BRG/08/2543 Assoc.Dr. Boris I. Kvasov

สำนักงานกองทุนสนับสนุนการวิจัย

THE THAILAND RESEARCH FUND

รายงานวิจัยฉบับสมบูรณ์

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Assoc. Prof. Dr. Boris I. Kvasov มหาวิทยาลัยเทคโนโลยีสุรนารี

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

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ชื่อโครงการ : Difference Method for Constructing Shape Preserving Splines

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งานวิจัยนี้ได้กล่าวถึงนิยามและการศึกษาสปลายน์เต็มหน่วยทั่วไปเป็นฟังก์ชันต่อเนื่องเป็นช่วงซึ่งสอดคล้องกับเงื่อนไขปรับเรียบสำหรับความแตกต่างผลหารที่หนึ่งและที่สอง ณ ตำแหน่งน็อตสปลายน์เต็มหน่วย เป็นการวางนัยทั่วไปของทั้งสปลายน์ปรับเรียบและสปลายน์กำลังสามเต็มหน่วยแบบฉบับ เราพิจารณาโครงการแบบทั่วไปสำหรับสตีปในผลต่าง เราเสนอขั้นตอนวิธีในการสร้างสปลายน์ทั่วไปเต็มหน่วยและบีสปลายน์ทั่วไปเต็มหน่วย (โดยเรียกสั้น ๆ ว่า จีบีสปลายน์เต็ม-หน่วย) นอกจากนี้ เราได้สูตรชัดแจ้งและความสัมพันธ์เวียนเกิดสำหรับจีบีสปลายน์และศึกษาคุณสมบัติของจีบีสปลายน์เต็มหน่วยและอนุกรมด้วย เราแสดงว่าจีบีสปลายน์เต็มหน่วยเป็นระบบเชบีเชฟและอนุกรมของจีบีสปลายน์เต็มหน่วยมีคุณสมบัติลดความแปรปรวน การเสนอนี้ได้แสดงให้เห็นโดยกราฟของจีบีสปลายน์ ตัวอย่างของการกำหนดฟังก์ชันได้รวบรวมไว้ด้วย

ไฮเพอร์โบลิกความตึงสปลายน์ ได้ถูกนิยามเป็นผลเฉลยของปัญหาค่าขอบเขตแบบหลายจุดสำหรับสมการเชิงอนุพันธ์ ไฮเพอร์โบลิกความตึงสปลายน์แบบไม่ต่อเนื่องสามารถหาได้โดยใช้ตัวดำเนินการที่แตกต่างออกไปโดยการคำนวณ ไม่จำเป็นต้องใช้ฟังก์ชันเอ็กซโพเนนเชียลแบบไฮเพอร์โบลิก ซึ่งสามารถขยายฟังก์ชันดังกล่าวไปเป็นแบบต่อเนื่องโดยยังเป็นสปลายน์แบบไฮเพอร์โบลิก ในงานวิจัยนี้เราให้ความสนใจแง่มุมพื้นฐานของวิธีคำนวณและสาธิตให้เห็นจุดสำคัญของวิธีการนี้

สำหรับการวิจัยในอนาคต ควรจะมีการนำผลที่ได้จากการวิจัยครั้งนี้ไปสร้างเป็นข้อสรุปทั่วไป เพื่อปรับใช้กับกรณีศึกษาที่มีมิติสำคัญที่หลากหลายต่อไป

คำหลัก : discrete generalized splines and GB-splines, recurrence relations, weak Chebyshev systems and variation diminishing property, multipoint boundary value problem, shape preserving interpolation.

Abstract

Project Code: BRG/8/2543

Project Title: Difference Method for Constructing Shape Preserving Splines

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This research report addresses the definition and the study of discrete generalized splines. Discrete generalized splines are continuous piecewise defined functions which meet some smoothness conditions for the first and second divided differences at the knots. They provide a generalization both of smooth generalized splines and of the classical discrete cubic splines. Completely general configurations for steps in divided differences are considered. Direct algorithms are proposed for constructing discrete generalized splines and discrete generalized B-splines (discrete GB-splines for short). Explicit formulae and recurrence relations are obtained for discrete GB-splines. Properties of discrete GB-splines and their series are studied. It is shown that discrete GB-splines form weak Chebyshev systems and that series of discrete GB-splines have a variation diminishing property. The presentation is illustrated by graphs of GB-splines. Examples of defining functions are also included.

A hyperbolic tension spline is defined as the solution of a differential multipoint boundary value problem. A discrete hyperbolic tension spline is obtained using the difference analogous of differential operators; its computation does not require exponential functions, even if its continuous extension is still a spline of hyperbolic type. We consider the basic computational aspects and show the main features of this approach.

In future research, one could try to generalize the results obtained here to the multidimensional case.

Keywords: Discrete generalized splines and GB-splines, recurrence relations, weak Chebyshev systems and variation diminishing property, multipoint boundary value problem, shape preserving interpolaion

Contents

Introduction

Univariate splines in their simplest and most useful form are nothing more than pieces of polynomials joined together smoothly at certain knots. They were studied intensely in the 60s, and by the mid-70s were sufficiently well understood to permit a fairly comprehensive treatment in book form (see [11,54,60]). Univariate splines remain very important tools in a multitude of applications involving curve fitting and design. The main reason for this is their excellent approximation properties. Splines are easy to store, manipulate, and evaluate on a digital computer.

However, polynomial splines do not retain the shape properties of the data. This problem is known as the *problem of shape preserving approximation.* During the last two decades different authors have developed various algorithms of spline approximation with both local and global shape control. Based on spline functions, such methods are usually called *methods of shape preserving spline approximation.*

One of the main applications of shape preserving spline approximation is computer aided geometric design (CAGD). The idea in CAGD is to find representations of curves and surfaces which are easy to treat on a computer, and which are easy to render on a graphical device such as a computer screen. To be of most use, these representations should have convenient *handles* consisting of a set of parameters which can be varied by the user to make well-defined changes in the curve or surface. The main challenge is to develop algorithms that select these parameters automatically. The design of curves and surfaces plays an important role not only in the construction of different products such as car bodies, ship hulls, airplane fuselages and wings, propellers blades, etc., but also in the description of geological, physical and even medical phenomena. New areas of CAGD applications include computer vision and inspection of manufactured parts, medical research (software for digital diagnostic equipment), image analysis, high resolution TV systems, cartography, the film industry, etc.

In the majority of these applications, it is important to construct curves and surfaces which have certain shape properties. For example, we may want the surface to be positive, monotone, or convex in some sense. Standard methods of spline functions do not preserve these properties of the data. Therefore, when fitting spline curves and surfaces to functions and data one needs to have more refined methods available which preserve the shape of the data. By introducing some parameters into the spline structure, one can preserve various characteristics of the data, including positivity, monotonicity, convexity, as well as linear and planar sections. By increasing one or more of these parameters the curve is pulled towards an inherent shape, usually a piecewise linear curve, at the same time keeping the smoothness of the curve.

Very strong requirements must be met in industrial design. Usually, a designer provides the envelopes of a car body, ship hull, airplane fuselage, engine details of complex shape, etc. as a discrete set of points. To produce the body one needs to describe these points as lying on some curve, or some surface. Any discontinuities of the first and even second derivative as well as large values of the second derivative (i.e. of the curvature) may lead to flow separation, that is, to an increase in friction. By this reason, the designer is often interested in a very smooth approximation which preserves the shape of the data.

In many cases one would also like to use the shape preserving representation of surves and surfaces to drive some manufacturing device (such as a lathe or another cutting tool). This is usually referred to as *computer aided manufacturing* (CAM).

Research on constructing shape-preserving interpolatory functions started with the spline in tension of Schweikert [55] where exponential splines were used as approximants. This was followed by the work of Späth [57,58], Nielson [42], Pruess [43,44] and de Boor [11] with various exponential and cubic spline interpolants containing "tension parameters" to control shape. All of these approximations were interpolatory and globally C^2, but strictly speaking were not local in the sense that changing data at one point meant the entire approximation had to be regenerated. This made automatic algorithms for choosing free parameters to control shape (especially monotonicity) fairly complicated. McAllister, Passow and Roulier [38] derived a method for generating shape-preserving curves of arbitrary smoothness based on the properties of Bernstein polynomials, but to achieve C^2 smoothness they had to use piecewise polynomials of degree at least four. There is also the possibility of using piecewise rational interpolants (e.g. see Delbourgo and Gregory [13], Gregory and Delbourgo [20]) although these are usually only C^1 or they are intended for strictly monotone or strictly convex data.

In 1980 Fritsch and Carlson [18] proposed a shape-preserving interpolatory cubic spline which was only C^1 globally, but consequently was local, and admitted much simpler algorithms for the choice of free parameters to control shape (Fritsch and Butland [17]). Renka [47] working on the exponential spline has produced an algorithm for authomatically choosing tension parameters in the C^1 case together with an iterative approach to extend this in a special manner to C^2. Costantini [6] also has families of shape-preserving interpolants based on Bernstein polynomials; these are very simple to use but are comonotone, i.e. the spline on the ith data interval is increasing or decreasing as is the data on that interval. Such splines have the disadvantage that they must have slope zero at a point where the neighboring secant lines have a sign change in their slope; hence, any local extrema of the underlying approximation are assumed to be in the data sample. Also, to get globally C^2 interpolants one must use quintic splines. Other examples of C^1 shape-preserving spline interpolants are found in Burmeister, Hess and Schmidt [3], and Schmidt and Hess [52]. There is the work of Dougherty, Edleman and Hyman [16] where C^2 quintic splines are used; a fairly complete algorithm is given there for preserving monotonicity and there is also a considerable discussion concerning convexity for the piecewise cubic case.

In the theory of splines mainly two approaches are used: algebraic and variational. In the first approach, splines are understood as smooth piecewise functions. In the second approach, splines are solutions of some minimization problems for quadratic functionals with equality and/or inequality constraints. Although less common, a third approach where splines are defined as the solutions of differential multipoint boundary value problems (DMBVP for short), has been considered, [22]. Even though some of the important classes of splines can be obtained from all these schemes, specific features make sometimes the last one an important tool in practical settings. This research report investiges this third approach and consists of two chapters.

Chapter 1 investigates the tool of discrete generalized tension splines. Such splines generalize the concept of discrete polynomial splines and reduce to them as the tension parameters go to zero. We propose direct algorithms and recurrence relations for constructing discrete generalized tension splines and generalized tension B-splines (discrete GB-splines for short). Properties of discrete GB-splines and their series are studied. It is shown that discrete GB-splines form weak Chebyshev systems and that series of discrete GB-splines have a variation diminishing property.

Chapter 2 illustrates the advantages of the finite difference approach by the example of hyperbolic tension splines. Using a finite difference approximation of DMBVP we obtain a system of linear equations with a pentadiagonal matrix. This permits us to easily find an approximate solution and avoid having to calculate hyperbolic functions. Even more, if we have a parallel machine, then we can share the computations of the solution of our pentadiagonal system among the processors. However, the extension of our approximate solution will be a discrete hyperbolic tension spline with continuous divided differences instead of derivatives.

The methods of shape preserving interpolation described in this research report were used in a package of computer programs which enable one to construct complex multivalued surfaces. A test of this package known as the "Viking boat" can be found in the appendix.

Chapter 1

Approximation by Discrete GB-Splines

This chapter addresses the definition and the study of discrete generalized splines. Discrete generalized splines are continuous piecewise defined functions which meet some smoothness conditions for the first and second divided differences at the knots. They provide a generalization both of smooth generalized splines and of the classical discrete cubic splines. Completely general configurations for steps in divided differences are considered. Direct algorithms are proposed for constructing discrete generalized splines and discrete generalized B-splines (discrete GB-splines for short). Explicit formulae and recurrence relations are obtained for discrete GB-splines. Properties of discrete GB-splines and their series are studied. It is shown that discrete GB-splines form weak Chebyshev systems and that series of discrete GB-splines have a variation diminishing property.

1.1 Introduction

The tools of generalized splines and GB-splines are widely used in solving problems of shape preserving approximation (e.g., see [3,20,22,23,31,36,39,40]). By introducing various parameters into the spline structure, one can preserve characteristics of the initial data such as positivity, monotonicity, convexity, presence of linear and planar sections, etc. Here, the main challenge is to develop algorithms that choose parameters automatically. Recently, in [5] a difference method for constructing shape preserving hyperbolic splines as solutions of multipoint boundary value problems was developed. Such an approach permits to avoid the computation of hyperbolic functions and has substantial other advantages. However, the extension of a mesh solution will be a discrete hyperbolic tension spline.

Discrete polynomial splines have been studied extensively. They were introduced in [28] as solutions to certain minimization problems involving differences instead of derivatives. They were connected to best summation formulae in [29], and have been used in [27] to compute nonlinear splines iteratively. Approximation properties of discrete splines have been studied by Lyche [25,26] and other authors (e.g., see [11,12,34,45]). Discrete L-splines were considered in [2]. Discrete B-splines on a uniform partition were introduced in [41]. Discrete B-splines on a nonuniform partition were defined in [8, p. 15]. In [4] discrete B-splines were applied to the general area of subdivision. While discrete polynomial splines are currently attracting widespread research interest (e.g., see [32,33,35]), discrete generalized splines and GB-splines have been less studied. The only results we know of concern discrete exponential Box-splines [7,38] and are therefore related to uniform partitions.

