Final Report

Higher-dimensional Soliton Dynamics

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April 2009

BRG/18/2543

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย

Project Code: BRG/18/2543

Project Title: Higher-dimensional Soliton Dynamics

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Project Period: 2000–2002

Abstract:

Higher-dimensional solitons are nonlinear waves localized in either 2 or 3 dimensions. This study examined the existence, formation, and interaction of higher-dimensional solitons in a number of systems applying to plasma physics. We mostly looked at modified Zakharov-Kuznetsov (ZK) equations which model weakly nonlinear ion-acoustic waves in strong magnetic fields, and in particular those modelling systems with non-isothermal electrons. We demonstrated that, as in the isothermal case, systems with non-isothermal electrons exhibit higher-dimensional soliton solutions with high symmetry and these evolve from perturbed plane (1-d) solitons. We also looked at modified Kadomtsev-Petviashvili (KP) equations. We showed that these have a similar behaviour to the original KP equations, with the modified version with positive dispersion also possessing lump solitons which form after perturbing a plane soliton. This demonstrated that the existence of lump solitons is not a consequence of integrability, a property the KP equations possess but the modified forms do not.

A number of the equations we examined have two nonlinear terms. We discovered new families of algebraic solitons to these in one, two, and three dimensions. In one dimension these appear to collide elastically with ordinary solitons but the collisions are inelastic in the higher-dimensional cases with the algebraic solitons decaying to ordinary ones. Higher-dimensional solitons can be formed as a result of instabilities of plane solitons. We developed techniques to obtain approximate analytical expressions for the growth rate of such instabilities when there are two nonlinear terms. As part of this, we also stumbled across a technique for performing certain types of improper integrals involving hyperbolic functions.

In addition to the interactions of algebraic solitons, we also studied the collisions of spherical solitons with each other and collisions of cylindrical solitons with plane solitons. Collisions between spherical solitons are inelastic. In the case of off-axis collisions, the energy loss was found to show little dependence on the distance between trajectories if the distance was small enough for identity exchange of the solitons to take place. Properties of the emerging solitons were accounted for by using conservation laws. Collisions of cylindrical solitons with plane solitons always resulted in the destruction of the plane soliton. For large amplitude plane solitons, additional cylindrical solitons are formed as a result of the collision.

Keywords: soliton, stability, nonlinear, Zakharov-Kuznetsov equation, non-isothermal electrons

รหัสโครงการ: BRG/18/2543

ชื่อโครงการ: โซลิตอนในมิติสูง

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บทคัดย่อ:

โซลิตอนในมิติสูงคือคลื่นไม่เชิงเส้นที่ดำรงอยู่ใน สอง และ สาม มิติ ในการศึกษานี้ได้ค้นหาการดำรงอยู่ รูปแบบ และ การมีปฏิสัมพันธ์ของโซลิตอนในมิติสูงในระบบต่าง ๆที่ประยุกต์กับพลาสมาฟิสิกส์ โดยเราพิจารณาสมการแบบปรับปรุงของซาคลารอฟ-คุซเนตซอฟเป็นส่วนใหญ่ซึ่งเป็นสมการที่บรรยายคลื่นไอออนแบบไม่เชิงเส้นอย่างอ่อนในสนามแม่เหล็กความเข้มสูง เราได้ทำการสาธิตว่าระบบที่มีอิเล็กตรอนอุณหภูมิต่างกันจะให้ผลเฉลยที่เป็นโซลิตอนในมิติสูงและมีความเป็นสมมาตรสูงเช่นกันโดยจะเกิดขึ้นเหมือนกันกับระบบที่อิเล็กตรอนมีอุณหภูมิใกล้เคียงกัน ผลดังกล่าวเกิดจากการวิวัฒน์ของระนาบโซลิตอน (หนึ่งมิติ) ที่ถูกรบกวน นอกจากนี้เรายังได้ศึกษาสมการคาโดมท์เซฟ-เพ็ทเวียร์ชวิลิแบบปรับปรุง ซึ่งก็แสดงพฤติกรรมแบบเดียวกับสมการแบบดั่งเดิม นอกจากนี้ สมการคาโดมท์เซฟ-เพ็ทเวียร์ชวิลิแบบปรับปรุงชนิดที่มีการกระจายเชิงบวกจะให้ลัมพ์โซลิตอนหลังจากโซลิตอนแบบระนาบถูกรบกวน ผลที่แสดงนี้แสดงว่าการดำรงอยู่ของล้มพ์โซลิตอนไม่ใช่สิ่งที่เกิดกับระบบที่ผลเฉลยได้เหมือนกับสมการแบบดั่งเดิม เพราะสมการแบบปรับปรุงนี้ไม่สามารถหาผลเฉลยได้

