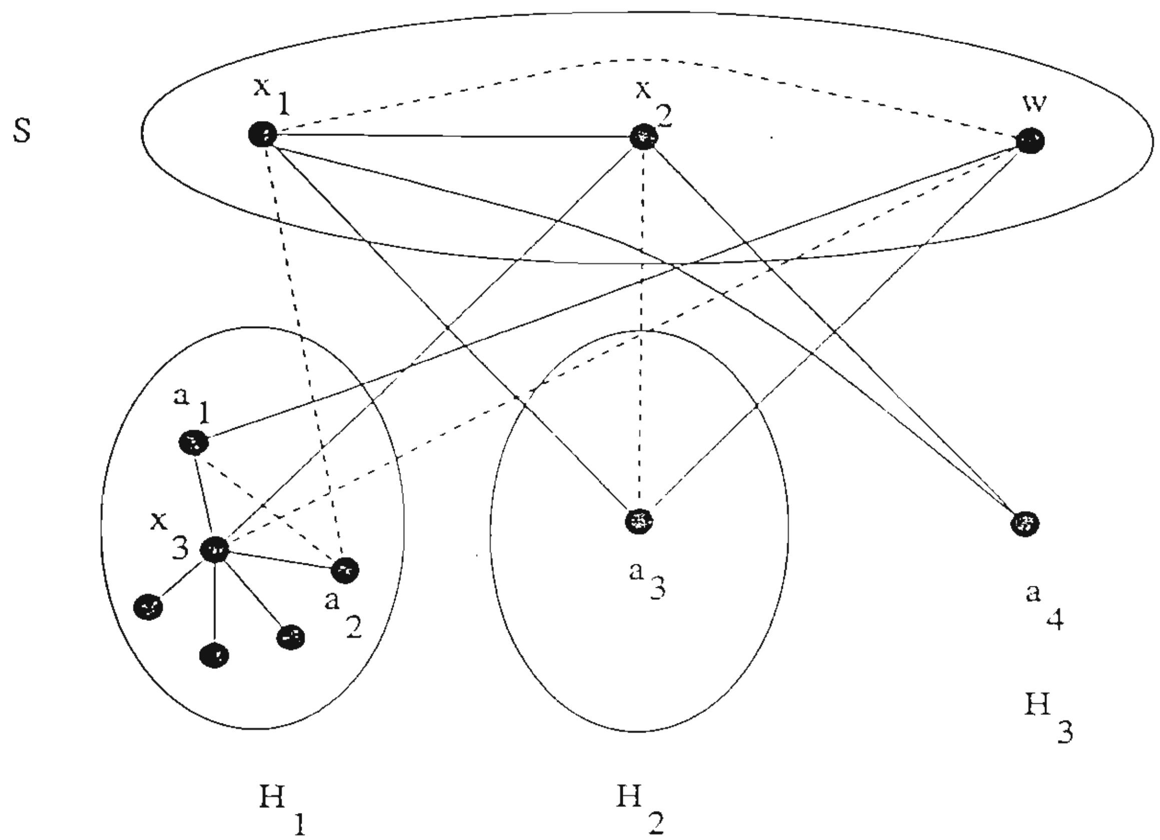
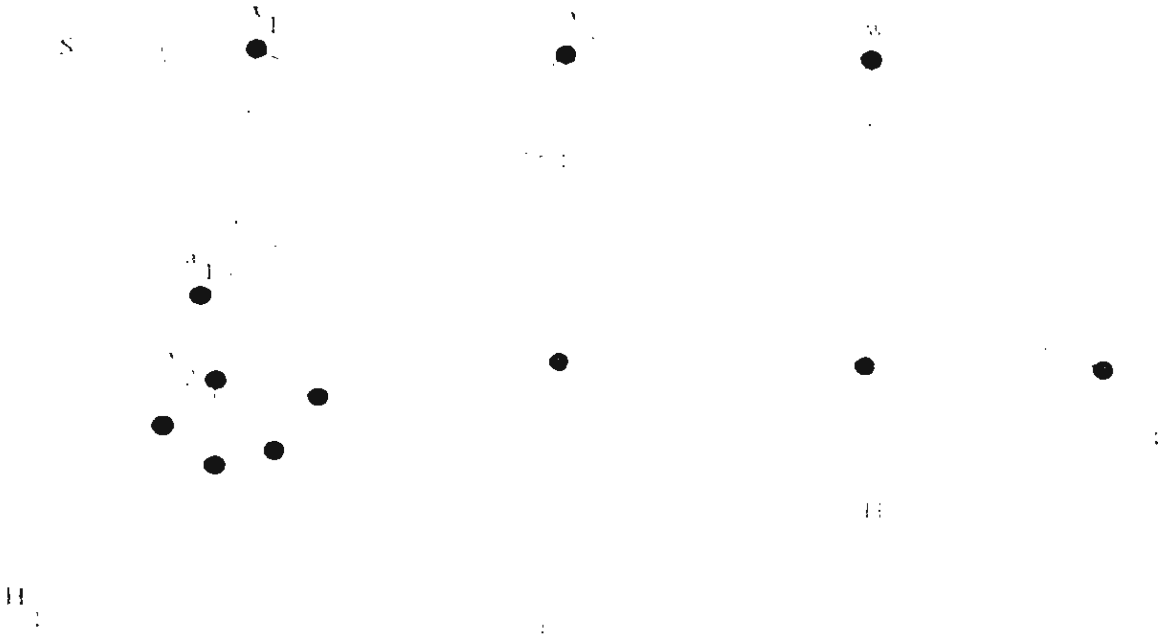
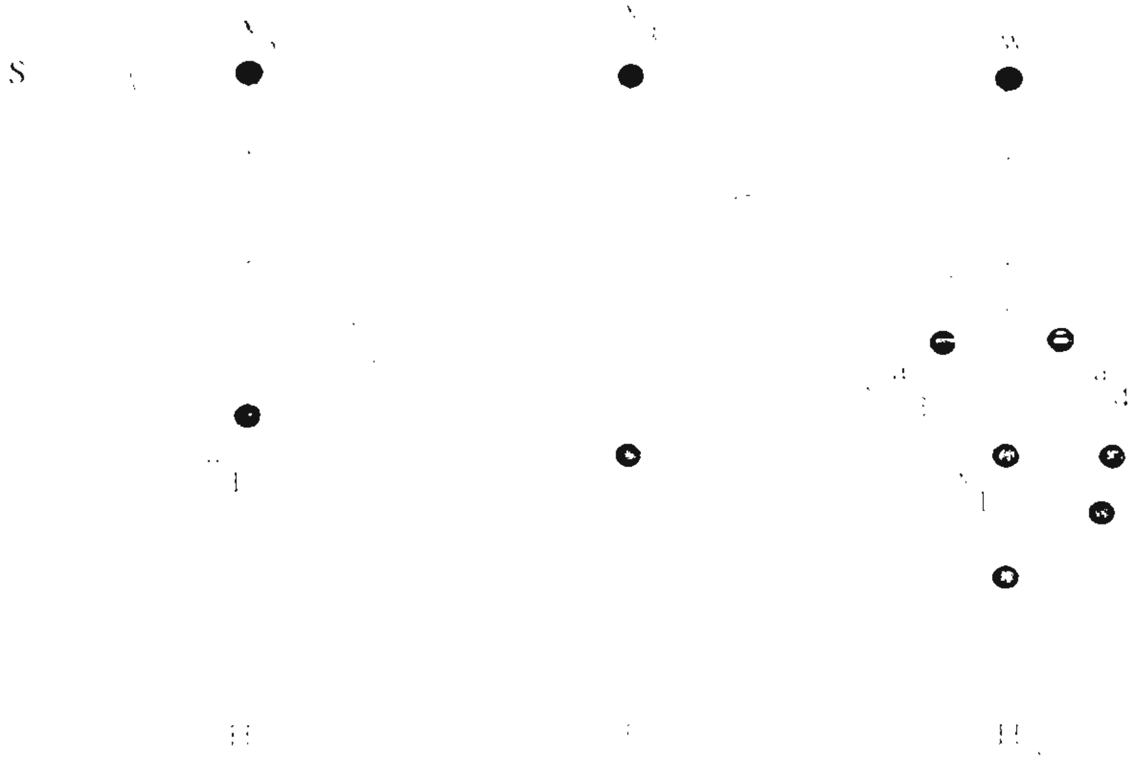


Figure 2.3





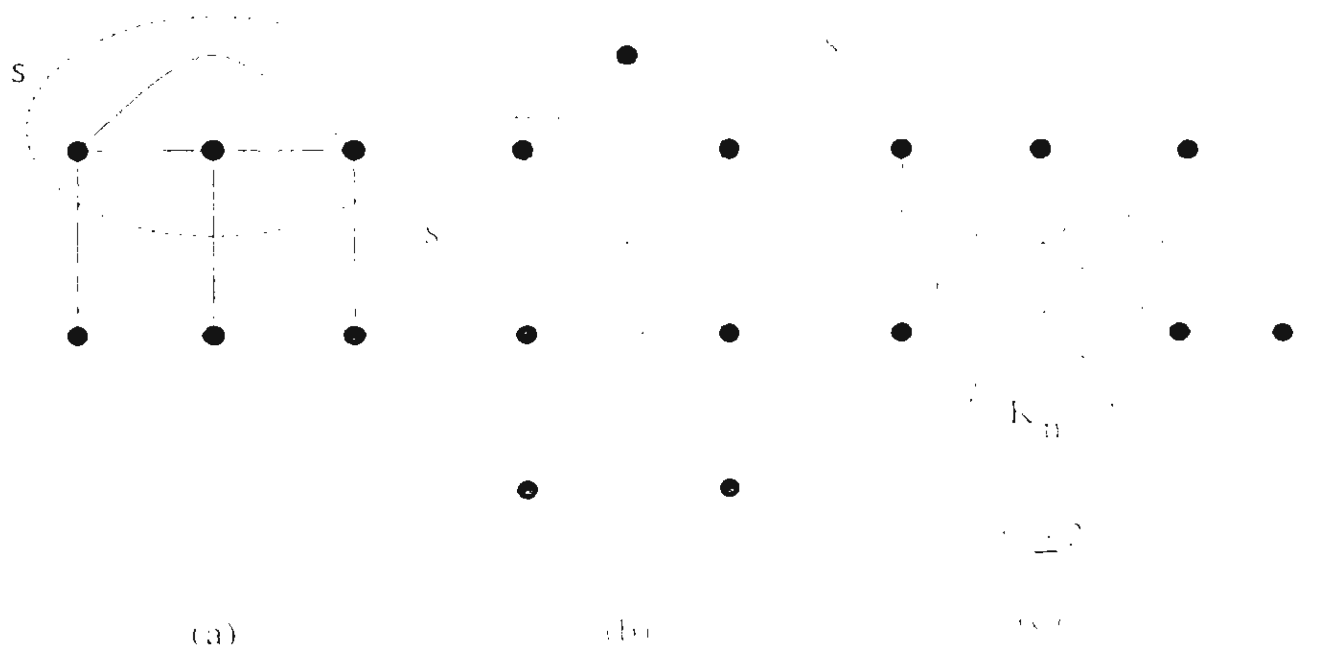


Figure 2.6

Research paper 2:

N.Ananchuen and M.D.Plummer, Matching properties in domination critical graphs. (submitted).

MATCHING PROPERTIES

IN

DOMINATION CRITICAL GRAPHS

by

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Abstract

A graph G is said to be k-γ-critical if the size of any minimum dominating set of vertices is k, but if any edge is added to G the resulting graph can be dominated with $k-1$ vertices. The structure of k-γ-critical graphs remains far from completely understood, even in the special case when the domination number $\gamma = 3$. In a 1983 paper, Sumner and Blitch proved a theorem which says that if S is any vertex cutset of such a graph, then $G - S$ has at most $|S| + 1$ components.

A graph G is factor-critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$ and is bicritical if $G - u - v$ has a perfect matching for every pair of distinct vertices $u, v \in V(G)$. In a previous paper [AP1], we improved and extended the Sumner-Blitch result above. Using these improvements, we show in the present paper that under certain assumptions regarding connectivity and minimum degree, a 3-γ-critical graph G will be either factor-critical (if $|V(G)|$ is odd) or bicritical (if $|V(G)|$ is even).

Keywords: domination, critical edge, matching, factor-critical, bicritical, claw-free

1. Introduction

Let G denote a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a **vertex dominating set** for G if every vertex of G either belongs to S

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is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the **vertex domination number** (or simply the domination number) of G and is denoted by $\gamma(G)$. Graph G is said to be k-γ-**domination critical** if $\gamma(G) = k$, but $\gamma(G+e) = k-1$ for each edge $e \notin E(G)$. In this paper, we will be concerned only with the case $k = 3$.

If u, v and w are vertices of G and u and v dominate $G - w$, we will follow previously accepted notation and write $[u, v] \longrightarrow w$. Suppose G is 3-γ-critical. If u and v are non-adjacent vertices of G, then $\gamma(G + uv) = 2$ and so there is a vertex $x \in V(G)$ such that either $[u, x] \longrightarrow v$ or $[v, x] \longrightarrow u$.

Sumner and Blitch [SB] initiated work on matchings in 3-γ-critical graphs. The following lemma from that paper will be very useful in our work to follow.

Lemma 1.1. Let G be a connected 3-γ-critical graph and let S be an independent set of $n \geq 2$ vertices in $V(G)$.

(i) Then the vertices of S can be ordered $a_1, a_2, \ldots, a_n$ in such a way that there exists a sequence of distinct vertices $x_1, x_2, \ldots, x_{n-1}$ so that $[a_i, x_i] \longrightarrow a_{i+1}$ for $i = 1, 2, \ldots, n-1$.

(ii) If, in addition, $n \geq 4$, then the x_i's can be chosen so that $x_1 x_2 \cdots x_{n-1}$ is a path and $S \cap \{x_1, \ldots, x_{n-1}\} = \emptyset$.

∎

In what is to follow, we shall also make frequent use of the following easy result.

Lemma 1.2. Let G be a 3-γ-critical graph and let u and v be non-adjacent vertices of G. If x is a vertex of G such that $[u, x] \longrightarrow v$, then $xv \notin E(G)$ and if x is a vertex of G with $[v, x] \longrightarrow u$ then $xu \notin E(G)$.

Proof: Suppose $[u, x] \longrightarrow v$. If $xv \in E(G)$, then u and x dominate G, contradicting the assumption that $\gamma(G) = 3$. Similarly, if $[v, x] \longrightarrow u$, then $xu \notin E(G)$.

∎

The next result which will prove useful to us was conjectured by Wojcicka [W] and in a series of three papers ([FTZ, FTWZ, TWZ]) proved by Favaron, Flandrin, Tian, Wei and Zhang. (In her survey [My], however, Mynhardt refers to this result as "Wojcicka's Theorem". See also [Mo].)

Theorem 1.3. Every connected 3-γ-critical graph having minimum degree at least 2 has a Hamiltonian cycle.

∎

The following lemma, which may be viewed as a toughness result, is due to Sumner and Blitch [SB] and leads to the first results on matchings in 3-γ-critical graphs which we then state as Lemma 1.5.

Lemma 1.4. Let G be a connected 3-γ-critical graph. Then if T is a separating set of vertices for G, it follows that $G - T$ has at most $|T| + 1$ components.

A **near-perfect matching** in a graph G is one which covers all but exactly one of the vertices of G. A **factor-critical** graph G is one with the property that $G - \{v\}$ contains a perfect matching for every vertex $v \in V(G)$. Throughout the rest of this paper, $c(G)$ (respectively $c_o(G)$) will denote the number of components (respectively *odd* components) of graph G.

Lemma 1.5. Let G be a connected 3-γ-critical graph.

(i) Then if $|V(G)|$ is even, G contains a perfect matching, while

(ii) if $|V(G)|$ is odd, G contains a near-perfect matching.

Proof: Part (i) is due to Sumner and Blitch [SB]. We prove only part (ii). Suppose G is a 3-γ-critical graph with an odd number of vertices and suppose G does not contain a near-perfect matching. Consider the Gallai-Edmonds decomposition of G. (See [LP].) That is, let $D(G)$ denote the set of all vertices $v \in V(G)$ such that some maximum matching of G does not cover v. Let $A(G)$ denote the set of all neighbors of vertices of $D(G)$ which are not themselves in $D(G)$ and finally, let $C(G) = V(G) - D(G) - A(G)$. Since G contains no near-perfect matching, the number of odd components of $D(G)$ is at least two larger than $|A(G)|$. If $A(G) = \emptyset$, then G is disconnected, a contradiction. So $A(G) \neq \emptyset$ and hence is a vertex cutset of G. But $c(G - A(G)) \geq |A(G)| + 2$ which contradicts Lemma 1.4. ∎

The next result proved in [AP1] (see also the related paper [CTW]) significantly sharpens Lemma 1.4 and will be of considerable importance in our work to follow.

Theorem 1.6. Let G be a connected 3-γ-critical graph and let S be a vertex cutset in G. Then

(i) if $|S| \geq 4$, $G - S$ has at most $|S| - 1$ components,

(ii) if $|S| = 3$, then $G - S$ contains at most $|S|$ components, and if $G - S$ has exactly three components, then each component is complete and at least one is a singleton.

(iii) if $|S| = 2$, then $G - S$ has at most three components and if $G - S$ has exactly three components, then G must have the structure shown below in Figure 1.1.

(iv) if $|S| = 1$, then $G - S$ has two components, exactly one of which is a singleton. Furthermore, G has exactly one or two cutvertices and if it has two, G is isomorphic to a graph of the type shown in Figure 1.1.

Figure 1.1.

A graph G is said to be **bicritical** if $G - u - v$ contains a perfect matching for every oice of two distinct vertices u and $v \in V(G)$. Bicritical graphs play an important role a canonical decomposition theory for arbitrary graphs in terms of their matchings. The erested reader is referred to [LP] for much more on this subject.

Our purpose in the present paper is to use the above assembled known results to help ove several new theorems which say that under certain assumptions on connectivity and inimum degree, a 3-γ-critical graph G either is factor-critical (when $|V(G)|$ is odd) or critical (when $|V(G)|$ is even).

2. 3-γ-criticality, Bicriticality and Factor-criticality

Our first main result shows that if the connectivity and minimum degree are suffiiently high in a 3-γ-critical graph of even order, then the graph must be bicritical.

Theorem 2.1. If G is a 3-connected 3-γ-critical graph with minimum degree at least and having even order, then G is bicritical.

Proof: Suppose, to the contrary, that G is a 3-connected 3-γ-critical graph with ninimum degree at least 4 and having even order, but G is not bicritical. Then there exist ertices u and v in G such that $G' = G - \{u, v\}$ has no perfect matching. By Tutte's 1-factor theorem, there then must exist an $S' \subseteq V(G')$ such that

$$c_o(G' - S') > |S'|.$$

Since $|V(G')|$ is even, by parity $c_o(G' - S') \geq |S'| + 2$. Put $S = S' \cup \{u, v\}$. Clearly, $c_o(G - S) = c_o(G' - S')$. But by Lemma 1.5(i), G has a perfect matching, so

$$|S| = |S'| + 2 \leq c_o(G' - S') = c_o(G - S) \leq |S|.$$

and hence $|S| = c_0(G - S)$.

By Theorem 1.6(i), $|S| \leq 3$. Since G is 3-connected, $|S| = c_0(G - S) = 3$ and $G - S$ has no even components. By Theorem 1.6(ii), at least one component of $G - S$ is a singleton. Let H_1 denote such a singleton component of $G - S$ and let $V(H_1) = \{x\}$. Then deg $(x) \leq 3$, a contradiction. Hence G is bicritical.

∎

The minimum degree bound in Theorem 2.1 is best possible as there exist 3-connected 3-γ-critical graphs having minimum degree 3 which are not bicritical. Two such graphs are shown in Figure 2.1. The first is due to Sumner and Blitch [SB].

Figure 2.1

On the other hand, if we consider planar graphs, then this minimum degree bound ɑn be relaxed.

Theorem 2.2. If G is a 3-connected 3-γ-critical planar graph having even order, then ('is bicritical.

Proof: Suppose G is not bicritical. Using exactly the same argument as in the proof of 'heorem 2.1, again we arrive at the conclusion that the Tutte Set S defined there has size and $c_o(G - S) = 3$ as well. Since G is 3-connected, each of the three (odd) components $G - S$ has edges to each of the three vertices of S. But then G is contractible to a $K_{3,3}$ ıd hence is non-planar, a contradiction.

∎

Let k be an integer such that $0 \leq k < |V(G)|/2$. G is said to be **k-extendable** every matching of size k in G extends to (i.e., is a subset of) a perfect matching in ;. ("0-extendable" will be taken to mean that G has a perfect matching.) Note that a -connected 3-γ-critical even graph is not necessarily 1-extendable. In Figure 2.1, graph a) is 1-extendable, but graph (b) is not.

In the case when G is not bipartite, 2-extendable is a stronger property than that of ›icriticality. More particularly, we have the following theorem [P1].

Theorem 2.3. If G is 2-extendable, then either G is bipartite or G is bicritical.

∎

Theorem 2.1 is also sharp in the sense that there exist graphs which are 3-connected (in fact 4-connected) with minimum degree at least 4, 3-γ-critical and even, but not 2-extendable. One such family of graphs is shown in Figure 2.2. (This graph family is a subclass of a larger class of 3-γ-critical graphs first discovered by Sumner and Blitch [SB].)

Figure 2.2.

Now let us turn our attention to the family of factor-critical graphs. (We refer the reader again to [LP] for a more extensive treatment of these graphs.) The following result s an immediate result of "Wojcicka's Theorem" (see above). (Note also that for a 3-γ-critical graph G, the assumptions that G is 2-connected and that G has minimum degree at least two are equivalent. This is an immediate consequence of Lemma 5.5.8 of [B].)

Theorem 2.4. Let G be a 2-connected 3-γ-critical graph having odd order. Then G is factor-critical.

∎

The graphs shown in Figure 1.1 (with n even) are 3-γ-critical and connected, but not factor-critical. Thus our lower bound on the connectivity stated in the hypotheses of Theorem 2.4 is best possible. More generally, if G is a 3-γ-critical graph with a cutvertex v, then v is adjacent to an endvertex (cf. [B]) and hence mindeg $G = 1$ and hence G is not factor-critical.

3. A Result About Claw-free Graphs

A graph is said to be **claw-free** if it contains no induced subgraph isomorphic to $K_{1,3}$. In [P2] the following result was proved.

Theorem 3.1. If G is a 3-connected claw-free graph of even order, then G is bicritical.

∎

If the even graphs involved are 3-γ-critical, we can lower the demand on connectivity and still obtain bicriticality. Before we state our result, however, we recall another result of Sumner and Blitch [SB] which will be useful in our proof.

Theorem 3.2. The diameter of a 3-γ-critical graph is at most 3.

∎

Theorem 3.3. Let G be a 3-γ-critical 2-connected claw-free graph of even order. Then if mindeg $G \geq 3$, G is bicritical.

Proof: Suppose to the contrary that G is not bicritical. Then there exist vertices u and v of G such that $G' = G - \{u, v\}$ has no perfect matching. By Tutte's theorem, there is a subset $S' \subseteq V(G')$ such that $c_o(G' - S') > |S'|$ and so by parity since $|V(G)|$ is even, $c_o(G' - S') \geq |S'| + 2$. Let $S = S' \cup \{u, v\}$. Clearly $|S| = |S'| + 2$ and $|S| = |S'| + 2 \leq c_o(G' - S') = c_o(G - S) \leq |S| = |S'| + 2$, since G contains a perfect matching by Lemma 1.5(i). Thus $c_o(G - S) = |S|$.

By Theorem 1.6(i), $|S| \leq 3$. Let $H_i, i = 1, \ldots, |S|$, denote the odd components of $G - S$. First suppose that $|S| = 3$. Clearly $G - S$ has no even components. Set $S = \{u, v, w\}$. By Theorem 1.6(ii), at least one component of $G - S$ is a singleton. Without loss of generality, we may assume that $|V(H_1)| = 1$ and that $V(H_1) = \{x\}$. Since mindeg $G \geq 3$, vertex x is adjacent to every vertex of S. Since G is 2-connected, there are at least two vertices of S which are adjacent to some vertex of H_2. Similarly, there are at least two vertices of S which are adjacent to some vertex of H_3. Because $|S| = 3$, there must be a vertex, say u, ch that u is adjacent to some vertex of H_2 and a vertex of H_3. Thus u is a claw center G, a contradiction. This proves that $|S| \leq 2$. Moreover, since G is 2-connected, $|S| = 2$.

Suppose $S = \{u, v\}$. If $G - S$ contains an even component then $c(G - S) = 3$. Thus, Theorem 1.6(iii), G must have the structure shown in Figure 1.1 and hence G is not connected, a contradiction. Therefore, $G - S$ has no even components. Thus we need ily consider the case when $G - S$ contains exactly two odd components and no even omponent.

Since mindeg $G \geq 3$, it follows that $|V(H_1)| \geq 3$ and $|V(H_2)| \geq 3$. Now $\gamma(G) = 3$, there exists a vertex $z \in V(G) - \{u, v\}$ such that $z \notin N(u) \cup N(v)$. Let $A = V(G) - S \cup N(u) \cup N(v))$. Thus $A \neq \emptyset$. Furthermore, suppose $z_1, z_2 \in A$. If $z_1 \in V(H_1)$ and $\in V(H_2)$, then $d(z_1, z_2) > 3$, contradicting Theorem 3.2. Thus z_1 and z_2 must belong the same component of $G - S$, say H_2. This implies that $V(H_1) = N_{H_1}(u) \cup N_{H_1}(v)$ nd $V(H_2) = N_{H_2}(u) \cup N_{H_2}(v) \cup A$. Moreover, since G is 2-connected and $|V(H_i)| \geq 3$ r $i = 1, 2$, it follows that $N(u) \cap V(H_i) \neq \emptyset$ and $N(v) \cap V(H_i) \neq \emptyset$, for $i = 1, 2$. Now uppose $x \in N_{H_1}(u)$ and $y \in N_{H_2}(v)$. Figure 3.1 illustrates the situation.

Figure 3.1.

Consider $G + xy$. Since G is 3-γ-critical, there exists a vertex $w \in V(G) - \{x, y\}$ such that either $[x, w] \longrightarrow y$ or $[y, w] \longrightarrow x$. We distinguish these two cases.

Case 1: Suppose $[x, w] \longrightarrow y$. Clearly, $w \in V(H_2)$ and $wy \notin E(G)$; otherwise $\{x, w\}$ dominates G. If $w \in N_{H_2}(v)$, then $G[\{v, w, y, v'\}]$ is a claw centered at v for some vertex $v' \in N_{H_1}(v)$, a contradiction. Thus $w \in N_{H_2}(u) - N_{H_2}(v)$ or $w \in A$.

Case 1.1: Suppose $w \in N_{H_2}(u) - N_{H_2}(v)$. Since $[x, w] \longrightarrow y$, w is adjacent to every vertex of $V(H_2) - \{y\}$ and x is adjacent to every vertex of $V(H_1) \cup \{v\}$. Figure 3.2 illustrates this situation.

Figure 3.2.

Now consider $G+vw$. There is a vertex $z \in V(G)-\{v,w\}$ such that either $[v,z] \longrightarrow w$ or $[w,z] \longrightarrow v$. Suppose $[v,z] \longrightarrow w$. Since $A \neq \emptyset$ and v is not adjacent to any vertex of A, it follows that $z \in V(H_2)$. Because $zw \notin E(G)$ and w is adjacent to every vertex of $V(H_2)-\{y\}$, it follows that $z=y$. Furthermore, since $[v,z] \longrightarrow w$ and $z=y$, vertex v is adjacent to every vertex of H_1. But then $\{v,w\}$ dominates G, a contradiction.

Thus $[w,z] \longrightarrow v$. Since $wy \notin E(G)$ and $[w,z] \longrightarrow v$, $z=u$. Hence $zy = uy \notin E(G)$ and $G[\{u,x,y,w\}]$ is a claw centered at vertex u, a contradiction. This proves that $w \notin N_{H_2}(u) - N_{H_2}(v)$.

Case 1.2: Suppose $w \in A$. Recall that $A = V(H_2) - (N_{H_2}(u) \cup N_{H_2}(v))$. Since $[x,w] \longrightarrow y$ and $w \in A$, x is adjacent to every vertex of $V(H_1) \cup \{u,v\}$ and w is adjacent to every vertex of $V(H_2)-\{y\}$. Figure 3.3 depicts this situation.

Figure 3.3 .

Since $d_G(w) \geq 3$ and $wy \notin E(G)$, $|V(H_2)| \geq 5$. We distinguish two subcases.

Subcase 1.2.1: Suppose $uy \notin E(G)$. We will show that $N_{H_2}(u) \cap N_{H_2}(v) = \emptyset$. Suppose not; say vertex $y_1 \in N_{H_2}(u) \cap N_{H_2}(v)$. If $y_1 y \in E(G)$, then $G[\{y_1,u,y,w\}]$ is a claw centered at y_1, a contradiction. Thus $y_1 y \notin E(G)$. But then $G[\{v,x,y,y_1\}]$ is a claw centered at v, again a contradiction. This proves that $N_{H_2}(u) \cap N_{H_2}(v) = \emptyset$. Since $N_{H_2}(u) \neq \emptyset$, there is a vertex $u_1 \in N_{H_2}(u)$. Since $N_{H_2}(u) \cap N_{H_2}(v) = \emptyset$, $u_1 v \notin E(G)$. If $u_1 y \in E(G)$, then $G[\{u_1,u,y,w\}]$ is a claw centered at u_1, a contradiction. Thus $u_1 y \notin E(G)$. Figure 3.4 illustrates this situation.

Figure 3.4.

Now consider $G+vu_1$. There exists a vertex $z \in V(G)-\{v, u_1\}$ such that either $[v, z] \longrightarrow u_1$ or $[u_1, z] \longrightarrow v$. First suppose that $[v, z] \longrightarrow u_1$. Then $zu_1 \notin E(G)$. Since v is not adjacent to any vertex of A and $[v, z] \longrightarrow u_1$, $z \in V(H_2) - \{u_1\}$. If $z \in N_{H_2}(u)$, then $G[\{u, x, u_1, z\}]$ is a claw centered at u since $u_1 z \notin E(G)$, a contradiction. Hence $z \in N_{H_2}(v) - N_{H_2}(u)$ or $z \in A$. In either case, $zu \notin E(G)$. Since $[v, z] \longrightarrow u_1$, v is adjacent to every vertex of $V(H_1) \cup \{u\}$. Because $yv \in E(G)$ and w is adjacent to every vertex of $V(H_2) - \{y\}$, $\{v, w\}$ dominates G, a contradiction. Hence $\{v, z\}$ does not dominate $G - u_1$. Therefore $[u_1, z] \longrightarrow v$. Since $u_1 y \notin E(G)$ and z must be adjacent to y and to every vertex of H_1, it follows that $z = u$. But this is impossible since $uy \notin E(G)$. This proves that $\{u_1, z\}$ does not dominate $G - v$ and contradicts the 3-γ-criticality of G. Hence $uy \in E(G)$.

Subcase 1.2.2: So suppose $uy \in E(G)$. We will show that $|A| \geq 2$. Suppose not. Then $|A| = 1$ and $A = \{w\}$.

Since $|V(H_2)| \geq 5$, it follows that $|N_{H_2}(u) \cup N_{H_2}(v)| \geq 4$. We will show that $G[N_{H_2}(u) \cup N_{H_2}(v)]$ is a complete graph. Suppose not. Then there exist a pair of vertices w_1 and w_2 in $N_{H_2}(u) \cup N_{H_2}(v)$ such that $w_1 w_2 \notin E(G)$. Since G is claw-free, we may assume without loss of generality that $w_1 \in N_{H_2}(u) - N_{H_2}(v)$ and $w_2 \in N_{H_2}(v) - N_{H_2}(u)$.

Now consider $G + uw_1$. By applying an argument similar to that presented in Case 1.2.1 for $G + vu_1$, but replacing u_1 with w_1 and y with w_2, we get a contradiction. Hence $G[N_{H_2}(u) \cup N_{H_2}(v)]$ is complete. But then if we choose any vertex $y_1 \in (N_{H_2}(u) \cup N_{H_2}(v)) - \{y\}$, we find that $\{x, y_1\}$ dominates G, a contradiction. This proves that $|A| \geq 2$.

Recall that $[x, w] \longrightarrow y$ and that $w \in A$. Since $|A| \geq 2$, there is a vertex $w_1 \in A - \{w\}$. Figure 3.5 depicts the situation.

Figure 3.5.

Consider $G + xw$. There exists a vertex $z \in V(G) - \{x, w\}$ such that $[w, z] \longrightarrow x$ or $[x, z] \longrightarrow w$. Suppose first that $[w, z] \longrightarrow x$. By Lemma 1.2, $zx \notin E(G)$. Since x is adjacent to every vertex of $V(H_1) \cup \{u, v\}$, $z \notin V(H_1) \cup \{u, v\}$. But then $\{z, w\}$ does not dominate $G - \{x\}$, a contradiction.

Hence we can suppose that $[x, z] \longrightarrow w$. By Lemma 1.2, $zw \notin E(G)$. Since $A - \{w\} \neq \emptyset$ and w is adjacent to every vertex of $V(H_2) - \{y\}$, $z = y$. Hence y is adjacent to every ertex of $V(H_2) - \{w\}$. Consequently, $uv \in E(G)$: otherwise $G[\{y, u, v, w_1\}]$ is a claw entered at y.

Next we consider $G + uw$. There exists a vertex $z \in V(G) - \{u, w\}$ such that $[w, z] \longrightarrow$ u or $[u, z] \longrightarrow w$.

Suppose first that $[w, z] \longrightarrow u$. Since $wy \notin E(G)$ and $[w, z] \longrightarrow u$, z must be adjacent to y and to every vertex of H_1. Thus $z = v$. This implies that $\{w, z\}$ dominates G since $wu = vu \in E(G)$, a contradiction. Hence $[u, z] \longrightarrow w$. since $A - \{w\} \neq \emptyset$ and $[u, z] \longrightarrow w$, it follows that $z \in V(H_2)$. Thus u is adjacent to every vertex of H_1. Recall that $uv \in E(G)$ and $uy \in E(G)$. Hence $\{u, w\}$ dominates G, contradicting the 3-γ-criticality of G. This completes the proof in Subcase 1.2.2 and consequently the proof of Case 1.

Case 2: Suppose $[y, w] \longrightarrow x$. Clearly $wx \notin E(G)$. This implies that $w \neq u$. Since $y \in V(H_2)$ and $[y, w] \longrightarrow x$, $w \in \{v\} \cup (V(H_1) - \{x\})$.

Case 2.1: Suppose $w = v$. Then $[y, v] \longrightarrow x$. Thus v is adjacent to every vertex of $V(H_1) - \{x\}$ and y is adjacent to every vertex of A. If y is not adjacent to some vertex of $V(H_2) - (A \cup \{y\})$, say y_1, then $vy_1 \in E(G)$ since $[y, v] \longrightarrow x$. But then $G[\{v, y, y_1, v'\}]$ is a claw centered at v for some vertex $v' \in N_{H_1}(v)$, a contradiction. Hence vertex y is adjacent to every vertex of $V(H_2) - (A \cup \{y\})$ and thus to every vertex of $V(H_2) - \{y\}$. Figure 3.6 illustrates this situation.

