(w) * (-1)^k (i)^{\frac{\alpha}{2}} T_{2k}(z)] * \delta. \end{aligned}$$

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By Lemma 2.1 and Lemma 2.2 (ii), we obtain (3.2) as required, we call the solution $K(x)$ in (3.2) the Green function of the operator \oplus^k we denote the Green function

$$G(x) = R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(w) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z). \quad (3.4)$$

Theorem 3.2. *Given the equation*

$$\oplus^k K(x) = f(x) \quad (3.5)$$

where \oplus^k is defined by (1.1) and $f(x)$ is a generalized function, then $K(x) = G(x) * f(x)$ is a weak solution for (3.5) where $G(x)$ is a Green function of \oplus^k defined by (3.4).

Proof. Convolving both sides of (3.5) by $G(x)$ defined by (3.4) we obtain

$$G(x) * \oplus^k K(x) = G(x) * f(x)$$

or

$$\oplus^k G(x) * K(x) = G(x) * f(x).$$

By Theorem 3.1, we have

$$\delta * K(x) = G(x) * f(x)$$

or

$$K(x) = G(x) * f(x)$$

as required.

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“On the distributional Kernel $K_{\alpha,\beta,\gamma,\delta}$ related to
the operator \oplus^k ”

ส่งไปลงตีพิมพ์ในวารสาร

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On the distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator \oplus^k

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ABSTRACT In this paper, we study the properties of the Kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator \oplus^k , iterated k -times where $\alpha, \beta, \gamma, \nu$ are complex parameter and \oplus^k is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

where k is a nonnegative integer and $p + g = n$ is the dimension of the Euclidean space \mathbb{R}^n , $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

The properties of $K_{\alpha,\beta,\gamma,\nu}$ are concerned with the conditions of α, β, γ and ν and moreover if $\alpha = \beta = \gamma = \nu = 2k$, we obtain the function $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator \oplus^k .

1. Introduction

The operator \oplus^k can be factorized in the form

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right]^k \quad (1.1)$$

where $p + g = n$ is the dimension of \mathbb{R}^n and $i = \sqrt{-1}$ and k a nonnegative integer. The operator $\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2$ is first introduced by A. Kananthai [4] and is named as the Diamond operator which is denoted by \diamond . Write

$$\diamond^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \quad (1.2)$$

Let we denote the operators

$$L_1 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad (1.3)$$

$$L_2 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad (1.4)$$

Thus (1.1) can be written by

$$\oplus^k = \diamond^k L_1^k L_2^k \quad (1.5)$$

Now consider the convolutions $R_\alpha^H(u) * R_\beta^e(u) * S_\gamma(w) * T_\nu(z)$ where $R_\alpha^H(u)$, $R_\beta^e(u)$, $S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.1), (2.2), (2.3) and (2.4) respectively. Now we defined the distributional Kernel

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H(u) * R_\beta^e(u) * S_\gamma(w) * T_\nu(z) \quad (1.6)$$

Since the function $R_\alpha^H(u)$, $R_\beta^e(u)$, $S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions see [1, p30 – 31] and [4, p154 – 155], then their convolutions of (1.5) exists and also is a tempered distributions. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also a tempered distribution.

In this work, we study the properties of $K_{\alpha,\beta,\gamma,\nu}$ related to the operator \oplus^k iterated k -times depending on the conditions $\alpha = \beta = \gamma = \nu = 0$, $\alpha = \beta = \gamma = \nu = -2k$ and $\alpha = \beta = \gamma = \nu = 2k$. Moreover, we obtain $(-1)^k K_{2k,2k,2k,2k}$ as elementary solution of the operator \oplus^k , that is $\oplus^k(-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution. \square

2. Preliminary

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathfrak{R}^n and write $u = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$, $p + q = n$. Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the set of and interior of the forward cone and $\overline{\Gamma}_+$ is the closure of Γ_+ . For any complex number α , define

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{(\alpha-n)}{2}}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+ \end{cases} \quad (2.1)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{(n-1)}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}$$

The function R_α^H is called The Ultra-Hyperbolic Rernel of Marcel Riesz and was introduced by Y. Nozaki [3, p72]. The function R_α^H is an ordinary function or classical function if $R_e(\alpha) \geq n$ and is a distribution of $R_e(\alpha) < n$. Let $\text{supp} R_\alpha^H(u) \subset \overline{\Gamma}_+$ where $\text{supp} R_\alpha^H(u)$ denote the support of R_α^H .

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathfrak{R}^n and write $v = x_1^2 + x_2^2 + \dots + x_n^2$. For any complex number β , define

$$R_\beta^e(v) = 2^{-\beta} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\beta}{2}\right) \frac{v^{\frac{\beta-n}{2}}}{\Gamma(\frac{\beta}{2})} \quad (2.2)$$

The function $R_\beta^e(v)$ is called the elliptic Kernel of Marcel Riesz and is an ordinary function for $R_e(\beta) \geq n$ and is a distribution of β if $R_e(\beta) < n$.

Definition 2.3 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathfrak{R}^n and write $w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$, $p+q = n$ is the dimension of \mathfrak{R}^n and $i = \sqrt{-1}$. For any complex number γ , define the function

$$S_\gamma(w) = 2^{-\gamma} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\gamma}{2}\right) \frac{w^{\frac{\gamma-n}{2}}}{\Gamma(\frac{\gamma}{2})} \quad (2.3)$$

The function $S_\gamma(w)$ is an ordinary function or classical function if $R_e(\gamma) \geq n$ and is a distribution of γ if $R_e(\gamma) < n$.

Definition 2.4 For any complex number ν , define the function

$$T_\nu(z) = 2^{-\nu} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\nu}{2}\right) \frac{z^{\frac{\nu-n}{2}}}{\Gamma(\frac{\nu}{2})} \quad (2.4)$$

where $z = x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$, $p+q = n$ is the dimension of \mathfrak{R}^n , $x = (x_1, x_2, \dots, x_n) \in \mathfrak{R}^n$ and $i = \sqrt{-1}$. We have $T_\nu(z)$ is an ordinary function if $R_e(\nu) \geq n$ and is a distribution of ν if $R_e(\nu) < n$.

Lemma 2.1 The convolution $R_{2k}^H(u) * (-1)^k R_{2k}^e(v)$ is an elementary solution of the operator \diamond^k iterated k -times where the function $R_{2k}^H(u)$ and $R_{2k}^e(v)$ are defined by (2.1) and (2.2) respectively with $\alpha = \beta = 2k$ and \diamond^k is defined by (1.2).

Proof We now show that The elementary solution of $\diamond^k[R_{2k}^H(u) * (-1)^k R_{2k}^e(v)] = \delta$ where δ is the Dirac-delta distribution.

To prove this, see [1, p33Theorem3.1]

Lemma2.2 (i) $\diamond^k[R_\alpha^H(u) * R_\beta^e(v)] = R_{\alpha-2k}^H(u) * (-1)^k R_{\beta-2k}^e(v)$

(ii) $R_{-2k}^H(u) = \square^k \delta$, $R_{-2k}^e(v) = \triangle^k (-1)^k \delta$

where \square^k is an ultra-hyperbolic operator iterate k -times defined by $\square^k = \left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k$,

$p+q = n$ and \triangle^k is the Laplacian iterated k -times defined by $\triangle^k = \left(\sum_{r=1}^n \frac{\partial^2}{\partial x_r^2} \right)^k$

Proof (i) From [1, p28], we have $\diamond^k = \square^k \Delta^k$ where \square and Δ are the ultra-hyperbolic and the Laplacian respectively. Thus

$$\begin{aligned}\diamond^k [R_\alpha^H(u) * R_\beta^e(v)] &= \square^k \Delta^k [R_\alpha^H(u) * R_\beta^e(v)] \\ &= \square^k R_\alpha^H(u) * \Delta^k R_\beta^e(v)\end{aligned}$$

Now by S.E. Trione [2, p11] we obtain $\square^k R_\alpha^H(u) = R_{\alpha-2k}^H(u)$ and also by Donoghue [4, p118] we obtain $\Delta^k R_\beta^e(v) = (-1)^k R_{\beta-2k}^e(v)$. Thus we obtain (i) as required.

(ii) We have $R_{-2k}^H(u) = \lim_{\alpha \rightarrow -2k} R_\alpha^H(u) = \square^k \delta$, see [2, p9] and $R_{-2k}^e = \lim_{\beta \rightarrow -2k} R_\beta^e(v) = \Delta^k (-1)^k \delta$, see [4, p118]

Lemma2.3 The functions $(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)$ and $(-1)^k (-i)^{\frac{k}{2}} T_{2k}(z)$ are the elementary solutions of the operators L_1^k and L_2^k respectively where $S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.3) and (2.4) respectively with $\gamma = \nu = 2k$ and the operators L_1 and L_2 are defined by (1.3) and (1.4) respectively.

Proof We need to show that $L_1^k [(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] = \delta$ and $L_2^k [(-1)^k (-i)^{\frac{k}{2}} T_{2k}(z)] = \delta$. Before showing such properties, at first we have to prove that

$$L_1^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w) \quad (2.5)$$

and

$$S_{-2k}(w) = (-1)^k (i)^{\frac{k}{2}} L_1^k \delta \quad ((2.6))$$

$$T_{-2k}(z) = (-1)^k (-i)^{\frac{k}{2}} L_2^k \delta \quad ((2.7))$$

Now consider the case $k = 1$, and by computing directly we have

$$L_1 S_\gamma(w) = 2^{-\gamma} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} L_1 w^{\frac{\gamma-n}{2}}$$

and

$$L_1 w^{\frac{\gamma-n}{2}} = (\gamma - n)(\gamma - 2) w^{\frac{\gamma-n-2}{2}}$$

Thus

$$L_1 S_\gamma(w) = 2^{-\gamma} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\gamma}{2}\right) \frac{(\gamma - n)(\gamma - 2)}{\Gamma\left(\frac{\gamma}{2}\right)} w^{\frac{\gamma-n-2}{2}}$$

By using the property $\Gamma(z+1) = z\Gamma(z)$, we obtain

$$\begin{aligned}L_1 S_\gamma(w) &= (-1) 2^{-(\gamma-2)} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-(\gamma-2)}{2}\right)}{\Gamma\left(\frac{\gamma-2}{2}\right)} w^{(\gamma-2)-n} \\ &= -S_{\gamma-2}(w)\end{aligned} \quad (2.8)$$

Thus we obtain (2.5) with $k = 1$

Now

$$\begin{aligned}
 L_1^2 S_\gamma(w) &= L_1(L_1 S_\gamma(w)) = L_1(-S_{\gamma-2}(w)) \\
 &= -L S_{\gamma-2}(w) = (-1)^2 S_{\gamma-2-2} \\
 &= (-1)^2 S_{\gamma-4}
 \end{aligned}
 \tag{2.8}$$

by keeping on operating L_1 k -times to $S_\gamma(w)$ and using (2.8) we obtain

$$L_1^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w)$$

Thus we obtain (2.5). Similarly, we obtain

$$L_2^k T_\nu(z) = (1)^k T_{\nu-3k}(z) \tag{2.9}$$

we next show that (2.6) holds. Now consider $w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$ by changing the variables $x_1 = y, x_2 = y, \dots, x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-1}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-1}}, \dots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-1}}, p+q = n$

Thus

$$\begin{aligned}
 w &= y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 + \dots + y_{p+q}^2, \quad p+q = n \\
 &= \sum_{j=1}^n y_j^2
 \end{aligned}$$

we know that

$$\begin{aligned}
 \langle w^\lambda, Q \rangle &= \langle r^{2\lambda}, Q \rangle \\
 &= \int_{\mathbb{R}^n} w^\lambda Q(x) dx \\
 &= \int_{\mathbb{R}^n} r^{2\lambda} Q \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} dy_1 dy_2 \dots dy_n
 \end{aligned}$$

where Q is a test function in the space D of infinitely differentiable with compact support and λ is a complex parameter.

Thus

$$\begin{aligned}
 \langle w^\lambda, Q \rangle &= \frac{1}{(-i)^{\frac{n}{2}}} \int_{\mathbb{R}^n} r^{2\lambda} Q dy \\
 &= \frac{1}{(-i)^{\frac{n}{2}}} \langle r^{2\lambda}, Q \rangle
 \end{aligned}$$

By Gelfand and Shilov [5, p72], $r^{2\lambda}$ have simple poles at $2\lambda = n - 2k$ or $\lambda = \frac{-n}{2} - k$, k is a nonnegative integer. Thus we can find the residues of $r^{2\lambda}$ at $\lambda = \frac{-n}{2} - k$. Now by [5, p271] for $k = 0$ the residue of $r^{2\lambda}$ at $\lambda = \frac{-n}{2}$ is given by

$$\text{res}_{\lambda = \frac{-n}{2}} r^{2\lambda} = \frac{\pi^{\frac{n}{2}} \delta(x)}{\Gamma(\frac{n}{2})}$$

Therefore

$$\operatorname{res}_{\lambda=\frac{-n}{2}} \langle w^\lambda, Q \rangle = \frac{1}{(-i)^{\frac{3}{2}}} \frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} \langle \delta(x), Q \rangle$$

or

$$\operatorname{res}_{\lambda=\frac{-n}{2}} w^\lambda = \frac{1}{(-i)^{\frac{3}{2}}} \frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} \delta(x) \quad (2.10)$$

We next find the residue of W^λ at $\lambda = \frac{-n}{2} - k$. By direct calculation, we obtain $L_1 w^{\lambda+1} = 4(\lambda+1)(\lambda + \frac{n}{2})w^\lambda$ and by iterating it k times. We arrive at

$$L_1^k w^{\lambda+k} = 4^k (\lambda+1)(\lambda+2)\dots(\lambda+k)(\lambda + \frac{n}{2})(\lambda + \frac{n}{2} + 1)\dots(\lambda + \frac{n}{2} + k-1)w^\lambda$$

or

$$w^\lambda = \frac{1}{4^k (\lambda+1)\dots(\lambda+k)(\lambda + \frac{n}{2})\dots(\lambda + \frac{n}{2} + k-1)} L_1^k w^{\lambda+k}$$

Thus

$$\operatorname{res}_{\lambda=\frac{-n}{2}} w^\lambda = \frac{1}{4^k (-1)^k (\lambda + \frac{n}{2} + k-1)(\lambda + \frac{n}{2} + k-2)\dots\frac{n}{2} (-1)^k k!} \operatorname{res}_{\lambda=\frac{-n}{2}} L_1^k w^\lambda$$

(2.5) and the property of Gamma functions, we obtain

$$\operatorname{res}_{\lambda=\frac{-n}{2}} w^\lambda = \frac{1}{(-i)^{\frac{3}{2}}} \frac{\pi^{\frac{3}{2}}}{4^k k! \Gamma(\frac{n}{2} + k)} L_1^k \delta(x) \quad (2.11)$$

Now, we have

$$S_\gamma(w) = 2^{-\gamma} \pi^{\frac{1}{2}} \frac{\Gamma(\frac{n-\gamma}{2})}{\Gamma(\frac{\gamma}{2})} w^{\frac{\gamma-n}{2}}$$

and

$$\operatorname{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \operatorname{res}_{\lambda=-2k} w^{\alpha-n/2} \quad (2.12)$$

Thus

$$\begin{aligned} S_{-2k}(w) &= \lim_{\gamma \rightarrow -2k} S_\gamma(w) \\ &= \frac{\lim_{\gamma \rightarrow -2k} w^{\frac{\gamma-n}{2}}}{\lim_{\gamma \rightarrow -2k} \Gamma(\frac{\gamma}{2})} \pi^{\frac{1}{2}} \lim_{\gamma \rightarrow -2k} 2^\gamma \Gamma(\frac{n-\gamma}{2}) \\ &= \frac{\lim_{\gamma \rightarrow -2k} (\gamma+2k) w^{\frac{\gamma-n}{2}}}{\lim_{\gamma \rightarrow -2k} (\gamma+2k) \Gamma(\frac{\gamma}{2})} \pi^{\frac{1}{2}} 2^{2k} \Gamma(\frac{n}{2} + k) \\ &= \frac{\operatorname{res}_{\gamma=-2k} w^{\frac{\gamma-n}{2}}}{\operatorname{res}_{\gamma=-2k} \Gamma(\frac{\gamma}{2})} 4^k \pi^{\frac{1}{2}} \Gamma(\frac{n}{2} + k) \end{aligned}$$

Now

$$\operatorname{res}_{\gamma=-2k} \Gamma(\frac{\gamma}{2}) = \frac{(-1)^k}{k!} \quad (2.14)$$

Thus, by (2.11), (2.12) and (2.13) we obtain

$$\begin{aligned} S_{-2k}(w) &= \frac{1}{(-i)^{\frac{n}{2}}} \frac{\pi^{\frac{n}{2}}}{4^k k! \Gamma(\frac{n}{2} + k)} \frac{k!}{(-1)^k} 4^k \pi^{-\frac{n}{2}} \Gamma(\frac{n}{2} + k) L_1^k \delta(x) \\ &= (-1)^k (i)^{\frac{n}{2}} L_1^k \delta(x) \end{aligned} \quad (2.14)$$

Thus, we obtain (2.6) as required. Similarly

$$T_{2k}(z) = (1)^k (-i)^{\frac{n}{2}} L_2^k \delta(x) \quad (2.15)$$

Now, from (2.14) $S_0(w) = (i)^{\frac{n}{2}} \delta$ for $k = 0$

Thus, from (2.5) $L_1^k S_{2k}(w) = (-1)^k S_0(w) = (-1)^k (i)^{\frac{n}{2}} \delta(x)$ or $L_1^k (-1)^k (-i)^{\frac{n}{2}} S_{2k}(w) = \delta$

Thus, we obtain $(-1)^k (i)^{\frac{n}{2}} S_{2k}(w)$ is an elementary solution of the operator L_1^k .

Similarly, the same process of proving we also obtain $(-1)^k (i)^{\frac{n}{2}} T_{2k}(z)$ is an elementary solution of the operator L_2^k . \square

3. Main results

Theorem Given the equation

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

where $p+g = n$ is the dimension of the n -dimensional Euclidean space \mathbb{R}^n , $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer, then the following formulae hold.

- (i) $\oplus^k K_{\alpha, \beta, \gamma, \nu}(x) = (-1)^k K_{\alpha-2k, \beta-2k, \gamma-2k, \nu-2k}(x)$
 - (ii) $K_{-2k, -2k, -2k, -2k}(x) = (-1)^k \oplus^k \delta$
 - (iii) $K_{0,0,0,0}(x) = \delta$
 - (iv) $(-1)^k K_{2k, 2k, 2k, 2k}(x)$ is an elementary solution of the operator \oplus^k .
- where $K_{\alpha, \beta, \gamma, \nu}$ is define by (1.6)

Proof (i) By (1.5) and (1.6) we have

$$\begin{aligned} \oplus^k K_{\alpha, \beta, \gamma, \nu}(x) &= \diamond^k L_1^k L_2^k K_{\alpha, \beta, \gamma, \nu}(x) \\ &= \diamond^k (R_{\alpha}^H(u) * R_{\beta}^e(v)) * L_1^k S_{\gamma}(w) * L_2^k T_{\nu}(z) \\ &= R_{\alpha-2k}^H * (-1)^k R_{\beta-2k}^e * (-1)^k S_{\gamma-2k}(w) * (-1)^k T_{\nu-2k}(z) \end{aligned}$$

by Lemma 2.2(i), (2.5) and (2.9)

Thus

$$\oplus^k K_{\alpha, \beta, \gamma, \nu}(x) = (-1)^k K_{\alpha-2k, \beta-2k, \gamma-2k, \nu-2k}(x)$$

(ii) By (1.6)

$$\begin{aligned} K_{-2k, -2k, -2k, -2k}(x) &= R_{-2k}^H * R_{-2k}^e * S_{-2k} * T_{-2k} \\ &= (\square^k \delta) * ((-1)^k \Delta^k \delta) * ((-1)^k (-i)^{\frac{k}{2}} L_1^k \delta) * ((-1)^k (-i)^{\frac{k}{2}} L_2^k \delta) \end{aligned}$$

by Lemma (2.2)(ii), (2.6) and (2.7).

Thus

$$\begin{aligned} K_{-2k, -2k, -2k, -2k} &= (-1)^{2k} (1)^{\frac{k}{2}} (-i)^k (\square^k \delta) * (\Delta^k \delta) * (L_1^k \delta) * (L_2^k \delta) \\ &= (-1)^k \square^k \Delta^k L_1^k L_2^k (\delta * \delta * \delta * \delta) \\ &= (-1)^k \diamond^k L_1^k L_2^k \delta = (-1)^k \oplus^k \delta \end{aligned}$$

(iii) Putting $k = 0$ in (ii), we obtain

$$K_{0,0,0,0}(x) = \delta$$

(iv) We now show that

$$\oplus^k ((-1)^k K_{2k, 2k, 2k, 2k}(x)) = \delta$$

that is $(-1)^k K_{2k, 2k, 2k, 2k}(x)$ is an elementary solution of the operator \oplus^k .

Now, by (1.5) and (1.6) we have

$$\begin{aligned} \oplus^k (-1)^k K_{2k, 2k, 2k, 2k}(x) &= (-1)^k \diamond^k (R_{2k}^H(u) * R_{2k}^e(v)) * (L_1^k S_{2k}(w)) * (L_2^k T_{2k}(z)) \\ &= \diamond^k (R_{2k}^H(u) * (-1)^k R_{2k}^e(v)) * (L_1^k (-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)) * (L_2^k (-1)^k (-i)^{\frac{k}{2}} T_{2k}(z)) \\ &= \delta * \delta * \delta = \delta \end{aligned}$$

by Lemma(2.1) and Lemma(2.3).

Equivalently, we can obtain $(-1)^k K_{2k, 2k, 2k, 2k}$ directly from (1) by putting $\alpha = \beta = \gamma = \nu = 2k$. We obtain

$$\begin{aligned} \oplus^k (-1)^k K_{2k, 2k, 2k, 2k}(x) &= (-1)^k K_{0,0,0,0}(x) \\ &= (-1)^k \delta \end{aligned} \quad \text{by (iii)}$$

Thus $\oplus^k (-1)^k K_{2k, 2k, 2k, 2k}(x) = \delta$, as required. \square

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On the operator \oplus^k related to the wave equation of Laplacian

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ABSTRACT In this paper, we study the Green function of the operator \oplus^k , iterated k -times and is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k$$

where $p + g = n$ is the dimension of the Euclidean space \mathbb{R}^n , $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer.

At first we study the elementary solution or the Green function of the operator \oplus^k and then such a solution is related to the solution of the wave equation and the Laplacian.

We found that the relationships of such solutions depending on the conditions of p , q and k .

1. Introduction

The operator \oplus^k can be expressed in the form

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right]^k \quad (1.1)$$

where $p + g = n$ is the dimension of \mathbb{R}^n , $i = \sqrt{-1}$ and k is a nonnegative integer. The operator $\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2$ is first introduced by A. Kananthai [1] and named as the Diamond operator and denoted by

$$\diamond = \left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \right)^2 \quad (1.2)$$

Let we denote the operators

$$L_1 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+g} \frac{\partial^2}{\partial x_j^2} \quad (1.3)$$

and

$$L_2 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad (1.4)$$

Thus we can be written by

$$\oplus^k = \diamond^k L_1^k L_2^k \quad (1.5)$$

Now, the operator \diamond can also expressed in the form $\diamond = \square \Delta = \square \Delta$ where \square is the ultra-hyperbolic operator defined by

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \quad (1.6)$$

where $p + q = n$ and Δ is the Laplacian defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad (1.7)$$

By putting $p + 1$ and $x_1 = t$ (times) is (1.6) then we obtain the wave operator

$$\square = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} \quad (1.8)$$

and from (1.1) with $q = 0$ and $k = 1$, we obtain

$$\oplus = \Delta_p^4 \quad (1.9)$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} \quad (1.7)$$

In this work, we can find the elementary solution $K(x)$ of the operator \oplus^k , that is $\oplus^k K(u) = \delta$ where δ is the Dirac-delta distribution. Moreover, we can find the relationship between $K(u)$ and the elementary solution of the wave operator defined by (1.8) depending on the conditions of p , q and k of (1.1) with $p = 1$, $q = n - 1$, $k = 1$ and $x_1 = f(\text{time})$.

Also, we found that $K(u)$ relates to the elementary solution of the Laplacian defined by (1.9) and (1.10) depending on the conditions of q and k of (1.1) with $q = 0$ and $k = 1$. \square

2. Preliminary

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$u = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of forward cone $\overline{\Gamma_+}$ denote its closure.

For any complex number α , we define the function

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.1)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{(n-1)}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}$$

The function R_α^H is called The Ultra-Hyperbolic Rernel of Marcel Riesz and was introduced by Y. Nozaki [4, p72] and is called the ultra-hyperbolic Kernel of Marcel Riesz. Now R_α^H lpha is an ordinary function if $R_e(\alpha) \geq n$ and is a distribution of α if $R_e(\alpha) < n$.

Now, if $p = 1$ then (2.1) reduces to the function $\Gamma_\alpha(u)$ say, and defined by

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.1)$$

where $u = x_1^2 + x_2^2 + \dots + x_n^2$ and $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right)$, the function $\Gamma_\alpha(u)$ is called the hyperbolic Kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$v = x_1^2 + x_2^2 + \dots + x_n^2$$

For any complex number β , define the function

$$R_\beta^e(v) = 2^{-\beta} \pi^{\frac{n}{2}} \Gamma\left(\frac{n-\beta}{2}\right) \frac{v^{\frac{\beta-n}{2}}}{\Gamma\left(\frac{\beta}{2}\right)} \quad (2.3)$$

The function $R_\beta^e(v)$ is called the elliptic Kernel of Marcel Riesz and is ordinary function if $Re(\beta) \geq n$ and is the is a distribution of β for $R_e(\beta) < n$.

Definition 2.3 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathbb{R}^n write

$$w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$$

and

$$z = x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2) \quad , p + q = n$$

For any number γ and ν , define

$$S_\gamma(w) = 2^{-\gamma} \pi^{\frac{-n}{2}} \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} w^{\frac{\gamma-n}{2}} \quad (2.4)$$

and

$$T_\nu(z) = 2^{-\nu} \pi^{\frac{-n}{2}} \frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} w^{\frac{\nu-n}{2}} \quad (2.5)$$

Lemma 2.1 The function $R_{2k}^H(u)$ and $(-1)^k R_{2k}^e(v)$ are the elementary solution of the operator \square^k and Δ^k respectively where \square^k and Δ^k are the operators iterated k -times defined by (1.6) and (1.7) respectively $R_{2k}^H(u)$ defined by (2.1) with $\alpha = 2k$ and $R_{2k}^e(v)$ defined by (2.3) with $\beta = 2k$.

Proof we have to show that $\square^k R_{2k}^H(u) = \delta$ and $\Delta^k (-1)^k R_{2k}^e(v) = \delta$

to prove $\square^k R_{2k}^H(u) = \delta$, see [2, p147]

also $\Delta^k (-1)^k R_{2k}^e(v) = \delta$, see [1, p31]

Lemma 2.2 The convolution $R_{2k}^H(u) * (-1)^k R_{2k}^e(v)$ is an elementary solution of the operator \diamond^k iterated k -times and is defined by (1.2)

Proof we need to show that $\diamond^k (R_{2k}^H(u) * (-1)^k R_{2k}^e(v)) = \delta$ to prove this, see [1, p33].

Lemma 2.3 The function $R_{-2k}^H(u)$ and $(-1)^k R_{-2k}^e(v)$ are the inverses in the convolution algebras of $R_{2k}^H(u)$ and $(-1)^k R_{2k}^e(v)$ respectively.

Proof we need to show that

$$R_{-2k}^H(u) * R_{2k}^H(u) = R_{-2k+2k}^H(u) = R_0^H(u) = \delta$$

and

$$(-1)^k R_{-2k}^e(v) * (-1)^k R_{2k}^e(v) = R_0^e(v) = \delta$$

To prove these, see [7, p123], [3, p10] and [5, p118, p158]

Lemma 2.4 (i) The function $(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)$ and $(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)$ are the elementary solutions of the operators L_1^k and L_2^k respectively where $S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.3) and (2.4) respectively with $\gamma = \nu = 2k$. The operators L_1^k and L_2^k are defined by (1.3) and (1.4) respectively.

