



สำนักงานกองทุนสนับสนุนการวิจัย
THE THAILAND RESEARCH FUND

รายงานวิจัยฉบับสมบูรณ์

Invariant and Partially Invariant Solutions
of the Navier-Stokes Equations

โดย Prof.Dr.Sergey V.Meleshko และคณะ

กุมภาพันธ์ 2545

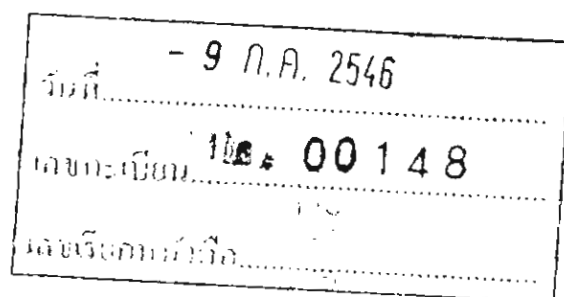


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รายงานวิจัยฉบับสมบูรณ์

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of the Navier-Stokes Equations

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**Invariant and partially invariant solutions
of the Navier-Stokes equations**

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บทคัดย่อ

โครงการนี้ได้ศึกษาการประยุกต์การวิเคราะห์เชิงกลุ่มกับสมการนาเวียร์-สโตกส์ ซึ่งมีบทบาทหลักในการวิจัยในสาขาคณิตศาสตร์ประยุกต์ ฟิสิกส์ และ วิศวกรรมศาสตร์

ปัญหาแรกที่ได้ศึกษาในโครงการนี้ คือความมีอยู่ของผลเฉลยที่เป็นประเทวอร์เทกซ์พิเศษของสมการนาเวียร์-สโตกส์และสมการพลศาสตร์ก๊าซหนืด ซึ่ง L.V. Ovsyannikov ได้เสนอผลเฉลยประเภทนี้ สำหรับพลศาสตร์ก๊าซและของเหลวหนืด ขอให้สังเกตว่าผลเฉลยนี้ยังเป็นบางส่วนโดยเทียบกับกลุ่มการหมุน $O(3)$ การวิเคราะห์ที่กระทำไปได้พิสูจน์ว่าผลเฉลยยังเป็นบางส่วนสำหรับสมการทั้งสองประเภท (สมการนาเวียร์-สโตกส์และสมการพลศาสตร์ก๊าซหนืดสมบูรณ์) เป็นผลเฉลยที่มีความสมมาตรเชิงทรงกลม ซึ่งต่างกับพลศาสตร์ของก๊าซไร้ความหนืดและของเหลวอุดมคติ การจำแนกเชิงกลุ่มของสมการพลศาสตร์ก๊าซหนืดสมบูรณ์ ซึ่งมีความสมมาตรเชิงทรงกลมได้กระทำเสร็จสิ้นแล้ว เพื่อที่จะอำนวยความสะดวกในการพิจารณาผลเฉลยนี้เป็นบางส่วนที่เกี่ยวข้องกับกลุ่มการหมุน $O(3)$ ได้สมบูรณ์ยิ่งขึ้น

การวิจัยอีกส่วนหนึ่งนั้น เกี่ยวข้องกับผลเฉลยอื่นๆที่เฉพาะทาง ซึ่งได้มีฐานของพีชคณิตย่อยสี่มิติหนึ่ง คือ H_4 ซึ่งมีตัวก่อกำเนิดดังต่อไปนี้

$$X_1 = \phi_1 \partial_x + \phi'_1 \partial_u - x \phi''_1 \partial_p, \quad X_2 = \phi_2 \partial_x + \phi'_2 \partial_u - x \phi''_2 \partial_p,$$

$$Y_1 = \psi_1 \partial_y + \psi'_1 \partial_v - y \psi''_1 \partial_p, \quad Y_2 = \psi_2 \partial_y + \psi'_2 \partial_v - y \psi''_2 \partial_p.$$

โครงการได้ศึกษากรณีที่สมการ Monge-Ampere นั้น ไฮเพอร์โบลิก คือ

$$Lf_x + k + l \geq 0$$

ซึ่งได้แสดงว่า ผลเฉลยอื่นๆนี้เป็นกรณีเฉพาะของผลเฉลยที่มีรูปแบบความเร็วเป็นเชิงเส้น โดยเทียบกับตัวแปรปริภูมิหนึ่งหรือสองตัวแปร

คำหลัก : ผลเฉลยอื่นๆและอื่นๆบางส่วน การจำแนกเชิงกลุ่ม การแบ่งชั้นเชิงกลุ่ม
สมการนาเวียร์-สโตกส์ และสมการก๊าซหนืด

Abstract

The project is devoted to the study two applications of group analysis to the Navier-Stokes equations. The Navier-Stokes equations play a central role in much of the research within applied mathematics, physics and engineering.

The first problem that we study in the project is the existence of solutions of special vortex type for the Navier-Stokes equations and viscous gas dynamics equations. This type of solutions for the inviscid gas and fluid dynamics equations was introduced by L.V.Ovsiannikov [36]. Note that this solution is partially invariant with respect to group of rotations $O(3)$. The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. For the completeness of consideration of partially invariant solutions that are connected with the group of rotations $O(3)$ the group classification of the full viscous gas dynamics equations with spherical symmetry has been done.

Another part of the research is devoted to a particular class of partially invariant solutions of the Navier-Stokes equations. This class of solutions is constructed on the base of the four-dimensional subalgebra H^4 with the generators

$$X_1 = \phi_1 \partial_x + \phi'_1 \partial_u - x \phi''_1 \partial_p, \quad X_2 = \phi_2 \partial_x + \phi'_2 \partial_u - x \phi''_2 \partial_p,$$

$$Y_1 = \psi_1 \partial_y + \psi'_1 \partial_v - y \psi''_1 \partial_p, \quad Y_2 = \psi_2 \partial_y + \psi'_2 \partial_v - y \psi''_2 \partial_p.$$

We systematically investigate the case, where the Monge-Ampere equation is hyperbolic ($Lf_z + k + l \geq 0$). It is shown that this class of solutions is a particular case of the solutions with linear profile of velocity with respect to one or two space variables.

Key words: Invariant and partially invariant solutions, group classification, group stratification, Navier-Stokes and viscous gas equations.

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3 Introduction

The mathematical models of many real world phenomena are formulated in the form of differential equations. One of the methods for studying the properties of differential equations is group analysis. Differential equations usually contain parameters or functions that are determined experimentally and hence are not strictly fixed. Group analysis not only helps to construct exact solutions, but also to classify the differential equations with respect to these arbitrary elements.

Many of the invariant solutions of the Navier-Stokes equations have been known for a long time; however their systematic analysis became possible only with the development¹ of the modern methods for the group analysis of differential equations [35]. The first group classification of the Navier-Stokes equations in the three-dimensional case was done in [13]. It was shown that the Lie group admitted by the Navier-Stokes equations is infinite-dimensional. There is still no classification of this group. Several papers [6–11]² are devoted to invariant solutions of the Navier-Stokes equations. Partially invariant solutions of the Navier-Stokes equations have been less studied [41, 33]. At the same time there has been progress in studying such classes of solutions of inviscid gas dynamics equations [35, 46, 29]. Recently, L.V.Ovsiannikov [36] found one class of partially invariant solutions, called a special vortex. This solution is based on the group of rotations $O(3)$. An ideal fluid and an inviscid gas have the same class of solutions. Therefore, it is natural to investigate the existence of special vortex type solutions for the Navier-Stokes equations and viscous gas dynamics equations.

As is well-known, the main difficulty in the study of partially invariant solutions is the analysis of the compatibility [15, 25] of the appearing overdetermined systems. The analysis of compatibility can be reduced to the consecutive performance of algebraic operations of symbolic nature. These operations are connected with a prolongation of the system, substitution of composite expressions (transition onto manifold), and finding ranks of matrices. Typically, the compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations. Here we used the system REDUCE [21].

Another part of our study is devoted to the group classification of spherically symmetric viscous gas dynamics equations. The group classification problem consists of searching for admitted groups of transformations admitted by the system for all arbitrary elements and all specifications of arbitrary elements. By special choice of the arbitrary elements one can extend the admitted group.

After finding the admitted group one can try to construct exact solutions: every subgroup of the admitted group can be a source of invariant or partially invariant solutions. There is an infinite number of subgroups³, even in cases where the admitted groups are finite-dimensional. But if two subgroups are similar, i.e., they are connected with each other by a symmetry transformation, then their corresponding invariant solutions are connected with each other by the same transformation. Since the set of subgroups can be divided into classes of similar subgroups, therefore, it is sufficient to find only one representative solution from each similar class of subgroups. A set of representatives of equivalent subgroup classes is called an optimal

¹A historical review of a group analysis development can be found in [22]. Many results of the group analysis are collected in [23]

²Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in [41, 16, 17, 28].

³Because there is a one-to-one correspondence between groups and Lie algebras one can study the Lie algebra of the admitted group.

system of subgroups. In this manuscript we give representations of all invariant solutions with respect to subgroups of two-dimensional admitted groups of spherically symmetric viscous gas dynamics equations.

We should also note here that, as for the Navier-Stokes equations, many of the invariant solutions of the viscous gas dynamics equations have been obtained by other methods [21–29]. The group classification of the viscous gas equations (in case when the first λ and the second μ coefficients of viscosity are related by the equation $\lambda = -2\mu/3$) was done in [8]. For some models of viscous gas dynamics equations, group analysis was used in [32, 7]. There also exist other similar approaches for constructing exact solutions of the Navier–Stokes equations. We note here two of them: nonclassical symmetry reductions [27, 28] and linear profile of velocity [45].

An unsteady motion of incompressible viscous fluid is governed by the Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

where $\mathbf{u} = (u, v, w)$ is the velocity field, p is the fluid pressure, ∇ is the gradient operator in the three-dimensional space $\mathbf{x} = (x, y, z)$ and Δ is the Laplacian. The Navier-Stokes equations contain complete information about the structure of flows under usual temperature and pressure. Despite progress in numerical methods and techniques, there is considerable interest in finding exact solutions of the Navier-Stokes equations. Each exact solution has value, first, as the exact description of the real process in the framework of a given model; secondly, as a model to compare various numerical methods; and thirdly, as theoretical tool to improve the models used.

