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สำนักงานกองทุนสนับสนุนการวิจัย

THE THAILAND RESEARCH FUND

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Invariant and Partially Invariant Solutions of the Navier-Stokes Equations

โดย Prof.Dr.Sergey V.Meleshko และคณะ

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Invariant and Partially Invariant Solutions of the Navier-Stokes Equations

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บทคัดย่อ

โครงการนี้ได้ศึกษาการประยุกต์การวิเคราะห์เชิงกลุ่มกับสมการนาเวียร์-สโตกส์ ซึ่งมีบทบาทหลักในการวิจัยในสาขาคณิตศาสตร์ประยุกต์ ฟิสิกส์ และ วิศวกรรมศาสตร์

ปัญหาแรกที่ได้ศึกษาในโครงการนี้ คือความมีอยู่ของผลเฉลยที่เป็นประเภทวอร์เทกซ์พิเศษของสมการนาเวียร์-สโตกส์และสมการพลศาสตร์ก๊าซหนืด ซึ่ง L.V. Ovsyannikov ได้เสนอผลเฉลยประเภทนี้ สำหรับพลศาสตร์ก๊าซและของเหลวหนืด ขอให้สังเกตว่า ผลเฉลยนี้ยืนยงเป็นบางส่วนโดยเทียบกับกลุ่มการหมุน O(3) การวิเคราะห์ที่กระทำไปได้พิสูจน์ว่าผลเฉลยยืนยงบางส่วนสำหรับสมการทั้งสองประเภท (สมการนาเวียร์-สโตกส์และสมการพลศาสตร์ก๊าซหนืดสมบูรณ์) เป็นผลเฉลยที่มีความสมมาตรเชิงทรงกลม ซึ่งต่างกับพลศาสตร์ของก๊าซไร้ความหนืดและของเหลวอุดมคติ การจำแนกเชิงกลุ่มของสมการพลศาสตร์ก๊าซหนืดสมบูรณ์ ซึ่งมีความสมมาตรเชิงทรงกลมได้กระทำเสร็จสิ้นแล้ว เพื่อที่จะอำนวยให้การพิจารณาผลเฉลยยืนยงเป็นบางส่วนที่เกี่ยวข้องกับกลุ่มการหมุน O(3) ได้สมบูรณ์ยิ่งขึ้น

การวิจัยอีกส่วนหนึ่งนั้น เกี่ยวข้องกับผลเฉลยยืนยงหมู่ที่เฉพาะทาง ซึ่งได้มีฐานของพีชคณิตย่อยสี่มิติหนึ่ง คือ H4 ซึ่งมีตัวก่อกำเนิดดังต่อไปนี้

$$X_1 = \phi_1\partial_x + \phi_1'\partial_u - x\phi_1''\partial_p, \quad X_2 = \phi_2\partial_x + \phi_2'\partial_u - x\phi_2''\partial_p,$$

$$Y_1 = \psi_1\partial_y + \psi_1'\partial_v - y\psi_1''\partial_p, \quad Y_2 = \psi_2\partial_y + \psi_2'\partial_v - y\psi_2''\partial_p.$$

โครงการได้ศึกษากรณีที่สมการ Monge-Ampere นั้น ไฮเพอร์โบลิก คือ

$$Lf_x + k + l \geq 0$$

ซึ่งได้แสดงว่า ผลเฉลยหมู่นี้ เป็นกรณีเฉพาะของผลเฉลยที่มีรูปแบบความเร็วเป็นเชิงเส้นโดยเทียบกับตัวแปรปริภูมิหนึ่งหรือสองตัวแปร

คำหลัก : ผลเฉลยยืนยงและยืนยงบางส่วน การจำแนกเชิงกลุ่ม การแบ่งชั้นเชิงกลุ่ม สมการนาเวียร์-สโตกส์ และสมการก๊าซหนืด

Abstract

The project is devoted to the study two applications of group analysis to the Navier-Stokes equations. The Navier-Stokes equations play a central role in much of the research within applied mathematics, physics and engineering.

The first problem that we study in the project is the existence of solutions of special vortex type for the Navier-Stokes equations and viscous gas dynamics equations. This type of solutions for the inviscid gas and fluid dynamics equations was introduced by L.V.Ovsiannikov [36]. Note that this solution is partially invariant with respect to group of rotations $O(3)$. The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. For the completeness of consideration of partially invariant solutions that are connected with the group of rotations $O(3)$ the group classification of the full viscous gas dynamics equations with spherical symmetry has been done.

Another part of the research is devoted to a particular class of partially invariant solutions of the Navier–Stokes equations. This class of solutions is constructed on the base of the four-dimensional subalgebra H^4 with the generators

$$X_1 = \phi_1\partial_x + \phi_1'\partial_u - x\phi_1''\partial_p, \quad X_2 = \phi_2\partial_x + \phi_2'\partial_u - x\phi_2''\partial_p,$$

$$Y_1 = \psi_1\partial_y + \psi_1'\partial_v - y\psi_1''\partial_p, \quad Y_2 = \psi_2\partial_y + \psi_2'\partial_v - y\psi_2''\partial_p.$$

We systematically investigate the case, where the Monge-Ampere equation is hyperbolic ($Lf_z + k + l \geq 0$). It is shown that this class of solutions is a particular case of the solutions with linear profile of velocity with respect to one or two space variables.

Key words: Invariant and partially invariant solutions, group classification, group stratification, Navier-Stokes and viscous gas equations.

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3 Introduction

The mathematical models of many real world phenomena are formulated in the form of differential equations. One of the methods for studying the properties of differential equations is group analysis. Differential equations usually contain parameters or functions that are determined experimentally and hence are not strictly fixed. Group analysis not only helps to construct exact solutions, but also to classify the differential equations with respect to these arbitrary elements.

Many of the invariant solutions of the Navier-Stokes equations have been known for a long time; however their systematic analysis became possible only with the development[1] of the modern methods for the group analysis of differential equations [35]. The first group classification of the Navier-Stokes equations in the three-dimensional case was done in [13]. It was shown that the Lie group admitted by the Navier-Stokes equations is infinite-dimensional. There is still no classification of this group. Several papers [6–11][2] are devoted to invariant solutions of the Navier-Stokes equations. Partially invariant solutions of the Navier–Stokes equations have been less studied [41, 33]. At the same time there has been progress in studying such classes of solutions of inviscid gas dynamics equations [35, 46, 29]. Recently, L.V.Ovsiannikov [36] found one class of partially invariant solutions, called a special vortex. This solution is based on the group of rotations $O(3)$. An ideal fluid and an inviscid gas have the same class of solutions. Therefore, it is natural to investigate the existence of special vortex type solutions for the Navier-Stokes equations and viscous gas dynamics equations.

As is well-known, the main difficulty in the study of partially invariant solutions is the analysis of the compatibility [15, 25] of the appearing overdetermined systems. The analysis of compatibility can be reduced to the consecutive performance of algebraic operations of symbolic nature. These operations are connected with a prolongation of the system, substitution of composite expressions (transition onto manifold), and finding ranks of matrices. Typically, the compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations. Here we used the system REDUCE [21].

Another part of our study is devoted to the group classification of spherically symmetric viscous gas dynamics equations. The group classification problem consists of searching for admitted groups of transformations admitted by the system for all arbitrary elements and all specifications of arbitrary elements. By special choice of the arbitrary elements one can extend the admitted group.

After finding the admitted group one can try to construct exact solutions: every subgroup of the admitted group can be a source of invariant or partially invariant solutions. There is an infinite number of subgroups[3], even in cases where the admitted groups are finite-dimensional. But if two subgroups are similar, i.e., they are connected with each other by a symmetry transformation, then their corresponding invariant solutions are connected with each other by the same transformation. Since the set of subgroups can be divided into classes of similar subgroups, therefore, it is sufficient to find only one representative solution from each similar class of subgroups. A set of representatives of equivalent subgroup classes is called an optimal

[1] A historical review of a group analysis develpoment can be found in [22]. Many results of the group analysis are collected in [23]

[2] Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in [41, 16, 17, 28].

[3] Because there is a one-to-one correspondence between groups and Lie algebras one can study the Lie algebra of the admitted group.

system of subgroups. In this manuscript we give representations of all invariant solutions with respect to subgroups of two-dimensional admitted groups of spherically symmetric viscous gas dynamics equations.

We should also note here that, as for the Navier-Stokes equations, many of the invariant solutions of the viscous gas dynamics equations have been obtained by other methods [21–29]. The group classification of the viscous gas equations (in case when the first λ and the second μ coefficients of viscosity are related by the equation $\lambda = -2\mu/3$) was done in [8]. For some models of viscous gas dynamics equations, group analysis was used in [32, 7]. There also exist other similar approaches for constructing exact solutions of the Navier–Stokes equations. We note here two of them: nonclassical symmetry reductions [27, 28] and linear profile of velocity [45].

An unsteady motion of incompressible viscous fluid is governed by the Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u} = (u, v, w)$ is the velocity field, p is the fluid pressure, ∇ is the gradient operator in the three-dimensional space $\mathbf{x} = (x, y, z)$ and Δ is the Laplacian. The Navier-Stokes equations contain complete information about the structure of flows under usual temperature and pressure. Despite progress in numerical methods and techniques, there is considerable interest in finding exact solutions of the Navier-Stokes equations. Each exact solution has value, first, as the exact description of the real process in the framework of a given model; secondly, as a model to compare various numerical methods; and thirdly, as theoretical tool to improve the models used.

One method of constructing exact solutions is group analysis [35][4]. This method is based on the symmetries of the given equations. Note that many of the invariant solutions of the Navier-Stokes equations have been known for a long time: these solutions were obtained by assuming a form of the representation of the solution. Group analysis gives a method for obtaining the representation of solution. The first group classification of the Navier-Stokes equations in the three-dimensional case was done in [13][5]. It was shown that the Lie algebra admitted by the Navier-Stokes equations is infinite-dimensional[6]. For each subalgebra of the admitted algebra one can try to find an invariant or partially invariant solution. Several papers [?, 14, 26, 6, 47, 49, 39][7] are devoted to invariant solutions of the Navier-Stokes equations. Another class of solutions that is suggested by group analysis is the class of partially invariant solutions [35, 34]. The theory of partially invariant solutions is still developing [37, 38]. While partially invariant solutions of the Navier–Stokes equations have been less studied [?], there has been substantial progress in studying such classes of solutions of inviscid gas dynamics equations [1,20–26].

We should note here that there are also other approaches for constructing exact solutions of the Navier–Stokes equations. We note here two of them: nonclassical symmetry reductions and direct methods [28, 27] and linear profile of velocity [45, 50].

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[4] A historical review of the development of group analysis can be found in [22]. Many of the results of group analysis are collected in [23].

[5] The first classification of the two-dimensional Navier-Stokes equations was studied in [40].

[6] The classification of infinite–dimensional subalgebras of this algebra was studied in [20]. There is still no full classification of the subalgebras of this algebra.

[7] Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in [?, 16, 17, 28].

4 The method of study

Let us first review the notations and techniques used in group analysis.

Let an l-th order system of differential equations

$$(S): \qquad F^k(x,u,p) = 0, \quad (k = 1,2,\ldots,s)$$

be given. Here $x = (x_i)$, $(i = 1,2,\ldots,n)$ are the independent variables, $u = (u^j)$ ($j = 1,2,\ldots,m)$ are the dependent variables, $p = (p^k_\alpha)$ are the derivatives up to l-th order, and $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ is a multiindex with $|\alpha| \equiv \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq l$. In the space J^l, which consists of the elements (x,u,p) the system of differential equations is considered as a regular assigned manifold

$$U_l = \{(x,u,p) \in J^l \mid F^k(x,u,p) = 0, \quad (k = 1,2,\ldots,s)\}. \tag{2}$$

The term "regular" means that the vector function $F(x,u,p)$ satisfies the condition

$$rank\ (\frac{\partial(F)}{\partial(x,u,p)}) = s.$$

4.1 Admitted Lie group of transformations

One of the main objects in group analysis is the local one–parameter Lie group G^1 of the transformations:

$$x_i' = f^{x_i}(x,u;a), \quad u^{j\prime} = f^{u^j}(x,u;a), \quad (i = 1,2,\ldots,n; \;\; j = 1,2,\ldots,m). \tag{3}$$

There is a one–to–one correspondence between such groups G^1 and infinitesimal generators

$$X = \xi^i(x,u)\partial_{x_i} + \zeta^j(x,u)\partial_{u^j},$$

where

$$\xi^i(x,u) = \frac{d\,f^{x_i}}{d\,a}_{|a=0}, \quad \zeta^j(x,u) = \frac{d\,f^{u^j}}{d\,a}_{|a=0}.$$

The operator

$$\tilde{X} = X + \sum_{j,\alpha}\zeta^j_\alpha\partial_{p^j_\alpha}$$

with coefficients

$$\zeta^j_{\alpha,k} = D_k\zeta^j_\alpha - \sum_i p^j_{\alpha,i}D_k\xi^i. \tag{4}$$

is called the l–th prolongation of a generator X. Here, the operators

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha}p^j_{\alpha,k}\frac{\partial}{\partial p^j_\alpha}$$

are the operators of total differentiation with respect to x_k, $(k = 1,2,\ldots,n)$. Formulae (4) for the coefficients of the prolonged operator are defined by the following fact. Let a function $u = u_o(x)$ be given. Substituting it into the first part of transformation (3) and using the inverse function theorem one has

$$x = g^x(x',a).$$

The transformed function $u_a(x')$ is given as follows

$$u_a(x') = f^u(g^x(x',a), u_o(g^x(x',a)); a).$$

Hence, the derivatives p' of the transformed function $u_a(x')$ are defined through the derivatives p of the function $u_o(x)$. The local Lie group of transformations in the space J^l

$$x' = f^x(x,u;a), u' = f^u(x,u;a), \quad p' = f^p(x,u,p;a) \tag{5}$$

with the operator $\underset{l}{X}$ is called the l-th prolongation of the group G^1 (3).

A function $\Phi(x,u,p)$ is called an invariant of the group G^1 if the equality

$$\Phi(x',u',p') = \Phi(x,u,p)$$

holds. If $\Phi_p \neq 0$, then the function Φ is called a differential invariant. The regular assigned manifold (2) is an invariant manifold with respect to the prolonged group (4) if and only if

$$DS: \qquad \underset{l}{X}F^k(x,u,p)_{|U_l} = 0, \quad (k = 1,2,\ldots,s). \tag{6}$$

A local Lie group G^1 of transformations (3) is said to be admitted by the system (S) (we also say that the system (S) admits the group G^1) if the manifold (S) is invariant with respect to the prolonged group. In this case, the generator X of the Lie group G^1 is also called admitted by the system (S). The algorithm for finding a local one-parameter Lie group (3) admitted by the system of differential equations (S) consists of the following four steps.

In the first step, the form of the generator

$$X = \xi^i(x,u)\partial_{x_i} + \zeta^j(x,u)\partial_{u^j},$$

is given, with unknown coefficients $\xi^i(x,u)$, $\zeta^j(x,u)$. In the second step the prolonged operator $\underset{l}{X}$ is applied to every equation of the system (S). In the next step the coefficients of the prolonged operator are substituted by using formulae (4). The equations obtained must be considered on the manifold U_l. As a result we obtain the system of differential equations (6). The system of equations DS is called a system of determining equations and it is an overdetermined system of linear homogeneous differential equations in the unknown coordinates $\xi^i(x,u)$, $\zeta^j(x,u)$. The general solution of the determining equations DS generates a full group GS of the system (S). The feature of the admitted group is that under action of any transformation of this group every solution $u = U(x)$ of the system (S) is transformed into a solution $u = U_a(x)$ of the same system (S). Therefore the admitted group allows constructing new solutions from known solutions. Note that the set of admitted generators generates a Lie algebra, which is called admitted by the system (S).

4.2 Equivalence transformations

Most systems of partially differential equations have arbitrary elements: arbitrary functions or arbitrary constants. These arbitrary elements can be separated into classes with respect to a group of equivalence transformations. An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. These transformations allow us to use the simplest representation of the given equations. Note that the admitted group depends on specialization of the arbitrary elements. The group classification problem consists in searching for an admitted group of transformations, which is admitted for all arbitrary elements of the system and all specializations of the arbitrary elements. The specialization of the arbitrary elements can extend the admitted group. For the calculation of equivalence transformations we follow the approach developed in [30, 31], which consists of the following.

Suppose, the system of differential equation

$$F^k(x,u,p,\phi)=0, (k=1,2,\dots,s) \tag{7}$$

has arbitrary elements $\phi=(\phi^1,\phi^2,...,\phi^t)$, which are functions (or constants) $\phi=\phi(x,u)$. A specific value of the arbitrary elements represents a concrete system of differential equations.