Figure 3.6.

Since G is claw-free and v is not adjacent to any vertex of A, $G[A]$ is complete. We will show that $G[N_{H_2}(u) \cup N_{H_2}(v)]$ is complete. Suppose, to the contrary, that there exist a pair of vertices y_1 and y_2 of $N_{H_2}(u) \cup N_{H_2}(v)$ such that $y_1y_2 \notin E(G)$. This implies that $y_1 \in N_{H_2}(u) - N_{H_2}(v)$ and $y_2 \in N_{H_2}(v) - N_{H_2}(u)$ or $y_1 \in N_{H_2}(v) - N_{H_2}(u)$ and $y_2 \in N_{H_2}(u) - N_{H_2}(v)$, since G is claw-free and $N_{H_1}(u) \neq \emptyset$ and $N_{H_1}(v) \neq \emptyset$. Without loss of generality, assume that $y_1 \in N_{H_2}(u) - N_{H_2}(v)$ and $y_2 \in N_{H_2}(v) - N_{H_2}(u)$. Now consider $G + uy_2$. There exists a vertex $z \in V(G) - \{u, y_2\}$ such that $[u, z] \longrightarrow y_2$ or $[y_2, z] \longrightarrow u$.

Suppose first that $[u, z] \longrightarrow y_2$. By Lemma 1.2, $zy_2 \notin E(G)$. Thus $z \neq v$. Further, since G is claw-free, $z \notin N_{H_2}(v)$. Since $[u, z] \longrightarrow y_2$ and $A \neq \emptyset$, $z \in N_{H_2}(u) - N_{H_2}(v)$ or $z \in A$. In either case, $zv \notin E(G)$. Thus u is adjacent to every vertex of $V(H_1) \cup \{v\}$. But then $\{u, y\}$ dominates G, a contradiction. Hence $\{u, z\}$ does not dominate $G - y_2$, a contradiction, so $[y_2, z] \longrightarrow u$. By Lemma 1.2, $uz \notin E(G)$. Since $y_1y_2 \notin E(G)$ and

$y_2, z] \longrightarrow u$. z must be adjacent to every vertex of $V(H_1) \cup \{y_1\}$. This implies that $z = v$. ut this is impossible since $vy_1 \notin E(G)$. This proves that $G[N_{H_2}(u) \cup N_{H_2}(v)]$ is complete.

Next we will show that $G[V(H_2)]$ is complete. Recall that $G[N_{H_2}(u) \cup N_{H_2}(v)]$ is omplete, $G[A]$ is complete, and y is adjacent to every vertex in A. Thus we need only now that each vertex of $[N_{H_2}(u) \cup N_{H_2}(v)] - \{y\}$ is adjacent to every vertex of A.

Suppose $y_1 \in [N_{H_2}(u) \cup N_{H_2}(v)] - \{y\}$. Consider $G + xy_1$. There exists a vertex $z \in V(G) - \{x, y_1\}$ such that $[x, z] \longrightarrow y_1$ or $[y_1, z] \longrightarrow x$.

Suppose first that $[x, z] \longrightarrow y_1$. Since $x \in V(H_1)$, $A \neq \emptyset$ and $G[N_{H_2}(u) \cup N_{H_2}(v)]$ is omplete, it follows that $z \in A$. Thus $zv \notin E(G)$ and since $xv \notin E(G)$, $\{x, z\}$ does not lominate $G - y_1$, a contradiction. Hence $[y_1, z] \longrightarrow x$. Consequently, y_1 is adjacent to very vertex of A as required. This proves that $G[V(H_2)]$ is complete.

Now consider the vertex x. Since mindeg $G \geq 3$ and $xv \notin E(G)$, x is adjacent to at east two vertices if $V(H_1) - \{x\}$. Let two such vertices be designated x_1 and x_2. Since $v, y] \longrightarrow x$ and $|V(H_1)| \geq 3$, it follows that $G[V(H_1) - \{x\}]$ is complete because of claw-reedom at vertex v. Choose $y_1 \in N_{H_2}(u)$. Then $\{x_1, y_1\}$ dominates G since x_1 is adjacent o every vertex of $V(H_1) \cup \{v\}$ and y_1 is adjacent to every vertex of $V(H_2) \cup \{u\}$. This ontradicts the fact that $\gamma(G) = 3$ and thus proves that $w \neq v$.

Case 2.2: So suppose $w \in V(H_1) - \{x\}$. Since $[y, w] \longrightarrow x$, w is adjacent to every vertex of $V(H_1) - \{x\}$ and y is adjacent to every vertex of $V(H_2)$. Figure 3.7 depicts this situation.

Figure 3.7.

Recall that $N_{H_1}(u) \cup N_{H_1}(v) = V(H_1)$ and $xu \in E(G)$. Since $wx \notin E(G)$ in the present Case 2.2 and $xu \in E(G)$, it follows that $wu \notin E(G)$, for otherwise $G[\{u, w, x, y_1\}]$ is a claw centered at u for some $y_1 \in N_{H_2}(u)$. Since $V(H_1) = N_{H_1}(u) \cup N_{H_1}(v)$ and $wu \notin E(G)$, it follows that $wv \in E(G)$. Because of claw-freedom at v, $xv \notin E(G)$. We will show that $N_{H_1}(u) \cap N_{H_1}(v) = \emptyset$. Suppose not. Then there is a vertex $w_1 \in N_{H_1}(u) \cap N_{H_1}(v)$. Clearly $w_1 \notin \{w, x\}$. Since w_1u and $w_1v \in E(G)$ and each vertex of $V(H_1)$ belongs to $N_{H_1}(u) \cup N_{H_1}(v)$, it follows that w_1 is adjacent to every vertex of $V(H_1)$ since G is claw-free. But then $\{w_1, y\}$ dominates G, a contradiction. This proves that $N_{H_1}(u) \cap N_{H_1}(v) = \emptyset$.

Since $|V(H_1)|$ is odd and is at least 3, there exists a vertex $w_2 \in V(H_1) - \{x, w\}$. Without loss of generality, we may assume that $w_2 \in N_{H_1}(v)$. Now consider $G + uw$. There xists a vertex $z \in V(G) - \{u, w\}$ such that $[u, z] \longrightarrow w$ or $[w, z] \longrightarrow u$. Suppose first that $[u, z] \longrightarrow w$. By Lemma 1.2, $zw \notin E(G)$. Since $[u, z] \longrightarrow w$ and u is not adjacent to any ertex of $N_{H_1}(v)$, vertex z must be adjacent to every vertex of $V(H_2) \cup (N_{H_1}(v) - \{w\})$. But this is impossible since $A \neq \emptyset$ and, since $w_2 \in N_{H_1}(v) - \{w\}$, $N_{H_1}(v) - \{w\} \neq \emptyset$. Hence $[w, z] \longrightarrow u$. Since $wx \notin E(G)$, vertex z must be adjacent to every vertex of $V(H_2) \cup \{x\}$. But this is impossible since $vx \notin E(G)$ and v is not adjacent to any vertex of A. This ontradiction completes Case 2.2 and the proof of the theorem.

∎

As a final remark, we point out that the preceding result is clearly best possible with espect to the minimum degree condition as the minimum degree of any bicritical graph nust be at least 3.

References

[AP1] N. Ananchuen and M.D. Plummer, Some results related to the toughness of 3-domination-critical graphs, *Discrete Math.*, 2001, (to appear).

[B] P. Blitch, Domination in Graphs, Ph.D. dissertation, Univ. of South Carolina, 1983.

[CTW] Y. Chen, F. Tian and B. Wei, The 3-domination-critical graphs with toughness one, *Utilitas Math.*, 2001, (to appear).

[FTWZ] E. Flandrin, F. Tian, B. Wei and L. Zhang, some properties of 3-domination-critical graphs, *Discrete Math.*, **205**, 1999, 65-76.

[FTZ] O. Favaron, F. Tian and L. Zhang, Independence and Hamiltonicity in 3-domination-critical graphs, *J. Graph Theory*, **25**, 1997, 173-184.

[LP] L. Lovász and M.D. Plummer, *Matching Theory*, Ann. Discrete Math. **29**, North-Holland, Amsterdam, 1986.

[Mo] L. Moodley, Wojcicka's theorem: complete, consolidated proof, *J. Combin. Math. Combin. Comput.*, **33**, 2000, 129-179.

[My] C.M. Mynhardt, On two conjectures concerning 3-domination-critical graphs, *Congr. Numer.*, **135**, 1998, 119-138.

[P1] M.D. Plummer, On n-extendable graphs, *Discrete Math.*, **31**, 1980, 201-210.

[P2] M.D. Plummer, Extending matchings in claw-free graphs, *Discrete Math.*, **125**, 1994, 301-307.

[SB] D.P. Sumner and P. Blitch, Domination critical graphs, *J. Combin. Theory Series B*, **34**, 1983, 65-76.

[TWZ] F. Tian, B. Wei and L. Zhang, Hamiltonicity in 3-domination-critical graphs with $\alpha = \delta + 2$, *Discrete Appl. Math.*, **92**, 1999, 57-70.

[W] E. Wojcicka, Hamiltonian properties of domination-critical graphs, *J. Graph Theory*, **14**, 1990, 205-215.

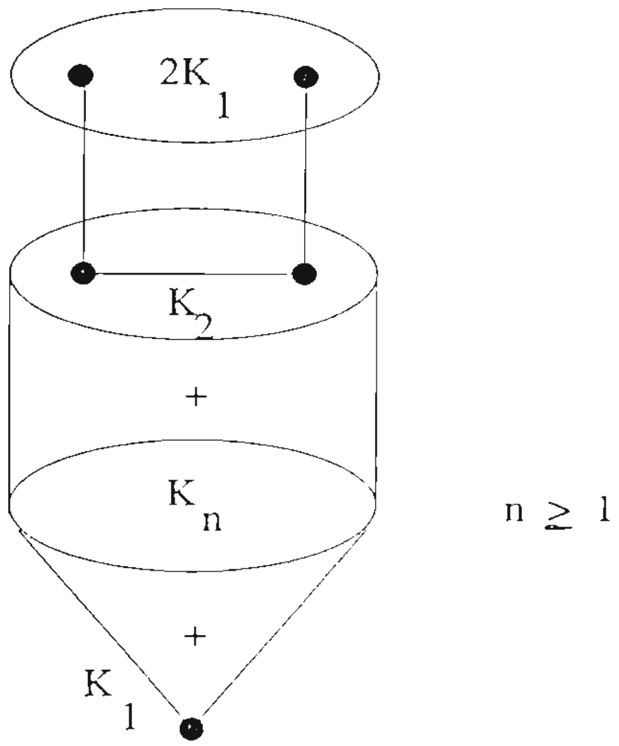
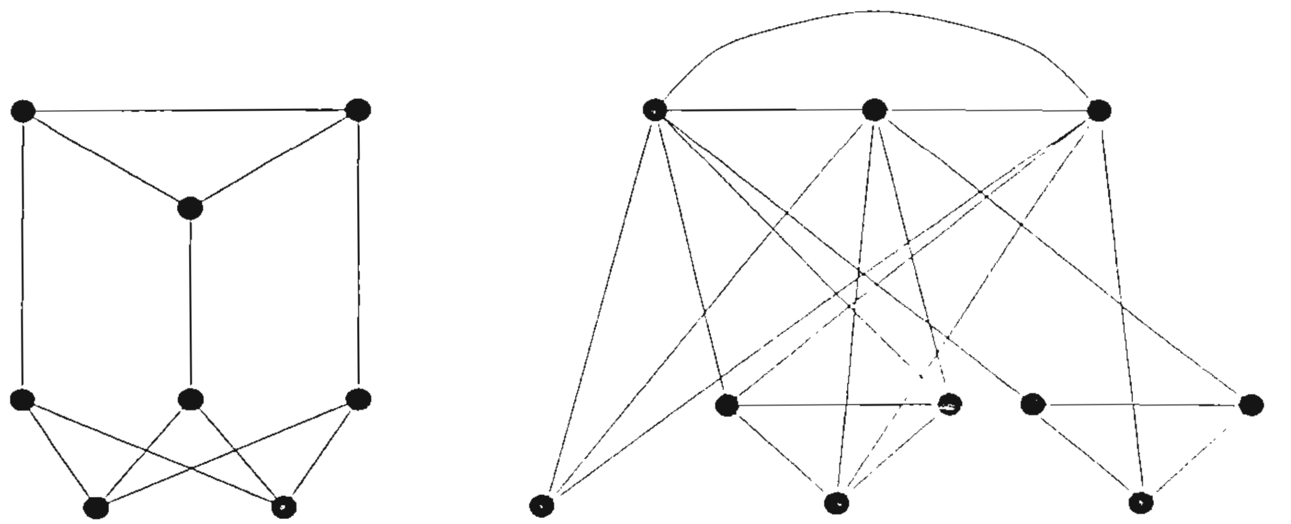


Figure 1.1



(a) (b)

Figure 2.1

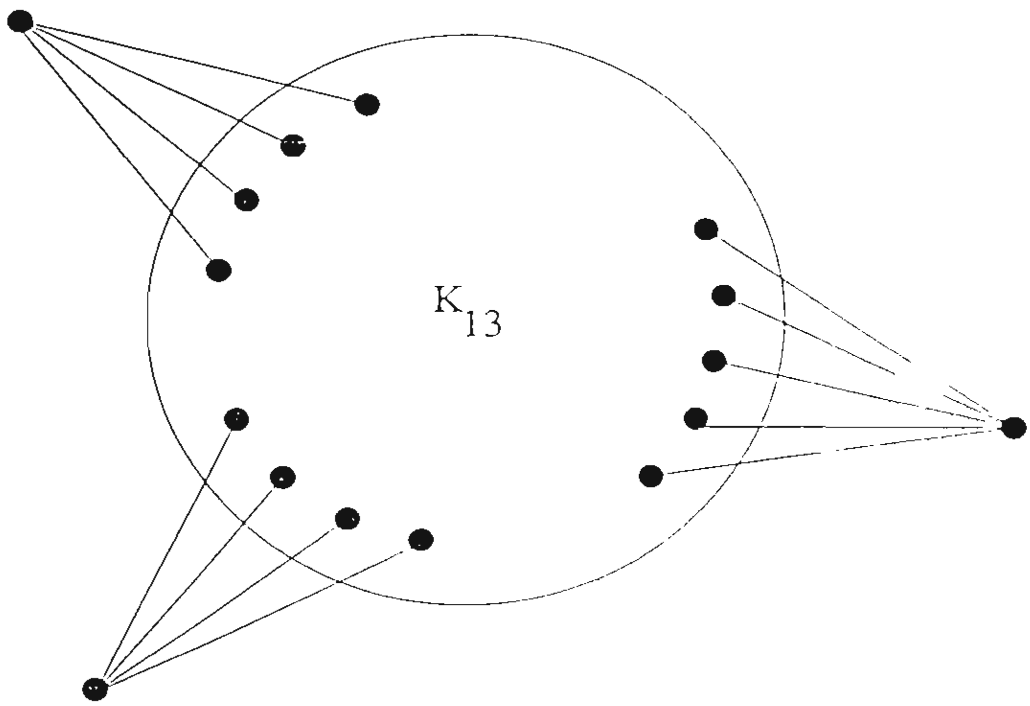


Figure 2.2

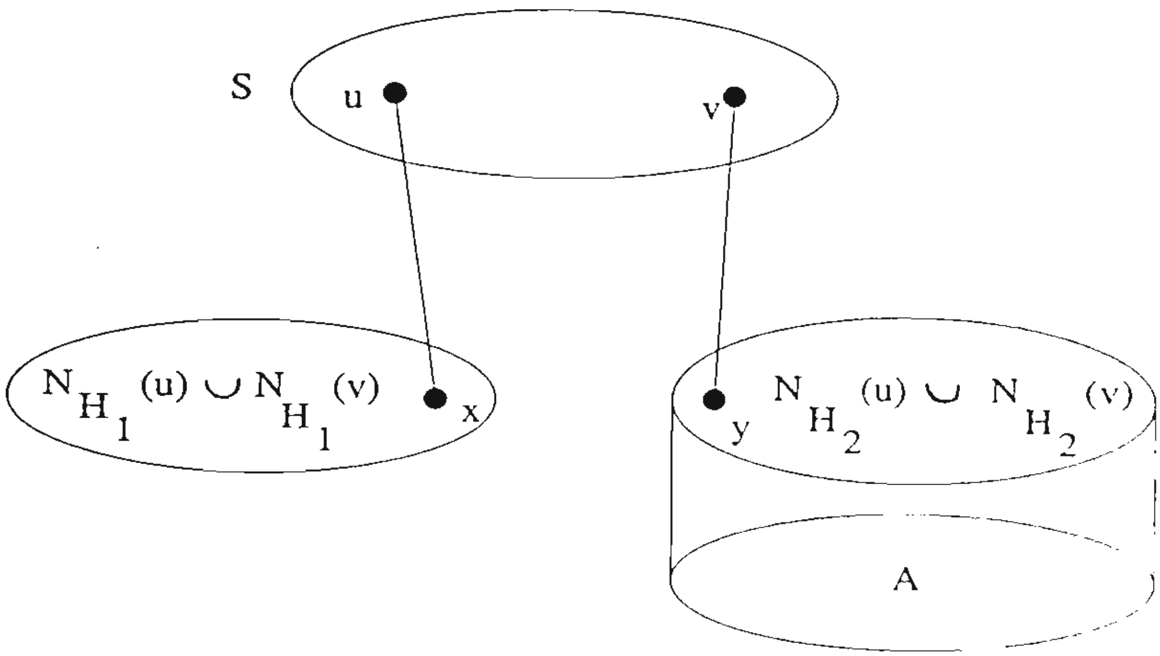


Figure 3.1

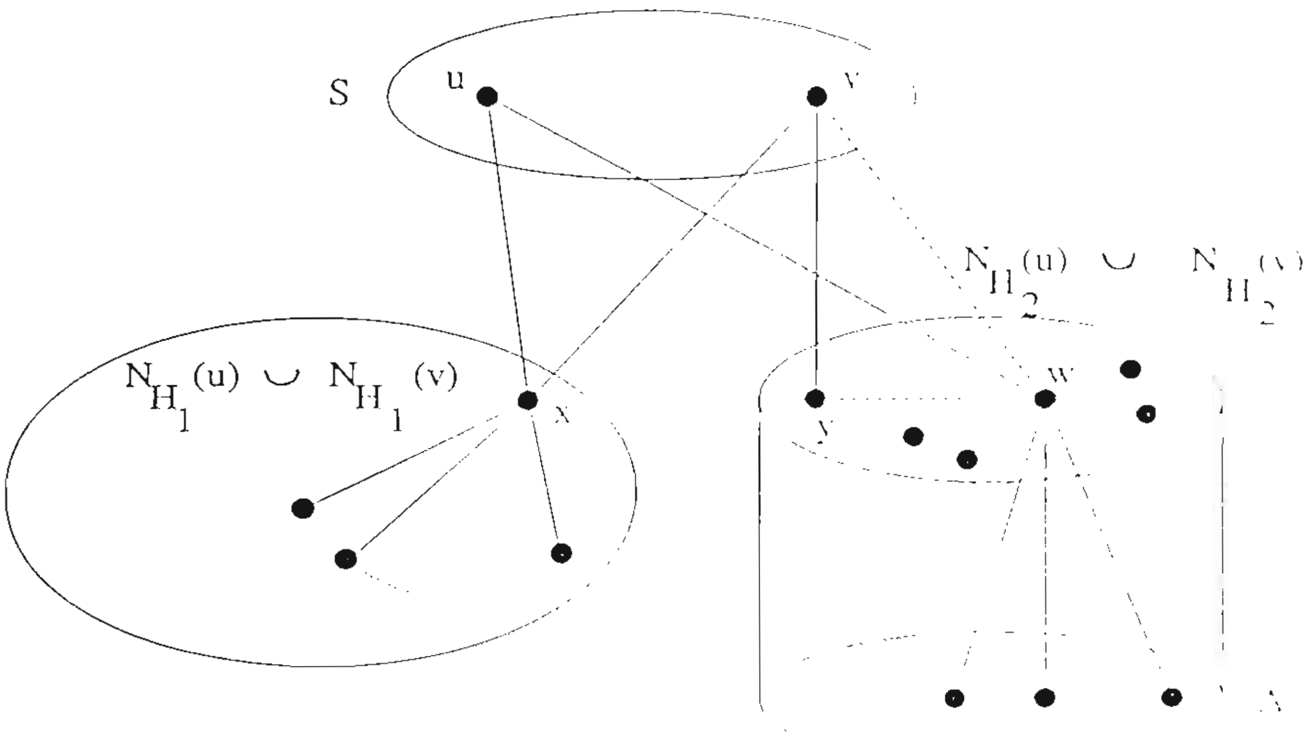


Figure 3.2

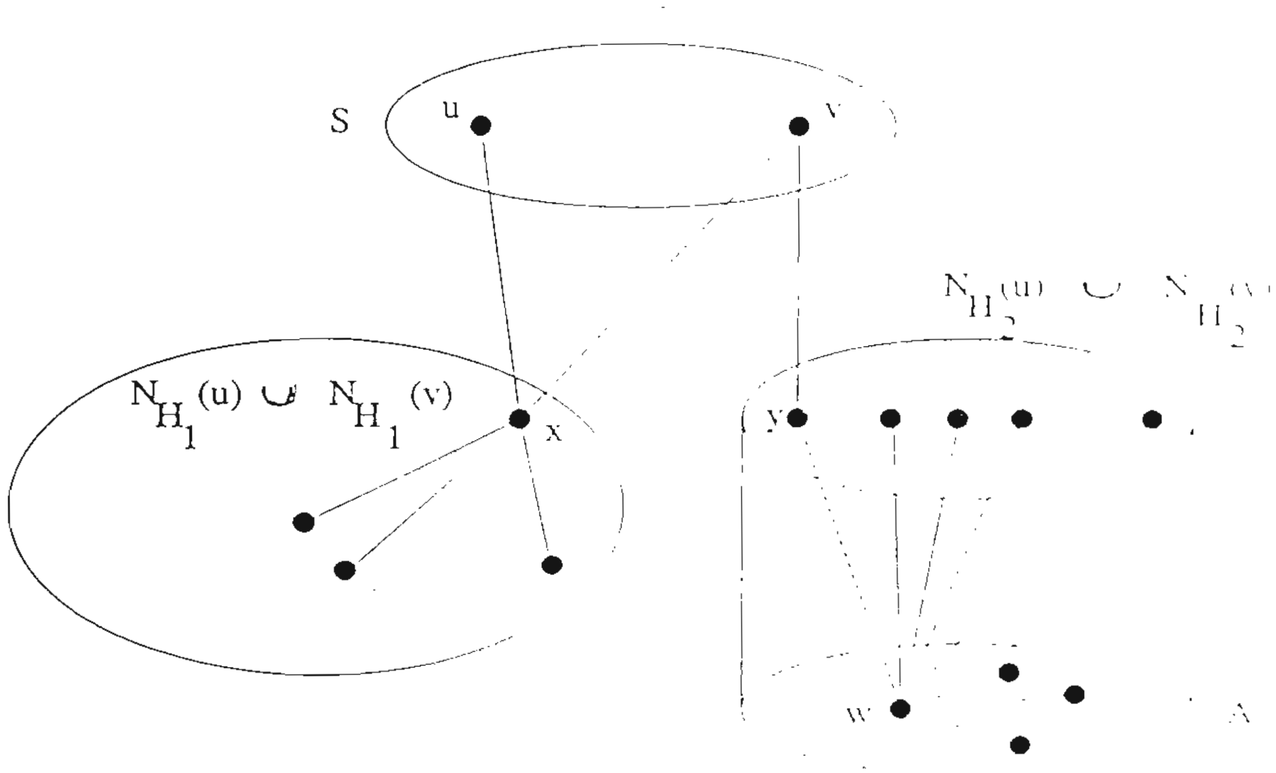


Figure 3.3

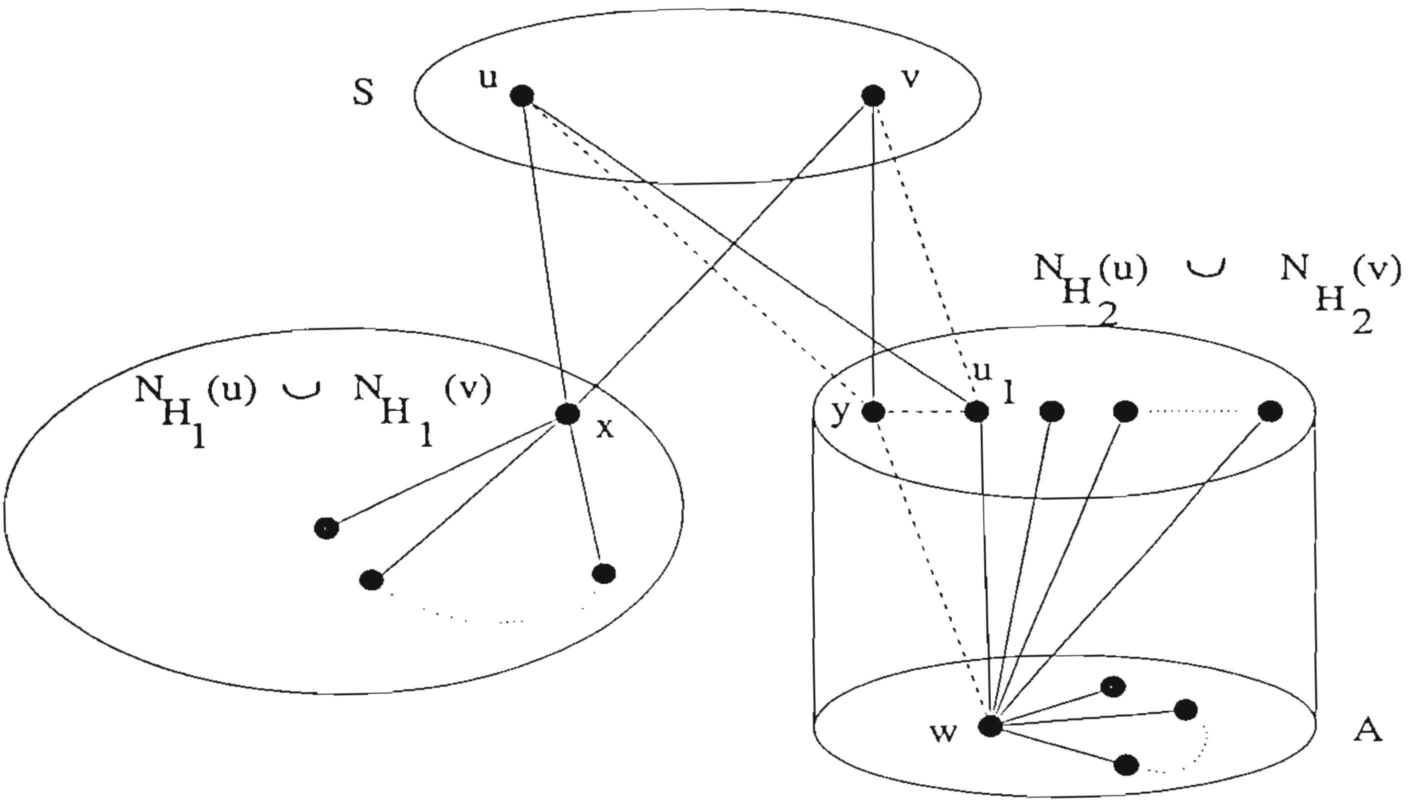


Figure 3.4

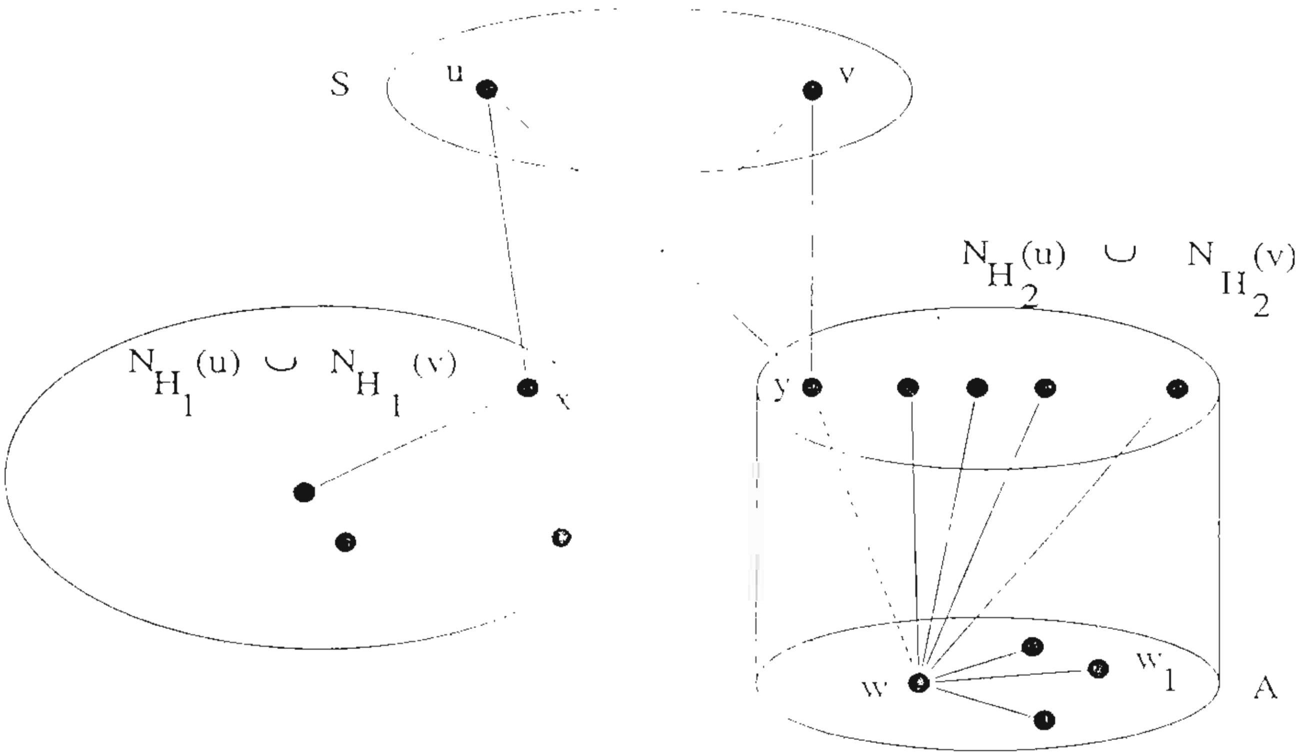


Figure 3.5

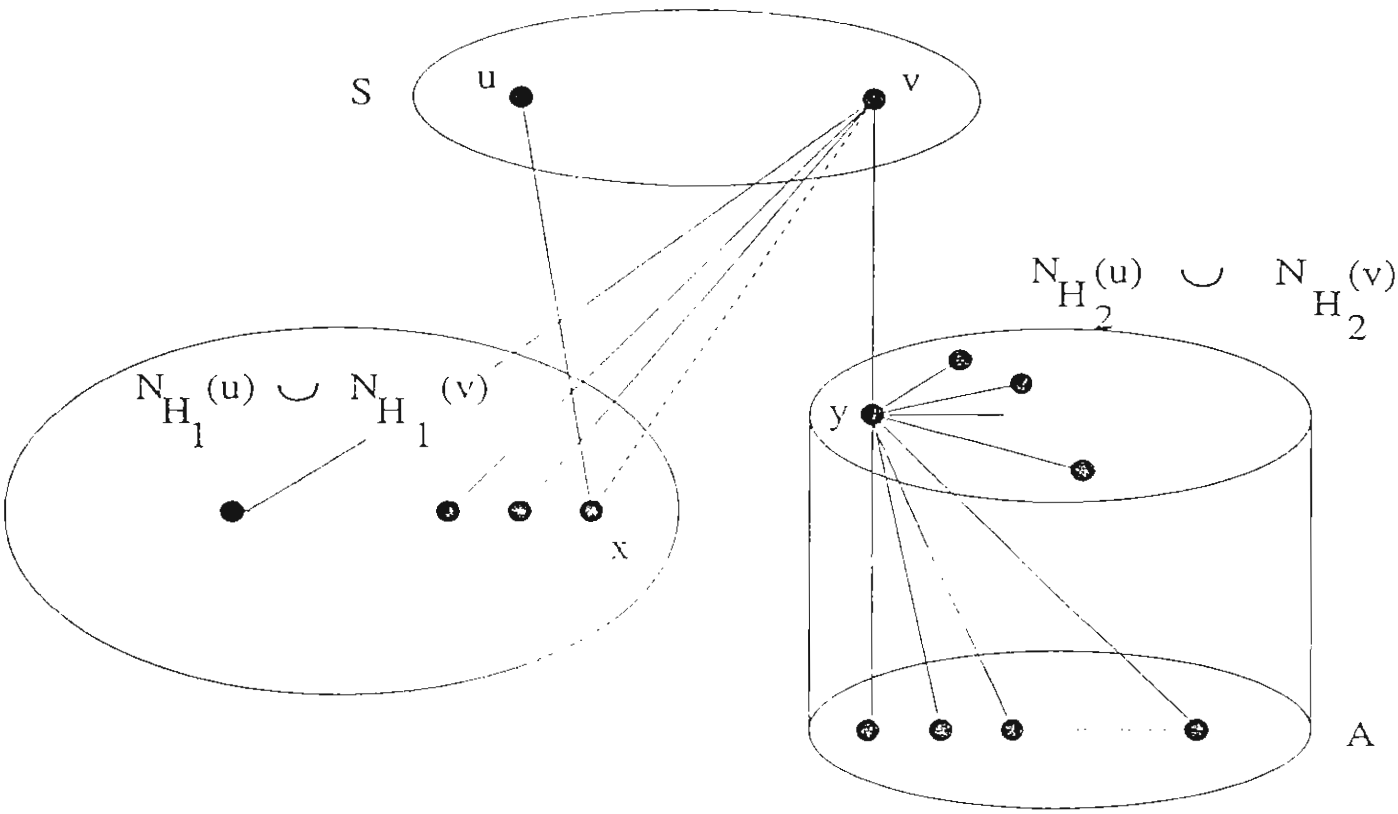


Figure 3.6

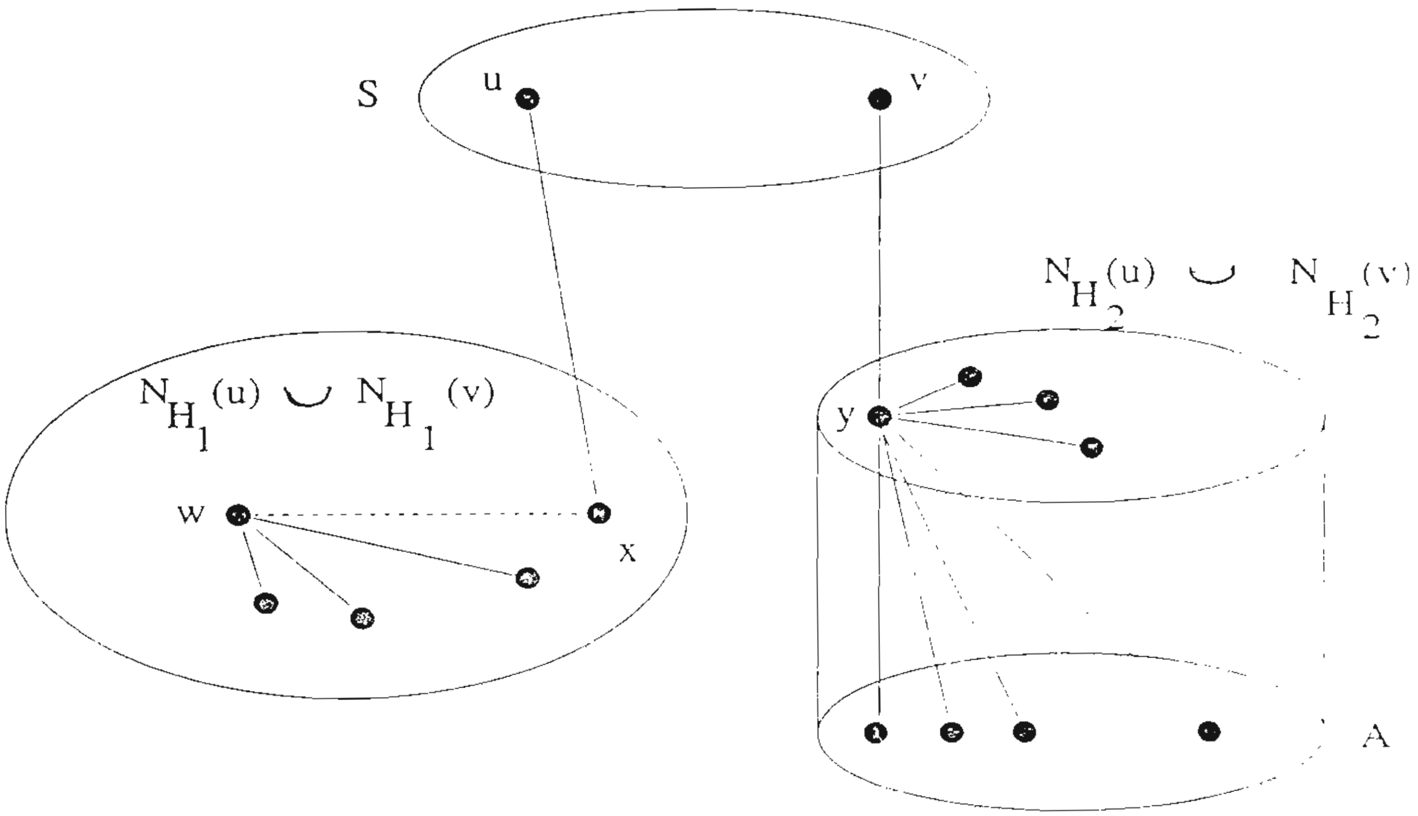


Figure 3.7

3-FACTOR-CRITICALITY

IN

DOMINATION CRITICAL GRAPHS

by

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Abstract

A graph G is said to be k-γ-critical if the size of any minimum dominating set of vertices is k, but if any edge is added to G the resulting graph can be dominated with $k-1$ vertices. The structure of k-γ-critical graphs remains far from completely understood when $\gamma \geq 3$.