One method of constructing exact solutions is group analysis [35]⁴. This method is based on the symmetries of the given equations. Note that many of the invariant solutions of the Navier-Stokes equations have been known for a long time: these solutions were obtained by assuming a form of the representation of the solution. Group analysis gives a method for obtaining the representation of solution. The first group classification of the Navier-Stokes equations in the three-dimensional case was done in [13]⁵. It was shown that the Lie algebra admitted by the Navier-Stokes equations is infinite-dimensional⁶. For each subalgebra of the admitted algebra one can try to find an invariant or partially invariant solution. Several papers [?, 14, 26, 6, 47, 49, 39]⁷ are devoted to invariant solutions of the Navier-Stokes equations. Another class of solutions that is suggested by group analysis is the class of partially invariant solutions [35, 34]. The theory of partially invariant solutions is still developing [37, 38]. While partially invariant solutions of the Navier–Stokes equations have been less studied [?], there has been substantial progress in studying such classes of solutions of inviscid gas dynamics equations [1, 20–26].

We should note here that there are also other approaches for constructing exact solutions of the Navier–Stokes equations. We note here two of them: nonclassical symmetry reductions and direct methods [28, 27] and linear profile of velocity [45, 50].

⁴A historical review of the development of group analysis can be found in [22]. Many of the results of group analysis are collected in [23].

⁵The first classification of the two-dimensional Navier-Stokes equations was studied in [40].

⁶The classification of infinite-dimensional subalgebras of this algebra was studied in [20]. There is still no full classification of the subalgebras of this algebra.

⁷Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in [?, 16, 17, 28].

4 The method of study

Let us first review the notations and techniques used in group analysis.

Let an l -th order system of differential equations

$$(S) : \quad F^k(x, u, p) = 0, \quad (k = 1, 2, \dots, s)$$

be given. Here $x = (x_i)$, $(i = 1, 2, \dots, n)$ are the independent variables, $u = (u^j)$ ($j = 1, 2, \dots, m$) are the dependent variables, $p = (p_\alpha^k)$ are the derivatives up to l -th order, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multiindex with $|\alpha| \equiv \alpha_1 + \alpha_2 + \dots + \alpha_n \leq l$. In the space J^l , which consists of the elements (x, u, p) the system of differential equations is considered as a regular assigned manifold

$$U_l = \{(x, u, p) \in J^l \mid F^k(x, u, p) = 0, \quad (k = 1, 2, \dots, s)\}. \quad (2)$$

The term "regular" means that the vector function $F(x, u, p)$ satisfies the condition

$$\text{rank} \left(\frac{\partial(F)}{\partial(x, u, p)} \right) = s.$$

4.1 Admitted Lie group of transformations

One of the main objects in group analysis is the local one-parameter Lie group G^1 of the transformations:

$$x'_i = f^{x_i}(x, u; a), \quad u^{j'} = f^{u^j}(x, u; a), \quad (i = 1, 2, \dots, n; \quad j = 1, 2, \dots, m). \quad (3)$$

There is a one-to-one correspondence between such groups G^1 and infinitesimal generators

$$X = \xi^i(x, u) \partial_{x_i} + \zeta^j(x, u) \partial_{u^j},$$

where

$$\xi^i(x, u) = \frac{d f^{x_i}}{d a} \Big|_{a=0}, \quad \zeta^j(x, u) = \frac{d f^{u^j}}{d a} \Big|_{a=0}.$$

The operator

$$X_l = X + \sum_{j, \alpha} \zeta_\alpha^j \partial_{p_\alpha^j}$$

with coefficients

$$\zeta_{\alpha, k}^j = D_k \zeta_\alpha^j - \sum_i p_{\alpha, i}^j D_k \xi^i. \quad (4)$$

is called the l -th prolongation of a generator X . Here, the operators

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j, \alpha} p_{\alpha, k}^j \frac{\partial}{\partial p_\alpha^j}$$

are the operators of total differentiation with respect to x_k , $(k = 1, 2, \dots, n)$. Formulae (4) for the coefficients of the prolonged operator are defined by the following fact. Let a function $u = u_o(x)$ be given. Substituting it into the first part of transformation (3) and using the inverse function theorem one has

$$x = g^x(x', a).$$

The transformed function $u_a(x')$ is given as follows

$$u_a(x') = f^u(g^x(x', a), u_o(g^x(x', a)); a).$$

Hence, the derivatives p' of the transformed function $u_a(x')$ are defined through the derivatives p of the function $u_o(x)$. The local Lie group of transformations in the space J^l

$$x' = f^x(x, u; a), u' = f^u(x, u; a), p' = f^p(x, u, p; a) \quad (5)$$

with the operator X is called the l -th prolongation of the group G^1 (3).

A function $\Phi(x, u, p)$ is called an invariant of the group G^1 if the equality

$$\Phi(x', u', p') = \Phi(x, u, p)$$

holds. If $\Phi_p \neq 0$, then the function Φ is called a differential invariant. The regular assigned manifold (2) is an invariant manifold with respect to the prolonged group (4) if and only if

$$DS : \quad X F^k(x, u, p)|_{U_l} = 0, \quad (k = 1, 2, \dots, s). \quad (6)$$

A local Lie group G^1 of transformations (3) is said to be admitted by the system (S) (we also say that the system (S) admits the group G^1) if the manifold (S) is invariant with respect to the prolonged group. In this case, the generator X of the Lie group G^1 is also called admitted by the system (S). The algorithm for finding a local one-parameter Lie group (3) admitted by the system of differential equations (S) consists of the following four steps.

In the first step, the form of the generator

$$X = \xi^i(x, u)\partial_{x_i} + \zeta^j(x, u)\partial_{u_j},$$

is given, with unknown coefficients $\xi^i(x, u)$, $\zeta^j(x, u)$. In the second step the prolonged operator X_l is applied to every equation of the system (S). In the next step the coefficients of the prolonged operator are substituted by using formulae (4). The equations obtained must be considered on the manifold U_l . As a result we obtain the system of differential equations (6). The system of equations DS is called a system of determining equations and it is an overdetermined system of linear homogeneous differential equations in the unknown coordinates $\xi^i(x, u)$, $\zeta^j(x, u)$. The general solution of the determining equations DS generates a full group GS of the system (S). The feature of the admitted group is that under action of any transformation of this group every solution $u = U(x)$ of the system (S) is transformed into a solution $u = U_a(x)$ of the same system (S). Therefore the admitted group allows constructing new solutions from known solutions. Note that the set of admitted generators generates a Lie algebra, which is called admitted by the system (S).

4.2 Equivalence transformations

Most systems of partially differential equations have arbitrary elements: arbitrary functions or arbitrary constants. These arbitrary elements can be separated into classes with respect to a group of equivalence transformations. An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. These transformations allow us to use the simplest representation of the given equations. Note that the

admitted group depends on specialization of the arbitrary elements. The group classification problem consists in searching for an admitted group of transformations, which is admitted for all arbitrary elements of the system and all specializations of the arbitrary elements. The specialization of the arbitrary elements can extend the admitted group. For the calculation of equivalence transformations we follow the approach developed in [30, 31], which consists of the following.

Suppose, the system of differential equation

$$F^k(x, u, p, \phi) = 0, (k = 1, 2, \dots, s) \quad (7)$$

has arbitrary elements $\phi = (\phi^1, \phi^2, \dots, \phi^t)$, which are functions (or constants) $\phi = \phi(x, u)$. A specific value of the arbitrary elements represents a concrete system of differential equations.

The problem of finding an equivalent transformation consists of constructing a transformation of the space $R^{n+m+t}(x, u, \phi)$ which preserves the equations by only changing their representative $\phi = \phi(x, u)$. For this purpose, we consider the one-parameter group of transformations of the space R^{n+m+t} :

$$x' = f^x(x, u, \phi; a), u' = f^u(x, u, \phi; a), \phi' = f^\phi(x, u, \phi; a). \quad (8)$$

A generator of this group has the form:

$$X^e = \xi^x \partial_x + \zeta^u \partial_u + \zeta^\phi \partial_\phi \quad (9)$$

with the coordinates⁸:

$$\begin{aligned} \xi^i &= \xi^i(x, u, \phi), \zeta^{u^j} = \zeta^{u^j}(x, u, \phi), \zeta^{\phi^k} = \zeta^{\phi^k}(x, u, \phi) \\ (i &= 1, \dots, n; j = 1, \dots, m; k = 1, \dots, t). \end{aligned}$$

We use the main feature of the Lie group that any solution $u_o(x)$ of system (7) with functions $\phi(x, u)$ is transformed by (8) into another solution $u = u_a(x')$ of system (7), but with different (transformed) functions $\phi_a(x, u)$, which are defined in the following way. Solving the relations

$$x' = f^x(x, u, \phi(x, u); a), \quad u' = f^u(x, u, \phi(x, u); a)$$

with respect to (x, u) , we obtain

$$x = g^x(x', u'; a), \quad u = g^u(x', u'; a). \quad (10)$$

Then the transformed function is

$$\phi_a(x', u') = f^\phi(x, u, \phi(x, u); a), \quad (11)$$

where instead of (x, u) we have to substitute their expressions (10). The transformed solution $u_a(x)$ is obtained by solving the relations

$$x' = f^x(x, u_o(x), \phi(x, u_o(x)); a)$$

with respect to (x) :

$$x = \psi^x(x'; a)$$

⁸Later the author discovered that similar assumptions about the coefficients of the operator were used in [48] for one class of ordinary differential equations with one nonessential restriction $\zeta^{\phi^k} = \zeta^{\phi^k}(x, \phi)$.

and substituting into

$$u_a(x') = f^u(x, u_o(x), \phi_a(x, u_o(x)); a). \quad (12)$$

The formulae for the transformations of the partial derivatives p_a and the derivatives of the functions ϕ are obtained by differentiating (11) and (12) with respect to x' and u' .

