เอกสารหมายเลข 1

Basic Concepts of Convolution of functions

1. Convolution of Functions
2. Convolution, Derivation and Regularization

(iii) The support of ρ_n is in $[-\varepsilon_n, \varepsilon_n]$, $\varepsilon_n > 0$, and $\lim_{n\to\infty} \varepsilon_n = 0$.

Definition 2.2 If $f \in L^1(R)$, the function $f * \rho_n$ are called regularization of f.

The relation between f and its regularizations is shown in next theorem.

Theorem 2.3 Let f be a function in $L^1(R)$. For $\varepsilon > 0$ there exists g_ε in $\mathcal{D}(R)$ such that $\|f - g_\varepsilon\|_1 \leq \varepsilon$.

Theorem 2.4 If $f \in L^1(R)$ and ρ_n is a regularizing sequence, then $\lim_{n\to\infty} \|f - f * \rho_n\|_1 = 0$.

Since $\mathcal{S}(R)$ is in $L^1(R)$, we know that the convolution of two functions in $\mathcal{S}(R)$ is in $L^1(R)$. However, theorem 2.5 gives a better result.

Theorem 2.5 If f and g are in $\mathcal{S}(R)$, then the folowing hold:

(i) $f * g$ in in $\mathcal{S}(R)$.

(ii) The convolution is a continuous operator from $\mathcal{S}(R) \times \mathcal{S}(R)$ to $\mathcal{S}(R)$.

Convolution of Distribution

1. The Convolution of a Distribution and a C^∞ Function

2. The Convolution $\mathcal{C}' * \mathcal{D}'$

3. The Convolution $\mathcal{C}' * \mathcal{S}'$

4. The Convolution $\mathcal{D}'_+ * \mathcal{D}'_+$

The Associativity of Convolution

Convolution of Distribution

In the first section we disscussed the convolution of functions. There we are saw that it is not always possible to take the convolution of two functions; it is the same for distributions. We will study the convolution of distributions and its basic properties for the more important cases.

1. The convolution of a distribution and a C^∞ function

When f and g are in $L^1(\Re)$, the convolution $f*g=\int_\Re f(x-t)g(t)dt$ is well-defined and $f*g\in L^1(\Re)$. Now consider $f*g$ as a distribution. For $\varphi\in\mathcal{D}(\Re)$ we have

$$
\begin{aligned}
<f*g,\varphi> &= \int_\Re\left(\int_\Re f(x-t)g(t)dt\right)\varphi(x)dx \\
&= \int_\Re\left(\int_\Re f(x-t)\varphi(x)dt\right)g(t)dx \qquad (1.1)\\
&= \int_\Re\left(\int_\Re g(x-u)\varphi(x)dt\right)f(u)dx \qquad (1.2)
\end{aligned}
$$

From this is appears that one should study the quatities $f_\sigma*\varphi$ and $g_\sigma*\varphi$ when f and g are distributions.

Prosposition 1.1 *Suppose that $\varphi\in C^\infty(\Re)$ and $T\in\mathcal{D}'(\Re)$ satisfy one of the follwing three conditions:*

(i) $\varphi\in\mathcal{D}(\Re)$ *and* $T\in\mathcal{D}'(\Re)$.

(ii) $\varphi\in\mathcal{S}(\Re)$ *and* $T\in\mathcal{S}'(\Re)$.

(iii) $\varphi\in C^\infty(\Re)$ *and* $T\in C'(\Re)$

Then the function φ define by

$$\psi(x)=<\tau_x T,\varphi> \qquad (1.3)$$

is infitely differentiable, and

$$\psi^{(k)}(x)=<\tau_x T,\varphi^{(k)}> \quad for \ \ k=1,2,... \qquad (1.4)$$

Proof.

(i) When $\varphi\in\mathcal{D}(\Re)$, the expression $<\tau_x T,\varphi>$ makes sense for all $x\in\Re$. We wish to show that the function $\phi(x)=<\tau_x T,\varphi>$ is differentiable. Thus let h_n be a sequence of nonzero reals that tends to 0 as $n\to\infty$. Define

$$\alpha_n(y)=\frac{1}{h_n}[\varphi(y+x+h_n)-\varphi(y+x)];$$

then

$$\frac{1}{h_n}[\psi(x+h_n)-\psi(x)]=<T,\alpha_n>.$$

Now, $\lim_{n\to\infty}\alpha_n(y)=\tau_{-x}\varphi'(y)$. To prove that ψ is differentiable it is sufficient to show that α_n converges to $\tau_{-x}\varphi'$ in $\mathcal{D}(\Re)$.

We consider the support of the α_n. If supp$(\varphi) \subset [-M, M]$ and $|h_n| \le 1$, then supp$\subset [-x-M-1, -x+M+1]$, which is a fixed compact interval K. The following inequality, which is based on the mean value theorem, show that α_n and of all its derivatives converges uniformly on K: For each $q \in N$,

$$|\alpha_n^{(q)}(y) - \varphi^{(q+1)}(x+y)| = |\varphi^{(q+1)}(x+y+\theta_n h_n) - \varphi^{(q+1)}(x+y)|$$
$$\le |h_n| ||\varphi^{(q+2)}||_\infty, \quad 0 < \theta_n < 1.$$

This prove that ψ is differentiable and that

$$\psi'(x) = < T, \tau_{-x}\varphi' > = < \tau_x T, \varphi' > .$$

Similary, one proves that $\psi \in \mathcal{C}^\infty(\Re)$ and that equation (1.4) holds for $k > 1$.

(ii) If $\varphi \in \mathcal{S}(\Re)$ and $T \in \mathcal{S}'(\Re)$, then $< \tau_x T, \varphi >$ makes sense. To show that ψ is differentiable it is sufficient to verify that α_n converges to $\tau_{-x}\varphi'$ in $\mathcal{S}(\Re)$. As in (i), the mean value theorem leads to the inequality

$$|y^p(\alpha_n - \tau_{-x}\varphi')^{(q)}(y)| \le |h_n||y^p\varphi^{(q+2)}(x+y+\rho_n h_n)|$$
$$= \frac{|h_n||y|^p}{1+|x+y+\rho_n h_n|^p}[(1+|x+y+\rho_n h_n|^p)\varphi^{(q+2)}(x+y+\rho_n h_n)]$$

where $0 < \rho_n < 1$. Consequently,

$$\sup_{y\in\Re}|y^p(\alpha_n - \tau_{-x}\varphi')^{(q)}(y)| \le C|h_n| \left(||\varphi^{(q+2)}||_\infty + \sup_{t\in\Re}|t^p\varphi^{(q+2)}(t)| \right)$$

for some constant C. Since φ is in $\mathcal{S}(\Re)$, we see that α_n converges in $\mathcal{S}(\Re)$ to $\tau_{-x}\varphi'$ as $n \to \infty$. That $\psi \in \mathcal{C}^\infty(\Re)$ and (1.4) are proved similaly.

(iii) Again, $< \tau_x T, \varphi >$ makes sense because $\varphi \in \mathcal{C}^\infty(\Re)$ and $T \in \mathcal{C}'(\Re)$. To prove that ψ is differentiable it is sufficient to show that $\alpha_n^{(q)}$ converges to $(\tau_{-x}\varphi')^{(q)}$ uniformly on all compact subsets of $\Re$. This is done using inequalities similar to those used in the proofs of (i) and (ii). □

Definition 1.2 Assume that $\varphi \in \mathcal{C}^\infty(\Re)$ and $T \in \mathcal{C}'(\Re)$ satisfy one of the conditions in Proposition 1.1. The convolution of φ and T is the function $\varphi * T$ define by

$$\varphi * T(x) = < T_y, \varphi(x-y) > . \tag{1.5}$$

The "y" in this definition indicates the variable of "integration." Written differently,

$$< T_y, \varphi(x-y) > = < \tau_{-x}Tv, \varphi_\sigma >,$$

which is the function we studied in Proposition 1.1. Thus we know the meaning of the convolutions

$$\mathcal{D} * \mathcal{D}', \quad \mathcal{S} * \mathcal{S}', \quad \mathcal{C}^\infty * \mathcal{C}'.$$

Proposition 1.3 (convolution $\mathcal{S} * \mathcal{S}'$) *Assume $\varphi \in \mathcal{S}(\Re)$ and $T \in \mathcal{S}(\Re)$. The convolution $\varphi * T$ and all of its derivatives are slowly increasing C^∞ functions.*

Proof. By definition $\varphi * T(x) =< T_y, \varphi(x-y) >$. By using a known result, we can write $T = \sum_{k=1}^{p} f_k^{(n_k)}$, where the continuous slowly increasing functions f_k satisfy $|f_k(x)| \leq C_k(1+x^2)^{N_k}$. Then

$$\begin{aligned}\varphi * T(x) &= \sum_{k=1}^{p} < f_k^{(n_k)}(y), \varphi(x-y) > \\ &= \sum_{k=1}^{p} \int_{\Re} f_k(y)\varphi^{(n_k)}(x-y)dy \\ &= \sum_{k=1}^{p} \int_{\Re} f_k(x-y)\varphi^{(n_k)}dy\end{aligned}$$

and,

$$\begin{aligned}|\varphi * T(x)| &\leq \sum_{k=1}^{p} C_k \int_{\Re} (1+(x-y)^2)^{N_k} |\varphi^{(n_k)}(y)|dy \\ &\leq \sum_{k=1}^{p} C_k \int_{\Re} \sum_{j=1}^{2N_k} |x|^j |\beta_{kj}(y)\varphi^{(n_k)}(y)|dy,\end{aligned}$$

where $\beta_{kj}(y)$ is a polynomail in y.

Since $\varphi \in \mathcal{S}(\Re)$, $\beta_{kj}\varphi^{(n_k)}$ is in $\mathcal{S}(\Re)$ and hence in $L^1(\Re)$. Thus $|\varphi * T(x)|$ is bounded by a polynomail in x, which proves that it is slowly increasing.

One obtain similar estimates for the derivatives of $\varphi * T(x)$ by repeating the computation for $(\varphi * T)^{(k)}(x) =< T_y, \varphi^{(k)}(x-y) >$. □

We are now going state a few of the convolution's essential properties. The convolution defined in Definition 1.2 is an operator that is continuous in each variable. The proof of this result , which we assume, depends on the topologies of the three spaces involved.

Proposition 1.4 (continuity) *The mapping $(\varphi, T) \to \varphi * T$ defined under one of the hypotheses of Proposition 1.1 is continuous with respect to each variable.*

Proposition 1.5 (derivative) *If $\varphi \in C^\infty(\Re)$ and $T \in \mathcal{D}'(\Re)$ satisfy one of the hypotheses of Proposition 32.1.1, then $\varphi * T \in C^\infty(\Re)$ and*

$$(\varphi * T)^{(k)} = \varphi^{(k)} * T = \varphi^{(k)} * T^{(k)}, \quad k = 1, 2, \ldots \tag{1.6}$$

Proof. From (1.4), the kth derivative of the function

$$H(x) = < T_y, \varphi(x-y) > = < \tau_{-x}T, \varphi_\sigma >$$

is

$$\begin{aligned} H^{(k)}(x) &= (-1)^k < \tau_{-x}T^{(k)}, (\varphi_\sigma) > \\ &= < \tau_{-x}T^{(k)}, (\varphi_\sigma)_\sigma^{(k)} > \\ &= < T_y, \varphi^{(k)}(x-y) > \\ &= \varphi^{(k)} * T(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi * T^{(k)}(x) &= < \tau_{-x}T^{(k)}, \varphi_\sigma > = (-1)^k < T, (\varphi_\sigma)^{(k)}(y-x) > \\ &= < T, \varphi^{(k)}(x-y) > = \varphi^{(k)} * T(x). \end{aligned}$$

□

Proposition 1.6 (support) *If $\varphi \in C^\infty(\Re)$ and $T \in C'(\Re)$ then* $\text{supp}(\varphi * T) \subset \text{supp}(\varphi) + \text{supp}(T)$, *where " + " denotes the algebraic sum of the two sets.*

Proof. Since $\text{supp}(T)$ is compact, $\text{supp}(\varphi) + \text{supp}(T)$ is closed. Define $\Omega = \Re \setminus (\text{supp}(\varphi) + \text{supp}(T))$. For $x \in \Omega$ and $y \in \text{supp}(\varphi), (x-y) \notin \text{supp}(T)$, and hence $< T_y, \varphi(x-y) > = 0$. This proves the required inclusion. □

Corollary 1.7 *If $\varphi \in \mathcal{D}(\Re)$ and $T \in C'(\Re)$ then the convolution $\varphi * T$ has compact support.*

The results of this section show that the convolution of a distribution and a C^∞ function is a smoothing operation. We will see in the next section, where we introduce the convolution of two distributions, that $\mathcal{D}(\Re)$ is even dense in $\mathcal{D}'(\Re)$.

2 The convolution $C' * \mathcal{D}'$

The Expression (1.1) and (1.2) suggest a way to generalize the convolution to distributions. Given two distributions S and T we write

$$\begin{aligned} < S * T, \varphi > &= < S_t, < T_x, \varphi(x+t) >> \\ &= < T_u, < S_x, \varphi(x+u) >> . \end{aligned}$$

We have seen that the function $\psi(t) = < T_x, \varphi(x+t) >$ is C^∞ when $T \in \mathcal{D}'(\Re)$. Thus the expression $< S_t, \psi(t) >$ makes sense when $S \in C'(\Re)$. Similarly, the function $\alpha(u) = < S_x, \varphi(x+u) >$ is in $\mathcal{D}(\Re)$ by Proposition 1.6, and the expression $< T_u, \alpha(u) >$ makes sense for all $T \in \mathcal{D}'(\Re)$. It is not clear, however, that $< S_t, \psi(t) >=< T_u, \alpha(u) >$, as is the case for functions. This is, in fact, true ; we state without proof the next result.

Theorem 2.1 $(C' * \mathcal{D}')$ *Assume* $S \in C'(\Re)$ *and* $T \in \mathcal{D}'(\Re)$.

(i) *There exists a distribution calle the convolution of* S *and* T *and denoted by* $S * T$ *such that for all* $\varphi \in \mathcal{D}(\Re)$,

$$< S * T, \varphi >=< S_t, < T_x, \varphi(x+t) >>=< T_u, < S_x, \varphi(x+u) >> . \tag{1.7}$$

(ii) *The maping* $(S, T) \mapsto S * T$ *from* $C'(\Re) \times \mathcal{D}'(\Re)$ *to* $\mathcal{D}'(\Re)$ *is continuous with respect to each variable.*

Formular (1.7) is important because it allows us to develop a calculus for the convolution. The convolution is a commutative operation; we next consider its other important properties.

Proposition 2.2 (Dirac distributions) *Take* $T \in \mathcal{D}'(\Re)$.

(i) *Then*

$$\sigma_a * T = T * \delta_a = \tau_a T, \tag{1.8}$$

and in particular, δ *acts like a unit element for convolution.*

(ii)

$$\delta^{(k)} * T = T * \delta^{(k)} = T^{(k)}, \quad k = 1, 2, \ldots \tag{1.9}$$

Proof. One needs to be careful not to confuse the index a in δ_a with the "dummy variables" u and x in (1.7). Both results follow from simple computations. To prove (i) we have

$$\begin{aligned} < \delta_a * T, \varphi > &=< T_u, < \delta_a, \varphi(x+u) >>=< T_u, \varphi(a+u) > \\ &=< T, \tau_{-a}\varphi >=< \tau_a T, \varphi > . \end{aligned}$$

For the proof of (ii) we have

$$\begin{aligned} < \delta^(k) * T, \varphi > &=< T_u, < \delta^{(k)}, \varphi(x+u) >>= (-1)^k < T_u, \varphi^{(k)}(u) > \\ &=< T^{(k)}, \varphi > . \end{aligned}$$

□

Proposition 2.3(derivatives) *If* $S \in C'(\Re)$ *and* $T \in \mathcal{D}'(\Re)$, *then*

$$(S * T)^{(k)} = S^{(k)} * T = S * T^{(k)}, \quad k = 1, 2, \ldots \tag{1.10}$$

Proof. For $\varphi \in \mathcal{D}(\Re)$,

$$\begin{aligned} < (S * T)^{(k)}, \varphi > &= (-1)^k < S * T, varphi^{(k)} >= (-1)^k < S_t, < T_x, \varphi^{(k)}(x+t) >> \\ &=< S_t, < T_x^{(k)}, \varphi(x+t) >>=< S * T^{(k)}, \varphi > . \end{aligned}$$

Similarly, $(S * T)^{(k)} = S^{(k)} * T$. □

Remark 2.4 If $T \in C'(\Re)$, one can find a primitive of T by writing $T * U$ where U is the Heaviside distribution. Indeed, from (1.10) we see that $(T * U)' = T * U' = T * \delta = T$.

Proposition 2.5 (support of a convolution)

(i) *Assume* $S \in C'(\Re)$ *and* $T \in \mathcal{D}'(\Re)$. *If* $\text{supp}(S) = A$ *and* $\text{supp}(T) = B$, *then* $\text{supp}(S * T) \subset A + B$.

(ii) *If* S *and* T *are in* $C'(\Re)$, *then* $S * T \in C'(\Re)$.

Proof.

(i) Since A is compact, $A + B$ is closed. Let $\Omega = \Re \setminus (A + B)$ and take $\varphi \in \mathcal{D}(\Re)$ with $\text{supp}(\varphi) \subset \Omega$. We will show that $< S * T, \varphi >= 0$. We have $< S * T, \varphi >=< T_u, < S_x, \varphi(x + u) >>$ and $\varphi(u) =< S_x, \varphi(x + u) >= S * \varphi_\sigma(-u)$. Thus we wish to show that $\text{supp}(\varphi) \cap B = \phi$.

If $u \in \text{supp}(\varphi) \cap B$, then $-u \in \text{supp}(S)$ (recall that by Proposition 1.6 $\text{supp}(S * \varphi_\sigma) \subset \text{supp}\varphi_\sigma + \text{supp}S$). This means that $-u = y + x$ with $-y \in \text{supp}(\varphi)$ and $x \in \text{supp}(S)$. But then $-y = u + x$ with $u \in B$, $x \in A$, and hence

$$\text{supp}(\varphi) \cap (A + B) \neq \phi,$$

which is a contradiction, since $\text{supp}(\varphi) \subset \Omega = \Re \setminus (A + B)$.

(ii) If S and T are in $C'(\Re)$, then $A + B$ is compact. □

Proposition 2.6 (density of $\mathcal{D}(\Re)$ in $\mathcal{D}'(\Re)$)
If $T \in \mathcal{D}'(\Re)$, *then there exists a regularizing sequence* $\theta_n \in \mathcal{D}(\Re)$ *such that* θ_n *converges to* T *in* $\mathcal{D}'(\Re)$.

Proof. Choose the usual regularizing sequence ρ_n. We know that ρ_n tends to δ in $\mathcal{D}'(\Re)$. Let $\alpha_n = \rho_n * T$. By Theorem 2.1 (ii), α_n converges to T in $\mathcal{D}'(\Re)$. This proves that $C^\infty(\Re)$ is dense in $\mathcal{D}'(\Re)$. To prove that $\mathcal{D}(\Re)$ is dense in $\mathcal{D}'(\Re)$, we must show that the functions α_n can be chosen with compact support. We fix this by multiplying α_n by a function $\beta_n \in \mathcal{D}(\Re)$ such that $\beta_n(x) = 1$ for $|x| \leq n$ and $\beta_n(x) = 0$ otherwise. Then $\theta_n = \alpha_n \beta_n \in \mathcal{D}(\Re)$, and it converges to T in $\mathcal{D}'(\Re)$. □

3. The convolution $C' * S'$

The convolution $C' * S'$ is a particular cast of the convolution $C' * \mathcal{D}'$. But in this case the distribution that one obtains is tempered.

Referring to (1.7) we see that the functions$\psi(t) =< T_x, \varphi(x + t) >$ and $\alpha(u) =< S_x, \varphi(x + u) >$ and well-defined when $\varphi \in \mathcal{S}(\Re)$, $T \in \mathcal{S}'(\Re)$, and $S \in \mathcal{S}'(\Re)$ (Proposition 1.1). We establish a preliminary result.

Proposition 3.1 *If* $S \in C'(\Re)$, *the mapping* $\varphi \mapsto \alpha$ *defined by* $\alpha(u) =< S_x, \varphi(x + u) >$ *is continuous from* $\mathcal{S}(\Re)$ *to* $\mathcal{S}(\Re)$.