A graph G is factor-critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$ and is bicritical if $G - u - v$ has a perfect matching for every pair of distinct vertices $u, v \in V(G)$. More generally, a graph is said to be k-factor-critical if $G - S$ has a perfect matching for every set of k vertices in G. In two previous papers [AP1, AP2], we explored respectively the toughness of 3-γ-critical graphs and some of their matching properties. In particular, we obtained some properties which are sufficient for a 3-γ-critical graph to be factor-critical and, respectively, bicritical. In the present work, we obtain similar results for k-factor-critical graphs when $k = 3$.

1. Introduction

Let G denote a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a **(vertex) dominating set** for G if every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in graph G is called the **(vertex) domination number** (or simply the **domination**

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umber) of G and is denoted by $\gamma(G)$. Graph G is said to be **k-γ-critical** if $\gamma(G) = k$. ıt $\gamma(G + e) = k - 1$ for each edge $e \notin E(G)$. In this paper, we will be concerned only ith the case $k = 3$.

If u, v and w are vertices of G and u and v dominate $G - w$, we will follow previously ccepted notation and write $[u, v] \longrightarrow w$. Suppose G is 3-γ-critical. If u and v are non-djacent vertices of G, then $\gamma(G + uv) = 2$ and so there is a vertex $x \in V(G)$ such that ither $[u, x] \longrightarrow v$ or $[v, x] \longrightarrow u$.

Sumner and Blitch [SB] initiated work on matchings in 3-γ-critical graphs. The fol-owing lemma from that paper will prove very useful in our work to follow.

Lemma 1.1. Let G be a connected 3-γ-critical graph and let S be an independent et of $n \geq 2$ vertices in $V(G)$.

(i) Then the vertices of S can be ordered $a_1, a_2, \ldots, a_n$ in such a way that there exists a equence of distinct vertices $x_1, x_2, \ldots, x_{n-1}$ so that $[a_i, x_i] \longrightarrow a_{i+1}$ for $i = 1, 2, \ldots, n-1$.

(ii) If, in addition, $n \geq 4$, then the x_i's can be chosen so that $x_1 x_2 \cdots x_{n-1}$ is a path and $S \cap \{x_1, \ldots, x_{n-1}\} = \emptyset$.

∎

In what is to follow, we shall also make frequent use of the following easy result.

Lemma 1.2. Let G be a 3-γ-critical graph and let u and v be non-adjacent vertices of G. If x is a vertex of G such that $[u, x] \longrightarrow v$, then $xv \notin E(G)$ and if x is a vertex of G with $[v, x] \longrightarrow u$ then $xu \notin E(G)$.

∎

In [AP1] the following result was obtained. (See also [CTW].)

Theorem 1.3. Let G be a connected 3-γ-critical graph and let S be a vertex cutset in G. Then

(i) if $|S| \geq 4$, $G - S$ has at most $|S| - 1$ components,

(ii) if $|S| = 3$, then $G - S$ contains at most $|S|$ components, and if $G - S$ has exactly three components, then each component is complete and at least one is a singleton,

(iii) if $|S| = 2$, then $G - S$ has at most three components and if $G - S$ has exactly three components, then G must have the structure shown below in Figure 1.1,

(iv) and if $|S| = 1$, then $G - S$ has two components, exactly one of which is a singleton. Furthermore, in case (iv), G has exactly one or two cutvertices and if it has two, G is isomorphic to a graph of the type shown in Figure 1.1.

Figure 1.1.

Finally, we refer the reader to [LP] for further notation, terminology and background or matching theory . In particular, we shall denote by $N(v)$ the neighborhood of vertex v, that is, the set of all vertices adjacent to v. In addition, we denote by $\omega(G)$ the number of components of the graph G and by $\omega_o(G)$, the number of components of odd order in G.

2. 3-γ-criticality and factor-criticality

The following result may be viewed as an extension of Theorem 1.3.

Theorem 2.1. If G is a connected 3-γ-critical graph and S is a vertex cutset in G, then if $|S| \geq 6$, it follows that $\omega(G - S) \leq |S| - 2$.

Proof: Suppose to the contrary that $G - S$ has at least $|S| - 1$ components for some vertex cut S. Then by Theorem 1.3(i), $G - S$ must have exactly $|S| - 1$ components and $|S| - 1 \geq 5$. Let the components of $G - S$ be denoted by $H_1, \ldots, H_k$. For each $i, 1 \leq i \leq k$, choose a vertex $w_i \in V(H_i)$. Clearly, $W = \{w_1, \ldots, w_k\}$ is an independent set. By Lemma 1.1, the vertices of W may be ordered as $a_1, \ldots, a_k$ in such a way that there exists a path $x_1 x_2 \cdots x_{k-1}$ in $G - W$ such that $[a_i, x_i] \longrightarrow a_{i+1}$ for each $i = 1, \ldots, k-1$. By Lemma 1.2, $x_i a_{i+1} \notin E(G)$ for each $i = 1, \ldots, k-1$. Clearly, $x_i \in S$, for $i = 1, \ldots, k-1$. Now let $S_0 = S - \{x_1, \ldots, x_{k-1}\}$. Then $|S_0| = 2$. So let $S_o = \{s_1, s_2\}$. Without any loss of generality, we may renumber the components of $G - S$ in such a way that $a_i \in V(H_i)$. In what is to follow, we make frequent use of the following four observations:

(O1) For $i = 1, \ldots, k-1$, vertex x_i is adjacent to every vertex of

$$[(\cup_{j=1}^{k} V(H_j)) - (V(H_i) \cup \{a_{i+1}\})]$$

since $[a_i, x_i] \longrightarrow a_{i+1}$.

(O2) By O1, vertex a_1 is adjacent to every vertex of $S - (S_0 \cup \{x_1\})$ and for $i = 2, \ldots, k$, vertex a_i is adjacent to every vertex of $S - (S_0 \cup \{x_{i-1}, x_i\})$.

(O3) By O1, O2, Lemma 1.2 and the fact that $|S_0| \geq 2$, if $[a_i, z] \longrightarrow a_j$ and $|i - j| \geq 2$, then $z = x_{j-1}$ or $z \in S_0$ for $j \geq 2$ and $z \in S_0$ for $j = 1$.

(O4) By O1, for $j \geq 2$ and $|i - j| \geq 2$, if $[a_i, x_{j-1}] \longrightarrow a_j$, then x_{j-1} dominates $(G - S) - \{a_j\}$ and thus $\{x_{j-1}, a_j\}$ dominates $G - S_0$ by O2.

Let us now begin by considering the graph $G + a_1 a_3$. Since $\gamma(G + a_1 a_3) = 2$, there is a vertex $z \in G - \{a_1, a_3\}$ such that $[a_1, z] \longrightarrow a_3$ or $[a_3, z] \longrightarrow a_1$. We distinguish two cases.

Case 1: Suppose $[a_1, z] \longrightarrow a_3$.

Then by O3, either $z = x_2$ or $z \in S_0$.

Subcase 1.1: Suppose $z = x_2$. That is, suppose we have $[a_1, x_2] \longrightarrow a_3$. Then t O4, $\{x_2, a_3\}$ dominates $G - S_0$. Since $\gamma(G) = 3$, there is a vertex of S_0, say without ls of generality s_1, such that $x_2 s_1 \notin E(G)$ and $a_3 s_1 \notin E(G)$. Thus $a_1 s_1 \in E(G)$, ce $[a_1, x_2] \longrightarrow a_3$. Furthermore, since $[a_i, x_i] \longrightarrow a_{i+1}$, edge $a_2 s_1 \in E(G)$ and edge $s_1 \in E(G)$. Figure 2.1 depicts this situation.

Figure 2.1.

Now consider $G + a_3 a_5$. Since $\gamma(G + a_3 a_5) = 2$, there is a vertex z_1 of $G - \{a_3, a_5\}$ ich that $[a_3, z_1] \longrightarrow a_5$ or $[a_5, z_1] \longrightarrow a_3$. Again we distinguish two cases.

Subcase 1.1.1: Suppose $[a_3, z_1] \longrightarrow a_5$.
By O3, it follows that $z_1 = x_4$ or $z_1 \in S_0$.

Subcase 1.1.1.1: Suppose $z_1 = x_4$. That is, $[a_3, x_4] \longrightarrow a_5$. Since $a_3 s_1 \notin E(G)$, edge $_4 s_1 \in E(G)$. If $x_4 s_2 \in E(G)$ or if $a_5 s_2 \in E(G)$, then by O4, the set $\{x_4, a_5\}$ dominates G, a contradiction. Thus $x_4 s_2 \notin E(G)$ and $a_5 s_2 \notin E(G)$.
Since $[a_4, x_4] \longrightarrow a_5$, edge $a_4 s_2 \in E(G)$. Figure 2.2 illustrates this situation.

Figure 2.2

Now let us consider $G + a_2 a_4$. Since $\gamma(G + a_2 a_4) = 2$, there is a vertex z_2 of $G - \{a_2, a_4\}$ such that $[a_2, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_2$. In either case, $z_2 \notin S_0$ since $a_3 s_1 \notin E(G)$ and $a_5 s_2 \notin E(G)$.
Suppose first that $[a_2, z_2] \longrightarrow a_4$. By O3 and the above we note that $z_2 = x_3$. Then by O4, $\{x_3, a_4\}$ dominates $G - S_0$. But then since $x_3 s_1 \in E(G)$ and $a_4 s_2 \in E(G)$, $\{x_3, a_4\}$ dominates G, a contradiction.
Hence $[a_4, z_2] \longrightarrow a_2$. By an argument similar to that above, $z_2 = x_1$. By O4, $\{x_1, a_2\}$ dominates $G - S_0$. Recall that $a_2 s_1 \in E(G)$. If $x_1 s_2$ or $a_2 s_2$ is an edge of G,

en $\{x_1, a_2\}$ dominates G, a contradiction. Thus $x_1 s_2 \notin E(G)$ and $a_2 s_2 \notin E(G)$. Because $_i, x_i] \longrightarrow a_{i+1}$, $a_1 s_2 \in E(G)$ and $x_2 s_2 \in E(G)$. Now consider $G + a_1 a_4$. There must be a ertex z_3 of $G - \{a_1, a_4\}$ such that $[a_1, z_3] \longrightarrow a_4$ or $[a_4, z_3] \longrightarrow a_1$. In either case, $z_3 \notin S_0$. nce $s_1 a_3 \notin E(G)$ and $s_2 a_5 \notin E(G)$. Thus by O3, the case $[a_4, z_3] \longrightarrow a_1$ is impossible.

Thus $[a_1, z_3] \longrightarrow a_4$. Then by O3 and O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since $x_3 s_1 \in$ (G) and $a_4 s_2 \in E(G)$, $\{x_3, a_4\}$ dominates G, a contradiction. This completes the proof Subcase 1.1.1.1.

Subcase 1.1.1.2: Suppose $z_1 \in S_0$. Since $x_2 s_1 \notin E(G)$ and $a_3 x_2 \notin E(G)$, it follows hat $z_1 \neq s_1$. Thus $z_1 = s_2$ and it then follows that $[a_3, s_2] \longrightarrow a_5$. Thus s_2 dominates $(G - S) \cup \{x_2\}) - (H_3 \cup \{a_5\})$. By Lemma 1.2, $s_2 a_5 \notin E(G)$. Figure 2.3 now depicts the resent situation.

Figure 2.3

Now consider $G + a_1 a_4$. Since $\gamma(G + a_1 a_4) = 2$, there is a vertex z_2 of $G - \{a_1, a_4\}$ such that $[a_1, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_1$. In either case, $z_2 \notin S_0$ since $s_1 a_3 \notin E(G)$ and $s_2 a_5 \notin E(G)$. Thus by O3, the case $[a_4, z_2] \longrightarrow a_1$ is impossible.

Thus $[a_1, z_2] \longrightarrow a_4$. Then by O3 and O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since s_2 dominates $((G - S) \cup \{x_2\}) - (H_3 \cup \{a_5\})$, edge $s_2 a_4 \in E(G)$. Recall that $x_3 s_1 \in E(G)$. Therefore, $\{x_3, a_4\}$ dominates G, a contradiction. This completes the proof of Subcase 1.1.1.2 and hence also the Subcase 1.1.1.

Subcase 1.1.2: Suppose $[a_5, z_1] \longrightarrow a_3$. By O3, it follows that either $z_1 = x_2$ or $z_1 \in S_0$.

Subcase 1.1.2.1: Suppose $z_1 = x_2$. That is, $[a_5, x_2] \longrightarrow a_3$. Recall that $x_2 s_1 \notin E(G)$ and $a_3 s_1 \notin E(G)$, but $x_3 s_1, a_1 s_1, a_2 s_1 \in E(G)$. Since $[a_5, x_2] \longrightarrow a_3$, by O4, $\{x_2, a_3\}$ dominates $G - S_0$. Because $x_2 s_1 \notin E(G)$ and $[a_5, x_2] \longrightarrow a_3$, it follows that $a_5 s_1 \in E(G)$. Furthermore, either $a_5 s_2 \in E(G)$ or $x_2 s_2 \in E(G)$.

First suppose that $a_5 s_2 \in E(G)$. Then a_5 is adjacent to both s_1 and s_2. Figure 2.4 now illustrates the present situation.

Figure 2.4

Consider now $G+a_1a_5$. Since $\gamma(G+a_1a_5)=2$, there must be a vertex z_2 of $G-\{a_1,a_5\}$ ch that $[a_1,z_2]\longrightarrow a_5$ or $[a_5,z_2]\longrightarrow a_1$.

Suppose first that $[a_1,z_2]\longrightarrow a_5$. Since a_5 is adjacent to s_1 and s_2, vertex $z_2\notin S_0$ by mma 1.2. Then by O3, $z=x_4$. Hence $\{x_4,a_5\}$ dominates G by O4 and the fact that a_5 is jacent to s_1 and s_2. This contradiction proves that for all $z_2\in G-\{a_1,a_5\}$, $\{a_1,z_2\}$ does t dominate $G-\{a_5\}$. Hence $[a_5,z_2]\longrightarrow a_1$. Then $z_2\in S_0$ by O3. Since $a_3s_1\notin E(G)$, it llows that $z_2=s_2$. Thus $[a_5,s_2]\longrightarrow a_1$ and s_2 dominates $((G-S)\cup\{x_4\})-(H_5\cup\{a_1\})$. ence a_2 is adjacent to both s_1 and s_2.

Next consider $G+a_2a_5$. Since $\gamma(G+a_2a_5)=2$, there is a vertex z_3 of $G-\{a_2,a_5\}$ such at $[a_2,z_3]\longrightarrow a_5$ or $[a_5,z_3]\longrightarrow a_2$. But in either case, $z_3\notin S_0$ by Lemma 1.2 and the fact at a_2 and a_5 are adjacent to both s_1 and s_2. If $[a_2,z_3]\longrightarrow a_5$, then $\{x_4,a_5\}$ dominates by O3 and O4 and the fact that a_5 is adjacent to s_1 and s_2. But this is a contradiction. imilarly, if $[a_5,z_3]\longrightarrow a_2$, then $\{x_1,a_2\}$ dominates G, again a contradiction. This proves at $a_5s_2\notin E(G)$. Therefore, edge $x_2s_2\in E(G)$ since $[a_5,x_2]\longrightarrow a_3$. Figure 2.5 now epicts the present situation.

Figure 2.5

Now consider $G+a_1a_4$. Then there is a vertex z_4 of $G-\{a_1,a_4\}$ such that $[a_1,z_4]\longrightarrow a_4$ or $[a_4,z_4]\longrightarrow a_1$. In either case, $z_4\notin S_0$ since $a_3s_1\notin E(G)$ and $a_5s_2\notin E(G)$. By O3, $[a_4,z_4]\longrightarrow a_1$ is impossible. Hence $[a_1,z_4]\longrightarrow a_4$ and $z_4=x_3$. Then by O4, $\{x_3,a_4\}$ dominates $G-S_0$. Since $x_3s_1\in E(G)$, it follows that $x_3s_2\notin E(G)$ and $a_4s_2\notin E(G)$, for otherwise $\{x_3,a_4\}$ would dominate G, a contradiction. Since $[a_4,x_4]\longrightarrow a_5$ and $a_4s_2\notin E(G)$, it follows that $x_4s_2\in E(G)$.

Now consider $G+a_1a_5$. There must be a vertex z_5 of $G-\{a_1,a_5\}$ such that $[a_1,z_5]\longrightarrow a_5$ or $[a_5,z_5]\longrightarrow a_1$. In either case, $z_5\notin S_0$ since $a_3s_1\notin E(G)$ and $a_4s_2\notin E(G)$. By O3, the case $[a_5,z_5]\longrightarrow a_1$ is impossible. Thus $[a_1,z_5]\longrightarrow a_5$. But then $z_5=x_4$ by O3 and hence $\{x_4,a_5\}$ dominates G by O4 and the facts that $a_5s_1\in E(G)$ and $x_4s_2\in E(G)$. This proves that $x_2s_2\notin E(G)$. Hence $[a_5,x_2]\longrightarrow a_3$ is impossible. This completes the proof in Subcase 1.1.2.1.

Subcase 1.1.2.2: Suppose $z_1\in S_0$ and $z_1=s_1$. That is, $[a_5,s_1]\longrightarrow a_3$. Then s_1 dominates $((G-S)\cup\{x_4\})-(H_5\cup\{a_3\})$.

Consider $G+a_2a_4$. Since $\gamma(G+a_2a_4)=2$, there is a vertex z_2 of $G-\{a_2,a_4\}$ such that $[a_2,z_2]\longrightarrow a_4$ or $[a_4,z_2]\longrightarrow a_2$. We distinguish two subcases.

Figure 2.4

Consider now $G+a_1a_5$. Since $\gamma(G+a_1a_5) = 2$, there must be a vertex z_2 of $G-\{a_1, a_5\}$ ch that $[a_1, z_2] \longrightarrow a_5$ or $[a_5, z_2] \longrightarrow a_1$.

Suppose first that $[a_1, z_2] \longrightarrow a_5$. Since a_5 is adjacent to s_1 and s_2, vertex $z_2 \notin S_0$ by mma 1.2. Then by O3, $z = x_4$. Hence $\{x_4, a_5\}$ dominates G by O4 and the fact that a_5 is ljacent to s_1 and s_2. This contradiction proves that for all $z_2 \in G-\{a_1, a_5\}$, $\{a_1, z_2\}$ does t dominate $G - \{a_5\}$. Hence $[a_5, z_2] \longrightarrow a_1$. Then $z_2 \in S_0$ by O3. Since $a_3s_1 \notin E(G)$, it llows that $z_2 = s_2$. Thus $[a_5, s_2] \longrightarrow a_1$ and s_2 dominates $((G-S) \cup \{x_4\}) - (H_5 \cup \{a_1\})$. ence a_2 is adjacent to both s_1 and s_2.

Next consider $G+a_2a_5$. Since $\gamma(G+a_2a_5) = 2$, there is a vertex z_3 of $G-\{a_2, a_5\}$ such iat $[a_2, z_3] \longrightarrow a_5$ or $[a_5, z_3] \longrightarrow a_2$. But in either case, $z_3 \notin S_0$ by Lemma 1.2 and the fact iat a_2 and a_5 are adjacent to both s_1 and s_2. If $[a_2, z_3] \longrightarrow a_5$, then $\{x_4, a_5\}$ dominates by O3 and O4 and the fact that a_5 is adjacent to s_1 and s_2. But this is a contradiction. imilarly, if $[a_5, z_3] \longrightarrow a_2$, then $\{x_1, a_2\}$ dominates G, again a contradiction. This proves iat $a_5s_2 \notin E(G)$. Therefore, edge $x_2s_2 \in E(G)$ since $[a_5, x_2] \longrightarrow a_3$. Figure 2.5 now epicts the present situation.

Figure 2.5

Now consider $G+a_1a_4$. Then there is a vertex z_4 of $G-\{a_1, a_4\}$ such that $[a_1, z_4] \longrightarrow a_4$ or $[a_4, z_4] \longrightarrow a_1$. In either case, $z_4 \notin S_0$ since $a_3s_1 \notin E(G)$ and $a_5s_2 \notin E(G)$. By O3, $[a_4, z_4] \longrightarrow a_1$ is impossible. Hence $[a_1, z_4] \longrightarrow a_4$ and $z_4 = x_3$. Then by O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since $x_3s_1 \in E(G)$, it follows that $x_3s_2 \notin E(G)$ and $a_4s_2 \notin E(G)$, for otherwise $\{x_3, a_4\}$ would dominate G, a contradiction. Since $[a_4, x_4] \longrightarrow a_5$ and $a_4s_2 \notin E(G)$, it follows that $x_4s_2 \in E(G)$.

Now consider $G+a_1a_5$. There must be a vertex z_5 of $G-\{a_1, a_5\}$ such that $[a_1, z_5] \longrightarrow a_5$ or $[a_5, z_5] \longrightarrow a_1$. In either case, $z_5 \notin S_0$ since $a_3s_1 \notin E(G)$ and $a_4s_2 \notin E(G)$. By O3, the case $[a_5, z_5] \longrightarrow a_1$ is impossible. Thus $[a_1, z_5] \longrightarrow a_5$. But then $z_5 = x_4$ by O3 and hence $\{x_4, a_5\}$ dominates G by O4 and the facts that $a_5s_1 \in E(G)$ and $x_4s_2 \in E(G)$. This proves that $x_2s_2 \notin E(G)$. Hence $[a_5, x_2] \longrightarrow a_3$ is impossible. This completes the proof in Subcase 1.1.2.1.

Subcase 1.1.2.2: Suppose $z_1 \in S_0$ and $z_1 = s_1$. That is, $[a_5, s_1] \longrightarrow a_3$. Then s_1 dominates $((G-S) \cup \{x_4\}) - (H_5 \cup \{a_3\})$.

Consider $G + a_2a_4$. Since $\gamma(G + a_2a_4) = 2$, there is a vertex z_2 of $G - \{a_2, a_4\}$ such that $[a_2, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_2$. We distinguish two subcases.

Subcase 1.1.2.2.1: Suppose $[a_2, z_2] \longrightarrow a_4$. Then $z_2 = x_3$ or $z_2 \in S_0$ by O3. Recall that $x_2 s_1 \notin E(G)$ and $a_3 s_1 \notin E(G)$, but $x_3 s_1, a_1 s_1$ and $a_2 s_1$ are all edges of G.

Suppose that $z_2 = x_3$. That is, $[a_2, x_3] \longrightarrow a_4$. By O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since $x_3 s_1 \in E(G)$, it follows that $x_3 s_2 \notin E(G)$ and $a_4 s_2 \notin E(G)$; otherwise $\{x_3, a_4\}$ dominates G, a contradiction. Since $[a_2, x_3] \longrightarrow a_4$ and $x_3 s_2 \notin E(G)$, it follows that $a_2 s_2 \in E(G)$. Thus a_2 is adjacent to both s_1 and s_2. Furthermore, since $[a_4, x_4] \longrightarrow a_5$ and $a_4 s_2 \notin E(G)$, it follows that $x_4 s_2 \in E(G)$. Because s_1 dominates $((G - S) \cup \{x_4\}) - H_5 \cup \{a_3\})$, it follows that $s_1 x_4 \in E(G)$. Thus x_4 is also adjacent to s_1 and s_2.

Consider $G + a_2 a_5$. Since $\gamma(G + a_2 a_5) = 2$, there must be a vertex z_3 of $G - \{a_2, a_5\}$ such that $[a_2, z_3] \longrightarrow a_5$ or $[a_5, z_3] \longrightarrow a_2$. In either case, $z_3 \notin S_0$ since $a_3 s_1 \notin E(G)$ and $a_4 s_2 \notin E(G)$. If $[a_5, z_3] \longrightarrow a_2$, then $\{x_1, a_2\}$ dominates G by O4 and the fact that a_2 is adjacent to s_1 and to s_2, a contradiction. Hence $[a_2, z_3] \longrightarrow a_5$. But then $\{x_4, a_5\}$ dominates G by O4 and the fact that x_4 is adjacent to s_1 and to s_2 and again we have a contradiction. This proves that if $[a_2, z_2] \longrightarrow a_4$, then $z_2 \neq x_3$. Thus $z_2 \in S_0$. Since $a_3 s_1 \notin E(G)$, $z_2 \neq s_1$. Therefore $z_2 = s_2$. That is, $[a_2, s_2] \longrightarrow a_4$. Now s_2 dominates $((G - S) \cup \{x_1\}) - (H_2 \cup \{a_4\})$. Hence a_1 is adjacent to both s_1 and s_2.

Now consider $G + a_1 a_5$. There must be a vertex z_3 of $G - \{a_1, a_5\}$ such that either $[a_1, z_3] \longrightarrow a_5$ or $[a_5, z_3] \longrightarrow a_1$. In either case, $z_3 \notin S_0$ since $a_3 s_1 \notin E(G)$ and $a_4 s_2 \notin E(G)$. But then the case $[a_5, z_3] \longrightarrow a_1$ is impossible by O3.

Therefore $[a_1, z_3] \longrightarrow a_5$. Thus $\{x_4, a_5\}$ dominates $G - S_0$ by O4. Since s_1 dominates $((G - S) \cup \{x_4\}) - (H_5 \cup \{a_3\})$, $s_1 x_4 \in E(G)$. Furthermore, since s_2 dominates $((G - S) \cup \{x_1\}) - (H_2 \cup \{a_4\})$, $s_2 a_5 \in E(G)$. Therefore $\{x_4, a_5\}$ dominates G, once more a contradiction. This contradiction proves that for every z_2 in $G - \{a_2, a_4\}$, $\{a_2, z_2\}$ does not dominate $G - \{a_4\}$ and completes the proof of Subcase 1.1.2.2.1.

Subcase 1.1.2.2.2: Suppose $[a_4, z_2] \longrightarrow a_2$. Then $z_2 = x_1$ or $z_2 \in S_0$ by O3. Recall that $x_2 s_1 \notin E(G)$ and $a_3 s_1 \notin E(G)$, but $x_3 s_1, a_1 s_1$ and $a_2 s_1$ are all edges of G. Furthermore, s_1 dominates $((G - S) \cup \{x_4\}) - (H_5 \cup \{a_3\})$.

Suppose $z_2 = x_1$. That is, $[a_4, x_1] \longrightarrow a_2$. By O4, $\{x_1, a_2\}$ dominates $G - S_0$. Since $a_2 s_1 \in E(G)$, it follows that $x_1 s_2 \notin E(G)$ and $a_2 s_2 \notin E(G)$; otherwise $\{x_1, a_2\}$ would dominate G, a contradiction.

Next consider $G + a_1 a_4$. Since $\gamma(G + a_1 a_4) = 2$, there must be a vertex z_3 of $G - \{a_1, a_4\}$ such that either $[a_1, z_3] \longrightarrow a_4$ or $[a_4, z_3] \longrightarrow a_1$. In either case, $z_3 \notin S_0$ since $s_1 a_3 \notin E(G)$ and $s_2 a_2 \notin E(G)$. By O3, the case $[a_4, z_3] \longrightarrow a_1$ is impossible. Thus $[a_1, z_3] \longrightarrow a_4$. By O3 and O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since $x_3 s_1 \in E(G)$, it follows that $x_3 s_2 \notin E(G)$ and $a_4 s_2 \notin E(G)$. Otherwise, $\{x_3, a_4\}$ would dominate G, a contradiction. Because $[a_4, x_4] \longrightarrow a_5$ and $a_4 s_2 \notin E(G)$, it follows that $x_4 s_2 \in E(G)$. So x_4 is adjacent to both s_1 and s_2.

Next we consider $G + a_1 a_5$. Since $\gamma(G + a_1 a_5) = 2$, there must be a vertex z_4 of $G - \{a_1, a_5\}$ such that either $[a_1, z_4] \longrightarrow a_5$ or $[a_5, z_4] \longrightarrow a_1$. In either case, $z_4 \notin S_0$, since $s_1 a_3 \notin E(G)$ and $s_2 a_2 \notin E(G)$. By O3, the case $[a_5, z_4] \longrightarrow a_1$ is impossible. Thus $[a_1, z_4] \longrightarrow a_5$. By O3 and O4, $\{x_4, a_5\}$ dominates $G - S_0$. Since x_4 is adjacent to both s_1 and s_2, $\{x_4, a_5\}$ dominates G. This contradiction proves that if $[a_4, z_2] \longrightarrow a_2$, then $z_2 \neq x_1$. Thus $z_2 \in S_0$. Since $s_1 a_3 \notin E(G)$, it follows that $z_2 \neq s_1$. Therefore,

$z_2 = s_2$. That is, $[a_4, s_2] \longrightarrow a_2$. Then s_2 dominates $((G-S)\cup\{x_3\}) - (H_4 \cup \{a_2\})$. Thus $s_2x_3 \in E(G)$ and $s_2a_2 \notin E(G)$. Then x_3 is adjacent to both s_1 and s_2 since $x_3s_1 \in E(G)$.

Now consider $G+a_1a_4$. Since $\gamma(G+a_1a_4) = 2$, there must be a vertex z_5 of $G-\{a_1,a_4\}$ such that either $[a_1, z_5] \longrightarrow a_4$ or $[a_4, z_5] \longrightarrow a_1$. In either case, $z_5 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_2 \notin E(G)$. By O3, the case $[a_4, z_5] \longrightarrow a_1$ is impossible. Hence $[a_1, z_5] \longrightarrow a_4$. Then $z_5 = x_3$ by O3. But then $\{x_3, a_4\}$ dominates G by O4 and the fact that x_3 is adjacent to s_1 and s_2. This contradiction completes the proof of Subcase 1.1.2.2.2 and thus of the Subcase 1.1.2.2.

Subcase 1.1.2.3: Suppose $z_1 \in S_0$ and $z_1 = s_2$. That is, $[a_5, s_2] \longrightarrow a_3$. Recall that $x_2s_1 \notin E(G)$ and $a_3s_1 \notin E(G)$, but x_3s_1, a_1s_1 and a_2s_1 are all edges of G. Since $[a_5, s_2] \longrightarrow a_3$, vertex s_2 dominates $((G-S)\cup\{x_4\}) - (H_5 \cup \{a_3\})$. Since $s_2a_3 \notin E(G)$, edge $x_3s_2 \in E(G)$, because $[a_3, x_3] \longrightarrow a_4$. Now x_3 and a_2 are adjacent to both s_1 and s_2.

Consider $G + a_2a_4$. Since $\gamma(G + a_2a_4) = 2$, there must be a vertex z_2 of $G - \{a_2, a_4\}$ such that either $[a_2, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_2$. In either case, $z_2 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_3 \notin E(G)$. If $[a_2, z_2] \longrightarrow a_4$, then $z_2 = x_3$ and hence $\{x_3, a_4\}$ dominates G by O3, O4 and the fact that x_3 is adjacent to both s_1 and s_2 and we have a contradiction. Hence $[a_4, z_2] \longrightarrow a_2$. But then a similar argument shows that $\{x_1, a_2\}$ dominates G. This contradiction completes the proof of Subcase 1.1.2.3 and hence of Subcase 1.1.2. But then $\gamma(G + a_3a_5) > 2$, contradicting the 3-γ-criticality of G. Thus Subcase 1.1 cannot occur.

Subcase 1.2: Suppose $z \in S_0$.

Without loss of generality, we may assume that $z = s_1$. That is, $[a_1, s_1] \longrightarrow a_3$. Then vertex s_1 dominates $(G-S) - (H_1 \cup \{a_3\})$. Since $s_1a_3 \notin E(G)$ by Lemma 1.2, edge $s_1x_3 \in E(G)$ because $[a_3, x_3] \longrightarrow a_4$.

