สำนักงานกองทุนสนับสนุนการวิจัย

THE   THAILAND   RESEARCH   FUND

รายงานวิจัยฉบับสมบูรณ์

Difference Method for Constructing

Shape Preserving Splines

โดย Assoc. Prof. Dr. Boris I. Kvasov

ตุลาคม 2544

รายงานวิจัยฉบับสมบูรณ์

Difference Method for Constructing

Shape Preserving Splines

โดย Assoc. Prof. Dr. Boris I. Kvasov

รายงานวิจัยฉบับสมบูรณ์

Difference Method for Constructing

Shape Preserving Splines

Assoc. Prof. Dr. Boris I. Kvasov   มหาวิทยาลัยเทคโนโลยีสุรนารี

List

of research papers published, in press and submitted for publication during the life time of the research project BRG/08/2543 "Difference Method for Constructing Shape Preserving Splines"

1. B. I. Kvasov, On discrete GB-splines, *Australian and New Zealand Industrial and Applied Mathematics Journal* 42(E), pp. C877–C899, 2000.
2. B. I. Kvasov and A. Luadsong, Difference method for solving shape-preserving interpolation problem, in: *Proceedings of the Fifth Annual Symposium on Computational Science and Engineering*, Bangkok, June 18–20, 2001, pp. 390–397.
3. B. I. Kvasov, Approximation by discrete GB-splines, to appear in *Int. J. Numerical Algorithms*, Kluwer Academic Publishers, 1–20, 2001.
4. B. I. Kvasov, Discrete GB-splines and their properties, submitted to *Proceedings of the Tenth International Conference on Approximation Theory*, Saint Louis, Missouri, 26–29 March, 2001, 10 pages.
5. B. I. Kvasov, Difference method for constructing hyperbolic tension splines, submitted to Int. J. Advances in Computational Mathematics, 23 pp.

On discrete GB-splines

Boris I. Kvasov*

(Received 7 August 2000)

Abstract

Explicit formulae and recurrence relations are obtained for discrete generalized B-splines (discrete GB-splines for short). Properties of discrete GB-splines and their series are studied. It is shown that the series of discrete GB-splines is a variation diminishing function and the systems of discrete GB-splines are weak Chebyshev systems.

* School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima 30000, THAILAND. mailto:boris@math.sut.ac.th

Contents	C878

Contents

1 Introduction

The tools of generalized splines and GB-splines are widely used in solving problems of shape-preserving approximation (e.g., see [7]). Recently, in [1] a difference method for constructing shape-preserving hyperbolic tension splines as solutions of multipoint boundary value problems was developed. Such an approach permits us to avoid the computation of hyperbolic func-

tions and has substantial other advantages. However, the extension of a mesh solution will be a discrete hyperbolic tension spline.

The contents of this paper is as follows. In Section 2 we give a definition of a discrete generalized spline. Next, we construct a minimum length local support basis (whose elements are denoted as discrete GB-splines) of the new spline; see Section 3. Properties of GB-splines are discussed in Section 4, while the local approximation by discrete GB-splines of a given continuous function from its samples is considered in Section 5. In Section 6 we derive recurrence formulae for calculations with discrete GB-splines. The properties of GB-spline series are summarized in Section 7.

2   Discrete generalized splines

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ be given. We will denote by S_4^{DG} the space of continuous functions whose restriction to a subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, N - 1$ is spanned by the system of four linearly independent functions $\{1, x, \Phi_i, \Psi_i\}$. In addition, we assume that each function in S_4^{DG} is smooth in the sense that for given $\tau_i^{L_j} > 0$ and $\tau_i^{R_j} > 0$, $j = i - 1, i$, the values of its first and second central divided differences with respect to the points $x_i - \tau_i^{L_{i-1}}$, x_i, $x_i + \tau_i^{R_{i-1}}$ and $x_i - \tau_i^{L_i}$, x_i, $x_i + \tau_i^{R_i}$ coincide.

Given a continuous function S we introduce the difference operators

$$D_1 S(x) \equiv D_{i,1} S(x) = (\lambda_i^{R_i} S[x - \tau_i^{L_i}, x] + \lambda_i^{L_i} S[x, x + \tau_i^{R_i}])(1 - t)$$
$$+ (\lambda_{i+1}^{R_i} S[x - \tau_{i+1}^{L_i}, x] + \lambda_{i+1}^{L_i} S[x, x + \tau_{i+1}^{R_i}])t,$$
$$D_2 S(x) \equiv D_{i,2} S(x) = 2 S[x - \tau_i^{L_i}, x, x + \tau_i^{R_i}](1 - t)$$
$$+ 2 S[x - \tau_{i+1}^{L_i}, x, x + \tau_{i+1}^{R_i}]t,$$
$$x \in [x_i, x_{i+1}), \quad i = 0, \ldots, N - 1,$$

where $\lambda_j^{R_i} = 1 - \lambda_j^{L_i} = \tau_j^{R_i}/(\tau_j^{L_i} + \tau_j^{R_i})$, $j = i, i + 1$ and $t = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$. The square parentheses denote the usual first and second divided differences of the function S with respect to the argument values $x_j - \tau_j^{L_i}$, x_j, $x_j + \tau_j^{R_i}$, $j = i, i + 1$.

Definition 1 *A discrete generalized spline is a function $S \in S_4^{DG}$ such that*

   1. for any $x \in [x_i, x_{i+1}]$, $i = 0, \ldots, N - 1$

$$S(x) \equiv S_i(x) = [S(x_i) - \Phi_i(x_i) M_i](1 - t)$$
$$+ [S(x_{i+1}) - \Psi_i(x_{i+1}) M_{i+1}]t$$
$$+ \Phi_i(x) M_i + \Psi_i(x) M_{i+1}, \tag{1}$$

   where $M_j = D_{i,2} S_i(x_j)$, $j = i, i + 1$, and the functions Φ_i and Ψ_i are subject to the constraints

$$\Phi_i(x_{i+1} - \tau_{i+1}^{L_i}) = \Phi_i(x_{i+1}) = \Phi_i(x_{i+1} + \tau_{i+1}^{R_i}) = 0,$$
$$\Psi_i(x_i - \tau_i^{L_i}) = \Psi_i(x_i) = \Psi_i(x_i + \tau_i^{R_i}) = 0, \tag{2}$$
$$D_{i,2} \Phi_i(x_i) = 1, \quad D_{i,2} \Psi_i(x_{i+1}) = 1;$$

2. S satisfies the following smoothness conditions

$$
\begin{aligned}
S_{i-1}(x_i) &= S_i(x_i), \\
D_{i-1,1}S_{i-1}(x_i) &= D_{i,1}S_i(x_i), \quad i = 1, \ldots, N-1. \\
D_{i-1,2}S_{i-1}(x_i) &= D_{i,2}S_i(x_i),
\end{aligned}
\tag{3}
$$

This definition generalizes the notion of a discrete polynomial spline in [9] and of a generalized spline in [5, 6]. The latter one can be obtained by setting $\tau_j^{L_i} = \tau_j^{R_i} = 0$, $j = i, i+1$ for all i. If $\tau_i^{L_j} = \tau_i^{L}$ and $\tau_i^{R_j} = \tau_i^{R}$, $j = i-1, i$ then according to smoothness conditions (3) the values of the functions S_{i-1} and S_i at the three consecutive points $x_i - \tau_i^{L}$, x_i, $x_i + \tau_i^{R}$ coincide. Setting $\tau_j^{L_i} = \tau_j^{R_i} = \tau_i$, $j = i, i+1$ we obtain $D_{1,i}S(x) = S[x - \tau_i, x + \tau_i]$ and $D_{2,i}S(x) = S[x - \tau_i, x, x + \tau_i]$, which is the case discussed in [1].

The functions Φ_i and Ψ_i depend on the tension parameters which influence the behaviour of S fundamentally. We call them the *defining functions*. In practice one takes $\Phi_i(x) = \Phi_i(p_i, x)$, $\Psi_i(x) = \Psi_i(q_i, x)$, $0 \le p_i, q_i < \infty$. In the limiting case when $p_i, q_i \to \infty$ we require that $\lim_{p_i \to \infty} \Phi_i(p_i, x) = 0$, $x \in (x_i, x_{i+1}]$ and $\lim_{q_i \to \infty} \Psi_i(q_i, x) = 0$, $x \in [x_i, x_{i+1})$ so that the function S in formula (1) turns into a linear function. Additionally, we require that if $p_i = q_i = 0$ for all i, then we get a discrete cubic spline. If $\tau_i^{L_j} = \tau_i^{R_j} = \tau_i$, $j = i-1, i$ for all i then this spline coincides with a discrete cubic spline of [10]. The case $\tau_i = \tau$ for all i was considered in [8].

3   Construction of discrete GB-splines

Let us construct a basis for the space of discrete generalized splines S_4^{DG} by using functions which have local supports of minimum length. Since $\dim(S_4^{DG}) = 4N - 3(N-1) = N + 3$ we extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$, such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$.

We demand that the discrete GB-splines B_i, $i = -3, \ldots, N-1$ have the properties

$$
\begin{aligned}
B_i(x) &> 0, \quad x \in (x_i + \tau_i^{R_i}, x_{i+4} - \tau_{i+4}^{L_{i+3}}), \\
B_i(x) &\equiv 0, \quad x \notin (x_i, x_{i+4}),
\end{aligned}
\tag{4}
$$

$$
\sum_{j=-3}^{N-1} B_j(x) \equiv 1, \quad x \in [a, b].
\tag{5}
$$

According to (1), on the interval $[x_j, x_{j+1}]$, $j = i, \ldots, i+3$, the discrete GB-spline B_i has the form

$$
B_i(x) \equiv B_{i,j}(x) = P_{i,j}(x) + \Phi_j(x)M_{j,B_i} + \Psi_j(x)M_{j+1,B_i},
\tag{6}
$$

where $P_{i,j}$ is a polynomial of the first degree and $M_{l,B_i} = D_{j,2}B_i(x_l)$, $l = j, j+1$ are constants to be determined. The smoothness conditions (3) together with the constraints (2) give the following relations

$$
\begin{aligned}
P_{i,j}(x_j) &= P_{i,j-1}(x_j) + z_j M_{j,B_i}, \\
D_{j,1}P_{i,j}(x_j) &= D_{j-1,1}P_{i,j-1}(x_j) + c_{j-1,2}M_{j,B_i},
\end{aligned}
$$

where

$$z_j \equiv z_j(x_j) = \Psi_{j-1}(x_j) - \Phi_j(x_j),$$
$$c_{j-1,2} = D_{j-1,1}\Psi_{j-1}(x_j) - D_{j,1}\Phi_j(x_j).$$

Thus in (6)

$$P_{i,j}(x) = P_{i,j-1}(x) + [z_j + c_{j-1,2}(x - x_j)]M_{j,B_i}. \tag{7}$$

By repeated use of this formula we get

$$P_{i,j}(x) = \sum_{l=i+1}^{j} [z_l + c_{l-1,2}(x - x_l)]M_{l,B_i} = -\sum_{l=j+1}^{i+3} [z_l + c_{l-1,2}(x - x_l)]M_{l,B_i}.$$

As B_i vanishes outside the interval (x_i, x_{i+4}), we have from (7) that $P_{i,j} \equiv 0$ for $j = i, i + 3$. In particular, the following identity is valid

$$\sum_{j=i+1}^{i+3} [z_j + c_{j-1,2}(x - x_j)]M_{j,B_i} \equiv 0,$$

from which one obtains the equalities

$$\sum_{j=i+1}^{i+3} c_{j-1,2}y_j^r M_{j,B_i} = 0, \quad r = 0, 1, \quad y_j = x_j - \frac{z_j}{c_{j-1,2}}. \tag{8}$$

Thus the formula for the discrete GB-spline B_i takes the form

$$B_i(x) = \begin{cases} \Psi_i(x)M_{i+1,B_i}, & x \in [x_i, x_{i+1}), \\ (x - y_{i+1})c_{i,2}M_{i+1,B_i} + \Phi_{i+1}(x)M_{i+1,B_i} \\ \quad + \Psi_{i+1}(x)M_{i+2,B_i}, & x \in [x_{i+1}, x_{i+2}), \\ (y_{i+3} - x)c_{i+2,2}M_{i+3,B_i} + \Phi_{i+2}(x)M_{i+2,B_i} \\ \quad + \Psi_{i+2}(x)M_{i+3,B_i}, & x \in [x_{i+2}, x_{i+3}), \\ \Phi_{i+3}(x)M_{i+3,B_i}, & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Substituting formula (9) into the normalization condition (5) written for $x \in [x_i, x_{i+1}]$, we obtain

$$\sum_{j=i-3}^{i} B_j(x) = \Phi_i(x) \sum_{j=i-3}^{i-1} M_{i,B_j} + \Psi_i(x) \sum_{j=i-2}^{i} M_{i+1,B_j}$$
$$+ (y_{i+1} - x)c_{i,2}M_{i+1,B_{i-2}} + (x - y_i)c_{i-1,2}M_{i,B_{i-1}} \equiv 1.$$

As according to (5)

$$\sum_{j=i-3}^{i-1} M_{i,B_j} = \sum_{j=i-2}^{i} M_{i+1,B_j} = 0 \tag{10}$$

the following identity is valid

$$(y_{i+1} - x)c_{i,2}M_{i+1,B_{i-2}} + (x - y_i)c_{i-1,2}M_{i,B_{i-1}} \equiv 1.$$

From here one gets the equalities

$$y_{i+1}^r c_{i,2} M_{i+1,B_{i-2}} - y_i^r c_{i-1,2} M_{i,B_{i-1}} \equiv \delta_{1,r}, \quad r = 0, 1,$$

where $\delta_{1,r}$ is the Kronecker symbol. Solving this system of equations and using (8) or (10), we obtain

$$M_{j,B_i} = \frac{y_{i+3} - y_{i+1}}{c_{j-1,2}\omega_{i+1}'(y_j)}, \quad j = i+1, i+2, i+3,$$

$$\omega_{i+1}(x) = (x - y_{i+1})(x - y_{i+2})(x - y_{i+3})$$

or with the notation $c_{j,3} = y_{j+2} - y_{j+1}$, $j = i, i+1$,

$$M_{i+1,B_i} = \frac{1}{c_{i,2}c_{i,3}},$$

$$M_{i+2,B_i} = -\frac{1}{c_{i+1,2}}\left(\frac{1}{c_{i,3}} + \frac{1}{c_{i+1,3}}\right), \tag{11}$$

$$M_{i+3,B_i} = \frac{1}{c_{i+2,2}c_{i+1,3}}.$$

4   Properties of discrete GB-splines

The functions B_j, $j = -3, \ldots, N-1$ possess many of the properties inherent in usual discrete polynomial B-splines. To provide inequality (4), in what follows we need to impose additional conditions on the functions Φ_j and Ψ_j.

The proofs of the following four assertions repeat those given in [5].

Lemma 2 *If the conditions*

$$0 < 2h_{j-1}^{-1}\Psi_{j-1}(x_j) < D_{j-1,1}\Psi_{j-1}(x_j),$$
$$0 < 2h_j^{-1}\Phi_j(x_j) < -D_{j,1}\Phi_j(x_j), \quad j = i+1, i+2, i+3 \tag{12}$$

are satisfied, then in (11) $c_{j,k} > 0$, $j = i, \ldots, i+4-k$; $k = 2, 3$, and

$$(-1)^{j-i-1}M_{j,B_i} > 0, \quad j = i+1, i+2, i+3. \tag{13}$$

Theorem 3 *Let the conditions of Lemma 2 be satisfied, the functions Φ_j and Ψ_j be convex and $D_{j,2}\Phi_j$ and $D_{j,2}\Psi_j$ be strictly monotone on the interval $[x_j, x_{j+1}]$ for all j. Then the functions B_j, $j = -3, \ldots, N-1$ have the following properties:*

1. $B_j(x) > 0$ *for* $x \in (x_j + \tau_j^{R_j}, x_{j+4} - \tau_{j+4}^{L_{j+3}})$, *and* $B_j(x) \equiv 0$ *if* $x \notin (x_j, x_{j+4})$;

2. B_j *satisfies the smoothness conditions (3);*

3. $\sum_{j=-3}^{N-1} y_{j+2}^r B_j(x) \equiv x^r$, $r = 0, 1$ *for* $x \in [a, b]$, $\Phi_j(x) = c_{j-1,2}c_{j-2,3}B_{j-3}(x)$, $\Psi_j(x) = c_{j,2}c_{j,3}B_j(x)$ *for* $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N-1$.

Lemma 4 *The function* B_i *has support of minimum length.*

Theorem 5 *The functions* B_i, $i = -3, \dots, N - 1$, *are linearly independent and form a basis of the space* S_4^{DG} *of discrete generalized splines.*

5 Local approximation by discrete GB-splines

According to Theorem 5, any discrete generalized spline $S \in S_4^{DG}$ can be uniquely written in the form

$$S(x) = \sum_{j=-3}^{N-1} b_j B_j(x) \tag{14}$$

for some constant coefficients b_j.

If the coefficients b_j in (14) are known, then by virtue of formula (9) we can write out an expression for the discrete generalized spline S on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$S(x) = b_{i-2} + b_{i-1}^{(1)}(x - y_i) + b_{i-1}^{(2)}\Phi_i(x) + b_i^{(2)}\Psi_i(x), \tag{15}$$

where

$$b_k^{(k)} = \frac{b_j^{(k-1)} - b_{j-1}^{(k-1)}}{c_{j,4-k}}, \quad k = 1, 2; \quad b_j^{(0)} = b_j. \tag{16}$$

The representations (14) and (15) allow us to find a simple and effective way to approximate a given continuous function f from its samples.

Theorem 6 *Let a continuous function* f *be given by its samples* $f(y_j)$, $j = -1, \dots, N + 1$. *Then for* $b_j = f(y_{j+2})$, $j = -3, \dots, N - 1$, *formula (14) is exact for polynomials of the first degree and provides a formula for local approximation.*

Proof: It suffices to prove that the identities

$$\sum_{j=-3}^{N-1} y_{j+2}^r B_j(x) \equiv x^r, \quad r = 0, 1 \tag{17}$$

hold for $x \in [a, b]$. Using formula (15) with the coefficients $b_{j-2} = 1$ and $b_{j-2} = y_j$, $j = i - 1, i, i + 1, i + 2$, for an arbitrary interval $[x_i, x_{i+1}]$, we find that identities (17) hold.

For $b_{j-2} = f(y_j)$, formula (15) can be rewritten as

$$S(x) = f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1})f[y_{i-1}, y_i, y_{i+1}]c_{i-1,2}^{-1}\Phi_i(x)$$
$$+ (y_{i+2} - y_i)f[y_i, y_{i+1}, y_{i+2}]c_{i,2}^{-1}\Psi_i(x), \quad x \in [x_i, x_{i+1}].$$

This is the formula of local approximation. The theorem is thus proved. ♠

Corollary 7 *Let a continuous function f be given by its samples $f_j = f(x_j)$, $j = -2, \ldots, N+2$. Then by setting*

$$b_{j-2} = f_j - \frac{1}{c_{j-1,2}}\Big(\Psi_{j-1}(x_j)f[x_j, x_{j+1}] - \Phi_j(x_j)f[x_{j-1}, x_j]\Big) \qquad (18)$$

in (14), we obtain a formula of three-point local approximation, which is exact for polynomials of the first degree.

Proof: To prove the corollary, it is sufficient to take the monomials 1 and x as f. Then according to (18), we obtain $b_{j-2} = 1$ and $b_{j-2} = y_j$ and it only remains to make use of identities (17). This proves the corollary.   ♠

Equation (15) permits us to write the coefficients of the spline S in its representation (14) of the form

$$b_{j-2} = \begin{cases} S(y_j) - D_{j-1,2}S(x_{j-1})\Phi_{j-1}(y_j) - D_{j,2}S(x_j)\Psi_{j-1}(y_j), & y_j < x_j, \\ S(y_j) - D_{j,2}S(x_j)\Phi_j(y_j) - D_{j+1,2}S(x_{j+1})\Psi_j(y_j), & y_j \geq x_j. \end{cases}$$

According to this formula we have $b_{j-2} = S(y_j) + O(\overline{h}_j^2)$, $\overline{h}_j = \max(h_{j-1}, h_j)$. Hence it follows that the control polygon (e.g., see [4]) converges quadratically to the function f when $b_{j-2} = f(y_j)$, or if the formula (18) is used.

6   Recurrence formulae for discrete GB-splines

Let us define functions

$$B_{j,2}(x) = \begin{cases} D_{j,2}\Psi_j(x), & x \in [x_j, x_{j+1}), \\ D_{j+1,2}\Phi_{j+1}(x), & x \in [x_{j+1}, x_{j+2}], \qquad j = i, i+1, i+2. \\ 0, & \text{otherwise,} \end{cases} \qquad (19)$$

We assume that the functions $D_{j,2}\Psi_j$ and $D_{j+1,2}\Phi_{j+1}$ are strictly monotone on $[x_j, x_{j+1})$ and $[x_{j+1}, x_{j+2}]$ respectively. The splines $B_{j,2}$ are a generalization of the "hat-functions" for polynomial B-splines. They are nonnegative and, furthermore, $B_{j,2}(x_{j+l}) = \delta_{1,l}$, $l = 0, 1, 2$.

According to (9), (11) and (19) the function D_2B_i can be written as

$$D_2B_i(x) = \sum_{j=i+1}^{i+3} M_{j,B_i}B_{j-1,2}(x)$$

$$= \frac{1}{c_{i,3}}\Big(\frac{B_{i,2}(x)}{c_{i,2}} - \frac{B_{i+1,2}(x)}{c_{i+1,2}}\Big) - \frac{1}{c_{i+1,3}}\Big(\frac{B_{i+1,2}(x)}{c_{i+1,2}} - \frac{B_{i+2,2}(x)}{c_{i+2,2}}\Big). \qquad (20)$$

The function D_1B_i satisfies the relation

$$D_1B_i(x) = \frac{B_{i,3}(x)}{c_{i,3}} - \frac{B_{i+1,3}(x)}{c_{i+1,3}}, \qquad (21)$$

where

$$
B_{j,3}(x) = \begin{cases}
\dfrac{D_{j,1}\Psi_j(x)}{c_{j,2}}, & x \in [x_j, x_{j+1}), \\[2mm]
1 + \dfrac{D_{j+1,1}\Phi_{j+1}(x)}{c_{j,2}} - \dfrac{D_{j+1,1}\Psi_{j+1}(x)}{c_{j+1,2}}, & x \in [x_{j+1}, x_{j+2}), \\[2mm]
-\dfrac{D_{j+2,1}\Phi_{j+2}(x)}{c_{j+1,2}}, & x \in [x_{j+2}, x_{j+3}), \\[2mm]
0, & \text{otherwise.}
\end{cases} \tag{22}
$$

Using formula (22) it is easy to show that functions $B_{j,3}$, $j = -2, \ldots, N-1$ satisfy the first and second smoothness conditions in (3), have supports of minimum length, are linearly independent and form a partition of unity,

$$
\sum_{j=1}^{N-1} B_{j,3}(x) \equiv 1, \quad x \in [a, b].
$$

Applying formulae (20) and (21) to the representation (14) we also obtain

$$
D_1 S(x) = \sum_{j=-2}^{N-1} b_j^{(1)} B_{j,3}(x), \quad D_2 S(x) = \sum_{j=-1}^{N-1} b_j^{(2)} B_{j,2}(x), \tag{23}
$$

where $b_j^{(k)}$, $k = 1, 2$ are defined in (16).

7   Series of discrete GB-splines (uniform case)

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ is an integer multiple of the same tabulation step, τ, of some detailed uniform refinement on $[a, b]$.

For $\theta \in \mathbb{R}$, $\tau > 0$ define

$$
\mathbb{R}_{\theta\tau} = \{\theta + i\tau \mid i \text{ is an integer}\}
$$

and let $\mathbb{R}_{\theta 0} = \mathbb{R}$. For any $a, b \in \mathbb{R}$ and $\tau > 0$ let

$$
[a, b]_\tau = [a, b] \cap \mathbb{R}_{a\tau}.
$$

The functions $B_{j,2}$, $B_{j,3}$, and B_j with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$ for all i are nonnegative on the discrete interval $[a, b]_\tau$. This permits us to reprove the main results for discrete polynomial splines of [9] for series of discrete generalized splines. Even more, one can obtain the results of generalized splines of [5] from the corresponding statements for discrete generalized splines as a limiting case when $\tau \to 0$.

In particular, if in (14) and (23) we have the coefficients $b_j^{(k)} > 0$, $k = 0, 1, 2$, $j = -3 + k, \ldots, N - 1$, then the spline S will be a positive, monotonically increasing and convex function on $[a, b]_\tau$.

Let f be a function defined on the discrete set $[a, b]_\tau$. We say that f has

a zero at the point $x \in [a, b]_\tau$ provided

$$f(x) = 0 \quad \text{or} \quad f(x - \tau) \cdot f(x) < 0.$$

When f vanishes at a set of consecutive points of $[a, b]_\tau$, say f is 0 at $x, \ldots, x + (r - 1)\tau$, but $f(x - \tau) \cdot f(x + r\tau) \neq 0$, then we call the set $X = \{x, x + \tau, \ldots, x + (r - 1)\tau\}$ a *multiple zero* of f, and we define its multiplicity by

$$Z_X(f) = \begin{cases} r, & \text{if } f(x - \tau) \cdot f(x + r\tau) < 0 \text{ and } r \text{ is odd,} \\ r, & \text{if } f(x - \tau) \cdot f(x + r\tau) > 0 \text{ and } r \text{ is even,} \\ r + 1, & \text{otherwise.} \end{cases}$$

This definition assures that f changes sign at a zero if and only if the zero is of odd multiplicity.

Let $Z_{[a,b]_\tau}(f)$ be the number of zeros of a function f on the discrete set $[a, b]_\tau$, counted according to their multiplicity. Let us denote $D_1^L S(x) = S[x - \tau, x]$.

Theorem 8 (Rolle's Theorem For Discrete Generalized Splines.) *For any $S \in S_4^{DG}$,*

$$Z_{[a,b]_\tau}(D_1^L S) \geq Z_{[a,b]_\tau}(S) - 1. \tag{24}$$

Proof: First, if S has a z-tuple zero on the set $X = \{x, \ldots, x + (r - 1)\tau\}$, it follows that $D_1^L S$ has a $(z - 1)$-tuple zero on the set $X' = \{x + \tau, \ldots, x + (r - 1)\tau\}$. Now if X^1 and X^2 are two consecutive zero sets of S, then it is trivially true that $D_1^L S$ must have a sign change at some point between X^1 and X^2. Counting all of these zeros, we arrive at the assertion (24). This completes the proof.   ♠

Lemma 9 *Let the function $D_{i,2}\Phi_i$ and $D_{i,2}\Psi_i$ be strictly monotone on the interval $[x_i, x_{i+1}]$ for all i. Then for every $S \in S_4^{DG}$ which is not identically zero on any interval $[x_i, x_{i+1}]_\tau$, $i = 0, \ldots, N - 1$,*

$$Z_{[a,b]_\tau}(S) \leq N + 2.$$

Proof: According to (19) and (23), the function $D_2 S$ has no more than one zero on $[x_i, x_{i+1}]$, because the functions $D_2 \Phi_i$ and $D_2 \Psi_i$ are strictly monotone and nonnegative on this interval. Hence $Z_{[a,b]_\tau}(D_2 S) \leq N$. Then according to the Rolle's Theorem 8, we find $Z_{[a,b]_\tau}(S) \leq N + 2$. This completes the proof.   ♠

Denote by $\text{supp}_\tau B_i = \{x \in \mathbb{R}_{a,\tau} \mid B_i(x) > 0\}$ the discrete support of the spline B_i, i.e. the discrete set $(x_i + \tau, x_{i+4} - \tau)_\tau$.

Theorem 10 *Assume that $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ are prescribed points on the discrete line $\mathbb{R}_{a,\tau}$. Then*

$$D = \det(B_i(\zeta_j)) \geq 0, \quad i, j = -3, \ldots, N - 1$$

and strict positivity holds if and only if

$$\zeta_i \in \operatorname{supp}_\tau B_i, \quad i = -3, \ldots, N - 1. \tag{25}$$

The proof of this theorem is based on Lemma 9 and repeats that of Theorem 8.66 in [9, p.355]. The following statements follow immediately from Theorem 10.