The contents of this chapter is as follows. In section 1.2 we give a general definition of a discrete generalized spline and prove sufficient conditions for its existence and uniqueness. Next, we construct a minimum length local support basis (whose elements are denoted as discrete GB-splines) of the new spline space; see section 1.3. Properties of GB-splines are discussed in section 1.4, while the local approximation by discrete GB-splines of a given continuous function from its samples is considered in section 1.5. In section 1.6 we derive recurrence formulae for calculations with discrete GB-splines. The properties of GB-spline series are summarized in section 1.7. Section 1.8 provides some examples of defining functions that conform to the sufficiency conditions derived earlier in the paper.

1.2 Discrete generalized splines. Conditions of existence and uniqueness

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ be given. For fixed $\tau_j^{L_i} > 0$ and $\tau_j^{R_i} > 0$, $j = i, i+1$, and a function S which is defined and continuous on the real line $\mathbb{R}$ we introduce the linear difference operators

$$\begin{aligned} D_{i,1}S(x) =& (\lambda_i^{R_i} S[x - \tau_i^{L_i}, x] + \lambda_i^{L_i} S[x, x + \tau_i^{R_i}])(1-t) \\ &+ (\lambda_{i+1}^{R_i} S[x - \tau_{i+1}^{L_i}, x] + \lambda_{i+1}^{L_i} S[x, x + \tau_{i+1}^{R_i}])t, \\ D_{i,2}S(x) =& 2S[x - \tau_i^{L_i}, x, x + \tau_i^{R_i}](1-t) + 2S[x - \tau_{i+1}^{L_i}, x, x + \tau_{i+1}^{R_i}]t, \\ & x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N-1, \end{aligned} \tag{1.1}$$

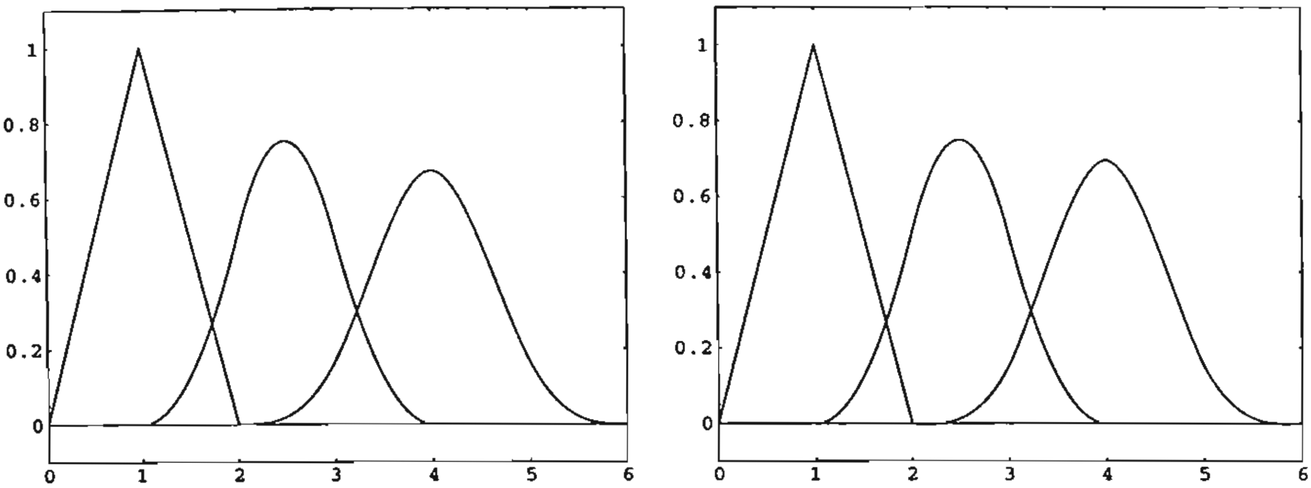
where $\lambda_j^{R_i} = 1 - \lambda_j^{L_i} = \tau_j^{R_i}/(\tau_j^{L_i} + \tau_j^{R_i})$, $j = i, i+1$ and $t = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$. The square parentheses denote the usual first and second divided differences of the function S.

We associate to Δ a system of functions $\{1, x, \Phi_i, \Psi_i\}$, $i = 0, \ldots, N-1$, which are defined and continuous on $\mathbb{R}$ and for given i are linearly independent on the interval $[x_i, x_{i+1}]$. The functions Φ_i and Ψ_i are subject to the constraints

$$\begin{aligned} &\Phi_i(x_{i+1} - \tau_{i+1}^{L_i}) = \Phi_i(x_{i+1}) = \Phi_i(x_{i+1} + \tau_{i+1}^{R_i}) = 0, \quad D_{i,2}\Phi_i(x_i) = 1, \\ &\Psi_i(x_i - \tau_i^{L_i}) = \Psi_i(x_i) = \Psi_i(x_i + \tau_i^{R_i}) = 0, \qquad D_{i,2}\Psi_i(x_{i+1}) = 1. \end{aligned} \tag{1.2}$$

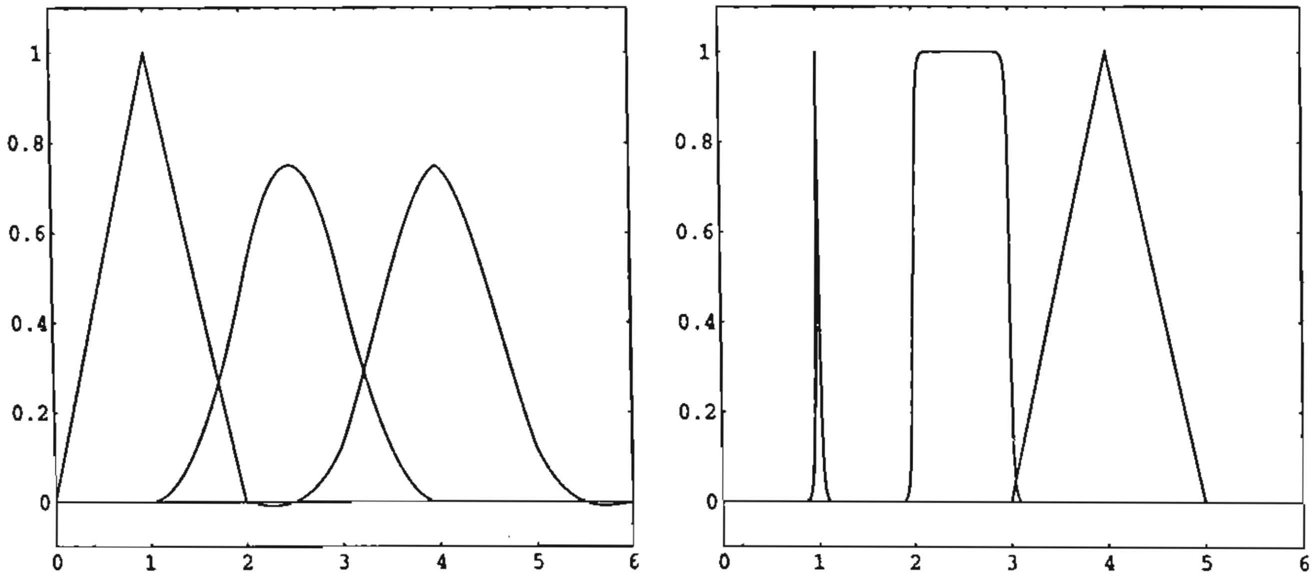
Any element S_i of the linear space Υ_i spanned by the four functions 1, x, Φ_i, Ψ_i can be uniquely written as follows

$$\begin{aligned} S_i(x) =& S_i(x_i)(1-t) + S_i(x_{i+1})t + D_{i,2}S_i(x_i)[\Phi_i(x) - \Phi_i(x_i)(1-t)] \\ &+ D_{i,2}S_i(x_{i+1})[\Psi_i(x) - \Psi_i(x_{i+1})t]. \end{aligned} \tag{1.3}$$



(a) (b)

Fig. 1.1. The discrete GB-splines $B_{j,2}$, $B_{j,3}$, and B_j (from left to right) on a uniform mesh with step size $h_i = 1$, no tension and discretization parameter $\tau = 0.1$ (a) and $\tau = 0.33$ (b).



(a) (b)

Fig. 1.2. Same as Fig. 1.1, but with discretization parameter $\tau = 0.5$ (a) and with tension parameters $q_i = 50$ for all i (b).

1.7 Series of discrete GB-splines (uniform case)

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ is an integer multiple of the same tabulation step, τ, of some uniform mesh refinement on $[a, b]$.

For $\theta \in \mathbb{R}$, $\tau > 0$ define

$$\mathbb{R}_{\theta\tau} = \{\theta + i\tau : \ i \text{ is an integer}\}$$

and let $\mathbb{R}_{\theta 0} = \mathbb{R}$. For any $a, b \in \mathbb{R}$ and $\tau > 0$ let

$$[a, b]_\tau = [a, b] \cap \mathbb{R}_{a\tau} .$$

The functions $B_{j,2}$, $B_{j,3}$, and B_j with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$ for all i are nonnegative on the discrete interval $[a,b]_\tau$. This permits us to reprove the main results for discrete polynomial splines in [42] for series of discrete generalized splines. Even more, one can obtain the results of generalized splines in [19] from the corresponding statements for discrete generalized splines as a limiting case when $\tau \to 0$.

In particular, if in (1.25) and (1.34) we have coefficients $b_j^{(k)} > 0$, $k = 0,1,2$, $j = -3+k, \ldots, N-1$, then the spline S will be a positive, monotonically increasing and convex function on $[a,b]_\tau$.

Let f be a function defined on the discrete set $[a,b]_\tau$. We say that f has a zero at the point $x \in [a,b]_\tau$ provided

$$f(x) = 0 \quad \text{or} \quad f(x-\tau)\cdot f(x) < 0.$$

When f vanishes at a set of consecutive points of $[a,b]_\tau$, say f is 0 at $x, \ldots, x+(r-1)\tau$, but $f(x-\tau)\cdot f(x+r\tau) \neq 0$, then we call the set $X = \{x, x+\tau, \ldots, x+(r-1)\tau\}$ a *multiple zero* of f, and we define its multiplicity by

$$Z_X(f) = \begin{cases} r, & \text{if } f(x-\tau)\cdot f(x+r\tau) < 0 \text{ and } r \text{ is odd,} \\ r, & \text{if } f(x-\tau)\cdot f(x+r\tau) > 0 \text{ and } r \text{ is even,} \\ r+1, & \text{otherwise.} \end{cases}$$

This definition assures that f changes sign at a zero if and only if the zero is of odd multiplicity.

Let $Z_{[a,b]_\tau}(f)$ be the number of zeros of a function f on the discrete set $[a,b]_\tau$, counted according to their multiplicity. Let us denote $D_1^L S(x) = S[x-\tau, x]$.

Theorem 4. *(Rolle's Theorem For Discrete Generalized Splines.) For any $S \in S_4^{DG}$,*

$$Z_{[a,b]_\tau}(D_1^L S) \geq Z_{[a,b]_\tau}(S) - 1. \tag{1.35}$$

Proof: First, if S has a z-tuple zero on the set $X = \{x, \ldots, x+(r-1)\tau\}$, it follows that $D_1^L S$ has a $(z-1)$-tuple zero on the set $X' = \{x+\tau, \ldots, x+(r-1)\tau\}$. Now if X^1 and X^2 are two consecutive zero sets of S, then it is trivially true that $D_1^L S$ must have a sign change at some point between X^1 and X^2. Counting all of these zeros, we arrive at the assertion (1.35). This completes the proof. □

Lemma 1.4. *Let the functions $D_{i,2}\Phi_i$ and $D_{i,2}\Psi_i$ be strictly monotone on the interval $[x_i, x_{i+1}]$ for all i. Then for every $S \in S_4^{DG}$ which is not identically zero on any interval $[x_i, x_{i+1}]_\tau$, $i = 0, \ldots, N-1$,*

$$Z_{[a,b]_\tau}(S) \leq N+2.$$

Proof: According to (1.30) and (1.34), the function $D_2 S$ has no more than one zero on $[x_i, x_{i+1}]$, because the functions $D_{i,2}\Phi_i$ and $D_{i,2}\Psi_i$ are strictly monotone and nonnegative on this interval. Hence $Z_{[a,b]_\tau}(D_2 S) \leq N$. Then according to the Rolle's Theorem 1.4, we find $Z_{[a,b]_\tau}(S) \leq N+2$. This completes the proof. □

Denote by $supp_\tau B_i = \{x \in \mathbb{R}_{a,\tau} \,|B_i(x) > 0\}$ the discrete support of the spline B_i, i.e. the discrete set $(x_i + \tau, x_{i+4} - \tau)_\tau$.

Theorem 1.5. *Assume that* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ *are prescribed points on the discrete line* $\mathbb{R}_{a,\tau}$. *Then*

$$D = \det(\mathrm{B}_i(\zeta_j)) \geq 0, \quad i,j = -3,\ldots,N-1$$

and strict positivity holds if and only if

$$\zeta_i \in supp_\tau \mathrm{B}_i, \quad i = -3,\ldots,N-1. \tag{1.36}$$

The proof of this theorem is based on Lemma 1.4 and repeats that of theorem 8.66 in [42, p. 355]. The following three statements follow immediately from Theorem 1.5.