Consider $G + a_2a_4$. Since $\gamma(G + a_2a_4) = 2$, there is a vertex z_1 of $G - \{a_2, a_4\}$ such that either $[a_2, z_1] \longrightarrow a_4$ or $[a_4, z_1] \longrightarrow a_2$. We distinguish two subcases.

Subcase 1.2.1: Suppose $[a_2, z_1] \longrightarrow a_4$. By O3 and the fact that $s_1a_3 \notin E(G)$, it follows that either $z_1 = x_3$ or $z_1 = s_2$.

Subcase 1.2.1.1: Suppose $z_1 = x_3$. That is, $[a_2, x_3] \longrightarrow a_4$. Then by O4, $\{x_3, a_4\}$ dominates $G - S_0$. Because s_1 dominates $(G-S) - (H_1 \cup \{a_3\})$, edges s_1x_3 and s_1a_4 belong to $E(G)$. If $x_3s_2 \in E(G)$ or if $a_4s_2 \in E(G)$, then $\{x_3, a_4\}$ dominates G, a contradiction. Hence $x_3s_2 \notin E(G)$ and $a_4s_2 \notin E(G)$. Since $[a_i, x_i] \longrightarrow a_{i+1}$ for all i, $1 \le i \le k-1$, it follows that $a_3s_2 \in E(G)$ and $x_4s_2 \in E(G)$.

Consider $G + a_2a_5$. Since $\gamma(G + a_2a_5) = 2$, there is a vertex z_2 of $G - \{a_2, a_5\}$ such that either $[a_2, z_2] \longrightarrow a_5$ or $[a_5, z_2] \longrightarrow a_2$. In either case, $z_2 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_4 \notin E(G)$. If $[a_2, z_2] \longrightarrow a_5$, then $\{x_4, a_5\}$ dominates $G - S_0$ by O3 and O4. Since s_1 dominates $(G-S) - (H_1 \cup \{a_3\})$, edge $s_1a_5 \in E(G)$. Because $x_4s_2 \in E(G)$, $\{x_4, a_5\}$ dominates G, a contradiction. Hence $[a_5, z_2] \longrightarrow a_2$. By O3 and the fact that $z_2 \notin S_0$, it follows that $z_2 = x_1$. Then by O4, $\{x_1, a_2\}$ dominates $G - S_0$. Since s_1 dominates $(G-S) - (H_1 \cup \{a_3\})$, edge $s_1a_2 \in E(G)$. If $a_2s_2 \in E(G)$ or if $x_1s_2 \in E(G)$, then $\{x_1, a_2\}$ dominates G, a contradiction. Hence $a_2s_2 \notin E(G)$ and $x_1s_2 \notin E(G)$. Because $[a_i, x_i] \longrightarrow a_{i+1}$ for $1 \le i \le k-1$, it follows that $x_2s_2 \in E(G)$ and $a_1s_2 \in E(G)$.

Now consider $G + a_1a_5$. Since $\gamma(G + a_1a_5) = 2$, there is a vertex z_3 of $G - \{a_1, a_5\}$ such that either $[a_1, z_3] \longrightarrow a_5$ or $[a_5, z_3] \longrightarrow a_1$. In either case $z_3 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_2 \notin E(G)$. Then by O3, the case $[a_5, z_3] \longrightarrow a_1$ is impossible. Thus $[a_1, z_3] \longrightarrow a_5$. Then $z_3 = x_4$. But then $\{x_4, a_5\}$ dominates G by O4 and the fact that $x_4s_2 \in E(G)$ and $a_5s_1 \in E(G)$. This contradiction completes the proof of Subcase 1.2.1.1.

Subcase 1.2.1.2: Suppose $z_1 = s_2$. That is, $[a_2, s_2] \longrightarrow a_4$. Then s_2 dominates $((G - S) \cup \{x_1\}) - (H_2 \cup \{a_4\})$. Recall that s_1 dominates $(G - S) - (H_1 \cup \{a_3\})$. More specifically, $s_2a_4 \notin E(G)$, $s_1a_3 \notin E(G)$, but a_5 is adjacent to both s_1 and s_2.

Consider $G + a_2a_5$. Since $\gamma(G + a_2a_5) = 2$, there is vertex z_2 of $G - \{a_2, a_5\}$ such that either $[a_2, z_2] \longrightarrow a_5$ or $[a_5, z_2] \longrightarrow a_2$. In either case, $z_2 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_4 \notin E(G)$. Suppose $[a_2, z_2] \longrightarrow a_5$. By O3 and O4, $\{x_4, a_5\}$ dominates $G - S_0$. Since a_5 is adjacent to s_1 and s_2, it follows that $\{x_4, a_5\}$ dominates G, a contradiction.

Hence $[a_5, z_2] \longrightarrow a_2$. By O3 and O4, $\{x_1, a_2\}$ dominates $G - S_0$. Since s_1 dominates $(G - S) - (H_1 \cup \{a_3\})$ and s_2 dominates $((G - S) \cup \{x_1\}) - (H_2 \cup \{a_4\})$, it follows that $s_1a_2 \in E(G)$ and $s_2x_1 \in E(G)$. But then $\{x_1, a_2\}$ dominates G, a contradiction. This completes the proof of Subcase 1.2.1.

Subcase 1.2.2: Suppose $[a_4, z_1] \longrightarrow a_2$. By O3 and the fact that $s_1a_3 \notin E(G)$, it follows that either $z_1 = x_1$ or $z_1 = s_2$.

Subcase 1.2.2.1: Suppose $z_1 = x_1$. That is, $[a_4, x_1] \longrightarrow a_2$. Then by O4, $\{x_1, a_2\}$ dominates $G - S_0$. Recall that s_1 dominates $(G - S) - (H_1 \cup \{a_3\})$. Since $s_1a_2 \in E(G)$, it follows that $a_2s_2 \notin E(G)$ and $x_1s_2 \notin E(G)$, for otherwise $\{x_1, a_2\}$ dominates G, a contradiction. Since $[a_i, x_i] \longrightarrow a_{i+1}$ for all $1 \leq i \leq k-1$, it follows that $x_2s_2 \in E(G)$ and $a_1s_2 \in E(G)$.

Consider $G + a_1a_4$. Since $\gamma(G + a_1a_4) = 2$, there must be a vertex z_2 of $G - \{a_1, a_4\}$ such that either $[a_1, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_1$. In either case, $z_2 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_2 \notin E(G)$. By O3, the case $[a_4, z_2] \longrightarrow a_1$ is impossible. Hence $[a_1, z_2] \longrightarrow a_4$. Then $\{x_3, a_4\}$ dominates $G - S_0$ by O3 and O4. Since s_1 dominates $(G - S) - (H_1 \cup \{a_3\})$, it follows that $s_1a_4 \in E(G)$. Then $a_4s_2 \notin E(G)$ and $x_3s_2 \notin E(G)$, for otherwise $\{x_3, a_4\}$ dominates G, a contradiction. Since $[a_i, x_i] \longrightarrow a_{i+1}$ for all $1 \leq i \leq k-1$, $x_4s_2 \in E(G)$ and $a_3s_2 \in E(G)$.

Now consider $G + a_1a_5$. Since $\gamma(G + a_1a_5) = 2$, there is a vertex z_3 of $G - \{a_1, a_5\}$ such that either $[a_1, z_3] \longrightarrow a_5$ or $[a_5, z_3] \longrightarrow a_1$. In either case, $z_3 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_4 \notin E(G)$. By O3, the case $[a_5, z_3] \longrightarrow a_1$ is impossible. Hence $[a_1, z_3] \longrightarrow a_5$. Then by O3, $z_3 = x_4$. But then $\{x_4, a_5\}$ dominates G by O4 and the fact that $s_1a_5 \in E(G)$ and $x_4s_2 \in E(G)$. This contradiction completes the proof of Subcase 1.2.2.1.

Subcase 1.2.2.2: Suppose $z_1 = s_2$. That is, $[a_4, s_2] \longrightarrow a_2$. Then s_2 dominates $((G - S) \cup \{x_3\}) - (H_4 \cup \{a_2\})$. Recall that s_1 dominates $(G - S) - (H_1 \cup \{a_3\})$. More specifically, $s_1a_4 \in E(G)$ and $s_2x_3 \in E(G)$, but $s_1a_3 \notin E(G)$ and $s_2a_2 \notin E(G)$.

Consider $G + a_1a_4$. Since $\gamma(G + a_1a_4) = 2$, there is a vertex z_2 of $G - \{a_1, a_4\}$ such that either $[a_1, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_1$. In either case, $z_2 \notin S_0$ since $s_1a_3 \notin E(G)$ and $s_2a_2 \notin E(G)$. By O3, the case $[a_4, z_2] \longrightarrow a_1$ is impossible. Hence $[a_1, z_2] \longrightarrow a_4$. By

C and O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since $s_1a_4 \in E(G)$ and $s_2x_3 \in E(G)$, it follows tht $\{x_3, a_4\}$ dominates G, a contradiction. This completes the proof of Subcase 1.2.2.2 al hence Subcase 1.2.2. Hence $\gamma(G + a_2a_4) > 2$, contradicting the 3-γ-criticality of G. Tus the Case 1.2 cannot occur. Hence for all z in $G - \{a_1, a_3\}$, $\{a_1, z\}$ does not dominate (- a_3.

Case 2: Suppose $[a_3, z] \longrightarrow a_1$.

By O3, $z \in S_0$. Without loss of generality, we may assume that $z = s_1$. Then $[\,, s_1] \longrightarrow a_1$ and s_1 dominates $((G - S) \cup \{x_2\}) - (H_3 \cup \{a_1\})$. Since $s_1a_1 \notin E(G)$ by mma 1.2, it follows that edge $x_1s_1 \in E(G)$ since $[a_1, x_1] \longrightarrow a_2$.

Consider $G + a_2a_4$. Since $\gamma(G + a_2a_4) = 2$, there is a vertex z_1 of $G - \{a_2, a_4\}$ such at either $[a_2, z_1] \longrightarrow a_4$ or $[a_4, z_1] \longrightarrow a_2$. We distinguish two cases.

Subcase 2.1: Suppose $[a_2, z_1] \longrightarrow a_4$. By O3 and the fact that $s_1a_1 \notin E(G)$, it llows that $z_1 = x_3$ or $z_1 = s_2$.

Subcase 2.1.1: Suppose first that $z_1 = x_3$. That is, $[a_2, x_3] \longrightarrow a_4$. By O4, x_3 ominates $(G - S) - \{a_4\}$. Since s_1 dominates $((G - S) \cup \{x_2\}) - (H_3 \cup \{a_1\})$, it follows that $a_4 \in E(G)$. If $x_3s_2 \in E(G)$ or $a_4s_2 \in E(G)$, then $\{x_3, a_4\}$ dominates G, a contradiction. ence $x_3s_2 \notin E(G)$ and $a_4s_2 \notin E(G)$. Since $[a_i, x_i] \longrightarrow a_{i+1}$ for $1 \leq i \leq k - 1$, it follows iat $a_3s_2 \in E(G)$ and $x_4s_2 \in E(G)$. Figure 2.6 now depicts our situation.

Figure 2.6

Now consider $G + a_3a_5$. Since $\gamma(G + a_3a_5) = 2$, there is a vertex z_2 of $G - \{a_3, a_5\}$ such that either $[a_3, z_2] \longrightarrow a_5$ or $[a_5, z_2] \longrightarrow a_3$. In either case, $z_2 \notin S_0$ since $s_1a_1 \notin E(G)$ and $s_2a_4 \notin E(G)$. Suppose $[a_3, z_2] \longrightarrow a_5$. By O3 and O4, $\{x_4, a_5\}$ dominates $G - S_0$. Since s_1 dominates $((G - S) \cup \{x_2\}) - (H_3 \cup \{a_1\})$, it follows that $s_1a_5 \in E(G)$. Because $x_4s_2 \in E(G)$, $\{x_4, a_5\}$ dominates G, a contradiction. Hence $[a_5, z_2] \longrightarrow a_3$. By O3 and O4, $\{x_2, a_3\}$ dominates $G - S_0$. Since s_1 dominates $((G - S) \cup \{x_2\}) - (H_3 \cup \{a_1\})$, $s_1x_2 \in E(G)$. But then $\{x_2, a_3\}$ dominates G since $a_3s_2 \in E(G)$. This contradiction completes the proof of Subcase 2.1.1.

Subcase 2.1.2: Suppose $z_1 = s_2$. That is, $[a_2, s_2] \longrightarrow a_4$. Then s_2 dominates $((G - S) \cup \{x_1\}) - (H_2 \cup \{a_4\})$. Recall that $[a_3, s_1] \longrightarrow a_1$ and $x_1s_1 \in E(G)$. Thus a_5 is adjacent to both s_1 and s_2, $s_1a_1 \notin E(G)$ and $s_2a_4 \notin E(G)$.

Consider $G + a_2a_5$. Since $\gamma(G + a_2a_5) = 2$, there is a vertex z_2 of $G - \{a_2, a_5\}$ such that $[a_2, z_2] \longrightarrow a_5$ or $[a_5, z_2] \longrightarrow a_2$. In either case, $z_2 \notin S_0$ since $s_1a_1 \notin E(G)$

and $s_2a_4 \notin E(G)$. Suppose $[a_2, z_2] \longrightarrow a_5$. By O3 and O4, $\{x_4, a_5\}$ dominates $G - S_0$. Thus $\{x_4, a_5\}$ dominates G since a_5 is adjacent to both s_1 and s_2, and again we have a contradiction. Hence $[a_5, z_2] \longrightarrow a_2$. By O3 and O4, $\{x_1, a_2\}$ dominates $G - S_0$. Since $x_1s_1 \in E(G)$ and $x_1s_2 \in E(G)$, $\{x_1, a_2\}$ dominates G. This contradiction completes the proof of Subcase 2.1.2 and therefore also Subcase 2.1.

Subcase 2.2: Suppose $[a_4, z_1] \longrightarrow a_2$. Recall that $[a_3, s_1] \longrightarrow a_1$, $s_1a_1 \notin E(G)$, $s_1x_2 \in E(G)$ and $x_1s_1 \in E(G)$. Since $[a_4, z_1] \longrightarrow a_2$ and $s_1a_1 \notin E(G)$, it follows from O3 that either $z_1 = x_1$ or $z_1 = s_2$.

Subcase 2.2.1: Suppose $z_1 = x_1$. That is, $[a_4, x_1] \longrightarrow a_2$. By O3 and O4, $\{x_1, a_2\}$ dominates $G - S_0$. Since $x_1s_1 \in E(G)$, if either $x_1s_2 \in E(G)$ or $a_2s_2 \in E(G)$, then $\{x_1, a_2\}$ dominates G, a contradiction. Hence $x_1s_2 \notin E(G)$ and $a_2s_2 \notin E(G)$. Since $[a_i, x_i] \longrightarrow a_{i+1}$ for $1 \leq i \leq k-1$, $a_1s_2 \in E(G)$ and $x_2s_2 \in E(G)$. Figure 2.7 illustrates our situation.

Figure 2.7

Now consider $G + a_3a_5$. Since $\gamma(G + a_3a_5) = 2$, there is a vertex z_2 of $G - \{a_3, a_5\}$ such that either $[a_3, z_2] \longrightarrow a_5$ or $[a_5, z_2] \longrightarrow a_3$. In either case, $z_2 \notin S_0$ since $s_1a_1 \notin E(G)$ and $s_2a_2 \notin E(G)$. Suppose $[a_5, z_2] \longrightarrow a_3$. Then by O3 and O4, $\{x_2, a_3\}$ dominates $G - S_0$. Since $x_2s_1 \in E(G)$ and $x_2s_2 \in E(G)$, $\{x_2, a_3\}$ dominates G, a contradiction. Hence $[a_3, z_2] \longrightarrow a_5$. Then by O3 and O4, $\{x_4, a_5\}$ dominates $G - S_0$. Since $[a_3, s_1] \longrightarrow a_1$, it follows that $s_1a_5 \in E(G)$. If $x_4s_2 \in E(G)$ or $a_5s_2 \in E(G)$, then $\{x_4, a_5\}$ dominates G, a contradiction. Thus $x_4s_2 \notin E(G)$ and $a_5s_2 \notin E(G)$.

Now consider $G + a_2a_5$. Since $\gamma(G + a_2a_5) = 2$, there is a vertex z_3 of $G - \{a_2, a_5\}$ such that either $[a_2, z_3] \longrightarrow a_5$ or $[a_5, z_3] \longrightarrow a_2$. In either case, $z_2 \neq s_1$ since $s_1a_1 \notin E(G)$.

Suppose $[a_2, z_3] \longrightarrow a_5$. By O3 and the fact that $z_3 \neq s_1$, it follows that either $z_3 = x_4$ or $z_3 = s_2$. Since $a_2s_2 \notin E(G)$ and $x_4s_2 \notin E(G)$, it follows that $z_3 \neq x_4$. Thus $z_3 = s_2$. But this is impossible since $s_2x_1 \notin E(G)$ and $a_2x_1 \notin E(G)$. Hence $[a_5, z_3] \longrightarrow a_2$. By O3 and the fact that $z_3 \neq s_1$, it follows that either $z_3 = x_1$ or $z_3 = s_2$. Since $x_1s_2 \notin E(G)$ and $a_5s_2 \notin E(G)$, $z_3 \neq x_1$. Thus $z_3 = s_2$. But this is impossible since $s_2x_4 \notin E(G)$ and $a_5x_4 \notin E(G)$. This contradiction completes the proof of Subcase 2.2.1.

Subcase 2.2.2: Suppose $z_1 = s_2$. That is, $[a_4, s_2] \longrightarrow a_2$. Then s_2 dominates $((G - S) \cup \{x_3\}) - (H_4 \cup \{a_2\})$. Recall that $[a_3, s_1] \longrightarrow a_1$, $s_1a_1 \notin E(G)$, $s_1x_2 \in E(G)$ and $x_1s_1 \in E(G)$. More specifically, $s_1a_1 \notin E(G)$ and $s_2a_2 \notin E(G)$, but $s_1a_4 \in E(G)$, $s_2x_3 \in E(G)$ and $s_2a_1 \in E(G)$.

Consider $G + a_1a_4$. Since $\gamma(G + a_1a_4) = 2$, there is a vertex z_2 of $G - \{a_1, a_4\}$ such that either $[a_1, z_2] \longrightarrow a_4$ or $[a_4, z_2] \longrightarrow a_1$. In either case, $z_2 \neq s_2$ since $s_2a_2 \notin E(G)$. Suppose

$_1, z_2] \longrightarrow a_4$. By O3, $z_2 = x_3$ or $z_2 = s_1$. By Lemma 1.2, $z_2 \neq s_1$ since $s_1a_4 \in E(G)$. hus $z_2 = x_3$. By O4, $\{x_3, a_4\}$ dominates $G - S_0$. Since $s_1a_4 \in E(G)$ and $s_2x_3 \in E(G)$, follows that $\{x_3, a_4\}$ dominates G, a contradiction. Hence $[a_4, z_2] \longrightarrow a_1$. By O3 and ie fact that $z_2 \neq s_2$, it follows that $z_2 = s_1$. That is, $[a_4, s_1] \longrightarrow a_1$. Since $[a_3, s_1] \longrightarrow a_1$, rtex s_1 dominates $(G - S) - \{a_1\}$. Because $s_1x_1 \in E(G)$ and $s_2a_1 \in E(G)$, $\{s_1, a_1\}$ ominates G. This contradiction completes the proof of Subcase 2.2.2 and hence also ibcase 2.2. Hence $\gamma(G + a_2a_4) > 2$, contradicting the 3-γ-criticality of G. Hence for all $\in V(G) - \{a_1, a_3\}$, $\{a_3, z\}$ does not dominate $G - \{a_1\}$. This implies that $\gamma(G + a_1a_3) > 2$, gain a contradiction. This completes the proof of the theorem.

∎

In order to prove the main theorem of this section, we shall also need the following esult which may be viewed as yet another extension of Theorem 1.3.

Theorem 2.2 Let G be a connected 3-γ-critical graph and let S be a vertex cutset 1 G with $4 \leq |S| \leq 5$. If each component of $G - S$ has at least three vertices, then $(G - S) \leq |S| - 2$.

Proof: Suppose, to the contrary, that $\omega(G - S) \geq |S| - 1$. By Theorem 1.3(i), it follows hat $\omega(G - S) = |S| - 1$. Put $t = |S| - 1$. Note that $3 \leq t \leq 4$ Let H_i be a component f $G - S$ for $i = 1, 2, \ldots, t$. Choose a vertex $w_i \in V(H_i)$ for $i = 1, 2, \ldots, t$. Clearly, $V = \{w_1, w_2, \ldots, w_t\}$ is an independent set. By Lemma 1.1(i), the vertices of W may e ordered as $a_1, a_2, \ldots, a_t$ in such a way that there exist distinct vertices $x_1, x_2, \ldots, x_{t-1}$ uch that $[a_i, x_i] \longrightarrow a_{i+1}$, for $i = 1, 2, \ldots, t - 1$. By Lemma 1.2, $x_ia_{i+1} \notin E(G)$ for $= 1, 2, \ldots, t - 1$. Thus $x_1x_2 \cdots x_{t-1}$ is a path. Without loss of generality, we may enumber the components of $G - S$ in such a way that $a_i \in V(H_i)$. Since $|V(H_i)| \geq 3$, it ollows that $x_i \in S$ for $i = 1, 2, \ldots, t - 1$.

Let $S_0 = S - \{x_1, x_2, \ldots, x_{t-1}\}$. Then $|S_0| = 2$ and so we may set $S_0 = \{s_1, s_2\}$. Note hat obvervations O1, O2, O3 and O4 made in the proof of Theorem 2.1 are still valid n the present situation. Furthermore, O4 is still true if we replace a_i with b_i, where b_i belongs to the same component as a_i.

Since $|V(H_t)| \geq 3$, there are vertices $b_t, c_t \in V(H_t) - \{a_t\}$. Consider $G + a_1b_t$. Since $\gamma(G + a_1b_t) = 2$, there is a vertex z of $G - \{a_1, b_t\}$ such that either $[a_1, z] \longrightarrow b_t$ or $[b_t, z] \longrightarrow a_1$. In either case, $z \in S$. We distinguish two cases.

Case 1: Suppose $[a_1, z] \longrightarrow b_t$.

By O1 and Lemma 1.2, $z \notin \{x_1, x_2, \ldots, x_{t-1}\}$. Then it follows that $z \in S_0$. Without loss of generality, we may assume that $z = s_1$. That is, $[a_1, s_1] \longrightarrow b_t$. Then s_1 dominates $\cup_{j=2}^{t} V(H_j) - \{b_t\}$. If $x_2s_2 \in E(G)$ or $s_1s_2 \in E(G)$, then $\{x_2, s_1\}$ dominates G by O1, together with the fact that $x_1x_2 \cdots x_{t-1}$ is a path with $2 \leq t - 1 \leq 3$ and s_1 dominates $\cup_{j=2}^{t} V(H_j) - \{b_t\}$. But this is a contradiction and so $x_2s_2 \notin E(G)$ and $s_1s_2 \notin E(G)$. Since $[a_2, x_2] \longrightarrow a_3$ and $x_2s_2 \notin E(G)$, $a_2s_2 \in E(G)$. Furthermore, since $[a_1, s_1] \longrightarrow b_t$ and $s_1s_2 \notin E(G)$, $a_1s_2 \in E(G)$.

Figure 2.8 illustrates the present situation.

Figure 2.8

Now consider $G + a_1c_t$. Since $\gamma(G + a_1c_t) = 2$, there must be a vertex z_1 of $G - \{a_1, c_t\}$ such that either $[a_1, z_1] \longrightarrow c_t$ or $[c_t, z_1] \longrightarrow a_1$. In either case, $z_1 \in S$. We distinguish two cases.

Subcase 1.1: Suppose $[a_1, z_1] \longrightarrow c_t$.

By O1 and Lemma 1.2, $z_1 \notin \{x_1, x_2, \ldots, x_{t-1}\}$. Since $s_1b_t \notin E(G)$ and $a_1b_t \notin E(G)$, it follows that $z_1 \neq s_1$. But then $z_1 = s_2$. That is, $[a_1, s_2] \longrightarrow c_t$. Hence s_2 dominates $\cup_{j=2}^{t} V(H_j) - \{c_t\}$. Since $s_1s_2 \notin E(G)$ and $[a_1, s_2] \longrightarrow c_t$, it follows that $a_1s_1 \in E(G)$. Now a_1 is adjacent to both s_1 and s_2. Further, a_t is also adjacent to both s_1 and s_2 since s_1 dominates $\cup_{j=2}^{t} V(H_j) - \{b_t\}$ and s_2 dominates $\cup_{j=2}^{t} V(H_j) - \{c_t\}$.

Consider $G + a_1a_t$. Since $\gamma(G + a_1a_t) = 2$, there must be a vertex z_2 of $G - \{a_1, a_t\}$ such that either $[a_1, z_2] \longrightarrow a_t$ or $[a_t, z_2] \longrightarrow a_1$. In either case, $z_2 \in S$. By Lemma 1.2 and the fact that a_1 and a_t are adjacent to both s_1 and s_2, $z_2 \notin S_0$. By O3, $[a_t, z_2] \longrightarrow a_1$ is impossible. Hence $[a_1, z_2] \longrightarrow a_t$. By O3 and O4, $\{x_{t-1}, a_t\}$ dominates $G - S_0$. Since $a_ts_1 \in E(G)$ and $a_ts_2 \in E(G)$, it follows that $\{x_{t-1}, a_t\}$ dominates G. This contradiction completes the proof of Subcase 1.1.

Subcase 1.2: $[c_t, z_1] \longrightarrow a_1$.

Since $x_1a_2 \notin E(G)$ and $a_2c_t \notin E(G)$, $z_1 \neq x_1$. By O1, $x_ia_1 \in E(G)$ for $2 \leq i \leq t-1$. Thus by Lemma 1.2, $z_1 \notin \{x_2, \ldots, x_{t-1}\}$. Furthermore, $z_1 \neq s_2$ since $a_1s_2 \in E(G)$. Thus $z_1 = s_1$. That is, $[c_t, s_1] \longrightarrow a_1$. Recall that $[a_1, s_1] \longrightarrow b_t$, $x_2s_2 \notin E(G)$, $s_1s_2 \notin E(G)$, but $a_2s_2 \in E(G)$ and $a_1s_2 \in E(G)$. Since $[c_t, s_1] \longrightarrow a_1$ and $[a_1, s_1] \longrightarrow b_t$, s_1 dominates $(G - S) - \{a_1, b_t\}$. Furthermore, since $s_1s_2 \notin E(G)$ and $[c_t, s_1] \longrightarrow a_1$, $c_ts_2 \in E(G)$. Now c_t is adjacent to every vertex of S by O1 and the fact that s_1 dominates $(G - S) - \{a_1, b_t\}$.

Since $|V(H_1)| \geq 3$, there are vertices $b_1, c_1 \in V(H_1) - \{a_1\}$. Consider $G + b_1c_t$. Since $\gamma(G + b_1c_t) = 2$, there is a vertex z_2 of $G - \{b_1, c_t\}$ such that either $[b_1, z_2] \longrightarrow c_t$ or $[c_t, z_2] \longrightarrow b_1$. In either case, $z_2 \in S$. Suppose $[b_1, z_2] \longrightarrow c_t$. By Lemma 1.2, $z_2c_t \notin E(G)$. Thus $z_2 \notin S$ since c_t is adjacent to every vertex of S, a contradiction. Hence $[c_t, z_2] \longrightarrow b_1$. Since $x_1a_2 \notin E(G)$ and $c_ta_2 \notin E(G)$, $z_2 \neq x_1$. By O1, $x_ib_1 \in E(G)$ for $2 \leq i \leq t-1$. Thus by Lemma 1.2, $z_2 \notin \{x_2, \ldots, x_{t-1}\}$. Furthermore, $z_2 \neq s_1$ since $s_1b_1 \in E(G)$. Thus $z_2 = s_2$. Now s_2 dominates $\cup_{j=1}^{t-1} V(H_j) - \{b_1\}$.

Now consider $G + c_1c_t$. Since $\gamma(G + c_1c_t) = 2$, there is a vertex z_3 of $G - \{c_1, c_t\}$ such that either $[c_1, z_3] \longrightarrow c_t$ or $[c_t, z_3] \longrightarrow c_1$. In either case, $z_3 \in S$. By applying the same argument as above, the case $[c_1, z_3] \longrightarrow c_t$ is impossible. Hence $[c_t, z_3] \longrightarrow c_1$. Since $x_1a_2 \notin E(G)$ and $c_ta_2 \notin E(G)$, $z_3 \neq x_1$. Then $z_3 \in \{s_1, s_2\} \cup \{x_2, \ldots, x_{t-1}\}$. But this contradicts Lemma 1.2 since for $2 \leq i \leq t-1, x_i, s_1$ and s_2 are all adjacent to c_1. This completes the proof of Subcase 1.2 and hence the proof of Case 1 is complete.

Case 2: Suppose, therefore, that $[b_t, z] \longrightarrow a_1$.

By O3, $z \in S_0$. Without loss of generality, we may assume that $z = s_1$. That is, $[b_t, s_1] \longrightarrow a_1$. Thus s_1 dominates $\cup_{j=1}^{t-1} V(H_j) - \{a_1\}$. Since $[a_1, x_1] \longrightarrow a_2$ and $s_1a_1 \notin E(G)$, $x_1s_1 \in E(G)$. Figure 2.9 depicts our situation.

Figure 2.9

Since $|V(H_1)| \geq 3$, there is a vertex b_1 of $V(H_1) - \{a_1\}$. Consider $G + b_1b_t$. Since $\gamma(G + b_1b_t) = 2$, there is a vertex z_1 of $G - \{b_1, b_t\}$ such that either $[b_1, z_1] \longrightarrow b_t$ or $[b_t, z_1] \longrightarrow b_1$. In either case, $z_1 \in S$. We distinguish two subcases.

Subcase 2.1: Suppose $[b_1, z_1] \longrightarrow b_t$.

Since $[a_i, x_i] \longrightarrow a_{i+1}$, $b_tx_i \in E(G)$ for $1 \leq i \leq t-1$ by O1. Then $z_1 \notin \{x_1, \ldots, x_{t-1}\}$ by Lemma 1.2. Hence $z_1 = s_1$ or $z_1 = s_2$.

Subcase 2.1.1: Suppose $z_1 = s_1$. That is, $[b_1, s_1] \longrightarrow b_t$. Since $[b_t, s_1] \longrightarrow a_1$, s_1 dominates $(G - S) - \{a_1, b_t\}$. Since $x_2a_1 \in E(G)$ and $x_2b_t \in E(G)$ by O1, if $x_2s_2 \in E(G)$ or $s_1s_2 \in E(G)$, then $\{x_2, s_1\}$ dominates G, a contradiction. Hence $x_2s_2 \notin E(G)$ and $s_1s_2 \notin E(G)$. Because $[a_2, x_2] \longrightarrow a_3$ and $x_2s_2 \notin E(G)$, $a_2s_2 \in E(G)$. Since $[b_1, s_1] \longrightarrow b_t$ and $s_1s_2 \notin E(G)$, $b_1s_2 \in E(G)$. Since s_1 dominates $(G - S) - \{a_1, b_t\}$, edges s_1b_1, s_1a_2 and s_1a_t belong to $E(G)$. Figure 2.10 illustrates our situation.

Figure 2.10

We will now show that $s_2a_t \notin E(G)$. Suppose, to the contrary, that $s_2a_t \in E(G)$. Consider $G + b_1a_t$. Since $\gamma(G + b_1a_t) = 2$, there is a vertex z_2 of $G - \{b_1, a_t\}$ such that either $[b_1, z_2] \longrightarrow a_t$ or $[a_t, z_2] \longrightarrow b_1$. In either case, $z_2 \in S$, but by Lemma 1.2, $z_2 \notin \{s_1, s_2\}$ since b_1 and a_t are adjacent to both s_1 and s_2. Suppose $[b_1, z_2] \longrightarrow a_t$. Then by O1 and Lemma 1.2, $z_2 = x_{t-1}$. That is, $[b_1, x_{t-1}] \longrightarrow a_t$. By O4, and the fact that a_t is adjacent to both s_1 and s_2, it follows that $\{x_{t-1}, a_t\}$ dominates G, a contradiction. Hence $[a_t, z_2] \longrightarrow b_1$. But then by O1 and Lemma 1.2, $z_2 \neq x_i$ for $2 \leq i \leq t-1$. Furthermore, $z_2 \neq x_1$ since $x_1a_2 \notin E(G)$. Hence $z_2 \notin S$, a contradiction. This proves that $s_2a_t \notin E(G)$. By applying similar arguments, we also obtain that $s_2c_t \notin E(G)$.

Now consider $G + a_2c_t$. Since $\gamma(G + a_2c_t) = 2$, there is a vertex z_3 of $G - \{a_2, c_t\}$ such that either $[a_2, z_3] \longrightarrow c_t$ or $[c_t, z_3] \longrightarrow a_2$. In either case, $z_3 \in S$. Suppose that $[a_2, z_3] \longrightarrow c_t$. By O1, x_i is adjacent to c_t for $1 \leq i \leq t-1$. Since s_1 dominates $(G - S) - \{a_1, b_t\}$, $s_1c_t \in E(G)$. By Lemma 1.2, $z_3 \notin \{s_1, x_1, x_2, \ldots, x_{t-1}\}$. Therefore, $z_3 = s_2$. But this is impossible, since $s_2a_t \notin E(G)$ and $a_2a_t \notin E(G)$. Hence $[c_t, z_3] \longrightarrow a_2$. Since a_2 is adjacent to s_1 and s_2, $z_3 \notin \{s_1, s_2\}$ by Lemma 1.2. Therefore, $z_3 \in \{x_1, x_2, \ldots, x_{t-1}\}$. Suppose $z_3 = x_1$. Then x_1 dominates $\cup_{j=1}^{t-1} V(H_j) - \{a_2\}$. Thus $\{x_1, a_2\}$ dominates G by O1, O2 and the fact that $x_1x_2 \cdots x_{t-1}$ is a path with $2 \leq t-1 \leq 3$ and a_2 is adjacent to both s_1 and s_2, a contradiction. Hence $z_3 \neq x_1$. Since $x_2s_2 \notin E(G)$ and $c_ts_2 \notin E(G)$, $z_3 \neq x_2$.

his implies that $t = 4$ and $z_3 = x_3 = x_{t-1}$. But this also contradicts Lemma 1.2 since $_3a_2 \in E(G)$ by O1. This completes the proof of Subcase 2.1.1.

Subcase 2.1.2: Suppose $z_1 = s_2$.

That is, $[b_1, s_2] \longrightarrow b_t$. Then s_2 dominates $\cup_{j=2}^{t} V(H_j) - \{b_t\}$. Recall that $[b_t, s_1] \longrightarrow$ $_1$. Since $s_2b_t \notin E(G)$, $s_1s_2 \in E(G)$. But then $\{x_2, s_2\}$ dominates G by O1, together with he facts that $x_1x_2 \cdots x_t$ is a path with $2 \leq t-1 \leq 3$, $s_1s_2 \in E(G)$ and s_2 dominates $_{j=2}^{t}V(H_j) - \{b_t\}$. This contradiction completes the proof of Subcase 2.1.2 and hence also f Subcase 2.1.