The problem of finding an equivalent transformation consists of constructing a transformation of the space $R^{n+m+t}(x,u,\phi)$ which preserves the equations by only changing their representative $\phi=\phi(x,u)$. For this purpose, we consider the one-parameter group of transformations of the space R^{n+m+t} :

$$x'=f^x(x,u,\phi;a), u'=f^u(x,u,\phi;a), \phi'=f^\phi(x,u,\phi;a). \tag{8}$$

A generator of this group has the form:

$$X^e=\xi^x\partial_x+\zeta^u\partial_u+\zeta^\phi\partial_\phi \tag{9}$$

with the coordinates[8]:

$$\xi^i=\xi^i(x,u,\phi), \zeta^{u^j}=\zeta^{u^j}(x,u,\phi), \zeta^{\phi^k}=\zeta^{\phi^k}(x,u,\phi)$$
$$(i=1,\dots,n; j=1,\dots,m; k=1,\dots,t).$$

We use the main feature of the Lie group that any solution $u_o(x)$ of system (7) with functions $\phi(x,u)$ is transformed by (8) into another solution $u=u_a(x')$ of system (7), but with different (transformed) functions $\phi_a(x,u)$, which are defined in the following way. Solving the relations

$$x'=f^x(x,u,\phi(x,u);a), \quad u'=f^u(x,u,\phi(x,u);a)$$

with respect to (x,u), we obtain

$$x=g^x(x',u';a), \quad u=g^u(x',u';a). \tag{10}$$

Then the transformed function is

$$\phi_a(x',u')=f^\phi(x,u,\phi(x,u);a), \tag{11}$$

where instead of (x,u) we have to substitute their expressions (10). The transformed solution $u_a(x)$ is obtained by solving the relations

$$x'=f^x(x,u_o(x),\phi(x,u_o(x));a)$$

with respect to (x):

$$x=\psi^x(x';a)$$

[8]Later the author discovered that similar assumptions about the coefficients of the operator were used in [48] for one class of ordinary differential equations with one nonessential restriction $\zeta^{\phi^k}=\zeta^{\phi^k}(x,\phi)$.

and substituting into

$$u_a(x') = f^u(x, u_o(x), \phi_a(x, u_o(x)); a). \tag{12}$$

The formulae for the transformations of the partial derivatives p_a and the derivatives of the functions ϕ are obtained by differentiating (11) and (12) with respect to x' and u'.

The method for finding a group of equivalence transformations is similar to the algorithm for finding an admitted group of transformations. The difference only consists of the prolongation of the infinitesimal generator X^e. In agreement with the construction of the functions $u_a(x')$ and $\phi_a(x', u')$, the prolonged operator

$$\bar{X}^e = X^e + \zeta^{u_x}\partial_{u_x} + \zeta^{\phi_x}\partial_{\phi_x} + \zeta^{\phi_u}\partial_{\phi_u} + \ldots$$

has the following coordinates

$$\zeta^{u_\lambda} = D^e_\lambda\zeta^u - u_x D^e_\lambda\xi^x, \ (\lambda = x_1, x_2, ...x_n)$$

with $D^e_\lambda = \partial_\lambda + u_\lambda\partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi$ and

$$\zeta^{\phi_\lambda} = \tilde{D}^e_\lambda\zeta^\phi - \phi_x\tilde{D}^e_\lambda\xi^x - \phi_u\tilde{D}^e_\lambda\zeta^u, \ \ (\lambda = u^1, u^2, ..., u^m, x_1, x_2, ...x_n)$$

with $\tilde{D}^e_\lambda = \partial_\lambda + \phi_\lambda\partial_\phi$.

An equivalence group GS^e of transformations is generated by $G^1(X^e)$.

Remark. In some cases one may have additional requirements for the arbitrary elements. For example, the arbitrary elements ϕ^μ may be supposed to be independent of the independent variables $\dfrac{\partial\phi^\mu}{\partial x_k} = 0$. When studying the equivalence group such conditions have to be added to the original system of differential equations (7), leading to additional determining equations.

Remark. Note that in case of the Navier-Stokes equations, kinematic viscosity is the arbitrary element and these equations can be transformed to equations (1) by scaling the independent and dependent variables.

4.3 Invariant and partially invariant solutions

For each subgroup of the admitted group GS one can try to find an invariant or partially invariant solution. Let $H \subset GS$ be a group admitted by the system of equations (S). Assume that $X_1, ..., X_r$ is a basis of the Lie algebra L^r which corresponds to the group H. An invariant or partially invariant solution with respect to the group H is called an H-solution. The method [35] for constructing H-solutions with respect to the group H requires to find an universal invariant of this group: a set of all functionally independent invariants. For this purpose one needs to solve the overdetermined linear system of differential equations:

$$X_i\phi\,(x, u) = 0, \ (i = 1, 2, ..., r). \tag{13}$$

Because $X_1, ..., X_r$ generate a Lie algebra, system (13) is complete. Its general solution can be expressed through the $m + n - r_*$ invariants

$$J = \left(J^1(x, u), J^2(x, u), ..., J^{m+n-r_*}(x, u)\right),$$

where r_* is the total rank of the matrix composed of the coefficients of the generators X_i, $(i = 1, 2, ..., r)$. If the rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^{m+n-r_*})}{\partial(u_1, ..., u_m)}$ is equal to q, then without loss of generality, one can choose the first $q \leq m$ invariants $J^1, ..., J^q$ such that the rank of the Jacobi matrix $\frac{\partial(J^1,...,J^q)}{\partial(u_1,...,u_m)}$ is equal to q and the remaining $k = m+n-r_* - q$ invariants J^{q+1}, $J^{q+2}, ..., J^{m+n-r_*}$ only depend on the independent variables x. H-solutions are characterized by two integers: the rank $\sigma = \delta + n - r_* \geq 0$ and the defect $\delta \geq 0$, thus one uses the notation $H(\sigma, \delta)$-solution. Rank and defect must satisfy the inequalities

$$k \leq \sigma < n, \ \max\{r_* - n, m - q, 0\} \leq \delta \leq \min\{r_* - 1, m - 1\}.$$

To construct a representation of $H(\sigma, \delta)$-solutions one needs to separate the universal invariant into two parts: $J = (\overline{J}, \overline{\overline{J}})$, where $l = m - \delta$ and

$$\overline{J} = (J^1, ..., J^l), \ \overline{\overline{J}} = (J^{l+1}, J^{l+2}, ..., J^{m+n-r_*}).$$

This means that one can choose the number l such that $1 \leq l \leq q \leq m$. The rank and the defect of the $H(\sigma, \delta)$-solution are $\delta = m - l$, $\sigma = m + n - r_* - l = \delta + n - r_*$. A solution is called invariant if $\delta = 0$, otherwise it is called a partially invariant solution. From the first l invariants $J^1, J^2, ..., J^l$ one can define the l dependent functions

$$u^i = \phi^i(\overline{J}, u^{l+1}, u^{l+2}, ..., u^m, x), \ (i = 1, ..., l). \tag{14}$$

The functions $u^{l+1}, u^{l+2}, ..., u^m$ are called superfluous. The representation of the $H(\sigma, \delta)$-solution is obtained by assuming that the first part of the universal invariant is a function of the second part:

$$\overline{J} = \Psi(\overline{\overline{J}}). \tag{15}$$

and substituting (15) into (14). Thus, the representation of an invariant or partially invariant solution is

$$u^i = \Phi^i(\overline{\overline{J}}, u^{l+1}, u^{l+2}, ..., u^m, x), \ (i = 1, ..., l), \tag{16}$$

where $\Phi^i = \phi^i(\Psi(\overline{\overline{J}}), u^{l+1}, u^{l+2}, ..., u^m, x)$.

If $\delta \neq 0$, then either $\sigma = k$ or $\sigma > k$. In the first case ($\sigma = k$) the partially invariant solution is called regular, otherwise it is called irregular [37]. The number $\sigma - k$ is called the measure of irregularity.

After constructing the representation of an invariant or partially invariant solution one needs to substitute it into the original system of equations. The system of equations in the functions Ψ^i and superfluous functions thus obtained is called the reduced system. This system is overdetermined and requires analysis of compatibility. Usually the compatibility analysis is easier for invariant solutions than for partially invariant ones.

If H' is a subgroup of H, then it may be possible that a partially invariant $H(\sigma, \delta)$-solution is a partially invariant $H'(\sigma', \delta')$-solution. In this case $\delta' \leq \delta$, $\sigma' \geq \sigma$ [35]. A solution is called reducible to a $H'(\sigma', \delta')$-solution. if there exists $H' \subset H$ such that $\delta' < \delta$, $\sigma' = \sigma$. In particular, a solution is called reducible to an invariant solution if there exists $H' \subset H$ with $\delta' = 0$. Thus, a natural problem is to reduce a partially invariant $H(\sigma, \delta)$-solution to an invariant $H'(\sigma, 0)$-solution.

5 Results

For the first problem the analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. Group classification of the full viscous gas dynamics equations with spherical symmetry shows that the kernel of admitted groups is extended for three types of the state equations. For each class the optimal system of subalgebras is constructed. Universial invariants of all subalgebras are constructed. Representations of all possible invariant solutions is given.

Another part of the research is devoted to a particular class of partially invariant solutions of the Navier–Stokes equations. This class of solutions is constructed on the base of the four-dimensional subalgebra H^4 with the generators

$$X_1 = \phi_1\partial_x + \phi_1'\partial_u - x\phi_1''\partial_p, \quad X_2 = \phi_2\partial_x + \phi_2'\partial_u - x\phi_2''\partial_p,$$

$$Y_1 = \psi_1\partial_y + \psi_1'\partial_v - y\psi_1''\partial_p, \quad Y_2 = \psi_2\partial_y + \psi_2'\partial_v - y\psi_2''\partial_p.$$

According to the classification [37], a partially invariant solution with minimum defect $\delta = 2$ is a regular partially invariant solution of $H(2,2)$. In this case a representation of the partially invariant solution is

$$w = 2f(z,t), \quad p = h(z,t) - k(t)x^2 - l(t)y^2, \quad u = u(x,y,z,t), \quad v = v(x,y,z,t),$$

where $k = \phi_i''/(2\phi_i)$, $l = \psi_i''/(2\psi_i)$. After substitution of the representation of solution into the Navier-Stokes equations the overdetermined system of equations is obtained. It is proven that in the process of compatibility study the Monge-Ampere equation is constructed. In the research a systematical investigation of the case, where the Monge-Ampere equation is hyperbolic. It is shown that this class of solutions is a particular case of the solutions with linear profile of velocity with respect to one or two space variables. Examples of solutions with elliptic Monge-Ampere equation are given.

5.1 Full Navier-Stokes Equations

5.1.1 Coordinateless form of the equations

In this manuscript we study unsteady viscous gas dynamics equations. These equations govern a three-dimensional motion of a compressible, thermal conductive, Newtonian viscous gas flow

$$\frac{d\mathbf{v}}{dt} = \tau\, div(P), \ \frac{d\tau}{dt} - \tau div(\mathbf{v}) = 0,$$

$$\frac{d\varepsilon}{dt} = \tau\, P : D + \tau\, div(\kappa\nabla T).$$

Here $\tau = 1/\rho$ is a specific volume, ρ is a density, $\mathbf{v}$ is a velocity, P is a stress tensor, $D = \frac{1}{2}\left(\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + (\frac{\partial\mathbf{v}}{\partial\mathbf{x}})^*\right)$ is a rate-of-strain tensor, ε is an internal energy, T is a temperature, κ is a coefficient of a heat conductivity. The Stokes axioms for a viscous gas give

$$P = (-p + \lambda div(v))I + 2\mu D,$$

where p is a pressure, λ and μ are the first and the second coefficients of viscosity, respectively. A viscous gas is a two parametric media. As the main thermodynamic variables we choose the pressure p and specific volume τ: the entropy η, the internal energy ε and the temperature T are functions of the pressure and specific volume

$$\eta = \eta(p,\tau), \ \varepsilon = \varepsilon(p,\tau), \ T = T(p,\tau).$$

The first and the second thermodynamic laws require for these functions to satisfy the equations

$$\eta_p = \frac{\varepsilon_p}{T},\ \eta_\tau = \frac{\varepsilon_\tau + p}{T},\ 3\lambda + 2\mu \geq 0,\ \mu \geq 0,\ \kappa \geq 0.$$

For the simplicity of classification we study case, which corresponds to an essentially viscous and heat conductive gas

$$\mu \neq 0,\ \kappa \neq 0.$$

Thus, the studying viscous gas dynamics equations are

$$\frac{d\mathbf{v}}{dt} + \tau \nabla p = \tau \left((\lambda + \mu)\nabla(div(\mathbf{v})) + (div(\mathbf{v}))\nabla\lambda + \mu\triangle\mathbf{v} + 2D(\nabla\mu) \right), \tag{17}$$

$$\frac{d\tau}{dt} - \tau div(\mathbf{v}) = 0,$$

$$\frac{dp}{dt} + A(p,\tau)div(\mathbf{v}) = B(p,\tau)\left(\lambda(div(\mathbf{v}))^2 + 2\mu D : D + (\nabla\kappa)(\nabla T) + \kappa\triangle T\right)$$

with the functions

$$A = \frac{\tau(\varepsilon_\tau + p)}{\varepsilon_p},\ B = \frac{\tau}{\varepsilon_p}.$$

Note that the internal energy and entropy can be expressed through the functions $A = A(p,\tau)$, $B = B(p,\tau)$ by formulae

$$\varepsilon_p = \frac{\tau}{B},\ \varepsilon_\tau = \frac{A}{B} - p,\ \eta_p = \frac{\tau}{BT},\ \eta_\tau = \frac{A}{BT}.$$

The conditions $\varepsilon_{p\tau} = \varepsilon_{\tau p}$, $\eta_{p\tau} = \eta_{\tau p}$ lead to the restrictions

$$\begin{array}{c} \tau B_\tau + BA_p - AB_p = B^2 + B, \\ \tau T_\tau = AT_p - TB. \end{array} \tag{18}$$

In the case of an ideal gas (i.e., the gas that obeys the Clapeyron equation $T = R^{-1}p\tau$) there are $B = B(\tau p)$, $A = p(1 + B(\tau p))$ with an arbitrary function $B(\tau p)$. For a polytropic gas $\varepsilon = (\gamma - 1)^{-1}\tau p$ and this one more simplifies the functions A and B: $B = (\gamma - 1)$, $A = \gamma p$. Here R is the gas constant and γ is a polytropic exponent. Note also that the Navier-Stokes equations are obtained from the viscous gas equations by assuming that the second coefficient of viscosity μ and the density ρ (or the specific volume τ) are constants.

Remark. In the case of constant τ and μ system (17) is split on two systems: the Navier-Stokes equations and the energy equation.

5.1.2 Spherical coordinate system

Equations (17) are written in coordinateless form. For applications one needs to write them in some coordinate system. Because our goal is to study solutions of viscous gas equations connected with the group of rotations, then it is convenient to use a spherical coordinate system.

The spherical coordinates (r, θ, φ) and the Cartesian coordinates (x, y, z) of the point $\mathbf{x} \in R^3$ are introduced by the formulae

$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\quad z = r\cos\theta$$

Corresponding physical components of the velocity vector $\mathbf{v}$ in the spherical coordinate system (U, V, W) and (u, v, w) in the Cartesian coordinate system are related by the expressions

$$u = U\sin\theta\cos\varphi + V\cos\theta\cos\varphi - W\sin\varphi,$$

$$v = U\sin\theta\sin\varphi + V\cos\theta\sin\varphi - W\cos\varphi,$$

$$w = U\sin\theta - V\sin\theta.$$

Note that the vector (V, W) can be described by its modules H and by the angle ω:

$$V = H\cos\omega,\ W = H\sin\omega. \tag{19}$$

For the spherical coordinate system the fundamental tensor is diagonal:

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix};\ (g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix};\ |g| = det(g_{ij}) = r^4\sin^2\theta.$$

The Christoffel's symbols

$$\Gamma^l_{ij} = \frac{1}{2}g^{ls}\left(\frac{\partial g_{is}}{\partial K^j} + \frac{\partial g_{js}}{\partial K^i} - \frac{\partial g_{ij}}{\partial K^s}\right)$$

are (we write down only nonvanishing symbols)

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\ \Gamma^1_{22} = -r,\ \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta,$$

$$\Gamma^1_{33} = -r\sin^2\theta,\ \Gamma^2_{33} = -\sin\theta\cos\theta.$$

Here $K^1 = r;\ K^2 = \theta;\ K^3 = \varphi$ and there is a summation with respect to a repeat index.