(ii) The function $(-1)^k (-i)^{\frac{k}{2}} S_{-2k}(w)$ and $(-1)^k (i)^{\frac{k}{2}} T_{-2k}(z)$ are the inverses in the convolution algebras of $(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)$ and $(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)$ respectively

Proof (i) We need to show that

$$L_1[(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] = \delta$$

and

$$L_2[(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)] = \delta$$

At first we have to show that

$$L_1^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w) \quad (2.6)$$

$$L_2^k T_\nu(z) = (-1)^k T_{\nu-2k}(z) \quad (2.7)$$

and also

$$S_{-2k}(w) = (-1)^k (-i)^{\frac{k}{2}} L_1^k \delta \quad (2.8)$$

$$T_{-2k}(z) = (-1)^k (-i)^{\frac{k}{2}} L_2^k \delta \quad (2.9)$$

Now for $k = 1$, $L_1 S_\gamma(w) = \left(\sum_{\gamma=1}^p \frac{\partial^2}{\partial x_\gamma^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) S_\gamma(w)$ where $S_\gamma(w)$ is defined by (2.4). By computing directly, we obtain

$$\begin{aligned} L_1 S_\gamma(w) &= 2^{-\gamma} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} (\gamma-n)(\gamma-2) w^{\frac{\gamma-2-n}{2}} \\ &= (-1) 2^{-(\gamma-2)} \frac{\Gamma\left(\frac{n-(\gamma-2)}{2}\right)}{\Gamma\left(\frac{\gamma-2}{2}\right)} w^{\frac{(\gamma-2)-n}{2}} \end{aligned} \quad \text{by the properties of Gamma function}$$

Thus $L_1 S_\gamma(w) = -S_{\gamma-2}(w)$ By repeating k -times in operating L_1 to $S_\gamma(w)$, we obtain

$$L_1^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w)$$

Similarly,

$$L_2^k T_\nu(z) = (-1)^k T_{\nu-2}(z)$$

Thus we obtain (2.5) and (2.7) as required. Now consider $w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$, $p+q = n$ by changing the variable $x_1 = y_1, x_2 = y_2, \dots, x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \dots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$. Thus we have $w = y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 + \dots + y_{p+q}^2$. Denote $w = \gamma^2 = y_1^2 + y_2^2 + \dots + y_n^2$ and consider the generalized function $w^\lambda = \gamma^{2\lambda}$ where λ is any complex number. Now $\langle w^\lambda, Q \rangle = \int_{\mathbb{R}^n} w^\lambda Q(x) dx$ where $Q \in D$ the space of infinitely differentiable functions with compact supports. Thus

$$\begin{aligned} \langle w^\lambda, Q \rangle &= \int_{\mathbb{R}^n} r^{2\lambda} \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} Q dy_1 dy_2 \dots dy_n \\ &= \frac{1}{(-i)^{\frac{q}{2}}} \int_{\mathbb{R}^n} r^{2\lambda} Q dy \\ &= \frac{1}{(-i)^{\frac{q}{2}}} \langle r^{2\lambda}, Q \rangle. \end{aligned}$$

By Gelfand and Shilov [6, p271], the functional $\gamma^{2\lambda}$ have simple poles at $\lambda = \frac{-n}{2} - k$, k is nonnegative and for $k = 0$ we can find the residue of $r^{2\lambda}$ at $\lambda = \frac{-n}{2}$ and by [6, p74 and p271], we obtain

$$\operatorname{res}_{\lambda=\frac{-n}{2}} r^{2\lambda} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \delta(x)$$

Thus

$$\operatorname{res}_{\lambda=\frac{-n}{2}} w^\lambda = (i)^{\frac{q}{2}} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \delta(x) \quad (2.10)$$

We next find the residues of w^λ at $\lambda = \frac{-n}{2} - k$. Now, by computing directly we have

$$L_1 w^{\lambda+1} = 2(\lambda+1)(2\lambda+n)w^\lambda$$

where w is defined by definition 2.3 and L_1 is defined by (1.3). By k -fold iteration, we obtain

$$L_1^k w^{\lambda+k} = 4^k (\lambda+1)(\lambda+2)\dots(\lambda+k) \left(\lambda + \frac{n}{2}\right) \left(\lambda + \frac{n}{2} + 1\right) \dots \left(\lambda + \frac{n}{2} + k - 1\right) w^\lambda$$

or

$$w^\lambda = \frac{1}{4^k (\lambda+1)(\lambda+2)\dots(\lambda+k) \left(\lambda + \frac{n}{2}\right) \left(\lambda + \frac{n}{2} + 1\right) \dots \left(\lambda + \frac{n}{2} + k - 1\right)} L_1^k w^{\lambda+k}$$

Thus

$$\operatorname{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \frac{1}{4^k k \left(\frac{n}{2} + k - 1\right) \left(\frac{n}{2} + k - 2\right) \dots \frac{n}{2}} \operatorname{res}_{\lambda=\frac{-n}{2}} L_1^k w^\lambda$$

By (2.10) and the properties of Gamma functions, we obtain

$$\operatorname{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \frac{(i)^{\frac{3}{2}} \pi^{\frac{3}{2}}}{4^k \Gamma\left(\frac{n}{2} + k\right)} L_1^k \delta(x) \quad (2.11)$$

Now we consider $S_{-2k}(w)$ we have

$$\begin{aligned} S_{-2k}(w) &= \lim_{\gamma \rightarrow -2k} S(w) \\ &= \pi^{-\frac{n}{2}} \frac{\lim_{\gamma \rightarrow -2k} w^{\frac{\gamma-n}{2}}}{\lim_{\gamma \rightarrow -2k} \Gamma\left(\frac{\gamma}{2}\right)} \lim_{\gamma \rightarrow -2k} \left(2^\gamma \Gamma\left(\frac{n-\gamma}{2}\right)\right) \\ &= \pi^{-\frac{n}{2}} \frac{\lim_{\gamma \rightarrow -2k} (\gamma+2k) w^{\frac{\gamma-n}{2}}}{\lim_{\gamma \rightarrow -2k} (\gamma+2k) \Gamma\left(\frac{\gamma}{2}\right)} 2^{2k} \Gamma\left(\frac{n+2k}{2}\right) \\ &= 4^k \pi^{-\frac{n}{2}} \frac{\operatorname{res}_{\gamma=-2k} w^{\frac{\gamma-n}{2}}}{\operatorname{res}_{\gamma=-2k} \Gamma\left(\frac{\gamma}{2}\right)} \Gamma\left(\frac{n+2k}{2}\right) \end{aligned}$$

Since $\operatorname{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \operatorname{res}_{\gamma=-2k} w^{\frac{\gamma-n}{2}}$ and $\operatorname{res}_{\gamma=-2k} \Gamma\left(\frac{\gamma}{2}\right) = \frac{(-i)^k}{k!}$ Thus by (2.11) and the properties of Gamma function we obtain

$$S_{2k}(w) = (-1)^k (i)^{\frac{3}{2}} L_1^k \delta(x)$$

Similarly,

$$T_{-2k}(z) = (-1)^k (-i)^{\frac{3}{2}} L_2^k \delta(x)$$

Thus we have

$$S_0(w) = (i)^{\frac{3}{2}} \delta(x) \quad (2.12)$$

and

$$T_0(w) = (-i)^{\frac{3}{2}} \delta(x) \quad (2.12)$$

Now, from (2.6) $L_1^k S_{2k}(w) = (-1)^k S_0(w)$ for $\gamma = 2k$. Thus, by (2.12) we obtain $L_1^k (-i)^k (-i)^{\frac{3}{2}} S_{2k}(w) = \delta(x)$. It follows that $(-1)^k (-i)^{\frac{3}{2}} S_{2k}(w)$ is an elementary solution of the operator L_1^k . Similarly $(-1)^k (i)^{\frac{3}{2}} T_{2k}(z)$ is also an elementary solution of L_2^k .

(ii) We need to show that

$$[(-1)^k(-i)^{\frac{\alpha}{2}}S_{-2k}(w)] * [(-1)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w)] = \delta$$

and

$$[(-1)^k(i)^{\frac{\alpha}{2}}T_{-2k}(z)] * [(-1)^k(i)^{\frac{\alpha}{2}}T_{2k}(z)] = \delta$$

Now, from (2.8) $(-1)^k(-i)^{\frac{\alpha}{2}}S_{-2k}(w) = L_1^k\delta$ convolving both sides by $(-1)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w)$ we obtain

$$\begin{aligned} [(-1)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w)] * [(-1)^k(-i)^{\frac{\alpha}{2}}S_{-2k}(w)] &= [(-i)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w)] * L_1^k\delta \\ &= L_1^k[(-i)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w)] * \delta \\ &= \delta * \delta = \delta \quad \text{by Lemma 2.4 (i)} \end{aligned}$$

Similarly $[(-1)^k(i)^{\frac{\alpha}{2}}T_{-2k}(z)] * [(-1)^k(i)^{\frac{\alpha}{2}}T_{2k}(z)] = \delta$ □

3. Main results

Theorem 3.1 Given the equation

$$\oplus^k K(x) = \delta \tag{3.1}$$

where \oplus^k is the operator iterated k -times defined by (1.1), δ is the Dirac-delta distribution, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. Then we obtain

$$K(u) = R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w) * (-1)^k(i)^{\frac{\alpha}{2}}T_{2k}(z) \tag{3.2}$$

as an elementary solution of (3.1) where $R_{2k}^H(u)$, $R_{2k}^e(v)$, $S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.1), (2.3), (2.4) and (2.5) respectively with $\alpha = \beta = \gamma = \nu = 2k$, k is a nonnegative integer.

Moreover, from (3.2) we obtain

$$R_{2k}^e(v) * [(-1)^k(-i)^{\frac{\alpha}{2}}S_{2k}(w)] * [(-1)^k(i)^{\frac{\alpha}{2}}T_{2k}(z)] * K(x) = R_{2k}^H(u) \tag{3.3}$$

as an elementary solution of the operator \square^k iterated k -times defined by (1.6) and in particular from (3.2) and (3.3) with $p = 1, q = n - 1, k = 1$ and $x_1 = t$, we obtain

$$(-1)R_{-2}^e(v) * [(-1)(-i)^{\frac{\alpha}{2}}S_{-2}(w)] * [(-1)(i)^{\frac{\alpha}{2}}T_{-2}(z)] * K(x) = M_2(u) \tag{3.4}$$

as an elementary solution of the wave operator defined by (1.8) where $M_2(u)$ is defined by (2.2) with $\alpha = 2$. Also, for $q = 0$ then (3.1) become

$$\Delta_p^{4k} K(x) = \delta \tag{3.5}$$

and by (3.2) we obtain

$$\begin{aligned} K(x) &= (-1)^k R_{2k}^c(v) * (-1)^k R_{2k}^c(v) * (-1)^k R_{2k}^c(v) * (-1)^k R_{2k}^c(v) \\ &= (-1)^{4k} R_{8k}^c(v) = R^c(v) \end{aligned} \quad (3.6)$$

is an elementary solution of (3.5)

where Δ_p^{4k} is the Laplacian of p -dimension, iterated $4k$ -times and is defined by (1.10) and $v = x_1^2 + x_2^2 + \dots + x_p^2$

Proof From (3.1) and (1.5) we have

$$\oplus_k K(x) = (\diamond^k L_1^k L_2^k) K(x) = \delta$$

convolving both sides of the above equation by the convolution $[R_{2k}^H(u) * (-1)^k R_{2k}^c(v)] * [(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] * [(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)]$ and the properties of convolution with derivatives, we obtain

$$\begin{aligned} &\diamond^k (R_{2k}^H(u) * (-1)^k R_{2k}^c(v)) * L_1((-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)) * L_2((-1)^k (i)^{\frac{k}{2}} T_{2k}(z)) * k(x) \\ &= [R_{2k}^H(u) * (-1)^k R_{2k}^c(v)] * [(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] * [(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)] \end{aligned}$$

Thus $\delta * \delta * K(x) = K(x) = [R_{2k}^H(u) * (-1)^k R_{2k}^c(v)] * [(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] * [(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)]$ by Lemma 2.2 and Lemma 2.4 (i)

Thus we obtain (3.2) as required Now we will relate the elementary solution $K(u)$ given by (3.2) to the elementary solution of the wave equation defined by (1.8) and the Laplacian defined by (1.10). Now from (3.2) and by Lemma 2.3, Lemma 2.5(ii) and the properties of inverses in the convolution algebra, we obtain

$$(-1)^k R_{2k}^c(v) * [(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] * [(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)] * K(x) = \delta * \delta * R_{2k}^H(u) = R_{2k}^H(u)$$

Actually, by Lemma 2.1 $R_{2k}^H(u)$ is an elementary solution of the operator \square^k iterated k -times defined by (1.6).

In particular, by putting $p = 1, q = n - 1, k = 1$ and $x_1 = t$ in (3.2) and (3.3) then $R_{2k}^H(u)$ reduces to $M_2(u)$ is defined by (2.2) with $\alpha = 2$. Thus we obtain

$$(-1)^k R_{2k}^c(v) * [(-1)^k (-i)^{\frac{k}{2}} S_{2k}(w)] * [(-1)^k (i)^{\frac{k}{2}} T_{2k}(z)] * K(x) = M_2(u)$$

is an elementary solution of the wave operator defined by (1.8) where $u = f^2 - X_1^2 - X_2^2 - \dots - X_{n-1}^2$.

Also, for $q = 0$ then (3.1) becomes

$$\Delta_p^{4k} K(x) = \delta \quad (3.7)$$

where $\Delta_p^{4k} K(x)$ is the Laplacian of p -dimension iterated $4k$ -times. By Lemma 2.1, we have

$$\begin{aligned} K(x) &= (-1)^{4k} R_{8k}^c(v) \\ &= R_{8k}^c(v) \end{aligned}$$

is an elementary solution of (3.7) where

$$v = X_1^2 + X_2^2 + \dots + X_p^2$$

On the otherhand, we can also find $K(x)$ from (3.2), since $q = 0$, we have $R_{2k}^H(u)$ reduces to $(-1)^k R_{2k}^e(v)$

$(-1)^k S_{2k}(w)$ reduces to $(-1)^k R_{2k}^e(v)$

also $(-1)^k T_{2k}(z)$ reduces to $(-1)^k R_{2k}^e(v)$

where $v = X_1^2 + X_2^2 + \dots + X_p^2$ Thus, by (3.2) for $q = 0$, we obtain

$$\begin{aligned} K(x) &= (-1)^k R_{2k}^e(v) * (-1)^k R_{2k}^e(v) * (-1)^k R_{2k}^e(v) * (-1)^k R_{2k}^e(v) \\ &= (-1)^{4k} R_{2k+2k+2k+2k}^e(v) \\ &= R_{8k}^e(v) \quad \text{by W.F. Donoghue [5, p158]} \end{aligned}$$

That complete the proofs □

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On the Green function of the operator \odot^k

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Abstract.

In this paper we study the Green function or the elementary solution of the operator \odot^k iterated k -times and is defined by

$$\odot^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k$$

where $p + q = n$ is the dimension of the Euclidean space \mathbb{R}^n , k is a nonnegative integer.

First we find the elementary solution of the operator \odot^k and later we obtain the solution of the equation $\odot^k u(x) = f(x)$ where $f(x)$ is a generalized function, $u(x)$ is an unknown and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

1. Introduction

The operator \odot^k can be factorized in the form

$$\odot^k = \square^k L^k = L^k \square^k \quad (1.1)$$

where \square^k is the ultra-hyperbolic operator iterated k -times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (1.2)$$

and L^k is the operator iterated k -times defined by

$$L^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) \right]^k \quad (1.3)$$

The operator \square^k has been first studied by *I.M Galfand* and *G.E shilov* [6, p279] and they obtained the elementary solution of \square^k in the very complicated form. Later *S.E Trione* [3, p11] has shown that the function $R_{2k}^H(u)$ defined by (2.1.1) with $\alpha = 2k$ is an elementary solution of the operator \square^k . Moreover *M. Aguirre Tellez* [4, p147] has proved that the solution $R_{2k}^H(u)$ exists only if the dimension n is odd with p odd and q even and $p + q = n$, we use the method of convolution of distributions and the Fourier transform of distribution to obtain the elementary solution of the operator \odot^k and the solution of $\odot^k u(x) = f(x)$.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathbb{R}^n and write $u = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$, $p + q = n$ and denoted by Γ_+ the interior of the forward cone given by

$$\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$$

and $\overline{\Gamma_+}$ denotes the closure of Γ_+ . Let $R_\alpha^H(u)$ be the function defined by

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.1.1)$$

Here α is a complex number and n is the dimension of the space \mathbb{R}^n . The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\Pi^{\frac{\alpha-n}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}$$

The function $R_\alpha^H(u)$ is first introduced by Y. Nozaki [5, p72] and is called the ultra-hyperbolic Kernel of Marcel Riesz. If $R(\alpha) \geq n$ then $R_\alpha^H(u)$ is an ordinary function and if $\text{Re}(\alpha) < n$, $R_\alpha^H(u)$ is a distribution of α .

Let $K \subset \overline{\Gamma_+}$ be a compact set and $K = \text{supp} R_\alpha^H(u)$ where $\text{supp} R_\alpha^H(u)$ denotes its support.

By A. Kanathai [1, p31],

$$R_\alpha^H(u) \text{ is a tempered} \quad (2.1.2)$$

distribution with compact support.

Definition 2.2 Let f be locally integrable function in \mathbb{R}^n and also absolutely integrable. The Fourier transform of f is defined by

$$\mathfrak{F}f(x) = \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx \quad (2.1.3)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$.

$$(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$$

If f is distribution with compact support, then by A.H Zemanian [7, p187], (2.1.3) can be written as

$$\mathfrak{F}f(x) = \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle f(x), e^{-i(\xi, x)} \rangle \quad (2.1.4)$$

The inverse Fourier transform of $\hat{f}(\xi)$ is also defined by

$$f(x) = \mathfrak{F}^{-1} \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} \hat{f}(\xi) d\xi \quad (2.1.5)$$

and for f is a tempered distribution with compact support,

$$f(x) = \mathfrak{F}^{-1} \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle \hat{f}(\xi), e^{-i(\xi, x)} \rangle$$

Lemma 2.1 The function $R_{2k}^H(u)$ defined by (2.1.1) with $\alpha = 2k$ is an elementary solution of the operator \square^k defined by (1.2). That is $\square^k R_{2k}^H(u) = \delta$ where δ is the Dirac-delta distribution.

Proof see [3, p11].

Lemma 2.2 The Fouier transform of the function $R_{2k}^H(u)$ is given by

$$\mathfrak{F}R_{2k}^H(u) = \frac{1}{(2\pi)^{\frac{n}{2}} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k}. \quad (2.1.6)$$

Proof see [2, p154 - 157].

Lemma 2.3

$$\mathfrak{F}(L^k \delta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 + \left(\sum_{i=1}^p \xi_i^2 \right)^2 \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right]^k \quad (2.1.7)$$

where L^k is defined by (1.3).

Proof Since δ is a tempered distribution with compact support $\{0\}$, thus we also obtain $L^k \delta$ as a tempered with compact support.

By (2.1.3), we have

$$\begin{aligned}
\mathfrak{F}(L^k \delta) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \langle L^k \delta, e^{-i(\xi, x)} \rangle \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \langle L^{k-1} \delta, L e^{-i(\xi, x)} \rangle \text{ by computing directly.} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \left\langle L^{k-1} \delta, \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 + \left(\sum_{i=1}^p \xi_i^2 \right) \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right) \right] e^{-i(\xi, x)} \right\rangle
\end{aligned}$$

By keeping on operating L with $k-1$ times, we obtain

$$\begin{aligned}
\mathfrak{F}(L^k \delta) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \left\langle \delta, \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 + \left(\sum_{i=1}^p \xi_i^2 \right) \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right) \right]^k e^{-i(\xi, x)} \right\rangle \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 + \left(\sum_{i=1}^p \xi_i^2 \right) \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right) \right]^k.
\end{aligned}$$

3. Main result

Theorem 3.1 Given the equation

$$\odot^k u(x) = \delta \quad (3.1.1)$$

where \odot^k is defined by (1.1), $x = (x_1, x_2, \dots, x_n) \in \mathfrak{R}^n$, δ is the Dirac-delta distribution and k is a nonnegative integer. Then

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \mathfrak{F}^{-1} \left(\frac{1}{\left[\left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 - \left(\sum_{i=1}^p \xi_i^2 \right)^3 \right]^k} \right)$$

is an elementary solution of the operator \odot^k where \mathfrak{F}^{-1} is an inverse Fourier transform defined by (2.1.5).

Proof We have

$$\odot^k u(x) = \square^k L^k u(x) = \delta$$

where \square^k and L^k are defined by (1.2) and (1.3) respectively. Convolving the above equation both sides by R_{2k}^H we obtain

$$R_{2k}^H(u) * \square^k L^k u(x) = R_{2k}^H(u) * \delta = R_{2k}^H(u)$$

$$\square^k R_{2k}^H(u) * L^k u(x) = \delta * L^k u(x) = R_{2k}^H(u)$$

Thus $L^k u(x) = R_{2k}^H(u)$ by Lemma 2.1. Now, we have

$$L^k u(x) = (L^k \delta) * u(x) = R_{2k}^H(u).$$

Taking the Fourier transform both sides of the above equation, we obtain

$$(2\pi)^{\frac{n}{2}} \Im(L^k \delta) \Im u(x) = \Im R_{2k}^H(u) = \frac{1}{(2\pi)^{\frac{n}{2}} [\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2]}$$

Thus

$$\begin{aligned} \Im u(x) &= \frac{1}{(2\pi)^{\frac{n}{2}} \Im(L^k \delta) (2\pi)^{\frac{n}{2}} [\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2]^k} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} \left[\left(\sum_{i=1}^p \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 + \left(\sum_{i=1}^p \xi_i^2 \right)^2 \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right]^k \times} \\ &\quad [\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2]^k \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} \left[\left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 - \left(\sum_{i=1}^p \xi_i^2 \right)^3 \right]^k} \end{aligned}$$

Since \Im is 1-1 continuous on the space S of tempered distribution. By taking the inverse \Im^{-1} both sides of the equation, we obtain

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \Im^{-1} \left(\frac{1}{\left[\left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 - \left(\sum_{i=1}^p \xi_i^2 \right)^3 \right]^k} \right)$$

as required. Denote by

$$G(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \mathfrak{F}^{-1} \left(\frac{1}{\left[\left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 - \left(\sum_{i=1}^p \xi_i^2 \right)^3 \right]^k} \right) \quad (3.1.2)$$

the Green function of (3.1.1)

Theorem 3.2 Given the equation

$$\odot^k u(x) = f(x) \quad (3.2.1)$$

where $f(x)$ is a generalized function, $x \in \mathbb{R}^n$ and \odot^k defined by (1.1). Then $u(x) = G(x) * f(x)$ is a solution of (3.2.1)

Proof Convolving both sides of (3.2.1) by the function $G(x)$ given by (3.1.2). Thus

$$G(x) * \odot^k u(x) = G(x) * f(x)$$

$$[\odot^k G(x)] * u(x) = \delta * u(x) = u(x) = G(x) * f(x)$$

since $G(x)$ given by (3.1.2) is a tempered distribution with compact support, thus $G(x) * f(x)$ exists and unique. Thus we obtain $u(x) = G(x) * f(x)$ is a solution of (3.1.3) \square

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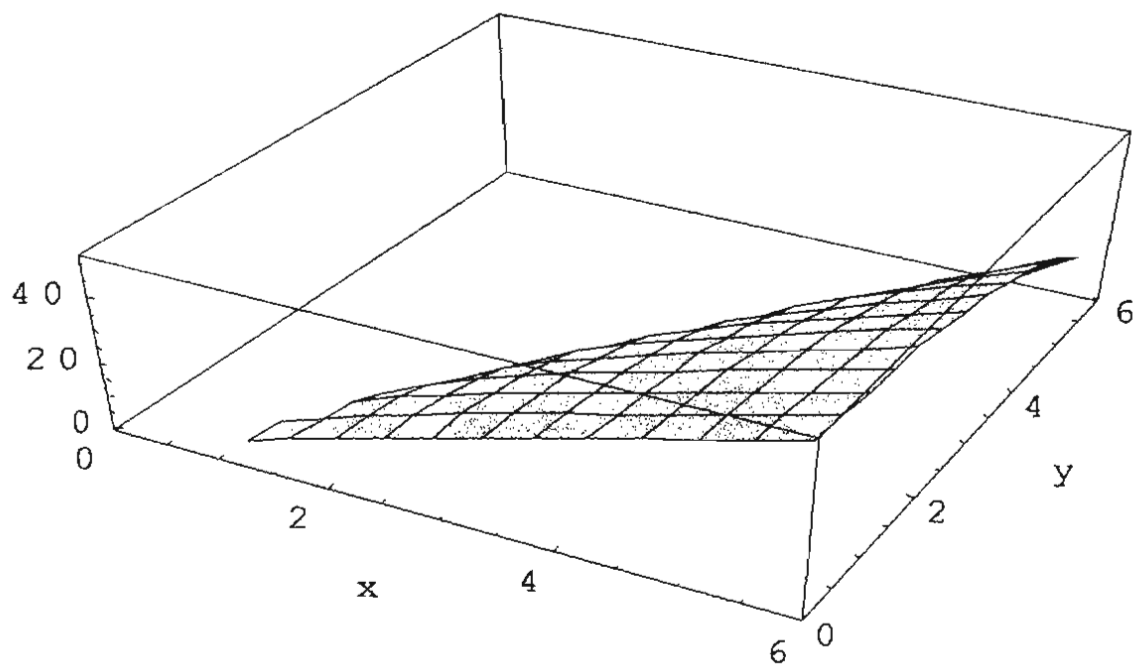
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กราฟของ $R_{2k}^H(u)$ $R_{2k}^e(v)$ และ $K_{2k,2k}(x)$

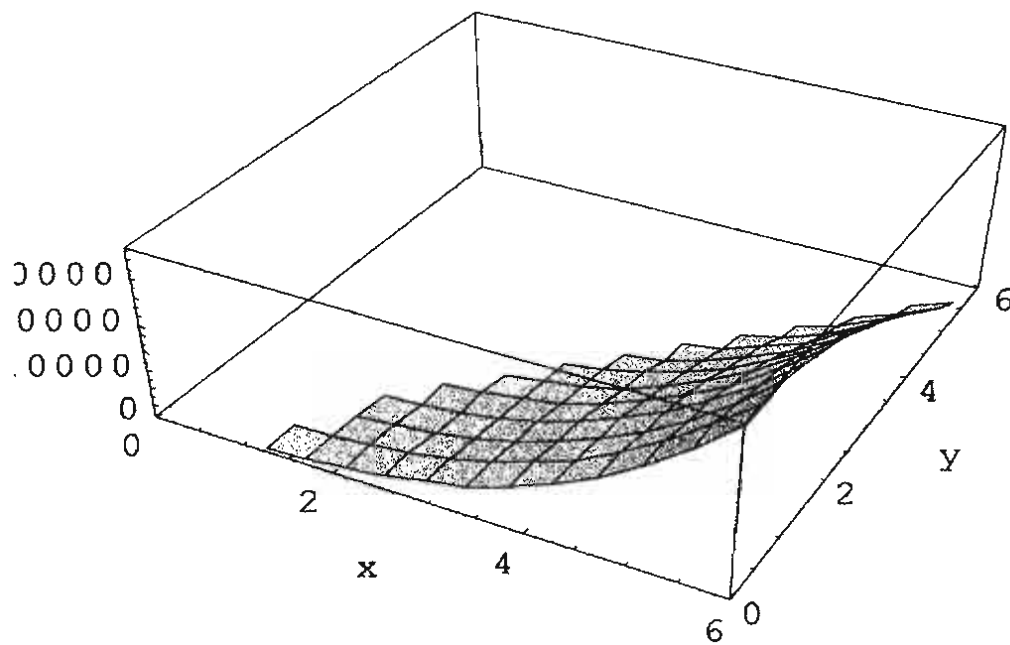
ตัวอย่างกราฟของฟังก์ชัน $R_{2k}^H(u)$ $R_{2k}^e(v)$ และ $K_{2k,2k}(x)$

$$R_{2k}^H(u) = \frac{\Gamma(1+2k)}{\pi \Gamma(\frac{2k-1}{2})} (x^2 - y^2 - z^2)^{\frac{2k-n}{2}} \quad \text{เมื่อ } u = x^2 - y^2 - z^2$$



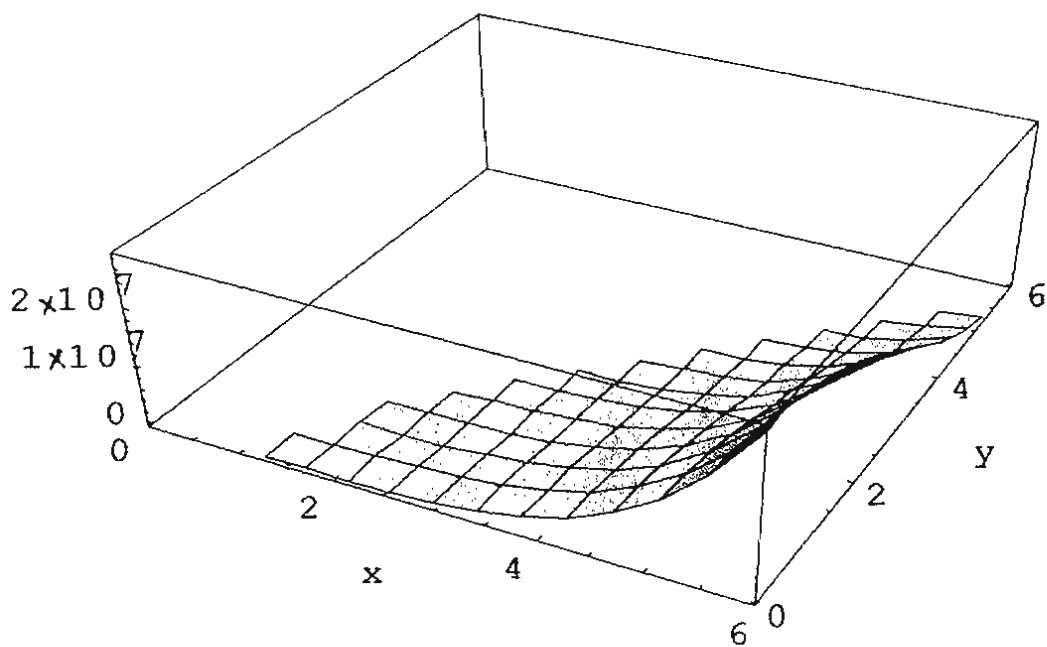
รูปที่ 1 กราฟของ $R_{2k}^H(u)$ เมื่อ $z=1$, $n=3$ และ $k=2$

$$R_{2k}^H(u) = \frac{\Gamma(1+2k)}{\pi \Gamma(\frac{2k-1}{2})} (x^2 - y^2 - z^2)^{\frac{2k-n}{2}} \quad \text{เมื่อ } u = x^2 - y^2 - z^2$$



รูปที่ 2 กราฟของ $R_{2k}^H(u)$ เมื่อ $z=1$, $n=3$ และ $k=3$

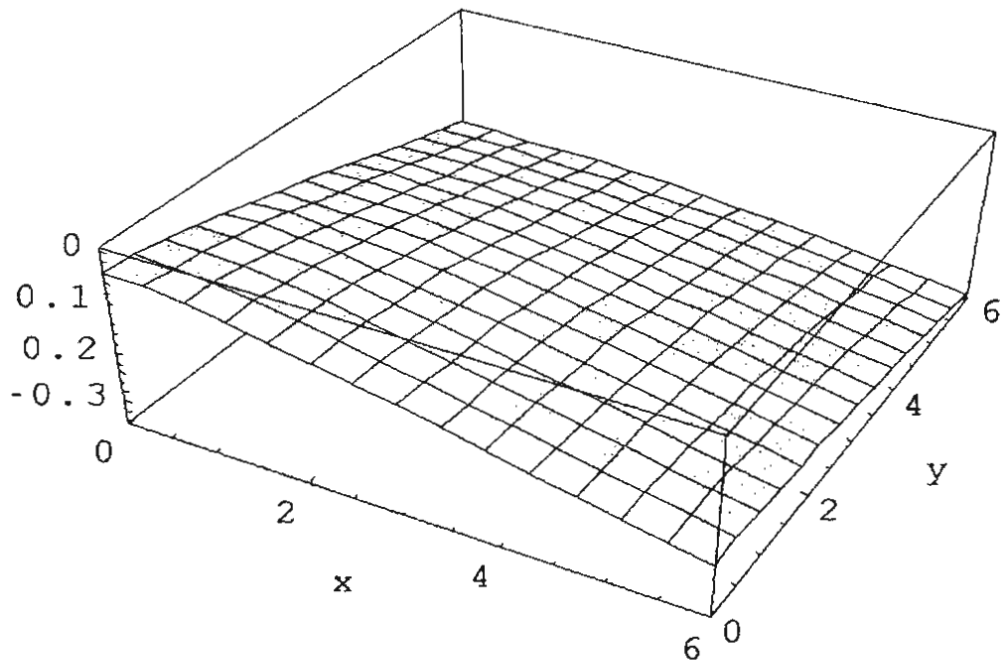
$$R_{2k}^H(u) = \frac{\Gamma(1+2k)}{\pi \Gamma(\frac{2k-1}{2})} (x^2 - y^2 - z^2)^{\frac{2k-n}{2}} \quad \text{เมื่อ } u = x^2 - y^2 - z^2$$



รูปที่ 3 กราฟของ $R_{2k}^H(u)$ เมื่อ $z=1$, $n=3$ และ $k=4$

$$R_{2k}^e(v) = \frac{2^{-2k} \pi^{-3/2} \Gamma(3/2 - k) (x^2 + y^2 + z^2)^{\frac{2k-n}{2}}}{\Gamma(k)}$$

