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On a generalized James constant ☆

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Abstract

We introduce a generalized James constant $J(a, X)$ for a Banach space X, and prove that, if $J(a, X) < (3+a)/2$ for some $a \in [0, 1]$, then X has uniform normal structure. The class of spaces X with $J(1, X) < 2$ is proved to contain all u-spaces and their generalizations. For the James constant $J(X)$ itself, we show that X has uniform normal structure provided that $J(X) < (1+\sqrt{5})/2$, improving the previous known upper bound at 3/2. Finally, we establish the stability of uniform normal structure of Banach spaces.

Keywords: James constant; Uniformly nonsquare space; Uniform normal structure

1. Introduction

It is well known that the notions of normal structure and uniform normal structure play important role in metric fixed point theory (see [13]). Various properties of Banach spaces have been known to imply uniform normal structure: $J(X) < 3/2$ [8], $R(X) > 0$ [6], and $C_{NJ}(X) < (3+\sqrt{5})/4$ or $C_{NJ}(1, X) < 2$ [3].

In this paper, we first show that the upper bound 3/2 of $J(X)$ above can be replaced by $(1+\sqrt{5})/2$. Next we introduce a new coefficient $J(\cdot, X)$ generalizing the James constant or nonsquare constant. The number $J(a, X)$ is computed for all $a \geqslant 0$ when X is a Hilbert space. For a general Banach space X, we show that if $J(a, X) < (3+a)/2$ for some

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$a \in (0, 1]$, then X possesses uniform normal structure. In particular, when $a = 1$, we give a class of spaces X having $J(1, X) < 2$. Following Gao and Lau [8], we extend the concept of the stability of the fixed point property of Banach spaces (see [1,15,17,20,21]) to the stability of uniform normal structure, and finally show, for example, that if the Banach–Mazur distance $d(X, H) < (1 + \sqrt{5})/(2\sqrt{2})$ for any Hilbert space H, then X has uniform normal structure.

Throughout the paper we let X and X^* stand for a Banach space and its dual space, respectively. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. $x_n \xrightarrow{w} x$ stands for weak convergence of a sequence $\{x_n\}$ in X to a point x in X. For $x \in X$, let ∇_x denote the set of norm 1 supporting functions at x. This is the subdifferential of the norm of point x. It is nonempty by the Hanh–Banach theorem. We will say that a nonempty weakly compact convex subset C of X has the *fixed point property* (fpp for short) if every nonexpansive mapping $T : C \to C$ (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$ for every $x, y \in C$) has a fixed point, i.e., there exists $x \in C$ such that $T(x) = x$. We will say that X has the *(weak) fixed point property* (fpp) if every weakly compact convex subset of X has the fpp. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y \in A} \|x - y\|: x \in A\}$ is called the *Chebyshev radius* of A. The number $\operatorname{diam} A = \sup_{x, y \in A} \|x - y\|$ is called the *diameter* of A. A Banach space X has *normal structure* if

$$r(A) < \operatorname{diam} A \tag{1.1}$$

for every bounded convex closed subset A of X with $\operatorname{diam} A > 0$. When (1.1) holds for every weakly compact convex subset A of X, X is said to have *weak normal structure*. The normal structure and weak normal structure coincide if X is reflexive. X is said to have *uniform normal structure* if $\inf\{\operatorname{diam} A / r(A)\} > 1$, where the infimum is taken over all bounded convex closed subsets A of X with $\operatorname{diam} A > 0$. Weak normal structure, as well as many other properties imply the fixed point property. The relevant papers are [9,11,18, 25,27].

The *modulus of convexity of* X (cf. [2,4,22–24]) is a function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\|: x, y \in S_X, \|x - y\| \geqslant \epsilon\right\}.$$

When X is nontrivial, i.e., $\dim X \geqslant 2$, we can deduce that

$$\begin{aligned}\delta_X(\epsilon) &= \inf\left\{1 - \left\|\frac{x + y}{2}\right\|: x, y \in B_X, \|x - y\| \geqslant \epsilon\right\}\\ &= \inf\left\{1 - \left\|\frac{x + y}{2}\right\|: x, y \in S_X, \|x - y\| = \epsilon\right\}\\ &= \inf\left\{1 - \left\|\frac{x + y}{2}\right\|: x, y \in B_X, \|x - y\| = \epsilon\right\}.\end{aligned}$$

If $\delta_X(1) > 0$, then X has uniform normal structure (see [12]). For the rest of the paper, we assume that all Banach spaces are nontrivial.

The *modulus of smoothness* of X (cf. [2,4,22,23]) is a function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned}\rho_X(\tau) &= \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1:\ x,y\in S_X\right\}\\ &= \sup\left\{\frac{\tau\epsilon}{2}-\delta_{X^*}(\epsilon):\ \epsilon\in[0,2]\right\}.\end{aligned}$$

A space X is called *uniformly convex* if $\delta_X(\epsilon)>0$ for all $0<\epsilon<2$. It is called *uniformly smooth* if $\rho_0(X)=\rho'_X(0)=\lim_{\tau\to 0}(\rho_X(\tau)/\tau)=0$. Examples of uniformly convex spaces are the spaces $L^p(\Omega)$, where Ω is a measure space such that $L^p(\Omega)$ is at least 2-dimensional. We know that X is uniformly convex if and only if $\varepsilon_0(X)=0$, where the *characteristic of convexity* $\varepsilon_0(X):=\sup\{\epsilon:\ \delta_X(\epsilon)=0\}$. Also, X is uniformly convex if and only if it is ϵ-InQ for all $0<\epsilon\leqslant 2$. Here X is said to be ϵ-InQ, for $0<\epsilon\leqslant 2$, if $\varepsilon_0(X)<\epsilon$. Clearly, X is ϵ-InQ if and only if $\delta_X(\epsilon)>0$. Uniformly convex spaces and uniformly smooth spaces are examples of u-spaces, where a space X is called a *u-space* if for any $\epsilon>0$, there exists $\delta>0$ such that for each $x,y\in S_X$,

$$\left\|\frac{x+y}{2}\right\|>1-\delta \quad\Rightarrow\quad f(y)>1-\epsilon \quad \text{for all } f\in\nabla_x.$$

The *modulus of u-convexity* is defined, for $0\leqslant\epsilon\leqslant 2$, as

$$\begin{aligned}u(\epsilon) &:= \inf\left\{1-\left\|\frac{x+y}{2}\right\|:\ x,y\in S_X \text{ and } f(x-y)\geqslant\epsilon \text{ for some } f\in\nabla_x\right\}\\ &= \inf\left\{1-\left\|\frac{x+y}{2}\right\|:\ x\in S_X,\ y\in B_X\setminus\{0\},\right.\\ &\qquad\qquad \left.\text{and } f(x-y)\geqslant\epsilon \text{ for some } f\in\nabla_x\right\}. \qquad (1.2)\end{aligned}$$

To verify Eq. (1.2), we let $x\in S_X$, $y\in B_X\setminus S_X\cup\{0\}$, and $f\in\nabla_x$ be such that $f(x-y)\geqslant\epsilon$. Thus $|f(y)|\neq 1$, i.e., $\epsilon<2$. We prove that $\|x+y\|\leqslant\|x+z\|$ for some $z\in S_X$ with $f(x-z)=f(x-y)$. Put $f(x-y)=\epsilon'\geqslant\epsilon$ and find $y'\in B_X\setminus S_X$, y and y' are independent, and $f(y')=f(y)$. Write $S_X\cap\{\alpha y'+(1-\alpha)y:\ \alpha\in\mathbb{R}\}=\{z',z''\}$. So $y=\lambda z'+(1-\lambda)z''$ for some $\lambda\in(0,1)$. Hence $\|x+y\|\leqslant\|x+z'\|$ or $\|x+y\|\leqslant\|x+z''\|$. Clearly, $f(z')=f(z'')=1-\epsilon'$ and we find $z\in S_X$ as desired. Clearly, X is a u-space if and only if $u(\epsilon)>0$ for all $\epsilon>0$. The notion of u-spaces was introduced by Lau [19].

A Banach space X is called *uniformly nonsquare* provided that there exists $\delta>0$ such that if $x,y\in S_X$, then $\|x+y\|/2\leqslant 1-\delta$ or $\|x-y\|/2\leqslant 1-\delta$. Uniformly nonsquare spaces are superreflexive (see [14]). Every u-space is uniformly nonsquare (see [19]), hence, it is superreflexive.

In [3], Dhompongsa et al. introduced a generalized Jordan–von Neumann constant $C_{\mathrm{NJ}}(a,X)$ for $a\geqslant 0$ as

$$\begin{aligned}C_{\mathrm{NJ}}(a,X)=\sup\left\{\frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}:\ x,y,z\in B_X \text{ of which at least one}\right.\\ \left.\text{belongs to } S_X \text{ and } \|y-z\|\leqslant a\|x\|\right\}.\end{aligned}$$

Some of its properties are:

(1) $1 + 4a/(4 + a^2) \leqslant C_{NJ}(a, X) \leqslant 2$ for all $a \geqslant 0$ and $C_{NJ}(a, X) = 2$ for all $a \geqslant 2$.
(2) $C_{NJ}(a, X) = 1 + 4a/(4 + a^2)$ whenever X is a Hilbert space.
(3) $C_{NJ}(a, X)$ is continuous as a function of a.
(4) If $C_{NJ}(1, X) < 2$ or $C_{NJ}(X) < (3 + \sqrt{5})/4$, then X has uniform normal structure. Here $C_{NJ}(X) = C_{NJ}(0, X)$ is the Jordan–von Neumann constant of X.
(5) Every u-space X has $C_{NJ}(a, X) < 2$ for all $0 < a < 2$.

2. James constant

Let X be a Banach space. The *James constant*, or the *nonsquare constant* is defined by Gao and Lau [7] as

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\|: x, y \in S_X\}$$
$$= \sup\{\|x + y\| \wedge \|x - y\|: x, y \in B_X\}.$$

Clearly, X is uniformly nonsquare if and only if $J(X) < 2$. In [8], Gao and Lau proved that, in general, $\sqrt{2} \leqslant J(X) \leqslant 2$ and X has uniform normal structure provided that $J(X) < 3/2$. We show that $3/2$ can be replaced by $(1 + \sqrt{5})/2$.

Theorem 2.1. *If $J(X) < (1 + \sqrt{5})/2$, then X has uniform normal structure.*

Proof. Since $J(X) < 2$, X is uniformly nonsquare, and consequently, X is reflexive. Thus, normal structure and weak normal structure coincide. By [8, Theorem 5.2], it suffices to prove that X has weak normal structure.

Suppose on the contrary that X does not have weak normal structure. Thus, there exists a weak null sequence $\{x_n\}$ in S_X such that for $C := \overline{\mathrm{co}}\{x_n: n \geqslant 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam} C = 1 \quad \text{for all } x \in C \tag{2.1}$$

(cf. [27]).

Let $r = (\sqrt{5} - 1)/2$. Thus $r(1 + r) = 1$. Let $0 < \epsilon < 1$. We choose first $x_0 \in C$ with $\|x_0\| > 1 - \epsilon/2$. We shall consider, without loss of generality, the following possibilities: *Case* 1 ($\|x_n + x_0\| \leqslant 1 + r$ for all large n) and *Case* 2 ($\|x_n + x_0\| > 1 + r$ for all large n). We subdivide *Case* 2 into *Case* 2.1 (for all large n, $\|x_n + x_m - x_0\| > 1 + r$ for all large m) and *Case* 2.2 (for all large n, $\|x_n + x_m - x_0\| \leqslant 1 + r$ for all large m).