Corollary 1.2. *The system of discrete GB-splines* $\{\mathrm{B}_j\}$, $j = -3,\ldots,N-1$, *associated with knots on* $\mathbb{R}_{a,\tau}$ *is a weak Chebyshev system according to the definition given in* [42, p. 36], *i.e. for any* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ *in* $\mathbb{R}_{a,\tau}$ *we have* $D \geq 0$ *and* $D > 0$ *if and only if condition (1.36) is satisfied. In the latter case the discrete generalized spline* $S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x)$ *has no more than* $N+2$ *zeros.*

Corollary 1.3. *If the conditions of Theorem 1.5 are satisfied, then the solution of the interpolation problem*

$$S(\zeta_i) = f_i, \quad i = -3,\ldots,N-1, \quad f_i \in \mathbb{R} \tag{1.37}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, be a rectangular $m \times n$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally positive) (e.g., see [16]) if the minors of all order of the matrix are nonnegative (positive), i.e. for all $1 \leq p \leq m$ we have

$$det(a_{i_k j_l}) \geq 0 \ (>0) \quad \text{for all} \quad \begin{array}{l} 1 \leq i_1 < \cdots < i_p \leq m, \\ 1 \leq j_1 < \cdots < j_p \leq n. \end{array}$$

Corollary 1.4. *For arbitrary integers* $-3 \leq \nu_{-3} < \cdots < \nu_{p-4} \leq N-1$ *and* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{p-4}$ *in* $\mathbb{R}_{a,\tau}$ *we have*

$$\bar{D}_p = det\{\mathrm{B}_{\nu_i}(\zeta_j)\} \geq 0, \quad i,j = -3,\ldots,p-4$$

and strict positivity holds if and only if

$$\zeta_i \in supp_\tau \, \mathrm{B}_{\nu_i}, \quad i = -3,\ldots,p-4$$

i.e. the matrix $\{\mathrm{B}_j(\zeta_i)\}$, $i,j = -3,\ldots,N-1$ *is totally nonnegative.*

The last statement is proved by induction based on Theorem 1.5 and the recurrence relations for the minors of the matrix $\{\mathrm{B}_j(\zeta_i)\}$. The proof does not differ from that of Theorem 8.67 described in [42, p. 356].

Since the supports of discrete GB-splines are finite, the matrix of system (1.37) is banded and has seven nonzero diagonals in general. The matrix is tridiagonal if $\zeta_i = x_{i+2}$, $i = -3, \ldots, N-1$.

An important particular case of the problem, in which $S'(x_i) = f_i'$, $i = 0, N$, can be obtained by passing to the limit as $\zeta_{-3} \to \zeta_{-2}$, $\zeta_{N-1} \to \zeta_{N-2}$.

De Boor and Pinkus [9] proved that linear systems with totally nonnegative matrices can be solved by Gaussian elimination without pivoting. Thus, the system (1.37) can be solved effectively by the conventional Gauss method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \cdots, v_n)$, with zeros being neglected. Karlin [16] showed that if a matrix A is totally nonnegative then it decreases the variation, i.e.

$$S^-(A\mathbf{v}) \le S^-(\mathbf{v}).$$

By virtue of Corollary 4, the totally nonnegative matrix $\{\mathrm{B}_j(\zeta_i)\}$, $i, j = -3, \ldots, N-1$, formed by discrete GB-splines decreases the variation.

For a bounded real function f, let $S^-(f)$ be the number of sign changes of the function f on the real axis $\mathbb{R}$, without taking into account the zeros

$$S^-(f) = \sup_n S^-[f(\zeta_1), \ldots, f(\zeta_n)], \quad \zeta_1 < \zeta_2 < \cdots < \zeta_n.$$

Theorem 1.6. *The discrete generalized spline $S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x)$ is a variation diminishing function, i.e. the number of sign changes of S does not exceed that in the sequence of its coefficients:*

$$S^-\left(\sum_{j=-3}^{N-1} b_j \mathrm{B}_j\right) \le S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-3}, \ldots, b_{N-1}).$$

The proof of this statement does not differ from that of Theorem 8.68 for discrete polynomial B-splines in [42, p. 356].

By Theorem 1.6, the spline

$$S_f(x) = \sum_{j=-3}^{N-1} f(y_{j+2}) \mathrm{B}_j(x)$$

is a variation diminishing function. It enables us to write the inequalities

$$S^-(S_f) \le S^-(\overline{\mathbf{f}}) \le S^-(f),$$

where $\overline{\mathbf{f}} = (f(y_{-1}), \ldots, f(y_{N+1}))$.

Since in addition by Theorem 1.3, the locally approximating discrete generalized spline S_f is also exact for polynomials l of first order, we arrive at the inequality

$$S^-(S_f - l) = S^-(S_{f-l}) \le S^-(f - l).$$

Thus, the following statement is true.

Theorem 1.7. *Let a continuous function f be given by its samples $f(y_j)$, $j = -1, \ldots, N+1$. If $b_j = f(y_{j+2})$, $j = -3, \ldots, N-1$, then the locally approximating discrete generalized spline S_f intersects an arbitrary straight line at most as often as the function f.*

1.8 Examples of defining functions

Let us give some choices of the defining functions Φ_i and Ψ_i for discrete generalized splines that conform to the sufficiency conditions derived earlier in the paper.

Putting

$$\Psi_i(x) = \psi_i(t)h_i^2 = \psi(q_i, \hat{\tau}_i^{L_i}, \hat{\tau}_i^{R_i}, t)h_i^2, \quad \Phi_i(x) = \psi(p_i, \hat{\tau}_{i+1}^{R_i}, \hat{\tau}_{i+1}^{L_i}, 1-t)h_i^2,$$
$$\hat{\tau}_j^{L_i} = \tau_j^{L_i}/h_i, \quad \hat{\tau}_j^{R_i} = \tau_j^{R_i}/h_i; \quad j = i, i+1; \quad 0 \le p_i, q_i < \infty,$$

we consider some possibilities for choosing the functions ψ_i which, due to the constraints (1.2), satisfy the conditions

$$\psi_i(-\hat{\tau}_i^{L_i}) = \psi_i(0) = \psi_i(\hat{\tau}_i^{R_i}) = 0, \; D_{i+1,2}\psi_i(1) = h_i^{-2}. \tag{1.38}$$

1. Discrete rational spline with linear denominator:

$$\psi_i(t) = C_i \frac{(t+\hat{\tau}_i^{L_i})t(t-\hat{\tau}_i^{R_i})}{1+q_i(1-t)}.$$

2. Discrete rational spline with quadratic denominator:

$$\psi_i(t) = C_i \frac{(t+\hat{\tau}_i^{L_i})t(t-\hat{\tau}_i^{R_i})}{1+q_i t(1-t)}.$$

3. Discrete exponential spline:

$$\psi_i(t) = C_i(t+\hat{\tau}_i^{L_i})t(t-\hat{\tau}_i^{R_i})exp\big(-q_i(1-t)\big).$$

4. Discrete hyperbolic spline:

$$\psi_i(t) = C_{i,1}\Big[\sinh q_i t - t\frac{\sinh q_i \hat{\tau}_i^{R_i}}{\hat{\tau}_i^{R_i}}\Big] + C_{i,2}\Big[\cosh q_i t - 1 - t\frac{\cosh q_i \hat{\tau}_i^{R_i} - 1}{\hat{\tau}_i^{R_i}}\Big].$$

5. Discrete cubic spline with additional knots:

$$\psi_i(t) = \frac{1}{2}\frac{(t-\beta_i+\hat{\tau}_i^{L_i})(t-\beta_i)_+(t-\beta_i-\hat{\tau}_i^{R_i})}{3(1-\beta_i)+\hat{\varepsilon}_{i+1}-\hat{\varepsilon}_i},$$
$$\hat{\varepsilon}_j = \hat{\tau}_j^{R_i} - \hat{\tau}_j^{L_i}, \quad j = i, i+1; \quad \beta_i = 1-(1+q_i)^{-1}, \quad E_+ = \max(0, E).$$

The points $x_i+\alpha_i h_i$ ($\alpha_i = (1+p_i)^{-1}$) and $x_i+\beta_i h_i$ fix the position of two additional knots of the spline on the interval $[x_i, x_{i+1}]$. By moving these knots one can perform a transfer from a discrete cubic spline to piecewise linear interpolation.

6. Discrete spline of variable order:

$$\psi_i(t) = C_i(t+\hat{\tau}_i^{L_i})t^{k_i}(t-\hat{\tau}^{R_i}), \quad k_i = 1+q_i.$$

The constants C_i in the expressions for the function ψ_i above are calculated from the condition (1.38) for the second divided difference of ψ_i. To find $C_{i,k}$, $k = 1, 2$, one needs additionally use the condition $\psi_i(-\hat{\tau}_i^{L_i}) = 0$. It is easy to check that in all cases 1.–6. we get the corresponding defining functions in [21] by setting $\hat{\tau}_j^{L_i} = \hat{\tau}_j^{R_i} = 0$, $j = i, i+1$.

Chapter 2

Difference Method for Constructing Hyperbolic Tension Splines

In this chapter a hyperbolic tension spline is defined as the solution of a differential multipoint boundary value problem. A discrete hyperbolic tension spline is obtained using the difference analogous of differential operators; its computation does not require exponential functions, even if its continuous extension is still a spline of hyperbolic type. We consider the basic computational aspects and show the main features of this approach.

2.1 Introduction

Spline theory is mainly grounded on two approaches: the algebraic one (where splines are understood as smooth piecewise functions, see e.g. [54,60]) and the variational one (where splines are obtained via minimization of quadratic functionals with equality and/or inequality constraints, see e.g. [31]). Although less common, a third approach where splines are defined as the solutions of differential multipoint boundary value problems (DMBVP for short), has been considered, [22]. Even though some of the important classes of splines can be obtained from all these schemes, specific features make sometimes the last one an important tool in practical settings. We want to illustrate this fact by the example of hyperbolic tension splines.

Introduced by Schweikert in 1966, [55], hyperbolic tension splines are solutions of DMBVP where the differential operators depend on *tension* parameters. Their tension properties (that is the possibility of pulling the curve toward a piecewise linear function) have kept hyperbolic splines popular (see for example [25,47,48,50] and references quoted therein) in shape-preserving interpolation and/or approximation. Unfortunately, it is difficult to work with hyperbolic splines for small or large values of the tension parameters. For this reason, in spite of the presence of refined algorithms for their calculation [48], hyperbolic tension splines were forced out by rational splines (see for example [13, 27]) in practical applications.

We observe that for practical purposes it is often neccessary to know the values of the solution S of a DMBVP only over a prescribed grid instead of its global

In addition, the function $D_1\mathrm{B}_i$ satisfies to the relation

$$D_1\mathrm{B}_i(x) = \frac{\mathrm{B}_{i,3}(x)}{c_{i,3}} - \frac{\mathrm{B}_{i+1,3}(x)}{c_{i+1,3}}, \tag{2.38}$$

where

$$\mathrm{B}_{j,3}(x) = \begin{cases} \frac{1}{c_{j,2}}\Psi_j[x-\tau_j, x+\tau_j], & x \in [x_j, x_{j+1}), \\ 1 + \frac{1}{c_{j,2}}\Phi_{j+1}[x-\tau_{j+1}, x+\tau_{j+1}] & \\ -\frac{1}{c_{j+1,2}}\Psi_{j+1}[x-\tau_{j+1}, x+\tau_{j+1}], & x \in [x_{j+1}, x_{j+2}), \\ -\frac{1}{c_{j+1,2}}\Phi_{j+2}[x-\tau_{j+2}, x+\tau_{j+2}], & x \in [x_{j+2}, x_{j+3}), \\ 0, & \text{otherwise}. \end{cases} \tag{2.39}$$

Functions $\mathrm{B}_{j,3}$ and $\mathrm{B}_{j,4} \equiv \mathrm{B}_j$ possess many of the properties inherent in usual discrete polynomial B-splines. We collect their characteristics in the next theorem which can be proved by using the explicit formulae (2.33), (2.36), and (2.39) for discrete HB-splines $\mathrm{B}_{j,k}$, $j = 2, 3, 4$, and the relations (2.37) and (2.38).