เราได้ศึกษาสมการที่ประกอบด้วยความไม่เชิงเส้นสองเทอม และได้ค้นพบโซลิตอนเชิงพีชคณิต ซึ่งมีอยู่ใน หนึ่งสอง และ สามมิติ สำหรับหนึ่งมิติเมื่อชนกับจะเป็นแบบยืดหยุนกับโซลิตอนทั่วไป แต่การชนกันจะเป็นแบบไม่ยืดหยุนในกรณีของโซลิตอนที่อยู่ในมิติที่สูงกว่าหนึ่ง โดยโซลิตอนเชิงพีชคณิตจะสลายตัวกลับไปเป็นโซลิตอนแบบดั้งเดิม โซลิตอนในมิติสูงสามารถเกิดขึ้นจากผลของความไม่เสถียรของระนาบโซลิตอน เรายังได้พัฒนาการประมาณของอัตราการโตเชิงวิเคราะห์ของความไม่เสถียรในกรณีสมการที่มีเทอมไม่เชิงเส้นสองเทอม ผลเหตุการณ์ดังกล่าวทำให้เราค้นการหาคำตอบของปริพันธ์ไม่ตรงแบบของฟังก์ชันไฮเปอร์โบลิค

นอกเหนือจากนี้การศึกษาปฏิสัมพันธ์ระหว่างโซลิตอนพีชคณิต ได้แสดงว่าการชนกันระหว่างโซลิตอนเชิงทรงกลมจะไม่เสถียร สำหรับกรณีของการชนแบบกระเจิง พลังงานที่สูญเสียกับจะขึ้นกับระยะทางของเส้นทางการเคลื่อนที่เพียงเล็กน้อยแต่ถ้าระยะทางสั้นเพียงพอก็จะเหมือนกับการแลกเปลี่ยนโซลิตอน นั่นคือคุณสมบัติของการเกิดโซลิตอนจะต้องให้กฎทรงพลังงานด้วย สำหรับการชนระหว่างโซลิตอนเชิงทรงกระบอกกับโซลิตอนเชิงระนาบจะเป็นการแสดงผลของการทำลายโซลิตอนเชิงระนาบ แต่ถ้าแอมปริจูดของโซลิตอนเชิงระนาบมีค่ามาก ๆ จะสามารถเกิดโซลิตอนเชิงทรงกระบอกได้หลังจากการชน

คำหลัก: โซลิตอน, ความเสถียร, ไม่เชิงเส้น, สมการ ซาคลารอฟ-คุซเนตซอฟ, อิเล็กตรอนอุณหภูมิต่างกัน

Introduction

Many of the ordered structures we see in the world around us, ranging from vortices in turbulent flow to life itself, are either a product of self-organization or are themselves self-organizing systems. By self-organization we mean the spontaneous appearance of a spatial pattern which has some degree of global cooperation. In other words, the parts which make up the spatial pattern or 'coherent structure' interact with each other to preserve the structure as a whole.

In spite of its ubiquity, self-organization is not well understood theoretically. This is a result of it necessarily being a nonlinear phenomenon. Although we inhabit a decidedly nonlinear universe, at least at the macroscopic level, most of physics has been concerned with the study of linear systems. There are two reasons for this. First, many of the simplest systems, particularly at the microscopic level, are governed by linear equations. The other reason is that linear systems are much more easy to understand and analyse than nonlinear systems. By definition, the behaviour of a linear system is simply the sum of the behaviours of its constituent parts. If we understand how the individual parts work, then we understand the whole. In contrast, for nonlinear systems, the whole is in a sense greater than the sum of the parts - we obtain unexpected or 'emergent' phenomena that cannot be understood by merely summing the actions of the parts. This is simply because the behaviour of one part affects the behaviour of one or more of the other parts of the system. In a linear system the parts act completely independently and it is for this reason that only a nonlinear system can show the self-organization defined above.