The numbers m and n can be chosen properly under any one of these situations and satisfy conditions (2.2)–(2.7) below. Since 0 belongs to the weak closed convex hull of $\{x_n\}$, which equals to the norm closed convex hull C, we can choose m by (2.1) so that

$$\left\|x_m - \frac{r}{1+r}x_0\right\| \geqslant 1 - \frac{\epsilon}{2}. \tag{2.2}$$

Choose n such that the following estimations are satisfied:

$$\left\|x_n - \frac{1-r}{1+r}x_0\right\| \geqslant 1 - \epsilon, \tag{2.3}$$

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$$\|(1-r)x_n-(1+r)x_0\|\geqslant(1+r)\|x_0\|-\frac{\epsilon}{2}\geqslant(1+r)(1-\epsilon) \tag{2.4}$$

(by lower semicontinuity of the space norm $\|\cdot\|$),

$$\|x_n-(x_m+x_0)\|\geqslant\|x_m+x_0\|-\frac{\epsilon}{2}, \tag{2.5}$$

$$\left\|x_n-\frac{1}{1+r}\big((1-r)x_m+rx_0\big)\right\|\geqslant 1-\epsilon, \tag{2.6}$$

and

$$\big\|(1-r)x_n-\big((1-r)x_m-rx_0\big)\big\|\geqslant\big\|(1+r)x_m-rx_0\big\|-\frac{\epsilon}{2}. \tag{2.7}$$

Our aim is to find $x,y\in B_X$ such that

$$\|x+y\|\geqslant(1-r)(1-\epsilon)\quad\text{and}\quad\|x-y\|\geqslant(1+r)(1-\epsilon). \tag{2.8}$$

Taking (2.8) for granted, we see that $J(X)\geqslant(1+r)(1-\epsilon)$ for all $\epsilon>0$. Therefore, $J(X)\geqslant(1+\sqrt{5})/2$, a contradiction.

For Case 1 ($\|x_n+x_0\|\leqslant 1+r$ for all large n), we let $x=x_n-x_0$, $y=r(x_n+x_0)$. Thus $x,y\in B_X$ and

$$\|x+y\|=\|(1-r)x_n-(1-r)x_0\|=(1+r)\left\|x_n-\frac{1-r}{1+r}x_0\right\|\geqslant(1+r)(1-\epsilon)$$

by (2.3).

Similarly,

$$\|x-y\|=\big\|(1-r)x_n-(1+r)x_0\big\|\geqslant(1+r)\|x_0\|-\frac{\epsilon}{2}\geqslant(1+r)(1-\epsilon)$$

by (2.4). Therefore (2.8) follows.

For Case 2.1 (for all large n, $\|x_n+x_0\|>1+r$, $\|x_n+x_m-x_0\|>1+r$ for all large m), we put $x=x_n-x_0$ and $y=x_m$. Clearly x and y belong to B_X. Moreover, we obtain

$$\|x+y\|=\|x_n+x_m-x_0\|\geqslant(1+r),$$

and by (2.5),

$$\|x-y\|=\big\|x_n-(x_m+x_0)\big\|\geqslant\|x_m+x_0\|-\frac{\epsilon}{2}\geqslant 1+r-\frac{\epsilon}{2}>(1+r)(1-\epsilon).$$

Thus, (2.8) is valid in Case 2.1 as well.

Finally, we let, in Case 2.2 (for all large n, $\|x_n+x_0\|>1+r$, $\|x_n+x_m-x_0\|\leqslant 1+r$ for all large m), $x=x_n-x_m$ and $y=r(x_n+x_m-x_0)$. By (2.6) we have

$$\begin{aligned}\|x+y\|&=\big\|(1+r)x_n-\big((1-r)x_m+rx_0\big)\big\|\\&=(1+r)\left\|x_n-\frac{1}{1+r}\big((1-r)x_m+rx_0\big)\right\|\\&\geqslant(1+r)(1-\epsilon),\end{aligned}$$

and by (2.7) and (2.2) we have

$$\begin{aligned}\|x-y\| &= \|(1-r)x_n - ((1+r)x_m - rx_0)\| \geqslant \|(1+r)x_m - rx_0\| - \frac{\epsilon}{2} \\ &= (1+r)\left\|x_m - \frac{r}{1+r}x_0\right\| - \frac{\epsilon}{2} \geqslant (1+r)(1-\epsilon).\end{aligned}$$

Now as (2.8) is established in all cases, the proof is complete. □

From the relation

$$\frac{J(X)^2}{2} \leqslant C_{\mathrm{NJ}}(X) \tag{2.9}$$

(see [16]), we obtain

Corollary 2.2 [5, Theorem 4.6]. *For a Banach space X, if $C_{\mathrm{NJ}}(X) < 5/4$, then X has uniform normal structure.*

We consider now the Bynum space $l_{p,\infty}$. Recall that $l_{p,\infty}$ is the space of all sequences $\{x_n\}$ whose norm defined by $\|\{x_n\}\| = \max\{\|x_n^+\|_p, \|x_n^-\|_p\}$. It is shown in [13] that this space fails to have uniform normal structure for $p \geqslant 1$. Thus $J(l_{p,\infty}) \geqslant (1+\sqrt{5})/2$. Indeed, we shall show that

$$\begin{aligned} J(l_{p,\infty}) &\geqslant 2^{1/p} \geqslant \frac{1+\sqrt{5}}{2} \quad \text{for } p \leqslant p_0, \\ J(l_{p,\infty}) &\geqslant 1 + \left(\frac{1}{2}\right)^{1/p} \geqslant \frac{1+\sqrt{5}}{2} \quad \text{for } p \geqslant p_0, \end{aligned} \tag{2.10}$$

where p_0 satisfies the equation $2^{1/p} = 1 + (1/2)^{1/p}$. Note that $2^{1/p_0} = (1+\sqrt{5})/2$.

For $p \leqslant p_0$, let $x = (1, 0, -1, 0, 0, \ldots)$ and $y = (0, 1, 0, -1, 0, 0, \ldots)$. Clearly $\|x + y\| = \|x - y\| = 2^{1/p}$. For $p \geqslant p_0$, let $x = (1, -1, 0, 0, \ldots)$ and $y = ((1/2)^{1/p}, (1/2)^{1/p}, 0, 0, \ldots)$. Hence $\|x+y\| = \|x-y\| = 1 + (1/2)^{1/p}$. Thus (2.10) follows. Note that

$$C_{\mathrm{NJ}}(l_{p,\infty}) \geqslant 2^{2/p-1} \geqslant \frac{3+\sqrt{5}}{4} \quad \text{or} \quad C_{\mathrm{NJ}}(l_{p,\infty}) \geqslant \frac{(1+(1/2)^{1/p})^2}{2} \geqslant \frac{3+\sqrt{5}}{4}$$

according to $p \leqslant p_0$ or $p \geqslant p_0$.

Conjecture. *$J(l_{p_0,\infty}) = (1+\sqrt{5})/2$ and $C_{\mathrm{NJ}}(l_{p_0,\infty}) = (3+\sqrt{5})/4$.*

We end this section by investigating the James constant of some well-known spaces. Define, for $\alpha > 0$,

$$J_\alpha(X) = \sup\{\|x+y\| \wedge \alpha\|x-y\|: x, y \in S_X\}.$$

Clearly, $J_1(X) = J(X)$, $J_\alpha(X)$ is a nondecreasing function with respect to α, and $J_\alpha(X) \leqslant 2$.

We first obtain

Proposition 2.3. $J_\alpha(X) = \sup\{\epsilon: \delta(\epsilon/\alpha) \leqslant 1 - \epsilon/2\}$.

Proof. Let $\eta > \epsilon_0 = \sup\{\epsilon\colon \delta(\epsilon/\alpha) \leqslant 1 - \epsilon/2\}$ and let x, y be any elements of S_X. If $\alpha\|x - y\| \geqslant \eta$, then $1 - \|x + y\|/2 \geqslant \delta(\eta/\alpha) > 1 - \eta/2$, i.e., $\|x + y\| < \eta$. Thus, $J_\alpha(X) \leqslant \eta$, and since $\eta > \epsilon_0$ is arbitrary, $J_\alpha(X) \leqslant \epsilon_0$. To show that $J_\alpha(X) \geqslant \epsilon_0$, we let $\eta > 0$ and $\epsilon > \epsilon_0 - \eta$ satisfying $\delta(\epsilon/\alpha) \leqslant 1 - \epsilon/2$. Choose $x, y \in S_X$ such that $\alpha\|x - y\| \geqslant \epsilon$ and $1 - \|x + y\|/2 < \delta(\epsilon/\alpha) + \eta$, i.e., $\|x + y\| > 2 - 2\delta(\epsilon/\alpha) - 2\eta$. Thus, $J_\alpha(X) \geqslant \|x + y\| \wedge \alpha\|x - y\| \geqslant (2 - 2\delta(\epsilon/\alpha) - 2\eta) \wedge \epsilon \geqslant (\epsilon - 2\eta) \wedge \epsilon \geqslant \epsilon_0 - 3\eta$, and hence $J_\alpha(X) \geqslant \epsilon_0$ as desired. □

Example 2.4. (1) For the l_p space $(2 \leqslant p < \infty)$, it is known (cf. [26]) that $\delta_{l_p}(\epsilon) = 1 - (1 - (\epsilon/2)^p)^{1/p}$. Hence, by Proposition 2.3 we have $J_\alpha(l_p) = 2(\alpha^p/(\alpha^p + 1))^{1/p}$.

(2) $J(l_\infty - l_p) = 1 + (1/2)^{1/p}$ for $1 \leqslant p \leqslant \infty$. Here $l_\infty - l_p$ is the 2-dimensional Day–James space whose norm is defined by $\|(x_1, x_2)\| = (|x_1|^p + |x_2|^p)^{1/p}$ or $\max\{|x_1|, |x_2|\}$ according to $x_1x_2 \geqslant 0$ or $x_1x_2 \leqslant 0$.

Proof. The case when $p = \infty$ is clear. Let $x, y \in S_X$. If both x and y are in the first quadrant, then we have $\|x + y\| \wedge \|x - y\| \leqslant 1 \leqslant 1 + (1/2)^{1/p}$. Suppose $x = (-1, a)$, $y = (-b, 1)$, where $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant 1$. If $a > (1/2)^{1/p}$, then $\|x - y\| = ((1 - b)^p + (1 - a)^p)^{1/p} \leqslant (1 + (1/2)^p)^{1/p} \leqslant 1 + (1/2)^p \leqslant 1 + (1/2)^{1/p}$. If $a \leqslant (1/2)^{1/p}$ and $b \leqslant (1/2)^{1/p}$, then $\|x + y\| \leqslant 1 + (1/2)^{1/p}$. Thus $\|x + y\| \wedge \|x - y\| \leqslant 1 + (1/2)^{1/p}$.

If $x = (a, b)$ and $y = (-c, 1)$, where a, b, c are all nonnegative, then $x + y = (a - c, b + 1)$ and $x - y = (a + c, b - 1)$. First let $c > a$. Since $f(a) = a + (1 - a^p)^{1/p}$ has the maximum value $2(1/2)^{1/p}$ on $[0, 1]$, we have

$$\|x + y\| \wedge \|x - y\| \leqslant (1 + b) \wedge (1 + a) \leqslant \frac{a + b + 2}{2} = 1 + \frac{a + b}{2} \leqslant 1 + \left(\frac{1}{2}\right)^{1/p}.$$

If $c \leqslant a$ and $a \leqslant (1 + (1/2)^{1/p})/2$, then $\|x - y\| = a + c \leqslant 1 + (1/2)^{1/p}$. If $c \leqslant a$ and $a \geqslant (1 + (1/2)^{1/p})/2$, then putting $a_0 = (1 + (1/2)^{1/p})/2$,

$$\|x + y\| \leqslant \|x_0 + y_0\| = \left(1 + \left(\frac{1 + (1/2)^{1/p}}{2}\right)^p\right)^{1/p} \leqslant 1 + \left(\frac{1}{2}\right)^{1/p},$$

where $x_0 = (a_0, b_0)$ and $y_0 = (0, 1)$. Thus

$$\|x + y\| \wedge \|x - y\| \leqslant 1 + \left(\frac{1}{2}\right)^{1/p}$$

for the case $c \leqslant a$.