The method for finding a group of equivalence transformations is similar to the algorithm for finding an admitted group of transformations. The difference only consists of the prolongation of the infinitesimal generator X^e . In agreement with the construction of the functions $u_a(x')$ and $\phi_a(x', u')$, the prolonged operator

$$\bar{X}^e = X^e + \zeta^{u_x} \partial_{u_x} + \zeta^{\phi_x} \partial_{\phi_x} + \zeta^{\phi_u} \partial_{\phi_u} + \dots$$

has the following coordinates

$$\zeta^{u_\lambda} = D_\lambda^e \zeta^u - u_x D_\lambda^e \zeta^x, \quad (\lambda = x_1, x_2, \dots, x_n)$$

with $D_\lambda^e = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda) \partial_\phi$ and

$$\zeta^{\phi_\lambda} = \tilde{D}_\lambda^e \zeta^\phi - \phi_x \tilde{D}_\lambda^e \zeta^x - \phi_u \tilde{D}_\lambda^e \zeta^u, \quad (\lambda = u^1, u^2, \dots, u^m, x_1, x_2, \dots, x_n)$$

with $\tilde{D}_\lambda^e = \partial_\lambda + \phi_\lambda \partial_\phi$.

An equivalence group GS^e of transformations is generated by $G^1(X^e)$.

Remark. In some cases one may have additional requirements for the arbitrary elements. For example, the arbitrary elements ϕ^μ may be supposed to be independent of the independent variables $\frac{\partial \phi^\mu}{\partial x_k} = 0$. When studying the equivalence group such conditions have to be added to the original system of differential equations (7), leading to additional determining equations.

Remark. Note that in case of the Navier-Stokes equations, kinematic viscosity is the arbitrary element and these equations can be transformed to equations (1) by scaling the independent and dependent variables.

4.3 Invariant and partially invariant solutions

For each subgroup of the admitted group GS one can try to find an invariant or partially invariant solution. Let $H \subset GS$ be a group admitted by the system of equations (S) . Assume that $X_1, \dots, X_r$ is a basis of the Lie algebra L^r which corresponds to the group H . An invariant or partially invariant solution with respect to the group H is called an H -solution. The method [35] for constructing H -solutions with respect to the group H requires to find an universal invariant of this group: a set of all functionally independent invariants. For this purpose one needs to solve the overdetermined linear system of differential equations:

$$X_i \phi(x, u) = 0, \quad (i = 1, 2, \dots, r). \quad (13)$$

Because $X_1, \dots, X_r$ generate a Lie algebra, system (13) is complete. Its general solution can be expressed through the $m + n - r_*$ invariants

$$J = (J^1(x, u), J^2(x, u), \dots, J^{m+n-r_*}(x, u)),$$

where r_* is the total rank of the matrix composed of the coefficients of the generators X_i , $(i = 1, 2, \dots, r)$. If the rank of the Jacobi matrix $\frac{\partial(J^1, \dots, J^{m+n-r_*})}{\partial(u_1, \dots, u_m)}$ is equal to q , then without loss

of generality, one can choose the first $q \leq m$ invariants $J^1, \dots, J^q$ such that the rank of the Jacobi matrix $\frac{\partial(J^1, \dots, J^q)}{\partial(u_1, \dots, u_m)}$ is equal to q and the remaining $k = m + n - r_* - q$ invariants $J^{q+1}, J^{q+2}, \dots, J^{m+n-r_*}$ only depend on the independent variables x . H -solutions are characterized by two integers: the rank $\sigma = \delta + n - r_* \geq 0$ and the defect $\delta \geq 0$, thus one uses the notation $H(\sigma, \delta)$ -solution. Rank and defect must satisfy the inequalities

$$k \leq \sigma < n, \max\{r_* - n, m - q, 0\} \leq \delta \leq \min\{r_* - 1, m - 1\}.$$

To construct a representation of $H(\sigma, \delta)$ -solutions one needs to separate the universal invariant into two parts: $J = (\bar{J}, \bar{\bar{J}})$, where $l = m - \delta$ and

$$\bar{J} = (J^1, \dots, J^l), \bar{\bar{J}} = (J^{l+1}, J^{l+2}, \dots, J^{m+n-r_*}).$$

This means that one can choose the number l such that $1 \leq l \leq q \leq m$. The rank and the defect of the $H(\sigma, \delta)$ -solution are $\delta = m - l$, $\sigma = m + n - r_* - l = \delta + n - r_*$. A solution is called invariant if $\delta = 0$, otherwise it is called a partially invariant solution. From the first l invariants $J^1, J^2, \dots, J^l$ one can define the l dependent functions

$$u^i = \phi^i(\bar{J}, u^{l+1}, u^{l+2}, \dots, u^m, x), \quad (i = 1, \dots, l). \quad (14)$$

The functions $u^{l+1}, u^{l+2}, \dots, u^m$ are called superfluous. The representation of the $H(\sigma, \delta)$ -solution is obtained by assuming that the first part of the universal invariant is a function of the second part:

$$\bar{J} = \Psi(\bar{\bar{J}}). \quad (15)$$

and substituting (15) into (14). Thus, the representation of an invariant or partially invariant solution is

$$u^i = \Phi^i(\bar{\bar{J}}, u^{l+1}, u^{l+2}, \dots, u^m, x), \quad (i = 1, \dots, l), \quad (16)$$

where $\Phi^i = \phi^i(\Psi(\bar{\bar{J}}), u^{l+1}, u^{l+2}, \dots, u^m, x)$.

If $\delta \neq 0$, then either $\sigma = k$ or $\sigma > k$. In the first case ($\sigma = k$) the partially invariant solution is called regular, otherwise it is called irregular [37]. The number $\sigma - k$ is called the measure of irregularity.

After constructing the representation of an invariant or partially invariant solution one needs to substitute it into the original system of equations. The system of equations in the functions Ψ^i and superfluous functions thus obtained is called the reduced system. This system is overdetermined and requires analysis of compatibility. Usually the compatibility analysis is easier for invariant solutions than for partially invariant ones.

If H' is a subgroup of H , then it may be possible that a partially invariant $H(\sigma, \delta)$ -solution is a partially invariant $H'(\sigma', \delta')$ -solution. In this case $\delta' \leq \delta$, $\sigma' \geq \sigma$ [35]. A solution is called reducible to a $H'(\sigma', \delta')$ -solution. if there exists $H' \subset H$ such that $\delta' < \delta$, $\sigma' = \sigma$. In particular, a solution is called reducible to an invariant solution if there exists $H' \subset H$ with $\delta' = 0$. Thus, a natural problem is to reduce a partially invariant $H(\sigma, \delta)$ -solution to an invariant $H'(\sigma, 0)$ -solution.

5 Results

For the first problem the analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and

the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. Group classification of the full viscous gas dynamics equations with spherical symmetry shows that the kernel of admitted groups is extended for three types of the state equations. For each class the optimal system of subalgebras is constructed. Universal invariants of all subalgebras are constructed. Representations of all possible invariant solutions is given.

Another part of the research is devoted to a particular class of partially invariant solutions of the Navier–Stokes equations. This class of solutions is constructed on the base of the four-dimensional subalgebra H^4 with the generators

$$\begin{aligned} X_1 &= \phi_1 \partial_x + \phi'_1 \partial_u - x \phi''_1 \partial_p, & X_2 &= \phi_2 \partial_x + \phi'_2 \partial_u - x \phi''_2 \partial_p, \\ Y_1 &= \psi_1 \partial_y + \psi'_1 \partial_v - y \psi''_1 \partial_p, & Y_2 &= \psi_2 \partial_y + \psi'_2 \partial_v - y \psi''_2 \partial_p. \end{aligned}$$

According to the classification [37], a partially invariant solution with minimum defect $\delta = 2$ is a regular partially invariant solution of $H(2, 2)$. In this case a representation of the partially invariant solution is

$$w = 2f(z, t), \quad p = h(z, t) - k(t)x^2 - l(t)y^2, \quad u = u(x, y, z, t), \quad v = v(x, y, z, t),$$

where $k = \phi''_i/(2\phi_i)$, $l = \psi''_i/(2\psi_i)$. After substitution of the representation of solution into the Navier-Stokes equations the overdetermined system of equations is obtained. It is proven that in the process of compatibility study the Monge-Ampere equation is constructed. In the research a systematical investigation of the case, where the Monge-Ampere equation is hyperbolic. It is shown that this class of solutions is a particular case of the solutions with linear profile of velocity with respect to one or two space variables. Examples of solutions with elliptic Monge-Ampere equation are given.

