รายงานวิจัยฉบับสมบูรณ์

โครงการ :

"FIXED POINT PROPERTY IN BANACH SPACES"

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัยสกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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ต้องยอมรับว่างานวิจัยทางคณิตศาสตร์ในประเทศไทยยังล้าหลังนานาประเทศอยู่มาก ปัญหาสำคัญคือการขาดบุคลากร และปัญหาที่สำคัญประการต่อมาสำหรับบุคลากรที่น้อยนิดนี้ก็คือการขาดอุปกรณ์การวิจัย ในที่นี้หมายถึงตำรา(text) และวารสาร(journal) สาเหตุของการขาดแคลนน่าจะมาจากการไม่เห็นหรือการไม่ให้ความสำคัญของผู้บริหารในหลายระดับ นักวิจัยจึงต้องช่วยตัวเองอย่างยากลำบากและไม่มีประสิทธิภาพ หลายต่อหลายคนเกิดความท้อแท้และเลิกทำวิจัยไปเลย ปัจจุบันนี้สถานการณ์ค่อยคลี่คลายลงในระดับหนึ่ง เพราะมีอินเตอร์เน็ต(internet) เข้ามาช่วยให้สามารถค้นหาบทความ (publication) ได้บ้าง การติดต่อกับอาจารย์ต่างประเทศทำได้ง่ายขึ้น แต่ถึงกระนั้นปัญหาเดิมก็ยังคงมีอยู่ หน่วยงานหนึ่งซึ่งถือว่าได้เข้ามาช่วยแก้ปัญหานี้ให้กับนักวิจัยได้อีกระดับหนึ่งคือ สำนักงานกองทุนสนับสนุนการวิจัย(สกว.) การสนับสนุนของสำนักงานกองทุนสนับสนุนการวิจัยที่ผ่านมาต้องยอมรับว่าทำให้บรรยากาศการวิจัยกลับมาคึกคักขึ้นอีกครั้งหนึ่ง ผู้วิจัยมีความหวังมากขึ้น

งานวิจัยทางคณิตศาสตร์ที่ทำอยู่ในประเทศไทยมีอยู่ 2 กลุ่ม กลุ่มหนึ่งเป็นทางประยุกต์ อีกกลุ่มหนึ่งเป็นทางคณิตศาสตร์บริสุทธิ์ (pure math) งานวิจัยทางคณิตศาสตร์บริสุทธิ์เองอาจจะแบ่งย่อยได้เป็นหลายระดับเปรียบเสมือนเหรียญรางวัลการแข่งขันที่ประกอบไปด้วยเหรียญทองแดง เหรียญเงิน และเหรียญทอง งานวิจัยก็เช่นกัน หากหวังผลงานที่ดี มีความลึกซึ้งได้ลงพิมพ์ในวารสารระดับสูงๆ อาจต้องใช้เวลาในการวิจัยนานๆ เป็นที่น่ายินดีที่สำนักงานกองทุนสนับสนุนการวิจัยเข้าใจและเปิดใจกว้างยินดีให้การสนับสนุนในทุก ๆ ระดับ งานวิจัยชิ้นนี้เป็นงานวิจัยทางคณิตศาสตร์บริสุทธิ์ ผลการวิจัยมีทั้งระดับง่ายและยาก แน่นอนเราต้องการผลิตในระดับยากเป็นเป้าหมายหลัก แต่ก็จำเป็นต้องแบ่งเวลาให้กับนักศึกษาได้ฝึกในระดับง่ายๆบ้าง ซึ่งถือว่างานในระดับนี้ประสบผลสำเร็จ 100 % ส่วนระดับที่สูงกว่าได้ดำเนินการไปเสร็จสมบูรณ์แล้วมีอยู่ 2 เรื่อง คือ

- Generalized Jordan-von Neumann constants and uniform normal structure, Bull. Austral. Math. Soc. 67(2003), 225-240.
- On a generalized James constant, J. Math. Anal. Appl. (in press)

ผลงานทั้งสองนี้ได้นำไปบรรยายที่ International conference on fixed point theory and its applications ประเทศสเปนระหว่างวันที่ 13-19 กรกฎาคม 2546 และ The Wladyslaw Orlicz centenary conference and Function spaces VII ประเทศโปแลนด์ระหว่างวันที่ 21-25 กรกฎาคม 2546

ขอขอบพระคุณ สำนักงานกองทุนสนับสนุนงานวิจัยเป็นอย่างสูงไว้ ณ ที่นี้ ขอขอบพระคุณภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ที่อำนวยความสะดวกทางด้านอุปกรณ์คอมพิวเตอร์และงานพิมพ์

คณะผู้วิจัย

30 พฤศจิกายน 2546

บทคัดย่อ

งานวิจัยนี้ได้ให้ลักษณะเฉพาะของสมบัติทางเรขาคณิตของปริภูมิลำดับ Musielak-Orlicz ภายใต้นอร์มแบบ Luxemburg เช่น

- Rotundity, Locally uniform rotundity, Weakly locally uniform rotundity
- k- Rotundity
- Property (H), Property (K)

นอกจากนี้ยังมีผลงานอีก 2 ชิ้น ซึ่งอาจถือเป็นผลงานเด่นของโครงการวิจัยนี้ คือ

- การวางนัยทั่วไปของ Jordan-von Neuman constant
- การวางนัยทั่วไปของ James constant

คำหลัก สมบัติทางเรขาคณิต ปริภูมิบานาค ปริภูมิลำดับ Musielak-Orlicz สมบัติจุดตรึง Jordan-von Neuman constant James constant

ABSTRACT

We characterize some geometric properties of the Musielak-Orlicz sequence spaces equipped with the Luxemburg norm:

- Rotundity, Locally uniform rotundity, Weakly locally uniform rotundity
- k- Rotundity
- Property (H), Property (K)

Moreover, we obtained 2 more results, which may be considered as the best results of this research project:

- Generalized Jordan-von Neuman constants
- Generalized James constants

Keywords geometric property, Banach space, Musielak-Orlicz sequence space, fixed point property, Jordan-von Neuman constant James constant

CONTENTS

EXECUTIVE SUMMARY

1. Project Title

FIXED POINT PROPERTY IN BANACH SPACES

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3. Research Field

FUNCTIONAL ANALYSIS

4. Problem statement and importance

The fixed point property (FPP) has been studied since J.Brouwer and S.Banach giving the celebrated theorems: Brouwer's Fixed Point Theorem and the principle of Banach's Contraction Mapping, respectively. The theory of fixed point property is one of the most important subject. It contributes a lot of applications in many fields of mathematics, for examples, theory of operator, control theory, approximation theory, theory of equations. Using geometric property to study FPP has been developed since W.K.Kirk who proved in 1965 that a Banach space with normal structure has weakly fixed point property (WFPP).

The fixed point property is still proven to be the most active area and is continuing to interest people worldwide. Here the emphasis will be on the relation between geometric properties (and geometric coefficients) and the FPP in Banach spaces.

5. Objective of Research

5.1 To investigate which geometric properties and geometric coefficients imply the FPP or WFPP of Banach spaces both in general and well-known spaces in particular.

5.2 To solve some open problems concerning the FPP.

6. Plan of Research

6.1 Introducing weakly normal structure.

6.2 Giving criterions for some classical Banach spaces to have the fixed point property.

6.3 Introducing some weaker geometric properties, for examples, weakly nearly uniformly convex (WNUC) and prove that a Banach space with (WNUC) has the fixed point property.

6.4 Calculating some geometric coefficients concerning the fixed point property in some classical Banach spaces.

6.5 Trying to solve some open questions, for examples, whether or not non-strictly Opial property implies the weak fixed point property.

7. Methodology of Research

(1) Characterize geometric properties of Banach spaces in general and of some specific well-known spaces.

(2) Find properties in (1) having the FPP as their necessary condition.

(3) Do the same for (1) and (2) above but for geometric coefficients instead of geometric properties.

Plan for each 6 month period :

October 1, 2000 - March 31, 2001:	Writing a paper on "Some geometric property in Musielak-Orlicz spaces".
April 1, 2001 - September 30, 2001:	Writing a paper on "Geometric coefficients and fixed point property in Musielak-Orlicz spaces".
October 1, 2001 - March 31, 2002:	Writing a paper on "Geometric coefficient in Banach spaces".
April 1, 2002 - September 30, 2002:	Writing a paper on "Geometric coefficient and fixed point property in Banach spaces".
October 1, 2002 - March 31, 2003:	Concentrating on open problems, e.g. "Whether or not nonstrict Opial property implies the weak fixed point property?".
April 1, 2003 - September 30, 2003:	Continuation on open problems, e.g. "Does the WNUC property imply the fixed point property?".

8. Expected output

We expect to publish at least 2 papers a year.

Tentative titles and journals:

(1) Title :The fixed point property in Banach spaces with WNUC property
Journal :Nonlinear Analysis

(2) Title :Fixed point property and some geometric coefficients in Banach spaces
Journal :Pacific J. Math.

(3) Title :Fixed point property in Musielak-Orlicz spaces

Journal :J. Math. Anal. Appl.

9. Selected published research papers related to research project matter.

1. L.P.Belluce and W.A.Kirk, Fixed point theorems for families of contraction mappings. *Pacific J. Math.*, **19** (1966), 213-217.

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21. W.A.Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly* **72** (1965), 1004-1006.

22. D.Kutzarova, E.Maluta, and S.Prus, On some l_p-type estimates for Banach spaces, *Bull. Austral. Math. Soc.* **48** (1993), 187-194.

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OUTPUT

We divide the output into 2 categories.

I. Published papers:

- Extreme points in Musielak-Orlicz sequence spaces, Acta Mathematica Vietnamica, 27, No.2 (2002), 219-229.

  Appendix 1

- Generalized Jordan-von Neumann constants and uniform normal structure, Bull. Austral. Math. Soc. 67(2003), 225-240.

  Appendix 2

- On some local geometry of Musielak-Orlicz sequence spaces, Commentationes Math. Prace. Mat. (already appeared)

  Appendix 3

- On a generalized James constant, J. Math. Anal. Appl. (in press)

  Appendix 4

II. Submitted papers:

- Remarks on convexity properties of Nakano spaces

  Appendix 5

- Preservation of uniform smoothness and U-convexity by ψ-direct sums

  Appendix 6

- Some convexity properties in Musielak–Orlicz sequence spaces endowed with the Luxemburg norm

  Appendix 7

- On the modulus of U–convexity

  Appendix 8

Appendix

Appendix 1: Extreme points in Musielak-Orlicz sequence spaces, Acta Mathematica Vietnamica., Vol.27, No.2, 2002, pp.219-229

ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 2, 2002, pp. 219-229

EXTREME POINTS IN MUSIELAK-ORLICZ SEQUENCE SPACES

S. SAEJUNG AND S. DHOMPONGSA

ABSTRACT. This paper establishes some characterizations of extreme points and strongly extreme points of the closed unit ball in a Musielak-Orlicz sequence space equipped with the Luxemburg norm. As a consequence of these results, we obtain some geometric properties such as rotundity and strong rotundity in Nakano sequence spaces and Orlicz sequence spaces.

1. INTRODUCTION

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and the closed unit ball of X, respectively. Recall that a point $x \in S(X)$ is an *extreme point* if $2x = y + z$ for $y, z \in B(X)$ implies $y = z$, and is a *strongly extreme point* if $2x = y_n + z_n$ for all $n \in \mathbb{N}$ and $\|y_n\| \to 1$, $\|z_n\| \to 1$ imply $\|y_n - z_n\| \to 0$. A Banach space X is said to be *rotund* if every point in its unit sphere is an extreme point. If every point in its unit sphere is a strongly extreme point, then X is said to be *strongly rotund*.

Clearly, every strongly extreme point is an extreme point. Thus every strongly rotund space is a rotund space. An example in [8] shows that there is a rotund Banach space which is not strongly rotund.

In this paper, we study extreme points and related properties in Musielak-Orlicz sequence spaces. Before stating our main result we first recall the following definitions:

Let $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and the set of real numbers, respectively. A function $\Phi : \mathbb{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ is even, convex, and vanishes at zero. A sequence $\Phi = (\Phi_k)$ of Orlicz functions Φ_k is called a *Musielak-Orlicz function*. If $\Phi = (\Phi_k)$ is a Musielak-Orlicz function, then the sequence $\Psi = (\Psi_k)$ defined by

$$\Psi_k(v) := \sup\{|v|u - \Phi_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots \tag{1.1}$$

is called the *complementary function* of Φ in the sense of Young (see [7]).

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Let $\mathbb{R}^{\mathbf{N}}$ denote the space of all real sequences $x = (x(k))$. For a given Musielak-Orlicz function Φ we define a *convex modular* $I_\Phi : \mathbb{R}^{\mathbf{N}} \to [0, \infty]$ by the formula

$$(1.2) \qquad I_\Phi(x) = \sum_{k=1}^{\infty} \Phi_k(x(k)) \quad \text{for } x \in \mathbb{R}^{\mathbf{N}}.$$

The *Musielak-Orlicz sequence space* l_Φ generated by $\Phi = (\Phi_k)$ is defined by

$$(1.3) \qquad l_\Phi := \{x \in \mathbb{R}^{\mathbf{N}} : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

In particular, if $\Phi_k = M$ for every $k \in \mathbb{N}$, then l_M is called the *Orlicz sequence space* generated by M. We consider two norms on l_Φ: The *Luxemburg norm:*

$$(1.4) \qquad \|x\| = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\}$$

and the *Orlicz norm:*

$$(1.5) \qquad \|x\|^o = \inf\left\{\frac{1}{\lambda}(1 + I_\Phi(\lambda x)) : \lambda > 0\right\},$$

where $I_\Phi(\cdot)$ is defined by (1.2).

Let $l_\Phi := (l_\Phi, \|\cdot\|)$ and $l_\Phi^o := (l_\Phi, \|\cdot\|^o)$ denote the space l_Φ equipped with the Luxemburg norm and the Orlicz norm, respectively. It is known (see [7]) that both are Banach spaces. The subspace h_Φ of l_Φ defined by

$$(1.6) \qquad h_\Phi := \{x \in l_\Phi : I_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

is called *the space of finite elements.* Let

$$(1.7) \qquad \theta(x) = \inf\{\lambda > 0 : I_\Phi(x/\lambda) < \infty\}.$$

It is clear that $x \in h_\Phi$ if and only if $\theta(x) = 0$. If Ψ is the complementary function (see (1.1)) of the Musielak-Orlicz function Φ, then by [7] the space $h_\Psi^o := (h_\Psi, \|\cdot\|^o)$ equipped with the Orlicz norm (1.5) is separable, and its dual is isometrically isomorphic to l_Φ.

We say that a Musielak-Orlicz function $\Phi = (\Phi_k)$ satisfies:

(1.8) *the δ_2-condition*, denoted $\Phi \in \delta_2$, if there exist constants $K \ge 2$, $u_0 > 0$ and a sequence (c_k) of positive numbers, with $\sum_{k=1}^{\infty} c_k < \infty$, such that for $\Phi_k(u) \le u_0$ we have

$$\Phi_k(2u) \le K\Phi_k(u) + c_k \quad \text{for every} \quad k \in \mathbb{N} \quad \text{and} \quad u \in \mathbb{R}.$$

(1.9) *the $(*)$-condition* (see [6]) if for any $\varepsilon \in (0,1)$ there exists a $\delta > 0$ such that $\Phi_k((1+\delta)u) \le 1$ whenever $\Phi_k(u) \le 1 - \varepsilon$ for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$.

The following theorem is known (see [5]).

Theorem 1.1. *$h_\Phi = l_\Phi$ if and only if $\Phi \in \delta_2$.*

By [5] and [6] if a Musielak-Orlicz function $\Phi = (\Phi_k)$ satisfies (1.8), (1.9) and $\Phi_k(u) = 0$ if and only if $u = 0$ for every k, then

(1.10) For each $\varepsilon > 0$ and each $c > 0$ there exists a $\delta > 0$ such that

$$|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon \text{ whenever } I_\Phi(x) \le c \text{ and } I_\Phi(y) < \delta.$$

(1.11) For any sequence $(x_n) \subset l_\Phi$, $\|x_n\| \to 1$ implies $I_\Phi(x_n) \to 1$, and

(1.12) $\|x\| = 1$ if and only if $I_\Phi(x) = 1$.