The other case when x and y belong to opposite quadrants are easy to handle. □

Example 2.4 (Continued). (3) $J(l_p - l_q) \leqslant 2(2^{p/q}/(2^{p/q} + 2))^{1/p}$ for $1 \leqslant q \leqslant p < \infty$ and $p \geqslant 2$. Here we define the norm $\|(x_1, x_2)\| = (|x_1|^p + |x_2|^p)^{1/p}$ or $(|x_1|^q + |x_2|^q)^{1/q}$ according to $x_1x_2 \geqslant 0$ or $x_1x_2 \leqslant 0$.

Proof. By the convexity of the function $f(u) = u^{p/q}$, we have $\|x\|_p \leqslant \|x\| \leqslant 2^{(1/q - 1/p)}\|x\|_p$. Let $x, y \in S_X$. If x, y are in the same quadrant, then

$$\|x + y\| \wedge \|x - y\| \leqslant J_{2^{(1/q-1/p)}}(l_p) \leqslant 2\left(\frac{2^{p/q}}{2^{p/q} + 2}\right)^{1/p}.$$

For the rest, it is enough to consider when $x = (a, b)$ and $y = (-c, d)$, where a, b, c, d are nonnegative.

Case 1 ($c \leqslant a$ and $d \leqslant b$).

$$\|x + y\| \wedge \|x - y\| = \|x + y\|_p \wedge \|x - y\|_p \leqslant J(l_p) \leqslant 2\left(\frac{2^{p/q}}{2^{p/q} + 2}\right)^{1/p}.$$

Case 2 ($c \leqslant a$ and $d \geqslant b$).

$$\begin{aligned}\|x + y\| \wedge \|x - y\| &= \|x + y\|_p \wedge \|x - y\| \leqslant \|x + y\|_p \wedge 2^{(1/q - 1/p)}\|x - y\|_p \\ &\leqslant J_{2^{(1/q-1/p)}}(l_p) = 2\left(\frac{2^{p/q}}{2^{p/q} + 2}\right)^{1/p}. \qquad \square\end{aligned}$$

In [16], it is shown that $J(l_p - l_q) \leqslant \min\{2, 2^{(1/q - 1/p)}J(l_p)\}$ which is not smaller than our bound. They are equal only when $p = q \geqslant 2$.

3. A generalized James constant

Let us begin with a generalization of the James constant. Define, for $a \geqslant 0$,

$$\begin{aligned}J(a, X) &= \sup\bigl\{\|x + y\| \wedge \|x - z\|:\ x, y, z \in B_X \text{ and } \|y - z\| \leqslant a\|x\|\bigr\} \\ &= \sup\bigl\{\|x + y\| \wedge \|x - z\|:\ x, y, z \in B_X \text{ of which at least one belongs to } S_X \\ &\qquad \text{and } \|y - z\| \leqslant a\|x\|\bigr\}.\end{aligned}$$

Note that

(1) $J(0, X) = J(X)$;
(2) $J(a, X)$ is a nondecreasing function with respect to a;
(3) If $J(a, X) < 2$, for some $a \geqslant 0$, then $J(X) < 2$ and consequently X is uniformly nonsquare.

Let us consider the case when X is a Hilbert space.

Proposition 3.1. *For a Hilbert space H, $J(a, H) = \sqrt{2 + a}$ for all $a \in [0, 2]$.*

Proof. Let $x, y, z \in B_H$ with $\|y - z\| \leqslant a\|x\|$. On one hand we have

$$\begin{aligned}\|x + y\|^2 \wedge \|x - z\|^2 &\leqslant \frac{\|x + y\|^2 + \|x - z\|^2}{2} \\ &= \frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2\langle x, y - z\rangle}{2} \\ &\leqslant \frac{4 + 2\|x\|\|y - z\|}{2} \leqslant \frac{4 + 2a}{2} = 2 + a.\end{aligned}$$

On the other hand, let e_1 and e_2 be orthonormal elements of S_H. Put

$$x = e_1, \qquad y = \frac{a}{2}e_1 + \sqrt{1 - \frac{a^2}{4}}\, e_2, \qquad z = \frac{-a}{2}e_1 + \sqrt{1 - \frac{a^2}{4}}\, e_2.$$

Thus we have $\|y - z\| = a\|x\|$ and $\|x + y\| \wedge \|x - z\| = \|x + y\| = \sqrt{2 + a}$ and the proof is complete. □

Later we will make use of the following

Lemma 3.2. *Let X be a Banach space. For $0 \leqslant a < 2$, if $C_{NJ}(a, X) = 2$, then there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in B_X satisfying*

(i) $\|x_n\|, \|y_n\|, \|z_n\| \to 1$;
(ii) $\|x_n + y_n\|, \|x_n - z_n\| \to 2$; *and*
(iii) $\|y_n - z_n\| \leqslant a\|x_n\|$ *for all n.*

Furthermore, the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ can be chosen from S_X.

Proof. Suppose $C_{NJ}(a, X) = 2$ for some $0 \leqslant a < 2$. Choose $x_n, y_n, z_n \in B_X$ which at least one of them belongs to S_X such that $\|y_n - z_n\| \leqslant a\|x_n\|$ for all n and $g(x_n, y_n, z_n) \nearrow 2$, where

$$g(x, y, z) = \frac{\|x + y\|^2 + \|x - z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2}.$$

Hence (iii) follows. Consider

$$\begin{aligned} g(x, y, z) &= \frac{\|x + y\|^2 + \|x - z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \\ &\leqslant \frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2(\|x\|\|y\| + \|x\|\|z\|)}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \\ &= 1 + \frac{2(\|x\|\|y\| + \|x\|\|z\|)}{2\|x\|^2 + \|y\|^2 + \|z\|^2}. \end{aligned}$$

Thus we have

$$\frac{2(\|x_n\|\|y_n\| + \|x_n\|\|z_n\|)}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \to 1$$

which implies

$$\frac{(\|x_n\| - \|y_n\|)^2 + (\|x_n\| - \|z_n\|)^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \to 0.$$

Since, for each n, at least one of x_n, y_n, z_n belongs to S_X, we can find a subsequence $\{n'\}$ of $\{n\}$ such that $\|x_{n'}\|, \|y_{n'}\|, \|z_{n'}\| \to 1$. It follows then that $\|x_{n'} + y_{n'}\|, \|x_{n'} - z_{n'}\| \to 2$. Thus (i) and (ii) hold. Next, put $x' = x/\|x\|$ for nonzero x. From the choice of x_n, y_n, and z_n we also have $\|x'_{n'} - x_{n'}\|, \|y'_{n'} - y_{n'}\|, \|z'_{n'} - z_{n'}\| \to 0$. As $2 \geqslant \|x' + y'\| \geqslant \|x + y\| -$

$\|x' - x\| - \|y' - y\|$, we can see that $\|x'_{n'} + y'_{n'}\|, \|x'_{n'} - z'_{n'}\| \to 2$ as well. Finally, $a\|x_{n'}\| \geqslant \|y_{n'} - z_{n'}\| \geqslant \|y'_{n'} - z'_{n'}\| - \|z'_{n'} - z_{n'}\| - \|y'_{n'} - y_{n'}\|$. Thus, $\limsup_{n'\to\infty} \|y'_{n'} - z'_{n'}\| \leqslant a$, completing the proof. □

With the same proof, relation (2.9) continues to hold in general case.

Proposition 3.3. *For a Banach space X, $J(a, X)^2/2 \leqslant C_{\mathrm{NJ}}(a, X)$ for all $a \in [0, \infty)$.*

Corollary 3.4. *For a Banach space X, $J(a, X) = 2$ if and only if $C_{\mathrm{NJ}}(a, X) = 2$ for all $a \in [0, 2]$.*

Proof. If $C_{\mathrm{NJ}}(a, X) = 2$, then by Lemma 3.2, there exist sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in S_X satisfying $\|x_n + y_n\|, \|x_n - z_n\| \to 2$ and $\|y_n - z_n\| \leqslant a$ for all n. Thus, $J(a, X) = 2$. The other direction is an easy consequence of Proposition 3.3. □

Corollary 3.5. *Let X be a Banach space. If $J(1, X) < 2$, then X has uniform normal structure.*

Proof. This is an immediate consequence of [3, Corollary 3.7] and Corollary 3.4. □

We now obtain the continuity of the function $J(\cdot, X)$.

Proposition 3.6. *For $0 \leqslant a \leqslant b$, $J(b, X) + a/2 \leqslant J(a, X) + b/2$. In particular, $J(\cdot, X)$ is continuous on $[0, \infty)$.*

Proof. Let $\epsilon > 0$. There exist $x, y, z \in B_X$ such that $\|y - z\| = b_1\|x\|$ and $J(b, X) - \epsilon \leqslant \|x + y\| \wedge \|x - z\|$. b_1 can be chosen so that $a < b_1$. Otherwise, the assertion is obviously true. We can choose $z_1, y_1 \in B_X$ such that $\|y - y_1\|, \|z - z_1\| \leqslant (b - a)/2$ and $\|y_1 - z_1\| \leqslant a\|x\|$. (Just put $y_1 = \alpha y + (1 - \alpha)z$ and $z_1 = \alpha z + (1 - \alpha)y$, where $1 - \alpha = (b_1 - a)/2b_1$.)
Combining all these, we have

$$\begin{aligned} J(b, X) - \epsilon &\leqslant \|x + y\| \wedge \|x - z\| \\ &\leqslant \big(\|x + y_1\| + \|y - y_1\|\big) \wedge \big(\|x - z_1\| + \|z - z_1\|\big) \\ &\leqslant \big(\|x + y_1\| \wedge \|x - z_1\|\big) + \frac{b - a}{2} \leqslant J(a, X) + \frac{b - a}{2}. \end{aligned}$$

To finish the proof, we let $\epsilon \to 0$. □

From the continuity of $J(\cdot, X)$, it is easy to see that the James constant $J(\cdot, X)$ of the space X is invariant when computed on any of its ultraproduct $\tilde{X}$: $J(\cdot, X) = J(\cdot, \tilde{X})$. As a consequence, if $J(a, X) < (3 + a)/2$ for some $a \in [0, 1]$, then X has uniform normal structure. It is worth noting that this upper bound does not give as strong as the result in Theorem 2.1 for small a. It only gives new information for a in $(0, 1]$.

In [3], it is shown that all u-spaces X satisfy $C_{\mathrm{NJ}}(1, X) < 2$, equivalently $J(1, X) < 2$, which in turn implies that all of these spaces have uniform normal structure, the implication that was previously known [8]. We shall give more examples of spaces satisfying