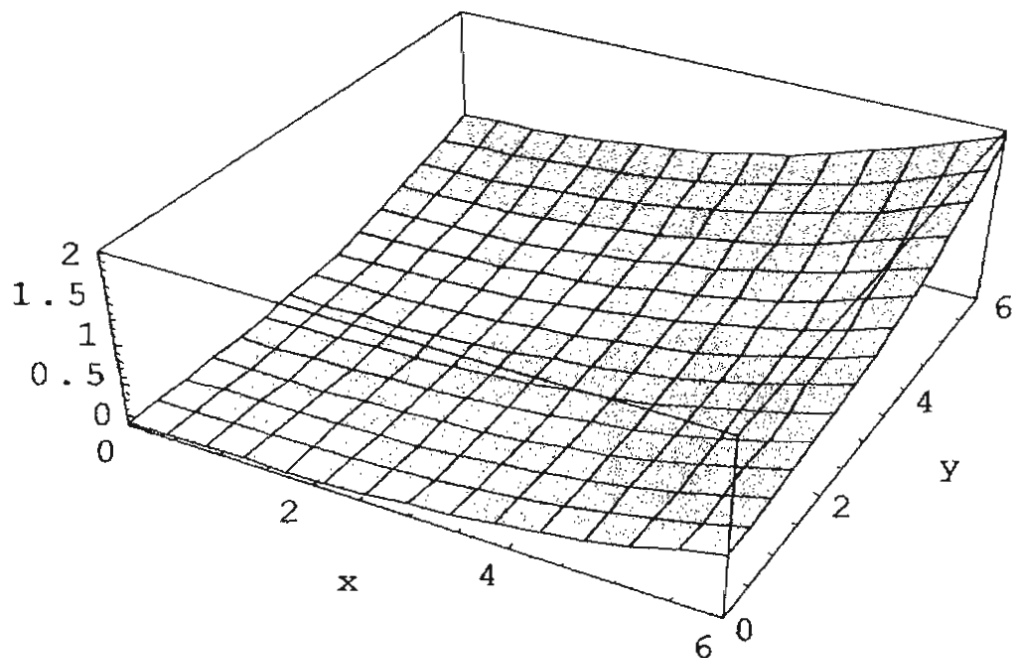
$$\text{เมื่อ } v = x^2 + y^2 + z^2$$



รูปที่ 4 กราฟของ $R_{2k}^e(v)$ เมื่อ $z=1$ $n=3$ และ $k=2$

$$R_{2k}^e(v) = \frac{2^{-2k} \pi^{-3/2} \Gamma(3/2 - k) (x^2 + y^2 + z^2)^{\frac{2k-n}{2}}}{\Gamma(k)}$$

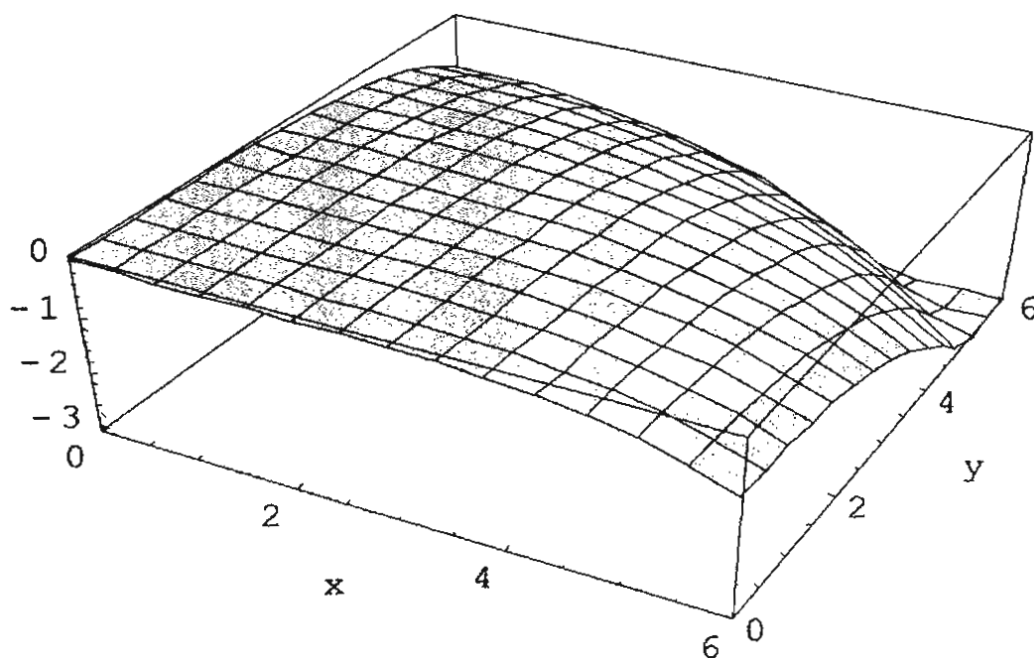
เมื่อ $v = x^2 + y^2 + z^2$



รูปที่ 5 กราฟของ $R_{2k}^e(v)$ เมื่อ $z=1$ $n=3$ และ $k=3$

$$R_{2k}^e(v) = \frac{2^{-2k} \pi^{-3/2} \Gamma(3/2 - k) (x^2 + y^2 + z^2)^{\frac{2k-n}{2}}}{\Gamma(k)}$$

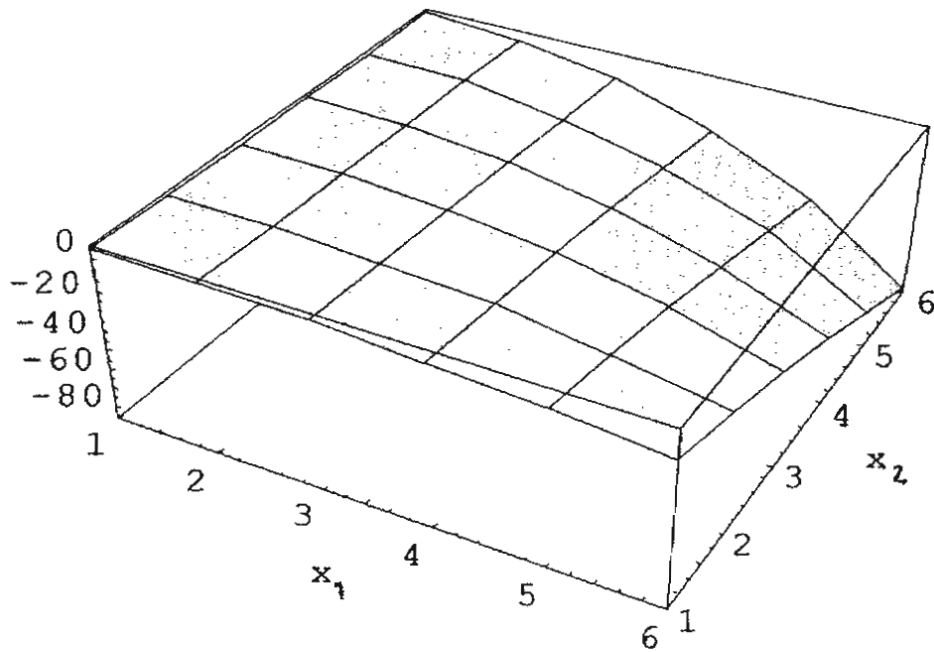
$$\text{เมื่อ } v = x^2 + y^2 + z^2$$



รูปที่ 6 กราฟของ $R_{2k}^e(v)$ เมื่อ $z=1$ $n=3$ และ $k=4$

$$K_{2k,2k}(x) = (-1)^k \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} R_{2k}^H(x) * R_{2k}^e(x-r) dr_1 dr_2 dr_3$$

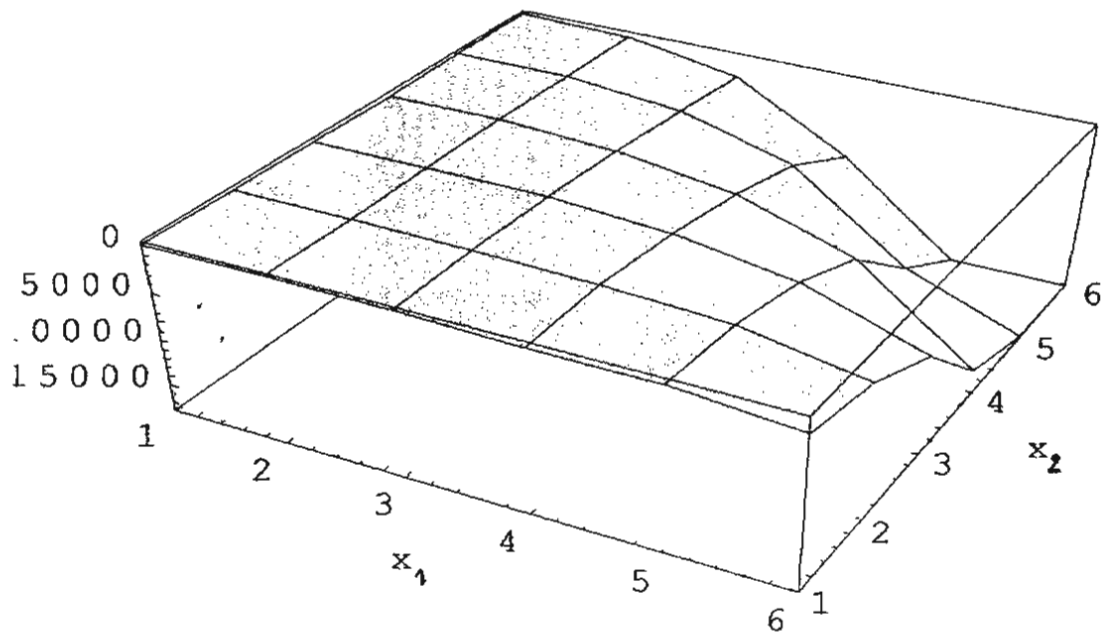
$$x = (x_1, x_2, x_3) \quad r = (r_1, r_2, r_3)$$



รูปที่ 7 กราฟของส่วนจริงของ $K_{2k,2k}(x)$ เมื่อ $x_3=1$ $n=3$ และ $k=2$

$$K_{2k,2k}(x) = (-1)^k \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} R_{2k}^H(x) * R_{2k}^a(x-r) dr_1 dr_2 dr_3$$

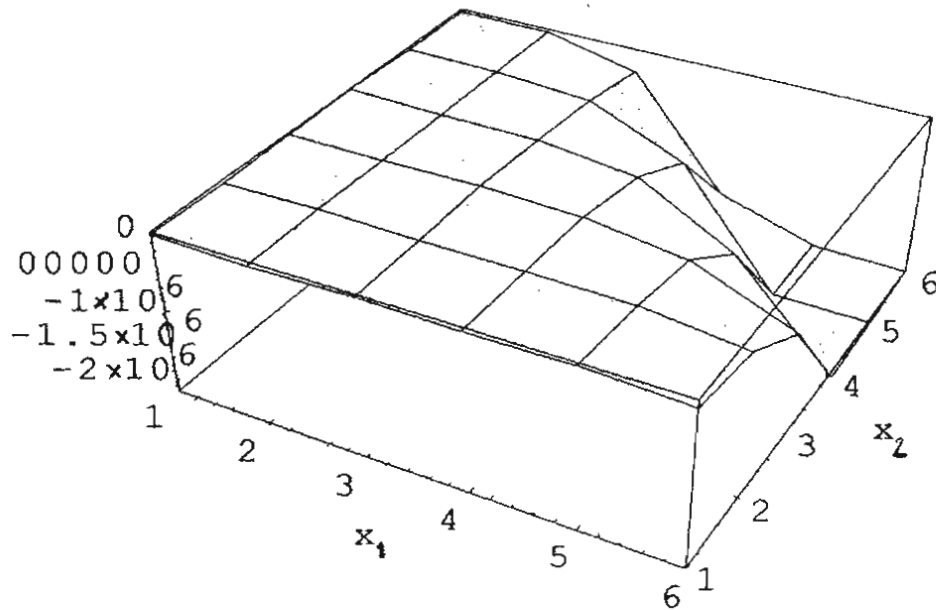
$$x = (x_1, x_2, x_3) \quad r = (r_1, r_2, r_3)$$



รูปที่ 8 กราฟของส่วนจริงของ $K_{2k,2k}(x)$ เมื่อ $x_3=1$ $n=3$ และ $k=3$

$$K_{2k,2k}(x) = (-1)^k \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} R_{2k}^H(x) * R_{2k}^e(x-r) dr_1 dr_2 dr_3$$

$$x = (x_1, x_2, x_3) \quad r = (r_1, r_2, r_3)$$



รูปที่ 9 กราฟของส่วนจริงของ $K_{2k,2k}(x)$ เมื่อ $x_3 = 1$ $n = 3$ และ $k = 4$



On the Diamond Operator related to the Wave Equation

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Abstract

In this paper, we study the solution of the equation $\diamond^k u(x) = f(x)$ where \diamond^k is the Diamond operator iterated k times and is defined by

$$\diamond^k = \left(\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k$$

where $p+q = n$ is the dimension of the n -dimensional Euclidean space R^n , $x = (x_1, x_2, \dots, x_n) \in R^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a generalized function.

It is found that the solution $u(x)$ depends on the conditions of p and q and moreover such a solution is related to the solution of the Laplace equation and the wave equation.

1 Introduction

The operator \diamond^k has been first introduced by A. Kananthai [3] and is named as the Diamond operator iterated k times and is defined by

$$\diamond^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \dots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right)^k \quad (1.1)$$

where $p+q = n$ is the dimension of the space R^n , $x = (x_1, x_2, \dots, x_n) \in R^n$ and k is a nonnegative integer.

Actually the operator \diamond^k is an extension of the ultrahyperbolic operator and the Laplacian. So the operator \diamond^k can be expressed as the product of the operator \square and \triangle , that is $\diamond^k = \square^k \triangle^k = \triangle^k \square^k$ where

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (1.2)$$

is the ultrahyperbolic operator iterated k -time with $p + q = n$, and

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k \quad (1.3)$$

is the Laplacian iterated k -times.

A. Kananthai ([3], Theorem 3.1 p33) has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementary solution of the operator \diamond^k , that is

$$\diamond^k ((-1)^k R_{2k}^e(x) * R_{2k}^H(x)) = \delta \quad (1.4)$$

where δ is the Dirac-delta distribution and the functions R_{2k}^e and R_{2k}^H are defined by (2.5) and (2.1) respectively with $\alpha = 2k$, k is nonnegative integer.

In this paper, we study the solution of the equation

$$\diamond^k u(x) = f(x) \quad (1.5)$$

This equation is the generalization of the ultrahyperbolic equation and it can be applied to the wave equation and potential that has been shown in the last part of this paper.

Let $K_{\alpha,\beta}(x)$ be a distributional family and is defined by

$$K_{\alpha,\beta}(x) = R_{\alpha}^e * R_{\beta}^H \quad (1.6)$$

where R_{α}^e is called the elliptic Kernel defined by (2.5) and R_{β}^H is called the ultra-hyperbolic Kernel defined by (2.1) and α, β are the complex parameters.

The family $K_{\alpha,\beta}(x)$ is well-defined and is a tempered distribution, since $R_{\alpha}^e * R_{\beta}^H$ is a tempered, see ([1], Lemma 2.2) and R_{β}^H has a compact support.

In this paper, we can show that

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} + (-1)^k K_{2k,2k}(x) * f(x)$$

is a solution of (1.5) where $m = \frac{n-4}{2}$, $n \geq 4$ and n is even number and $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$. Moreover, we can show that the solution $u(x)$ relates to the solution of Laplace operator \triangle^{2k} defined by (1.3) and also the wave operator defined by (1.2) with $k = 1$ and $p = 1$.

2 Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional Euclidean space R^n .

Denote by $v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, p+q = n$ the nondegenerated quadratic form. By Γ_+ we designate the interior of the forward cone.

$\Gamma_+ = \{x \in R^n : x_1 > 0 \text{ and } v > 0\}$, and by $\bar{\Gamma}_+$ designate its closure. For any complex number α , define

$$R_\alpha^H(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+ \end{cases} \quad (2.1)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}. \quad (2.2)$$

The function $R_\alpha^H(x)$ was introduced by Nozaki ([4], p.72).

It is well known that $R_\alpha^H(x)$ is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and it is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$. Suppose that $\operatorname{supp} R_\alpha^H(x) \subset \bar{\Gamma}_+$.

From S.E Trione ([5], p11), $R_{2k}^H(v)$ is an elementary solution of the operator \square^k that is

$$\square^k R_{2k}^H(v) = \delta \quad (2.3)$$

where \square^k is defined by (1.2).

By putting $p = 1$ in (2.1) and (2.2) and remembering the Legendre's duplication of $\Gamma(z)$.

$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$ then the formula (2.1) reduces to

$$M_\alpha(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.4)$$

Here $v = x_1^2 - x_2^2 - \dots - x_n^2$ and $H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha-n+2}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)$.

$M_\alpha(v)$ is, precisely, the hyperbolic kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n)$ be a point of R^n and the function $R_\alpha^e(x)$ be defined by

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \quad (2.5)$$

where $W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$, α is a complex parameter and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$.

It can be shown that $R_{-2k}^e(x) = (-1)^k \Delta^k \delta(x)$ where Δ^k is defined by (1.3). It follows that $R_0^e(x) = \delta$, see ([2], p118).

Moreover, we obtain $(-1)^k R_{2k}^e(x)$ is an elementary solution of the operator Δ^k , that is

$$\Delta^k \{(-1)^k R_{2k}^e(x)\} = \delta \quad (2.6)$$

see ([3], Lemma 2.4 p31).

Lemma 2.1 Given P is a hyper-surface then

$$P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.

Proof. See ([1], P233).

Lemma 2.2 Given the equation

$$\Delta^k u(x) = 0 \quad (2.7)$$

where Δ^k is defined by (1.3) and $x = (x_1, x_2, \dots, x_n) \in R^n$ then

$u(x) = (-1)^{(k-1)} \left(R_{2(k-1)}^e(x) \right)^{(m)}$ is a solution of (2.7) where m is a nonnegative integer with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even and $\left(R_{2(k-1)}^e(x) \right)^{(m)}$ is a function defined by (2.5) with m derivatives with $\alpha = 2(k-1)$

Proof. We first show that the generalized function $u(x) = \delta^{(m)}(r^2)$ where $r^2 = |x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$ is a solution of

$$\Delta u(x) = 0 \quad (2.8)$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is a Laplace operator. Now

$$\begin{aligned} \frac{\partial}{\partial x_i} \delta^{(m)}(r^2) &= 2x_i \delta^{(m+1)}(r^2) \\ \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2) &= 2\delta^{(m+1)}(r^2) + 4x_i^2 \delta^{(m+2)}(r^2). \end{aligned}$$

Thus

$$\begin{aligned} \Delta \delta^{(m)}(r^2) &= \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2) \\ &= 2n\delta^{(m+1)}(r^2) + 4r^2 \delta^{(m+2)}(r^2) \\ &= 2n\delta^{(m+1)}(r^2) - 4(m+2)\delta^{(m+1)}(r^2) \end{aligned}$$

by Lemma 2.1 with $P = r^2$. We have

$$\begin{aligned} \Delta \delta^{(m)}(r^2) &= [2n - 4(m+2)]\delta^{(m+1)}(r^2) \\ &= 0 \quad \text{if } 2n - 4(m+2) = 0 \end{aligned}$$

or $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of (2.8) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Now $\Delta^k u(x) = \Delta(\Delta^{k-1} u(x)) = 0$ then from the above proof $\Delta^{k-1} u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{k-1}R_{2(k-1)}^e(x)$, we obtain

$$\begin{aligned} (-1)^{k-1}R_{2(k-1)}^e(x) * \Delta^{k-1}u(x) &= (-1)^{k-1}R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) \\ \text{or } \Delta^{k-1} \left((-1)^{k-1}R_{2(k-1)}^e(x) \right) * u(x) &= (-1)^{k-1}R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) \end{aligned}$$

$$\text{or } \delta * u(x) = u(x) = (-1)^{k-1}R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) \quad \text{by (2.6)}$$

Now from (2.1)

$$\begin{aligned} R_{2(k-1)}^e(x) &= \frac{|x|^{2(k-1)-n}}{W_n(\alpha)} \\ &= \frac{(|x|^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} = \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Hence

$$\begin{aligned} R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) &= \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} * \delta^{(m)}(r^2) \\ &= \left[\frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \right]^{(m)} = \left[R_{2(k-1)}^e(x) \right]^{(m)}. \end{aligned}$$

It follows that $u(x) = (-1)^{k-1} \left[R_{2(k-1)}^e(x) \right]^{(m)}$ is a solution of (2.7) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension of R^n .

Lemma 2.3 Given the equation

$$\square^k u(x) = 0 \quad (2.9)$$

where \square^k is defined by (1.2) and $x = (x_1, x_2, \dots, x_n) \in R^n$ then $u(x) = \left[R_{2(k-1)}^{II}(v) \right]^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension and v is defined by Definition 2.1. The function $\left[R_{2(k-1)}^{II}(v) \right]^{(m)}$ is defined by (2.1) with m -derivatives and $\alpha = 2(k-1)$.

Proof At first we show that the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, $p+q = n$, is a solution of the equation

$$\square u(x) = 0 \quad (2.10)$$

where \square is defined by (1.2) with $k=1$ and $x = (x_1, x_2, \dots, x_n) \in R^n$. Now

$$\begin{aligned}
\frac{\partial}{\partial x_i} \delta^{(m)}(r^2 - s^2) &= 2x_i \delta^{(m+1)}(r^2 - s^2) \\
\frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) &= 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2 \delta^{(m+2)}(r^2 - s^2) \\
\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) &= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2 \delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&\quad + 4s^2 \delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) \\
&\quad - 4(m+2)\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4s^2 \delta^{(m+2)}(r^2 - s^2) \\
&= [2p - 4(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4s^2 \delta^{(m+2)}(r^2 - s^2)
\end{aligned}$$

by Lemma 2.1 with $P = r^2 - s^2$.

Similarly,

$$\begin{aligned}
\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \delta^{(m)}(r^2 - s^2) &= [-2q + 4(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4r^2 \delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\Box \delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \delta^{(m)}(r^2 - s^2) \\
&= [2(p+q) - 8(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad - 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&= [2n - 8(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4(m+2)\delta^{(m+1)}(r^2 - s^2) \quad \text{by Lemma 2.1} \\
&= [2n - 4(m+2)]\delta^{(m+1)}(r^2 - s^2).
\end{aligned}$$

If $2n - 4(m+2) = 0$, we have $\Box \delta^{(m)}(r^2 - s^2) = 0$. That is $u(x) = \delta^{(m)}(r^2 - s^2)$ is a solution of (2.10) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension. Now $\Box^k u(x) = \Box(\Box^{k-1} u(x)) = 0$.

From (2.10) we have $\Box^{k-1} u(x) = \delta^{(m)}(r^2 - s^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension.

Convolving the above equation by $R_{2(k-1)}^H(v)$, we obtain

$$\begin{aligned}
R_{2(k-1)}^H(v) * \Box^{k-1} u(x) &= R_{2(k-1)}^H(v) * \delta^{(m)}(r^2 - s^2) \\
\Box^{k-1} [R_{2(k-1)}^H(v)] * u(x) &= [R_{2(k-1)}^H(v)]^{(m)} \quad \text{where } v = r^2 - s^2 \\
\delta * u(x) = u(x) &= [R_{2(k-1)}^H(v)]^{(m)}
\end{aligned}$$

by (2.3) and $v = r^2 - s^2$ is defined by Definition 2.1.

Thus $u(x) = [R_{2(k-1)}^H(v)]^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension.

Lemma 2.4 Given the equation

$$\diamond^k u(x) = 0 \quad (2.11)$$

where \diamond^k is the Diamond operator iterated k -times defined by (1.1) and $u(x)$ is an unknown generalized function. Then

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} \quad (2.12)$$

is a solution of (2.11), $(R_{2(k-1)}^H(v))^{(m)}$ is a function with m -derivatives defined by (2.1) and v is defined by definition 2.1.

Proof Now $\diamond^k u(x) = \square^k \Delta^k u(x) = 0$. By Lemma 2.3, $\Delta^k u(x) = (R_{2(k-1)}^H(v))^{(m)}$.

Convolving both sides by $(-1)^k R_{2k}^e(x)$, we have

$$(-1)^k R_{2k}^e(x) * \Delta^k u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}.$$

By (2.6), $\Delta^k ((-1)^k R_{2k}^e(x)) * u(x) = \delta * u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}$.

It follows that

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}. \quad (2.13)$$

3 Main results

Theorem Given The equation

$$\diamond^k u(x) = f(x) \quad (3.1)$$

where \diamond^k is the Diamond operator iterated k -times defined by (1.1), $f(x)$ is a generalized function, $u(x)$ is an unknown generalized function and $x = (x_1, x_2, \dots, x_n) \in R^n$ -the n -dimensional Euclidean space and n is even, then (3.1) has the general solution

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} + (-1)^k K_{2k,2k}(x) * f(x) \quad (3.2)$$

where $(R_{2(k-1)}^H(v))^{(m)}$ is a function with m -derivatives defined by (2.1) and $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$.

Proof Convolving (3.1) both sides by $(-1)^k K_{2k,2k}(x)$, we obtain

$$(-1)^k K_{2k,2k}(x) * \diamond^k u(x) = (-1)^k K_{2k,2k}(x) * f(x).$$

By (1.4) $\diamond^k ((-1)^k K_{2k,2k}(x)) * u(x) = \delta * u(x) = (-1)^k K_{2k,2k}(x) * f(x)$. We obtain $u(x) = (-1)^k K_{2k}(x) * f(x)$. Since, for a Homogeneous equation $\diamond^k u(x) = 0$ we have a solution $u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}$.

Thus the general solution of (3.1) is

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} + (-1)^k K_{2k,2k}(x) * f(x).$$

In particular, if $q = 0$ the equation (3.1) becomes the Laplace equation $\Delta^{2k}u(x) = f(x)$ where $x = (x_1, x_2, x_3, \dots, x_p) \in R^p$ and p is even. Using the formulae $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + \frac{1}{2})$ and $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi \sec(\pi z)$. Then for $\alpha = 2k$ the function of (2.1) becomes $(-1)^k R_{2k}^e(x)$ where $R_{2k}^e(x)$ is defined by (2.5) and $x = (x_1, x_2, \dots, x_p) \in R^p$. Thus by (1.6)

$$\begin{aligned} (-1)^k K_{2k,2k}(x) &= (-1)^k R_{2k}^e(x) * (-1)^k R_{2k}^e(x) \\ &= R_{4k}^e(x) \text{ see ([2], p156-159)} \end{aligned}$$

where $x = (x_1, x_2, \dots, x_p) \in R^p$ and p is even.

Now, from (2.12) for $q = 0$ we have

$$\begin{aligned} u(x) &= (-1)^k R_{2k}^e(x) * (-1)^{k-1} (R_{2(k-1)}^e(x))^{(m)} \\ &= (-1)^{2k-1} (R_{4k-2}^e(x))^{(m)} \text{ for } x = (x_1, x_2, \dots, x_p) \in R^p. \end{aligned}$$

Thus the equation (3.2) becomes

$$u(x) = (-1)^{2k-1} (R_{4k-2}^e(x))^{(m)} + R_{4k}^e(x) * f(x) \quad (3.3)$$

for $x = (x_1, x_2, \dots, x_p) \in R^p$ and p is even.

It follows that (3.3) is the general solution of the Laplace equation $\Delta^{2k}u(x) = f(x)$ where Δ^{2k} is the Laplace operator iterated $2k$ -times defined by (1.3) for $x = (x_1, x_2, \dots, x_n) \in R^n$ and n is even.

Now consider the case for the Wave Equation. Given the equation

$$\square^k V(x) = f(x) \quad (3.4)$$

where $f(x)$ is a generalized function, \square^k is defined by (1.2) and $V(x)$ is an unknown function. By ([5], p11) we obtain $V(x) = R_{2k}^H(v) * f(x)$ is a solution of (3.4) where $R_{2k}^H(v)$ is defined by (2.1).

Now, from (3.1) we have $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ is a solution where $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$ or

$$u(x) = [(-1)^k R_{2k}^e(x) * R_{2k}^e(v)] * f(x) \quad (3.5)$$

Convolving both sides of (3.5) by $(-1)^k R_{-2k}^e(x)$, we obtain

$$\begin{aligned} (-1)^k R_{-2k}^e(x) * u(x) &= ((-1)^{2k} R_{-2k}^e(x) * R_{2k}^e(x)) * R_{2k}^H(v) * f(x) \\ &= (R_0^e(x) * R_{2k}^H(v)) * f(x) \\ &= (\delta * R_{2k}^H(v)) * f(x) \\ &= R_{2k}^H(v) * f(x) \text{ see ([2], p156-159)} \end{aligned}$$

Thus it follows that

$$V(x) = (-1)^k R_{-2k}^e(x) * u(x) \quad (3.6)$$

In particular, for $k = 1$, we obtain $V(x) = R_2^H(v) * f(x)$ is a solution of the equation

$$\square V(x) = f(x). \quad (3.7)$$

If we put $p = 1$ and $x_1 = t(\text{time})$, then $\square = \frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2}$ is the wave operator. Thus (3.7) becomes wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2} \right) V(x) = f(x) \quad (3.8)$$

Thus $V(x) = M_2(v) * f(x)$ is a solution of (3.8) and the general solution of (3.8) is $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ where $\delta^{(m)}(v)$ is a solution for $f(x) = 0$ $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$ and $M_2(v)$ is defined by (2.4) with $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$.

Now in (3.1), put $k = 1$ and by (3.2) we obtain

$$\begin{aligned} u(x) &= (-1) R_2^e(x) * (R_0^H(v))^{(m)} + (-1) K_{2,2}(x) * f(x) \\ &= (-1) R_2^e(x) * \delta^{(m)}(v) + (-1) K_{2,2}(x) * f(x) \end{aligned} \quad (3.9)$$

is a solution of the equation $\diamond u(x) = f(x)$ and by (3.6) with $k = 1$, $V(x) = (-1) R_{-2}^e(x) * u(x)$ is a solution of (3.7) where $u(x)$ is defined by (3.9). Thus

$$\begin{aligned} V(x) &= (-1) R_{-2}^e(x) * ((-1) R_2^e(x) * \delta^{(m)}(v)) \\ &\quad + (-1) R_{-2}^e(x) * ((-1) K_{2,2}(x) * f(x)) \\ &= (R_{-2}^e(x) * R_2^e(x)) * \delta^{(m)}(v) \\ &\quad + (R_{-2}^e(x) * R_2^e(x)) * R_2^H(x) * f(x) \\ &= R_0^e(x) * \delta^{(m)}(v) + R_0^e(x) * (R_2^H(x) * f(x)) \\ &= \delta^{(m)}(v) + R_2^H(x) * f(x) \quad (R_0^e(x) = \delta) \end{aligned} \quad (3.10)$$

where $v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$, $p + q = n$.

Now, if we put $p = 1$ and $x_1 = t$, then (3.10) becomes $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ since $R_2^H(x)$ becomes $M_2(v)$ for $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$ where $M_2(v)$ is defined by (2.4) with $\alpha = 2$. Thus $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ is the general solution of the wave equation (3.8) and $\delta^{(m)}(v)$ is a solution of

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2} \right) V(x) = 0 \quad (3.11)$$

Now $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$, let $r^2 = x_2^2 + x_3^2 + \dots + x_n^2$. Thus by ([1], p234-236) we obtain $V(x, t) = \delta^{(m)}(t^2 - r^2)$ is the solution of (3.11) with the initial conditions $V(x, 0) = 0$ and $\frac{\partial V(x, 0)}{\partial t} = (-1)^m 2\pi^{m+1} \delta(x)$ at $t = 0$ and $x = (x_2, x_3, \dots, x_n) \in R^{n-1}$.