Our paper is organized as follows: In Section 2, we characterize extreme points in Musielak-Orlicz sequence spaces. Strongly extreme points in some subspaces of a Musielak-Orlicz sequence space are investigated in Section 3. Finally, in Section 4 we study geometric properties related to rotundity, strong rotundity and H-points.

2. Extreme points in Musielak-Orlicz sequence spaces

Let M be an Orlicz function. An interval $[a, b]$, $a < b$, is called an *affine interval* of M if

$$M(\lambda a + (1-\lambda)b) = \lambda M(a) + (1-\lambda)M(b) \quad \text{for all } \lambda \in [0,1]. \tag{2.1}$$

In addition, if M is neither affine on $[a-\varepsilon, b]$ nor on $[a, b+\varepsilon]$ for any $\varepsilon > 0$ we call $[a, b]$ a *structural affine interval* of M. Let $\{[a_i, b_i] : i \in I\}$ be the family of all the structural affine intervals of M. The set

$$S_M := \mathbb{R} \setminus \bigcup_{i \in I}(a_i, b_i) \tag{2.2}$$

is called the set of *strictly convex points* of M. Let

$$a_M = \sup\{u \ge 0 : M(u) = 0\}. \tag{2.3}$$

Theorem 2.1. *A point $x = (x(k)) \in S(l_\Phi)$ is an extreme point if and only if*

(i) *$I_\Phi(x) = 1$ and*

(ii) *$\#\{k : |x(k)| \in [0, a_{\Phi_k})\} = 0$ and $\#\{k : x(k) \notin S_{\Phi_k}\} \le 1$, where a_{Φ_k} and S_{Φ_k} are defined by* (2.3) *and* (2.2) *respectively, and $\#A$ denotes the cardinality of a set A.*

Proof. Necessity. Let $x = (x(k))$ be an extreme point of $S(l_\Phi)$. We will show that (i) and (ii) must hold. Suppose (i) does not hold, i.e. $I_\Phi(x) = r < 1$. Since Φ_1 is continuous we can choose $\varepsilon > 0$ so small that

$$\Phi_1(x(1) \pm \varepsilon) < \Phi_1(x(1)) + \frac{1-r}{2}.$$

Define sequences $y = (y(k)), z = (z(k)) \in l_\Phi$ by $y(1) = x(1) + \varepsilon, z(1) = x(1) - \varepsilon$ and $y(k) = z(k) = x(k)$ for all $k \ge 2$. Obviously, $y \ne z$ and $2x = y + z$. Moreover,

$$I_\Phi(y) < I_\Phi(x) + \frac{1-r}{2} = \frac{1+r}{2} < 1.$$

Thus $\|y\| \le 1$. Similarly, we also have $\|z\| \le 1$. This contradiction shows that (i) must hold.

Suppose the first condition in (ii) does not hold, i.e. $j \in \{k : |x(k)| \in [0, a_{\Phi_k})\}$. Choose $\varepsilon \neq 0$ such that $x(j) \pm \varepsilon \in (-a_{\Phi_k}, a_{\Phi_k})$. Define $y = (y(k)) \in l_\Phi$ by $y(j) = x(j) + \varepsilon$, $y(k) = x(k)$ for all $k \neq j$ and $z = 2x - y$. It is easy to verify that $I_\Phi(y) = I_\Phi(z) = I_\Phi(x) = 1$. Since $y \neq z$, x can not be an extreme point.

Suppose the second condition in (ii) does not hold, i.e. $\#\{k : x(k) \notin S_{\Phi_k}\} \geq 2$. Without loss of generality we assume that $x(1) \notin S_{\Phi_1}$ and $x(2) \notin S_{\Phi_2}$. Then $x(1) \in (a_1, b_1)$ and $x(2) \in (a_2, b_2)$ for some structural affine intervals $[a_1, b_1]$ and $[a_2, b_2]$ of Φ_1 and Φ_2, respectively. Let $\Phi_1(u) = k_1 u + \beta_1$ $(u \in (a_1, b_1))$ and $\Phi_2(u) = k_2 u + \beta_2$ $(u \in (a_2, b_2))$ where $k_1 \neq 0$ and $k_2 \neq 0$. Choose $\varepsilon_1 \neq 0, \varepsilon_2 \neq 0$ such that

$$k_1 \varepsilon_1 = k_2 \varepsilon_2 \quad \text{and} \quad x(k) \pm \varepsilon_k \in (a_k, b_k) \quad \text{for } k = 1, 2.$$

Define $y = (y(k)) \in l_\Phi$ by $y(1) = x(1) + \varepsilon_1, y(2) = x(2) - \varepsilon_2, y(k) = x(k)$ for all $k \geq 3$, and $z = 2x - y$. Then we have $\Phi_1(y(1)) + \Phi_2(y(2)) = k_1 x(1) + \beta_1 + k_2 x(2) + \beta_2 = \Phi_1(x(1)) + \Phi_2(x(2))$. This implies $I_\Phi(y) \leq 1$, so $\|y\| \leq 1$. Similarly we have $\|z\| \leq 1$. This is a contradiction.

Sufficiency. If $2x = y + z$ for some $y, z \in B(l_\Phi)$ then, by (i) and the convexity of the modular $I_\Phi(\cdot)$,

$$1 = I_\Phi(x) \leq \frac{1}{2} I_\Phi(y) - \frac{1}{2} I_\Phi(z) \leq 1.$$

This implies $\Phi_k(x(k)) = \frac{1}{2}\Phi_k(y(k)) + \frac{1}{2}\Phi_k(z(k))$ for all $k \in \mathbb{N}$. By the first condition of (ii), there exists at most one $k \in \mathbb{N}$ such that $x(k) \notin S_{\Phi_k}$. If $x(k) \in S_{\Phi_k}$ then $x(k) = y(k) = z(k)$. Now suppose that there exists $j \in \mathbb{N}$ such that $x(j) \notin S_{\Phi_j}$. Then we have $x(k) = y(k) = z(k)$ for all $k \neq j$ and $x(j), y(j), z(j)$ belong to the same structural affine intervals of Φ_j. Since $\sum_{k=1}^{\infty} \Phi_k(y(k)) = 1 = \sum_{k=1}^{\infty} \Phi_k(z(k))$, we have $\Phi_j(y(j)) = \Phi_j(z(j)) = \Phi_j(x(j))$. If $y(j) \neq z(j)$, then $x(j) \in [-a_{\Phi_j}, a_{\Phi_j}]$. Since $a_{\Phi_j} \in S_{\Phi_j}$, $x(j) \in (-a_{\Phi_j}, a_{\Phi_j})$. This contradicts the second condition of (ii). Hence $y(j) = z(j)$. Therefore x is an extreme point. $\square$

Recall that a *Nakano sequence space* $l^{\{p_k\}}$ is a Musielak-Orlicz sequence space with $\Phi_k(u) = |u|^{p_k}$ for some sequence $\{p_k\}$ in $[1, \infty)$.

Corollary 2.1. ([4, Theorem 1]) *A point $x \in S(l^{\{p_k\}})$ is an extreme point if and only if $I_\Phi(x) = 1$ and $\#\{k : x(k) \neq 0$ and $p_k = 1\} \leq 1$.*

Corollary 2.2. ([1, Theorem 2.6]) *A point $x \in S(l_M)$ is an extreme point if and only if $I_M(x) = 1$, $\#\{k : x(k) \notin S_M\} \leq 1$ and $\#\{k : |x(k)| \in [0, a_M)\} = 0$.*

Observe that Corollary 2.1 was proved in [4] under the assumption that $\{p_k\}$ is bounded and Corollary 2.2 was proved in [4] under the assumption that the Orlicz function is an N-function. Our Corollaries 2.1 and 2.2 say that these assumptions can be removed.

3. STRONGLY EXTREME POINTS IN MUSIELAK-ORLICZ SEQUENCE SPACES

In this section, we investigate strongly extreme points in the Musielak-Orlicz sequence space h_Φ.

Theorem 3.1. *If $x \in S(l_\Phi)$ is a strongly extreme point and $\theta(x) < 1$ (see (1.7)), then $\Phi \in \delta_2$.*

Suppose, in addition, that Φ satisfies the $(*)$-condition (see (1.9)) and each Φ_k vanishes only at zero. Then a point $x \in S(h_\Phi)$ is a strongly extreme point of $B(h_\Phi)$ if and only if it is an extreme point and $\Phi \in \delta_2$. In particular, if $h_\Phi = l_\Phi$, then a point $x \in S(l_\Phi)$ is a strongly extreme point if and only if it is an extreme point.

Proof. Suppose that $\Phi \notin \delta_2$, then by [5] there exists $x_0 = (x_0(k))$ such that

$$I_\Phi(x_0) \le 1 \quad \text{and} \quad I_\Phi(\lambda x_0) = \infty \text{ for any } \lambda > 1.$$

Since $\theta(x) < 1$, we have $I_\Phi(\lambda_0 x) < \infty$ for some $\lambda_0 > 1$. We define (y_n) and (z_n) by

$$\begin{aligned} y_n &= (x(1), \dots, x(n), x(n+1) + \varepsilon_0 x_0(n+1), x(n+2) + \varepsilon_0 x_0(n+2), \dots), \\ z_n &= (x(1), \dots, x(n), x(n+1) - \varepsilon_0 x_0(n+1), x(n+2) - \varepsilon_0 x_0(n+2), \dots), \end{aligned}$$

where $\varepsilon_0 = 1 - 1/\lambda_0$. Clearly, $2x = y_n + z_n$ for all $n = 1, 2, \dots$. Moreover,

$$I_\Phi\left(\frac{y_n - z_n}{\varepsilon_0}\right) = \sum_{k=n+1}^{\infty} \Phi_k(2x_0(k)) = \infty.$$

It follows that $\|y_n - z_n\| > \varepsilon_0$ for all $n \in \mathbb{N}$. We will prove that $\|y_n\| \to 1$ and $\|z_n\| \to 1$. For $\varepsilon \in (0,1)$ let $\lambda = 1 + \varepsilon$. Observe that for each $n \in \mathbb{N}$ we have

$$\begin{aligned} I_\Phi\left(\frac{y_n}{\lambda}\right) &= \sum_{k=1}^{n} \Phi_k\left(\frac{x(k)}{\lambda}\right) + \sum_{k=n+1}^{\infty} \Phi_k\left(\frac{1}{\lambda\lambda_0}\lambda_0 x(k) + \frac{\varepsilon_0}{\lambda} x_0(k)\right) \\ &\le \sum_{k=1}^{n} \Phi_k\left(\frac{x(k)}{\lambda}\right) + \frac{1}{\lambda\lambda_0} \sum_{k=n+1}^{\infty} \Phi_k(\lambda_0 x(k)) + \frac{\varepsilon_0}{\lambda} \sum_{k=n+1}^{\infty} \Phi_k(x_0(k)). \end{aligned}$$

Note that $I_\Phi(x/\lambda) < 1$. Choose $N > 0$ so that for each $n \ge N$

$$\frac{1}{\lambda\lambda_0} \sum_{k=n+1}^{\infty} \Phi_k(\lambda_0 x(k)) < \frac{1 - I_\Phi(x/\lambda)}{2},$$

$$\frac{\varepsilon_0}{\lambda} \sum_{k=n+1}^{\infty} \Phi_k(x_0(k)) < \frac{1 - I_\Phi(x/\lambda)}{2}.$$

So $I_\Phi(y_n/\lambda) \le 1$ for all $n \ge N$. Then $\|y_n\| \le \lambda = 1 + \varepsilon$ for all $n \ge N$. Therefore $\limsup_{n\to\infty} \|y_n\| \le 1$. Similarly, $\limsup_{n\to\infty} \|z_n\| \le 1$. Hence $\liminf_{n\to\infty} \|y_n\| \ge 2 - \limsup_{n\to\infty} \|z_n\| \ge 1$ which yields $\|y_n\| \to 1$. Similarly, $\|z_n\| \to 1$. Hence, x can not be a strongly extreme point. This contradiction proves the first part of the theorem.

To prove the second part of the theorem observe that, since $\theta(x) = 0$ for every $x \in S(h_\Phi)$, the necessity of the theorem is trivial. To demonstrate the sufficiency of the theorem, assume that x is an extreme point and $\Phi \in \delta_2$. Let (x_n) and (y_n) be sequences in h_Φ such that $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $2x = x_n + y_n$ for all $n \in \mathbb{N}$. By the Banach-Alaoglu Theorem, the unit ball of l_Φ is weakly star compact. Therefore, by passing to subsequences if necessary, we may assume that $x_n \overset{w*}{\to} x_0$, and $y_n \overset{w*}{\to} y_0$, for some $\|x_0\| \le 1$ and $\|y_0\| \le 1$. But since $x_n + y_n = 2x$ we have $x_0 + y_0 = 2x$, which implies $x_0 = y_0 = x$. Therefore

$$x_n(k) \to x(k) \text{ and } y_n(k) \to x(k) \quad \text{for each} \quad k = 1, 2, \ldots. \tag{3.1}$$

Given $\varepsilon \in (0,1)$, by (1.10) we can find $\delta \in (0, \varepsilon)$ such that

$$|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon \quad \text{whenever} \quad I_\Phi(x) \le 1 \quad \text{and} \quad I_\Phi(y) < \delta. \tag{3.2}$$

We choose m_0 so that $\sum\limits_{k=m_0+1}^{\infty} \Phi_k(x(k)) < \delta/3$.

By (1.11) and (1.12), we have $I_\Phi(x_n) \to 1 = I_\Phi(x)$. Then $I_\Phi(x_n) < I_\Phi(x) + \delta/3$ for sufficiently large n. From (3.1) we have

$$\left| \sum_{k=1}^{m_0} (\Phi_k(x_n(k)) - \Phi_k(x(k))) \right| < \delta/3 \quad \text{for sufficiently large} \quad n. \tag{3.3}$$

Consequently, for n large enough, we have

$$\begin{aligned} \sum_{k=m_0+1}^{\infty} \Phi_k(x_n(k)) &= I_\Phi(x_n) - \sum_{k=1}^{m_0} \Phi_k(x_n(k)) \\ &< I_\Phi(x) + \delta/3 - \left(\sum_{k=1}^{m_0} \Phi_k(x(k)) - \delta/3 \right) \\ &= \sum_{k=m_0+1}^{\infty} \Phi_k(x(k)) + 2\delta/3 < \delta. \end{aligned}$$

Let

$$\begin{aligned} x' &= (0, \ldots, 0, x(m_0+1). x(m_0+2), \ldots), \\ x'_n &= (0, \ldots, 0, x_n(m_0+1), x_n(m_0+2), \ldots). \end{aligned}$$

Then we have $I_\Phi(x') < \delta$ and $I_\Phi(x'_n) < \delta$ for all large n. Again, from (3.1) it follows that $\sum\limits_{k=1}^{m_0} \Phi_k(x_n(k) - x(k)) < \varepsilon$ for sufficiently large n.

By (3.2) and (3.3), for all large n we have

$$\begin{aligned} I_\Phi(x_n - x) &= \sum_{k=1}^{m_0} \Phi_k(x_n(k) - x(k)) + I_\Phi(x'_n - x') \\ &< \varepsilon + I_\Phi(x'_n) + \varepsilon < 3\varepsilon. \end{aligned}$$

This implies $I_\Phi(x_n - x) \to 0$, i.e. $x_n \to x$. Therefore $\|x_n - y_n\| \to 0$, so x is a strongly extreme point. The proof is complete. □

Remark 3.1. (1) By [3], if $x \in l_M$ is a strongly extreme point then $\theta(x) = 0$.

(2) The assumption $\theta(x) < 1$ in Theorem 3.1 is essential as we can see in the following example.