Proof. We use the theorem about the structure of elements in $C'(\Re)$. Thus we write $S = \sum_{j=1}^{p} f_j^{(n_j)}$, where the f_j are continuous with support in some compact set K. Then

$$\alpha(u) = \sum_{j=1}^{p} < f_j^{(n_j)}(x), \varphi(x+u) >= \sum_{j=1}^{p}(-1)^{n_j}\int_K f_j(x)\varphi^{(n_j)}(x+u)dx,$$

and it is not difficult to verify that $\alpha \in \mathcal{S}(\Re)$.

To prove continuity, we assume that $\varphi_n \to 0$ in $\mathcal{S}(\Re)$ and show that the corresponding sequence α_n converges to 0 in $\mathcal{S}(\Re)$. It is clear that we can differentiate under the integral sign, and consequently

$$\begin{aligned} |u^m \alpha_n^{(q)}(u)| &\leq \sum_{j=1}^{p}\int_K |f_j(x)||u^m\varphi_n^{(n_j+q)}(x+u)|dx \\ &\leq \sum_{j=1}^{p}\int_K |f_j(x)|\frac{|u|^m}{1+|x+u|^m}(1+|x+u|^m)|\varphi_n^{(n_j+q)}(x+u)|dx \\ &\leq C\sum_{j=1}^{p}\left(||\varphi_n^{(n_j+q)}||_\infty + \sup_{t\in\Re}|t^m\varphi_n^{(n_j+q)}(t)|\right) \end{aligned}$$

for some constant C. This shows that $\varphi_n \to 0$ in $\mathcal{S}(\Re)$ implies that α_n converges to 0 in $\mathcal{S}(\Re)$.

We use this results to prove the next one.

Proposition 3.2 *If $S \in C'(\Re)$ and $T \in \mathcal{S}(\Re)$, then the convolution $S * T$ is a tempered distribution.*

Proof. We know that $S*T$ is a distribution. Let φ_n be a sequence in $\mathcal{D}(\Re)$ that tends to 0 in $\mathcal{S}(\Re)$. Then $< S*T, \varphi_n >=< T_u, < S_x, \varphi_n(x+u) >>$, and by Proposition 3.1 the sequence $\alpha_n(u) =< S_x, \varphi(x+u) >$ is in $\mathcal{S}(\Re)$ and converges to 0. Hence $\lim_{n\to\infty} < T, \alpha_n >= 0$. The result follows from Proposition 1.3.

The next step is to examine continuity.

Proposition 3.3

(i) *Let S_n be a sequence in $C'(\Re)$ that converges to 0 in $C'(\Re)$; that is, $< S_n, \varphi >\to 0$ for all $\varphi \in C^\infty(\Re)$. Then $S_n * T \to 0$ in $\mathcal{S}'(\Re)$, and hence in $\mathcal{D}'(\Re)$, for all $T \in \mathcal{S}'(\Re)$.*

(ii) *Let T_n be a sequence in $\mathcal{S}'(\Re)$ that converges to 0 in $\mathcal{S}'$. Then for all $S \in C'(\Re)$, $S * T_n \to 0$ in $\mathcal{S}'(\Re)$, and hence in $\mathcal{D}'(\Re)$.*

Proof.

(i) By Proposition 1.1, the function $\psi(t) =< T_x, \varphi(x+t) >$ is in $C^\infty(\Re)$ for all φ is $\mathcal{S}(\Re)$. Thus

$$< S_n * T, \varphi >=< S_n, \psi >\to 0 \quad \text{as} \quad n \to \infty.$$

(ii) Similarly, $\alpha(u) =< S_x, \varphi(x+u) >$ is in $\mathcal{S}(\Re)$ for all φ is $\mathcal{S}(\Re)$ (Proposition 3.1). Hence $\lim_{n\to\infty} < S * T_n, \varphi >= \lim_{n\to\infty} < T_n, \varphi >= 0$. □

It is necessary to pay attention to the various notions of convergence. Here is a simple example;

$$\delta_n * \mathbf{1} = \mathbf{1}$$

for all n, and δ_n converges to $\mathbf{0}$ in $\mathcal{D}'(\Re)$. In this case the result of Proposition 3.3 is not true. This is because δ_n does not converge in $C'(\Re)$.

Proposition 3.4

(i) *Let S_n be a sequence in $C'(\Re)$ that converges to 0 in $\mathcal{D}'(\Re)$. Assume that there exists a compact set K such that* $\mathrm{supp}(S_n) \subset K$ *for all n. Then $S_n * T \to 0$ in $\mathcal{D}'(\Re)$ for all $T \in \mathcal{S}'(\Re)$.*

(ii) *Let T_n a sequence in $\mathcal{S}'(\Re)$ that converges to 0 in $\mathcal{D}'(\Re)$. Then for all $S \in C'(\Re)$, $S * T_n \to 0$ in $\mathcal{D}'(\Re)$.*

Proof.

(i) Take $\varphi \in \mathcal{D}(\Re)$. The function $\psi(t) =< T_x, \varphi(x+t) >$ is in $C^\infty(\Re)$. Since $\mathrm{supp}(S_n) \subset K$, $< S_n, \psi >=< S_n, \theta\psi >$, where θ is a function in $\mathcal{D}(\Re)$ such that $\theta(x) = 1$ for $x \in K$. Then $\theta\psi$ is in $\mathcal{D}(\Re)$ and $< S_n, \theta\psi > \to 0$.

(ii) If $\varphi \in \mathcal{D}(\Re)$, then the function $\alpha(u) =< S_x, \varphi(x+u) >$ is in $\mathcal{D}(\Re)$, and hence $< T_n, \alpha > \to 0$.

4 The convolution $\mathcal{D}'_+ * \mathcal{D}'_+$

We have studied the convolution of two distributions where at least one of them has compact support. Without this condition on the support, the convolution is not generally defined. However, as is the case for functions, the convolution is defined when both distributions are in $\mathcal{D}'_+$ (or $\mathcal{D}'_-$).

We begine with a preliminary result about $\mathcal{D}'_+$.

Proposition 4.1 *Suppose that $T \in \mathcal{D}'_+$ and $\varphi \in C^\infty(\Re)$ and that* $\mathrm{supp}(T) \subset [a, +\infty)$ *and* $\mathrm{supp}(\varphi) \subset (-\infty, b]$. *Then $< T, \varphi >$, defined by*

$$< T, \varphi >=< T, \theta\varphi >, \tag{1.11}$$

where θ is a function in $\mathcal{D}(\Re)$ equal to 1 on an interval $[-M, M]$ containing a and b in its interior, is well-defined.

Proof. $\theta\varphi \in \mathcal{D}(\Re)$, so $< T, \theta\varphi >$ makes sense. We must show that the definition of $< T, \varphi >$ does not depend on the choice of θ. Let θ_1 be another function in $\mathcal{D}(\Re)$ equal to 1 on $[-M_1, M_1]$ containing a and b. Then $(\theta - \theta_1)\varphi$ vanishes on $[-m, +\infty)$, where $m = \min\{M, M_1\}$. Since $\mathrm{supp}(T) \subset [a, +\infty)$, we have $\mathrm{supp}(T) \cap \mathrm{supp}((\theta - \theta_1)\varphi) = \phi$ and $< T, (\theta - \theta_1)\varphi >= 0$. □

To define the convolution, it is necessary to give meaning to the expressions $< S_t, < T_x, \varphi(x+t) >>$ and $< T_u, < S_x, \varphi(x+u) >>$ for S and T in $\mathcal{D}'_+$ and φ in $\mathcal{D}(\Re)$.

Proposition 4.2 *Suppose that $T \in \mathcal{D}'_+$ and $\varphi \in C^\infty(\Re)$ and that* $\mathrm{supp}(T) \subset [a, +\infty)$ *and* $\mathrm{supp}(\varphi) \subset (-\infty, b]$. *Then $\psi(t) =< T_x, \varphi(x+t) >$ is defined of all $t \in \Re$,* $\mathrm{supp}(\psi) \subset (-\infty, b-a]$, *and $\psi \in C^\infty(\Re)$.*

Proof. The function $\tau_{-t}\varphi$ is in $C^\infty(\Re)$ with $\mathrm{supp}(\tau_{-t}\varphi) \subset (-\infty, b-t]$ for all $t \in \Re$. Thus by Proposition 4.1, $< T_x, \varphi(x+t) >$ is well-defined.

Now, $\psi(t) = 0$ if $\mathrm{supp}(\tau_{-t}\varphi) \cap \mathrm{supp}(T) = \phi$, which is the case when $b - t < a$. Hence, $\mathrm{supp}(\psi) \subset (-\infty, b-a]$. That $\psi \in C^\infty$ is a consequence of (1.11) and Proposition 1.1(i). □

These preliminary results lead to the next theorem, which we state without proof.

Theorem 4.3(the convolution $\mathcal{D}'_+ * \mathcal{D}'_+$) *Suppose that S and T are in $\mathcal{D}'_+$.*

(i) *There exists a distribution called the convolution of S and T and denoted by $S * T$ such that for all $\varphi \in \mathcal{D}(\Re)$,*

$$< S * T, \varphi >=< S_t, < T_x, \varphi(x+t) >>=< T_u, < S_x, \varphi(x+u) >> . \tag{1.12}$$

(ii) $(S * T)^{(k)} = S^{(k)} * T = S * T^{(k)}, \quad k = 1, 2, 3, ...$

(iii) *The mapping $(S, T) \mapsto S * T$ of $\mathcal{D}'_+ \times \mathcal{D}'_+$ into $\mathcal{D}'$ is continuous with respect to each variable. (The convergence of S_n to 0 in $\mathcal{D}'_+$ means that $S_n \to 0$ in $\mathcal{D}'(\Re)$ and that there exists a c such that for all n,* $\mathrm{supp}(S_n) \subset [c, +\infty)$.*)*

Proposition 4.4 (support of $S * T$) *If S and $T \in \mathcal{D}'_+$ with* $\mathrm{supp}(S) \subset [a_1, +\infty)$ *and* $\mathrm{supp}(T) \subset [a_2, +\infty)$, *then*

$$\mathrm{supp}(S * T) \subset [a_1 + a_2, +\infty),$$

and hence $S * T$ is in $\mathcal{D}'_+$.

Proof. Take $\varphi \in \mathcal{D}(\Re)$ with $\mathrm{supp}(\varphi) \subset (-\infty, a_1 + a_2)$. The support of $\varphi(t) =< T_x, \varphi(x+t) >$ is in $(-\infty, a_1)$ by Proposition 4.2. Thus

$$\mathrm{supp}(\psi) \cap \mathrm{supp}(S) = \phi$$

and $< S * T, \varphi >= 0$, which proves that $\mathrm{supp}(S * T) \subset [a_1 + a_2, \infty)$. □

Remark 4.5 As in Section 2, we can obtain a primitive of T in $\mathcal{D}'_+$ by taking the convolution of T with the Heaviside distribution U. Since $U \in \mathcal{D}'_+$, the primitive of T is in $\mathcal{D}'_+$.

5. The associativity of convolution

We have defined the convolution of two distribtions and seen that it is a commutative operation. If we wish to convolve three or more distributions, we run into two problems: existence and associativity. Here is a classic example: We wish to compute $\mathbf{1} * \delta' * u$. From Proposition 2.3,

$$\mathbf{1} * \delta' = \mathbf{1}' * \delta = 0$$

Thus $(\mathbf{1} * \delta') * u$ makes sense and is equal to 0. On the other hand, by Proposition 2.2,

$$\delta' * u = \delta * u' = \delta * \delta = \delta;$$

hence $\mathbf{1} * (\delta' * u)$ make sense and is equal to δ. This shows that the convolution product is not associative in general. Nevertheless, convolution is associative in several cases.

Proposition 5.1 *The convolution of n distributions of which at least $n-1$ have compact support is associative and commutative.*

Proof. The proof is based directly on the definitions. We take S and T in $\mathcal{C}'(\Re)$ and U in $\mathcal{D}'(\Re)$; we show, for example, that $(S*T)*U = S*(T*U)$. First, these convolutions make sense, since $S*T \in \mathcal{C}'(\Re)$ (Proposition 32.2.5(ii)) and $(S*T)*U$ is an $\mathcal{C}' * \mathcal{D}'$ convolution. Similarly, $T*U$ is as $\mathcal{C}' * \mathcal{D}'$ convolution, so $T*U \in \mathcal{D}'$; and $S*(T*U)$ is another $\mathcal{C}' * \mathcal{D}'$ convolution. Take $\varphi \in \mathcal{D}(\Re)$. From equation (1.7),

$$< (S*T)*U, \varphi > = < (S*T)_t, < U_x, \varphi(x+t) >>$$

Now, $\varphi(t) = < U_x, \varphi(x+t) >$ is in C^∞, and $S*T \in \mathcal{C}'(\Re)$ then

$$\begin{aligned} < S*T, \varphi > &= < S_z, < T_u, \varphi(u+z) >> \\ &= < S_z, < T_u, < U_x, \varphi(x+u+z) >>> \\ &= < S_z, < (T*U)_t, \varphi(t+z) >> \\ &= < S*(T*U), \varphi >, \end{aligned}$$

from which we see that $(S*T)*U = S*(T*U)$. □

The space$\mathcal{C}'(\Re)$ endowed with the convolution operation is a *convolution algebra,* since T, $S \in \mathcal{C}'(\Re)$ implies that $T*S \in \mathcal{C}'(\Re)$. This algebra is commutative and associative, and it contains a unit element, *delta*. $\mathcal{D}'_+(\Re)$ is another important convolution algebra.

Proposition 5.2 *The convolution in $\mathcal{D}'_+(\Re)$ is assosicative.*

The proof is similar to that of Proposition 32.5.1, and δ is also a unit element for this algebra.

Since these two distribution algebras have unit elements, it is natural to ask whether a distribution S in $\mathcal{C}'(\Re)$ or $\mathcal{D}'_+(\Re)$ has an inverse; that is, is there a T such that

$$S * T = \delta$$

This question is important for solving differential equations with constant coefficients. With this in mind, we introduce the differential operator

$$P = \sum_{m=0}^{p} a_m D^{(m)},$$

where $a_m \in C$ and $D^{(m)}$ denotes the mth derivative. Given $U \in \mathcal{D}'(\Re)$, we wish to find a $T \in \mathcal{D}'(\Re)$ such that

$$P(T) = U. \tag{1.13}$$

We saw in Proposition 2.2(ii) that $T^{(k)} = \delta^{(k)} * T$, and hence we can write (1.13) as

$$\sum_{m=0}^{p} a_m \delta^{(m)} * T = U. \tag{1.14}$$

A distribution E is said to be an *elementary solution* of (1.14) if

$$\left(\sum_{m=0}^{p} a_m \delta^{(m)} \right) * E = \delta$$

This means that E is the inverse of $S = \sum_{m=0}^{p} a_m \delta^{(m)}$. If we have associativity, then knowing E yields all the solutions of (1.13), since

$$S * E * U = (S * E) * U = \delta * U = U,$$

and hence $T = E * U$. Furthermore, this solution is unique: If T_1 and T_2 satisfy (1.13), then we have

$$S * T_1 = S * T_2 = U$$

By taking the convolution with E this becomes, thanks to associativity,

$$(E * S) * T_1 = (E * S) * T_2,$$

and since $E * S = \delta$, we are left with

$$T_1 = T_2$$

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On the convolutions of the diamond kernel of Marcel Riesz

A. Kananthai

Department of Mathematics, Chiang Mai University, Chiang Mai 50200, Thailand

Abstract

In this paper, we consider the equation $\diamondsuit^k u(x) = \delta$ where $\diamondsuit^k$ is introduced and named as the diamond operator iterated k-times and is defined by $\diamondsuit^k = ((\sum_{i=1}^{p}(\partial^2/\partial x_i^2))^2 - (\sum_{j=p+1}^{p+q}(\partial^2/\partial x_j^2))^2)^k$, $u(x)$ is a generalized function, $x = (x_1, x_2, \ldots, x_n) \in R^n$ the n-dimensional Euclidean space, $p + q = n$, $k = 0, 1, 2, 3, \ldots$ and δ is the Dirac-delta distribution. Now $u(x)$ is the elementary solution of the operator $\diamondsuit^k$ and is called the diamond kernel of Marcel Riesz. The main part of this work is studying the convolution of $u(x)$. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Diamond kernel; Ultra-hyperbolic kernel; Elliptic kernel; Tempered distribution

1. Introduction

Consider the equation

$$\diamondsuit^k u(x) = \delta, \tag{1.1}$$

where $\diamondsuit^k$ is the Diamond operator iterated k-times defined by

$$\diamondsuit^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right)^k. \tag{1.2}$$

E-mail address: malamnka@science.cmu.ac.th (A. Kananthai).

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Kananthai [2, Theorem 3.1] has proved that the convolution solution $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the elementary solution of (1.1) where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.2) and (2.4), respectively with $\alpha = 2k$. Now $u(x)$ is called the diamond kernel of Marcel Riesz and defines such a kernel by

$$T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x), \quad m = 0, 1, 2, \ldots \tag{1.3}$$

In this work we study the existence of $T_m(x) * T_n(x)$ and moreover the inverse T_m^{*-1} of $T_m(x)$ in the convolution algebra $\mathscr{A}'$ is also considered.

2. Preliminaries

Definition 2.1. Let $E(x)$ be a function defined by

$$E(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}, \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$, $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ and $\omega_n = (2\pi^{n/2})/(\Gamma(n/2))$ is a surface area of the unit sphere.

It is well known that $E(x)$ is an elementary solution of the Laplace operator Δ, that $\Delta E(x) = \delta$ where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ and δ is the Dirac-delta distribution.

Definition 2.2. Let $S_\alpha(x)$ be a function defined by

$$S_\alpha(x) = 2^{-\alpha}\pi^{-n/2}\Gamma\left(\frac{n-\alpha}{2}\right)\frac{|x|^{\alpha-n}}{\Gamma(\alpha/2)}, \tag{2.2}$$

where α is a complex parameter, $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$, $x = (x_1, x_2, \ldots, x_n) \in R^n$. $S_\alpha(x)$ is called the elliptic kernel of Marcel Riesz. Now $S_\alpha(x)$ is an ordinary function for $\mathrm{Re}(\alpha) \geqslant n$ and is a distribution of α for $\mathrm{Re}(\alpha) < n$.

From (2.1) and (2.2) we obtain

$$E(x) = -S_2(x) \tag{2.3}$$

and it can be shown that

$$\underbrace{E(x) * E(x) * \cdots * E(x)}_{k\text{-times}} = (-1)^k S_{2k}(x)$$

is the elementary solution of the operator Δ^k iterated k-times that is $\Delta^k(-1)^k S_{2k}(x) = \delta$, see [3].

Definition 2.3. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of R^n and write

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \quad p + q = n.$$

$\Gamma_+ = \{x \in R^n: x_1 > 0 \text{ and } V > 0\}$ designates the interior of the forward cone and denotes $\overline{\Gamma}_+$ by its closure and the following function introduced by Nozaki [4, p. 72] that

$$R_\alpha(x) = \begin{cases} V^{(\alpha-n)/2}/K_n(\alpha) & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+. \end{cases} \tag{2.4}$$

Here, $R_\alpha(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and α is a complex parameter and n is the dimension of the space R^n.

The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}.$$

Now $R_\alpha(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geqslant n$ and is a distribution of α if $\mathrm{Re}(\alpha) < n$.

Let supp $R_\alpha(x) \subset \overline{\Gamma}_+$ where supp $R_\alpha(x)$ denotes the support of $R_\alpha(x)$.