Tensor components of the vector $\mathbf{v}$ are

$$(v^1, v^2, v^3) = (U, \frac{V}{r}, \frac{W}{r\sin\theta}),\quad (v_1, v_2, v_3) = (U, rV, r\sin\theta\, W).$$

Coordinates of a gradient of any scalar function F are

$$(\nabla F)_1 = (\nabla F)^1 = \frac{\partial F}{\partial r},\ (\nabla F)_2 = \frac{\partial F}{\partial\theta},\ (\nabla F)^2 = \frac{1}{r^2}\frac{\partial F}{\partial\theta},$$

$$(\nabla F)_3 = \frac{\partial F}{\partial\varphi},\ (\nabla F)^3 = \frac{1}{r^2\sin^2\theta}\frac{\partial F}{\partial\varphi}.$$

A matrix of the covariant derivatives is (here i is a number of a row, j is a number of a column)

$$\left(v^i_{,j}\right) = \begin{bmatrix} \frac{\partial U}{\partial r} & \frac{\partial U}{\partial\theta} - V & \frac{\partial U}{\partial\varphi} - \sin\theta W \\ \frac{1}{r}\frac{\partial V}{\partial r} & \frac{1}{r}\frac{\partial V}{\partial\theta} + \frac{U}{r} & \frac{1}{r}\frac{\partial V}{\partial\varphi} - \frac{\cos\theta}{r}W \\ \frac{1}{r\sin\theta}\frac{\partial W}{\partial r} & \frac{1}{r\sin\theta}\frac{\partial W}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial W}{\partial\varphi} + \frac{U}{r} + \frac{\cot\theta}{r}V \end{bmatrix}$$

Coordinates of the rate-of-strain tensor $D = \frac{1}{2}\left(\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + (\frac{\partial\mathbf{v}}{\partial\mathbf{x}})^*\right)$ are

$$2D^{j}_{.i} = v^j_{,i} + v^\alpha_{,\beta}g_{i\alpha}g^{j\beta}.$$

Hence,

$$2D : D = v^j_{,i} v^i_{,j} + v^i_{,j} v^\alpha_{,\beta} g_{i\alpha} g^{j\beta}.$$

For the divergency there is

$$div\, \mathbf{v} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial K^i} \left(\sqrt{|g|} v^i \right) = \frac{1}{r^2} \frac{\partial (r^2 U)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \sin\theta\, V}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial W}{\partial \varphi}.$$

The Laplace operator of a scalar function is

$$\Delta F = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial K^i} \left(\sqrt{|g|} g^{is} \frac{\partial F}{\partial K^s} \right) =$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 F}{\partial \varphi^2}.$$

The Laplace operator $\Delta \mathbf{v}$ of the vector $\mathbf{v}$ has coordinates

$$(\Delta \mathbf{v})^l = \Delta \left(v^l \right) + 2\, g^{ij} \Gamma^l_{is} \frac{\partial v^s}{\partial K^j} + g^{ij} \left(\frac{\partial \Gamma^l_{ip}}{\partial K^j} + \Gamma^s_{ip} \Gamma^l_{js} - \Gamma^s_{ij} \Gamma^l_{ps} \right) v^p,$$

which for the spherical coordinate system are

$$(\Delta \mathbf{v})^1 = \Delta (U) - \frac{2}{r} \frac{\partial V}{\partial \theta} - \frac{2}{r^2 \sin\theta} \frac{\partial W}{\partial \varphi} - \frac{2U}{r^2} - \frac{2 \cot\theta}{r^2} V,$$

$$(\Delta \mathbf{v})^2 = \Delta \left(\frac{V}{r} \right) + \frac{2}{r^2} \frac{\partial V}{\partial r} + \frac{2}{r^3} \frac{\partial U}{\partial \theta} - \frac{2 \cot\theta}{r^3 \sin\theta} \frac{\partial W}{\partial \varphi} - \frac{V}{r^3 \sin^2\theta},$$

$$(\Delta \mathbf{v})^3 = \Delta \left(\frac{W}{r \sin\theta} \right) + \frac{2}{r \sin\theta} \frac{\partial}{\partial r} \left(\frac{W}{r} \right) + \frac{2 \cot\theta}{r^3} \frac{\partial}{\partial \theta} \left(\frac{W}{\sin\theta} \right) +$$
$$+ \frac{2}{r^3 \sin^2\theta} \frac{\partial U}{\partial \varphi} + \frac{2 \cot\theta}{r^3 \sin^2\theta} \frac{\partial V}{\partial \varphi}.$$

The acceleration vector $\dfrac{d\mathbf{v}}{dt}$ has the components

$$\left(\frac{d\mathbf{v}}{dt} \right)^1 = \mathcal{D}(U) - \frac{W^2 + V^2}{r}, \quad \left(\frac{d\mathbf{v}}{dt} \right)^2 = \frac{1}{r} \mathcal{D}(V) + \frac{UV - \cot\theta\, W^2}{r^2},$$

$$\left(\frac{d\mathbf{v}}{dt} \right)^3 = \frac{1}{r \sin\theta} \mathcal{D}(W) + \frac{UW + \cot\theta\, WV}{r^2 \sin\theta},$$

where

$$\mathcal{D}(f) = \frac{\partial f}{\partial t} + U \frac{\partial f}{\partial r} + \frac{V}{r} \frac{\partial f}{\partial \theta} + \frac{W}{r \sin\theta} \frac{\partial f}{\partial \varphi}.$$

A substitution of the presented coordinates of the tensors and the vectors into system (17) gives equations of a viscous thermoconductive gas in the spherical coordinate system. These equations are very cumbersome[9].

[9]All symbolic calculations for the coordinates of the tensors and the vectors were made on computer with the help of the system REDUCE [21].

5.2 Partially Invariant Solution with respect to $SO(3)$

In the space of variables $t, r, \theta, \varphi, U, H, \omega, \tau, p$ the group of rotations $O(3)$ has the generators [36]

$$X = -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}\partial_\omega,$$
$$Y = \cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}\partial_\omega,$$
$$Z = \partial_\varphi.$$

Invariants of this group are t, r, U, H, τ, p.

The rank of the Jacobi matrix of the invariants with respect to the dependent functions is equal to four. Therefore, according to [35] there are no nonsingular invariant solutions that are invariant with respect to group of rotations $O(3)$. A minimal possible defect of a partially invariant with respect to $O(3)$ solution is equal to one. In this case a representation of the partially invariant solution is

$$\tau = \tau(t,r),\ U = U(t,r),\ H = H(t,r),\ p = p(t,r),\ \omega = \omega(t,r,\theta,\varphi). \tag{20}$$

The function $\omega(t,r,\theta,\varphi)$ is "superfluous": it depends on all independent variables. Note that if $H = 0$, then by virtue of (19) the tangent component of the velocity vector is equal to zero and it corresponds to spherically symmetric flows that are considered in the next subsection. In this subsection it is assumed that $H \neq 0$.

All analytic calculations for the viscous gas dynamics and the Navier-Stokes equations are done in the REDUCE system [21]. The result of these calculation is: class of solutions that is partially invariant with respect to $O(3)$ is confined by spherically symmetric solutions.

5.2.1 Analysis of compatibility of partially invariant solutions

For the sake of simplicity we present here analysis of compatibility of partially invariant solution for the Navier-Stokes equations, i.e., when τ and μ are constants. Analysis of compatibility for the viscous gas dynamics equations is similar, but it needs more cumbersome symbolic calculations.

After substituting the representation of the partially invariant solution (20) into the Navier-Stokes equations[10] and some combinations of the second and the third equations the initial system can be split on two subsystems: the invariant system

$$D_0U + p_r = r^{-1}H^2 + (U_{rr} + 4r^{-1}U_r + 2r^{-2}U) \tag{21}$$

with the operator $D_0 = \partial_t + U\partial_r$ and the supplementary system

$$\begin{gathered} D_0(rH) = (rH)_{rr} - (r\sin^2\theta)^{-1}H - rH(\omega_r^2 + r^{-2}\omega_\theta^2 + \\ +(r\sin\theta)^{-2}\omega_\varphi^2 + 2(r^2\sin\theta)^{-1}\cot\theta\omega_\varphi), \\ D_0\omega + (r\sin\theta)^{-1}H(\sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi + \cos\theta\sin\omega) = \\ = \omega_{rr} + 2(rH)^{-1}(rH)_r\omega_r + r^{-2}\omega_{\theta\theta} + r^{-2}\cot\theta\omega_\theta + (r\sin\theta)^{-2}\omega_{\varphi\varphi}, \\ \sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi = \cos\theta\cos\omega + \sin\theta(rH)^{-1}(r^2U)_r. \end{gathered} \tag{22}$$

[10]Here we use dimensionless representation of the Navier-Stokes equations in which one can account that $\mu = 1$ and $\tau = 1$.

For the analysis of compatibility of system (21),(22) it is convenient to use implicit representation for the function $\omega = \omega(t, r, \theta, \varphi)$ in the form

$$F(\omega, t, r, \theta, \varphi) = 0, \quad (F_\omega \neq 0).$$

All derivatives of the function $\omega(t, r, \theta, \varphi)$ can be calculated through the derivatives of the function $F(\omega, t, r, \theta, \varphi)$. For example, for the first derivatives we have

$$\omega_t = -F_t/F_\omega, \ \omega_r = -F_r/F_\omega, \ \omega_\theta = -F_\theta/F_\omega, \ \omega_\varphi = -F_\varphi/F_\omega.$$

Then the last equation of (22) becomes

$$\sin\theta \sin\omega F_\theta - \cos\omega F_\varphi + F_\omega(\cos\theta\cos\omega + k\sin\theta) = 0,$$

where the function $k = (rH)^{-1}(r^2U)_r$ only depends on t and r. Note that for a viscous gas dynamics equations there is the same equation with the function $k(t,r) = (Hr\tau)^{-1}(-rD_0\tau + \tau(r^2U)_r$. The general solution of the last equation is

$$F = \Phi\left(\varphi + arctan(\frac{\sin\omega}{k\sin\theta + \cos\theta\cos\omega}), \sin\theta\cos\omega - k\cos\theta, t, r\right).$$

Here the function $\Phi = \Phi(y_1, y_2, t, r)$ is an arbitrary function of the arguments t, r and

$$y_1 = \varphi + arctan(\frac{\sin\omega}{k\sin\theta + \cos\theta\cos\omega}), \ y_2 = \sin\theta\cos\omega - k\cos\theta.$$

All further intermediate calculations in studying the compatibility of overdetermined system (21), (22) were made on computer in the system REDUCE [21]. Here we give the way of computing and the final results.

Note that the Jacobian $\frac{\partial(y_1,y_2,\theta,t,r)}{\partial(\omega,\theta,\varphi,t,r)} \neq 0$, therefore one can choose (y_1, y_2, θ, t, r) as the new independent variables. All derivatives of the function $\omega(t, r, \theta, \varphi)$ can be written through the derivatives of the function $\Phi(y_1, y_2, t, r)$. After that the second equation of (8) accepts the form

$$\sin\omega G_1(y_1, y_2, t, r, \theta) + G_2(y_1, y_2, t, r, \theta) = 0,$$

where the functions $G_1(y_1, y_2, t, r, \theta)$ and $G_2(y_1, y_2, t, r, \theta)$ do not include ω and its derivatives. In the last equation $\sin\omega$ can be excluded by using the trigonometry identity:

$$G_1^2(1 - (y_2 + k\cos\theta)^2) - G_2^2(1 - \cos^2\theta) = 0,$$

where the equality $\cos\omega = \sin^{-1}\theta(y_2 + k\cos\theta)$ found from the representation of y_2 was applied.

Further calculations show that the last equation depends on θ as the polynomial of the degree 8 with respect to $\cos\theta$:

$$P_8 = \sum_{k=0}^{8} a_k \cos^k\theta = 0.$$

The coefficients a_k, $(k = 0, 1, \ldots, 8)$ only depend on y_1, y_2, t, r and do not depend on θ. This allows splitting the equation with respect to $\cos\theta$: $a_k = 0$, $(k = 1, 2, \ldots, 8)$.

The equality $a_8 = 0$ gives

$$D_0 h = h_{rr} + h(k^2+1)^{-1}h_r, \tag{23}$$

where $h = rH$. Substituting h_t found from (23) into $a_6 = 0$, we obtain

$$k_r\left((k^2+1)\Phi_r + kk_r y_2 \Phi_{y_2}\right) = 0. \tag{24}$$

If $(k^2+1)\Phi_r + kk_r y_2\Phi_{y_2} = 0$, then the equation $a_4 = 0$ gives the equation $y_2^2 - (k^2+1) = 0$ or

$$(\sin\theta\cos\omega - k\cos\theta)^2 = k^2 + 1.$$

Note that substituting the representation of the function $\omega(t, r, \theta, \varphi)$ found from this equation into (21), (22) and splitting them with respect to $\cos\theta$ gives the expression $H = 0$ that contradicts the assumption about H.

For the second case in (24), when $k_r = 0$ we will obtain a contradiction with the help of the first equation of (22). Really, the same study of the first equation of (22) as for the second equation leads to the polynomial of the degree 10 with respect to $\cos\theta$:

$$P_{10} = \sum_{k=0}^{10} b_k \cos^k\theta = 0,$$

where the coefficients b_k, $(k = 0, 1, ..., 10)$ only depend on y_1, y_2, t, r. The equality $b_{10} = 0$ gives

$$k_t = r^{-2}h(k^2+1). \tag{25}$$

By virtue of $k_r = 0$, (25) and the definition of $k = (r^2U)_r/h$ one can obtain that

$$h(t, r) = 3c(t)r^2, \quad r^2U(t, r) = k(t)c(t)r^3 + \lambda(t),$$

where $c(t) = (k^2(t)+1)^{-1}k'(t)/3$. Substitution of this representation into (23) and splitting it with respect to r gives $c(t) = 0$ that contradicts the assumption $H \neq 0$.

Similar calculations have been done for the viscous gas dynamics equations.

The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal incompressible inviscid fluid dynamics equations, are only spherically symmetric solutions.

5.3 Spherically Symmetric Flows of a Viscous Gas

The case $H = 0$ corresponds to a spherically symmetric flow of a viscous gas. According to the definitions of the group analysis it is a singular invariant solution with respect to group of rotations $O(3)$. The viscous gas dynamics equations in this case are

$$\begin{aligned} D_0\tau - \tau(U_r + 2r^{-1}U) &= 0, \\ D_0U + \tau p_r = \tau(\lambda+2\mu)(U_{rr} + 2r^{-1}U_r - 2r^{-2}U) &+ 6\tau(\mu_\tau\tau_r + \mu_p p_r) + \\ &+\tau(U_r + 2r^{-1}U)(\lambda_\tau\tau_r + \lambda_p p_r), \\ D_0 p + A(U_r + 2r^{-1}U) = B[\lambda(U_r + 2r^{-1}U)^2 &+ 2\mu(U_r^2 + 2r^{-2}U^2) + \\ +\kappa(T_{\tau\tau}\tau_r^2 + 2T_{\tau p}\tau_r p_r &+ T_{pp}p_r^2 + T_\tau(\tau_{rr} + 2r^{-1}\tau_r) + \\ +T_p(p_{rr} + 2r^{-1}p_r) &+ (\kappa_p p_r + \kappa_\tau\tau_r)(T_\tau\tau_r + T_p p_r)], \end{aligned} \tag{26}$$

where $D_0 = \partial_t + U\partial_r$. In this subsection we study a group classification of equations (26) with respect to the arbitrary elements A , B, λ, μ, κ, T.

5.3.1 Equivalence transformations

The first stage of group classification requires determining a group of equivalence transformations of equations (26). An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to the system of equations of the same class. It allows using the simplest representation of given equations. Here we give a construction of the group of equivalence transformations without restrictions on the representation of equivalence transformations [35]. We follow the approach for the calculation of equivalence transformations developed in [30].

Since arbitrary elements satisfy restrictions (18) and $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$, then for calculating an equivalence group of transformations we have to append the equations

$$\begin{gathered} A_r = 0, A_t = 0, A_U = 0, B_r = 0, B_t = 0, B_U = 0, \\ \lambda_r = 0, \lambda_t = 0, \lambda_U = 0, \mu_r = 0, \mu_t = 0, \mu_U = 0, \\ \kappa_r = 0, \kappa_t = 0, \kappa_U = 0, T_r = 0, T_t = 0, T_U = 0 \end{gathered}$$

to equations (26). All coefficients of the infinitesimal generator of the equivalence group

$$X^e = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^U\partial_U + \zeta^\tau\partial_\tau + \zeta^p\partial_p + \zeta^A\partial_A + \zeta^B\partial_B + \zeta^\lambda\partial_\lambda + \zeta^\mu\partial_\mu + \zeta^\kappa\partial_\kappa + \zeta^T\partial_T$$

are dependent on all independent, dependent variables and arbitrary elements

$$r,\ t,\ U,\ \tau,\ p,\ A,\ B,\ \lambda,\ \mu,\ \kappa,\ T.$$

With the following notation:

$$u^1 = U,\ u^2 = \tau,\ u^3 = p,\ a^1 = A,\ a^2 = B,\ a^3 = \lambda,\ a^4 = \mu,\ a^5 = \kappa,\ a^6 = T$$

and

$$z^1 = r,\ z^2 = t,\ z^3 = U,\ z^4 = \tau,\ z^5 = p, a^k_\beta = \frac{\partial a^k}{\partial z^\beta}, a^k_{j\beta} = \frac{\partial^2 a^k}{\partial z^j \partial z^\beta},$$

the coefficients of the prolonged operator

$$\tilde{X}^e = X^e + \sum_i (\zeta^{u^i_r}\partial_{u^i_r} + \zeta^{u^i_t}\partial_{u^i_t}) + \sum_{k,j}\zeta^{a^k_{zj}}\partial_{a^k_{zj}} + \dots$$

can be constructed with the prolongation formulae:

$$\zeta^{u^i_r} = D_r\zeta^{u^i} - u^i_r D_r\zeta^r - u^i_t D_r\zeta^t,\ \zeta^{u^i_t} = D_t\zeta^{u^i} - u^i_r D_t\zeta^r - u^i_t D_t\zeta^t,$$

$$\zeta^{u^i_{rr}} = D_r\zeta^{u^i_r} - u^i_{rr} D_r\zeta^r - u^i_{rt} D_r\zeta^t.$$

$$\zeta^{a^k_\beta} = D^e_{z^\beta}\zeta^{a^k} - \sum_{\alpha=1}^{5} a^k_\alpha D^e_{z^\beta}\zeta^{z^\alpha},\ \zeta^{a^k_{j\beta}} = D^e_{z^\beta}\zeta^{a^k_j} - \sum_{\alpha=1}^{5} a^k_{j\alpha} D^e_{z^\beta}\zeta^{z^\alpha}.$$

Here the operators $D_r,\ D_t$ denote the total derivative operators with respect to r and t, respectively. For example,

$$D_r = \partial_r + \sum_\alpha u^\alpha_r \partial_{u^\alpha} + \sum_i (a^i_r + \sum_j a^i_{u^j} u^j_r)\partial_{a^i} + \dots$$

When we use the operator $D^e_{z^j}$ we consider $z^1, \dots, z^5$ as independent variables and $a^1, \dots, a^6$ as dependent variables, we obtain:

$$D^e_{z^j} = \partial_{z^j} + \sum_i a^i_{z^j} \partial_{a^i} + \dots.$$

All necessary calculations here as in the previous subsections were carried on a computer using the symbolic manipulation program REDUCE [21].