5.1 Full Navier-Stokes Equations

5.1.1 Coordinateless form of the equations

In this manuscript we study unsteady viscous gas dynamics equations. These equations govern a three-dimensional motion of a compressible, thermal conductive, Newtonian viscous gas flow

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \tau \operatorname{div}(\mathbf{P}), \quad \frac{d\tau}{dt} - \tau \operatorname{div}(\mathbf{v}) = 0, \\ \frac{d\varepsilon}{dt} &= \tau \mathbf{P} : \mathbf{D} + \tau \operatorname{div}(\kappa \nabla T). \end{aligned}$$

Here $\tau = 1/\rho$ is a specific volume, ρ is a density, $\mathbf{v}$ is a velocity, $\mathbf{P}$ is a stress tensor, $\mathbf{D} = \frac{1}{2} \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^* \right)$ is a rate-of-strain tensor, ε is an internal energy, T is a temperature, κ is a coefficient of a heat conductivity. The Stokes axioms for a viscous gas give

$$\mathbf{P} = (-p + \lambda \operatorname{div}(\mathbf{v}))\mathbf{I} + 2\mu \mathbf{D},$$

where p is a pressure, λ and μ are the first and the second coefficients of viscosity, respectively. A viscous gas is a two parametric media. As the main thermodynamic variables we choose the pressure p and specific volume τ : the entropy η , the internal energy ε and the temperature T are functions of the pressure and specific volume

$$\eta = \eta(p, \tau), \quad \varepsilon = \varepsilon(p, \tau), \quad T = T(p, \tau).$$

The first and the second thermodynamic laws require for these functions to satisfy the equations

$$\eta_p = \frac{\varepsilon_p}{T}, \quad \eta_\tau = \frac{\varepsilon_\tau + p}{T}, \quad 3\lambda + 2\mu \geq 0, \quad \mu \geq 0, \quad \kappa \geq 0.$$

For the simplicity of classification we study case, which corresponds to an essentially viscous and heat conductive gas

$$\mu \neq 0, \quad \kappa \neq 0.$$

Thus, the studying viscous gas dynamics equations are

$$\frac{d\mathbf{v}}{dt} + \tau \nabla p = \tau ((\lambda + \mu) \nabla (\operatorname{div}(\mathbf{v})) + (\operatorname{div}(\mathbf{v})) \nabla \lambda + \mu \Delta \mathbf{v} + 2D(\nabla \mu)), \quad (17)$$

$$\frac{d\tau}{dt} - \tau \operatorname{div}(\mathbf{v}) = 0,$$

$$\frac{dp}{dt} + A(p, \tau) \operatorname{div}(\mathbf{v}) = B(p, \tau) (\lambda (\operatorname{div}(\mathbf{v}))^2 + 2\mu D : D + (\nabla \kappa)(\nabla T) + \kappa \Delta T)$$

with the functions

$$A = \frac{\tau(\varepsilon_\tau + p)}{\varepsilon_p}, \quad B = \frac{\tau}{\varepsilon_p}.$$

Note that the internal energy and entropy can be expressed through the functions $A = A(p, \tau)$, $B = B(p, \tau)$ by formulae

$$\varepsilon_p = \frac{\tau}{B}, \quad \varepsilon_\tau = \frac{A}{B} - p, \quad \eta_p = \frac{\tau}{BT}, \quad \eta_\tau = \frac{A}{BT}.$$

The conditions $\varepsilon_{p\tau} = \varepsilon_{\tau p}$, $\eta_{p\tau} = \eta_{\tau p}$ lead to the restrictions

$$\begin{aligned} \tau B_\tau + B A_p - A B_p &= B^2 + B, \\ \tau T_\tau &= A T_p - T B. \end{aligned} \quad (18)$$

In the case of an ideal gas (i.e., the gas that obeys the Clapeyron equation $T = R^{-1}p\tau$) there are $B = B(\tau p)$, $A = p(1 + B(\tau p))$ with an arbitrary function $B(\tau p)$. For a polytropic gas $\varepsilon = (\gamma - 1)^{-1}\tau p$ and this one more simplifies the functions A and B : $B = (\gamma - 1)$, $A = \gamma p$. Here R is the gas constant and γ is a polytropic exponent. Note also that the Navier-Stokes equations are obtained from the viscous gas equations by assuming that the second coefficient of viscosity μ and the density ρ (or the specific volume τ) are constants.

Remark. In the case of constant τ and μ system (17) is split on two systems: the Navier-Stokes equations and the energy equation.

5.1.2 Spherical coordinate system

Equations (17) are written in coordinateless form. For applications one needs to write them in some coordinate system. Because our goal is to study solutions of viscous gas equations connected with the group of rotations, then it is convenient to use a spherical coordinate system.

The spherical coordinates (r, θ, φ) and the Cartesian coordinates (x, y, z) of the point $\mathbf{x} \in R^3$ are introduced by the formulae

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta$$

Corresponding physical components of the velocity vector $\mathbf{v}$ in the spherical coordinate system (U, V, W) and (u, v, w) in the Cartesian coordinate system are related by the expressions

$$\begin{aligned} u &= U \sin \theta \cos \varphi + V \cos \theta \cos \varphi - W \sin \varphi, \\ v &= U \sin \theta \sin \varphi + V \cos \theta \sin \varphi - W \cos \varphi, \\ w &= U \sin \theta - V \sin \theta. \end{aligned}$$

Note that the vector (V, W) can be described by its modules H and by the angle ω :

$$V = H \cos \omega, \quad W = H \sin \omega. \quad (19)$$

For the spherical coordinate system the fundamental tensor is diagonal:

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}; \quad (g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix}; \quad |g| = \det(g_{ij}) = r^4 \sin^2 \theta.$$

The Christoffel's symbols

$$\Gamma_{ij}^l = \frac{1}{2} g^{ls} \left(\frac{\partial g_{is}}{\partial K^j} + \frac{\partial g_{js}}{\partial K^i} - \frac{\partial g_{ij}}{\partial K^s} \right)$$

are (we write down only nonvanishing symbols)

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \cot \theta,$$

$$\Gamma_{33}^1 = -r \sin^2 \theta, \quad \Gamma_{33}^2 = -\sin \theta \cos \theta.$$

Here $K^1 = r$; $K^2 = \theta$; $K^3 = \varphi$ and there is a summation with respect to a repeat index.

Tensor components of the vector $\mathbf{v}$ are

$$(v^1, v^2, v^3) = (U, \frac{V}{r}, \frac{W}{r \sin \theta}), \quad (v_1, v_2, v_3) = (U, rV, r \sin \theta W).$$

Coordinates of a gradient of any scalar function F are

$$(\nabla F)_1 = (\nabla F)^1 = \frac{\partial F}{\partial r}, \quad (\nabla F)_2 = \frac{\partial F}{\partial \theta}, \quad (\nabla F)^2 = \frac{1}{r^2} \frac{\partial F}{\partial \theta},$$

$$(\nabla F)_3 = \frac{\partial F}{\partial \varphi}, \quad (\nabla F)^3 = \frac{1}{r^2 \sin^2 \theta} \frac{\partial F}{\partial \varphi}.$$

A matrix of the covariant derivatives is (here i is a number of a row, j is a number of a column)

$$(v^i_j) = \begin{bmatrix} \frac{\partial U}{\partial r} & \frac{\partial U}{\partial \theta} - V & \frac{\partial U}{\partial \varphi} - \sin \theta W \\ \frac{1}{r} \frac{\partial V}{\partial r} & \frac{1}{r} \frac{\partial V}{\partial \theta} + \frac{U}{r} & \frac{1}{r} \frac{\partial V}{\partial \varphi} - \frac{\cos \theta}{r} W \\ \frac{1}{r \sin \theta} \frac{\partial W}{\partial r} & \frac{1}{r \sin \theta} \frac{\partial W}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial W}{\partial \varphi} + \frac{U}{r} + \frac{\cot \theta}{r} V \end{bmatrix}$$

Coordinates of the rate-of-strain tensor $D = \frac{1}{2} \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^* \right)$ are

$$2D^j_i = v^j_{,i} + v^\alpha_{,\beta} g_{i\alpha} g^{j\beta}.$$

Hence,

$$2D : D = v_{,i}^j v_{,j}^i + v_{,j}^i v_{,\beta}^\alpha g_{i\alpha} g^{j\beta}.$$

For the divergency there is

$$\text{div } \mathbf{v} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial K^i} \left(\sqrt{|g|} v^i \right) = \frac{1}{r^2} \frac{\partial(r^2 U)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \sin \theta V}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial W}{\partial \varphi}.$$

The Laplace operator of a scalar function is

$$\begin{aligned} \Delta F &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial K^i} \left(\sqrt{|g|} g^{is} \frac{\partial F}{\partial K^s} \right) = \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \varphi^2}. \end{aligned}$$

The Laplace operator $\Delta \mathbf{v}$ of the vector $\mathbf{v}$ has coordinates

$$(\Delta \mathbf{v})^l = \Delta(v^l) + 2 g^{ij} \Gamma_{is}^l \frac{\partial v^s}{\partial K^j} + g^{ij} \left(\frac{\partial \Gamma_{ip}^l}{\partial K^j} + \Gamma_{ip}^s \Gamma_{js}^l - \Gamma_{ij}^s \Gamma_{ps}^l \right) v^p,$$

which for the spherical coordinate system are

$$\begin{aligned} (\Delta \mathbf{v})^1 &= \Delta(U) - \frac{2}{r} \frac{\partial V}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial W}{\partial \varphi} - \frac{2U}{r^2} - \frac{2 \cot \theta}{r^2} V, \\ (\Delta \mathbf{v})^2 &= \Delta\left(\frac{V}{r}\right) + \frac{2}{r^2} \frac{\partial V}{\partial r} + \frac{2}{r^3} \frac{\partial U}{\partial \theta} - \frac{2 \cot \theta}{r^3 \sin \theta} \frac{\partial W}{\partial \varphi} - \frac{V}{r^3 \sin^2 \theta}, \\ (\Delta \mathbf{v})^3 &= \Delta\left(\frac{W}{r \sin \theta}\right) + \frac{2}{r \sin \theta} \frac{\partial}{\partial r} \left(\frac{W}{r} \right) + \frac{2 \cot \theta}{r^3} \frac{\partial}{\partial \theta} \left(\frac{W}{\sin \theta} \right) + \\ &\quad + \frac{2}{r^3 \sin^2 \theta} \frac{\partial U}{\partial \varphi} + \frac{2 \cot \theta}{r^3 \sin^2 \theta} \frac{\partial V}{\partial \varphi}. \end{aligned}$$

The acceleration vector $\frac{d\mathbf{v}}{dt}$ has the components

$$\begin{aligned} \left(\frac{d\mathbf{v}}{dt} \right)^1 &= \mathcal{D}(U) - \frac{W^2 + V^2}{r}, \quad \left(\frac{d\mathbf{v}}{dt} \right)^2 = \frac{1}{r} \mathcal{D}(V) + \frac{UV - \cot \theta W^2}{r^2}, \\ \left(\frac{d\mathbf{v}}{dt} \right)^3 &= \frac{1}{r \sin \theta} \mathcal{D}(W) + \frac{UW + \cot \theta WV}{r^2 \sin \theta}, \end{aligned}$$

where

$$\mathcal{D}(f) = \frac{\partial f}{\partial t} + U \frac{\partial f}{\partial r} + \frac{V}{r} \frac{\partial f}{\partial \theta} + \frac{W}{r \sin \theta} \frac{\partial f}{\partial \varphi}.$$

A substitution of the presented coordinates of the tensors and the vectors into system (17) gives equations of a viscous thermoconductive gas in the spherical coordinate system. These equations are very cumbersome⁹.