Theorem 2.1. *The functions $\mathrm{B}_{j,k}$, $k = 3, 4$ have the following properties:*

(a) $\mathrm{B}_{j,4}(x) > 0$ *for* $x \in (x_j + \tau_j, x_{j+4} - \tau_{j+4})$, *and* $\mathrm{B}_{j,4}(x) \equiv 0$ *if* $x \notin (x_j, x_{j+4})$, $\mathrm{B}_{j,3}(x) > 0$ *for* $x \in (x_j, x_{j+3})$, *and* $\mathrm{B}_{j,3}(x) \equiv 0$ *if* $x \notin (x_j, x_{j+3})$;

(b) $\mathrm{B}_{j,4}$ *satisfies the smoothness conditions* (2.14);

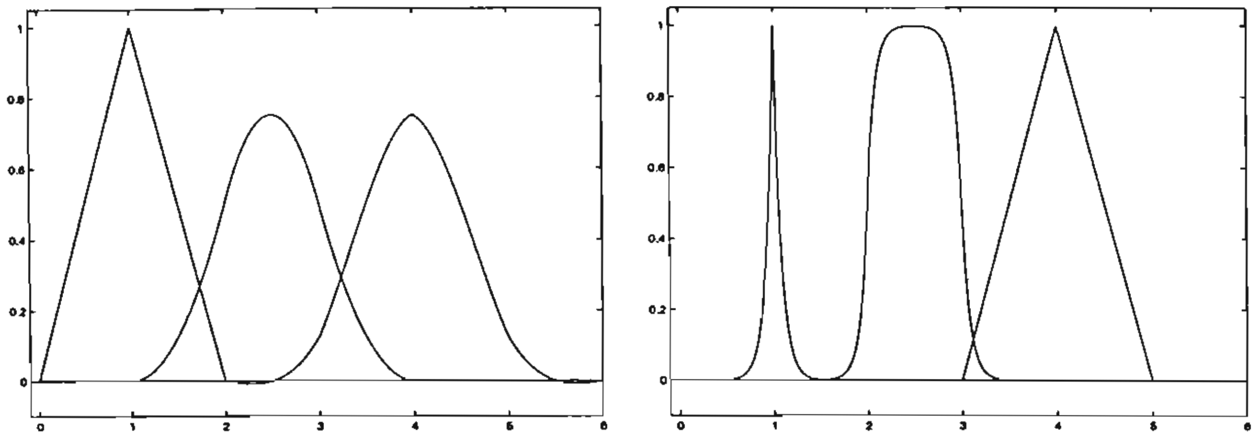
(c) $\mathrm{B}_{j,3}$ *satisfies the first and second smoothness conditions* (2.14);

(d) $\sum_{j=-2}^{N} \mathrm{B}_{j,3}(x) \equiv 1$ *for* $x \in [a, b]$,
$\Phi_j[x-\tau_j, x+\tau_j] = -c_{j-1,2}\mathrm{B}_{j-2,3}(x)$, $\Psi_j[x-\tau_j, x+\tau_j] = c_{j,2}\mathrm{B}_{j,3}(x)$
for $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N$;

(e) $\sum_{j=-3}^{N} y_{j+2}^r \mathrm{B}_{j,4}(x) \equiv x^r$, $r = 0, 1$ *for* $x \in [a, b]$,
$\Phi_j(x) = c_{j-1,2}c_{j-2,3}\mathrm{B}_{j-3,4}(x)$, $\Psi_j(x) = c_{j,2}c_{j,3}\mathrm{B}_{j,4}(x)$
for $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N$.

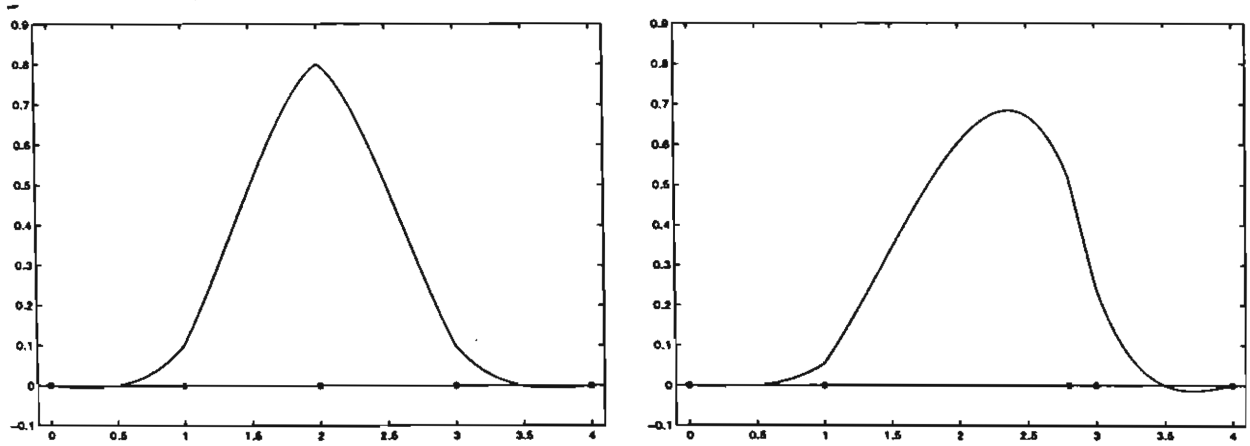
(a) (b)

Figure 2.1. The discrete HB-splines $\mathrm{B}_{j,k}$, $k = 2, 3, 4$ (from left to right) on a uniform mesh with step size $h_i = 1$, no tension and discretization parameter $\tau = 0.1$ (a) and $\tau = 0.33$ (b).



(a) (b)

Figure 2.2. Same as Figure 2.1, but with discretization parameter $\tau = 0.5$ (a) and with tension parameters $p_i = 50$ for all i (b).



(a) (b)

Figure 2.3. The discrete HB-splines $B_{j,4}$ on a uniform mesh (a) and on a nonuniform mesh (b). The asterisk $*$ denotes the x_i. For both plots $p_i = 2$ and $n_i = 2$.

Figures 2.1 and 2.2 show the graphs of discrete HB-splines $B_{j,k}$, $k = 2, 3, 4$ (from left to right) on a uniform mesh with step size $h_i = 1$ and with $\tau_i = \tau$ for all i. We have chosen discretization parameters $\tau = 0.1$ (Figures 2.1(a) and 2.2(b)), $\tau = 0.33$ (Figure 2.1(b)) and $\tau = 0.5$ (Figure 2.2(a)) for $\psi_i(t)$ from (2.28). In figures 2.1 and 2.2(a) we have parameters $p_i = 0$, that is, we have conventional discrete cubic B-splines (e.g., see [33]). Visually, the presence of intervals where the B-spline $B_{j,4}$ is negative is more visible with growing discretization parameter τ. In figure 2.2(b) the tension parameters are $p_i = 50$ for all i, whence the shape of the graphs is practically unchanged when τ increases from 0.1 to 0.5. As the limit for $p_i \to \infty$ we obtain the impulse function for $B_{j,2}$, the "step-function" for $B_{j,3}$ and the "hat-function" for $B_{j,4}$ (all of height 1).

Figure 2.3 shows the graphs of discrete HB-splines $B_{j,4}$ on a uniform mesh (a) and on a nonuniform mesh (b), where the asterisk $*$ denotes the x_i. For both plots $p_i = 2$ and $n_i = 2$.

Using the approach of chapter 6, it is easy to show that the functions B_j,

$j = -3, \ldots, N$, have supports of minimum length, are linearly independent and form a basis in the space S_4^{DH}. So any discrete hyperbolic tension spline $\mathrm{U} \in S_4^{DH}$ can be uniquely represented in the form

$$\mathrm{U}(x) = \sum_{j=-3}^{N} b_j \mathrm{B}_j(x) \tag{2.40}$$

with some constant coefficients b_j.

Applying formulae (2.37) and (2.38) to the representation (2.40) we obtain

$$D_1\mathrm{U}(x) = \sum_{j=-2}^{N} b_j^{(1)} \mathrm{B}_{j,3}(x), \quad \Lambda\mathrm{U}(x) = \sum_{j=-1}^{N} b_j^{(2)} \mathrm{B}_{j,2}(x), \tag{2.41}$$

where

$$b_j^{(k)} = \frac{b_j^{(k-1)} - b_{j-1}^{(k-1)}}{c_{j,4-k}}, \quad k = 1, 2; \quad b_j^{(0)} \equiv b_j. \tag{2.41a}$$

2.5.3 Formulae for Local Approximation by Discrete HB-Splines

If the coefficients b_j in (2.40) are known then by virtue of formula (2.33) we can write out an expression for the discrete hyperbolic tension spline U on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$\mathrm{U}(x) = b_{i-2} + b_{i-1}^{(1)}(x - y_i) + b_{i-1}^{(2)}\Phi_i(x) + b_i^{(2)}\Psi_i(x), \tag{2.42}$$

where $b_j^{(k)}$, $k = 1, 2$, are defined in (2.41a).

The representations (2.40) and (2.42) allow us to find a simple and effective way to approximate a given continuous function f from its samples.

Theorem 2.2. *Let a continuous function f be given by its samples $f(y_j)$, $j = -1, \ldots, N+2$. Then for $b_j = f(y_{j+2})$, $j = -3, \ldots, N$, formula (2.40) is exact for polynomials of the first degree and provides a formula for local approximation.*

Proof: It suffices to prove that the identities

$$\sum_{j=-3}^{N} y_{j+2}^r \mathrm{B}_j(x) \equiv x^r, \quad r = 0, 1 \tag{2.43}$$

hold for $x \in [a, b]$. Using formula (2.42) with the coefficients $b_{j-2} = 1$ and $b_{j-2} = y_j$, $j = i-1, i, i+1, i+2$, for an arbitrary interval $[x_i, x_{i+1}]$, we find that identities (2.43) hold.

For $b_{j-2} = f(y_j)$, formula (2.42) can be rewritten as

$$\begin{aligned}\mathrm{U}(x) = & f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1}) f[y_{i-1}, y_i, y_{i+1}] c_{i-1,2}^{-1} \Phi_i(x) \\ & + (y_{i+2} - y_i) f[y_i, y_{i+1}, y_{i+2}] c_{i,2}^{-1} \Psi_i(x), \quad x \in [x_i, x_{i+1}].\end{aligned}$$

This is the formula of local approximation. The theorem is thus proved. □

Corollary 2.1. *Let a continuous function f be given by its samples $f_j = f(x_j)$, $j = -2, \ldots, N+3$. Then by setting*

$$b_{j-2} = f_j - \frac{1}{c_{j-1,2}}\Big(\Psi_{j-1}(x_j) f[x_j, x_{j+1}] - \Phi_j(x_j) f[x_{j-1}, x_j]\Big) \tag{2.44}$$

in (2.40), we obtain a formula of three-point local approximation, which is exact for polynomials of the first degree.

Proof: To prove the corollary, it is sufficient to take the monomials 1 and x as f. Then according to (2.44), we obtain $b_{j-2} = 1$ and $b_{j-2} = y_j$ and it only remains to make use of identities (2.43). This proves the corollary. □

Equation (2.42) permits us to write the coefficients of the spline U in its representation (2.40) of the form

$$b_{j-2} = \begin{cases} \mathrm{U}(y_j) - \Lambda_{j-1}\mathrm{U}(x_{j-1})\Phi_{j-1}(y_j) - \Lambda_j \mathrm{U}(x_j)\Psi_{j-1}(y_j), & y_j < x_j, \\ \mathrm{U}(y_j) - \Lambda_j \mathrm{U}(x_j)\Phi_j(y_j) - \Lambda_{j+1}\mathrm{U}(x_{j+1})\Psi_j(y_j), & y_j \geq x_j. \end{cases} \tag{2.45}$$

Using this formula we obtain $b_{j-2} = \mathrm{U}(y_j) + O(\overline{h}_j^2)$, $\overline{h}_j = \max(h_{j-1}, h_j)$. Hence it follows that the control polygon (e.g., see [21]) converges quadratically to the function f when $b_{j-2} = f(y_j)$, or if the formula (2.44) is used. Formulae (2.42), (2.43), and (2.49) generalize their continuous equivalents developed in chapter 6.

2.6 Computational Aspects

The aim of this section is to investigate the practical aspects related to the numerical evaluation of the mesh solution defined in (2.9).

A standard approach, [48], consists of solving the tridiagonal system (2.15) and then evaluating (2.13) at the mesh points as is usually done for the evaluation of continuous hyperbolic splines. At first sight, this approach based on the solution of a tridiagonal system seems preferable because of the limited waste of computational time and the good classical estimates for the condition number of the matrix in (2.15). However, it should be observed that, as in the continuous case, we have to perform a large number of numerical computations of hyperbolic functions of the form $\sinh(k_i t)$ and $\cosh(k_i t)$ both to define system (2.15) and to tabulate functions (2.13). This is a very difficult task, both for cancellation errors (when $k_i \to 0$) and for overflow problems (when $k_i \to \infty$). A stable computation of the hyperbolic functions was proposed in [48], where different formulae for the cases $k_i \leq 0.5$ and $k_i > 0.5$ were considered and a specialized polynomial approximation for $\sinh(\cdot)$ was used.

However, we note that this approach is the only one possible if we want a continuous extension of the discrete solution beyond the mesh point.

In contrast, the discretized structure of our construction provides us with a much cheaper and simpler approach to compute the mesh solution (2.9). This can be achieved both by following the system splitting approach presented in Section 2.3. or by a direct computation of the solution of the linear system (2.6)–(2.8).

As for the system splitting approach presented in Section 2.3, the following algorithm can be considered.

Step 1. Solve the 3-diagonal system (2.15) for M_i, $i = 1, \dots, N$.
Step 2. Solve $N+1$ 3-diagonal systems (2.11) for M_{ij}, $j = 1, \dots, n_i - 1$, $i = 0, \dots, N$.
Step 3. Solve $N+1$ 3-diagonal systems (2.12) for u_{ij}, $j = 1, \dots, n_i - 1$, $i = 0, \dots, N$.

In this algorithm, hyperbolic functions need only be computed in step 1. Furthermore, the solution of any system (2.11) or (2.12) requires $8q$ arithmetic operations, namely, $3q$ additions, $3q$ multiplications, and $2q$ divisions [60], where q is the number of unknowns, and is thus substantially cheaper than direct computation by formula (2.13).

Steps 2 and 3 can be replaced by a direct splitting of the system (2.6)–(2.8) into $N+1$ systems with 5-diagonal matrices

$$
\begin{aligned}
&u_{i,0} = f_i, \quad \Lambda_i u_{i,0} = M_i, \\
&\Lambda_i^2 u_{i,j} - \left(\frac{p_i}{h_i}\right)^2 \Lambda_i u_{i,j} = 0, \quad j = 1, \dots, n_i - 1, \quad i = 0, \dots, N. \\
&u_{i,n_i} = f_{i+1}, \quad \Lambda_i u_{i,n_i} = M_{i+1},
\end{aligned}
\tag{2.46}
$$

Also, in this case the calculations for steps 2 and 3 or for system (2.46) can be tailored for a multiprocessor computer system.