Lemma 2.1. *The functions $S_\alpha(n)$ and $R_\alpha(x)$ defined by* (2.2) *and* (2.4) *respectively, for* $\mathrm{Re}(\alpha) < n$ *are Homogeneous distribution of order $\alpha - n$ and also a tempered distribution.*

Proof. Since $R_\alpha(x)$ and $S_\alpha(x)$ satisfy the Euler equation, that is

$$(\alpha - n)R_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_\alpha(x) \quad \text{and} \quad (\alpha - n)S_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_\alpha(x).$$

we have $R_\alpha(x)$ and $S_\alpha(x)$ are homogeneous distributions of order $\alpha - n$ and Donoghue [1, pp. 154–155] has proved that every homogeneous distribution is a tempered distribution. That completes the proof. □

Lemma 2.2 (The convolution of tempered distributions). *The convolution $S_\alpha(x) * R_\alpha(x)$ exists and is a tempered distribution.*

Proof. Choose supp $R_\alpha(x) = K \subset \overline{\Gamma}_+$ where K is a compact set. Then $R_\alpha(x)$ is a tempered distribution with compact support and by Donoghue [1, pp. 156-159]. $S_\alpha(x) * R_\alpha(x)$ exists and is a tempered distribution. □

Lemma 2.3. *Given the equation $\diamondsuit^k u(x) = \delta$ where $\diamondsuit^k$ is the operator defined by* (1.2), $x = (x_1, x_2, \ldots, x_n) \in R^n$, *$K$ is a nonnegative integer and δ is the Dirac-delta distribution. Then $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the equation where $S_\alpha(x)$ and $R_\alpha(x)$ are defined by* (2.2) *and* (2.4), *respectively with $\alpha = 2k$. Moreover $u(x)$ is a tempered distribution.*

Proof. See [3, Theorem 3.1] and by Lemma 2.2, $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a tempered distribution. □

Lemma 2.4 (The convolutions of $R_\alpha(x)$ and $S_\alpha(x)$). *Let $S_\alpha(x)$ and $R_\alpha(x)$ be defined by* (2.2) *and* (2.4) *respectively, then we obtain the following formulas*:
1. $S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x)$, *where α and β are complex parameters.*
2. $R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x)$, *for α and β are both integers and except only the case both α and β are odd integers.*

Proof. Proof of first formula, see [1, p. 158].

Proof of second formula, for the case α and β are both even integers, see [3] and for the case α is odd and β is even or α is even and β is odd, we know from Trione [5] that

$$\Box^k R_\alpha(x) = R_{\alpha-2k}(x) \tag{2.5}$$

and

$$\Box^k R_{2k} = \delta, \quad k = 0, 1, 2, \ldots \tag{2.6}$$

where $\Box^k$ is and ultra-hyperbolic operator iterated k-times $(k = 0, 1, 2, \ldots)$ defined by

$$\Box^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k.$$

Now let m be an odd integer, we have

$$\Box^k R_m(x) = R_{m-2k}(x)$$

and

$$R_{2k}(x) * \Box^k R_m(x) = R_{2k}(x) * R_{m-2k}(x)$$

or

$$(\Box^k R_{2k}(x)) * R_m(x) = R_{2k}(x) * R_{m-2k}(x).$$

$$\delta * R_m(x) = R_{2k}(x) * R_{m-2k}(x) \quad \text{by (2.6)}.$$

or

$$R_m(x) = R_{2k}(x) * R_{m-2k}(x).$$

Since m is odd, hence $m - 2k$ is odd and $2k$ is a positive even. Put $\alpha = 2k$, $\beta = m - 2k$ we obtain

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x)$$

for α is a nonnegative even and β is odd.

For the case α is a negative even and β is odd, by (2.5) we have

$$\square^k R_0(x) = R_{-2k}(x) \quad \text{or} \quad \square^k \delta = R_{-2k}(x)$$

where $R_0(x) = \delta$. Now

$$R_{-2k}(x) * \square^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x) \quad \text{for } m \text{ is odd},$$

or

$$(\square^k \delta) * \square^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$\delta * \square^{2k} R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$R_{m-2(2k)}(x) = R_{-2k}(x) * R_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m - 2k$, now α is a negative even and β is odd. Then we obtain

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x).$$

That completes the proofs. $\square$

3. Main results

Theorem 3.1. *Let $T_m(x)$ the diamond kernel of Marcel Riesz defined by* (1.3), *then T_m is a tempered distribution and can be expressed by*

$$T_m(x) = T_{m-r}(x) * T_r(x),$$

where r is a nonnegative integer and $r < m$. Moreover if we put $\ell = m - r$, $n = r$ we obtain

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) \quad \textit{for } \ell + n = m.$$

Proof. Since $T_m = (-1)^m S_{2m}(x) * R_{2m}(x)$, $(m = 0, 1, 2, \ldots)$, by Lemma 2.2 T_m is a tempered distribution. Now by Lemma 2.3, $\diamond^m T_m = \delta$, then $\diamond^r \diamond^{m-r} T_m = \delta$ for $m > r$ and by Lemma 2.3 again, we obtain $\diamond^{m-r} T_m = (-1)^r S_2 r(x) * R_{2r}(x)$. Convolving both sides by $(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)$, we obtain

$$\begin{aligned}
&[(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * \diamond^{m-r} T_m \\
&\quad = [(-1)^{m-r} S_{2m-2r}(x) * R_{2m-2r}(x)] * [(-1)^r S_{2r}(x) * R_{2r}(x)]
\end{aligned} \tag{3.1}$$

or

$$\begin{aligned}
&\diamond^{m-r}[(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * T_m \\
&\quad = (-1)^m (S_{2m-2r}(x) * S_{2r}(x) * (R_{2m-2r}(x) * R_{2r}(x)).
\end{aligned}$$

since $S_{2m}(x)$ and $R_{2m}(x)$ are tempered distributions and are the elements of the space of convolution algebra, $\mathscr{a}'$.

By Lemmas 2.3 and 2.4 we obtain

$$\delta * T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x),$$
$$T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x).$$

From (3.1) we have $T_m(x) = T_{m-r}(x) * T_r(x)$, put $\ell = m - r$, $n = r$, it follows that

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) = T_m(x)$$

as required. □

Theorem 3.2. *Let $T_m(x)$ be defined by* (1.3) *then T_m is an element of the space $\mathscr{a}'$ of convolution algebra and there exist an inverse T_m^{*-1} of T_m such that*

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

Proof. Since $T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x)$ is a tempered distribution by Lemma 2.2. Now the supports of $S_{2m}(x)$ and $R_{2m}(x)$ are compact. Then they are the elements of the space of convolution algebra $\mathscr{a}'$ of distribution. By Zemanian [6, Theorem 6.2.1, p. 151] there exist a unique inverse T_m^{*-1} such that

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

That completes the proof. □

Acknowledgements

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The convolution product of the distributional families related to the Diamond operator

1. Introduction

A. Kananthai [4] has first introduce the Diamond operator $\diamondsuit^k$ iterated k-times which is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \tag{1.1}$$

where $p+q=n$ is the dimension of the n-dimentional Euclidean space $\Re^n$ and k is a nonegative integer. Actually (1.1) can be rewrite in the following form

$$\diamondsuit^k = \square^k \triangle^k = \triangle^k \square^k \tag{1.2}$$

where the operator $\square^k$ and $\triangle^k$ are defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k \tag{1.3}$$

and,

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_{p+q}^2}\right)^k \tag{1.4}$$

In this paper, the family $K_{\alpha,\beta}$ is defined by $K_{\alpha,\beta}(x) = R_\alpha^e * R_\beta^H$ where R_α^ℓ is the elliptic kernal defined by (2.1) and R_β^H is the hyperbolic kernal defined by (2.4) and the symbol $*$ denotes the convolution and $x \in \Re^n$. In this paper, we study the convolution $K_{\alpha,\beta} * K_{\alpha',\beta'}$ where α, β, α' and β' are complex parameters.

2. Preliminaries

Definition 2.1 Let the function $R_\alpha^e(x)$ be defined by

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{2.1}$$

where $x = (x_1, x_2, ..., x_n) \in \Re^n, \alpha$ is a complex parameter, n is the dimension of $\Re^n$ and $|x| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$ and $W_n(\alpha)$ is defined by the formula $W_n(\alpha) = \frac{\Pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$. The function $R_\alpha^e(x)$ is precisely the definition of elliptic kernal of Marcel Rieze [2] and the following formula is valid

$$R_\alpha^e(x) * R_\beta^e(x) = R_{\alpha+\beta}^e(x) \tag{2.2}$$

which hold for $\alpha > 0, \beta > 0$ and $\alpha + \beta \leq n$ see ([2], p.20)

Definition 2.2 Let $x = (x_1, x_2, ..., x_n) \in \Re^n$ and write

$$u = u(x) = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2, p + q = n \tag{2.3}$$

Denote by Γ_+ the interior of the forward cone defined by $\Gamma_+ = \{x \in \Re^n : x_1 < 0 \text{ and } u > 0\}$ and $\overline{\Gamma_+}$ denote it closure. Similarly, define $\Gamma_- = \{x \in \Re^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma_-}$ denote it closure.

For any complex number α, define

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.4}$$

where $K_n(\alpha)$ is given by the formula

$$K_n(/alpha) = \Pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{p-\alpha}{2})} \tag{2.5}$$

The function R_α^H was introduced by Y. Nozaki ([3], p.72) $R_\alpha(u)$, which is an ordinary function if $Re(\alpha) \geq n$, is a distribution of α. We shall call R_α^H the Marcel Riesz's ultra-hyperbolic kernal. By putting $p = 1$ in (2.4) and (2.5) and using the Legendre's duplication formula of $\Gamma(z)$.

$$\Gamma(2z) = 2^{2z-1}\Pi^{\frac{-1}{2}}\Gamma(z)\Gamma(z + \frac{1}{2}) \tag{2.6}$$

see ([5], Vol 1. p.5) the formula (2.4) reduces to

$$M_\alpha = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.7}$$

where $u = u(x) = x_1^2 - x_2^2 - ... - x_n^2$ and

$$H_n(\alpha) = 2^{\alpha-1}\Pi^{\frac{n-2}{2}}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n+2}{2}) \tag{2.8}$$

M_α is called the hyperbolic kernel of Marcel Riesz ([2], p.31).

Lemma 2.1 *The function $R_\alpha^e(x)$ has the following properties*

(i) $R_0^e = \delta(x)$

(ii) $R_{-2k}^e(x) = (-1)^k \triangle^k \delta(x)$

(iii) $\triangle^k R_\alpha^e(x) = (-1)^k R_{\alpha-2k}^e(x)$, *where* $\triangle^k$ *is the Laplace operator iterated k-times defined by (1.3).*

The proof Lemma is given by S.E Trione [5].

Lemma 2.2

(i) $R_\alpha^H * R_\beta^H = \frac{\cos\alpha\frac{\Pi}{2}\cos\beta\frac{\Pi}{2}}{\cos(\frac{\alpha+\beta}{2})\Pi} R_{\alpha+\beta}^H$ *where* R_α^H *is defined by (2.4) and (2.5) with p is an even.*

(ii) $R_\alpha^H * R_\beta^H = R_{\alpha+\beta}^H + T_{\alpha,\beta}$ *for p ia an odd where*

$$T_{\alpha,\beta} = T_{\alpha,\beta}(u \pm i\ 0, n)$$
$$= \frac{\frac{2\pi i}{4} C(-\frac{\alpha-\beta}{2})}{C(-\frac{\alpha}{2})C(-\frac{\beta}{2})}[H^+_{\alpha+\beta} - H^-_{\alpha+\beta}] \tag{2.9}$$

$$C(r) = \Gamma(r)\Gamma(1-r)$$

$$H^{\pm}_r = H_r(u \pm i\ 0, n) = e^{\mp r\frac{\pi}{2}i} e^{\pm q\frac{\pi}{2}i} a(\frac{r}{2})(u \pm i\ 0)^{\frac{r-n}{2}}$$
$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2}]^{-1}$$

$$(u \pm i\ 0)^{\lambda} = \lim_{\epsilon \to 0}(u + i\epsilon|x|^2)^{\lambda}$$

see([6], p.275)

$u = u(x)$ is defined by (2.3) and $|x| = (x_1^2 = x_2^2 + ... + x_n^2)^{\frac{1}{2}}$

In particular $R^H_\alpha * R^H_{-2k} = R^H_{\alpha-2k}$, $R^H_\alpha * R^H_{2k} = R^H_{\alpha+2k}$

Proof see([1], p.121-123)

Lemma 2.3

(i) $R^H_{-2k} = \Box^k \delta$

(ii) $\Box^k R^H_\alpha = R^H_{\alpha-2k}$

(iii) $\Box^k R^H_{2k} = R^H_0 = \delta$ *where* $\Box^k$ *is defined by (1.2)*

Proof see ([1], p.123)

3. The family of distribution $K_{\alpha,\beta}(x)$

Let $K_{\alpha,\beta}(x)$ be a distributional family defined by

$$K_{\alpha,\beta}(x) = R^e_\alpha * R^H_\beta \tag{2.10}$$

where the function R^e_α and R^H_β are defined by (2.1) and (2.4) respectively. The right hand side of (2.10) is well-defined and also a tempered distribution see ([4], Lemma 2.3, Lemma 2.4)

Lemma 3.1 *The following formulas are valid*

(i) $K_{0,0}(x) = \delta(x)$

(ii) $K_{-2k,-2k}(x) = (-1)^k \Diamond^k \delta(x)$

(iii) $\Diamond^k(K_{\alpha,\beta}(x)) = (-1)^k K_{\alpha-2k,\beta-2k}(x)$

(iv) $\diamondsuit^k(K_{2k,2k}(x)) = (-1)^k\delta(x)$

Proof (i)By (2.10), $K_{0,0}(x) = R_0^e * R_0^H$. By Lemma 2.1(i) and Lemma 2.3(i), we obtain

$$K_{0,0}(x) = \delta * \delta = \delta.$$

(ii) We have

$$\begin{aligned}
\diamondsuit^k(K_{\alpha,\beta}(x)) &= \diamondsuit^k(R_\alpha^e * R_\beta^H) \\
&= \Box^k\triangle^k(R_\alpha^e * R_\beta^H) \\
&= \triangle^k R_\alpha^e * \Box^k R_\beta^H \\
&= (-1)^k R_{\alpha-2k}^e * R_{\beta-2k}^H \\
&= (-1)^k K_{\alpha-2k,\beta-2k}(x)
\end{aligned}$$

by Lemma 2.1(iii) and Lemma 2.3(ii). Putting $\alpha = \beta = 0$ and (i) we obtain

$$K_{-2k,-2k}(x) = (-1)^k\diamondsuit^k\delta.$$

(iii) Similarly as (ii).

(iv) Putting $\alpha = \beta = 2k$ in (iii), we obtain

$$\diamondsuit^k(K_{-2k,-2k}(x)) = (-1)^k K_{0,0}(x) = (-1)^k\delta(x)$$

4. Main results

Theorem *Let the family $K_{\alpha,\beta}(x)$ and $K_{\alpha',\beta'}(x)$ be defined by (2.10) then the convolution product $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}$ can be obtained by the following formula*

(i) $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = B_{\beta,\beta'}R_{\beta+\beta'}^H * R_{\alpha+\alpha'}^e$ where R_β^H and R_α^e are defined by (2.4) and (2.1) respectively which p is an even and $B_{\beta,\beta'} = \frac{\cos\beta\frac{\Pi}{2}\cos\beta'\frac{\Pi}{2}}{\cos(\frac{\beta+\beta'}{2})\Pi}$.

(ii) $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'} + T_{\beta,\beta'}) * R_{\alpha+\alpha'}$ if p is an odd and $T_{\beta,\beta'}$ is defined by (2.9)

(iii) $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k\diamondsuit^k K_{\alpha,\beta}(x)$

Proof (i) we have

$$\begin{aligned}
K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) &= (R_\alpha^e * R_\beta^H) * (R_{\alpha'}^e * R_{\beta'}^H) \\
&= R_{\alpha+\alpha'}^e * (R_\beta^H * R_\beta^H) \text{ by (2.2)} \\
&= B_{\beta,\beta'}R_{\beta+\beta'}^H * R_{\alpha+\alpha'} \text{ by Lemma2.2}(i)
\end{aligned}$$

for p ia even and $B_{\beta,\beta'} = \frac{\cos\beta\frac{\Pi}{2}\cos\beta'\frac{\Pi}{2}}{\cos(\frac{\beta+\beta'}{2})\Pi}$.

(ii) From (i),

$$K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R^H_\beta * R^H_{\beta'}) * R^e_{\alpha+\alpha'}$$
$$= (R^H_{\beta+\beta'} + T_{\beta,\beta'}) * R^e_{\alpha+\beta'}.$$

by Lemma 2.2(ii) for p is odd and $T_{\beta,\beta'}$ is defined by (2.9)

(iii) We have $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = B_{\beta,-2k} R^H_{\beta-2k} * R^\ell_{\alpha-2k}$ for p is even. Since $B_{\beta,-2k} = \frac{\cos\beta\frac{\Pi}{2}\cos(-2k)\frac{\Pi}{2}}{\cos(\frac{\beta-2k}{2})\Pi} = 1$. Thus

$$K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = R^H_{\beta-2k} * R^\ell_{\alpha-2k} = K_{\alpha-2k,\beta-2k}(x) = (-1)^k \Diamond^k K_{\alpha,\beta}(x)$$

by Lemma 3.1(iii) Now for p is odd, we have

$$K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (R^H_{\beta-2k} + T_{\beta,-2k}) * R^e_{\beta-2k}$$

by Lemma 2.2(ii) and we have

$$T_{\beta,-2k} = \frac{\frac{2\Pi i}{4} C(\frac{-\beta+2k}{2})}{C(\frac{-\beta}{2})C(\frac{2k}{2})}[H^+_{\beta-2k} + H^-_{\beta-2k}]$$

where $C(r) = \Gamma(r)\Gamma(1-r)$ and $H^\pm_r = \ell^{\mp r\frac{\Pi}{2}i}\ell^{\pm q\frac{\Pi}{2}i}a(\frac{r}{2})(a \pm i0)^{\frac{r-n}{2}}$ and $a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r\Pi^{\frac{n}{2}}\Gamma(\frac{r}{2})]^{-1}$. Applying the formula $\Gamma(z)\Gamma(1-z) = \frac{\Pi}{\sin z\Pi}$ to $C(-\frac{\beta+2k}{2}), C(-\frac{\beta}{2})$ and $C(k)$ and also the formular $H^\pm_{\beta-2k}$ and $a(\frac{\beta-2k}{2})$, we obtain

$$T_{\beta,-2k} = 0 \quad \text{and} \quad T_{-2k,\beta} = 0$$

It follow that

$$K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = R^H_{\beta-2k} * R^e_{\alpha-2k}$$
$$= K_{\alpha-2k,\beta-2k}(x)$$

for p is odd. Thus by Lemma 3.1(iii) we obtain $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \Diamond^k K_{\alpha,\beta}(x)$ □

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On the Convolutions of the diamond kernel of Marcel Riesz

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On the Convolution Product of the Distributional Kernel $K_{\alpha,\beta,\gamma,\upsilon}$

On the Convolution product of the Distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$

A. Kananthai, S. Suantai, V. Longani
Department of mathematics, Chiangmai University
Chiangmai 50200, Thailand

ABSTRACT

In this paper, we introduce a distributional Kernel $K_{\alpha,\beta,\gamma,\nu}$ which is related to the opertor $\oplus^k$ iterated k-time and is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k$$

where $p+q=n$ is the dimension of the space C^n of n dimensional complex space, $x = (x_1, x_2, ..., x_n) \in C^n, k$ is a nonnegative integer, α, β, γ and ν are complex parameters. It is found that the existence of the convolution $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$ depending on the conditions of p and q.

1. Introduction

The operator $\oplus^k$ can be factorized in the form

$$\oplus^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \tag{1.1}$$

where $p+q=n$ is the dimension of the space C^n, $i=\sqrt{-1}$ and k is a nonnegative integer. The operator $\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2$ is first introduced by A. Kananthai [1] and named the Diamond operator denoted by

$$\diamondsuit = \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 \tag{1.2}$$

Let us denote the operator L_1 and L_2 by

$$L_1 = \sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \tag{1.3}$$

$$L_2 = \sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \tag{1.4}$$

Thus (1.1) can be written by

$$\oplus^k = \diamondsuit^k L_1^k L_2^k \tag{1.5}$$

Now consider the convolution $R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z)$ where $R_\alpha^H(u), R_\beta^e(v), S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.1), (2.2), (2.3) and (2.4) respectively.

We defined the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z) \tag{1.6}$$

Since the functions $R_\alpha^H(u), R_\beta^e(v), S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions see [1, p30-31] and [6, p154-155] then the convolution on the right hand side of (1.6) exists and also is a tempered distributions. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also is a tempered distribution.