⁹All symbolic calculations for the coordinates of the tensors and the vectors were made on computer with the help of the system REDUCE [21].

5.2 Partially Invariant Solution with respect to $SO(3)$

In the space of variables $t, r, \theta, \varphi, U, H, \omega, \tau, p$ the group of rotations $O(3)$ has the generators [36]

$$\begin{aligned} X &= -\sin \varphi \partial_\theta - \cos \varphi \cot \theta \partial_\varphi + \cos \varphi (\sin \theta)^{-1} \partial_\omega, \\ Y &= \cos \varphi \partial_\theta - \sin \varphi \cot \theta \partial_\varphi + \sin \varphi (\sin \theta)^{-1} \partial_\omega, \\ Z &= \partial_\varphi. \end{aligned}$$

Invariants of this group are t, r, U, H, τ, p .

The rank of the Jacobi matrix of the invariants with respect to the dependent functions is equal to four. Therefore, according to [35] there are no nonsingular invariant solutions that are invariant with respect to group of rotations $O(3)$. A minimal possible defect of a partially invariant with respect to $O(3)$ solution is equal to one. In this case a representation of the partially invariant solution is

$$\tau = \tau(t, r), \quad U = U(t, r), \quad H = H(t, r), \quad p = p(t, r), \quad \omega = \omega(t, r, \theta, \varphi). \quad (20)$$

The function $\omega(t, r, \theta, \varphi)$ is "superfluous": it depends on all independent variables. Note that if $H = 0$, then by virtue of (19) the tangent component of the velocity vector is equal to zero and it corresponds to spherically symmetric flows that are considered in the next subsection. In this subsection it is assumed that $H \neq 0$.

All analytic calculations for the viscous gas dynamics and the Navier-Stokes equations are done in the REDUCE system [21]. The result of these calculation is: class of solutions that is partially invariant with respect to $O(3)$ is confined by spherically symmetric solutions.

5.2.1 Analysis of compatibility of partially invariant solutions

For the sake of simplicity we present here analysis of compatibility of partially invariant solution for the Navier-Stokes equations, i.e., when τ and μ are constants. Analysis of compatibility for the viscous gas dynamics equations is similar, but it needs more cumbersome symbolic calculations.

After substituting the representation of the partially invariant solution (20) into the Navier-Stokes equations¹⁰ and some combinations of the second and the third equations the initial system can be split on two subsystems: the invariant system

$$D_0 U + p_r = r^{-1} H^2 + (U_{rr} + 4r^{-1} U_r + 2r^{-2} U) \quad (21)$$

with the operator $D_0 = \partial_t + U \partial_r$ and the supplementary system

$$\begin{aligned} D_0(rH) &= (rH)_{rr} - (r \sin^2 \theta)^{-1} H - rH(\omega_r^2 + r^{-2} \omega_\theta^2 + \\ &\quad + (r \sin \theta)^{-2} \omega_\varphi^2 + 2(r^2 \sin \theta)^{-1} \cot \theta \omega_\varphi), \\ D_0 \omega &+ (r \sin \theta)^{-1} H (\sin \theta \cos \omega \omega_\theta + \sin \omega \omega_\varphi + \cos \theta \sin \omega) = \\ &= \omega_{rr} + 2(rH)^{-1} (rH)_r \omega_r + r^{-2} \omega_{\theta\theta} + r^{-2} \cot \theta \omega_\theta + (r \sin \theta)^{-2} \omega_{\varphi\varphi}, \\ \sin \theta \sin \omega \omega_\theta - \cos \omega \omega_\varphi &= \cos \theta \cos \omega + \sin \theta (rH)^{-1} (r^2 U)_r. \end{aligned} \quad (22)$$

¹⁰Here we use dimensionless representation of the Navier-Stokes equations in which one can account that $\mu = 1$ and $\tau = 1$.

For the analysis of compatibility of system (21),(22) it is convenient to use implicit representation for the function $\omega = \omega(t, r, \theta, \varphi)$ in the form

$$F(\omega, t, r, \theta, \varphi) = 0, \quad (F_\omega \neq 0).$$

All derivatives of the function $\omega(t, r, \theta, \varphi)$ can be calculated through the derivatives of the function $F(\omega, t, r, \theta, \varphi)$. For example, for the first derivatives we have

$$\omega_t = -F_t/F_\omega, \quad \omega_r = -F_r/F_\omega, \quad \omega_\theta = -F_\theta/F_\omega, \quad \omega_\varphi = -F_\varphi/F_\omega.$$

Then the last equation of (22) becomes

$$\sin \theta \sin \omega F_\theta - \cos \omega F_\varphi + F_\omega (\cos \theta \cos \omega + k \sin \theta) = 0,$$

where the function $k = (\tau H)^{-1}(\tau^2 U)_r$ only depends on t and r . Note that for a viscous gas dynamics equations there is the same equation with the function $k(t, r) = (H\tau\tau)^{-1}(-\tau D_0\tau + \tau(r^2 U)_r)$. The general solution of the last equation is

$$F = \Phi \left(\varphi + \arctan \left(\frac{\sin \omega}{k \sin \theta + \cos \theta \cos \omega} \right), \sin \theta \cos \omega - k \cos \theta, t, r \right).$$

Here the function $\Phi = \Phi(y_1, y_2, t, r)$ is an arbitrary function of the arguments t, r and

$$y_1 = \varphi + \arctan \left(\frac{\sin \omega}{k \sin \theta + \cos \theta \cos \omega} \right), \quad y_2 = \sin \theta \cos \omega - k \cos \theta.$$

All further intermediate calculations in studying the compatibility of overdetermined system (21), (22) were made on computer in the system REDUCE [21]. Here we give the way of computing and the final results.

Note that the Jacobian $\frac{\partial(y_1, y_2, \theta, t, r)}{\partial(\omega, \theta, \varphi, t, r)} \neq 0$, therefore one can choose (y_1, y_2, θ, t, r) as the new independent variables. All derivatives of the function $\omega(t, r, \theta, \varphi)$ can be written through the derivatives of the function $\Phi(y_1, y_2, t, r)$. After that the second equation of (8) accepts the form

$$\sin \omega G_1(y_1, y_2, t, r, \theta) + G_2(y_1, y_2, t, r, \theta) = 0,$$

where the functions $G_1(y_1, y_2, t, r, \theta)$ and $G_2(y_1, y_2, t, r, \theta)$ do not include ω and its derivatives. In the last equation $\sin \omega$ can be excluded by using the trigonometry identity:

$$G_1^2(1 - (y_2 + k \cos \theta)^2) - G_2^2(1 - \cos^2 \theta) = 0,$$

where the equality $\cos \omega = \sin^{-1} \theta(y_2 + k \cos \theta)$ found from the representation of y_2 was applied.

Further calculations show that the last equation depends on θ as the polynomial of the degree 8 with respect to $\cos \theta$:

$$P_8 = \sum_{k=0}^8 a_k \cos^k \theta = 0.$$

The coefficients a_k , ($k = 0, 1, \dots, 8$) only depend on y_1, y_2, t, r and do not depend on θ . This allows splitting the equation with respect to $\cos \theta$: $a_k = 0$, ($k = 1, 2, \dots, 8$).

The equality $a_8 = 0$ gives

$$D_0 h = h_{rr} + h(k^2 + 1)^{-1} h_r, \quad (23)$$

where $h = rH$. Substituting h_t found from (23) into $a_6 = 0$, we obtain

$$k_r \left((k^2 + 1)\Phi_r + k k_r y_2 \Phi_{y_2} \right) = 0. \quad (24)$$

If $(k^2 + 1)\Phi_r + k k_r y_2 \Phi_{y_2} = 0$, then the equation $a_4 = 0$ gives the equation $y_2^2 - (k^2 + 1) = 0$ or

$$(\sin \theta \cos \omega - k \cos \theta)^2 = k^2 + 1.$$

Note that substituting the representation of the function $\omega(t, r, \theta, \varphi)$ found from this equation into (21), (22) and splitting them with respect to $\cos \theta$ gives the expression $H = 0$ that contradicts the assumption about H .

For the second case in (24), when $k_r = 0$ we will obtain a contradiction with the help of the first equation of (22). Really, the same study of the first equation of (22) as for the second equation leads to the polynomial of the degree 10 with respect to $\cos \theta$:

$$P_{10} = \sum_{k=0}^{10} b_k \cos^k \theta = 0,$$

where the coefficients b_k , ($k = 0, 1, \dots, 10$) only depend on y_1, y_2, t, r . The equality $b_{10} = 0$ gives

$$k_t = \tau^{-2} h(k^2 + 1). \quad (25)$$

By virtue of $k_r = 0$, (25) and the definition of $k = (r^2 U)_r / h$ one can obtain that

$$h(t, r) = 3c(t)r^2, \quad \tau^2 U(t, r) = k(t)c(t)r^3 + \lambda(t),$$

where $c(t) = (k^2(t) + 1)^{-1} k'(t) / 3$. Substitution of this representation into (23) and splitting it with respect to r gives $c(t) = 0$ that contradicts the assumption $H \neq 0$.

Similar calculations have been done for the viscous gas dynamics equations.

The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal incompressible inviscid fluid dynamics equations, are only spherically symmetric solutions.