Let us discuss now the direct solution of system (2.6)–(2.8) which, of course, only involves rational computations on the given data. In order to do this in the next subsections we investigate in some details the structure of the mentioned system.

2.6.1 The Pentadiagonal System

Eliminating the unknowns $\{u_{i,-1},\ i = 1, \dots, N,\}$ and $\{u_{i,n_i+1},\ i = 0, \dots, N-1\}$ from (2.7), determining the values of the mesh solution at the data sites x_i by the interpolation conditions and eliminating $u_{0,-1}$, u_{N,n_N+1} from the end conditions (2.8) we can collect (2.6)–(2.8) into the system

$$\mathbf{A}\mathbf{u} = \mathbf{b}, \tag{2.47}$$

where

$$\mathbf{u} = (u_{01}, \dots, u_{0,n_0-1}, u_{11}, \dots, u_{21}, \dots, u_{N1}, \dots, u_{N,n_N-1})^T.$$

$\mathbf{A}$ is the following pentadiagonal matrix (see also Figure 2.3(a)):

$$
\begin{bmatrix}
b_0-1 & a_0 & 1 & & & & & & & & & & \\
a_0 & b_0 & a_0 & 1 & & & & & & & & & \\
1 & a_0 & b_0 & a_0 & 1 & & & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & & & \\
 & & 1 & a_0 & b_0 & a_0 & & & & & & & \\
 & & & 1 & a_0 & \eta_{0,n_0-1} & \delta_{0,n_0-1} & & & & & & \\
 & & & & & \delta_{1,1} & \eta_{1,1} & a_1 & 1 & & & & \\
 & & & & & & a_1 & b_1 & a_1 & 1 & & & \\
 & & & & & & & \ddots & \ddots & \ddots & & & \\
 & & & & & & & 1 & a_N & b_N & a_N & 1 \\
 & & & & & & & & 1 & a_N & b_N & a_N \\
 & & & & & & & & & 1 & a_N & b_N-1
\end{bmatrix}
$$

with

$$
\begin{aligned}
a_i &= -(4+\omega_i)\ ,\ b_i = 6+2\omega_i\ ,\ \omega_i = \left(\frac{p_i}{n_i}\right)^2 ;\quad i=0,\ldots,N,\\
\eta_{i-1,n_{i-1}-1} &= 6+2\omega_{i-1}+\frac{1-\rho_i}{1+\rho_i}\ ,\ \eta_{i,1} = 6+2\omega_i+\frac{\rho_i-1}{\rho_i+1}\\
\delta_{i-1,n_{i-1}-1} &= \frac{2}{\rho_i(\rho_i+1)}\ ,\ \delta_{i,1} = 2\frac{\rho_i^2}{\rho_i+1}\ ,\\
\rho_i &= \frac{\tau_i}{\tau_{i-1}}, \qquad i=1,\ldots,N;
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{b} =& (-(a_0+2)f_0-\tau_0^2 f_0'', -f_0, 0, \ldots, 0, -f_1, -\gamma_{0,n_0-1}f_1, -\gamma_{1,1}f_1, -f_1, 0,\\
&\ldots, 0, -f_{N+1}, -(a_N+2)f_{N+1}-\tau_N^2 f_{N+1}'')^T,
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_{i-1,n_{i-1}-1} &= -(4+\omega_{i-1}+2\frac{1-\rho_i}{\rho_i}),\\
\gamma_{i,1} &= -(4+\omega_i+2(\rho_i-1)),\quad i=1,\ldots,N.
\end{aligned}
$$

2.6.2 The Uniform Case

From the practical point of view it is interesting to examine the structure of $\mathbf{A}$ when we are dealing with a uniform mesh, that is $\tau_i = \tau$. In such a case it is immediately seen that $\mathbf{A}$ is symmetric. In addition, following [34] we observe that $\mathbf{A} = \mathbf{C} + \mathbf{D}$, where both $\mathbf{C}$ and $\mathbf{D}$ are symmetric block diagonal matrices. To be more specific,

$$
\mathbf{C} = \begin{bmatrix} \mathbf{C}_0 & & & \\ & \mathbf{C}_1 & & \\ & & \ddots & \\ & & & \mathbf{C}_N \end{bmatrix},\quad \mathbf{C}_i = \mathbf{B}_i^2 - \omega_i \mathbf{B}_i,
$$

where $\mathbf{B}_i$ is the $(n_i - 1) \times (n_i - 1)$ tridiagonal matrix

$$\mathbf{B}_i = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix};$$

and

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & & & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & & \\ & & & 0 & & & & & & & & & & \\ & & & & 1 & 1 & & & & & & & & \\ & & & & 1 & 1 & & & & & & & & \\ & & & & & & 0 & & & & & & & \\ & & & & & & & \ddots & & & & & & \\ & & & & & & & & 0 & & & & & \\ & & & & & & & & & 1 & 1 & & & \\ & & & & & & & & & 1 & 1 & & & \\ & & & & & & & & & & & 0 & & \\ & & & & & & & & & & & & \ddots & \\ & & & & & & & & & & & & & 0 \quad 0 \\ & & & & & & & & & & & & & 0 \quad 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{C}$, $\lambda_k(\mathbf{C})$, are the collection of the eigenvalues of $\mathbf{C}_i$. Since, (see [34]),

$$\lambda_j(\mathbf{B}_i) = -2\Big(1 - \cos\frac{j\pi}{n_i}\Big), \; j = 1, \ldots, n_i - 1,$$

we have

$$\lambda_j(\mathbf{C}_i) = 4\Big(1 - \cos\frac{j\pi}{n_i}\Big)^2 + 2\omega_i\Big(1 - \cos\frac{j\pi}{n_i}\Big) \quad j = 1, \ldots, n_i - 1.$$

In addition, the eigenvalues of $\mathbf{D}$ are 0 and 2, thus we deduce from a corollary of the Courant-Fisher theorem [19] that the eigenvalues of $\mathbf{A}$ satisfy the following inequalities

$$\lambda_k(\mathbf{A}) \geq \lambda_k(\mathbf{C}) = \min_{i,j} \lambda_j(\mathbf{C}_i) = \min_i \Big[4\Big(1 - \cos\frac{\pi}{n_i}\Big)^2 + 2\omega_i\Big(1 - \cos\frac{\pi}{n_i}\Big)\Big].$$

Hence, $\mathbf{A}$ is a positive matrix and we directly obtain that the pentadiagonal linear system has a unique solution.

In addition, by Gershgorin's theorem, $\lambda_k(\mathbf{A}) \leq \max_i[16+4\omega_i]$. Then we obtain the following upper bound for the condition number of $\mathbf{A}$ which is independent of the number of data points, $N+2$, and which recovers the result presented in [34] for the limit case $p_i = 0, \quad i = 0, \ldots, N$,

$$\|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty \leq \frac{\max_i \left[16 + 4(p_i/n_i)^2\right]}{\min_i \left[4(1-\cos(\pi/n_i))^2 + 2(p_i/n_i)^2(1-\cos(\pi/n_i))\right]}$$
$$\simeq \frac{\max_i \left[16 + 4(p_i/n_i)^2\right]}{\min_i (1/n_i)^4 [\pi^4 + (\pi p_i)^2]}. \tag{2.48}$$

Summarizing, in the particular but important uniform case we can compute the mesh solution by solving a symmetric, pentadiagonal, positive definite system and therefore, we can use specialized algorithms, with a computational cost of $17q$ arithmetic operations, namely, $7q$ additions, $7q$ multiplications, and $3q$ divisions [60], where q is the number of unknowns.

Moreover, since the upper bound (2.48) for the condition number of the matrix $\mathbf{A}$ does not depend on the number of interpolation points, such methods can be used with some confidence.

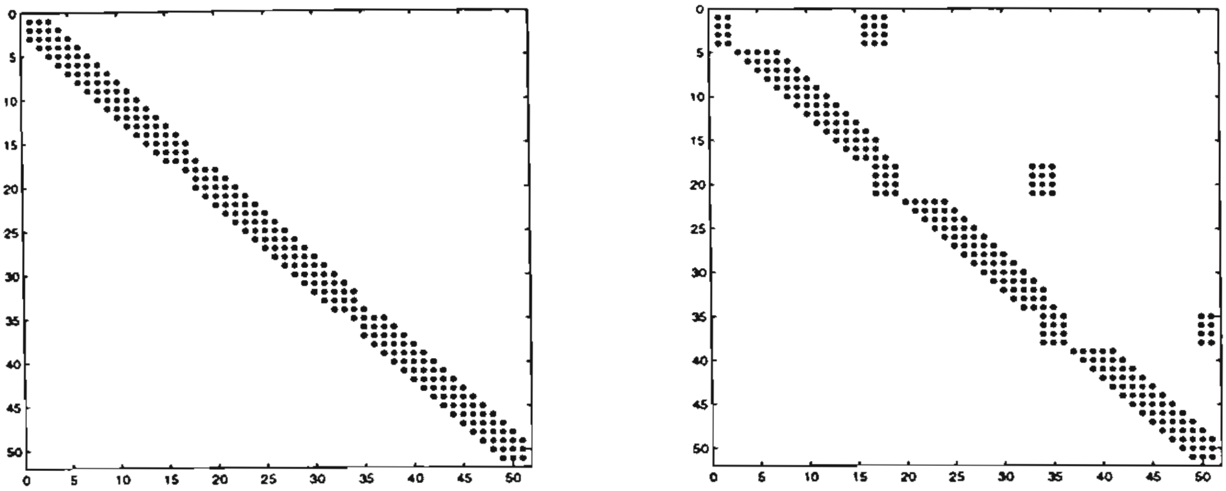
In the general case of a non–uniform mesh, the matrix $\mathbf{A}$ is no longer symmetric, and an analysis of its condition number cannot be carried out analytically. However, several numerical experiments have shown that the condition number is not influenced by the non–symmetric structure, but does depend on the maximum number of grid points in each subinterval, exactly as in the symmetric case. In other words, symmetric and nonsymmetric matrices, with the same dimension and produced by difference equations with the same largest n_i, produce very close condition numbers. Non–uniform discrete hyperbolic tension splines have in fact been used for the graphical tests of the following section.

2.6.3 System Splitting

Sometimes the number of unknowns in (2.47) can be very large (for example for generating a grid in bivariate interpolation) and then even the linear computational cost of the solution of the pentadiagonal system may turn out to be too expensive. However, as for the two first approaches proposed at the beginning of this section for evaluating the mesh solution, if we have a parallel machine we can easily share the computation of the solution of our pentadiagonal system among the processors as outlined below.

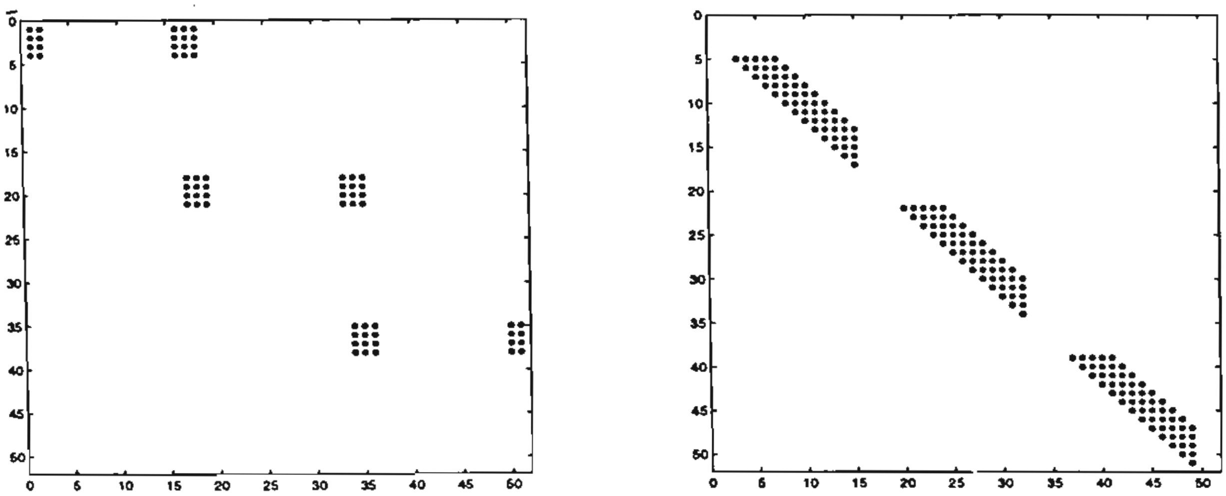
The basic idea is to transform $\mathbf{A}$, which, for $N=2$, $n_i = 18$ has the form shown in Figure 2.4(a), into the form $\mathbf{K}$ (see Figure 2.4(b)). Setting $r_i = \sum_{\nu=0}^{i-1}(n_\nu - 1)$, we note that the rows $r_i + 1, \ldots, r_i + n_i - 1$ of $\mathbf{A}$ describe equations (2.6) for the subinterval $[x_i, x_{i+1}]$. If we extract from $\mathbf{K}$ the rows $r_i+1, \ldots, r_i+4$, $i = 0, \ldots, N$, we get a block matrix $\mathbf{E}$ of the form shown in Figure 2.5(a). The corresponding linear system has few equations, and having solved it, it is possible to solve in parallel the $N+1$ linear systems obtained from the "remaining" matrix $\mathbf{F}$ of Figure 2.5(b) by extracting its independent blocks.