Subcase 2.2: Suppose, then, that $[b_t, z_1] \longrightarrow b_1$.

By O1 and Lemma 1.2, $z_1 \neq x_i$ for $2 \leq i \leq t-1$. Since $x_1a_2 \notin E(G)$ and $s_1a_1 \notin E(G)$, $_1 \notin \{x_1, s_1\}$. But then $z_1 = s_2$. Thus s_2 dominates $\cup_{j=1}^{t-1} V(H_j) - \{b_1\}$. Recall that $b_t, s_1] \longrightarrow a_1$ and $x_1s_1 \in E(G)$. Then y and w must be adjacent to s_1 and s_2 for all $\in V(H_1) - \{a_1, b_1\}$ and for all $w \in V(H_{t-1})$. Since $|V(H_i)| \geq 3$, for $1 \leq i \leq t$, there nust be a vertex $c_1 \in V(H_1) - \{a_1, b_1\}$ and a vertex $b_{t-1} \in V(H_{t-1}) - \{a_{t-1}\}$. Then c_1 nd b_{t-1} are adjacent to both s_1 and s_2.

Consider, finally, $G + c_1b_{t-1}$. Since $\gamma(G + c_1b_{t-1}) = 2$, there must be a vertex z_2 of $G - \{c_1, b_{t-1}\}$ such that either $[c_1, z_2] \longrightarrow b_{t-1}$ or $[b_{t-1}, z_2] \longrightarrow c_1$. In either case, $z_2 \in S$. ut $z_2 \notin \{s_1, s_2\}$ by Lemma 1.2 and the fact that c_1 and b_{t-1} are adjacent to s_1 and s_2. Suppose $[c_1, z_2] \longrightarrow b_{t-1}$. By O1 and Lemma 1.2, $z_2 \neq x_1$. Since $x_2a_3 \notin E(G)$, $z_2 \neq x_2$. Then $t = 4$ and $z_2 = x_3$. But this is impossible since $c_1a_4 \notin E(G)$ and $x_3a_4 \notin E(G)$. Hence $[b_{t-1}, z_2] \longrightarrow c_1$. Since $z_2 \notin \{s_1, s_2\}$, $|(t-1) - 1| < 2$ by O3. This implies that $t - 1 = 2$. Then by O1 and Lemma 1.2, $z_2 \neq x_2$. Thus $z_2 = x_1$. But then x_1 dominates $(G - S) - \{c_1, a_2\}$. Hence $\{x_1, s_2\}$ dominates G, since $x_1s_1 \in E(G), x_1x_2 \in E(G)$ and s_2 dominates $(H_1 \cup H_2) - \{b_1\}$. This contradiction completes the proof of Subcase 2.2 and hence Case 2. This proves that $\gamma(G + a_1b_t) > 2$, contradicting the 3-γ-criticality of G. This completes the proof of Theorem 2.2.

∎

Finally, the following two results will also be useful in proving our main theorem. The first is due to Favaron [F] and the second is proved in [AP2].

Theorem 2.3 A graph G is n-factor-critical if and only if $\omega_o(G - S) \leq |S| - n$, for every $S \subseteq V(G)$ and $|S| \geq n$.

∎

Theorem 2.4 Let G be a 2-connected 3-γ-critical graph having odd order. Then G is factor-critical.

∎

We now present our main result of this section.

Theorem 2.5 If G is a 4-connected 3-γ-critical graph of odd order and having minimum degree at least 5, then G is 3-factor-critical.

Proof: Suppose, by way of contradiction, that G is not 3-factor-critical. By Theorem 3, there is a set $S \subset V(G)$ with $|S| \geq 3$ such that $\omega_o(G-S) > |S| - 3$.

Since G is factor-critical, by Theorem 2.4, $\omega_o(G-S) \leq |S|-1$. Since G has odd order, $_o(G-S) \neq |S|-2$. But then $\omega_o(G-S) = |S|-1$. By Theorem 2.1 and our connectivity ypothesis, $4 \leq |S| \leq 5$. Since $G-S$ has $|S|-1$ odd components, there is a component $G-S$, say H_1, such that $|V(H_1)| = 1$ by Theorem 2.2. Let $V(H_1) = \{w_1\}$. If $|S| = 4$, ien $d(w_1) \leq 4$, a contradiction of our minimum degree hypothesis. Hence $|S| = 5$. By heorem 1.3(i), $G-S$ has no even components. For $i = 1, \ldots, 4$, let H_i be the components $G-S$. Choose a vertex $w_i \in V(H_i)$, for each $i = 1, \ldots, 4$. Clearly $W = \{w_1, w_2, w_3, w_4\}$ an independent set. By Lemma 1.1, the vertices of W may be ordered as a_1, a_2, a_3 and $_4$ in such a way that there exists a path $x_1x_2x_3$ in $G-W$ such that $[a_i, x_i] \longrightarrow a_{i+1}$, or each $i = 1, 2, 3$. By Lemma 1.2, $x_ia_{i+1} \notin E(G)$ for each $i = 1, 2, 3$. Clearly, $x_i \in S$ or $i = 1, 2, 3$. Let $S_0 = S - \{x_1, x_2, x_3\}$. So $|S_0| = 2$. Let $S_0 = \{s_1, s_2\}$. With loss of enerality, we may renumber the components of $G-S$ in such a way that $a_i \in V(H_i)$. Note that obvervations O1, O2, O3 and O4, stated in the proof of Theorem 2.1, remain alid in the present situation. Since for each $i = 1, 2, 3$, $x_ia_{i+1} \notin E(G)$, it follows that $V(H_i)| \geq 3$ for $2 \leq i \leq 4$ because mindeg $(G) \geq 5$ and $|S| = 5$. Further, $V(H_1) = \{a_1\}$, by Theorem 2.2 and a_1 is adjacent to every vertex of S. But then for each i, $2 \leq i \leq 4$, there exist two distinct vertices b_i and c_i in $V(H_i) - \{a_i\}$.

Let y be a vertex of $(H_3 \cup H_4) - \{a_3, a_4\}$. Consider $G + a_1y$. Since $\gamma(G + a_1y) = 2$, there is a vertex z of $G - \{a_1, y\}$ such that either $[a_1, z] \longrightarrow y$ or $[y, z] \longrightarrow a_1$. In either case, $z \in S$. Since a_1 is adjacent to every vertex of S_0, the case $[y, z] \longrightarrow a_1$ is impossible by O3 and Lemma 1.2. Hence, $[a_1, z] \longrightarrow y$. By O1 and Lemma 1.2, $z \notin \{x_1, x_2\}$. Since $x_3a_4 \notin E(G)$ and $a_1a_4 \notin E(G)$, it follows that $z \neq x_3$. Thus $z \in S_0$.

Now let $y = b_3$ and consider $G + a_1b_3$. Then, by the above argument, there is a vertex $z \in S_0$ such that $[a_1, z] \longrightarrow b_3$. Without any loss of generality, we may assume that $z = s_1$. That is, $[a_1, s_1] \longrightarrow b_3$. By Lemma 1.2, $s_1b_3 \notin E(G)$. Next let $y = c_3$ and consider $G + a_1c_3$. Again there is a vertex $z_1 \in S_0$ such that $[a_1, z_1] \longrightarrow c_3$. Since $s_1b_3 \notin E(G)$ and $a_1b_3 \notin E(G)$, $z_1 \neq s_1$. Thus $z_1 = s_2$. That is, $[a_1, s_2] \longrightarrow c_3$ and $s_2c_3 \notin E(G)$. Finally, we let $y = b_4$ and consider $G + a_1b_4$. Then there is a vertex $z_2 \in S_0$ such that $[a_1, z_2] \longrightarrow b_4$. But $z_2 \neq s_1$ since $s_1b_3 \notin E(G)$ and $a_1b_3 \notin E(G)$. Thus $z_2 = s_2$. But this is impossible since $s_2c_3 \notin E(G)$ and $a_1c_3 \notin E(G)$. This proves that $\gamma(G + a_1b_4) > 2$, contradicting the 3-criticality of G. This completes the proof of our theorem.

∎

The bound on the minimum degree stated in the hypotheses of Theorem 2.5 is best possible since there is a 4-connected 3-γ-critical graph with minimum degree 4 and having odd order, but which is not 3-factor-critical. Such a graph G_0 is shown in Figure 2.11 below.

Figure 2.11

3. A Result About 3-γ-criticality in Claw-free Graphs

A graph is said to be **claw-free** if it contains no induced subgraph isomorphic to $K_{1,3}$. In [P] the following result was proved.

Theorem 3.1. If G is a 3-connected claw-free graph of even order, then G is bicritical. ∎

If the even graphs under consideration are 3-γ-critical, we can lower the demand on connectivity and still obtain bicriticality. More particularly, we have the following result proved in [AP2].

Theorem 3.2. Let G be a 3-γ-critical 2-connected claw-free graph of even order. Then if mindeg $G \geq 3$, G is bicritical.

We now prove a similar result involving 3-factor criticality. First, however, we state a result of Sumner and Blitch [SB, B] which will be useful in this regard.

Theorem 3.3 The diameter of a 3-γ-critical graph is at most 3. ∎.

We now present the main result of this section.

Theorem 3.4 Let G be a 3-γ-critical 3-connected claw-free graph of odd order. Then if mindeg $G \geq 4$, G is 3-factor-critical.

Proof: Suppose, to the contrary, that G is not 3-factor-critical. Then by Theorem 2.3, there is a subset S of $V(G)$ such that $|S| \geq 3$ and $\omega_o(G-S) > |S| - 3$. But by Theorem 2.4, G is factor-critical. Thus $\omega_o(G-S) \leq |S|-1$. Since $|V(G)|$ is odd, it follows by parity that $\omega_o(G-S) = |S|-1$. Then by Theorem 2.1, $|S| \leq 5$. Since G is 3-connected, $3 \leq |S| \leq 5$.

We first suppose that $|S| = 4$. By Theorem 1.3(i), $G - S$ has no even components. Since G is 3-connected, there are at least three vertices of S which are adjacent to some vertices of each component of $G - S$. Because $|S| = 4$, there must be a vertex of S, say u, such that u is adjacent to at least one vertex of each component of $G - S$. Thus u is a claw center in G which contradicts the assumption that G is claw-free. Hence $|S| \neq 4$. By a similar argument, $|S| \neq 5$. Thus $|S| = 3$.

Since G is claw-free, it is easy to see that $G - S$ has no even components. Furthermore, since G is 3-connected, S is a *minimum* cutset in G. Because mindeg $G \geq 4$, each component of $G - S$ has at least three vertices. Let H_1 and H_2 be the odd components of $G - S$ and let $S = \{u_1, u_2, u_3\}$.

We now define several sets of vertices in G as follows. For $1 \leq i \leq 3$, let

$$A_i = V(H_1) \cap N(u_i),$$

$$B_i = V(H_2) \cap N(u_i),$$

$$C = V(H_1) - \cup_{i=1}^{3} A_i,$$

d

$$D = V(H_2) - \cup_{i=1}^{3} B_i.$$

Claim 1: For $1 \leq i \leq 3$, $A_i \neq \emptyset \neq B_i$. Furthermore, both $G[A_i]$ and $G[B_i]$ are mplete.

This claim follows directly from the fact that S is a minimum cutset and G is claw-free.

Claim 2: Either $C = \emptyset$ or $D = \emptyset$.

Suppose, to the contrary, that there is a vertex $x \in C$ and a vertex $y \in D$. Then the istance between vertices x and y is at least 4, since $x \notin \cup_{i=1}^{3} A_i$ and $y \notin \cup_{i=1}^{3} B_i$. But this ontradicts Theorem 3.3, thus proving the Claim.

So without loss of generality, let us assume that $C = \emptyset$. We now distinguish two cases. ccording to whether D is empty or not.

Case 1: Suppose $D \neq \emptyset$.
Choose vertices $a_1 \in A_1$ and $b_1 \in B_1$. Consider $G + a_1 b_1$. Since G is 3-γ-critical, there s a vertex $z_1 \in V(G) - \{a_1, b_1\}$ such that either $[a_1, z_1] \longrightarrow b_1$ or $[b_1, z_1] \longrightarrow a_1$.

Subcase 1.1: Suppose $[a_1, z_1] \longrightarrow b_1$. By Lemma 1.2, $z_1 b_1 \notin E(G)$. Since $D \neq \emptyset$ and B_1 is complete, $z_1 \in V(H_2) - B_1$. Thus a_1 dominates $V(H_1) \cup \{u_1\}$.

Subcase 1.1.1: Suppose $z_1 \in (B_2 \cup B_3) - B_1$.
Without loss of generality, we may assume that $z_1 \in B_2 - B_1$. Then z_1 dominates $V(H_2) - \{b_1\}$. Since u_2 is not adjacent to any vertex of D, $G[D]$ is complete. To see this, just suppose that $|V(D)| > 1$ and suppose that x_1 and x_2 are two non-adjacent vertices in D. Then $G[z_1; u_2, x_1, x_2]$ is a claw centered at z_1. Furthermore, since $z_1 b_1 \notin E(G)$, it follows that $z_1 u_1 \notin E(G)$ by Claim 1. Similarly, $u_2 b_1 \notin E(G)$. Figure 3.1 depicts our present situation.

Figure 3.1

Since $[a_1, z_1] \longrightarrow b_1$ either $z_1 u_3 \in E(G)$ or $a_1 u_3 \in E(G)$.

Suppose first that $z_1u_3 \in E(G)$. Then there is a vertex $v \in A_2 \cup A_3$ such that u_1 is not adjacent to v, for otherwise $\{u_1, z_1\}$ dominates G, a contradiction.

Now consider $G + u_1z_1$. There must be a vertex z_2 of $G - \{u_1, z_1\}$ such that either $[u_1, z_2] \longrightarrow z_1$ or $[z_1, z_2] \longrightarrow u_1$. First suppose that $[u_1, z_2] \longrightarrow z_1$. By Lemma 1.2, $z_1z_2 \notin E(G)$. Since z_1 dominates $(V(H_2) - \{b_1\}) \cup \{u_2, u_3\}$, $z_2 \notin (V(H_2) - \{b_1\}) \cup \{u_2, u_3\}$. Thus $z_2 = b_1$ or $z_2 \in V(H_1)$. In either case, $\{u_1, z_2\}$ does not dominate $G - \{z_1\}$ since $u_1v \notin E(G)$ and $D \neq \emptyset$. Hence $[z_1, z_2] \longrightarrow u_1$. Since $z_1b_1 \notin E(G)$, $b_1u_2 \notin E(G)$ and $V(H_1) \neq \emptyset$, it follows that $z_2 = u_3$. Thus $u_3b_1 \in E(G)$. But then $G[u_3; b_1, z_1, a_3]$ is a claw centered at u_3 for any $a_3 \in A_3$. Thus $z_1u_3 \notin E(G)$.

Hence $a_1u_3 \in E(G)$.

Now consider $G + b_1u_2$. There must be a vertex z_3 of $G - \{b_1, u_2\}$ such that either $[b_1, z_3] \longrightarrow u_2$ or $[u_2, z_3] \longrightarrow b_1$. Suppose first that $[b_1, z_3] \longrightarrow u_2$. Since $b_1z_1 \notin E(G)$ and $V(H_1) \neq \emptyset$, $z_3 \in \{u_1, u_3\}$. But this is impossible since $z_1u_1 \notin E(G)$ and $z_1u_3 \notin E(G)$. Thus $\{b_1, z_3\}$ does not dominate $G - \{u_2\}$.

Hence $[u_2, z_3] \longrightarrow b_1$. By Lemma 1.2, $z_3b_1 \notin E(G)$. Since $D \neq \emptyset$, $z_3 \in V(H_2) - B_1$, since B_1 is complete. But then u_2 must dominate $V(H_1) \cup \{u_1\}$. Suppose there is a vertex $x \in V(H_1)$ such that u_1 is not adjacent to x. Then $G[u_2; u_1, x, z_1]$ is a claw centered at u_2, a contradiction. Hence u_1 also dominates $V(H_1)$.

Next we will show that $u_2u_3 \notin E(G)$. Suppose, to the contrary, that $u_2u_3 \in E(G)$. By an argument similar to the one immediately above, u_3 dominates $V(H_1)$, for otherwise G contains a claw centered at u_2. But then $u_1u_3 \in E(G)$, for otherwise $G[u_2; u_1, u_3, z_1]$ is a claw centered at u_2. Consequently, $\{u_1, z_1\}$ dominates G, a contradiction. Hence $u_2u_3 \notin E(G)$.

Since $[u_2, z_3] \longrightarrow b_1$, $z_3u_3 \in E(G)$. Because $z_1u_3 \notin E(G)$, $z_3 \neq z_1$. Now choose $a_3 \in A_3$. Since u_2 dominates $V(H_1) \cup \{u_1\}$, $G[V(H_1)]$ is complete by Claim 1. Thus a_3 dominates every vertex of $V(H_1) \cup \{u_1, u_2, u_3\}$. Consider now $G + a_3z_1$. There must be a vertex z_4 of $G - \{a_3, z_1\}$ such that either $[a_3, z_4] \longrightarrow z_1$ or $[z_1, z_4] \longrightarrow a_3$. Suppose $[a_3, z_4] \longrightarrow z_1$. By Lemma 1.2, $z_4z_1 \notin E(G)$. Since z_1 dominates $V(H_2) - \{b_1\}$, $z_4 \notin V(H_2) - \{b_1\}$. But $D \neq \emptyset$, so $z_4 = b_1$. This is impossible since $b_1z_3 \notin E(G)$ and $a_3z_3 \notin E(G)$. Thus $\{a_3, z_4\}$ does not dominate $G - \{z_1\}$. Hence $[z_1, z_4] \longrightarrow a_3$. By Lemma 1.2, $z_4a_3 \notin E(G)$. Since $V(H_1) - \{a_3\} \neq \emptyset$ (recall that $|V(H_1)| \geq 3$), $z_4 \in (V(H_1) - \{a_3\}) \cup \{u_1, u_2, u_3\}$. But this is impossible since a_3 dominates $V(H_1) \cup \{u_1, u_2, u_3\}$. This proves that Subcase 1.1.1 cannot occur.

Subcase 1.1.2: Suppose $z_1 \in D$.

Recall first that $[a_1, z_1] \longrightarrow b_1$. Since $z_1 \in D$, z_1 is adjacent to every vertex of $V(H_2) - \{b_1\}$ and a_1 dominates $V(H_1) \cup \{u_1, u_2, u_3\}$. Consider $G + a_1z_1$. There must be a vertex z_2 of $G - \{a_1, z_1\}$ such that either $[z_1, z_2] \longrightarrow a_1$ or $[a_1, z_2] \longrightarrow z_1$. Suppose first that $[z_1, z_2] \longrightarrow a_1$. By Lemma 1.2, $z_2a_1 \notin E(G)$. Since $z_1 \in D$ and $V(H_1) - \{a_1\} \neq \emptyset$, $z_2 \in (V(H_1) \cup \{u_1, u_2, u_3\}) - \{a_1\}$. But this is impossible since a_1 dominates $V(H_1) \cup \{u_1, u_2, u_3\}$. Thus $\{z_1, z_2\}$ does not dominate $G - \{a_1\}$. Hence $[a_1, z_2] \longrightarrow z_1$. By Lemma 1.2, $z_2z_1 \notin E(G)$. Since z_1 is adjacent to every vertex of $H_2 - \{b_1\}$ and $V(H_2) - \{z_1\} \neq \emptyset$, $z_2 = b_1$ or $z_2 \in \{u_1, u_2, u_3\}$. Suppose $z_2 \in \{u_1, u_2, u_3\}$. Then $D = \{z_1\}$ and $G[V(H_2) - D]$ must be complete by Claim 1. But then $\{a_1, b_2\}$ dominates G, where b_2 is any vertex in B_2 different from b_1, since $G[V(H_2) - D]$ is complete and $b_2z_1 \in E(G)$, a contradiction.

ence $z_2 = b_1$. That is, $[a_1, b_1] \longrightarrow z_1$. Thus b_1 is adjacent to every vertex of $H_2 - \{b_1, z_1\}$. or $i = 2, 3$, if $b_1 u_i \notin E(G)$, then $G[b_i; b_1, u_i, z_1]$ is a claw centered at b_i where $b_i \in B_i$, a ontradiction. Hence $b_1 u_i \in E(G)$ for $i = 2, 3$. Furthermore, since $b_1 \in B_1$, $b_1 u_i \in E(G)$, or $i = 1, 2, 3$. Figure 3.2 now depicts our situation.

Figure 3.2

Choose $x \in V(H_1) - \{a_1\}$ and consider $G + xb_1$. There must be a vertex z_3 of $G - \{x, b_1\}$ uch that either $[x, z_3] \longrightarrow b_1$ or $[b_1, z_3] \longrightarrow x$. Suppose that $[b_1, z_3] \longrightarrow x$. Since $b_1 z_1 \notin$ $E(G)$, $z_3 z_1 \in E(G)$. But because $V(H_1) - \{x\} \neq \emptyset$, z_3 must dominate $H_1 - \{x\}$ as well. But this is clearly impossible. Thus $\{b_1, z_3\}$ does not dominate $G - \{x\}$. So $[x, z_3] \longrightarrow b_1$. By Lemma 1.2, $z_3 b_1 \notin E(G)$. Since b_1 dominates $(V(H_2) - \{z_1\}) \cup \{u_1, u_2, u_3\}$, $z_3 = z_1$. Then x dominates $V(H_1) \cup \{u_1, u_2, u_3\}$. But x is an arbitrary member of $V(H_1) - \{a_1\}$ and also a_1 is adjacent to all vertices of $V(H_1) \cup \{u_1, u_2, u_3\}$. Consequently, $G[V(H_1)]$ is complete and every vertex of H_1 is adjacent to every vertex of $\{u_1, u_2, u_3\}$. Note that $G[\{u_1, u_2, u_3\}]$ must contain an edge, for otherwise any vertex of H_1 becomes the center of a claw in G. In fact, $G[\{u_1, u_2, u_3\}]$ contains exactly one edge; otherwise G is dominated by z_1 and one of the u_i, for $1 \leq i \leq 3$. Then there exist vertices u_i and u_j for some i, j, $1 \leq i < j \leq 3$ such that $u_i u_j \notin E(G)$. This implies that z_1 is the only vertex of D as otherwise b_1 is a claw center in G.

Without loss of generality, we may suppose that $u_1 u_2 \in E(G)$. Then $u_1 u_3 \notin E(G)$ and $u_2 u_3 \notin E(G)$. Thus no vertex of $B_2 - \{b_1\}$ can be adjacent to u_3, otherwise $G[b_2; u_2, u_3, z_1]$ is a claw centered at b_2 for every vertex $b_2 \in B_2 - \{b_1\}$. Similarly, each vertex of $B_3 - \{b_1\}$ is not adjacent to u_2. Note that u_1 dominates $V(H_1) \cup B_1 \cup \{u_2\}$.

Now first suppose that $B_3 = \{b_1\}$. Recall that $[a_1, b_1] \longrightarrow z_1$, so b_1 is adjacent to every vertex of $B_1 \cup B_2 \cup B_3$. If any two vertices of $B_1 \cup B_2$ are non-adjacent, we obtain a claw at b_1, a contradiction. Hence $G[B_1 \cup B_2 \cup B_3]$ is complete. But then choose any $b_2 \neq b_1, b_2 \in B_1 \cup B_2$ and see that b_2, together with any vertex in H_1, dominates G, again a contradiction. So $B_3 - \{b_1\} \neq \emptyset$. So choose a vertex $b_3 \in B_3 - \{b_1\}$. If $G[B_2 \cup B_3]$ is complete, then $\{u_1, b_3\}$ dominates G, since b_3 dominates $B_2 \cup B_3 \cup \{u_3, z_1\}$ and u_1 dominates $H_1 \cup \{u_2\}$. This contradiction implies that $G[B_2 \cup B_3]$ is not complete. This, in turn, implies that there are vertices $w \in B_2 - B_3$ and $y \in B_3 - B_2$ such that $wy \notin E(G)$, since both $G[B_2]$ and $G[B_3]$ are complete.

Now consider $G + wu_3$. There must be a vertex z_4 of $G - \{w, u_3\}$ such that either $[u_3, z_4] \longrightarrow w$ or $[w, z_4] \longrightarrow u_3$. Suppose first that $[u_3, z_4] \longrightarrow w$. By Lemma 1.2, $z_4 w \notin E(G)$. Since u_3 is not adjacent to any vertex of $\{u_1, u_2, z_1\}$, it follows that $z_4 \in$ $(B_1 \cap B_2) - \{w\}$. But this is impossible since $\{z_4, w\} \subseteq B_2$ and $G[B_2]$ is complete. Thus $\{u_3, z_4\}$ does not dominate $G - \{w\}$. Hence $[w, z_4] \longrightarrow u_3$. By Lemma 1.2, $z_4 u_3 \notin E(G)$.

call that each vertex of $V(H_1)$ is adjacent to every vertex of $\{u_1, u_2, u_3\}$. Thus $z_4 \notin$ $H_1)$. But then $z_4 \in \{u_1, u_2\}$ since $V(H_1) \neq \emptyset$. Since $wy \notin E(G)$ and $yu_2 \notin E(G)$, this plies that $z_4 = u_1$ and then $u_1 y \in E(G)$. Hence $G[y; u_1, u_3, z_1]$ is a claw centered at y. his contradiction completes the proof of Subcase 1.1.2 and hence also Subcase 1.1.

Subcase 1.2: So $[b_1, z_1] \longrightarrow a_1$.
By Lemma 1.2, $a_1 z_1 \notin E(G)$. Then $z_1 \notin A_1 \cup \{u_1\}$. Since $V(H_1) - \{a_1\} \neq \emptyset$, $\in A_2 \cup A_3$ or $z_1 \in \{u_2, u_3\}$. We distinguish two cases.

Subcase 1.2.1 Suppose $z_1 \in A_2 \cup A_3$. Without loss of generality we may assume that $_1 \in A_2$. Since $[b_1, z_1] \longrightarrow a_1$ and $b_1 \in V(H_2)$, z_1 dominates $H_1 - \{a_1\}$ and b_1 dominates I_2. Since $a_1 z_1 \notin E(G)$, $u_1 z_1 \notin E(G)$ by Claim 1. Similarly, $a_1 u_2 \notin E(G)$.

Now consider $G + u_1 z_1$. There must be a vertex z_2 of $G - \{u_1, z_1\}$ such that either $_1, z_2] \longrightarrow u_1$ or $[u_1, z_2] \longrightarrow z_1$. Suppose first that $[z_1, z_2] \longrightarrow u_1$. By Lemma 1.2, $_2 u_1 \notin E(G)$. Since $z_1 a_1 \notin E(G)$ and $D \neq \emptyset$, z_2 must be adjacent to a_1 and every vertex f D. But this is impossible since $D \cap N(u_i) = \emptyset$, for $1 \leq i \leq 3$. Then $\{z_1, z_2\}$ does ot dominate $G - \{u_1\}$. Hence $[u_1, z_2] \longrightarrow z_1$. By Lemma 1.2, $z_1 z_2 \notin E(G)$. Since $D \neq \emptyset$, it follows that $z_2 \in V(H_2)$. This implies that u_1 dominates $V(H_1) - \{z_1\}$ and hen $G[V(H_1) - \{z_1\}]$ is complete because of Claim 1. Thus $G[V(H_1)] = K_t - a_1 z_1$, where $= |V(H_1)|$, since z_1 is adjacent to every vertex of $V(H_1) - \{a_1\}$. (See Figure 3.3.)

Figure 3.3

Now choose $a_2 \in V(H_1) - \{a_1, z_1\}$. Consider $G + a_2 b_1$. Then there is a vertex z_3 of $G - \{a_2, b_1\}$ such that either $[a_2, z_3] \longrightarrow b_1$ or $[b_1, z_3] \longrightarrow a_2$. Suppose that $[a_2, z_3] \longrightarrow b_1$. By Lemma 1.2, $z_3 b_1 \notin E(G)$. Then $z_3 \notin V(H_2)$ since b_1 dominates $V(H_2)$. Because $D \neq \emptyset$, $z_3 \in V(H_2) - \{b_1\}$. This contradiction proves that $\{a_2, z_3\}$ does not dominate $G - \{b_1\}$. Hence it must be the case that $[b_1, z_3] \longrightarrow a_2$. By Lemma 1.2, $z_3 a_2 \notin E(G)$. Since $G[V(H_1)] = K_t - a_1 z_1$ and u_1 dominates $H_1 - \{a_1\}$, it follows that $z_3 \notin V(H_1) \cup \{u_1\}$. Since $V(H_1) - \{a_2\} \neq \emptyset$ and $b_1 \in V(H_2)$, it follows that $z_3 \in \{u_2, u_3\}$. But $z_3 \neq u_2$ since $u_2 a_1 \notin E(G)$ and $b_1 a_1 \notin E(G)$. Hence $z_3 = u_3$. Then $u_3 a_1 \in E(G)$ and $u_3 z_1 \in E(G)$. This contradicts Claim 1 since $a_1 z_1 \notin E(G)$. But then it is false that $[b_1, z_3] \longrightarrow a_2$. Hence Subcase 1.2.1 cannot occur.

Subcase 1.2.2: So $z_1 \in \{u_2, u_3\}$.
Without loss of generality, we may assume that $z_1 = u_2$. That is, $[b_1, u_2] \longrightarrow a_1$. By Lemma 1.2, $u_2 a_1 \notin E(G)$, u_2 dominates $H_1 - \{a_1\}$ and b_1 dominates $(B_1 \cup B_3 \cup D) - B_2$. By Claim 1, $H_1 - \{a_1\}$ is complete. Choose $a_3 \in V(H_1) - \{a_1\}$. Then $a_3 \in A_2$ since u_2

minates $H_1 - \{a_1\}$. Now consider $G + a_3b_1$. There must be a vertex z_2 of $G - \{a_3, b_1\}$ ch that either $[a_3, z_2] \longrightarrow b_1$ or $[b_1, z_2] \longrightarrow a_3$.

Subcase 1.2.2.1: Suppose $[a_3, z_2] \longrightarrow b_1$. By Lemma 1.2, $z_2b_1 \notin E(G)$. Since b_1 minates $(B_1 \cup B_3 \cup D) - B_2$ and $D \neq \emptyset$, $z_2 \in B_2$ and $z_2b_1 \notin E(G)$. Furthermore, dominates $H_2 - \{b_1\}$. By Claim 1, $b_1u_2 \notin E(G)$ and $u_1z_2 \notin E(G)$. Recall that $a_1 \notin E(G)$. Since $u_2b_1 \notin E(G)$, $u_1u_2 \notin E(G)$; otherwise, $G[u_1; u_2, b_1, a_1]$ is a claw ntered at u_1.

Now consider $G + u_1z_2$. There must be a vertex z_3 of $G - \{u_1, z_2\}$ such that either $_1, z_3] \longrightarrow z_2$ or $[z_2, z_3] \longrightarrow u_1$. Suppose first that $[u_1, z_3] \longrightarrow z_2$. By Lemma 1.2, $z_2 \notin E(G)$. Since z_2 dominates $H_2 - \{b_1\}$ and $D \neq \emptyset$, $z_3 = b_1$. This is impossible nce $b_1u_2 \notin E(G)$ and $u_1u_2 \notin E(G)$. Thus $\{u_1, z_3\}$ does not dominate $G - \{z_2\}$. Hence $_2, z_3] \longrightarrow u_1$. By Lemma 1.2, $z_3u_1 \notin E(G)$. Then $z_3 \notin A_1 \cup \{b_1\}$. Since $z_2b_1 \notin$ (G), $z_3b_1 \in E(G)$. Because $V(H_1) \neq \emptyset$ and $b_1 \in V(H_2)$, it follows that $z_3 \in \{u_2, u_3\}$. ut $u_2b_1 \notin E(G)$, so $z_3 = u_3$. By Lemma 1.2, $u_3u_1 \notin E(G)$. Since b_1 dominates D, $[b_1; u_1, u_3, w]$ is a claw centered at b_1, for any $w \in D$. This contradiction completes the roof of Subcase 1.2.2.1.

Subcase 1.2.2.2: So suppose that $[b_1, z_2] \longrightarrow a_3$. By Lemma 1.2, $z_2a_3 \notin E(G)$. Then, since $H_1 - \{a_1\}$ is complete, $z_2 \notin (V(H_1) - \{a_1\}) \cup \{u_2\}$. Since $V(H_1) - \{a_3\} \neq \emptyset$, $z_2 \in \{a_1, u_1, u_3\}$.

Suppose $z_2 = a_1$. That is, $[b_1, a_1] \longrightarrow a_3$. Then a_1 dominates $H_1 - \{a_3\}$ and b_1 dominates H_2. Recall that $u_2a_1 \notin E(G)$. Therefore, $b_1u_2 \in E(G)$. Since $H_1 - \{a_1\}$ is complete, $G[V(H_1)] = K_t - a_1a_3$, where $t = |V(H_1)|$. Since $|V(H_1)| \geq 3$, there is a vertex $y \in V(H_1) - \{a_1, a_3\}$. Clearly y dominates H_1. If $b_1u_3 \in E(G)$ or if $yu_3 \in E(G)$, then $\{b_1, y\}$ dominates G, a contradiction. Hence $b_1u_3 \notin E(G)$ and $yu_3 \notin E(G)$ for every choice of $y \in V(H_1) - \{a_1, a_3\}$. Since $[b_1, a_1] \longrightarrow a_3$ and $b_1u_3 \notin E(G)$, $a_1u_3 \in E(G)$. Because of Claim 1, $a_3u_3 \notin E(G)$. Recall that $[b_1, u_2] \longrightarrow a_1$. Since $b_1u_3 \notin E(G)$, $u_2u_3 \in E(G)$. But then $G[u_2; u_3, a_3, b_1]$ is a claw, a contradiction. Hence $z_2 \neq a_1$.

Next we suppose that $z_2 = u_1$. That is, $[b_1, u_1] \longrightarrow a_3$. Then b_1 dominates $(B_2 \cup B_3 \cup D) - B_1$. Since $b_1 \in B_1$ and $G[B_1]$ is complete, b_1 also dominates B_1. Thus now b_1 dominates H_2. Since $[b_1, u_1] \longrightarrow a_3$, it follows that u_1 dominates $H_1 - \{a_3\}$. By Claim 1, $H_1 - \{a_3\}$ is complete. Since $H_1 - \{a_1\}$ is complete, we have $G[V(H_1)] = K_t$ or $K_t - a_1a_3$ where $t = |V(H_1)|$. By a similar argument, there is a vertex y of $H_1 - \{a_1, a_3\}$ such that y dominates H_1. Furthermore, $y \in A_1 \cap A_2$ since $[b_1, u_1] \longrightarrow a_3$ and $[b_1, u_2] \longrightarrow a_1$. If $yu_3 \in E(G)$ or $b_1u_3 \in E(G)$, then $\{y, b_1\}$ dominates G, a contradiction. Hence $yu_3 \notin E(G)$ and $b_1u_3 \notin E(G)$. Since $[b_1, u_1] \longrightarrow a_3$ and $b_1u_3 \notin E(G)$, $u_1u_3 \in E(G)$. But then $G[u_1; u_3, y, b_1]$ is a claw centered at u_1, a contradiction. Hence $z_2 \neq u_1$.