Corollary 11 *The system of discrete GB-splines* $\{B_j\}$, $j = -3, \ldots, N - 1$, *associated with knots on* $\mathbb{R}_{a,\tau}$ *is a weak Chebyshev system according to the definition given in [9, p. 36], i.e. for any* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ *in* $\mathbb{R}_{a,\tau}$ *we have* $D \geq 0$ *and* $D > 0$ *if and only if condition (25) is satisfied. In the latter case the discrete generalized spline* $S(x) = \sum_{j=-3}^{N-1} b_j B_j(x)$ *has no more than* $N + 2$ *zeros.*

Corollary 12 *If the conditions of Theorem 5 are satisfied, then the solution of the interpolation problem*

$$S(\zeta_i) = f_i, \quad i = -3, \ldots, N - 1, \quad f_i \in \mathbb{R} \tag{26}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be a rectangular $m \times n$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally

positive) (e.g., see [3]) if the minors of all order of the matrix are nonnegative (positive), i.e. for all $1 \leq p \leq m$ we have

$$\det(a_{i_k j_l}) \geq 0 \ (> 0) \quad \text{for all} \quad \begin{array}{l} 1 \leq i_1 < \cdots < i_p \leq m, \\ 1 \leq j_1 < \cdots < j_p \leq n. \end{array}$$

Corollary 13 *For arbitrary integers* $-3 \leq \nu_{-3} < \cdots < \nu_{p-4} \leq N - 1$ *and* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{p-4}$ *in* $\mathbb{R}_{a,\tau}$ *we have*

$$D_p = \det\{B_{\nu_i}(\zeta_j)\} \geq 0, \quad i, j = -3, \ldots, p - 4$$

and strict positivity holds if and only if

$$\zeta_i \in \operatorname{supp}_\tau B_{\nu_i}, \quad i = -3, \ldots, p - 4$$

i.e. the matrix $\{B_j(\zeta_i)\}$, $i, j = -3, \ldots, N - 1$ *is totally nonnegative.*

The last statement is proved by induction based on Theorem 5 and the recurrence relations for the minors of the matrix $\{B_j(\zeta_i)\}$. The proof does not differ from that of Theorem 8.67 described by [9, p.356].

Since the supports of discrete GB-splines are finite, the matrix of system (26) is banded and has seven nonzero diagonals in general. The matrix is tridiagonal if $\zeta_i = x_{i+2}$, $i = -3, \ldots, N - 1$.

An important particular case of the problem, in which $S'(x_i) = f_i'$, $i = 0, N$, can be obtained by passing to the limit as $\zeta_{-3} \to \zeta_{-2}$, $\zeta_{N-1} \to \zeta_{N-2}$.

De Boor and Pinkus [2] proved that linear systems with totally nonnegative matrices can be solved by Gaussian elimination without choosing a pivot element. Thus, the system (26) can be solved effectively by the conventional Gauss method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \cdots, v_n)$, with zeros being neglected. Karlin [3] showed that if a matrix A is totally nonnegative then it decreases the variation, i.e.

$$S^-(A\mathbf{v}) \leq S^-(\mathbf{v}).$$

By virtue of Corollary 4, the totally nonnegative matrix $\{B_j(\zeta_i)\}$, $i, j = -3, \ldots, N-1$, formed by discrete GB-splines decreases the variation.

For a bounded real function f, let $S^-(f)$ be the number of sign changes of the function f on the real axis $\mathbb{R}$, without taking into account the zeros

$$S^-(f) = \sup_n S^-[f(\zeta_1), \ldots, f(\zeta_n)], \quad \zeta_1 < \zeta_2 < \cdots < \zeta_n.$$

Theorem 14 *The discrete generalized spline $S(x) = \sum_{j=-3}^{N-1} b_j B_j(x)$ is a variation diminishing function, i.e. the number of sign changes of S does*

not exceed that in the sequence of its coefficients:

$$S^-\left(\sum_{j=-3}^{N-1} b_j B_j\right) \leq S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-3}, \ldots, b_{N-1}).$$

The proof of this statement does not differ from that of Theorem 8.68 for discrete polynomial B-splines in [9, p.356].

References

[1] P. Costantini, B. I. Kvasov, and C. Manni. On discrete hyperbolic tension splines. *Advances in Computational Mathematics*, 11:331–354, 1999. C878, C881

[2] C. De Boor and A. Pinkus. Backward error analysis for totally positive linear systems. *Numer. Math.*, 27:485–490, 1977. C897

[3] S. Karlin. *Total Positivity, Volume 1*. Stanford University Press, Stanford, 1968. C896, C897

[4] P. E. Koch and T. Lyche. Exponential B-splines in tension. In C.K. Chui, L.L. Schumaker, and J.D. Ward, editors, *Approximation Theory VI: Proceedings of the Sixth International Symposium on Approximation Theory. Vol. II*, pages 361–364, 1989. Academic Press. C889

[5] B. I. Kvasov. Local bases for generalized cubic splines. *Russ. J. Numer. Anal. Math. Modelling*, 10:49–80, 1995. C881, C886, C892

[6] B. I. Kvasov. GB-splines and their properties. *Annals of Numerical Mathematics*, 3:139–149, 1996. C881

[7] B. I. Kvasov. Algorithms for shape preserving local approximation with automatic selection of tension parameters. *Computer Aided Geometric Design*, 17:17–37, 2000. C878

[8] T. Lyche. *Discrete polynomial spline approximation methods*. PhD thesis, University of Texas, Austin, 1975. For a summary, see *Spline Functions*, K. Böhmer, G. Meinardus, and W. Schimpp, editors, Karlsruhe 1975, Lectures Notes in Mathematics No. 501, pages 144–176, Springer-Verlag, Berlin, 1976. C881

[9] L. L. Schumaker. *Spline Functions: Basic Theory*. John Wiley & Sons, New York, 1981. C881, C892, C895, C895, C896, C898

[10] Yu. S. Zav'yalov, B. I. Kvasov, and V. L. Miroshnichenko. *Methods of Spline Functions*. Nauka, Moscow (in Russian), 1980. C881

Approximation by discrete GB-splines

Boris I. Kvasov *

*School of Mathematics, Suranaree University of Technology, University Avenue 111,
Nakhon Ratchasima 30000, Thailand*
E-mail: boris@math.sut.ac.th

This paper addresses the definition and the study of discrete generalized splines. Discrete
generalized splines are continuous piecewise defined functions which meet some smoothness
conditions for the first and second divided differences at the knots. They provide a generaliza-
tion both of smooth generalized splines and of the classical discrete cubic splines. Completely
general configurations for steps in divided differences are considered. Direct algorithms are
proposed for constructing discrete generalized splines and discrete generalized B-splines (dis-
crete GB-splines for short). Explicit formulae and recurrence relations are obtained for dis-
crete GB-splines. Properties of discrete GB-splines and their series are studied. It is shown
that discrete GB-splines form weak Chebyshev systems and that series of discrete GB-splines
have a variation diminishing property.

Keywords: discrete generalized splines, discrete GB-splines, recurrence relations, weak
Chebyshev systems, variation diminishing property

AMS subject classification: 41A05, 65D05

1. Introduction

The tools of generalized splines and GB-splines are widely used in solving prob-
lems of shape preserving approximation (e.g., see [3,14,16,17,23,28,32,33]). By intro-
ducing various parameters into the spline structure, one can preserve characteristics of
the initial data such as positivity, monotonicity, convexity, presence of linear and planar
sections, etc. Here, the main challenge is to develop algorithms that choose parameters
automatically. Recently, in [5] a difference method for constructing shape preserving
hyperbolic splines as solutions of multipoint boundary value problems was developed.
Such an approach permits to avoid the computation of hyperbolic functions and has sub-
stantial other advantages. However, the extension of a mesh solution will be a discrete
hyperbolic tension spline.

---

* The author is on leave from the Institute of Computational Technologies, Russian Academy of Sciences,
Novosibirsk 630090, Russia.

Discrete polynomial splines have been studied extensively. They were introduced in [21] as solutions to certain minimization problems involving differences instead of derivatives. They were connected to best summation formulae in [22], and have been used in [20] to compute nonlinear splines iteratively. Approximation properties of discrete splines have been studied by Lyche [18,19] and other authors (e.g., see [9,10,26,37]). Discrete L-splines were considered in [2]. Discrete B-splines on a uniform partition were introduced in [34]. Discrete B-splines on a nonuniform partition were defined in [7, p. 15]. In [4] discrete B-splines were applied to the general area of subdivision. While discrete polynomial splines are currently attracting widespread research interest (e.g., see [24,25,27]), discrete generalized splines and GB-splines have been less studied. The only results we know of concern discrete exponential Box-splines [6,31] and are therefore related to uniform partitions.

The contents of this paper is as follows. In section 2 we give a general definition of a discrete generalized spline and prove sufficient conditions for its existence and uniqueness. Next, we construct a minimum length local support basis (whose elements are denoted as discrete GB-splines) of the new spline space; see section 3. Properties of GB-splines are discussed in section 4, while the local approximation by discrete GB-splines of a given continuous function from its samples is considered in section 5. In section 6 we derive recurrence formulae for calculations with discrete GB-splines. The properties of GB-spline series are summarized in section 7. Section 8 provides some examples of defining functions that conform to the sufficiency conditions derived earlier in the paper.

2.   Discrete generalized splines. Conditions of existence and uniqueness

Let a partition Δ: $a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ be given. For fixed $\tau_j^{L_i} > 0$ and $\tau_j^{R_i} > 0$, $j = i, i + 1$, and a function S which is defined and continuous on the real line $\mathbb{R}$ we introduce the linear difference operators

$$
\begin{aligned}
D_{i,1}S(x) &= \left(\lambda_i^{R_i} S\left[x - \tau_i^{L_i}, x\right] + \lambda_i^{L_i} S\left[x, x + \tau_i^{R_i}\right]\right)(1 - t) \\
&\quad + \left(\lambda_{i+1}^{R_i} S\left[x - \tau_{i+1}^{L_i}, x\right] + \lambda_{i+1}^{L_i} S\left[x, x + \tau_{i+1}^{R_i}\right]\right)t, \\
D_{i,2}S(x) &= 2S\left[x - \tau_i^{L_i}, x, x + \tau_i^{R_i}\right](1 - t) + 2S\left[x - \tau_{i+1}^{L_i}, x, x + \tau_{i+1}^{R_i}\right]t, \\
&\quad x \in [x_i, x_{i+1}], \ i = 0, \dots, N - 1,
\end{aligned}
\tag{1}
$$

where $\lambda_j^{R_i} = 1 - \lambda_j^{L_i} = \tau_j^{R_i}/(\tau_j^{L_i} + \tau_j^{R_i})$, $j = i, i + 1$, and $t = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$. The square parentheses denote the usual first and second divided differences of the function S.

We associate to Δ a system of functions $\{1, x, \Phi_i, \Psi_i\}$, $i = 0, \dots, N - 1$, which are defined and continuous on $\mathbb{R}$ and for given i are linearly independent on the interval $[x_i, x_{i+1}]$. The functions Φ_i and Ψ_i are subject to the constraints

$$
\begin{aligned}
\Phi_i\left(x_{i+1} - \tau_{i+1}^{L_i}\right) = \Phi_i(x_{i+1}) = \Phi_i\left(x_{i+1} + \tau_{i+1}^{R_i}\right) = 0, \quad & D_{i,2}\Phi_i(x_i) = 1, \\
\Psi_i\left(x_i - \tau_i^{L_i}\right) = \Psi_i(x_i) = \Psi_i\left(x_i + \tau_i^{R_i}\right) = 0, \quad & D_{i,2}\Psi_i(x_{i+1}) = 1.
\end{aligned}
\tag{2}
$$

Any element S_i of the linear space Υ_i spanned by the four functions 1, x, Φ_i, Ψ_i can be uniquely written as follows

$$S_i(x) = S_i(x_i)(1 - t) + S_i(x_{i+1})t + D_{i,2}S_i(x_i)\big[\Phi_i(x) - \Phi_i(x_i)(1 - t)\big]$$
$$+ D_{i,2}S_i(x_{i+1})\big[\Psi_i(x) - \Psi_i(x_{i+1})t\big]. \tag{3}$$

Definition 1. A function $S : [a, b] \to \mathbb{R}$ is called a discrete generalized spline if:

(1) for any integer i, $0 \leqslant i \leqslant N - 1$, there exist a unique function $S_i \in \Upsilon_i$ such that

$$S(x) \equiv S_i(x), \quad x \in [x_i, x_{i+1}]; \tag{4}$$

- (2) for all integers $i = 1, \ldots, N - 1$, S satisfies the following smoothness conditions

$$S_{i-1}(x_i) = S_i(x_i),$$
$$D_{i-1,1}S_{i-1}(x_i) = D_{i,1}S_i(x_i), \tag{5}$$
$$D_{i-1,2}S_{i-1}(x_i) = D_{i,2}S_i(x_i).$$

The set of discrete generalized splines satisfying definition 1 will be denoted by S_4^{DG}. It is clear that this set consists of functions whose restrictions to $[x_i, x_{i+1}]$, $i = 0, \ldots, N - 1$ belong to the linear space Υ_i and that each function in S_4^{DG} is smooth in the sense of (5), that is, it is continuous and the values of its first and second divided differences with respect to the points $x_i - \tau_i^{L_{i-1}}, x_i, x_i + \tau_i^{R_{i-1}}$ and $x_i - \tau_i^{L_i}, x_i, x_i + \tau_i^{R_i}$ coincide. The usual operations of addition of elements from S_4^{DG} and their multiplication by real numbers give again elements in the set S_4^{DG} which hence is a linear space.

Definition 1 generalizes the notion of discrete polynomial splines in [35] and of generalized splines in [13,15]. The latter type can be obtained by setting $\tau_j^{L_i} = \tau_j^{R_i} = 0$, $j = i, i + 1$ for all i. If $\tau_i^{L_j} = \tau_i^L$ and $\tau_i^{R_j} = \tau_i^R$, $j = i - 1, i$, then according to smoothness conditions (5) the values of the functions S_{i-1} and S_i at the three consecutive points $x_i - \tau_i^L, x_i, x_i + \tau_i^R$ coincide. Setting $\tau_j^{L_i} = \tau_j^{R_i} = \tau_i$, $j = i, i + 1$ we obtain $D_{i,1}S(x) = S[x - \tau_i, x + \tau_i]$ and $D_{i,2}S(x) = 2S[x - \tau_i, x, x + \tau_i]$, which is the case discussed in [5].

According to the conditions (5), the discrete generalized spline S defined by (3) and (4) can be written as follows

$$S(x) = S(x_i)(1 - t) + S(x_{i+1})t + M_i\big[\Phi_i(x) - \Phi_i(x_i)(1 - t)\big]$$
$$+ M_{i+1}[\Psi_i(x) - \Psi_i(x_{i+1})t], \quad x \in [x_i, x_{i+1}], \ i = 0, \ldots, N - 1, \tag{6}$$

where $M_j = D_{i,2}S(x_j)$, $j = i, i + 1$.

The functions Φ_i and Ψ_i depend on tension parameters which influence the behaviour of S fundamentally. We call them the *defining functions*. In practice one takes Φ_i to depend on a parameter p_i and Ψ_i to depend on a parameter q_i, $0 \leqslant p_i, q_i < \infty$. In the limiting case when $p_i, q_i \to \infty$ we require that $\lim_{p_i \to \infty} \Phi_i(x) = 0$, $x \in (x_i, x_{i+1}]$ and $\lim_{q_i \to \infty} \Psi_i(x) = 0$, $x \in [x_i, x_{i+1})$ so that the function S in formula (6) turns into

a linear function. Additionally, we require that if $p_i = q_i = 0$ for all i, then we get a discrete cubic spline with

$$\Phi_i(x) = \frac{1}{2} \frac{(x_{i+1} - x - \tau_{i+1}^{L_i})(x_{i+1} - x)(x_{i+1} - x + \tau_{i+1}^{R_i})}{3h_i + \varepsilon_{i+1} - \varepsilon_i},$$

$$\Psi_i(x) = \frac{1}{2} \frac{(x - x_i + \tau_i^{L_i})(x - x_i)(x - x_i - \tau_i^{R_i})}{3h_i + \varepsilon_{i+1} - \varepsilon_i}, \tag{7}$$

$$\varepsilon_j = \tau_j^{R_i} - \tau_j^{L_i}, \quad j = i, i + 1.$$

If in (7) $\tau_i^{L_j} = \tau_i^{R_j} = \tau_i$, $j = i - 1, i$, for all i then this spline coincides with the discrete cubic spline in [37]. The case $\tau_i = \tau$ for all i was considered in [19].

Introducing the notation $m_j \equiv D_{i,1}S(x_j)$, $j = i, i + 1$, $i = 0, \ldots, N - 1$, we can obtain less restrictive conditions on the functions Φ_i and Ψ_i to guarantee existence and uniqueness of the discrete generalized spline S than could be achieved by using the unknowns M_i.

From the equations $D_{i,1}S(x_j) = m_j$, $j = i, i + 1$, we find using (6)

$$M_i = \frac{h_i}{T_i} \{[\bar{b}_i + \Psi_i(x_{i+1})] S[x_i, x_{i+1}] - \bar{b}_i m_i - \Psi_i(x_{i+1}) m_{i+1}\},$$

$$M_{i+1} = \frac{h_i}{T_i} \{-[\bar{a}_i + \Phi_i(x_i)] S[x_i, x_{i+1}] + \Phi_i(x_i) m_i + \bar{a}_i m_{i+1}\}, \tag{8}$$

$$T_i = \bar{a}_i \bar{b}_i - \Phi_i(x_i) \Psi_i(x_{i+1}),$$

where

$$\bar{a}_i = -\Phi_i(x_i) - h_i D_{i,1}\Phi_i(x_i),$$
$$\bar{b}_i = -\Psi_i(x_{i+1}) + h_i D_{i,1}\Psi_i(x_{i+1}). \tag{9}$$

The third equation in the smoothness conditions (5) together with (8) and the boundary relations $M_0 = D_{0,2}S(a)$ and $M_N = D_{N-1,2}S(b)$ result in the following system of equations

$$\bar{b}_0 m_0 + \Psi_0(x_1) m_1 = [\bar{b}_0 + \Psi_0(x_1)] S[x_0, x_1] - D_{0,2}S(a) \frac{T_0}{h_0},$$

$$\Phi_{i-1}(x_{i-1}) \frac{h_{i-1}}{T_{i-1}} m_{i-1} + \left(\bar{a}_{i-1} \frac{h_{i-1}}{T_{i-1}} + \bar{b}_i \frac{h_i}{T_i}\right) m_i + \Psi_i(x_{i+1}) \frac{h_i}{T_i} m_{i+1}$$

$$= [\bar{a}_{i-1} + \Phi_{i-1}(x_{i-1})] \frac{h_{i-1}}{T_{i-1}} S[x_{i-1}, x_i] + [\bar{b}_i + \Psi_i(x_{i+1})] \frac{h_i}{T_i} S[x_i, x_{i+1}], \tag{10}$$

$$i = 1, \ldots, N - 1,$$

$$\Phi_{N-1}(x_{N-1}) m_{N-1} + \bar{a}_{N-1} m_N$$

$$= D_{N-1,2}S(b) \frac{T_{N-1}}{h_{N-1}} + [\bar{a}_{N-1} + \Phi_{N-1}(x_{N-1})] S[x_{N-1}, x_N].$$

Let us find constraints on the defining functions Φ_i and Ψ_i which ensure that the discrete generalized spline S exists and is unique.

Lemma 1. Let the values $S(x_i)$, $i = 0, \ldots, N$, of a discrete generalized spline S be fixed together with the boundary relations $M_0 = D_{0.2}S(a)$ and $M_N = D_{N-1.2}S(b)$. If the conditions

$$0 < \Phi_i(x_i) < \bar{a}_i, \qquad 0 < \Psi_i(x_{i+1}) < \bar{b}_i, \quad i = 0, \ldots, N - 1,$$

are satisfied, where $\bar{a}_i$ and $\bar{b}_i$ are as defined in (9), then the discrete generalized spline S exists and is unique.

Proof. It follows from the conditions of the lemma that

$$0 < \Phi_i(x_i)\Psi_i(x_{i+1}) < \bar{a}_i\bar{b}_i, \quad i = 0, \ldots, N - 1.$$

Then

$$T_i = \bar{a}_i\bar{b}_i - \Phi_i(x_i)\Psi_i(x_{i+1}) > 0, \quad i = 0, \ldots, N - 1.$$

Therefore, by virtue of the conditions of the lemma, the matrix of system (10) is diagonally dominant:

$$\bar{r}_0 = \bar{b}_0 - \Psi_0(x_1) > 0,$$
$$\bar{r}_i = \left[\bar{a}_{i-1} - \Phi_{i-1}(x_{i-1})\right]\frac{h_{i-1}}{T_{i-1}} + \left[\bar{b}_i - \Psi_i(x_{i+1})\right]\frac{h_i}{T_i} > 0, \quad i = 1, \ldots, N - 1,$$
$$\bar{r}_N = \bar{a}_{N-1} - \Phi_{N-1}(x_{N-1}) > 0.$$

Now, the Hadamard criterion (e.g., see [37]) implies that the matrix of system (10) is nonsingular and that the discrete generalized spline S exists and is unique. This proves the lemma.   □

In practical examples, the conditions of lemma 1 are satisfied for the majority of discrete generalized splines. The reader can readily verify that these conditions are fulfilled by all of the defining functions presented in section 8. This allows one to construct the splines, i.e., to solve the tridiagonal linear system (10), efficiently by a special version of Gaussian elimination that avoids pivoting (e.g., see [36]).

3. Construction of discrete GB-splines

Let us construct a basis for the space of discrete generalized splines S_4^{DG} by using functions which have local supports of minimum length. According to lemma 1, the space S_4^{DG} is of dimension $N + 3$. We extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$, such that $x_{-3} < x_{-2} < x_{-1} < a, b < x_{N+1} < x_{N+2} < x_{N+3}$. As in section 2 for each interval $[x_i, x_{i+1}]$, $i = -3, -2, -1, N, N+1, N+2$, we introduce the linear space Υ_i. This permits us to define the discrete generalized spline S on the extended interval $[x_{-3}, x_{N+3}]$.

We demand that the discrete GB-splines B_i, $i = -3, \ldots, N-1$, have the properties

$$B_i(x) > 0, \quad x \in \left(x_i + \tau_i^{R_i}, x_{i+4} - \tau_{i+4}^{L_{i+3}}\right), \tag{11}$$

$$B_i(x) \equiv 0, \quad x \notin (x_i, x_{i+4}),$$

$$\sum_{j=-3}^{N-1} B_j(x) \equiv 1, \quad x \in [a, b]. \tag{12}$$

According to (6), on the interval $[x_j, x_{j+1}]$, $j = i, \ldots, i + 3$, the discrete GB-spline B_i, $i = -3, \ldots, N - 1$, has the form

$$B_i(x) \equiv B_{i,j}(x) = P_{i,j}(x) + \Phi_j(x)M_{j,B_i} + \Psi_j(x)M_{j+1,B_i}, \tag{13}$$

where $P_{i,j}$ is a polynomial of the first degree and $M_{l,B_i} = D_{j,2}B_i(x_l)$, $l = j, j + 1$, are constants to be determined.

The smoothness conditions (5) together with the constraints (2) give the following relations

$$P_{i,j}(x_j) = P_{i,j-1}(x_j) + z_j M_{j,B_i},$$
$$D_{j,1}P_{i,j}(x_j) = D_{j-1,1}P_{i,j-1}(x_j) + c_{j-1,2}M_{j,B_i},$$

where

$$z_j \equiv z_j(x_j) = \Psi_{j-1}(x_j) - \Phi_j(x_j),$$
$$c_{j-1,2} = D_{j-1,1}\Psi_{j-1}(x_j) - D_{j,1}\Phi_j(x_j).$$

Thus

$$P_{i,j}(x) = P_{i,j-1}(x) + \left[z_j + c_{j-1,2}(x - x_j)\right]M_{j,B_i}. \tag{14}$$

According to (5), the condition $B_i(x) \equiv 0$ for $x \notin (x_i, x_{i+4})$ is satisfied if and only if

$$B_i(x_i) = D_{i,1}B_i(x_i) = D_{i,2}B_i(x_i) = 0,$$
$$B_i(x_{i+4}) = D_{i+3,1}B_i(x_{i+4}) = D_{i+3,2}B_i(x_{i+4}) = 0,$$

Due to (13) and (2), the latter relations are equivalent to

$$P_{i,i} \equiv 0, \qquad M_{i,B_i} = 0 \quad \text{and} \quad P_{i,i+3} \equiv 0, \qquad M_{i+4,B_i} = 0. \tag{15}$$

Therefore, by repeated use of (14) we obtain

$$P_{i,j}(x) = \sum_{l=i+1}^{j} \left[z_l + c_{l-1,2}(x - x_l)\right]M_{l,B_i} = -\sum_{l=j+1}^{i+3} \left[z_l + c_{l-1,2}(x - x_l)\right]M_{l,B_i}.$$

In particular, the following identity is valid,

$$\sum_{j=i+1}^{i+3} \left[z_j + c_{j-1,2}(x - x_j)\right]M_{j,B_i} \equiv 0.$$

from which one obtains the equalities

$$\sum_{j=i+1}^{i+3} c_{j-1,2} y_j^r M_{j,B_i} = 0, \quad r = 0, 1, \qquad y_j = x_j - \frac{z_j}{c_{j-1,2}}. \tag{16}$$

Thus the formula for the discrete GB-spline B_i takes the form

$$B_i(x) \doteq \begin{cases} \Psi_i(x) M_{i+1,B_i}, & x \in [x_i, x_{i+1}), \\ (x - y_{i+1}) c_{i,2} M_{i+1,B_i} + \Phi_{i+1}(x) M_{i+1,B_i} + \Psi_{i+1}(x) M_{i+2,B_i}, \\ \quad x \in [x_{i+1}, x_{i+2}), \\ (y_{i+3} - x) c_{i+2,2} M_{i+3,B_i} + \Phi_{i+2}(x) M_{i+2,B_i} + \Psi_{i+2}(x) M_{i+3,B_i}, \\ \quad x \in [x_{i+2}, x_{i+3}), \\ \Phi_{i+3}(x) M_{i+3,B_i}, & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Substituting formula (17) into the normalization condition (12) written for $x \in [x_i, x_{i+1}]$, $i = 0, \dots, N - 1$, we obtain

$$\sum_{j=i-3}^{i} B_j(x) = \Phi_i(x) \sum_{j=i-3}^{i-1} M_{i,B_j} + \Psi_i(x) \sum_{j=i-2}^{i} M_{i+1,B_j}$$
$$+ (y_{i+1} - x) c_{i,2} M_{i+1,B_{i-2}} + (x - y_i) c_{i-1,2} M_{i,B_{i-1}}$$
$$\equiv 1.$$

Due to the linear independence of functions $1, x, \Phi_i$, and Ψ_i on $[x_i, x_{i+1}]$, the latter relation is satisfied if and only if

$$\sum_{j=i-3}^{i-1} M_{i,B_j} = \sum_{j=i-2}^{i} M_{i+1,B_j} = 0, \tag{18}$$

$$y_{i+1} c_{i,2} M_{i+1,B_{i-2}} - y_i c_{i-1,2} M_{i,B_{i-1}} = 1,$$
$$c_{i,2} M_{i+1,B_{i-2}} - c_{i-1,2} M_{i,B_{i-1}} = 0. \tag{19}$$

In particular, from (19) we derive the following identity

$$(y_{i+1} - x) c_{i,2} M_{i+1,B_{i-2}} + (x - y_i) c_{i-1,2} M_{i,B_{i-1}} \equiv 1.$$

Solving system (19) and using (16) or (18), we obtain

$$M_{j,B_i} = \frac{y_{i+3} - y_{i+1}}{c_{j-1,2} \omega_{i+1}'(y_j)}, \quad j = i+1, i+2, i+3,$$

$$\omega_{i+1}(x) = (x - y_{i+1})(x - y_{i+2})(x - y_{i+3})$$

or with the notation $c_{j,3} = y_{j+2} - y_{j+1}$, $j = i, i + 1$,

$$M_{i+1,B_i} = \frac{1}{c_{i,2}c_{i,3}},$$

$$M_{i+2,B_i} = -\frac{1}{c_{i+1,2}}\left(\frac{1}{c_{i,3}} + \frac{1}{c_{i+1,3}}\right), \tag{20}$$

$$M_{i+3,B_i} = \frac{1}{c_{i+2,2}c_{i+1,3}}.$$

4.   Properties of discrete GB-splines

If the inequalities of lemma 1 are satisfied at only the interior nodes of the support interval of the discrete GB-spline B_i, then we obtain the following result.