For $\alpha = \beta = \gamma = \nu = 2k$, we obtain $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator $\oplus^k$, see [2]. That is $\oplus^k (-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution and $\oplus^k$ is defined by (1.5). In this work, we study the convolutions $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$ we found that such a convolution depending on p and q.

(i) If p is an even number and q can be either odd or even. We obtain

$$K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'}.R^H_{\alpha+\alpha'} * R^e_{\beta+\beta'} * S_{\gamma+\gamma'} * T_{\nu+\nu'}$$

where $B_{\alpha,\alpha'} = \frac{\cos(\frac{\alpha\pi}{2})\cos(\frac{\alpha'\pi}{2})}{\cos(\frac{\alpha+\alpha'}{2})\pi}$

In particularly, if $\alpha = \alpha' = 2k \quad (k = 0, 1, 2, 3, ...)$ we obtain

$$K_{2k,\beta,\gamma,\nu} * K_{2k,\beta',\gamma',\nu'} = R^H_{4k} * R^e_{\beta+\beta'} * S_{\gamma+\gamma'} * T_{\nu+\nu'}$$
$$= K_{4k,\beta+\beta',\gamma+\gamma',\nu+\nu'}$$

(ii) If p is an odd and q is either odd or even. We obtain

$$K_{2k,\beta,\gamma,\nu} * K_{2k,\beta',\gamma',\nu'} = (R^H_{\alpha+\alpha'} + A_{\alpha+\alpha'}) * R^e_{\beta+\beta'} * S_{\gamma+\gamma'} * T_{\nu+\nu'}$$

where $A_{\alpha,\alpha'} = \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})} . \frac{2\pi i}{4}[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}]$
$C(r) = \Gamma(r)\Gamma(1-r)$ and $H^{\pm}_r = H_r(u \pm i0, n) = e^{\mp r\frac{\pi}{2}i} e^{\pm q\frac{\pi}{2}i} q(\frac{r}{2})(u \pm i0)^{\frac{r-n}{2}}$
and $a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, ..., x_n) \in \Re^n$ and write

$$\hat{x} = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \ , \ \ p + q = n \tag{2.1}$$

Denote by $\Gamma_+ = \{x \in \Re^n : x_1 > 0 \text{ and } u > 0\}$ the interrior of forward cone and $\bar{\Gamma}_+$ denote its closure. For any complex number α, we define the function

$$R^H_\alpha(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}$$

the function R^H_α is first introduced by Y. Nozaki [5, p72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Hence $R^H_\alpha(x)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$.

Definition 2.2 Let $x = (x_1, x_2, ..., x_n) \in \Re^n$ and write $v = x_1^2 + x_2^2 + ... + x_n^2$.

For any complex number β, define the function

$$R^e_\beta(v) = \frac{v^{\frac{\beta-n}{2}}}{W_n(\beta)} \tag{2.3}$$

where $W_n(\beta) = \frac{\pi^{\frac{n}{2}} 2^\beta \Gamma(\beta)}{\Gamma(\frac{n-\beta}{2})}$ the function $R^e_\beta(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $Re(\beta) \geq n$ and is a distribution of β if $Re(\beta) < n$.

Definition 2.3 Let $x = (x_1, x_2, ..., x_n)$ be a point of the space C^n of n dimensional complex space. Write $w = x_1^2 + x_2^2 + ... + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q})$ and

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}) \ , \quad p+q = n \quad i = \sqrt{-1}$$

For any complex number γ and ν, define

$$S_\gamma(\omega) = \frac{\omega^{\frac{\gamma-n}{2}}}{W_n(\gamma)} \tag{2.4}$$

and

$$T_\nu(z) = \frac{z^{\frac{\nu-n}{2}}}{W_n(\nu)} \tag{2.5}$$

where $W_n(\gamma) = \frac{\pi^{\frac{n}{2}} 2^\gamma \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}$, $W_n(\nu) = \frac{\pi^{\frac{n}{2}} 2^\nu \Gamma(\frac{\nu}{2})}{\Gamma(\frac{n-\nu}{2})}$.

Lemma 2.1 (The convolution product of $R^e_\beta(v)$)

The convolution $R^e_\beta * R^e_{\beta'} = R^e_{\beta+\beta'}$ where R^e_β and $R_{\beta'}$ are given by (2.2)

Proof See [3, p20].

Lemma 2.2 (The convolution product of $R^H_\alpha(x)$)

(i) $R^H_\alpha * R^H_{\alpha'} = \frac{\cos(\alpha\frac{\pi}{2})\cos(\alpha'\frac{\pi}{2})}{\cos(\frac{\alpha+\beta}{2})\pi} . R^H_{\alpha+\alpha'}$ where R^H_α and $R^H_{\alpha'}$ are defined by (2.1) with p is an even.

(ii) $R^H_\alpha * R^H_\alpha = R^H_{\alpha+\alpha'} + T_{\alpha,\alpha'}$ for p is an odd, where

$$T_{\alpha,\alpha'} = T_{\alpha,\alpha'}(u \pm i0, n) = \frac{2\pi i}{4} \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})}[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}]$$

$$C(r) = \Gamma(r)\Gamma(1-r)$$
$$H_r^{\pm} = H_r(x \pm i0, n) = e^{\mp r\frac{\pi}{2}i}e^{\pm q\frac{\pi}{2}i}a(\frac{r}{2})(u \pm i0)^{\frac{r-n}{2}}$$
$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})(2^r\pi^{\frac{n}{2}}\Gamma(\frac{r}{2}))^{-1}$$
$$(u \pm i0, n)^{\lambda} = \lim_{\epsilon\to 0}(u \pm i\epsilon, n)^{\lambda}$$

$u = u(x)$ is defined by (2.1) and $|x| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$ in particular $R_\alpha^H * R_{-2k}^H = R_{\alpha-2k}^H$ and $R_\alpha^H * R_{2k}^H = R_{\alpha+2k}^H$. The proof of this Lemma is given by M. Aguirre Tellez [4, p121-123].

Lemma 2.3 *(The convolutions product of $S_\gamma(w)$ and $T_\nu(z)$)*

(i) $S_\gamma * S_{\gamma'} = (i)^{\frac{q}{2}} S_{\gamma+\gamma'}$

(ii) $T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$ where S_γ and T_ν are defined by (2.4) and (2.5) respectively.

Proof (i) Now

$$< S_\gamma(w), \varphi(x) >= \frac{1}{W_n(\gamma)}\int_{\Re^n} \omega^{\frac{\gamma-n}{2}}\varphi(x)dx \tag{2.6}$$

where $\varphi \in \mathcal{D}$ the space of infinitely differentialble function with compact suports. We have $\omega = x_1^2 + x_2^2 + ... + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2)$, $p + q = n$. By changing the variables $x_1 = y_1, x_2 = y_2, ..., x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, ..., x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$. Thus we obtain $\omega = y_1^2 + y_2^2 + ... + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + ... + y_{p+q}^2$. Let $r^2 = y_1^2 + y_2^2 + ... + y_{p+q}^2$, $p + q = n$. Thus (2.6) can be written in the form

$$\begin{aligned} < S_\gamma(w), \varphi(x) > &= \frac{1}{W_n(\gamma)}\int_{\Re^n} r^{\gamma-n}\varphi\frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)}dy_1dy_2...dy_n \\ &= \frac{1}{(-i)^{\frac{q}{2}}}\frac{1}{W_n(\gamma)}\int_{\Re^n} r^{\gamma-n}\varphi dy \\ &= \frac{(i)^{\frac{q}{2}}}{W_n(\gamma)} < r^{\gamma-n}, \varphi > \end{aligned}$$

Consider the convolution $S_\gamma * S_{\gamma'}$. We have

$$\begin{aligned} S_\gamma * S_{\gamma'} &= (i)^{\frac{q}{2}} \frac{r^{\gamma-n}}{W_n(\gamma)} * (i)^{\frac{q}{2}} \frac{r^{\gamma'-n}}{W_n(\gamma')} \\ &= (i)^q \frac{r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \quad \text{Lemma 2.1} \\ &= (i)^{\frac{q}{2}} S_{\gamma+\gamma'} \end{aligned}$$

Similarly, for (ii) we also have

$$T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$$

Lemma 2.4 *Given the equation*

$$\oplus^k K(x) = \delta \tag{2.7}$$

where $\oplus^k$ *is the operator iterated k-times defined by (2.5) and* $x = (x_1, x_2, ..., x_n) \in C^n$. *Then*

$$K(x) = R_{2k}^H(x) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z)$$

is an elementary solution of the operatoe $\oplus^k$.

Proof See [2].

3. Main Results

Theorem *Let* $K_{\alpha,\beta,\gamma,\nu}$ *be the distributional kernel defined by (1.6). Then we obtain*

(i) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ *for* $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ *and* ν' *are positive even numbers with* $\alpha = \beta = \gamma = \nu, \alpha' = \beta' = \gamma' = \nu'$.

(ii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'} . K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ *for p is an even* $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ *and* ν' *are any complex numbers and* $\beta_{\alpha,\alpha'} = \frac{\cos(\frac{\alpha\pi}{2})\cos(\frac{\alpha'\pi}{2})}{\cos(\frac{\alpha+\alpha'}{2})\pi}$

(iii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R^e_{\beta+\beta'} * S_{\gamma+\gamma'} * T_{\nu+\nu'}$ *for* p *is an odd* $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ *and* ν' *are any complex numbers* R^e_β, S_γ *and* T_ν *are defined by (2.3), (2.4) and (2.5) respectively. And*

$$A_{\alpha,\alpha'} = \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})} C(\frac{-\alpha'}{2}).\frac{2\pi i}{4}[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}]$$

$$C(r) = \Gamma(r)\Gamma(1-r), H^\pm_r = H_r(u \pm i0, n) = e^{\mp r\frac{\pi}{2}i} e^{\pm q\frac{\pi}{2}i} a(\frac{r}{2})(u \pm i0)^{r-\frac{n}{2}}$$

and

$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$$

$$(u \pm i0)^\lambda = \lim_{\epsilon \to 0} (u + i \in |x|^2)^\lambda$$

$u = u(x)$ *is defined by (2.1) and*

$$|x| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$$

Proof (i) Consider the equation

$$\oplus^k V(x) = \delta$$

where $\oplus^k$ is the operator iterated k-times defined by (1.5) and $x = (x_1, x_2, ..., x_n) \in C^n$. By Lemma 2.4, we obtain the elementary solution

$$\begin{aligned} V(x) &= R^H_{2k}(x) * (-1)^k R^e_{2k}(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z) \\ &= (-1)^k R^H_{2k}(u) * R^e_{2k}(v) * S_{2k}(\omega) * T_{2k}(z) \qquad (3.1) \\ &= (-1)^k K_{2k,2k,2k,2k}(x) \quad \text{by (1.6)} \end{aligned}$$

Thus

$$V(x) = (-1)^k K_{2k,2k,2k,2k}(x) \qquad (3.2)$$

Now $\oplus^k V(x) = \oplus^m \oplus^{k-m} V(x) = \delta$ for $k > m$. Thus

$\oplus^{k-m} V(x) = R^H_{2m}(x) * (-1)^m R^e_{2m}(v) * (-1)^m (-i)^{\frac{q}{2}} S_{2m}(\omega) * (-1)^m (i)^{\frac{q}{2}} T_{2m}(z)$ by Lemma 2.4.

Convolving both sides of the above equation by $R^H_{2k-2m}(x) * (-1)^{k-m}R^e_{2k-2m}(v) * (-1)^{k-m}(-i)^{\frac{q}{2}}S_{2k-2m}(\omega) * (-1)^{k-m}(i)^{\frac{q}{2}}T_{2k-2m}(z)$ and the properties of convolutions and Lemma 2.4 again, we obtain

$$\begin{aligned}V(x) &= [R^H_{2k-2m}(x) * R^H_{2m}(x)] * [(-1)^{k-m}R^e_{2k-2m}(v) * (-1)^m R^e_{2m}(v)] \\ &\quad * [(-1)^{k-m}(-i)^{\frac{q}{2}}S_{2k-2m}(\omega) * (-1)^m(-i)^{\frac{q}{2}}S_{2m}(\omega)] * [(-1)^{k-m}(i)^{\frac{q}{2}}T_{2k-2m}(z) * (-1)^m(i)^{\frac{q}{2}}T_{2m} \\ &= (-1)^k(R^H_{2k-2m}(x) * R^H_{2m}(x)) * (R^e_{2k-2m}(v) * R^e_{2m}(v)) \\ &\quad * (S_{2k-2m}(\omega) * S_{2m}(\omega)) * (T_{2k-2m}(z) * T_{2m}(z))\end{aligned}$$

Thus by (3.1), (3.3) and (3.2) we obtain

$$K_{2k-2m,2k-2m,2k-2m,2k-2m}*K_{2m,2m,2m,2m} = K_{2k-2m+2m,2k-2m+2m,2k-2m+2m,2k-2m+2m}.$$

Since $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma'$ and ν' are all even, thus by putting $\alpha = \beta = \gamma = \nu = 2k - 2m$ and $\alpha' = \beta' = \gamma' = \nu' = 2m$ for $k > m$ we obtain

$$K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$$

(ii) We have

$$K_{\alpha,\beta,\gamma,\nu}*K_{\alpha',\beta',\gamma',\nu'} = [R^H_\alpha(u)*R^H_{\alpha'}(u)]*[R^e_\beta(v)*R^e_{\beta'}(v)]*[S_\gamma(\omega)*S_{\gamma'}(\omega)]*[T_\nu(z)*T_{\nu'}(z)]$$

By Lemma 2.2 (i) for p is an even and Lemma 2.3 (i), (ii), we obtain

$$\begin{aligned}K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} &= B_{\alpha,\alpha'}R^H_{\alpha+\alpha'}(u) * R^e_{\beta+\beta'}(v) * (i)^{\frac{q}{2}}S_{\gamma+\gamma'}(\omega) * (-i)^{\frac{q}{2}}T_{\nu+\nu'}(z) \\ &= B_{\alpha,\alpha'}K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}\end{aligned}$$

where $B_{\alpha,\alpha'} = \frac{\cos(\frac{\alpha}{2}\pi)\cos\frac{\alpha'}{2}\pi)}{\cos(\frac{\alpha+\alpha'}{2})\pi}$. If $\alpha = \beta = \gamma = \nu$ and $\alpha' = \beta' = \gamma' = \nu'$ and $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma', \nu'$ are all even, then $B_{\alpha,\alpha'} = 1$. Thus we obtain case (i).

(iii) By Lemma 2.2 (ii) for p is an odd and Lemma 2.3 (i), (ii) again we obtain

$$\begin{aligned} K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} &= (R^H_{\alpha+\alpha'}(u) + A_{\alpha,\alpha'}) * R^e_{\beta+\beta'}(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'} \\ &= (R^H_{\alpha+\alpha'}(u) * R^e_{\beta+\beta'}(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'}) + A_{\alpha,\alpha'} * R^e_{\beta+\beta'}(v) * S_{\gamma+\gamma'}(\omega) * T \\ &= K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R^e_{\beta+\beta'}(v) * S_{\gamma+\gamma'}(\omega) * T_{\nu+\nu'} \end{aligned}$$

where $A_{\alpha,\alpha'} = \frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})}.\frac{2\pi i}{4}[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}]$, $\quad C(r) = \Gamma(r)\Gamma(1-r)$

$$\begin{aligned} H^{\pm}_r &= H_r(u \pm i0, n) \\ &= e^{\mp r\frac{\pi}{2}i} e^{\pm q\frac{\pi}{2}i} a(\frac{r}{2})(u \pm i0)^{\frac{r-n}{2}} \end{aligned}$$

$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$$

$$(u \pm i0)^{\lambda} = \lim_{\epsilon \to 0}(u + i \in |x|^2)^{\lambda}$$

$u = u(x)$ is defined by (2.1) and

$$|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$$

□

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On the Inversion of Kernel $K_{\alpha,\beta,\gamma,\upsilon}$ related to the operator $\oplus^{k}$

On the Inversion of the Kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$

A. Kananthai, S. Suantai, V. Longani
Department of mathematics, Chiangmai University
Chiangmai 50200, Thailand

ABSTRACT

In this paper, we study the inversion of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$ iterated k-times and is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial X_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k$$

$p+q = n$ is the dimension of the n-dimensional complex space C^n and k is a nonnegative integer α, β, γ and ν are complex parameters. We found that the inverse $[K_{\alpha,\beta,\gamma,\nu}]^{-1}$ of $K_{\alpha,\beta,\gamma,\nu}$ exists depending on the conditions of p and q whether they are odd or even numbers.

1. Introduction

The operator $\oplus^k$ can be factorized in the form

$$\oplus^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \cdot \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + i\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k$$
$$\cdot \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - i\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \qquad (1.1)$$

$p+q = n$ is the dimension of the space C^n, $i = \sqrt{-1}$ and k is a nonnegative integer. The operator $\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2$ is first introduced by A. Kananthai [1] and named the Diamond operator denoted by

$$\diamondsuit = \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 \tag{1.2}$$

Let us denote the operator L_1 and L_2 by

$$L_1 = \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial X_r^2}\right)^2 + i\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 \tag{1.3}$$

$$L_2 = \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial X_r^2}\right)^2 - i\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 \tag{1.4}$$

Thus (1.1) can be written by

$$\oplus^k = \diamondsuit^k L_1^k L_2^k \tag{1.5}$$

Let us define the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z) \tag{1.6}$$

where $R_\alpha^H(u), R_\beta^e(v), S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.2), (2.3), (2.4) and (2.5) respectively and the symbol $*$ denotes the convolution. Since $R_\alpha^H(u), R_\beta^e(v), S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions see [1, p 30-31] and [5, p 154-155], then the convolutions on the right hand side of (1.6) exists and also is a tempered distribution. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also is a tempered distribution. For $\alpha = \beta = \gamma = \nu = 2k$, we obtained $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator $\oplus^k$, see [2].

That is $\oplus^k(-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution and $\oplus^k$ is defined by (1.5). Let $A^{\alpha,\beta,\gamma,\nu}$ be the operator defined by the formula

$$A^{\alpha,\beta,\gamma,\nu}(f) = K_{\alpha,\beta,\gamma,\nu} * f \tag{1.7}$$

where α, β, γ and ν are complex parameters and the symbol $*$ denotes the convolution product and $f \in S$ where S is the Schwartz space, see [8, p 223]. Our objective is to obtain the operator

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1}$$

such that if $\varphi = A^{\alpha,\beta,\gamma,\nu}(f)$ then $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$. We found that, from (1.7) for p and q are both odd we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = K_{-\alpha,-\beta,-\gamma,-\nu}.$$

If p is odd and q is even, we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = (1 + (\sin \alpha\frac{\pi}{2})^2)^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

and if p is even for all complex α we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [(\cos \alpha\frac{\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

such that $\alpha \neq 2s+1 \quad (s = 0, 1, 2, 3, ...)$.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, ..., x_n)$ be a point in the space $\Re^n$ of n-dimensional Euclidean space. Write

$$u = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2, \; p + q = n \tag{2.1}$$

Denote by $\Gamma_+ = \{x \in \Re^n : x_1 > 0 \text{ and } u > 0\}$ the interior of the forward cone and $\bar{\Gamma}_+$ denotes it closure. For any complex number α, we define the function.

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \neq \Gamma_+ \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is give by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}$$

the function R_α^H is first introduced by Y. Nozaki [5, p72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Here $R_\alpha^H(x)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$.

Definition 2.2 Let $x = (x_1, x_2, ..., x_n) \in \Re^n$ and write $v = x_1^2 + x_2^2 + ... + x_n^2$.

For any complex number β, define the function

$$R^e_\beta(v) = \frac{v^{\frac{\beta-n}{2}}}{W_n(\beta)} \tag{2.3}$$

where $W_n(\beta) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\beta)}{\Gamma(\frac{n-\beta}{2})}$ the function $R^e_\beta(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $Re(\beta) \geq n$ and is a distribution of β if $Re(\beta) < n$.

Definition 2.3 Let $x = (x_1, x_2, ..., x_n)$ be a point of the space C^n of n dimensional complex space. Write $w = x_1^2 + x_2^2 + ... + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q})$ and

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q})\ ,\quad p+q=n\quad i=\sqrt{-1}$$

For any complex number γ and ν, define

$$S_\gamma(\omega) = \frac{\omega^{\frac{\gamma-n}{2}}}{W_n(\gamma)} \tag{2.4}$$

and

$$T_\nu(z) = \frac{z^{\frac{\nu-n}{2}}}{W_n(\nu)} \tag{2.5}$$

where $W_n(\gamma) = \frac{\pi^{\frac{n}{2}} 2^\gamma \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}$, $W_n(\nu) = \frac{\pi^{\frac{n}{2}} 2^\nu \Gamma(\frac{\nu}{2})}{\Gamma(\frac{n-\nu}{2})}$.