Due to the mutual interaction of the component parts, nonlinear systems are in general very difficult to analyse. Most can only be solved numerically and hence it is only with the advent of fast computers that many nonlinear systems could be studied in detail. The reward of studying these systems is the rich and unexpected behaviour they can yield. These emergent phenomena fall into two categories - chaotic and self-organizing. Chaotic behaviour is characterized by similar starting conditions leading to very different states after a relatively short time. This means that the long term behaviour of a chaotic system cannot be predicted. Previously people assumed that the world around us only appears unpredictable because it is complicated. It is now known that even the simplest nonlinear systems can show chaotic behaviour and that as a result of such chaotic processes, the universe is unpredictable in a fundamental sense.

In many aspects of our universe, the unpredictability decreed by chaos theory is far from apparent. This is because nonlinearity can also result in self-organization, and if present, chaos is generally found in the fine details of the coherent structures, or in their motion as a whole. One the simplest examples of self-organizing phenomena is the soliton. Solitons are long-lived localized distortions of a medium governed by a nonlinear partial differential equation and often take the form of solitary moving pulses. In dispersive media, ordinary linear pulses spread out, losing their shape. Nonlinearity, on the other hand, tends to steepen a waveform. When these two opposing effects balance, a soliton results.

Solitons have successfully been used to model a wide variety of phenomena ranging from fluxons in superconducting Josephson junctions (Lonngren & Scott, 1978) to tidal waves (Yeh *et al.*, 1994). As well as being of theoretical interest, solitons have been to shown to have very important practical applications. Optical solitons have been sent down fibres for over 4000 km without any of the electronic regeneration required by conventional pulses (Mollenauer & Smith, 1988). In addition, again because of the lack of problems with dispersion, very narrow soliton pulses can be used which means that data transmission rates a hundred times larger than is possible using ordinary pulses have been achieved.

A key attraction of solitons is their amenability to analysis. Many of the soliton-bearing equations in one spatial dimension (1-d) are integrable - that is, the time evolution of a large family of solutions can be obtained in closed form via an inverse scattering transform (Infeld & Rowlands, 2000). As a result, the interaction of these solitons has been studied in detail. However, with the notable exception of the Kadomtsev-Petviashvili (KP) equations in two dimensions (Kadomtsev & Petviashvili, 1970; Pelinovsky & Stepanyants, 1993), the higher-dimensional analogues of these integrable 1-d equations are nonintegrable, and so less analysis can be performed. Numerical studies have shown that 1-d solitons are in general unstable to higher-dimensional perturbations and will decay into higher-dimensional solitons or other coherent structures.

This decay process seems worthy of study because similar transitions between coherent structures take place within the realm of more complicated self-organizing systems that are also nonintegrable. For such systems, any study, whether numerical or analytical, would be far more involved. It is hoped that the results from a study of the simpler transitions between solitons and the interactions of the higher-dimensional solitons will provide a foundation on which to base a future investigation of more complex phenomena.

For convenience, we now list the equations that we studied. In all cases the dependent variable u is proportional to the electrostatic potential which is itself proportional to the deviation from the mean ion density. They are all given in reduced (dimensionless) variables. The Korteweg-de Vries (KdV) equation, of which all the equations we mention are generalizations, takes the form

$$u_t + uu_x + u_{xxx} = 0$$

in which the subscripts denote partial differentiation. The generalizations of it that we looked at are the modified KdV (mKdV) equation,

$$u_t + u^2 u_x + u_{xxx} = 0,$$

the extended KdV (eKdV) equation,

$$u_t + (u + bu^2)u_x + u_{xxx} = 0,$$

where b is a (real) parameter, the Schamel equation,

$$u_t + u^{1/2} u_x + u_{xxx} = 0,$$

and the Schamel-KdV (SKdV) equation,

$$u_t + (u + bu^{1/2})u_x + u_{xxx} = 0,$$

where b depends on the temperatures of the free and trapped electrons. The Zakharov-Kuznetsov (ZK) equation is