Lemma 2. If the conditions

$$0 < \Psi_{j-1}(x_j) < \bar{b}_{j-1}, \qquad 0 < \Phi_j(x_j) < \bar{a}_j, \quad j = i+1, i+2, i+3,$$

are satisfied, where $\bar{b}_{j-1}$ and $\bar{a}_j$ are as defined in (9), then in (20) $c_{j,k} > 0$, $j = i, \ldots, i + 4 - k$, $k = 2, 3$, and

$$(-1)^{j-i-1}M_{j,B_i} > 0, \quad j = i+1, i+2, i+3. \tag{21}$$

Proof. The conditions of the lemma can be rewritten in the form

$$0 < \frac{2}{h_{j-1}}\Psi_{j-1}(x_j) < D_{j-1,1}\Psi_{j-i}(x_j),$$

$$0 < \frac{2}{h_j}\Phi_j(x_j) < -D_{j,1}\Phi_j(x_j), \quad j = i+1, i+2, i+3. \tag{22}$$

Adding these inequalities we obtain

$$c_{j-1,2} = D_{j-1,1}\Psi_{j-i}(x_j) - D_{j,1}\Phi_j(x_j) > 0, \quad j = i+1, i+2, i+3.$$

The inequalities

$$x_j - \frac{h_{j-1}}{2} < y_j < x_j + \frac{h_j}{2}, \quad j = i+1, i+2, i+3, \tag{23}$$

are equivalent to the relations

$$0 < \left[-\Psi_{j-1}(x_j) + \frac{h_{j-1}}{2}D_{j-1,1}\Psi_{j-1}(x_j)\right] + \left[\Phi_j(x_j) - \frac{h_{j-1}}{2}D_{j,1}\Phi_j(x_j)\right],$$

$$0 < \left[\Psi_{j-1}(x_j) + \frac{h_j}{2}D_{j-1,1}\Psi_{j-1}(x_j)\right] - \left[\Phi_j(x_j) + \frac{h_j}{2}D_{j,1}\Phi_j(x_j)\right],$$

$$j = i+1, i+2, i+3,$$

which are obviously satisfied by conditions (22). From (23) we obtain $c_{j,3} = y_{j+2} - y_{j+1} > 0$, $j = i, i + 1$. Now it follows from (20) that if the conditions of the lemma are satisfied, then the inequalities (21) hold. This proves the lemma. $\quad\square$

The functions B_j, $j = -3, \ldots, N - 1$, possess many of the properties inherent in the usual discrete polynomial B-splines. To provide inequality (11), in what follows we need to impose additional conditions on the functions Φ_j and Ψ_j which, as the reader may readily check, are satisfied by all the defining functions given in section 8.

Theorem 1. Let the conditions of lemma 1 be satisfied, the functions Φ_j and Ψ_j be convex and $D_{j,2}\Phi_j$ and $D_{j,2}\Psi_j$ be strictly monotone on the interval $[x_j, x_{j+1}]$ for all j. Then the functions B_j, $j = -3, \ldots, N - 1$ have the following properties:

(1) $B_j(x) > 0$ for $x \in (x_j + \tau_j^{R_j}, x_{j+4} - \tau_{j+4}^{L_{j+3}})$, and $B_j(x) \equiv 0$ if $x \notin (x_j, x_{j+4})$;

(2) B_j satisfies the smoothness conditions (5);

(3)

$$\sum_{j=-3}^{N-1} y_{j+2}^r B_j(x) \equiv x^r, \quad r = 0, 1 \text{ for } x \in [a, b],$$

$$\Phi_j(x) = c_{j-1,2} c_{j-2,3} B_{j-3}(x),$$

$$\Psi_j(x) = c_{j,2} c_{j,3} B_j(x) \quad \text{for } x \in [x_j, x_{j+1}], \ j = 0, \ldots, N - 1.$$

Proof. Since the functions Φ_j and Ψ_j are convex and satisfy the constraints (2) we conclude that Φ_j is positive and monotonically decreasing on the interval $[x_j, x_{j+1} - \tau_{j+1}^{L_j})$ and Ψ_j is positive and monotonically increasing on the interval $(x_j + \tau_j^{R_j}, x_{j+1}]$.

Using formula (17) and inequalities (21) we see that the function B_j is positive and monotonically increasing on the interval $(x_j + \tau_j^{R_j}, x_{j+1}]$ and positive and monotonically decreasing on the interval $[x_{j+3}, x_{j+4} - \tau_{j+4}^{L_{j+3}})$.

According to (13), for $x \in [x_l, x_{l+1})$, $l = j + 1, j + 2$, we have

$$D_{l,2}B_j(x) = D_{l,2}\Phi_l(x)M_{l,B_j} + D_{l,2}\Psi_l(x)M_{l+1,B_j}.$$

As the sum of two strictly monotonically decreasing (increasing) functions, the function $D_{l,2}B_j$ decreases monotonically on $[x_{j+1}, x_{j+2}]$ and increases monotonically on $[x_{j+2}, x_{j+3}]$. Thus, there exist two points $\zeta_l \in (x_l, x_{l+1})$, $l = j + 1, j + 2$, such that $D_{l,2}B_j(\zeta_l) = 0$. Since by (17), (21), and (22) $D_{j+1,1}B_j(x_{j+1}) > 0$ and $D_{j+3,1}B_j(x_{j+3}) < 0$ we find that the function B_j can not cross the x-axis and therefore is positive on the interval $[x_{j+1}, x_{j+3}]$.

All the remaining properties of the discrete GB-splines B_j, which are formulated in (2) and (3) of theorem 1, follow directly from formula (17). This proves the theorem. $\quad\square$

Lemma 3. The function B_i has support of minimum length.

Proof. It is clear that the function B_i cannot be different from zero on only a part of the interval $[x_j, x_{j+1}]$, $j = i, i + 3$. If we assume that B_i vanishes outside the interval (x_{i+1}, x_{i+4}), then due to (15), we have $M_{i+1,B_i} = 0$. But then it follows from the equalities (16) that $M_{j,B_i} = 0$, $j = i + 2, i + 3$, and according to (17) we obtain $B_i \equiv 0$. If we assume that B_i vanishes outside the interval (x_i, x_{i+3}), we arrive at the same result. This proves the lemma. $\square$

Theorem 2. The functions B_i, $i = -3, \ldots, N - 1$, are linearly independent on $[a, b]$ and form a basis of the space S_4^{DG} of discrete generalized splines.

Proof. Let us assume on the contrary that there exist constants $\bar{c}_i$, $i = -3, \ldots, N - 1$, which are not all equal to zero and such that

$$\bar{c}_{-3} B_{-3}(x) + \cdots + \bar{c}_{N-1} B_{N-1}(x) = 0, \quad x \in [a, b]. \tag{24}$$

Because the functions B_i, $i = -3, \ldots, N - 1$ have finite supports, in the sum (24) only the four terms with subscripts $i - 3, \ldots, i$ do not vanish on the interval $[x_i, x_{i+1}]$, $i = 1, \ldots, N - 1$. Hence taking into account formulae (17) and (20), we have

$$\sum_{j=i-3}^{i} \bar{c}_j B_j(x) = \Phi_i(x) \sum_{j=i-3}^{i-1} \bar{c}_j M_{i,B_j} + \Psi_i(x) \sum_{j=i-2}^{i} \bar{c}_j M_{i+1,B_j}$$

$$+ \bar{c}_{i-2} \frac{y_{i+1} - x}{y_{i+1} - y_i} + \bar{c}_{i-1} \frac{x - y_i}{y_{i+1} - y_i} = 0, \quad x \in [x_i, x_{i+1}].$$

Since the functions 1, x, Φ_i, Ψ_i are linearly independent on the interval $[x_i, x_{i+1}]$ and $M_{k,B_j} \neq 0$, $k = i, i + 1$, it follows that $\bar{c}_{i-3} = \cdots = \bar{c}_i = 0$. Continuing this process, we find that $\bar{c}_i = 0$ for all i.

Since $\dim(S_4^{DG}) = N + 3$, we see that the discrete GB-splines B_i, $i = -3, \ldots, N - 1$, all of which are elements of the space S_4^{DG}, form a basis of this space. This proves the theorem. $\square$

5. Local approximation by discrete GB-splines

According to theorem 2, any discrete generalized spline $S \in S_4^{DG}$ can be uniquely written in the form

$$S(x) = \sum_{j=-3}^{N-1} b_j B_j(x), \quad x \in [a, b] \tag{25}$$

for some constant coefficients b_j.

If the coefficients b_j in (25) are known, then by virtue of formula (17) we can write out an expression for the discrete generalized spline S on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$S(x) = b_{i-2} + b_{i-1}^{(1)}(x - y_i) + b_{i-1}^{(2)} \Phi_i(x) + b_i^{(2)} \Psi_i(x), \tag{26}$$

where

$$b_j^{(k)} = \frac{b_j^{(k-1)} - b_{j-1}^{(k-1)}}{c_{j,4-k}}, \quad k = 1, 2; \ b_j^{(0)} = b_j. \tag{27}$$

The representations (25) and (26) allow us to find a simple and effective way to approximate a given continuous function f from its samples.

Theorem 3. Let a continuous function f be given by its samples $f(y_j)$, $j = -1, \ldots, N + 1$. Then for $b_j = f(y_{j+2})$, $j = -3, \ldots, N - 1$, formula (25) is exact for polynomials of the first degree and provides a formula for local approximation.

Proof. It suffices to prove that the identities

$$\sum_{j=-3}^{N-1} y_{j+2}^r B_j(x) \equiv x^r, \quad r = 0, 1, \tag{28}$$

hold for $x \in [a, b]$. Using formula (26) with the coefficients $b_{j-2} = 1$ and $b_{j-2} = y_j$, $j = i - 1, i, i + 1, i + 2$, for an arbitrary interval $[x_i, x_{i+1}]$, we find that identities (28) hold.

For $b_{j-2} = f(y_j)$, formula (26) can be rewritten as

$$S(x) = f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1})f[y_{i-1}, y_i, y_{i+1}]c_{i-1,2}^{-1}\Phi_i(x)$$
$$+ (y_{i+2} - y_i)f[y_i, y_{i+1}, y_{i+2}]c_{i,2}^{-1}\Psi_i(x), \quad x \in [x_i, x_{i+1}].$$

This is the formula of local approximation. The theorem is thus proved. □

Corollary 1. Let a continuous function f be given by its samples $f_j = f(x_j)$, $j = -2, \ldots, N + 2$. Then by setting

$$b_{j-2} = f_j - \left(\Psi_{j-1}(x_j)f[x_j, x_{j+1}] - \Phi_j(x_j)f[x_{j-1}, x_j]\right)c_{j-1,2}^{-1} \tag{29}$$

in (25), we obtain a formula of three-point local approximation, which is exact for polynomials of the first degree.

Proof. To prove the corollary, it is sufficient to take the monomials 1 and x as f. Then according to (29), we obtain $b_{j-2} = 1$ and $b_{j-2} = y_j$ and it only remains to make use of the identities (28). This proves the corollary. □

Equation (26) permits us to write the coefficients of the spline S in its representation (25) of the form

$$b_{j-2} = \begin{cases} S(y_j) - D_{j-1,2}S(x_{j-1})\Phi_{j-1}(y_j) - D_{j,2}S(x_j)\Psi_{j-1}(y_j), & y_j < x_j, \\ S(y_j) - D_{j,2}S(x_j)\Phi_j(y_j) - D_{j+1,2}S(x_{j+1})\Psi_j(y_j), & y_j \geqslant x_j. \end{cases}$$

According to this formula we have $b_{j-2} = S(y_j) + O(\bar{h}_j^2)$, $\bar{h}_j = \max(h_{j-1}, h_j)$. Hence it follows that the control polygon (e.g., see [12]) converges quadratically to the function f when $b_{j-2} = f(y_j)$, or if formula (29) is used.

6. Recurrence formulae for discrete GB-splines

Let us define functions

$$B_{j,2}(x) = \begin{cases} D_{j,2}\Psi_j(x), & x \in [x_j, x_{j+1}), \\ D_{j+1,2}\Phi_{j+1}(x), & x \in [x_{j+1}, x_{j+2}], \quad j = i, i+1, i+2. \\ 0, & \text{otherwise}; \end{cases} \tag{30}$$

We assume that the functions $D_{j,2}\Psi_j$ and $D_{j+1,2}\Phi_{j+1}$ are strictly monotone on $[x_j, x_{j+1})$ and $[x_{j+1}, x_{j+2}]$, respectively. The splines $B_{j,2}$ are a generalization of the "hat-functions" for polynomial B-splines. They are nonnegative and, furthermore, $B_{j,2}(x_j) = B_{j,2}(x_{j+2}) = 0$, $B_{j,2}(x_{j+1}) = 1$.

Let us denote

$$\begin{aligned} D_1 S(x) &\equiv D_{i,1} S_i(x), \\ D_2 S(x) &\equiv D_{i,2} S_i(x), \end{aligned} \quad x \in [x_i, x_{i+1}], \ i = 0, \ldots, N-1;$$

then from (5) $D_1 S$ and $D_2 S$ are well-defined if $S \in S_4^{DG}$. With previous notation, according to (17), (20) and (30) we obtain

$$\begin{aligned} D_2 B_i(x) &= \sum_{j=i+1}^{i+3} M_{j,B_i} B_{j-1,2}(x) \\ &= \frac{1}{c_{i,3}} \left(\frac{B_{i,2}(x)}{c_{i,2}} - \frac{B_{i+1,2}(x)}{c_{i+1,2}} \right) - \frac{1}{c_{i+1,3}} \left(\frac{B_{i+1,2}(x)}{c_{i+1,2}} - \frac{B_{i+2,2}(x)}{c_{i+2,2}} \right). \end{aligned} \tag{31}$$

In addition, the function $D_1 B_i$ satisfies the relation

$$D_1 B_i(x) = \frac{B_{i,3}(x)}{c_{i,3}} - \frac{B_{i+1,3}(x)}{c_{i+1,3}}, \tag{32}$$

where

$$B_{j,3}(x) = \begin{cases} \dfrac{D_{j,1}\Psi_j(x)}{c_{j,2}}, & x \in [x_j, x_{j+1}), \\ 1 + \dfrac{D_{j+1,1}\Phi_{j+1}(x)}{c_{j,2}} - \dfrac{D_{j+1,1}\Psi_{j+1}(x)}{c_{j+1,2}}, & x \in [x_{j+1}, x_{j+2}), \\ -\dfrac{D_{j+2,1}\Phi_{j+2}(x)}{c_{j+1,2}}, & x \in [x_{j+2}, x_{j+3}), \\ 0, & \text{otherwise}. \end{cases} \tag{33}$$

Using formula (33) it is easy to show that the functions $B_{j,3}$, $j = -2, \ldots, N-1$, satisfy the first and second smoothness conditions in (5), have supports of minimum length, are linearly independent and form a partition of unity,
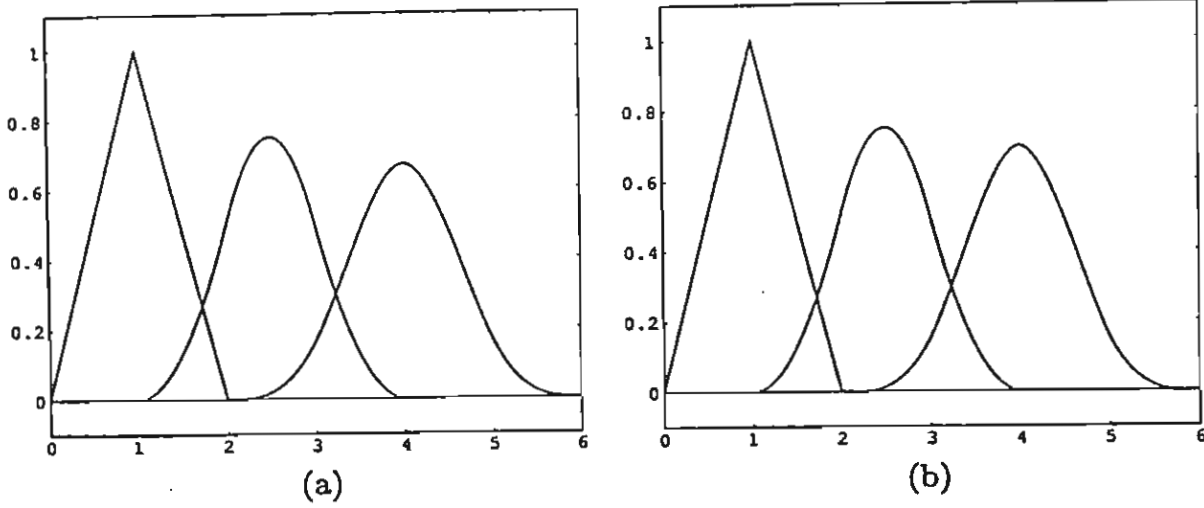
Figure 1. The discrete GB-splines $B_{j,2}$, $B_{j,3}$, and B_j (from left to right) on a uniform mesh with step size $h_i = 1$, no tension and discretization parameter $\tau = 0.1$ (a) and $\tau = 0.33$ (b).

$$\sum_{j=1}^{N-1} B_{j,3}(x) \equiv 1, \quad x \in [a, b].$$

Figures 1 and 2 show the graphs of the discrete GB-splines $B_{j,2}$, $B_{j,3}$, and B_j (from left to right) on a uniform mesh with step size $h_i = 1$ and with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$ for all i. We have chosen discretization parameter $\tau = 0.1$ (figures 1(a) and 2(b)), $\tau = 0.33$ (figure 1(b)) and $\tau = 0.5$ (figure 2(a)) for

$$\Psi_i(x) = \psi_i(q_i, t)h_i^2 = \frac{\hat{\tau}_i \sinh q_i t - t \sinh(q_i \hat{\tau}_i)}{(4/\hat{\tau}_i) \sinh^2(q_i \hat{\tau}_i/2) \sinh q_i} h_i^2, \quad \hat{\tau}_i = \frac{\tau}{h_i},$$

$$\Phi_i(x) = \psi_i(q_i, 1 - t)h_i^2.$$

In figures 1 and 2(a) we have parameters $q_i = 0$, i.e., we have conventional discrete cubic B-splines (e.g., see [18]). Visually, the presence of intervals where the B-spline B_j is negative is more visible with growing discretization parameter τ. In figure 2(b) the tension parameters are $q_i = 50$ for all i, whence the shape of the graphs is practically unchanged when τ increases from 0.1 to 0.5. As the limit for $q_i \to \infty$ we obtain the pulse function for $B_{j,2}$, the "step-function" for $B_{j,3}$ and the "hat-function" for B_j (all of height 1).

Applying formulae (31) and (32) to the representation (25) we also obtain

$$D_1 S(x) = \sum_{j=-2}^{N-1} b_j^{(1)} B_{j,3}(x), \qquad D_2 S(x) = \sum_{j=-1}^{N-1} b_j^{(2)} B_{j,2}(x), \tag{34}$$

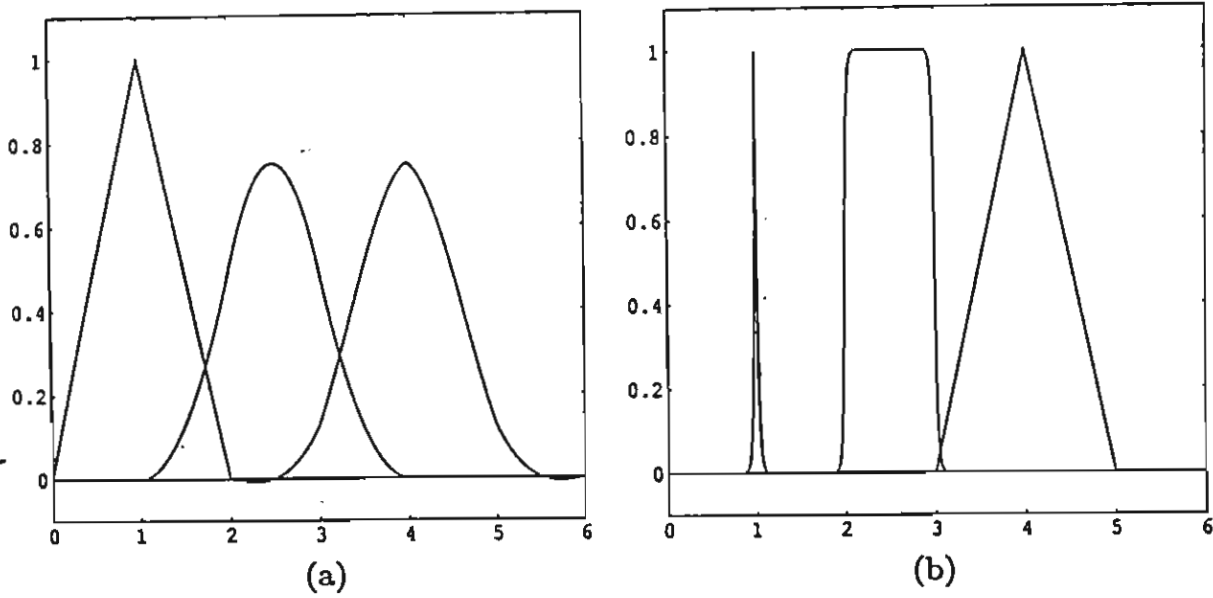where $b_j^{(k)}$, $k = 1, 2$, are defined in (27).

Figure 2. Same as figure 1, but with discretization parameter $\tau = 0.5$ (a) and with tension parameters $q_i = 50$ for all i (b).

7. Series of discrete GB-splines (uniform case)

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh Δ: $a = x_0 < x_1 < \cdots < x_N = b$ is an integer multiple of the same tabulation step, τ, of some uniform mesh refinement on $[a, b]$.

For $\theta \in \mathbb{R}, \tau > 0$ define

$$\mathbb{R}_{\theta\tau} = \{\theta + i\tau: i \text{ is an integer}\}$$

and let $\mathbb{R}_{\theta 0} = \mathbb{R}$. For any $a, b \in \mathbb{R}$ and $\tau > 0$ let

$$[a, b]_\tau = [a, b] \cap \mathbb{R}_{a\tau}.$$

The functions $B_{j,2}$, $B_{j,3}$ and B_j with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i + 1$, for all i are nonnegative on the discrete interval $[a, b]_\tau$. This permits us to reprove the main results for discrete polynomial splines in [35] for series of discrete generalized splines. Even more, one can obtain the results of generalized splines in [13] from the corresponding statements for discrete generalized splines as a limiting case when $\tau \to 0$.

In particular, if in (25) and (34) we have coefficients $b_j^{(k)} > 0$, $k = 0, 1, 2$, $j = -3 + k, \ldots, N - 1$, then the spline S will be a positive, monotonically increasing and convex function on $[a, b]_\tau$.

Let f be a function defined on the discrete set $[a, b]_\tau$. We say that f has a zero at the point $x \in [a, b]_\tau$ provided

$$f(x) = 0 \quad \text{or} \quad f(x - \tau)f(x) < 0.$$

When f vanishes at a set of consecutive points of $[a, b]_\tau$, say f is 0 at $x, \ldots, x + (r-1)\tau$, but $f(x-\tau)f(x+r\tau) \neq 0$, then we call the set $X = \{x, x+\tau, \ldots, x+(r-1)\tau\}$ a *multiple zero* of f, and we define its multiplicity by

$$Z_X(f) = \begin{cases} r, & \text{if } f(x-\tau)f(x+r\tau) < 0 \text{ and } r \text{ is odd,} \\ r, & \text{if } f(x-\tau)f(x+r\tau) > 0 \text{ and } r \text{ is even,} \\ r+1, & \text{otherwise.} \end{cases}$$

This definition assures that f changes sign at a zero if and only if the zero is of odd multiplicity.

Let $Z_{[a,b]_\tau}(f)$ be the number of zeros of a function f on the discrete set $[a, b]_\tau$, counted according to their multiplicity. Let us denote $D_1^L S(x) = S[x - \tau, x]$.

Theorem 4 (Rolle's theorem for discrete generalized splines). For any $S \in S_4^{DG}$,

$$Z_{[a,b]_\tau}(D_1^L S) \geqslant Z_{[a,b]_\tau}(S) - 1. \tag{35}$$

Proof. First, if S has a z-tuple zero on the set $X = \{x, \ldots, x + (r-1)\tau\}$, it follows that $D_1^L S$ has a $(z-1)$-tuple zero on the set $X' = \{x+\tau, \ldots, x + (r-1)\tau\}$. Now if X^1 and X^2 are two consecutive zero sets of S, then it is trivially true that $D_1^L S$ must have a sign change at some point between X^1 and X^2. Counting all of these zeros, we arrive at the assertion (35). This completes the proof. $\square$

Lemma 4. Let the functions $D_{i,2}\Phi_i$ and $D_{i,2}\Psi_i$ be strictly monotone on the interval $[x_i, x_{i+1}]$ for all i. Then for every $S \in S_4^{DG}$ which is not identically zero on any interval $[x_i, x_{i+1}]_\tau$, $i = 0, \ldots, N-1$,

$$Z_{[a,b]_\tau}(S) \leqslant N + 2.$$

Proof. According to (30) and (34), the function $D_2 S$ has no more than one zero on $[x_i, x_{i+1}]$, because the functions $D_{i,2}\Phi_i$ and $D_{i,2}\Psi_i$ are strictly monotone and nonnegative on this interval. Hence $Z_{[a,b]_\tau}(D_2 S) \leqslant N$. Then according to the Rolle's theorem 4, we find $Z_{[a,b]_\tau}(S) \leqslant N + 2$. This completes the proof. $\square$

Denote by $\text{supp}_\tau B_i = \{x \in \mathbb{R}_{a,\tau} \mid B_i(x) > 0\}$ the discrete support of the spline B_i, i.e., the discrete set $(x_i + \tau, x_{i+4} - \tau)_\tau$.

Theorem 5. Assume that $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ are prescribed points on the discrete line $\mathbb{R}_{a,\tau}$. Then

$$D = \det(B_i(\zeta_j)) \geqslant 0, \quad i, j = -3, \ldots, N-1,$$

and strict positivity holds if and only if

$$\zeta_i \in \text{supp}_\tau B_i, \quad i = -3, \ldots, N-1. \tag{36}$$

The proof of this theorem is based on lemma 4 and repeats that of [35, theorem 8.66, p. 355]. The following three statements follow immediately from theorem 5.

Corollary 2. The system of discrete GB-splines $\{B_j\}$, $j = -3, \ldots, N - 1$, associated with knots on $\mathbb{R}_{a,\tau}$ is a weak Chebyshev system according to the definition given in [35, p. 36], i.e., for any $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ in $\mathbb{R}_{a,\tau}$ we have $D \geqslant 0$ and $D > 0$ if and only if condition (36) is satisfied. In the latter case the discrete generalized spline $S(x) = \sum_{j=-3}^{N-1} b_j B_j(x)$ has no more than $N + 2$ zeros.