The calculations showed that the group of equivalence transformations of equations (26) corresponds to Lie algebra with generators

$$X^e_1 = \partial_t,\ X^e_2 = \partial_p,\ X^e_3 = r\partial_r + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa,$$
$$X^e_4 = r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa,\ X^e_5 = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

Remark. If instead of the functions $A(p,\tau), B(p,\tau)$ one considers the internal energy $\varepsilon(p,\tau)$, then the operators X^e_2, X^e_4, and X^e_5 are changed to

$$X^e_2 = \partial_p - \tau\partial_\varepsilon,\ X^e_4 = r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon,$$

$$X^e_5 = -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

and there is one more generator $X^e_6 = \partial_\varepsilon$.

Remark. By a direct checking one can obtain that in the general case[11] (equations (17)) the equivalence group includes the transformations with the generators

$$X^e_1 = \partial_t,\ X^e_2 = \partial_p,$$
$$X^e_3 = \mathbf{x}\partial_\mathbf{x} + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa,$$
$$X^e_4 = \mathbf{x}\partial_\mathbf{x} + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa,$$
$$X^e_5 = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

There are also other generators, for example, that correspond to the Galilei transformations and to the rotations in the three-dimensional case.

5.3.2 Admitted group

Finding an admitted group consists of seeking solutions of determining equations [35]. We are looking for the generator

$$X = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^U\partial_U + \zeta^\tau\partial_\tau + \zeta^p\partial_p$$

with the coefficients depending on r, t, U, τ, p. Calculations lead to the following result.

The kernel of the fundamental Lie algebra is made up of the generator

$$X = \partial_t.$$

Extension of the kernel of the main Lie algebra occurs by specializing the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. Note that the functions $A = A(p,\tau), B = B(p,\tau), T = T(p,\tau)$ have to satisfy equations (18). There are three types of the generators admitted by system (26). Further α, β and δ are arbitrary constants.

[11] Group classification of three-dimensional viscous gas dynamics equations with $\lambda = -2\mu/3$ was studied in [8].

Type (a). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{gathered}\alpha\tau A_\tau + A_p = 0,\ \alpha\tau B_\tau + B_p = 0,\\ \alpha\tau\mu_\tau + \mu_p = \beta\mu,\ \alpha\tau\lambda_\tau + \lambda_p = \beta\lambda,\\ \alpha\tau T_\tau + T_p = \delta T,\ \alpha\tau\kappa_\tau + \kappa_p = (-\delta+\alpha+\beta)\kappa,\end{gathered} \tag{27}$$

then there is one more admitted generator:

$$Y_a = \alpha U\partial_U + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha+2\beta)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (27) is

$$\begin{gathered}A = A(\tau e^{-\alpha p}),\ B = B(\tau e^{-\alpha p}),\ \mu = e^{\beta p}M(\tau e^{-\alpha p}),\ \lambda = e^{\beta p}\Lambda(\tau e^{-\alpha p}),\\ T = e^{\delta p}\Theta(\tau e^{-\alpha p}),\ \kappa = e^{(-\delta+\alpha+\beta)p}K(\tau e^{-\alpha p}),\end{gathered}$$

where the functions $A(z), B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau e^{-\alpha p}$)

$$-\alpha zBA' + zB'(1+\alpha A) = B^2 + B, \quad (1+\alpha A)z\Theta' = (\delta A - B)\Theta. \tag{28}$$

The internal energy is represented by the formula

$$\varepsilon = e^{\alpha p}(\varphi(z) - zp) + \psi(p),\ \psi'(p) = Ce^{\alpha p},$$

where the function $\varphi(z)$ and the constant C can be accounted as arbitrary and they are related with the functions $A(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{A(z)}{B(z)}, \quad C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - \alpha\varphi(z).$$

In this case the function $\Theta(z)$ has to satisfy the equation

$$(C - z + \alpha\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

Type (b). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{gathered}\alpha\tau A_\tau + pA_p = A,\ \alpha\tau B_\tau + pB_p = 0,\\ \alpha\tau\mu_\tau + p\mu_p = (\beta+1)\mu,\ \alpha\tau\lambda_\tau + p\lambda_p = (\beta+1)\lambda,\\ \alpha\tau T_\tau + pT_p = \delta T,\ \alpha\tau\kappa_\tau + pT_p = (-\delta+2+\alpha+\beta)\kappa,\end{gathered} \tag{29}$$

then there is an extension by the generator

$$Y_b = (1+\alpha)U\partial_U + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha+2\beta+1)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (29) is

$$\begin{gathered}A = p\widehat{A}(\tau p^{-\alpha}),\ B = B(\tau p^{-\alpha}),\ \mu = p^{\beta+1}M(\tau p^{-\alpha}),\ \lambda = p^{\beta+1}\Lambda(\tau p^{-\alpha}),\\ T = p^{\delta}\Theta(\tau p^{-\alpha}),\ \kappa = p^{-\delta+\alpha+\beta+2}K(\tau p^{-\alpha}),\end{gathered}$$

where the functions $\widehat{A}(z), B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau p^{-\alpha}$)

$$-\alpha zB\widehat{A}' + zB'(1+\alpha\widehat{A}) = B^2 + B - B\widehat{A}, \quad (1+\alpha\widehat{A})z\Theta' = (\delta\widehat{A} - B)\Theta. \tag{30}$$

The internal energy is represented by the formula

$$\varepsilon = p^{(\alpha+1)}(\varphi(z) - z) + \psi(p), \ \psi'(p) = Cp^{\alpha},$$

where the function $\varphi(z)$ and the constant C are arbitrary and they are related with the functions $\widehat{A}(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{\widehat{A}(z)}{B(z)}, \ C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - (\alpha+1)\varphi(z)$$

The function $\Theta(z)$ is represented through the function $\varphi(z)$ by the formula

$$(C - z + (\alpha+1)\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z)$$

Note that an ideal gas belongs to this type in case of $\delta = \alpha + 1$ and the function $\varphi(z)$ satisfies the equation

$$\delta(z\varphi' - \varphi) = C.$$

Type (c). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{gathered} A_\tau = 0, \ B_\tau = 0, \ \tau\mu_\tau = \beta\mu, \ \tau\lambda_\tau = \beta\lambda, \\ \tau T_\tau = \delta T, \ \tau\kappa_\tau = (-\delta + 1 + \beta)\kappa, \end{gathered} \tag{31}$$

then there is one more admitted generator:

$$Y_c = U\partial_U + 2\tau\partial_\tau + (1+2\beta)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (31) is

$$\begin{gathered} A = A(p), \ B = B(p), \ \mu = \tau^{\beta}M(p), \ \lambda = \tau^{\beta}\Lambda(p), \\ T = \tau^{\delta}\Theta(p), \ \kappa = \tau^{-\delta+\beta+1}K(p), \end{gathered}$$

where the functions $A(p), B(p)$ and $\Theta(p)$ satisfy the equations

$$BA' - AB' = B^2 + B, \quad A\Theta' = (\delta + B)\Theta. \tag{32}$$

The internal energy is represented by the formula

$$\varepsilon = \tau\varphi(p) - \tau p,$$

where the function $\varphi(p)$ is an arbitrary function and it is related with the functions $A(p)$ and $B(p)$ by the formula

$$\varphi(p) = \frac{A(p)}{B(p)}.$$

In this case the function $\Theta(p)$ is related with the function $\varphi(z)$ by the formula

$$\varphi(p)\Theta'(p) = (1 - \delta + \delta\varphi'(p))\Theta(p).$$

Note that if $\delta = 1$ and $\varphi = Cp$, then the gas is ideal.

The final results of the group classification are presented in Table I. In this table the first column means the type of the extension of the algebra $\{X\}$: the types a, b, or c, respectively. The last column means conditions for the state functions.

Therefore, there are three kinds of admitted by equations (26) groups, which depend on the specifications of the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. These groups are one-dimensional, two-dimensional and three-dimensional.

The two-dimensional admitted groups are groups with the generators either $\{X, Y_a\}$ or $\{X, Y_b\}$ or $\{X, Y_c\}$. The three-dimensional admitted groups are the groups with the generators either $\{X, Y_a, Y_b\}$ or $\{X, Y_a, Y_c\}$ or $\{X, Y_b, Y_c\}$.

The group with the generators $\{X, Y_a, Y_b\}$ is admitted by equations (26) if

$$A = A_0\tau^\alpha,\ B = -1,\ \mu = \mu_0\tau^{\beta+\alpha},\ \lambda = \lambda_0\tau^{\beta+\alpha},\ \kappa = \kappa_0\tau^{\beta+2\alpha},\ T = T_0\tau,\ \alpha \neq 0.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0 \int \tau^\alpha\, d\tau)$. Instead the operators Y_a and Y_b one can use their linear combinations:

$$\widehat{Y}_a = \partial_p,\ \widehat{Y}_b = (1+\alpha)U\partial_U + 2\tau\partial_\tau + (\alpha + 2\beta + 1)r\partial_r + 2\beta t\partial_t.$$

The algebra of the type $\{X, Y_a, Y_c\}$ is admitted by equations (26) if

$$A = A_0,\ B = -1,\ \mu = \mu_0\tau^\beta e^{\alpha p},\ \lambda = \lambda_0\tau^\beta e^{\alpha p},\ \kappa = \kappa_0\tau^{\beta - A_0\sigma}e^{(\alpha-\sigma)p},\ T = T_0\tau^{1+A_0\sigma}e^{\sigma p}.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\tau)$ and by taking linear combinations of the operators Y_a and Y_c one obtains another basis of the generators:

$$\widehat{Y}_a = \partial_p + \alpha(r\partial_r + t\partial_t),\ \widehat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta+1)r\partial_r + 2\beta t\partial_t.$$

The third type of the algebras $\{X, Y_b, Y_c\}$ is admitted by (26) if

$$\begin{gathered} A = \gamma p,\ B = \gamma - 1,\ \mu = \mu_0\tau^\beta p^{1+\alpha},\ \lambda = \lambda_0\tau^\beta p^{1+\alpha}, \\ \kappa = \kappa_0\tau^{\gamma(1-\alpha)+\beta}p^{\alpha-\delta+2},\ T = T_0\tau^{\gamma(\delta-1)+1}p^\delta,\ \gamma \neq 1. \end{gathered}$$

The internal the energy in this case is

$$\varepsilon = \frac{\tau p}{\gamma - 1}$$

and linear combinations of the operators Y_b and Y_c are:

$$\widehat{Y}_b = U\partial_U + 2p\partial_p + (2\alpha+1)r\partial_r + 2\alpha t\partial_t,\ \widehat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta+1)r\partial_r + 2\beta t\partial_t$$

Note that a polytropic gas belongs to the last case of gases, where γ is a polytropic exponent.

In the formulas above $A_0, \mu_0, \lambda_0, \kappa_0, T_0, \alpha, \beta, \gamma, \delta, \sigma$ are arbitrary constants; the commutators

$$[\widehat{Y}_a, \widehat{Y}_b] = 0,\ [\widehat{Y}_a, \widehat{Y}_c] = 0,\ [\widehat{Y}_b, \widehat{Y}_c] = 0.$$

5.3.3 Optimal systems of subalgebras

Here we study subalgebras of the two-dimensional admitted algebras $\{X, Y_a\}$, $\{X, Y_b\}$, $\{X, Y_c\}$.

The commutator $[X, Y]$ of the generators X and Y is

$$[X, Y] = zX.$$

Here either $Y = Y_a$ or $Y = Y_b$ or $Y = Y_c$ and $z = 2\beta$. Automorphisms are recovered by the table of commutators and consists of the automorphisms

$$\begin{aligned} A_1 &: x' = x + zya_1,\ y' = y, \\ A_2 &: x' = e^{-za_2}x,\ y' = y, \end{aligned}$$

where x and y are coordinates of the operator $Z = xX + yY$, x' and y' are coordinates of the operator Z' after actions of the automorphisms, and a_1, a_2 are parameters of the automorphisms. There is also one involution

$$E: \; x' = -x, \; y' = y,$$

which corresponds to the change of the variables $t \to -t$ and $U \to -U$ without changes of equations (26). Note that if $z = 0$, then the automorphisms are identical transformations. This leads to two optimal systems of subalgebras.

If $z = 0$ (or $\beta = 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\}, \; \{Y + hX\}, \; \{X, Y\},$$

where h is an arbitrary positive constant.

If $z \neq 0$ (or $\beta \neq 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\}, \; \{Y\}, \; \{X, Y\}.$$

Therefore, one can summarize: optimal systems of subalgebras for the two-dimensional algebras are described by the following system of subalgebras

$$\{X\}, \; \{Y + hX\}, \; \{X, Y\}, \; \beta h = 0. \tag{33}$$

5.3.4 Representations of invariant solutions

A next step in the construction of representations of invariant solutions is a finding universal invariants. Note that invariant solutions corresponding to the case of the subalgebra $\{X\}$ are the well-known stationary solutions. The universal invariants for the other subalgebras of the optimal system (33) of the algebras $\{X, Y_a\}$, $\{X, Y_b\}$ and $\{X, Y_c\}$ are presented in Tables II, III and IY, respectively.

According to the theory of the group analysis [35] on the next step in constructing of invariant solutions one needs to separate the universal invariant on two parts: one part has to be solvable with respect to the dependent variables U, τ, p. After that the representations of invariant solutions are obtained by supposing that the first part of the universal invariant depends on the second part. Because of this requirement there are no invariant solutions in the cases: $a.1$ if $h = 0$, $a.4$, $b.1$, $b.5$ and $c.3$. The cases $a.5$, $b.6$ and $c.4$ correspond to the special cases of stationary solutions, which we also exclude from our consideration[12].

All possible representations of invariant solutions of equations (26) are presented in Table Y, where the functions f^u, f^τ, f^p are functions of one independent variable presented in the last column. These functions must satisfy ordinary differential equations, which are obtained after substituting the representation of solution into system (26).

Remark. Invariant solutions $b.2$, $b.4$, $c.2$ are self-similar solutions.

Remark. One of the well-known solutions of the Boltzmann equation (the BKW-solution[13]) has the representation [5, 24]

$$f = \phi(|u|e^{ct}),$$

[12] If an universal invariant is three-dimensional (consists of three invariants), such as in the cases of $a.5$, $b.6$, $c.4$, then the representation of the invariant solution is obtained by supposing that all invariants of the universal invariant are constants.

[13] This solution is constructed for the Maxwell molecules.

where f is a distribution function, $|u|$ is a modulus of the velocity. The invariant solution of the viscous gas equations, which corresponds to the case $b.3$ gives

$$|u|e^{-t(\alpha+1)/h} = qf^u(q),$$

with $q = re^{-t(\alpha+1)/h}$. Therefore this solution can correspond to the BKW-solution and generalize it on molecules with an exponent intermolecular potentials. For the molecules with an exponent intermolecular potentials the coefficients of viscosity and conductivity are [4]

$$\mu = \mu_0 T^k, \kappa = \kappa_0 T^k,$$

where $T = T_0 p\tau$, $k = (n-1)/m + 1/2$, n is dimension of the problem, m is the exponent of intermolecular potentials. In this case $\alpha = 1/k - 1 = (m+2n-2)^{-1}(m-2n+2)$. For the Maxwell molecules, for which the BKW-solution was constructed, the exponent of intermolecular potentials is $m = 4$, and hence, in the three-dimensional case $\alpha = 0$ and $k = 1$.

5.4 Spherically Symmetric Flows of the Navier-Stokes Equations

For the complete consideration of solutions connected with the group of rotations $O(3)$ we present solutions of the Navier-Stokes equations with spherical symmetry[14]. Substituting the value of $V = 0$, $W = 0$ in the last equation of (22) one obtains that $r^2 U = h(t, \theta, \varphi)$. From the remained equations of (21),(22) all space derivatives of the pressure can be found

$$p_r = r^{-4}(\cot\theta h_\theta + \sin^{-2}\theta h_{\varphi\varphi} + h_{\theta\theta} - r^2 h_t + 2r^{-1}h^2),$$
$$p_\theta = 2r^{-3}h_\theta,\ p_\varphi = 2r^{-3}h_\varphi,$$

where $g(t)$ and $h(t)$ are arbitrary functions.

Using symmetry property of the mixed derivatives $p_{\theta r} = p_{r\theta}$, $p_{\varphi r} = p_{r\varphi}$, $p_{\varphi\theta} = p_{\theta\varphi}$ and spliting these equalities with respect to r one can get that $h = h(t)$ and the general solution of the Navier-Stokes equations in this case is

$$p = r^{-1}h'(t) - r^{-4}h^2(t)/2 + g(t),\ U = r^{-2}h(t),\ V = 0,\ W = 0.$$

5.5 Conclusion of the first part of the research

The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. For the completeness of consideration of partially invariant solutions that are connected with the group of rotations $O(3)$ the group classification of the full viscous gas dynamics equations with spherical symmetry has been done.

5.6 One class of partially invariant solutions

The class of solutions studied in [33] is a class of partially invariant solutions with respect to the group H with generators

$$X = \partial_x,\quad Y = \partial_y,\quad U = t\partial_x + \partial_u,\quad V = t\partial_y + \partial_v.$$

[14]These solutions are irregular invariant solutions of the Navier-Stokes equations with respect to rotations. We think that they are known, but, unfortunately, we do not know any reference on this subject.

Table 1: Group classification.