Example 3.1. We consider a Nakano sequence space $l^{\{k^2\}}$. Observe that $\Phi_k(u) = |u|^{k^2}$. Let $x = (x(k))$, where $x(k) = (1/2)^{1/k}$. Clearly, $\Phi = (\Phi_k) \notin \delta_2$. We also have $I_\Phi(x) = 1$ and $I_\Phi(\lambda x) = \sum\limits_{k=1}^{\infty} \frac{\lambda^{k^2}}{2^k} = \sum\limits_{k=1}^{\infty} \left(\frac{\lambda^k}{2}\right)^k = \infty$ for any $\lambda > 1$, so $\theta(x) = 1$. By Corollary 2.1, x is an extreme point. Next, we prove that x is a strongly extreme point. Suppose $(x_n), (y_n) \subset l_\Phi$, $x_n + y_n = 2x$ for all $n \in \mathbb{N}$, $\|x_n\| \to 1$ and $\|y_n\| \to 1$. As in the proof of Theorem 3.1 we may assume that

$$x_n(k) \to x(k) \quad \text{and} \quad y_n(k) \to x(k) \quad \text{for each} \quad k = 1, 2, \ldots .$$

It suffices to prove that $\|x_n - x\| \to 0$. Given $\varepsilon > 0$, we choose integers K and N_1 so that

$$1/K < \varepsilon \text{ and } \|x_n\| < 1 + \varepsilon \quad \text{for all} \quad n > N_1. \tag{3.4}$$

This implies $\sum\limits_{k=1}^{\infty} \left|\frac{x_n(k)}{1+\varepsilon}\right|^{k^2} < 1$ for all $n > N_1$. In particular,

$$|x_n(k)| < 1 + \varepsilon \quad \text{for all } n > N_1 \quad \text{and} \quad k = 1, 2, \ldots . \tag{3.5}$$

Again, choose $N_2 > N_1$ so that

$$|x_n(k) - x(k)| < \varepsilon \text{ and } |y_n(k) - x(k)| < \varepsilon \tag{3.6}$$

for all $n > N_2$, and $k = 1, \ldots, K$. Let $\Gamma_n = \{k \in \mathbb{N} : x_n(k) > 1 \text{ or } y_n(k) > 1\}$. We consider two cases.

Case 1. $k \in \Gamma_n$. If $x_n(k) > 1$, then $x_n(k) - 1 < \varepsilon$ for all $n > N_1$. Note that

$$1 - \left(\frac{1}{2}\right)^{1/k} \leq \frac{1}{k} \quad \text{for all} \quad k \in \mathbb{N}.$$

This means for all $n > N_1$ and $k \in \Gamma_n$ we have

$$|x_n(k) - x(k)| \leq |x_n(k) - 1| + |1 - x(k)| < \varepsilon + 1/k. \tag{3.7}$$

Similarly, if $y_n(k) > 1$ then

$$|y_n(k) - x(k)| < \varepsilon + 1/k \quad \text{for all} \quad n > N_1.$$

Case 2. $k \notin \Gamma_n$. In this case we have

$$|x_n(k) - x(k)| \le 1/k \quad \text{and} \quad |y_n(k) - x(k)| \le 1/k \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.8}$$

If $n > N_2$ and $\lambda > 8\varepsilon$, then from (3.4)-(3.8) we obtain

$$\begin{aligned} I_\Phi\Big(\frac{x_n - x}{\lambda}\Big) &= \sum_{k=1}^{\infty}\Big(\frac{|x_n(k) - x(k)|}{\lambda}\Big)^{k^2} \\ &= \Bigg(\sum_{k=1}^{K} + \sum_{k\in\Gamma_n\setminus\{1,\ldots,K\}} + \sum_{k\notin(\Gamma_n\cup\{1,\ldots,K\})}\Bigg)\Big(\frac{|x_n(k) - x(k)|}{\lambda}\Big)^{k^2} \\ &< \sum_{k=1}^{K}\Big(\frac{\varepsilon}{\lambda}\Big)^{k^2} + \sum_{k=K+1}^{\infty}\Big(\frac{\varepsilon + 1/k}{\lambda}\Big)^{k^2} \\ &< \sum_{k=1}^{K}\Big(\frac{1}{8}\Big)^{k^2} + \sum_{k=K+1}^{\infty}\Big(\frac{1}{4}\Big)^{k^2} < 1. \end{aligned}$$

This means $\|x_n - x\| \le \lambda$ for all $n > N_2$. Letting $\lambda \downarrow 8\varepsilon$ we get $\|x_n - x\| \le 8\varepsilon$ for all $n > N_2$, i.e. $\|x_n - x\| \to 0$.

Let

$$h^{\{p_k\}} = \Big\{x = (x(k)) \in l^{\{p_k\}} : \sum_{k=1}^{\infty}|\lambda x(k)|^{p_k} < \infty \text{ for all } \lambda > 0\Big\}.$$

From Theorem 3.1 we get

Corollary 3.1. *A point $x \in S(h^{\{p_k\}})$ is a strongly extreme point if and only if it is an extreme point and the sequence $\{p_k\}$ is bounded.*

Proof. It is easy to verify that the δ_2-condition is equivalent to the boundedness of the sequence $\{p_k\}$ (see [5]). □

The following corollary follows immediately from Remark 3.1(1) and Theorem 3.1.

Corollary 3.2. ([1, Theorem 2.10] and [3, Corollary 1]) *Suppose that M vanishes only at zero. Then $x \in S(l_M)$ is a strongly extreme point if and only if x is an extreme point and $M \in \delta_2$.*

4. The rotundity and strong rotundity in Musielak-Orlicz sequence spaces

Theorem 4.1. *The Musielak-Orlicz sequence space l_Φ is rotund if and only if*

(i) $\Phi \in \delta_2$,
(ii) *each Φ_k vanishes only at zero, and*
(iii) *there exists at most one k such that $[0, \Phi_k^{-1}(\frac{1}{2})]$ contains an affine interval and if $[0, \Phi_{k_0}^{-1}(\frac{1}{2})]$ contains an affine interval $[a, b]$ for some k_0, then*

$[0, \Phi_k^{-1}(1 - \Phi_{k_0}(a))]$ does not contain any affine interval for any $k \neq k_0$, i.e., $[0, \Phi_k^{-1}(1 - \Phi_{k_0}(a))] \subset S_{\Phi_k}$ for every $k \neq k_0$.

Proof. Necessity. If (i) does not hold, then we can construct an element $x = (x_k)$ such that $\|x\| = 1$ but $I_\Phi(x) < 1$. By Theorem 2.1 x is not an extreme point.

If (ii) does not hold, then we can construct an element $x \in S(l_\Phi)$ which is not an extreme point. If (iii) does not hold, then we can construct an element $x \in S(l_\Phi)$ such that $\#\{k : x(k) \notin S_{\Phi_k}\} \geq 2$.

Sufficiency. It suffices to prove that $\#\{k : x(k) \notin S_{\Phi_k}\} \leq 1$ for any $x \in S(l_\Phi)$. From (i) we have $I_\Phi(x) = 1$. Then $\Phi_k(x(k)) > \frac{1}{2}$ for at most one k. By (iii) we conclude that x is an extreme point. □

Remark 4.1. (1) In [5], condition (iii) in Theorem 4.1 is replaced by

(iii′) there exists a sequence $\{a_k\} \subset [0, \infty)$ such that $\Phi_n(a_n) + \Phi_m(a_m) \geq 1$ for all $n \neq m$ and Φ_k is strictly convex on $[0, a_k]$ for all $k \in \mathbb{N}$.

(2) By Theorem 1.1, l_Φ is rotund if and only if $l_\Phi = h_\Phi$ and h_Φ is rotund.

(3) Observe that for every $x \in S(h_\Phi)$ we have $I_\Phi(x) = 1$. Therefore h_Φ is rotund if and only if (ii) and (iii) are satisfied.

Corollary 4.1. *The Nakano sequence space $l^{\{p_k\}}$ is rotund if and only if $\{p_k\}$ is bounded and $\#\{k : p_k = 1\} \leq 1$.*

Corollary 4.2. ([1, Theorem 2.7]) *The Orlicz sequence space l_M is rotund if and only if $M \in \delta_2$, M vanishes only at zero and M is strictly convex on $[0, M^{-1}(1/2)]$.*

Corollary 4.3. *Suppose that Φ satisfies the $(*)$-condition (see (1.9)). Then the Musielak-Orlicz sequence space l_Φ is strongly rotund if and only if it is rotund.*

Proof. The necessity of the condition is obvious. We prove the sufficiency. Let $x \in S(l_\Phi)$. By Theorem 4.1 and the definition of rotundity, we have $\Phi \in \delta_2$ and x is an extreme point. By Theorem 3.1, x is a strongly extreme point. □

Corollary 4.4. ([1, Theorem 2.30] and [4. Theorem 21]) *The rotundity and strong rotundity are equivalent in Orlicz sequence spaces and in Nakano sequence spaces.*

A point $x \in S(X)$ is called an *H-point* if for any sequence $(x_n) \subset X$, $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ we have $x_n \to x$.

Theorem 4.2. *Suppose that a Musielak-Orlicz function Φ satisfies the $(*)$-condition (see (1.9)) and each Φ_k vanishes only at zero, then $x \in S(l_\Phi)$ is an H-point if and only if $\Phi \in \delta_2$.*

Proof. Sufficiency. Suppose $\Phi \in \delta_2$. Let $(x_n) \subset l_\Phi$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$. Then $x_n \to x$ coordinatewise. From the proof of Theorem 3.1 we have $I_\Phi(x_n - x) \to 0$, which implies $\|x_n - x\| \to 0$.

Necessity. Suppose $x = (x(k))$ is an H-point, but $\Phi \notin \delta_2$. Then there exists an $x_0 = (x_0(k)) \in S(l_\Phi)$ such that $I_\Phi(x_0) \leq 1$ and $I_\Phi(\lambda x_0) = \infty$ for all $\lambda > 1$. Consequently, there is a sequence $i_1 < i_2 < \cdots$ such that

$$\|(0, 0, \ldots, x_0(i_n + 1), \ldots, x_0(i_{n+1}), 0, \ldots)\| \geq \frac{1}{2},$$

for all $n = 1, 2, \ldots$. Let

$$u_n = (x(1), \ldots, x(i_n), x(i_n + 1) - |x_0(i_n + 1)|(\mathrm{sgn} x(i_n + 1), \ldots, \\ x(i_{n+1}) - |x_0(i_{n+1})|(\mathrm{sgn} x(i_n + 1), x(i_{n+1} + 1), \ldots).$$

It was proved in [2] that $u_n \xrightarrow{w} x_0$ and $\|u_n - x_0\| \geq \frac{1}{2}$. Moreover,

$$\|x_0\| \leq \liminf_{n\to\infty} \|u_n\| \leq \limsup_{n\to\infty} \|u_n\| \leq \|x_0\|.$$

So $\|u_n\| \to 1$. This contradicts to the definition of an H-point. $\square$

Recall that a Banach space X is said to possess *property* (H) if every point in $S(X)$ is an H-point.

Corollary 4.5. ([2, Theorem 2]) *Suppose that a Musielak-Orlicz function Φ satisfies the $(*)$-condition (see (1.9)) and each Φ_k vanishes only at zero. Then the Musielak-Orlicz sequence space l_Φ possesses property (H) if and only if $\Phi \in \delta_2$.*

Corollary 4.6. ([4, Theorem 6]) *The Nakano sequence space $l^{\{p_k\}}$ possesses property (H) if and only if the sequence $\{p_k\}$ is bounded. In fact, $x \in S(l^{\{p_k\}})$ is an H-point if and only if the sequence $\{p_k\}$ is bounded.*

Corollary 4.7. ([1, Theorem 3.17, 3.18]) *Suppose that M vanishes only at zero. Then the Orlicz sequence space l_M possesses property (H) if and only if $M \in \delta_2$. Furthermore, if $M \notin \delta_2$ then $S(l_M)$ contains no H-points.*

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GENERALISED JORDAN-VON NEUMANN CONSTANTS AND UNIFORM NORMAL STRUCTURE

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We introduce a new geometric coefficient related to the Jordan-von Neumann constant. This leads to improved versions of known results and yields new ones on super-normal structure for Banach spaces.

1. INTRODUCTION

The notions of normal structure and uniform normal structure play an important role in metric fixed point theory (see Goebel and Kirk [10]). A number of Banach space properties have been shown to imply uniform normal structure. Some sufficient properties for a Banach space X to have uniform normal structure are:

(i) $J(X) < 3/2$ (see Gao and Lau [6]),
(ii) $R(X) > 0$ (see Gao [5]),
(iii) $C_{NJ}(X) < 5/4$ (see Kato, Maligranda and Takahashi [13]), and
(iv) X is a u-space, a class of spaces that includes uniformly convex spaces and uniformly smooth spaces (see Gao and Lau [6]).

Recently, Kirk and Sims [17] introduced a new variant, ϕ-uniform normal structure, which lies strictly between normal structure and uniform normal structure.

In this paper we introduce a parameterised coefficient $C_{NJ}(\cdot, X)$ generalising the Jordan-von Neumann constant $C_{NJ}(X)$. Utilising ultraproduct techniques, the coefficient $C_{NJ}(\cdot, X)$ enables us to establish new sufficient conditions for a Banach space to have uniform normal structure. To achieve this, we first show that the coefficients $C_{NJ}(\cdot, X)$ of the space X and $C_{NJ}(\cdot, \tilde{X})$ of its ultrapower $\tilde{X}$ coincide. From this and some other new results, which also improve the number appearing in property (iii) from $5/4$ to $(3+\sqrt{5})/4$, we can apply the powerful ultraproduct technique to show that X has uniform normal structure whenever $C_{NJ}(1, X) < 2$. An example of a Banach space X is given which has $C_{NJ}(1, X) < 2$ and hence uniform normal structure, but for which neither (i) or (iii) apply. An exact determination of the coefficient $C_{NJ}(\cdot, X)$ is obtained when X is a Hilbert space. More generally, a connection between $C_{NJ}(\cdot, X)$ and the modulus of convexity δ_X is established. Finally, we investigate the constants $C_{NJ}(\cdot, X)$ when X is a u-space. This leads to an alternative proof of (iv).

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2. Preliminaries

Throughout the paper we let X and X^* stand for a Banach space and its dual space, respectively. By a non-trivial Banach space X we shall mean that either X is a real space with dim X $\geqslant 2$, or a complex space with dim X $\geqslant 1$. We shall denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. For a sequence (x_n) in X, $x_n \overset{w}{\to} x$ stands for weak convergence to x. For $x \in X\backslash\{0\}$, let ∇_x denote the set of norm 1 supporting functionals at x. This is the subdifferential of the norm at the point x, which is nonempty by the Hanh-Banach Theorem.

We shall say that a nonempty weakly compact convex subset C of X has the *fixed point property* (*fpp* for short) if every nonexpansive mapping $T : C \to C$ has a fixed point (that is, there exists $x \in C$ such that $T(x) = x$). Recall that T is nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for every $x, y \in C$. We shall say that X has the *fixed point property* (*fpp*) if every weakly compact convex subset of X has the fpp. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y\in A} \|x - y\| : x \in A\}$ is called the *Chebyshev radius* of A. The number $\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$ is called the *diameter* of A. A Banach space X has *normal structure* if

$$r(A) < \operatorname{diam} A \tag{2.1}$$

for every bounded convex closed subset A of X with $\operatorname{diam} A > 0$. When (2.1) holds for every weakly compact convex subset A of X with $\operatorname{diam} A > 0$, we say X has *weak normal structure*. Normal structure and weak normal structure coincide if X is reflexive. A space X is said to have *uniform normal structure* if $\inf\left\{(\operatorname{diam} A)/(r(A))\right\} > 1$, where the infimum is taken over all bounded convex closed subsets A of X with diam $A > 0$. Weak normal structure, as well as many other properties imply the fixed point property. Some relevant papers are Opial [22], Kirk [16], Sims [24], Garcia-Falset [7], and Gacia-Falset and Sims [8].

The *modulus of convexity of* X (see [3, 4, 19, 20, 21]) is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| \geqslant \varepsilon\right\}. \tag{2.2}$$

When X is non-trivial, we can deduce that

$$\begin{aligned}
\delta_X(\varepsilon) &= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \geqslant \varepsilon\right\} \\
&= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| = \varepsilon\right\} \\
&= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| = \varepsilon\right\}.
\end{aligned}$$

If $\delta_X(1) > 0$, then X has uniform normal structure (see [9]).

The *modulus of smoothness* of X (see [3, 4, 19, 20]) is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(\tau) = \sup\Big\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in S_X\Big\} \tag{2.3}$$
$$= \sup\Big\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : \varepsilon \in [0,2]\Big\}.$$

A space X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon < 2$. It is called *uniformly smooth* if $\rho'_X(0) = \lim_{\tau\to 0}(\rho_X(\tau))/\tau = 0$. Uniformly convex spaces and uniformly smooth spaces are examples of u-spaces, where a space X is called a *u-space* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in S_X$,

$$\Big\|\frac{x+y}{2}\Big\| > 1 - \delta \Rightarrow f(y) > 1 - \epsilon \text{ for all } f \in \nabla_x. \tag{2.4}$$

The notion of u-spaces was introduced by Lau [18]. Examples of uniformly convex spaces are the spaces $L^p(\Omega)$ where Ω is a measure space such that $L^p(\Omega)$ is at least two dimensional and $1 < p < \infty$.