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this condition. Before that we give first an example showing that $J(X)$ can be arbitrarily close to 2, whereas $J(1, X) < 2$. Let $1 < p < 2$ and let $X = \mathbb{R}^2$ be equipped with the norm defined by $\|x\| = \|(x_1, x_2)\| = \|x\|_1$ or $\|x\|_p$ according to $x_1x_2 \geqslant 0$ or $x_1x_2 \leqslant 0$. In [3], it is seen that $C_{\mathrm{NJ}}(X) = 1 + 2^{2/p-2}$, $J(X) \geqslant 2^{1/p}$, and $C_{\mathrm{NJ}}(1, X) < 2$. Let $X_2 = l_2(X)$, the space of sequences $\{x_k\}$ in X with $\{\|x_k\|\} \in l_2$. We define a norm for each $x = \{x_k\}$ in X_2 as the l_2-norm of the sequence $\{\|x_k\|\}$, that is, $\|x\| = (\sum_{n=1}^{\infty} \|x_k\|^2)^{1/2}$. Clearly, $J(X_2) \geqslant 2^{1/p}$. To show $J(1, X_2) < 2$ we assume on the contrary and obtain, by Lemma 3.2, sequences $\{x^n\}$, $\{y^n\}$, and $\{z^n\}$ in S_{X_2} with $\|x^n + y^n\|, \|x^n - z^n\| \to 2$ and $\|y^n - z^n\| \leqslant 1$ for all n. Put, by the continuity of $C_{\mathrm{NJ}}(\cdot, X)$, $E = C_{\mathrm{NJ}}(t, X) < 2$ for some $t > 1$, $A_n = \{k \in \mathbb{N}\colon \|y_k^n - z_k^n\| > t\|x_k^n\|\}$, and $B_n = \{k \in \mathbb{N}\colon \|y_k^n - z_k^n\| \leqslant t\|x_k^n\|\}$. Observe that, if $\sum_{k\in B_n} \|x_k^n\|^2 \to 0$ as $n \to \infty$, then from the estimation $1 \geqslant \|y^n - z^n\|^2 = (\sum_{k\in A_n} + \sum_{k\in B_n})\|y_k^n - z_k^n\|^2 \geqslant t^2 \sum_{k\in A_n} \|x_k^n\|^2$, we can deduce that $1 \geqslant t^2$ which is impossible. So we assume without loss of generality that, for some $\epsilon_0 > 0$, $\sum_{k\in B_n} \|x_k^n\|^2 \geqslant \epsilon_0$ for all n. Put $a_n = \sum_{k\in A_n} (2\|x_k^n\|^2 + \|y_k^n\|^2 + \|z_k^n\|^2)$ and $b_n = \sum_{k\in B_n} (2\|x_k^n\|^2 + \|y_k^n\|^2 + \|z_k^n\|^2)$. Thus, $a_n + b_n = 4$ and $b_n \geqslant 2\epsilon_0$ for each n. Consider the estimation

$$\begin{aligned}
\|x^n + y^n\|^2 + \|x^n - z^n\|^2 &= \Big(\sum_{k\in A_n} + \sum_{k\in B_n}\Big)\big(\|x_k^n + y_k^n\|^2 + \|x_k^n - z_k^n\|^2\big) \\
&\leqslant 2\sum_{k\in A_n} \big(2\|x_k^n\|^2 + \|y_k^n\|^2 + \|z_k^n\|^2\big) \\
&\quad + E\sum_{k\in B_n} \big(2\|x_k^n\|^2 + \|y_k^n\|^2 + \|z_k^n\|^2\big) \\
&\leqslant 2a_n + Eb_n = 8 - (2 - E)b_n \leqslant 8 - 2(2 - E)\epsilon_0.
\end{aligned}$$

Therefore,

$$\frac{\|x^n + y^n\|^2 + \|x^n - z^n\|^2}{2\|x^n\|^2 + \|y^n\|^2 + \|z^n\|^2} \leqslant 2 - \frac{(2 - E)\epsilon_0}{2} \quad \text{for all } n,$$

a contradiction since the left-hand side of the last inequality tends to 2 as $n \to \infty$.

This example shows that the notion of the generalized James constant $J(a, X)$ is a step forward.

Before we turn to some new classes of spaces, we consider one more sufficient condition for a Banach space to have uniform normal structure.

Theorem 3.7. *Let X be a Banach space. Then $C_{\mathrm{NJ}}(a, X) \geqslant (1 + a)^2/(1 + a^2)$ for all $a \in (0, 1]$ if and only if $J(1, X) = 2$.*

Proof. ($\Rightarrow$) Since $C_{\mathrm{NJ}}(a, X)$ is continuous,

$$C_{\mathrm{NJ}}(1, X) = \lim_{a\to 1} C_{\mathrm{NJ}}(a, X) \geqslant \lim_{a\to 1} \frac{(1 + a)^2}{1 + a^2} = 2.$$

($\Leftarrow$) Suppose that $J(1, X) = 2$. By Lemma 3.2, there exist sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in S_X satisfying $\|x_n + y_n\|$, $\|x_n - z_n\| \to 2$ and $\|y_n - z_n\| \leqslant 1$ for all n. Therefore $\|ay_n - az_n\| \leqslant a$ for all n. Consider inequalities

$$\|x_n + y_n\| - \|y_n - ay_n\| \leqslant \|x_n + ay_n\| \leqslant 1 + a$$

and

$$\|x_n - z_n\| - \|z_n - az_n\| \leqslant \|x_n - az_n\| \leqslant 1 + a.$$

Hence

$$\lim_{n\to\infty} \|x_n + ay_n\| = 1 + a, \qquad \lim_{n\to\infty} \|x_n - az_n\| = 1 + a.$$

Combining together we get

$$C_{\mathrm{NJ}}(a, X) \geqslant \lim_{n\to\infty} \frac{\|x_n + ay_n\|^2 + \|x_n - az_n\|^2}{2\|x_n\|^2 + \|ay_n\|^2 + \|az_n\|^2} = \frac{(1+a)^2}{1+a^2},$$

and the proof is complete. □

Sometimes it is more convenient to recognize Theorem 3.7 in the following form.

Corollary 3.8. *Let X be a Banach space. If $C_{\mathrm{NJ}}(a, X) < (1+a)^2/(1+a^2)$ for some $a \in (0, 1]$, then X has uniform normal structure.*

In [3] it is shown that $C_{\mathrm{NJ}}(a, X) = 1 + 4a/(4 + a^2)$ whenever X is a Hilbert space. We do not know if the converse is true, however, as a consequence of Corollary 3.8, we clearly have the following.

If, for a Banach space X, $C_{\mathrm{NJ}}(a, X) = 1 + 4a/(4 + a^2)$ for some $a \in (0, 1]$, then X has uniform normal structure.

Note that

$$\frac{(1+a)^2 + (3a-1)^2}{2(1+a^2)} \leqslant \frac{(1+a)^2}{1+a^2} = 1 + \frac{2a}{1+a^2} \leqslant 2 \quad \text{for all } a \in [0, 1].$$

Observe that

$$\frac{(1+a)^2}{1+a^2} > \frac{(1+a)^2 + (3a-1)^2}{2(1+a^2)} \quad \text{for all } a \in (0, 1).$$

The bigger function is strictly concave and the smaller one is strictly convex. Thus, Corollary 3.8 gives a strong improvement of [3, Theorem 3.6].

We introduce now new classes of spaces.

Definition 3.9. A Banach space X is said to be ε-uniformly smooth for $0 < \varepsilon \leqslant 1$, if $\rho_0(X) < \varepsilon$. X is said to be an ε-u-space, for $0 < \varepsilon \leqslant 2$, if there exists a $\delta > 0$ such that for $x, y \in S_X$,

$$\left\|\frac{x+y}{2}\right\| > 1 - \delta \quad \Rightarrow \quad f(y) > 1 - \varepsilon \quad \text{for all } f \in \nabla_x.$$

Clearly, if $0 < \varepsilon_1 < \varepsilon_2$ and X is ε_1-uniformly smooth (ε_1-u-space), then X is ε_2-uniformly smooth (ε_2-u-space, respectively). Also, if X is ε-uniformly smooth (or an ε-u-space) for every $\varepsilon > 0$, then X is uniformly smooth (or a u-space, respectively). It is also well known and easy to see from the equation connecting the moduli of smoothness and convexity that (i) $X(X^*)$ is ϵ-uniformly smooth if and only if X^* (X, respectively) is 2ϵ-InQ and (ii) X is ε-uniformly smooth for some $0 < \varepsilon < 1$ if and only if $J(X) < 2$.

Lemma 3.10. *$u(\cdot)$ is a continuous function on $[0, 2)$.*

Proof. Suppose $u(\cdot)$ is not continuous at $\varepsilon \geqslant 0$. If $\varepsilon > 0$, then there exist α, β, and γ such that $\sup_{b<\varepsilon} u(b) = \alpha < \beta < \gamma = \inf_{b>\varepsilon} u(b)$. Choose $\gamma_n \uparrow \varepsilon$ and $x_n, y_n \in S_X$, $f_n \in \nabla_{x_n}$ such that $f_n(x_n - y_n) = \gamma_n$, and $1 - \|(x_n + y_n)/2\| \leqslant \beta$. Therefore, $f_n(y_n) = 1 - \gamma_n \downarrow 1 - \varepsilon$. Choose $\eta_n \downarrow 1$ such that $f_n(y_n/\eta_n) = (1 - \gamma_n)/\eta_n < 1 - \varepsilon$ for all n (e.g., $\eta_n = (1 - \gamma_{n-1})/(1 - \epsilon)$ for all $n > 1$). This implies, by (1.2), that

$$1 - \left\| \frac{x_n + y_n/\eta_n}{2} \right\| \geqslant \gamma \quad \text{for all } n.$$

Finally,

$$1 - \gamma \geqslant \limsup_{n\to\infty} \left(\frac{1}{2}\right) \left\| x_n + \frac{y_n}{\eta_n} \right\| = \liminf_{n\to\infty} \left(\frac{1}{2}\right) \|x_n + y_n\| \geqslant 1 - \beta,$$

a contradiction.

For $\varepsilon = 0$, choose $\alpha_n \downarrow 0$ and take $x_n, y_n \in S_X$ and $f_n \in \nabla_{x_n}$ such that $f_n(x_n - y_n) = \alpha_n$ for all n. Since $\|x_n + y_n\| \geqslant f_n(x_n + y_n) = 1 + f_n(y_n)$ for all n,

$$\limsup_{n\to\infty} \left(1 - \frac{\|x_n + y_n\|}{2}\right) \leqslant \lim_{n\to\infty} \left(1 - \frac{1 + f_n(y_n)}{2}\right) = \lim_{n\to\infty} \frac{f_n(x_n - y_n)}{2} = 0.$$

This shows that $\lim_{n\to\infty} u(\alpha_n) = 0 = u(0)$. Since u is monotone, the proof is complete. □

Lemma 3.11. *X is an ε-u-space if and only if for any $r > 0$, and any sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\|x_n\|, \|y_n\|, \|(x_n + y_n)/2\| \to r$ and $f_n \in S_{X^*}$ satisfying $f_n(x_n) = \|x_n\|$ for all n, imply that*

$$\liminf_{n\to\infty} f_n(y_n) > r(1 - \varepsilon).$$

Proof. ($\Rightarrow$) Let X be an ε-u-space. Thus by Lemma 3.10, X is an ε'-u-space for some $\varepsilon' < \varepsilon$. Let $r > 0$, $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\|x_n\|, \|y_n\|, \|(x_n + y_n)/2\| \to r$ and $f_n \in S_{X^*}$ satisfying $f_n(x_n) = \|x_n\|$ for all n.

Let $x_n' = x_n/\|x_n\|$, $y_n' = y_n/\|y_n\|$. We then have $x_n', y_n' \in S_X$, $f_n(x_n') = 1$ for all n, and $\|(x_n' + y_n')/2\| \to 1$. This implies that $\liminf_{n\to\infty} f_n(y_n') \geqslant 1 - \varepsilon'$. Thus $\liminf_{n\to\infty} f_n(y_n) \geqslant r(1 - \varepsilon') > r(1 - \varepsilon)$.