$$u_t + uu_x + \nabla^2 u_x = 0$$

and the generalizations of it that we looked at are the modified ZK (mZK) equation

$$u_t + u^2 u_x + \nabla^2 u_x = 0,$$

the extended ZK (eZK) equation,

$$u_t + (u + bu^2)u_x + \nabla^2 u_x = 0,$$

the Schamel-ZK (SZK) equation,

$$u_t + u^{1/2} u_x + \nabla^2 u_x = 0,$$

and the Schamel-KdV-ZK (SKdVZK) equation,

$$u_t + (u^{1/2} + bu)u_x + \nabla^2 u_x = 0.$$

Finally, we also mention the Kadomtsev-Petviashvili equations

$$(u_t + uu_x + u_{xxx})_x = \sigma u_{yy}$$

where $\sigma = \pm 1$. They are referred to as the KP^+ and KP^- depending on the sign of σ which in turn signifies the sign of the dispersion. We looked the generalization,

$$(u_t + u^{1/2} u_x + u_{xxx})_x = \sigma u_{yy}$$

which we referred to as the Schamel-KP (SKP) equation and also at

$$(u_t + (u + bu^{1/2})u_x + u_{xxx})_x = \sigma u_{yy}$$

which we called the (1,1/2)-mKP equation.

Rather than repeat what we have already written in the published works attached at the end of this report, the following sections are intended as a guide to that literature, to which the reader may refer for further details.

Ion-acoustic nonlinear waves in magnetized plasma with non-isothermal electrons

For an introduction to non-isothermal electrons in weakly nonlinear ion-acoustic waves, see the introductory section of Allen *et al.* (2007). The derivation of the SKdV equation is given in Phibanchon & Allen (2003). This paper also gives an expression for the first order growth rate of transverse instabilities of the plane solitons of the SKdVZK equation. This result is improved upon in Allen *et al.* (2007) where an approximate analytical expression is obtained for the growth rate over all wavenumbers for which the plane soliton is unstable. Also in this paper we give two hitherto undiscovered solutions to the SKdVZK equation. One of these is a family of algebraically decaying solitons (known as algebraic solitons) and will be dealt with in a later section.

In obtaining the growth rate of instabilities for the SKdVZK equation, it was necessary to evaluate some improper integrals involving hyperbolic functions. We found a simple new method for doing this – the results are presented in Allen (2007).

We showed that as in the case of the ZK equation, the SZK equation also has cylindrical soliton solutions which evolve from perturbed plane solitons (Phibanchon & Allen, 2002b). The number of cylindrical solitons produced depends on the perturbation wavelength (Phibanchon & Allen, 2002a).

Ion-acoustic nonlinear waves in magnetized plasma with mixtures of ions at or near critical density

Ion-acoustic nonlinear waves in magnetized plasma with mixtures of ions at or near critical density are governed by the mZK and eZK equations, respectively. The cylindrical solitons of the mZK equation are unstable – they grow in size without limit, eventually rendering the equation invalid – all these equations are only valid for moderate sized u. These solitons are produced by perturbing plane solitons. If a periodic train of plane solitons is perturbed, this blowing up can be prevented if the wavelength is short enough (Buppha & Allen, 2002; Buppha, 2004).

The techniques used for obtaining growth rate curves for equations with two nonlinear terms were applied to the eZK equation (Uppaman, 2007). Further work is being done to test whether variational methods can be used instead. These are impractical in the case of the SKdVZK equation due to difficulties with evaluation of the necessary integrals resulting from the square root term. Again, the method of Allen (2007) is necessary for evaluating the variational integrals in the case of the eZK equation. A paper dealing with this (in which this grant will be acknowledged) is in preparation.

Algebraic solitons

The following, although presented as a poster, has not yet been published so we give more details. It will form part of a forthcoming paper (in which this grant will be acknowledged).