Thus $z_2 = u_3$. That is, $[b_1, u_3] \longrightarrow a_3$.Then u_3 dominates $H_1 - \{a_3\}$. By an argument similar to that above, we have $G[V(H_1)] = K_t$ or $K_t - a_1a_3$, where $t = |V(H_1)|$ and there is a vertex y of $H_1 - \{a_1, a_3\}$ such that y dominates H_1. Since $[b_1, u_3] \longrightarrow a_3$ and $[b_1, u_2] \longrightarrow a_1$, $y \in A_2 \cap A_3$. Furthermore, b_1 dominates $V(H_2) - (B_2 \cap B_3)$. If b_1 dominates H_2, then $\{y, b_1\}$ dominates G, a contradiction. Hence there is a vertex $w \in B_2 \cap B_3$ such that $b_1w \notin E(G)$. Because of Claim 1, $u_1w \notin E(G)$, $b_1u_2 \notin E(G)$ and $b_1u_3 \notin E(G)$. Consequently, $u_1u_2 \notin E(G)$.

Now consider $G + u_1w$. There must be a vertex z_3 such that either $[u_1, z_3] \longrightarrow w$ or $[w, z_3] \longrightarrow u_1$. Suppose first that $[u_1, z_3] \longrightarrow w$. By Lemma 1.2, $z_3w \notin E(G)$. Then $z_3 \notin B_2 \cup B_3 \cup \{u_2, u_3\}$. Since $u_1u_2 \notin E(G)$, $z_3u_2 \in E(G)$. Thus $z_3 \in B_2 \cup \{u_2\}$, a contradiction. So it must be the case that $[w, z_3] \longrightarrow u_1$. Since $wb_1 \notin E(G)$ and $V(H_1) \neq \emptyset$, $z_3 \in \{u_2, u_3\}$. But this is also impossible since $b_1u_2 \notin E(G)$ and $b_1u_3 \notin E(G)$. This contradiction completes the proof in Subcase 1.2.2.2 and hence Case 1 is settled.

Case 2: So suppose now that $D = \emptyset$.

Choose $a_1 \in A_1$ and $b_1 \in B_1$ and consider $G + a_1b_1$. There must be a vertex $z_1 \in G - \{a_1, b_1\}$ such that either $[a_1, z_1] \longrightarrow b_1$ or $[b_1, z_1] \longrightarrow a_1$. Without loss of generality, suppose $[a_1, z_1] \longrightarrow b_1$. By Lemma 1.2, $z_1b_1 \notin E(G)$. So $z_1 \in (V(H_2) \cup \{u_2, u_3\}) - B_1$. Since B_1 is complete and $V(H_2) - \{b_1\} \neq \emptyset$

Subcase 2.1: Suppose $z_1 \in V(H_2) - B_1$. Then a_1 dominates $V(H_1)$. Since $D = \emptyset$, without loss of generality we may assume that z_1 is adjacent to u_2. Then z_1 dominates $H_2 - \{b_1\}$. By Claim 1, $u_2b_1 \notin E(G)$ and $z_1a_1 \notin E(G)$. (See Figure 3.4.)

Figure 3.4

Consider $G + a_1z_1$. There must be a vertex z_2 in $G - \{a_1, z_1\}$ such that either $[a_1, z_2] \longrightarrow z_1$ or $[z_1, z_2] \longrightarrow a_1$.

Subcase 2.1.1: Suppose $[z_1, z_2] \longrightarrow a_1$. By Lemma 1.2, $z_2a_1 \notin E(G)$. But then $z_2 \notin V(H_1) \cup \{u_1\}$. Furthermore, $z_2 \neq u_2$ since $z_1b_1 \notin E(G)$ and $u_2b_1 \notin E(G)$. Hence $z_2 = u_3$. Hence u_3 dominates $(V(H_1) - \{a_1\}) \cup \{u_1, b_1\}$. So H_1 must be complete since a_1 is adjacent to all vertices of $V(H_1) - \{a_1\}$ and because of Claim 1.

If u_1 dominates $V(H_1)$, then $\{u_1, z_1\}$ dominates G, a contradiction. So there exists a vertex $a_2 \in V(H_1) - \{a_1\}$ such that a_2 is not adjacent to u_1. Hence u_3 is not adjacent to z_1 by claw freedom. (See now Figure 3.5.)

Figure 3.5

Now choose $a_3 \in V(H_1) - \{a_1, a_2\}$ and consider $G + b_1a_3$. There must be a vertex z_3 in $G - \{b_1, a_3\}$ such that either $[b_1, z_3] \longrightarrow a_3$ or $[a_3, z_3] \longrightarrow b_1$.

Subcase 2.1.1.1: Suppose $[b_1, z_3] \longrightarrow a_3$. By Lemma 1.2, z_3 is not adjacent to a_3 and hence $z_3 \in \{u_1, u_2\}$. But $z_3 \neq u_1$, since $u_1a_2 \notin E(G)$ and $b_1a_2 \notin E(G)$, so $z_3 = u_2$. Then u_2 dominates $V(H_1) - \{a_3\}$. If $u_1u_2 \in E(G)$, then $G[u_2; u_1, a_2, z_1]$ is a claw centered at u_2, a contradiction. Hence $u_1u_2 \notin E(G)$.

Now consider $G + u_1u_2$. There must be a vertex z_4 in $G - \{u_1, u_2\}$ such that either $[u_1, z_4] \longrightarrow u_2$ or $[u_2, z_4] \longrightarrow u_1$. Suppose first that $[u_1, z_4] \longrightarrow u_2$. Since $u_1a_2 \notin E(G)$ and $u_1z_1 \notin E(G)$, it follows that $z_4 = u_3$. But this is impossible since $u_3z_1 \notin E(G)$. Thus $\{u_1, z_4\}$ does not dominate $G - \{u_2\}$. Hence $[u_2, z_4] \longrightarrow u_1$. By Lemma 1.2, $z_4u_1 \notin E(G)$. Since $u_1u_3 \in E(G)$, $z_4 \neq u_3$. Because $u_2a_3 \notin E(G)$ and $u_2b_1 \notin E(G)$, $z_4 = u_3$, a contradiction. This completes the proof in Subcase 2.1.1.1.

Subcase 2.1.1.2: So $[a_3, z_3] \longrightarrow b_1$. By Lemma 1.2, $z_3b_1 \notin E(G)$ and hence $z_3 \notin B_1 \cup \{u_1, u_3\}$. But then $z_3 = u_2$ or $z_3 \in V(H_2) - B_1$.

Subcase 2.1.1.2.1: Suppose $z_3 = u_2$. Then u_2 dominates $V(H_2) - \{b_1\}$. Hence $G[V(H_2) - \{b_1\}]$ is complete by Claim 1. If $u_2a_1 \in E(G)$, then $\{u_2, u_3\}$ dominates G, a contradiction. Hence u_2 is not adjacent to a_1. Choose $b_2 \in V(H_2) - \{b_1\}$ such that $b_2b_1 \in E(G)$. So $b_2 \neq z_1$. Consider $G + a_3b_2$. There is a vertex z_4 in $G - \{a_3, b_2\}$ such that either $[a_3, z_4] \longrightarrow b_2$ or $[b_2, z_4] \longrightarrow a_3$.

Subcase 2.1.1.2.1.1: Suppose $[a_3, z_4] \longrightarrow b_2$. So $z_4 \notin V(H_2) \cup \{u_2\}$ by Lemma 1.2. So $z_4 \in \{u_1, u_3\}$. But this is impossible because u_1 is not adjacent to z_1 and u_3 is not adjacent to z_1.

Subcase 2.1.1.2.1.2: So $[b_2, z_4] \longrightarrow a_3$. By Lemma 1.2, $z_4 \notin V(H_1)$, since H_1 is complete. So $z_4 \in \{u_1, u_2, u_3\}$. But $z_4 \neq u_1$ because u_1 is not adjacent to a_2. Moreover, $z_4 \neq u_2$ or u_3 either, because u_2 is not adjacent to a_1 and u_3 is not adjacent to a_1. So we have a contradiction.

Subcase 2.1.1.2.2: So suppose $z_3 \in V(H_2) - B_1$. Then u_1 is not adjacent to z_3 by Claim 1 and the fact that $z_3b_1 \notin E(G)$. Therefore, $a_3u_1 \in E(G)$, since $[a_3, z_3] \longrightarrow b_1$. Recall that u_3 dominates $(V(H_1) - \{a_1\}) \cup \{u_1, b_1\}$. Now a_3 dominates $V(H_1) \cup \{u_1, u_3\}$.

We claim that $B_1 \subseteq B_3$. Suppose not. Choose $y \in B_1 - B_3$. Then we have a claw $G[u_1; u_3, y, a_1]$, a contradiction. Similarly, we claim that $B_3 \subseteq B_1$. Suppose not. Choose $y \in B_3 - B_1$. Then $G[u_3; u_1, y, a_2]$ is a claw and again we have a contradiction.

Thus $B_1 = B_3$.

Next we claim that $B_1 \cap B_2 = \emptyset$. Suppose not. Choose $y \in B_1 \cap B_2$. Then y is adjacent to u_1, u_2 and u_3. Furthermore, y dominates $B_1 \cup B_2 = V(H_2)$. Thus $\{y, a_3\}$ dominates G, a contradiction. Hence $B_1 \cap B_2 = \emptyset$. Figure 3.6 illustrates the present situation.

Figure 3.6

Now consider $G + b_1u_2$. There must be a vertex z_5 in $G - \{b_1, u_2\}$ such that either $[b_1, z_5] \longrightarrow u_2$ or $[u_2, z_5] \longrightarrow b_1$. Suppose first that $[b_1, z_5] \longrightarrow u_2$. Then, by Lemma 1.2, $z_5u_2 \notin E(G)$ and so $z_5 \notin B_2$. So $z_5 \in V(H_1) \cup \{u_1, u_3\}$. But then it is false that $[b_1, z_5] \longrightarrow u_2$, since neither b_1 nor z_5 is adjacent to z_1.

So we may suppose that $[u_2, z_5] \longrightarrow b_1$. Then by Lemma 1.2, $z_5b_1 \notin E(G)$. Therefore, $z_5 \notin B_1 \cup \{u_1, u_3\}$. Since $B_1 \cap B_2 = \emptyset$, y is not adjacent to u_2, for all $y \in B_1$. If $|B_1| \geq 2$, then $z_5 \in B_2$. But then u_2 dominates $V(H_1)$ and hence $\{u_2, b_1\}$ dominates G, a contradiction. Hence $|B_1| = 1$. Since H_2 is connected, there is a vertex $b_2 \in B_2$ such that $b_1b_2 \in E(G)$. Because B_2 is complete, vertex b_2 dominates $V(H_2) \cup \{u_2\}$. Thus $\{a_3, b_2\}$ dominates G since a_3 dominates $V(H_1) \cup \{u_1, u_3\}$. This contradiction settles Subcase 2.1.1.2.2 and hence also Subcase 2.1.1.

Subcase 2.1.2: Suppose $[a_1.z_2] \longrightarrow z_1$. Then, by Lemma 1.2, $z_2z_1 \notin E(G)$ and so $z_2 \notin (V(H_2) - \{b_1\}) \cup \{u_2\}$. Hence $z_2 \in \{u_1, u_3, b_1\}$.

Subcase 2.1.2.1: Suppose $z_2 = u_1$. That is, $[a_1, u_1] \longrightarrow z_1$. Hence u_1 dominates $V(H_2) - \{z_1\}$. But then by Claim 1, $G[V(H_2) - \{z_1\}]$ is complete. Recall that z_1 dominates $V(H_2) - \{b_1\}$. Hence $G[V(H_2)] = K_t - b_1z_1$, where $t = |V(H_2)|$.

If $a_1u_2 \notin E(G)$, then $u_1u_2 \in E(G)$ since $[a_1, u_1] \longrightarrow z_1$. But then $G[u_1; a_1, u_2, b_1]$ is a claw centered at u_1, a contradiction. Hence $a_1u_2 \in E(G)$.

Now if $a_1u_3 \in E(G)$, then $\{a_1, y_1\}$ dominates G for any choice of $y_1 \in V(H_2) - \{b_1, z_1\}$, a contradiction. Thus $a_1u_3 \notin E(G)$. But then, because $[a_1, u_1] \longrightarrow z_1$, $u_1u_3 \in E(G)$. Moreover, because of claw freedom at u_1, u_3 is adjacent to every $y \in V(H_2) - \{z_1\}$. This implies that $\{a_1, y_1\}$ dominates G for every choice of $y_1 \in V(H_2) - \{b_1, z_1\}$, a contradiction. Hence $z_2 \neq u_1$.

Subcase 2.1.2.2: Suppose $z_2 = u_3$. That is, $[a_1, u_3] \longrightarrow z_1$. Then, by Lemma 1.2, $u_3z_1 \notin E(G)$ and u_3 dominates $V(H_2) - \{z_1\}$. So $G[V(H_2) - \{z_1\}]$ is complete because of Claim 1. Recall that z_1 dominates $V(H_2) - \{b_1\}$. Thus $G[V(H_2)] = K_t - b_1z_1$, where $t = |V(H_2)|$. If $a_1u_2 \in E(G)$, then $\{a_1, y_2\}$ dominates G for any choice of $y_2 \in V(H_2) - \{b_1, z_1\}$, a contradiction. So $a_1u_2 \notin E(G)$. But then $u_3u_2 \in E(G)$ since $[a_1, u_3] \longrightarrow z_1$. By claw freedom at u_3, a_1 is not adjacent to u_3. But this contradicts our assumption that $[a_1, z_1] \longrightarrow b_1$. Hence $z_2 \neq u_3$.

Subcase 2.1.2.3: Suppose $z_2 = b_1$. That is, $[a_1, b_1] \longrightarrow z_1$. Then b_1 dominates $V(H_2) - \{z_1\}$. Since $b_1u_2 \notin E(G)$, a_1 is adjacent to u_2. (See now Figure 3.7.)

Figure 3.7

Consider $G + u_1z_1$. There must be a vertex z_3 in $G - \{u_1, z_1\}$ such that either $[u_1, z_3] \longrightarrow z_1$ or $[z_1, z_3] \longrightarrow u_1$.

Subcase 2.1.2.3.1: Suppose $[u_1, z_3] \longrightarrow z_1$. Hence by Lemma 1.2, $z_3z_1 \notin E(G)$. Thus $z_3 \notin (V(H_2) - \{b_1\}) \cup \{u_2\}$. But then $z_3 \in \{b_1, u_3\} \cup V(H_1)$.

Subcase 2.1.2.3.1.1: Suppose $z_3 = b_1$; that is, $[u_1, b_1] \longrightarrow z_1$. Then u_1 dominates $V(H_1) \cup \{u_2\}$. By Claim 1, $G[V(H_1)]$ is complete. If $u_1u_3 \in E(G)$, then $\{u_1, z_1\}$ dominates G, a contradiction. Hence u_1 is not adjacent to u_3. Therefore, b_1 is adjacent to u_3. Now since there is no claw at u_1, u_2 dominates $V(H_1)$. Therefore $\{b_1, u_2\}$ dominates G, a contradiction.

Subcase 2.1.2.3.1.2: Next, suppose that $z_3 = u_3$. That is, $[u_1, u_3] \longrightarrow z_1$. By Lemma 1.2, u_3 is not adjacent to z_1. But since $[a_1, z_1] \longrightarrow b_1$ and z_1 is not adjacent to u_3, a_1 must be adjacent to u_3.

We now claim that $G[V(H_1)]$ is complete. Suppose not. Say $xy \notin E(G)$ for some choice of x and y in $V(H_1)$. Then $\{x, y\} \cap \{a_1\} = \emptyset$. Consider $G + xy$. There must be a vertex z_4 of $G - \{x, y\}$ such that either $[x, z_4] \longrightarrow y$ or $[y, z_4] \longrightarrow x$. Without loss of generality, suppose $[x, z_4] \longrightarrow y$. Clearly, $z_4 \in \{u_1, u_2, u_3\} \cup V(H_2)$. But $z_4 \neq u_1$ because u_1 is not adjacent to z_1, $z_4 \neq u_2$, since u_2 is not adjacent to b_1, and $z_4 \neq u_3$, since u_3 is not adjacent to z_1. Thus $z_4 \in V(H_2)$. But then z_4 dominates H_2 and hence $\{z_4, a_1\}$ dominates G, a contradiction. This completes the proof of the claim.

We now have the situation depicted in Figure 3.8.

Figure 3.8

Now since mindeg $G \geq 4$, $|V(H_2)| \geq 5$. Choose $x_1 \in V(H_2) - \{b_1, z_1\}$. Consider $G + a_1x_1$. There exists a vertex z_5 in $G - \{a_1, x_1\}$ such that either $[a_1, z_5] \longrightarrow x_1$ or $[x_1, z_5] \longrightarrow a_1$. Suppose $[x_1, z_5] \longrightarrow a_1$. Then, by Lemma 1.2, z_5 is not adjacent to a_1 and hence $z_5 \in V(H_2)$ which is impossible since $V(H_1) - \{a_1\} \neq \phi$. So we may assume that $[a_1, z_5] \longrightarrow x_1$. Therefore $z_5 \in \{u_1, u_2, u_3\} \cup (V(H_2) - \{x_1\})$. But $z_5 \neq u_1$, since u_1 is not adjacent to z_1, $z_5 \neq u_2$, since u_2 is not adjacent to b_1, $z_5 \neq u_3$, since u_3 is not adjacent to z_1, $z_5 \neq b_1$, since b_1 is not adjacent to z_1, and $z_5 \neq z_1$, since z_1 is not adjacent to b_1. So $z_5 \in V(H_2) - \{x_1, b_1, z_1\}$. Let $z_5 = y_1$. Then y_1 dominates $V(H_2) - \{x_1\}$.

Similarly, there is a vertex z_6 such that $[a_1, z_6] \longrightarrow y_1$ and by Lemma 1.2, z_6 is not adjacent to y_1. So $z_6 = x_1$ and x_1 dominates $V(H_2) - \{y_1\}$. Continuing in this manner, we can get a sequence of distinct vertices $x_1, y_1, x_2, y_2, \ldots$, such that $x_i y_i \notin E(G)$, for all i, but $x_i b_1 \in E(G), x_i z_1 \in E(G)$, for all i, $x_i x_j \in E(G)$, for all $j \neq i$, $x_i y_j \in E(G)$, for all $j \neq i$, $y_i x_j \in E(G)$, for all $j \neq i$, $y_i y_j \in E(G)$, for all $j \neq i$, $y_j b_1 \in E(G)$ and $y_j z_1 \in E(G)$, for all j, But $|V(H_2)|$ is odd and this contradiction settles Case 2.1.2.3.1.2.

Subcase 2.1.2.3.1.3: So $z_3 \in V(H_1)$. So since $[u_1, z_3] \longrightarrow z_1$, vertex u_1 dominates $V(H_2) - \{z_1\}$. Hence $G[V(H_2)] = K_t - b_1 z_1$, where $t = |V(H_2)|$, because of Claim 1 and because z_1 dominates $H_2 - \{b_1\}$. If there exists a vertex $y \in V(H_2) - \{b_1, z_1\}$ such that y is adjacent to u_3, then $\{a_1, y\}$ dominates G, a contradiction. So $B_3 \subseteq \{b_1, z_1\}$. But $B_3 \neq \{b_1, z_1\}$ because of Claim 1. So $B_3 = \{b_1\}$ or $B_3 = \{z_1\}$.

Choose $c_1 \in V(H_2) - \{b_1, z_1\}$ and consider $G + a_1 c_1$. There is a vertex $z_4 \in G - \{a_1, c_1\}$ such that either $[a_1, z_4] \longrightarrow c_1$ or $[c_1, z_4] \longrightarrow a_1$. Suppose first that $[a_1, z_4] \longrightarrow c_1$. By Lemma 1.2, z_4 is not adjacent to c_1. Hence $z_4 \notin V(H_2)$ and therefore $z_4 \in \{u_1, u_2, u_3\}$. But $z_4 \neq u_1$ since u_1 is not adjacent to z_1, $z_4 \neq u_2$ since u_2 is not adjacent to b_1 and $z_4 \neq u_3$ since either u_3 is not adjacent to z_1 or u_3 is not adjacent to b_1. So we have a contradiction.

Hence $[c_1, z_4] \longrightarrow a_1$. By Lemma 1.2, z_4 is not adjacent to a_1. So $z_4 \notin V(H_1) \cup \{u_1, u_2\}$. So $z_4 = u_3$ and so u_3 dominates $V(H_1) - \{a_1\}$. But then $G[V(H_1)]$ is complete because of Claim 1 and because a_1 is adjacent to x, for every $x \in V(H_1)$. Now recall that u_1 dominates $V(H_2) - \{z_1\}$. Thus u_1 is not adjacent to u_3 by claw freedom at u_1.

We now claim that for all $y \in V(H_2) - \{b_1, z_1\}$, u_2 is not adjacent to y. Suppose not. That is, suppose there is a $y_1 \in V(H_2) - \{b_1, z_1\}$ such that $y_1 u_2 \in E(G)$. Then $\{y_1, a_2\}$ dominates G for any $a_2 \in V(H_1) - \{a_1\}$, a contradiction. This proves the claim.

Since $u_2 b_1 \notin E(G)$, $B_2 = \{z_1\}$. Now if u_2 is adjacent to a_2 for some $a_2 \in V(H_1) - \{a_1\}$, then $\{a_2, c_2\}$ dominates G for any choice of $c_2 \in V(H_2) - \{b_1, z_1\}$, a contradiction. So u_2 is adjacent to no vertex in $V(H_1) - \{a_1\}$. Therefore, since mindeg $G \geq 4$, u_2 must be adjacent to both u_1 and u_3. Now if $B_3 = \{b_1\}$, we get a claw $G[u_2; u_1, u_3, z_1]$. Hence $B_3 = \{z_1\}$. But then $\{u_1, u_3\}$ dominates G, a contradiction. This completes the proof of Subcase 2.1.2.3.1.3, and hence also 2.1.2.3.1.

Subcase 2.1.2.3.2: So suppose $[z_1, z_3] \longrightarrow u_1$. Recall that $[a_1, z_1] \longrightarrow b_1$, $[a_1, b_1] \longrightarrow z_1$, $u_1 z_1 \notin E(G)$ and $u_2 b_1 \notin E(G)$. Since $z_1 \in V(H_2)$ and $z_1 b_1 \notin E(G)$, vertex z_3 must dominate $H_1 \cup \{b_1\}$. So $z_3 = u_3$. Therefore, H_1 is complete by Claim 1. Note that u_3 is adjacent to a_1 and b_1. Then u_3 is not adjacent to z_1 or we would have a claw at u_3. So the situation is similar to that depicted in Figure 3.8 and an argument analogous to the one given there results in a contradiction.

So Subcase 2.1.2.3.2 is settled and hence also Subcase 2.1.

Subcase 2.2: Suppose $[a_1, z_1] \longrightarrow b_1$ and $z_1 \in \{u_2, u_3\}$. Without loss of generality, we may suppose that $z_1 = u_2$. That is, $[a_1, u_2] \longrightarrow b_1$. By Lemma 1.2, u_2 is not adjacent to b_1 and u_2 dominates $H_2 - \{b_1\}$. Hence $G[V(H_2) - \{b_1\}]$ is complete by Claim 1. Now choose $b_2 \in V(H_2) - \{b_1\}$ such that $b_1 b_2 \in E(G)$. Consider $G + a_1 b_2$. There must be a vertex $z_2 \in G - \{a_1, b_2\}$ such that either $[a_1, z_2] \longrightarrow b_2$ or $[b_2, z_2] \longrightarrow a_1$.

Subcase 2.2.1: Suppose $[a_1, z_2] \longrightarrow b_2$. By Lemma 1.2, z_2 is not adjacent to b_2. Then $z_2 \notin V(H_2) \cup \{u_2\}$. So $z_2 \in \{u_1, u_3\}$.

Subcase 2.2.1.1: Suppose $z_2 = u_1$. That is, $[a_1, u_1] \longrightarrow b_2$. So a_1 dominates $V(H_1) - A_1$. Since $a_1 \in A_1$ and A_1 is complete, a_1 dominates $V(H_1)$. Also u_1 dominates $V(H_2) - \{b_2\}$, since $[a_1, u_1] \longrightarrow b_2$. So $G[V(H_2) - \{b_2\}]$ is complete by Claim 1. But since $G[V(H_2) - \{b_1\}]$ is also complete and since b_1 is adjacent to b_2, it must be the case that H_2 is complete. In fact, $B_1 \cap B_2 = V(H_2) - \{b_1, b_2\}$. Then a_1 is not adjacent to u_3, for otherwise $\{a_1, b_3\}$ dominates G, for any choice of $b_3 \in V(H_2) - \{b_1, b_2\}$, a contradiction. Now because $[a_1, u_1] \longrightarrow b_2$, vertex u_1 is adjacent to u_3.

We now claim that $B_3 = \{b_1\}$. Since $B_3 \neq \emptyset$ by Claim 1, so suppose there is a $y \in V(H_2) - \{b_1\}$ such that y is adjacent to u_3. Then $\{a_1, y\}$ dominates G, a contradiction, and the claim is proved.

Choose $b_3 \in V(H_2) - \{b_1, b_2\}$. Then b_3 is adjacent to u_1 and u_1 is adjacent to both u_3 and a_1, so we obtain a claw at u_1, a contradiction and hence Subcase 2.2.1.1 is settled.

Subcase 2.2.1.2: So $z_2 = u_3$. That is, $[a_1, u_3] \longrightarrow b_2$. Therefore, u_3 dominates $V(H_2) - \{b_2\}$; that is, $B_3 = V(H_2) - \{b_2\}$. By an argument similar to that used in Subcase 2.2.1.1, $G[V(H_2)]$ is complete. Now if a_1 dominates $V(H_1)$, then $\{a_1, b_3\}$ dominates G, for any choice of $b_3 \in V(H_2) - \{b_1, b_2\}$, a contradiction. Hence a_1 does not dominate $V(H_1)$. But $[a_1, u_2] \longrightarrow b_1$ and $[a_1, u_3] \longrightarrow b_2$, so a_1 dominates $V(H_1) - (A_2 \cap A_3)$. Therefore, there exists a vertex $a_2 \in A_2 \cap A_3$ such that a_2 is not adjacent to a_1. Now, by Claim 1, $a_1u_2 \notin E(G)$, $a_1u_3 \notin E(G)$ and $a_2u_1 \notin E(G)$. Thus $u_1u_2 \notin E(G)$ because of claw freedom at u_1.

Consider $G + a_2b_1$. There is a vertex z_3 in $G - \{a_2, b_1\}$ such that either $[a_2, z_3] \longrightarrow b_1$ or $[b_1, z_3] \longrightarrow a_2$. Suppose $[a_2, z_3] \longrightarrow b_1$. By Lemma 1.2, z_3 is not adjacent to b_1. Then $z_3 \notin V(H_2) \cup \{u_1, u_3\}$. But then $z_3 = u_2$. However, this is impossible since u_2 is not adjacent to a_1 and a_2 is not adjacent to a_1. Hence $[b_1, z_3] \longrightarrow a_2$. By Lemma 1.2, z_3 is not adjacent to a_2. But then $z_3 \notin \{u_2, u_3\} \cup A_2 \cup A_3$. But b_1 is not adjacent to u_2, so z_3 must dominate u_2. Because $V(H_1) - \{a_2\} \neq \emptyset$, $z_3 \in A_2 \cup \{u_2\}$, a contradiction. This settles Subcase 2.2.1.2 and hence Subcase 2.2.1.

Subcase 2.2.2: So $[b_2, z_2] \longrightarrow a_1$. By Lemma 1.2, $z_2a_1 \notin E(G)$. Then $z_2 \neq u_1$. Since $V(H_1) - \{a_1\} \neq \emptyset$, it follows that $z_2 \in \{u_2, u_3\} \cup (V(H_1) - \{a_1\})$.

Subcase 2.2.2.1: Suppose $z_2 = u_2$. That is, $[b_2, u_2] \longrightarrow a_1$. By Lemma 1.2, u_2 is not adjacent to a_1. So u_2 dominates $V(H_1) - \{a_1\}$. Therefore, $G[V(H_1) - \{a_1\}]$ is complete by Claim 1. If $u_1u_3 \in E(G)$ or if $u_2u_3 \in E(G)$, then $\{u_1, u_2\}$ dominates G, a contradiction. So u_3 is adjacent to neither u_1 nor u_2. But $[a_1, u_2] \longrightarrow b_1$ and u_2 is not adjacent to u_3, so a_1 is adjacent to u_3. Moreover, $[b_2, u_2] \longrightarrow a_1$ and u_2 is not adjacent to u_3, so b_2 is adjacent to u_3. So we have the configuration depicted in Figure 3.9.

Figure 3.9

Now since $V(H_1)$ is connected, there is a vertex $a_2 \in V(H_1) - \{a_1\}$ such that a_2 is djacent to a_1. If u_1 is adjacent to b_2, then $\{b_2, a_2\}$ dominates G, a contradiction. Hence $_1$ is not adjacent to b_2. Since $[b_2, u_2] \longrightarrow a_1$, and $b_2u_1 \notin E(G)$, u_2 must be adjacent to $_1$. But then $G[u_1; a_1, u_2, b_1]$ is a claw at u_1. This completes the proof in Subcase 2.2.2.1.

Subcase 2.2.2.2: Suppose $z_2 = u_3$. That is, $[b_2, u_3] \longrightarrow a_1$. By Lemma 1.2, $u_3a_1 \notin$ $E(G)$. So u_3 dominates $H_1 - \{a_1\}$. Because of Claim 1, $V(H_1) - \{a_1\}$ is complete. Recall that $[a_1, u_2] \longrightarrow b_1$ and $G[V(H_2) - \{b_1\}]$ is complete. Since $a_1u_3 \notin E(G)$, u_2 must be adjacent to u_3. Furthermore, vertex a_1 dominates $V(H_1) - A_2$. Also since H_1 is connected, there is a vertex $a_2 \in V(H_1) - \{a_1\}$ such that a_1 is adjacent to a_2.

First we claim that u_1 is not adjacent to y, for all $y \in V(H_2) - \{b_1\}$. Suppose not. Then there is a $y_1 \in V(H_2) - \{b_1\}$ such that y_1 is adjacent to u_1. Then y_1 is adjacent to b_1 also, by Claim 1. Hence y_1 dominates $V(H_2) \cup \{u_1, u_2\}$. Thus $\{y_1, a_2\}$ dominates G, a contradiction and the claim is proved. Hence $B_1 = \{b_1\}$.

Next we claim that $u_1x \notin E(G)$, for all $x \in V(H_1) - \{a_1\}$. For suppose not. Then there is a vertex $x_1 \in V(H_1) - \{a_1\}$ such that x_1 is adjacent to u_1. But then x_1 is adjacent to a_1 also, by Claim 1. Hence x_1 dominates $V(H_1) \cup \{u_1, u_3\}$. But then $\{x_1, b_2\}$ dominates G, a contradiction and this claim is proved also.

Hence $A_1 = \{a_1\}$. But then since mindeg $G \geq 4$, $N(u_1) = \{a_1, b_1, u_2, u_3\}$. Thus $a_1u_2 \in E(G)$, for otherwise $G[u_1; a_1, u_2, b_1]$ is a claw. So $a_1 \in A_2$. Since A_2 is complete and a_1 dominates $V(H_1) - A_2$, a_1 dominates $V(H_1)$.

So by claw freedom at u_2, $b_2u_3 \in E(G)$. So b_2 dominates $V(H_2) \cup \{u_2, u_3\}$. Hence $\{a_1, b_2\}$ dominates G, a contradiction, and Subcase 2.2.2.2 is settled.

Subcase 2.2.2.3: So suppose finally that $z_2 \in V(H_1) - \{a_1\}$.

Recall that $[a_1, u_2] \longrightarrow b_1$. So a_1 dominates $V(H_1) - A_2$ and u_2 dominates $V(H_2) - \{b_1\}$. Since $[b_2, z_2] \longrightarrow a_1$, z_2 is not adjacent to a_1 by Lemma 1.2 and then z_2 dominates $V(H_1) - \{a_1\}$. Because a_1 dominates $V(H_1) - A_2$ and by Claim 1, it follows that $z_2 \in A_2 - A_1$. Furthermore, a_1 is not adjacent to u_2. Now if $u_1u_2 \in E(G)$, then $G[u_1; a_1, u_2, b_1]$ is a claw. So $u_1u_2 \notin E(G)$. Moreover, since $[b_2, z_2] \longrightarrow a_1$ and $z_2u_1 \notin E(G)$, it follows that b_2 is adjacent to u_1. So our current situation is depicted in Figure 3.10.

Figure 3.10

Next we claim that u_2 is not adjacent to u_3. Suppose, by way of contradiction, that u_2 is adjacent to u_3. Consider $G + z_2b_2$. There must be a vertex z_3 in $G - \{z_2, b_2\}$ such that either $[z_2, z_3] \longrightarrow b_2$ or $[b_2, z_3] \longrightarrow z_2$. Suppose that $[z_2, z_3] \longrightarrow b_2$. By Lemma 1.2, $z_3b_2 \notin E(G)$. Then $z_3 \notin \{u_1, u_2\} \cup V(H_2)$. Since $V(H_2) - \{b_2\} \neq \emptyset$, it follows that $z_3 = u_3$. Thus $[z_2, u_3] \longrightarrow b_2$. But then u_3 dominates $H_2 - \{b_2\}$ and u_3 is adjacent to a_1. But then $G[u_3; u_2, a_1, b_1]$ is a claw.

So $[b_2, z_3] \longrightarrow z_2$. By Lemma 1.2, $z_3z_2 \notin E(G)$. But then $z_3 \notin \{u_2\} \cup (V(H_1) - \{a_1\})$. Since $V(H_1) - \{z_2\} \neq \emptyset$, $z_3 \in \{u_1, u_3, a_1\}$. If $z_3 = u_1$, then u_1 dominates $H_1 - \{z_2\}$, and then $\{u_1, u_2\}$ dominates G, a contradiction. So $z_3 \neq u_1$. If $z_3 = u_3$, then u_3 dominates $H_1 - \{z_2\}$. So $G[V(H_1) - \{z_2\}]$ is complete because of Claim 1. Since z_2 dominates $V(H_1) - \{a_1\}$, $G[V(H_1)] = K_t - a_1z_2$ where t $= |V(H_1)|$.

Now choose $a_2 \in V(H_1) - \{a_1, z_2\}$. Then $\{a_2, b_2\}$ dominates G, a contradiction. So $z_3 \neq u_3$. Hence $z_3 = a_1$; that is, $[b_2, a_1] \longrightarrow z_2$. So a_1 dominates $H_1 - \{z_2\}$.

Suppose a_1 is adjacent to u_3. Then u_3 is not adjacent to b_1 by claw freedom at u_3. Also by Claim 1, $z_2u_3 \notin E(G)$. Moreover, claw freedom at u_2 together with the fact that $u_2z_2 \in E(G)$ implies that u_3 must dominate $V(H_2) - \{b_1\}$.