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บทที่ 3

คอนวอลูชันของตัวดำเนินการ \oplus^k

ในบทนี้เราจะกล่าวถึงสมบัติพื้นฐานเกี่ยวกับ Convolution ของ functions และ convolution ของ Distribution หลังจากนั้นเป็นการให้ทฤษฎีใหม่ๆ เกี่ยวกับ Convolution product $K_{\alpha,\beta} * K_{\alpha',\beta'}$ เมื่อ $K_{\alpha,\beta} = R_\alpha^\ell * R_\beta^\ell$ โดยที่ R_α^ℓ เป็น Elliptic kernel ที่นิยามโดย

$$R_\alpha^\ell = \frac{|x|^{\alpha-n}}{W_n(\alpha)}$$

ทฤษฎีที่สำคัญที่ค้นพบคือ ทฤษฎีเกี่ยวกับ Convolution ของ Diamond kernel of Marcel Riesz ทฤษฎีเกี่ยวกับ Convolution และ Inversion ของ distributional kernel ของ $K_{\alpha,\beta,\gamma,\nu}$ นอกจากนี้ยังได้ทฤษฎีที่เกี่ยวกับตัวดำเนินการโดมอนด์ที่เกี่ยวกับตัวดำเนินการคลื่น

Basic Concepts of Convolution of functions

1. Convolution of Functions
2. Convolution, Derivation and Regularization

(iii) The support of ρ_n is in $[-\varepsilon_n, \varepsilon_n]$, $\varepsilon_n > 0$, and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

Definition 2.2 If $f \in L^1(R)$, the function $f * \rho_n$ are called regularization of f .

The relation between f and its regularizations is shown in next theorem.

Theorem 2.3 Let f be a function in $L^1(R)$. For $\varepsilon > 0$ there exists g_ε in $\mathcal{D}(R)$ such that $\|f - g_\varepsilon\|_1 \leq \varepsilon$.

Theorem 2.4 If $f \in L^1(R)$ and ρ_n is a regularizing sequence, then $\lim_{n \rightarrow \infty} \|f - f * \rho_n\|_1 = 0$.

Since $\mathcal{S}(R)$ is in $L^1(R)$, we know that the convolution of two functions in $\mathcal{S}(R)$ is in $L^1(R)$. However, theorem 2.5 gives a better result.

Theorem 2.5 If f and g are in $\mathcal{S}(R)$, then the following hold:

- (i) $f * g$ is in $\mathcal{S}(R)$.
- (ii) The convolution is a continuous operator from $\mathcal{S}(R) \times \mathcal{S}(R)$ to $\mathcal{S}(R)$.

Convolution of Distribution

1. The Convolution of a Distribution and a C^∞ Function
2. The Convolution $\mathcal{C}' * \mathcal{D}'$
3. The Convolution $\mathcal{C}' * \mathcal{S}'$
4. The Convolution $\mathcal{D}'_+ * \mathcal{D}'_+$

The Associativity of Convolution

Convolution of Distribution

In the first section we discussed the convolution of functions. There we saw that it is not always possible to take the convolution of two functions; it is the same for distributions. We will study the convolution of distributions and its basic properties for the more important cases.

1. The convolution of a distribution and a C^∞ function

When f and g are in $L^1(\mathbb{R})$, the convolution $f * g = \int_{\mathbb{R}} f(x-t)g(t)dt$ is well-defined and $f * g \in L^1(\mathbb{R})$. Now consider $f * g$ as a distribution. For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned} \langle f * g, \varphi \rangle &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-t)g(t)dt \right) \varphi(x)dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-t)\varphi(x)dt \right) g(t)dx \end{aligned} \quad (1.1)$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x-u)\varphi(x)dt \right) f(u)dx \quad (1.2)$$

From this it appears that one should study the quantities $f_\sigma * \varphi$ and $g_\sigma * \varphi$ when f and g are distributions.

Proposition 1.1 Suppose that $\varphi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$ satisfy one of the following three conditions:

- (i) $\varphi \in \mathcal{D}(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$.
- (ii) $\varphi \in \mathcal{S}(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R})$.
- (iii) $\varphi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{C}'(\mathbb{R})$

Then the function ψ defined by

$$\psi(x) = \langle \tau_x T, \varphi \rangle \quad (1.3)$$

is infinitely differentiable, and

$$\psi^{(k)}(x) = \langle \tau_x T, \varphi^{(k)} \rangle \quad \text{for } k = 1, 2, \dots \quad (1.4)$$

Proof.

(i) When $\varphi \in \mathcal{D}(\mathbb{R})$, the expression $\langle \tau_x T, \varphi \rangle$ makes sense for all $x \in \mathbb{R}$. We wish to show that the function $\psi(x) = \langle \tau_x T, \varphi \rangle$ is differentiable. Thus let h_n be a sequence of nonzero reals that tends to 0 as $n \rightarrow \infty$. Define

$$\alpha_n(y) = \frac{1}{h_n} [\varphi(y+x+h_n) - \varphi(y+x)];$$

then

$$\frac{1}{h_n} [\psi(x+h_n) - \psi(x)] = \langle T, \alpha_n \rangle.$$

Now, $\lim_{n \rightarrow \infty} \alpha_n(y) = \tau_{-x} \varphi'(y)$. To prove that ψ is differentiable it is sufficient to show that α_n converges to $\tau_{-x} \varphi'$ in $\mathcal{D}(\mathbb{R})$.

We consider the support of the α_n . If $\text{supp}(\varphi) \subset [-M, M]$ and $|h_n| \leq 1$, then $\text{supp} \subset [-x - M - 1, -x + M + 1]$, which is a fixed compact interval K . The following inequality, which is based on the mean value theorem, show that α_n and of all its derivatives converges uniformly on K : For each $q \in \mathbb{N}$,

$$\begin{aligned} |\alpha_n^{(q)}(y) - \varphi^{(q+1)}(x+y)| &= |\varphi^{(q+1)}(x+y+\theta_n h_n) - \varphi^{(q+1)}(x+y)| \\ &\leq |h_n| \|\varphi^{(q+2)}\|_\infty, \quad 0 < \theta_n < 1. \end{aligned}$$

This prove that ψ is differentiable and that

$$\psi'(x) = \langle T, \tau_{-x} \varphi' \rangle = \langle \tau_x T, \varphi' \rangle.$$

Similary, one proves that $\psi \in C^\infty(\mathbb{R})$ and that equation (1.4) holds for $k > 1$.

(ii) If $\varphi \in \mathcal{S}(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R})$, then $\langle \tau_x T, \varphi \rangle$ makes sense. To show that ψ is differentiable it is sufficient to verify that α_n converges to $\tau_{-x} \varphi'$ in $\mathcal{S}(\mathbb{R})$. As in (i), the mean value theorem leads to the inequality

$$\begin{aligned} |y^p (\alpha_n - \tau_{-x} \varphi')^{(q)}(y)| &\leq |h_n| |y^p \varphi^{(q+2)}(x+y+\rho_n h_n)| \\ &= \frac{|h_n| |y|^p}{1 + |x+y+\rho_n h_n|^p} [(1 + |x+y+\rho_n h_n|^p) \varphi^{(q+2)}(x+y+\rho_n h_n)] \end{aligned}$$

where $0 < \rho_n < 1$. Consequently,

$$\sup_{y \in \mathbb{R}} |y^p (\alpha_n - \tau_{-x} \varphi')^{(q)}(y)| \leq C |h_n| \left(\|\varphi^{(q+2)}\|_\infty + \sup_{t \in \mathbb{R}} |t^p \varphi^{(q+2)}(t)| \right)$$

for some constant C . Since φ is in $\mathcal{S}(\mathbb{R})$, we see that α_n converges in $\mathcal{S}(\mathbb{R})$ to $\tau_{-x} \varphi'$ as $n \rightarrow \infty$. That $\psi \in C^\infty(\mathbb{R})$ and (1.4) are proved similary.

(iii) Again, $\langle \tau_x T, \varphi \rangle$ makes sense because $\varphi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{C}'(\mathbb{R})$. To prove that ψ is differentiable it is sufficient to show that $\alpha_n^{(q)}$ converges to $(\tau_{-x} \varphi')^{(q)}$ uniformly on all compact subsets of \mathbb{R} . This is done using inequalities similar to those used in the proofs of (i) and (ii). \square

Definition 1.2 Assume that $\varphi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{C}'(\mathbb{R})$ satisfy one of the conditions in Proposition 1.1. The convolution of φ and T is the function $\varphi * T$ define by

$$\varphi * T(x) = \langle T_y, \varphi(x-y) \rangle. \quad (1.5)$$

The "y" in this definition indicates the variable of "integration." Written differently,

$$\langle T_y, \varphi(x-y) \rangle = \langle \tau_{-x} T v, \varphi_\sigma \rangle,$$

which is the function we studied in Proposition 1.1. Thus we know the meaning of the convolutions

$$\mathcal{D} * \mathcal{D}', \quad \mathcal{S} * \mathcal{S}', \quad C^\infty * C'.$$

Proposition 1.3 (convolution $\mathcal{S} * \mathcal{S}'$) Assume $\varphi \in \mathcal{S}(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R})$. The convolution $\varphi * T$ and all of its derivatives are slowly increasing C^∞ functions.

Proof. By definition $\varphi * T(x) = \langle T_y, \varphi(x-y) \rangle$. By using a known result, we can write $T = \sum_{k=1}^p f_k^{(n_k)}$, where the continuous slowly increasing functions f_k satisfy $|f_k(x)| \leq C_k(1+x^2)^{N_k}$. Then

$$\begin{aligned}\varphi * T(x) &= \sum_{k=1}^p \langle f_k^{(n_k)}(y), \varphi(x-y) \rangle \\ &= \sum_{k=1}^p \int_{\mathbb{R}} f_k(y) \varphi^{(n_k)}(x-y) dy \\ &= \sum_{k=1}^p \int_{\mathbb{R}} f_k(x-y) \varphi^{(n_k)} dy\end{aligned}$$

and,

$$\begin{aligned}|\varphi * T(x)| &\leq \sum_{k=1}^p C_k \int_{\mathbb{R}} (1+(x-y)^2)^{N_k} |\varphi^{(n_k)}(y)| dy \\ &\leq \sum_{k=1}^p C_k \int_{\mathbb{R}} \sum_{j=1}^{2N_k} |x|^j |\beta_{kj}(y) \varphi^{(n_k)}(y)| dy,\end{aligned}$$

where $\beta_{kj}(y)$ is a polynomial in y .

Since $\varphi \in \mathcal{S}(\mathbb{R})$, $\beta_{kj} \varphi^{(n_k)}$ is in $\mathcal{S}(\mathbb{R})$ and hence in $L^1(\mathbb{R})$. Thus $|\varphi * T(x)|$ is bounded by a polynomial in x , which proves that it is slowly increasing.

One obtain similar estimates for the derivatives of $\varphi * T(x)$ by repeating the computation for $(\varphi * T)^{(k)}(x) = \langle T_y, \varphi^{(k)}(x-y) \rangle$. \square

We are now going state a few of the convolution's essential properties. The convolution defined in Definition 1.2 is an operator that is continuous in each variable. The proof of this result, which we assume, depends on the topologies of the three spaces involved.

Proposition 1.4 (continuity) The mapping $(\varphi, T) \rightarrow \varphi * T$ defined under one of the hypotheses of Proposition 1.1 is continuous with respect to each variable.

Proposition 1.5 (derivative) If $\varphi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$ satisfy one of the hypotheses of Proposition 32.1.1, then $\varphi * T \in C^\infty(\mathbb{R})$ and

$$(\varphi * T)^{(k)} = \varphi^{(k)} * T = \varphi^{(k)} * T^{(k)}, \quad k = 1, 2, \dots \quad (1.6)$$

Proof. From (1.4), the k th derivative of the function

$$H(x) = \langle T_y, \varphi(x - y) \rangle = \langle \tau_{-x}T, \varphi_\sigma \rangle$$

is

$$\begin{aligned} H^{(k)}(x) &= (-1)^k \langle \tau_{-x}T^{(k)}, (\varphi_\sigma) \rangle \\ &= \langle \tau_{-x}T^{(k)}, (\varphi_\sigma)_\sigma^{(k)} \rangle \\ &= \langle T_y, \varphi^{(k)}(x - y) \rangle \\ &= \varphi^{(k)} * T(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi * T^{(k)}(x) &= \langle \tau_{-x}T^{(k)}, \varphi_\sigma \rangle = (-1)^k \langle T, (\varphi_\sigma)^{(k)}(y - x) \rangle \\ &= \langle T, \varphi^{(k)}(x - y) \rangle = \varphi^{(k)} * T(x). \end{aligned}$$

□

Proposition 1.6 (support) *If $\varphi \in C^\infty(\mathbb{R})$ and $T \in \mathcal{C}'(\mathbb{R})$ then $\text{supp}(\varphi * T) \subset \text{supp}(\varphi) + \text{supp}(T)$, where "+" denotes the algebraic sum of the two sets.*

Proof. Since $\text{supp}(T)$ is compact, $\text{supp}(\varphi) + \text{supp}(T)$ is closed. Define $\Omega = \mathbb{R} \setminus (\text{supp}(\varphi) + \text{supp}(T))$. For $x \in \Omega$ and $y \in \text{supp}(\varphi)$, $(x - y) \notin \text{supp}(T)$, and hence $\langle T_y, \varphi(x - y) \rangle = 0$. This proves the required inclusion. □

Corollary 1.7 *If $\varphi \in \mathcal{D}(\mathbb{R})$ and $T \in \mathcal{C}'(\mathbb{R})$ then the convolution $\varphi * T$ has compact support.*

The results of this section show that the convolution of a distribution and a C^∞ function is a smoothing operation. We will see in the next section, where we introduce the convolution of two distributions, that $\mathcal{D}(\mathbb{R})$ is even dense in $\mathcal{D}'(\mathbb{R})$.

2 The convolution $\mathcal{C}' * \mathcal{D}'$

The Expression (1.1) and (1.2) suggest a way to generalize the convolution to distributions. Given two distributions S and T we write

$$\begin{aligned} \langle S * T, \varphi \rangle &= \langle S_t, \langle T_x, \varphi(x + t) \rangle \rangle \\ &= \langle T_u, \langle S_x, \varphi(x + u) \rangle \rangle. \end{aligned}$$

We have seen that the function $\psi(t) = \langle T_x, \varphi(x + t) \rangle$ is C^∞ when $T \in \mathcal{D}'(\mathbb{R})$. Thus the expression $\langle S_t, \psi(t) \rangle$ makes sense when $S \in \mathcal{C}'(\mathbb{R})$. Similarly, the function $\alpha(u) = \langle S_x, \varphi(x + u) \rangle$ is in $\mathcal{D}(\mathbb{R})$ by Proposition 1.6, and the expression $\langle T_u, \alpha(u) \rangle$ makes sense for all $T \in \mathcal{D}'(\mathbb{R})$. It is not clear, however,

that $\langle S_t, \psi(t) \rangle = \langle T_u, \alpha(u) \rangle$, as is the case for functions. This is, in fact, true ; we state without proof the next result.

Theorem 2.1 ($\mathcal{C}' * \mathcal{D}'$) Assume $S \in \mathcal{C}'(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$.

(i) There exists a distribution called the convolution of S and T and denoted by $S * T$ such that for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle S * T, \varphi \rangle = \langle S_t, \langle T_x, \varphi(x+t) \rangle \rangle = \langle T_u, \langle S_x, \varphi(x+u) \rangle \rangle. \quad (1.7)$$

(ii) The mapping $(S, T) \mapsto S * T$ from $\mathcal{C}'(\mathbb{R}) \times \mathcal{D}'(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R})$ is continuous with respect to each variable.

Formular (1.7) is important because it allows us to develop a calculus for the convolution. The convolution is a commutative operation; we next consider its other important properties.

Proposition 2.2 (Dirac distributions) Take $T \in \mathcal{D}'(\mathbb{R})$.

(i) Then

$$\sigma_a * T = T * \delta_a = \tau_a T, \quad (1.8)$$

and in particular, δ acts like a unit element for convolution.

(ii)

$$\delta^{(k)} * T = T * \delta^{(k)} = T^{(k)}, \quad k = 1, 2, \dots \quad (1.9)$$

Proof. One needs to be careful not to confuse the index a in δ_a with the "dummy variables" u and x in (1.7). Both results follow from simple computations. To prove (i) we have

$$\begin{aligned} \langle \delta_a * T, \varphi \rangle &= \langle T_u, \langle \delta_a, \varphi(x+u) \rangle \rangle = \langle T_u, \varphi(a+u) \rangle \\ &= \langle T, \tau_{-a} \varphi \rangle = \langle \tau_a T, \varphi \rangle. \end{aligned}$$

For the proof of (ii) we have

$$\begin{aligned} \langle \delta^{(k)} * T, \varphi \rangle &= \langle T_u, \langle \delta^{(k)}, \varphi(x+u) \rangle \rangle = (-1)^k \langle T_u, \varphi^{(k)}(u) \rangle \\ &= \langle T^{(k)}, \varphi \rangle. \end{aligned} \quad \square$$

Proposition 2.3(derivatives) If $S \in \mathcal{C}'(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$, then

$$(S * T)^{(k)} = S^{(k)} * T = S * T^{(k)}, \quad k = 1, 2, \dots \quad (1.10)$$

Proof. For $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned} \langle (S * T)^{(k)}, \varphi \rangle &= (-1)^k \langle S * T, \varphi^{(k)} \rangle = (-1)^k \langle S_t, \langle T_x, \varphi^{(k)}(x+t) \rangle \rangle \\ &= \langle S_t, \langle T_x^{(k)}, \varphi(x+t) \rangle \rangle = \langle S * T^{(k)}, \varphi \rangle. \end{aligned}$$

Similarly, $(S * T)^{(k)} = S^{(k)} * T$. □

Remark 2.4 If $T \in \mathcal{C}'(\mathbb{R})$, one can find a primitive of T by writing $T * U$ where U is the Heaviside distribution. Indeed, from (1.10) we see that $(T * U)' = T * U' = T * \delta = T$.

Proposition 2.5 (support of a convolution)

- (i) Assume $S \in \mathcal{C}'(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$. If $\text{supp}(S) = A$ and $\text{supp}(T) = B$, then $\text{supp}(S * T) \subset A + B$.
- (ii) If S and T are in $\mathcal{C}'(\mathbb{R})$, then $S * T \in \mathcal{C}'(\mathbb{R})$.

Proof.

(i) Since A is compact, $A + B$ is closed. Let $\Omega = \mathbb{R} \setminus (A + B)$ and take $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\varphi) \subset \Omega$. We will show that $\langle S * T, \varphi \rangle = 0$. We have $\langle S * T, \varphi \rangle = \langle T_u, \langle S_x, \varphi(x + u) \rangle \rangle$ and $\varphi(u) = \langle S_x, \varphi(x + u) \rangle = S * \varphi_\sigma(-u)$. Thus we wish to show that $\text{supp}(\varphi) \cap B = \emptyset$.

If $u \in \text{supp}(\varphi) \cap B$, then $-u \in \text{supp}(S)$ (recall that by Proposition 1.6 $\text{supp}(S * \varphi_\sigma) \subset \text{supp} \varphi_\sigma + \text{supp}(S)$). This means that $-u = y + x$ with $-y \in \text{supp}(\varphi)$ and $x \in \text{supp}(S)$. But then $-y = u + x$ with $u \in B$, $x \in A$, and hence

$$\text{supp}(\varphi) \cap (A + B) \neq \emptyset,$$

which is a contradiction, since $\text{supp}(\varphi) \subset \Omega = \mathbb{R} \setminus (A + B)$.

- (ii) If S and T are in $\mathcal{C}'(\mathbb{R})$, then $A + B$ is compact. □

Proposition 2.6 (density of $\mathcal{D}(\mathbb{R})$ in $\mathcal{D}'(\mathbb{R})$)

If $T \in \mathcal{D}'(\mathbb{R})$, then there exists a regularizing sequence $\theta_n \in \mathcal{D}(\mathbb{R})$ such that θ_n converges to T in $\mathcal{D}'(\mathbb{R})$.

Proof. Choose the usual regularizing sequence ρ_n . We know that ρ_n tends to δ in $\mathcal{D}'(\mathbb{R})$. Let $\alpha_n = \rho_n * T$. By Theorem 2.1 (ii), α_n converges to T in $\mathcal{D}'(\mathbb{R})$. This proves that $\mathcal{C}^\infty(\mathbb{R})$ is dense in $\mathcal{D}'(\mathbb{R})$. To prove that $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{D}'(\mathbb{R})$, we must show that the functions α_n can be chosen with compact support. We fix this by multiplying α_n by a function $\beta_n \in \mathcal{D}(\mathbb{R})$ such that $\beta_n(x) = 1$ for $|x| \leq n$ and $\beta_n(x) = 0$ otherwise. Then $\theta_n = \alpha_n \beta_n \in \mathcal{D}(\mathbb{R})$, and it converges to T in $\mathcal{D}'(\mathbb{R})$. □

3. The convolution $\mathcal{C}' * \mathcal{S}'$

The convolution $\mathcal{C}' * \mathcal{S}'$ is a particular cast of the convolution $\mathcal{C}' * \mathcal{D}'$. But in this case the distribution that one obtains is tempered.

Referring to (1.7) we see that the functions $\psi(t) = \langle T_x, \varphi(x + t) \rangle$ and $\alpha(u) = \langle S_x, \varphi(x + u) \rangle$ and well-defined when $\varphi \in \mathcal{S}(\mathbb{R})$, $T \in \mathcal{S}'(\mathbb{R})$, and $S \in \mathcal{S}'(\mathbb{R})$ (Proposition 1.1). We establish a preliminary result.

Proposition 3.1 If $S \in \mathcal{C}'(\mathbb{R})$, the mapping $\varphi \mapsto \alpha$ defined by $\alpha(u) = \langle S_x, \varphi(x + u) \rangle$ is continuous from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$.

Proof. We use the theorem about the structure of elements in $C'(\mathbb{R})$. Thus we write $S = \sum_{j=1}^p f_j^{(n_j)}$, where the f_j are continuous with support in some compact set K . Then

$$\alpha(u) = \sum_{j=1}^p \langle f_j^{(n_j)}(x), \varphi(x+u) \rangle = \sum_{j=1}^p (-1)^{n_j} \int_K f_j(x) \varphi^{(n_j)}(x+u) dx,$$

and it is not difficult to verify that $\alpha \in \mathcal{S}(\mathbb{R})$.

To prove continuity, we assume that $\varphi_n \rightarrow 0$ in $\mathcal{S}(\mathbb{R})$ and show that the corresponding sequence α_n converges to 0 in $\mathcal{S}(\mathbb{R})$. It is clear that we can differentiate under the integral sign, and consequently

$$\begin{aligned} |u^m \alpha_n^{(q)}(u)| &\leq \sum_{j=1}^p \int_K |f_j(x)| |u^m \varphi_n^{(n_j+q)}(x+u)| dx \\ &\leq \sum_{j=1}^p \int_K |f_j(x)| \frac{|u|^m}{1+|x+u|^m} (1+|x+u|^m) |\varphi_n^{(n_j+q)}(x+u)| dx \\ &\leq C \sum_{j=1}^p \left(\|\varphi_n^{(n_j+q)}\|_\infty + \sup_{t \in \mathbb{R}} |t^m \varphi_n^{(n_j+q)}(t)| \right) \end{aligned}$$

for some constant C . This shows that $\varphi_n \rightarrow 0$ in $\mathcal{S}(\mathbb{R})$ implies that α_n converges to 0 in $\mathcal{S}(\mathbb{R})$.

We use this results to prove the next one.

Proposition 3.2 *If $S \in C'(\mathbb{R})$ and $T \in \mathcal{S}(\mathbb{R})$, then the convolution $S * T$ is a tempered distribution.*

Proof. We know that $S * T$ is a distribution. Let φ_n be a sequence in $\mathcal{D}(\mathbb{R})$ that tends to 0 in $\mathcal{S}(\mathbb{R})$. Then $\langle S * T, \varphi_n \rangle = \langle T_u, \langle S_x, \varphi_n(x+u) \rangle \rangle$, and by Proposition 3.1 the sequence $\alpha_n(u) = \langle S_x, \varphi(x+u) \rangle$ is in $\mathcal{S}(\mathbb{R})$ and converges to 0. Hence $\lim_{n \rightarrow \infty} \langle T, \alpha_n \rangle = 0$. The result follows from Proposition 1.3.

The next step is to examine continuity.

Proposition 3.3

(i) *Let S_n be a sequence in $C'(\mathbb{R})$ that converges to 0 in $C'(\mathbb{R})$; that is, $\langle S_n, \varphi \rangle \rightarrow 0$ for all $\varphi \in C^\infty(\mathbb{R})$. Then $S_n * T \rightarrow 0$ in $\mathcal{S}'(\mathbb{R})$, and hence in $\mathcal{D}'(\mathbb{R})$, for all $T \in \mathcal{S}'(\mathbb{R})$.*

(ii) *Let T_n be a sequence in $\mathcal{S}'(\mathbb{R})$ that converges to 0 in \mathcal{S}' . Then for all $S \in C'(\mathbb{R})$, $S * T_n \rightarrow 0$ in $\mathcal{S}'(\mathbb{R})$, and hence in $\mathcal{D}'(\mathbb{R})$.*

Proof.

(i) By Proposition 1.1, the function $\psi(t) = \langle T_x, \varphi(x+t) \rangle$ is in $C^\infty(\mathbb{R})$ for all φ is $\mathcal{S}(\mathbb{R})$. Thus

$$\langle S_n * T, \varphi \rangle = \langle S_n, \psi \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(ii) Similarly, $\alpha(u) = \langle S_x, \varphi(x+u) \rangle$ is in $\mathcal{S}(\mathbb{R})$ for all φ is $\mathcal{S}(\mathbb{R})$ (Proposition 3.1). Hence $\lim_{n \rightarrow \infty} \langle S * T_n, \varphi \rangle = \lim_{n \rightarrow \infty} \langle T_n, \varphi \rangle = 0$. \square

It is necessary to pay attention to the various notions of convergence. Here is a simple example;

$$\delta_n * 1 = 1$$

for all n , and δ_n converges to 0 in $\mathcal{D}'(\mathbb{R})$. In this case the result of Proposition 3.3 is not true. This is because δ_n does not converge in $\mathcal{C}'(\mathbb{R})$.

Proposition 3.4

(i) Let S_n be a sequence in $\mathcal{C}'(\mathbb{R})$ that converges to 0 in $\mathcal{D}'(\mathbb{R})$. Assume that there exists a compact set K such that $\text{supp}(S_n) \subset K$ for all n . Then $S_n * T \rightarrow 0$ in $\mathcal{D}'(\mathbb{R})$ for all $T \in \mathcal{S}'(\mathbb{R})$.

(ii) Let T_n a sequence in $\mathcal{S}'(\mathbb{R})$ that converges to 0 in $\mathcal{D}'(\mathbb{R})$. Then for all $S \in \mathcal{C}'(\mathbb{R})$, $S * T_n \rightarrow 0$ in $\mathcal{D}'(\mathbb{R})$.

Proof.

(i) Take $\varphi \in \mathcal{D}(\mathbb{R})$. The function $\psi(t) = \langle T_x, \varphi(x+t) \rangle$ is in $C^\infty(\mathbb{R})$. Since $\text{supp}(S_n) \subset K$, $\langle S_n, \psi \rangle = \langle S_n, \theta\psi \rangle$, where θ is a function in $\mathcal{D}(\mathbb{R})$ such that $\theta(x) = 1$ for $x \in K$. Then $\theta\psi$ is in $\mathcal{D}(\mathbb{R})$ and $\langle S_n, \theta\psi \rangle \rightarrow 0$.

(ii) If $\varphi \in \mathcal{D}(\mathbb{R})$, then the function $\alpha(u) = \langle S_x, \varphi(x+u) \rangle$ is in $\mathcal{D}(\mathbb{R})$, and hence $\langle T_n, \alpha \rangle \rightarrow 0$.

4 The convolution $\mathcal{D}'_+ * \mathcal{D}'_+$

We have studied the convolution of two distributions where at least one of them has compact support. Without this condition on the support, the convolution is not generally defined. However, as is the case for functions, the convolution is defined when both distributions are in \mathcal{D}'_+ (or \mathcal{D}'_-).

We begin with a preliminary result about \mathcal{D}'_+ .

Proposition 4.1 Suppose that $T \in \mathcal{D}'_+$ and $\varphi \in C^\infty(\mathbb{R})$ and that $\text{supp}(T) \subset [a, +\infty)$ and $\text{supp}(\varphi) \subset (-\infty, b]$. Then $\langle T, \varphi \rangle$, defined by

$$\langle T, \varphi \rangle = \langle T, \theta\varphi \rangle, \quad (1.11)$$

where θ is a function in $\mathcal{D}(\mathbb{R})$ equal to 1 on an interval $[-M, M]$ containing a and b in its interior, is well-defined.

Proof. $\theta\varphi \in \mathcal{D}(\mathbb{R})$, so $\langle T, \theta\varphi \rangle$ makes sense. We must show that the definition of $\langle T, \varphi \rangle$ does not depend on the choice of θ . Let θ_1 be another function in $\mathcal{D}(\mathbb{R})$ equal to 1 on $[-M_1, M_1]$ containing a and b . Then $(\theta - \theta_1)\varphi$ vanishes on $[-m, +\infty)$, where $m = \min\{M, M_1\}$. Since $\text{supp}(T) \subset [a, +\infty)$, we have $\text{supp}(T) \cap \text{supp}((\theta - \theta_1)\varphi) = \emptyset$ and $\langle T, (\theta - \theta_1)\varphi \rangle = 0$. \square

To define the convolution, it is necessary to give meaning to the expressions $\langle S_t, \langle T_x, \varphi(x+t) \rangle \rangle$ and $\langle T_u, \langle S_x, \varphi(x+u) \rangle \rangle$ for S and T in \mathcal{D}'_+ and φ in $\mathcal{D}(\mathbb{R})$.

Proposition 4.2 Suppose that $T \in \mathcal{D}'_+$ and $\varphi \in C^\infty(\mathbb{R})$ and that $\text{supp}(T) \subset [a, +\infty)$ and $\text{supp}(\varphi) \subset (-\infty, b]$. Then $\psi(t) = \langle T_x, \varphi(x+t) \rangle$ is defined of all $t \in \mathbb{R}$, $\text{supp}(\psi) \subset (-\infty, b-a]$, and $\psi \in C^\infty(\mathbb{R})$.

Proof. The function $\tau_{-t}\varphi$ is in $C^\infty(\mathbb{R})$ with $\text{supp}(\tau_{-t}\varphi) \subset (-\infty, b-t]$ for all $t \in \mathbb{R}$. Thus by Proposition 4.1, $\langle T_x, \varphi(x+t) \rangle$ is well-defined.

Now, $\psi(t) = 0$ if $\text{supp}(\tau_{-t}\varphi) \cap \text{supp}(T) = \emptyset$, which is the case when $b-t < a$. Hence, $\text{supp}(\psi) \subset (-\infty, b-a]$. That $\psi \in C^\infty$ is a consequence of (1.11) and Proposition 1.1(i). \square

These preliminary results lead to the next theorem, which we state without proof.

Theorem 4.3 (the convolution $\mathcal{D}'_+ * \mathcal{D}'_+$) Suppose that S and T are in \mathcal{D}'_+ .

(i) There exists a distribution called the convolution of S and T and denoted by $S * T$ such that for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle S * T, \varphi \rangle = \langle S_t, \langle T_x, \varphi(x+t) \rangle \rangle = \langle T_u, \langle S_x, \varphi(x+u) \rangle \rangle. \quad (1.12)$$

(ii) $(S * T)^{(k)} = S^{(k)} * T = S * T^{(k)}$, $k = 1, 2, 3, \dots$

(iii) The mapping $(S, T) \mapsto S * T$ of $\mathcal{D}'_+ \times \mathcal{D}'_+$ into \mathcal{D}' is continuous with respect to each variable. (The convergence of S_n to 0 in \mathcal{D}'_+ means that $S_n \rightarrow 0$ in $\mathcal{D}'(\mathbb{R})$ and that there exists a c such that for all n , $\text{supp}(S_n) \subset [c, +\infty)$.)