We consider a modified ZK equation,

$$u_t + (u^p - bu^q)u_x + \nabla^2 u_x = 0, \quad p > q > 0, \tag{1}$$

in which the subscripts denote partial differentiation. We refer to (1) as the (p, q)-mZK equation, or, if ∇^2 is replaced by ∂_x^2, the (p, q)-mKdV equation. It reduces to the original ZK and KdV equations which have a single quadratic nonlinearity when $p = 1$ and $b = 0$. A number of instances of the equation have been derived and studied. The $(2, 1)$-mKdV equation, also known as the extended KdV (eKdV) equation, describes such systems as ion-acoustic waves for which the effect of three wave mode coupling has been included (Konno & Ichikawa, 1974), and wave propagation in a nonlinear lattice (Wadati, 1975). The higher-dimensional version, the extended

ZK (eZK) equation, applies to ion-acoustic waves in magnetized plasmas at critical density (Verheest *et al.*, 2002) and to the continuum limit of waves propagating in an array of nonlinear transmission lines (Duan, 2004). The $(1, \frac{1}{2})$-mKdV and $(1, \frac{1}{2})$-mZK equations have been obtained as descriptions of ion-acoustic modes in plasmas with a slightly non-isothermal electron distribution (Schamel, 1972; Shukla & Bharuthram, 1986).

The (p, q)-mZK equations have solitary pulse solutions that decay exponentially to zero for large x and travel faster than linear waves. As well as planar waves (which are also solutions to the (p, q)-mKdV equations and can be obtained in closed form when $b = 0$ or $p = 2q$), there are solitary pulse solutions with cylindrical and spherical symmetry (Zakharov & Kuznetsov, 1974; Shukla & Bharuthram, 1986). It has been shown that in addition to these conventional pulses, the eKdV and $(1, \frac{1}{2})$-mKdV equations also admit solitary pulses that decay to zero algebraically at large x and travel at the same speed as linear waves (Konno & Ichikawa, 1974; Allen *et al.*, 2007). Unlike conventional solitary pulses which are stable if $p < 4$ in the case of systems with one spatial dimension and $p < 2$ for cylindrical or spherical pulses (Laedke & Spatschek, 1984; Pelinovsky & Grimshaw, 1996), algebraic solitons of the eKdV equation are unstable with respect to multiplicative perturbations (Pelinovsky & Grimshaw, 1997).

Because the speed of a conventional solitary pulse depends on its amplitude, collisions are possible. For an integrable system, these collisions are elastic and can be described analytically via an inverse scattering transform. Of the equations we consider here, only the KdV, eKdV and modified KdV (mKdV) equations (for which $b = 0,\ p = 2$) are integrable. For the remaining non-integrable equations, numerical studies have shown that collisions are very nearly elastic in 1-d cases (Schamel, 1973), but are rather less so when cylindrical or spherical pulses collide (Iwasaki *et al.*, 1990; Infeld *et al.*, 2000).

We show that when $b > 0$, the (p, q)-mKdV equation admits bounded algebraic solitary wave solutions that decay to zero and have a velocity equal to that of linear waves. For the (p, q)-mZK we demonstrate for the first time that cylindrical and spherical algebraic solitary pulse solutions exist and, like their planar counterparts, can be expressed in closed form in some instances. We also investigate the collisions of conventional solitary pulses with algebraic solitary pulses.

Algebraic solitary wave solutions

To demonstrate the existence of planar, cylindrical and spherical solitary waves, we first transform (1) to a frame $x' \equiv x - Vt$ moving at speed V above the speed of linear waves and assume time independence. After dropping the primes, integrating once with respect to x, setting the constant of integration to zero (since the solitary wave derivatives vanish at infinity) one obtains

$$\frac{1}{r^{d-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^{d-1} \frac{\mathrm{d}u}{\mathrm{d}r} \right) = Vu + \frac{bu^{q+1}}{q+1} - \frac{u^{p+1}}{p+1}, \tag{2}$$

where $d = 1, 2, 3$ is the dimension of the solitary wave, and r is the radial coordinate which is the same as x when $d = 1$. Algebraic solitary waves that vanish at infinity will behave as $u \sim r^{-\mu}$, $\mu > 0$ for large r. It can be seen that this can only be compatible with (2) if $V = 0$ in which case one then obtains