($\Leftarrow$) This is trivial by letting $r = 1$. □

Combining Lemmas 3.2, 3.10, and 3.11, we can easily see that (i) all ε-u-spaces X have $J(2 - 2\varepsilon, X) < 2$, and (ii) [3, Corollary 3.8], all u-spaces X have $J(2 - \delta, X) < 2$ for all $\delta > 0$. In general, we have

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Proposition 3.12. *$J(2-\delta, X) < 2$ for all $\delta > J(X)$.*

Proof. Suppose the assertion is not true for some $\delta > J(X)$. Let $\delta > \epsilon > J(X)$. By Lemma 3.2, there exist sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in S_X satisfying

$$\|x_n + y_n\|, \|x_n - z_n\| \to 2 \quad \text{and} \quad \|y_n - z_n\| \leqslant 2-\delta \quad \text{for all } n. \tag{3.1}$$

Since $J(X) = \sup\{\varepsilon \in (0,2)\colon \delta_X(\varepsilon) \leqslant 1-\varepsilon/2\}$, by Proposition 2.3, $\|x_n - y_n\| < \varepsilon$ for all large n. It follows that for these n, $\|x_n - z_n\| \leqslant \|x_n - y_n\| + \|y_n - z_n\| \leqslant \varepsilon + 2 - \delta$. Hence $\delta \leqslant \varepsilon$, a contradiction. □

Proposition 3.13. *For a Banach space X, $J(2-\delta, X) < 2$ for all $\delta > \varepsilon_0(X)$. In particular, all ε-InQ spaces X have $J(2-\varepsilon, X) < 2$.*

Proof. Suppose the conclusion is not true. By Lemma 3.2, there exist sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in S_X satisfying (3.1) for some $\delta > \varepsilon_0(X)$. Thus, $\limsup_{n\to\infty} \|x_n - y_n\| < \delta$ and since $\|x_n - z_n\| \leqslant \|x_n - y_n\| + \|y_n - z_n\|$, letting $n \to \infty$, we have $2 < \delta + (2-\delta)$, a contradiction. □

For ε-uniformly smooth spaces, we have

Proposition 3.14. *All ε-uniformly smooth spaces X have $J(2-2\varepsilon, X) < 2$.*

Proof. Suppose that $J(2-2\varepsilon, X) = 2$. By Lemma 3.2, there exist sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in S_X such that $\|x_n + y_n\|, \|x_n - z_n\| \to 2$, and $\limsup_{n\to\infty} \|y_n - z_n\| = 2-\delta \leqslant 2-2\varepsilon$ for some $\delta \geqslant 2\epsilon$. Let $0 < t < 1$, we have $\|x_n + ty_n\| \to 1+t$, $\|x_n - tz_n\| \to 1+t$, and $\liminf_{n\to\infty} \|x_n - ty_n\| \geqslant 1 + t - t(2-\delta)$. Thus,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2}\left(\frac{\|x_n + ty_n\| + \|x_n - ty_n\| - 2}{t}\right) \\
&\quad \geqslant \frac{1}{2}\left(\frac{(1+t) + (1+t) - t(2-\delta) - 2}{t}\right) = \frac{\delta}{2} \geqslant \epsilon.
\end{aligned}$$

Hence $\rho_0(X) \geqslant \epsilon$, a contradiction. □

We close the paper by considering the stability of uniform normal structure of Banach spaces. This concept follows from the concept of the stability of the fixed point property of Banach spaces. In 1980, Bynum [1], showed that if for any $p > 1$ the Banach–Mazur distance $d(X, l_p) < 2^{1/p}$ or $= 2^{1/p}$, then X has normal structure, or X has the fpp, respectively. Using a result of Lin [20] it is known that X has the fpp if $d(X, l_p) < (\sqrt{33}-3)/2$. Khamsi [17] improved this number to c_p satisfying

$$c_p \geqslant c_2 = \sqrt{\frac{3+\sqrt{5}}{2}} > \frac{\sqrt{33}-3}{2}.$$

Finally, Lin [21] showed in 1999 that if

$$d(X, H) < \sqrt{\frac{5+\sqrt{13}}{2}}$$

for a Hilbert space H, then X has the fpp. In getting this result, Lin first gave a simple proof of Jiménez-Melado and Llorens-Fuster's result in [15], the paper that came out later. Then used it to prove the above result. We imitate this concept and work on uniform normal structure property. For more on this topic, we refer to [10, Chapter 7] and references therein.

Theorem 3.15. *Let X, Y be isomorphic spaces and $d(X, Y)$ their Banach–Mazur distance. If, for some $a \in [0, 1]$,*

$$d(X, Y) < \frac{3+a}{2J(ad(X, Y), Y)} \quad \text{or} \quad \frac{1+\sqrt{5}}{2J(Y)},$$

then $J(a, X) < (3+a)/2$ or $J(X) < (1+\sqrt{5})/2$, respectively. In particular, if Y is a Hilbert space and

$$d(X, Y) < \frac{1+\sqrt{5}}{2\sqrt{2}},$$

then X has uniform normal structure.

Proof. Let $a \in [0, 1]$ satisfy the given condition in the theorem. For each $\epsilon > 0$, there exists an isomorphism $\phi : (X, \|\cdot\|) \to (Y, \|\cdot\|)$ such that $M := \|\phi\|\|\phi^{-1}\| \leqslant (1+\epsilon)d(X, Y)$. We define a norm on Y by $|y| = \|\phi\|\|\phi^{-1}(y)\|$. Thus, $\|y\| \leqslant |y| \leqslant M\|y\|$ for all $y \in Y$ and $J(a, Y_{|\cdot|}) = J(a, X)$. Let x, y, and z be elements in $B_{(Y, |\cdot|)}$ with $|y - z| \leqslant a|x|$. Hence $x, y, z \in B_{(Y, \|\cdot\|)}$ and $\|y - z\| \leqslant aM\|x\|$. Since $|x + y| \wedge |x - z| \leqslant M(\|x + y\| \wedge \|x - z\|) \leqslant MJ(aM, Y_{\|\cdot\|})$, $J(a, X) = J(a, Y_{|\cdot|}) \leqslant MJ(aM, Y_{\|\cdot\|})$. Consequently, by the definition of the Banach–Mazur distance, we have $J(a, X) \leqslant d(X, Y)J(ad(X, Y), Y)$. The rest of the proof is clear. □

Corollary 3.16. *If, for some $a \in [0, 1]$,*

$$d(X, Y) < \frac{a - 2J(a, Y) + \sqrt{(2J(a, Y) - a)^2 + 4a(3+a)}}{2a},$$

then $J(a, X) < (3+a)/2$.

Proof. From the proof of Theorem 3.15 we have $J(a, X) \leqslant d(X, Y)J(ad(X, Y), Y)$. Using Proposition 3.6 and the fact that

$$d(X, Y)\left(J(a, X) + \frac{a(d(X, Y) - 1)}{2}\right) < \frac{3+a}{2}$$

if and only if

$$d(X, Y) < \frac{a - 2J(a, Y) + \sqrt{(2J(a, Y) - a)^2 + 4a(3+a)}}{2a},$$

we then have

$$d(X, Y) < \frac{3+a}{2J(ad(X, Y), Y)}$$

and Theorem 3.15 can then be applied. □

Remark. (1) In [8], it is proved that if $J(Y) < 3/2$, and if

$$d(X, Y) < \frac{7}{2(J(Y)+2)},$$

then X has uniform normal structure. This result is contained in Theorem 3.15, since it is seen that, when $a = 0$,

$$\frac{7}{2(J(Y)+2)} < \frac{3}{2J(Y)}$$

if and only if $J(Y) < 3/2$.

(2) Since

$$1 < \frac{a - 2J(a, Y) + \sqrt{(2J(a, Y) - a)^2 + 4a(3+a)}}{2a}$$

if and only if $J(a, Y) < (3+a)/2$, we note that, when $X = Y$, $J(a, X) < (3+a)/2$.

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Appendix 5: Remarks on convexity properties of Nakano spaces, (submitted)

REMARKS ON CONVEXITY PROPERTIES OF NAKANO SPACES

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Abstract

We characterize all geometric properties of Nakano spaces $l^{\{p_i\}}$ appeared in [3] without assuming the boundedness of the sequence $\{p_i\}$.

AMS Subject Classification(1991): 46B20, 46E30

Keywords: geometric property, Nakano space

1 Introduction

Geometric properties are important tools for studying the nonlinear functional analysis. In [3], Dhompongsa investigated many geometric properties of the Nakano space $l^{\{p_i\}}$ under the assumption that the sequence $\{p_i\}$ is bounded. We prove that this condition turns out to be a necessary condition of some of these geometric properties.

Recall that the Nakano space $l^{\{p_i\}}$, where $1 \le p_i < \infty$, is the space of all real sequences $x = (x(i))$ such that

$$\varrho(\lambda x) < \infty$$

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for some $\lambda > 0$. Here and elsewhere, the modular $\varrho(x)$ of x is defined by

$$\varrho(x) = \sum_{i=1}^{\infty} |x(i)|^{p_i}.$$

We equip this space with the norm defined by

$$\|x\| = \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

In fact, Nakano defined the norm, for each $x = (x(i))$ such that $\sum_{i=1}^{\infty} \frac{1}{p_i}|\frac{x(i)}{\lambda}|^{p_i} < \infty$ for some $\lambda > 0$, by

$$\|x\|' = \inf\left\{\lambda > 0 : \sum_{i=1}^{\infty} \frac{1}{p_i}\left|\frac{x(i)}{\lambda}\right|^{p_i} \leq 1\right\}.$$

However, both spaces are isometrically equal (see [4]).

Throughout this paper, we let e_i stand for the standard basis for $\mathbb{R}^\infty$. That is, $e_i = (\delta_{ij})_j$ for all i.

2 Preliminary

Lemma 1 *The following statements are equivalent:*

(i) *the sequence $\{p_i\}$ is unbounded;*

(ii) *there exists a norm-one element x such that*

$$\varrho(\lambda x) = \infty$$

for all $\lambda > 1$.

Proof (i)$\Rightarrow$(ii) We may assume that $p_i > i^2$ for all $i \in \mathbb{N}$. Define $x = (x(i))$ by $x(i) = \left(\frac{1}{2}\right)^{1/i}$. It is easy to see that $\|x\| = 1$ and $\varrho(\lambda x) = \infty$ for all $\lambda > 1$.

(ii)$\Rightarrow$(i) Let $p = \sup_i p_i$. Then, for each $x \in S(l^{\{p_i\}})$,

$$\varrho(\lambda x) \leq \lambda^p \varrho(x) \leq \lambda^p < \infty$$

for all $\lambda > 1$. $\square$

We consider the closed subspace

$$h^{\{p_i\}} = \{x \in l^{\{p_i\}} : \varrho(\lambda x) < \infty \text{ for all } \lambda > 0\}$$

and we define

$$\theta(x) = \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) < \infty\right\}.$$

Thus $x \in h^{\{p_i\}}$ if and only if $\theta(x) = 0$. Moreover, we have

Lemma 2 $\theta(x) = \inf\{\|x - y\| : y \in h^{\{p_i\}}\}$.

Proof See [1]. □

Note that $h^{\{p_i\}} = l^{\{p_i\}}$ if and only if $\limsup_{i\to\infty} p_i < \infty$. Thus, in [3], only geometric properties of $h^{\{p_i\}}$ are characterized.

3 Results

A careful reading allows us to prove only four following geometric properties, namely, k-rotundity, reflexivity, property (H) and uniform λ-property. For each of the first three properties, we prove that the boundedness of the sequence $\{p_i\}$ is its necessary condition. While the last property, the characterization in [3] still holds even if we drop the assumption on the boundedness of $\{p_i\}$.

3.1 k-Rotundity

A Banach space X is said to be *k-rotund* if for any $x_1, \ldots, x_{k+1} \in S(X)$ with $\|x_1 + \cdots + x_{k+1}\| = k + 1$ implies $x_1, \ldots, x_{k+1}$ are linearly dependent. Here $S(X) = \{x \in X : \|x\| = 1\}$.

Theorem 3 *If the Nakano space $l^{\{p_i\}}$ is k-rotund, then* $\limsup_{i\to\infty} p_i < \infty$.

Proof Suppose the assertion does not hold. Thus, by Lemma 1, there exists a norm-one element $x = (x(i))$ such that

$$\varrho(\lambda x) = \infty$$

for all $\lambda > 1$. Without loss of generality, we may assume that $x(1), \ldots, x(k+1) \neq 0$. For each $j \in \mathbb{N}$, we put $x_j = \sum_{i=j}^{\infty} x(i)e_i$. Then $\{x_1, \ldots, x_{k+1}\}$ is a linearly independent subset of $S(l^{\{p_i\}})$. Furthermore,

$$1 = \|x_{k+1}\| \leq \left\| \frac{x_1 + \cdots + x_{k+1}}{k+1} \right\| \leq 1.$$

This is a contradiction. □

3.2 Reflexivity

A Banach space X is said to be *reflexive* if the natural map from X into its second dual X^{**} is surjective. Equivalently, every bounded sequence in X has a weakly convergent subsequence.

Theorem 4 *If the Nakano space $l^{\{p_i\}}$ is reflexive, then* $\limsup_{i\to\infty} p_i < \infty$.