A Banach space X is called *uniformly nonsquare* provided that there exists $\delta > 0$ such that if $x, y \in S_X$, then $\|x+y\|/2 \leqslant 1-\delta$ or $\|x-y\|/2 \leqslant 1-\delta$. Uniformly nonsquare spaces are superreflexive (see James [11]). Every u-space is uniformly nonsquare (see Lau [18]), hence, it is superreflexive.

The Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of a Banach space X is defined by

$$C_{\mathrm{NJ}}(X) = \sup\Big\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero }\Big\} \tag{2.5}$$
$$= \sup\Big\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\Big\}.$$

REMARK 2.1. We collect together some properties of the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ (see [2, 12, 13, 14, 15, 25]):

(1) $1 \leqslant C_{\mathrm{NJ}}(X) \leqslant 2$.
(2) X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$.
(3) $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*)$.
(4) X is uniformly nonsquare if and only if $C_{\mathrm{NJ}}(X) < 2$ and this happens if and only if $\delta_X(\varepsilon) > 0$ for some $\varepsilon \in (0,2)$.
(5) If $C_{\mathrm{NJ}}(X) < 5/4$ then X, as well as its dual X^*, have uniform normal structure, and hence both X and X^* have the fixed point property.

One technique used in this paper is the "ultraproduct" technique. We refer to Askoy and Khamsi [1] and Sims [23] for a complete discussion on the topic. However, let us briefly recall the construction of an ultrapower of a Banach space X. As a first step we consider the space $l_\infty(X)$ consisting of all bounded sequences (x_n) of elements of X. The norm in $l_\infty(X)$ is given by the formula $\|(x_n)\| = \sup_{n\in\mathbb{N}} \|x_n\|$, where $\mathbb{N}$ is the set of positive integers. Now, let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. The set $\mathcal{N} = \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$ is a closed linear subspace of $l_\infty(X)$. Here, $\lim_{\mathcal{U}}$ stands for the limit over the ultrafilter $\mathcal{U}$. The *ultrapower* $\widetilde{X}$ of X with respect to $\mathcal{U}$ is defined to be the quotient space $l_\infty(X)/\mathcal{N}$. By $\widetilde{x}$ we denote the equivalent class of $x = (x_n)$. From the definition of the quotient norm, we can derive the following canonical formula $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$. Identifying an element $x \in X$ with the equivalence class of the constant sequence $(x, x, \dots)$, we can treat X as a subspace of $\widetilde{X}$. In what follow, we shall consider only non-trivial ultrafilters on the set of positive integers. Under this setting, the ultrapower $\widetilde{X}$ is finitely representable in X. Consequently, $\widetilde{X}$ inherits all finite-dimensional geometrical properties of X.

DEFINITION 2.2: Let $\mathcal{P}$ be a Banach space property. We say that a Banach space X has the property *super-$\mathcal{P}$* if every Banach space finitely representable in X has property $\mathcal{P}$.

THEOREM 2.3. (See [1, Theorem 3.5].) *Let X and Y be Banach spaces and suppose that Y is finitely representable in X. Then there is an ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ such that Y is isometrically isomorphic to a subspace of $\widetilde{X}$.*

We remark that when the property $\mathcal{P}$ is hereditary: that is, any subspace of a space with $\mathcal{P}$ also has $\mathcal{P}$, one has the following stronger conclusion.

COROLLARY 2.4. (See [1].) *Let $\mathcal{P}$ be a Banach space property which is inherited by subspaces. Then a Banach space X has super-$\mathcal{P}$ if and only if every ultrapower $\widetilde{X}$ of X has $\mathcal{P}$.*

THEOREM 2.5. (See [1].) *Let X be a Banach space. If X has super-normal structure, then X has uniform normal structure.*

3. RESULTS

Let us begin with our generalisation of the Jordan-von Neumann constant. For $a \geqslant 0$ define,

$$C_{NJ}(a, X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X \text{ not all zero} \right. \\ \left. \text{and } \|y - z\| \leqslant a\|x\|\right\}$$

$$= \sup \left\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in B_X \text{ not all zero} \right.$$
$$\left. \text{and } \|y - z\| \leqslant a\|x\| \right\}$$
$$= \sup \left\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in B_X \text{ of which at least one} \right.$$
$$\left. \text{belongs to } S_X \text{ and } \|y - z\| \leqslant a\|x\| \right\}.$$

REMARK 3.1.

(1) Obviously, $C_{NJ}(0, X) = C_{NJ}(X)$ (see (2.5)).
(2) $C_{NJ}(a, X)$ is a nondecreasing function with respect to a.
(3) If $C_{NJ}(a, X) < 2$, for some $a \geqslant 0$, then $C_{NJ}(X) < 2$ and consequently X is uniformly nonsquare (see Remark 2.1(4)).
(4) $1 + \left(4a/4 + a^2\right) \leqslant C_{NJ}(a, X) \leqslant 2$ for all $a \geqslant 0$ and $C_{NJ}(a, X) = 2$ for all $a \geqslant 2$.

To see that (4) is true, we begin by proving the left inequality. For this, we take any $x \in S_X$ and put $y = (a/2)x = -z$. We then have $y - z = ax$ and so,

$$C_{NJ}(a, X) \geqslant \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} = \frac{(1 + (a/2))^2\|x\|^2 + (1 + (a/2))^2\|x\|^2}{2\|x\|^2 + 2(a^2/4)\|x\|^2}$$
$$= \frac{2(1 + (a/2))^2}{2(1 + (a^2/4))} = \frac{4 + 4a + a^2}{4 + a^2} = 1 + \frac{4a}{4 + a^2}.$$

Next, we show that $C_{NJ}(a, X) \leqslant 2$. By the triangle inequality, we have

$$\begin{aligned} \|x+y\|^2 + \|x-z\|^2 &\leqslant \left(\|x\|^2 + 2\|x\|\|y\| + \|y\|^2\right) + \left(\|x\|^2 + 2\|x\|\|z\| + \|z\|^2\right) \\ &\leqslant \left(2\|x\|^2 + 2\|y\|^2\right) + \left(2\|x\|^2 + 2\|z\|^2\right) \\ &= 4\|x\|^2 + 2\|y\|^2 + 2\|z\|^2, \end{aligned}$$

from which it is clear that $C_{NJ}(a, X) \leqslant 2$. Finally, we observe that the function $a \mapsto 1 + \left(4a/4 + a^2\right)$ is strictly increasing on $[0, 2]$ and attains its maximum of 2 at $a = 2$. It follows that $C_{NJ}(a, X) = 2$ for all $a \geqslant 2$.

EXAMPLES 3.2. (1) ($l_\infty - l_1$ norm) Let $X = \mathbf{R}^2$ be equipped with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \geqslant 0, \\ \|x\|_1 & \text{if } x_1x_2 \leqslant 0. \end{cases}$$

Take $x = (1, 1), y = (0, 1)$ and $z = (-1, 0)$. Then we have $y - z = (1, 1) = x$ and $\|x + y\| = \|(1, 2)\|_\infty = 2$, $\|x - z\| = \|(2. 1)\|_\infty = 2, \|z\| = 1$. So $2 = (4 + 4)/4$

$= \left(\|x+y\|^2 + \|x-z\|^2\right)/\left(2\|x\|^2 + \|y\|^2 + \|z\|^2\right) \leqslant C_{\mathrm{NJ}}(1,X) \leqslant 2$. Hence $C_{\mathrm{NJ}}(1,X) = 2$. It is not difficult to see that $\delta_X(\varepsilon) = \max\{0, (\varepsilon-1)/2\}$ and so $\delta_X(1) = 0$. We shall shortly see (Remark 3.12(1)) that this implies $C_{\mathrm{NJ}}(0,X) \geqslant 5/4$, however, we do not know its exact value. This example shows that sometimes it is easy to compute $C_{\mathrm{NJ}}(a,X)$ at some point $a \in (0,2)$, but not at $a = 0$.

(2) Let $1 < p < 2$ and let the norm on $X = \mathbf{R}^2$ now be defined by

$$\|x\| = \begin{cases} \|x\|_1 & \text{if } x_1x_2 \geqslant 0, \\ \|x\|_p & \text{if } x_1x_2 \leqslant 0. \end{cases}$$

Under this norm, it can be shown that $\delta_X(1) = 0$, $C_{\mathrm{NJ}}(X) = 1 + 2^{2/p-2}$, $J(X) \geqslant 2^{1/p}$ and $C_{\mathrm{NJ}}(1,X) < 2$, where James' nonsquare constant $J(X)$ is defined by $J(X) = \sup\left\{\min\{\|x+y\|, \|x-y\} : x, y \in S_X\right\}$. The verification that $C_{\mathrm{NJ}}(1,X) < 2$ follows by an argument similar to that given later in the proof of Theorem 3.15. We shall shortly see that all spaces X with $C_{\mathrm{NJ}}(1,X) < 2$ have uniform normal structure (Corollary 3.7). This example also reveals that we may have $C_{\mathrm{NJ}}(X)$ close to 2 but still have uniform normal structure (also see the observation given later at the beginning of Remark 3.16).

These examples show that information on $C_{\mathrm{NJ}}(a,X)$ for general a proves to be useful. We note in passing that $C_{\mathrm{NJ}}\left(1, l_2(X)\right) < 2$ whenever $C_{\mathrm{NJ}}(1,X) < 2$, where $l_2(X)$ is the space of sequences (x_n) of elements of X for which the sequence of norms $\left(\|x_n\|\right)$ is in l_2, with the norm of (x_n) defined to be the l_2-norm of $\left(\|x_n\|\right)$.

We aim to show that the generalised Jordan-von Neumann constants $C_{\mathrm{NJ}}(a,X)$ of the space X and $C_{\mathrm{NJ}}\left(a,\widetilde{X}\right)$ of its ultrapower coincide. Before that we need to establish the continuity of the function $C_{\mathrm{NJ}}(\cdot,X)$.

Proposition 3.3. *$C_{\mathrm{NJ}}(\cdot,X)$ is a continuous function on $[0,\infty)$.*

Proof: We have already noted that $C_{\mathrm{NJ}}(\cdot,X)$ is nondecreasing, thus suppose that for some $a > 0$,

$$\sup_{b<a} C_{\mathrm{NJ}}(b,X) = \alpha < \beta < \gamma = \inf_{b>a} C_{\mathrm{NJ}}(b,X).$$

Choose $\gamma_n \downarrow a$ and $x_n, y_n, z_n \in B_X$ of which at least one belongs to S_X and such that $\|y_n - z_n\| = \gamma_n\|x_n\|$ and $g(x_n,y_n,z_n) \geqslant \beta$ for all $n \in \mathbb{N}$. Here $g(x,y,z) = \left(\|x+y\|^2 + \|x-z\|^2\right)/\left(2\|x\|^2 + \|y\|^2 + \|z\|^2\right)$. Choose $\eta_n \downarrow 1$ such that $\gamma_n/\eta_n < a$ for all n. Thus, $g(\eta_n x_n, y_n, z_n) = g\left(x_n, (y_n/\eta_n), (z_n/\eta_n)\right) \leqslant \alpha$ for all $n \in \mathbb{N}$. Take a subsequence (n') of (n) such that all the sequences

$$\|x_{n'} + y_{n'}\|, \|x_{n'} - z_{n'}\|, \|x_{n'}\|, \|y_{n'}\| \text{ and } \|z_{n'}\|$$

converge. As $\|x_n + w\| - (\eta_n - 1)\|x_n\| \leqslant \|\eta_n x_n + w\| \leqslant \|x_n + w\| + (\eta_n - 1)\|x_n\|$ for any $w \in X$ and $\eta_n \to 1$, we have $\lim_{n'} \|\eta_{n'}x_{n'} + y_{n'}\| = \lim_{n'} \|x_{n'} + y_{n'}\|$ and $\lim_{n'} \|\eta_{n'}x_{n'}$

$-z_{n'}\| = \lim_{n'} \|x_{n'} - z_{n'}\|$. Consequently, $\beta - \alpha \leqslant g(x_{n'}, y_{n'}, z_{n'}) - g(\eta_{n'}x_{n'}, y_{n'}, z_{n'}) \to 0$, a contradiction. This finishes the proof when $a > 0$.

For $a = 0$, given $\varepsilon > 0$ we take a triple (x_n, y_n, z_n) in B_X^3 with at least one of x_n, y_n, z_n belonging to S_X, $\|y_n - z_n\| = \alpha_n \|x_n\|$, $\alpha_n \downarrow 0$, and

$$C_{\mathrm{NJ}}(0+, X) - \varepsilon := \inf_{a>0} C_{\mathrm{NJ}}(a, X) - \varepsilon < \lim_{n\to\infty} g(x_n, y_n, z_n).$$

Put $\varepsilon_n = 4\alpha_n + \alpha_n^2$ and $\gamma_n = \alpha_n\|x_n\|(\|y_n\| - \alpha_n\|x_n\|)$. Thus $\varepsilon_n, \gamma_n \to 0$. Passing through subsequences if necessary, we may assume that $\lim_{n\to\infty} (\|x_n\|^2 + \|y_n\|^2) = b$ exists. By the choice of (x_n, y_n, z_n) we see that $b \neq 0$. Next we observe that, for all large n,

$$\begin{aligned} g(x_n, y_n, z_n) &\leqslant \frac{\|x_n + y_n\|^2 + \|x_n - y_n\|^2 + \varepsilon_n}{2\|x_n\|^2 + 2\|y_n\|^2 - \gamma_n} \\ &\leqslant g(x_n, y_n, y_n) + \frac{\varepsilon_n + \gamma_n g(x_n, y_n, y_n)}{2\|x_n\|^2 + 2\|y_n\|^2 - \gamma_n} \\ &\leqslant C_{\mathrm{NJ}}(X) + \frac{\varepsilon_n + \gamma_n C_{\mathrm{NJ}}(X)}{2\|x_n\|^2 + 2\|y_n\|^2 - \gamma_n}. \end{aligned}$$

Thus $C_{\mathrm{NJ}}(0+, X) - \varepsilon < C_{\mathrm{NJ}}(X) \leqslant C_{\mathrm{NJ}}(0+, X)$ for all $\varepsilon > 0$. Therefore $C_{\mathrm{NJ}}(0+, X) = C_{\mathrm{NJ}}(X)$ which implies that $C_{\mathrm{NJ}}(\cdot, X)$ is continuous at 0. Hence the continuity of $C_{\mathrm{NJ}}(\cdot, X)$ is established. $\square$

We are now ready to obtain an important tool.

COROLLARY 3.4. $C_{\mathrm{NJ}}(a, X) = C_{\mathrm{NJ}}(a. \widetilde{X})$.

PROOF: Clearly, $C_{\mathrm{NJ}}(a, X) \leqslant C_{\mathrm{NJ}}(a, \widetilde{X})$. To show $C_{\mathrm{NJ}}(a, X) \geqslant C_{\mathrm{NJ}}(a, \widetilde{X})$, let $\delta > 0$, $\alpha \in [0, a]$ and suppose $\widetilde{x}, \widetilde{y}, \widetilde{z} \in \widetilde{X}$ not all of which are zero and for which $\|\widetilde{y} - \widetilde{z}\| = \alpha\|\widetilde{x}\|$. If $\widetilde{x} = 0$, then $g(\widetilde{x}, \widetilde{y}, \widetilde{z}) = 1 \leqslant C_{\mathrm{NJ}}(a, X)$. If $\widetilde{x} \neq 0$, choose $\varepsilon > 0$ such that $\varepsilon < \delta\|\widetilde{x}\|$. Since

$$c := \frac{\|\widetilde{x} + \widetilde{y}\|^2 + \|\widetilde{x} - \widetilde{z}\|^2}{2\|\widetilde{x}\|^2 + \|\widetilde{y}\|^2 + \|\widetilde{z}\|^2} = \lim_{\mathcal{U}} \frac{\|x_n + y_n\|^2 + \|x_n - z_n\|^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} := \lim_{\mathcal{U}} c_n,$$

the set $\{n \in \mathbb{N} : |c_n - c| < \delta \text{ and } \|y_n - z_n\| \leqslant \alpha\|x_n\| + \varepsilon < (\alpha + \delta)\|x_n\|\}$ belongs to $\mathcal{U}$. In particular,

$$\begin{aligned} c &< g(x_n, y_n, z_n) + \delta \\ &\leqslant C_{\mathrm{NJ}}(a + \delta, X) + \delta \quad \text{for some } n. \end{aligned}$$

The inequality $C_{\mathrm{NJ}}(a, \widetilde{X}) \leqslant C_{\mathrm{NJ}}(a, X)$ follows from the arbitrariness of δ and the continuity of $C_{\mathrm{NJ}}(\cdot, X)$. $\square$

This result also follows from the fact that the parameterised Jordan-von Neumann constant is finitely determined.