Proposition 4.4 (support of $S * T$) If S and $T \in \mathcal{D}'_+$ with $\text{supp}(S) \subset [a_1, +\infty)$ and $\text{supp}(T) \subset [a_2, +\infty)$, then

$$\text{supp}(S * T) \subset [a_1 + a_2, +\infty),$$

and hence $S * T$ is in \mathcal{D}'_+ .

Proof. Take $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\varphi) \subset (-\infty, a_1 + a_2)$. The support of $\varphi(t) = \langle T_x, \varphi(x+t) \rangle$ is in $(-\infty, a_1)$ by Proposition 4.2. Thus

$$\text{supp}(\psi) \cap \text{supp}(S) = \emptyset$$

and $\langle S * T, \varphi \rangle = 0$, which proves that $\text{supp}(S * T) \subset [a_1 + a_2, \infty)$. \square

Remark 4.5 As in Section 2, we can obtain a primitive of T in \mathcal{D}'_+ by taking the convolution of T with the Heaviside distribution U . Since $U \in \mathcal{D}'_+$, the primitive of T is in \mathcal{D}'_+ .

5. The associativity of convolution

We have defined the convolution of two distributions and seen that it is a commutative operation. If we wish to convolve three or more distributions, we run into two problems: existence and associativity. Here

is a classic example: We wish to compute $\mathbf{1} * \delta' * u$. From Proposition 2.3,

$$\mathbf{1} * \delta' = \mathbf{1}' * \delta = 0$$

Thus $(\mathbf{1} * \delta') * u$ makes sense and is equal to 0. On the other hand, by Proposition 2.2,

$$\delta' * u = \delta * u' = \delta * \delta = \delta;$$

hence $\mathbf{1} * (\delta' * u)$ make sense and is equal to δ . This shows that the convolution product is not associative in general. Nevertheless, convolution is associative in several cases.

Proposition 5.1 *The convolution of n distributions of which at least $n - 1$ have compact support is associative and commutative.*

Proof. The proof is based directly on the definitions. We take S and T in $\mathcal{C}'(\mathbb{R})$ and U in $\mathcal{D}'(\mathbb{R})$; we show, for example, that $(S * T) * U = S * (T * U)$. First, these convolutions make sense, since $S * T \in \mathcal{C}'(\mathbb{R})$ (Proposition 32.2.5(ii)) and $(S * T) * U$ is an $\mathcal{C}' * \mathcal{D}'$ convolution. Similarly, $T * U$ is as $\mathcal{C}' * \mathcal{D}'$ convolution, so $T * U \in \mathcal{D}'$; and $S * (T * U)$ is another $\mathcal{C}' * \mathcal{D}'$ convolution. Take $\varphi \in \mathcal{D}(\mathbb{R})$. From equation (1.7),

$$\langle (S * T) * U, \varphi \rangle = \langle (S * T)_t, \langle U_x, \varphi(x + t) \rangle \rangle$$

Now, $\varphi(t) = \langle U_x, \varphi(x + t) \rangle$ is in C^∞ , and $S * T \in \mathcal{C}'(\mathbb{R})$ then

$$\begin{aligned} \langle S * T, \varphi \rangle &= \langle S_z, \langle T_u, \varphi(u + z) \rangle \rangle \\ &= \langle S_z, \langle T_u, \langle U_x, \varphi(x + u + z) \rangle \rangle \rangle \\ &= \langle S_z, \langle (T * U)_t, \varphi(t + z) \rangle \rangle \\ &= \langle S * (T * U), \varphi \rangle, \end{aligned}$$

from which we see that $(S * T) * U = S * (T * U)$. □

The space $\mathcal{C}'(\mathbb{R})$ endowed with the convolution operation is a *convolution algebra*, since $T, S \in \mathcal{C}'(\mathbb{R})$ implies that $T * S \in \mathcal{C}'(\mathbb{R})$. This algebra is commutative and associative, and it contains a unit element, *delta*. $\mathcal{D}'_+(\mathbb{R})$ is another important convolution algebra.

Proposition 5.2 *The convolution in $\mathcal{D}'_+(\mathbb{R})$ is assosicative.*

The proof is similar to that of Proposition 32.5.1, and δ is also a unit element for this algebra.

Since these two distribution algebras have unit elements, it is natural to ask whether a distribution S in $\mathcal{C}'(\mathbb{R})$ or $\mathcal{D}'_+(\mathbb{R})$ has an inverse; that is, is there a T such that

$$S * T = \delta$$

This question is important for solving differential equations with constant coefficients. With this in mind, we introduce the differential operator

$$P = \sum_{m=0}^p a_m D^{(m)},$$

where $a_m \in C$ and $D^{(m)}$ denotes the m th derivative. Given $U \in \mathcal{D}'(\mathcal{R})$, we wish to find a $T \in \mathcal{D}'(\mathcal{R})$ such that

$$P(T) = U. \quad (1.13)$$

We saw in Proposition 2.2(ii) that $T^{(k)} = \delta^{(k)} * T$, and hence we can write (1.13) as

$$\sum_{m=0}^p a_m \delta^{(m)} * T = U. \quad (1.14)$$

A distribution E is said to be an *elementary solution* of (1.14) if

$$\left(\sum_{m=0}^p a_m \delta^{(m)} \right) * E = \delta$$

This means that E is the inverse of $S = \sum_{m=0}^p a_m \delta^{(m)}$. If we have associativity, then knowing E yields all the solutions of (1.13), since

$$S * E * U = (S * E) * U = \delta * U = U,$$

and hence $T = E * U$. Furthermore, this solution is unique: If T_1 and T_2 satisfy (1.13), then we have

$$S * T_1 = S * T_2 = U$$

By taking the convolution with E this becomes, thanks to associativity,

$$(E * S) * T_1 = (E * S) * T_2,$$

and since $E * S = \delta$, we are left with

$$T_1 = T_2$$



On the convolutions of the diamond kernel of Marcel Riesz

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Abstract

In this paper, we consider the equation $\diamond^k u(x) = \delta$ where \diamond^k is introduced and named as the diamond operator iterated k -times and is defined by $\diamond^k = ((\sum_{i=1}^p (\partial^2/\partial x_i^2))^2 - (\sum_{j=p+1}^{p+q} (\partial^2/\partial x_j^2))^2)^k$, $u(x)$ is a generalized function, $x = (x_1, x_2, \dots, x_n) \in R^n$ the n -dimensional Euclidean space, $p + q = n$, $k = 0, 1, 2, 3, \dots$ and δ is the Dirac-delta distribution. Now $u(x)$ is the elementary solution of the operator \diamond^k and is called the diamond kernel of Marcel Riesz. The main part of this work is studying the convolution of $u(x)$. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Diamond kernel; Ultra-hyperbolic kernel; Elliptic kernel; Tempered distribution

1. Introduction

Consider the equation

$$\diamond^k u(x) = \delta, \quad (1.1)$$

where \diamond^k is the Diamond operator iterated k -times defined by

$$\diamond^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \dots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right)^k. \quad (1.2)$$

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Kananthai [2, Theorem 3.1] has proved that the convolution solution $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the elementary solution of (1.1) where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.2) and (2.4), respectively with $\alpha = 2k$. Now $u(x)$ is called the diamond kernel of Marcel Riesz and defines such a kernel by

$$T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x), \quad m = 0, 1, 2, \dots \quad (1.3)$$

In this work we study the existence of $T_m(x) * T_n(x)$ and moreover the inverse T_m^{*-1} of $T_m(x)$ in the convolution algebra \mathcal{A}' is also considered.

2. Preliminaries

Definition 2.1. Let $E(x)$ be a function defined by

$$E(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}, \quad (2.1)$$

where $x = (x_1, x_2, \dots, x_n) \in R^n$, $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$ and $\omega_n = (2\pi^{n/2})/(\Gamma(n/2))$ is a surface area of the unit sphere.

It is well known that $E(x)$ is an elementary solution of the Laplace operator Δ , that $\Delta E(x) = \delta$ where $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$ and δ is the Dirac-delta distribution.

Definition 2.2. Let $S_\alpha(x)$ be a function defined by

$$S_\alpha(x) = 2^{-\alpha} \pi^{-n/2} \Gamma\left(\frac{n-\alpha}{2}\right) \frac{|x|^{\alpha-n}}{\Gamma(\alpha/2)}, \quad (2.2)$$

where α is a complex parameter, $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$, $x = (x_1, x_2, \dots, x_n) \in R^n$. $S_\alpha(x)$ is called the elliptic kernel of Marcel Riesz. Now $S_\alpha(x)$ is an ordinary function for $\operatorname{Re}(\alpha) \geq n$ and is a distribution of α for $\operatorname{Re}(\alpha) < n$.

From (2.1) and (2.2) we obtain

$$E(x) = -S_2(x) \quad (2.3)$$

and it can be shown that

$$\underbrace{E(x) * E(x) * \dots * E(x)}_{k\text{-times}} = (-1)^k S_{2k}(x)$$

is the elementary solution of the operator Δ^k iterated k -times that is $\Delta^k (-1)^k S_{2k}(x) = \delta$, see [3].

Definition 2.3. Let $x = (x_1, x_2, \dots, x_n)$ be a point of R^n and write

$$V = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p+q = n.$$

$\Gamma_+ = \{x \in \mathbb{R}^n: x_1 > 0 \text{ and } V > 0\}$ designates the interior of the forward cone and denotes $\bar{\Gamma}_+$ by its closure and the following function introduced by Nozaki [4, p. 72] that

$$R_z(x) = \begin{cases} V^{(z-n)/2}/K_n(\alpha) & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+. \end{cases} \quad (2.4)$$

Here, $R_z(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and α is a complex parameter and n is the dimension of the space \mathbb{R}^n .

The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2+\alpha-n)/2) \Gamma((1-\alpha)/2) \Gamma(z)}{\Gamma((2+\alpha-p)/2) \Gamma((p-\alpha)/2)}.$$

Now $R_z(x)$ is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and is a distribution of α if $\operatorname{Re}(\alpha) < n$.

Let $\operatorname{supp} R_z(x) \subset \bar{\Gamma}_+$ where $\operatorname{supp} R_z(x)$ denotes the support of $R_z(x)$.

Lemma 2.1. *The functions $S_z(n)$ and $R_z(x)$ defined by (2.2) and (2.4) respectively, for $\operatorname{Re}(z) < n$ are Homogeneous distribution of order $\alpha - n$ and also a tempered distribution.*

Proof. Since $R_z(x)$ and $S_z(x)$ satisfy the Euler equation, that is

$$(z-n)R_z(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} S_z(x) \quad \text{and} \quad (z-n)S_z(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_z(x).$$

we have $R_z(x)$ and $S_z(x)$ are homogeneous distributions of order $z-n$ and Donoghue [1, pp. 154–155] has proved that every homogeneous distribution is a tempered distribution. That completes the proof. \square

Lemma 2.2 (The convolution of tempered distributions). *The convolution $S_z(x) * R_z(x)$ exists and is a tempered distribution.*

Proof. Choose $\operatorname{supp} R_z(x) = K \subset \bar{\Gamma}_+$ where K is a compact set. Then $R_z(x)$ is a tempered distribution with compact support and by Donoghue [1, pp. 156–159], $S_z(x) * R_z(x)$ exists and is a tempered distribution. \square

Lemma 2.3. *Given the equation $\diamond^k u(x) = \delta$ where \diamond^k is the operator defined by (1.2), $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer and δ is the Dirac-delta distribution. Then $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the equation where $S_z(x)$ and $R_z(x)$ are defined by (2.2) and (2.4), respectively with $z = 2k$. Moreover $u(x)$ is a tempered distribution.*

Proof. See [3, Theorem 3.1] and by Lemma 2.2, $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a tempered distribution. \square

Lemma 2.4 (The convolutions of $R_\alpha(x)$ and $S_\alpha(x)$). *Let $S_\alpha(x)$ and $R_\alpha(x)$ be defined by (2.2) and (2.4) respectively, then we obtain the following formulas:*

1. $S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x)$, where α and β are complex parameters.
2. $R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x)$, for α and β are both integers and except only the case both α and β are odd integers.

Proof. Proof of first formula, see [1, p. 158].

Proof of second formula, for the case α and β are both even integers, see [3] and for the case α is odd and β is even or α is even and β is odd, we know from Trione [5] that

$$\square^k R_\alpha(x) = R_{\alpha-2k}(x) \quad (2.5)$$

and

$$\square^k R_{2k} = \delta, \quad k = 0, 1, 2, \dots \quad (2.6)$$

where \square^k is an ultra-hyperbolic operator iterated k -times ($k = 0, 1, 2, \dots$) defined by

$$\square^k = \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k.$$

Now let m be an odd integer, we have

$$\square^k R_m(x) = R_{m-2k}(x)$$

and

$$R_{2k}(x) * \square^k R_m(x) = R_{2k}(x) * R_{m-2k}(x)$$

or

$$(\square^k R_{2k}(x)) * R_m(x) = R_{2k}(x) * R_{m-2k}(x).$$

$$\delta * R_m(x) = R_{2k}(x) * R_{m-2k}(x) \quad \text{by (2.6).}$$

or

$$R_m(x) = R_{2k}(x) * R_{m-2k}(x).$$

Since m is odd, hence $m - 2k$ is odd and $2k$ is a positive even. Put $\alpha = 2k$, $\beta = m - 2k$ we obtain

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x)$$

for α is a nonnegative even and β is odd.

For the case α is a negative even and β is odd, by (2.5) we have

$$\square^k R_0(x) = R_{-2k}(x) \quad \text{or} \quad \square^k \delta = R_{-2k}(x)$$

where $R_0(x) = \delta$. Now

$$R_{-2k}(x) * \square^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x) \quad \text{for } m \text{ is odd,}$$

or

$$(\square^k \delta) * \square^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$\delta * \square^{2k} R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$R_{m-2(2k)}(x) = R_{-2k}(x) * R_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m - 2k$, now α is a negative even and β is odd. Then we obtain

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x).$$

That completes the proofs. \square

3. Main results

Theorem 3.1. Let $T_m(x)$ the diamond kernel of Marcel Riesz defined by (1.3), then T_m is a tempered distribution and can be expressed by

$$T_m(x) = T_{m-r}(x) * T_r(x),$$

where r is a nonnegative integer and $r < m$. Moreover if we put $\ell = m - r$, $n = r$ we obtain

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) \quad \text{for } \ell + n = m.$$

Proof. Since $T_m = (-1)^m S_{2m}(x) * R_{2m}(x)$, ($m = 0, 1, 2, \dots$), by Lemma 2.2 T_m is a tempered distribution. Now by Lemma 2.3, $\diamond^m T_m = \delta$, then $\diamond^r \diamond^{m-r} T_m = \delta$ for $m > r$ and by Lemma 2.3 again, we obtain $\diamond^{m-r} T_m = (-1)^r S_{2r}(x) * R_{2r}(x)$. Convoluting both sides by $(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)$, we obtain

$$\begin{aligned} & [(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * \diamond^{m-r} T_m \\ &= [(-1)^{m-r} S_{2(m-2r)}(x) * R_{2(m-2r)}(x)] * [(-1)^r S_{2r}(x) * R_{2r}(x)] \end{aligned} \quad (3.1)$$

or

$$\begin{aligned} & \diamond^{m-r} [(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * T_m \\ &= (-1)^m (S_{2(m-2r)}(x) * S_{2r}(x) * (R_{2(m-2r)}(x) * R_{2r}(x))). \end{aligned}$$

since $S_{2m}(x)$ and $R_{2m}(x)$ are tempered distributions and are the elements of the space of convolution algebra, \mathcal{A}' .

By Lemmas 2.3 and 2.4 we obtain

$$\begin{aligned}\delta * T_m(x) &= (-1)^m S_{2m}(x) * R_{2m}(x), \\ T_m(x) &= (-1)^m S_{2m}(x) * R_{2m}(x).\end{aligned}$$

From (3.1) we have $T_m(x) = T_{m-r}(x) * T_r(x)$, put $\ell = m - r$, $n = r$, it follows that

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) = T_m(x)$$

as required. \square

Theorem 3.2. *Let $T_m(x)$ be defined by (1.3) then T_m is an element of the space \mathcal{A}' of convolution algebra and there exist an inverse T_m^{*-1} of T_m such that*

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

Proof. Since $T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x)$ is a tempered distribution by Lemma 2.2. Now the supports of $S_{2m}(x)$ and $R_{2m}(x)$ are compact. Then they are the elements of the space of convolution algebra \mathcal{A}' of distribution. By Zemanian [6, Theorem 6.2.1, p. 151] there exist a unique inverse T_m^{*-1} such that

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

That completes the proof. \square

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The convolution product of the distributional families related to the Diamond operator

1. Introduction

A. Kananthai [4] has first introduce the Diamond operator \diamond^k iterated k -times which is defined by

$$\diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \quad (1.1)$$

where $p+q=n$ is the dimension of the n -dimensional Euclidean space \mathbb{R}^n and k is a nonegative integer. Actually (1.1) can be rewrite in the following form

$$\diamond^k = \square^k \Delta^k = \Delta^k \square^k \quad (1.2)$$

where the operator \square^k and Δ^k are defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (1.3)$$

and,

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (1.4)$$

In this paper, the family $K_{\alpha,\beta}$ is defined by $K_{\alpha,\beta}(x) = R_{\alpha}^e * R_{\beta}^H$ where R_{α}^e is the elliptic kernel defined by (2.1) and R_{β}^H is the hyperbolic kernel defined by (2.4) and the symbol $*$ denotes the convolution and $x \in \mathbb{R}^n$. In this paper, we study the convolution $K_{\alpha,\beta} * K_{\alpha',\beta'}$ where α, β, α' and β' are complex parameters.

2. Preliminaries

Definition 2.1 Let the function $R_{\alpha}^e(x)$ be defined by

$$R_{\alpha}^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \quad (2.1)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, α is a complex parameter, n is the dimension of \mathbb{R}^n and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$ and $W_n(\alpha)$ is defined by the formula $W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^{\alpha} \Gamma(\frac{n}{2})}{\Gamma(\frac{n-\alpha}{2})}$. The function $R_{\alpha}^e(x)$ is precisely the definition of elliptic kernel of Marcel Rieze [2] and the following formula is valid

$$R_{\alpha}^e(x) * R_{\beta}^e(x) = R_{\alpha+\beta}^e(x) \quad (2.2)$$

which hold for $\alpha > 0, \beta > 0$ and $\alpha + \beta \leq n$ see ([2], p.20)

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$u = u(x) = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, p + q = n \quad (2.3)$$

Denote by Γ_+ the interior of the forward cone defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 < 0 \text{ and } u > 0\}$ and $\overline{\Gamma}_+$ denote its closure. Similarly, define $\Gamma_- = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma}_-$ denote its closure.

For any complex number α , define

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.4)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \Pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{p-\alpha}{2})} \quad (2.5)$$

The function R_α^H was introduced by Y. Nozaki ([3], p.72) $R_\alpha(u)$, which is an ordinary function if $\operatorname{Re}(\alpha) \geq n$, is a distribution of α . We shall call R_α^H the Marcel Riesz's ultra-hyperbolic kernel. By putting $p = 1$ in (2.4) and (2.5) and using the Legendre's duplication formula of $\Gamma(z)$.

$$\Gamma(2z) = 2^{2z-1} \Pi^{\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2}) \quad (2.6)$$

see ([5], Vol 1. p.5) the formula (2.4) reduces to

$$M_\alpha = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.7)$$

where $u = u(x) = x_1^2 - x_2^2 - \dots - x_n^2$ and

$$H_n(\alpha) = 2^{\alpha-1} \Pi^{\frac{n-2}{2}} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{\alpha-n+2}{2}) \quad (2.8)$$

M_α is called the hyperbolic kernel of Marcel Riesz ([2], p.31).

Lemma 2.1 The function $R_\alpha^e(x)$ has the following properties

- (i) $R_0^e = \delta(x)$
- (ii) $R_{-2k}^e(x) = (-1)^k \Delta^k \delta(x)$
- (iii) $\Delta^k R_\alpha^e(x) = (-1)^k R_{\alpha-2k}^e(x)$, where Δ^k is the Laplace operator iterated k -times defined by (1.3).

The proof Lemma is given by S.E Trione [5].

Lemma 2.2

- (i) $R_\alpha^H * R_\beta^H = \frac{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}}{\cos(\frac{\alpha+\beta}{2}) \Pi} R_{\alpha+\beta}^H$ where R_α^H is defined by (2.4) and (2.5) with p is an even.
- (ii) $R_\alpha^H * R_\beta^H = R_{\alpha+\beta}^H + T_{\alpha,\beta}$ for p is an odd where

$$\begin{aligned}
T_{\alpha,\beta} &= T_{\alpha,\beta}(u \pm i 0, n) \\
&= \frac{\frac{2\pi i}{4} C(-\frac{\alpha-\beta}{2})}{C(-\frac{\alpha}{2})C(-\frac{\beta}{2})} [H_{\alpha+\beta}^+ - H_{\alpha+\beta}^-]
\end{aligned} \tag{2.9}$$

$$C(r) = \Gamma(r)\Gamma(1-r)$$

$$H_r^\pm = H_r(u \pm i 0, n) = e^{\mp r \frac{\pi}{2} i} e^{\pm q \frac{\pi}{2} i} a(\frac{r}{2})(u \pm i 0)^{\frac{r-n}{2}}$$

$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2}) [2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$$

$$(u \pm i 0)^\lambda = \lim_{\epsilon \rightarrow 0} (u + i\epsilon |x|^2)^\lambda$$

see([6], p.275)

$$u = u(x) \text{ is defined by (2.3) and } |x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$$

In particular $R_\alpha^H * R_{-2k}^H = R_{\alpha-2k}^H$, $R_\alpha^H * R_{2k}^H = R_{\alpha+2k}^H$

Proof see([1], p.121-123)

Lemma 2.3

- (i) $R_{-2k}^H = \square^k \delta$
- (ii) $\square^k R_\alpha^H = R_{\alpha-2k}^H$
- (iii) $\square^k R_{2k}^H = R_0^H = \delta$ where \square^k is defined by (1.2)

Proof see ([1], p.123)

3. The family of distribution $K_{\alpha,\beta}(x)$

Let $K_{\alpha,\beta}(x)$ be a distributional family defined by

$$K_{\alpha,\beta}(x) = R_\alpha^e * R_\beta^H \tag{2.10}$$

where the function R_α^e and R_β^H are defined by (2.1) and (2.4) respectively. The right hand side of (2.10) is well-defined and also a tempered distribution see ([4], Lemma 2.3, Lemma 2.4)

Lemma 3.1 *The following formulas are valid*

- (i) $K_{0,0}(x) = \delta(x)$
- (ii) $K_{-2k,-2k}(x) = (-1)^k \diamond^k \delta(x)$
- (iii) $\diamond^k (K_{\alpha,\beta}(x)) = (-1)^k K_{\alpha-2k,\beta-2k}(x)$

$$(iv) \diamond^k(K_{2k,2k}(x)) = (-1)^k \delta(x)$$

Proof (i) By (2.10), $K_{0,0}(x) = R_0^e * R_0^H$. By Lemma 2.1(i) and Lemma 2.3(i), we obtain

$$K_{0,0}(x) = \delta * \delta = \delta.$$

(ii) We have

$$\begin{aligned} \diamond^k(K_{\alpha,\beta}(x)) &= \diamond^k(R_\alpha^e * R_\beta^H) \\ &= \square^k \triangle^k(R_\alpha^e * R_\beta^H) \\ &= \triangle^k R_\alpha^e * \square^k R_\beta^H \\ &= (-1)^k R_{\alpha-2k}^e * R_{\beta-2k}^H \\ &= (-1)^k K_{\alpha-2k,\beta-2k}(x) \end{aligned}$$

by Lemma 2.1(iii) and Lemma 2.3(ii). Putting $\alpha = \beta = 0$ and (i) we obtain

$$K_{-2k,-2k}(x) = (-1)^k \diamond^k \delta.$$

(iii) Similarly as (ii).

(iv) Putting $\alpha = \beta = 2k$ in (iii), we obtain

$$\diamond^k(K_{-2k,-2k}(x)) = (-1)^k K_{0,0}(x) = (-1)^k \delta(x)$$

4. Main results

Theorem Let the family $K_{\alpha,\beta}(x)$ and $K_{\alpha',\beta'}(x)$ be defined by (2.10) then the convolution product $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x)$ can be obtained by the following formula

- (i) $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = B_{\beta,\beta'} R_{\beta+\beta'}^H * R_{\alpha+\alpha'}^e$ where R_β^H and R_α^e are defined by (2.4) and (2.1) respectively which p is an even and $B_{\beta,\beta'} = \frac{\cos \beta \frac{p}{2} \cos \beta' \frac{p}{2}}{\cos(\frac{\beta+\beta'}{2})\Pi}$.
- (ii) $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'} + T_{\beta,\beta'}) * R_{\alpha+\alpha'}$ if p is an odd and $T_{\beta,\beta'}$ is defined by (2.9)
- (iii) $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \diamond^k K_{\alpha,\beta}(x)$

Proof (i) we have

$$\begin{aligned} K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) &= (R_\alpha^e * R_\beta^H) * (R_{\alpha'}^e * R_{\beta'}^H) \\ &= R_{\alpha+\alpha'}^e * (R_\beta^H * R_{\beta'}^H) \text{ by (2.2)} \\ &= B_{\beta,\beta'} R_{\beta+\beta'}^H * R_{\alpha+\alpha'}^e \text{ by Lemma 2.2(i)} \end{aligned}$$

for p is even and $B_{\beta,\beta'} = \frac{\cos \beta \frac{p}{2} \cos \beta' \frac{p}{2}}{\cos(\frac{\beta+\beta'}{2})\Pi}$.

(ii) From (i),

$$\begin{aligned} K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) &= (R_{\beta}^H * R_{\beta'}^H) * R_{\alpha+\alpha'}^e \\ &= (R_{\beta+\beta'}^H + T_{\beta,\beta'}) * R_{\alpha+\alpha'}^e. \end{aligned}$$

by Lemma 2.2(ii) for p is odd and $T_{\beta,\beta'}$ is defined by (2.9)

(iii) We have $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = B_{\beta,-2k} R_{\beta-2k}^H * R_{\alpha-2k}^e$ for p is even.

Since $B_{\beta,-2k} = \frac{\cos \beta \frac{\pi}{2} \cos(-2k) \frac{\pi}{2}}{\cos(\frac{\beta-2k}{2})\pi} = 1$. Thus

$$K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = R_{\beta-2k}^H * R_{\alpha-2k}^e = K_{\alpha-2k,\beta-2k}(x) = (-1)^k \diamond^k K_{\alpha,\beta}(x)$$

by Lemma 3.1(iii) Now for p is odd, we have

$$K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (R_{\beta-2k}^H + T_{\beta,-2k}) * R_{\alpha-2k}^e$$

by Lemma 2.2(ii) and we have

$$T_{\beta,-2k} = \frac{\frac{2\pi i}{4} C(-\frac{\beta+2k}{2})}{C(-\frac{\beta}{2})C(\frac{2k}{2})} [H_{\beta-2k}^+ + H_{\beta-2k}^-]$$

where $C(r) = \Gamma(r)\Gamma(1-r)$ and $H_r^{\pm} = \ell^{\mp r \frac{\pi}{2}} i \ell^{\pm r \frac{\pi}{2}} a(\frac{r}{2})(a \pm i0)^{\frac{r-n}{2}}$ and $a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{r}{2}} \Gamma(\frac{r}{2})]^{-1}$. Applying the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$ to $C(-\frac{\beta+2k}{2})$, $C(-\frac{\beta}{2})$ and $C(k)$ and also the formula $H_{\beta-2k}^{\pm}$ and $a(\frac{\beta-2k}{2})$, we obtain

$$T_{\beta,-2k} = 0 \quad \text{and} \quad T_{-2k,\beta} = 0$$

It follow that

$$\begin{aligned} K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) &= R_{\beta-2k}^H * R_{\alpha-2k}^e \\ &= K_{\alpha-2k,\beta-2k}(x) \end{aligned}$$

for p is odd. Thus by Lemma 3.1(iii) we obtain $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \diamond^k K_{\alpha,\beta}(x)$ \square

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On the Convolutions of the diamond kernel
of Marcel Riesz

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On the Convolution Product of the Distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$

On the Convolution product of the Distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$

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ABSTRACT

In this paper, we introduce a distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$ which is related to the operator \oplus^k iterated k -time and is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

where $p+q=n$ is the dimension of the space C^n of n dimensional complex space, $x = (x_1, x_2, \dots, x_n) \in C^n$, k is a nonnegative integer, α, β, γ and ν are complex parameters. It is found that the existence of the convolution $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$ depending on the conditions of p and q .

1. Introduction

The operator \oplus^k can be factorized in the form

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \quad (1.1)$$

where $p+q=n$ is the dimension of the space C^n , $i = \sqrt{-1}$ and k is a nonnegative integer. The operator $\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2$ is first introduced by A. Kananthai [1]

and named the Diamond operator denoted by

$$\diamond = \left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \quad (1.2)$$

Let us denote the operator L_1 and L_2 by

$$L_1 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad (1.3)$$

$$L_2 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad (1.4)$$

Thus (1.1) can be written by

$$\oplus^k = \diamond^k L_1^k L_2^k \quad (1.5)$$

Now consider the convolution $R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z)$ where $R_\alpha^H(u)$, $R_\beta^e(v)$, $S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.1), (2.2), (2.3) and (2.4) respectively.

We defined the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z) \quad (1.6)$$

Since the functions $R_\alpha^H(u)$, $R_\beta^e(v)$, $S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions see [1, p30-31] and [6, p154-155] then the convolution on the right hand side of (1.6) exists and also is a tempered distributions. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also is a tempered distribution.

For $\alpha = \beta = \gamma = \nu = 2k$, we obtain $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator \oplus^k , see [2]. That is $\oplus^k(-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution and \oplus^k is defined by (1.5). In this work, we study the convolutions $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$ we found that such a convolution depending on p and q .

(i) If p is an even number and q can be either odd or even. We obtain

$$K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'} \cdot R_{\alpha+\alpha'}^H * R_{\beta+\beta'}^e * S_{\gamma+\gamma'} * T_{\nu+\nu'}$$

where $B_{\alpha,\alpha'} = \frac{\cos(\frac{\alpha\pi}{2}) \cos(\frac{\alpha'\pi}{2})}{\cos(\frac{\alpha+\alpha'}{2})\pi}$

In particularly, if $\alpha = \alpha' = 2k$ ($k = 0, 1, 2, 3, \dots$) we obtain

$$\begin{aligned} K_{2k,\beta,\gamma,\nu} * K_{2k,\beta',\gamma',\nu'} &= R_{4k}^H * R_{\beta+\beta'}^e * S_{\gamma+\gamma'} * T_{\nu+\nu'} \\ &= K_{4k,\beta+\beta',\gamma+\gamma',\nu+\nu'} \end{aligned}$$

(ii) If p is an odd and q is either odd or even. We obtain

$$K_{2k,\beta,\gamma,\nu} * K_{2k,\beta',\gamma',\nu'} = (R_{\alpha+\alpha'}^H + A_{\alpha+\alpha'}) * R_{\beta+\beta'}^e * S_{\gamma+\gamma'} * T_{\nu+\nu'}$$

where $A_{\alpha,\alpha'} = \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})} \cdot \frac{2\pi i}{4} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-]$

$C(r) = \Gamma(r)\Gamma(1-r)$ and $H_r^\pm = H_r(u \pm i0, n) = e^{\mp r \frac{\pi}{2} i} e^{\pm q \frac{\pi}{2} i} q(\frac{r}{2})(u \pm i0)^{\frac{r-n}{2}}$

and $a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$\hat{x} = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n \quad (2.1)$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of forward cone and $\bar{\Gamma}_+$ denote its closure. For any complex number α , we define the function

$$R_\alpha^H(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}$$

the function R_α^H is first introduced by Y. Nozaki [5, p72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Hence $R_\alpha^H(x)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$.