	λ	μ	T	κ	A	B	z	Cond.
a	$e^{\beta p}\Lambda(z)$	$e^{\beta p}M(z)$	$e^{\delta p}\Theta(z)$	$e^{(-\delta+\alpha+\beta)p}K(z)$	$A(z)$	$B(z)$	$\tau e^{-\alpha p}$	(28)
b	$p^{\beta+1}\Lambda(z)$	$p^{\beta+1}M(z)$	$p^{\delta}\Theta(z)$	$p^{-\delta+\alpha+\beta+2}K(z)$	$p\widehat{A}(z)$	$B(z)$	$\tau p^{-\alpha}$	(30)
c	$\tau^{\beta}\Lambda(p)$	$\tau^{\beta}M(p)$	$\tau^{\delta}\Theta(p)$	$\tau^{-\delta+\beta+1}K(p)$	$A(p)$	$B(p)$	p	(32)

Table 2: Universial invariants of subalgebras of the algebra $\{X, Y_a\}$.

N	Subalgebra	consts	Universal invariant
$a.1$	$Y_a + hX$	$\beta = 0, \alpha = 0$	$U, \tau, t - hp/2, r$
$a.2$	$\beta h = 0$	$\beta = 0, \alpha \neq 0$	$Ur^{-1}, \tau r^{-2}, p - 2\alpha^{-1}\ln r, t - h\alpha^{-1}\ln r$
$a.3$		$\beta \neq 0$	$Ut^{-\alpha/(2\beta)}, \tau t^{-\alpha/\beta}, p - \beta^{-1}\ln t, rt^{-(\alpha+2\beta)/(2\beta)}$
$a.4$	X, Y_a	$\alpha + 2\beta = 0$	$Ue^{-\alpha p/2}, \tau e^{-\alpha p}, r$
$a.5$		$\alpha + 2\beta \neq 0$	$Ur^{-\alpha/(\alpha+2\beta)}, \tau r^{-2\alpha/(\alpha+2\beta)}, p - 2(\alpha+2\beta)^{-1}\ln r$

Table 3: Universial invariants of subalgebras of the algebra $\{X, Y_b\}$ ($k \equiv \alpha + 2\beta + 1$).

N	Subalgebra	consts	Universal invariant
$b.1$	$Y_b + hX$	$\beta = 0, \alpha = -1, h = 0$	$U, p/\tau, r, t$
$b.2$	$\beta h = 0$	$\beta = 0, \alpha \neq -1, h = 0$	$Ur^{-1}, \tau r^{-2\alpha/(\alpha+1)}, pr^{-2/(\alpha+1)}, t$
$b.3$		$\beta = 0, h \neq 0$	$Ue^{-t(\alpha+1)/h}, \tau e^{-2t\alpha/h}, pe^{-2t/h}, re^{-t(\alpha+1)/h}$
$b.4$		$\beta \neq 0$	$Ut^{-(\alpha+1)/(2\beta)}, \tau t^{-\alpha/\beta}, pt^{-1/\beta}, rt^{-k/(2\beta)}$
$b.5$	X, Y_b	$k = 0$	$Up^{-(\alpha+1)/2}, \tau p^{-\alpha}, r$
$b.6$		$k \neq 0$	$Ur^{-(\alpha+1)/k}, \tau r^{-2\alpha/k}, pr^{-2/k}$

Table 4: Universial invariants of subalgebras of the algebra $\{X, Y_c\}$.

N	Subalgebra	consts	Universal invariant
c.1	$Y_c + hX$	$\beta = 0$	$Ur^{-1}, \tau r^{-2}, p, t - h\ln r$
c.2	$\beta h = 0$	$\beta \neq 0$	$Ut^{-1/(2\beta)}, \tau t^{-1/\beta}, p, rt^{-(2\beta+1)/(2\beta)}$
c.3	X, Y_c	$2\beta + 1 = 0$	$U\tau^{-1/2}, p, r$
c.4		$2\beta + 1 \neq 0$	$Ur^{-1/(2\beta+1)}, \tau r^{-2/(2\beta+1)}, p$

Table 5: Representations of invariant solutions.

N	Representation of invariant solution	Ind. variable	Model
1	$U = f^u, \tau = f^\tau, p = 2th^{-1} + f^p$	r	$a.1$
2	$U = rf^u, \tau = r^2 f^\tau, p = 2\alpha^{-1}\ln r + f^p$	$t - h\alpha^{-1}\ln r$	$a.2$
3	$U = t^{\alpha/(2\beta)}f^u, \tau = t^{\alpha/\beta}f^\tau, p = \beta^{-1}\ln t + f^p$	$rt^{-(\alpha+2\beta)/(2\beta)}$	$a.3$
4	$U = rf^u, \tau = r^{2\alpha/(\alpha+1)}f^\tau, p = r^{2/(\alpha+1)}f^p$	t	$b.2$
5	$U = e^{t(\alpha+1)/h}f^u, \tau = e^{2t\alpha/h}f^\tau, p = e^{2t/h}f^p,$	$re^{-t(\alpha+1)/h}$	$b.3$
6	$U = t^{(\alpha+1)/(2\beta)}f^u, \tau = t^{\alpha/\beta}f^\tau, p = t^{1/\beta}f^p$	$rt^{-(\alpha+2\beta+1)/(2\beta)}$	$b.4$
7	$U = rf^u, \tau = r^2 f^\tau, p = f^p$	$t - h\ln r$	c.1
8	$U = t^{1/(2\beta)}f^u, \tau = t^{1/\beta}f^\tau, p = f^p$	$rt^{-(2\beta+1)/(2\beta)}$	c.2

This group is a subgroup of the group admitted by the Navier-Stokes equations. There exist no invariant solutions that corresponds to this group. In fact, the universal invariant of this algebra is t, z, w, p, hence, the rank of the Jacobi matrix of the universal invariant with respect to the dependent variables q is equal to two. Therefore, $\delta \geq 2$ and one can only construct partially invariant solutions with respect to this group. According to the classification [37], a partially invariant solution with minimum defect $\delta = 2$ is a regular partially invariant solution of $H(2,2)$. In this case a representation of the partially invariant solution is

$$w = 2f(z,t), \quad p = h(z,t), \quad u = u(x,y,z,t), \quad v = v(x,y,z,t).$$

For the gas dynamics equations such a class of solutions was studied in [38]. Pukhnachov V.V.[15] noted that for the Navier-Stokes equations this representation can be generalized by including two arbitrary functions $k = k(t)$ and $l = l(t)$:

$$w = 2f(z,t), \quad p = h(z,t) - k(t)x^2 - l(t)y^2, \quad u = u(x,y,z,t), \quad v = v(x,y,z,t). \tag{34}$$

The arbitrariness of the functions $k(t)$ and $l(t)$ gives additional possibilities for satisfying boundary conditions. Representation (34) can also be explained from the group point of view. In fact, let us consider the four-dimensional group H^4, which is generated by the operators

$$X_1 = \phi_1\partial_x + \phi_1'\partial_u - x\phi_1''\partial_p, \quad X_2 = \phi_2\partial_x + \phi_2'\partial_u - x\phi_2''\partial_p,$$

$$Y_1 = \psi_1\partial_y + \psi_1'\partial_v - y\psi_1''\partial_p, \quad Y_2 = \psi_2\partial_y + \psi_2'\partial_v - y\psi_2''\partial_p.$$

Here the functions $\phi_i = \phi_i(t)$, $\psi_i = \psi_i(t)$, $(i = 1,2)$ satisfy the natural conditions for the algebra $H^4 = \{X_1, X_2, Y_1, Y_2\}$ to be a four-dimensional algebra:

$$\phi_1\phi_2' - \phi_1'\phi_2 \neq 0, \quad \psi_1\psi_2' - \psi_1'\psi_2 \neq 0,$$

$$\phi_1\phi_2'' - \phi_1''\phi_2 = 0, \quad \psi_1\psi_2'' - \psi_1''\psi_2 = 0.$$

A regular partially invariant solution with respect to the Lie group H^4 has representation (34), where $k = \phi_i''/(2\phi_i)$, $l = \psi_i''/(2\psi_i)$. In the literature there are solutions which are particular cases of (34). Examples are the solution in [42] and one of the solutions in [27].

5.7 Compatibility conditions

As is well-known, the main difficulty in the study of partially invariant solutions is the compatibility analysis of the reduced systems. The compatibility analysis can be reduced to a consecutive performance of algebraic operations of symbolic nature [15, 25]. These operations are related with the prolongation of the system, substitution of composite expressions (transition onto manifold), and finding ranks of matrices. Typically, the compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations. Here we have used the system REDUCE [21].

For the case $k = 0$, $l = 0$ the analysis of compatibility was done in [33]. As mentioned earlier the arbitrariness of the functions $k(t)$ and $l(t)$ gives additional possibilities, however the

[15] An oral communication.

compatibility analysis of the overdetermined system obtained after substitution of representation (34) into the Navier-Stokes equations (17) becomes more difficult. Here the compatibility analysis of this overdetermined system is given.

Note that after introducing the functions $\hat{u}(x,y,z,t)$, $\hat{v}(x,y,z,t)$ by the formulae:

$$u = \hat{u} - x\frac{\partial f}{\partial z}, \quad v = \hat{v} - y\frac{\partial f}{\partial z},$$

the second equation of (17) becomes

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0,$$

which shows that for the compatibility analysis it is more convenient to use the functions $\hat{u}(x,y,z,t)$, $\hat{v}(x,y,z,t)$ instead of $u(x,y,z,t)$ and $v(x,y,z,t)$. The general solution of the last equation can be given through the analog of the stream function $\psi = \psi(x,y,z,t)$ by the formulae

$$\hat{u} = \frac{\partial \psi}{\partial y}, \quad \hat{v} = -\frac{\partial \psi}{\partial x}.$$

Then the first two scalar equations of (17) have the form

$$\begin{aligned}\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} + 2f\psi_{yz} - x(f_{zt} + f_z\psi_{xy} + 2ff_{zz} - f_z^2) - yf_z\psi_{yy} = \\ = \Delta\psi_y - xf_{zzz} + 2xk, \\ -\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} - 2f\psi_{xz} - y(f_{zt} - f_z\psi_{xy} + 2ff_{zz} - f_z^2) + xf_z\psi_{xx} = \\ = -\Delta\psi_x - yf_{zzz} + 2yl,\end{aligned} \tag{35}$$

and the third equation serves for determining the function $h(z,t)$ (if the function $f(z,t)$ is known):

$$h_z + 2f_t - 2f_{zz} + 4ff_z = 0.$$

The compatibility conditions are derived with respect to the following equivalence transformations: representation (34) is invariant with respect to rotations in the (x,y)-plane and shifts in (x,y,z) and t.

5.7.1 Preliminary analysis of compatibility.

Let us consider some solutions of (17), which we will call simple.

The first solution is a solution of the form

$$\psi(x,y,z,t) = \frac{1}{2}(x^2\gamma(z,t) + y^2c(z,t)) + x\lambda(z,t) + yb(z,t) + xy\alpha(z,t). \tag{36}$$

This representation is a particular case of the solutions with linear profile of velocity[16]

$$u = x(\alpha - f_z) + yc + b, \ v = -x\gamma - y(\alpha + f_z) - \lambda.$$

After substituting the representation (36) of the solution into (35) and splitting with respect to x and y, one obtains the compatibility conditions:

$$\begin{aligned}Lf_z + k + l = -c\gamma + \alpha^2, \ L\alpha = \alpha f_z + k - l, \\ L\gamma = f_z\gamma, \ L\lambda = \lambda\alpha - b\gamma, \ Lc = f_zc, \ \ Lb = \lambda c - \alpha b,\end{aligned} \tag{37}$$

[16]Solutions with a linear profile of velocity with respect to one, two or three space variables were studied in [45, 50].

where L is the linear operator

$$LF \equiv F_t + 2fF_z - F_{zz} - f_zF.$$

The second type of solutions has the representation[17]

$$\psi(x,y,z,t) = x^2a(y,z,t) + xb(y,z,t) + g(y,z,t). \tag{38}$$

As in the previous case, after substituting the representation of the solution into (35) and splitting with respect to x, one obtains the compatibility conditions. Two of these conditions are $a_y = 0$, $b_{yy} = 0$. Hence, the function $b(y,z,t)$ is linear with respect to y: $b(y,z,t) = y\alpha(z,t) + \lambda(z,t)$. If $a \neq 0$, then $g_{yyy} = 0$, but this case corresponds to (36), which was considered earlier. Hence, $a = 0$. The remaining compatibility conditions are

$$\begin{gathered} Lf_z + k + l = \alpha^2, \ L\alpha = \alpha f_z + k - l, \\ L\lambda = \alpha\lambda, \ L\varphi - \varphi_{yy} - (y(\alpha + f_z) + \lambda)\varphi_y + \alpha\varphi = 0, \end{gathered} \tag{39}$$

where $\varphi = g_y$. This solution has a linear profile of velocity with respect to x

$$u = x(\alpha - f_z) + \varphi, \ v = -y(\alpha + f_z) - \lambda.$$

Note that the case $\psi_{xx} = 0$ is a particular case of the representation (38).

Let us consider the representation

$$\psi(x,y,z,t) = a(x,z,t) + b(y,z,t) + xy\alpha(z,t). \tag{40}$$

After substitution of this representation into the Navier-Stokes equations one obtains

$$a_{xxx}b_{yy} = 0, \ a_{xx}b_{yyy} = 0.$$

Without loss of generality, this case can be considered as a particular case of representation (38).

We exclude the above considered solutions from the further study of the compatibility conditions of system (35).

Remark. A solution of the form

$$\psi(x,y,z,t) = x^2\varphi(z,t) + x\lambda(z,t) + y^2c(z,t) + yb(z,t) + xy\alpha(z,t) + Q(x + yq(z,t), z, t) \tag{41}$$

is a particular case of (36) if the function $Q = Q(\xi, z, t)$ is a quadratic function with respect to the first argument. This case corresponds to a linear profile of velocity, which was studied before. If $Q_{\xi\xi\xi} \neq 0$, then the compatibility conditions require that q is a constant. By rotating in the (x,y)–plane this case can be transformed to (38), which was also regarded earlier.

5.7.2 Monge-Ampere equation.

Note that after adding the first equation of (35) differentiated with respect to x to the second equation differentiated with respect to y, one obtains

$$\psi_{xy}^2 - \psi_{xx}\psi_{yy} = Lf_z + k + l. \tag{42}$$

[17]Because the Navier-Stokes equations are symmetric with respect to rotations, the case $\psi_{yyy} = 0$ is similar to the case $\psi_{xxx} = 0$.

The right side of this equation only depends on z and t, therefore it can be regarded as the Monge-Ampere equation with a constant (depending on the parameters z and t) right side. The method for solving the Monge-Ampere equation depends on the sign of the right side.

The next theorem is one of the main results of this manuscript.

Theorem. Any solution of system (35) satisfies the Monge-Ampere equation (42). If the right side of the Monge-Ampere equation is non negative, $Lf_z + k + l \geq 0$, then the solution of the overdetermined system (35) and of the Monge-Ampere equation (42) is either a solution of system (37) or system (39).

Before proving the theorem let us give some comments.

There are known particular solutions of the Navier-Stokes equations of type (34) with both positive and negative right sides. For example, solutions with linear profile of velocity (36) with respect to x and y can be of both types, depending on the value of $\alpha^2 - c\gamma$. For solutions that are linear with respect to one independent variable x (and essentially nonlinear with respect to another y) (41) the right side of the Monge-Ampere equation is positive. In case (38) the type of the Monge-Ampere equation is hyperbolic.

Here we also present two known solutions [27, 42].

As the first example one can consider a slight generalization of the solution [42][18]

$$u = -\Omega(y - g_1(z,t)), \ u = \Omega(x - g_2(z,t)), \ w = w_0,$$

where w_0 is a constant and Ω denotes a constant angular velocity. Compatibility conditions for this solution are

$$(g_{1t} + w_0 g_{1z} - g_{1zz} + \Omega g_2)_z = 0, \ (g_{2t} + w_0 g_{2z} - g_{2zz} - \Omega g_1)_z = 0.$$

This solution can be represented as type (34) if $k = -\Omega^2/2$, $l = -\Omega^2/2$, $h = h(t)$, $2f = w_0$ and

$$g_{1t} + w_0 g_{1z} - g_{1zz} + \Omega g_2 = 0, \ g_{2t} + w_0 g_{2z} - g_{2zz} - \Omega g_1 = 0.$$

In this case

$$Lf_z + k + l = -\Omega^2 \leq 0. \tag{43}$$

The second example is the class of steady solutions studied in [27], where

$$f = f(z), \ \ h = 2(f'(z) - f^2(z)), \ u = x\tilde{u}(z), \ v = -y(\tilde{u}(z) + 2(f'(z))^2,$$

and constants k and l. The functions $f(z)$ and $\tilde{u}(z)$ satisfy the equations

$$\tilde{u}'' - 2f\tilde{u}' - \tilde{u}^2 + 2l = 0, \ f''' - 2ff'' + 2(f')^2 + 2\tilde{u}f' + \tilde{u}^2 = k + l.$$

For this solution

$$Lf_z + k + l = (f' + \tilde{u})^2 \geq 0. \tag{44}$$

The next subsections are devoted to proof of the theorem: we study compatibility conditions for equations (35) with non negative right side (the hyperbolic case) of the Monge-Ampere equation (42).

[18]In [42] the functions g and f do not depend on time t. But this is not significant, because without loss of generality one can include dependence of this functions on time.