The problem now is how to move from $\mathbf{A}$ to $\mathbf{K}$. From Sections 2.2 and 2.3 we have the following two facts. Having in mind the structure of $\mathbf{A}$ and the corresponding Figure 2.4, let us consider the section given by rows $r_i + 1, \ldots, r_{i+1}$. We



(a) (b)

Figure 2.4. (a) The form of A for $N = 2$, $n_i = 18$. (b) The matrix K.



(a) (b)

Figure 2.5. (a) The block matrix E. (b) The block matrix F.

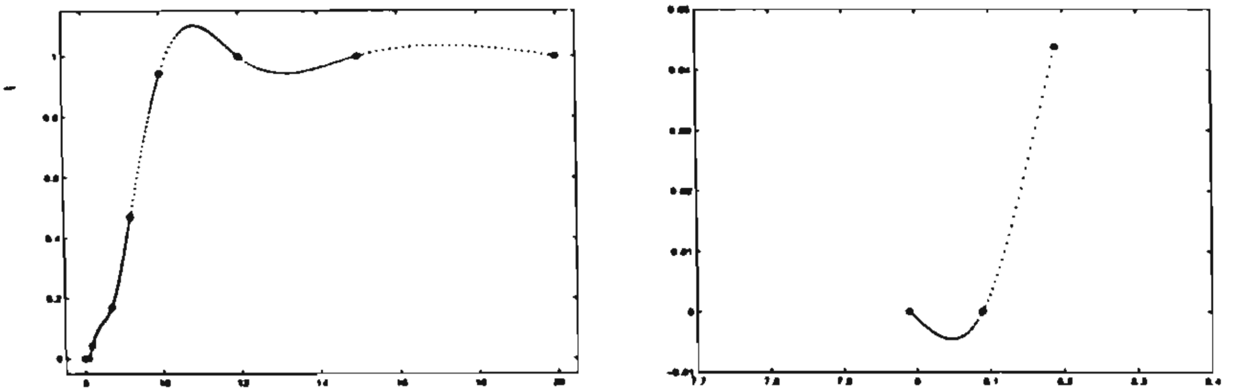
note that the entries of the columns with index $r_i + 3, \ldots, r_{i+1} - 2$ are $1, a_i, b_i, a_i, 1$ which are the coefficients of the difference equation (2.6). On the other hand, it is shown in Section 2.3 that any function of the form

$$\Upsilon_i(x) = c_1(1-t) + c_2 t + c_3\varphi_i(1-t) + c_4\varphi_i(t) \ , \tag{2.49}$$

is a solution for (2.6); therefore if we multiply the row of index $r_i + j$, $j = 1, \ldots, n_i - 1$, by $\Upsilon_i(x_{i,j}) = \Upsilon_i(x_i + j\tau_i)$ and then add all these rows, then the contribution of all the columns from r_i+3 to $r_{i+1}-2$ sums up to zero. The idea for obtaining the matrix $\mathbf{K}$ from $\mathbf{A}$ is the following: we replace the four rows of index $r_i+1, r_i+2, r_i+3, r_i+4$ with the sum of the rows from $r_i + 1$ to r_{i+1} multiplied by the values assumed in x_{ij} by four linearly independent functions of the form (2.49). The remaining question is how to choose these functions. Several numerical experiments have shown that the lowest condition number of the matrix $\mathbf{K}$ (which is in general larger than that of $\mathbf{A}$) is achieved when we use the cardinal functions for Lagrange interpolation at the points $x_{i\nu}$ closest to $x_i, x_i + h_i/3, x_{i+1} - h_i/3, x_{i+1}$.

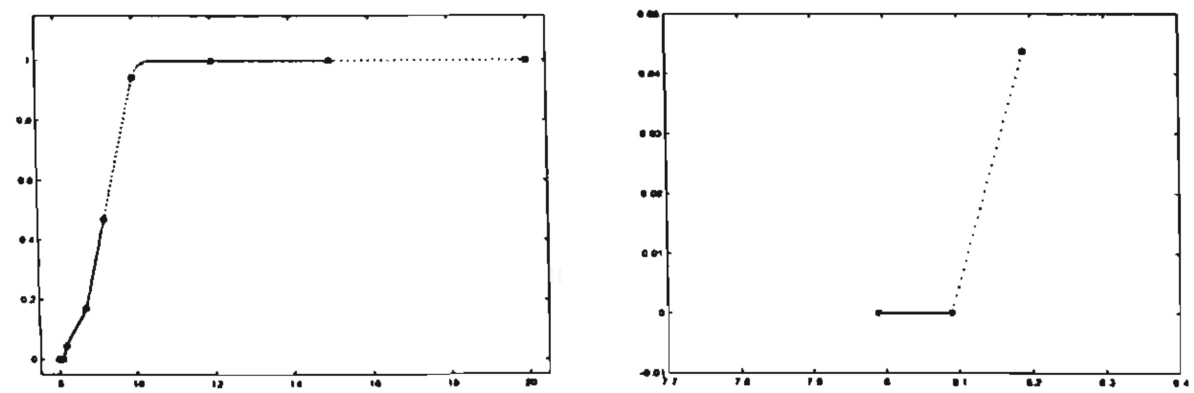
2.7 Graphical Examples

The aim of this final section is to illustrate the tension features of discrete hyperbolic tension splines with some (famous) examples. Before, we want to notice that the continuous form U_i of our solution given in (2.13) has the good shape-preserving properties of cubics (see e.g. [48]) in the sense that U_i is convex (concave) in $[x_i, x_{i+1}]$ if and only if $M_{i+j} \geq 0$ (≤ 0), $j = 0, 1$, and has at most one inflection point in $[x_i, x_{i+1}]$. In order to preserve the shape of the data, we therefore simply have to analyze the values $\Lambda_i u_{i,0}$ and $\Lambda_i u_{i,n_i}$ and increase the tension parameters if necessary. All the strategies proposed for the automatic choice of tension parameters in continuous hyperbolic tension spline interpolation can be used in our discrete context, see e.g. [30].



(a) (b)

Figure 2.6. (a) The radio chemical data with natural end conditions $M_0 = M_{N+1} = 0$. Interpolation by discrete cubic spline ($p_i = 0$). (b) A magnification of the lower left corner.



(a) (b)

Figure 2.7. The same as Figure 2.6 with $p_0 = p_1 = 300$, $p_i = 15, i = 2, \ldots, 7$.

In our first example we have interpolated the *radio chemical* data reported in Table 2.1. The effects of changing the tension values p_i are depicted in Figures 2.6–2.7. We have adopted a non-uniform mesh, assigning the same number of points

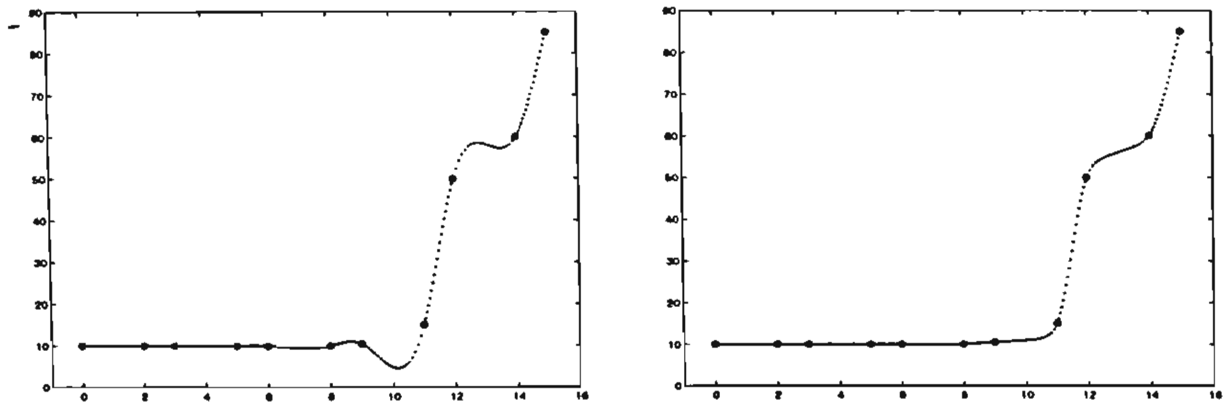
(30) to each interval of the main mesh, and imposed *natural* end conditions, that is, following formulae (2.15), $M_0 = M_{N+1} = 0$.

Table 2.1. Radio chemical data:

x_i	7.99	8.09	8.19	8.7	9.2
f_i	0	2.76429E-5	4.37498E-2	0.169183	0.469428

x_i	10	12	15	20
f_i	0.943740	0.998636	0.999916	0.999994

Figure 2.6 is obtained setting $p_i = 0$, that is considering the discrete cubic spline interpolating the data. In Figure 2.7 a new discrete interpolant with $p_0 = p_1 = 300$, $p_i = 15$, $i = 2, \ldots, 7$, is displayed for the same data, and the stretching effect of the increase in tension parameters is evident.



(a) (b)

Figure 2.8. Akima's data with natural end conditions. (a) Discrete interpolating cubic spline ($p_i = 0$). (b) Discrete hyperbolic spline with $p_5 = p_6 = p_8 = 10$.

In the second example we have taken *Akima's* data of Table 2.2 and constructed discrete interpolants with 20 points for each interval, with natural end conditions $M_0 = M_{N+1} = 0$.

Table 2.2. Akima's data [1]:

x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	50	60	85

Figure 2.8(a) shows the plot produced by a uniform choice of tension factors, namely $p_i = 0$. Figure 2.8(b) shows a second mesh solution, which perfectly reproduces the data shape, where we have set $p_5 = p_6 = p_8 = 10$ while the remaining p_i are unchanged.

Critique

In this research report we considered 1-D problem of shape-preserving interpolation only. However in many practical application it is important to constract a surface defined on a two-dimensional domain which has certain shape properties. For example, we may want the surface to be positive, monotone, or convex in some sense. The results obtained in this research report can be also used for solving such problem. By this reason 2-D problem of shape-preserving interpolation is formulated here as a Differential Multipoint Boundary Value Problem (DMBVP for short) for thin plate tension spline. For a numerical treatment of this problem, one can consider its finite-difference approximation. This gives a system of linear algebraic equations which can be solved either by direct and iterative methods. As a direct method, we suggest to consider a block Gaussian elimination. For iterative solution of the obtained linear system one can apply Successive Over-Relaxation (SOR) method. Finite-difference schemes in fractional steps [45] should also prove their efficiency in the numerical treatment of this DMBVP.

Let a rectangular domain $\overline{\Omega} = \Omega \cup \Gamma$ be given where

$$\Omega = \{(x, y) \mid a < x < b, \; c < y < d\}$$

and Γ is the boundary of Ω. We consider on $\overline{\Omega}$ a regular mesh $\Delta = \Delta_x \times \Delta_y$ with

$$\Delta_x : a = x_0 < x_1 < \cdots < x_{N+1} = b,$$
$$\Delta_y : c = y_0 < y_1 < \cdots < y_{M+1} = d,$$

which divides the domain $\overline{\Omega}$ into the rectangles $\overline{\Omega}_{ij} = \Omega_{ij} \bigcup \Gamma_{ij}$ where

$$\Omega_{ij} = \{(x, y) \mid x \in (x_i, x_{i+1}), \; y \in (y_j, y_{j+1})\}$$

and Γ_{ij} is the boundary of Ω_{ij}, $i = 0, \ldots, N$, $j = 0, \ldots, M$.

Let us associate to the mesh Δ the data

$$\begin{array}{lll} (x_i, y_j, f_{ij}), & i = 0, \ldots, N+1, & j = 0, \ldots, M+1, \\ f_{ij}^{(2,0)}, & i = 0, N+1, & j = 0, \ldots, M+1, \\ f_{ij}^{(0,2)}, & i = 0, \ldots, N+1, & j = 0, M+1, \\ f_{ij}^{(2,2)}, & i = 0, N+1, & j = 0, M+1 \end{array}$$

where

$$f_{ij}^{(r,s)} = \frac{\partial^{r+s} f(x_i, y_j)}{\partial x^r \partial y^s}.$$

It is convenient for us to collect this data in the following table.