Lemma 2.1

(i) $R^e_\beta * R^e_{\beta'} = R^e_{\beta+\beta'}$ *where* R^e_β *and* $R^e_{\beta'}$ *are given by (2.3).*

(ii) $S_\gamma * S_{\gamma'} = (i)^{\frac{q}{2}} S_{\gamma+\gamma'}$ *and* $T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$ *where* S_γ *and* T_ν *are defined by (2.4) and (2.5) respectively.*

Proof (i) see [6, p. 20].

(ii) Now $< S_\gamma(\omega), \varphi(x) > = \frac{1}{W_n(\gamma)} \int_{\Re^n} \omega^{\frac{\gamma-n}{2}} \varphi(x) dx$. By changing the variable $x_1 = y_1, x_2 = y_2, ..., x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, ..., x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$ then we obtain

$$\begin{aligned}\omega &= x_1^2 + x_2^2 + ... + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}) \\ &= y_1^2 + y_2^2 + ... + y_p^2 + y_{p+1}^2 + ... + y_{p+q}^2,\ p+q=n.\end{aligned}$$

Let $r^2 = y_1^2 + y_2^2 + ... + y_{p+q}^2$. Thus

$$
\begin{aligned}
< S_\gamma(\omega), \varphi(x) > &= \frac{1}{W_n(\gamma)} \int_{\Re^n} r^{\gamma-n} \varphi \left| \frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)} \right| dy_1 dy_2 ... dy_n \\
&= \frac{(i)^{\frac{q}{2}}}{W_n(\gamma)} \int_{\Re^n} r^{\gamma-n} \varphi dy \\
&= < \frac{(i)^{\frac{q}{2}} r^{\gamma-n}}{W_n(\gamma)}, \varphi > .
\end{aligned}
$$

Now

$$
\begin{aligned}
S_\gamma(\omega) * S_{\gamma'}(\omega) &= \frac{(i)^{\frac{q}{2}} r^{\gamma-n}}{W_n(\gamma)} * \frac{(i)^{\frac{q}{2}} r^{\gamma'-n}}{W_n(\gamma')} \\
&= (i)^q \frac{r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \quad \text{by (i)} \\
&= (i)^{\frac{q}{2}} [\frac{(i)^{\frac{q}{2}} r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')}] \\
&= (i)^{\frac{q}{2}} S_{\gamma+\gamma'}.
\end{aligned}
$$

Thus $S_\gamma * S_{\gamma'} = (i)^{\frac{q}{2}} S_{\gamma+\gamma'}$. Simlarly, we obtain $T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$.

Lemma 2.2

(i) $L_1(-1)^k(-i)^{\frac{q}{2}} S_{2k}(\omega) = \delta$ *that is* $(-1)^k(-i)^{\frac{q}{2}} S_{2k}(\omega)$ *is the elementary solution of the operator* L_1 *defined by (1.3).*

(ii) $L_2(-1)^k(i)^{\frac{q}{2}} T_{2k}(z) = \delta$ *that is* $(-1)^k(i)^{\frac{q}{2}} T_{2k}(z)$ *is the elementary solution of the operator* L_2 *defined by (1.4). Moreover from (i), we obtain* $S_{-2k}(\omega) = (-1)^k(i)^{\frac{q}{2}} L_1^k \delta(x)$ *and*
$T_{-2k}(z) = (-1)^k(-i)^{\frac{q}{2}} L_2^k \delta(x)$. *It follows that*

$$S_0(\omega) = (i)^{\frac{q}{2}} \delta(x) \tag{2.6}$$

and

$$T_0(\omega) = (i)^{\frac{q}{2}} \delta(x) \tag{2.7}$$

Proof See [2, Lemma 2.2 (ii)].

Lemma 2.3 (The convolutions of $R^H_\alpha(u)$)

(i) $R^H_\alpha * R^H_{\alpha'} = \frac{\cos\frac{\alpha\pi}{2}\cos\frac{\alpha'\pi}{2}}{\cos(\frac{\alpha+\alpha'}{2})\pi}.R^H_{\alpha+\alpha'}$ *where* R^H_α *is defined by (2.2) with* p *is an even.*

(ii) $R^H_\alpha * R^H_{\alpha'} = R^H_{\alpha+\alpha'} + T_{\alpha,\alpha'}$ *for* p *is an odd where*

$$T_{\alpha,\alpha'} = T_{\alpha,\alpha'}(u \pm i0, n) = \frac{2\pi i}{4}\frac{C(\frac{-\alpha-\alpha'}{2})}{C(\frac{-\alpha}{2})C(\frac{-\alpha'}{2})}[H^+_{\alpha+\alpha'} - H^-_{\alpha+\alpha'}]$$

$$C(r) = \Gamma(r)\Gamma(1-r)$$

$$H^\pm_r = H_r(u \pm i0, n) = e^{\mp\frac{r\pi i}{2}}e^{\mp\frac{q\pi i}{2}}a(\frac{r}{2})(u \pm i0)^{\frac{r-n}{2}}$$

$$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r\pi^{\frac{n}{2}}\Gamma(\frac{r}{2})]^{-1}$$

$$(u \pm i0)^\lambda = \lim_{\epsilon\to 0}(u + i \in |x|^2)^\lambda$$

$u = u(x)$ *is defined by (2.1)*

Proof See [4, p 121-123].

Lemma 2.4 (The inverse of convolution algebra)

(i) $R^H_\alpha * R^H_{-\alpha} = R^H_{\alpha-\alpha} = R^H_0 = \delta$ *if* p *and* q *are both odd.*

(ii) $R^H_\alpha * R^H_{-\alpha} = (1 + (\sin\frac{\alpha\pi}{2})^2)\delta(x)$ *if* p *is odd and* q *is even.*

(iii) $R^H_\alpha * R^H_{-\alpha} = ((\cos\frac{\alpha\pi}{2})^2)\delta(x)$ *for* p *is even and* $\alpha \neq 2s+1,\ s = 0, 1, 2, 3, ...$

(iv) $R^e_\beta * R^e_{-\beta} = R^e_{\beta-\beta} = R^e_0 = \delta$

(v) $S_\gamma * S_{-\gamma} = (i)^q\delta$

(vi) $T_\nu * T_{-\nu} = (-i)^q\delta$

Proof (i) By Lemma 2.3 (ii), for p is an odd, we obtain $R^H_\alpha * R^H_{-\alpha} = R^H_0 + T_{\alpha,-\alpha}$. Since $R^H_0 = \delta$ and by computing directly $T_{\alpha,-\alpha} = 0$ for p and q are both odd. Thus $R^H_\alpha * R^H_{-\alpha} = \delta$ for p and q are both odd.

(ii) By Lemma 2.3 (ii) again, for p is an odd, we obtain $R^H_\alpha * R^H_{-\alpha} = R^H_0 + T_{\alpha,-\alpha}$. Now, for p is an odd and q is an even, we obtain $T_{\alpha,-\alpha} = (\sin\frac{\alpha\pi}{2})^2\delta$. Thus

$$\begin{aligned} R^H_\alpha * R^H_{-\alpha} &= \delta + (\sin\frac{\alpha\pi}{2})^2\delta \\ &= [1 + (\sin\frac{\alpha\pi}{2})^2]\delta(x) \end{aligned}$$

(iii) By Lemma 2.3 (i), for p is an even

$$\begin{aligned} R^H_\alpha * R^H_{-\alpha} &= \frac{\cos\frac{\alpha\pi}{2}\cos\frac{(-\alpha)\pi}{2}}{\cos\frac{\alpha-\alpha}{2}\pi}.R^H_0 \\ &= [\cos\frac{\alpha\pi}{2}]^2 R^H_0 \\ &= (\cos\frac{\alpha\pi}{2})^2\delta \quad ,\alpha \neq 2s+1, \quad s=0,1,2,3,... \end{aligned}$$

(iv) By Lemma 2.1 (i), $R^e_\beta * R^e_{-\beta} = R^e_{\beta-\beta} = R^e_0 = \delta$ since $R^e_0 = \delta$, see [5, p 118].
(v) By Lemma 2.1 (ii),

$$\begin{aligned} S_\gamma * S_{-\gamma} &= (i)^{\frac{q}{2}} S_{\gamma-\gamma} = (i)^{\frac{q}{2}} S_0 \\ &= (i)^{\frac{q}{2}}(i)^{\frac{q}{2}}\delta = (i)^q\delta \quad \text{by (2.6)} \end{aligned}$$

(vi) By Lemma 2.1 (ii) again, we obtain

$$\begin{aligned} T_\nu * T_{-\nu} &= (-i)^{\frac{q}{2}} T_{\nu-\nu} = (-i)^{\frac{q}{2}} T_0 \\ &= (-i)^{\frac{q}{2}}(-i)^{\frac{q}{2}}\delta = (-i)^q\delta \quad \text{by (2.7)} \end{aligned}$$

Lemma 2.5 *Let the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by defined by (1.6) then the following formulas hold.*

(i) $K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = K_{0,0,0,0} = \delta$ *for p and q are both odd numbers.*

(ii) $K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = [1 + (\sin\frac{\alpha\pi}{2})^2]\delta$ *for p is odd and q is even.*

(iii) $K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = (\cos\frac{\alpha\pi}{2})^2\delta$ *for p is even.*

Proof (i) By (1.6) and properties of convolutions,

$$K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = (R^H_\alpha * R^H_{-\alpha}) * (R^e_\beta * R^e_{-\beta}) * (S_\gamma * S_{-\gamma}) * (T_\nu * T_{-\nu}) \tag{2.8}$$

$$\begin{aligned} &= \delta * \delta * (i)^q\delta * (-i)^q\delta \\ &= \delta * \delta * \delta * \delta = \delta \qquad \text{by Lemma 2.4 (i), (iv), (v), (vi)} \end{aligned}$$

(ii) From (2.8),

$$K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = [1 + (\sin \frac{\alpha\pi}{2})^2]\delta * \delta * (i)^q\delta * (-i)^q\delta$$
$$= [1 + (\sin \frac{\alpha\pi}{2})^2]\delta * \delta * \delta * \delta$$
$$= [1 + (\sin \frac{\alpha\pi}{2})^2]\delta \quad \text{by Lemma 2.4 (ii), (iv), (v), (vi)}$$

(iii) From (2.8) again, we obtain

$$K_{\alpha,\beta,\gamma,\nu} * K_{-\alpha,-\beta,-\gamma,-\nu} = (\cos \frac{\alpha\pi}{2})^2\delta * \delta * (i)^q\delta * (-i)^q\delta$$
$$= (\cos \frac{\alpha\pi}{2})^2\delta \quad \text{for} \quad \alpha \neq 2s + 1, s = 0, 1, 2, 3, \ldots$$

by Lemma 2.4 (iii), (iv), (v), (vi).

3. Main results

Theorem Given $\varphi = A^{\alpha,\beta,\gamma,\nu}(f)$ where $A^{\alpha,\beta,\gamma,\nu}(f)$ is defined by (1.7) for every $f \in S$ where S is the Schwartz space of function, then there exists the operator $B^{\alpha,\beta,\gamma,\nu}$ such that

(i) $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = K_{-\alpha,-\beta,-\gamma,-\nu}$$

if p and q are both odd numbers.

(ii) $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = (1 + (\sin \frac{\alpha\pi}{2})^2)^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

if p is odd and q is even.

(iii) $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [(\cos \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

if p is even for all α such that $\alpha \neq 2s + 1, \ s = 0, 1, 2, 3, \ldots$.

Proof (i) From (1.7), we have

$$A^{\alpha,\beta,\gamma,\nu}(f) = K_{\alpha,\beta,\gamma,\nu} * f = \varphi$$

where $K_{\alpha,\beta,\gamma,\nu}$ is defined by (1.6). Then by Lemma 2.5 (i) we obtain

$$\begin{aligned} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) &= (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f \\ &= K_{0,0,0,0} * f = \delta * f = f \end{aligned}$$

if p and q are both odd. Thus

$$B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = K_{-\alpha,-\beta,-\gamma,-\nu}$$

It follow that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

(ii) By Lemma 2.5 (ii), for p is an odd and q is an even, we have

$$\begin{aligned} [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) &= [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} \\ &\quad \times (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f \\ &= [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} [1 + (\sin \frac{\alpha\pi}{2})^2] \\ &\quad \times K_{0,0,0,0} * f \\ &= \delta * f = f \end{aligned}$$

Thus $B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [1 + (\sin \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$. It follow that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

(iii) By Lemma 2.5 (iii), for p is even

$$\begin{aligned} [(\cos \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) &= [(\cos \frac{\alpha\pi}{2})^2]^{-1} \\ &\quad \times (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f \\ &= [(\cos \frac{\alpha\pi}{2})^2]^{-1} [(\cos \frac{\alpha\pi}{2})^2] \\ &\quad \times K_{0,0,0,0} * f \\ &= \delta * f = f \end{aligned}$$

Thus $B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = [(\cos \frac{\alpha\pi}{2})^2]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$. It follow that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

In particular, we obtain for $\alpha = \beta = \gamma = \nu = 2k$ and p and q are both odd numbers.

$$(-1)^k K_{2k,2k,2k,2k}(x) = (-1)^k R^H_{2k}(u)*(-1)^k R^e_{2k}(v)*(-1)^k(-i)^{\frac{q}{2}} S_{2k}(\omega)*(-1)^k(i)^{\frac{q}{2}} T_{2k}(z)$$

is an elementary solution of the operator $\oplus^k$ defined by (1.5) and by Lemma 2.2 (i), (ii). Now

$$\begin{aligned}
(-1)^k K_{2k,2k,2k,2k} * (-1)^k K_{-2k,-2k,-2k,-2k} &= [(-1)^k R^H_{2k}(u) * (-1)^k R^H_{-2k}] \\
&\quad * [(-1)^k R^e_{2k}(v) * (-1)^k R^e_{-2k}(v)] \\
&\quad * [(-1)^k(-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k(-i)^{\frac{q}{2}} S_{-2k}] \\
&\quad * [(-1)^k(i)^{\frac{q}{2}} T_{2k}(z) * (-1)^k(i)^{\frac{q}{2}} T_{-2k}(z)] \\
&= R^H_0 * R^e_0 * (-i)^q S_0(\omega) * (i)^q T_0(z) \\
&= \delta * \delta * (-i)^q (i)^{\frac{q}{2}} \delta * (i)^q (-i)^{\frac{q}{2}} \delta \\
&= \delta * \delta * \delta * \delta \\
&= \delta \qquad \text{by (2.6), (2.7)}
\end{aligned}$$

Thus $(-1)^k K_{-2k,-2k,-2k,-2k}$ is an inverse of the convolution algebra of the elementary solution of the operator $\oplus^k$.

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On the Diamond Operator Related to the Wave Equation

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On the nonlinear Diamond operator related to the Wave Equation

Amnuay Kananthai, G. Sritanratana

Department of Mathematics, Chiangmai University, Chiangmai, 50200 Thailand.
E-mail:malamnaka@science.cmu.ac.th

Abstract

In this paper, we study the solution of nonlinear equation $\diamondsuit^k u(x) = f(x, \triangle^{k-1}\Box^k u(x))$ where $\diamondsuit^k$ is the Diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left((\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2})^2 - (\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2})^2 \right)^k$$

or the operator $\diamondsuit^k$ can be expressed by $\diamondsuit^k = \Box^k \triangle^k = \triangle^k \Box^k$. The operators $\triangle^k$ and $\Box^k$ are Laplacian and the ultrahyperbolic operator iterated k times, defined by

$$\Box^k = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2})^k$$

and $\triangle^k = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2})^k$, $p+q=n$ is the dimension of the n-dimensional Euclidean space $\Re^n, x = (x_1, x_2, \dots, x_n) \in \Re^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a given function. It is found that, the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\triangle^{k-1}\Box^k u(x)$ and moreover such solution $u(x)$ related to the wave equation depending on the conditions of p, q and k.

1. Introduction

The operator $\diamondsuit^k$ was first introduced by A. Kananthai [3] and is named the Diamond operator iterated k times and is defined by

$$\diamondsuit^k = ((\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_p^2})^2 - (\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + ... + \frac{\partial^2}{\partial x_{p+q}^2})^2)^k \quad (1.1)$$

For the linear equation $\diamondsuit^k u(x) = f(x)$, see [4], it has been already studied and obtained the convolution $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ as a solution of such equation where $K_{2k,2k}(x) = R^H_{2k}(x) * R^e_{2k}(x)$. The functions $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by (2.2) and (2.5) respectively with $\alpha = \beta = 2k$.

In this work, we study the nonlinear equation of the form

$$\diamondsuit^k u(x) = f(x, \triangle^{k-1}\square^k u(x)) \quad (1.2)$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of $\Re^n$ and $\partial\Omega$ denotes the boundary of Ω. We can find the solution $u(x)$ of (1.2) and is unique under the condition $|f(x, \triangle^{k-1}\square^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\triangle^{k-1}\square^k u(x) = 0$ for $x \in \partial\Omega$. If we put $k = 1, p = 1$ and $q = n-1$ in (1.2), we obtain $u(x) = M_2(s) * W(x)$ is a solution of nonhomogeneous wave equation where $M_2(s)$ is defined by (2.4) with $\alpha = 2$ and $W(x)$ is continuous function.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, ..., x_n) \in \Re^n$ and denote by

$$v = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \ , \ p + q = n \quad (2.1)$$

the nondegenerated quadratic form. Denotes the interior of the forward cone by

$$\Gamma_+ = \{x \in \Re^n : x_1 > 0 \text{ and } v > 0\}$$

and by $\overline{\Gamma_+}$ the closure of Γ_+. For any complex number α, define

$$R^H_\alpha(v) = \begin{cases} \frac{v^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+ \\ 0, & \text{for } x \notin \Gamma_+ \end{cases} \quad (2.2)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})} \tag{2.3}$$

The function $R^H_\alpha(v)$ was introduced by Y. Nozaki [5, p.72]. It is well known that $R^H_\alpha(v)$ is an ordinary function if $Re(\alpha) \geq n$ and it is a distribution of α if $Re(\alpha) < n$. Let *supp* $R^H_\alpha(v)$ denote the support of $R^H_\alpha(v)$. By putting $p = 1$ in (2.1) and (2.2) and remembering the Legendre's duplication of $\Gamma(z)$. $\Gamma(2z) = 2^{2z-1}\pi^{\frac{-1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$ then (2.2) reduces to

$$M_\alpha(s) = \begin{cases} \frac{s^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.4}$$

Here $s = x_1^2 - x_2^2 - \ldots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{n-1}{2}}2^{\alpha-1}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{\alpha}{2})$, $M_\alpha(s)$ is, precisely, the hyperbolic kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\Re^n$ and the function $R^e_\beta(x)$ be defined by

$$R^e_\beta(x) = \frac{|x|^{\beta-n}}{W_n(\beta)} \tag{2.5}$$

where $W_n(\beta) = \frac{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$, β is a complex parameter and $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$.

Lemma 2.1 *Given the equation*

$$\triangle^k u(x) = 0 \tag{2.6}$$

where $\triangle^k$ is the Laplacian iterated k times, defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k \quad , x = (x_1, x_2, \ldots, x_n) \in \Re^n \tag{2.7}$$

then, we obtain $u(x) = (-1)^{k-1}(R^{e}_{2(k-1)}(x))^{(m)}$ *as a solution of (2.5) where* m *is a nonnegative integer with* $m = \frac{n-4}{2}$, $n \geq 4$ *and* n *is an even and* $(R_{2(k-1)^e(x)})^{(m)}$ *is a function defined by (2.4) with* m *derivatives and* $\beta = 2(k-1)$.

Proof see [4, Lemma 2.2].