5.7.3 The hyperbolic case

Further we consider the hyperbolic case, where the right side of the Monge–Ampere equation (42) is non negative. By virtue of this assumption we denote

$$\alpha^2(z,t) \equiv Lf_z + k + l.$$

It is well known [19], that in this case the Monge–Ampere equation can be integrated over time[19]

$$g_y = 2\alpha x + G\left(g_x, z, t\right). \tag{45}$$

where $g(x,y,z,t) = \psi(x,y,z,t) + xy\alpha(z,t)$, and $G = G(z,t,\xi)$ is an arbitrary function. After substituting this representation into the first equation (35) with the help of the second equation can exclude the third order derivatives:

$$S \equiv b_4 g_{xx}^2 + b_5 g_{xz}^2 + b_1 g_{xx} + b_2 g_{xz} - b_3 = 0, \tag{46}$$

where

$$b_1 = 4\alpha G_\xi G_{\xi\xi},\ b_2 = 2G_{\xi z},\ b_4 = (G_\xi^2 + 1)G_{\xi\xi},\ b_5 = G_{\xi\xi},$$
$$b_3 = x(\widehat{\alpha} - 2(k-l)) + y\widehat{\alpha}G_\xi + (f_z - \alpha)\left(\xi G_\xi - G\right) + G_t + 2fG_z - G_{zz} - 4\alpha^2 G_{\xi\xi},$$
$$\widehat{\alpha} \equiv L\alpha - \alpha f_z + k - l$$

By direct calculations one can rewrite the expression $D_y S - G_\xi D_x S - 2g_{xx}G_{\xi\xi}S = 0$ as a polynomial of second order with respect to the derivatives g_{xx}, g_{xz}:

$$\alpha(1 + G_\xi^2)G_{\xi\xi\xi}g_{xx}^2 + \alpha G_{\xi\xi\xi}g_{xz}^2 + f_1 g_{xx} + f_2 g_{xz} + f_3 = 0. \tag{47}$$

Here D_x and D_y are the total derivatives with respect to x and y, respectively,

$$f_1 = \left(x(\widehat{\alpha} - 2(k-l)) + y\widehat{\alpha}G_\xi\right)G_{\xi\xi} + \widehat{f}_1,\quad f_2 = 2\left(\alpha_z G_{\xi\xi} + \alpha G_{\xi\xi z}\right),$$
$$f_3 = -y\alpha\widehat{\alpha}G_{\xi\xi} + \widehat{f}_3$$

with some functions $\widehat{f}_i$, $(i = 1, 3)$, which are not explicitly dependent on x and y. Because the expressions of the functions $\widehat{f}_1$ and $\widehat{f}_1$ are very cumbersome we omit their representations here. For the treatment of complicated mathematical expressions we used the system REDUCE [21].

Note that if $G_{\xi\xi} = 0$, then this is a particular case of the representation (40) or (41). In fact, assume that $G_{\xi\xi} = 0$ or $G = qg_x + \beta$ for some functions $q = q(z,t), \beta = \beta(z,t)$. By (45) the function $g(x,y,z,t)$ has to satisfy the equation

$$g_y - qg_x = 2\alpha x + \beta. \tag{48}$$

If $q = 0$, then the general solution of (48) is

$$g = 2\alpha xy + y\beta + \varphi(x,z,t),$$

which is a particular case of (40). If $q \neq 0$, then the general solution of (48) is

$$g = q^{-1}(\alpha x^2 + \beta x) + \varphi(x - qy, z, t),$$

which is a particular case of (41).

[19] There are some studies of an elliptic case of the Monge-Ampere equation, for example [18, 1].

5.7.4 The non-linear case ($G_{\xi\xi} \neq 0$)

Let $G_{\xi\xi} \neq 0$, then equation (47) with the help of (46) can be rewritten as the quasilinear equation

$$a_1 g_{xx} + a_2 g_{xz} + a_3 = 0, \tag{49}$$

with coefficients $a_i = b_i G_{\xi\xi\xi} - f_i G_{\xi\xi}$, $(i = 1, 2, 3)$. The last equation and equation (46) can be regarded as a system of linear equations with respect to x and y. The determinant of this system is equal to $G_{\xi\xi}\alpha\widehat{\alpha}(\widehat{\alpha} - 2(k-l))$.

If $\alpha = 0$, then by virtue of the definition of $\widehat{\alpha}$ we get $\widehat{\alpha} = (k-l)$, and the following prolongation

$$D_y H - G_\xi D_x H - g_{xx} G_{\xi\xi} H = -6(k-l) g_{xx} G_\xi G_{\xi\xi} = 0,$$

where $H = D_y S - G_\xi D_x S - 2 g_{xx} G_{\xi\xi} S$. Because $g_{xx} G_\xi G_{\xi\xi} \neq 0$, then $k - l = 0$. This means, that $\widehat{\alpha} = 0$. In this case

$$H + 2 g_{xx} G_{\xi\xi} S = -2 g_{xx} \left((g_{xz} G_{\xi\xi} + G_{\xi z})^2 + g_{xx}^2 G_{\xi\xi}^2 (1 + G_\xi^2) \right) = 0.$$

The last equation contradicts the condition $g_{xx} G_{\xi\xi} \neq 0$. Therefore, $\alpha \neq 0$.

5.7.5 Case $\widehat{\alpha}(\widehat{\alpha} - 2(k-l)) \neq 0$

If $\widehat{\alpha}(\widehat{\alpha} - 2(k-l)) \neq 0$, then $\alpha\widehat{\alpha}(\widehat{\alpha} - 2(k-l)) G_{\xi\xi} \neq 0$ and, hence, the two equations (46) and (49) can be solved with respect to x and y:

$$x = \Phi_1(g_{xx}, g_{xz}, g_x, z, t), \; y = \Phi_2(g_{xx}, g_{xz}, g_x, z, t). \tag{50}$$

After differentiating the last equations with respect to x and y, substituting the expressions of $g_y, g_{xy}, g_{xyz}, g_{xxy}$ into them and taking linear combinations, one obtains

$$D_y \Phi_1 - G_\xi D_x \Phi_1 = g_{xx}^2 \Phi_{1,1} G_{\xi\xi} + \Phi_{1,2}(2\alpha_z + g_{xx} g_{xz} G_{\xi\xi} + g_{xx} G_{\xi z}) + 2\Phi_{1,3}\alpha + G_\xi = 0, \tag{51}$$

$$\begin{aligned} H(g_{xx}, g_{xz}, g_x, z, t) &\equiv D_y \Phi_2 - G_\xi D_x \Phi_2 = \\ &= g_{xx}^2 \Phi_{2,1} G_{\xi\xi} + \Phi_{2,2}(2\alpha_z + g_{xx} g_{xz} G_{\xi\xi} + g_{xx} G_{\xi z}) + 2\Phi_{2,3}\alpha - 1 = 0, \end{aligned} \tag{52}$$

$$\begin{aligned} H_3(g_{xx}, g_{xz}, g_x, z, t) &\equiv -\Phi_{2,1} D_x \Phi_1 + \Phi_{1,1} D_x \Phi_2 - \Phi_{2,2} D_z \Phi_1 + \Phi_{1,2} D_z \Phi_2 = \\ &= g_{xx}(\Phi_{1,1}\Phi_{2,3} - \Phi_{1,3}\Phi_{2,1}) + g_{xz}(\Phi_{1,2}\Phi_{2,3} - \Phi_{1,3}\Phi_{2,2}) + \Phi_{1,2}\Phi_{2,4} - \Phi_{1,4}\Phi_{2,2} + \Phi_{2,1} = 0 \end{aligned}$$

where $\Phi_{i,1} = \dfrac{\partial \Phi_i}{\partial g_{xx}}$, $\Phi_{i,2} = \dfrac{\partial \Phi_i}{\partial g_{xz}}$, $\Phi_{i,3} = \dfrac{\partial \Phi_i}{\partial z}$. Note that after substituting the expressions of the functions Φ_i, $(i = 1, 2)$ into the last equations, equation (51) is a consequence of equation (52) and the function $H(g_{xx}, g_{xz}, g_x, z, t)$ is a polynomial of fourth degree with respect to g_{xx} and second degree with respect to g_{xz}

$$H = h_2 g_{xz}^2 + h_1 g_{xz} + h_0,$$

where

$$h_2 = 3 g_{xx}^2 G_{\xi\xi}^4 + 4\alpha g_{xx} G_{\xi\xi\xi} G_{\xi\xi}^2 + 2\alpha^2 (G_{\xi\xi\xi\xi} G_{\xi\xi} - G_{\xi\xi\xi}^2)$$

The coefficient of the polynomial H with respect to g_{xx}^4 is $3G_{\xi\xi}^4(1 + G_\xi^2) \neq 0$ and does not depend on g_{xz}. Hence, the equation $H(g_{xx}, g_{xz}, g_x, z, t) = 0$ can be rewritten as $H_1 \equiv g_{xx} -$

$\chi(g_{xz}, g_x, z, t) = 0$. In the same way, after differentiating the last equation with respect to x and y, substituting the expressions of $g_y, g_{xy}, g_{xyz}, g_{xxy}$ into them, one obtains

$$D_y H - G_\xi D_x H = g_{xx}^2 H_{,1} G_{\xi\xi} + H_{,2}(2\alpha_z + g_{xx} g_{xz} G_{\xi\xi} + g_{xx} G_{\xi z}) + 2H_{,3}\alpha = 0.$$

Since $H(g_{xx}, g_{xz}, g_x, z, t) = 0$, the left side of the last equation can be rewritten as a polynomial of degree three with respect to g_{xx}:

$$H_2(g_{xx}, g_{xz}, g_x, z, t) = 0. \tag{53}$$

If the Jacobian $\dfrac{\partial(H_1, H_2)}{\partial(g_{xx}, g_{xz})}$ is not equal to zero, then from the equations

$$H_1(g_{xx}, g_{xz}, g_x, z, t) = 0,\ H_2(g_{xx}, g_{xz}, g_x, z, t) = 0.$$

one can define

$$g_{xx} = \Psi_1(g_x, z, t), \quad g_{xz} = \Psi_2(g_x, z, t).$$

Substitution of these derivatives into (50) gives the contradictory equalities

$$x = \widehat{\Phi}_1(g_x, z, t),\ y = \widehat{\Phi}_2(g_x, z, t). \tag{54}$$

If the Jacobian $\dfrac{\partial(H_1, H_2)}{\partial(g_{xx}, g_{xz})} = \dfrac{\partial H_2}{\partial g_{xx}} \dfrac{\partial \chi}{\partial g_{xz}} - \dfrac{\partial H_2}{\partial g_{xz}} = 0$, then this means that the function $\widehat{H_2} = H_2(\chi(g_{xz}, g_x, z, t), g_{xz}, g_x, z, t)$ does not depend on g_{xz}. Furthermore $\widehat{H_2}_{g_x} = 0$, because otherwise one can define g_x as a function of z and t, which contradicts the condition $g_{xx} \neq 0$. Therefore $H_2 = F(H_1)$. In our case,

$$H_2 = \widehat{a}_3 H_1^3 + \widehat{a}_2 H_1^2 + \widehat{a}_1 H_1 + \widehat{a}_0.$$

Thus, the coefficients $\widehat{a}_i$ must be constants and $\widehat{a}_0 = 0$. Note that

$$\widehat{a}_2 = \widehat{b}_1 \chi + \widehat{b}_2,\ \widehat{a}_1 = \widehat{b}_3 \chi^2 + \widehat{b}_4 \chi + \widehat{b}_5 g_{xz}^2 + \widehat{b}_6 g_{xz} + \widehat{b}_7,$$

where $\widehat{b}_i$ are functions of the variables g_x, z, t and

$$\widehat{b}_3 = \widehat{b}_1 = 3(1 + G_\xi^2)\widehat{b}_5,\ \widehat{b}_5 = 3G_{\xi\xi}G_{\xi\xi\xi\xi} - 5G_{\xi\xi\xi}^2.$$

If $\widehat{b}_1 \neq 0$, then from the equation $\widehat{a}_2 = const$ we have $\chi = -\widehat{b}_1^{-1}(\widehat{b}_2 - \widehat{a}_2)$, which does not depend on g_{xz}. In this case the equation $\widehat{a}_1 = const$ is a polynomial of degree two with respect to g_{xz} with coefficient $\widehat{b}_5 \neq 0$. This means that one can obtain contradictory equations of type (54). Therefore, $\widehat{b}_1 = 0$ or

$$3G_{\xi\xi}G_{\xi\xi\xi\xi} - 5G_{\xi\xi\xi}^2 = 0.$$

This equation can be integrated twice with respect to ξ:

$$G_{\xi\xi} = \lambda(G_\xi + q)^3,$$

Two more integrations with respect to ξ give:

$$\lambda(G + \xi q + \gamma)^2 + 2\xi + \beta = 0.$$

Here the functions $\lambda = \lambda(z,t)$, $q = q(z,t)$, $\gamma = \gamma(z,t)$, $\beta = \beta(z,t)$ are arbitrary and $\lambda \neq 0$. Note that in this case $\widehat{a}_3 = 0$, $b_2 = b_3 = b_4 = b_5 = b_6 = 0$, $\widehat{a}_3 = b_7$

$$\widehat{a}_0 = \varphi_1(g_x, z, t)\chi + \varphi_0(g_x, z, t), \tag{55}$$

and

$$h_2 = 3\lambda^2(G_\xi + q)^5(g_{xx}(G_\xi + q) + 2\alpha)^2.$$

Assume that the function $\chi(g_{xz}, g_x, z, t)$ does not depend on g_{xz} : $\chi = \chi(g_x, z, t)$. Because of the prohibition of obtaining equations of the type (54), the coefficients h_i $(i = 1, 2)$ of the polynomial H have to be equal to zero. Hence, as $G_{\xi\xi} \neq 0$, we have

$$g_{xx}(G_\xi + q) = -2\alpha.$$

The left side of this expression is the total derivative with respect to x of $G(g_x, z, t) + q(z,t)g_x$. Thus,

$$G(g_x, z, t) + q(z,t)g_x + 2\alpha(z,t)x = \phi(y, z, t). \tag{56}$$

Because $g_{xy} = G_\xi g_{xx} + 2\alpha$, then

$$\phi_y = g_{xy}(G_\xi + q) = (G_\xi + q)G_\xi g_{xx} + 2\alpha(G_\xi + q) = 2\alpha q.$$

After integrating the last equation with respect to y, there is $\phi(y,z,t) = 2y\alpha(z,t)q(z,t) + h(z,t)$. Substituting the function $\phi(y,z,t)$ and g_y into (56), one obtains

$$g_y + qg_x = 2y\alpha q + h.$$

The general solution of this equation is

$$g(x,y,z,t) = yh(z,t) + y^2\alpha(z,t)q(z,t) + \Phi(x - yq(z,t), z, t)$$

or

$$\psi(x,y,z,t) = -xy\alpha(z,t) + yh(z,t) + y^2\alpha(z,t)q(z,t) + \Phi(x - yq(z,t), z, t).$$

This is a particular case of (41). Therefore, we need to study the case $\dfrac{\partial\chi}{\partial g_{xz}} \neq 0$.

Assume that $\dfrac{\partial\chi}{\partial g_{xz}} \neq 0$. From the expression for the function $\widehat{a}_0 = 0$ (55) we conclude that

$$\varphi_1(g_x, z, t) = 0, \ \varphi_0(g_x, z, t) = 0.$$

After splitting these equations with respect to g_x, one obtains

$$q_z = 0, \ \alpha q_t + q(k - l) = 0,$$
$$2\alpha^2\lambda(\lambda_t + 2f\lambda_z - \lambda_{zz} + f_z\lambda - \alpha\lambda) - (\alpha_z\lambda + \alpha\lambda_z)^2 + \lambda^2\alpha(\widehat{\alpha} - 4(k - l)) + 4\lambda_z^2\alpha^2 = 0.$$

The same analysis of the equation $H_3(g_{xx}, g_{xz}, g_x, z, t) = 0$ as for the equation $H_2 = 0$ leads to a contradiction. Therefore, we have to study the case $\widehat{\alpha}(\widehat{\alpha} - 2(k - l)) = 0$.

5.7.6 The case $\hat{\alpha} = 0$

Let us consider $\hat{\alpha} = 0$ or

$$\frac{\partial \alpha}{\partial t} + 2f\frac{\partial \alpha}{\partial z} - \frac{\partial^2 \alpha}{\partial z^2} - 2\alpha\frac{\partial f}{\partial z} + k - l = 0.$$

The coefficients a_i, b_i, f_i, $(i = 1, 2, 3)$, b_4, b_5 do not explicitly depend on y.

Assume first that $k \neq l$. In this case one can define the value of x from (46) and substitute it into (49), which is a polynomial of degree three with respect to g_{xx}

$$H_1 = h_3 g_{xx}^3 + h_2 g_{xx}^2 + h_1 g_{xx} + h_0,$$

where[20]

$$h_3 = G_{\xi\xi}^2(1 + G_\xi^2) \neq 0$$

This means that one can define $g_{xx} = \chi(g_{xz}, g_x, z, t)$ from this equation. Note that the coefficient in H_1, which is related with the maximal degree (second) with respect to g_{xz}, is equal to

$$g_{xx}G_{\xi\xi}^2 + \alpha G_{\xi\xi\xi}. \tag{57}$$

By the equation $H_1(g_{xx}, g_{xz}, g_x, z, t) = 0$, the left side of the expression

$$H_2 \equiv D_y H_1 - G_\xi D_x H_1 = 0.$$

is a polynomial of second degree with respect to g_{xx}:

$$H_2 = a_2 g_{xx}^2 + a_1 g_{xx} + a_0 = 0.$$

Before further consideration, we note that if from the equations

$$H_1(g_{xx}, g_{xz}, g_x, z, t) = 0, \;\; H_2(g_{xx}, g_{xz}, g_x, z, t) = 0.$$

one can define

$$g_{xx} = \Psi_1(g_x, z, t), \quad g_{xz} = \Psi_2(g_x, z, t),$$

then after substitution of these derivatives into (46) one has the equality

$$x = \Phi(g_x, z, t). \tag{58}$$

Differentiating the last equality with respect to y we have $g_{xy}\Phi_\xi = 0$. If $\Phi_\xi = 0$, then (58) is a contradictory equality between the independent variables. The case $g_{xy} = 0$ was considered earlier.