Now choose $a_2 \in V(H_1) - \{a_1, z_2\}$ and consider $G + a_2b_2$. There must be a vertex $z_4 \in V(G) - \{a_2, b_2\}$ such that either $[a_2, z_4] \longrightarrow b_2$ or $[b_2, z_4] \longrightarrow a_2$. Suppose $[a_2, z_4] \longrightarrow b_2$. By Lemma 1.2, $z_4b_2 \notin E(G)$. So $z_4 \notin \{u_1, u_2, u_3\} \cup V(H_2)$. But $V(H_2) - \{b_2\} \neq \emptyset$, so $z_4 \in \{u_1, u_2, u_3\} \cup (V(H_2) - \{b_2\})$, a contradiction. So $[b_2, z_4] \longrightarrow a_2$. Since $V(H_1) - \{a_2\} \neq \emptyset$, $z_4 \in \{u_1, u_2, u_3\} \cup (V(H_1) - \{a_2\})$. But $z_4 \neq u_1$ since u_1 is not adjacent to z_2, $z_4 \neq u_2$ since u_2 is not adjacent to a_1, $z_4 \neq u_3$ since u_3 is not adjacent to z_2, $z_4 \neq a_1$ and $z_4 \neq z_2$, since a_1 and z_2 are not adjacent. So $z_4 \in V(H_1) - \{a_2, z_2, a_1\}$ and z_4 dominates $V(H_1) - \{a_2\}$. Let $z_4 = c_2$. Then c_2 dominates $V(H_1) - \{a_2\}$. Similarly, there is a vertex z_5 such that $[b_2, z_5] \longrightarrow c_2$. By Lemma 1.2, z_5 is not adjacent to c_2. So $z_5 = a_2$ and a_2 dominates $V(H_1) - \{c_2\}$.

Now choose $a_3 \in V(H_1) - \{a_1, z_2, a_2, c_2\}$ and repeating a previous argument, we produce a sequence of distinct vertices $a_1, z_2, a_2, c_2, a_3, c_3, \ldots$ such that a_1 is not adjacent to z_2, a_i is not adjacent to c_i, for all $i \geq 2$, but a_i is adjacent to a_j for all $i \neq j$, c_i is adjacent to c_j for all $i \neq j$, a_i is adjacent to c_j for all $i \neq j$ and z_2 is adjacent a_i and c_j for all $i \geq 2$ and for all $j \geq 1$. But this is a contradiction to the fact that $|V(H_1)|$ is odd. So a_1 is not adjacent to u_3. Since $[b_2, a_1] \longrightarrow z_2$, vertex b_2 is adjacent to u_3.

Now choose $a_2 \in V(H_1) - \{a_1, z_2\}$ and consider $G + a_2b_2$. There must be a vertex $z_6 \in V(G) - \{a_2, b_2\}$ such that either $[a_2, z_6] \longrightarrow b_2$ or $[b_2, z_6] \longrightarrow a_2$. Suppose $[a_2, z_6] \longrightarrow b_2$. Then by Lemma 1.2, $z_6b_2 \notin E(G)$. Then $z_6 \notin \{u_1, u_2, u_3\} \cup V(H_2)$. But this is impossible since z_6 must dominate $V(H_2) - \{b_2\}$.

So $[b_2, z_6] \longrightarrow a_2$. Since $V(H_1) - \{a_2\} \neq \emptyset$, $z_6 \in \{u_1, u_2, u_3\} \cup (V(H_1) - \{a_2\})$. But $z_6 \neq u_1$ since u_1 is not adjacent to z_2, $z_6 \neq u_2$ or u_3 since u_2 and u_3 are not adjacent to a_1, $z_6 \neq a_1$ or z_2 since a_1 and z_2 are not adjacent. But then $z_6 \in V(H_1) - \{a_1, a_2, z_2\}$. Let $z_6 = c_2$. Note that c_2 dominates $V(H_1) - \{a_2\}$. So again we argue as above to get a sequence of distinct vertices $a_1, z_2, a_2, c_2, a_3, c_3, \ldots$ contradicting the fact that $|V(H_1)|$ is odd.

So the claim is proved; that is, u_2 is not adjacent to u_3.

Since $[a_1, u_2] \longrightarrow b_1$ and $u_2u_3 \notin E(G)$, it follows that a_1 is adjacent to u_3. Because of Claim 1 and since a_1 is not adjacent to z_2, it follows also that z_2 is not adjacent to u_3. But $[z_2, b_2] \longrightarrow a_1$, so b_2 is adjacent to u_3. But then because of claw freedom at b_2, u_1 is djacent to u_3. (Figure 3.11 now represents the present situation.)

Figure 3.11

Now consider $G + b_2z_2$. There must be a vertex z_3 in $G - \{b_2, z_2\}$, such that either $[b_2, z_3] \longrightarrow z_2$ or $[z_2, z_3] \longrightarrow b_2$. Suppose $[z_2, z_3] \longrightarrow b_2$. By Lemma 1.2, z_3 is not adjacent to b_2. So $z_3 \notin \{u_1, u_2, u_3\} \cup V(H_2)$. But $V(H_2) - \{b_2\} \neq \emptyset$, so $z_3 \in (V(H_2) - \{b_2\}) \cup \{u_1, u_2, u_3\}$ and we have a contradiction.

Hence $[b_2, z_3] \longrightarrow z_2$. By Lemma 1.2, z_3 is not adjacent to z_2 and so $z_3 \notin (V(H_1) - \{a_1\}) \cup \{u_2\}$. Furthermore, $z_3 \notin V(H_2)$ because $V(H_1) - \{a_1\} \neq \emptyset$. Thus $z_3 \in \{u_1, u_3, a_1\}$.

Suppose first that $z_3 = u_1$. So $[b_2, u_1] \longrightarrow z_2$. Then u_1 dominates $V(H_1) - \{z_2\}$. But then $\{u_1, u_2\}$ dominates G, a contradiction. So $z_3 \neq u_1$.

Suppose next that $z_3 = u_3$. So $[b_2, u_3] \longrightarrow z_2$ and so u_3 dominates $V(H_1) - \{z_2\}$. So $G[V(H_1) - \{z_2\}]$ is complete because of Claim 1. But z_2 is adjacent to every vertex of $V(H_1) - \{a_1\}$, so $G[V(H_1)] = K_t - a_1z_2$, where $t = |V(H_1)|$. Now choose $x \in V(H_1) - \{a_1, z_2\}$. We then have $\{x, b_2\}$ dominates G, a contradiction. So $z_3 \neq u_3$.

So $z_3 = a_1$. That is, $[b_2, a_1] \longrightarrow z_2$. But then a_1 dominates $V(H_1) - \{z_2\}$. Now choose $a_2 \in V(H_1) - \{a_1, z_2\}$ and consider $G + a_2b_2$. There must be a vertex z_4 in $G - \{a_2, b_2\}$ such that either $[a_2, z_4] \longrightarrow b_2$ or $[b_2, z_4] \longrightarrow a_2$. Suppose $[a_2, z_4] \longrightarrow b_2$. Then $z_4 \notin V(H_2) \cup \{u_1, u_2, u_3\}$ by Lemma 1.2 and so $z_4 \in V(H_1)$. But $V(H_2) - \{b_2\} \neq \emptyset$, so we have a contradiction.

So $[b_2, z_4] \longrightarrow a_2$. But then $z_4 \neq u_1$ since u_1 is not adjacent to z_2, $z_4 \neq u_2$ since u_2 is not adjacent to a_1, $z_4 \neq u_3$ since u_3 is not adjacent to z_2, $z_4 \neq a_1$ since a_1 is not adjacent to z_1, and $z_4 \neq z_2$ since z_2 is not adjacent to a_1. Furthermore, $z_4 \notin V(H_2)$. So it follows that $z_4 \in V(H_1) - \{a_1, z_2, a_2\}$. Let $z_4 = c_2$ and if we argue as before, we obtain a sequence of distinct vertices $a_1, z_1, a_2, c_2, a_3, c_3, \ldots$, such that a_1 is not adjacent to z_2, a_i is not adjacent to c_i for all $i \geq 2$, $a_ia_j \in E(G)$, for all $i \neq j$, $c_ia_j \in E(G)$, for all $i \neq j$, $c_ic_j \in E(G)$, for all $i \neq j$ and z_2 is adjacent to a_i and c_j for all $i \geq 2$ and for all $j \geq 1$. That each time we obtain the vertex a_i, we can always get the next vertex c_i in the sequence. But once again this contradicts the fact that $|V(H_1)|$ is odd and the proof of the theorem is complete.

∎

It should be noted that both the connectivity bound and the minimum degree bound stated as hypotheses in the preceding theorem are sharp. Indeed, Favaron has proved [F: Theorems 2.5 and 2.6] that for all $k \geq 0$, every k-factor-critical graph of order $n > k$ is k-(vertex)-connected and for all $k \geq 1$, every k-factor-critical graph of order $n > k$ is $(k+1)$-edge-connected (and hence has minimum degree at least $k+1$).

References

[AP1] N. Ananchuen and M.D. Plummer, Some results related to the toughness of 3-domination-critical graphs, *Discrete Math.*, 2002, (to appear).

[AP2] N. Ananchuen and M.D. Plummer, Matching properties in domination critical graphs, (submitted).

[B] P. Blitch, Domination in Graphs, Ph.D. dissertation, Univ. of South Carolina, 1983.

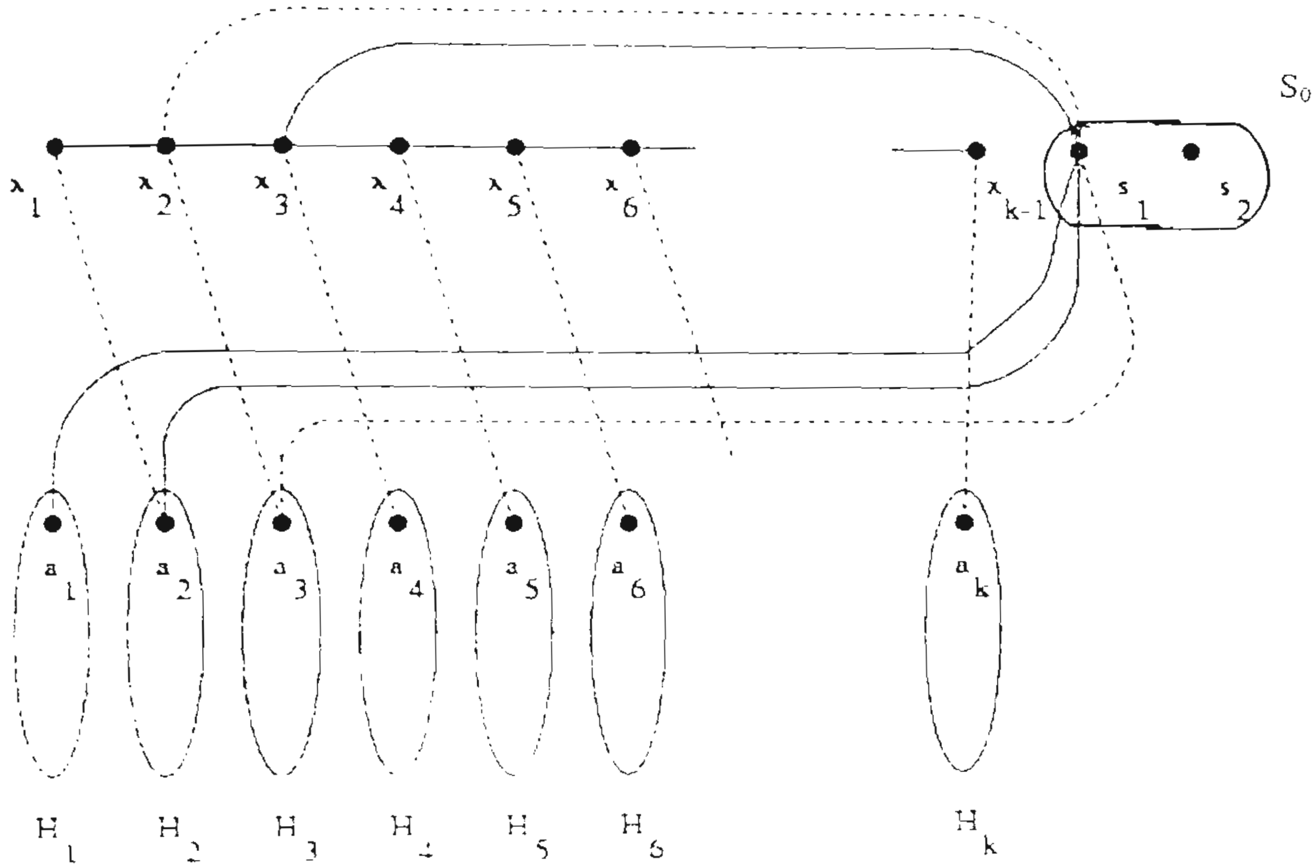
[CTW] Y. Chen, F. Tian and B. Wei, The 3-domination-critical graphs with toughness one, *Utilitas Math.*, 2002, (to appear).

[F] O. Favaron, On k-factor-critical graphs, *Discuss. Math.*, **16**, 1996, 41-51.

[LP] L. Lovász and M.D. Plummer, *Matching Theory*, Ann. Discrete Math. **29**, North-Holland, Amsterdam, 1986.

[P] M.D. Plummer, Extending matchings in claw-free graphs, *Discrete Math.*, **125**, 1994, 301-307.

[SB] D.P. Sumner and P. Blitch, Domination critical graphs, *J. Combin. Theory Series B*, **34**, 1983, 65-76.



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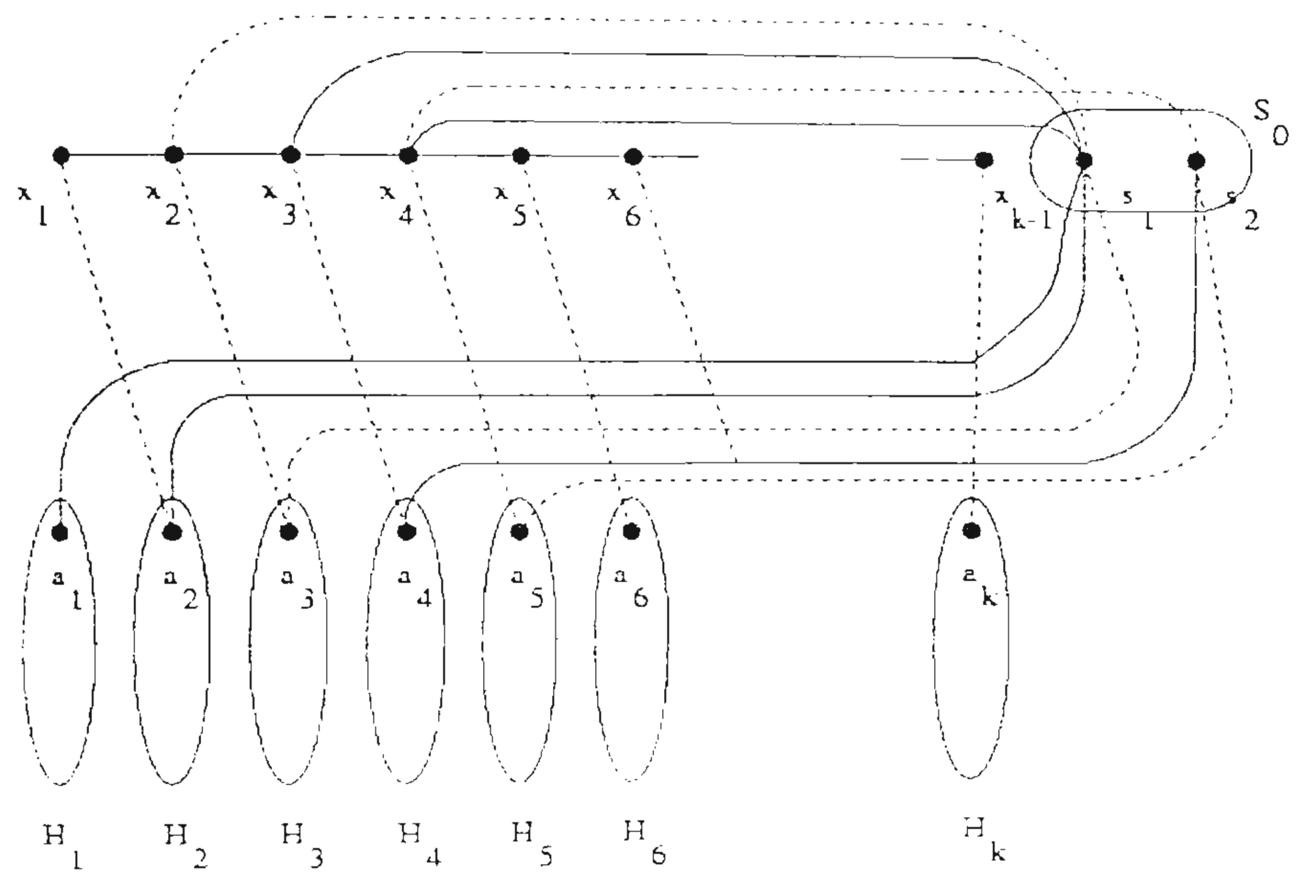

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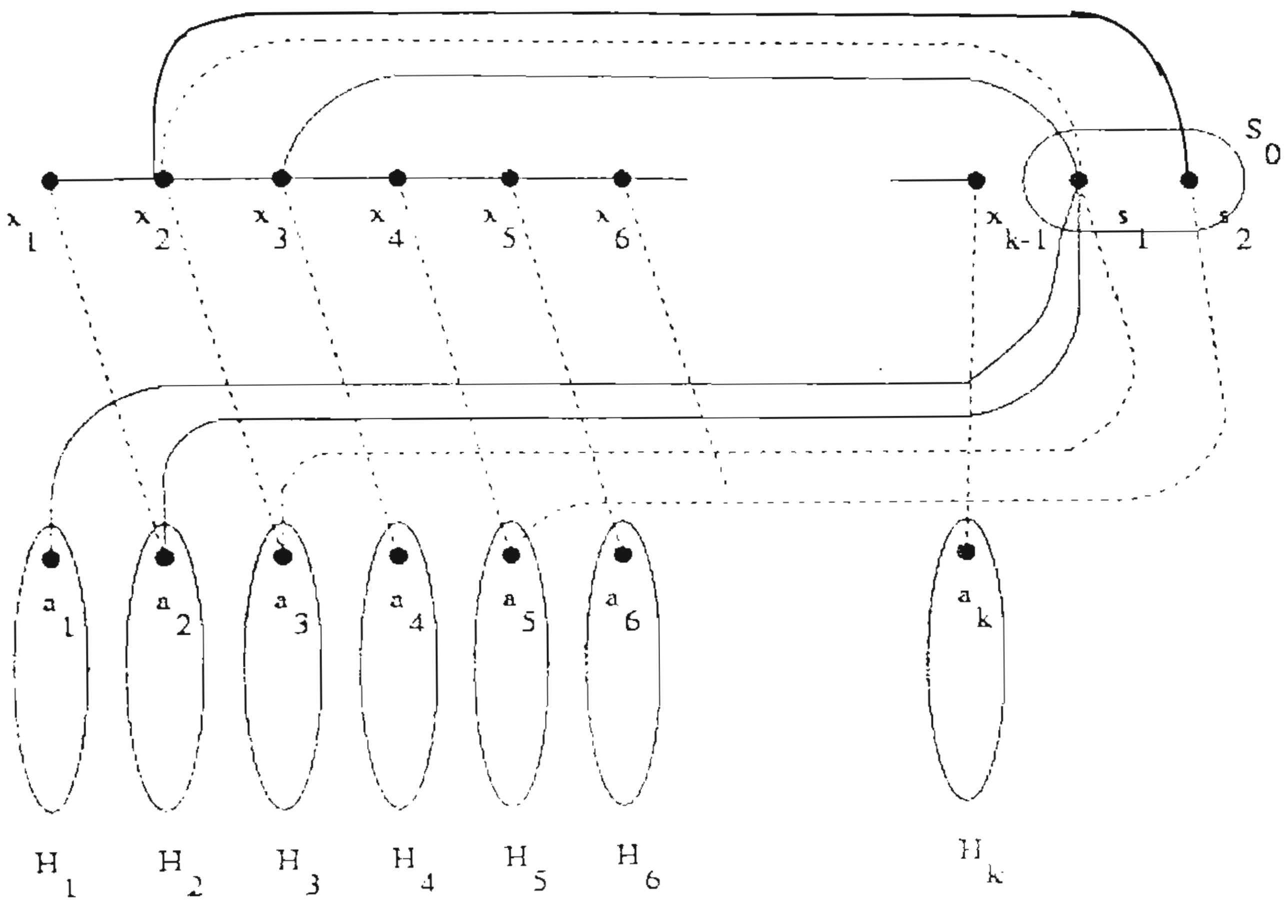

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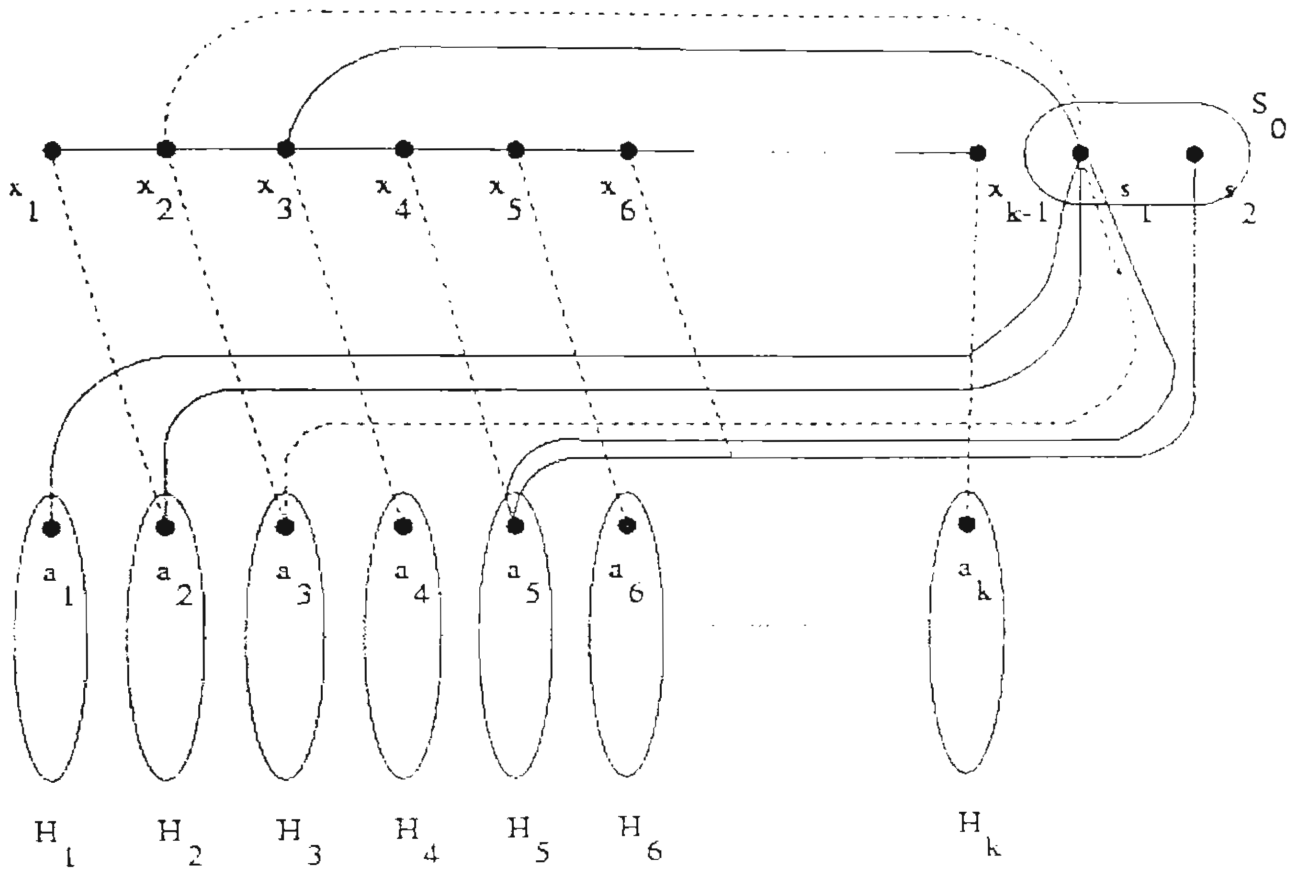

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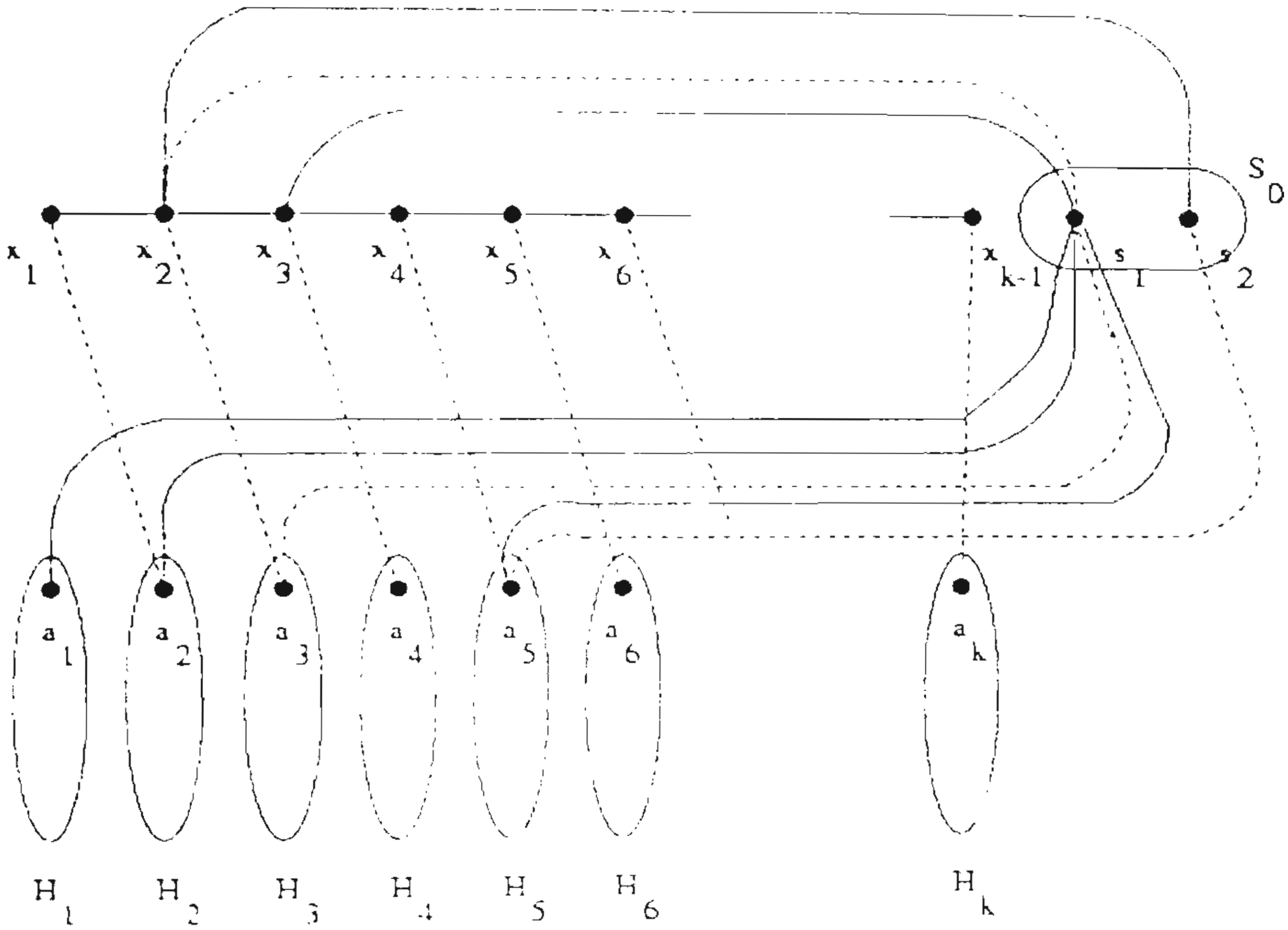

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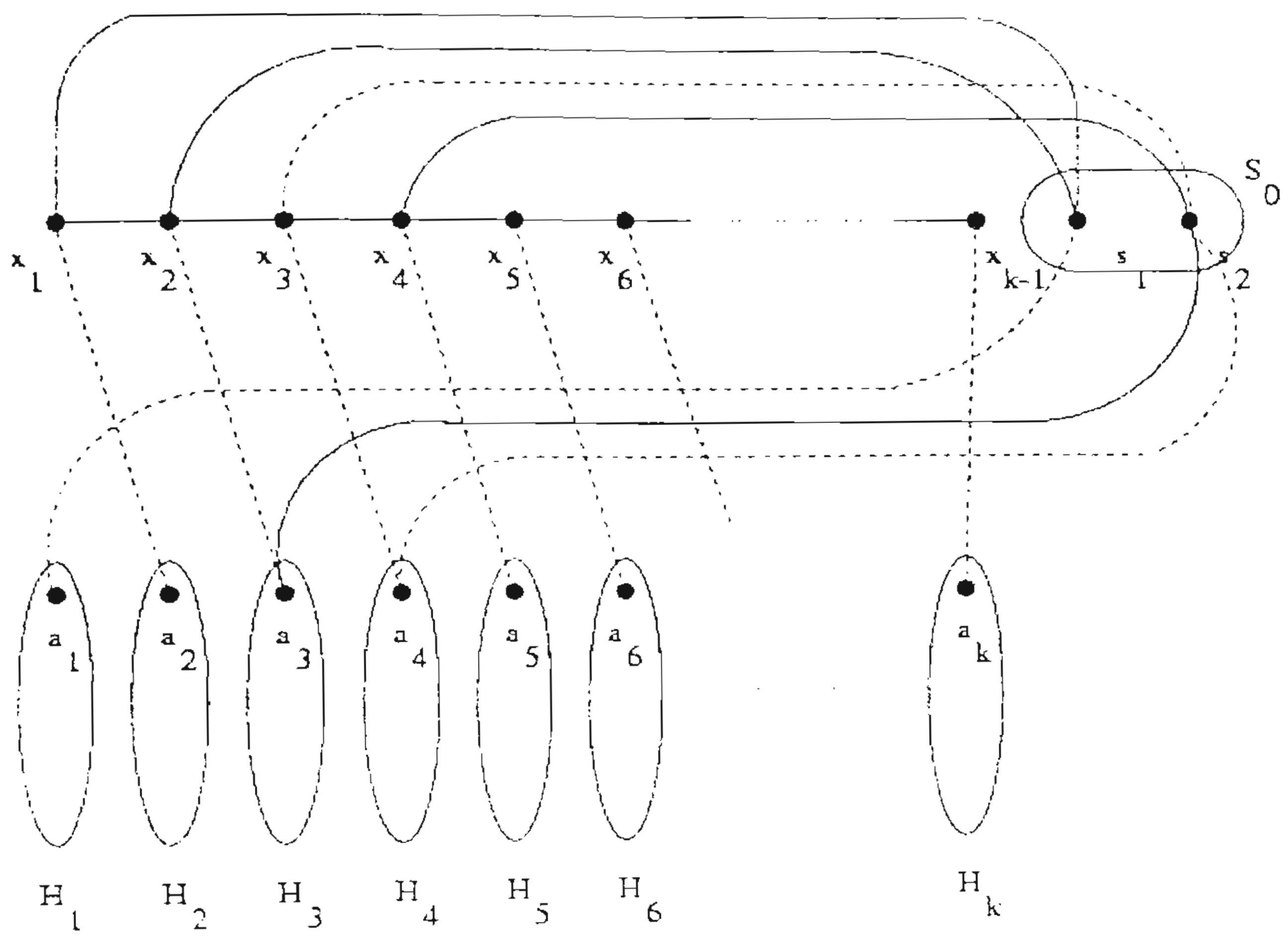

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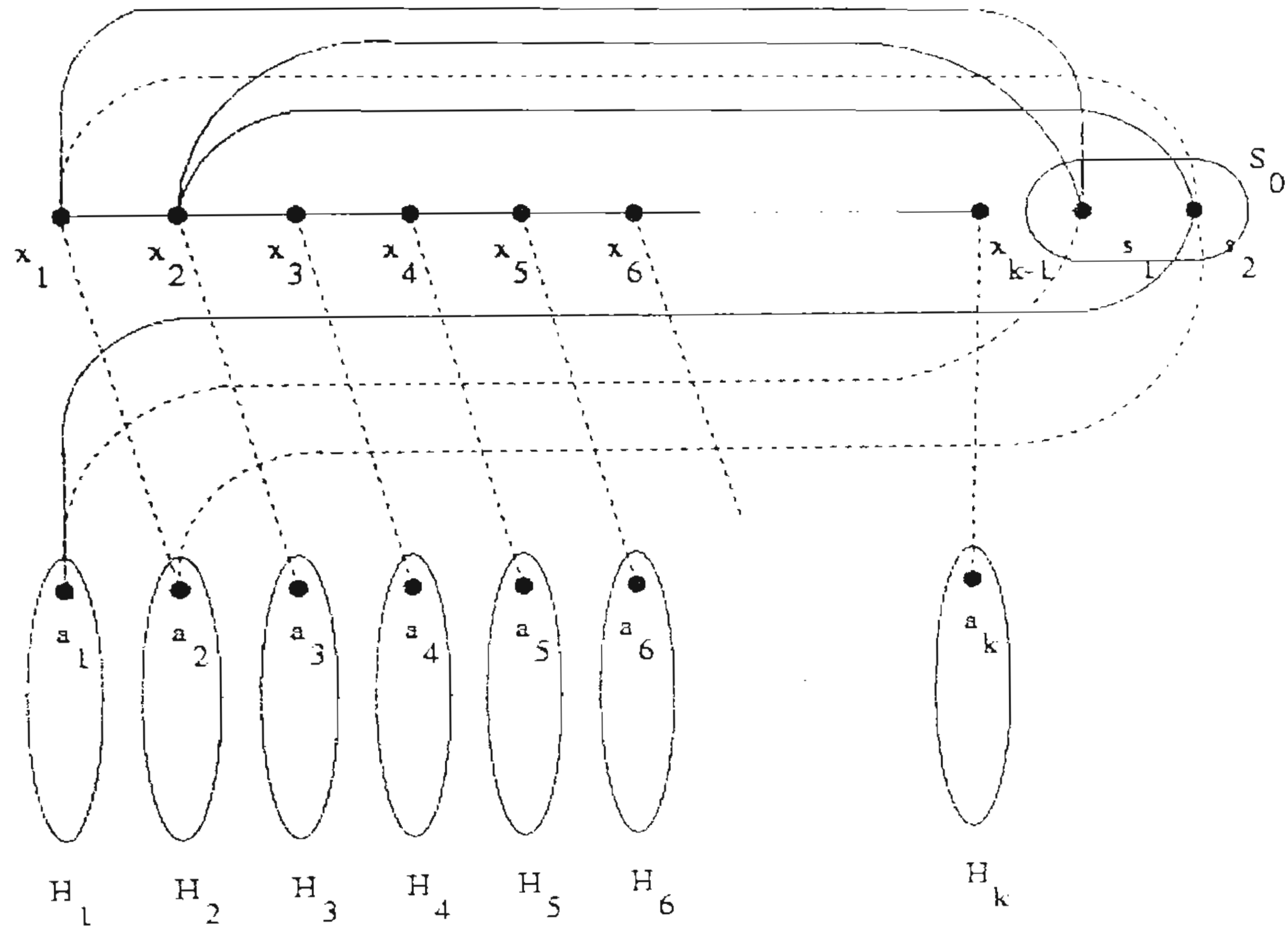