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write $v = x_1^2 + x_2^2 + \dots + x_n^2$.

For any complex number β , define the function

$$R_{\beta}^e(v) = \frac{v^{\frac{\beta-n}{2}}}{W_n(\beta)} \quad (2.3)$$

where $W_n(\beta) = \frac{\pi^{\frac{n}{2}} 2^{\beta} \Gamma(\beta)}{\Gamma(\frac{n-\beta}{2})}$ the function $R_{\beta}^e(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $Re(\beta) \geq n$ and is a distribution of β if $Re(\beta) < n$.

Definition 2.3 Let $x = (x_1, x_2, \dots, x_n)$ be a point of the space C^n of n dimensional complex space. Write $w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$ and

$$z = x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2), \quad p + q = n \quad i = \sqrt{-1}$$

For any complex number γ and ν , define

$$S_{\gamma}(\omega) = \frac{\omega^{\frac{\gamma-n}{2}}}{W_n(\gamma)} \quad (2.4)$$

and

$$T_{\nu}(z) = \frac{z^{\frac{\nu-n}{2}}}{W_n(\nu)} \quad (2.5)$$

where $W_n(\gamma) = \frac{\pi^{\frac{n}{2}} 2^{\gamma} \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}$, $W_n(\nu) = \frac{\pi^{\frac{n}{2}} 2^{\nu} \Gamma(\frac{\nu}{2})}{\Gamma(\frac{n-\nu}{2})}$.

Lemma 2.1 (The convolution product of $R_{\beta}^e(v)$)

*The convolution $R_{\beta}^e * R_{\beta'}^e = R_{\beta+\beta'}^e$, where R_{β}^e and $R_{\beta'}^e$ are given by (2.2)*

Proof See [3, p20].

Lemma 2.2 (The convolution product of $R_{\alpha}^H(x)$)

(i) $R_{\alpha}^H * R_{\alpha'}^H = \frac{\cos(\alpha \frac{\pi}{2}) \cos(\alpha' \frac{\pi}{2})}{\cos(\frac{\alpha+\alpha'}{2})\pi} R_{\alpha+\alpha'}^H$, where R_{α}^H and $R_{\alpha'}^H$ are defined by (2.1)

with p is an even.

(ii) $R_{\alpha}^H * R_{\alpha'}^H = R_{\alpha+\alpha'}^H + T_{\alpha, \alpha'}$ for p is an odd, where

$$T_{\alpha, \alpha'} = T_{\alpha, \alpha'}(u \pm i0, n) = \frac{2\pi i}{4} \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-]$$

$$C(r) = \Gamma(r)\Gamma(1-r)$$

$$H_r^\pm = H_r(x \pm i0, n) = e^{\mp r \frac{\pi}{2} i} e^{\pm q \frac{\pi}{2} i} a\left(\frac{r}{2}\right) (u \pm i0)^{\frac{r-n}{2}}$$

$$a\left(\frac{r}{2}\right) = \Gamma\left(\frac{n-r}{2}\right) (2^r \pi^{\frac{n}{2}} \Gamma\left(\frac{r}{2}\right))^{-1}$$

$$(u \pm i0, n)^\lambda = \lim_{\epsilon \rightarrow 0} (u \pm i\epsilon, n)^\lambda$$

$u = u(x)$ is defined by (2.1) and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$ in particular $R_\alpha^H * R_{-2k}^H = R_{\alpha-2k}^H$ and $R_\alpha^H * R_{2k}^H = R_{\alpha+2k}^H$. The proof of this Lemma is given by M. Aguirre Tellez [4, p121-123].

Lemma 2.3 (The convolutions product of $S_\gamma(w)$ and $T_\nu(z)$)

$$(i) S_\gamma * S_{\gamma'} = (i)^{\frac{\gamma}{2}} S_{\gamma+\gamma'}$$

(ii) $T_\nu * T_{\nu'} = (-i)^{\frac{\gamma}{2}} T_{\nu+\nu'}$ where S_γ and T_ν are defined by (2.4) and (2.5) respectively.

Proof (i) Now

$$\langle S_\gamma(w), \varphi(x) \rangle = \frac{1}{W_n(\gamma)} \int_{\mathbb{R}^n} \omega^{\frac{\gamma-n}{2}} \varphi(x) dx \quad (2.6)$$

where $\varphi \in \mathcal{D}$ the space of infinitely differentialble function with compact supports. We have $\omega = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$, $p+q=n$. By changing the variables $x_1 = y_1, x_2 = y_2, \dots, x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \dots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$. Thus we obtain $\omega = y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + \dots + y_{p+q}^2$. Let $r^2 = y_1^2 + y_2^2 + \dots + y_{p+q}^2$, $p+q=n$. Thus (2.6) can be written in the form

$$\begin{aligned} \langle S_\gamma(w), \varphi(x) \rangle &= \frac{1}{W_n(\gamma)} \int_{\mathbb{R}^n} r^{\gamma-n} \varphi \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} dy_1 dy_2 \dots dy_n \\ &= \frac{1}{(-i)^{\frac{\gamma}{2}}} \frac{1}{W_n(\gamma)} \int_{\mathbb{R}^n} r^{\gamma-n} \varphi dy \\ &= \frac{(i)^{\frac{\gamma}{2}}}{W_n(\gamma)} \langle r^{\gamma-n}, \varphi \rangle \end{aligned}$$

Consider the convolution $S_\gamma * S_{\gamma'}$. We have

$$\begin{aligned} S_\gamma * S_{\gamma'} &= (i)^{\frac{q}{2}} \frac{r^{\gamma-n}}{W_n(\gamma)} * (i)^{\frac{q}{2}} \frac{r^{\gamma'-n}}{W_n(\gamma')} \\ &= (i)^q \frac{r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \quad \text{Lemma 2.1} \\ &= (i)^{\frac{q}{2}} S_{\gamma+\gamma'} \end{aligned}$$

Similarly, for (ii) we also have

$$T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$$

Lemma 2.4 *Given the equation*

$$\oplus^k K(x) = \delta \tag{2.7}$$

where \oplus^k is the operator iterated k -times defined by (2.5) and $x = (x_1, x_2, \dots, x_n) \in C^n$. Then

$$K(x) = R_{2k}^H(x) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z)$$

is an elementary solution of the operator \oplus^k .

Proof See [2].

3. Main Results

Theorem Let $K_{\alpha,\beta,\gamma,\nu}$ be the distributional kernel defined by (1.6). Then we obtain

(i) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ for $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ and ν' are positive even numbers with $\alpha = \beta = \gamma = \nu, \alpha' = \beta' = \gamma' = \nu'$.

(ii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'} \cdot K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ for p is an even $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ and ν' are any complex numbers and $\beta_{\alpha,\alpha'} = \frac{\cos(\frac{\alpha\pi}{2}) \cos(\frac{\alpha'\pi}{2})}{\cos(\frac{\alpha+\alpha'}{2})\pi}$

(iii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R_{\beta+\beta'}^e * S_{\gamma+\gamma'} * T_{\nu+\nu'}$
 for p is an odd $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ and ν' are any complex numbers R_{β}^e, S_{γ} and T_{ν} are defined by (2.3), (2.4) and (2.5) respectively. And

$$A_{\alpha,\alpha'} = \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})} C(\frac{-\alpha'}{2}) \cdot \frac{2\pi i}{4} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-]$$

$$C(r) = \Gamma(r)\Gamma(1-r), H_r^{\pm} = H_r(u \pm i0, n) = e^{\mp r \frac{\pi}{2} i} e^{\pm q \frac{\pi}{2} i} a(\frac{r}{2})(u \pm i0)^{r-\frac{n}{2}}$$

and

$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2}) [2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$$

$$(u \pm i0)^{\lambda} = \lim_{\epsilon \rightarrow 0} (u + i \epsilon |x|^2)^{\lambda}$$

$u = u(x)$ is defined by (2.1) and

$$|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$$

Proof (i) Consider the equation

$$\oplus^k V(x) = \delta$$

where \oplus^k is the operator iterated k -times defined by (1.5) and $x = (x_1, x_2, \dots, x_n) \in C^n$.

By Lemma 2.4, we obtain the elementary solution

$$\begin{aligned} V(x) &= R_{2k}^H(x) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z) \\ &= (-1)^k R_{2k}^H(u) * R_{2k}^e(v) * S_{2k}(\omega) * T_{2k}(z) \\ &= (-1)^k K_{2k,2k,2k,2k}(x) \quad \text{by (1.6)} \end{aligned} \tag{3.1}$$

Thus

$$V(x) = (-1)^k K_{2k,2k,2k,2k}(x) \tag{3.2}$$

Now $\oplus^k V(x) = \oplus^m \oplus^{k-m} V(x) = \delta$ for $k > m$. Thus

$$\oplus^{k-m} V(x) = R_{2m}^H(x) * (-1)^m R_{2m}^e(v) * (-1)^m (-i)^{\frac{q}{2}} S_{2m}(\omega) * (-1)^m (i)^{\frac{q}{2}} T_{2m}(z) \quad \text{by Lemma 2.4.}$$

Convolving both sides of the above equation by $R_{2k-2m}^H(x) * (-1)^{k-m} R_{2k-2m}^e(v) * (-1)^{k-m} (-i)^{\frac{q}{2}} S_{2k-2m}(\omega) * (-1)^{k-m} (i)^{\frac{q}{2}} T_{2k-2m}(z)$ and the properties of convolutions and Lemma 2.4 again, we obtain

$$\begin{aligned} V(x) &= [R_{2k-2m}^H(x) * R_{2m}^H(x)] * [(-1)^{k-m} R_{2k-2m}^e(v) * (-1)^m R_{2m}^e(v)] \\ &\quad * [(-1)^{k-m} (-i)^{\frac{q}{2}} S_{2k-2m}(\omega) * (-1)^m (-i)^{\frac{q}{2}} S_{2m}(\omega)] * [(-1)^{k-m} (i)^{\frac{q}{2}} T_{2k-2m}(z) * (-1)^m (i)^{\frac{q}{2}} T_{2m}(z)] \\ &= (-1)^k (R_{2k-2m}^H(x) * R_{2m}^H(x)) * (R_{2k-2m}^e(v) * R_{2m}^e(v)) \\ &\quad * (S_{2k-2m}(\omega) * S_{2m}(\omega)) * (T_{2k-2m}(z) * T_{2m}(z)) \end{aligned}$$

Thus by (3.1), (3.3) and (3.2) we obtain

$$K_{2k-2m, 2k-2m, 2k-2m, 2k-2m} * K_{2m, 2m, 2m, 2m} = K_{2k-2m+2m, 2k-2m+2m, 2k-2m+2m, 2k-2m+2m}.$$

Since $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ and ν' are all even, thus by putting $\alpha = \beta = \gamma = \nu = 2k - 2m$ and $\alpha' = \beta' = \gamma' = \nu' = 2m$ for $k > m$ we obtain

$$K_{\alpha, \beta, \gamma, \nu} * K_{\alpha', \beta', \gamma', \nu'} = K_{\alpha+\alpha', \beta+\beta', \gamma+\gamma', \nu+\nu'}$$

(ii) We have

$$K_{\alpha, \beta, \gamma, \nu} * K_{\alpha', \beta', \gamma', \nu'} = [R_{\alpha}^H(u) * R_{\alpha'}^H(u)] * [R_{\beta}^e(v) * R_{\beta'}^e(v)] * [S_{\gamma}(\omega) * S_{\gamma'}(\omega)] * [T_{\nu}(z) * T_{\nu'}(z)]$$

By Lemma 2.2 (i) for p is an even and Lemma 2.3 (i), (ii), we obtain

$$\begin{aligned} K_{\alpha, \beta, \gamma, \nu} * K_{\alpha', \beta', \gamma', \nu'} &= B_{\alpha, \alpha'} R_{\alpha+\alpha'}^H(u) * R_{\beta+\beta'}^e(v) * (i)^{\frac{q}{2}} S_{\gamma+\gamma'}(\omega) * (-i)^{\frac{q}{2}} T_{\nu+\nu'}(z) \\ &= B_{\alpha, \alpha'} K_{\alpha+\alpha', \beta+\beta', \gamma+\gamma', \nu+\nu'} \end{aligned}$$

where $B_{\alpha, \alpha'} = \frac{\cos(\frac{q}{2}\pi) \cos \frac{q}{2}\pi}{\cos(\frac{\alpha+\alpha'}{2}\pi)}$. If $\alpha = \beta = \gamma = \nu$ and $\alpha' = \beta' = \gamma' = \nu'$ and $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma', \nu'$ are all even, then $B_{\alpha, \alpha'} = 1$. Thus we obtain case (i).

(iii) By Lemma 2.2 (ii) for p is an odd and Lemma 2.3 (i), (ii) again we obtain

$$\begin{aligned} K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} &= (R_{\alpha+\alpha'}^H(u) + A_{\alpha,\alpha'}) * R_{\beta+\beta'}^e(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'} \\ &= (R_{\alpha+\alpha'}^H(u) * R_{\beta+\beta'}^e(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'}) + A_{\alpha,\alpha'} * R_{\beta+\beta'}^e(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'} \\ &= K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R_{\beta+\beta'}^e(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'} \end{aligned}$$

$$\text{where } A_{\alpha,\alpha'} = \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})} \cdot \frac{2\pi i}{4} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-], \quad C(r) = \Gamma(r)\Gamma(1-r)$$

$$\begin{aligned} H_r^\pm &= H_r(u \pm i0, n) \\ &= e^{\mp r \frac{\pi}{2} i} e^{\pm q \frac{\pi}{2} i} a\left(\frac{r}{2}\right) (u \pm i0)^{\frac{r-n}{2}} \\ a\left(\frac{r}{2}\right) &= \Gamma\left(\frac{n-r}{2}\right) [2^r \pi^{\frac{n}{2}} \Gamma\left(\frac{r}{2}\right)]^{-1} \\ (u \pm i0)^\lambda &= \lim_{\epsilon \rightarrow 0} (u + i\epsilon)^{\lambda} \in |x|^2)^\lambda \end{aligned}$$

$u = u(x)$ is defined by (2.1) and

$$|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$$

□

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On the Inversion of Kernel $K_{\alpha,\beta,\gamma,\upsilon}$ related to
the operator \oplus^k

On the Inversion of the Kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator \oplus^k

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ABSTRACT

In this paper, we study the inversion of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator \oplus^k iterated k -times and is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

$p+q = n$ is the dimension of the n -dimensional complex space C^n and k is a nonnegative integer α, β, γ and ν are complex parameters. We found that the inverse $[K_{\alpha,\beta,\gamma,\nu}]^{-1}$ of $K_{\alpha,\beta,\gamma,\nu}$ exists depending on the conditions of p and q whether they are odd or even numbers.

1. Introduction

The operator \oplus^k can be factorized in the form

$$\begin{aligned} \oplus^k &= \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \cdot \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 + i \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \\ &\quad \cdot \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - i \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \end{aligned} \quad (1.1)$$

$p+q = n$ is the dimension of the space C^n , $i = \sqrt{-1}$ and k is a nonnegative integer.

The operator $\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2$ is first introduced by A. Kananthai [1] and

named the Diamond operator denoted by

$$\diamond = \left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \quad (1.2)$$

Let us denote the operator L_1 and L_2 by

$$L_1 = \left(\sum_{r=1}^p \frac{\partial^2}{\partial X_r^2} \right)^2 + i \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \quad (1.3)$$

$$L_2 = \left(\sum_{r=1}^p \frac{\partial^2}{\partial X_r^2} \right)^2 - i \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \quad (1.4)$$

Thus (1.1) can be written by

$$\oplus^k = \diamond^k L_1^k L_2^k \quad (1.5)$$

Let us define the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_{\alpha}^H(u) * R_{\beta}^e(v) * S_{\gamma}(w) * T_{\nu}(z) \quad (1.6)$$

where $R_{\alpha}^H(u)$, $R_{\beta}^e(v)$, $S_{\gamma}(w)$ and $T_{\nu}(z)$ are defined by (2.2), (2.3), (2.4) and (2.5) respectively and the symbol $*$ denotes the convolution. Since $R_{\alpha}^H(u)$, $R_{\beta}^e(v)$, $S_{\gamma}(w)$ and $T_{\nu}(z)$ are all tempered distributions see [1, p 30-31] and [5, p 154-155], then the convolutions on the right hand side of (1.6) exists and also is a tempered distribution. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also is a tempered distribution. For $\alpha = \beta = \gamma = \nu = 2k$, we obtained $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator \oplus^k , see [2].

That is $\oplus^k (-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution and \oplus^k is defined by (1.5). Let $A^{\alpha,\beta,\gamma,\nu}$ be the operator defined by the formula

$$A^{\alpha,\beta,\gamma,\nu}(f) = K_{\alpha,\beta,\gamma,\nu} * f \quad (1.7)$$

where α, β, γ and ν are complex parameters and the symbol $*$ denotes the convolution product and $f \in S$ where S is the Schwartz space, see [8, p 223]. Our objective is to obtain the operator

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1}$$

such that if $\varphi = A^{\alpha,\beta,\gamma,\nu}(f)$ then $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$. We found that, from (1.7) for p and q are both odd we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = K_{-\alpha,-\beta,-\gamma,-\nu}.$$

If p is odd and q is even, we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = (1 + (\sin \alpha \frac{\pi}{2})^2)^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

and if p is even for all complex α we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [(\cos \alpha \frac{\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

such that $\alpha \neq 2s + 1$ ($s = 0, 1, 2, 3, \dots$).

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point in the space \mathbb{R}^n of n -dimensional Euclidean space. Write

$$u = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n \quad (2.1)$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of the forward cone and $\bar{\Gamma}_+$ denotes its closure. For any complex number α , we define the function.

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}$$

the function R_α^H is first introduced by Y. Nozaki [5, p72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Here $R_\alpha^H(x)$ is an ordinary function if $\text{Re}(\alpha) \geq n$ and is a distribution of α if $\text{Re}(\alpha) < n$.

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write $v = x_1^2 + x_2^2 + \dots + x_n^2$.

For any complex number β , define the function

$$R_\beta^e(v) = \frac{v^{\frac{\beta-n}{2}}}{W_n(\beta)} \quad (2.3)$$

where $W_n(\beta) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\beta)}{\Gamma(\frac{n-\beta}{2})}$ the function $R_\beta^e(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $Re(\beta) \geq n$ and is a distribution of β if $Re(\beta) < n$.

Definition 2.3 Let $x = (x_1, x_2, \dots, x_n)$ be a point of the space C^n of n dimensional complex space. Write $w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$ and

$$z = x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2), \quad p+q=n \quad i = \sqrt{-1}$$

For any complex number γ and ν , define

$$S_\gamma(\omega) = \frac{\omega^{\frac{\gamma-n}{2}}}{W_n(\gamma)} \quad (2.4)$$

and

$$T_\nu(z) = \frac{z^{\frac{\nu-n}{2}}}{W_n(\nu)} \quad (2.5)$$

where $W_n(\gamma) = \frac{\pi^{\frac{n}{2}} 2^\gamma \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}$, $W_n(\nu) = \frac{\pi^{\frac{n}{2}} 2^\nu \Gamma(\frac{\nu}{2})}{\Gamma(\frac{n-\nu}{2})}$.

Lemma 2.1

- (i) $R_\beta^e * R_{\beta'}^e = R_{\beta+\beta'}^e$, where R_β^e and $R_{\beta'}^e$ are given by (2.3).
- (ii) $S_\gamma * S_{\gamma'} = (i)^{\frac{\gamma}{2}} S_{\gamma+\gamma'}$ and $T_\nu * T_{\nu'} = (-i)^{\frac{\nu}{2}} T_{\nu+\nu'}$ where S_γ and T_ν are defined by (2.4) and (2.5) respectively.

Proof (i) see [6, p. 20].

(ii) Now $\langle S_\gamma(\omega), \varphi(x) \rangle = \frac{1}{W_n(\gamma)} \int_{\mathbb{R}^n} \omega^{\frac{\gamma-n}{2}} \varphi(x) dx$. By changing the variable $x_1 = y_1, x_2 = y_2, \dots, x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \dots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$ then we obtain

$$\begin{aligned} \omega &= x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2) \\ &= y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 + \dots + y_{p+q}^2, \quad p+q=n. \end{aligned}$$

Let $r^2 = y_1^2 + y_2^2 + \dots + y_{p+q}^2$. Thus

$$\begin{aligned} \langle S_\gamma(\omega), \varphi(x) \rangle &= \frac{1}{W_n(\gamma)} \int_{\mathbb{R}^n} r^{\gamma-n} \varphi \left| \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} \right| dy_1 dy_2 \dots dy_n \\ &= \frac{(i)^{\frac{q}{2}}}{W_n(\gamma)} \int_{\mathbb{R}^n} r^{\gamma-n} \varphi dy \\ &= \langle \frac{(i)^{\frac{q}{2}} r^{\gamma-n}}{W_n(\gamma)}, \varphi \rangle. \end{aligned}$$

Now

$$\begin{aligned} S_\gamma(\omega) * S_{\gamma'}(\omega) &= \frac{(i)^{\frac{q}{2}} r^{\gamma-n}}{W_n(\gamma)} * \frac{(i)^{\frac{q}{2}} r^{\gamma'-n}}{W_n(\gamma')} \\ &= (i)^q \frac{r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \text{ by (i)} \\ &= (i)^{\frac{q}{2}} \left[\frac{(i)^{\frac{q}{2}} r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \right] \\ &= (i)^{\frac{q}{2}} S_{\gamma+\gamma'}. \end{aligned}$$

Thus $S_\gamma * S_{\gamma'} = (i)^{\frac{q}{2}} S_{\gamma+\gamma'}$. Similarly, we obtain $T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$.

Lemma 2.2

- (i) $L_1(-1)^k(-i)^{\frac{q}{2}} S_{2k}(\omega) = \delta$ that is $(-1)^k(-i)^{\frac{q}{2}} S_{2k}(\omega)$ is the elementary solution of the operator L_1 defined by (1.3).
- (ii) $L_2(-1)^k(i)^{\frac{q}{2}} T_{2k}(z) = \delta$ that is $(-1)^k(i)^{\frac{q}{2}} T_{2k}(z)$ is the elementary solution of the operator L_2 defined by (1.4). Moreover from (i), we obtain $S_{-2k}(\omega) = (-1)^k(i)^{\frac{q}{2}} L_1^k \delta(x)$ and $T_{-2k}(z) = (-1)^k(-i)^{\frac{q}{2}} L_2^k \delta(x)$. It follows that

$$S_0(\omega) = (i)^{\frac{q}{2}} \delta(x) \tag{2.6}$$

and

$$T_0(\omega) = (i)^{\frac{q}{2}} \delta(x) \tag{2.7}$$

Proof See [2, Lemma 2.2 (ii)].

Lemma 2.3 (The convolutions of $R_\alpha^H(u)$)

- (i) $R_\alpha^H * R_{\alpha'}^H = \frac{\cos \frac{\alpha\pi}{2} \cos \frac{\alpha'\pi}{2}}{\cos(\frac{\alpha+\alpha'}{2}\pi)} R_{\alpha+\alpha'}^H$, where R_α^H is defined by (2.2) with p is an even.
(ii) $R_\alpha^H * R_{\alpha'}^H = R_{\alpha+\alpha'}^H + T_{\alpha,\alpha'}$ for p is an odd where

$$T_{\alpha,\alpha'} = T_{\alpha,\alpha'}(u \pm i0, n) = \frac{2\pi i}{4} \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-]$$

$$C(r) = \Gamma(r)\Gamma(1-r)$$

$$H_r^\pm = H_r(u \pm i0, n) = e^{\mp \frac{r\pi i}{2}} e^{\mp \frac{q\pi i}{2}} a(\frac{r}{2})(u \pm i0)^{\frac{r-n}{2}}$$

$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2}) [2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$$

$$(u \pm i0)^\lambda = \lim_{\epsilon \rightarrow 0} (u + i\epsilon |x|^2)^\lambda$$

$u = u(x)$ is defined by (2.1)

Proof See [4, p 121-123].

Lemma 2.4 (The inverse of convolution algebra)

- (i) $R_\alpha^H * R_{-\alpha}^H = R_{\alpha-\alpha}^H = R_0^H = \delta$ if p and q are both odd.
(ii) $R_\alpha^H * R_{-\alpha}^H = (1 + (\sin \frac{\alpha\pi}{2})^2)\delta(x)$ if p is odd and q is even.
(iii) $R_\alpha^H * R_{-\alpha}^H = ((\cos \frac{\alpha\pi}{2})^2)\delta(x)$ for p is even and $\alpha \neq 2s + 1$, $s = 0, 1, 2, 3, \dots$
(iv) $R_\beta^e * R_{-\beta}^e = R_{\beta-\beta}^e = R_0^e = \delta$
(v) $S_\gamma * S_{-\gamma} = (i)^q \delta$
(vi) $T_\nu * T_{-\nu} = (-i)^q \delta$

Proof (i) By Lemma 2.3 (ii), for p is an odd, we obtain $R_\alpha^H * R_{-\alpha}^H = R_0^H + T_{\alpha,-\alpha}$. Since $R_0^H = \delta$ and by computing directly $T_{\alpha,-\alpha} = 0$ for p and q are both odd. Thus $R_\alpha^H * R_{-\alpha}^H = \delta$ for p and q are both odd.

(ii) By Lemma 2.3 (ii) again, for p is an odd, we obtain $R_\alpha^H * R_{-\alpha}^H = R_0^H + T_{\alpha,-\alpha}$. Now, for p is an odd and q is an even, we obtain $T_{\alpha,-\alpha} = (\sin \frac{\alpha\pi}{2})^2 \delta$. Thus

$$\begin{aligned} R_\alpha^H * R_{-\alpha}^H &= \delta + (\sin \frac{\alpha\pi}{2})^2 \delta \\ &= [1 + (\sin \frac{\alpha\pi}{2})^2] \delta(x) \end{aligned}$$

(iii) By Lemma 2.3 (i), for p is an even

$$\begin{aligned} R_\alpha^H * R_{-\alpha}^H &= \frac{\cos \frac{\alpha\pi}{2} \cos \frac{(-\alpha)\pi}{2}}{\cos \frac{\alpha-\alpha}{2}\pi} R_0^H \\ &= [\cos \frac{\alpha\pi}{2}]^2 R_0^H \\ &= (\cos \frac{\alpha\pi}{2})^2 \delta, \alpha \neq 2s+1, s=0,1,2,3,\dots \end{aligned}$$

(iv) By Lemma 2.1 (i), $R_\beta^e * R_{-\beta}^e = R_{\beta-\beta}^e = R_0^e = \delta$ since $R_0^e = \delta$, see [5, p 118].

(v) By Lemma 2.1 (ii),

$$\begin{aligned} S_\gamma * S_{-\gamma} &= (i)^{\frac{q}{2}} S_{\gamma-\gamma} = (i)^{\frac{q}{2}} S_0 \\ &= (i)^{\frac{q}{2}} (i)^{\frac{q}{2}} \delta = (i)^q \delta \text{ by (2.6)} \end{aligned}$$

(vi) By Lemma 2.1 (ii) again, we obtain

$$\begin{aligned} T_\nu * T_{-\nu} &= (-i)^{\frac{q}{2}} T_{\nu-\nu} = (-i)^{\frac{q}{2}} T_0 \\ &= (-i)^{\frac{q}{2}} (-i)^{\frac{q}{2}} \delta = (-i)^q \delta \text{ by (2.7)} \end{aligned}$$

Lemma 2.5 *Let the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ be defined by (1.6) then the following formulas hold.*

- (i) $K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = K_{0,0,0,0} = \delta$ for p and q are both odd numbers.
- (ii) $K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = [1 + (\sin \frac{\alpha\pi}{2})^2] \delta$ for p is odd and q is even.
- (iii) $K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = (\cos \frac{\alpha\pi}{2})^2 \delta$ for p is even.

Proof (i) By (1.6) and properties of convolutions,

$$\begin{aligned} K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} &= (R_\alpha^H * R_{-\alpha}^H) * (R_\beta^e * R_{-\beta}^e) * (S_\gamma * S_{-\gamma}) * (T_\nu * T_{-\nu}) \\ &= \delta * \delta * (i)^q \delta * (-i)^q \delta \\ &= \delta * \delta * \delta * \delta = \delta \text{ by Lemma 2.4 (i), (iv), (v), (vi)} \end{aligned} \tag{2.8}$$

(ii) From (2.8),

$$\begin{aligned}
 K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} &= [1 + (\sin \frac{\alpha\pi}{2})^2] \delta * \delta * (i)^q \delta * (-i)^q \delta \\
 &= [1 + (\sin \frac{\alpha\pi}{2})^2] \delta * \delta * \delta * \delta \\
 &= [1 + (\sin \frac{\alpha\pi}{2})^2] \delta \quad \text{by Lemma 2.4 (ii), (iv), (v), (vi)}
 \end{aligned}$$

(iii) From (2.8) again, we obtain

$$\begin{aligned}
 K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} &= (\cos \frac{\alpha\pi}{2})^2 \delta * \delta * (i)^q \delta * (-i)^q \delta \\
 &= (\cos \frac{\alpha\pi}{2})^2 \delta \quad \text{for } \alpha \neq 2s + 1, s = 0, 1, 2, 3, \dots
 \end{aligned}$$

by Lemma 2.4 (iii), (iv), (v), (vi).

3. Main results

Theorem Given $\varphi = A^{\alpha,\beta,\gamma,\nu}(f)$ where $A^{\alpha,\beta,\gamma,\nu}(f)$ is defined by (1.7) for every $f \in S$ where S is the Schwartz space of function, then there exists the operator $B^{\alpha,\beta,\gamma,\nu}$ such that

(i) $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = K_{-\alpha,-\beta,-\gamma,-\nu}$$

if p and q are both odd numbers.

(ii) $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = (1 + (\sin \frac{\alpha\pi}{2})^2)^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

if p is odd and q is even.