Assume that the function $\chi(g_{xz}, g_x, z, t)$ does not depend on g_{xz}. In this case all coefficients of the equation $H_1 = 0$ with respect to g_{xz} have to be equal to zero. Hence, from (57) we obtain

$$\chi = -\alpha G_{\xi\xi}^{-2} G_{\xi\xi\xi}. \tag{59}$$

Because one can find the derivatives g_{xxx} and g_{xxz}, one obtains the equation $H_3(g_{xz}, g_x, z, t) \equiv D_x S = 0$. Note that $G_{\xi\xi\xi} \neq 0$ (because $g_{xx} \neq 0$) and because it is prohibited to define g_{xz} from the equation $H_1 = 0$, $H_2 = 0$, $H_3 = 0$, all coefficients of this polynomial with respect to g_{xz}

[20]The analysis is similar to the previous case. For the polynomials and their coefficients we use the same symbols as in the previous case. However, the functions H, H_2, H_3 and etc. are now different.

have to be equal to zero. Particularly, from the coefficient related with the highest (second) degree of the equation $H_3 = 0$ we have

$$G_{\xi\xi\xi\xi} = \frac{3G_{\xi\xi\xi}^2}{2G_{\xi\xi}}.$$

Because $G_{\xi\xi\xi} \neq 0$ (the equation $G_{\xi\xi\xi} \neq 0$ leads to the case $g_{xx} = 0$, which is excluded from our consideration as it has already been studied), the general solution of the last equation is

$$G = -\lambda^{-1}\ln(\lambda g_x + \beta) + \mu g_x + \gamma,$$

where $\beta, \lambda, \mu, \gamma$ are arbitrary functions of the independent variables z, t. In this case

$$g_{xx} = 2\alpha(\lambda g_x + \beta). \tag{60}$$

The general solution of equation (60) is

$$g = -\frac{\beta}{2\alpha\lambda^2}(1 + 2\alpha\lambda x) + \varphi_1 e^{2\alpha\lambda x} + \varphi_2,$$

where $\varphi_1 = \varphi_1(y, z, t)$, $\varphi_2 = \varphi_2(y, z, t)$. All coefficients of the polynomials H_1 and H_3 with respect to g_{xz}, which have to be equal to zero, are polynomials with respect to g_x. This allows splitting them with respect to g_x, otherwise g_x can be defined as a function only of z and t. Further study of all these coefficients leads in particular to the equality $\mu = 0$.

By virtue of $\mu = 0$ and substituting g into the equation

$$g_y = G(g_x, z, t) + 2\alpha x,$$

one obtains $\varphi_{1,y} = 0, \varphi_{2,yy} = 0$. This means that $g_{yy} = 0$ or $\psi_{yy} = 0$. This case was studied earlier.

Assume that $\chi_{g_{xz}} \neq 0$. The study of this case is similar to the previous case where $\widehat{\alpha}(\widehat{\alpha} - 2(k - l)) \neq 0$. Because the Jacobian $\dfrac{\partial(H_1, H_2)}{\partial(g_{xx}, g_{xz})}$ has to be equal to zero, then $H_2 = F(g_{xx} - \chi(g_{xz}, g_x, z, t))$. In our case

$$H_2 = \widehat{a}_2 H_1^2 + \widehat{a}_1 H_1 + \widehat{a}_0,$$

The coefficients $\widehat{a}_i$ must be constant and $\widehat{a}_0 = 0$. Note that

$$\widehat{a}_1 = \widehat{b}_1\chi + \widehat{b}_2,\ \widehat{a}_0 = \widehat{b}_3\chi^2 + \widehat{b}_4\chi + \widehat{b}_5 g_{xz}^2 + \widehat{b}_6 g_{xz} + \widehat{b}_7,$$

where $\widehat{b}_i$ are functions of the variables g_x, z, t and

$$\widehat{a}_2 = \widehat{b}_1 = \widehat{b}_3 = (1 + G_\xi^2)\widehat{b}_5,\ \widehat{b}_5 = (2G_{\xi\xi}G_{\xi\xi\xi\xi} - 3G_{\xi\xi\xi}^2).$$

If $\widehat{b}_1 \neq 0$, then $\chi = -\widehat{b}_1^{-1}(\widehat{b}_2 - \widehat{a}_2)$ does not depend on g_{xz}. This case has already been studied. Also note that if $G_{\xi\xi\xi} = 0$, then $\widehat{a}_2 = \widehat{b}_1 = \widehat{b}_3 = \widehat{b}_5 = \widehat{b}_6 = 0$. This requires $\widehat{b}_4 = \widehat{b}_7 = 0$. Analysis of these coefficients by splitting them with respect to g_x leads to the condition that $a_2 = a_3 = 0$ in equation (49) and that a_1 is linear with respect to g_x: $a_1 = a(z, t)g_x + \phi(x, z, t)$, where $a \neq 0$. This contradicts $g_{xx}g_{xy} = 0$. Therefore, $\widehat{b}_1 = 0$ and $G_{\xi\xi\xi} \neq 0$ or

$$G = -\frac{1}{\lambda}\ln(\lambda\xi + \beta) + \mu\xi + \gamma.$$

In this case

$$\widehat{a}_0 = \widehat{b}_4\chi + \widehat{b}_6 g_{xz} + \widehat{b}_7 = 0.$$

If the coefficient $\widehat{b}_4 = 0$, then as done earlier, an analysis of the coefficients $\widehat{b}_4 = \widehat{b}_6 = \widehat{b}_7 = 0$ by splitting them with respect to g_x leads to the condition that equation (49) be written as

$$a_1(g_{xx} - 2\alpha(\lambda g_x + \beta)) = 0,$$

where $a_1 = a(z,t)g_x + \phi(x,z,t)$ with $a \neq 0$. These cases have already been studied.

If $\widehat{b}_4 \neq 0$, then

$$\chi = -\widehat{b}_4^{-1}(\widehat{b}_6 g_{xz} + \widehat{b}_7).$$

Returning to the equation $H_1 = 0$, which becomes a cubic polynomial with respect to g_{xz} and analyzing the coefficients of this polynomial, which have to be equal to zero, leads to a contradiction. This completes the study of the case $k \neq l$.

Assume that $k = l$. Note that if $a_1 = 0$ in equation (49), then equation (46) is reduced to

$$(g_{xz}G_{\xi\xi} + G_{\xi z})^2 + G_{\xi\xi}^2(g_{xx}G_\xi + 2\alpha)^2 + (g_{xx}G_{\xi\xi})^2 = 0.$$

Hence, $a_1 \neq 0$ and from equation (49) one can define $g_{xx} = -a_1^{-1}(a_2 g_{xz} + a_0)$. Substituting g_{xx} into (46) gives a polynomial of second degree with respect to g_{xz}:

$$S = a_1^{-2}G_{\xi\xi}(a_1^2 + a_2^2(1 + G_\xi^2))g_{xz}^2 + \widehat{b}_1 g_{xz} + \widehat{b}_0 = 0.$$

This means that equations (46), (49) can be solved with respect to g_{xx} and g_{xz}:

$$g_{xx} = \widehat{\Phi}_1(g_x, z, t), \quad g_{xz} = \widehat{\Phi}_2(g_x, z, t). \tag{61}$$

Because $g_{xx} \neq 0$, then the first equation of (61) can be integrated

$$\widehat{\Phi}(g_x, z, t) = x + q(y, z, t)$$

or

$$g_x = \Phi(x + q(y,z,t), z, t).$$

Here the function $q = q(y,z,t)$ is an arbitrary function. The general solution of the last equation is expressed by the formula

$$g(x,y,z,t) = \Phi_1(x + q(y,z,t), z, t) + \Phi_2(y,z,t)$$

Note that

$$G(g_x, z, t) = \widehat{G}(x + q(y,z,t), z, t)$$

and the equation $g_y - (2\alpha x + G) = 0$ is rewritten as

$$q_y\Phi_{1,x'}(x', z, t) + \Phi_{2,y}(y,z,t) = 2\alpha x' + \widehat{G}(x', z, t) - 2\alpha q,$$

where $x' = x + q(y,z,t)$. Differentiating the last equation with respect to y one obtains

$$q_{yy}\Phi_{1,x'}(x', z, t) + \Phi_{2,yy}(y,z,t) = -2\alpha q_y. \tag{62}$$

Differentiating once more with respect to x' gives

$$q_{yy}\Phi_{1,x'x'} = 0.$$

If $\Phi_{1,x'x'} = 0$, then this is a particular case of the representation (38). If $q_{yy} = 0$ or $q = yk_1(z,t) + k_2(z,t)$, then integrating equation (62) we have

$$\Phi_2 = -\alpha k_1 y^2 + y\psi_1(z,t) + \psi_2(z,t).$$

This is a particular case of the representation (41).

The case $\widehat{\alpha} = 2(k-l)$ is studied in a similar way as the previous case $\widehat{\alpha} = 0$. Note that in this case $\alpha(k-l) \neq 0$. A detailed analysis leads either to contradictions or to the already studied cases.

5.8 Group classification of system (39)

System (39) is split into three parts: the system of the first two equations

$$Lf_z + k + l = \alpha^2, \ L\alpha = \alpha f_z + k - l \tag{63}$$

is determined and can be studied independently; the equation

$$L\lambda = \alpha\lambda$$

is for determining the function $\lambda(z,t)$: and the equation

$$L\varphi - \varphi_{yy} - (y(\alpha + f_z) + \lambda)\varphi_y + \alpha\varphi = 0$$

is for the function $\varphi(y,z,t)$. In this subsection system (63) is studied.

5.8.1 Equivalence transformations

The first stage of group classification requires determining a group of equivalence transformations of equations (63). An equivalence transformation [35] is a nondegenerate change of the dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. It allows using the simplest representation of the given equations.

Since the arbitrary elements are $k = k(t)$, $l = l(t)$, then for calculating group of equivalence transformations we have to append the equations

$$\begin{gathered} k_z = 0, k_f = 0, k_\alpha = 0, \\ l_z = 0, l_f = 0, l_\alpha = 0 \end{gathered}$$

to equations (63). All coefficients of the infinitesimal generator of the equivalence group

$$X^e = \zeta^t\partial_t + \zeta^z\partial_z + \zeta^f\partial_f + \zeta^\alpha\partial_\alpha + \zeta^k\partial_k + \zeta^l\partial_l$$

are dependent on all independent, dependent variables and arbitrary elements

$$t, z, f, \alpha, k, l.$$

Our calculations show that the group of equivalence transformations of equations (63) corresponds to the Lie algebra with generators

$$\begin{gathered} X_1^e = \partial_t, \ X_2^e = 2\xi(t)\partial_z + \xi'(t)\partial_f, \\ X_3^e = -2t\partial_t - z\partial_z + f\partial_f + 2\alpha\partial_\alpha + 4k\partial_k + 4l\partial_l. \end{gathered}$$

5.8.2 Admitted group

Finding an admitted group consists of seeking solutions of determining equations [35]. We are looking for the generator

$$X = \zeta^t\partial_t + \zeta^z\partial_z + \zeta^f\partial_f + \zeta^\alpha\partial_\alpha$$

with the coefficients depending on t, z, f, α. Calculations lead to the following result.

The equations that determine the extensions are

$$c_1(tk' + 2k) + c_2k' = 0, \ c_1(tl' + 2l) + c_2l' = 0,$$

where c_1 and c_2 are constant. The analysis of these equations is similar to the analysis of the group classification of the gas dynamics equations [35]. Let us consider the vectors $\mathbf{v}_1(t) = (tk' + 2k, k')$ and $\mathbf{v}_2(t) = (tl' + 2l, l')$. If they generate a two-dimensional space (where t is changed), then $c_1 = 0$, $c_2 = 0$. This corresponds to the kernel of the fundamental Lie algebra that is made up of the generators

$$X_1 = 2\xi(t)\partial_z + \xi'(t)\partial_f.$$

An extension of the kernel of the main Lie algebra occurs by specializing the functions $k = k(t)$, $l = l(t)$.

Let the vectors $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$ generate a one-dimensional space

$$\mathbf{v}_1(t) = s_1(k_1, k_2), \ \mathbf{v}_2(t) = s_2(k_1, k_2),$$

with some scalars $s_1 = s_1(t), s_2 = s_2(t)$. Note that in this case $s_1^2 + s_2^2 \neq 0$ and $k_1^2 + k_2^2 \neq 0$.

If $k_2 = 0$, then $k(t)$, $l(t)$ are constants and $k \neq l$ (otherwise the space is zero-dimensional). Hence, $c_1 = 0$ and the kernel is extended by the generator

$$X_2 = \partial_t.$$

If $k_2 \neq 0$, then

$$(t - k_2^{-1}k_1)k' + 2k = 0, \ (t - k_2^{-1}k_1)l' + 2l = 0.$$

By virtue of an equivalent transformation (shift with respect to t), one can without loss of generality assume that $k_1 = 0$. The general solution of the last equations is

$$k = q_1t^{-2}, \ l = q_2t^{-2}, \ (q_1^2 + q_2^2 \neq 0).$$

In this case $c_2 = 0$ and the extension of the kernel is

$$X_3 = 2t\partial_t + z\partial_z - f\partial_f - 2\alpha\partial_\alpha.$$

Assume that the vectors $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$ generate a zero-dimensional space. This gives that $k(t) = l(t) = const$. If this constant is not equal to zero, then the kernel is extended by the generator $X_2 = \partial_t$. If $k(t) = l(t) = 0$, the kernel is extended by the generators X_2, X_3.

The result of the group classification is given in the following table

	functions	extension
1.	$k = q_1t^{-2}$, $l = q_2t^{-2}$	X_3
2.	$k = const$, $l = const$	X_2
3.	$k = l = 0$	X_2, X_3

Remark. A detailed analysis of the invariant solutions of the case $k = l = 0$ has been done in [33].

5.8.3 Group stratification and invariant solutions

The group admitted by equations (63) is infinite-dimensional. The classification of an infinite-dimensional group is more difficult than that of a finite-dimensional group. This obstacle can be overcome by studying the group stratification of an infinite-dimensional group [35]. Group stratification allows splitting the initial system into automorphic and resolving systems. Any solution of the automorphic system is obtained from one fixed solution by a transformation belonging to the group.

The infinite-dimensional group with the operator X_1 has the prolonged operator

$$X_1 = 2\xi(t)\partial_z + \xi'(t)(\partial_f - 2f_z\partial_{f_t} - 2\alpha_z\partial_{\alpha_t} - 2\beta_z\partial_{\beta_t}) + \xi''(t)\partial_{f_t},$$

where $\beta = f_z$. The universal invariant of the first order of the operators, which are obtained as coefficients of ξ, ξ', ξ'' is

$$J = (t, \beta, \alpha, \alpha_z, \beta_z, \beta_t + 2f\beta_z, \alpha_t + 2f\alpha_z).$$

Hence, the automorphic system AG of rank 2 can be written in the form

$$\alpha = \alpha(t,\beta),\ \alpha_z = \varphi(t,\beta),\ \beta_z = \gamma(t,\beta),\ \beta_t + 2f\beta_z = \varsigma_1(t,\beta),\ \alpha_t + 2f\alpha_z = \varsigma_2(t,\beta), \tag{64}$$

where $\alpha(t,\beta)$, $\varphi(t,\beta)$, $\gamma(t,\beta)$, $\varsigma_1(t,\beta)$ and $\varsigma_2(t,\beta)$ are unknown functions. The compatibility conditions for the last system and the initial system (63) are

$$\varphi = \gamma\alpha_\beta,\ \varsigma_1 = \gamma\gamma_\beta + \alpha^2 + \beta^2, \tag{65}$$

$$\begin{gathered}\alpha_t + (\alpha^2+\beta^2-k-l)\alpha_\beta - \gamma^2\alpha_{\beta\beta} - 2\alpha\beta - k + l = 0,\\ \gamma_t + (\alpha^2+\beta^2-k-l)\gamma_\beta - \gamma^2\gamma_{\beta\beta} - 2\alpha\gamma = 0.\end{gathered} \tag{66}$$

Thus, the group stratification of system (63) with respect to the infinite-dimensional group with the operator X_1 is the union of the automorphic system (64) with the functions (65) and the resolving system, which consists of equations (66).

The group of equivalence transformations of equations (66) corresponds to the Lie algebra with generators

$$Y_1^e = \partial_t,\ Y_2^e = -2t\partial_t + 2\beta\partial_\beta + 2\alpha\partial_\alpha + 3\gamma\partial_\gamma + 4k\partial_k + 4l\partial_l.$$

The kernel of the admitted group is empty. The group classification with respect to the arbitrary elements $k = k(t)$ and $l = l(t)$ is summarized in the following table

	functions	extension
1.	$k = q_1t^{-2},\ l = q_2t^{-2}$	Y_2
2.	$k = const,\ l = const$	Y_1
3.	$k = l = 0$	Y_1, Y_2

where

$$Y_1 = \partial_t,\ Y_2 = 2t\partial_t - 2\beta\partial_\beta - 2\alpha\partial_\alpha - 3\gamma\partial_\gamma.$$

System (64), (66) is equivalent to the initial system (63) provided that $f_{zz} \neq 0$. Let us consider the degenerate case $f_{zz} = 0$. In this case the function $f = f(t,z)$ has the representation

$f = zq(t) + q_1(t)$, where the functions $q = q(t)$ and $q_1 = q_1(t)$ are arbitrary. After substituting this representation into system (63) one obtains that the function α depends only on t, and

$$(q - \alpha)' - (q - \alpha)^2 = -2k,$$
$$(q + \alpha)' - (q - \alpha)^2 = -2l.$$

One can consider these equations either as equations for the functions $\alpha = \alpha(t)$ and $q = q(t)$ with known functions $k = k(t)$ and $l = l(t)$, or the functions $\alpha = \alpha(t)$ and $q = q(t)$ can be considered as arbitrary functions, and the functions $k = k(t)$ and $l = l(t)$ are defined by these equations.