Corollary 3. If the conditions of theorem 5 are satisfied, then the solution of the interpolation problem

$$S(\zeta_i) = f_i, \quad i = -3, \ldots, N - 1, \ f_i \in \mathbb{R}, \tag{37}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be a rectangular $m \times n$ matrix with $m \leqslant n$. The matrix A is said to be totally nonnegative (totally positive) (e.g., see [11]) if the minors of all order of the matrix are nonnegative (positive), i.e., for all $1 \leqslant p \leqslant m$ we have

$$\det(a_{i_k j_l}) \geqslant 0 \ (> 0) \quad \text{for all } 1 \leqslant i_1 < \cdots < i_p \leqslant m, \ 1 \leqslant j_1 < \cdots < j_p \leqslant n.$$

Corollary 4. For arbitrary integers $-3 \leqslant \nu_{-3} < \cdots < \nu_{p-4} \leqslant N - 1$ and $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{p-4}$ in $\mathbb{R}_{a,\tau}$ we have

$$\overline{D}_p = \det\{B_{\nu_i}(\zeta_j)\} \geqslant 0, \quad i, j = -3, \ldots, p - 4,$$

and strict positivity holds if and only if

$$\zeta_i \in \mathrm{supp}_\tau B_{\nu_i}, \quad i = -3, \ldots, p - 4,$$

i.e., the matrix $\{B_j(\zeta_i)\}$, $i, j = -3, \ldots, N - 1$, is totally nonnegative.

The last statement is proved by induction based on theorem 5 and the recurrence relations for the minors of the matrix $\{B_j(\zeta_i)\}$. The proof does not differ from that of [35, theorem 8.67, p. 356].

Since the supports of discrete GB-splines are finite, the matrix of system (37) is banded and has seven nonzero diagonals in general. The matrix is tridiagonal if $\zeta_i = x_{i+2}$, $i = -3, \ldots, N - 1$.

An important particular case of the problem, in which $S'(x_i) = f_i'$, $i = 0, N$, can be obtained by passing to the limit as $\zeta_{-3} \to \zeta_{-2}$, $\zeta_{N-1} \to \zeta_{N-2}$.

De Boor and Pinkus [8] proved that linear systems with totally nonnegative matrices can be solved by Gaussian elimination without pivoting. Thus, the system (37) can be solved effectively by the conventional Gauss method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \ldots, v_n)$, with zeros being neglected. Karlin [11] showed that if a matrix A is totally nonnegative then it decreases the variation, i.e.,

$$S^-(A\mathbf{v}) \leqslant S^-(\mathbf{v}).$$

By virtue of corollary 4, the totally nonnegative matrix $\{B_j(\zeta_i)\}$, $i, j = -3, \ldots, N - 1$, formed by discrete GB-splines decreases the variation.

For a bounded real function f, let $S^-(f)$ be the number of sign changes of the function f on the real axis $\mathbb{R}$, without taking into account the zeros

$$S^-(f) = \sup_n S^-\big[f(\zeta_1), \ldots, f(\zeta_n)\big], \quad \zeta_1 < \zeta_2 < \cdots < \zeta_n.$$

Theorem 6. The discrete generalized spline $S(x) = \sum_{j=-3}^{N-1} b_j B_j(x)$ is a variation diminishing function, i.e. the number of sign changes of S does not exceed that in the sequence of its coefficients:

$$S^-\left(\sum_{j=-3}^{N-1} b_j B_j\right) \leqslant S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-3}, \ldots, b_{N-1}).$$

The proof of this statement does not differ from that of [35, theorem 8.68, p. 356] for discrete polynomial B-splines.

By theorem 6, the spline

$$S_f(x) = \sum_{j=-3}^{N-1} f(y_{j+2}) B_j(x)$$

is a variation diminishing function. It enables us to write the inequalities

$$S^-(S_f) \leqslant S^-(\bar{\mathbf{f}}) \leqslant S^-(f),$$

where $\bar{\mathbf{f}} = (f(y_{-1}), \ldots, f(y_{N+1}))$.

Since, in addition, by theorem 3, the locally approximating discrete generalized spline S_f is also exact for polynomials l of first order, we arrive at the inequality

$$S^-(S_f - l) = S^-(S_{f-l}) \leqslant S^-(f - l).$$

Thus, the following statement is true.

Theorem 7. Let a continuous function f be given by its samples $f(y_j)$, $j = -1$, $\ldots, N + 1$. If $b_j = f(y_{j+2})$, $j = -3, \ldots, N - 1$, then the locally approximating discrete generalized spline S_f intersects an arbitrary straight line at most as often as the function f.

8.   Examples of defining functions

Let us give some choices of the defining functions Φ_i and Ψ_i for discrete generalized splines that conform to the sufficiency conditions derived earlier in the paper.

Putting

$$\Psi_i(x) = \psi_i(t)h_i^2 = \psi\big(q_i, \hat{\tau}_i^{L_i}, \hat{\tau}_i^{R_i}, t\big)h_i^2, \qquad \Phi_i(x) = \psi\big(p_i, \hat{\tau}_{i+1}^{R_i}, \hat{\tau}_{i+1}^{L_i}, 1-t\big)h_i^2,$$

$$\hat{\tau}_j^{L_i} = \frac{\tau_j^{L_i}}{h_i}, \qquad \hat{\tau}_j^{R_i} = \frac{\tau_j^{R_i}}{h_i}, \quad j = i, i+1, \ 0 \leqslant p_i, q_i < \infty,$$

we consider some possibilities for choosing the functions ψ_i which, due to the constraints (2), satisfy the conditions

$$\psi_i\big(-\hat{\tau}_i^{L_i}\big) = \psi_i(0) = \psi_i\big(\hat{\tau}_i^{R_i}\big) = 0, \qquad D_{i+1,2}\psi_i(1) = h_i^{-2}. \tag{38}$$

1. Discrete rational spline with linear denominator:

$$\psi_i(t) = C_i\frac{(t + \hat{\tau}_i^{L_i})t(t - \hat{\tau}_i^{R_i})}{1 + q_i(1-t)}.$$

2. Discrete rational spline with quadratic denominator:

$$\psi_i(t) = C_i\frac{(t + \hat{\tau}_i^{L_i})t(t - \hat{\tau}_i^{R_i})}{1 + q_i t(1-t)}.$$

3. Discrete exponential spline:

$$\psi_i(t) = C_i\big(t + \hat{\tau}_i^{L_i}\big)t\big(t - \hat{\tau}_i^{R_i}\big)\exp\big(-q_i(1-t)\big).$$

4. Discrete hyperbolic spline:

$$\psi_i(t) = C_{i,1}\left[\sinh q_i t - t\frac{\sinh q_i\hat{\tau}_i^{R_i}}{\hat{\tau}_i^{R_i}}\right]$$

$$+ C_{i,2}\left[\cosh q_i t - 1 - t\frac{\cosh q_i\hat{\tau}_i^{R_i} - 1}{\hat{\tau}_i^{R_i}}\right].$$

5. Discrete cubic spline with additional knots:

$$\psi_i(t) = \frac{1}{2}\frac{(t - \beta_i + \hat{\tau}_i^{L_i})(t - \beta_i)_+(t - \beta_i - \hat{\tau}_i^{R_i})}{3(1 - \beta_i) + \hat{\varepsilon}_{i+1} - \hat{\varepsilon}_i},$$

$$\hat{\varepsilon}_j = \hat{\tau}_j^{R_i} - \hat{\tau}_j^{L_i}, \quad j = i, i+1, \ \beta_i = 1 - (1 + q_i)^{-1}, \ E_+ = \max(0, E).$$

The points $x_i + \alpha_i h_i$ ($\alpha_i = (1+p_i)^{-1}$) and $x_i + \beta_i h_i$ fix the position of two additional knots of the spline on the interval $[x_i, x_{i+1}]$. By moving these knots one can perform a transfer from a discrete cubic spline to piecewise linear interpolation.

6. Discrete spline of variable order:

$$\psi_i(t) = C_i\big(t + \hat{\tau}_i^{L_i}\big)t^{k_i}\big(t - \hat{\tau}^{R_i}\big), \quad k_i = 1 + q_i.$$

The constants C_i in the expressions for the function ψ_i above are calculated from the condition (38) for the second divided difference of ψ_i. To find $C_{i,k}$, $k = 1, 2$, one needs additionally use the condition $\psi_i(-\hat{\tau}_i^{L_i}) = 0$. It is easy to check that in all cases 1–6. we get the corresponding defining functions in [15] by setting $\hat{\tau}_j^{L_i} = \hat{\tau}_j^{R_i} = 0$, $j = i, i + 1$.

Acknowledgements

References

[1] H. Akima, A new method of interpolation and smooth curve fitting based on local procedures, J. Assoc. Comput. Mech. 17 (1970) 589–602.

[2] P.H. Astor and C.S. Duris, Discrete *L*-splines, Numer. Math. 22 (1974) 393–402.

[3] J.C. Clement, Convexity-preserving piecewise rational cubic interpolation, SIAM J. Numer. Anal. 27 (1990) 1016–1023.

[4] E. Cohen, T. Lyche and R. Riesenfeld, Discrete B-splines and subdivision techniques in computer aided geometric design and computer graphics, Comput. Graph. Image Process. 14 (1980) 87–111.

[5] P. Costantini, B.I. Kvasov and C. Manni, On discrete hyperbolic tension splines, Adv. Comput. Math. 11 (1999) 331–354.

[6] W. Dahmen and Ch.A. Micchelli, Algebraic properties of discrete box splines, Constr. Approx. 3 (1987) 209–221.

[7] C. de Boor, Splines as linear combinations of B-splines: A survey, in: *Approximation Theory II*, eds. G.G. Lorentz, C.K. Chui and L.L. Schumaker (Academic Press, New York, 1976) pp. 1–47.

[8] C. de Boor and A. Pinkus, Backward error analysis for totally positive linear systems, Numer. Math. 27 (1977) 485–490.

[9] H.P. Dikshit and P. Powar, Discrete cubic spline interpolation, Numer. Math. 40 (1982) 71–78.

[10] H.P. Dikshit and S.S. Rana, Discrete cubic spline interpolation over a nonuniform mesh, Rocky Mountain J. Math. 17 (1987) 709–718.

[11] S. Karlin, *Total Positivity*, Vol. 1 (Stanford Univ. Press, Stanford, CA, 1968).

[12] P.E. Koch and T. Lyche, Exponential B-splines in tension, in: *Approximation Theory VI: Proc. of the 6th Internat. Symposium on Approximation Theory*, Vol. II, eds. C.K. Chui, L.L. Schumaker and J.D. Ward (Academic Press, Boston, 1989) pp. 361–364.

[13] B.I. Kvasov, Local bases for generalized cubic splines, Russian J. Numer. Anal. Math. Modelling 10 (1995) 49–80.

[14] B.I. Kvasov, Shape preserving spline approximation via local algorithms, in: *Advanced Topics in Multivariate Approximation*, eds. F. Fontanella, K. Jetter and P.J. Laurent (World Scientific, Singapore, 1996) pp. 181–196.

[15] B.I. Kvasov, GB-splines and their properties, Ann. Numer. Math. 3 (1996) 139–149.

[16] B.I. Kvasov, Algorithms for shape preserving local approximation with automatic selection of tension parameters, Comput. Aided Geom. Design 17 (2000) 17–37.

[17] B.I. Kvasov, *Methods of Shape-Preserving Spline Approximation* (World Scientific, Singapore, 2000).

[18] T. Lyche, Discrete polynomial spline approximation methods, Thesis, University of Texas, Austin (1975). For a summary, see *Spline Functions*, eds. K. Böhmer, G. Meinardus and W. Schimpp, Karlsruhe, 1975, Lectures Notes in Mathematics, Vol. 501 (Springer, Berlin, 1976) pp. 144–176.

[19] T. Lyche, Discrete cubic spline interpolation, BIT 16 (1976) 281–290.

[20] M.A. Malcolm, On computation of nonlinear spline functions, SIAM J. Numer. Anal. 14 (1977) 254–282.

[21] O.L. Mangasarian and L.L. Schumaker, Discrete splines via mathematical programming, SIAM J. Control 9 (1971) 174–183.

[22] O.L. Mangasarian and L.L. Schumaker, Best summation formulae and discrete spline, SIAM J. Numer. Anal. 10 (1973) 448–459.

[23] B.J. McCartin, Computation of exponential splines, SIAM J. Sci. Statist. Comput. 11 (1990) 242–262.

[24] A.A. Melkman, Another proof of the total positivity of the discrete spline collocation matrix, J. Approx. Theory 84 (1996) 247–264.

[25] K.M. Mørken, On total positivity of the discrete spline collocation matrix, J. Approx. Theory 84 (1996) 247–264.

[26] S.S. Rana, Convergence of a class of deficient interpolatory splines, Rocky Mountain J. Math. 14 (1988) 825–831.

[27] S.S. Rana and Y.P. Dubey, Local behaviour of the deficient discrete cubic spline interpolator, J. Approx. Theory 86 (1996) 120–127.

[28] R. Renka, Interpolatory tension splines with automatic selection of tension factors, SIAM J. Sci. Statist. Comput. 8 (1987) 393–415.

[29] P. Rentrop, An algorithm for the computation of exponential splines, Numer. Math. 35 (1980) 81–93.

[30] P. Rentrop and U. Wever, Computational strategies for the tension parameters of the exponential spline, in: *Lecture Notes in Control and Information Science*, Vol. 95 (1987) pp. 122–134.

[31] A. Ron, Exponential box splines, Constructive Approx. 4 (1988) 357–378.

[32] N.S. Sapidis and P.D. Kaklis, An algorithm for constructing convexity and monotonicity-preserving splines in tension, Comput. Aided Geom. Design 5 (1988) 127–137.

[33] N.S. Sapidis, P.D. Kaklis and T.A. Loukakis, A method for computing the tension parameters in convexity-preserving spline-in-tension interpolation, Numer. Math. 54 (1988) 179–192.

[34] L.L. Schumaker, Constructive aspects of discrete polynomial spline functions, in: *Approximation Theory*, ed. G.G. Lorentz (Academic Press, New York, 1973) pp. 469–476.

[35] L.L. Schumaker, *Spline Functions: Basic Theory* (Wiley, New York, 1981).

[36] A.F. Shampine, R.C. Allen, Jr. and S. Pruess, *Fundamentals of Numerical Computing* (Wiley, New York, 1997).

[37] Yu.S. Zav'yalov, B.I. Kvasov and V.L. Miroshnichenko, *Methods of Spline Functions* (Nauka, Moscow, 1980) (in Russian).

Discrete GB-Splines and Their Properties

Boris I. Kvasov

Abstract. Discrete generalized splines are continuous piecewise defined functions which meet some smoothness conditions for the first and second divided differences at the knots. Direct algorithms and recurrence relations are proposed for constructing discrete generalized B-splines (discrete GB-splines for short). Properties of discrete GB-splines and their series are studied. It is shown that discrete GB-splines form weak Chebyshev systems and that series of discrete GB-splines have a variation diminishing property.

§1. Introduction

The tools of generalized splines and GB-splines are widely used in solving problems of shape preserving approximation (e.g., see [6]). By introducing various parameters into the spline structure, one can preserve characteristics of the initial data such as positivity, monotonicity, convexity, presence of linear sections, etc. Here, the main challenge is to develop algorithms that choose parameters automatically. Recently, in [2] a difference method for constructing shape preserving hyperbolic splines as solutions of multipoint boundary value problems was developed. Such an approach permits to avoid the computation of hyperbolic functions and has substantial other advantages. However, the extension of a mesh solution will be a discrete hyperbolic spline. In this paper we consider more general construction of discrete generalized splines and discrete GB-splines and investigate their main properties.

§2. Discrete Generalized Splines

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ be given. For fixed $\tau_j^{L_i} > 0$ and $\tau_j^{R_i} > 0$, $j = i, i+1$, and a function S which is defined and continuous on the real line $\mathbb{R}$ we introduce the linear difference operators

$$
\begin{aligned}
D_{i,1}S(x) =& (\lambda_i^{R_i} S[x - \tau_i^{L_i}, x] + \lambda_i^{L_i} S[x, x + \tau_i^{R_i}])(1 - t) \\
&+ (\lambda_{i+1}^{R_i} S[x - \tau_{i+1}^{L_i}, x] + \lambda_{i+1}^{L_i} S[x, x + \tau_{i+1}^{R_i}])t, \\
D_{i,2}S(x) =& 2S[x - \tau_i^{L_i}, x, x + \tau_i^{R_i}](1 - t) + 2S[x - \tau_{i+1}^{L_i}, x, x + \tau_{i+1}^{R_i}]t, \\
& x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N - 1,
\end{aligned}
$$

where $\lambda_j^{R_i} = 1 - \lambda_j^{L_i} = \tau_j^{R_i}/(\tau_j^{L_i} + \tau_j^{R_i})$, $j = i, i+1$ and $t = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$. The square parentheses denote the usual first and second divided differences of the function S.

We associate to Δ a system of functions $\{1, x, \Phi_i, \Psi_i\}$, $i = 0, \ldots, N-1$, which are defined and continuous on $\mathbb{R}$ and for given i are linearly independent on the interval $[x_i, x_{i+1}]$. The functions Φ_i and Ψ_i are subject to the constraints

$$\begin{aligned}
\Phi_i(x_{i+1} - \tau_{i+1}^{L_i}) = \Phi_i(x_{i+1}) = \Phi_i(x_{i+1} + \tau_{i+1}^{R_i}) = 0, \quad & D_{i,2}\Phi_i(x_i) = 1, \\
\Psi_i(x_i - \tau_i^{L_i}) = \Psi_i(x_i) = \Psi_i(x_i + \tau_i^{R_i}) = 0, \quad & D_{i,2}\Psi_i(x_{i+1}) = 1.
\end{aligned} \tag{1}$$

Any element S_i of the linear space Υ_i spanned by the four functions 1, x, Φ_i, Ψ_i can be uniquely written as follows

$$\begin{aligned}
S_i(x) = & S_i(x_i)(1-t) + S_i(x_{i+1})t + D_{i,2}S_i(x_i)[\Phi_i(x) - \Phi_i(x_i)(1-t)] \\
& + D_{i,2}S_i(x_{i+1})[\Psi_i(x) - \Psi_i(x_{i+1})t].
\end{aligned} \tag{2}$$

Definition 1. *A function $S : [a, b] \to \mathbb{R}$ is called a discrete generalized spline if:*

(i) *for any integer i, $0 \le i \le N-1$, there exist a unique function $S_i \in \Upsilon_i$ such that*
$$S(x) \equiv S_i(x), \quad x \in [x_i, x_{i+1}]; \tag{3}$$

(ii) *for all integers $i = 1, \ldots, N-1$, S satisfies the following smoothness conditions*
$$\begin{aligned}
S_{i-1}(x_i) &= S_i(x_i), \\
D_{i-1,1}S_{i-1}(x_i) &= D_{i,1}S_i(x_i), \\
D_{i-1,2}S_{i-1}(x_i) &= D_{i,2}S_i(x_i).
\end{aligned} \tag{4}$$

The set of discrete generalized splines satisfying definition 1 will be denoted by S_4^{DG}. It is clear that this set consists of functions whose restrictions to $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$ belong to the linear space Υ_i and that each function in S_4^{DG} is smooth in the sense of (4), that is, it is continuous and the values of its first and second divided differences with respect to the points $x_i - \tau_i^{L_{i-1}}$, x_i, $x_i + \tau_i^{R_{i-1}}$ and $x_i - \tau_i^{L_i}$, x_i, $x_i + \tau_i^{R_i}$ coincide. The usual operations of addition of elements from S_4^{DG} and their multiplication by real numbers give again elements in the set S_4^{DG} which hence is a linear space.

The definition 1 generalizes the notion of discrete polynomial splines in [7] and of generalized splines in [5]. The latter type can be obtained by setting $\tau_j^{L_i} = \tau_j^{R_i} = 0$, $j = i, i+1$ for all i. If $\tau_i^{L_j} = \tau_i^L$ and $\tau_i^{R_j} = \tau_i^R$, $j = i-1, i$ then according to smoothness conditions (4) the values of the functions S_{i-1} and S_i at the three consecutive points $x_i - \tau_i^L$, x_i, $x_i + \tau_i^R$ coincide. Setting $\tau_j^{L_i} = \tau_j^{R_i} = \tau_i$, $j = i, i+1$ we obtain $D_{i,1}S(x) = S[x - \tau_i, x + \tau_i]$ and $D_{i,2}S(x) = 2S[x - \tau_i, x, x + \tau_i]$, which is the case discussed in [2].

According to the conditions (4), the discrete generalized spline S defined by (2) and (3) can be written as follows

$$\begin{aligned}
S(x) = & S(x_i)(1-t) + S(x_{i+1})t + M_i[\Phi_i(x) - \Phi_i(x_i)(1-t)] \\
& + M_{i+1}[\Psi_i(x) - \Psi_i(x_{i+1})t], \quad x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N-1,
\end{aligned} \tag{5}$$

where $M_j = D_{i,2}S(x_j)$, $j = i, i+1$.

The functions Φ_i and Ψ_i depend on tension parameters which influence the behaviour of S fundamentally. We call them the defining functions. In practice one takes Φ_i to depend on a parameter p_i and Ψ_i to depend on a parameter q_i, $0 \leq p_i, q_i < \infty$. In the limiting case when $p_i, q_i \to \infty$ we require that $\lim_{p_i \to \infty} \Phi_i(x) = 0$, $x \in (x_i, x_{i+1}]$ and $\lim_{q_i \to \infty} \Psi_i(x) = 0$, $x \in [x_i, x_{i+1})$ so that the function S in formula (5) turns into a linear function. Additionally, we require that if $p_i = q_i = 0$ for all i, then we get a discrete cubic spline.

§3. Construction of Discrete GB-Splines

Let us construct a basis for the space of discrete generalized splines S_4^{DG} by using functions which have local supports of minimum length. Since $\dim(S_4^{DG}) = N + 3$ we extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$, such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$. As in section 2 for each interval $[x_i, x_{i+1}]$, $i = -3, -2, -1, N, N+1, N+2$ we introduce the linear space Υ_i. This permits us to define the discrete generalized spline S on the extended interval $[x_{-3}, x_{N+3}]$.

We demand that the discrete GB-splines B_i, $i = -3, \ldots, N - 1$ have the properties

$$B_i(x) > 0, \quad x \in (x_i + \tau_i^{R_i}, x_{i+4} - \tau_{i+4}^{L_{i+3}}),$$
$$B_i(x) \equiv 0, \quad x \notin (x_i, x_{i+4}),$$
$$\sum_{j=-3}^{N-1} B_j(x) \equiv 1, \quad x \in [a, b]. \tag{6}$$

According to (5), on the interval $[x_j, x_{j+1}]$, $j = i, \ldots, i+3$, the discrete GB-spline B_i, $i = -3, \ldots, N - 1$ has the form

$$B_i(x) \equiv B_{i,j}(x) = P_{i,j}(x) + \Phi_j(x)M_{j,B_i} + \Psi_j(x)M_{j+1,B_i}, \tag{7}$$

where $P_{i,j}$ is a polynomial of the first degree and $M_{l,B_i} = D_{j,2}B_i(x_l)$, $l = j, j+1$ are constants to be determined.

The smoothness conditions (4) together with the constraints (1) give the following relations

$$P_{i,j}(x_j) = P_{i,j-1}(x_j) + z_j M_{j,B_i},$$
$$D_{j,1}P_{i,j}(x_j) = D_{j-1,1}P_{i,j-1}(x_j) + c_{j-1,2}M_{j,B_i},$$

where

$$z_j \equiv z_j(x_j) = \Psi_{j-1}(x_j) - \Phi_j(x_j),$$
$$c_{j-1,2} = D_{j-1,1}\Psi_{j-1}(x_j) - D_{j,1}\Phi_j(x_j).$$

Thus

$$P_{i,j}(x) = P_{i,j-1}(x) + [z_j + c_{j-1,2}(x - x_j)]M_{j,B_i}. \tag{8}$$

According to (4), the condition $B_i(x) \equiv 0$ for $x \notin (x_i, x_{i+4})$ is satisfied if and only if

$$B_i(x_i) = D_{i,1}B_i(x_i) = D_{i,2}B_i(x_i) = 0,$$
$$B_i(x_{i+4}) = D_{i+3,1}B_i(x_{i+4}) = D_{i+3,2}B_i(x_{i+4}) = 0,$$

Due to (7) and (1), the latter relations are equivalent to

$$P_{i,i} \equiv 0, \quad M_{i,B_i} = 0 \quad \text{and} \quad P_{i,i+3} \equiv 0, \quad M_{i+4,B_i} = 0.$$

Therefore, by repeated use of (8) we obtain

$$P_{i,j}(x) = \sum_{l=i+1}^{j} [z_l + c_{l-1,2}(x - x_l)]M_{l,B_i} = - \sum_{l=j+1}^{i+3} [z_l + c_{l-1,2}(x - x_l)]M_{l,B_i}.$$

In particular, the following identity is valid,

$$\sum_{j=i+1}^{i+3} [z_j + c_{j-1,2}(x - x_j)]M_{j,B_i} \equiv 0,$$

from which one obtains the equalities

$$\sum_{j=i+1}^{i+3} c_{j-1,2}y_j^r M_{j,B_i} = 0, \quad r = 0, 1, \quad y_j = x_j - \frac{z_j}{c_{j-1,2}}. \tag{9}$$

Thus the formula for the discrete GB-spline B_i takes the form

$$B_i(x) = \begin{cases} \Psi_i(x)M_{i+1,B_i}, & x \in [x_i, x_{i+1}), \\ (x - y_{i+1})c_{i,2}M_{i+1,B_i} + \Phi_{i+1}(x)M_{i+1,B_i} + \Psi_{i+1}(x)M_{i+2,B_i}, \\ \qquad\qquad x \in [x_{i+1}, x_{i+2}), \\ (y_{i+3} - x)c_{i+2,2}M_{i+3,B_i} + \Phi_{i+2}(x)M_{i+2,B_i} + \Psi_{i+2}(x)M_{i+3,B_i}, \\ \qquad\qquad x \in [x_{i+2}, x_{i+3}), \\ \Phi_{i+3}(x)M_{i+3,B_i}, & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Substituting formula (10) into the normalization condition (6) written for $x \in [x_i, x_{i+1}]$, $i = 0, \ldots, N - 1$, we obtain

$$\sum_{j=i-3}^{i} B_j(x) = \Phi_i(x) \sum_{j=i-3}^{i-1} M_{i,B_j} + \Psi_i(x) \sum_{j=i-2}^{i} M_{i+1,B_j}$$
$$+ (y_{i+1} - x)c_{i,2}M_{i+1,B_{i-2}} + (x - y_i)c_{i-1,2}M_{i,B_{i-1}} \equiv 1.$$

Due to the linear independence of functions $1, x, \Phi_i$, and Ψ_i on $[x_i, x_{i+1}]$, the latter relation is satisfied if and only if

$$\sum_{j=i-3}^{i-1} M_{i,B_j} = \sum_{j=i-2}^{i} M_{i+1,B_j} = 0, \tag{11}$$

$$y_{i+1}c_{i,2}M_{i+1,B_{i-2}} - y_i c_{i-1,2}M_{i,B_{i-1}} = 1,$$
$$c_{i,2}M_{i+1,B_{i-2}} - c_{i-1,2}M_{i,B_{i-1}} = 0. \tag{12}$$

In particular, from (12) we derive the following identity

$$(y_{i+1} - x)c_{i,2}M_{i+1,B_{i-2}} + (x - y_i)c_{i-1,2}M_{i,B_{i-1}} \equiv 1.$$

Solving system (12) and using (9) or (11), we obtain

$$M_{j,B_i} = \frac{y_{i+3} - y_{i+1}}{c_{j-1,2}\omega'_{i+1}(y_j)}, \quad j = i+1, i+2, i+3,$$

$$\omega_{i+1}(x) = (x - y_{i+1})(x - y_{i+2})(x - y_{i+3})$$

or with the notation $c_{j,3} = y_{j+2} - y_{j+1}, j = i, i+1$,

$$M_{i+1,B_i} = \frac{1}{c_{i,2}c_{i,3}},$$

$$M_{i+2,B_i} = -\frac{1}{c_{i+1,2}}\left(\frac{1}{c_{i,3}} + \frac{1}{c_{i+1,3}}\right), \tag{13}$$

$$M_{i+3,B_i} = \frac{1}{c_{i+2,2}c_{i+1,3}}.$$

§4. Local Approximation by Discrete GB-Splines

The functions $B_j, j = -3, \ldots, N - 1$ possess many of the properties inherent in the usual discrete polynomial B-splines.