Lemma 2.2 *Given the equation*

$$\Box^k u(x) = 0 \tag{2.8}$$

where $\Box^k$ *is the ultrahyperbolic operator iterated* k *times defined by*

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \tag{2.9}$$

$x = (x_1, x_2, \dots, x_n) \in \Re^n$. *Then we obtain* $u(x) = [R^H_{2(k-1)}(v)]^{(m)}$ *as a solution of (2.8) with* $m = \frac{n-4}{2}$, $n \geq 4$ *and* n *is an even,* v *is defined by (2.1). The functions* $[R^H_{2(k-1)}(v)]^{(m)}$ *is defined by (2.2) with* m *derivatives and* $\alpha = 2(k-1)$.

Proof see [4, Lemma 2.3].

Lemma 2.3 *Given the equation*

$$\Delta u(x) = f(x, u(x)) \tag{2.10}$$

where f *defined and having continuous first derivatives for all* $x \in \Omega \cup \partial\Omega$, Ω *is an open subset of* $\Re^n$ *and* $\partial\Omega$ *is the boundary of* Ω. *Assume that* f *is bounded, that is* $|f(x, u)| \leq N$ *and the boundary condition* $u(x) = 0$ *for* $x \in \partial\Omega$. *Then we obtain* $u(x)$ *as a unique solution of (2.10).*

Proof We can prove the existence of the solution $u(x)$ of (2.10) by the method of iterations and the Schauder's estimates. The details of proving is given by R. Courant and D. Hilbert, see [1, pp 369-372].

3. Main results

Theorem *Consider the nonlinear equation*

$$\diamondsuit^k u(x) = f(x, \triangle^{k-1}\square^k u(x)) \tag{3.1}$$

where $\diamondsuit^k$ *is the Diamond operator iterated* k *times, defined by (1.1),* $\triangle^{k-1}$ *is the Laplacian iterated* $k-1$ *times defined by (2.7) and* $\square^k$ *is the ultrahyperbolic operator iterated* k *times, defined by (2.9). Let* f *be defined and continuous first derivative for all* $x \in \Omega \cup \partial\Omega$*,* Ω *is an open subset of* $\Re^n$ *and* $\partial\Omega$ *denotes the boundary of* Ω *and* n *is even with* $n \geq 4$*. Let suppose* f *be bounded function, that is*

$$|f(x, \triangle^{k-1}\square^k u(x))| \leq N \text{ for all } x \in \Omega \tag{3.2}$$

and the boundary condition

$$\triangle^{k-1}\square^k u(x) = 0 \quad \text{for } x \in \partial\Omega \tag{3.3}$$

then we obtain

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * R^H_{2k}(v) * W(x) \tag{3.4}$$

as a solution of (3.1) with the boundary condition

$$u(x) = R^H_{2k}(v) * (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)}$$

for $x \in \partial\Omega$ *,* $m = \frac{n-4}{2}$ *,* $k = 2, 3, 4, 5, \ldots$ *and* v *is given by (2.1),* $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$*,* $R^e_{2(k-2)}(x)$ *and* $R^H_{2k}(v)$ *are given by (2.5) and (2.2) respectively with* $\beta = 2(k-2)$ *and* $\alpha = 2k$*. Moreover, for* $k = 1$ *we obtain*

$$u(x) = R^H_2(v) * W(x)$$

as a solution of (3.1) with the boundary condition

$$u(x) = \delta^{(m)}(v) \quad \text{for } x \in \partial\Omega$$

where $\delta^{(m)}(v)$ *is the Dirac-delta distribution with* m *derivatives and* $m = \frac{n-4}{2}$*. Also if we put* $k = 1, p = 1$ *and* $q = n-1$*. We obtain* $u(x) = M_2(s) * W(x)$ *as a solution* *of nonhomogeneous wave equation*

$$\Box^* u(x) = W(x) \tag{3.5}$$

with the boundary condition $u(x) = \delta^{(m)}(s)$ *for* $x \in \partial\Omega$ *where* $\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \ldots - \frac{\partial^2}{\partial x_n^2}$ *and*
$s = x_1^2 - x_2^2 - \ldots - x_n^2$ *and* $M_2(s)$ *is defined by (2.4) with* $\alpha = 2$.

Proof We have

$$\Diamond^k u(x) = \triangle(\triangle^{k-1}\Box^k u(x)) = f(x, \triangle^{k-1}\Box^k u(x)) \tag{3.6}$$

Since $u(x)$ has continuous derivatives up to order $4k$ for $k = 1, 2, 3, \ldots$, thus we can assume

$$\triangle^{k-1}\Box^k u(x) = W(x) \quad \text{for all } x \in \Omega \tag{3.7}$$

Thus (3.6) can be written in the form

$$\Diamond^k u(x) = \triangle W(x) = f(x, W(x)) \tag{3.8}$$

by (3.2),

$$|f(x, w)| \le N \quad \text{for all} \quad x \in \Omega \tag{3.9}$$

and by (3.3), $W(x) = 0$ or

$$\triangle^{k-1}\Box^k u(x)) = 0 \tag{3.10}$$

for $x \in \partial\Omega$. Thus by Lemma 2.3 there exist a unique solution $W(x)$ of (3.8) which satisfies (3.9). Now consider the equation (3.7), we have $\triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x) = \delta$ and $\Box^k R^H_{2k}(v) = \delta$ where δ is the Dirac-delta distribution, that is $R^H_{2k}(v)$ and $(-1)^{k-1}R^e_{2(k-1)}(x)$ are the elementary solutions of the operators $\Box^k$ and $\triangle^{k-1}$ respectively, see [6, p11] and [2, p118]. The functions $R^e_{2(k-1)}(x)$ and $R^H_{2k}(v)$ are defined by (2.5) and (2.2) respectively, with $\beta = 2(k-1), \alpha = 2k$. Thus convolving both sides of (3.7) by $(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v)$, we obtain

$$[(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v)] * \triangle^{k-1}\Box^k u(x) = [(-1)^{k-1}R^e_{2(k-1)}(x) * R^H_{2k}(v)] * W(x)$$

By the properties of convolution, we obtain

$$(\triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x))*(\square^k R^H_{2k}(v))*u(x) = (-1)^{k-1}R^e_{2(k-1)}(x)*R^H_{2k}(v)*W(x)$$

$$\delta*\delta*u(x) = (-1)^{k-1}R^e_{2(k-1)}(x)*R^H_{2k}(v)*W(x)$$

Thus

$$u(x) = (-1)^{k-1}R^e_{2(k-1)}(x)*R^H_{2k}(v)*W(x) \tag{3.11}$$

as required. Consider $\triangle^{k-1}\square^k u(x) = 0$ for $x \in \partial\Omega$. By Lemma 2.1, we have

$$\square^k u(x) = (-1)^{k-2}(R^e_{2(k-2)}(x))^{(m)}.$$

Convoling both sides of the above equation by $R^H_{2k}(v)$, we obtain $R^H_{2k}(v) * \square^k u(x) = R^H_{2k}(v)*(-1)^{k-2}(R^e_{2k}(x))^{(m)}$ or $\square^k R^H_{2k}(v)*u(x) = R^H_{2k}(v)*(-1)^{(k-2)}(R^e_{2(k-2)}(x)$. It follows that $\delta*u(x) = u(x) = R^H_{2k}(v)*(-1)^{(k-2)}(R^e_{2(k-2)}(x))^{(m)}$ for $x \in \partial\Omega$ and $k = 2, 3, 4, 5, \ldots$.

Now, for $k = 1$ in (3.11), we obtain

$$\begin{aligned} u(x) &= R^e_0(x) * R^H_2(v) * W(x) \\ &= \delta * R^H_2(v) * W(x) \\ &= R^H_2(v) * W(x) \end{aligned} \tag{3.12}$$

since $R^e_0(x) = \delta$, see [2, p118]. Now consider the boundary condition for $k = 1$ in (3.10) we have $\square u(x) = 0$ for $x \in \partial\Omega$. Thus, by Lemma 2.2, for $k = 1$. we obtain

$$u(x) = \delta^{(m)}(v) \quad \text{for} \quad x \in \partial\Omega \tag{3.13}$$

where $R^H_0(v) = \delta$, see [6]. Now consider the cases $k = 1, p = 1$ and $q = n-1$. thus from (3.12) $R^H_2(v)$ reduces to $M_2(s)$ where $M_2(s)$ is defined by (2.4) with $\alpha = 2$ and $s = x_1^2 - x_2^2 - \ldots - x_n^2$ and also the operator $\square$ defined by (2.9) reduces to the wave operator

$$\square^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Thus the solution $u(x)$ of (3.12) reduces to $u(x) = M_2(s) * W(x)$ which is the solution of the wave equation $\Box^* u(x) = W(x)$ with the boundary condition $\Box^* u(x) = 0$ for $x \in \partial\Omega$ or $u(x) = \delta^{(m)}(s)$ for $x \in \partial\Omega$ by (3.13). $\Box$

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บทที่ 4

การแปลงฟูริเยร์ของตัวดำเนินการ $\oplus^k$

Fourier Transform of the Operator $\oplus^k$

ในส่วนแรกของบทนี้เป็นการศึกษาการแปลงฟูริเยร์ของฟังก์ชันปกติ สมบัติต่างๆ ที่เกี่ยวกับการแปลงฟูริเยร์ของตัวดำเนินการไดมอนด์ และ ตัวดำเนินการอัลตรา ไฮเปอร์โบลิค และได้ศึกษาการแปลงฟูริเยร์ของ Distributional Kernel $K_{\alpha,\beta,\gamma,\upsilon}$ ที่เกี่ยวข้องกับตัวดำเนินการ $\oplus^k$

On the Fourier transform of the Diamond Kernel of Marcel Riesz

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On the Fourier transform of the Diamond Kernel of Marcel Riesz

Amnuay Kananthai

Department of Mathematics, Chiang Mai University, Chiang Mai 50200, Thailand

Abstract

In this paper, the operator $\diamondsuit^k$ is introduced and named as the Diamond operator iterated k-times and is defined by $\diamondsuit^k = [(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_p^2)^2 - (\partial^2/\partial x_{p+1}^2 + \partial^2/\partial x_{p+2}^2 + \cdots + \partial^2/\partial x_{p+q}^2)^2]^k$, where n is the dimension of the Euclidean space $\mathbb{R}^n$, k is a nonnegative integer and $p + q = n$. The elementary solution of the operator $\diamondsuit^k$ is called the Diamond Kernel of Marcel Riesz. In this work we study the Fourier transform of the elementary solution and also the Fourier transform of their convolutions.

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1. Introduction

Consider the equation

$$\diamondsuit^k u(x) = \delta, \tag{1}$$

where $\diamondsuit^k$ is the Diamond operator iterated k-times ($k = 0, 1, 2, \ldots$) with $\diamondsuit^0 u(x) = u(x)$ and is defined by

$$\diamondsuit^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right)^k, \tag{2}$$

where $p + q = n$, the dimension of the Euclidean space $\mathbb{R}^n$ and $u(x)$ is the generalized function, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and δ is the Dirac-delta distribution.

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Kananthai ([1], Theorem 1.3) has shown that the solution of convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is an unique elementary solution of Eq. (1) where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by Eqs. (4) and (6), respectively, with $\alpha = 2k$. Now $(-1)^k S_{2k}(x) * R_{2k}(x)$ is a generalized function, see [1], and is called the Diamond Kernel of Marcel Riesz. In this paper we study the Fourier transform of $(-1)^k S_{2k}(x) * R_{2k}(x)$ and the Fourier transform of $[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]$ where k and m are nonnegative integers.

2. Preliminaries

Definition 2.1. Let $E(x)$ be a function defined by

$$E(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}, \tag{3}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\omega_n = (2\pi^{n/2})/\Gamma(n/2)$ is a surface area of the unit sphere.

It is well known that $E(x)$ is an elementary solution of the Laplace operator Δ, that is $\Delta E(x) = \delta$ where $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$ and δ is the Dirac-delta distribution.

Definition 2.2. Let $S_\alpha(x)$ be a function defined by

$$S_\alpha(x) = 2^{-\alpha}\pi^{-n/2}\Gamma\left(\frac{n-\alpha}{2}\right)\frac{|x|^{\alpha-n}}{\Gamma(\frac{\alpha}{2})}, \tag{4}$$

where α is a complex parameter, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

$S_\alpha(x)$ is called the Elliptic Kernel of Marcel Riesz. Now $S_\alpha(x)$ is an ordinary function for $\mathrm{Re}(\alpha) \geq n$ and is a distribution of α for $\mathrm{Re}(\alpha) < n$.

From Eqs. (3) and (4) we obtain

$$E(x) = -S_2(x). \tag{5}$$

Definition 2.3. Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n$ and write

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,$$

where $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n: x_1 > 0 \text{ and } V > 0\}$ designating the interior of the forward cone and denote $\bar{\Gamma}_+$ by its closure and the following function introduced by Nozaki ([2], p. 72),

$$R_\alpha(x) = \begin{cases} \dfrac{V^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{6}$$

Here $R_\alpha(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and α is a complex parameter and n is the dimension of the space $\mathbb{R}^n$.

The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}.$$

Here $R_\alpha(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geqslant n$ and is a distribution of α if $\mathrm{Re}(\alpha) < n$.

Let $\operatorname{supp} R_\alpha(x) \subset \bar{\Gamma}_+$ where $\operatorname{supp} R_\alpha(x)$ denote the support of $R_\alpha(x)$.

Definition 2.4. Let f be a continuous function, the Fourier transform of f denoted by

$$\mathscr{F}f = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi x}\,dx, \tag{7}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$. Sometimes we write $\mathscr{F}f(x) \equiv \hat{f}(\xi)$. By Eq. (7), we can define the inverse of the Fourier transform of $\hat{f}(\xi)$ by

$$f(x) = \mathscr{F}^{-1}\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}} e^{i\xi x}\hat{f}(\xi)\,d\xi. \tag{8}$$

If f is a distribution with compact supports by [3], Theorem 7.4-3, p. 187 Eq. (7) can be written as

$$\mathscr{F}f = \frac{1}{(2\pi)^{n/2}}\langle f(x), e^{-i\xi x}\rangle. \tag{9}$$

Lemma 2.1. *The functions $S_\alpha(x)$ and $R_\alpha(x)$ defined by Eqs. (4) and (6), respectively, for* $\mathrm{Re}(\alpha) < n$ *are homogeneous distributions of order $\alpha - n$.*

Proof. Since $R_\alpha(x)$ and $S_\alpha(x)$ satisfy the Euler equation, that is

$$(\alpha - n)R_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_\alpha(x) \text{ and } (\alpha - n)S_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_\alpha(x),$$

we have that $R_\alpha(x)$ and $S_\alpha(x)$ are homogeneous distributions of order $\alpha - n$.

Donoghue ([4], pp. 154 and 155) has proved that every homogeneous distribution is a tempered distribution.

That completes the proof.

Lemma 2.2 (The convolution of tempered distributions). *The convolution $S_\alpha(x) * R_\alpha(x)$ exists and is a tempered distribution.*

Proof. Choose supp $R_\alpha(x) = K \subset \bar{\Gamma}$, where K is a compact set. Then $R_\alpha(x)$ is a tempered distribution with compact support and by [3], pp. 156–159 $S_\alpha(x) * R_\alpha(x)$ exists and is a tempered distribution.

Lemma 2.3. *Given the equation $\diamondsuit^k u(x) = \delta$ where the operator $\diamondsuit^k$ is defined by Eq. (2), $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer and δ is the Dirac-delta distribution, then $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the equation where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by Eqs. (4) and (6), respectively, with $\alpha = 2k$.*

Proof. By Lemma 2.2, for $\alpha = 2k$, the distribution $(-1)^k S_{2k}(x) * R_{2k}(x)$ exists and is a tempered distribution.

Now the distribution $(-1)^k S_{2k}(x)$ is obtained by the convolution

$$\underbrace{E(x) * E(x) * \cdots * E(x)}_{k\text{-times}} = \underbrace{(-S_2(x)) * (-S_2(x)) * \cdots * (-S_2(x))}_{k\text{-times}},$$

where $E(x)$ is defined by Eq. (3) and by Eq. (5).

Kananthai ([5], Lemma 2.5) has shown that

$$\underbrace{-S_2(x) * (-S_2(x)) * \cdots * (-S_2(x))}_{k\text{-times}} = (-1)^k S_{2k}(x)$$

is an elementary solution of the Laplace operator Δ^k iterated k-times. By Eq. (2), $\diamondsuit^k$ can be written as

$$\diamondsuit^k = \square^k \Delta^k, \tag{10}$$

where

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial^2_{p+q}}\right)^k$$

and

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k, \quad p + q = n.$$

By [1], Theorem 3.1 $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the operator $\diamondsuit^k$ as required.

Lemma 2.4 (The Fourier transform of $\diamondsuit^k \delta$).

$$\mathscr{F} \diamondsuit^k \delta = \frac{1}{(2\pi)^{n/2}} \left((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2)^2\right)^k,$$

where $\mathscr{F}$ is the Fourier transform defined by Eq. (7) and if the norm of ξ is given by $\|\xi\| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{1/2}$ then

$$|\mathscr{F}\Diamond^k\delta| \leqslant \frac{1}{(2\pi)^{n/2}}\|\xi\|^{4k} \tag{11}$$

that is $\mathscr{F}\Diamond^k\delta$ is bounded and continuous on the space S' of the tempered distribution. Moreover, by Eq. (8)

$$\Diamond^k\delta = \mathscr{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2)^2\right)^k.$$

Proof. By Eq. (9)

$$\begin{aligned}
\mathscr{F}\Diamond^k\delta &= \frac{1}{(2\pi)^{n/2}}\langle\Diamond^k\delta, e^{-i\xi x}\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, \Diamond^k e^{-i\xi x}\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, \Box^k\Delta^k e^{-i\xi x}\rangle \quad \text{by (10)} \\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, (-1)^k(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^k\Box^k e^{-i\xi x}\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, (-1)^k(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^k(-1)^k \\
&\quad\times(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_{p+2}^2)^k e^{-i\xi x}\rangle \\
&= \frac{1}{(2\pi)^{n/2}}(-1)^{2k}((\xi_1^2 + \cdots + \xi_n^2)^k \times (\xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2)^k \\
&= \frac{1}{(2\pi)^{n/2}}\left((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2\right)^k.
\end{aligned}$$

Now

$$\begin{aligned}
|\mathscr{F}\Diamond^k\delta| &= \frac{1}{(2\pi)^{n/2}}\left(|\xi_1^2 + \cdots + \xi_n^2||\xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots \xi_{p+q}^2|\right)^k \\
&\leqslant \frac{1}{(2\pi)^{n/2}}\left(|\xi_1^2 + \cdots + \xi_n^2|\right)^k \\
&= \frac{1}{(2\pi)^{n/2}}\|\xi\|^{4k}.
\end{aligned}$$

where $\|\xi\| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$, ξ_i $(i = 1, 2, \ldots, n) \in \mathbb{R}$. Hence we obtain Eq. (11) and $\mathscr{F}\Diamond^k\delta$ is bounded and continuous on the space S' of the tempered distribution.

Since $\mathscr{F}$ is 1–1 transformation from the space S' of the tempered distribution to the real space R, then by Eq. (8)

$$\diamondsuit^k \delta = \frac{1}{(2\pi)^{n/2}} \mathscr{F}^{-1}\left((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2 \right)^k.$$

That completes the proof.

3. Main results

Theorem 3.1.

$$\mathscr{F}((-1)^k S_{2k}(x) * R_{2k}(x)) = \frac{1}{(2\pi)^{n/2}[(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^k}$$

and

$$|\mathscr{F}((-1)^k S_{2k}(x) * R_{2k}(x))| \leqslant \frac{1}{(2\pi)^{n/2}} M \text{ for a large } \xi_i \in R, \tag{12}$$

where M is a constant. That is $\mathscr{F}$ is bounded and continuous on the space S' of the tempered distributions.

Proof. By Lemma 2.3 $\diamondsuit^k((-1)^k S_{2k}(x) * R_{2k}(x) = \delta$ or $(\diamondsuit^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta$.