$$\mu = 2/q. \tag{3}$$

When $d = 1$, we can obtain at least implicit expressions for the algebraic solitary waves. After reinstating $x \equiv r$, multiplying (2) by $2u_x$ and integrating once more we have

$$u_x^2 = \frac{2bu^{q+2}}{(q+1)(q+2)} - \frac{2u^{p+2}}{(p+1)(p+2)}. \tag{4}$$

From considering the phase plane corresponding to (4) it is clear that solitary wave solutions will only occur if bu^{q+2} and u^{p+2} are both real and positive. The solutions can be found explicitly when $p = 2q$ in which case one obtains

$$u = \left(\frac{B}{1 + a^2x^2}\right)^{1/q} \tag{5}$$

where

$$B = \frac{b(p+1)(p+2)}{(q+1)(q+2)}, \quad a = \frac{qB}{\sqrt{2(p+1)(p+2)}}.$$

Note that the form of (5) for large $|x|$ agrees with (3). This solution can also be obtained by considering the appropriate limit of the conventional soliton solution of the $(2q, q)$-mKdV equation.

For $d = 2, 3$, (2) cannot be integrated further. Instead, based on our knowledge of the asymptotic dependence on r of the algebraic solitary wave solution we find by trial substitution that when $p = 2q$

$$u(r) = \left(\frac{\beta}{1 + \alpha^2r^2}\right)^{1/q} \tag{6}$$

is a solution. When $d = 2$,

$$\alpha = \frac{bq}{2}\sqrt{\frac{2q+1}{q+1}}, \quad \beta = \frac{4(q+1)\alpha^2}{q^2b},$$

and when $d = 3$,

$$\alpha^2 = \frac{b^2q^2(2q+1)}{(2-q)^2(q+1)}, \quad \beta = \frac{2\alpha^2}{q^2b}(2 + q - q^2).$$

Collisions

To investigate collisions between conventional and algebraic solitary waves we solve (1) numerically using a semi-implicit leap-frog spectral method (Feng *et al.*, 1999). Initial conditions take the form of the sum of sufficiently separated conventional and algebraic soliton solutions. This is permissible in spite of the nonlinear nature of the equations as for well separated solitons, the values of u where one solution is non-zero are very close to zero for the other solution.

The results for the eKdV and $(1, \frac{1}{2})$-mKdV equations are shown in Figs. 1 and 2. In both cases the collision leaves both solitons unscathed except that the algebraic soliton moves to the left of its original position.

In collisions of higher-dimensional conventional and algebraic solitary waves, the algebraic solitary wave is destroyed and replaced by conventional solitary wave.