Theorem 1. *Let the conditions*

$$0 < 2\Phi_j(x_j) < -h_i D_{j,1}\Phi_j(x_j), \quad 0 < 2\Psi_j(x_{j+1}) < h_j D_{j,1}\Psi_j(x_{j+1})$$

be satisfied, the functions Φ_j and Ψ_j be convex and $D_{j,2}\Phi_j$ and $D_{j,2}\Psi_j$ be strictly monotone on the interval $[x_j, x_{j+1}]$ for all j. Then the functions B_j, $j = -3, \ldots, N - 1$ have the following properties:

(a) $B_j(x) > 0$ *for* $x \in (x_j + \tau_j^{R_j}, x_{j+4} - \tau_{j+4}^{L_{j+3}})$, *and* $B_j(x) \equiv 0$ *if* $x \notin (x_j, x_{j+4})$;

(b) B_j *satisfies the smoothness conditions (4) and has support of minimum length;*

(c) $\sum_{j=-3}^{N-1} y_{j+2}^r B_j(x) \equiv x^r, r = 0, 1$ *for* $x \in [a, b], \Phi_j(x) = c_{j-1,2}c_{j-2,3}B_{j-3}(x)$, $\Psi_j(x) = c_{j,2}c_{j,3}B_j(x)$ *for* $x \in [x_j, x_{j+1}], j = 0, \ldots, N - 1$;

(d) B_j, $j = -3, \ldots, N - 1$, are linearly independent on $[a, b]$ and form a basis of the space S_4^{DG} of discrete generalized splines.

The proof of this theorem repeats that given in [6].

According to Theorem 1, any discrete generalized spline $S \in S_4^{DG}$ can be uniquely written in the form

$$S(x) = \sum_{j=-3}^{N-1} b_j B_j(x), \quad x \in [a, b] \tag{14}$$

for some constant coefficients b_j.

If the coefficients b_j in (14) are known, then by virtue of formula (10) we can write out an expression for the discrete generalized spline S on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$S(x) = b_{i-2} + b_{i-1}^{(1)}(x - y_i) + b_{i-1}^{(2)}\Phi_i(x) + b_i^{(2)}\Psi_i(x), \tag{15}$$

where

$$b_j^{(k)} = (b_j^{(k-1)} - b_{j-1}^{(k-1)})c_{j,4-k}^{-1}, \quad k = 1, 2; \quad b_j^{(0)} = b_j. \tag{16}$$

The representations (14) and (15) allow us to find a simple and effective way to approximate a given continuous function f from its samples.

Theorem 2. *Let a continuous function f be given by its samples $f(y_j)$, $j = -1, \ldots, N + 1$. Then for $b_j = f(y_{j+2})$, $j = -3, \ldots, N - 1$, formula (14) is exact for polynomials of the first degree and provides a formula for local approximation.*

Proof: It suffices to prove that the identities

$$\sum_{j=-3}^{N-1} y_{j+2}^r B_j(x) \equiv x^r, \quad r = 0, 1, \tag{17}$$

hold for $x \in [a, b]$. Using formula (15) with the coefficients $b_{j-2} = 1$ and $b_{j-2} = y_j$, $j = i - 1, i, i + 1, i + 2$, for an arbitrary interval $[x_i, x_{i+1}]$, we find that identities (17) hold.

For $b_{j-2} = f(y_j)$, formula (15) can be rewritten as

$$S(x) = f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1})f[y_{i-1}, y_i, y_{i+1}]c_{i-1,2}^{-1}\Phi_i(x)$$
$$+ (y_{i+2} - y_i)f[y_i, y_{i+1}, y_{i+2}]c_{i,2}^{-1}\Psi_i(x), \quad x \in [x_i, x_{i+1}].$$

This is the formula of local approximation. The theorem is thus proved. $\square$

Corollary 1. *Let a continuous function f be given by its samples $f_j = f(x_j)$, $j = -2, \ldots, N + 2$. Then by setting*

$$b_{j-2} = f_j - (\Psi_{j-1}(x_j)f[x_j, x_{j+1}] - \Phi_j(x_j)f[x_{j-1}, x_j])c_{j-1,2}^{-1} \tag{18}$$

in (14), we obtain a formula of three-point local approximation, which is exact
for polynomials of the first degree.

Proof: To prove the corollary, it is sufficient to take the monomials 1 and x
as f. Then according to (18), we obtain $b_{j-2} = 1$ and $b_{j-2} = y_j$ and it only
remains to make use of the identities (17). This proves the corollary. $\quad\square$

Equation (15) permits us to write the coefficients of the spline S in its
representation (14) of the form

$$b_{j-2} = \begin{cases} S(y_j) - D_{j-1,2}S(x_{j-1})\Phi_{j-1}(y_j) - D_{j,2}S(x_j)\Psi_{j-1}(y_j), & y_j < x_j, \\ S(y_j) - D_{j,2}S(x_j)\Phi_j(y_j) - D_{j+1,2}S(x_{j+1})\Psi_j(y_j), & y_j \geq x_j. \end{cases}$$

According to this formula we have $b_{j-2} = S(y_j) + O(\overline{h}_j^2)$, $\overline{h}_j = \max(h_{j-1}, h_j)$.
Hence it follows that the control polygon (e.g., see [3]) converges quadratically
to the function f when $b_{j-2} = f(y_j)$, or if the formula (18) is used.

§5. Recurrence Formulae for Discrete GB-Splines

Let us define functions

$$B_{j,2}(x) = \begin{cases} D_{j,2}\Psi_j(x), & x \in [x_j, x_{j+1}), \\ D_{j+1,2}\Phi_{j+1}(x), & x \in [x_{j+1}, x_{j+2}], \quad j = i, i+1, i+2. \\ 0, & \text{otherwise}, \end{cases} \quad (19)$$

We assume that the functions $D_{j,2}\Psi_j$ and $D_{j+1,2}\Phi_{j+1}$ are strictly monotone on
$[x_j, x_{j+1})$ and $[x_{j+1}, x_{j+2}]$ respectively. The splines $B_{j,2}$ are a generalization
of the "hat-functions" for polynomial B-splines. They are nonnegative and,
furthermore, $B_{j,2}(x_j) = B_{j,2}(x_{j+2}) = 0$, $B_{j,2}(x_{j+1}) = 1$.

Let us denote

$$\begin{aligned} D_1 S(x) &\equiv D_{i,1}S_i(x), \\ D_2 S(x) &\equiv D_{i,2}S_i(x), \end{aligned} \quad x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N-1;$$

then from (4) $D_1 S$ and $D_2 S$ are well defined if $S \in S_4^{DG}$. With previous
notation, according to (10), (13) and (19) we obtain

$$\begin{aligned} D_2 B_i(x) &= \sum_{j=i+1}^{i+3} M_{j,B_i} B_{j-1,2}(x) \\ &= \frac{1}{c_{i,3}}\left(\frac{B_{i,2}(x)}{c_{i,2}} - \frac{B_{i+1,2}(x)}{c_{i+1,2}}\right) - \frac{1}{c_{i+1,3}}\left(\frac{B_{i+1,2}(x)}{c_{i+1,2}} - \frac{B_{i+2,2}(x)}{c_{i+2,2}}\right). \end{aligned} \quad (20)$$

In addition, the function $D_1 B_i$ satisfies the relation

$$D_1 B_i(x) = \frac{B_{i,3}(x)}{c_{i,3}} - \frac{B_{i+1,3}(x)}{c_{i+1,3}}, \quad (21)$$

where

$$B_{j,3}(x) = \begin{cases} \dfrac{D_{j,1}\Psi_j(x)}{c_{j,2}}, & x \in [x_j, x_{j+1}), \\[2mm] 1 + \dfrac{D_{j+1,1}\Phi_{j+1}(x)}{c_{j,2}} - \dfrac{D_{j+1,1}\Psi_{j+1}(x)}{c_{j+1,2}}, & x \in [x_{j+1}, x_{j+2}), \\[2mm] -\dfrac{D_{j+2,1}\Phi_{j+2}(x)}{c_{j+1,2}}, & x \in [x_{j+2}, x_{j+3}), \\[2mm] 0, & \text{otherwise.} \end{cases} \quad (22)$$

Using formula (22) it is easy to show that the functions $B_{j,3}$, $j = -2, \ldots, N-1$ satisfy the first and second smoothness conditions in (4), have supports of minimum length, are linearly independent and form a partition of unity,

$$\sum_{j=1}^{N-1} B_{j,3}(x) \equiv 1, \quad x \in [a, b].$$

Applying formulae (20) and (21) to the representation (14) we also obtain

$$D_1 S(x) = \sum_{j=-2}^{N-1} b_j^{(1)} B_{j,3}(x), \quad D_2 S(x) = \sum_{j=-1}^{N-1} b_j^{(2)} B_{j,2}(x), \quad (23)$$

where $b_j^{(k)}$, $k = 1, 2$ are defined in (16).

§6. Series of Discrete GB-Splines (Uniform Case)

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ is an integer multiple of the same tabulation step, τ, of some uniform mesh refinement on $[a, b]$. For $\theta \in \mathbb{R}$, $\tau > 0$ define $\mathbb{R}_{\theta\tau} = \{\theta + i\tau : i \text{ is an integer}\}$ and let $\mathbb{R}_{\theta 0} = \mathbb{R}$. For any $a, b \in \mathbb{R}$ and $\tau > 0$ let $[a, b]_\tau = [a, b] \cap \mathbb{R}_{a\tau}$.

The functions $B_{j,2}$, $B_{j,3}$, and B_j with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$ for all i are nonnegative on the discrete interval $[a, b]_\tau$. This permits us to reprove the main results for discrete polynomial splines in [7] for series of discrete generalized splines. In particular, if in (14) and (23) we have coefficients $b_j^{(k)} > 0$, $k = 0, 1, 2$, $j = -3 + k, \ldots, N-1$, then the spline S will be a positive, monotonically increasing and convex function on $[a, b]_\tau$.

Denote by $supp_\tau B_i = \{x \in \mathbb{R}_{a,\tau} | B_i(x) > 0\}$ the discrete support of the spline B_i, i.e. the discrete set $(x_i + \tau, x_{i+4} - \tau)_\tau$.

Theorem 3. *Assume that* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ *are prescribed points on the discrete line* $\mathbb{R}_{a,\tau}$. *Then*

$$D = \det(B_i(\zeta_j)) \geq 0, \quad i, j = -3, \ldots, N-1$$

and strict positivity holds if and only if

$$\zeta_i \in supp_\tau B_i, \quad i = -3, \ldots, N-1. \quad (24)$$

The proof of this theorem repeats that of theorem 8.66 in [7, p. 355]. The following three statements follow immediately from Theorem 3.

Corollary 2. *The system of discrete GB-splines* $\{B_j\}$, $j = -3, \ldots, N - 1$, *associated with knots on* $\mathbb{R}_{a,\tau}$ *is a weak Chebyshev system according to the definition given in [7, p. 36], i.e. for any* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ *in* $\mathbb{R}_{a,\tau}$ *we have* $D \geq 0$ *and* $D > 0$ *if and only if condition (24) is satisfied. In the latter case the discrete generalized spline* $S(x) = \sum_{j=-3}^{N-1} b_j B_j(x)$ *has no more than* $N + 2$ *zeros.*

Corollary 3. *If the conditions of Theorem 3 are satisfied, then the solution of the interpolation problem*

$$S(\zeta_i) = f_i, \quad i = -3, \ldots, N - 1, \quad f_i \in \mathbb{R} \tag{25}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be a rectangular $m \times n$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally positive) (e.g., see [4]) if the minors of all order of the matrix are nonnegative (positive), i.e. for all $1 \leq p \leq m$ we have

$$det(a_{i_k j_l}) \geq 0 \ (> 0) \quad \text{for all} \quad \begin{array}{l} 1 \leq i_1 < \cdots < i_p \leq m, \\ 1 \leq j_1 < \cdots < j_p \leq n. \end{array}$$

Corollary 4. *For arbitrary integers* $-3 \leq \nu_{-3} < \cdots < \nu_{p-4} \leq N - 1$ *and* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{p-4}$ *in* $\mathbb{R}_{a,\tau}$ *we have*

$$\bar{D}_p = det\{B_{\nu_i}(\zeta_j)\} \geq 0, \quad i, j = -3, \ldots, p - 4$$

and strict positivity holds if and only if

$$\zeta_i \in supp_\tau B_{\nu_i}, \quad i = -3, \ldots, p - 4$$

i.e. the matrix $\{B_j(\zeta_i)\}$, $i, j = -3, \ldots, N - 1$ *is totally nonnegative.*

The last statement is proved by induction based on Theorem 3 and the recurrence relations for the minors of the matrix $\{B_j(\zeta_i)\}$. The proof does not differ from that of Theorem 8.67 described in [7, p. 356].

Since the supports of discrete GB-splines are finite, the matrix of system (25) is banded and has seven nonzero diagonals in general. The matrix is tridiagonal if $\zeta_i = x_{i+2}$, $i = -3, \ldots, N - 1$.

An important particular case of the problem, in which $S'(x_i) = f'_i$, $i = 0, N$, can be obtained by passing to the limit as $\zeta_{-3} \to \zeta_{-2}$, $\zeta_{N-1} \to \zeta_{N-2}$.

De Boor and Pinkus [1] proved that linear systems with totally nonnegative matrices can be solved by Gaussian elimination without pivoting. Thus, the system (25) can be solved effectively by the conventional Gauss method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \cdots, v_n)$, with zeros being neglected. Karlin [4] showed that if a matrix A is totally nonnegative then it decreases the variation, i.e.

$$S^-(A\mathbf{v}) \leq S^-(\mathbf{v}).$$

By virtue of Corollary 4, the totally nonnegative matrix $\{B_j(\zeta_i)\}$, $i, j = -3, \ldots, N-1$, formed by discrete GB-splines decreases the variation.

For a bounded real function f, let $S^-(f)$ be the number of sign changes of the function f on the real axis $\mathbb{R}$, without taking into account the zeros

$$S^-(f) = \sup_n S^-[f(\zeta_1), \ldots, f(\zeta_n)], \quad \zeta_1 < \zeta_2 < \cdots < \zeta_n.$$

Theorem 4. *The discrete generalized spline* $S(x) = \sum_{j=-3}^{N-1} b_j B_j(x)$ *is a variation diminishing function, i.e. the number of sign changes of S does not exceed that in the sequence of its coefficients:*

$$S^-\left(\sum_{j=-3}^{N-1} b_j B_j\right) \leq S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-3}, \ldots, b_{N-1}).$$

The proof of this statement does not differ from that of Theorem 8.68 for discrete polynomial B-splines in [7, p. 356].

References

1. de Boor, C., and A. Pinkus, Backward error analysis for totally positive linear systems, Numer. Math. **27** (1977), 485–490.

2. Costantini, P., B. I. Kvasov, and C. Manni, On discrete hyperbolic tension splines, Advances in Computational Mathematics **11** (1999), 331–354.

3. Hoschek, J., and D. Lasser, *Fundamentals of Computer Aided Geometric Design*, A K Peters, Wellesley, Massachusetts, 1993.

4. Karlin, S., *Total Positivity, Vol. 1*, Stanford University Press, Stanford, CA, 1968.

5. Kvasov, B. I., GB-splines and their properties, Annals of Numerical Mathematics **3** (1996), 139–149.

6. Kvasov, B. I., *Methods of Shape-Preserving Spline Approximation*, World Scientific Publ. Co. Pte. Ltd., Singapore, 2000.

7. Schumaker, L. L., *Spline Functions: Basic Theory*, Wiley, New York, 1981.

Boris I. Kvasov
School of Mathematics
Suranaree University of Technology
111 University Avenue
Nakhon Ratchasima 30000, Thailand
boris@math.sut.ac.th

Difference Method for Solving Shape-Preserving Interpolation Problem

Boris I. Kvasov and Anirut Luadsong

School of Mathematics, Suranaree University of Technology
111 University Avenue, Nakhon Ratchasima 30000, Thailand
Email:{boris,anirut}@math.sut.ac.th

Abstract. Hyperbolic tension splines are defined as solutions of differential multipoint boundary value problem. For computations we use a difference approximation of that problem. This permits to avoid calculations of hyperbolic functions, however, the extension of a mesh solution will be a discrete tension spline. We consider the basic computational aspects of this approach.

Keywords: Hyperbolic tension splines, multipoint boundary value problem, discrete tension splines, shape-preserving interpolation.

1. Introduction

In the theory of splines mainly two approaches are used: algebraic and variational. In the first approach [5] splines are understood as smooth piecewise defined functions. In the second approach [3] splines are solutions of some minimization problems for quadratic functionals with restrictions of equality and/or unequality type. But a third approach is also known [2] where splines are defined as solutions of differential multipoint boundary value problems. In some important cases all three approaches give the same solutions. However the third approach has substantial computational advantages which are illustrated here by the example of hyperbolic tension splines.

For the numerical treatment of differential multipoint boundary value problems we replace the differential operator by its difference approximation. This permits us to avoid calculating hyperbolic functions and to find easily mesh solution whose extension will, however, be a discrete tension spline with continuous differences instead of derivatives.

2. Problem formulation

Let the data

$$(x_i, f_i), \quad i = 0, \ldots, N+1, \tag{1}$$

be given, where: $a = x_0 < x_1 < \cdots < x_{N+1} = b$. Let us put

$$h_i = x_{i+1} - x_i, \quad i = 0, \ldots, N.$$

Interpolating tension spline S with a set of tension parameters $\{p_i \geq 0 \mid i = 0, \ldots, N\}$ is a solution of the differential multipoint boundary value problem

$$\frac{d^4 S}{dx^4} - \left(\frac{p_i}{h_i}\right)^2 \frac{d^2 S}{dx^2} = 0, \quad \text{in each} \quad (x_i, x_{i+1}), \quad i = 0, \ldots, N. \quad (2)$$

$$S \in C^2[a, b], \quad (3)$$

with the interpolation conditions

$$S(x_i) = f_i, \quad i = 0, \ldots, N+1 \quad (4)$$

and the endpoint constraints

$$S''(a) = f_0'' \quad \text{and} \quad S''(b) = f_{N+1}''. \quad (5)$$

For practical purposes it is often more interesting to know the values of the solution over a given tabulation of $[a, b]$ than its global analytic expression. In this paper we do not consider directly a tabulation of S but we study a natural discretization of the previous problem. We prove that the discretized problem has a unique solution, called *mesh solution*, and we study its properties. Of course it turns out that the mesh solution is not a tabulation of S but it can be extended on $[a, b]$ to a function u, with properties very similar to those of S and which approaches S as the discretization step goes to zero. Due to these properties we will refer to u as *discrete* tension spline interpolation of the data (1).

Let us assume that each h_i is an integer multiple of the same tabulation step, τ. Putting $n_i = h_i/\tau$, we look for a mesh solution $\bar{u} = \{u_{i,j} \mid j = -1, \ldots, n_i + 1, \quad i = 0, \ldots, N\}$, satisfying the difference equations:

$$\left[\Lambda^2 - \left(\frac{p_i}{h_i}\right)^2 \Lambda\right] u_{i,j} = 0, \quad j = 1, \ldots, n_i - 1, \quad i = 0, \ldots, N, \quad (6)$$

where

$$\Lambda u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\tau^2}.$$

The smoothness condition (3) is changed for the equations

$$u_{i-1,n_{i-1}} = u_{i,0},$$

$$\frac{u_{i-1,n_{i-1}-1} - u_{i-1,n_{i-1}+1}}{2\tau} = \frac{u_{i,1} - u_{i,-1}}{2\tau}, \quad i = 1, \ldots, N, \quad (7)$$

$$\Lambda u_{i-1,n_{i-1}} = \Lambda u_{i,0},$$

which are equivalent to

$$u_{i-1,n_{i-1}+j} = u_{i,j}, \quad j = -1, 0, 1. \quad (8)$$

The interpolation conditions (4) take the form

$$u_{i,0} = f_i, \quad u_{i,n_i} = f_{i+1}, \quad i = 0, \ldots, N, \tag{9}$$

and for the end conditions (5) we have

$$\Lambda u_{0,0} = f_0'' \quad \text{and} \quad \Lambda u_{N,n_N} = f_{N+1}''. \tag{10}$$

The equalities (8) permit to eliminate the redundant unknowns in the difference equations (6). The values $u_{0,-1}$ and u_{N,n_N+1} are not explicitly computed but are introduced into the formulation to accommodate the two necessary end conditions. Putting $m = \sum_{i=0}^{N} n_i + 3$, the previous equations can be collected in the $m \times m$ linear system

$$A\hat{u} = b, \tag{11}$$

where

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & a_0 & b_0 & a_0 & 1 \\ & 1 & a_0 & b_0 & a_0 & 1 \\ & & & \cdots \\ & & 1 & a_0 & b_0 & a_0 & 1 \\ & & & 0 & 0 & 1 & 0 & 0 \\ & & & & 1 & a_1 & b_1 & a_1 & 1 \\ & & & & & 1 & a_1 & b_1 & a_1 & 1 \\ & & & & & & \cdots \\ & & & & & & & \cdots \\ & & & & & & & 1 & a_N & b_N & a_N & 1 \\ & & & & & & & & 1 & a_N & b_N & a_N & 1 \\ & & & & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & & & & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$a_i = -(4 + \omega_i), \quad b_i = 6 + 2\omega_i, \quad \omega_i = \left(\frac{p_i \tau}{h_i}\right)^2, \quad i = 0, \ldots, N,$$

and

$$\hat{u} = (u_{0,-1}, u_{00}, u_{01}, \ldots, u_{0n_0-1}, u_{10}, \ldots, u_{20}, \ldots, u_{N0}, \ldots, u_{Nn_N}, u_{Nn_N+1})^T,$$

$$b = (\tau^2 f_0'', f_0, 0, \ldots, 0, f_1, \ldots, f_2, \ldots, f_N, \ldots, f_{N+1}, \tau^2 f_{N+1}'')^T.$$

In the previous system the unknowns $u_{i,0}$, $i = 0, \ldots, N+1$, can be immediately determined from the interpolation conditions while the expression of $u_{0,-1}$ (u_{N,n_N+1}) can be obtained from the first (last) equation and substituted in

the third (third to last) equation. Then in practice we deal with the $m^* \times m^*$ linear system ($m^* = m - N - 4$)

$$A^* u^* = b^*, \tag{12}$$

where

$$A^* = \begin{bmatrix} b_0 - 1 & a_0 & 1 & & & & & & & & \\ a_0 & b_0 & a_0 & 1 & & & & & & & \\ 1 & a_0 & b_0 & a_0 & 1 & & & & & & \\ & & \cdots & & & & & & & & \\ & 1 & a_0 & b_0 & a_0 & & & & & & \\ & & 1 & a_0 & b_0 & 1 & & & & & \\ & & & 1 & b_1 & a_1 & 1 & & & & \\ & & & & a_1 & b_1 & a_1 & 1 & & & \\ & & & & & \cdots & & & & & \\ & & & & & 1 & a_N & b_N & a_N & 1 & \\ & & & & & & 1 & a_N & b_N & a_N & \\ & & & & & & & 1 & a_N & b_N - 1 \end{bmatrix},$$

and u^*, b^* are correspondingly deduced from $\hat{u}$ and b.

Following [4] we observe that

$$A^* = C^* + D^*,$$

where both C^* and D^* are symmetric block diagonal matrices; to be more specific,

$$C^* = \begin{bmatrix} C_0 & & & & & \\ & C_1 & & & & \\ & & C_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & C_N \end{bmatrix},$$

$$C_i = \begin{bmatrix} b_i - 1 & a_i & 1 & & & & & \\ a_i & b_i & a_i & 1 & & & & \\ 1 & a_i & b_i & a_i & 1 & & & \\ & & \cdots & & & & & \\ & & & \cdots & & & & \\ & & & 1 & a_i & b_i & a_i & 1 \\ & & & & 1 & a_i & b_i & a_i \\ & & & & & 1 & a_i & b_i - 1 \end{bmatrix}, \tag{13}$$

$$D^{*} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ & & 0 \\ & & & 1 & 1 \\ & & & 1 & 1 \\ & & & & & 0 \\ & & & & & & 0 \\ & & & & & & & 1 & 1 \\ & & & & & & & 1 & 1 \\ & & & & & & & & & 0 \\ & & & & & & & & & & 0 \\ & & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 & 0 \end{bmatrix}$$

Since the eigenvalues of D^{*} are 0 and 2, from a corollary of the Courant-Fisher theorem [1] we have that the eigenvalues of A^{*}, $\lambda_{k}(A^{*})$, satisfy the following inequalities

$$\lambda_{k}(A^{*}) \geq \lambda_{k}(C^{*}), \quad k = 1, \ldots, m^{*}.$$

The eigenvalues of C^{*} are the collection of the eigenvalues of C_i and we have

$$C_{i} = B_{i}^{2} - \omega_{i} B_{i},$$

where B_i is the $(n_i - 1) \times (n_i - 1)$ tridiagonal matrix

$$B_{i} = \begin{bmatrix} -2 & 1 \\ 1 & -2 & 1 \\ & 1 & -2 & 1 \\ & & & \cdots \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$

It is well known, (see also [4]), that

$$\lambda_{j}(B_{i}) = -2\left(1 - \cos\frac{j\pi}{n_{i}}\right), \quad j = 1, \ldots, n_{i} - 1,$$

then

$$\lambda_{j}(C_{i}) = 4\left(1 - \cos\frac{j\pi}{n_{i}}\right)^{2} + 2\omega_{i}\left(1 - \cos\frac{j\pi}{n_{i}}\right).$$

It follows that

$$\lambda_{k}(A^{*}) \geq \min_{i,j} \lambda_{j}(C_{i}) = \min_{i}\left[4\left(1 - \cos\frac{\tau\pi}{h_{i}}\right)^{2} + 2\omega_{i}\left(1 - \cos\frac{\tau\pi}{h_{i}}\right)\right].$$

Hence, A^{*} is a positive matrix and the linear system (12) (and (11) as well) has unique solution.

In addition, from Gershgorin's theorem, $\lambda_k(A^*) \leq \max_i[16 + 4\omega_i]$, then for the condition number, $\mu_2(A^*)$, with respect to the 2 norm of A^*, we have the following upper bound not depending on the number of data points, $N+2$:

$$\mu_2(A^*) \leq \frac{\max_i \left[16 + 4(\frac{\tau p_i}{h_i})^2\right]}{\min_i \left[4(1 - \cos\frac{\tau\pi}{h_i})^2 + 2(\frac{\tau p_i}{h_i})^2(1 - \cos\frac{\tau\pi}{h_i})\right]}$$

$$\leq \frac{\max_i \left[16 + 4(\frac{\tau p_i}{h_i})^2\right]}{\min_i(\frac{\tau}{h_i})^4[\pi^4 + (\pi p_i)^2]} .$$

(14)

We remark that, for $p_i = 0$, $i = 0, \ldots, N$, we recover the results presented in [4].

From the structure of A^*, the linear system (12) can be solved efficiently using a direct method for band matrices. Since A^* is positive band matrix of band width 2, the classical Cholesky factorization, $A^* = LL^T$, provides a lower triangular band matrix L of band 2 and it can be performed in $O(2m^*)$ operations, [1].

3. System Splitting and Mesh Solution Extension

In order to solve numerically the differential multipoint boundary value problem (2)–(5) we consider the system of difference equations (6) completed with the smoothness conditions (7) (or (8)), interpolation conditions (9) and end conditions (10).