Taking the Fourier transform on both sides, we obtain

$$\mathscr{F}((\diamondsuit^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)]) = \mathscr{F}\delta = \frac{1}{(2\pi)^{n/2}}.$$

By Eq. (9)

$$\frac{1}{(2\pi)^{n/2}} \langle (\diamondsuit^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)], \mathrm{e}^{-\mathrm{i}\xi x} \rangle = \frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\frac{1}{(2\pi)^{n/2}} \langle (\diamondsuit^k \delta), \langle [(-1)^k S_{2k}(r) * R_{2k}(r)], \mathrm{e}^{-\mathrm{i}\xi(x+r)} \rangle \rangle = \frac{1}{(2\pi)^{n/2}}.$$

$$\frac{1}{(2\pi)^{n/2}} \langle [(-1)^k S_{2k}(r) * R_{2k}(r)], \mathrm{e}^{-\mathrm{i}\xi r} \rangle \langle (\diamondsuit^k \delta), \mathrm{e}^{-\mathrm{i}\xi x} \rangle = \frac{1}{(2\pi)^{n/2}}.$$

$$\mathscr{F}([(-1)^k S_{2k}(r) * R_{2k}(r)])(2\pi)^{n/2} \mathscr{F}(\diamondsuit^k \delta) = \frac{1}{(2\pi)^{n/2}}.$$

By Lemma 2.4,

$$\mathscr{F}([(-1)^k S_{2k}(x) * R_{2k}(x)])((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^k = \frac{1}{(2\pi)^{n/2}}.$$

It follows that

$$\mathscr{F}([(-1)^k S_{2k}(x) * R_{2k}(x)]) = \frac{1}{(2\pi)^{n/2}[(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^k}.$$

Now

$$\begin{aligned}&\frac{1}{[(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^k}\\ &\quad = \frac{1}{(\xi_1^2 + \cdots + \xi_n^2)^k}\,\frac{1}{(\xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2)^k}.\end{aligned} \tag{13}$$

Let $\xi = (\xi_1, \ldots, \xi_n) \in \Gamma_+$ with Γ_+ defined by Definition 2.3. Then $(\xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2) > 0$ and for a large ξ_i and a large k, the right-hand side of Eq. (13) tends to zero. It follows that it is bounded by a positive constant M say, that is we obtain Eq. (12) as required and also by Eq. (12) $\mathscr{F}$ is continuous on the space S' of the tempered distribution.

Theorem 3.2.

$$\begin{aligned}&\mathscr{F}([(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)])\\ &\quad = (2\pi)^{n/2}\mathscr{F}([(-1)^k S_{2k}(x) * R_{2k}(x)]\mathscr{F}[(-1)^m S_{2m}(x) * R_{2m}(x)]\\ &\quad = \frac{1}{(2\pi)^{n/2}}\,\frac{1}{[(\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2]^{k+m}},\end{aligned}$$

where k and m are nonnegative integers and $\mathscr{F}$ is bounded and continuous on the space S' of the tempered distribution.

Proof. Since $R_{2k}(x)$ and $S_{2k}(x)$ are tempered distributions with compact supports, we have

$$\begin{aligned}&[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]\\ &\quad = (-1)^{k+m}(S_{2k}(x) * S_{2m}(x)) * (R_{2k}(x) * R_{2m}(x))\\ &\quad = (-1)^{k+m}(S_{2(k+m)}(x) * R_{2(k+m)}(x))\end{aligned}$$

by [3], pp. 156–159 and [2], Lemma 2.5. Taking the Fourier transform on both sides and using Theorem 3.1 we obtain

$$\begin{aligned}
&\mathscr{F}([(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]) \\
&= \frac{1}{(2\pi)^{n/2}((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^{k+m}} \\
&= \frac{1}{(2\pi)^{n/2}((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p-1}^2 + \cdots + \xi_{p-q}^2)^2)^{k}} \\
&\quad \times \frac{(2\pi)^{n/2}}{(2\pi)^{n/2}((\xi_1^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \cdots + \xi_{p+q}^2)^2)^{m}} \\
&= (2\pi)^{n/2}\mathscr{F}[(-1)^k S_{2k}(x) * R_{2k}(x)]\mathscr{F}[(-1)^m S_{2m}(x) * R_{2m}(x)].
\end{aligned}$$

Since $(-1)^k S_{2(k+m)}(x) * R_{2(k+m)}(x) \in S'$, the space of tempered distribution, and by Theorem 3.1 we obtain that $\mathscr{F}$ is bounded and continuous on S'.

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On the Fourier transform of Marcel Riesz's Ultra-hyperbolic Kernel

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On the Fourier Transform of Marcel Riesz's Ultra-hyperbolic Kernel

Amnuay Kananthai
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand.

ABSTRACT

In this paper, we study the Fourier transform of Marcel Riesz's ultra-hyperbolic kernel $R_\alpha(x)$ for $\alpha = 2k$ $(k = 0, 1, 2, 3, \dots)$ and $x = (x_1, x_2, \dots, x_n) \in R^n$ the n-dimensional Euclidean space. Moreover, the Fourier transform of the convolution $R_\alpha * R_\beta$ is also determined.

1. INTRODUCTION

The function $R_\alpha(x)$ defined by (2.1) is first introduced by Y. Nozaki [1]. Later S.E Trione [2] has shown that, for $\alpha = 2k$ $(k = 0, 1, 2, 3, \dots)$, $R_{2k}(x)$ is the elementary solution of the ultra-hyperbolic operator $\Box^k$ iterated k-times. That is $\Box^k R_{2k}(x) = \delta$ where $\Box^k$ is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k \quad \text{.......} (1.1)$$

$p + q = n$, the dimension of the space R^n and δ is the Dirac-delta distribution. Moreover M. Aguirre Tellez [3] has proved that $R_{2k}(x)$ exists only for the case p is odd with n is odd or even, where $p + q = n$.

For the convolution $R_\alpha * R_\beta$, A. Kananthai [4] has shown that $R_\alpha * R_\beta = R_{\alpha+\beta}$ for α and β are even integers. In this paper, we study the Fourier transform of the function $R_\alpha(x)$ with $\alpha = 2k$ where k is a non-negative integer. Moreover we also obtain the formula form of the Fourier transform of $R_\alpha * R_\beta$.

2. MATERIALS AND METHODS

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n-dimensional Euclidean space R^n.

Denote $v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$, $p + q = n$.

The set $\Gamma_+ = \{ x \in R^n : x_1 > 0 \text{ and } v > 0 \}$ is called an interior of the forward cone. By $\overline{\Gamma}_+$ we denote its closure. For any complex number α, define

$$R_\alpha(x) = \begin{cases} \dfrac{v^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+ \end{cases} \quad \text{.......} (2.1)$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}$$

The function $R_\alpha(x)$ was introduced by Y. Nozaki [1] is called the Marcel Riesz's ultra-hyperbolic kernel. It is well known that $R_\alpha(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and it is a distribution of α if $\mathrm{Re}(\alpha) < n$. Let supp $R_\alpha(x)$ denote the support of $R_\alpha(x)$.

Suppose that supp $R_\alpha(x) \subset \overline{\Gamma}_+$.

Definition 2.2 Let f be continuous function, the Fourier transform of f denoted by $\mathcal{F}$ f and is defined by

$$\mathcal{F} f = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-i\xi.x} f(x)\, dx \qquad \text{....... (2.2)}$$

where $x = (x_1, x_2, \dots, x_n) \in R^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in R^n$ and $\xi.x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$. Sometimes we write $\mathcal{F} f(x) \equiv \hat{f}(\xi)$.

By (2.2), we can define the inverse of Fourier transform of $\hat{f}(\xi)$ by

$$f(x) = \mathcal{F}^{-1} \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{i\xi.x} \hat{f}(\xi)\, d\xi \qquad \text{....... (2.3)}$$

From definition 2.2, if f is a distribution with compact support, by A.H Zemanian [5], we can define

$$\mathcal{F} f = \frac{1}{(2\pi)^{n/2}} < f(x)\, ,\, e^{-i\xi.x} > \qquad \text{....... (2.4)}$$

Lemma 2.1 Given the equation $\Box^k u(x) = \delta$ where $\Box^k$ is defined by (1.1) and $x \in R^n$. Then $u(x) = R_{2k}(x)$ is a unique elementary solution of the equation.

Proof See [2].

Lemma 2.2 (The existence of the convolution $R_\alpha * R_\beta$)

Let $R_\alpha(x)$ be defined by (2.1) and supp $R_\alpha(x) = K \subset \overline{\Gamma}_+$ where K is a compact set, then $R_\alpha * R_\beta$ exists and is a tempered distribution and particularly $R_\alpha * R_\beta = R_{\alpha+\beta}$ for α and β are even integers.

Proof The proof of existence of $R_\alpha * R_\beta$ has been given by W.F Donoghue [6] and A. Kananthai [4] also proved that $R_\alpha * R_\beta = R_{\alpha+\beta}$ for α and β are even integers.

Lemma 2.3 $\mathcal{F} \Box^k \delta = \frac{1}{(2\pi)^{n/2}}(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \dots - \xi_p^2)^k$

where $\Box^k$ is defined by (1.1) and $|\mathcal{F} \Box^k \delta| \leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{2k}$ where $\|\xi\| = (\sum_{i=1}^{n} |\xi_i|^2)^{1/2}$. That is $\mathcal{F}$ is continuous on the space S' of tempered distribution.

Proof Since $\square^k\delta$ is a tempered distribution with compact support, hence by (2.4) we have

$$\mathcal{F}\,\square^k\delta = \frac{1}{(2\pi)^{n/2}} < \square^k\delta \,,\, e^{-i\xi.x} >$$

$$= \frac{1}{(2\pi)^{n/2}} < \delta \,,\, \square^k e^{-i\xi.x} >$$

$$= \frac{1}{(2\pi)^{n/2}} < \delta, ((i\xi_1)^2 + (i\xi_2)^2 + ... + (i\xi_p)^2 - (i\xi_{p+1})^2 - (i\xi_{p+2})^2 - ... - (i\xi_{p+q})^k) e^{-i\xi.x} >$$

$$= \frac{1}{(2\pi)^{n/2}} (\xi_{p+1}^2 + \xi_{p+2}^2 + ... + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - ... - \xi_p^2)^k.$$

Since $|\mathcal{F}\,\square^k\delta| = \frac{1}{(2\pi)^{n/2}} |(\xi_{p+1}^2 + \xi_{p+2}^2 + ... + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - ... - \xi_p^2)|^k$, $p + q = n$

$$\leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{2k}.$$

We have $|\mathcal{F}\,\square^k\delta| \leq \frac{1}{(2\pi)^{n/2}} \|\xi\|^{2k}$. That is $\mathcal{F}$ is bounded on S′. It follows that $\mathcal{F}$ is continuous on the space S′ of tempered distribution.

3. RESULTS AND DISCUSSION

Theorem 3.1 Let $R_{2k}(x)$ be defined by (2.1) with $\alpha = 2k$, k is a non-negative. Then the Fourier transform of $R_{2k}(x)$ is given by

$$\mathcal{F}\, R_{2k}(x) = \frac{1}{(2\pi)^{n/2} (\xi_{p+1}^2 + \xi_{p+2}^2 + ... + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - ... - \xi_p^2)^k}$$

and $|\mathcal{F}\, R_{2k}(x)| \leq \frac{M}{(2\pi)^{n/2}}$ where M is a constant and $\mathcal{F}$ is continuous on the space S′ of tempered distribution.

Proof We can show that $R_\alpha(x)$ is a tempered distribution, see [4] and we choose supp $R_\alpha(x) = K \subset \overline{\Gamma}_+$ and K is a compact set. Then we obtain $R_{2k}(x)$ where $\alpha = 2k$ is a tempered distribution with compact support. Then by (2.4)

$$\mathcal{F}\, R_{2k}(x) = \frac{1}{(2\pi)^{n/2}} < R_{2k}(x) \,,\, e^{-i\xi.x} >$$

To compute $< R_{2k}(x) \,,\, e^{-i\xi.x} >$ directly is difficult. So we apply Lemma 2.1

$$\square^k R_{2k}(x) = \delta$$

or

$$(\square^k\delta) * R_{2k}(x) = \delta.$$

By taking the Fourier transform both sides, we obtain

$$\mathcal{F}\,[(\square^k\delta) * R_{2k}(x)] = \mathcal{F}\,\delta = \frac{1}{(2\pi)^{n/2}}.$$

Now $\mathcal{F}\,[(\square^k\delta) * R_{2k}(x)] = < \square^k\delta * R_{2k}(x) \,,\, e^{-i\xi.x} >$

$$= \frac{1}{(2\pi)^{n/2}} < \Box^k\delta\,,\, < R_{2k}(r)\,,\, e^{-i\xi.(x+r)} >>$$
$$= \frac{1}{(2\pi)^{n/2}} < \Box^k\delta\,,\, < R_{2k}(r)\,,\, e^{-i\xi.r}.e^{-i\xi.x} >>$$
$$= \frac{1}{(2\pi)^{n/2}} < \Box^k\delta\,,\, e^{-i\xi.x} < R_{2k}(r)\,,\, e^{-i\xi.r} >>$$
$$= \frac{1}{(2\pi)^{n/2}} < R_{2k}(r)\,,\, e^{-i\xi.r} > < \Box^k\delta\,,\, e^{-i\xi.x} >$$
$$= \frac{1}{(2\pi)^{n/2}} (2\pi)^{n/2} \mathcal{F}\, R_{2k}(r).(2\pi)^{n/2} \mathcal{F}\, (\Box^k\delta) \qquad \text{by (2.4)}$$

$= \mathcal{F}\, R_{2k}(r).(2\pi)^{n/2}.\frac{1}{(2\pi)^{n/2}}(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \ldots - \xi_p^2)^k$ by Lemma 2.3. It follows that

$$\mathcal{F}\, R_{2k}(x) = \frac{1}{(2\pi)^{n/2}(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \ldots - \xi_p^2)^k} \qquad \ldots\ldots. (3.1)$$

Now, the right hand side of (3.1) tend to zero for a large ξ_i and a large k, that is bounded by a constant M say, then $|\mathcal{F}\, R_{2k}(x)| \leq \frac{M}{(2\pi)^{n/2}}$. It follows that $\mathcal{F}$ is a continuous on the space S′. Moreover, by (2.3) we obtain

$$R_{2k}(x) = \mathcal{F}^{-1} \left(\frac{1}{(2\pi)^{n/2}(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \ldots - \xi_p^2)^k}\right).$$

Theorem 3.2 Let $R_{2k}(x)$ and $R_{2m}(x)$ are defined by (2.1) with $\alpha = 2k, 2m$ where k and m are non-negative integers. Then the Fourier transform of the convolution $R_{2k}(x) * R_{2m}(x)$ is given by

$$\mathcal{F}\, (R_{2k}(x) * R_{2m}(x)) = (2\pi)^{n/2}(\mathcal{F}\, R_{2k}(x))(\mathcal{F}\, R_{2m}(x)).$$

Proof By Lemma 2.2

$$R_{2k} * R_{2m} = R_{2k+2m}.$$

Thus
$$\mathcal{F}\, (R_{2k}(x) * R_{2m}(x)) = \mathcal{F}\, R_{2k+2m}(x)$$

$$= \frac{1}{(2\pi)^{n/2}(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \ldots - \xi_p^2)^{k+m}}$$

by Theorem 3.1.

Now
$$\frac{1}{(2\pi)^{n/2}(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 - \xi_1^2 - \xi_2^2 - \ldots - \xi_p^2)^{k+m}}$$

For the case α is a negative even and β is odd, by (2.5) we have

$$\Box^k R_0(x) = R_{-2k}(x) \quad \text{or} \quad \Box^k \delta = R_{-2k}(x)$$

where $R_0(x) = \delta$. Now

$$R_{-2k}(x) * \Box^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x) \quad \text{for } m \text{ is odd},$$

or

$$(\Box^k \delta) * \Box^k R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$\delta * \Box^{2k} R_m(x) = R_{-2k}(x) * R_{m-2k}(x),$$

$$R_{m-2(2k)}(x) = R_{-2k}(x) * R_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m - 2k$, now α is a negative even and β is odd. Then we obtain

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x).$$

That completes the proofs. □

3. Main results

Theorem 3.1. *Let $T_m(x)$ the diamond kernel of Marcel Riesz defined by* (1.3), *then T_m is a tempered distribution and can be expressed by*

$$T_m(x) = T_{m-r}(x) * T_r(x).$$

where r is a nonnegative integer and $r < m$. Moreover if we put $\ell = m - r$, $n = r$ we obtain

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) \quad \text{for } \ell + n = m.$$

Proof. Since $T_m = (-1)^m S_{2m}(x) * R_{2m}(x)$. $(m = 0, 1, 2, \ldots)$, by Lemma 2.2 T_m is a tempered distribution. Now by Lemma 2.3, $\Diamond^m T_m = \delta$, then $\Diamond^r \Diamond^{m-r} T_m = \delta$ for $m > r$ and by Lemma 2.3 again, we obtain $\Diamond^{m-r} T_m = (-1)^r S_2 r(x) * R_{2r}(x)$. Convolving both sides by $(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)$, we obtain

$$\begin{aligned}&[(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * \Diamond^{m-r} T_m \\ &\quad = [(-1)^{m-r} S_{2m-2r}(x) * R_{2m-2r}(x)] * [(-1)^r S_{2r}(x) * R_{2r}(x)]\end{aligned} \tag{3.1}$$

or

$$\begin{aligned}&\Diamond^{m-r}[(-1)^{m-r} S_{2(m-r)}(x) * R_{2(m-r)}(x)] * T_m \\ &\quad = (-1)^m (S_{2m-2r}(x) * S_{2r}(x) * (R_{2m-2r}(x) * R_{2r}(x)),\end{aligned}$$

since $S_{2m}(x)$ and $R_{2m}(x)$ are tempered distributions and are the elements of the space of convolution algebra, $\mathscr{a}'$.

By Lemmas 2.3 and 2.4 we obtain

$$\delta * T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x),$$
$$T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x).$$

From (3.1) we have $T_m(x) = T_{m-r}(x) * T_r(x)$, put $\ell = m - r$, $n = r$, it follows that

$$T_\ell(x) * T_n(x) = T_{\ell+n}(x) = T_m(x)$$

as required. □

Theorem 3.2. *Let $T_m(x)$ be defined by* (1.3) *then T_m is an element of the space $\mathscr{a}'$ of convolution algebra and there exist an inverse T_m^{*-1} of T_m such that*

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

Proof. Since $T_m(x) = (-1)^m S_{2m}(x) * R_{2m}(x)$ is a tempered distribution by Lemma 2.2. Now the supports of $S_{2m}(x)$ and $R_{2m}(x)$ are compact. Then they are the elements of the space of convolution algebra $\mathscr{a}'$ of distribution. By Zemanian [6, Theorem 6.2.1, p. 151] there exist a unique inverse T_m^{*-1} such that

$$T_m(x) * T_m^{*-1} = T_m^{*-1} * T_m(x) = \delta.$$

That completes the proof. □

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On the Diamond Operator related to the Wave Equation

Amnuay Kananthai

Department of Mathematics, Chiang Mai University,
Chiang Mai, 50200 Thailand
E-mail : malamnaka@science.cmu.ac.th

Abstract

In this paper, we study the solution of the equation $\diamondsuit^k u(x) = f(x)$ where $\diamondsuit^k$ is the Diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k$$

where $p+q=n$ is the dimension of the n-dimensional Euclidean space $R^n, x = (x_1, x_2, ..., x_n) \in R^n, k$ is a nonnegative integer, $u(x)$ is an unknown and f is a generalized function.

It is found that the solution $u(x)$ depends on the conditions of p and q and moreover such a solution is related to the solution of the Laplace equation and the wave equation.

1 Introduction

The operator $\diamondsuit^k$ has been first introduced by A. Kananthai [3] and is named as the Diamond operater iterated k times and is defined by

$$\begin{aligned}\diamondsuit^k \quad = \quad & ((\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2} + \cdots + \tfrac{\partial^2}{\partial x_p^2})^2 - \\ & (\tfrac{\partial^2}{\partial x_{p+1}^2} + \tfrac{\partial^2}{\partial x_{p+2}^2} + \cdots + \tfrac{\partial^2}{\partial x_{p+q}^2})^2)^k\end{aligned} \tag{1.1}$$

where $p+q=n$ is the dimension of the space $R^n, x = (x_1, x_2, ..., x_n) \in R^n$ and k is a nonnegative integer.

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Actually the operator $\diamondsuit^k$ is an extension of the ultrahyperbolic operator and the Laplacian. So the operator $\diamondsuit^k$ can be expressed as the product of the operator $\Box$ and $\triangle$, that is $\diamondsuit^k = \Box^k \triangle^k = \triangle^k \Box^k$ where

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \quad (1.2)$$

is the ultrahyperbolic operator iterated k-time with $p + q = n$, and

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k \quad (1.3)$$

is the Laplacian iterated k-times.