Proof Suppose not, by Lemma 1, there exists a norm-one element $x = (x(i))$ such that

$$\varrho(\lambda x) = \infty$$

for all $\lambda > 1$. For each $j \in \mathbb{N}$, we put $x_j = \sum_{i=j}^{\infty} x(i)e_i$. If the sequence $\{x_n\}$ has a weakly convergent subsequence, then $x_n \xrightarrow{w} 0$. By the Hahn-Banach Theorem, there exists a norm-one functional f such that

$$f(x) = \inf_{y \in h^{\{p_i\}}} \|x - y\| = \theta(x) = 1$$

and $f(y) = 0$ for all $y \in h^{\{p_i\}}$. On the other hand, we have

$$f(x_n) = f(x) + f(x_n - x) = 1$$

for all $n \in \mathbb{N}$ which is a contradiction since $x_n \xrightarrow{w} 0$. □

3.3 Property (H)

A Banach space X is said to have *property (H)* if weak convergence and norm convergence coincide on $S(X)$, i.e. for any $x_n, x \in S(X)$ with $x_n \xrightarrow{w} x$ implies $x_n \longrightarrow x$.

Theorem 5 *If the Nakano space $l^{\{p_i\}}$ has property (H), then* $\limsup_{i\to\infty} p_i < \infty$.

Proof See [2]. □

3.4 Uniform λ-property

A point $x \in S(X)$ is said to be an *extreme point* if it cannot be a midpoint of any two distinct points in $S(X)$. This means that if $x = \frac{y+z}{2}$ and $y, z \in S(X)$, then $y = z$. We define

$$\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y, e \text{ is an extreme point}, \|y\| \leq 1\}.$$

A Banach space X is said to have *uniform λ-property* if

$$\lambda(X) := \inf\{\lambda(x) : x \in S(X)\} > 0.$$

It is easy to see that if x is an extreme point, then $\lambda(x) = 1$. But the converse does not hold. However, it is not difficult to prove that if $\lambda(x) = 1$, then x is a limit point of the set of extreme points.

Theorem 6 *For the Nakano space $l^{\{p_i\}}$,*

$$\lambda(l^{\{p_i\}}) = \inf\{\lambda(x) : \varrho(x) = 1\}.$$

Proof It suffices to prove that

$$\lambda(l^{\{p_i\}}) \geq \inf\{\lambda(x) : \varrho(x) = 1\} := \lambda_0.$$

Let $x = (x(i)) \in S(l^{\{p_i\}})$ be such that $\varrho(x) < 1$. Then, for any $\alpha \in (0,1)$,

$$\varrho\Big(\frac{x}{1-\alpha}\Big) = \infty.$$

For every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$\sum_{i=1}^{k_n} \left|\frac{x(i)}{1-\frac{1}{n}}\right|^{p_i} > 1.$$

We can choose $\alpha_n \in (0, \frac{1}{n})$ so that

$$\sum_{i=1}^{k_n} \left|\frac{x(i)}{1-\alpha_n}\right|^{p_i} + \sum_{i=k_n+1}^{\infty} |x(i)|^{p_i} = 1.$$

Define

$$y = \sum_{i=1}^{k_n} \frac{x(i)}{1-\alpha_n} e_i + \sum_{i=k_n+1}^{\infty} x(i) e_i$$

and

$$z = \sum_{i=k_n+1}^{\infty} x(i)e_i.$$

Then $\varrho(y) = 1$ and $\|z\| \le 1$. Moreover,

$$x = (1-\alpha_n)y + \alpha_n z.$$

Hence, by Proposition 2.12 of [1],

$$\lambda(x) \ge (1-\alpha_n)\lambda(y) \ge (1-\alpha_n)\lambda_0.$$

Letting $n \to \infty$ yields $\lambda(x) \ge \lambda_0$ and then $\lambda(l^{\{p_i\}}) \ge \lambda_0$. This completes the proof. □

The following is also proved in [5] without assuming the boundedness of the sequence $\{p_i\}$.

Proposition 7 *For the Nakano sequence space $l^{\{p_i\}}$ and $x = (x(i)) \in S(l^{\{p_i\}})$, x is an extreme point if and only if*

(i) $\varrho(x) = 1$ *and*

(ii) $\#\{i \in \mathbb{N} : x(i) \neq 0$ *and* $p_i = 1\} \le 1$.

Here $\#A$ denotes the cardinality of a set A.

Supplement to the original proof of Theorem 5 in [3], we have

Theorem 8 *The Nakano space $l^{\{p_i\}}$ has uniform λ-property if and only if $w := \#\{i \in \mathbb{N} : p_i = 1\} < \infty$.*
Furthermore,

$$\lambda(l^{\{p_i\}}) = \begin{cases} \frac{1}{w} & \text{if } w \ge 1, \\ 1 & \text{if } w = 0. \end{cases} \tag{$*$}$$

Proof We need only prove $(*)$ when $w = 0$ and 1. In these cases, by Proposition 7, $\{x \in S(l^{\{p_i\}}) : \varrho(x) = 1\}$ is just the set of all extreme points. Therefore, by the observation before Theorem 4, $\lambda(x) = 1$ for all $x \in S(l^{\{p_i\}})$ with $\varrho(x) = 1$. □

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Appendix 6: Preservation of uniform smoothness and U-convexity by ψ-direct sums, (submitted)

Preservation of uniform smoothness and U-convexity by ψ-direct sums*

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Abstract

We study the ψ-direct sum, introduced by K.-S. Saito and M. Kato, of U-spaces, introduced by K. S. Lau. For Banach spaces X and Y and a continuous convex function ψ on the unit interval $[0,1]$ satisfying certain conditions, let $X \oplus_\psi Y$ be the ψ-direct sum of X and Y equipped with the norm associated with ψ. We first show that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is isometric to the space $X^* \oplus_\varphi Y^*$ for some continuous convex function φ satisfying the same conditions as of ψ. We introduce the so-called u-spaces and show that: (1) $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function. We also show that (2) $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function. As consequences, using the notion of ultrapower, we obtain : (3) $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function, and (4) $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

MSC: primary 46B20; secondary 46B08

Keywords: ψ-direct sums; Smooth spaces; u-spaces; Uniformly smooth spaces; U-spaces

1 Introduction

For every continuous convex function ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$, there corresponds a unique absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ (see Bonsall and Duncan [4], also [18]). Recently, in [15] the authors introduced the ψ-direct sums $X \oplus_\psi Y$ of Banach spaces X

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and Y equipped with the norm associated with ψ, and proved that $X \oplus_\psi Y$ is uniformly convex if and only if X, Y are uniformly convex and ψ is strictly convex. We write $X \simeq Y$ to indicate that X and Y are isometric (or Banach isomorphism, see [11]).

The purposes of this paper are to characterize uniform smoothness and U-convexity of $X \oplus_\psi Y$. In Section 2 we shall recall some fundamental facts on the ψ-direct sums of Banach spaces and introduce the dual function φ of ψ so that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is $X^* \oplus_\varphi Y^*$. In Section 3 we shall show that the ultrapower of $X \oplus_\psi Y$ is the ψ-direct sum of the ultrapowers of X and of Y. In Section 4 we shall prove that $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function, and by using the ultrapower technique we obtain that $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function. In Section 5 we introduce new spaces, namely u-spaces, and prove that $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function, and again by using the ultrapower we have $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-space and ψ is a u-function.

2 The ψ-direct sums

Let X be a Banach space. Throughout this paper, let X^* be the dual space of X, $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \leq 1\}$, and for $x \neq 0$, $\nabla_x = \{f \in S_{X^*} : f(x) = \|x\|\}$. In this section we shall recall the definition of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y. A norm on $\mathbb{C}^2$ is called *absolute* if $\|(z, w)\| = \|(|z|, |w|)\|$ for all $(z, w) \in \mathbb{C}^2$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{C}^2$ is denoted by N_a. The l_p-norms $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ on $\mathbb{C}^2$ are examples of such norms, and for any norm $\|\cdot\| \in N_a$,

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$

Let Ψ be the set of all continuous convex functions ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$. N_c and Ψ are in one-to-one correspondence under the following equations. For each $\|\cdot\| \in N_a$, the function ψ defined by $\psi(t) = \|(1-t, t)\|$ $(0 \leq t \leq 1)$ belongs to Ψ. Conversely, for each $\psi \in \Psi$, let $\|(0,0)\|_\psi = 0$, and $\|(z, w)\|_\psi = (|z| + |w|)\psi(\frac{|w|}{|w|+|z|})$ for $(z, w) \neq (0,0)$ and this norm belongs to N_a (see [4] and [18]). For Banach spaces X and Y, we denote by $X \oplus_\psi Y$ the direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\| = \|(\|x\|, \|y\|)\|_\psi \text{ for } (x, y) \in X \oplus Y.$$

Thus, under this norm, $X \oplus_\psi Y$, which will be called the ψ-direct sum of X and Y, is a Banach space and for all $(x, y) \in X \oplus Y$ we also have (see [15])

$$\|(x, y)\|_\infty \leq \|(x, y)\|_\psi \leq \|(x, y)\|_1.$$

Now we show that the dual space of this ψ-direct sum is a direct sum $X \oplus_\varphi Y$ of the same kind for some $\varphi \in \Psi$. We first define

$$\varphi_\psi(s) = \varphi(s) := \sup_{t \in [0,1]} \frac{st + (1-s)(1-t)}{\psi(t)}$$

for $s \in [0,1]$. We show that $\varphi \in \Psi$ and call it *the dual function* of ψ.

Proposition 1 *The above function φ is continuous, convex on $[0,1]$ and satisfies $\varphi(s) \geq \max\{s, 1-s\}$ for all $s \in [0,1]$.*

Proof. It is easy to see that $\varphi(\cdot)$ is continuous. To show that φ is convex, we let $s_1, s_2 \in [0,1]$ and consider

$$\begin{aligned}
\varphi\Big(\frac{s_1+s_2}{2}\Big) &= \sup_{t\in[0,1]} \frac{\frac{s_1+s_2}{2}t + (1 - \frac{s_1+s_2}{2})(1-t)}{\psi(t)} \\
&= \sup_{t\in[0,1]} \frac{1}{2}\frac{s_1t + s_2t + (1-s_1)(1-t) + (1-s_2)(1-t)}{\psi(t)} \\
&\leq \frac{1}{2}(\varphi(s_1) + \varphi(s_2)),
\end{aligned}$$

which verifies the convexity of $\varphi(\cdot)$. Next we prove the last assertion. Since $\psi(t) \leq 1$ for all $t \in [0,1]$,

$$\varphi(s) \geq \sup_{t\in[0,1]} \{st + (1-s)(1-t)\} \geq \max\{s, 1-s\}$$

for all $s \in [0,1]$, and the proof is complete. $\square$

Theorem 2 *The dual space $(X \oplus_\psi Y)^*$ is isometric to $X^* \oplus_\varphi Y^*$, where φ is the dual function of ψ. Moreover, each bounded linear functional F in $(X \oplus_\psi Y)^*$ can be (uniquely) represented by (f,g) where $f \in X^*$ and $g \in Y^*$ and*

$$F(x,y) = f(x) + g(y)$$

for all $(x,y) \in X \oplus_\psi Y$. In this case, $\|F\| \leq \|(f,g)\|_\varphi \|(x,y)\|_\psi$.

Proof. Define $T : X^* \oplus_\varphi Y^* \to (X \oplus_\psi Y)^*$ by

$$T(f,g)(x,y) = f(x) + g(y)$$

where $f \in X^*$, $g \in Y^*$, $x \in X$, and $y \in Y$. It is easy to see that T is linear. Moreover, by the definition of φ, we have, recalling that the norm of each nonzero element (f,g) of the φ-direct sum $X^* \oplus_\varphi Y^*$ is defined by

$$\|(f,g)\|_\varphi = (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big),$$

$$\begin{aligned}
|T(f,g)(x,y)| &\leq \|f\|\|x\| + \|g\|\|y\| \\
&= (\|f\| + \|g\|)(\|x\| + \|y\|)\frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\| + \|g\|)(\|x\| + \|y\|)} \\
&\leq (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big)(\|x\| + \|y\|)\psi\Big(\frac{\|y\|}{\|x\| + \|y\|}\Big) \\
&= \|(f,g)\|_\varphi\|(x,y)\|_\psi,
\end{aligned}$$

for all nonzero (f,g). Thus, $T(f,g)$ is actually an element of $(X \oplus_\psi Y)^*$. For each $F \in (X \oplus_\psi Y)^*$, $F(\cdot,0)$ and $F(0,\cdot)$ are bounded linear functionals on X and Y, respectively. Put $f(x) = F(x,0)$ and $g(y) = F(0,y)$, then $T(f,g) = F$ and the surjectivity of T is proved.