The following Lemma is a modification of [6, Lemma 2.3].

Lemma 3.5. *Let X be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$ and each $1/2 < r \leqslant 1$, there exist $x_1 \in S_X$ and $x_2, x_3 \in rS_X$ satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a - r| < \varepsilon$,

(ii) $\|x_1 - x_2\| > 1 - \varepsilon$, *and*

(iii) $\|x_1 + x_2\| > (1 + r) - \varepsilon, \|x_3 + (-x_1)\| > (3r - 1) - \varepsilon$.

Proof: Put $\eta = \min\{(\varepsilon/12r), 2 - (1/r)\}$, and let z_n be a sequence in S_X with $z_n \xrightarrow{w} 0$ and

$$1 - \eta < \|z_{n+1} - z\| < 1 + \eta$$

for sufficiently large n and for any $z \in \operatorname{co}\{z_k\}_{k=1}^{n}$. Take $n_0 \in \mathbb{N}$, $y \in \operatorname{co}\{z_n\}_{n=1}^{n_0}$ and a norm 1 supporting functional f of z_1 such that

$$\|y\| < \eta, \ \left|\langle f, z_{n_0}\rangle\right| < \eta, \ 1 - \eta < \|z_{n_0} - z_1\|, \left\|z_{n_0} - \frac{z_1}{2}\right\| < 1 + \eta,$$

and

$$\left\|\frac{z_1 - z_{n_0}}{\|z_1 - z_{n_0}\|} - z_{n_0}\right\| > 2 - 3\eta.$$

Put $x_1 = (z_1 - z_{n_0})/(\|z_1 - z_{n_0}\|)$, $x_2 = rz_1$ and $x_3 = rz_{n_0}$. We show that (i), (ii) and (iii) hold. We first note that $x_2 - x_3 = r(z_1 - z_{n_0}) = r\|z_1 - z_{n_0}\|x_1$. Observe that $1 - \eta < \|z_1 - z_{n_0}\| < 1 + \eta$, so $\left|r\|z_1 - z_{n_0}\| - r\right| < r\eta < \varepsilon$, hence (i) holds. Next, since $1/2 < r \leqslant 1$,

$$\left|r(1 + \|z_1 - z_{n_0}\|) - 1\right| = r(1 + \|z_1 - z_{n_0}\|) - 1 < r(2 + \eta) - 1 = (2r - 1) + r\eta.$$

This implies

$$\begin{aligned}
\|x_1 - x_2\| &= \left\|rx_1 + (1 - r)x_1 - r\|z_1 - z_{n_0}\|x_1 - rz_{n_0}\right\| \\
&\geqslant r\|x_1 - z_{n_0}\| - \left|1 - r - r\|z_1 - z_{n_0}\|\right| \\
&> r(2 - 3\eta) - (2r - 1) - r\eta \\
&= 2r - 3r\eta - 2r + 1 - r\eta \\
&> 1 - \varepsilon.
\end{aligned}$$

Thus (ii) follows.

To verify (iii) we first note the estimate $\|rz_1 - rz_{n_0} - x_1\| = \Big\|(1-r)x_1 + r\big(x_1 - (z_1 - z_{n_0})\big)\Big\| \leqslant (1-r) + r\eta < (1-r) + r\eta$. Using this we have,

$$\begin{aligned}\|x_1 - x_3\| &= \|x_1 - rz_{n_0}\| \\ &\geqslant \|rz_{n_0} - (rz_1 - rz_{n_0})\| - \|rz_1 - rz_{n_0} - x_1\| \\ &\geqslant 2r\Big\|z_{n_0} - \frac{z_1}{2}\Big\| - (1-r) - r\eta \\ &> 2r - 2r\eta - (1-r) - r\eta \\ &> (3r-1) - \varepsilon.\end{aligned}$$

We now estimate $\|x_1 + x_2\|$. From the definition of f, we have

$$\begin{aligned}\|x_1 + x_2\| \geqslant \langle f, x_1 + rz_1\rangle &= r + \langle f, x_1\rangle \\ &= r + \frac{\langle f, z_1\rangle - \langle f, z_{n_0}\rangle}{\|z_1 - z_{n_0}\|} \\ &> r + \frac{1-\eta}{1+\eta} \\ &= (r+1) - \frac{2\eta}{1+\eta} \\ &> (r+1) - \varepsilon.\end{aligned}$$

The proof of the Lemma is now complete. □

We now obtain sufficient conditions for X to have uniform normal structure, the second of which improves [13, Corollary 4] which states that "A Banach space X with $C_{NJ}(X) < 5/4$ has uniform normal structure."

THEOREM 3.6. *Let X be a Banach space. If*

$$C_{NJ}(r, X) < \frac{(1+r)^2 + (3r-1)^2}{2(1+r^2)}, \quad \text{for some } r \in \left(\frac{1}{2}, 1\right].$$

or

$$C_{NJ}(0, X) < \frac{3+\sqrt{5}}{4},$$

then X has uniform normal structure.

PROOF: It suffices to show that these conditions imply X has normal structure. As then, by Corollary 3.4, it follows that $\tilde{X}$ also has normal structure, so X has super-normal structure, by Corollary 2.4, and hence X has uniform normal structure by Theorem 2.5.

For the case $C_{\mathrm{NJ}}(r,X) < \left((1+r)^2+(3r-1)^2\right)/\left(2(1+r^2)\right)$ we first observe that from Remark 3.1(3), X is uniformly nonsquare and so in turn is reflexive. Thus, normal structure and weak normal structure coincide. It then suffices to prove that X has weak normal structure.

By the continuity of $C_{\mathrm{NJ}}(\cdot,X)$, $C_{\mathrm{NJ}}(r',X) < \left((1+r)^2+(3r-1)^2\right)/\left(2(1+r^2)\right)$ for some $r' > r$. Choose $m \in \mathbb{N}$ such that $r + (1/m) \leqslant r'$. Suppose X does not have weak normal structure. By Lemma 3.5 there exist $x_n \in S_X$ and $y_n, z_n \in rS_X$ such that, for each $n \in \mathbb{N}$,

$$y_n - z_n = \alpha_n x_n \text{ with } |\alpha_n - r| < \frac{1}{n+m},$$
$$\|x_n - y_n\|^2 > \left(1 - \frac{1}{n+m}\right)^2, \ \|x_n + y_n\|^2 > \left(1 + r - \frac{1}{n+m}\right)^2,$$

and

$$\|x_n - z_n\|^2 > \left((3r-1) - \frac{1}{n+m}\right)^2.$$

Observe that $\|y_n - z_n\| = \alpha_n < r + (1/n+m) < r + (1/m) \leqslant r'$ and

$$\liminf_{n\to\infty} \|x_n + y_n\|^2 \geqslant (1+r)^2 \text{ and } \liminf_{n\to\infty} \|x_n - z_n\|^2 \geqslant (3r-1)^2.$$

Thus

$$\begin{aligned} \frac{(1+r)^2+(3r-1)^2}{2(1+r^2)} &\leqslant \liminf_{n\to\infty} \frac{\|x_n+y_n\|^2+\|x_n-z_n\|^2}{2\|x_n\|^2+\|y_n\|^2+\|z_n\|^2} \\ &\leqslant C_{\mathrm{NJ}}(r',X) \\ &< \frac{(1+r)^2+(3r-1)^2}{2(1+r^2)}. \end{aligned} \tag{3.1}$$

This contradiction shows that X must have weak normal structure as desired.

For the case $C_{\mathrm{NJ}}(0,X) < (3+\sqrt{5})/4$, we first show that $C_{\mathrm{NJ}}(0,X) < \left((1+r)^2+1\right)/\left(2(1+r^2)\right)$ for any $r \in (1/2,1]$. The proof of this is the same as above except that here we consider the lower bound $\left(1-(1/m+n)\right)^2$ for $\|x_n - y_n\|^2$ instead of the one for $\|x_n - z_n\|^2$. Thus (3.1) becomes

$$\frac{(1+r)^2+1}{2(1+r^2)} \leqslant \liminf_{n\to\infty} \frac{\|x_n+y_n\|^2+\|x_n-y_n\|^2}{2(\|x_n\|^2+\|y_n\|^2)} \leqslant C_{\mathrm{NJ}}(0,X) < \frac{(1+r)^2+1}{2(1+r^2)}$$

which is impossible. The conclusion now follows by noting that $\left((1+r)^2+1\right)/\left(2(1+r^2)\right)$ achieves a maximum of $(3+\sqrt{5})/4$ at $r = (\sqrt{5}-1)/2 \in (1/2,1]$. $\square$

NOTE. The restriction $r \in (1/2, 1]$ in the first inequality of Theorem 3.6 reflects the fact that for $r \leqslant 1/2$ the right hand side is less than or equal to one. Indeed, from Remark 3.1(4) the first inequality in Theorem 3.6 is only possible if

$$\frac{(1+r)^2 + (3r-1)^2}{2(1+r^2)} \geqslant 1 + \frac{4r}{4+r^2},$$

that is, if $r \in (r_1, 1]$ where $r_1 \doteq 0.87$ is the real root of the polynomial $2x^3 - 3x^2 + 8x - 6$. Thus, Theorem 3.6 only gives us information near $r = 1$.

COROLLARY 3.7. *Let X be a Banach space. If $C_{\mathrm{NJ}}(1, X) < 2$, then X has uniform normal structure.*

PROOF: This follows immediately from Theorem 3.6 with $r = 1$. □

Utilising Corollary 3.7, Tasena [26] has shown "$C_{\mathrm{NJ}}(a, X) < (1+a)^2/(1+a^2)$ for some $a \in (0, 1]$ implies X has uniform normal structure". This improvement of Theorem 3.6 is quite strong since

$$\frac{(1+a)^2}{1+a^2} > \max\left(1 + \frac{4a}{4+a^2}, \frac{(1+a)^2 + (3a-1)^2}{2(1+a^2)}\right) \text{ for } a \in (0, 1).$$

We now consider the case when X is a Hilbert space, thereby extending Remark 2.1(2).

THEOREM 3.8. *Let H be a Hilbert space. Then*

$$C_{\mathrm{NJ}}(a, H) = 1 + \frac{4a}{4+a^2}$$

for all $a \in [0, 2]$.

PROOF: Let $a \in [0, 2]$ and $x, y, z \in H$ with $x \neq 0$ and $\|y - z\| = \alpha\|x\|$ for some $\alpha \in [0, a]$. Then

$$\begin{aligned}
\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} &\leqslant \frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2\|x\|\|y-z\|}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \\
&\leqslant 1 + \frac{2\alpha\|x\|^2}{2\|x\|^2 + (\|y-z\|^2 + \|y+z\|^2)/2} \\
&\leqslant 1 + \frac{2\alpha\|x\|^2}{2\|x\|^2 + \|y-z\|^2/2} \\
&= 1 + \frac{4\alpha}{4+\alpha^2} \\
&\leqslant 1 + \frac{4a}{4+a^2}.
\end{aligned}$$

Thus, by Remark 3.1(4), $C_{\mathrm{NJ}}(a, H) = 1 + (4a)/(4+a^2)$. □

QUESTION. Is X a Hilbert space if $C_{\mathrm{NJ}}(a, X) = 1 + (4a)/(4 + a^2)$ for some $a \in (0, 2)$?

Theorem 3.8 and Corollary 3.7 give us the following

COROLLARY 3.9. *Every Hilbert space has uniform normal structure.*

We now give a connection between the constant $C_{\mathrm{NJ}}(\cdot, X)$ and the modulus of convexity $\delta_X(\cdot)$ (see (2.2)).

THEOREM 3.10. *Let X be a Banach space, $\varepsilon \in [0, 2]$, and $\beta \geqslant 0$. If $C_{\mathrm{NJ}}(\beta, X) < (4 + (\varepsilon - \beta)^2)/(3 + (\beta + 1)^2)$, then $\delta_X(\varepsilon) > 0$.*

PROOF: Suppose $\delta_X(\varepsilon) = 0$, then there exist $x_n, y_n \in S_X$ such that $\|x_n - y_n\| = \varepsilon$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Put $z_n = y_n - \beta x_n$. Then, for each $n \in \mathbb{N}$, we have $y_n - z_n = \beta x_n$, $\|z_n\| = \|y_n - \beta x_n\| \leqslant 1 + \beta$ and $\|x_n - z_n\| \geqslant \big|\|x_n - y_n\| - \|\beta x_n\|\big| = |\varepsilon - \beta|$. Thus

$$\frac{4 + (\varepsilon - \beta)^2}{3 + (\beta + 1)^2} \leqslant \liminf_{n\to\infty} \frac{\|x_n + y_n\|^2 + \|x_n - z_n\|^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \leqslant C_{\mathrm{NJ}}(\beta, X) < \frac{4 + (\varepsilon - \beta)^2}{3 + (\beta + 1)^2},$$

a contradiction. ∎

Note that Theorem 3.10 is applicable for all $\beta \in [0, \beta_1]$ where β_1 is the root of the equation

$$1 + \frac{4\beta}{4 + \beta^2} = \frac{4 - (\varepsilon - \beta)^2}{3 - (1 + \beta)^2}.$$

The above theorem immediately yields the following.

COROLLARY 3.11. *If, for $\varepsilon \in [0, 2]$, $C_{\mathrm{NJ}}(0, X) < (4 + \varepsilon^2)/4$, then $\delta_X(\varepsilon) > 0$. In particular, every Hilbert space is uniformly convex, that is, $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0, 2)$.*

REMARK 3.12.

(1) Corollary 3.11 shows that if $C_{\mathrm{NJ}}(X) < 5/4$, then $\delta_X(1) > 0$.

(2) $C_{\mathrm{NJ}}(0, X) < 2$ if and only if $C_{\mathrm{NJ}}(0, X) < (4 + \varepsilon^2)/4$ for some $\varepsilon \in (0, 2)$. Thus, this gives us a simpler proof of [**13**, Theorem 1] which states that "$C_{\mathrm{NJ}}(0, X) < 2$ if and only if X is uniformly nonsquare."

(3) Since $C_{\mathrm{NJ}}(0, X) = C_{\mathrm{NJ}}(0, X^*)$, the corresponding results in Theorem 3.6 and Corollary 3.11 hold for X^* as well.

QUESTION. Does the equality $C_{\mathrm{NJ}}(a, X) = C_{\mathrm{NJ}}(a, X^*)$ hold for $a \in (0, 2]$?

COROLLARY 3.13. *If $C_{\mathrm{NJ}}(\cdot, X)$ is concave and $C_{\mathrm{NJ}}(a, X) < (3 + \sqrt{5} + (5 - \sqrt{5})a)/4$ for some $a \in [0, 1]$, then X has uniform normal structure.*

PROOF: If $C_{\mathrm{NJ}}(1, X) < 2$, we are done by Corollary 3.7. Let $C_{\mathrm{NJ}}(1, X) = 2$ and suppose that X does not have uniform normal structure. Therefore $C_{\mathrm{NJ}}(0, X)$

$\geqslant (3+\sqrt{5})/4$ by Theorem 3.6. By the concavity of $C_{\mathrm{NJ}}(\cdot, X)$, we have for all a $\in [0,1]$,

$$C_{\mathrm{NJ}}(a,X) \geqslant (1-a)C_{\mathrm{NJ}}(0,X) + aC_{\mathrm{NJ}}(1,X) \geqslant \frac{3+\sqrt{5}+\left(5-\sqrt{5}\right)a}{4},$$

a contradiction. □

QUESTION. Is Corollary 3.13 still valid if we drop the assuption of concavity?