A. Kananthai ([3], Theorem 3.1 p33) has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementory solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k \left((-1)^k R_{2k}^e(x) * R_{2k}^H(x) \right) = \delta \quad (1.4)$$

where δ is the Dirac-delta distribution and the functions R_{2k}^e and R_{2k}^H are defined by (2.5) and (2.1) respectively with $\alpha = 2k, k$ is nonnegative integer.

In this paper, we study the solution of the equation

$$\diamondsuit^k u(x) = f(x) \quad (1.5)$$

This equation is the geheralization of the ultrahyperbolic equation and it can be applied to the wave equation and potential that has been shown in the last part of this paper.

Let $K_{\alpha,\beta}(x)$ be a distributional family and is defined by

$$K_{\alpha,\beta}(x) = R_\alpha^e * R_\beta^H \quad (1.6)$$

where R_α^e is called the elliptic Kernel defined by (2.5) and R_β^H is called the ultra-hyperbolic Kernel defined by (2.1) and α, β are the complex parameters.

The family $K_{\alpha,\beta}(x)$ is well-defined and is a tempered distribution, since $R_\alpha^e * R_\beta^H$ is a tempered, see ([1], Lemma 2.2) and R_β^H has a compact support.

In this paper, we can show that

$$u(x) = (-1)^k R_{2k}^e(x) * \left(R_{2(k-1)}^H(v) \right)^{(m)} + (-1)^k K_{2k,2k}(x) * f(x)$$

is a solution of (1.5) where m $= \frac{n-4}{2}, n \geq 4$ and n is even number and $k_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$. Moreover, we can show that the solution u(x) relates to the solution of Laplace operator $\triangle^{2k}$ defined by (1.3)and also the wave operator defined by (1.2) with $k = 1$ and $p = 1$.

2 Preliminaries

Definition 2.1 Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space R^n.

Denote by $v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, p + q = n$ the nondegenerated quadratic form. By Γ_+ we designate the interior of the forward cone.

$\Gamma_+ = \{x \in R^n : x_1 > 0\, and\, v > 0\}$, and by $\overline{\Gamma}_+$ designate its closure. For any complex number α, define

$$R_\alpha^H(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+ \end{cases} \tag{2.1}$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{2.2}$$

The function $R_\alpha^H(x)$ was introduced by Nozaki ([4], p.72).

It is well known that $R_\alpha^H(x)$ is an ordinary function if $Re(\alpha) \geq n$ and it is a distribution of α if $Re(\alpha) < n$. Let supp$R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$. Suppose that supp$R_\alpha^H(x) \subset \overline{\Gamma}_+$.

From S.E Trione ([5], p11), $R_{2k}^H(v)$ is an elementary solution of the operator $\Box^k$ that is

$$\Box^k R_{2k}^H(v) = \delta \tag{2.3}$$

where $\Box^k$ is defined by (1.2).

By putting $p = 1$ in (2.1) and (2.2) and remembering the Legendre's duplication of $\Gamma(z)$.

$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right)$ then the formula (2.1) reduces to

$$M_\alpha(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.4}$$

Here $v = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1}\Gamma\left(\frac{\alpha-n+2}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$.

$M_\alpha(v)$ is, precisely, the hyperbolic kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, ..., x_n)$ be a point of R^n and the function $R_\alpha^e(x)$ be defined by

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{2.5}$$

where $W_n(\alpha) = \frac{\pi^{n/2}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$, α is a complex parameter and $|x| = (x_1^2 + x + 2^2 + \cdots + x_n^2)^{1/2}$.

It can be shown that $R_{-2k}^e(x) = (-1)^k\triangle^k\delta(x)$ where $\triangle^k$ is defined by (1.3). It follows that $R_0^e(x) = \delta$, see ([2], p118).

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$, that is

$$\triangle^k \left((-1)^k R^e_{2k}(x)\right) = \delta \tag{2.6}$$

see ([3] , Lemma 2.4 p31).

Lemma 2.1 Given P is a hyper-surface then

$$P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.
Proof. See ([1], P233).

Lemma 2.2 Given the equation

$$\triangle^k u(x) = 0 \tag{2.7}$$

where $\triangle^k$ is defined by (1.3) and $x = (x_1, x_2, ..., x_n) \in R^n$ then $u(x) = (-1)^{(k-1)} \left(R^e_{2(k-1)}(x)\right)^{(m)}$ is a solution of (2.7) where m is a nonnegative integer with $m = \frac{n-4}{2}, n \geq 4$ and n is even and $\left(R^e_{2(k-1)}(x)\right)^{(m)}$ is a function defined by (2.5) with m derivatives with $\alpha = 2(k-1)$
Proof. We first show that the generalized function $u(x) = \delta^{(m)}(r^2)$ where $r^2 = |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ is a solution of

$$\triangle u(x) = 0 \tag{2.8}$$

where $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is a Laplace operator.Now

$$\begin{aligned} \frac{\partial}{\partial x_i}\delta^{(m)}(r^2) &= 2x_i\delta^{(m+1)}(r^2) \\ \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2) &= 2\delta^{(m+1)}(r^2) + 4x_i^2\delta^{(m+2)}(r^2). \end{aligned}$$

Thus

$$\begin{aligned} \triangle\delta^{(m)}(r^2) &= \textstyle\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2) \\ &= 2n\delta^{(m+1)}(r^2) + 4r^2\delta^{(m+2)}(r^2) \\ &= 2n\delta^{(m+1)}(r^2) - 4(m+2)\delta^{(m+1)}(r^2) \end{aligned}$$

by Lemma 2.1 with $P = r^2$.We have

$$\begin{aligned} \triangle\delta^{(m)}(r^2) &= [2n - 4(m+2)]\delta^{(m+1)}(r^2) \\ &= 0 \quad \text{if } 2n - 4(m+2) = 0 \end{aligned}$$

or $m = \frac{n-4}{2}, n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of (2.8) with $m = \frac{n-4}{2}, n \geq 4$ and n is even. Now $\triangle^k u(x) = \triangle(\triangle^{k-1}u(x)) = 0$ then from the above proof $\triangle^{k-1}u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{k-1}R^e_{2(k-1)}(x)$, we obtain

$$\begin{aligned}(-1)^{k-1}R^e_{2(k-1)}(x)*\triangle^{k-1}u(x) &= (-1)^{k-1}R^e_{2(k-1)}(x)*\delta^{(m)}(r^2)\\ \text{or } \triangle^{k-1}\left((-1)^{k-1}R^e_{2(k-1)}(x)\right)*u(x) &= (-1)^{k-1}R^e_{2(k-1)}(x)*\delta^{(m)}(r^2)\end{aligned}$$

$$\text{or} \qquad \delta * u(x) = u(x) = (-1)^{k-1}R^e_{2(k-1)}(x)*\delta^{(m)}(r^2) \quad \text{by (2.6)}$$

Now from (2.1)

$$\begin{aligned}R^e_{2(k-1)}(x) &= \frac{|x|^{2(k-1)-n}}{W_n(\alpha)}\\ &= \frac{\left(|x|^2\right)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} = \frac{\left(r^2\right)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}\end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Hence

$$\begin{aligned}R^e_{2(k-1)}(x)*\delta^{(m)}(r^2) &= \frac{\left(r^2\right)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}*\delta^{(m)}(r^2)\\ &= \left[\frac{\left(r^2\right)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}\right]^{(m)} = \left[R^e_{2(k-1)}(x)\right]^{(m)}.\end{aligned}$$

It follows that $u(x) = (-1)^{k-1}\left[R^e_{2(k-1)}(x)\right]^{(m)}$ is a solution of (2.7) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension of R^n.

Lemma 2.3 Given the equation

$$\square^k u(x) = 0 \tag{2.9}$$

where $\square^k$ is defined by (1.2) and $x = (x_1, x_2, ..., x_n) \in R^n$ then $u(x) = \left[R^H_{2(k-1)}(v)\right]^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{n}, n \geq 4$ and n is even dimension and v is defined by Definition 2.1. The function $\left[R^H_{2(k-1)}(v)\right]^{(m)}$ is defined by (2.1) with m-derivatives and $\alpha = 2(k-1)$.

Proof At first we show that the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2, p + q = n$, is a solution of the equation

$$\square u(x) = 0 \tag{2.10}$$

where $\square$ is defined by (1.2) with $k = 1$ and $x = (x_1, x_2, ..., x_n) \in R^n$. Now

$$\begin{aligned}
\frac{\partial}{\partial x_i}\delta^{(m)}(r^2-s^2) &= 2x_i\delta^{(m+1)}(r^2-s^2)\\
\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2-s^2) &= 2\delta^{(m+1)}(r^2-s^2)+4x_i^2\delta^{(m+2)}(r^2-s^2)\\
\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2-s^2) &= 2p\delta^{(m+1)}(r^2-s^2)+4r^2\delta^{(m+2)}(r^2-s^2)\\
&= 2p\delta^{(m+1)}(r^2-s^2)\\
&\quad +4(r^2-s^2)\delta^{(m+2)}(r^2-s^2)\\
&\quad +4s^2\delta^{(m+2)}(r^2-s^2)\\
&= 2p\delta^{(m+1)}(r^2-s^2)\\
&\quad -4(m+2)\delta^{(m+1)}(r^2-s^2)\\
&\quad +4s^2\delta^{(m+2)}(r^2-s^2)\\
&= [2p-4(m+2)]\delta^{(m+1)}(r^2-s^2)\\
&\quad +4s^2\delta^{(m+2)}(r^2-s^2)
\end{aligned}$$

by Lemma 2.1 with $P = r^2 - s^2$.
Similarly,

$$\begin{aligned}
\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2-s^2) &= [-2q+4(m+2)]\delta^{(m+1)}(r^2-s^2)\\
&\quad +4r^2\delta^{(m+2)}(r^2-s^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\Box\delta^{(m)}(r^2-s^2) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2-s^2)-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2-s^2)\\
&= [2(p+q)-8(m+2)]\delta^{(m+1)}(r^2-s^2)\\
&\quad -4(r^2-s^2)\delta^{(m+2)}(r^2-s^2)\\
&= [2n-8(m+2)]\delta^{(m+1)}(r^2-s^2)\\
&\quad +4(m+2)\delta^{(m+1)}(r^2-s^2) \quad \text{by Lemma 2.1}\\
&= [2n-4(m+2)]\delta^{(m+1)}(r^2-s^2).
\end{aligned}$$

If $2n - 4(m+2) = 0$, we have $\Box\delta^{(m)}(r^2-s^2) = 0$. That is $u(x) = \delta^{(m)}(r^2-s^2)$ is a solution of (2.10) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension. Now $\Box^k u(x) = \Box\left(\Box^{k-1}u(x)\right) = 0$.

From (2.10) we have $\Box^{k-1}u(x) = \delta^{(m)}(r^2-s^2)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Convolving the above equation by $R^H_{2(k-1)}(v)$, we obtain

$$\begin{aligned}
R^H_{2(k-1)}(v) * \Box^{k-1}u(x) &= R^H_{2(k-1)}(v) * \delta^{(m)}(r^2-s^2)\\
\Box^{k-1}\left[R^H_{2(k-1)}(v)\right] * u(x) &= \left[R^H_{2(k-1)}(v)\right]^{(m)} \text{ where } v = r^2-s^2\\
\delta * u(x) = u(x) &= \left[R^H_{2(k-1)}(v)\right]^{(m)}
\end{aligned}$$

by (2.3) and $v = r^2 - s^2$ is defined by Definition 2.1.

Thus $u(x) = \left[R^H_{2(k-1)}(v)\right]^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Lemma 2.4 Given the equation

$$\diamondsuit^k u(x) = 0 \tag{2.11}$$

where $\diamondsuit^k$ is the Diamond operator iterated k-times defined by (1.1) and $u(x)$ is an unknown generalized function.Then

$$u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)} \tag{2.12}$$

is a solution of (2.11), $\left(R^H_{2(k-1)}(v)\right)^{(m)}$ is a function with m-derivatives defined by (2.1) and v is defined by definition 2.1.

Proof Now $\diamondsuit^k u(x) = \Box^k \triangle^k u(x) = 0$. By Lemma 2.3 ,$\triangle^k u(x) = \left(R^H_{2(k-1)}(v)\right)^{(m)}$.

Convolving both sides by $(-1)^k R^e_{2k}(x)$, we have

$$(-1)^k R^e_{2k}(x) * \triangle^k u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)}.$$

By (2.6), $\triangle^k \left((-1)^k R^e_{2k}(x)\right) * u(x) = \delta * u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)}$.
It follows that

$$u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)}. \tag{2.13}$$

3 Main results

Theorem Given The equation

$$\diamondsuit^k u(x) = f(x) \tag{3.1}$$

where $\diamondsuit^k$ is the Diamond operator iterated k-times defined by (1.1), $f(x)$ is a generalized function, $u(x)$ is an unknown generalized function and $x = (x_1, x_2, ...x_n) \in R^n$-the n-dimensional Euclidean space and n is even, then (3.1) has the general solution

$$u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)} + (-1)^k K_{2k,2k}(x) * f(x) \tag{3.2}$$

where $\left(R^H_{2(k-1)}(v)\right)^{(m)}$ is a function with m-derivatives defined by (2.1) and $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$.

Proof Convolving (3.1) both sides by $(-1)^k K_{2k,2k}(x)$, we obtain

$$(-1)^k K_{2k,2k}(x) * \diamondsuit^k u(x) = (-1)^k K_{2k,2k}(x) * f(x).$$

By (1.4) $\diamondsuit^k \left((-1)^k K_{2k,2k}(x)\right) * u(x) = \delta * u(x) = (-1)^k K_{2k,2k}(x) * f(x)$. We obtain $u(x) = (-1)^k K_{2k}(x) * f(x)$. Since, for a Homogeneous equation $\diamondsuit^k u(x) = 0$ we have a solution $u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)}$.

Thus the general solution of (3.1) is

$$u(x) = (-1)^k R^e_{2k}(x) * \left(R^H_{2(k-1)}(v)\right)^{(m)} + (-1)^k K_{2k,2k}(x) * f(x).$$

In particular, if $q = 0$ the equation (3.1) becomes the Laplace equation $\triangle^{2k} u(x) = f(x)$ where $x = (x_1, x_2, x_3, ..., x_p) \in R^p$ and p is even. Using the formulae $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2})$ and $\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \pi sec(\pi z)$. Then for $\alpha = 2k$ the function of (2.1) becomes $(-1)^k R^e_{2k}(x)$ where $R^e_{2k}(x)$ is defined by (2.5) and $x = (x_1, x_2, ..., x_p) \in R^p$. Thus by (1.6)

$$\begin{aligned} (-1)^k K_{2k,2k}(x) &= (-1)^k R^e_{2k}(x) * (-1)^k R^e_{2k}(x) \\ &= R^e_{4k}(x) \text{ see ([2], p156-159)} \end{aligned}$$

where $x = (x_1, x_2, ..., x_p) \in R^p$ and p is even.

Now, from (2.12) for $q = 0$ we have

$$\begin{aligned} u(x) &= (-1)^k R^e_{2k}(x) * (-1)^{k-1} \left(R^e_{2(k-1)}(x)\right)^{(m)} \\ &= (-1)^{2k-1} \left(R^e_{4k-2}(x)\right)^{(m)} \quad \text{for } x = (x_1, x_2, ..., x_p) \in R^p. \end{aligned}$$

Thus the equation (3.2) becomes

$$u(x) = (-1)^{2k-1} \left(R^e_{4k-2}(x)\right)^{(m)} + R^e_{4k}(x) * f(x) \tag{3.3}$$

for $x = (x_1, x_2, ..., x_p) \in R^p$ and p is even.

It follows that (3.3) is the general solution of the Laplace equation $\triangle^{2k} u(x) = f(x)$ where $\triangle^{2k}$ is the Laplace operator iterated $2k$-times defined by (1.3) for $x = (x_1, x_2, ..., x_n) \in R^n$ and n is even.

Now consider the case for the Wave Equation. Given the equation

$$\square^k V(x) = f(x) \tag{3.4}$$

where $f(x)$ is a generalized function, $\square^k$ is defined by (1.2) and $V(x)$ is an unknown function. By ([5],p11) we obtain $V(x) = R^H_{2k}(v) * f(x)$ is a solution of (3.4) where $R^H_{2k}(v)$ is defined by (2.1).

Now, from (3.1) we have $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ is a solution where $K_{2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = 2k$ or

$$u(x) = \left[(-1)^k R^e_{2k}(x) * R^e_{2k}(v)\right] * f(x) \tag{3.5}$$

Convolving both sides of (3.5) by $(-1)^k R^e_{-2k}(x)$, we obtain

$$\begin{aligned} (-1)^k R^e_{-2k}(x) * u(x) &= \left((-1)^{2k} R^e_{-2k}(x) * R^e_{2k}(x)\right) * R^H_{2k}(v) * f(x) \\ &= \left(R^e_0(x) * R^H_{2k}(v)\right) * f(x) \\ &= \left(\delta * R^H_{2k}(v)\right) * f(x) \\ &= R^H_{2k}(v) * f(x) \quad \text{see ([2],p156-159)} \end{aligned}$$

Thus it follows that

$$V(x) = (-1)^k R^e_{-2k}(x) * u(x) \tag{3.6}$$

In particular, for $k = 1$, we obtain $V(x) = R_2^H(v) * f(x)$ is a solution of the equation

$$\Box V(x) = f(x). \tag{3.7}$$

If we put $p = 1$ and $x_1 = t$(time),then $\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2}$ is the wave operator. Thus (3.7) becomes wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}\right) V(x) = f(x) \tag{3.8}$$

Thus $V(x) = M_2(v) * f(x)$ is a solution of (3.8) and the general solution of (3.8) is $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ where $\delta^{(m)}(v)$ is a solution for $f(x) = 0$ $v = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ and $M_2(v)$ is defined by (2.4) with $v = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$.

Now in (3.1),put $k = 1$ and by (3.2) we obtain

$$\begin{aligned} u(x) &= (-1)R_2^e(x) * \left(R_0^H(v)\right)^{(m)} + (-1)K_{2,2}(x) * f(x) \\ &= (-1)R_2^e(x) * \delta^{(m)}(v) + (-1)K_{2,2}(x) * f(x) \end{aligned} \tag{3.9}$$

is a solution of the equation $\Diamond u(x) = f(x)$ and by (3.6) with $k = 1, V(x) = (-1)R^e_{-2}(x) * u(x)$ is a solution of (3.7) where $u(x)$ is defined by (3.9). Thus

$$\begin{aligned} V(x) &= (-1)R^e_{-2}(x) * ((-1)R_2^e(x) * \delta^{(m)}(v)) \\ &\quad +(-1)R^e_{-2}(x) * ((-1)K_{2,2}(x) * f(x)) \\ &= \left(R^e_{-2}(x) * R_2^e(x)\right) * \delta^{(m)}(v) \\ &\quad +(R^e_{-2}(x) * R_2^e(x)) * R_2^H(x) * f(x) \\ &= R_0^e(x) * \delta^{(m)}(v) + R_0^e(x) * (R_2^H(x) * f(x)) \\ &= \delta^{(m)}(v) + R_2^H(x) * f(x) \qquad (R_0^e(x) = \delta) \end{aligned} \tag{3.10}$$

where $v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$, $p + q = n$.
Now, if we put $p = 1$ and $x_1 = t$, then (3.10) becomes $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ since $R_2^H(x)$ becomes $M_2(v)$ for $v = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ where $M_2(v)$ is defined by (2.4) with $\alpha = 2$. Thus $V(x) = \delta^{(m)}(v) + M_2(v) * f(x)$ is the general solution of the wave equation (3.8) and $\delta^{(m)}(v)$ is a solution of

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}\right) V(x) = 0 \tag{3.11}$$

Now $v = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, let $r^2 = x_2^2 + x_3^2 + \cdots + x_n^2$. Thus by ([1],p234-236) we obtain $V(x,t) = \delta^{(m)}(t^2 - r^2)$ is the solution of (3.11) with the initial conditions $V(x,0) = 0$ and $\frac{\partial V(x,0)}{\partial t} = (-1)^m 2\pi^{m+1}\delta(x)$ at $t = 0$ and $x = (x_2, x_3, ..., x_n) \in R^{n-1}$.

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