Finally we prove that T is an isometry, i.e., $\|T(f,g)\| = \|(f,g)\|_\varphi$. From the above calculation, we always have $\|T(f,g)\| \leq \|(f,g)\|_\varphi$. Now we prove the reverse inequality. We choose sequences $\{t_n\} \subset [0,1]$, $\{x_n\} \subset S_X$, and $\{y_n\} \subset S_Y$ so that

$$\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)\|f\|}{\|f\| + \|g\|} + \frac{t_n\|g\|}{\|f\| + \|g\|}\Big) \to \varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big),$$

$$f(x_n) \to \|f\|, \quad \text{and} \quad g(y_n) \to \|g\| \quad \text{as } n \to \infty.$$

Therefore, since $\frac{1}{\psi(t_n)}((1-t_n)x_n, t_ny_n) \in S_{X\oplus_\psi Y}$,

$$\begin{aligned}
\|T(f,g)\| &\geq \frac{1}{\psi(t_n)}\Big(f((1-t_n)x_n) + g(t_ny_n)\Big) \\
&= (\|f\| + \|g\|)\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)f(x_n)}{\|f\| + \|g\|} + \frac{t_ng(y_n)}{\|f\| + \|g\|}\Big).
\end{aligned}$$

The last expression tends to $\|(f,g)\|_\varphi$ as $n \to \infty$, proving that $\|T(f,g)\| \geq \|(f,g)\|_\varphi$ and this completes the proof. □

Our first application of Theorem 2 is to show that reflexivity is preserved under the ψ-direct sums.

Corollary 3 *For each $\psi \in \Psi$, $X \oplus_\psi Y$ is reflexive if and only if X and Y are reflexive.*

Proof. We only proof the sufficiency. We first show, without using reflexivity, that $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\psi Y^{**}$, i.e., they are isometric. For this, we let φ and then θ be the dual functions of ψ and of φ, respectively. Thus $(X \oplus_\psi Y)^* \simeq X^* \oplus_\varphi Y^*$ by the isometry T where $TF = (F_1, F_2)$, $F_1 = F(\cdot,0)$ and $F_2 = F(0,\cdot)$; and $(X^* \oplus_\varphi Y^*)^* \simeq X^{**} \oplus_\theta Y^{**}$ by the isometry U where $UG = (G_1, G_2)$, $G_1 = G(\cdot,0)$ and $G_2 = G(0,\cdot)$. Hence $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\theta Y^{**}$ via the isometry which maps $L \in (X \oplus_\psi Y)^{**}$ to $ULT^{-1} = (LT^{-1}(\cdot,0), LT^{-1}(0,\cdot)) \in X^{**} \oplus_\theta Y^{**}$ so that $ULT^{-1}(x^*,y^*) = (LT^{-1}(x^*,0), LT^{-1}(0,y^*)) = (L(x^*,0), L(0,y^*)) = (L_1(x^*), L_2(y^*))$. In particular, when $L = L_{(x,y)}$, the evaluation map at (x,y), i.e., $L_{(x,y)}(F) = F(x,y) = F_1(x) + F_2(y)$ for $F \in (X \oplus_\psi Y)^*$, $UL_{(x,y)}T^{-1}(x^*,y^*) =$

$x^*(x)+y^*(y) = L_x(x^*)+L_y(y*) = (L_x, L_y)(x^*, y^*)$. This shows that $\|(x,y)\|_\psi = \|L_{(x,y)}\| = \|(L_x, L_y)\|_\theta$ for $(x,y) \in X \oplus Y$. Therefore, $\psi(\frac{\|y\|}{\|x\|+\|y\|}) = \theta(\frac{\|L_y\|}{\|L_x\|+\|L_y\|}) = \theta(\frac{\|y\|}{\|x\|+\|y\|})$ for $\|x\| + \|y\| \neq 0$. From this we can easily see that $\psi = \theta$.

Now suppose that X and Y are reflexive. Thus elements in X^{**} and Y^{**} are of the form L_x and L_y for some $x \in X$ and $y \in Y$. To show that $(X \oplus_\psi Y)^{**}$ is reflexive, let $L \in (X \oplus_\psi Y)^{**}$ and consider, for each $F \in (X \oplus Y)^*$, $L(F) = L(F_1, 0) + L(0, F_2) = L_x(F_1) + L_y(F_2) = F_1(x) + F_2(y) = L_{(x,y)}(F)$, for some $x \in X$ and $y \in Y$. That is $L = L_{(x,y)}$ showing that $X \oplus_\psi Y$ is reflexive and the proof is complete. $\square$

We observe that $X \oplus_\psi Y$ is super-reflexive when (and only when) X and Y are super-reflexive. By Henson and Moore [7], this is equivalent to showing that the ultrapower $\widetilde{X \oplus_\psi Y}$ is reflexive. But this follows from Proposition 4 below and Corollary 3.

3 Ultrapowers of the ψ-direct sums

The ultrapower of a Banach space is proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. In this section we prove that every ultrapower of a ψ-direct sum is isometric to the ψ-direct sum of their ultrapowers. First we recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i \in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i \in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i \in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i \in I} X_i$ equipped with the norm $\|(x_i)\| := \sup_{i \in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from remark (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [16]).

Proposition 4

$$\widetilde{X \oplus_\psi Y} \simeq \widetilde{X} \oplus_\psi \widetilde{Y}.$$

Proof. Define $T : \widetilde{X \oplus_\psi Y} \to \widetilde{X} \oplus_\psi \widetilde{Y}$ by $T(\widetilde{x,y}) = (\widetilde{x}, \widetilde{y})$ for $(\widetilde{x,y}) \in \widetilde{X \oplus_\psi Y}$. Let $(\widetilde{x,y}) = (x_n, y_n)_\mathcal{U}$ and $(\widetilde{x',y'}) = (x'_n, y'_n)_\mathcal{U}$. If $(\widetilde{x,y}) = (\widetilde{x',y'})$, then

$$\begin{aligned} 0 &= \lim_\mathcal{U} \|(x_n - x'_n, y_n - y'_n)\|_\psi \\ &= \lim_\mathcal{U} (\|x_n - x'_n\| + \|y_n - y'_n\|)\psi\Big(\frac{\|y_n - y'_n\|}{\|x_n - x'_n\| + \|y_n - y'_n\|}\Big). \end{aligned}$$

This implies

$$\lim_\mathcal{U} (\|x_n - x'_n\| + \|y_n - y'_n\|) = 0,$$

and hence

$$\lim_\mathcal{U} \|x_n - x'_n\| = \lim_\mathcal{U} \|y_n - y'_n\| = 0.$$

This means $(\widetilde{x}, \widetilde{y}) = (\widetilde{x'}, \widetilde{y'})$ and T is well-defined. It is easy to see that T is linear and onto. Now we prove that T is an isometry. To this end, let $(\widetilde{x,y}) = (x_n, y_n)_\mathcal{U}$, so

$$\begin{aligned} \|T(\widetilde{x,y})\| &= \|(\widetilde{x}, \widetilde{y})\|_\psi \\ &= (\|\widetilde{x}\| + \|\widetilde{y}\|)\psi\Big(\frac{\|\widetilde{y}\|}{\|\widetilde{x}\| + \|\widetilde{y}\|}\Big) \\ &= (\lim_\mathcal{U} \|x_n\| + \lim_\mathcal{U} \|y_n\|)\psi\Big(\frac{\lim_\mathcal{U} \|y_n\|}{\lim_\mathcal{U} \|x_n\| + \lim_\mathcal{U} \|y_n\|}\Big) \\ &= \lim_\mathcal{U} (\|x_n\| + \|y_n\|)\psi\Big(\frac{\|y_n\|}{\|x_n\| + \|y_n\|}\Big) \\ &= \lim_\mathcal{U} \|(x_n, y_n)\|_\psi \\ &= \|(\widetilde{x,y})\|, \end{aligned}$$

completing the proof. □

It is known that X is uniformly convex if and only if $\widetilde{X}$ is strictly convex (see [16]). Also, $X \oplus_\psi Y$ is strictly convex if and only if X and Y are strictly convex and ψ is strictly convex (see [18, Theorem 6]). Combining these results and Proposition 4 gives

Corollary 5 *[15, Theorem 1] Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is uniformly convex if and only if X and Y are uniformly convex and ψ is strictly convex.*

Following T. Landes [10], a normed space Z is a *substitution* space (with index $I \neq \emptyset$ with any cardinality) whenever Z has a (Shauder) basis $(e_i)_{i\in I}$ (unconditional if I is uncountable) and the norm of Z is *monotone*, i.e., $\|z\| \le \|z'\|$ whenever $0 \le z_i \le z_i'$ for all $i \in I$ ($z, z' \in Z$), where we write $z = \sum_{i\in I} z_i e_i$ for $z \in Z$. Given a family $(X_i)_{i\in I}$ of normed spaces, then the Z *direct sum* $\bigoplus_{i\in Z} X_i$ of the family (X_i) is defined to be the space $\{x = (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\| e_i \in Z\}$ endowed with the norm $\| \sum_{i\in I} \|x_i\| e_i \|_Z$.

A property P defined for normed spaces is said to be preserved under the Z-direct-sum-operation, if the Z-direct sums of a family $(X_i)_{i\in I}$ of normed spaces satisfies P whenever all X_i do so.

The following proposition shows that, under some conditions, "normal structure" is preserved under the Z-direct-sum-operation. This result improves the first permanence result for normal structure obtained by Belluce, Kirk, and Steiner [3].

Proposition 6 *[10, Theorem 2, Corollary 3 and Corollary 4] Let Z be a substitution space with index set $I = \{1, \ldots, N\}$ such that*

$$\|z + z'\| < 2 \text{ whenever } \|z\| = \|z'\| = 1, z_i \ge 0, z_i' \ge 0 \text{ for all } i \in I,$$
$$\text{and } z_i = z_i' \text{ only for these } i \in I \text{ for which } z_i = z_i' = 0.$$

Thus, normal structure is preserved under the Z-direct-sum-operation. In particular, if Z is strictly convex or $Z = l_p^N$ for any p with $1 < p \le \infty$.

In case $I = \{1, 2\}$ and ψ is strictly convex, it follows from [4, Lemma 2] and [18, Theorem 5] that the norm $\| \cdot \|_\psi$ is monotone and strictly convex on $\mathbb{C}^2$. Thus, in the light of super-reflexivity, we can extend "normal structure" to "uniform normal structure" for ψ-direct sums whenever ψ is strictly convex.

Corollary 7 *Let X and Y be super-reflexive Banach spaces and $\psi \in \Psi$. Suppose ψ is strictly convex. Then, the ψ-direct sum $X \oplus_\psi Y$ has uniform normal structure if and only if X and Y have uniform normal structure.*

Proof. Note that, by Khamsi [9], it suffices to show that the ultrapower $\widetilde{(X \oplus_\psi Y)}$ has normal structure. But this is an immediate consequence of Proposition 4 together with Proposition 6. □

It is well-known that every uniformly nonsquare space is super-reflexive (see [8]). Thus, Corollary 7 and [5, Corollary 3.7] give

Corollary 8 *Let X and Y be Banach spaces and $\psi \in \Psi$ be strictly convex. Then, if $C_{\mathrm{NJ}}(1, X) < 2$ and $C_{\mathrm{NJ}}(1, Y) < 2$, the ψ-direct sum $X \oplus_\psi Y$ has uniform normal structure.*

It is interesting to see if we can conclude that $C_{\mathrm{NJ}}(1, X \oplus_\psi Y) < 1$ in Corollary 8.