In the notation

$$M_{ij} = \Lambda u_{ij}, \quad j = 0, \ldots, n_i, \quad i = 0, \ldots, N, \tag{15}$$

on the interval $[x_i, x_{i+1}]$ the system (6) takes the form

$$M_{i0} = M_i,$$
$$\frac{M_{ij-1} - 2M_{ij} + M_{ij+1}}{\tau^2} - \left(\frac{p_i}{h_i}\right)^2 M_{ij} = 0, \quad j = 1, \ldots, n_i - 1, \tag{16}$$
$$M_{i,n_i} = M_{i+1},$$

where M_i and M_{i+1} are prescribed numbers. The system (16) has a unique solution, which can be represented as follows

$$M_{ij} = m_i(x_{ij}), \quad x_{ij} = x_i + j\tau, \quad j = 0, \ldots, n_i$$

with

$$m_i(x) = M_i \frac{\sinh k_i(1-t)}{\sinh(k_i)} + M_{i+1} \frac{\sinh k_i t}{\sinh(k_i)},$$

$$t = \frac{x - x_i}{h_i}, \quad \frac{2}{\hat{\tau}_i}\sinh\frac{k_i\hat{\tau}_i}{2} = p_i \geq 0, \quad \hat{\tau}_i = \frac{\tau}{h_i}.$$

From the equation (15) and the interpolation conditions (9) we have

$$u_{i0} = f_i,$$

$$\frac{u_{ij-1} - 2u_{ij} + u_{ij+1}}{\tau^2} = M_{ij}, \quad j = 1, \ldots, n_i - 1, \tag{17}$$

$$u_{i,n_i} = f_{i+1}.$$

Let us consider the function

$$u_i(x) = f_i(1-t) + f_{i+1}t + \hat{\varphi}_i(1-t)h_i^2 M_i + \hat{\varphi}_i(t)h_i^2 M_{i+1}. \tag{18}$$

where

$$\hat{\varphi}_i(t) = \frac{\sinh(k_i t) - t\sinh(k_i)}{p_i^2 \sinh(k_i)}.$$

The function u_i satisfies to the conditions

$$u_i(x_j) = f_j, \quad \Lambda u_i(x_j) = M_j, \quad j = i, i+1,$$

where

$$\Lambda u_i(x) = \frac{u_i(x-\tau) - 2u_i(x) + u_i(x+\tau)}{\tau^2}.$$

The mesh restriction of the function u_i gives us the solution of the system (17) with $u_{ij} = u_i(x_{ij})$, $j = 0, \ldots, n_i$. The smoothness conditions (7) can be rewritten as

$$u_{i-1}(x_i) = u_i(x_i),$$

$$\Delta_{\bar{\tau},\tau} u_{i-1}(x_i) = \Delta_{\bar{\tau},\tau} u_{i-1}(x_i), \tag{19}$$

$$\Lambda u_{i-1}(x_i) = \Lambda u_i(x_i),$$

where

$$\Delta_{\bar{\tau},\tau} u_j(x) = \frac{u_j(x+\tau) - u_j(x-\tau)}{2\tau};$$

(19) are equivalent to

$$u_{i-1}(x_i + j\tau) = u_i(x_i + j\tau), \quad j = -1, 0, 1.$$

Using (18) and the second condition (19) we obtain a linear system with 3-diagonal matrix

$$M_0 = f_0'',$$

$$\alpha_{i-1}h_{i-1}M_{i-1} + (\beta_{i-1}h_{i-1} + \beta_i h_i)M_i + \alpha_i h_i M_{i+1} = d_i, \quad i = 1, \ldots, N, \tag{20}$$

$$M_{N+1} = f_{N+1}'',$$

where

$$d_i = \frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}},$$

$$\alpha_i = -\frac{\hat{\varphi}_i(\hat{\tau}_i) - \hat{\varphi}_i(-\hat{\tau}_i)}{2\hat{\tau}_i} = -\frac{\sinh(k_i\hat{\tau}_i) - \hat{\tau}_i\sinh(k_i)}{p_i^2\hat{\tau}_i\sinh(k_i)},$$

$$\beta_i = \frac{\hat{\varphi}_i(1+\hat{\tau}_i) - \hat{\varphi}_i(1-\hat{\tau}_i)}{2\hat{\tau}_i} = \frac{\cosh(k_i)\sinh(k_i\hat{\tau}_i) - \hat{\tau}_i\sinh(k_i)}{p_i^2\hat{\tau}_i\sinh(k_i)}.$$

Using an expansion of the hyperbolic functions in the above expressions as power series we obtain

$$\beta_i > 2\alpha_i > 0, \quad i = 0, \ldots, N, \quad \text{for all } \tau > 0, \quad p_i > 0.$$

Therefore the system (20) is diagonal dominant and has a unique solution.

We can now conclude that the function u which coincides with u_i for $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, N$, is a *discrete tension interpolation spline*. A mesh restriction of the spline u gives us a solution of the system (6). The spline u can also be easy recovered from the solution of the system (6).

Instead of looks for a direct solution for the system (6) we recommend the following algorithm.

Step 1. *Solve 3-diagonal system (20) for M_i, $i = 1, \ldots, N$.*

Step 2. *Solve $N + 1$ 3-diagonal systems (16) for M_{ij}, $j = 1, \ldots, n_i - 1$, $i = 0, \ldots, N$,*

Step 3. *Solve $N + 1$ 3-diagonal systems (17) for u_{ij}, $j = 1, \ldots, n_i - 1$, $i = 0, \ldots, N$.*

Steps 2 and 3 can be replaced by a direct splitting of the system (11) into $N + 1$ systems with 5-diagonal matrices

$$C_i u_i = c_i, \quad i = 0, \ldots, N, \tag{21}$$

where the $(n_i - 1) \times (n_i - 1)$ matrix C_i has the form (13),

$$u_i = (u_{i1}, u_{i2}, \ldots, u_{i,n_i-1})^T,$$

$$c_i = \big((2 + \omega_i)f_i - M_i, -f_i, 0, \ldots, 0, -f_{i+1}, (2 + \omega_i)f_{i+1} - M_{i+1}\big)^T.$$

The calculations to solve the systems (16) and (17) or (21) can be performed by using a multi-processing parallel computer system. If $n_i = n$ for all i, we can first store a triangular factorization of the matrices of the systems and then use parallel computations.

References

1. Golub, G. H. and C. F. Van Loan (1991), *Matrix Computations*, John Hopkins University Press, Baltimore.

2. Kvasov, B. I. (2000), *Methods of Shape-Preserving Spline Approximation*, World Scientific Publ. Co. Pte. Ltd., Singapore.

3. Laurent, P. J. (1972), *Approximation et optimization*, Hermann, Paris.

4. Malcolm, M. A. (1977), On the computation of nonlinear spline functions, *SIAM J. Numer. Anal.* 14, 254–282.

5. Schumaker, L. L. (1981), *Spline functions: Basic theory*, Wiley, New York.

Difference Method for Constructing Hyperbolic Tension Splines

Boris I. Kvasov

School of Mathematics, Suranaree University of Technology
University Avenue 111, Nakhon Ratchasima 30000, Thailand
Email: boris@math.sut.ac.th

Abstract. A hyperbolic tension spline is defined as the solution of a differential multipoint boundary value problem. A discrete hyperbolic tension spline is obtained using the difference analogous of differential operators; its computation does not require exponential functions, even if its continuous extension is still a spline of hyperbolic type. We consider the basic computational aspects and show the main features of this approach.

Keywords: Hyperbolic tension splines, multipoint boundary value problem, discrete hyperbolic tension splines and B-splines, shape preserving interpolation.

1. Introduction

Spline theory is mainly grounded on two approaches: the algebraic one (where splines are understood as smooth piecewise functions, see e.g. [29,31]) and the variational one (where splines are obtained via minimization of quadratic functionals with equality and/or inequality constraints, see e.g. [15]). Although less common, a third approach where splines are defined as the solutions of differential multipoint boundary value problems (DMBVP for short), has been considered, [9]. Even though some of the important classes of splines can be obtained from all three schemes, specific features make sometimes the last one an important tool in practical settings. We want to illustrate this fact by the example of hyperbolic tension splines.

Introduced by Schweikert in 1966, [30], hyperbolic tension splines are solutions of DMBVP where the differential operators depend on *tension* parameters. Their tension properties (that is the possibility of pulling the curve toward a piecewise linear function) have kept hyperbolic splines popular (see for example [11,24,25,27] and references quoted therein) in shape-preserving interpolation and/or approximation. Unfortunately, it is difficult to work with hyperbolic splines for small or large values of the tension parameters. For this reason, in spite of the presence of refined algorithms for their calculation [25], hyperbolic tension splines were forced out by rational splines (see for example [6, 12]) in practical applications.

We observe that for practical purposes, it is often neccessary to know the values of the solution S of a DMBVP only over a prescribed grid instead of its global analytic expression. In this paper, we study a natural discretization of the DMBVP

replacing, in the given interval $[a, b]$, the differential operator by its difference approximation. This provides a linear system with a pentadiagonal matrix. It turns out that the solution of the discretized problem, called *mesh solution*, is not a tabulation of S but can be extended on $[a, b]$ to a smooth function U which has shape properties very similar to those of S and which provides a second order approximation of S as the discretization step goes to zero. Due to these properties we will refer to U as a *discrete* hyperbolic tension spline.

In contrast with the continuous case, an important fact here is that the values of a discrete hyperbolic tension spline over a prescribed grid in $[a, b]$ (basically the mesh solution) can be obtained solving a pentadiagonal system. This construction is substantially cheaper than performing calculations by the standard algorithm [25], which involves the solution of a simple 3-diagonal system, but with hyperbolic coefficients. In addition, the classical construction requires the evaluation of hyperbolic functions over the prescribed grid with much larger computational cost.

Moreover, just as cubic splines can be seen as a subclass of the exponential ones in the continuous setting, our discrete hyperbolic splines generalize the concept of discrete polynomial splines and reduce to them as the tension parameters go to zero.

Discrete polynomial splines have been studied extensively. They were introduced in [18] as solutions to certain minimization problems involving differences instead of derivatives. They are connected to best summation formulas [19] and have been used in [17] for the computation of nonlinear splines by iteration. Approximation properties of discrete splines have been studied in [16]. Discrete B-splines on a uniform partition were introduced in [28] and discrete B-splines on a non–uniform partition were defined in [2, p.15]. In [3] discrete B-splines were applied to the general area of subdivision. While discrete polynomial splines are currently attracting widespread research interest [21,22,23], discrete tension splines and B-splines have been less studied. The only results we know regarding this topic concern discrete exponential Box-splines [5, 26] and are therefore related to uniform partitions.

The content of this paper is as follows. In Section 2 we formulate the problem. In Section 3 we prove the existence of a mesh solution by constructing its extension as a discrete hyperbolic tension spline. An upper bound for the distance between a discrete hyperbolic tension spline and the corresponding continuous one is established in Section 4. In Section 5 we give direct and recurrence algorithms for constructing discrete hyperbolic tension B-splines. Section 6, with its subsections, is devoted to the discussion of practical aspects and computational advantages of our discrete spline. Finally, Section 7 gives some graphical examples to illustrate the main properties of discrete hyperbolic tension splines.

2. Finite Difference Approximation

Let the data

$$(x_i, f_i), \quad i = 0, \ldots, N + 1, \tag{2.1}$$

be given, where: $a = x_0 < x_1 < \cdots < x_{N+1} = b$. Let us put

$$h_i = x_{i+1} - x_i, \quad i = 0, \ldots, N.$$

An *interpolating hyperbolic tension spline* S with a set of tension parameters $\{p_i \geq 0 \mid i = 0, \ldots, N\}$ is a solution of the DMBVP

$$\frac{d^4S}{dx^4} - \left(\frac{p_i}{h_i}\right)^2 \frac{d^2S}{dx^2} = 0, \quad \text{in each} \quad (x_i, x_{i+1}), \quad i = 0, \ldots, N, \tag{2.2}$$

$$S \in C^2[a, b], \tag{2.3}$$

with the interpolation conditions

$$S(x_i) = f_i, \quad i = 0, \ldots, N+1, \tag{2.4}$$

and some end constraints. For the sake of simplicity we only consider the following classical end conditions

$$S''(a) = f_0'' \quad \text{and} \quad S''(b) = f_{N+1}''. \tag{2.5}$$

Let us now consider a discretized version of the previous DMBVP. Let $n_i \in \mathbb{N}$, $i = 0, \ldots, N$, be given; we look for

$$\{u_{ij}, \quad j = -1, \ldots, n_i + 1, \quad i = 0, \ldots, N\},$$

satisfying the difference equations:

$$\left[\Lambda_i^2 - \left(\frac{p_i}{h_i}\right)^2 \Lambda_i\right] u_{ij} = 0, \quad j = 1, \ldots, n_i - 1, \quad i = 0, \ldots, N, \tag{2.6}$$

where

$$\Lambda_i u_{ij} = \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{\tau_i^2}, \quad \tau_i = \frac{h_i}{n_i}.$$

The smoothness condition (2.3) is changed into

$$u_{i-1,n_{i-1}} = u_{i0},$$

$$\frac{u_{i-1,n_{i-1}+1} - u_{i-1,n_{i-1}-1}}{2\tau_{i-1}} = \frac{u_{i,1} - u_{i,-1}}{2\tau_i}, \quad i = 1, \ldots, N, \tag{2.7}$$

$$\Lambda_{i-1} u_{i-1,n_{i-1}} = \Lambda_i u_{i,0}$$

while conditions (2.4)–(2.5) take the form

$$u_{i,0} = f_i, \quad i = 0, \ldots, N, \quad u_{N,n_N} = f_{N+1},$$
$$\Lambda_0 u_{0,0} = f_0'', \quad \Lambda_N u_{N,n_N} = f_{N+1}''. \tag{2.8}$$

Our discrete *mesh solution* will be then defined as

$$\{u_{ij}, \quad j = 0, \ldots, n_i, \quad i = 0, 1, \ldots, N\}. \tag{2.9}$$

In the next section we prove the existence of the solution of the previous linear system while we postpone to Section 6 the comments on the practical computation of the mesh solution.

3. System Splitting and Mesh Solution Extension

In order to analyze the solution of system (2.6)–(2.8) we introduce the notation

$$m_{ij} = \Lambda_i u_{ij}, \quad j = 0, \ldots, n_i, \quad i = 0, \ldots, N. \tag{3.1}$$

Then, on the interval $[x_i, x_{i+1}]$, (2.6) takes the form

$$m_{i0} = m_i,$$
$$\frac{m_{i,j-1} - 2m_{ij} + m_{i,j+1}}{\tau_i^2} - \left(\frac{p_i}{h_i}\right)^2 m_{ij} = 0, \quad j = 1, \ldots, n_i - 1, \tag{3.2}$$
$$m_{i,n_i} = m_{i+1},$$

where m_i and m_{i+1} are prescribed numbers. The system (3.2) has a unique solution, which can be represented as follows

$$m_{ij} = M_i(x_{ij}), \quad x_{ij} = x_i + j\tau_i, \quad j = 0, \ldots, n_i,$$

with

$$M_i(x) = m_i \frac{\sinh k_i(1-t)}{\sinh(k_i)} + m_{i+1} \frac{\sinh k_i t}{\sinh(k_i)}, \quad t = \frac{x - x_i}{h_i},$$

and where the parameters k_i are the solutions of the transcendental equations

$$2n_i \sinh \frac{k_i}{2n_i} = p_i, \quad p_i \geq 0,$$

that is

$$k_i = 2n_i \ln \left(\frac{p_i}{2n_i} + \sqrt{\left(\frac{p_i}{2n_i}\right)^2 + 1}\right) \geq 0, \quad i = 0, \ldots, N.$$

From (3.1) and from the interpolation conditions (2.8) we have

$$u_{i0} = f_i,$$
$$\frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{\tau_i^2} = m_{ij}, \quad j = 0, \ldots, n_i, \tag{3.3}$$
$$u_{i,n_i} = f_{i+1}.$$

For each sequence m_{ij}, $j = 0, \ldots, n_i$, system (3.3) has a unique solution which can be represented as follows

$$u_{ij} = U_i(x_{ij}), \quad j = -1, \ldots, n_i + 1,$$

where

$$U_i(x) = f_i(1-t) + f_{i+1}t + \varphi_i(1-t)h_i^2 m_i + \varphi_i(t)h_i^2 m_{i+1}, \tag{3.4}$$

with

$$\varphi_i(t) = \frac{\sinh(k_i t) - t \sinh(k_i)}{p_i^2 \sinh(k_i)}.$$

In order to solve system (2.6)–(2.8), we only need to determine the values m_i, $i = 0, \ldots, N+1$, so that the smoothness conditions (2.7) and the end conditions in (2.8) are verified. From (3.3)–(3.4), conditions (2.7) can be rewritten as

$$U_{i-1}(x_i) = U_i(x_i),$$

$$\frac{U_{i-1}(x_i + \tau_{i-1}) - U_{i-1}(x_i - \tau_{i-1})}{2\tau_{i-1}} = \frac{U_i(x_i + \tau_i) - U_i(x_i - \tau_i)}{2\tau_i}, \qquad (3.5)$$

$$\Lambda_{i-1}U_{i-1}(x_i) = \Lambda_i U_i(x_i),$$

where

$$\Lambda_j U_j(x) = \frac{U_j(x + \tau_j) - 2U_j(x) + U_j(x - \tau_j)}{\tau_j^2}, \quad x \in [x_j, x_{j+1}].$$

Then, from (3.1)–(3.2) and (3.4), the first and the third equalities in (3.5) are immediately satisfied, while, using (3.4) and the end conditions in (2.8), the second equality provides the following linear system with a 3-diagonal matrix for the unknown values m_i:

$$m_0 = f_0'',$$

$$\alpha_{i-1}h_{i-1}m_{i-1} + (\beta_{i-1}h_{i-1} + \beta_i h_i)m_i + \alpha_i h_i m_{i+1} = d_i, \quad i = 1, \ldots, N, \qquad (3.6)$$

$$m_{N+1} = f_{N+1}'',$$

where

$$d_i = \frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}},$$

$$\alpha_i = -\frac{\varphi_i(\frac{1}{n_i}) - \varphi_i(-\frac{1}{n_i})}{\frac{2}{n_i}} = -\frac{n_i \sinh(\frac{k_i}{n_i}) - \sinh(k_i)}{p_i^2 \sinh(k_i)},$$

$$\beta_i = \frac{\varphi_i(1 + \frac{1}{n_i}) - \varphi_i(1 - \frac{1}{n_i})}{\frac{2}{n_i}} = \frac{n_i \cosh(k_i) \sinh(\frac{k_i}{n_i}) - \sinh(k_i)}{p_i^2 \sinh(k_i)}.$$

Expanding the hyperbolic functions in the above expressions as power series we obtain

$$\beta_i \geq 2\alpha_i > 0, \quad i = 0, \ldots, N, \quad \text{for all} \quad n_i > 1, \quad p_i \geq 0.$$

Therefore, the system (3.6) is diagonally dominant and has a unique solution. We can now conclude that system (2.6)–(2.8) has a unique solution which can be represented as $U_i(x_{ij})$, $j = -1, \ldots, n_i + 1$, $i = 0, \ldots, N$, whenever the constants m_i are solution of (3.6).

Let us put

$$U(x) := U_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, \ldots, N. \qquad (3.7)$$

Due to the previous construction we will refer to U as *discrete hyperbolic tension spline* interpolating the data (2.1). We observe that we recover the result of [17] for discrete cubics since

$$\lim_{p_i \to 0} \alpha_i = \frac{1}{6}\left(1 - \frac{1}{n_i^2}\right), \quad \lim_{p_i \to 0} \beta_i = \frac{1}{6}\left(2 + \frac{1}{n_i^2}\right), \quad \lim_{p_i \to 0} \varphi_i(t) = \frac{t(t^2 - 1)}{6}. \qquad (3.8)$$

4. Error Estimates

In this section, we present a bound for the distance between the discrete hyperbolic tension spline defined in (3.7) and the corresponding continuous one interpolating the same set of data and having the same end conditions.

As mentioned in section 2, the classical C^2 hyperbolic tension spline interpolating the data (2.1) is a function S satisfying (2.2)–(2.5). It is well known that we can express $S_i(x) := S(x)_{|[x_i, x_{i+1}]}$ as

$$S_i(x) = f_i(1 - t) + f_{i+1}t + \tilde{\varphi}_i(1 - t)h_i^2 \tilde{m}_i + \tilde{\varphi}_i(t)h_i^2 \tilde{m}_{i+1}, \tag{4.1}$$

where

$$\tilde{m}_{i+j} = \frac{d^2 S}{dx^2}(x_{i+j}), \quad \tilde{\varphi}_i(t) = \frac{\sinh(p_i t) - t \sinh(p_i)}{p_i^2 \sinh(p_i)}, i = 0, \ldots, N, \quad j = 0, 1,$$

and the constants $\tilde{m}_i$ are solutions of the linear system

$$\tilde{m}_0 = f_0'',$$
$$\tilde{\alpha}_{i-1}h_{i-1}\tilde{m}_{i-1} + (\tilde{\beta}_{i-1}h_{i-1} + \tilde{\beta}_i h_i)\tilde{m}_i + \tilde{\alpha}_i h_i \tilde{m}_{i+1} = d_i, \ i = 1, \ldots, N, \tag{4.2}$$
$$\tilde{m}_{N+1} = f_{N+1}'',$$

where

$$\tilde{\alpha}_i = -\tilde{\varphi}_i'(0) = \frac{\sinh(p_i) - p_i}{p_i^2 \sinh(p_i)}, \quad \tilde{\beta}_i = \tilde{\varphi}_i'(1) = \frac{p_i \cosh(p_i) - \sinh(p_i)}{p_i^2 \sinh(p_i)}.$$

It is easy to verify that $\tilde{\beta}_i \geq 2\tilde{\alpha}_i > 0$, $\forall p_i \geq 0$, so that the 3-diagonal linear system (4.2) is diagonally dominant. In addition, as $n_i \to +\infty$, systems (3.6) and (4.2) coincide since

$$\lim_{n_i \to +\infty} \alpha_i = \tilde{\alpha}_i, \quad \lim_{n_i \to +\infty} \beta_i = \tilde{\beta}_i.$$

Let us put

$$\tilde{\mathcal{A}} := \min_{i=1,\ldots,N} (\tilde{\beta}_i - \tilde{\alpha}_i)h_i + (\tilde{\beta}_{i-1} - \tilde{\alpha}_{i-1})h_{i-1} > 0. \tag{4.3}$$

For notational purposes, let us consider systems (3.6) and (4.2) where the first and the last equation have been multiplied by $\tilde{\mathcal{A}}$; let $\mathbf{T}$ and $\tilde{\mathbf{T}}$ be the corresponding matrices and let $\mathbf{m}$, $\tilde{\mathbf{m}}$ be the corresponding solutions. We have $\tilde{\mathbf{T}} = \mathbf{T} + \delta\mathbf{T}$, where

$$\delta\mathbf{T} = \begin{bmatrix} 0 & 0 & & & & \\ \frac{a_0\tau_0^2}{h_0} & \frac{b_0\tau_0^2}{h_0} + \frac{b_1\tau_1^2}{h_1} & \frac{a_1\tau_1^2}{h_1} & & & \\ 0 & \frac{a_1\tau_1^2}{h_1} & \frac{b_1\tau_1^2}{h_1} + \frac{b_2\tau_2^2}{h_2} & \frac{a_2\tau_2^2}{h_2} & & \\ & & \cdots & & & \\ & & \frac{a_{N-1}\tau_{N-1}^2}{h_{N-1}} & \frac{b_{N-1}\tau_{N-1}^2}{h_{N-1}} + \frac{b_N\tau_N^2}{h_N} & \frac{a_N\tau_N^2}{h_N} \\ & & & 0 & 0 \end{bmatrix},$$

and

$$a_i = \frac{n_i^2}{p_i}\left[\frac{\cosh\frac{k_i}{2n_i}}{\sinh(k_i)} - \frac{1}{\sinh(p_i)}\right], \quad b_i = -\frac{n_i^2}{p_i}\left[\frac{\cosh(k_i)\cosh\frac{k_i}{2n_i}}{\sinh(k_i)} - \frac{\cosh(p_i)}{\sinh(p_i)}\right].$$

After some computations we obtain that a_i, b_i are bounded functions of n_i, more precisely

$$|a_i|, |b_i| \le \mathcal{A}_i := \lim_{n_i \to +\infty} |b_i| = \frac{3p_i\cosh(p_i)\sinh(p_i) + p_i^2}{24\sinh^2(p_i)}. \tag{4.4}$$

Then, following [17], [16] and [4]

$$\|\mathbf{m} - \tilde{\mathbf{m}}\|_\infty \le \|\tilde{\mathbf{T}}^{-1}\|_\infty \|\delta\mathbf{T}\|_\infty \|\mathbf{m}\|_\infty.$$

Since $\|\tilde{\mathbf{T}}\mathbf{r}\|_\infty \ge \bar{\mathcal{A}}\|\mathbf{r}\|_\infty$ for all $\mathbf{r} \in \mathbb{R}^{N+2}$, then $\|\tilde{\mathbf{T}}^{-1}\|_\infty \le \bar{\mathcal{A}}^{-1}$. In addition,

$$\|\delta\mathbf{T}\|_\infty \le \tau^2 \max_{i=0,\dots,N}\frac{4\mathcal{A}_i}{h_i}, \quad \tau = \max_{i=0,\dots,N}\tau_i.$$

Therefore

$$\|\mathbf{m} - \tilde{\mathbf{m}}\|_\infty \le \|\mathbf{m}\|_\infty\tau^2\frac{1}{\bar{\mathcal{A}}}\left[\max_{i=0,\dots,N}\frac{4\mathcal{A}_i}{h_i}\right]. \tag{4.5}$$

Then, setting

$$\mathcal{B}_i := 2\max_{t\in[0,1]}|\tilde{\varphi}_i(t)|, \quad \mathcal{C}_i := 2n_i^2\max_{t\in[0,1]}|\tilde{\varphi}_i(t) - \varphi_i(t)|, \tag{4.6}$$

we obtain from the expressions of S_i and U_i, see (4.1) and (3.4), and from (4.5)

$$\|S_i - U_i\|_\infty = \max_{x\in[x_i,x_{i+1}]}|S_i(x) - U_i(x)| \le h_i^2\left[\|\mathbf{m} - \tilde{\mathbf{m}}\|_\infty\mathcal{B}_i + \|\mathbf{m}\|_\infty\mathcal{C}_i\frac{\tau_i^2}{h_i^2}\right]$$

$$\le h_i^2\tau^2\|\mathbf{m}\|_\infty\left[\frac{\mathcal{B}_i}{\bar{\mathcal{A}}}\max_{i=0,\dots,N}\frac{4\mathcal{A}_i}{h_i} + \frac{\mathcal{C}_i}{h_i^2}\right]. \tag{4.7}$$

Since (see [4] for details) $\mathcal{C}_i$ is a bounded function of n_i, then from (4.7), for each fixed sequence of the values $p_0,\dots,p_N$, we have a second order convergence of the discrete hyperbolic tension splines to the corresponding continuous spline. The results agree with the order of approximation of the discretization which we have used for the first, second and fourth derivatives. For example, let us consider in detail the upper bound (4.7) in the limit case $p_i = 0$, $i = 0,\dots,N$. From (3.8), (4.4) and (4.6) we obtain

$$\lim_{p_i\to 0}\mathcal{A}_i = \frac{1}{6}, \quad \lim_{p_i\to 0}(\tilde{\beta}_i - \tilde{\alpha}_i) = \frac{1}{6}, \quad \lim_{p_i\to 0}\mathcal{B}_i \le 1, \quad \lim_{p_i\to 0}\mathcal{C}_i = 0,$$

so that from (4.3) and (4.7)

$$\|S_i - U_i\| \le 4h_i^2\tau^2\|\mathbf{m}\|_\infty\max_{i=1,\dots,N}\frac{1}{h_i + h_{i-1}}\left[\max_{i=0,\dots,N}\frac{1}{h_i}\right],$$

and we recover, with some improvements, the corresponding result of [17].

Finally, we observe that (4.7) can be used to estimate the rate of convergence of a discrete hyperbolic tension spline towards a function generating the interpolation points as $\max_i h_i \to 0$. To do this, it suffices to combine, via the triangle inequality, (4.7) with the results of [20] where the convergence of a continuous hyperbolic tension spline towards a function generating the interpolation points is studied.

5. Discrete Hyperbolic Tension B-Splines

In this section, we use the strategy outlined in [13,14], where generalized B-splines and their properties are discussed in more detail.

Let us associate with a partition $\Delta : a = x_0 < x_1 < \cdots < x_{N+1} = b$ of the interval $[a, b]$ a space of functions S_4^{DH} whose restriction to an interval $[x_i, x_{i+1}]$, $i = 0, \ldots, N$ is spanned by the system of four linearly independent functions $\{1, x, \Phi_i, \Psi_i\}$ and where every function in S_4^{DH} satisfies smoothness conditions (3.5) for discrete hyperbolic tension splines.