(iii) $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [(\cos \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

if p is even for all α such that $\alpha \neq 2s + 1, s = 0, 1, 2, 3, \dots$

Proof (i) From (1.7), we have

$$A^{\alpha,\beta,\gamma,\nu}(f) = K_{\alpha,\beta,\gamma,\nu} * f = \varphi$$

where $K_{\alpha,\beta,\gamma,\nu}$ is defined by (1.6). Then by Lemma 2.5 (i) we obtain

$$\begin{aligned} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) &= (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f \\ &= K_{0,0,0,0} * f = \delta * f = f \end{aligned}$$

if p and q are both odd. Thus

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = K_{-\alpha,-\beta,-\gamma,-\nu}$$

It follow that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

(ii) By Lemma 2.5 (ii), for p is an odd and q is an even, we have

$$\begin{aligned} [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) &= [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} \\ &\quad \times (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f \\ &= [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} [1 + (\sin \frac{\alpha\pi}{2})^2] \\ &\quad \times K_{0,0,0,0} * f \\ &= \delta * f = f \end{aligned}$$

$$\text{Thus } B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}.$$

It follow that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

(iii) By Lemma 2.5 (iii), for p is even

$$\begin{aligned} [(\cos \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) &= [(\cos \frac{\alpha\pi}{2})^2]^{-1} \\ &\quad \times (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f \\ &= [(\cos \frac{\alpha\pi}{2})^2]^{-1} [(\cos \frac{\alpha\pi}{2})^2] \\ &\quad \times K_{0,0,0,0} * f \\ &= \delta * f = f \end{aligned}$$

$$\text{Thus } B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [(\cos \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}.$$

It follow that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

In particular, we obtain for $\alpha = \beta = \gamma = \nu = 2k$ and p and q are both odd numbers.

$$(-1)^k K_{2k,2k,2k,2k}(x) = (-1)^k R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z)$$

is an elementary solution of the operator \oplus^k defined by (1.5) and by Lemma 2.2 (i), (ii). Now

$$\begin{aligned} (-1)^k K_{2k,2k,2k,2k} * (-1)^k K_{-2k,-2k,-2k,-2k} &= [(-1)^k R_{2k}^H(u) * (-1)^k R_{-2k}^H] \\ &\quad * [(-1)^k R_{2k}^e(v) * (-1)^k R_{-2k}^e(v)] \\ &\quad * [(-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (-i)^{\frac{q}{2}} S_{-2k}] \\ &\quad * [(-1)^k (i)^{\frac{q}{2}} T_{2k}(z) * (-1)^k (i)^{\frac{q}{2}} T_{-2k}(z)] \\ &= R_0^H * R_0^e * (-i)^q S_0(\omega) * (i)^q T_0(z) \\ &= \delta * \delta * (-i)^q (i)^{\frac{q}{2}} \delta * (i)^q (-i)^{\frac{q}{2}} \delta \\ &= \delta * \delta * \delta * \delta \\ &= \delta \quad \text{by (2.6), (2.7)} \end{aligned}$$

Thus $(-1)^k K_{-2k,-2k,-2k,-2k}$ is an inverse of the convolution algebra of the elementary solution of the operator \oplus^k .

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On the Diamond Operator Related to the Wave Equation

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On the nonlinear Diamond operator related to the Wave Equation

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Abstract

In this paper, we study the solution of nonlinear equation $\diamond^k u(x) = f(x, \Delta^{k-1} \square^k u(x))$ where \diamond^k is the Diamond operator iterated k times and is defined by

$$\diamond^k = \left(\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k$$

or the operator \diamond^k can be expressed by $\diamond^k = \square^k \Delta^k = \Delta^k \square^k$. The operators Δ^k and \square^k are Laplacian and the ultrahyperbolic operator iterated k times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k$$

and $\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^k$, $p+q = n$ is the dimension of the n -dimensional Euclidean space \mathbb{R}^n , $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a given function. It is found that, the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\Delta^{k-1} \square^k u(x)$ and moreover such solution $u(x)$ related to the wave equation depending on the conditions of p, q and k .

1. Introduction

The operator \diamond^k was first introduced by A. Kananthai [3] and is named the Diamond operator iterated k times and is defined by

$$\diamond^k = ((\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2})^2 - (\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \dots + \frac{\partial^2}{\partial x_{p+q}^2})^2)^k \quad (1.1)$$

For the linear equation $\diamond^k u(x) = f(x)$, see [4], it has been already studied and obtained the convolution $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ as a solution of such equation where $K_{2k,2k}(x) = R_{2k}^H(x) * R_{2k}^e(x)$. The functions $R_{2k}^H(x)$ and $R_{2k}^e(x)$ are defined by (2.2) and (2.5) respectively with $\alpha = \beta = 2k$.

In this work, we study the nonlinear equation of the form

$$\diamond^k u(x) = f(x, \Delta^{k-1} \square^k u(x)) \quad (1.2)$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of \mathbb{R}^n and $\partial\Omega$ denotes the boundary of Ω . We can find the solution $u(x)$ of (1.2) and is unique under the condition $|f(x, \Delta^{k-1} \square^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\Delta^{k-1} \square^k u(x) = 0$ for $x \in \partial\Omega$. If we put $k = 1, p = 1$ and $q = n - 1$ in (1.2), we obtain $u(x) = M_2(s) * W(x)$ is a solution of nonhomogeneous wave equation where $M_2(s)$ is defined by (2.4) with $\alpha = 2$ and $W(x)$ is continuous function.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and denote by

$$v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n \quad (2.1)$$

the nondegenerated quadratic form. Denotes the interior of the forward cone by

$$\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$$

and by $\overline{\Gamma_+}$ the closure of Γ_+ . For any complex number α , define

$$R_\alpha^H(v) = \begin{cases} \frac{v^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+ \\ 0, & \text{for } x \notin \Gamma_+ \end{cases} \quad (2.2)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})} \quad (2.3)$$

The function $R_\alpha^H(v)$ was introduced by Y. Nozaki [5, p.72]. It is well known that $R_\alpha^H(v)$ is an ordinary function if $\text{Re}(\alpha) \geq n$ and it is a distribution of α if $\text{Re}(\alpha) < n$. Let $\text{supp } R_\alpha^H(v)$ denote the support of $R_\alpha^H(v)$. By putting $p = 1$ in (2.1) and (2.2) and remembering the Legendre's duplication of $\Gamma(z)$. $\Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2})$ then (2.2) reduces to

$$M_\alpha(s) = \begin{cases} \frac{s^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.4)$$

Here $s = x_1^2 - x_2^2 - \dots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{n-1}{2}} 2^{\alpha-1} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{\alpha}{2})$, $M_\alpha(s)$ is, precisely, the hyperbolic kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathbb{R}^n and the function $R_\beta^e(x)$ be defined by

$$R_\beta^e(x) = \frac{|x|^{\beta-n}}{W_n(\beta)} \quad (2.5)$$

where $W_n(\beta) = \frac{\pi^{\frac{n}{2}} 2^\beta \Gamma(\frac{\beta}{2})}{\Gamma(\frac{n-\beta}{2})}$, β is a complex parameter and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$.

Lemma 2.1 Given the equation

$$\Delta^k u(x) = 0 \quad (2.6)$$

where Δ^k is the Laplacian iterated k times, defined by

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^k, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \quad (2.7)$$

then, we obtain $u(x) = (-1)^{k-1} (R_{2(k-1)}^e(x))^{(m)}$ as a solution of (2.5) where m is a nonnegative integer with $m = \frac{n-4}{2}$, $n \geq 4$ and n is an even and $(R_{2(k-1)}^e(x))^{(m)}$ is a function defined by (2.4) with m derivatives and $\beta = 2(k-1)$.

Proof see [4, Lemma 2.2].

Lemma 2.2 Given the equation

$$\square^k u(x) = 0 \quad (2.8)$$

where \square^k is the ultrahyperbolic operator iterated k times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (2.9)$$

$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then we obtain $u(x) = [R_{2(k-1)}^H(v)]^{(m)}$ as a solution of (2.8) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is an even, v is defined by (2.1). The functions $[R_{2(k-1)}^H(v)]^{(m)}$ is defined by (2.2) with m derivatives and $\alpha = 2(k-1)$.

Proof see [4, Lemma 2.3].

Lemma 2.3 Given the equation

$$\Delta u(x) = f(x, u(x)) \quad (2.10)$$

where f defined and having continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of \mathbb{R}^n and $\partial\Omega$ is the boundary of Ω . Assume that f is bounded, that is $|f(x, u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (2.10).

Proof We can prove the existence of the solution $u(x)$ of (2.10) by the method of iterations and the Schauder's estimates. The details of proving is given by R. Courant and D. Hilbert, see [1, pp 369-372].

3. Main results

Theorem Consider the nonlinear equation

$$\diamond^k u(x) = f(x, \Delta^{k-1} \square^k u(x)) \quad (3.1)$$

where \diamond^k is the Diamond operator iterated k times, defined by (1.1), Δ^{k-1} is the Laplacian iterated $k-1$ times defined by (2.7) and \square^k is the ultrahyperbolic operator iterated k times, defined by (2.9). Let f be defined and continuous first derivative for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of \mathbb{R}^n and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Let suppose f be bounded function, that is

$$|f(x, \Delta^{k-1} \square^k u(x))| \leq N \text{ for all } x \in \Omega \quad (3.2)$$

and the boundary condition

$$\Delta^{k-1} \square^k u(x) = 0 \text{ for } x \in \partial\Omega \quad (3.3)$$

then we obtain

$$u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{2k}^H(v) * W(x) \quad (3.4)$$

as a solution of (3.1) with the boundary condition

$$u(x) = R_{2k}^H(v) * (-1)^{k-2} (R_{2(k-2)}^e(x))^{(m)}$$

for $x \in \partial\Omega$, $m = \frac{n-4}{2}$, $k = 2, 3, 4, 5, \dots$ and v is given by (2.1), $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $R_{2(k-2)}^e(x)$ and $R_{2k}^H(v)$ are given by (2.5) and (2.2) respectively with $\beta = 2(k-2)$ and $\alpha = 2k$. Moreover, for $k = 1$ we obtain

$$u(x) = R_2^H(v) * W(x)$$

as a solution of (3.1) with the boundary condition

$$u(x) = \delta^{(m)}(v) \text{ for } x \in \partial\Omega$$

where $\delta^{(m)}(v)$ is the Dirac-delta distribution with m derivatives and $m = \frac{n-4}{2}$. Also if we put $k = 1, p = 1$ and $q = n - 1$. We obtain $u(x) = M_2(s) * W(x)$ as a solution

of nonhomogeneous wave equation

$$\square^* u(x) = W(x) \quad (3.5)$$

with the boundary condition $u(x) = \delta^{(m)}(s)$ for $x \in \partial\Omega$ where $\square^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2}$ and $s = x_1^2 - x_2^2 - \dots - x_n^2$ and $M_2(s)$ is defined by (2.4) with $\alpha = 2$.

Proof We have

$$\diamond^k u(x) = \Delta(\Delta^{k-1} \square^k u(x)) = f(x, \Delta^{k-1} \square^k u(x)) \quad (3.6)$$

Since $u(x)$ has continuous derivatives up to order $4k$ for $k = 1, 2, 3, \dots$, thus we can assume

$$\Delta^{k-1} \square^k u(x) = W(x) \quad \text{for all } x \in \Omega \quad (3.7)$$

Thus (3.6) can be written in the form

$$\diamond^k u(x) = \Delta W(x) = f(x, W(x)) \quad (3.8)$$

by (3.2),

$$|f(x, w)| \leq N \quad \text{for all } x \in \Omega \quad (3.9)$$

and by (3.3), $W(x) = 0$ or

$$\Delta^{k-1} \square^k u(x) = 0 \quad (3.10)$$

for $x \in \partial\Omega$. Thus by Lemma 2.3 there exist a unique solution $W(x)$ of (3.8) which satisfies (3.9). Now consider the equation (3.7), we have $\Delta^{k-1}(-1)^{k-1} R_{2(k-1)}^e(x) = \delta$ and $\square^k R_{2k}^H(v) = \delta$ where δ is the Dirac-delta distribution, that is $R_{2k}^H(v)$ and $(-1)^{k-1} R_{2(k-1)}^e(x)$ are the elementary solutions of the operators \square^k and Δ^{k-1} respectively, see [6, p11] and [2, p118]. The functions $R_{2(k-1)}^e(x)$ and $R_{2k}^H(v)$ are defined by (2.5) and (2.2) respectively, with $\beta = 2(k-1)$, $\alpha = 2k$. Thus convolving both sides of (3.7) by $(-1)^{k-1} R_{2(k-1)}^e(x) * R_{2k}^H(v)$, we obtain

$$[(-1)^{k-1} R_{2(k-1)}^e(x) * R_{2k}^H(v)] * \Delta^{k-1} \square^k u(x) = [(-1)^{k-1} R_{2(k-1)}^e(x) * R_{2k}^H(v)] * W(x)$$

By the properties of convolution, we obtain

$$(\Delta^{k-1}(-1)^{k-1}R_{2(k-1)}^e(x)) * (\Box^k R_{2k}^H(v)) * u(x) = (-1)^{k-1}R_{2(k-1)}^e(x) * R_{2k}^H(v) * W(x)$$

$$\delta * \delta * u(x) = (-1)^{k-1}R_{2(k-1)}^e(x) * R_{2k}^H(v) * W(x)$$

Thus

$$u(x) = (-1)^{k-1}R_{2(k-1)}^e(x) * R_{2k}^H(v) * W(x) \quad (3.11)$$

as required. Consider $\Delta^{k-1}\Box^k u(x) = 0$ for $x \in \partial\Omega$. By Lemma 2.1, we have

$$\Box^k u(x) = (-1)^{k-2}(R_{2(k-2)}^e(x))^{(m)}.$$

Convolving both sides of the above equation by $R_{2k}^H(v)$, we obtain $R_{2k}^H(v) * \Box^k u(x) = R_{2k}^H(v) * (-1)^{k-2}(R_{2k}^e(x))^{(m)}$ or $\Box^k R_{2k}^H(v) * u(x) = R_{2k}^H(v) * (-1)^{(k-2)}(R_{2(k-2)}^e(x))$. It follows that $\delta * u(x) = u(x) = R_{2k}^H(v) * (-1)^{(k-2)}(R_{2(k-2)}^e(x))^{(m)}$ for $x \in \partial\Omega$ and $k = 2, 3, 4, 5, \dots$

Now, for $k = 1$ in (3.11), we obtain

$$\begin{aligned} u(x) &= R_0^e(x) * R_2^H(v) * W(x) \\ &= \delta * R_2^H(v) * W(x) \\ &= R_2^H(v) * W(x) \end{aligned} \quad (3.12)$$

since $R_0^e(x) = \delta$, see [2, p118]. Now consider the boundary condition for $k = 1$ in (3.10) we have $\Box u(x) = 0$ for $x \in \partial\Omega$. Thus, by Lemma 2.2, for $k = 1$, we obtain

$$u(x) = \delta^{(m)}(v) \quad \text{for } x \in \partial\Omega \quad (3.13)$$

where $R_0^H(v) = \delta$, see [6]. Now consider the cases $k = 1, p = 1$ and $q = n - 1$. thus from (3.12) $R_2^H(v)$ reduces to $M_2(s)$ where $M_2(s)$ is defined by (2.4) with $\alpha = 2$ and $s = x_1^2 - x_2^2 - \dots - x_n^2$ and also the operator \Box defined by (2.9) reduces to the wave operator

$$\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2}.$$

Thus the solution $u(x)$ of (3.12) reduces to $u(x) = M_2(s) * W(x)$ which is the solution of the wave equation $\square^* u(x) = W(x)$ with the boundary condition $\square^* u(x) = 0$ for $x \in \partial\Omega$ or $u(x) = \delta^{(m)}(s)$ for $x \in \partial\Omega$ by (3.13). \square

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บทที่ 4

การแปลงฟูรีเยร์ของตัวดำเนินการ \oplus^k Fourier Transform of the Operator \oplus^k

ในส่วนแรกของบทนี้เป็นการศึกษาการแปลงฟูรีเยร์ของฟังก์ชันปกติ สมบัติต่างๆ ที่เกี่ยวกับการแปลงฟูรีเยร์ของตัวดำเนินการไดมอนด์ และ ตัวดำเนินการอัลตราไฮเปอร์โบลิค และได้ศึกษาการแปลงฟูรีเยร์ของ Distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$ ที่เกี่ยวข้องกับตัวดำเนินการ \oplus^k

On the Fourier transform of the Diamond Kernel of
Marcel Riesz



On the Fourier transform of the Diamond Kernel of Marcel Riesz

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Abstract

In this paper, the operator \diamond^k is introduced and named as the Diamond operator iterated k -times and is defined by $\diamond^k = [(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_p^2)^2 - (\partial^2/\partial x_{p+1}^2 + \partial^2/\partial x_{p+2}^2 + \cdots + \partial^2/\partial x_{p+q}^2)^2]^k$, where n is the dimension of the Euclidean space \mathbb{R}^n , k is a nonnegative integer and $p+q=n$. The elementary solution of the operator \diamond^k is called the Diamond Kernel of Marcel Riesz. In this work we study the Fourier transform of the elementary solution and also the Fourier transform of their convolutions. © 1999 Elsevier Science Inc. All rights reserved.

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Keywords: Diamond operator; Fourier transform; Kernel of Marcel Riesz; Dirac delta distributions; Tempered distribution

1. Introduction

Consider the equation

$$\diamond^k u(x) = \delta, \quad (1)$$

where \diamond^k is the Diamond operator iterated k -times ($k=0,1,2,\dots$) with $\diamond^0 u(x) = u(x)$ and is defined by

$$\diamond^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right)^k, \quad (2)$$

where $p+q=n$, the dimension of the Euclidean space \mathbb{R}^n and $u(x)$ is the generalized function, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and δ is the Dirac-delta distribution.

Kananthai ([1], Theorem 1.3) has shown that the solution of convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is an unique elementary solution of Eq. (1) where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by Eqs. (4) and (6), respectively, with $\alpha = 2k$. Now $(-1)^k S_{2k}(x) * R_{2k}(x)$ is a generalized function, see [1], and is called the Diamond Kernel of Marcel Riesz. In this paper we study the Fourier transform of $(-1)^k S_{2k}(x) * R_{2k}(x)$ and the Fourier transform of $[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]$ where k and m are nonnegative integers.

2. Preliminaries

Definition 2.1. Let $E(x)$ be a function defined by

$$E(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}, \quad (3)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ and $\omega_n = (2\pi^{n/2})/\Gamma(n/2)$ is a surface area of the unit sphere.

It is well known that $E(x)$ is an elementary solution of the Laplace operator Δ , that is $\Delta E(x) = \delta$ where $\Delta = \sum_{i=1}^n (\partial^2/\partial x_i^2)$ and δ is the Dirac-delta distribution.

Definition 2.2. Let $S_\alpha(x)$ be a function defined by

$$S_\alpha(x) = 2^{-\alpha} \pi^{-n/2} \Gamma\left(\frac{n-\alpha}{2}\right) \frac{|x|^{\alpha-n}}{\Gamma(\frac{\alpha}{2})}, \quad (4)$$

where α is a complex parameter, $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

$S_\alpha(x)$ is called the Elliptic Kernel of Marcel Riesz. Now $S_\alpha(x)$ is an ordinary function for $\operatorname{Re}(\alpha) > n$ and is a distribution of α for $\operatorname{Re}(\alpha) < n$.

From Eqs. (3) and (4) we obtain

$$E(x) = -S_2(x). \quad (5)$$

Definition 2.3. Let $x = (x_1, \dots, x_n)$ be a point in \mathbb{R}^n and write

$$V = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2,$$

where $p+q=n$. Define $\Gamma_- = \{x \in \mathbb{R}^n: x_1 > 0 \text{ and } V > 0\}$ designating the interior of the forward cone and denote $\bar{\Gamma}_-$ by its closure and the following function introduced by Nozaki ([2], p. 72),

$$R_\alpha(x) = \begin{cases} \frac{V^{(n-\alpha)/2}}{K_n(\alpha)}, & \text{if } x \in \Gamma_-, \\ 0, & \text{if } x \notin \Gamma_-. \end{cases} \quad (6)$$

Here $R_z(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and α is a complex parameter and n is the dimension of the space \mathbb{R}^n .

The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{n-\alpha}{2})}.$$

Here $R_z(x)$ is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and is a distribution of α if $\operatorname{Re}(\alpha) < n$.

Let $\operatorname{supp} R_z(x) \subset \bar{F}_+$ where $\operatorname{supp} R_z(x)$ denote the support of $R_z(x)$.

Definition 2.4. Let f be a continuous function, the Fourier transform of f denoted by

$$\mathcal{F}f = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\zeta x} dx, \quad (7)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$ and $\zeta x = \zeta_1 x_1 + \zeta_2 x_2 + \dots + \zeta_n x_n$. Sometimes we write $\mathcal{F}f(x) \equiv \hat{f}(\zeta)$. By Eq. (7), we can define the inverse of the Fourier transform of $\hat{f}(\zeta)$ by

$$f(x) = \mathcal{F}^{-1} \hat{f}(\zeta) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\zeta x} \hat{f}(\zeta) d\zeta. \quad (8)$$

If f is a distribution with compact supports by [3], Theorem 7.4-3, p. 187 Eq. (7) can be written as

$$\mathcal{F}f = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\zeta x} \rangle. \quad (9)$$

Lemma 2.1. The functions $S_z(x)$ and $R_z(x)$ defined by Eqs. (4) and (6), respectively, for $\operatorname{Re}(\alpha) < n$ are homogeneous distributions of order $\alpha - n$.

Proof. Since $R_z(x)$ and $S_z(x)$ satisfy the Euler equation, that is

$$(\alpha - n)R_z(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_z(x) \text{ and } (\alpha - n)S_z(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} S_z(x),$$

we have that $R_z(x)$ and $S_z(x)$ are homogeneous distributions of order $\alpha - n$.

Donoghue ([4], pp. 154 and 155) has proved that every homogeneous distribution is a tempered distribution.

That completes the proof.

Lemma 2.2 (The convolution of tempered distributions). *The convolution $S_z(x) * R_z(x)$ exists and is a tempered distribution.*

Proof. Choose $\text{supp } R_x(x) = K \subset \bar{I}$, where K is a compact set. Then $R_x(x)$ is a tempered distribution with compact support and by [3], pp. 156–159 $S_x(x) * R_x(x)$ exists and is a tempered distribution.

Lemma 2.3. Given the equation $\diamond^k u(x) = \delta$ where the operator \diamond^k is defined by Eq. (2), $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer and δ is the Dirac-delta distribution, then $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the equation where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by Eqs. (4) and (6), respectively, with $\alpha = 2k$.

Proof. By Lemma 2.2, for $\alpha = 2k$, the distribution $(-1)^k S_{2k}(x) * R_{2k}(x)$ exists and is a tempered distribution.

Now the distribution $(-1)^k S_{2k}(x)$ is obtained by the convolution

$$\underbrace{E(x) * E(x) * \dots * E(x)}_{k\text{-times}} = \underbrace{(-S_2(x)) * (-S_2(x)) * \dots * (-S_2(x))}_{k\text{-times}},$$

where $E(x)$ is defined by Eq. (3) and by Eq. (5).

Kananthai ([5], Lemma 2.5) has shown that

$$\underbrace{(-S_2(x)) * (-S_2(x)) * \dots * (-S_2(x))}_{k\text{-times}} = (-1)^k S_{2k}(x)$$

is an elementary solution of the Laplace operator Δ^k iterated k -times. By Eq. (2), \diamond^k can be written as

$$\diamond^k = \square^k \Delta^k, \quad (10)$$

where

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k$$

and

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^k, \quad p + q = n.$$

By [1], Theorem 3.1 $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the operator \diamond^k as required.

Lemma 2.4 (The Fourier transform of $\diamond^k \delta$).

$$\mathcal{F} \diamond^k \delta = \frac{1}{(2\pi)^{n/2}} \left((\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2 \right)^k,$$

where \mathcal{F} is the Fourier transform defined by Eq. (7) and if the norm of ξ is given by $\|\xi\| = (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^{1/2}$ then

$$|\mathcal{F} \diamond^k \delta| \leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{4k} \quad (11)$$

that is $\mathcal{F} \diamond^k \delta$ is bounded and continuous on the space S' of the tempered distribution. Moreover, by Eq. (8)

$$\diamond^k \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left((\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2 \right)^k.$$

Proof. By Eq. (9)

$$\begin{aligned} \mathcal{F} \diamond^k \delta &= \frac{1}{(2\pi)^{n/2}} \langle \diamond^k \delta, e^{-i\xi x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} \langle \delta, \diamond^k e^{-i\xi x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} \langle \delta, \square^k \Delta^k e^{-i\xi x} \rangle \quad \text{by (10)} \\ &= \frac{1}{(2\pi)^{n/2}} \langle \delta, (-1)^k (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^k \square^k e^{-i\xi x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} \langle \delta, (-1)^k (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^k (-1)^k \\ &\quad \times (\xi_1^2 + \xi_2^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \dots - \xi_{p+q}^2)^k e^{-i\xi x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} (-1)^{2k} ((\xi_1^2 + \dots + \xi_n^2)^k \times (\xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \dots - \xi_{p+q}^2)^k) \\ &= \frac{1}{(2\pi)^{n/2}} \left((\xi_1^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \dots + \xi_{p+q}^2)^2 \right)^k. \end{aligned}$$

Now

$$\begin{aligned} |\mathcal{F} \diamond^k \delta| &= \frac{1}{(2\pi)^{n/2}} \left(|\xi_1^2 + \dots + \xi_n^2| |\xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \dots - \xi_{p+q}^2| \right)^k \\ &\leq \frac{1}{(2\pi)^{n/2}} (|\xi_1^2 + \dots + \xi_n^2|)^k \\ &= \frac{1}{(2\pi)^{n/2}} \|\xi\|^{4k}. \end{aligned}$$

where $\|\xi\| = (\xi_1^2 + \dots + \xi_n^2)^{1/2}$, ξ_i ($i = 1, 2, \dots, n$) $\in \mathbb{R}$. Hence we obtain Eq. (11) and $\mathcal{F} \diamond^k \delta$ is bounded and continuous on the space S' of the tempered distribution.

Since \mathcal{F} is 1-1 transformation from the space S' of the tempered distribution to the real space R , then by Eq. (8)

$$\diamond^k \delta = \frac{1}{(2\pi)^{n/2}} \mathcal{F}^{-1} \left((\xi_1^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \dots + \xi_{p+q}^2)^2 \right)^k.$$

That completes the proof.

3. Main results

Theorem 3.1.

$$\begin{aligned} & \mathcal{F}((-1)^k S_{2k}(x) * R_{2k}(x)) \\ &= \frac{1}{(2\pi)^{n/2} [(\xi_1^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \dots + \xi_{p+q}^2)^2]^k} \end{aligned}$$

and

$$|\mathcal{F}((-1)^k S_{2k}(x) * R_{2k}(x))| \leq \frac{1}{(2\pi)^{n/2}} M \text{ for a large } \xi_i \in R. \quad (12)$$

where M is a constant. That is \mathcal{F} is bounded and continuous on the space S' of the tempered distributions.

Proof. By Lemma 2.3 $\diamond^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta$ or $(\diamond^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta$.

Taking the Fourier transform on both sides, we obtain

$$\mathcal{F}((\diamond^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)]) = \mathcal{F} \delta = \frac{1}{(2\pi)^{n/2}}.$$

By Eq. (9)

$$\frac{1}{(2\pi)^{n/2}} \langle (\diamond^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)], e^{-i\tilde{\xi}x} \rangle = \frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\frac{1}{(2\pi)^{n/2}} \langle (\diamond^k \delta), \langle [(-1)^k S_{2k}(r) * R_{2k}(r)], e^{-i\tilde{\xi}(x+r)} \rangle \rangle = \frac{1}{(2\pi)^{n/2}},$$

$$\frac{1}{(2\pi)^{n/2}} \langle [(-1)^k S_{2k}(r) * R_{2k}(r)], e^{-i\tilde{\xi}r} \rangle \langle (\diamond^k \delta), e^{-i\tilde{\xi}x} \rangle = \frac{1}{(2\pi)^{n/2}},$$

$$\mathcal{F}([(-1)^k S_{2k}(r) * R_{2k}(r)])(2\pi)^{n/2} \mathcal{F}(\diamond^k \delta) = \frac{1}{(2\pi)^{n/2}}.$$

By Lemma 2.4,

$$\mathcal{F}([(-1)^k S_{2k}(x) * R_{2k}(x)])((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^k = \frac{1}{(2\pi)^{n/2}}.$$

It follows that

$$\mathcal{F}([(-1)^k S_{2k}(x) * R_{2k}(x)]) = \frac{1}{(2\pi)^{n/2} [(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^k}.$$

Now

$$\begin{aligned} & \frac{1}{[(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^k} \\ &= \frac{1}{(\xi_1^2 + \cdots + \xi_p^2)^k} \frac{1}{(\xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2)^k}. \end{aligned} \quad (13)$$

Let $\xi = (\xi_1, \dots, \xi_n) \in \Gamma_+$ with Γ_+ defined by Definition 2.3. Then $(\xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2) > 0$ and for a large ξ_i and a large k , the right-hand side of Eq. (13) tends to zero. It follows that it is bounded by a positive constant M say, that is we obtain Eq. (12) as required and also by Eq. (12) \mathcal{F} is continuous on the space S' of the tempered distribution.

Theorem 3.2.

$$\begin{aligned} & \mathcal{F}([(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]) \\ &= (2\pi)^{n/2} \mathcal{F}([(-1)^k S_{2k}(x) * R_{2k}(x)]) \mathcal{F}([(-1)^m S_{2m}(x) * R_{2m}(x)]) \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{[(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^{k+m}}, \end{aligned}$$

where k and m are nonnegative integers and \mathcal{F} is bounded and continuous on the space S' of the tempered distribution.

Proof. Since $R_{2k}(x)$ and $S_{2k}(x)$ are tempered distributions with compact supports, we have

$$\begin{aligned} & [(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)] \\ &= (-1)^{k+m} (S_{2k}(x) * S_{2m}(x)) * (R_{2k}(x) * R_{2m}(x)) \\ &= (-1)^{k+m} (S_{2(k+m)}(x) * R_{2(k+m)}(x)) \end{aligned}$$

by [3], pp. 156–159 and [2], Lemma 2.5. Taking the Fourier transform on both sides and using Theorem 3.1 we obtain

$$\begin{aligned}
& \mathcal{F}([(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]) \\
&= \frac{1}{(2\pi)^{n/2} ((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^{k+m}} \\
&= \frac{1}{(2\pi)^{n/2} ((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^k} \\
&\quad \times \frac{(2\pi)^{n/2}}{(2\pi)^{n/2} ((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^m} \\
&= (2\pi)^{n/2} \mathcal{F}[(-1)^k S_{2k}(x) * R_{2k}(x)] \mathcal{F}[(-1)^m S_{2m}(x) * R_{2m}(x)].
\end{aligned}$$

Since $(-1)^k S_{2(k+m)}(x) * R_{2(k+m)}(x) \in S'$, the space of tempered distribution, and by Theorem 3.1 we obtain that \mathcal{F} is bounded and continuous on S' .

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On the Fourier transform of Marcel Riesz's Ultra-
hyperbolic Kernel



On the Fourier Transform of Marcel Riesz's Ultra-hyperbolic Kernel

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ABSTRACT

In this paper, we study the Fourier transform of Marcel Riesz's ultra-hyperbolic kernel $R_\alpha(x)$ for $\alpha = 2k$ ($k = 0, 1, 2, 3, \dots$) and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ the n -dimensional Euclidean space. Moreover, the Fourier transform of the convolution $R_\alpha * R_\beta$ is also determined.

1. INTRODUCTION

The function $R_\alpha(x)$ defined by (2.1) is first introduced by Y. Nozaki [1]. Later S.E Trione [2] has shown that, for $\alpha = 2k$ ($k = 0, 1, 2, 3, \dots$), $R_{2k}(x)$ is the elementary solution of the ultra-hyperbolic operator \square^k iterated k -times. That is $\square^k R_{2k}(x) = \delta$ where \square^k is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad \dots\dots (1.1)$$

$p + q = n$, the dimension of the space \mathbb{R}^n and δ is the Dirac-delta distribution. Moreover M. Aguirre Tellez [3] has proved that $R_{2k}(x)$ exists only for the case p is odd with n is odd or even, where $p + q = n$.