REMARK 3.14. In the definition of a u-space (see (2.4)), we can replace x, y in S_X by $x, y \in B_X$. To see this, we first observe that, $\|x\| \geqslant \|x+y\| - \|y\|$. Thus,

$$\text{(3.2)} \qquad \begin{aligned} &\text{if } x, y \in B_X \text{ and } \left\|\frac{x+y}{2}\right\| > 1-\delta \text{ for some } \delta > 0, \\ &\text{then } \|x\| \geqslant 1-2\delta \text{ and } \|y\| \geqslant 1-2\delta. \end{aligned}$$

From (3.2) if we put $x' = x/\|x\|$ and $y' = y/\|y\|$ we obtain

$$\text{(3.3)} \qquad \left\|\frac{x'+y'}{2}\right\| > 1-3\delta, \text{ whenever } \left\|\frac{x+y}{2}\right\| > 1-\delta.$$

Indeed, (3.3) follows from the fact that $\|x'-x\| < 2\delta$ and $\|y'-y\| < 2\delta$, together with the inequality

$$\|x'+y'\| \geqslant \|x+y\| - \|x'-x\| - \|y'-y\|.$$

Now, given any $\varepsilon > 0$, choose $\delta \in \left(0, (3\varepsilon)/4\right)$ so that for $x', y' \in S_X$,

$$\left\|\frac{x'+y'}{2}\right\| > 1-\delta \Rightarrow f(y') > 1-\frac{\varepsilon}{2} \text{ for all } f \in \nabla_{x'}.$$

Then, if $x, y \in B_X$, and $\|(x+y)/2\| > 1-(\delta/3)$, (3.3) implies that $\|(x'+y')/2\| > 1-\delta$ where $x' = x/\|x\|$ and $y' = y/\|y\|$. Note. by (3.2), that $\|y'-y\| < (2\delta)/3$. Fix $f \in \nabla_x = \nabla_{x'}$ and consider the inequalities

$$f(y)+\frac{\varepsilon}{2} > f(y)+\frac{2\delta}{3} \geqslant f(y)+\|y'-y\| \geqslant f(y)+f(y'-y) = f(y') > 1-\frac{\varepsilon}{2}.$$

Consequently, $f(y) > 1-\varepsilon$ as required.

THEOREM 3.15. *For $1 < p < \infty$, all $L^p(\Omega)$ spaces satisfy $C_{\mathrm{NJ}}\left(1, L^p(\Omega)\right) < 2$. Indeed, all u-spaces X have $C_{\mathrm{NJ}}(a,X) < 2$ for all $0 < a < 2$.*

PROOF: Suppose $C_{\mathrm{NJ}}(2-\delta, X) = 2$ for all sufficiently small $\delta > 0$. For one such δ choose $x_n, y_n, z_n \in B_X$ of which at least one belongs to S_X and such that

$\|y_n - z_n\| \leqslant (2-\delta)\|x_n\|$ for each n and $g(x_n, y_n, z_n) \nearrow 2$. Consider

$$g(x,y,z) = \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \tag{3.4}$$
$$\leqslant \frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2(\|x\|\|y\| + \|x\|\|z\|)}{2\|x\|^2 + \|y\|^2 + \|z\|^2}$$
$$= 1 + \frac{2(\|x\|\|y\| + \|x\|\|z\|)}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \leqslant 2.$$

This implies

$$\frac{2\|x_n\|\|y_n\| + 2\|x_n\|\|z_n\|}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \to 1$$

and then

$$\frac{(\|x_n\| - \|y_n\|)^2 + (\|x_n\| - \|z_n\|)^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \to 0.$$

Since, for each n, one of x_n, y_n, z_n belongs to S_X, we must have $\|x_{n'}\|, \|y_{n'}\|, \|z_{n'}\| \to 1$ for some subsequence (n') of (n). From this, together with (3.4), one can conclude that

$$\|x_{n'} + y_{n'}\|, \|x_{n'} - z_{n'}\| \to 2. \tag{3.5}$$

Take $f_{n'} \in \nabla_{x_{n'}}$ for each n. Since X is a u-space, we have, by (3.5) and (2.4), $f_{n'}(x_{n'} - y_{n'}) \to 0$ and $f_{n'}(x_{n'} + z_{n'}) \to 0$. Therefore,

$$\begin{aligned} 2\|x_{n'}\| = 2f_{n'}(x_{n'}) &= f_{n'}(x_{n'} - y_{n'}) + f_{n'}(x_{n'} + z_{n'}) + f_{n'}(y_{n'} - z_{n'}) \\ &\leqslant f_{n'}(x_{n'} - y_{n'}) + f_{n'}(x_{n'} + z_{n'}) + \|y_{n'} - z_{n'}\| \\ &\leqslant f_{n'}(x_{n'} - y_{n'}) + f_{n'}(x_{n'} + z_{n'}) + 2 - \delta. \end{aligned}$$

Thus, $2 \leqslant 2 - \delta$ a contradiction. □

REMARK 3.16.

(1) In [2], it is shown that $C_{\mathrm{NJ}}(L^p) = 2^{(2/t)-1}$, for $1 \leqslant p \leqslant \infty$, where $t = \min\{p, q\}$ and $(1/p) + (1/q) = 1$. Thus, while $C_{\mathrm{NJ}}(L^p)$ is close to 2 for p large, or near 1, Theorem 3.15 still applies and says that for $1 < p < \infty$, all L^p spaces have uniform normal structure.

(2) As a measure of uniform nonsquareness, we say X is *ϵ-inquadrate* (ϵ-InQ), for $0 \leqslant \epsilon \leqslant 2$, if for any sequences $(x_n), (y_n)$ in B_X,

$$\|x_n + y_n\| \to 2 \text{ implies } \limsup_{n\to\infty} \|x_n - y_n\| \leqslant \epsilon.$$

In [26], Tasena introduces ε-u-spaces and ε-u-smooth spaces and proves that "all ε-u-spaces have $C_{\mathrm{NJ}}(2-\delta, X) < 2$ for all $\delta > 2\varepsilon$". He also observes that $\varepsilon - InQ$ spaces are ε-u-spaces.

(3) A long standing open problem is whether $C_{\mathrm{NJ}}(0, X) < 2$ implies the fixed point property. It now appears that $C_{\mathrm{NJ}}(1, X) < 2$ implies uniform normal structure which in turn implies the fpp. Concerning this open problem, it is interesting to ask what is the smallest $a \in (0,1)$ for which the fpp follows whenever $C_{\mathrm{NJ}}(a, X) < 2$.

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Appendix 3: On some local geometry of Musielak-Orlicz sequence spaces, Commentationes Math. Prace. Mat. (already appeared)

ON SOME LOCAL GEOMETRY OF MUSIELAK-ORLICZ SEQUENCE SPACES*

S. DHOMPONGSA and S. SAEJUNG

Abstract

Criterions for strong U-points, LUR-points, WLUR-points, CLUR-points and WCLUR-points in Musielak-Orlicz sequence spaces endowed with the Luxemburg norm are given.

Key words and phrases: strong U-point, LUR-point, WLUR-point, CLUR-point, WCLUR-point and Musielak-Orlicz sequence space

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1 Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers, and the set of real numbers, respectively. For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and the closed unit ball of X, respectively. A point $x \in S(X)$ is called an *extreme point* if $2x = y + z$ and $y, z \in B(X)$ imply $y = z$. A Banach space X is said to be *rotund* if every point in its unit sphere is an extreme point. A point $x \in S(X)$ is called a *strong U-point* if $\|\frac{x+y}{2}\| = \|y\| = 1$ implies $x = y$. It is easy to see that every strong U-point is an extreme point. But the converse is not true (see Theorem 2.1 and Example 2.3 to follow). This also follows from Cui, Hudzik and Meng [2] and Grząślewicz, Hudzik and Kurc [5]. However, if X is rotund, both notion coincide.

A point $x \in S(X)$ is called an *H-point* if for any sequence $\{x_n\}$ in $S(X)$ with $x_n \xrightarrow{w} x$ implies $x_n \to x$. A point $x \in S(X)$ is called a *locally uniformly rotund point* (LUR-point, for short) if for any sequence $\{x_n\}$ in $S(X)$ such that $\|x + x_n\| \to 2$ implies

$$(1.1) \qquad x_n \to x.$$

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If (1.1) is replaced by $x_n \xrightarrow{w} x$, we call x a *weakly locally uniformly rotund point* (WLUR-point). Again, if (1.1) is replaced by $\{x_n : n \in \mathbb{N}\}$ is relatively compact with respect to the norm topology (weak topology, resp.) in $S(X)$, we call x a *compactly locally uniformly rotund point* (*weakly compactly locally uniformly rotund point*, resp.) (CLUR-point, WCLUR-point, resp.). It is easy to see that $x \in S(X)$ is an LUR-point if and only if it is a strong U-point and a CLUR-point, if and only if it is an H-point and a WLUR-point (see [2]). We say that a Banach space X has one of the above properties if every point of $S(X)$ has the same property. For instance, X is locally uniformly rotund (LUR) if and only if every point of $S(X)$ is an LUR-point. From the above observation, we can conclude that, X is LUR if and only if it is rotund and CLUR, if and only if it is WCLUR and has property (H).

In this paper we follow the "local" approach to the geometry of Musielak-Orlicz sequence spaces. We are mainly interested in properties of the unit sphere instead of properties of the whole unit ball. More precisely, we shall investigate those points that appear to be strong U-points, LUR-points, WLUR-points, CLUR-points and WCLUR-points.

Since every Orlicz space is an example of a Musielak-Orlicz space, this paper contains all results in [2] even without assuming that the corresponding Orlicz function is an N-function.

A function $\Phi : \mathbb{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ vanishes at zero and Φ is even, convex and not identically equal to zero. A sequence $\Phi = (\Phi_i)$ of Orlicz functions Φ_i is called a *Musielak-Orlicz function*. In addition, a function $\Psi = (\Psi_i)$ is called a *complementary function* of a Musielak-Orlicz function Φ in the sense of Young if

$$\Psi_i(v) := \sup\{|v|u - \Phi_i(u) : u \geq 0\},$$

$i \in \mathbb{N}$. Denote by l the space of all real sequences $x = (x(i))$. For a given Musielak-Orlicz function Φ, we define a *convex modular* $I_\Phi : l \to [0, \infty]$ by the formula

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x(i)).$$

The *Musielak-Orlicz sequence space* l_Φ is the space

$$l_\Phi := \{x \in l : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

We consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\}.$$

To simplify notation, we put $l_\Phi := (l_\Phi, \|\cdot\|)$. Moreover, l_Φ is a Banach space (see [8]).

The subspace h_Φ, called *the space of finite* (or *order continuous*) *elements*, is defined by

$$h_\Phi := \{x \in l_\Phi : I_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

Define

$$\theta(x) = \inf\{\lambda > 0 : I_\Phi(x/\lambda) < \infty\}.$$

It is clear that $x \in h_\Phi$ if and only if $\theta(x) = 0$.

We say a Musielak-Orlicz function Φ *satisfies the δ_2-condition* ($\Phi \in \delta_2$) if there exist constants $K \geq 2$, $u_0 > 0$ and a sequence (c_i) of positive numbers such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \leq K\Phi_i(u) + c_i$$

holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_i(u) \leq u_0$.

It is well known that $h_\Phi = l_\Phi$ if and only if $\Phi \in \delta_2$ (see [6]).

Moreover, we say a Musielak-Orlicz function Φ *satisfies the $(*)$-condition* if for any $\varepsilon \in (0,1)$ there exists a $\delta > 0$ such that, for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, $\Phi_i((1+\delta)u) \leq 1$ whenever $\Phi_i(u) \leq 1 - \varepsilon$ (see [7]).

In order to obtain some results, we will use the following well-known facts.

Lemma 1.1 (See [7]) *If a Musielak-Orlicz function $\Phi = (\Phi_i)$ satisfies the $(*)$-condition, $\Phi \in \delta_2$, and each Φ_i vanishes only at zero, then*

(i) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|I_\Phi(x) - I_\Phi(y)| < \varepsilon$$

whenever $I_\Phi(x) \leq 1$, $I_\Phi(y) \leq 1$ and $I_\Phi(x-y) < \delta$,

and

(ii) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that $I_\Phi(x) \geq 1 - \varepsilon$ whenever $\|x\| \geq 1 - \delta$.*

Lemma 1.2 (See [3, Lemma 3]) *If $\Psi \in \delta_2$, then there exist $\theta \in (0,1)$ and a sequence $(h_i) \subset \mathbb{R}_+$ such that $\sum_{i=1}^{\infty} \Phi_i(h_i) < \infty$ and*

$$\Phi_i\left(\frac{u}{2}\right) \leq \frac{1-\theta}{2}\Phi_i(u)$$

for every $i \in \mathbb{N}$ and u satisfying $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$.

Lemma 1.3 (See [10, Lemma 10]) *If $\Psi \notin \delta_2$ then there exist an increasing sequence of natural numbers $0 = I_0 < I_1 < \cdots$ and a set $\{u_i^k\}$ of positive numbers such that*

$$\Phi_i(u_i^k) \leq \frac{1}{k}, \qquad \Phi_i\left(\frac{u_i^k}{2}\right) > \left(1 - \frac{1}{k}\right)\frac{\Phi_i(u_i^k)}{2}$$

for all $i = I_{k-1}+1, \ldots, I_k$, and

$$\sum_{i=I_{k-1}+1}^{I_k} \Phi_i(u_i^k) > 1$$

for all $k \in \mathbb{N}$.

Lemma 1.4 (See [9]) *Suppose that a Musielak-Orlicz function $\Phi = (\Phi_i)$ satisfies the $(*)$-condition, $\Phi \in \delta_2$, and each Φ_i vanishes only at zero. If $\{x_n\}$ is a sequence in $B(l_\Phi)$ such that $\|x_n\| \to 1$ and $x_n \to x$ coordinatewise where $x \in S(l_\Phi)$, then $x_n \to x$ in norm.*

2 Results

An interval $[a, b]$ is called a *structural affine interval* (SAI) of an Orlicz function M if M is affine on $[a, b]$, i.e.

$$M(\lambda a + (1 - \lambda)b) = \lambda M(a) + (1 - \lambda)M(b)$$

for all $\lambda \in [0, 1]$, but not affine either on $[a - \varepsilon, b]$ or $[a, b + \varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_n, b_n]\}_n$ be the set of all SAIs of M. Define

$$SC_M = \mathbb{R} \setminus \cup_n (a_n, b_n).$$

We also define

$$\begin{aligned} SC_M^- &= \{u \in SC_M : \text{ there exists } \varepsilon > 0 \text{ such that } M \text{ is affine on} [|u| - \varepsilon, |u|]\}, \\ SC_M^+ &= \{u \in SC_M : \text{ there exists } \varepsilon > 0 \text{ such that } M \text{ is affine on} [|u|, |u| + \varepsilon]\}, \end{aligned}$$

and

$$SC_M^0 = SC_M \setminus (SC_M^- \cup SC_M^+).$$

Let $a_M = \sup\{u \in \mathbb{R} : M(u) = 0\}$ for any Orlicz function M.

Our first result is a generalization of Theorem 5 from [2] from Orlicz spaces into Musielak-Orlicz spaces. However, we do not assume about Φ that they are N-functions as it was done in [2].

Theorem 2.1 *Let $x = (x(i)) \in S(l_\Phi)$. Then x is a strong U-point if and only if*

(i) $I_\Phi(x) = 1$,

(ii) *there do not exist index i such that $a_{\Phi_i} > 0$ and $|x(i)| \in [0, a_{\Phi_i}]$,*

(iii) $\theta(x) < 1$,

(iv) *if $x(i) \in SC_{\Phi_i}$ for all i, then there do not exist two distinct indices j, k such that $x(j) \in SC_{\Phi_j}^+$ and $0 \neq x(k) \in SC_{\Phi_k}^-$,*
and

(v) *if $x(i_0) \notin SC_{\Phi_{i_0}}$ for some i_0, then $x(i) \in SC_{\Phi_i}^0$ for all $i \neq i_0$.*

Proof. *Necessity.* Since every strong U-point is an extreme point, (i) is satisfied (see [9]). It is easy to see that (ii) is also satisfied.

Let us prove (iii). Suppose that condition (iii) is not satisfied, i.e. $\theta(x) = 1$. Thus $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$. Without loss of generality, we may assume that $x(1) \neq 0$. Define

$$y = (0, x(2), x(3), \ldots).$$

Obviously $x \neq y$. By the monotonicity of the norm and the equalities $\theta(y) = \theta(\frac{x+y}{2}) = 1$, we have $\|y\| = \|\frac{x+y}{2}\| = 1$. Hence x can not be a strong U-point.