Following [14] let us rewrite formula (3.4) on the interval $[x_i, x_{i+1}]$, $i = 0, \ldots, N$, in the form

$$U(x) \equiv U_i(x) = [f_i - \Phi_i(x_i)m_i](1 - t) + [f_{i+1} - \Psi_i(x_{i+1})m_{i+1}]t \\ + \Phi_i(x)m_i + \Psi_i(x)m_{i+1}, \tag{5.1}$$

where $t = (x - x_i)/h_i$, $m_j = \Lambda_i U_i(x_j)$, $j = i, i + 1$, and

$$\Psi_i(x) = \psi_i(t)h_i^2 = \psi(p_i, t)h_i^2, \quad \Phi_i(x) = \psi_i(1 - t)h_i^2, \\ \psi_i(t) = \frac{\sinh(k_i t) - t n_i \sinh(k_i/n_i)}{p_i^2 \sinh(k_i)}. \tag{5.2}$$

Functions Φ_i and Ψ_i satisfy the conditions

$$\Psi_i(x_i + j\tau_i) = \Phi_i(x_{i+1} + j\tau_i) = 0, \quad j = -1, 0, 1, \\ \Lambda_i \Phi_i(x_i) = \Lambda_i \Psi_i(x_{i+1}) = 1. \tag{5.3}$$

Let us construct a basis for the space of discrete hyperbolic tension splines S_4^{DH} by using functions which have local supports of minimum length. Since $dim(S_4^{DH}) = 4(N + 1) - 3N = N + 4$ we extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N + 2, N + 3, N + 4$, such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+2} < x_{N+3} < x_{N+4}$.

We demand that the discrete hyperbolic tension B-splines (HB-splines for short) B_i, $i = -3, \ldots, N$ have the following properties

$$B_i(x) > 0, \quad x \in (x_i + \tau_i, x_{i+4} - \tau_{i+4}), \\ B_i(x) \equiv 0, \quad x \notin (x_i, x_{i+4}), \\ \sum_{j=-3}^{N} B_j(x) \equiv 1, \quad x \in [a, b]. \tag{5.4}$$

5.1 Construction of HB-Splines

According to (5.1), on the interval $[x_j, x_{j+1}]$, $j = i, \ldots, i + 3$, the discrete HB-spline B_i has the form

$$B_i(x) \equiv B_{i,j}(x) = P_{i,j}(x) + \Phi_j(x)m_{j,B_i} + \Psi_j(x)m_{j+1,B_i},$$

where $P_{i,j}$ is a polynomial of the first degree and $m_{l,B_i} = \Lambda_l B_i(x_l)$, $l = j, j+1$ are constants to be determined. The smoothness conditions (3.5) and constraints (5.3) give the following relations

$$P_{i,j}(x_j) = P_{i,j-1}(x_j) + z_j \dot{m}_{j,B_i},$$
$$P_{i,j}[x_j - \tau_j, x_j + \tau_j] = P_{i,j-1}[x_j - \tau_{j-1}, x_j + \tau_{j-1}] + c_{j-1,2} m_{j,B_i},$$

where

$$z_j \equiv z_j(x_j) = \Psi_{j-1}(x_j) - \dot{\Phi}_j(x_j),$$
$$c_{j-1,2} = \Psi_{j-1}[x_j - \tau_{j-1}, x_j + \tau_{j-1}] - \Phi_j[x_j - \tau_j, x_j + \tau_j].$$

Thus

$$P_{i,j}(x) = P_{i,j-1}(x) + [z_j + c_{j-1,2}(x - x_j)] m_{j,B_i}. \tag{5.5}$$

As B_i vanishes outside the interval (x_i, x_{i+4}), we have from (5.5), in particular, $P_{i,j} \equiv 0$ for $j = i, i+3$. By repeated use of formula (5.5) we get

$$P_{i,j}(x) = \sum_{l=i+1}^{j} [z_l + c_{l-1,2}(x - x_l)] m_{l,B_i} = -\sum_{l=j+1}^{i+3} [z_l + c_{l-1,2}(x - x_l)] m_{l,B_i}.$$

In particular, the following identity is valid

$$\sum_{j=i+1}^{i+3} [z_j + c_{j-1,2}(x - x_j)] m_{l,B_i} \equiv 0,$$

from which one obtains the equalities

$$\sum_{j=i+1}^{i+3} c_{j-1,2} y_j^r m_{j,B_i} = 0, \quad r = 0, 1, \quad y_j = x_j - \frac{z_j}{c_{j-1,2}}. \tag{5.6}$$

Thus, the formula for the discrete HB-spline B_i takes the form

$$B_i(x) = \begin{cases} \Psi_i(x) m_{i+1,B_i}, & x \in [x_i, x_{i+1}), \\ (x - y_{i+1}) c_{i,2} m_{i+1,B_i} \\ \quad + \Phi_{i+1}(x) m_{i+1,B_i} + \Psi_{i+1}(x) m_{i+2,B_i}, \\ \qquad\qquad\qquad x \in [x_{i+1}, x_{i+2}), \\ (y_{i+3} - x) c_{i+2,2} m_{i+3,B_i} \\ \quad + \Phi_{i+2}(x) m_{i+2,B_i} + \Psi_{i+2}(x) m_{i+3,B_i}, \\ \qquad\qquad\qquad x \in [x_{i+2}, x_{i+3}), \\ \Phi_{i+3}(x) m_{i+3,B_i}, & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise.} \end{cases} \tag{5.7}$$

Substituting formula (5.7) into the normalization condition (5.4) written for $x \in [x_i, x_{i+1}]$, we obtain

$$\sum_{j=i-3}^{i} B_j(x) = \Phi_i(x) \sum_{j=i-3}^{i-1} m_{i,B_j} + \Psi_i(x) \sum_{j=i-2}^{i} m_{i+1,B_j}$$
$$+ (y_{i+1} - x)c_{i,2}m_{i+1,B_{i-2}} + (x - y_i)c_{i-1,2}m_{i,B_{i-1}} \equiv 1.$$

As according to (5.4)

$$\sum_{j=i-3}^{i-1} m_{i,B_j} = \sum_{j=i-2}^{i} m_{i+1,B_j} = 0 \qquad (5.8)$$

the following identity is valid

$$(y_{i+1} - x)c_{i,2}m_{i+1,B_{i-2}} + (x - y_i)c_{i-1,2}m_{i,B_{i-1}} \equiv 1.$$

From here one gets the equalities

$$y_{i+1}^r c_{i,2}m_{i+1,B_{i-2}} - y_i^r c_{i-1,2}m_{i,B_{i-1}} \equiv \delta_{1,r}, \quad r = 0, 1,$$

where $\delta_{1,r}$ is the Kronecker symbol. Solving this system of equations and using (5.6) or (5.8), we obtain

$$m_{j,B_i} = \frac{y_{i+3} - y_{i+1}}{c_{j-1,2}\omega'_{i+1}(y_j)}, \quad j = i+1, i+2, i+3,$$
$$\omega_{i+1}(x) = (x - y_{i+1})(x - y_{i+2})(x - y_{i+3})$$

or with the notation $c_{j,3} = y_{j+2} - y_{j+1}, j = i, i+1$,

$$m_{i+1,B_i} = \frac{1}{c_{i,2}c_{i,3}},$$
$$m_{i+2,B_i} = -\frac{1}{c_{i+1,2}}\left(\frac{1}{c_{i,3}} + \frac{1}{c_{i+1,3}}\right), \qquad (5.9)$$
$$m_{i+3,B_i} = \frac{1}{c_{i+2,2}c_{i+1,3}}.$$

5.2 Recurrence Formulas for HB-Splines

Let us define functions

$$B_{j,2}(x) = \begin{cases} \Lambda_j \Psi_j(x), & x \in [x_j, x_{j+1}), \\ \Lambda_{j+1}\Phi_{j+1}(x), & x \in [x_{j+1}, x_{j+2}], \\ 0, & \text{otherwise}, \end{cases} \quad j = i, i+1, i+2. \qquad (5.10)$$

Using (5.2) one can readily check that $\Lambda_j\Phi_j$ and $\Lambda_j\Psi_j$ are strictly monotonic functions on the interval $[x_j, x_{j+1}]$. The splines $B_{j,2}$ are a generalization of the "hat-functions" for polynomial B-splines. They are nonnegative and, furthermore, $B_{j,2}(x_{j+l}) = \delta_{1,l}$, $l = 0, 1, 2$.

Let us denote

$$\Lambda U(x) \equiv \Lambda_i U_i(x),$$
$$D_1 U(x) \equiv U_i[x - \tau_i, x + \tau_i], \qquad x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N;$$

then from (3.5) ΛU and $D_1 U$ are well defined if $U \in S_4^{DH}$. With the previous notation, according to (5.7), (5.9), and (5.10) we obtain

$$\Lambda B_i(x) = \sum_{j=i+1}^{i+3} m_{j,B_i} B_{j-1,2}(x)$$

$$= \frac{1}{c_{i,3}}\left(\frac{B_{i,2}(x)}{c_{i,2}} - \frac{B_{i+1,2}(x)}{c_{i+1,2}}\right) - \frac{1}{c_{i+1,3}}\left(\frac{B_{i+1,2}(x)}{c_{i+1,2}} - \frac{B_{i+2,2}(x)}{c_{i+2,2}}\right). \qquad (5.11)$$

In addition the function $D_1 B_i$ satisfies to the relation

$$D_1 B_i(x) = \frac{B_{i,3}(x)}{c_{i,3}} - \frac{B_{i+1,3}(x)}{c_{i+1,3}}, \qquad (5.12)$$

where

$$B_{j,3}(x) = \begin{cases} \frac{1}{c_{j,2}}\Psi_j[x - \tau_j, x + \tau_j], & x \in [x_j, x_{j+1}), \\ 1 + \frac{1}{c_{j,2}}\Phi_{j+1}[x - \tau_{j+1}, x + \tau_{j+1}] \\ \quad - \frac{1}{c_{j+1,2}}\Psi_{j+1}[x - \tau_{j+1}, x + \tau_{j+1}], & x \in [x_{j+1}, x_{j+2}), \\ -\frac{1}{c_{j+1,2}}\Phi_{j+2}[x - \tau_{j+2}, x + \tau_{j+2}], & x \in [x_{j+2}, x_{j+3}), \\ 0, & \text{otherwise.} \end{cases} \qquad (5.13)$$

Functions $B_{j,3}$ and $B_{j,4} \equiv B_j$ possess many of the properties inherent in usual discrete polynomial B-splines. We collect their characteristics in the next theorem which can be proved by using the explicit formulae (5.7), (5.10), and (5.13) for discrete HB-splines $B_{j,k}$, $j = 2, 3, 4$, and the relations (5.11) and (5.12).

Theorem 1. *The functions $B_{j,k}$, $k = 3, 4$ have the following properties:*

1. *$B_{j,4}(x) > 0$ for $x \in (x_j + \tau_j, x_{j+4} - \tau_{j+4})$, and $B_{j,4}(x) \equiv 0$ if $x \notin (x_j, x_{j+4})$, $B_{j,3}(x) > 0$ for $x \in (x_j, x_{j+3})$, and $B_{j,3}(x) \equiv 0$ if $x \notin (x_j, x_{j+3})$;*

2. *$B_{j,4}$ satisfies the continuity conditions (3.5);*

3. *$B_{j,3}$ satisfies the first and second continuity conditions in (3.5);*

4. *$\sum_{j=-2}^{N} B_{j,3}(x) \equiv 1$ for $x \in [a, b]$,*
   $\Phi_j[x - \tau_j, x + \tau_j] = -c_{j-1,2}B_{j-2,3}(x)$, $\Psi_j[x - \tau_j, x + \tau_j] = c_{j,2}B_{j,3}(x)$
   for $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N$;

5. $\sum_{j=-3}^{N} y_{j+2}^r B_{j,4}(x) \equiv x^r$, $r = 0, 1$ for $x \in [a, b]$,

   $\Phi_j(x) = c_{j-1,2}c_{j-2,3}B_{j-3,4}(x)$, $\quad \Psi_j(x) = c_{j,2}c_{j,3}B_{j,4}(x)$

   for $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N$.

Figures 1 and 2 show the graphs of discrete HB-splines $B_{j,k}$, $k = 2, 3, 4$ (from left to right) on a uniform mesh with step size $h = 1$ and with $\tau_i = \tau$ for all i. We have chosen discretization parameters $\tau = 0.1$ (Fig. 1, left and Fig. 2, right), $\tau = 0.33$ (Fig. 1, right) and $\tau = 0.5$ (Fig. 2, left) for $\psi_i(t)$ from (5.2). In figures 1 and 2 (left) we have parameters $p_i = 0$, i.e. we have conventional discrete cubic B-splines (e.g., see [16]). Visually, the presence of intervals where the B-spline $B_{j,4}$ is negative is more visible with growing discretization parameter τ. In figure 2 (right) the tension parameters are $p_i = 50$ for all i, whence the shape of the graphs is practically unchanged when τ increases from 0.1 to 0.5. As the limit for $p_i \to \infty$ we obtain the pulse function for $B_{j,2}$, the "step-function" for $B_{j,3}$ and the "hat-function" for $B_{j,4}$ (all of height 1).

Figure 3 shows the graphs of discrete HB-splines $B_{j,4}$ on a uniform mesh (left) and on a nonuniform mesh (right), where the asterisk $*$ denotes the x_i. For both plots $p_i = 2$ and $n_i = 2$.

Using the approach of [14], it is easy to show that the functions B_j, $j = -3, \ldots, N$ have supports of minimum length, are linearly independent and form a basis in the space S_4^{DH}. So any discrete hyperbolic tension spline $U \in S_4^{DH}$ can be uniquely represented in the form

$$U(x) = \sum_{j=-3}^{N} b_j B_j(x) \qquad (5.14)$$

with some constant coefficients b_j.

Applying formulae (5.11) and (5.12) to the representation (5.14) we obtain

$$D_1 U(x) = \sum_{j=-2}^{N} b_{j,3} B_{j,3}(x), \quad \Lambda U(x) = \sum_{j=-1}^{N} b_{j,2} B_{j,2}(x), \qquad (5.15)$$

where

$$b_{j,3-k} = \frac{b_{j,4-k} - b_{j-1,4-k}}{c_{j,3-k}}, \quad k = 0, 1; \quad b_{j,4} \equiv b_j.$$

5.3 Formulas for Local Approximation by HB-Splines

If the coefficients b_j in (5.14) are known then by virtue of formula (5.7) we can write out an expression for the discrete hyperbolic tension spline U on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$U(x) = b_{i-2} + \tilde{\Delta}_i b_{i-2}(x - y_i) + c_i \Phi_i(x) + c_{i+1} \Psi_i(x), \qquad (5.16)$$

where

$$c_j = \frac{\tilde{\Delta}_j b_{j-2} - \tilde{\Delta}_{j-1} b_{j-3}}{c_{j-1,2}}, \quad j = i, i+1, \quad \tilde{\Delta}_j b_{j-2} = \frac{b_{j-1} - b_{j-2}}{c_{j-1,3}}.$$

The representations (5.14) and (5.16) allow us to find a simple and effective way to approximate a given function f from its samples.
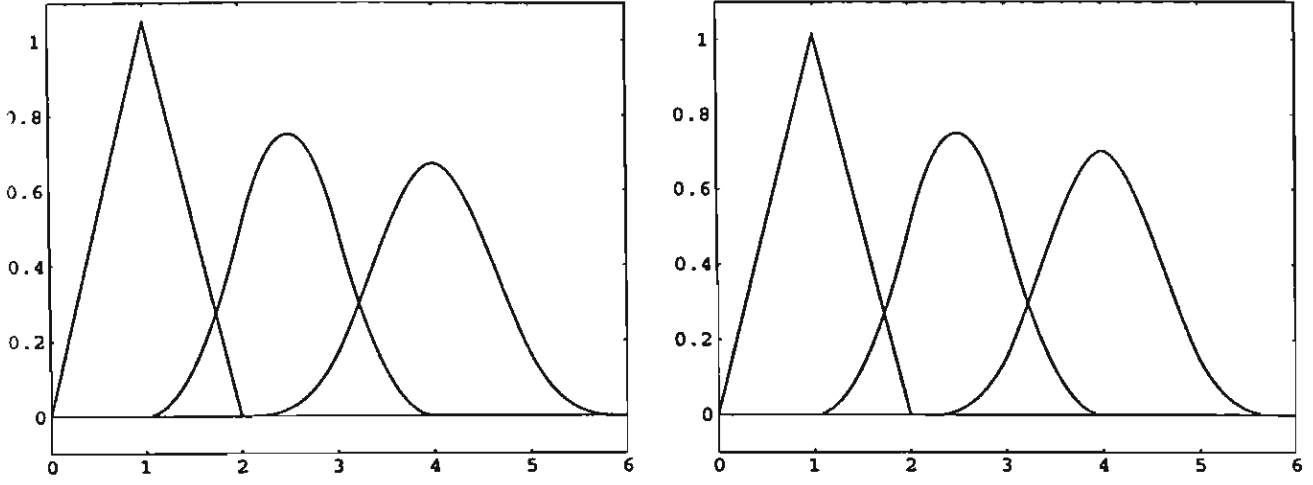
Fig. 1. The discrete HB-splines $B_{j,k}$, $k = 2, 3, 4$ (from left to right)
on a uniform mesh with step size $h = 1$, no tension and
discretization parameter $\tau = 0.1$ (left) and $\tau = 0.33$ (right).
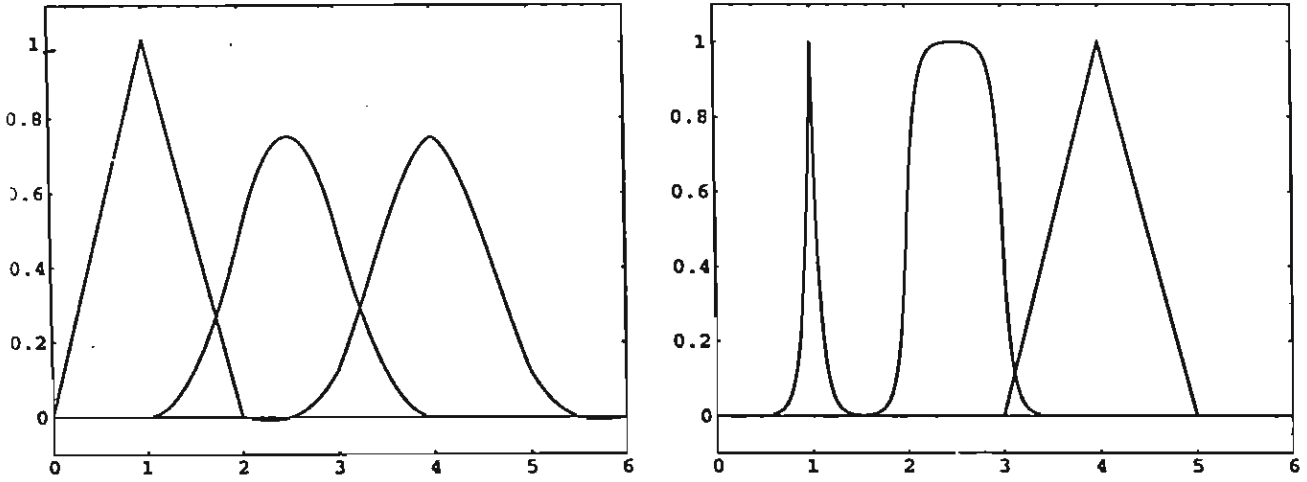
Fig. 2. Same as Fig. 1, but with discretization parameter $\tau = 0.5$
(left) and with tension parameters $p_i = 50$ for all i (right).
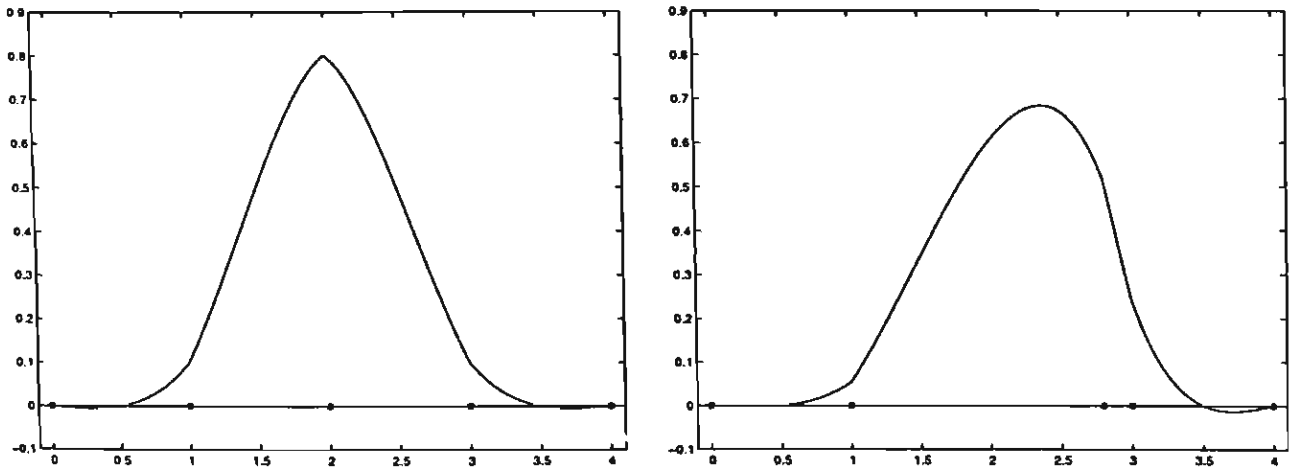
Fig. 3. The discrete HB-splines $B_{j,4}$
on a uniform mesh (left) and on a nonuniform mesh (right).
The asterisk $*$ denotes the x_i. For both plots $p_i = 2$ and $n_i = 2$.

Theorem 2. *For $b_j = f(y_{j+2})$, $j = -3, \ldots, N$, formula (5.14) is exact for polynomials of the first degree and provides a formula for local approximation.*

Proof: It suffies to prove that the identities

$$\sum_{j=-3}^{N} y_{j+2}^r \mathrm{B}_j(x) \equiv x^r, \quad r = 0, 1 \tag{5.17}$$

hold for $x \in [a, b]$. Using formula (5.16) with the coefficients $b_{j-2} = 1$ and $b_{j-2} = y_j$, $j = i-1, i, i+1, i+2$, for an arbitrary interval $[x_i, x_{i+1}]$, we find that identities (5.17) hold.

For $b_{j-2} = f(y_j)$, formula (5.16) can be rewritten as

$$\mathrm{U}(x) = f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1})f[y_{i-1}, y_i, y_{i+1}]c_{i-1,2}^{-1}\Phi_i(x)$$
$$+ (y_{i+2} - y_i)f[y_i, y_{i+1}, y_{i+2}]c_{i,2}^{-1}\Psi_i(x), \quad x \in [x_i, x_{i+1}].$$

This is the formula of local approximation. The theorem is proved.

Corollary 1. *By setting*

$$b_{j-2} = f_j - \frac{1}{c_{j-1,2}} \left[\Psi_{j-1}(x_j) \frac{f_{j+1} - f_j}{h_j} - \Phi_j(x_j) \frac{f_j - f_{j-1}}{h_{j-1}} \right] \tag{5.18}$$

in (5.14), we obtain a formula of three-point local approximation, which is exact for polynomials of the first degree.

Proof: To prove the corollary, it is sufficient to take the monomials 1 and x as f. Then according to (5.18), we obtain $b_{j-2} = 1$ and $b_{j-2} = y_j$ and it only remains to make use of identities (5.17). This proves the corollary.

Equation (5.16) permits us to write the coefficients of the spline U in its representation (5.14) in the form

$$b_{j-2} = \begin{cases} \mathrm{U}(y_j) - \Lambda_{j-1}\mathrm{U}(x_{j-1})\Phi_{j-1}(y_j) - \Lambda_j\mathrm{U}(x_j)\Psi_{j-1}(y_j), & y_j < x_j, \\ \mathrm{U}(y_j) - \Lambda_j\mathrm{U}(x_j)\Phi_j(y_j) - \Lambda_{j+1}\mathrm{U}(x_{j+1})\Psi_j(y_j), & y_j \geq x_j. \end{cases} \tag{5.19}$$

Using this formula we obtain $b_{j-2} = \mathrm{U}(y_j) + O(\overline{h}_j^2)$, $\overline{h}_j = \max(h_{j-1}, h_j)$. Hence it follows that the control polygon (e.g., see [8]) converges quadratically to the function f for $b_{j-2} = f(y_j)$, or if the formula (5.18) is used. Formulas (5.16), (5.17), and (5.19) generalize their continuous equivalents developed in [10].

6. Computational Aspects

The aim of this section is to investigate the practical aspects related to the numerical evaluation of the mesh solution defined in (2.9).

A standard approach, [25], consists of solving the tridiagonal system (3.6) and then evaluating (3.4) at the mesh points as is usually done for the evaluation of continuous hyperbolic splines. At first sight, this approach based on the solution of a

tridiagonal system seems preferable because of the limited waste of computational time and the good classical estimates for the condition number of the matrix in (3.6). However, it should be observed that, as in the continuous case, we have to perform a large number of numerical computations of hyperbolic functions of the form $\sinh(k_i t)$ and $\cosh(k_i t)$ both to define system (3.6) and to tabulate functions (3.4). This is a very difficult task, both for cancellation errors (when $k_i \to 0$) and for overflow problems (when $k_i \to \infty$). A stable computation of the hyperbolic functions was proposed in [25], where different formulas for the cases $k_i \leq 0.5$ and $k_i > 0.5$ were considered and a specialized polynomial approximation for $\sinh(\cdot)$ was used.

However, we note that this approach is the only one possible if we want a continuous extension of the discrete solution beyond the mesh point.

In contrast, the discretized structure of our construction provides us with a much cheaper and simpler approach to compute the mesh solution (2.9). This can be achieved both by following the system splitting approach presented in Section 3, or by a direct computation of the solution of the linear system (2.6)–(2.8).

- As for the system splitting approach, presented in Section 3, the following algorithm can be considered.

Step 1. *Solve the 3-diagonal system* (3.6) *for* m_i, $i = 1, \ldots, N$.

Step 2. *Solve* $N + 1$ *3-diagonal systems* (3.2) *for* m_{ij}, $j = 1, \ldots, n_i - 1$, $i = 0, \ldots, N$,

Step 3. *Solve* $N + 1$ *3-diagonal systems* (3.3) *for* u_{ij}, $j = 1, \ldots, n_i - 1$, $i = 0, \ldots, N$.

In this algorithm, hyperbolic functions need only be computed in step 1. Furthermore, the solution of any system (3.2) or (3.3) requires $8q$ arithmetic operations, namely, $3q$ additions, $3q$ multiplications, and $2q$ divisions [31], where q is the number of unknowns, and is thus substantially cheaper than direct computation by formula (3.4).

Steps 2 and 3 can be replaced by a direct splitting of the system (2.6)–(2.8) into $N + 1$ systems with 5-diagonal matrices

$$u_{i,0} = f_i, \quad \Lambda_i u_{i,0} = M_i,$$

$$\Lambda_i^2 u_{i,j} - \left(\frac{p_i}{h_i}\right)^2 \Lambda_i u_{i,j} = 0, \quad j = 1, \ldots, n_i - 1, \quad i = 0, \ldots, N. \tag{6.1}$$

$$u_{i,n_i} = f_{i+1}, \quad \Lambda_i u_{i,n_i} = M_{i+1},$$

Also, in this case the calculations for steps 2 and 3 or for system (6.1) can be tailored for a multiprocessor computer system.

Let us discuss now the direct solution of system (2.6)–(2.8) which, of course, only involves rational computations on the given data. In order to do this in the next subsections we investigate in some details the structure of the mentioned system.