$f_{0,M+1}^{(2,2)}$	$f_{0,M+1}^{(0,2)}$	$f_{1,M+1}^{(0,2)}$	$\cdots$	$f_{N,M+1}^{(0,2)}$	$f_{N+1,M+1}^{(0,2)}$	$f_{N+1,M+1}^{(2,2)}$
$f_{0,M+1}^{(2,0)}$	$f_{0,M+1}$	$f_{1,M+1}$	$\cdots$	$f_{N,M+1}$	$f_{N+1,M+1}$	$f_{N+1,M+1}^{(2,0)}$
$f_{0,M}^{(2,0)}$	$f_{0,M}$	$f_{1,M}$	$\cdots$	$f_{N,M}$	$f_{N+1,M}$	$f_{N+1,M}^{(2,0)}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$f_{0,1}^{(2,0)}$	$f_{0,1}$	$f_{1,1}$	$\cdots$	$f_{N,1}$	$f_{N+1,1}$	$f_{N+1,1}^{(2,0)}$
$f_{0,0}^{(2,2)}$	$f_{0,0}^{(0,2)}$	$f_{1,0}^{(0,2)}$	$\cdots$	$f_{N,0}^{(0,2)}$	$f_{N+1,0}^{(0,2)}$	$f_{N+1,0}^{(2,2)}$

We introduce the following notations for divided differences

$$\begin{aligned}
f[x_i, y_j] &= f(x_i, y_j) = f_{ij}, \\
f[x_i, \ldots, x_{i+k}, y_j] &= \frac{f[x_{i+1}, \ldots, x_{i+k}, y_j] - f[x_i, \ldots, x_{i+k-1}, y_j]}{x_{i+k} - x_i}, \\
&k = 1, \ldots, N+1,\ i = 0, \ldots, N+1-k,\ j = 0, \ldots, M+1, \\
f[x_i; y_j \ldots, y_{j+l}] &= \frac{f[x_i; y_{j+1}, \ldots, y_{j+l}] - f[x_i; y_j, \ldots, y_{j+l-1}]}{y_{j+l} - y_j}, \\
&l = 1, \ldots, M+1,\ i = 0, \ldots, N+1,\ j = 0, \ldots, M+1-l.
\end{aligned}$$

In particular, one has for the first order divided differences

$$\begin{aligned}
f[x_i, x_{i+1}; y_j] &= (f_{i+1,j} - f_{ij})/h_i, \quad h_i = x_{i+1} - x_i, \\
&\qquad i = 0, \ldots, N,\ j = 0, \ldots, M+1, \\
f[x_i; y_j, y_{j+1}] &= (f_{i,j+1} - f_{ij})/l_j, \quad l_j = y_{j+1} - y_j, \\
&\qquad i = 0, \ldots, N+1,\ j = 0, \ldots, M.
\end{aligned}$$

Definition 1. *The data f_{ij}, $i = 0, \ldots, N+1$, $j = 0, \ldots, M+1$ is said to be* **positive (negative)** *if*

$$f_{ij} > 0 \quad (< 0) \quad \text{for all} \quad i \text{ and } j;$$

monotonically increasing (decreasing) *by x if*

$$f[x_i, x_{i+1}; y_j] > 0 \quad (< 0), \quad i = 0, \ldots, N,\ j = 0, \ldots, M+1;$$

monotonically increasing (decreasing) *by y if*

$$f[x_i; y_j, y_{j+1}] > 0 \quad (< 0), \quad i = 0, \ldots, N+1,\ j = 0, \ldots, M;$$

convex (concave) *by x if*

$$f[x_i, x_{i+1}; y_j] - f[x_{i-1}, x_i; y_j] > 0 \quad (< 0), \quad i = 1, \ldots, N, \ j = 0, \ldots, M+1;$$

convex (concave) *by y if*

$$f[x_i; y_j; y_{j+1}] - f[x_i; y_{j-1}, y_j] > 0 \quad (< 0), \quad i = 0, \ldots, N+1, \ j = 1, \ldots, M.$$

We denote by $C^{2,2}[\overline{\Omega}]$ the set of all continuous on $\overline{\Omega}$ functions f having continuous partial and mixed derivatives up to the order 2. We say that the problem of searching for a function $S \in C^{2,2}[\overline{\Omega}]$ such that $S(x_i, y_j) = f_{ij}$, $i = 0, \ldots, N+1$, $j = 0, \ldots, M+1$, and S preserves the form of the initial data is *the 2-D shape-preserving interpolation problem.* This means that where the data increases (decreases) monotonically, S has the same behaviour, and S is convex (concave) over intervals where the data is convex (concave).

Evidently, the solution of 2-D shape-preserving interpolation problem is not unique. We are looking for a solution of this problem as a thin plate tension spline.

Definition 2. *An interpolating thin plate tension spline S with two sets of tension parameters $\{p_i \geq 0 \mid i = 0, \ldots, N\}$ and $\{q_j \geq 0 \mid j = 0, \ldots, M\}$ is a solution of the DMBVP*

$$\frac{\partial^4 S}{\partial x^4} + 2\frac{\partial^4 S}{\partial x^2 \partial y^2} + \frac{\partial^4 S}{\partial y^4} - \left(\frac{p_i}{h_i}\right)^2 \frac{\partial^2 S}{\partial x^2} - \left(\frac{q_j}{l_j}\right)^2 \frac{\partial^2 S}{\partial y^2} = 0, \quad \text{in each} \quad \Omega_{ij},$$
$$i = 0, \ldots, N; \quad j = 0, \ldots, M,$$

$$\frac{\partial^4 S}{\partial x^4} - \left(\frac{p_i}{h_i}\right)^2 \frac{\partial^2 S}{\partial x^2} = 0, \quad x \in (x_i, x_{i+1}), \quad y = y_j,$$
$$i = 0, \ldots, N; \quad j = 0, \ldots, M+1,$$

$$\frac{\partial^4 S}{\partial y^4} - \left(\frac{q_j}{l_j}\right)^2 \frac{\partial^2 S}{\partial y^2} = 0, \quad x = x_i, \quad y \in (y_j, y_{j+1}),$$
$$i = 0, \ldots, N+1; \quad j = 0, \ldots, M,$$

$$S \in C^{2,2}[\overline{\Omega}],$$

with the interpolation conditions

$$S(x_i, y_j) = f_{ij}, \quad i = 0, \ldots, N+1; \ j = 0, \ldots, M+1,$$

and the boundary conditions

$$D^{(2,0)} S\,(x_i, y_j) = f_{ij}^{(2,0)}, \quad i = 0, N+1; \quad j = 0, ..., M+1,$$
$$D^{(0,2)} S\,(x_i, y_j) = f_{ij}^{(0,2)}, \quad i = 0, ..., N+1; \quad j = 0, M+1,$$
$$D^{(2,2)} S\,(x_i, y_j) = f_{ij}^{(2,2)}, \quad i = 0, N+1; \quad j = 0, M+1,$$

where

$$D^{(r,s)} S(x, y) = \frac{\partial^{r+s} S(x, y)}{\partial x^r \partial y^s}.$$

If all tension parameters of the thin plate tension spline S are zero then one obtains a smooth thin plate spline [14] interpolating the data (x_i, y_j, f_{ij}), $i = 0, \ldots, N+1$, $j = 0, \ldots, M+1$. If tension parameters p_i and q_j approach the infinity then in the rectangle $\overline{\Omega}_{ij}$, $i = 0, \ldots, N$, $j = 0, \ldots, M$, thin plate spline S turns into a linear function separately by x and y, and obviously preserves on $\overline{\Omega}_{ij}$ shape properties of the data. So, by changing values of the shape control parameters p_i and q_j one can preserve various characteristics of the data including positivity, monotonicity, convexity, as well as linear and planar sections. By increasing one or more of these parameters the surface is pulled towards an inherent shape at the same time keeping its smoothness. Thus, DMBVP gives a reasonable mathematical formulation of 2-D shape-preserving interpolation problem. This problem can be investigated in the next research project with TRF.

The problem of convergence and the orders of approximation for shape preserving splines were considered in detail only for a discrete hyperbolic tension spline in chapter 2. However, it is obvious how to approach this problem in general. We have to prove that under a refinement of the mesh, the norm of the highest derivative of our shape preserving spline with best chosen shape control parameters remains bounded, or that at least the order of its growth is known. This question is still not solved in general. It can be considered in a separate research project with TRF.

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Applications

A large variety of applications now requires the use of curve/surface description, especially in fields such as computer aided design and machining, and computer vision and inspection of manufactured parts. The design of curves and surfaces plays an important role not only in the construction of different products such as car bodies, ship hulls, airplane fuselages and wings, propellers blades, etc., but also in the description of geological, physical and even medical phenomena. Other areas where the description of curves/surfaces is of interest include many fields of science, medical research (software for digital diagnostic equipment), image analysis, high resolution TV systems, cartography, the film industry, etc. This diversity and the wide range of applications of the subject enables us to consider the problem of constructing shape preserving curve/surface interpolation splines as very valuable.

The PI offers formal graduate level courses in Computer Aided Geometric Design and Geometric Modeling, and is available for training of Thai researchers, using methodology and results of this project, and thereby effecting advanced technology transfer of this very important tool to Thai scientists.

Appendix

Example: Reconstruction of a Ship Surface

The methods of shape-preserving spline approximation of curves and surfaces described in this research report were used in a package of computer programs which enables one to construct complex multivalued surfaces.

As a numerical test of this package, we tried to reconstruct the surface of a "Viking boat". The initial data, which the author obtained from Professor Lyche of Oslo University, was defined pointwise in the form of the envelopes of the sides and the keel of the boat, as well as six ribs. Three-dimensional view of the data is given in Figure A.1. Figures A.2 – A.4 show the main projections of the data. After partial selection of the data, a system of non-intersecting, generally speaking curvilinear, pointwise assigned loft sections was constructed from this data. Each section, except the sections for ribs, contained 4 points.

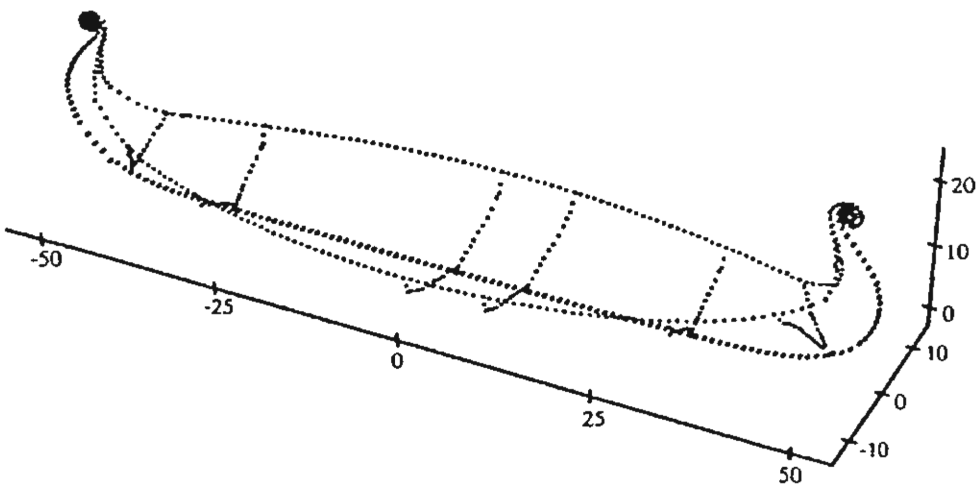

Figure A.1. Three-dimensional view of the data.

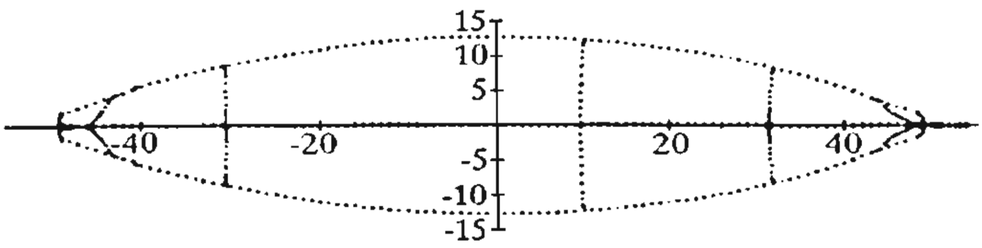

Figure A.2. Projection of the data onto the xy-plane.

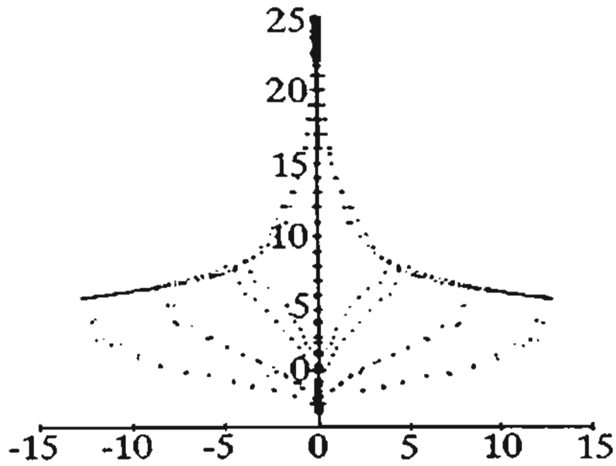

Figure A.3. Projection of the data onto the xz-plane.

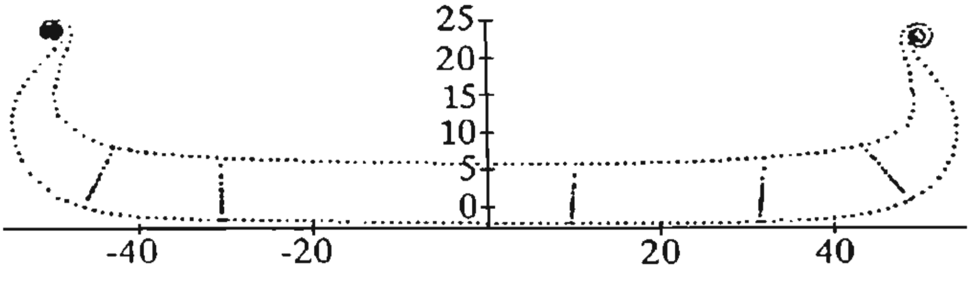

Figure A.4. Projection of the data onto the yz-plane.