5.3 Spherically Symmetric Flows of a Viscous Gas

The case $H = 0$ corresponds to a spherically symmetric flow of a viscous gas. According to the definitions of the group analysis it is a singular invariant solution with respect to group of rotations $O(3)$. The viscous gas dynamics equations in this case are

$$\begin{aligned} D_0 \tau - \tau(U_r + 2r^{-1}U) &= 0, \\ D_0 U + \tau p_r &= \tau(\lambda + 2\mu)(U_{rr} + 2r^{-1}U_r - 2r^{-2}U) + 6\tau(\mu_r \tau_r + \mu_p p_r) + \\ &\quad + \tau(U_r + 2r^{-1}U)(\lambda_r \tau_r + \lambda_p p_r), \\ D_0 p + A(U_r + 2r^{-1}U) &= B[\lambda(U_r + 2r^{-1}U)^2 + 2\mu(U_r^2 + 2r^{-2}U^2) + \\ &\quad + \kappa(T_{rr} \tau_r^2 + 2T_{rp} \tau_r p_r + T_{pp} p_r^2 + T_r(\tau_{rr} + 2r^{-1}\tau_r) + \\ &\quad + T_p(p_{rr} + 2r^{-1}p_r) + (\kappa_p p_r + \kappa_r \tau_r)(T_r \tau_r + T_p p_r)], \end{aligned} \quad (26)$$

where $D_0 = \partial_t + U\partial_r$. In this subsection we study a group classification of equations (26) with respect to the arbitrary elements $A, B, \lambda, \mu, \kappa, T$.



5.3.1 Equivalence transformations

The first stage of group classification requires determining a group of equivalence transformations of equations (26). An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to the system of equations of the same class. It allows using the simplest representation of given equations. Here we give a construction of the group of equivalence transformations without restrictions on the representation of equivalence transformations [35]. We follow the approach for the calculation of equivalence transformations developed in [30].

Since arbitrary elements satisfy restrictions (18) and $A = A(p, \tau)$, $B = B(p, \tau)$, $\lambda = \lambda(p, \tau)$, $\mu = \mu(p, \tau)$, $\kappa = \kappa(p, \tau)$, $T = T(p, \tau)$, then for calculating an equivalence group of transformations we have to append the equations

$$\begin{aligned} A_r &= 0, A_t = 0, A_U = 0, B_r = 0, B_t = 0, B_U = 0, \\ \lambda_r &= 0, \lambda_t = 0, \lambda_U = 0, \mu_r = 0, \mu_t = 0, \mu_U = 0, \\ \kappa_r &= 0, \kappa_t = 0, \kappa_U = 0, T_r = 0, T_t = 0, T_U = 0 \end{aligned}$$

to equations (26). All coefficients of the infinitesimal generator of the equivalence group

$$X^e = \zeta^r \partial_r + \zeta^t \partial_t + \zeta^U \partial_U + \zeta^\tau \partial_\tau + \zeta^p \partial_p + \zeta^A \partial_A + \zeta^B \partial_B + \zeta^\lambda \partial_\lambda + \zeta^\mu \partial_\mu + \zeta^\kappa \partial_\kappa + \zeta^T \partial_T$$

are dependent on all independent, dependent variables and arbitrary elements

$$r, t, U, \tau, p, A, B, \lambda, \mu, \kappa, T.$$

With the following notation:

$$u^1 = U, u^2 = \tau, u^3 = p, a^1 = A, a^2 = B, a^3 = \lambda, a^4 = \mu, a^5 = \kappa, a^6 = T$$

and

$$z^1 = r, z^2 = t, z^3 = U, z^4 = \tau, z^5 = p, a_\beta^k = \frac{\partial a^k}{\partial z^\beta}, a_{j\beta}^k = \frac{\partial^2 a^k}{\partial z^j \partial z^\beta},$$

the coefficients of the prolonged operator

$$\bar{X}^e = X^e + \sum_i (\zeta^{u^i} \partial_{u^i} + \zeta^{u^i} \partial_{u^i}) + \sum_{k,j} \zeta^{a_{z^j}^k} \partial_{a_{z^j}^k} + \dots$$

can be constructed with the prolongation formulae:

$$\zeta^{u^i} = D_r \zeta^{u^i} - u_r^i D_r \zeta^r - u_t^i D_r \zeta^t, \quad \zeta^{u^i} = D_t \zeta^{u^i} - u_r^i D_t \zeta^r - u_t^i D_t \zeta^t,$$

$$\zeta^{u_{rr}^i} = D_r \zeta^{u_r^i} - u_{rr}^i D_r \zeta^r - u_{rt}^i D_r \zeta^t.$$

$$\zeta^{a_\beta^k} = D_{z^\beta}^e \zeta^{a^k} - \sum_{\alpha=1}^5 a_\alpha^k D_{z^\beta}^e \zeta^{z^\alpha}, \quad \zeta^{a_{j\beta}^k} = D_{z^\beta}^e \zeta^{a_j^k} - \sum_{\alpha=1}^5 a_{j\alpha}^k D_{z^\beta}^e \zeta^{z^\alpha}.$$

Here the operators D_r , D_t denote the total derivative operators with respect to r and t , respectively. For example,

$$D_r = \partial_r + \sum_\alpha u_r^\alpha \partial_{u^\alpha} + \sum_i (a_r^i + \sum_j a_{w^j}^i u_r^j) \partial_{a^i} + \dots$$

When we use the operator $D_{z^j}^e$, we consider $z^1, \dots, z^5$ as independent variables and $a^1, \dots, a^6$ as dependent variables, we obtain:

$$D_{z^j}^e = \partial_{z^j} + \sum_i a_{z^j}^i \partial_{a^i} + \dots$$

All necessary calculations here as in the previous subsections were carried on a computer using the symbolic manipulation program REDUCE [21].

The calculations showed that the group of equivalence transformations of equations (26) corresponds to Lie algebra with generators

$$\begin{aligned} X_1^e &= \partial_t, \quad X_2^e = \partial_p, \quad X_3^e = r\partial_r + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa, \\ X_4^e &= r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa, \quad X_5^e = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa. \end{aligned}$$

Remark. If instead of the functions $A(p, \tau), B(p, \tau)$ one considers the internal energy $\varepsilon(p, \tau)$, then the operators X_2^e, X_4^e , and X_5^e are changed to

$$\begin{aligned} X_2^e &= \partial_p - \tau\partial_\varepsilon, \quad X_4^e = r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon, \\ X_5^e &= -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa. \end{aligned}$$

and there is one more generator $X_6^e = \partial_\varepsilon$.

Remark. By a direct checking one can obtain that in the general case¹¹ (equations (17)) the equivalence group includes the transformations with the generators

$$\begin{aligned} X_1^e &= \partial_t, \quad X_2^e = \partial_p, \\ X_3^e &= x\partial_x + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa, \\ X_4^e &= x\partial_x + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa, \\ X_5^e &= -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa. \end{aligned}$$

There are also other generators, for example, that correspond to the Galilei transformations and to the rotations in the three-dimensional case.

5.3.2 Admitted group

Finding an admitted group consists of seeking solutions of determining equations [35]. We are looking for the generator

$$X = \zeta^r \partial_r + \zeta^t \partial_t + \zeta^U \partial_U + \zeta^\tau \partial_\tau + \zeta^p \partial_p$$

with the coefficients depending on r, t, U, τ, p . Calculations lead to the following result.

The kernel of the fundamental Lie algebra is made up of the generator

$$X = \partial_t.$$

Extension of the kernel of the main Lie algebra occurs by specializing the functions $A = A(p, \tau), B = B(p, \tau), \lambda = \lambda(p, \tau), \mu = \mu(p, \tau), \kappa = \kappa(p, \tau), T = T(p, \tau)$. Note that the functions $A = A(p, \tau), B = B(p, \tau), T = T(p, \tau)$ have to satisfy equations (18). There are three types of the generators admitted by system (26). Further α, β and δ are arbitrary constants.

¹¹Group classification of three-dimensional viscous gas dynamics equations with $\lambda = -2\mu/3$ was studied in [8].

Type (a). If the functions $A(\tau, p), B(\tau, p), \lambda(\tau, p), \mu(\tau, p), \kappa(\tau, p), T(\tau, p)$ satisfy the equations

$$\begin{aligned} \alpha\tau A_\tau + A_p &= 0, \quad \alpha\tau B_\tau + B_p = 0, \\ \alpha\tau\mu_\tau + \mu_p &= \beta\mu, \quad \alpha\tau\lambda_\tau + \lambda_p = \beta\lambda, \\ \alpha\tau T_\tau + T_p &= \delta T, \quad \alpha\tau\kappa_\tau + \kappa_p = (-\delta + \alpha + \beta)\kappa, \end{aligned} \quad (27)$$

then there is one more admitted generator:

$$Y_a = \alpha U \partial_U + 2\alpha\tau \partial_\tau + 2\partial_p + (\alpha + 2\beta)\tau \partial_\tau + 2\beta t \partial_t.$$

The general solution of equations (27) is

$$\begin{aligned} A &= A(\tau e^{-\alpha p}), \quad B = B(\tau e^{-\alpha p}), \quad \mu = e^{\beta p} M(\tau e^{-\alpha p}), \quad \lambda = e^{\beta p} \Lambda(\tau e^{-\alpha p}), \\ T &= e^{\delta p} \Theta(\tau e^{-\alpha p}), \quad \kappa = e^{(-\delta + \alpha + \beta)p} K(\tau e^{-\alpha p}), \end{aligned}$$

where the functions $A(z), B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau e^{-\alpha p}$)

$$-\alpha z B A' + z B' (1 + \alpha A) = B^2 + B, \quad (1 + \alpha A) z \Theta' = (\delta A - B) \Theta. \quad (28)$$

The internal energy is represented by the formula

$$\varepsilon = e^{\alpha p} (\varphi(z) - zp) + \psi(p), \quad \psi'(p) = C e^{\alpha p},$$

where the function $\varphi(z)$ and the constant C can be accounted as arbitrary and they are related with the functions $A(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{A(z)}{B(z)}, \quad C = z + \frac{z}{B(z)} + \alpha z \varphi'(z) - \alpha \varphi(z).$$