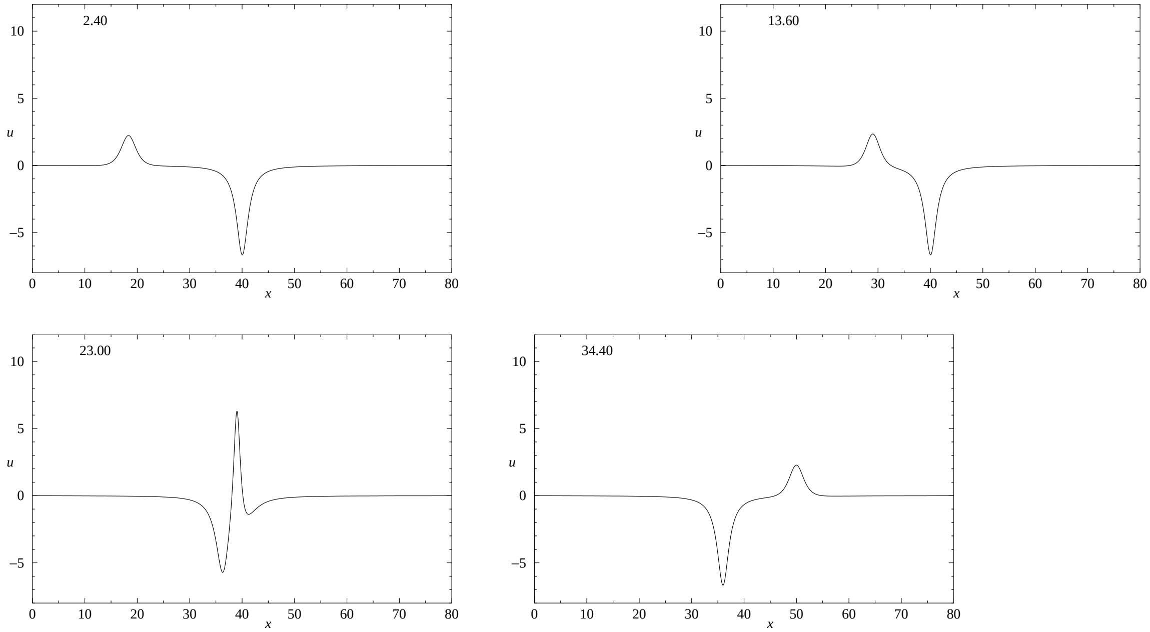

Figure 1: Collision of conventional soliton (initially on the left) with a rarefactive algebraic soliton governed by the eKdV equation. The numbers inside the plot indicate the value of t.

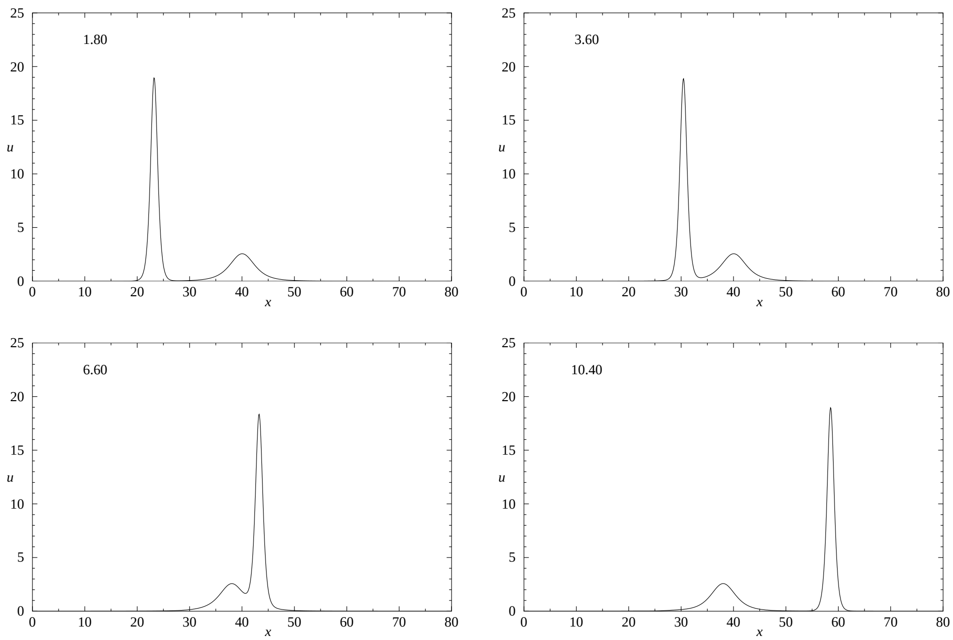

Figure 2: Collision of conventional soliton with a compressive algebraic soliton governed by the $(1, \frac{1}{2})$-mKdV equation.

Lump solitons

The KP^+ equation has lump soliton solutions – these are 2-d solitons without cylindrical symmetry. We showed that both the SKP^+ and (1,1/2)-mKP^+ equations also exhibit this type of soliton and in both cases they evolve from perturbed plane solitons (Phibanchon & Allen, 2004b, 2007).

Collisions of solitons

The solitons we looked only travel in the positive x-direction. Collisions can therefore only occur if the solitons are travelling at different speeds. The collisions are described as direct if the paths of the solitons before collision are on the same straight line. Otherwise they are called off-axis collisions. Aside from the cases mentioned previously, we looked at three further types of collisions. Collisions between spherical solitons in the ZK equation are discussed in Allen (2002b). Collisions between cylindrical and plane ZK solitons are described in Allen (2002a). Finally, results on the collisions between lump solitons are given in Phibanchon & Allen (2004a).