Let us consider invariant solutions of the resolving system with $f_{zz} \neq 0$ (or $\gamma \neq 0$). Because the case $k = l = 0$ has been studied in [33], then we only need to consider the two cases: a) $k = const$, $l = const$ $(k^2 + l^2 \neq 0)$; b) $k = q_1 t^{-2}$, $l = q_2 t^{-2}$, $(q_2^2 - q_1^2 \neq 0)$.

The case[21] $k = const$, $l = const$. The admitted algebra of the resolving system consists of the generator $Y_1 = \partial_t$. An invariant solution has the representation

$$\alpha = \alpha(\beta), \quad \gamma = \gamma(\beta), \tag{67}$$

where these functions have to satisfy the equations

$$(\beta^2 + \alpha^2 - k - l)\alpha' - \gamma^2\alpha'' - 2\alpha\beta - k - l = 0,$$
$$(\beta^2 + \alpha^2 - k - l)\gamma' - \gamma^2\gamma'' - 2\alpha\gamma\alpha' = 0.$$

Note that $\gamma \neq 0$, because the case $\gamma = 0$ corresponds to $f_{zz} = 0$. In order to find a solution of the initial system (63) one has to solve the automorphic system. One of the equations of the automorphic system is $\beta_z = \gamma(\beta)$. By virtue of $\gamma \neq 0$ and $\beta = f_z$, the function $f = f(t, z)$ has the representation $f = H(z + q(t)) + s(t)$ with arbitrary functions $q = q(t)$ and $s = s(t)$. Hence, the solution of system (63), which corresponds to the invariant solution (67) has the representation

$$\alpha = \alpha(z + q(t)), \quad f = H(z + q(t)) + s(t). \tag{68}$$

After substitution of this representation into system (63) one has

$$(2H + q' + 2s)H'' - H''' - (H')^2 + k + l = \alpha^2,$$
$$(2H + q' + 2s)\alpha' - \alpha'' - 2\alpha H' - k + l = 0.$$

From the first equation (by considering $\hat{z} = z + q(t)$ and $\hat{t} = t$ as new independent variables and differentiating the first equation with respect to $\hat{t}$) one can obtain $H''(q'' + 2s') = 0$. Because $f_{zz} = H'' \neq 0$, then $q' + 2s = s_0 = const$ and the last system becomes

$$(H' - \alpha)'' - (2H + s_0)(H' - \alpha)' + (H' - \alpha)^2 = 2k,$$
$$(H' + \alpha)'' - (2H + s_0)(H' + \alpha)' + (H' + \alpha)^2 = 2l.$$

The case $k = t^{-2}q_1$, $l = t^{-2}q_1$ (the case $q_1^2 - q_2^2 = 0$ is included). The admitted group of the resolving system consists of the generator $Y_2 = 2t\partial_t - 2\beta\partial_\beta - 2\alpha\partial_\alpha - 3\gamma\partial_\gamma$. An invariant solution has the representation

$$\alpha = t^{-1}\Lambda(t\beta), \quad \gamma = t^{-3/2}\Gamma(t\beta). \tag{69}$$

[21] Further study is also valid for $k = l = 0$.

By the same way as in the previous case one can obtain that the solution of system (63), which corresponds to the invariant solution (69) has the representation

$$\alpha = t^{-1}\Lambda(\xi),\ f = t^{-1/2}H(\xi) + s(t), \tag{70}$$

where $\xi = t^{-1/2}(z + q(t))$ and $q = q(t)$ is an arbitrary function. After substitution of this representation into system (63) one has that $t^{-1/2}(q' + 2s) = s_0 = const$ and the function $\Lambda(\xi), H(\xi)$ must satisfy the equations

$$\begin{gathered}(H' - \Lambda)'' + (\tfrac{\xi}{2} - 2H - s_0)(H' - \Lambda)' + (H' - \Lambda)^2 + (H' - \Lambda) = 2q_1,\\ (H' + \Lambda)'' + (\tfrac{\xi}{2} - 2H - s_0)(H' + \Lambda)' + (H' + \Lambda)^2 + (H' + \Lambda) = 2q_2.\end{gathered}$$

5.9 Group classification of system (37)

System (37) is split into two parts: the system of the four equations

$$\begin{gathered}Lf_z + k + l = -c\gamma + \alpha^2,\ L\alpha = \alpha f_z + k - l,\\ L\gamma = f_z\gamma,\ Lc = f_z c\end{gathered} \tag{71}$$

is closed and can be studied independently; the two equations

$$L\lambda = \lambda\alpha - b\gamma,\ Lb = \lambda c - \alpha b \tag{72}$$

are for determining the functions $\lambda(z,t)$ and $b(z,t)$.

Calculations showed that the group of equivalence transformations of equations (71) corresponds to the Lie algebra with the generators

$$\begin{gathered}X_1^e = \partial_t,\ X_2^e = 2\xi(t)\partial_z + \xi'(t)\partial_f,\ X_3^e = \gamma\partial_\gamma - c\partial_c\\ X_4^e = -2t\partial_t - z\partial_z + f\partial_f + 2\alpha\partial_\alpha + 2\gamma\partial_\gamma + 2c\partial_c + 4k\partial_k + 4l\partial_l.\end{gathered}$$

The equations that determine the admitted Lie group are

$$c_1(tk' + 2k) + c_2k' = 0,\ c_1(tl' + 2l) + c_2l' = 0, c_3(k - l) = 0,\ c_4(k - l) = 0,$$

where c_1, c_2, c_3 and c_4 are constants. The same analysis as in the previous case gives that the kernel of the fundamental Lie algebra is made up of the generator

$$X_1 = 2\xi(t)\partial_z + \xi'(t)\partial_f,\ X_2 = \gamma\partial_\gamma - c\partial_c.$$

An extension of the kernel of the main Lie algebra occurs by specializing the functions $k(t)$ and $l(t)$:

	functions	extension
	$k \neq l$	
1.	$k = q_1t^{-2},\ l = q_2t^{-2}\ (q_1 \neq q_2)$	X_5
2.	$k = const,\ l = const$	X_6
	$k = l$	
3.		X_3, X_4
4.	$k = l = qt^{-2}$	X_3, X_4, X_5
5.	$k = l = const \neq 0$	X_3, X_4, X_6
6.	$k = l = 0$	X_3, X_4, X_5, X_6

where

$$X_3 = \gamma\partial_\alpha + 2\alpha\partial_c,\ X_4 = c\partial_\alpha + 2\alpha\partial_\gamma,\ X_5 = 2t\partial_t + z\partial_z - f\partial_f - 2\alpha\partial_\alpha - 4\gamma\partial_\gamma,\ X_6 = \partial_t.$$

5.9.1 Group stratification and invariant solutions

The group admitted by equations (71) is infinite-dimensional. The infinite-dimensional group with the operator X_1 has the prolonged operator

$$X_1 = 2\xi(t)\partial_z + \xi'(t)(\partial_f - 2f_z\partial_{f_t} - 2\alpha_z\partial_{\alpha_t} - 2\beta_z\partial_{\beta_t} - 2\gamma_z\partial_{\gamma_t} - 2c_z\partial_{c_t}) + \xi''(t)\partial_{f_t}.$$

where $\beta = f_z$. The universal invariant of first order is

$$J = (t, \beta, \alpha, \alpha_z, \beta_z, \gamma_z, c_z, \beta_t + 2f\beta_z, \alpha_t + 2f\alpha_z, \gamma_t + 2f\gamma_z, c_t + 2fc_z).$$

Hence, the automorphic system AG of rank 2 can be written in the form

$$\begin{gathered}\alpha = \alpha(t,\beta), \gamma = \gamma(t,\beta), c = c(t,\beta),\ \alpha_z = \varphi_1(t,\beta),\ \beta_z = \varphi_2(t,\beta),\ \gamma_z = \varphi_3(t,\beta),\ c_z = \varphi_4(t,\beta),\\ \beta_t + 2f\beta_z = \varphi_5(t,\beta),\ \alpha_t + 2f\alpha_z = \varphi_6(t,\beta),\ \gamma_t + 2f\gamma_z = \varphi_7(t,\beta),\ c_t + 2fc_z = \varphi_8(t,\beta),\end{gathered} \tag{73}$$

where $\alpha(t,\beta)$, $\gamma(t,\beta)$, $c(t,\beta)$, $\varphi_i(t,\beta)$ $(i = 1, 2, ..., 8)$ are unknown functions. The compatibility conditions for the last system and the initial system (71) are

$$\begin{gathered}\varphi_1 = \varphi_2\alpha_\beta,\ \varphi_3 = \varphi_2\gamma_\beta,\ \varphi_4 = \varphi_2 c_\beta,\\ \varphi_5 = \varphi_2\varphi_{2\beta} + \beta^2 + \alpha^2 - c\gamma - k - l,\\ \varphi_6 = \alpha_t + \alpha_\beta\varphi_5,\ \varphi_7 = \gamma_t + \gamma_\beta\varphi_5,\ \varphi_8 = c_t + c_\beta\varphi_5,\end{gathered} \tag{74}$$

$$\begin{gathered}\alpha_t + (\alpha^2 + \beta^2 - c\gamma - k - l)\alpha_\beta - \varphi_2^2\alpha_{\beta\beta} - 2\alpha\beta - k + l = 0,\\ \varphi_{2t} + (\alpha^2 + \beta^2 - c\gamma - k - l)\varphi_{2\beta} - \varphi_2^2\varphi_{2\beta\beta} - 2\alpha\varphi_2\alpha_\beta + \gamma\varphi_2 c_\beta + c\varphi_2\gamma_\beta = 0,\\ \gamma_t + (\alpha^2 + \beta^2 - c\gamma - k - l)\gamma_\beta - \varphi_2^2\gamma_{\beta\beta} - 2\gamma\beta = 0,\\ c_t + (\alpha^2 + \beta^2 - c\gamma - k - l)c_\beta - \varphi_2^2 c_{\beta\beta} - 2c\beta = 0.\end{gathered} \tag{75}$$

Thus, the group stratification of system (71) with respect to the infinite-dimensional group with the operator X_1 is the union of the automorphic system (73) with the functions (65) and the resolving system, which consists of equations (75).

The group of equivalence transformations of equations (75) corresponds to the Lie algebra with generators

$$\begin{gathered}Y_1^e = \partial_t,\ Y_2^e = -2t\partial_t + 2\beta\partial_\beta + 2\alpha\partial_\alpha + 2\gamma\partial_\gamma + 2c\partial_c + 3\varphi_2\partial_{\varphi_2} + 4k\partial_k + 4l\partial_l,\\ Y_3^e = \gamma\partial_\gamma - c\partial_c.\end{gathered}$$

The kernel of the admitted group is one-dimensional and consists of the group, corresponding to the generator

$$Y_1 = \gamma\partial_\gamma - c\partial_c.$$

The group classification with respect to the arbitrary elements $k = k(t)$ and $l = l(t)$ is summarized in the following table,

	functions	extension
	$k \neq l$	
1.	$k = q_1 t^{-2},\ l = q_2 t^{-2}\ (q_1 \neq q_2)$	Y_2
2.	$k = const,\ l = const$	Y_3
	$k = l$	
3.		Y_3, Y_4
4.	$k = l = qt^{-2}$	Y_1, Y_3, Y_4
5.	$k = l = const \neq 0$	Y_2, Y_3, Y_4
6.	$k = l = 0$	Y_1, Y_2, Y_3, Y_4

where

$$Y_2 = -2t\partial_t + 2\beta\partial_\beta + 2\alpha\partial_\alpha + 3\varphi_2\partial_{\varphi_2} + 2\gamma\partial_\gamma + 2c\partial_c,$$
$$Y_3 = \partial_t,\ Y_4 = \gamma\partial_\alpha + 2\alpha\partial_c,\ Y_5 = c\partial_\alpha + 2\alpha\partial_\gamma.$$

5.9.2 Conclusion of the second part of the research

In this article we have systematically investigated the class of partially invariant solutions of the Navier-Stokes equations, where the Monge-Ampere equation (42) is hyperbolic ($Lf_z+k+l \geq 0$). It was shown that this class of solutions is a particular case of a solution either of system (37) or system (39). Note that the representation (34) is very rich and includes some solutions that were studied earlier. The presence of two arbitrary functions $k(t)$ and $l(t)$ gives additional possibilities for satisfying boundary conditions. The problem of describing all solutions of the given representation (34) where the Monge-Ampere equation (42) is elliptic ($Lf_z + k + l < 0$) still remains, although there are examples of solutions of such type of the Navier-Stokes equations(constructed here and known before).

In this paper the group classifications of systems (39) and (37) was discussed. These systems have infinite-dimensional admitted groups. Infinite-dimensionality is an obstacle for classification of such groups. To overcome this difficulties, group stratification of these groups was done. Group stratification allows splitting the initial system into automorphic and resolving systems. Any solution of the automorphic system is obtained from one fixed solution by a transformation belonging to the group. Therefore the problem of constructing solutions is reduced to finding solutions of the resolving systems. Group classification of resolving systems was done. The admitted groups are finite-dimensional. All invariant solutions of system (39) were presented.

Note that we did not present here a comprehensive study of invariant solutions of the group admitted by (39). This study is a subject for the construction of new solutions of the Navier-Stokes equations.

6 Discussion

In this section we discuss two problems that are related with topics studied in the project. These problems have not been solved.

The first problem is. One of the main problems after obtaining an admitted group is a constuction of invariant or partially invariant solutions. In order to construct essentially different solutions one needs to classify the set of subgroups of the admitted group. The Lie group admitted by the Navier-Stokes equations is infinite. Up to now there is no theory for classifying infinite Lie groups. For some infinite groups one can apply group splitting: in the project we used this method. But for the group admitted by the Navier-Stokes equations this method was not applied: even for simple equations this method is very cumbersome. For the Navier-Stokes equations there is only one article devoted to the classification of infinite subgroups of the Lie group admitted by the Navier-Stokes equations. For finite subgroups there is no any study. For understanding the structure of the finite subgroups admitted by the Navier-Stokes equations one needs to accumulate more results about finite subgroups. In the project we have studied two finite subgroups of the admitted by the Navier-Stokes equations group. There is still big interest to the study more examples of finite subgroups.

Another problem related with the second part of our reseach is the problem of solving the Monge-Ampere equation. In our study this equation was obtained as intermidiate result. If the Monge-Ampere has a hyperbolic type, then it has a first integral. This property helps to construct the general solution of the partially invariant solution for the studied four-parameter subgroup. In the project we considered this case. If the Monge-Ampere equation is elliptic, then there are some methods, by using integral or hodograph-like transformations. The integral transformation is known for a long time, but there is no way for applying it to our problem. The second method (hodograph-like) found recently allows transforming the elliptic Monge-Ampere equation to the Laplace equation.

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7 Obtained output

1. Article accepted in the International journal "Nonlinear Dynamics".

A.HEMATULIN and S.V.MELESHKO **Rotationally Invariant and Partially Invariant Flows of a Viscous Incompressible Fluid and a Viscous Gas** (the manuscript is presented in Appendix).

2. Article submitted to the International journal "European Journal of Applied Mathematics".

S.V.MELESHKO **A particular class of partially invariant solutions of the Navier–Stokes equations** (the manuscript is presented in Appendix).

3. Article submitted to the Proceedings of the conference "Progress in Mathematics".

A.P.CHAIYASENA **Differential Constraints Through the Wave and Boussinesq Equations** (the manuscript is presented in Appendix).

Presentations:

1. International Workshop and Conference on Analysis and Applications (Chiangmai, 15-19 May 2000).

Hematulin A. and Meleshko S.V. **Singular Vortex of the Navier-Stokes Equations.**

2. The 5-th Conference in Mathematics, Department of Mathematics, Institute of Science, Khon Kaen University (Khon Kaen, 2-3 November 2000);

Hematulin A. and Meleshko S.V. **One class of invariant solutions and partially invariant solutions of Viscous gas Dynamics Equations.**

3. The 1-st Conference on Mathematics, Department of Mathematics, Rachapart College (Mahasarakam, 4-6 December 2000).

Hematulin A. **Application of group analysis to the Navier-Stokes equations**

4. The conference "Progress in Mathematics", Department of Mathematics, Mahidol University, 12-13 December, 2000.

A.P.Chaiyasena **Differential Constraints Through the Wave and Boussinesq Equations.**

8 Use

8.1 Public use

Partially invariant solutions of the Navier-Stokes equations are special interests of the academic School of Professor V.V.Pukhnachov (Institute of Hydrodynamics, Novosibirsk, Russia). The research continued created links with this School.

8.2 Academic use

New Ph.D. thesis in frame of this research was defended:
A.HEMATULIN
Invariant and partially invariant solutions of the Navier-Stokes equations related with the group of rotations. School of Mathematics, Institute of Science, Suranaree University of Technology. August, 2001.