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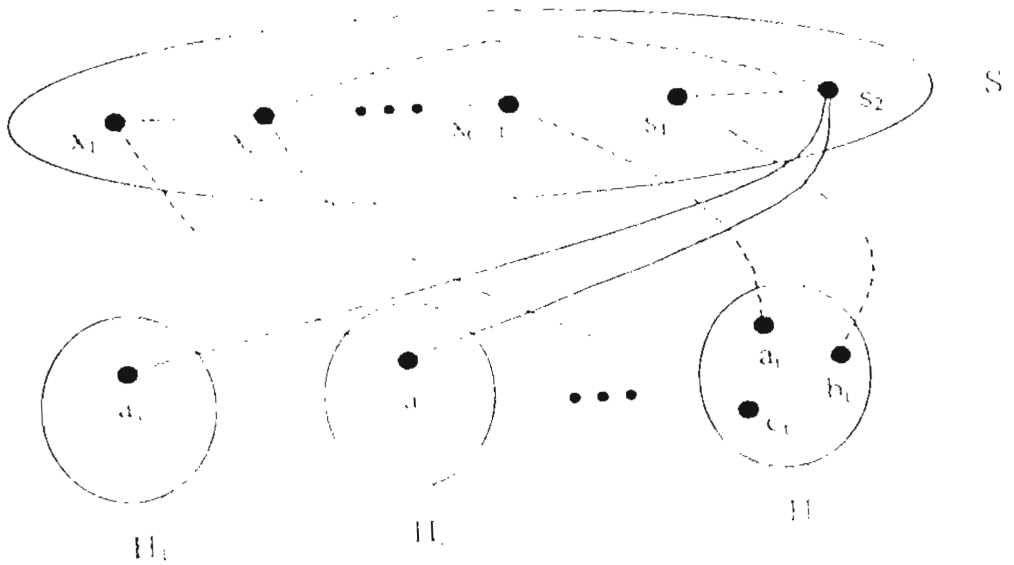

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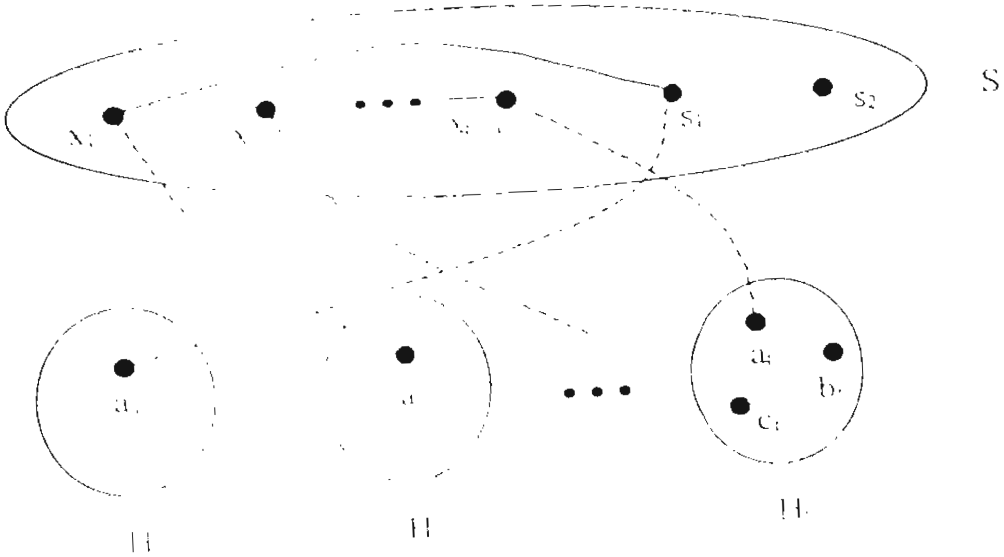

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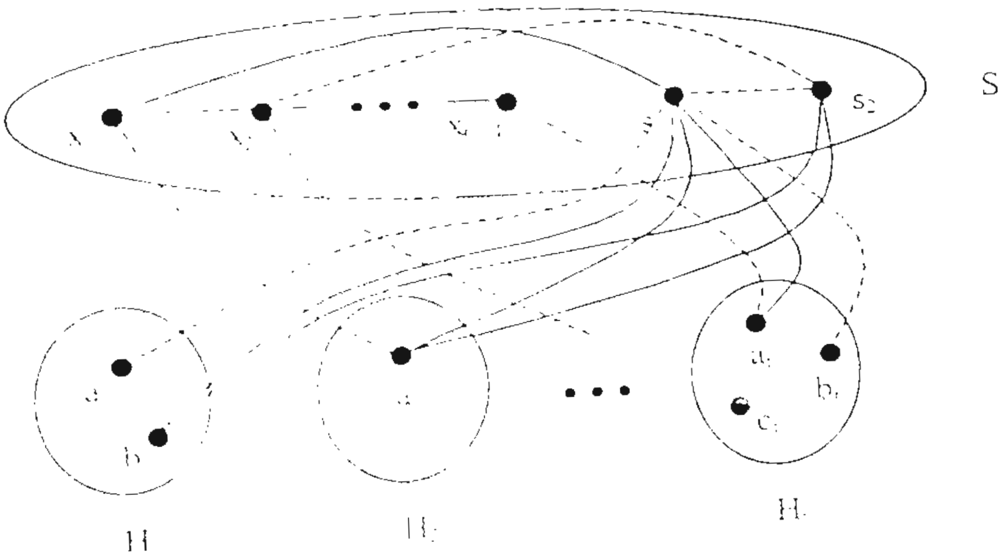
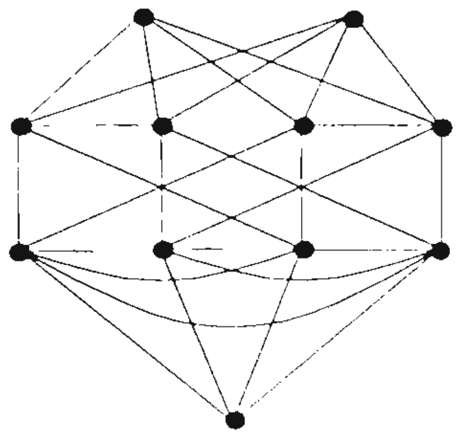


Figure 2.10

G.



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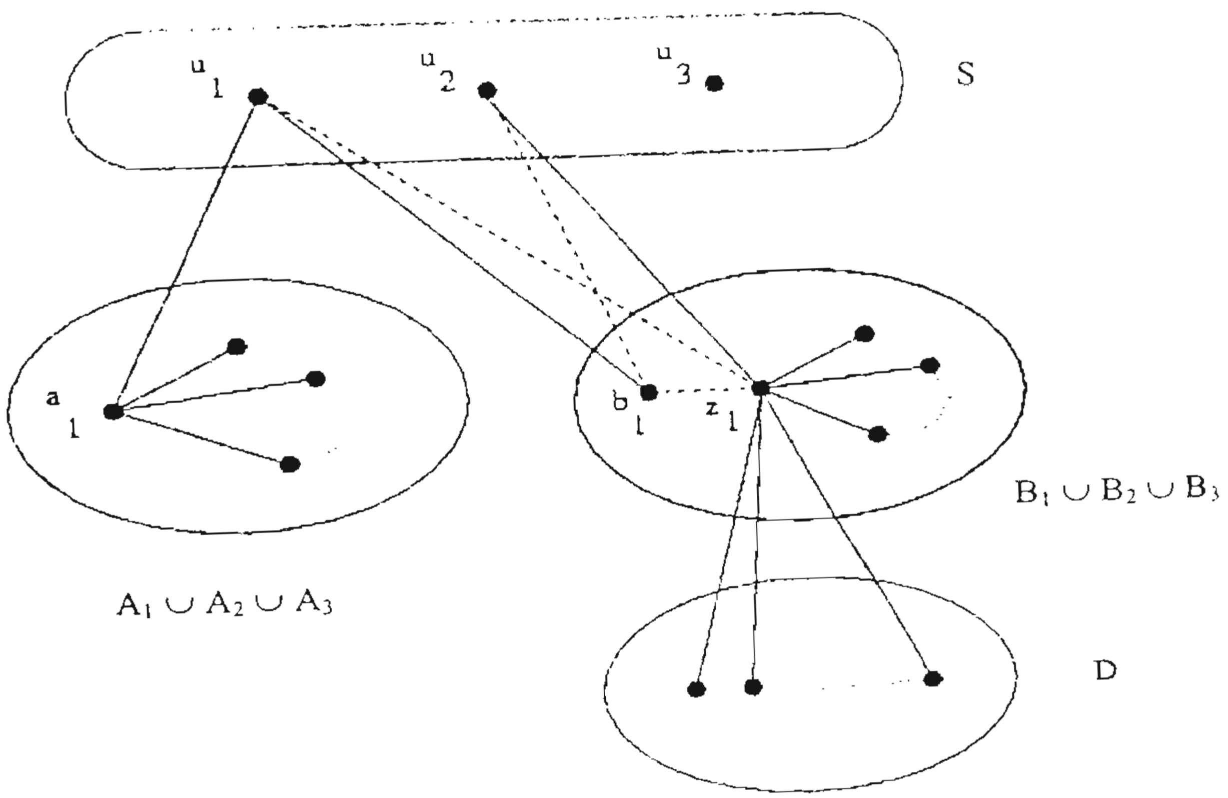

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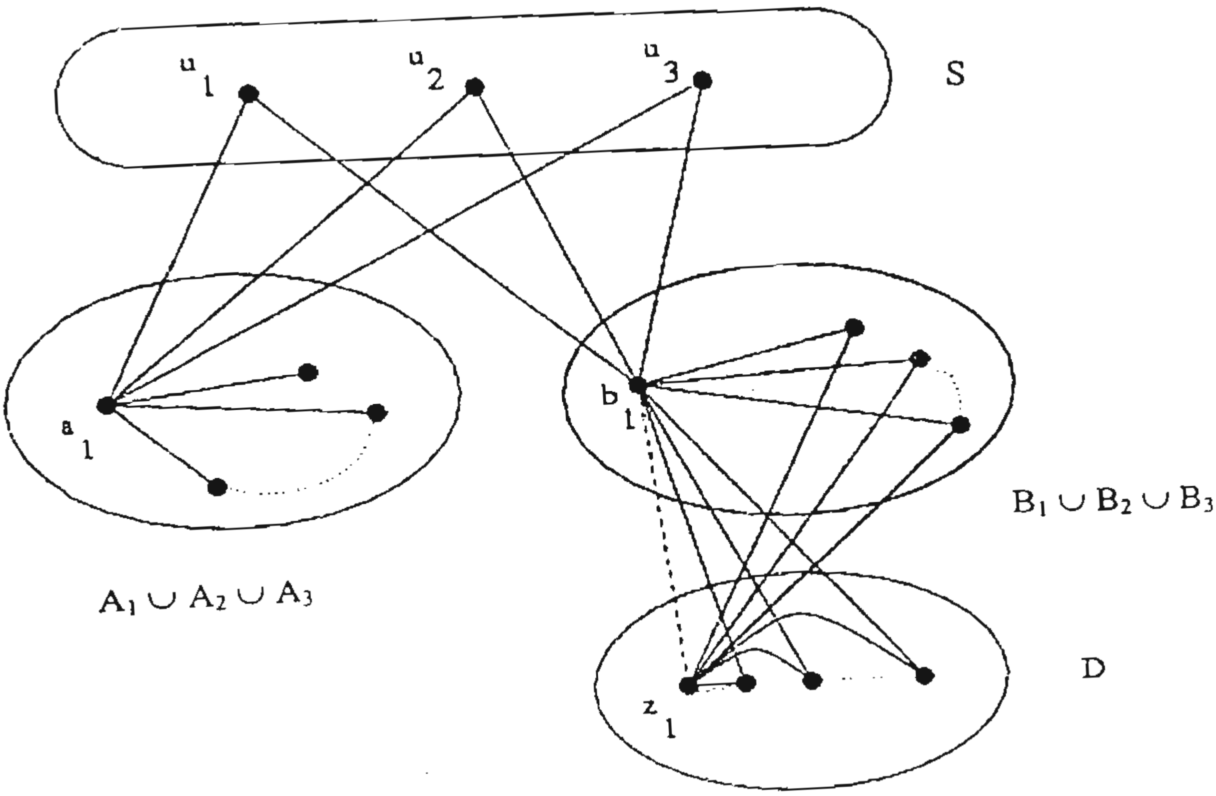

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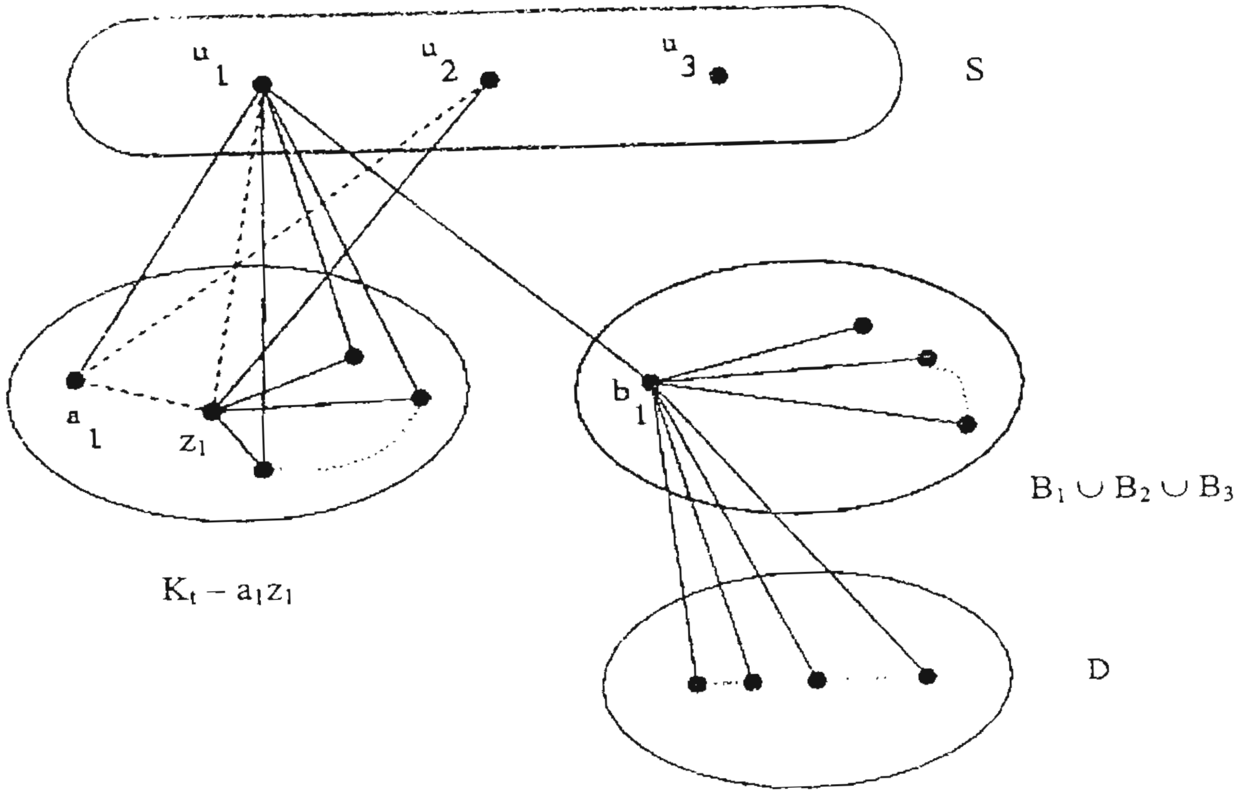

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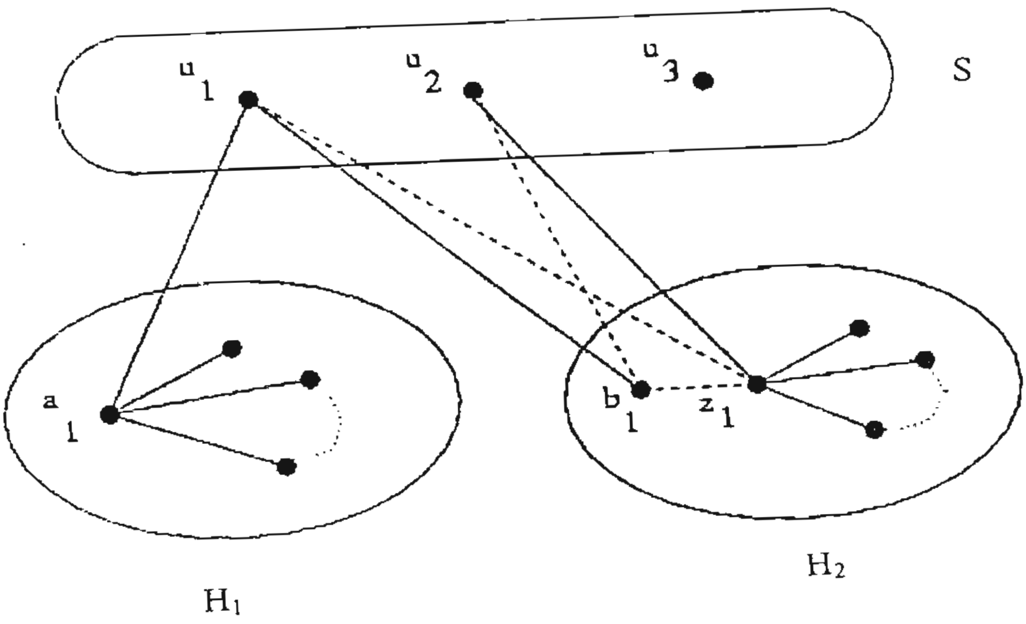

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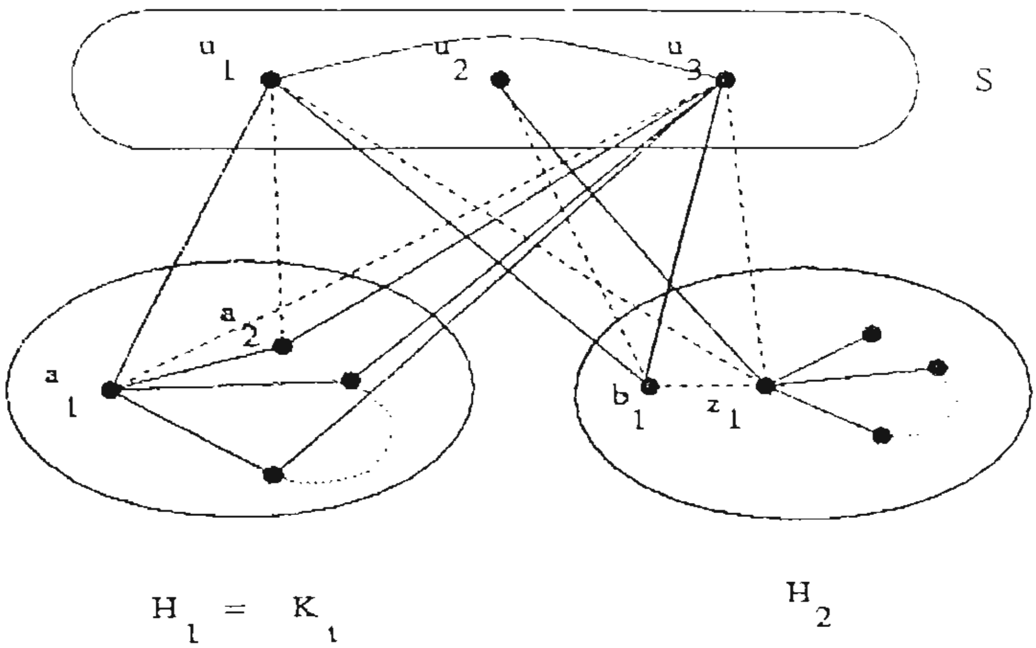

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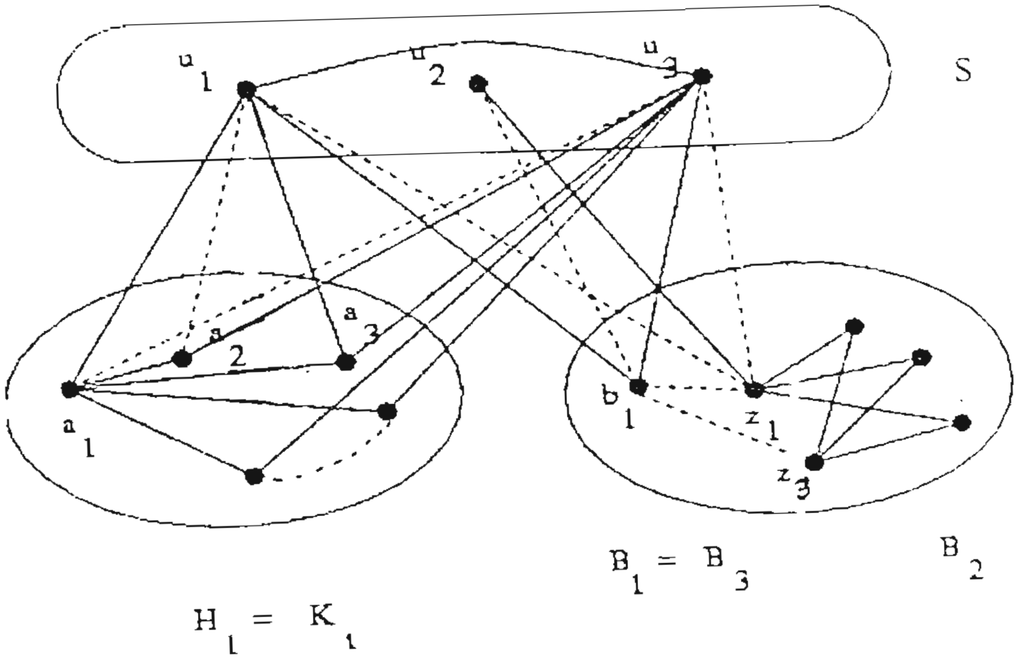

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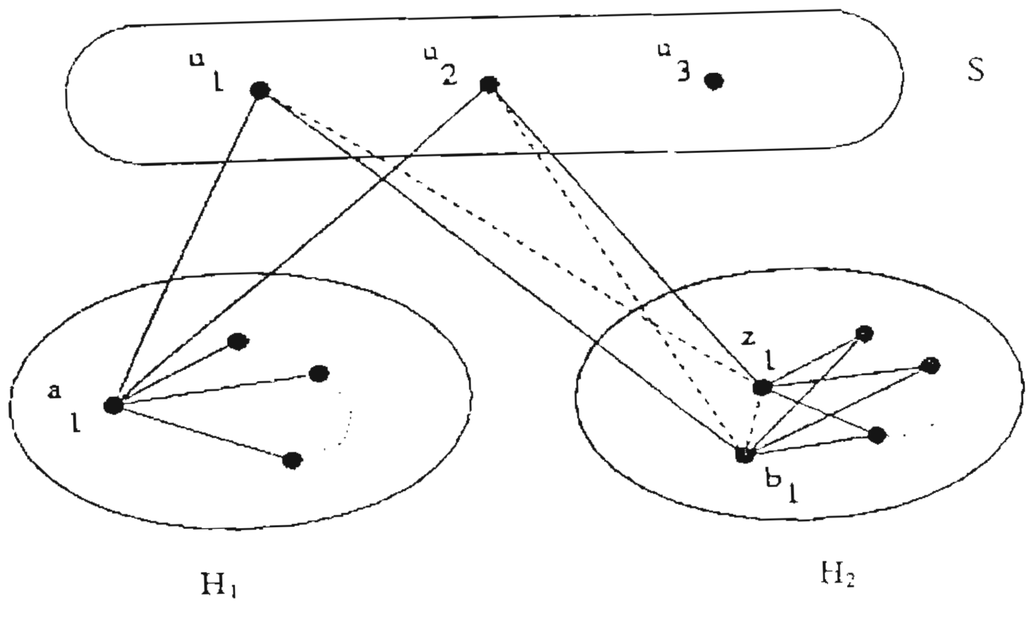

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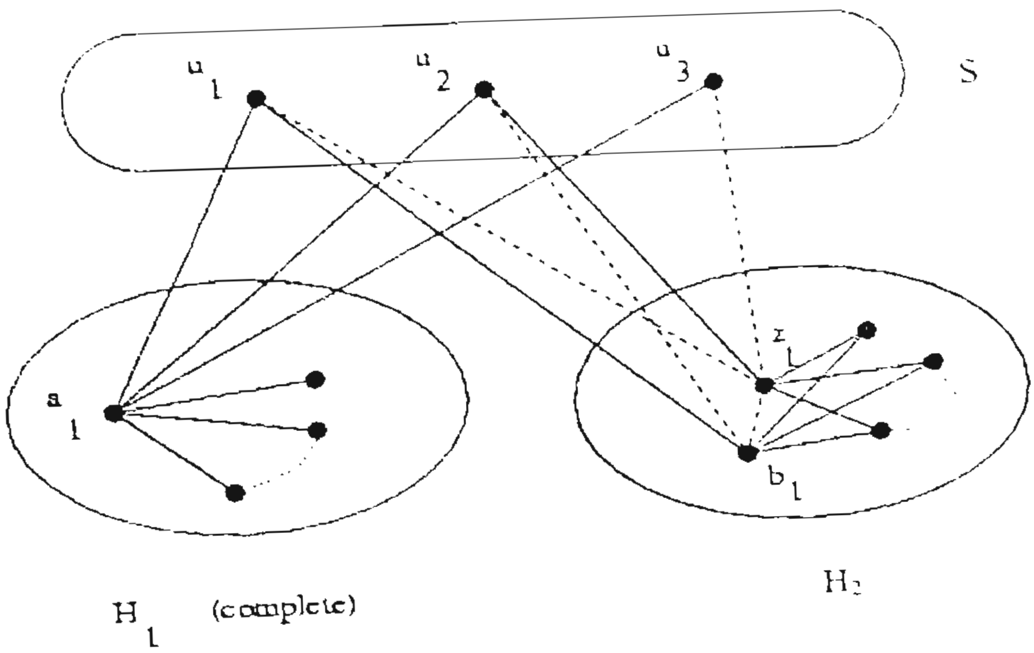

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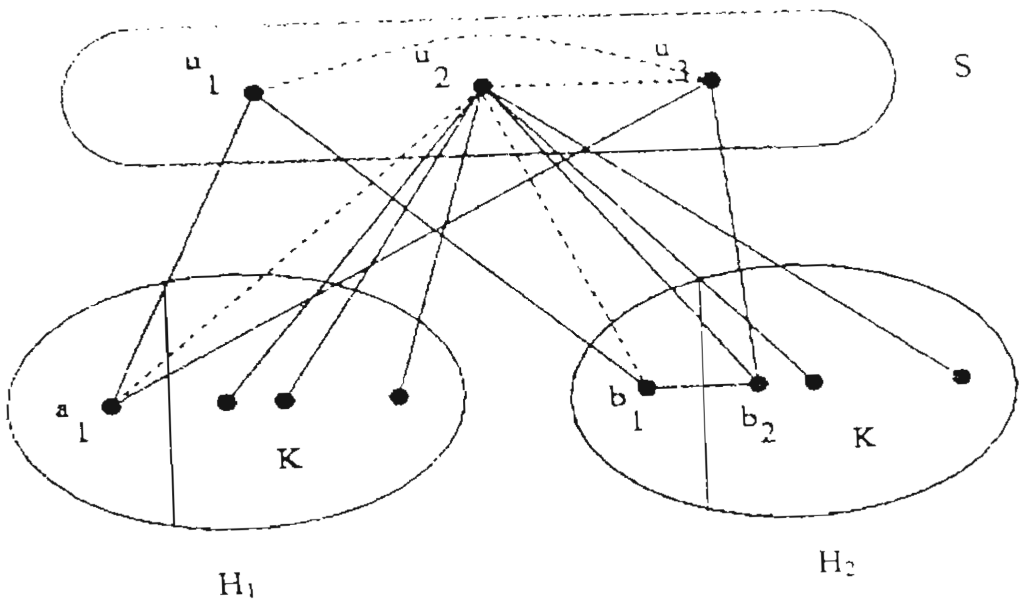

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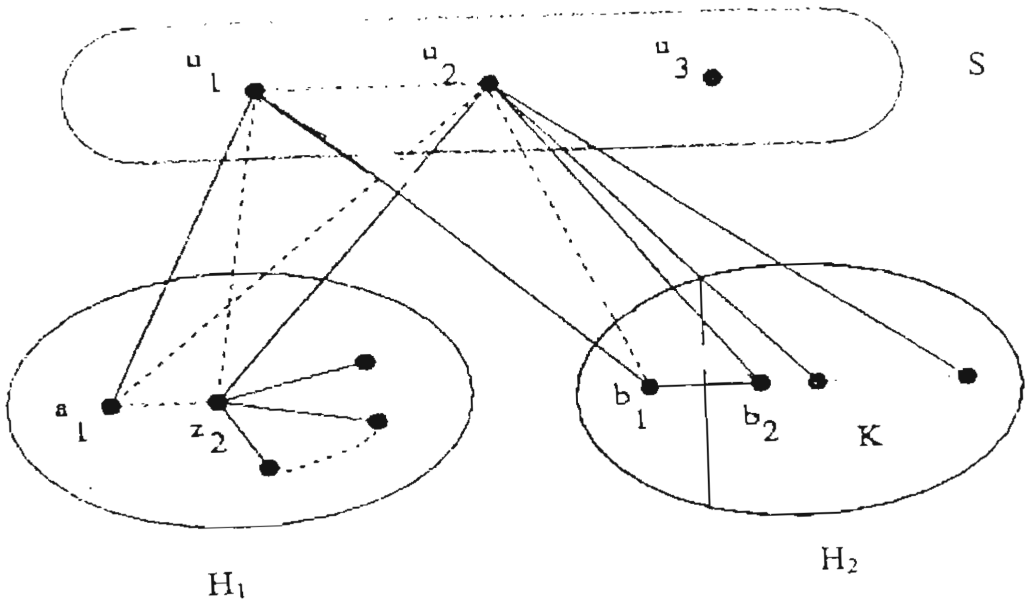

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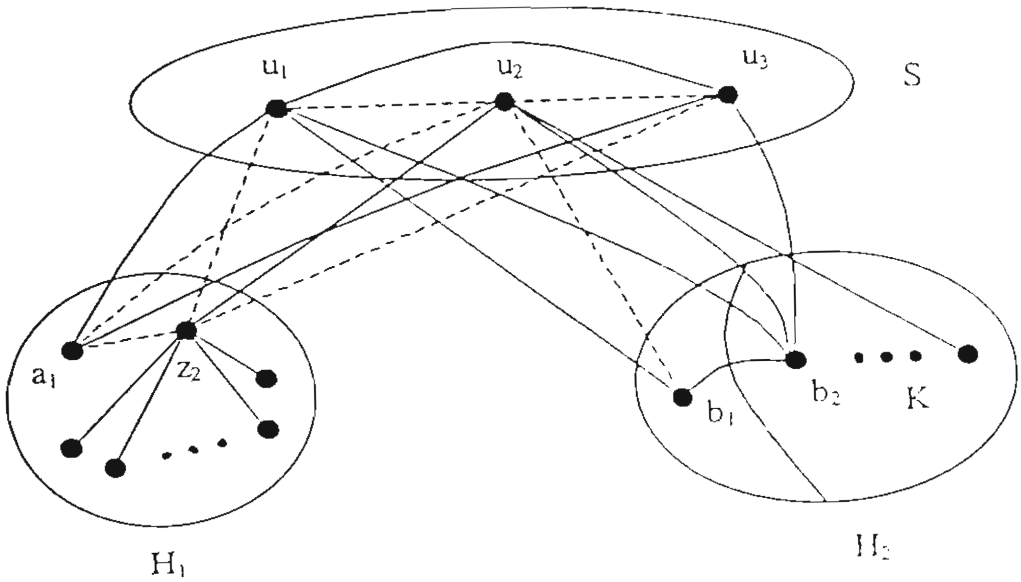

Figure 3.11

Research paper 4:

N.Ananchuen and M.D.Plummer, Two conjectures on matching in 3-domination-critical graphs, (submitted).

TWO CONJECTURES ON MATCHING

IN

3-DOMINATION-CRITICAL GRAPHS

by

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A set of vertices S dominates a graph G if every vertex of G either belongs to S or is adjacent to a vertex of S. The size of any smallest dominating set is called the *domination number* of the graph G and is denoted by $\gamma(G)$. Graph G is said to be *k-domination-critical* if $\gamma(G) = k$, but $\gamma(G+e) = k-1$, for any edge e not an edge of G. For $k \geq 3$, the structure of k-domination-critical graphs is far from understood.

This note sets forth two conjectures involving matching in 3-domination-critical graphs. Suppose G is a graph and $k \geq 1$. G is said to be *k-factor-critical* if for every set of vertices S with $|S| = k$, $G - S$ contains a perfect matching. (See [F].)

Conjecture 1: Suppose G is a graph with $k \geq 2$ and suppose $k-1$ and $|V(G)|$ have the same parity. Then if G is k-connected and 3-γ-critical with mindeg $G \geq k+1$, then G is $(k-1)$-factor-critical.

Conjecture 2: Suppose G is a graph with $k \geq 2$ and suppose k and $|V(G)|$ have the same parity. Then if G is k-connected and 3-γ-critical with mindeg $G \geq k+1$ and G is claw-free, then G is k-factor-critical.

Conjecture 1 is known to be true when $k = 2$ ([AP1; Theorem 2.4]), when $k = 3$ ([AP1; Theorem 2.1]) and when $k = 4$ ([AP2; Theorem 2.5]). Conjecture 2 is known to be true when $k = 2$ ([AP1; Theorem 3.3]) and when $k = 3$ ([AP2; Theorem 3.4]). However, the proofs of Conjecture 1 when $k = 4$ and Conjecture 2 when $k = 3$ are quite long and difficult. This leads us to think that settling either of these conjectures for any further values of k will very difficult, if not impossible, using the methods we employ for the small

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values of k mentioned above. In our opinion, some new methods must be discovered and utilized.

References

[F] O. Favaron, On k-factor-critical graphs, *Discuss. Math.*, **16**, 1996, 41-51.

[AP1] N. Ananchuen and M.D. Plummer, Matching properties in domination critical graphs, (submitted).

[AP2] N. Ananchuen and M.D. Plummer, 3-factor-criticality in domination critical graphs, (submitted).