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Output

International refereed publications

1. M. A. Allen, S. Phibanchon, G. Rowlands (2007) Weakly nonlinear waves in magnetized plasma with a slightly non-Maxwellian electron distribution. Part 1. Stability of solitary waves. *Journal of Plasma Physics* **73**, 215–29.

2. M. A. Allen (2007) Evaluation of some improper integrals involving hyperbolic functions. *American Mathematical Monthly* **114**, 341–3.

3. S. Phibanchon, M. A. Allen (2007) Time evolution of perturbed solitons of modified Kadomtsev-Petviashvili equations. *The 2007 International Conference on Computational Science and its Applications*, pp. 20–3.

Other benefits

1. Discovery of a technique for the evaluation of certain types of integrals.

2. Training of new researchers.

3. Two MSc theses:

   (a) Poompat Buppha, "Soliton Solutions of a Perturbed KdV and the Modified Zakharov-Kuznetsov equations", Mahidol University, 2004

   (b) Jeerawan Uppaman, "Stability of Solitary Wave Solutions of the Extended Zakharov-Kuznetsov Equation", Mahidol University, 2007

Conference proceedings, poster and oral presentations

1. M. A. Allen (2001) *The energetics of cylindrical soliton formation.* 27th Congress on Science and Technology in Thailand, Hat Yai, p. 370 [extended abstract and poster presentation].

2. M. A. Allen (2002) *Collisions between cylindrical and plane solitons.* The 28th Congress on Science and Technology of Thailand, Bangkok, p. 278 [extended abstract and poster presentation].

3. P. Buppha, M. A. Allen (2002) *Evolution of perturbed solitons of a modified Zakharov-Kuznetsov equation.* The 28th Congress on Science and Technology of Thailand, Bangkok, p. 279 [extended abstract and poster presentation].

4. S. Phibanchon, M. A. Allen (2002) *The effect of the perturbation wavelength on Schamel-Zakharov-Kuznetsov cylindrical soliton generation.* The 28th Congress on Science and Technology of Thailand, Bangkok, p. 281 [extended abstract and poster presentation].

5. M. A. Allen (2002) *Spherical soliton collisions.* The Sixth Annual National Symposium on Computational Science and Engineering, Walailuk University, Nakhon Si Thammarat, pp. C98–103 [conference paper and oral presentation].

6. S. Phibanchon, M. A. Allen (2002) *Time evolution of perturbed plane solitons in a modified Zakharov-Kuznetsov equation.* The Sixth Annual National Symposium on Computational Science and Engineering, Walailuk University, Nakhon Si Thammarat, pp. C104–9 [conference paper and oral presentation].

7. S. Phibanchon, M. A. Allen (2003) *Stability of solitons in a slightly non-isothermal acoustic plasma.* The 29th Congress on Science and Technology of Thailand, Khon Kaen, p. SD69 [conference paper and poster presentation].

8. S. Phibanchon, M. A. Allen (2004) *Instabilities of Schamel-Kadomtsev-Petviashvili plane solitons.* The Eighth Annual National Symposium on Computational Science and Engineering, Suranaree University of Technology, Nakhon Ratchasima, pp. 165–8 [conference paper and oral presentation].

9. S. Phibanchon, M. A. Allen (2004) *Collisions of lump solitons.* The 30th Congress on Science and Technology of Thailand, Bangkok, p. A0019 [extended abstract and poster presentation].

10. M. A. Allen and G. Rowlands (2006) *Non-singular algebraic solitary wave solutions to generalized KdV equations.* SIAM Conference on Nonlinear Waves and Coherent Structures, University of Washington, Seattle [poster presentation].

11. S. Phibanchon, M. A. Allen (2007) *Time evolution of perturbed plane soliton solutions of the* $(1, 1/2)$*-modified Kadomtsev-Petviashvili equations.* The 33rd Congress on Science and Technology of Thailand, Walailuk University, Nakorn Si Thammarat, p. 213 [extended abstract and poster presentation].

12. M. A. Allen, J. Uppaman, and G. Rowlands (2008) *Determination of the growth rate curve for transverse instabilities of plane soliton solutions of the extended Zakharov-Kuznetsov equation.* Nonlinear Waves – Theory and Applications, Tsinghua University, Beijing [oral presentation].