Next, we suppose that (iv) does not hold. Then $x(i) \in SC_{\Phi_i}$ for all $i \in \mathbb{N}$ but there exist two distinct indices j, k such that $x(j) \in SC^+_{\Phi_j}$ and $0 \neq x(k) \in SC^-_{\Phi_k}$. For convenience, we may assume that $j = 1, k = 2$. Choose $\varepsilon > 0$ such that

$$\begin{aligned}
\Phi_1(u) &= A_1 u + B_1 \quad \text{for } u \in [|x(1)|, |x(1)| + \varepsilon] \quad \text{and} \\
\Phi_2(u) &= A_2 u + B_2 \quad \text{for } u \in [|x(2)| - \varepsilon, |x(2)|].
\end{aligned}$$

Choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\max\{\varepsilon_1, \varepsilon_2\} < \varepsilon$ and $A_1\varepsilon_1 = A_2\varepsilon_2$. Define

$$y = ((|x(1)| + \varepsilon_1)\operatorname{sgn} x(1), (|x(2)| - \varepsilon_2)\operatorname{sgn} x(2), x(3), x(4), \ldots).$$

Then

$$\begin{aligned}
I_\Phi(y) &= \Phi_1(|x(1)| + \varepsilon_1) + \Phi_2(|x(2)| - \varepsilon_2) + \sum_{k=3}^{\infty} \Phi_k(x(k)) \\
&= A_1(|x(1)| + \varepsilon_1) + B_1 + A_2(|x(2)| - \varepsilon_2) + B_2 + \sum_{k=3}^{\infty} \Phi_k(x(k)) \\
&= \Phi_1(x(1)) + \Phi_2(x(2)) + \sum_{k=3}^{\infty} \Phi_k(x(k)) \\
&= I_\Phi(x) = 1.
\end{aligned}$$

Similarly, we have $I_\Phi\left(\frac{x+y}{2}\right) = 1$. Hence $\|y\| = \|\frac{x+y}{2}\| = 1$ and $x \neq y$. This is a contradiction.

Condition (v) can be proved as condition (iv).

Sufficiency. Let $x \in S(l_\Phi)$ and conditions (i)-(v) hold. Take any $y \in S(l_\Phi)$ with $\|x + y\| = 2$. We put $\left(0, \frac{1}{\theta(x)}\right) = (0, \infty)$ if $\theta(x) = 0$. Observe that $\lambda \mapsto I_\Phi(\lambda x)$ is continuous on $\left(0, \frac{1}{\theta(x)}\right)$. Let $\varepsilon \in \left(0, \frac{1-\theta(x)}{1+\theta(x)}\right)$. Then $\frac{1-\varepsilon}{1+\varepsilon} > \theta(x)$ and so $I_\Phi\left(\frac{1+\varepsilon}{1-\varepsilon}x\right) < \infty$. Moreover,

$$\begin{aligned}
1 &\leq I_\Phi\left((1+\varepsilon)\frac{x+y}{2}\right) \\
&= I_\Phi\left(\frac{1-\varepsilon}{2}\frac{1+\varepsilon}{1-\varepsilon}x + \frac{1+\varepsilon}{2}y\right) \\
&\leq \frac{1-\varepsilon}{2} I_\Phi\left(\frac{1+\varepsilon}{1-\varepsilon}x\right) + \frac{1+\varepsilon}{2} I_\Phi(y).
\end{aligned}$$

Letting $\varepsilon \searrow 0$ yields $I_\Phi(y) = 1$. Since $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$ and the norm is convex,

$$\left\| \frac{1}{2}\left(\frac{x+y}{2} + x\right) \right\| = 1.$$

Thus, the procedure can be repeated by replacing y by $\frac{x+y}{2}$ and obtaining $I_\Phi(\frac{1}{2}(\frac{x+y}{2}+x)) = 1$ as well. Consequently, for each $i \in \mathbb{N}$, either $x(i)$ and $y(i)$ fall in the same SAI or $x(i) = y(i)$.

We have two cases to consider, namely, the occurrence of conditions in (iv) and (v). First, let us assume that $x(i) \in SC'_{\Phi_i}$ for all $i \in \mathbb{N}$. We may assume that $x(i) \geq 0$ for all $i \in \mathbb{N}$. There are two subcases to consider:

Subcase 1: $0 \neq x(i) \in SC'_{\Phi_i}$ or $x(i) \in SC^0_{\Phi_i}$ for all $i \in \mathbb{N}$.

Let $A = \{i \in \mathbb{N} : 0 \neq x(i) \in SC'_{\Phi_i}\}$. Then, for each $i \in A$, there are constants $\varepsilon_i > 0$, $A_i, B_i \in \mathbb{R}$ such that

$$\Phi_i(u) = A_i u + B_i \quad \text{for} \quad u \in [x(i) - \varepsilon_i, x(i)] \quad \text{and} \quad y(i) \in [x(i) - \varepsilon_i, x(i)].$$

Observe that $x(i) > a_{\Phi_i}$ if $i \in A$. Hence $A_i > 0$ for all $i \in A$. Since $x(i) = y(i)$ for all $i \in \mathbb{N} \setminus A$ and $I_\Phi(\frac{x+y}{2}) = I_\Phi(x) = 1$,

$$\begin{aligned}
\sum_{i \in A} \left(A_i \frac{x(i) + y(i)}{2} + B_i \right) &= \sum_{i \in A} \Phi_i \left(\frac{x(i) + y(i)}{2} \right) \\
&= \sum_{i \in A} \Phi_i(x(i)) \\
&= \sum_{i \in A} (A_i x(i) + B_i).
\end{aligned}$$

and thus

$$\sum_{i \in A} \left(A_i \frac{x(i) - y(i)}{2} \right) = 0.$$

Since $A_i > 0$ and $x(i) \geq y(i)$,

$$x(i) = y(i)$$

for all $i \in A$. This implies that $x(i) = y(i)$ for all $i \in \mathbb{N}$, and so $x = y$.

Subcase 2: $x(i) \in SC^+_{\Phi_i}$ or $x(i) = 0$ or $x(i) \notin SC_{\Phi_i}$ for all $i \in \mathbb{N}$. If $x(i) \neq 0$, then $x(i) \in SC^+_{\Phi_i} \cup SC^0_{\Phi_i}$. On the other hand, if $x(i) = 0$, we see by condition (ii) that $a_{\Phi_i} = 0$, from which we can conclude that $x(i) \in SC^+_{\Phi_i} \cup SC^0_{\Phi_i}$ as well. Hence, in any case, either $x(i) \in SC^+_{\Phi_i}$ or $x(i) \in SC^0_{\Phi_i}$ holds. Let $B = \{i \in \mathbb{N} : x(i) \in SC^+_{\Phi_i}\}$. Similarly, we conclude that $x(i) \leq y(i)$ for all $i \in B$ and $x(i) = y(i)$ for all $i \in \mathbb{N} \setminus B$. Then, for each $i \in B$, there are constants $A_i, \varepsilon_i > 0$, $B_i \in \mathbb{R}$ such that

$$\Phi_i(u) = A_i u + B_i \quad \text{for} \quad u \in [x(i), x(i) + \varepsilon_i] \quad \text{and} \quad y(i) \in [x(i), x(i) + \varepsilon_i].$$

We again have

$$\sum_{i\in B}\left(A_i\frac{x(i)+y(i)}{2}+B_i\right) = \sum_{i\in B}(A_i x(i)+B_i).$$

This implies that $x(i)=y(i)$ for all $i\in B$. Hence $x=y$.

Finally, suppose that $x(i_0)\not\in SC_{\Phi_{i_0}}$ and $x(i)\in SC^0_{\Phi_i}$ for all $i\neq i_0$. We can conclude that $x(i)=y(i)$ for all $i\neq i_0$ and hence $\Phi_{i_0}\left(\frac{x(i_0)+y(i_0)}{2}\right)=\Phi_{i_0}(x(i_0))$. From $x(i_0)\not\in SC_{\Phi_{i_0}}$ and the condition (ii), we must have $x(i_0)=0=a_{\Phi_{i_0}}$ and hence $y(i_0)=0$. The proof is now complete. $\square$

Recall that a *Nakano sequence space* $l^{(p_i)}$ is a Musielak-Orlicz sequence space with

$$\Phi_i(u)=|u|^{p_i}$$

where $1\le p_i<\infty$.

Corollary 2.2 *Let $x=(x(i))\in S(l^{(p_i)})$. Then x is a strong U-point if and only if*

(i) $\sum_{i=1}^{\infty}|x(i)|^{p_i}=1$,

(ii) $\theta(x)<1$, *and*

(iii) *if there exists i_0 such that $x(i_0)\neq 0$ and $p_{i_0}=1$, then $p_i>1$ for all $i\neq i_0$.*

Example 2.3 We consider a Nakano sequence space $l^{(i^2)}$. Recall that, in this case, $\Phi_i(u)=|u|^{i^2}$. Let $x=(x(i))$, where $x(i)=(1/2)^{1/i}$. It is easy to see that x is an extreme point. But $I_\Phi(\lambda x)=\sum_{i=1}^{\infty}\frac{\lambda^{i^2}}{2^i}=\sum_{i=1}^{\infty}(\frac{\lambda^i}{2})^i=\infty$ for any $\lambda>1$, so $\theta(x)=1$. Hence x is not a strong U-point.

Theorem 2.4 *Suppose that Φ satisfies the $(*)$-condition and each Φ_i vanishes only at zero. Then the following statements are equivalent for $x=(x(i))\in S(l_\Phi)$:*

(1) *x is a CLUR-point;*

(2) *x is a WCLUR-point;*

(3) $\Phi\in\delta_2$ *and either* $\Psi\in\delta_2$ *or* $x(i)\in\{0\}\cup SC_{\Phi_i}\setminus SC^-_{\Phi_i}$ *for every* $i\in\mathbb{N}$.

Proof. Clearly, (1)$\Rightarrow$(2). To prove (2)$\Rightarrow$(1), it suffices to prove that (2) implies that l_Φ has property (H), i.e. $\Phi\in\delta_2$ (see [9]). First, we prove that $\theta(x)<1$ whenever x is a WCLUR-point. Otherwise, we have $I_\Phi(x)\le 1$ and $I_\Phi(\lambda x)=\infty$ for any $\lambda>1$. Let $\{x_n\}$ be a sequence in $S(l_\Phi)$ defined by

$$x_n=\sum_{i=n}^{\infty}x(i)e_i.$$

It is easy to see that $\|x_n + x\| = 2$ for all $n \in \mathbb{N}$. Since x is a WCLUR-point, we may assume that there exists an $x' \in S(l_\Phi)$ such that $x_n \xrightarrow{w} x'$. This implies $x' = 0$, because weak convergence implies coordinatewise convergence and $\{x_n\}$ tends to zero coordinatewise, which is impossible. So, the necessity of $\theta(x) < 1$ is proved. We now prove that $\Phi \in \delta_2$ is necessary. Suppose that $\Phi \notin \delta_2$, then there exists $u = (u'(i)\mathrm{sgn}x(i))$ such that $u'(i) \geq 0$, $I_\Phi(u) \leq 1$ and $I_\Phi(\lambda u) = \infty$ for any $\lambda > 1$ (see[6]). Since $\theta(x) < 1$,

$$I_\Phi(\alpha x) < \infty$$

for some $\alpha > 1$. For each $n \in \mathbb{N}$, we define the sequence $\{x_n\}$ by

$$x_n = x + \sum_{i=n+1}^{\infty} \varepsilon_0 u(i) e_i$$

where $\varepsilon_0 = 1 - \frac{1}{\alpha}$.

Next, we prove that $\|x_n\| \to 1$. For this let $\varepsilon \in (0,1)$. Setting $\lambda = 1 + \varepsilon$, we have $\frac{1}{\lambda\alpha} + \frac{\varepsilon_0}{\lambda} = \frac{1}{\lambda} < 1$ and

$$\begin{aligned} I_\Phi\left(\frac{x_n}{\lambda}\right) &= \sum_{i=1}^{n} \Phi_i\left(\frac{x(i)}{\lambda}\right) + \sum_{i=n+1}^{\infty} \Phi_i\left(\frac{1}{\lambda\alpha}\alpha x(i) + \frac{\varepsilon_0}{\lambda} u(i)\right) \\ &\leq \sum_{i=1}^{n} \Phi_i\left(\frac{x(i)}{\lambda}\right) + \frac{1}{\lambda\alpha}\sum_{i=n+1}^{\infty} \Phi_i(\alpha x(i)) + \frac{\varepsilon_0}{\lambda}\sum_{i=n+1}^{\infty} \Phi_i(u(i)) \end{aligned}$$

for all $n \in \mathbb{N}$. Note that $I_\Phi(x/\lambda) \leq \frac{1}{\lambda} I_\Phi(x) \leq \frac{1}{\lambda} < 1$. Choose $N > 0$ so that

$$\frac{1}{\lambda\alpha}\sum_{i=n+1}^{\infty} \Phi_i(\alpha x(i)) < \frac{1 - I_\Phi(x/\lambda)}{2}$$

and

$$\frac{\varepsilon_0}{\lambda}\sum_{i=n+1}^{\infty} \Phi_i(u(i)) < \frac{1 - I_\Phi(x/\lambda)}{2}$$

for all $n \geq N$. Then $\|x_n\| \leq \lambda = 1 + \varepsilon$ for all $n \geq N$. This shows that $\limsup_{n\to\infty} \|x_n\| \leq 1$. By the monotonicity of the norm, we have $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$. Hence $\|x_n\| \to 1$. Obviously, $1 \leq \|\frac{x_n + x}{2}\| \leq \frac{\|x_n\| + \|x\|}{2} \to 1$. Thus $\|x_n + x\| \to 2$. Since x is a WCLUR-point, we conclude that there is $x' \in S(l_\Phi)$ such that $x_{n'} \xrightarrow{w} x'$ for some subsequence $\{x_{n'}\}$ of $\{x_n\}$. Consequently $x_{n'} \to x'$ coordinatewise. However, by the definition of x_n, it is easy to see that $x_n \to x$ coordinatewise. Consequently $x' = x$. Now we may assume without loss of generality, that $x_n \xrightarrow{w} x$.

Finally, since it can be shown in an analogous way for Orlicz spaces in [1] that

$$\inf_{y \in h_\Phi} \|u - y\| = \theta(u) = 1,$$

there exists $u^* \in S(l_\Phi^*)$ such that $u^*(y) = 0$ for all $y \in h_\Phi$ and $u^*(u) = 1$. This implies

$$u^*(x_n - x) = u^*\left(\sum_{i=n+1}^{\infty} \varepsilon_0 u(i)e_i\right) = u^*\left(\sum_{i=1}^{\infty} \varepsilon_0 u(i)e_i\right) = \varepsilon_0 u^*(u) = \varepsilon_0,$$

contradicting to the fact that $x_n \xrightarrow{w} x$. Hence the necessity of $\Phi \in \delta_2$ for (2) and so the implication (2)⇒(1) are proved.