In this case the function $\Theta(z)$ has to satisfy the equation

$$(C - z + \alpha \varphi(z)) \Theta'(z) = (\delta \varphi'(z) - 1) \Theta(z).$$

Type (b). If the functions $A(\tau, p), B(\tau, p), \lambda(\tau, p), \mu(\tau, p), \kappa(\tau, p), T(\tau, p)$ satisfy the equations

$$\begin{aligned} \alpha\tau A_\tau + p A_p &= A, \quad \alpha\tau B_\tau + p B_p = 0, \\ \alpha\tau\mu_\tau + p\mu_p &= (\beta + 1)\mu, \quad \alpha\tau\lambda_\tau + p\lambda_p = (\beta + 1)\lambda, \\ \alpha\tau T_\tau + p T_p &= \delta T, \quad \alpha\tau\kappa_\tau + p\kappa_p = (-\delta + 2 + \alpha + \beta)\kappa, \end{aligned} \quad (29)$$

then there is an extension by the generator

$$Y_b = (1 + \alpha) U \partial_U + 2\alpha\tau \partial_\tau + 2p \partial_p + (\alpha + 2\beta + 1) \tau \partial_\tau + 2\beta t \partial_t.$$

The general solution of equations (29) is

$$\begin{aligned} A &= p \hat{A}(\tau p^{-\alpha}), \quad B = B(\tau p^{-\alpha}), \quad \mu = p^{\beta+1} M(\tau p^{-\alpha}), \quad \lambda = p^{\beta+1} \Lambda(\tau p^{-\alpha}), \\ T &= p^\delta \Theta(\tau p^{-\alpha}), \quad \kappa = p^{-\delta + \alpha + \beta + 2} K(\tau p^{-\alpha}), \end{aligned}$$

where the functions $\hat{A}(z), B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau p^{-\alpha}$)

$$-\alpha z B \hat{A}' + z B' (1 + \alpha \hat{A}) = B^2 + B - B \hat{A}, \quad (1 + \alpha \hat{A}) z \Theta' = (\delta \hat{A} - B) \Theta. \quad (30)$$

The internal energy is represented by the formula

$$\varepsilon = p^{(\alpha+1)}(\varphi(z) - z) + \psi(p), \quad \psi'(p) = Cp^\alpha,$$

where the function $\varphi(z)$ and the constant C are arbitrary and they are related with the functions $\hat{A}(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{\hat{A}(z)}{B(z)}, \quad C = z + \frac{z}{B(z)} + \alpha z \varphi'(z) - (\alpha + 1)\varphi(z)$$

The function $\Theta(z)$ is represented through the function $\varphi(z)$ by the formula

$$(C - z + (\alpha + 1)\varphi(z)) \Theta'(z) = (\delta \varphi'(z) - 1)\Theta(z)$$

Note that an ideal gas belongs to this type in case of $\delta = \alpha + 1$ and the function $\varphi(z)$ satisfies the equation

$$\delta(z\varphi' - \varphi) = C.$$

Type (c). If the functions $A(\tau, p), B(\tau, p), \lambda(\tau, p), \mu(\tau, p), \kappa(\tau, p), T(\tau, p)$ satisfy the equations

$$\begin{aligned} A_\tau &= 0, \quad B_\tau = 0, \quad \tau\mu_\tau = \beta\mu, \quad \tau\lambda_\tau = \beta\lambda, \\ \tau T_\tau &= \delta T, \quad \tau\kappa_\tau = (-\delta + 1 + \beta)\kappa, \end{aligned} \quad (31)$$

then there is one more admitted generator:

$$Y_c = U\partial_U + 2\tau\partial_\tau + (1 + 2\beta)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (31) is

$$\begin{aligned} A &= A(p), \quad B = B(p), \quad \mu = \tau^\beta M(p), \quad \lambda = \tau^\beta \Lambda(p), \\ T &= \tau^\delta \Theta(p), \quad \kappa = \tau^{-\delta+\beta+1} K(p), \end{aligned}$$

where the functions $A(p), B(p)$ and $\Theta(p)$ satisfy the equations

$$BA' - AB' = B^2 + B, \quad A\Theta' = (\delta + B)\Theta. \quad (32)$$

The internal energy is represented by the formula

$$\varepsilon = \tau\varphi(p) - \tau p,$$

where the function $\varphi(p)$ is an arbitrary function and it is related with the functions $A(p)$ and $B(p)$ by the formula

$$\varphi(p) = \frac{A(p)}{B(p)}.$$

In this case the function $\Theta(p)$ is related with the function $\varphi(z)$ by the formula

$$\varphi(p)\Theta'(p) = (1 - \delta + \delta\varphi'(p))\Theta(p).$$

Note that if $\delta = 1$ and $\varphi = Cp$, then the gas is ideal.

The final results of the group classification are presented in Table I. In this table the first column means the type of the extension of the algebra $\{X\}$: the types a, b , or c , respectively. The last column means conditions for the state functions.

Therefore, there are three kinds of admitted by equations (26) groups, which depend on the specifications of the functions $A = A(p, \tau)$, $B = B(p, \tau)$, $\lambda = \lambda(p, \tau)$, $\mu = \mu(p, \tau)$, $\kappa = \kappa(p, \tau)$, $T = T(p, \tau)$. These groups are one-dimensional, two-dimensional and three-dimensional.

The two-dimensional admitted groups are groups with the generators either $\{X, Y_a\}$ or $\{X, Y_b\}$ or $\{X, Y_c\}$. The three-dimensional admitted groups are the groups with the generators either $\{X, Y_a, Y_b\}$ or $\{X, Y_a, Y_c\}$ or $\{X, Y_b, Y_c\}$.

The group with the generators $\{X, Y_a, Y_b\}$ is admitted by equations (26) if

$$A = A_0 \tau^\alpha, B = -1, \mu = \mu_0 \tau^{\beta+\alpha}, \lambda = \lambda_0 \tau^{\beta+\alpha}, \kappa = \kappa_0 \tau^{\beta+2\alpha}, T = T_0 \tau, \alpha \neq 0.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0 \int \tau^\alpha d\tau)$. Instead the operators Y_a and Y_b one can use their linear combinations:

$$\hat{Y}_a = \partial_p, \hat{Y}_b = (1 + \alpha)U\partial_U + 2\tau\partial_\tau + (\alpha + 2\beta + 1)r\partial_r + 2\beta t\partial_t.$$

The algebra of the type $\{X, Y_a, Y_c\}$ is admitted by equations (26) if

$$A = A_0, B = -1, \mu = \mu_0 \tau^\beta e^{\alpha p}, \lambda = \lambda_0 \tau^\beta e^{\alpha p}, \kappa = \kappa_0 \tau^{\beta-A_0\sigma} e^{(\alpha-\sigma)p}, T = T_0 \tau^{1+A_0\sigma} e^{\sigma p}.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0 \tau)$ and by taking linear combinations of the operators Y_a and Y_c one obtains another basis of the generators:

$$\hat{Y}_a = \partial_p + \alpha(r\partial_r + t\partial_t), \hat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta + 1)r\partial_r + 2\beta t\partial_t.$$

The third type of the algebras $\{X, Y_b, Y_c\}$ is admitted by (26) if

$$A = \gamma p, B = \gamma - 1, \mu = \mu_0 \tau^\beta p^{1+\alpha}, \lambda = \lambda_0 \tau^\beta p^{1+\alpha}, \\ \kappa = \kappa_0 \tau^{\gamma(1-\alpha)+\beta} p^{\alpha-\delta+2}, T = T_0 \tau^{\gamma(\delta-1)+1} p^\delta, \gamma \neq 1.$$

The internal the energy in this case is

$$\varepsilon = \frac{\tau p}{\gamma - 1}$$

and linear combinations of the operators Y_b and Y_c are:

$$\hat{Y}_b = U\partial_U + 2p\partial_p + (2\alpha + 1)r\partial_r + 2\alpha t\partial_t, \hat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta + 1)r\partial_r + 2\beta t\partial_t$$

Note that a polytropic gas belongs to the last case of gases, where γ is a polytropic exponent.

In the formulas above $A_0, \mu_0, \lambda_0, \kappa_0, T_0, \alpha, \beta, \gamma, \delta, \sigma$ are arbitrary constants; the commutators

$$[\hat{Y}_a, \hat{Y}_b] = 0, [\hat{Y}_a, \hat{Y}_c] = 0, [\hat{Y}_b, \hat{Y}_c] = 0.$$

5.3.3 Optimal systems of subalgebras

Here we study subalgebras of the two-dimensional admitted algebras $\{X, Y_a\}$, $\{X, Y_b\}$, $\{X, Y_c\}$.

The commutator $[X, Y]$ of the generators X and Y is

$$[X, Y] = zX.$$

Here either $Y = Y_a$ or $Y = Y_b$ or $Y = Y_c$ and $z = 2\beta$. Automorphisms are recovered by the table of commutators and consists of the automorphisms

$$A_1 : x' = x + zy a_1, y' = y, \\ A_2 : x' = e^{-za_2} x, y' = y,$$

where x and y are coordinates of the operator $Z = xX + yY$, x' and y' are coordinates of the operator Z' after actions of the automorphisms, and a_1, a_2 are parameters of the automorphisms. There is also one involution

$$E : x' = -x, y' = y,$$

which corresponds to the change of the variables $t \rightarrow -t$ and $U \rightarrow -U$ without changes of equations (26). Note that if $z = 0$, then the automorphisms are identical transformations. This leads to two optimal systems of subalgebras.

If $z = 0$ (or $\beta = 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\}, \{Y + hX\}, \{X, Y\},$$

where h is an arbitrary positive constant.

If $z \neq 0$ (or $\beta \neq 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\}, \{Y\}, \{X, Y\}.$$

Therefore, one can summarize: optimal systems of subalgebras for the two-dimensional algebras are described by the following system of subalgebras

$$\{X\}, \{Y + hX\}, \{X, Y\}, \beta h = 0. \quad (33)$$

5.3.4 Representations of invariant solutions

A next step in the construction of representations of invariant solutions is a finding universal invariants. Note that invariant solutions corresponding to the case of the subalgebra $\{X\}$ are the well-known stationary solutions. The universal invariants for the other subalgebras of the optimal system (33) of the algebras $\{X, Y_a\}$, $\{X, Y_b\}$ and $\{X, Y_c\}$ are presented in Tables II, III and IV, respectively.