For the convolution $R_\alpha * R_\beta$, A. Kananthai [4] has shown that $R_\alpha * R_\beta = R_{\alpha+\beta}$ for α and β are even integers. In this paper, we study the Fourier transform of the function $R_\alpha(x)$ with $\alpha = 2k$ where k is a non-negative integer. Moreover we also obtain the formula form of the Fourier transform of $R_\alpha * R_\beta$.

2. MATERIALS AND METHODS

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional Euclidean space \mathbb{R}^n .

Denote $v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$, $p + q = n$.

The set $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$ is called an interior of the forward cone. By $\bar{\Gamma}_+$ we denote its closure. For any complex number α , define

$$R_\alpha(x) = \begin{cases} \frac{V^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+ \end{cases} \quad \dots\dots (2.1)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}$$

The function $R_\alpha(x)$ was introduced by Y. Nozaki [1] is called the Marcel Riesz's ultra-hyperbolic kernel. It is well known that $R_\alpha(x)$ is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and it is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha(x)$ denote the support of $R_\alpha(x)$.

Suppose that $\operatorname{supp} R_\alpha(x) \subset \overline{\Gamma}_+$.

Definition 2.2 Let f be continuous function, the Fourier transform of f denoted by

$$\mathcal{F} f \text{ and is defined by } \mathcal{F} f = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx \quad \dots\dots (2.2)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ and $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$. Sometimes we write $\mathcal{F} f(x) \equiv \hat{f}(\xi)$.

By (2.2), we can define the inverse of Fourier transform of $\hat{f}(\xi)$ by

$$f(x) = \mathcal{F}^{-1} \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \hat{f}(\xi) d\xi \quad \dots\dots (2.3)$$

From definition 2.2, if f is a distribution with compact support, by A.H Zemanian [5], we can define

$$\mathcal{F} f = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\xi \cdot x} \rangle \quad \dots\dots (2.4)$$

Lemma 2.1 Given the equation $\square^k u(x) = \delta$ where \square^k is defined by (1.1) and $x \in \mathbb{R}^n$. Then $u(x) = R_{2k}(x)$ is a unique elementary solution of the equation.

Proof - See [2].

Lemma 2.2 (The existence of the convolution $R_\alpha * R_\beta$)

Let $R_\alpha(x)$ be defined by (2.1) and $\operatorname{supp} R_\alpha(x) = K \subset \overline{\Gamma}_+$ where K is a compact set, then $R_\alpha * R_\beta$ exists and is a tempered distribution and particularly $R_\alpha * R_\beta = R_{\alpha+\beta}$ for α and β are even integers.

Proof The proof of existence of $R_\alpha * R_\beta$ has been given by W.F Donoghue [6] and A. Kananthai [4] also proved that $R_\alpha * R_\beta = R_{\alpha+\beta}$ for α and β are even integers.

$$\text{Lemma 2.3 } \mathcal{F} \square^k \delta = \frac{1}{(2\pi)^{n/2}} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k$$

where \square^k is defined by (1.1) and $|\mathcal{F} \square^k \delta| \leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{2k}$ where $\|\xi\| = \left(\sum_{i=1}^n |\xi_i|^2\right)^{1/2}$. That is

\mathcal{F} is continuous on the space S' of tempered distribution.

Proof Since $\square^k \delta$ is a tempered distribution with compact support, hence by (2.4) we have

$$\begin{aligned}\mathcal{F} \square^k \delta &= \frac{1}{(2\pi)^{n/2}} \langle \square^k \delta, e^{-i\xi \cdot x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} \langle \delta, \square^k e^{-i\xi \cdot x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} \langle \delta, ((i\xi_1)^2 + (i\xi_2)^2 + \dots + (i\xi_p)^2 - (i\xi_{p+1})^2 - (i\xi_{p+2})^2 - \dots - (i\xi_{p+q})^2) e^{-i\xi \cdot x} \rangle \\ &= \frac{1}{(2\pi)^{n/2}} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k.\end{aligned}$$

$$\begin{aligned}\text{Since } |\mathcal{F} \square^k \delta| &= \frac{1}{(2\pi)^{n/2}} |(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)|^k, p+q=n \\ &\leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{2k}.\end{aligned}$$

We have $|\mathcal{F} \square^k \delta| \leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{2k}$. That is \mathcal{F} is bounded on S' . It follows that \mathcal{F} is continuous on the space S' of tempered distribution.

3. RESULTS AND DISCUSSION

Theorem 3.1 Let $R_{2k}(x)$ be defined by (2.1) with $\alpha = 2k$, k is a non-negative. Then the Fourier transform of $R_{2k}(x)$ is given by

$$\mathcal{F} R_{2k}(x) = \frac{1}{(2\pi)^{n/2} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k}$$

and $|\mathcal{F} R_{2k}(x)| \leq \frac{M}{(2\pi)^{n/2}}$ where M is a constant and \mathcal{F} is continuous on the space S' of tempered distribution.

Proof We can show that $R_\alpha(x)$ is a tempered distribution, see [4] and we choose $\text{supp } R_\alpha(x) = K \subset \overline{\Gamma}_+$ and K is a compact set. Then we obtain $R_{2k}(x)$ where $\alpha = 2k$ is a tempered distribution with compact support. Then by (2.4)

$$\mathcal{F} R_{2k}(x) = \frac{1}{(2\pi)^{n/2}} \langle R_{2k}(x), e^{-i\xi \cdot x} \rangle$$

To compute $\langle R_{2k}(x), e^{-i\xi \cdot x} \rangle$ directly is difficult. So we apply Lemma 2.1

$$\begin{aligned}\square^k R_{2k}(x) &= \delta \\ \text{or } (\square^k \delta) * R_{2k}(x) &= \delta.\end{aligned}$$

By taking the Fourier transform both sides, we obtain

$$\mathcal{F} [(\square^k \delta) * R_{2k}(x)] = \mathcal{F} \delta = \frac{1}{(2\pi)^{n/2}}.$$

$$\text{Now } \mathcal{F} [(\square^k \delta) * R_{2k}(x)] = \langle \square^k \delta * R_{2k}(x), e^{-i\xi \cdot x} \rangle$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^{n/2}} \langle \square^k \delta, \langle R_{2k}(r), e^{-i\xi \cdot (x+r)} \rangle \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle \square^k \delta, \langle R_{2k}(r), e^{-i\xi \cdot r} \cdot e^{-i\xi \cdot x} \rangle \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle \square^k \delta, e^{-i\xi \cdot x} \langle R_{2k}(r), e^{-i\xi \cdot r} \rangle \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle R_{2k}(r), e^{-i\xi \cdot r} \rangle \langle \square^k \delta, e^{-i\xi \cdot x} \rangle \\
&= \frac{1}{(2\pi)^{n/2}} (2\pi)^{n/2} \mathcal{F} R_{2k}(r) \cdot (2\pi)^{n/2} \mathcal{F} (\square^k \delta) \quad \text{by (2.4)} \\
&= \mathcal{F} R_{2k}(r) \cdot (2\pi)^{n/2} \cdot \frac{1}{(2\pi)^{n/2}} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k \quad \text{by}
\end{aligned}$$

Lemma 2.3. It follows that

$$\mathcal{F} R_{2k}(x) = \frac{1}{(2\pi)^{n/2} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k} \quad \dots (3.1)$$

Now, the right hand side of (3.1) tend to zero for a large ξ_i and a large k , that is bounded by a constant M say, then $|\mathcal{F} R_{2k}(x)| \leq \frac{M}{(2\pi)^{n/2}}$. It follows that \mathcal{F} is a continuous on the space S' . Moreover, by (2.3) we obtain

$$R_{2k}(x) = \mathcal{F}^{-1} \left(\frac{1}{(2\pi)^{n/2} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k} \right).$$

Theorem 3.2 Let $R_{2k}(x)$ and $R_{2m}(x)$ are defined by (2.1) with $\alpha = 2k, 2m$ where k and m are non-negative integers. Then the Fourier transform of the convolution $R_{2k}(x) * R_{2m}(x)$ is given by

$$\mathcal{F} (R_{2k}(x) * R_{2m}(x)) = (2\pi)^{n/2} (\mathcal{F} R_{2k}(x)) (\mathcal{F} R_{2m}(x)).$$

Proof By Lemma 2.2

$$R_{2k} * R_{2m} = R_{2k+2m}.$$

Thus
$$\mathcal{F} (R_{2k}(x) * R_{2m}(x)) = \mathcal{F} R_{2k+2m}(x)$$

$$= \frac{1}{(2\pi)^{n/2} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^{k+m}}$$

by Theorem 3.1.

Now
$$\frac{1}{(2\pi)^{n/2} (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^{k+m}}$$

For the case α is a negative even and β is odd, by (2.5) we have

$$\square^k R_0(x) = R_{-2k}(x) \quad \text{or} \quad \square^k \delta = R_{-2k}(x)$$

where $R_0(x) = \delta$. Now

$$R_{-2k}(x) * \square^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x) \quad \text{for } m \text{ is odd,}$$

or

$$(\square^k \delta) * \square^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$\delta * \square^{2k} R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$R_{m-2(2k)}(x) = R_{-2k}(x) * R_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m - 2k$, now α is a negative even and β is odd. Then we obtain

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x).$$

That completes the proofs. \square

3. Main results

Theorem 3.1. Let $T_m(x)$ the diamond kernel of Marcel Riesz defined by (1.3), then T_m is a tempered distribution and can be expressed by

$$T_m(x) = T_{m-r}(x) * T_r(x).$$

where r is a nonnegative integer and $r < m$. Moreover if we put $\ell = m - r$, $n = r$ we obtain

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) \quad \text{for } \ell + n = m.$$

Proof. Since $T_m = (-1)^m S_{2m}(x) * R_{2m}(x)$, ($m = 0, 1, 2, \dots$), by Lemma 2.2 T_m is a tempered distribution. Now by Lemma 2.3, $\diamond^m T_m = \delta$, then $\diamond^r \diamond^{m-r} T_m = \delta$ for $m > r$ and by Lemma 2.3 again, we obtain $\diamond^{m-r} T_m = (-1)^r S_{2r}(x) * R_{2r}(x)$. Convoluting both sides by $(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)$, we obtain

$$\begin{aligned} & [(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * \diamond^{m-r} T_m \\ &= [(-1)^{m-r} S_{2m-2r}(x) * R_{2m-2r}(x)] * [(-1)^r S_{2r}(x) * R_{2r}(x)] \end{aligned} \quad (3.1)$$

or

$$\begin{aligned} & \diamond^{m-r} [(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * T_m \\ &= (-1)^m (S_{2m-2r}(x) * S_{2r}(x) * (R_{2m-2r}(x) * R_{2r}(x))), \end{aligned}$$

since $S_{2m}(x)$ and $R_{2m}(x)$ are tempered distributions and are the elements of the space of convolution algebra, \mathcal{a}' .

By Lemmas 2.3 and 2.4 we obtain

$$\begin{aligned}\delta * T_m(x) &= (-1)^m S_{2m}(x) * R_{2m}(x), \\ T_m(x) &= (-1)^m S_{2m}(x) * R_{2m}(x).\end{aligned}$$

From (3.1) we have $T_m(x) = T_{m-r}(x) * T_r(x)$, put $\ell = m - r$, $n = r$, it follows that

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) = T_m(x)$$

as required. \square

Theorem 3.2. Let $T_m(x)$ be defined by (1.3) then T_m is an element of the space \mathcal{a}' of convolution algebra and there exist an inverse T_m^{*-1} of T_m such that

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

Proof. Since $T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x)$ is a tempered distribution by Lemma 2.2. Now the supports of $S_{2m}(x)$ and $R_{2m}(x)$ are compact. Then they are the elements of the space of convolution algebra \mathcal{a}' of distribution. By Zemanian [6, Theorem 6.2.1, p. 151] there exist a unique inverse T_m^{*-1} such that

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

That completes the proof. \square

Acknowledgements

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On the Diamond Operator related to the Wave Equation

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Abstract

In this paper, we study the solution of the equation $\diamond^k u(x) = f(x)$ where \diamond^k is the Diamond operator iterated k times and is defined by

$$\diamond^k = \left(\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k$$

where $p+q = n$ is the dimension of the n -dimensional Euclidean space R^n , $x = (x_1, x_2, \dots, x_n) \in R^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a generalized function.

It is found that the solution $u(x)$ depends on the conditions of p and q and moreover such a solution is related to the solution of the Laplace equation and the wave equation.

1 Introduction

The operator \diamond^k has been first introduced by A. Kananthai [3] and is named as the Diamond operator iterated k times and is defined by

$$\diamond^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \dots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right)^k \quad (1.1)$$

where $p+q = n$ is the dimension of the space R^n , $x = (x_1, x_2, \dots, x_n) \in R^n$ and k is a nonnegative integer.

Actually the operator \diamond^k is an extension of the ultrahyperbolic operator and the Laplacian. So the operator \diamond^k can be expressed as the product of the operator \square and \triangle , that is $\diamond^k = \square^k \triangle^k = \triangle^k \square^k$ where

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (1.2)$$

is the ultrahyperbolic operator iterated k -time with $p + q = n$, and

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k \quad (1.3)$$

is the Laplacian iterated k -times.

A. Kananthai ([3], Theorem 3.1 p33) has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementary solution of the operator \diamond^k , that is

$$\diamond^k ((-1)^k R_{2k}^e(x) * R_{2k}^H(x)) = \delta \quad (1.4)$$

where δ is the Dirac-delta distribution and the functions R_{2k}^e and R_{2k}^H are defined by (2.5) and (2.1) respectively with $\alpha = 2k$, k is nonnegative integer.

In this paper, we study the solution of the equation

$$\diamond^k u(x) = f(x) \quad (1.5)$$

This equation is the generalization of the ultrahyperbolic equation and it can be applied to the wave equation and potential that has been shown in the last part of this paper.

Let $K_{\alpha,\beta}(x)$ be a distributional family and is defined by

$$K_{\alpha,\beta}(x) = R_{\alpha}^e * R_{\beta}^H \quad (1.6)$$

where R_{α}^e is called the elliptic Kernel defined by (2.5) and R_{β}^H is called the ultra-hyperbolic Kernel defined by (2.1) and α, β are the complex parameters.

The family $K_{\alpha,\beta}(x)$ is well-defined and is a tempered distribution, since $R_{\alpha}^e * R_{\beta}^H$ is a tempered, see ([1], Lemma 2.2) and R_{β}^H has a compact support.

In this paper, we can show that

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} + (-1)^k K_{2k,2k}(x) * f(x)$$

is a solution of (1.5) where $m = \frac{n-4}{2}$, $n \geq 4$ and n is even number and $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$. Moreover, we can show that the solution $u(x)$ relates to the solution of Laplace operator \triangle^{2k} defined by (1.3) and also the wave operator defined by (1.2) with $k = 1$ and $p = 1$.

2 Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional Euclidean space R^n .

Denote by $v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$, $p + q = n$ the nondegenerated quadratic form. By Γ_+ we designate the interior of the forward cone.

$\Gamma_+ = \{x \in R^n : x_1 > 0 \text{ and } v > 0\}$, and by $\bar{\Gamma}_+$ designate its closure. For any complex number α , define

$$R_\alpha^H(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+ \end{cases} \quad (2.1)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}. \quad (2.2)$$

The function $R_\alpha^H(x)$ was introduced by Nozaki ([4], p.72).

It is well known that $R_\alpha^H(x)$ is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and it is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$. Suppose that $\operatorname{supp} R_\alpha^H(x) \subset \bar{\Gamma}_+$.

From S.E Trione ([5], p11), $R_{2k}^H(v)$ is an elementary solution of the operator \square^k that is

$$\square^k R_{2k}^H(v) = \delta \quad (2.3)$$

where \square^k is defined by (1.2).

By putting $p = 1$ in (2.1) and (2.2) and remembering the Legendre's duplication of $\Gamma(z)$.

$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$ then the formula (2.1) reduces to

$$M_\alpha(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \quad (2.4)$$

Here $v = x_1^2 - x_2^2 - \dots - x_n^2$ and $H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha-n+2}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)$.

$M_\alpha(v)$ is, precisely, the hyperbolic kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, \dots, x_n)$ be a point of R^n and the function $R_\alpha^e(x)$ be defined by

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \quad (2.5)$$

where $W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$, α is a complex parameter and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$.

It can be shown that $R_{-2k}^e(x) = (-1)^k \Delta^k \delta(x)$ where Δ^k is defined by (1.3). It follows that $R_0^e(x) = \delta$, see ([2], p118).

Moreover, we obtain $(-1)^k R_{2k}^e(x)$ is an elementary solution of the operator Δ^k , that is

$$\Delta^k ((-1)^k R_{2k}^e(x)) = \delta \quad (2.6)$$

see ([3], Lemma 2.4 p31).

Lemma 2.1 Given P is a hyper-surface then

$$P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.

Proof. See ([1], P233).

Lemma 2.2 Given the equation

$$\Delta^k u(x) = 0 \quad (2.7)$$

where Δ^k is defined by (1.3) and $x = (x_1, x_2, \dots, x_n) \in R^n$ then

$u(x) = (-1)^{(k-1)} \left(R_{2(k-1)}^e(x) \right)^{(m)}$ is a solution of (2.7) where m is a nonnegative integer with $m = \frac{n-4}{2}, n \geq 4$ and n is even and $\left(R_{2(k-1)}^e(x) \right)^{(m)}$ is a function defined by (2.5) with m derivatives with $\alpha = 2(k-1)$

Proof. We first show that the generalized function $u(x) = \delta^{(m)}(r^2)$ where $r^2 = |x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$ is a solution of

$$\Delta u(x) = 0 \quad (2.8)$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is a Laplace operator. Now

$$\begin{aligned} \frac{\partial}{\partial x_i} \delta^{(m)}(r^2) &= 2x_i \delta^{(m+1)}(r^2) \\ \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2) &= 2\delta^{(m+1)}(r^2) + 4x_i^2 \delta^{(m+2)}(r^2). \end{aligned}$$

Thus

$$\begin{aligned} \Delta \delta^{(m)}(r^2) &= \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2) \\ &= 2n\delta^{(m+1)}(r^2) + 4r^2 \delta^{(m+2)}(r^2) \\ &= 2n\delta^{(m+1)}(r^2) - 4(m+2)\delta^{(m+1)}(r^2) \end{aligned}$$

by Lemma 2.1 with $P = r^2$. We have

$$\begin{aligned} \Delta \delta^{(m)}(r^2) &= [2n - 4(m+2)]\delta^{(m+1)}(r^2) \\ &= 0 \quad \text{if } 2n - 4(m+2) = 0 \end{aligned}$$

or $m = \frac{n-4}{2}, n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of (2.8) with $m = \frac{n-4}{2}, n \geq 4$ and n is even. Now $\Delta^k u(x) = \Delta(\Delta^{k-1} u(x)) = 0$ then from the above proof $\Delta^{k-1} u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{k-1}R_{2(k-1)}^e(x)$, we obtain

$$\begin{aligned} (-1)^{k-1}R_{2(k-1)}^e(x) * \Delta^{k-1}u(x) &= (-1)^{k-1}R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) \\ \text{or } \Delta^{k-1} \left((-1)^{k-1}R_{2(k-1)}^e(x) \right) * u(x) &= (-1)^{k-1}R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) \end{aligned}$$

$$\text{or } \delta * u(x) = u(x) = (-1)^{k-1}R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) \quad \text{by (2.6)}$$

Now from (2.1)

$$\begin{aligned} R_{2(k-1)}^e(x) &= \frac{|x|^{2(k-1)-n}}{W_n(\alpha)} \\ &= \frac{(|x|^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} = \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Hence

$$\begin{aligned} R_{2(k-1)}^e(x) * \delta^{(m)}(r^2) &= \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} * \delta^{(m)}(r^2) \\ &= \left[\frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} \right]^{(m)} = \left[R_{2(k-1)}^e(x) \right]^{(m)}. \end{aligned}$$

It follows that $u(x) = (-1)^{k-1} \left[R_{2(k-1)}^e(x) \right]^{(m)}$ is a solution of (2.7) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension of R^n .

Lemma 2.3 Given the equation

$$\square^k u(x) = 0 \tag{2.9}$$

where \square^k is defined by (1.2) and $x = (x_1, x_2, \dots, x_n) \in R^n$ then $u(x) = \left[R_{2(k-1)}^H(v) \right]^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension and v is defined by Definition 2.1. The function $\left[R_{2(k-1)}^H(v) \right]^{(m)}$ is defined by (2.1) with m -derivatives and $\alpha = 2(k-1)$.

Proof At first we show that the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, $p+q = n$, is a solution of the equation

$$\square u(x) = 0 \tag{2.10}$$

where \square is defined by (1.2) with $k = 1$ and $x = (x_1, x_2, \dots, x_n) \in R^n$. Now

$$\begin{aligned}
\frac{\partial}{\partial x_i} \delta^{(m)}(r^2 - s^2) &= 2x_i \delta^{(m+1)}(r^2 - s^2) \\
\frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) &= 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2 \delta^{(m+2)}(r^2 - s^2) \\
\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) &= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2 \delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&\quad + 4s^2 \delta^{(m+2)}(r^2 - s^2) \\
&= 2p\delta^{(m+1)}(r^2 - s^2) \\
&\quad - 4(m+2)\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4s^2 \delta^{(m+2)}(r^2 - s^2) \\
&= [2p - 4(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4s^2 \delta^{(m+2)}(r^2 - s^2)
\end{aligned}$$

by Lemma 2.1 with $P = r^2 - s^2$.

Similarly,

$$\begin{aligned}
\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \delta^{(m)}(r^2 - s^2) &= [-2q + 4(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4r^2 \delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\Box \delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \delta^{(m)}(r^2 - s^2) \\
&= [2(p+q) - 8(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad - 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&= [2n - 8(m+2)]\delta^{(m+1)}(r^2 - s^2) \\
&\quad + 4(m+2)\delta^{(m+1)}(r^2 - s^2) \quad \text{by Lemma 2.1} \\
&= [2n - 4(m+2)]\delta^{(m+1)}(r^2 - s^2).
\end{aligned}$$

If $2n - 4(m+2) = 0$, we have $\Box \delta^{(m)}(r^2 - s^2) = 0$. That is $u(x) = \delta^{(m)}(r^2 - s^2)$ is a solution of (2.10) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension. Now $\Box^k u(x) = \Box(\Box^{k-1} u(x)) = 0$.

From (2.10) we have $\Box^{k-1} u(x) = \delta^{(m)}(r^2 - s^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension.

Convolving the above equation by $R_{2(k-1)}^H(v)$, we obtain

$$\begin{aligned}
R_{2(k-1)}^H(v) * \Box^{k-1} u(x) &= R_{2(k-1)}^H(v) * \delta^{(m)}(r^2 - s^2) \\
\Box^{k-1} [R_{2(k-1)}^H(v)] * u(x) &= [R_{2(k-1)}^H(v)]^{(m)} \quad \text{where } v = r^2 - s^2 \\
\delta * u(x) = u(x) &= [R_{2(k-1)}^H(v)]^{(m)}
\end{aligned}$$

by (2.3) and $v = r^2 - s^2$ is defined by Definition 2.1.

Thus $u(x) = [R_{2(k-1)}^H(v)]^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension.

Lemma 2.4 Given the equation

$$\diamond^k u(x) = 0 \quad (2.11)$$

where \diamond^k is the Diamond operator iterated k -times defined by (1.1) and $u(x)$ is an unknown generalized function. Then

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} \quad (2.12)$$

is a solution of (2.11), $(R_{2(k-1)}^H(v))^{(m)}$ is a function with m -derivatives defined by (2.1) and v is defined by definition 2.1.

Proof Now $\diamond^k u(x) = \square^k \triangle^k u(x) = 0$. By Lemma 2.3, $\triangle^k u(x) = (R_{2(k-1)}^H(v))^{(m)}$.

Convolving both sides by $(-1)^k R_{2k}^e(x)$, we have

$$(-1)^k R_{2k}^e(x) * \triangle^k u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}.$$

By (2.6), $\triangle^k ((-1)^k R_{2k}^e(x)) * u(x) = \delta * u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}$. It follows that

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}. \quad (2.13)$$

3 Main results

Theorem Given The equation

$$\diamond^k u(x) = f(x) \quad (3.1)$$

where \diamond^k is the Diamond operator iterated k -times defined by (1.1), $f(x)$ is a generalized function, $u(x)$ is an unknown generalized function and $x = (x_1, x_2, \dots, x_n) \in R^n$ -the n -dimensional Euclidean space and n is even, then (3.1) has the general solution

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} + (-1)^k K_{2k,2k}(x) * f(x) \quad (3.2)$$

where $(R_{2(k-1)}^H(v))^{(m)}$ is a function with m -derivatives defined by (2.1) and $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$.

Proof Convolving (3.1) both sides by $(-1)^k K_{2k,2k}(x)$, we obtain

$$(-1)^k K_{2k,2k}(x) * \diamond^k u(x) = (-1)^k K_{2k,2k}(x) * f(x).$$

By (1.4) $\diamond^k ((-1)^k K_{2k,2k}(x)) * u(x) = \delta * u(x) = (-1)^k K_{2k,2k}(x) * f(x)$. We obtain $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$. Since, for a Homogeneous equation $\diamond^k u(x) = 0$ we have a solution $u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)}$.

Thus the general solution of (3.1) is

$$u(x) = (-1)^k R_{2k}^e(x) * (R_{2(k-1)}^H(v))^{(m)} + (-1)^k K_{2k,2k}(x) * f(x).$$

In particular, if $q = 0$ the equation (3.1) becomes the Laplace equation $\Delta^{2k}u(x) = f(x)$ where $x = (x_1, x_2, x_3, \dots, x_p) \in R^p$ and p is even. Using the formulae $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + \frac{1}{2})$ and $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi \sec(\pi z)$. Then for $\alpha = 2k$ the function of (2.1) becomes $(-1)^k R_{2k}^e(x)$ where $R_{2k}^e(x)$ is defined by (2.5) and $x = (x_1, x_2, \dots, x_p) \in R^p$. Thus by (1.6)

$$\begin{aligned} (-1)^k K_{2k,2k}(x) &= (-1)^k R_{2k}^e(x) * (-1)^k R_{2k}^e(x) \\ &= R_{4k}^e(x) \text{ see ([2], p156-159)} \end{aligned}$$

where $x = (x_1, x_2, \dots, x_p) \in R^p$ and p is even.

Now, from (2.12) for $q = 0$ we have

$$\begin{aligned} u(x) &= (-1)^k R_{2k}^e(x) * (-1)^{k-1} (R_{2(k-1)}^e(x))^{(m)} \\ &= (-1)^{2k-1} (R_{4k-2}^e(x))^{(m)} \text{ for } x = (x_1, x_2, \dots, x_p) \in R^p. \end{aligned}$$

Thus the equation (3.2) becomes

$$u(x) = (-1)^{2k-1} (R_{4k-2}^e(x))^{(m)} + R_{4k}^e(x) * f(x) \quad (3.3)$$

for $x = (x_1, x_2, \dots, x_p) \in R^p$ and p is even.

It follows that (3.3) is the general solution of the Laplace equation $\Delta^{2k}u(x) = f(x)$ where Δ^{2k} is the Laplace operator iterated $2k$ -times defined by (1.3) for $x = (x_1, x_2, \dots, x_n) \in R^n$ and n is even.

Now consider the case for the Wave Equation. Given the equation

$$\square^k V(x) = f(x) \quad (3.4)$$

where $f(x)$ is a generalized function, \square^k is defined by (1.2) and $V(x)$ is an unknown function. By ([5], p11) we obtain $V(x) = R_{2k}^H(v) * f(x)$ is a solution of (3.4) where $R_{2k}^H(v)$ is defined by (2.1).

Now, from (3.1) we have $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ is a solution where $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$ or

$$u(x) = [(-1)^k R_{2k}^e(x) * R_{2k}^e(v)] * f(x) \quad (3.5)$$

Convolving both sides of (3.5) by $(-1)^k R_{-2k}^e(x)$, we obtain

$$\begin{aligned} (-1)^k R_{-2k}^e(x) * u(x) &= ((-1)^{2k} R_{-2k}^e(x) * R_{2k}^e(x)) * R_{2k}^H(v) * f(x) \\ &= (R_0^e(x) * R_{2k}^H(v)) * f(x) \\ &= (\delta * R_{2k}^H(v)) * f(x) \\ &= R_{2k}^H(v) * f(x) \text{ see ([2], p156-159)} \end{aligned}$$

Thus it follows that

$$V(x) = (-1)^k R_{-2k}^e(x) * u(x) \quad (3.6)$$

In particular, for $k = 1$, we obtain $V(x) = R_2^H(v) * f(x)$ is a solution of the equation

$$\square V(x) = f(x). \quad (3.7)$$

If we put $p = 1$ and $x_1 = t(\text{time})$, then $\square = \frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2}$ is the wave operator. Thus (3.7) becomes wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2} \right) V(x) = f(x) \quad (3.8)$$

Thus $V(x) = M_2(v) * f(x)$ is a solution of (3.8) and the general solution of (3.8) is $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ where $\delta^{(m)}(v)$ is a solution for $f(x) = 0$ $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$ and $M_2(v)$ is defined by (2.4) with $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$.

Now in (3.1), put $k = 1$ and by (3.2) we obtain

$$\begin{aligned} u(x) &= (-1) R_2^e(x) * (R_0^H(v))^{(m)} + (-1) K_{2,2}(x) * f(x) \\ &= (-1) R_2^e(x) * \delta^{(m)}(v) + (-1) K_{2,2}(x) * f(x) \end{aligned} \quad (3.9)$$

is a solution of the equation $\diamond u(x) = f(x)$ and by (3.6) with $k = 1$, $V(x) = (-1) R_{-2}^e(x) * u(x)$ is a solution of (3.7) where $u(x)$ is defined by (3.9). Thus

$$\begin{aligned} V(x) &= (-1) R_{-2}^e(x) * ((-1) R_2^e(x) * \delta^{(m)}(v)) \\ &\quad + (-1) R_{-2}^e(x) * ((-1) K_{2,2}(x) * f(x)) \\ &= (R_{-2}^e(x) * R_2^e(x)) * \delta^{(m)}(v) \\ &\quad + (R_{-2}^e(x) * R_2^e(x)) * R_2^H(x) * f(x) \\ &= R_0^e(x) * \delta^{(m)}(v) + R_0^e(x) * (R_2^H(x) * f(x)) \\ &= \delta^{(m)}(v) + R_2^H(x) * f(x) \quad (R_0^e(x) = \delta) \end{aligned} \quad (3.10)$$

where $v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$, $p + q = n$.

Now, if we put $p = 1$ and $x_1 = t$, then (3.10) becomes $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ since $R_2^H(x)$ becomes $M_2(v)$ for $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$ where $M_2(v)$ is defined by (2.4) with $\alpha = 2$. Thus $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ is the general solution of the wave equation (3.8) and $\delta^{(m)}(v)$ is a solution of

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2} \right) V(x) = 0 \quad (3.11)$$

Now $v = t^2 - x_2^2 - x_3^2 - \dots - x_n^2$, let $r^2 = x_2^2 + x_3^2 + \dots + x_n^2$. Thus by ([1], p234-236) we obtain $V(x, t) = \delta^{(m)}(t^2 - r^2)$ is the solution of (3.11) with the initial conditions $V(x, 0) = 0$ and $\frac{\partial V(x, 0)}{\partial t} = (-1)^m 2\pi^{m+1} \delta(x)$ at $t = 0$ and $x = (x_2, x_3, \dots, x_n) \in R^{n-1}$.

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