(1)⇒(3): Assume that $x \in S(l_\Phi)$ is a CLUR-point. Thus, as it has been shown above, $\Phi \in \delta_2$ holds. If the second statement is not satisfied, without loss of generality, we may assume that $\Psi \notin \delta_2$ and $0 < x(1) \notin SC_{\Phi_1} \setminus SC_{\Phi_1}^-$. So there exist constants $\varepsilon, A > 0$ and B such that

$$\Phi_1(u) = Au + B \quad \text{for} \quad u \in [x(1) - \varepsilon, x(1)]$$

and $A\varepsilon \le 1$. Since $\Psi \notin \delta_2$, by Lemma 1.3, there exist an increasing sequence of natural numbers $0 = I_0 < I_1 < \cdots$ and a set $\{u_i^k > 0\}$ such that

$$\Phi_i(u_i^k) \le \frac{1}{k}, \quad \Phi_i\left(\frac{u_i^k}{2}\right) > \left(1 - \frac{1}{k}\right)\frac{\Phi_i(u_i^k)}{2}$$

for all $i = I_{k-1} + 1, \ldots, I_k$ and

$$\sum_{i=I_{k-1}+1}^{I_k} \Phi_i(u_i^k) > 1$$

for all $k \in \mathbb{N}$. For large $k \in \mathbb{N}$, choose $J_k \in \mathbb{N}$ so that

$$A\varepsilon - \frac{1}{k} < \sum_{i=I_{k-1}+1}^{J_k} \Phi_i(u_i^k) \le A\varepsilon.$$

Define a sequence (x_n) by

$$x_n = (x(1) - \varepsilon)e_1 + \sum_{k=2}^{n-1} x(k)e_k + \sum_{k=I_{n-1}+1}^{J_n} u_k^n e_k \operatorname{sgn} x(k).$$

Observe that

$$I_\Phi(x_n) = \sum_{i=1}^{n-1} \Phi_i(x(i)) - A\varepsilon + \sum_{i=I_{n-1}+1}^{J_n} \Phi_i(u_i^n) \le 1.$$

Hence

$$1 \quad \ge \quad I_\Phi\left(\frac{x_n + x}{2}\right)$$

$$\begin{aligned}
&\geq \sum_{i=1}^{n-1} \Phi_i(x(i)) - \frac{A\varepsilon}{2} + \sum_{i=I_{n-1}+1}^{J_n} \Phi_i\left(\frac{u_i^n}{2}\right) \\
&\geq \sum_{i=1}^{n-1} \Phi_i(x(i)) - \frac{A\varepsilon}{2} + \frac{1}{2}\left(1-\frac{1}{n}\right) \sum_{i=I_{n-1}+1}^{J_n} \Phi_i(u_i^n) \\
&\geq \sum_{i=1}^{n-1} \Phi_i(x(i)) - \frac{A\varepsilon}{2} + \frac{1}{2}\left(1-\frac{1}{n}\right)\left(A\varepsilon - \frac{1}{n}\right) \\
&\to 1
\end{aligned}$$

as $n \to \infty$. This implies that $\|x_n + x\| \to 2$. However, if $m > n$,

$$I_\Phi(x_m - x_n) \geq \sum_{i=I_{m-1}+1}^{J_m} \Phi_i(u_i^m) > A\varepsilon - \frac{1}{m},$$

so $\|x_m - x_n\| \geq A\varepsilon - \frac{1}{m}$. This means $\{x_n : n \in \mathbb{N}\}$ is not relatively compact in $S(l_\Phi)$ which finishes the proof of the implication (1)⇒(3).

(3)⇒(1) Let $x \in S(l_\Phi)$. We first prove that if $\Phi, \Psi \in \delta_2$, then x is a CLUR-point. Let $(x_n) \subset S(l_\Phi)$ be such that $\operatorname{sep}\{x_n\} \geq \varepsilon > 0$. Since $\Phi \in \delta_2$, there exists $\delta' > 0$ such that

$$\|y\| \geq \frac{\varepsilon}{2} \Rightarrow I_\Phi(y) \geq \delta'.$$

By Lemma 1.2, there exist a constant $\theta \in (0,1)$ and a sequence $(h_i) \subset \mathbb{R}_+$ such that $\sum_{i=1}^{\infty} \Phi_i(h_i) < \infty$ and

$$\Phi_i\left(\frac{u}{2}\right) \leq \frac{1-\theta}{2}\Phi_i(u)$$

for every $i \in \mathbb{N}$ and u satisfying $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$. And, by Lemma 1.1(1), there exists $\delta > 0$ such that

$$|I_\Phi(x) - I_\Phi(y)| < \frac{\theta\delta'}{6}$$

whenever $I_\Phi(x) \leq 1$ and $I_\Phi(x-y) \leq \delta$. Choose i_0 such that

$$\sum_{i=i_0+1}^{\infty} \Phi_i(x(i)) \leq \delta \text{ and } \sum_{i=i_0+1}^{\infty} \Phi_i(h_i) \leq \frac{\theta\delta'}{6}.$$

Since the set $\{\sum_{i=1}^{i_0} x_n(i)e_i : n \in \mathbb{N}\}$ is compact,

$$\operatorname{sep}\left\{\sum_{i=i_0+1}^{\infty} x_n(i)e_i : n \in \mathbb{N}\right\} \geq \operatorname{sep}\{x_n\} \geq \varepsilon.$$

Then we can find k such that

$$\left\|\sum_{i=i_0+1}^{\infty} x_k(i)e_i\right\| \geq \frac{\varepsilon}{2}.$$

Hence

$$\sum_{i=i_0+1}^{\infty} \Phi_i(x_k(i)) = I_\Phi\left(\sum_{i=i_0+1}^{\infty} x_k(i)e_i\right) \geq \delta'.$$

Then

$$\begin{aligned}
& I_\Phi\left(\frac{x_k+x}{2}\right) \\
= \ & \sum_{i=1}^{i_0} \Phi_i\left(\frac{x_k(i)+x(i)}{2}\right) + \sum_{i=i_0+1}^{\infty} \Phi_i\left(\frac{x_k(i)+x(i)}{2}\right) \\
\leq \ & \frac{1}{2}\left(\sum_{i=1}^{i_0} \Phi_i(x_k(i)) + \Phi_i(x(i))\right) + \sum_{i=i_0+1}^{\infty} \Phi_i\left(\frac{x_k(i)}{2}\right) + \frac{\theta\delta'}{6} \\
\leq \ & \frac{1}{2}\left(\sum_{i=1}^{i_0} \Phi_i(x_k(i)) + \Phi_i(x(i))\right) + \frac{1-\theta}{2}\sum_{i=i_0+1}^{\infty} \Phi_i(x_k(i)) + \sum_{i=i_0+1}^{\infty} \Phi_i(h_i) + \frac{\theta\delta'}{6} \\
\leq \ & \frac{1}{2}\left(\sum_{i=1}^{\infty} \Phi_i(x_k(i)) + \Phi_i(x(i))\right) - \frac{\theta}{2}\sum_{i=i_0+1}^{\infty} \Phi_i(x_k(i)) + \frac{\theta\delta'}{3} \\
\leq \ & 1 - \frac{\theta\delta'}{6}.
\end{aligned}$$

This implies that $\|\frac{x_k+x}{2}\| \leq 1-\delta''$ for some $\delta''>0$, which does not depend on k, so $\|x_n+x\| \not\to 2$.

Finally, we assume that $\Phi \in \delta_2$ and $x(i) \in \{0\} \cup SC_{\Phi_i} \setminus SC^-_{\Phi_i}$ for every $i \in \mathbb{N}$. Let $(x_n) \subset S(l_\Phi)$ be such that $\|x_n + x\| \to 2$. We prove that

(2.1) the set $\{n \in \mathbb{N} : |x(i)| \geq |x_n(i)| \text{ and } |x_n(i) - x(i)| \geq \varepsilon\}$ is a finite set

for all $i \in \mathbb{N}$ and $\varepsilon > 0$. Otherwise, there are $\varepsilon_0 > 0$, $i_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ such that

$$|x(i)| \geq |x_{n_k}(i)| \text{ and } |x_{n_k}(i_0) - x(i_0)| \geq \varepsilon_0$$

for all $k \in \mathbb{N}$. Since $x(i_0) \in SC_{\Phi_{i_0}} \setminus SC^-_{\Phi_{i_0}}$, there exists $\delta > 0$ such that

$$\Phi_{i_0}\left(\frac{x_{n_k}(i_0)+x(i_0)}{2}\right) \leq \frac{1-\delta}{2}(\Phi_{i_0}(x_{n_k}(i_0)) + \Phi_{i_0}(x(i_0)))$$

for all $k \in \mathbb{N}$. By the $(*)$-condition and $\Phi \in \delta_2$, we get $I_\Phi(x) = I_\Phi(x_{n_k}) = 1$ and $I_\Phi(\frac{x_n+x}{2}) \to 1$. We have

$$\begin{aligned}
I_\Phi\left(\frac{x_{n_k}+x}{2}\right) &= \sum_{i=1, i\neq i_0}^{\infty} \Phi_i\left(\frac{x_{n_k}(i)+x(i)}{2}\right) + \Phi_{i_0}\left(\frac{x_{n_k}(i_0)+x(i_0)}{2}\right) \\
&\leq \frac{1}{2}\sum_{i=1}^{\infty} \Phi_i(x_{n_k}(i)) + \Phi_i(x(i)) - \frac{\delta}{2}(\Phi_{i_0}(x_{n_k}(i_0)) + \Phi_{i_0}(x(i_0)))
\end{aligned}$$

$$\begin{aligned} &\leq 1 - \delta\Phi_{i_0}\left(\frac{x_{n_k}(i_0) - x(i_0)}{2}\right) \\ &\leq 1 - \delta\Phi_{i_0}(\frac{\varepsilon_0}{2}), \end{aligned}$$

for all $k \in \mathbb{N}$. This implies $\|\frac{x_{n_k}+x}{2}\| \leq 1-\delta'$ for some $\delta' > 0$ which is a contradiction proving that condition (2.1) holds true.

Now we reach the position to prove that x is a CLUR-point. It suffices to prove that

$$x_n(i) \to x(i)$$

for all $i \in \mathbb{N}$ (see Lemma 1.4). Otherwise, we may assume, by (2.1), that

$$\Phi_{j_0}(x_n(j_0)) - \Phi_{j_0}(x(j_0)) \geq \varepsilon$$

for some $j_0 \in \mathbb{N}$, $\varepsilon > 0$ and for all $n \in \mathbb{N}$. Choose $j_1 > j_0$ such that

$$\sum_{i=j_1+1}^{\infty} \Phi_i(x(i)) \leq \frac{\varepsilon}{2}.$$

By (2.1) again, there exists $N \in \mathbb{N}$ such that

$$\Phi_i(x_n(i)) \geq \Phi_i(x(i)) - \frac{\varepsilon}{3j_1}$$

for all $i = 1, \ldots, j_1$ and $n > N$. Then, if $n > N$,

$$\begin{aligned} 1 &= I_\Phi(x_n) \\ &\geq \Phi_{j_0}(x_n(j_0)) + \sum_{i\leq j_1, i\neq j_0} \Phi_i(x_n(i)) \\ &\geq \Phi_{j_0}(x(j_0)) + \varepsilon + \sum_{i\leq j_1, i\neq j_0} \Phi_i(x(i)) - \frac{\varepsilon}{3j_1} \\ &\geq \sum_{i\leq j_1} \Phi_i(x(i)) + \frac{2\varepsilon}{3} \\ &\geq 1 - \frac{\varepsilon}{2} + \frac{2\varepsilon}{3} \\ &= 1 + \frac{\varepsilon}{6}. \end{aligned}$$

This contradiction completes our proof. □

Corollary 2.5 *Let* $x = (x(i)) \in S(l^{(p_i)})$. *Then the following statements are equivalent:*

(1) x *is a CLUR-point;*

(2) x *is a WCLUR-point;*

(3) *the following conditions hold true:*

(i) $\limsup_{i\to\infty} p_i < \infty$, *and*

(ii) *either* $\liminf_{i\to\infty} p_i > 1$ *or* $\{i \in \mathbb{N} : x(i) \neq 0$ *and* $p_i = 1\} = \emptyset$.

Theorem 2.6 *Suppose that* $\Phi = (\Phi_i)$ *satisfies the* $(*)$*-condition and each* Φ_i *vanishes only at zero. Let* $x = (x(i)) \in S(l_\Phi)$. *Then the following statements are equivalent:*

(1) x *is an LUR-point;*

(2) x *is a WLUR-point;*

(3) *the following conditions hold true:*

(i) $\Phi \in \delta_2$,

(ii) *if* $x(i) \in SC_{\Phi_i}$ *for all* $i \in \mathbb{N}$ *and if there exists* i_0 *such that* $0 \neq x(i_0) \in SC_{\Phi_{i_0}}$, *then* $\Psi \in \delta_2$ *and there exists no index* $i \neq i_0$ *such that* $x(i) \in SC^+_{\Phi_i}$, *and*

(iii) *if* $x(i_0) \notin SC_{\Phi_{i_0}}$, *then* $\Psi \in \delta_2$ *and* $x(i) \in SC^0_{\Phi_i}$ *for all* $i \neq i_0$.

Proof. Obviously, (1)$\Rightarrow$(2). To prove (2)$\Rightarrow$(1), let $x \in S(l_\Phi)$ be a WLUR-point. Thus x is a WCLUR-point and then $\Phi \in \delta_2$. It follows that x is an LUR-point. For the equivalence of (1) and (3), we observe from [2] that a point x in the unit sphere of a Banach space is an LUR-point if and only if it is a strong U-point and a CLUR-point. Then apply Theorem 2.1 and 2.4. $\square$

Corollary 2.7 *Let* $x = (x(i)) \in S(l^{(p_i)})$. *Then the following statements are equivalent:*

(1) x *is an LUR-point;*

(2) x *is a WLUR-point;*

(3) *the following conditions hold true:*

(i) $\limsup_{i\to\infty} p_i < \infty$, *and*

(ii) *if there exists an index* i_0 *such that* $x(i_0) \neq 0$ *and* $p_{i_0} = 1$, *then* $\liminf_{i\to\infty} p_i > 1$ *and* $p_i > 1$ *for every* $i \neq i_0$.

3 Global Geometry

The following results are consequences of results in Section 2.

Theorem 3.1 *Suppose that* Φ *satisfies the* $(*)$*-condition and each* Φ_i *vanishes only at zero. The following statements are equivalent:*

(1) l_Φ *is CLUR;*

(2) l_Φ *is WCLUR;*

(3) $\Phi \in \delta_2$ *and either* $\Psi \in \delta_2$ *or* Φ_i *is strictly convex on* $[0, \Phi_i^{-1}(1)]$ *for every* $i \in \mathbb{N}$.

Proof. It suffices to prove (1)$\Leftrightarrow$(3). To prove (1)$\Rightarrow$(3) we assume that l_Φ is CLUR. It is easy to see that $\Phi \in \delta_2$ holds true. Suppose that $\Psi \notin \delta_2$ and there is $i_0 \in \mathbb{N}$ such that Φ_{i_0} is strictly convex on $[0, \Phi_{i_0}^{-1}(1)]$. We can construct a point $x = (x(i)) \in S(l_\Phi)$ such that $x(i_0) \notin \{0\} \cup SC_{\Phi_{i_0}} \setminus SC^-_{\Phi_{i_0}}$. Hence x is not a CLUR-point.

(3)$\Rightarrow$(1) is obvious. □

Corollary 3.2 *The following statements are equivalent for the Nakano sequence space* $l^{\{p_i\}}$:

(1) $l^{\{p_i\}}$ *is CLUR;*

(2) $l^{\{p_i\}}$ *is WCLUR;*

(3) *the following conditions hold true:*

  (i) $\limsup_{i\to\infty} p_i < \infty$, *and*

  (ii) *either* $\liminf_{i\to\infty} p_i > 1$ *or* $p_i > 1$ *for all* $i \in \mathbb{N}$.

Define $\sigma_i = \sup\{u \geq 0 : \Phi_i$ is strictly convex on $[0, u]$ and $\Phi_i(u) \leq 1\}$. It is known from [6] that l_Φ is rotund if and only if $\Phi \in \delta_2$, each Φ_i vanishes only at zero, and $\Phi_i(\sigma_i) + \Phi_j(\sigma_j) \geq 1$ for all $i \neq j$. Combining this result and Theorem 3.1, we obtain

Theorem 3.3 *Suppose that* Φ *satisfies the* $(*)$*-condition. Then the following statements are equivalent:*

(1) l_Φ *is LUR;*

(2) l_Φ *is WLUR;*

(3) *the following conditions hold true:*

  (i) $\Phi \in \delta_2$,

  (ii) *each* Φ_i *vanishes only at zero, and*

  (iii) *either* $\Psi \in \delta_2$ *and* $\Phi_i(\sigma_i) + \Phi_j(\sigma_j) \geq 1$ *for all* $i \neq j$ *or* $\Phi_i(\sigma_i) = 1$ *for all* $i \in \mathbb{N}$.

Corollary 3.4 *The following statements are equivalent for the Nakano sequence space* $l^{\{p_i\}}$:

(1) $l^{\{p_i\}}$ *is LUR;*

(2) $l^{\{p_i\}}$ *is WLUR;*

(3) *the following conditions hold true:*

   (i) $\limsup_{i\to\infty} p_i < \infty$, *and*

   (ii) *if there exists an index i_0 such that $p_{i_0} = 1$, then* $\liminf_{i\to\infty} p_i > 1$ *and $p_i > 1$ for every $i \neq i_0$.*

It is worthwhile to mention that the last corollary improves Theorem 16 and Corollary 19 of [4] without assuming the boundedness of the sequence $\{p_i\}$.

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Appendix 4: On a generalized James constant, J. Math. Anal. Appl. (in press)