According to the theory of the group analysis [35] on the next step in constructing of invariant solutions one needs to separate the universal invariant on two parts: one part has to be solvable with respect to the dependent variables U, τ, p . After that the representations of invariant solutions are obtained by supposing that the first part of the universal invariant depends on the second part. Because of this requirement there are no invariant solutions in the cases: *a.1* if $h = 0$, *a.4*, *b.1*, *b.5* and *c.3*. The cases *a.5*, *b.6* and *c.4* correspond to the special cases of stationary solutions, which we also exclude from our consideration¹².

All possible representations of invariant solutions of equations (26) are presented in Table Y, where the functions f^u, f^τ, f^p are functions of one independent variable presented in the last column. These functions must satisfy ordinary differential equations, which are obtained after substituting the representation of solution into system (26).

Remark. Invariant solutions *b.2*, *b.4*, *c.2* are self-similar solutions.

Remark. One of the well-known solutions of the Boltzmann equation (the BKW-solution¹³) has the representation [5, 24]

$$f = \phi(|u|e^{ct}),$$

¹²If an universal invariant is three-dimensional (consists of three invariants), such as in the cases of *a.5*, *b.6*, *c.4*, then the representation of the invariant solution is obtained by supposing that all invariants of the universal invariant are constants.

¹³This solution is constructed for the Maxwell molecules.

where f is a distribution function, $|u|$ is a modulus of the velocity. The invariant solution of the viscous gas equations, which corresponds to the case $b.3$ gives

$$|u|e^{-t(\alpha+1)/h} = qf^u(q),$$

with $q = re^{-t(\alpha+1)/h}$. Therefore this solution can correspond to the BKW-solution and generalize it on molecules with an exponent intermolecular potentials. For the molecules with an exponent intermolecular potentials the coefficients of viscosity and conductivity are [4]

$$\mu = \mu_0 T^k, \kappa = \kappa_0 T^k,$$

where $T = T_0 p\tau$, $k = (n - 1)/m + 1/2$, n is dimension of the problem, m is the exponent of intermolecular potentials. In this case $\alpha = 1/k - 1 = (m + 2n - 2)^{-1}(m - 2n + 2)$. For the Maxwell molecules, for which the BKW-solution was constructed, the exponent of intermolecular potentials is $m = 4$, and hence, in the three-dimensional case $\alpha = 0$ and $k = 1$.

5.4 Spherically Symmetric Flows of the Navier-Stokes Equations

For the complete consideration of solutions connected with the group of rotations $O(3)$ we present solutions of the Navier-Stokes equations with spherical symmetry¹⁴. Substituting the value of $V = 0$, $W = 0$ in the last equation of (22) one obtains that $r^2 U = h(t, \theta, \varphi)$. From the remained equations of (21),(22) all space derivatives of the pressure can be found

$$\begin{aligned} p_r &= r^{-4}(\cot \theta h_\theta + \sin^{-2} \theta h_{\varphi\varphi} + h_{\theta\theta} - r^2 h_t + 2r^{-1} h^2), \\ p_\theta &= 2r^{-3} h_\theta, \quad p_\varphi = 2r^{-3} h_\varphi, \end{aligned}$$

where $g(t)$ and $h(t)$ are arbitrary functions.

Using symmetry property of the mixed derivatives $p_{\theta r} = p_{r\theta}$, $p_{\varphi r} = p_{r\varphi}$, $p_{\varphi\theta} = p_{\theta\varphi}$ and splitting these equalities with respect to r one can get that $h = h(t)$ and the general solution of the Navier-Stokes equations in this case is

$$p = r^{-1} h'(t) - r^{-4} h^2(t)/2 + g(t), \quad U = r^{-2} h(t), \quad V = 0, \quad W = 0.$$

5.5 Conclusion of the first part of the research

The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. For the completeness of consideration of partially invariant solutions that are connected with the group of rotations $O(3)$ the group classification of the full viscous gas dynamics equations with spherical symmetry has been done.

5.6 One class of partially invariant solutions

The class of solutions studied in [33] is a class of partially invariant solutions with respect to the group H with generators

$$X = \partial_x, \quad Y = \partial_y, \quad U = t\partial_x + \partial_u, \quad V = t\partial_y + \partial_v.$$

¹⁴These solutions are irregular invariant solutions of the Navier-Stokes equations with respect to rotations. We think that they are known, but, unfortunately, we do not know any reference on this subject.

Table 1: Group classification.

	λ	μ	T	κ	A	B	z	Cond.
a	$e^{\beta p} \Lambda(z)$	$e^{\beta p} M(z)$	$e^{\delta p} \Theta(z)$	$e^{(-\delta+\alpha+\beta)p} K(z)$	$A(z)$	$B(z)$	$\tau e^{-\alpha p}$	(28)
b	$p^{\beta+1} \Lambda(z)$	$p^{\beta+1} M(z)$	$p^{\delta} \Theta(z)$	$p^{-\delta+\alpha+\beta+2} K(z)$	$p \hat{A}(z)$	$B(z)$	$\tau p^{-\alpha}$	(30)
c	$\tau^{\beta} \Lambda(p)$	$\tau^{\beta} M(p)$	$\tau^{\delta} \Theta(p)$	$\tau^{-\delta+\beta+1} K(p)$	$A(p)$	$B(p)$	p	(32)

Table 2: Universal invariants of subalgebras of the algebra $\{X, Y_a\}$.

N	Subalgebra	consts	Universal invariant
$a.1$	$Y_a + hX$	$\beta = 0, \alpha = 0$	$U, \tau, t - hp/2, r$
$a.2$	$\beta h = 0$	$\beta = 0, \alpha \neq 0$	$U r^{-1}, \tau r^{-2}, p - 2\alpha^{-1} \ln r, t - h\alpha^{-1} \ln r$
$a.3$		$\beta \neq 0$	$U t^{-\alpha/(2\beta)}, \tau t^{-\alpha/\beta}, p - \beta^{-1} \ln t, r t^{-(\alpha+2\beta)/(2\beta)}$
$a.4$	X, Y_a	$\alpha + 2\beta = 0$	$U e^{-\alpha p/2}, \tau e^{-\alpha p}, r$
$a.5$		$\alpha + 2\beta \neq 0$	$U r^{-\alpha/(\alpha+2\beta)}, \tau r^{-2\alpha/(\alpha+2\beta)}, p - 2(\alpha + 2\beta)^{-1} \ln r$

Table 3: Universal invariants of subalgebras of the algebra $\{X, Y_b\}$ ($k \equiv \alpha + 2\beta + 1$).

N	Subalgebra	consts	Universal invariant
$b.1$	$Y_b + hX$	$\beta = 0, \alpha = -1, h = 0$	$U, p/\tau, r, t$
$b.2$	$\beta h = 0$	$\beta = 0, \alpha \neq -1, h = 0$	$U r^{-1}, \tau r^{-2\alpha/(\alpha+1)}, p r^{-2/(\alpha+1)}, t$
$b.3$		$\beta = 0, h \neq 0$	$U e^{-t(\alpha+1)/h}, \tau e^{-2t\alpha/h}, p e^{-2t/h}, r e^{-t(\alpha+1)/h}$
$b.4$		$\beta \neq 0$	$U t^{-(\alpha+1)/(2\beta)}, \tau t^{-\alpha/\beta}, p t^{-1/\beta}, r t^{-k/(2\beta)}$
$b.5$	X, Y_b	$k = 0$	$U p^{-(\alpha+1)/2}, \tau p^{-\alpha}, r$
$b.6$		$k \neq 0$	$U r^{-(\alpha+1)/k}, \tau r^{-2\alpha/k}, p r^{-2/k}$

Table 4: Universal invariants of subalgebras of the algebra $\{X, Y_c\}$.

N	Subalgebra	consts	Universal invariant
$c.1$	$Y_c + hX$	$\beta = 0$	$U r^{-1}, \tau r^{-2}, p, t - h \ln r$
$c.2$	$\beta h = 0$	$\beta \neq 0$	$U t^{-1/(2\beta)}, \tau t^{-1/\beta}, p, r t^{-(2\beta+1)/(2\beta)}$
$c.3$	X, Y_c	$2\beta + 1 = 0$	$U \tau^{-1/2}, p, r$
$c.4$		$2\beta + 1 \neq 0$	$U r^{-1/(2\beta+1)}, \tau r^{-2/(2\beta+1)}, p$

Table 5: Representations of invariant solutions.

N	Representation of invariant solution	Ind. variable	Model
1	$U = f^u, \tau = f^\tau, p = 2th^{-1} + f^p$	r	$a.1$
2	$U = r f^u, \tau = r^2 f^\tau, p = 2\alpha^{-1} \ln r + f^p$	$t - h\alpha^{-1} \ln r$	$a.2$
3	$U = t^{\alpha/(2\beta)} f^u, \tau = t^{\alpha/\beta} f^\tau, p = \beta^{-1} \ln t + f^p$	$r t^{-(\alpha+2\beta)/(2\beta)}$	$a.3$
4	$U = r f^u, \tau = r^{2\alpha/(\alpha+1)} f^\tau, p = r^{2/(\alpha+1)} f^p$	t	$b.2$
5	$U = e^{t(\alpha+1)/h} f^u, \tau = e^{2t\alpha/h} f^\tau, p = e^{2t/h} f^p$	$r e^{-t(\alpha+1)/h}$	$b.3$
6	$U = t^{(\alpha+1)/(2\beta)} f^u, \tau = t^{\alpha/\beta} f^\tau, p = t^{1/\beta} f^p$	$r t^{-(\alpha+2\beta+1)/(2\beta)}$	$b.4$
7	$U = r f^u, \tau = r^2 f^\tau, p = f^p$	$t - h \ln r$	$c.1$
8	$U = t^{1/(2\beta)} f^u, \tau = t^{1/\beta} f^\tau, p = f^p$	$r t^{-(2\beta+1)/(2\beta)}$	$c.2$