6.1 The Pentadiagonal System

Eliminating the unknowns $\{u_{i,-1}, i = 1, \ldots, N, \}$ and $\{u_{i,n_i+1}, i = 0, \ldots, N - 1\}$, from (2.7) determining the values of the mesh solution at the data sites x_i by the

interpolation conditions and eliminating $u_{0,-1}$, u_{N,n_N+1} from the end conditions (2.8) we can collect (2.6)–(2.8) into the system

$$\mathbf{Au = b,} \tag{6.2}$$

where

$$\mathbf{u} = (u_{01}, \ldots, u_{0,n_0-1}, u_{11}, \ldots, u_{21}, \ldots, u_{N1}, \ldots, u_{N,n_N-1})^T,$$

$\mathbf{A}$ is the following pentadiagonal matrix (see also Fig. 1, left):

$$
\begin{bmatrix}
b_0 - 1 & a_0 & 1 & & & & & & & & & \\
a_0 & b_0 & a_0 & 1 & & & & & & & & \\
1 & a_0 & b_0 & a_0 & 1 & & & & & & & \\
& & \ddots & & & & & & & & & \\
& & 1 & a_0 & b_0 & & a_0 & & & & & \\
& & & 1 & a_0 & \eta_{0,n_0-1} & \delta_{0,n_0-1} & & & & & \\
& & & & & \delta_{1,1} & \eta_{1,1} & a_1 & 1 & & & \\
& & & & & & a_1 & b_1 & a_1 & 1 & & \\
& & & & & & & & \ddots & & & \\
& & & & & & & 1 & a_N & b_N & a_N & 1 \\
& & & & & & & & 1 & a_N & b_N & a_N \\
& & & & & & & & & 1 & a_N & b_N - 1
\end{bmatrix}
$$

with

$$a_i = -(4 + \omega_i) \,, \ b_i = 6 + 2\omega_i \,, \ \omega_i = \left(\frac{p_i}{n_i}\right)^2 ; \quad i = 0, 1, \ldots, N,$$

$$\eta_{i-1,n_{i-1}-1} = 6 + 2\omega_{i-1} + \frac{1 - \rho_i}{1 + \rho_i} \,, \ \eta_{i,1} = 6 + 2\omega_i + \frac{\rho_i - 1}{\rho_i + 1}$$

$$\delta_{i-1,n_{i-1}-1} = \frac{2}{\rho_i(\rho_i + 1)} \,, \ \delta_{i,1} = 2\frac{\rho_i^2}{\rho_i + 1} \,,$$

$$\rho_i = \frac{\tau_i}{\tau_{i-1}}, \qquad i = 1, 2, \ldots, N;$$

and

$$\mathbf{b} = (-(a_0 + 2)f_0 - \tau_0^2 f_0'', -f_0, 0, \ldots, 0, -f_1, -\gamma_{0,n_0-1}f_1, -\gamma_{1,1}f_1, -f_1, 0,$$
$$\ldots, 0, -f_{N+1}, -(a_N + 2)f_{N+1} - \tau_N^2 f_{N+1}'')^T,$$

with

$$\gamma_{i-1,n_{i-1}-1} = -(4 + \omega_{i-1} + 2\frac{1 - \rho_i}{\rho_i}), \quad i = 1, 2, \ldots, N.$$
$$\gamma_{i,1} = -(4 + \omega_i + 2(\rho_i - 1)),$$

6.2 The Uniform Case

From the practical point of view it is interesting to examine the structure of $\mathbf{A}$ when we are dealing with a uniform mesh, that is $\tau_i = \tau$. In such a case it is immediately seen that $\mathbf{A}$ is symmetric. In addition, following [17] we observe that $\mathbf{A} = \mathbf{C} + \mathbf{D}$, where both $\mathbf{C}$ and $\mathbf{D}$ are symmetric block diagonal matrices. To be more specific,

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_0 & & & \\ & \mathbf{C}_1 & & \\ & & \ddots & \\ & & & \mathbf{C}_N \end{bmatrix}, \quad \mathbf{C}_i = \mathbf{B}_i^2 - \omega_i \mathbf{B}_i,$$

where $\mathbf{B}_i$ is the $(n_i - 1) \times (n_i - 1)$ tridiagonal matrix

$$\mathbf{B}_i = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \cdots & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix};$$

and

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & \\ & & & 0 & & & & & & & & & \\ & & & & 1 & 1 & & & & & & & \\ & & & & 1 & 1 & & & & & & & \\ & & & & & & 0 & & & & & & \\ & & & & & & & \ddots & & & & & \\ & & & & & & & & 0 & & & & \\ & & & & & & & & & 1 & 1 & & \\ & & & & & & & & & 1 & 1 & & \\ & & & & & & & & & & & 0 & \\ & & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & & & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{C}$, $\lambda_k(\mathbf{C})$, are the collection of the eigenvalues of $\mathbf{C}_i$. Since, (see [17]),

$$\lambda_j(\mathbf{B}_i) = -2\left(1 - \cos\frac{j\pi}{n_i}\right), \quad j = 1, \ldots, n_i - 1,$$

we have

$$\lambda_j(\mathbf{C}_i) = 4\left(1 - \cos\frac{j\pi}{n_i}\right)^2 + 2\omega_i\left(1 - \cos\frac{j\pi}{n_i}\right) \quad j = 1, \ldots, n_i - 1.$$

In addition, the eigenvalues of $\mathbf{D}$ are 0 and 2, thus we deduce from a corollary of the Courant-Fisher theorem [7] that the eigenvalues of $\mathbf{A}$ satisfy the following inequalities

$$\lambda_k(\mathbf{A}) \geq \lambda_k(\mathbf{C}) = \min_{i,j} \lambda_j(\mathbf{C}_i) = \min_i \left[4\left(1 - \cos\frac{\pi}{n_i}\right)^2 + 2\omega_i\left(1 - \cos\frac{\pi}{n_i}\right) \right].$$

Hence, $\mathbf{A}$ is a positive matrix and we directly obtain that the pentadiagonal linear system has a unique solution.

In addition, by Gershgorin's theorem, $\lambda_k(\mathbf{A}) \leq \max_i[16+4\omega_i]$. Then we obtain the following upper bound for the condition number of $\mathbf{A}$ which is independent of the number of data points, $N + 2$, and which recovers the result presented in [17] for the limit case $p_i = 0, \quad i = 0, \ldots, N$,

$$\|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty \leq \frac{\max_i \left[16 + 4(\frac{p_i}{n_i})^2\right]}{\min_i \left[4(1 - \cos\frac{\pi}{n_i})^2 + 2(\frac{p_i}{n_i})^2(1 - \cos\frac{\pi}{n_i})\right]}$$
$$\simeq \frac{\max_i \left[16 + 4(\frac{p_i}{n_i})^2\right]}{\min_i(\frac{1}{n_i})^4[\pi^4 + (\pi p_i)^2]}. \tag{6.3}$$

Summarizing, in the particular but important uniform case we can compute the mesh solution by solving a symmetric, pentadiagonal, positive definite system and therefore, we can use specialized algorithms, with a computational cost of $17q$ arithmetic operations, namely, $7q$ additions, $7q$ multiplications, and $3q$ divisions [31], where q is the number of unknowns.

Moreover, since the upper bound (6.3) for the condition number of the matrix $\mathbf{A}$ does not depend on the number of interpolation points, such methods can be used with some confidence.

In the general case of a non–uniform mesh, the matrix $\mathbf{A}$ is no longer symmetric, and an analysis of its condition number cannot be carried out analytically. However, several numerical experiments have shown that the condition number is not influenced by the non–symmetric structure, but does depend on the maximum number of grid points in each subinterval, exactly as in the symmetric case. In other words, symmetric and nonsymmetric matrices, with the same dimension and produced by difference equations with the same largest n_i, produce very close condition numbers. Non–uniform discrete hyperbolic tension splines have in fact been used for the graphical tests of the following section.

6.3 System Splitting

Sometimes the number of unknowns in (6.2) can be very large (for example for generating a grid in bivariate interpolation) and then even the linear computational cost of the solution of the pentadiagonal system may turn out to be too expensive. However, as for the two first approaches proposed at the beginning of this section for evaluating the mesh solution, if we have a parallel machine we can easily share
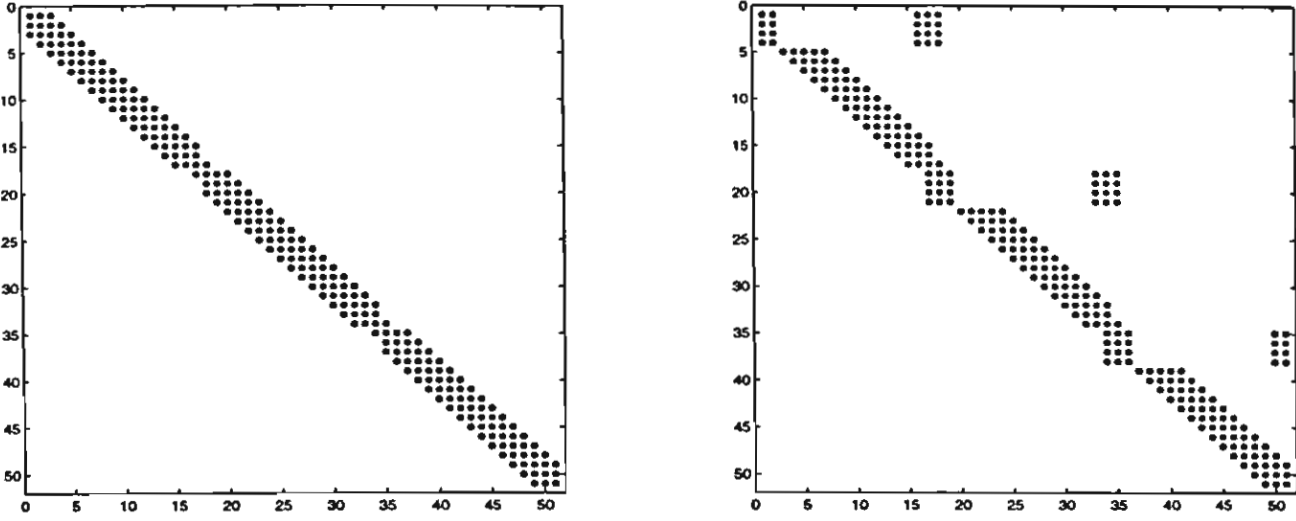
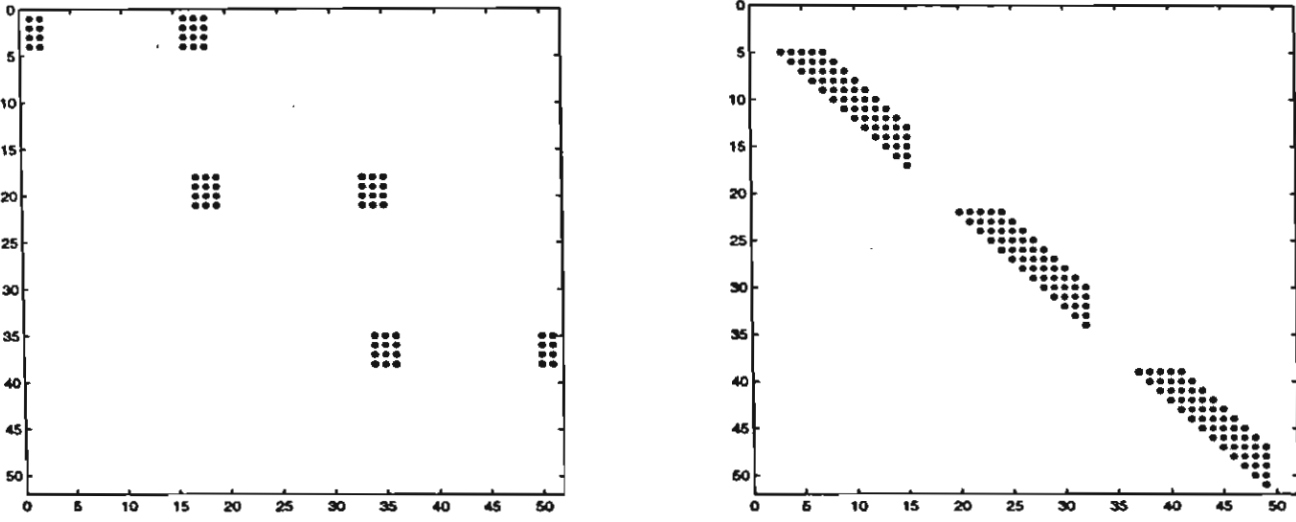Fig. 4. Left: the form of $\mathbf{A}$ for $N = 2$, $n_i = 18$. Right: the matrix $\mathbf{K}$.

Fig. 5. Left: The block matrix $\mathbf{E}$. Right: the block matrix $\mathbf{F}$.

the computation of the solution of our pentadiagonal system among the processors as outlined below.

The basic idea is to transform $\mathbf{A}$, which, for $N = 2$, $n_i = 18$ has the form shown in Fig. 4 left, into the form $\mathbf{K}$ (see Fig. 4 right). Setting $r_i = \sum_{\nu=0}^{i-1}(n_\nu - 1)$, we note that the rows $r_i + 1, \ldots, r_i + n_i - 1$ of $\mathbf{A}$ describe equations (2.6) for the subinterval $[x_i, x_{i+1}]$. If we extract from $\mathbf{K}$ the rows $r_i+1, \ldots, r_i+4$, $i = 0, 1, \ldots, N$, we get a block matrix $\mathbf{E}$ of the form shown in Fig. 5 left. The corresponding linear system has few equations, and having solved it, it is possible to solve in parallel the $N + 1$ linear systems obtained from the "remaining" matrix $\mathbf{F}$ of Fig. 5 right by extracting its independent blocks.

The problem now is how to move from $\mathbf{A}$ to $\mathbf{K}$. From Sections 2 and 3 we have the following two facts. Having in mind the structure of $\mathbf{A}$ and the corresponding Fig. 4, let us consider the section given by rows $r_i + 1, \ldots, r_{i+1}$. We note that the entries of the columns with index $r_i + 3, \ldots, r_{i+1} - 2$ are $1, a_i, b_i, a_i, 1$ which are the coefficients of the difference equation (2.6). On the other hand, it is shown in

Section 3 that any function of the form

$$\Upsilon_i(x) = c_1(1-t) + c_2 t + c_3 \varphi_i(1-t) + c_4 \varphi_i(t) , \qquad (6.4)$$

is a solution for (2.6); therefore if we multiply the row of index $r_i + j$, $j = 1, \ldots, n_i - 1$, by $\Upsilon_i(x_{i,j}) = \Upsilon_i(x_i + j\tau_i)$ and then add all these rows, then the contribution of all the columns from $r_i + 3$ to $r_{i+1} - 2$ sums up to zero. The idea for obtaining the matrix K from A is the following: we replace the four rows of index $r_i + 1, r_i + 2, r_i + 3, r_i + 4$ with the sum of the rows from $r_i + 1$ to r_{i+1} multiplied by the values assumed in x_{ij} by four linearly independent functions of the form (6.4). The remaining question is how to choose these functions. Several numerical experiments have shown that the lowest condition number of the matrix K (which is in general larger than that of A) is achieved when we use the cardinal functions for Lagrange interpolation at the points $x_{i\nu}$ closest to $x_i, x_i + h_i/3, x_{i+1} - h_i/3, x_{i+1}$.

7. Graphical Examples

The aim of this final section is to illustrate the tension features of discrete hyperbolic tension splines with some (famous) examples. Before, we want to notice that the continuous form U_i of our solution given in (3.4) has the good shape-preserving properties of cubics (see e.g. [25]) in the sense that U_i is convex (concave) in $[x_i, x_{i+1}]$ if and only if $m_{i+j} \geq 0$ (≤ 0), $j = 0, 1$, and has at most one inflection point in $[x_i, x_{i+1}]$. In order to preserve the shape of the data, we therefore simply have to analyze the values $\Lambda_i u_{i,0}$ and $\Lambda_i u_{i,n_i}$ and increase the tension parameters if necessary. All the strategies proposed for the automatic choice of tension parameters in continuous hyperbolic tension spline interpolation can be used in our discrete context, see e.g. [24, 25].

In our first example we have interpolated the *radio chemical* data reported in Table 1. The effects of changing the tension values p_i are depicted in Figs. 6–7. We have adopted a non–uniform mesh, assigning the same number of points (30) to each interval of the main mesh, and imposed *natural* end conditions, that is, following formulas (3.6), $m_0 = m_{N+1} = 0$.

Table 1. *Radio chemical* data:

x_i	7.99	8.09	8.19	8.7	9.2
f_i	0	2.76429E-5	4.37498E-2	0.169183	0.469428

x_i	10	12	15	20
f_i	0.943740	0.998636	0.999916	0.999994

Fig. 6 is obtained setting $p_i = 0$, that is considering the discrete cubic spline interpolating the data. In Fig 7 a new discrete interpolant with $p_0 = p_1 = 300$, $p_i = 15$, $i = 2, \ldots, 7$, is displayed for the same data, and the stretching effect of the increase in tension parameters is evident.

In the second example we have taken *Akima's* data of Table 2 and constructed discrete interpolants with 20 points for each interval, with natural end conditions $m_0 = m_{N+1} = 0$. Fig. 8 left shows the plot produced by a uniform choice of tension
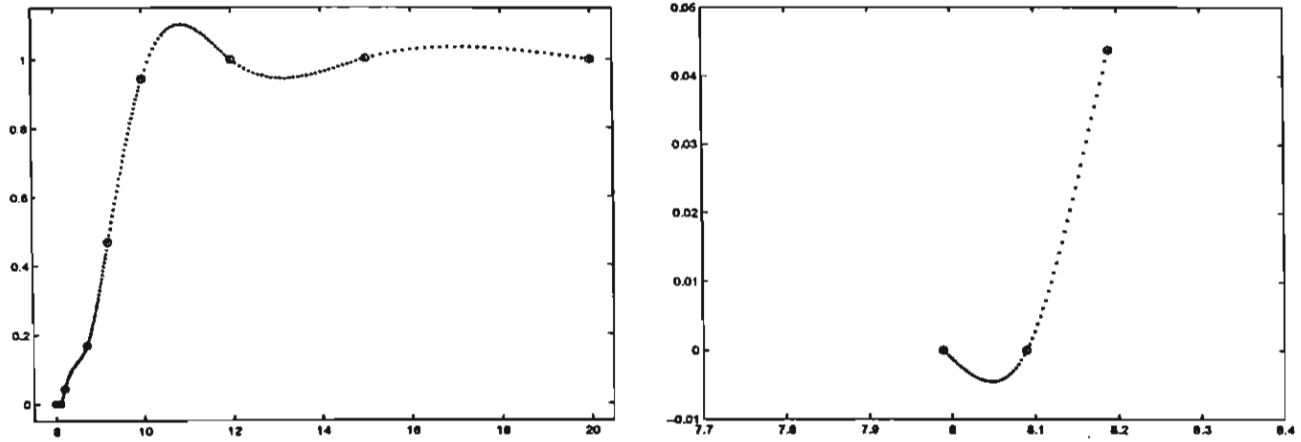
Fig. 6. The radio chemical data with natural end conditions $m_0 = m_{N+1} = 0$
Interpolation by discrete cubic spline. ($p_i = 0$)
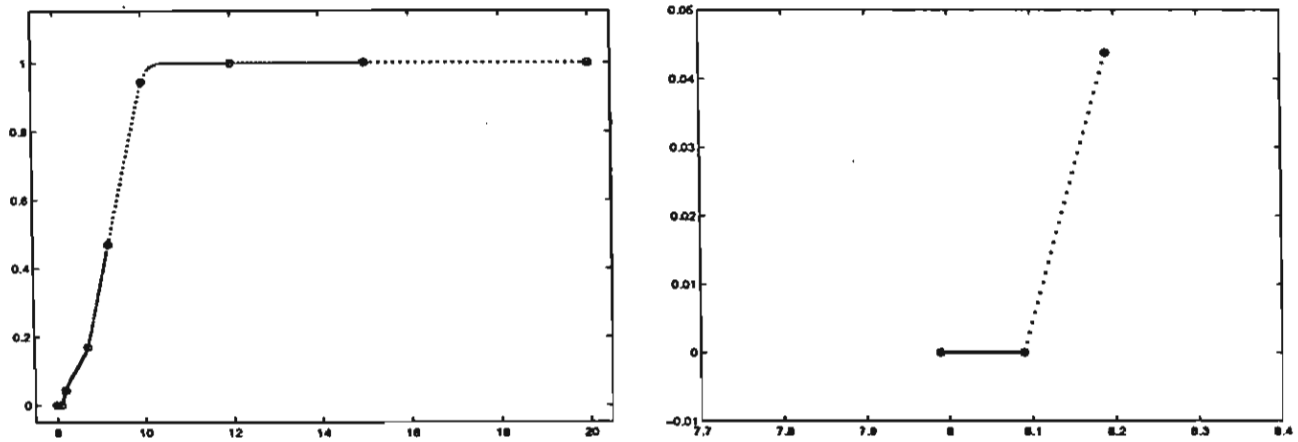Right: a magnification of the lower left corner.

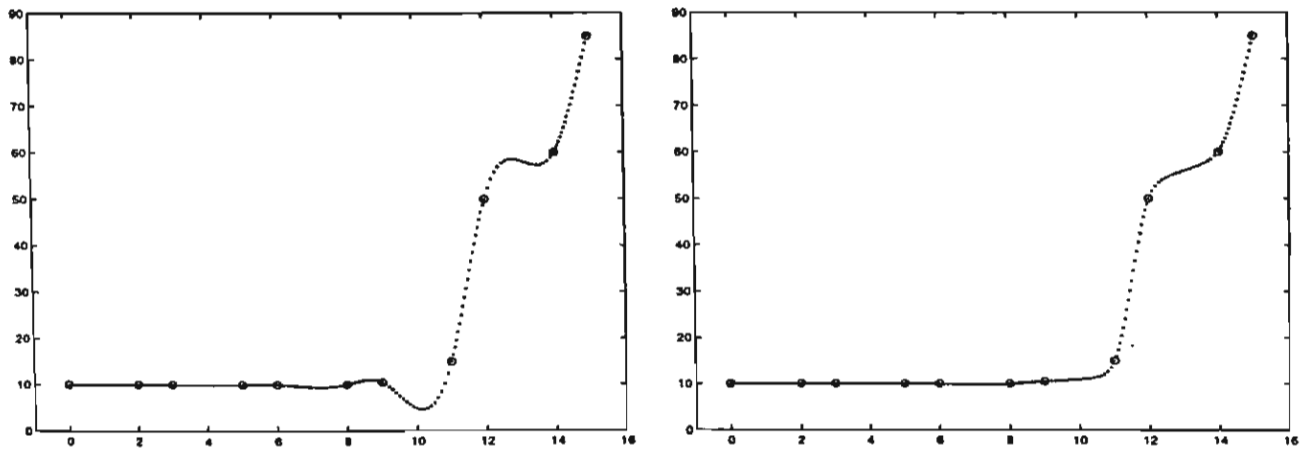Fig. 7. The same as Fig. 6 with $p_0 = p_1 = 300$, $p_i = 15, i = 2, \ldots, 7$.

Fig. 8. Akima's data with natural end conditions.
Left: Discrete interpolating cubic spline ($p_i = 0$)
Right: discrete hyperbolic spline with $p_5 = p_6 = p_8 = 10$.

factors, namely $p_i = 0$. The right part of the same figure shows a second mesh solution, which perfectly reproduces the data shape, where we have set $p_5 = p_6 = p_8 = 10$ while the remaining p_i are unchanged.

Table 2. *Akima's* data [1]:

x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	50	60	85

References

1. H. Akima, A new method of interpolation and smooth curve fitting based on local procedures, J. Assoc. Comput. Mech. **17** (1970) 589–602.

2. de Boor, C. (1976) Splines as linear combinations of B-splines: A survey, Approximation Theory II. G. G. Lorentz, C. K. Chui, and L. L. Schumaker (Eds.). Academic Press, New York, 1–47.

3. Cohen, E., T. Lyche, and R. Riesenfeld (1980) Discrete B-splines and subdivision techniques in computer aided geometric design and computer graphics, Computer Graphics and Image Processing **14**, 87–111.

4. P. Costantini, B. I. Kvasov and C. Manni, Difference Method for Constructing Hyperbolic Tension Splines, Rapporto Interno 341/1998, Università di Siena.

5. W. Dahmen and C. A. Micchelli, On multivariate E-splines, Advances in Mathematics **76** (1989) 33–93.

6. R. Delbourgo and J. A. Gregory, Shape preserving piecewise rational interpolation, SIAM J. Sci. and Statist. Comput. **6** (1985) 967–976.

7. G. H. Golub and C. F. Van Loan, *Matrix Computations* (John Hopkins University Press, Baltimore, 1991).

8. Hoschek, J. and D. Lasser, *Fundamentals of Computer Aided Geometric Design* (A K Peters, Wellesley, Massachusetts, 1993).

9. N. N. Janenko and B. I. Kvasov, An iterative method for the construction of polycubic spline functions, Soviet Math. Dokl. **11** (1970) 1643–1645.

10. P. E. Koch and T. Lyche, Exponential B-splines in tension, in: *Approximation Theory VI: Proceedings of the Sixth International Symposium on Approximation Theory*, Vol. II. Chui C. K., Schumaker L. L., and Ward J. D. (eds.) (Academic Press, Boston, 1989) pp. 361–364.

11. P. E. Koch and T. Lyche, Interpolation with Exponential B-splines in Tension, in: *Geometric Modelling*, Computing/Supplementum 8. Farin G. et al. (eds.) (Springer-Verlag, Wien, 1993) pp. 173–190.

12. B. I. Kvasov, Shape Preserving Spline Approximation via Local Algorithms, in: *Advanced Topics in Multivariate Approximation*, F. Fontanella, K. Jetter, and P. J. Laurent (eds.) (World Scientific Publ. Co., Inc., Singapore, 1996) pp. 181–196.

13. B. I. Kvasov, Local bases for generalized cubic splines, Russ. J. Numer. Anal. Math. Modelling **10** (1995), 1, 49–80.

14. B. I. Kvasov, GB-splines and their properties, Annals of Numerical Mathematics **3** (1996) 139–149.

15. P. J. Laurent, *Approximation et optimization* (Hermann, Paris, 1972).

16. T. Lyche, Discrete cubic spline interpolation, BIT **16** (1976) 281–290.

17. M. A. Malcolm, On the computation of nonlinear spline functions, SIAM J. Numer. Anal. **14** (1977) 254–282.

18. O. L. Mangasarian and L. L. Schumaker, Discrete splines via mathematical programming, SIAM J. Control **9** (1971) 174–183.

19. O. L. Mangasarian and L. L. Schumaker, Best summation formulae and discrete spline, SIAM J. Numerical Analysis **10** (1973) 448–459.

20. M. Marušić and M. Rogina, Sharp error bounds for interpolating splines in tension, J. of Comp. Appl. Math. **61** (1995) 205–223.

21. A. A. Melkman, Another proof of the total positivity of the discrete spline collocation matrix, J. Approx. Theory **84** (1996) 265–273.

22. K. M. Mørken, On total positivity of the discrete spline collocation matrix, J. Approx. Theory **84** (1996) 247–264.

23. S. S. Rana and Y. P. Dubey, Local behaviour of the deficient discrete cubic spline interpolator, J. Approx. Theory **86** (1996) 120–127.

24. R. J. Renka, Interpolation tension splines with automatic selection of tension factors, SIAM J. Sci. Stat. Comp. **8** (1987) 393–415.

25. P. Rentrop, An algorithm for the computation of exponential splines, Numer. Math. **35** (1980) 81–93.

26. A. Ron, Exponential Box splines, Constructive Approximation **4** (1988) 357–378.

27. N. S. Sapidis and P. D. Kaklis, An algorithm for constructing convexity and monotonicity-preserving splines in tension, Computer Aided Geometric Design **5** (1988) 127–137.

28. Schumaker, L. L. (1973) Constructive aspects of discrete polynomial spline functions, Approximation Theory, G. G. Lorentz (ed.). Academic Press, New York, 469–476.

29. L. L. Schumaker, *Spline functions: Basic theory* (John Wiley & Sons, New York, 1981).

30. D. G. Schweikert, An interpolating curve using a spline in tension, J. Math. Phys. **45** (1966) 312–317.

31. Yu. S. Zav'yalov, B. I. Kvasov, and V. L. Miroshnichenko, *Methods of Spline Functions* (Nauka, Moscow, 1980).