First, using the shape-preserving interpolation algorithm of Chapter 2, we construct a system of space curves along the selected sections. A twodimensional spline is defined as the tensor product of one-dimensional splines, generating a family of local approximation generalized splines in the orthogonal direction by

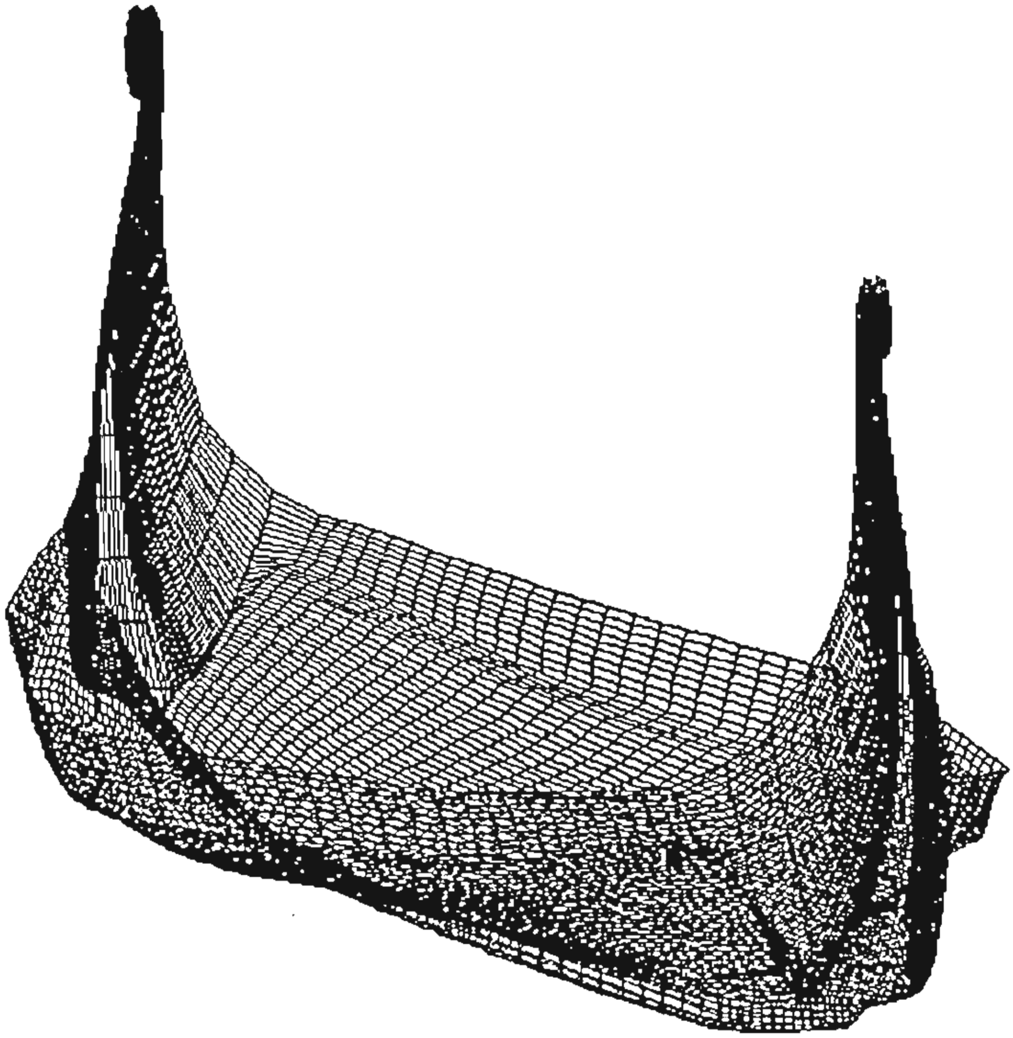

Figure A.5. Resulting shape-preserving surface. First projection.

algorithm of Chapter 1. This yields a finite system of curvilinear coordinate lines on the surface which form a regular grid. Properties of the initial data such as convexity, monotonicity, the presence of linear and plane segments, angles and non-smoothness are preserved along those lines.

The Euler coordinates of the multivalued shape-preserving surface were computed by the standard parametrization

$$x = S^x(w,u), \quad y = S^y(w,u), \quad z = S^z(w,u), \quad 0 \leq w, u \leq 1.$$

In Figures A.5 and A.6 the resulting shape-preserving surface is given in two different projections with a mesh of lines 100×100.

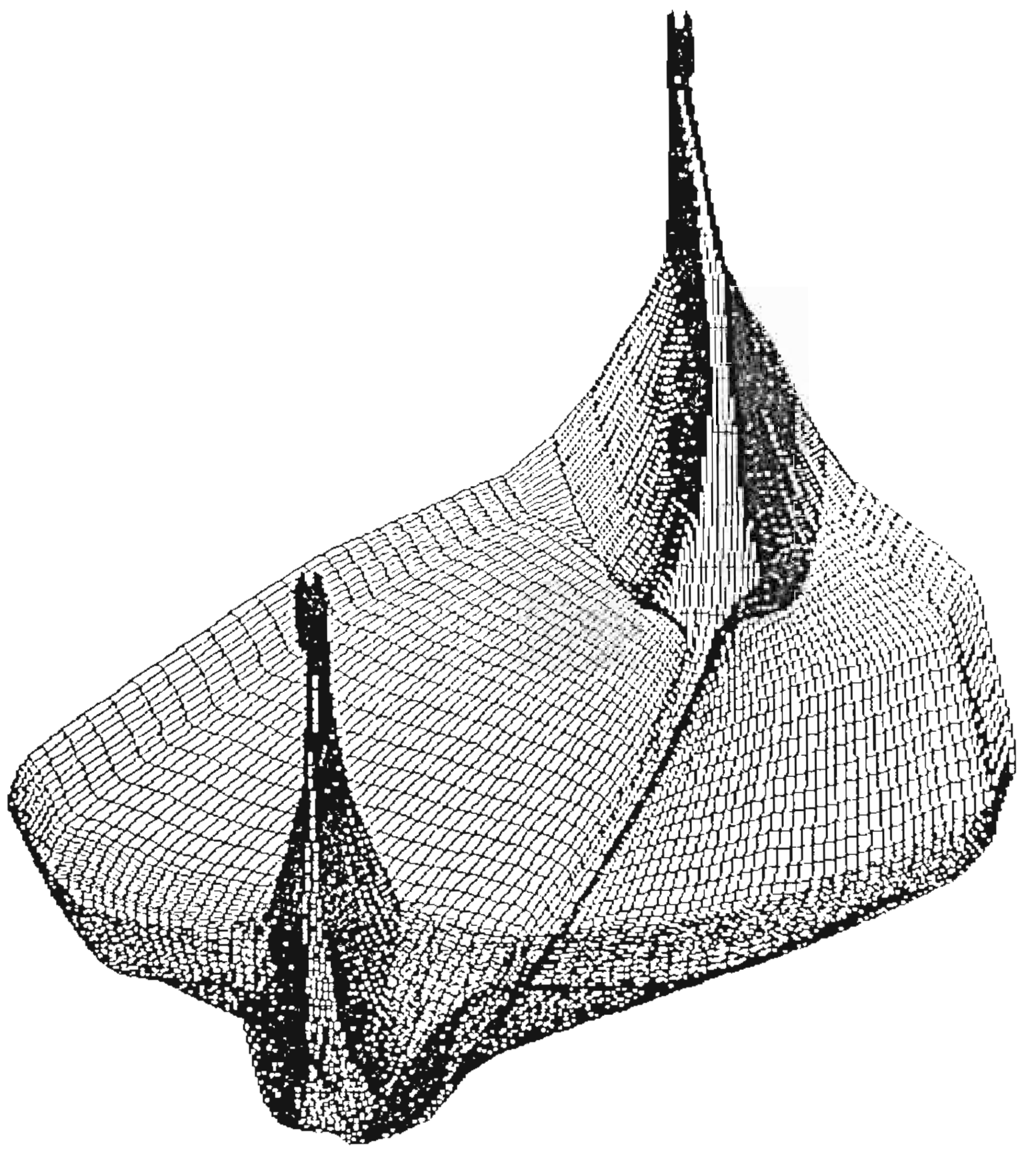

Figure A.6. Resulting shape-preserving surface. Second projection.

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D.Sc., Habilitation Degree in Computational Mathematics (CM for short), Russian Academy of Sciences (RAS for short), Moscow, 1997
Senior Researcher Degree in CM, RAS, Moscow, 1986
Associate Professor Degree in CM, Ministry of Higher Education, Moscow, 1986
Ph.D. in CM, Institut of Mathematics RAS, Novosibirsk, 1973
M.Sc. in CM, Novosibirsk State University (NSU for short), Novosibirsk, 1969

POSITIONS:

1995/to date	Suranaree University of Technology, Associate Professor of Mathematics
1991–1995	Institute of Computational Technology RAS, Senior Researcher, Leading Researcher
1978–1991	Institute of Theoretical and Applied Mechanics RAS, Senior Researcher
1976–1978	Institute of Mathematics Belorus Acad. of Sci., Gomel, Head of Laboratary
1974–1976	Institute of Mathematics RAS, Novosibirsk, Assistant Researcher
1975 (2 mo.)	Stefan Banach Mathematical Center, Warsaw, Poland, Visiting Research Fellow

ACADEMIC EXPERIENCE:

1996 (1 mo.)	University degli Studi di Firenze, Italy, Visiting Professor of Mathematics
1985–1995	NSU, Associate Professor of CM
1978–1985	NSU, Assistant Professor of CM
1983 (2 mo.)	Grodno State University, Visiting Assistant Professor of CM
1976–1978	Gomel State University, Senior Lecturer
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1972–1974	Krasnoyarsk State University, Lecturer

Graduate/Postgraduate students: I supervised 15 M.Sc. and 2 Ph.D. students in CM.

PRINCIPAL RESEARCH INTERESTS:

Numerical Analysis, Mathematical Methods in CAGD, Approximation Theory, Spline Based Curve and Surface Approximation, Scientific Visualization

CURRENT RESEARCH INTERESTS:

Shape Preserving Approximation, Subdivisions, Difference Methods for Constructing Splines, Tension and Discrete Splines and GB-splines, Curve and Surface Parametrization

RESEARCH GRANTS:

Difference Method for Constructing Shape Preserving Splines / Principal Investigator, The Thailand Research Fund, Thailand, November 1, 1999 to October 31, 2001 (code BRG/08/2543)

Shape Preserving Parametrization for Spline Interpolation / Principal Investigator, Suranaree University of Technology, Thailand, October 1, 1999 to April 30, 2001

Discrete B-Spline Approximation through Lagrange-Newton Polynomials / Principal Investigator, Suranaree University of Technology, Thailand, October 1, 1998 to December 31, 1999

Algorithms of Shape Preserving Spline Approximation / Coinvestigator, The Thailand Research Fund, Thailand, July 1, 1997 to June 30, 1999 (code BRG/16/2540)

Difference Method for Constructing Tension Splines / Principal investigator, MURST, Università Degli Studi di Firenze, Italy, 20 November to 20 December 1996

Geometric Splines for Curves and Surfaces Design / Principal Investigator, State Committee on Higher Education, Saint-Petersburg Technical University, January 1994 to December 1995 (code PG-13)

Shape Preserving Approximation for Curves and Surfaces / Principal Investigator, The Russian Foundation for Basic Research, RAS, Moscow, January 1993 to December 1995 (code 93–012–495)

PROFESSIONAL ACTIVITIES:

Thailand Research Fund's Royal Golden Jubilee Ph.D. Grantee, 1998, 2000

Organizer, Third All-union Conference on Approximation Theory and Problems of CM, Novosibirsk, January 28 - February 1, 1991

Guest Editor, *Aproximation Theory and Problems of CM*, Modelirovanie v Mekh., **5(22)** (1991) No. 5

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Reviewer for Mathematical Reviews (in the field of Numerical Analysis, 1975-1985)

CONFERENCE PRESENTATIONS: (recent years)

Difference Method for Constructing Shape-Preserving Splines, Fifth Annual National Symposium on Computational Science and Engineering, Bangkok Convention Center, June 19–20, 2001

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Approximation by Lagrange splines, Fourth Annual National Symposium on Computational Science and Engineering, Kasetsart University, Bangkok, March 27–29, 2000

On generalized discrete tension splines, Computational Techniques and Applications Conference and Workshops, Canberra, Australian National University, September 20–24, 1999

A general approach to discrete splines, Third Annual National Symposium on Computational Science and Engineering, Chulalongkorn University, Bangkok, March 24–26, 1999

INVITED LECTURES:

- National University of Singapore, Singapore, 1998
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- Università di Siena, Italy, 1996
- Institutt for Informatikk, Universitetet i Oslo, Norway, 1991

LANGUAGE PROFICIENCY:

Fluent in English and Russian. Read, understand and can translate from German and French into English and Russian. I translated from English and French into Russian four books on splines and the finite element method: (Publ. House "Mir", Moscow)

1. Laurent, P. J., *Approximation et Optimisation*, 1975, 496 pp.
2. Descloux, J., *Methode des Elements Finis*, 1976, 96 pp.
3. Ciarlet, Ph., *The Finite Element Method for Elliptic Problems*, 1980, 512 pp.
4. Zienkiewicz, O. C. and K. Morgan, *Finite Elements and Approximation*, 1986. 320 pp.

SCIENTIFIC PUBLICATIONS: Two books and more than 70 research publications.