7

4 Smoothness of the ψ-direct sums

A Banach space X is said to be *smooth* if for any $x \in S_X$, ∇_x is a singleton. We recall that a continuous convex function ψ has left and right derivatives ψ'_L, ψ'_R. Let G be defined on $[0,1]$ by

$$\begin{aligned} G(0) &= [-1, \psi'_R(0)],\ G(1) = [\psi'_L(1), 1], \\ G(t) &= [\psi'_L(t), \psi'_R(t)]\ (0 < t < 1). \end{aligned}$$

Given $\psi \in \Psi$, $t \in [0,1]$, let

$$x(t) = \frac{1}{\psi(t)}(1-t, t)$$

so that $\|x(t)\|_\psi = 1$. In [4], the authors identified the dual of $(\mathbb{C}^2, \|.\|_\psi)$ with $\mathbb{C}^2$ and used this fact to provide a proof of the following lemma.

Lemma 9 *[4, Lemma 4] For ψ, G, and x defined above,*

(1) $\nabla_{x(t)} = \{(\psi(t) - t\gamma, \psi(t) + (1-t)\gamma) : \gamma \in G(t)\}$ *for* $0 < t < 1$,

(2) $\nabla_{x(0)} = \{(1, z(1+\gamma)) : \gamma \in G(0), |z| = 1\}$, *and*

(3) $\nabla_{x(1)} = \{(z(1-\gamma), 1) : \gamma \in G(1), |z| = 1\}$.

In general, using Theorem 2 and Lemma 9, we have the following:

Lemma 10 *Let $(x, y) \in S_{X\oplus_\psi Y}$ and $t = \frac{\|y\|}{\|x\|+\|y\|}$. Thus*

$$\begin{aligned} (1)\ \nabla_{(x,y)} &= \{((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|} \text{ and } \\ &\quad\ g \in \nabla_{y/\|y\|}\} \text{ for } 0 < t < 1, \\ (2)\ \nabla_{(x,y)} &= \{(f, (1+\gamma)g) : \gamma \in G(0), f \in \nabla_x \text{ and } g \in S_{Y^*}\} \text{ for } t = 0, \text{ and} \\ (3)\ \nabla_{(x,y)} &= \{((1-\gamma)f, g) : \gamma \in G(1), g \in \nabla_y \text{ and } f \in S_{X^*}\} \text{ for } t = 1. \end{aligned}$$

Proof. We prove (1). Let $F = (f, g) \in \nabla_{(x,y)}$, then

$$\begin{aligned} F((x,y)) &= f(x) + g(y) \\ &\le \|f\|\|x\| + \|g\|\|y\| \\ &= \frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\| + \|g\|)(\|x\| + \|y\|)}(\|f\| + \|g\|)(\|x\| + \|y\|) \\ &\le \varphi\left(\frac{\|g\|}{\|f\| + \|g\|}\right)\psi\left(\frac{\|y\|}{\|x\| + \|y\|}\right)(\|f\| + \|g\|)(\|x\| + \|y\|) \\ &= \|F\|_\varphi\|(x,y)\|_\psi = 1. \end{aligned}$$

Thus, we have $\|f\|\|x\| + \|g\|\|y\| = 1$ and $f(x) = \|f\|\|x\|$ $g(y) = \|g\|\|y\|$, hence $(\|f\|, \|g\|) \in \nabla_{(\|x\|,\|y\|)}$ and $\frac{f}{\|f\|} \in \nabla_{\frac{x}{\|x\|}}$, $\frac{g}{\|g\|} \in \nabla_{\frac{y}{\|y\|}}$. We observe that $(\|x\|, \|y\|) = \frac{1}{\psi(t)}(1-t, t)$, thus it follows from Lemma 9 that

$$\|f\| = \psi(t) - t\gamma \text{ and } \|g\| = \psi(t) + (1-t)\gamma, \text{ for some } \gamma \in G(t).$$

Consequently, we have $(f, g) = (\|f\|\frac{f}{\|f\|}, \|g\|\frac{g}{\|g\|}) = ((\psi(t) - t\gamma)\frac{f}{\|f\|}, (\psi(t) + (1-t))\gamma\frac{g}{\|g\|})$. Thus, we have proved that $\nabla_{(x,y)} \subset \{((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|} \text{ and } g \in \nabla_{y/\|y\|}\}$. On the other hand, let $F = ((\psi(t)-t\gamma)f, (\psi(t)+(1-t)\gamma)g)$ where $\gamma \in G(t), f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}$. Consider, by using Lemma 9,

$$
\begin{aligned}
F((x, y)) &= (\psi(t) - t\gamma)f(x) + (\psi(t) + (1-t)\gamma)g(y) \\
&= (\psi(t) - t\gamma)\|x\| + (\psi(t) + (1-t)\gamma)\|y\| \\
&= (\|x\| + \|y\|)((\psi(t) - t\gamma)(1-t) + (\psi(t) + (1-t)\gamma)t) \\
&= \frac{1}{\psi(t)}((\psi(t) - t\gamma)(1-t) + (\psi(t) + (1-t)\gamma)t) \\
&= 1.
\end{aligned}
$$

Hence, (1) has been proved. The proof of (2) and (3) can be proceeded similarly. □

We say that a function ψ is *smooth* if the following conditions hold:

(1) ψ is *smooth* at every $t \in (0, 1)$, i.e., the derivative of ψ exists at t,

(2) the right derivative of ψ at 0 is -1, and

(3) the left derivative of ψ at 1 is 1.

Theorem 11 *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is smooth if and only if X and Y are smooth and ψ is smooth.*

Proof. *Necessity.* Assume that $X \oplus_\psi Y$ is smooth. Because X is isometric to $X \oplus_\psi \{0\}$ which is a subspace of $X \oplus_\psi Y$, then X and similarly Y must be smooth. It remains to prove that ψ is smooth, but by Lemma 10, if ψ is not smooth, there exists $(x, y) \in S_{X \oplus_\psi Y}$ such that $\nabla_{(x,y)}$ contains more than one point which can not happen, and the smoothness of ψ is proved

Sufficiency. This follows from Lemma 10. □

Again, since, for every Banach space X, X is uniformly smooth if and only of $\tilde{X}$ is smooth, we obtain

Corollary 12 *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is uniformly smooth if and only if X and Y are uniformly smooth and ψ is smooth.*

5 U-spaces and u-spaces

A Banach space X is called a *U-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in S_X$, we have $\|x + y\| \leq 2(1 - \delta)$ whenever $f(y) < 1 - \varepsilon$ for some $f \in \nabla_x$ (see [12]). A Banach space X is called a *u-space* if for any $x, y \in S_X$ with $\|x + y\| = 2$, we have $\nabla_x = \nabla_y$. Obviously, every U-space is a u-space.

Remark 13 *Let us collect together some properties of u-spaces and U-spaces:*

(1) If X^* is a u-space, then X is a u-space. The converse holds whenever X is reflexive.

(2) If X is a U-space, then X is a u-space. The converse holds whenever $\dim X < \infty$.

(3) $\widetilde{X}$ is a u-space if and only if X is a U-space.

Proof. (1) Let $x, y \in S_X$ be such that $\|x + y\| = 2$. We prove that $\nabla_x = \nabla_y$. Let $f \in \nabla_x$, and $h \in \nabla_{x+y}$. It follows that $h(x) = h(y) = 1$ and $\|f + h\| = 2$. By the assumption that X^* is a u-space and $h(y) = 1$, we have $f(y) = 1$. This implies that $\nabla_x \subset \nabla_y$, and then $\nabla_x = \nabla_y$ as required.

(2) The first assertion is obvious and the latter one follows from the compactness of the unit ball.

(3) It is known that $\widetilde{X}$ is a U-space if and only if X is a U-space (see [6] or [14]). In virtue of (2), it suffices to prove that X is a U-space whenever $\widetilde{X}$ is a u-space. Suppose that X is not a U-space. Then there exist an $\epsilon_0 > 0$ and sequences $\{x_n\}, \{y_n\} \subset S_X$, and $\{f_n\} \subset S_{X^*}$ such that $f_n(x_n) = 1$ and $f_n(x_n - y_n) \geq \epsilon_0$ for all $n \in \mathbb{N}$, and $\|x_n + y_n\| \to 2$ as $n \to \infty$. We put $\widetilde{x} = (x_n)_{\mathcal{U}}$, $\widetilde{y} = (y_n)_{\mathcal{U}}$ and $\widetilde{f} = (f_n)_{\mathcal{U}}$. Thus $\|x + \widetilde{y}\| = 2$, $\widetilde{f}(\widetilde{x}) = 1$ and $\widetilde{f}(\widetilde{y}) \leq 1 - \epsilon_0 < 1$. This means that $\nabla_{\widetilde{x}} \neq \nabla_{\widetilde{y}}$ which implies that $\widetilde{X}$ is not a u-space. $\square$

U-spaces can be considered as the "uniform" version of u-spaces. The following diagram explains this claim as well as it shows how the u-spaces are well-placed (see [1], [5], [6], [13], and [14]):

$$X \text{ is UC} \;\Leftrightarrow\; \widetilde{X} \text{ is UC} \;\Leftrightarrow\; \widetilde{X} \text{ is SC}$$

$$X \text{ is US} \;\Leftrightarrow\; \widetilde{X} \text{ is US} \;\Leftrightarrow\; \widetilde{X} \text{ is S}$$

$$X \text{ is UNC} \;\Leftrightarrow\; \widetilde{X} \text{ is UNC} \;\Leftrightarrow\; \widetilde{X} \text{ is NC}$$

$$X \text{ is a U-space} \;\Leftrightarrow\; \widetilde{X} \text{ is a U-space} \;\Leftrightarrow\; \widetilde{X} \text{ is a u-space}$$

$$\begin{array}{ccccc}
 & & C_{\mathrm{NJ}}(1, X) < 2 & \Rightarrow & \text{UNS} \\
 & & \Uparrow & & \\
\text{UC} & \Rightarrow & U & \Rightarrow & \text{UNSQ} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{SC} & \Rightarrow & u & \Rightarrow & \text{NSQ}
\end{array}
\qquad
\begin{array}{ccccc}
\text{US} & \Rightarrow & U & \Rightarrow & \text{UNSQ} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{S} & \Rightarrow & u & \Rightarrow & \text{NSQ}
\end{array}$$

UC $\equiv$ Uniformly Convex, SC $\equiv$ Strictly Convex, US $\equiv$ Uniformly Smooth, S $\equiv$ Smooth, UNC $\equiv$ Uniformly Noncreasy , NC $\equiv$ Noncreasy, $C_{\mathrm{NJ}}(\cdot)$ $\equiv$ a generalized Jordan-von Neumann constant, UNS $\equiv$ Uniform Normal Structure, UNSQ $\equiv$ Uniformly Nonsquare, NSQ $\equiv$ Nonsquare, U $\equiv$ a U-space, u $\equiv$ a u-space

Examples of u-spaces which are not U-spaces can be obtained from the direct product spaces $(\mathbb{R}^2_{p_1} \oplus \mathbb{R}^2_{p_2} \oplus \mathbb{R}^2_{p_3} \oplus \cdots)_2$ where (p_n) is a sequence of positive numbers strictly decreasing to 1, and $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ where each l_n is the l_n-space. Actually, both spaces are strictly convex, but with the James constant and the Jordan-von Neumann constant are both equal to 2, i.e., the spaces are not uniformly nonsquare, and hence can not be U-spaces. Sims and Smith [17] have shown that the space $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ has asymptotic property (P) but not property (P).

Examples of infinite dimensional u-spaces that are not strictly convex or smooth are easily established.

Let $\psi \in \Psi$. We say that ψ is a *u-function,* if for any interval $[a,b] \subset (0,1)$, we have ψ is smooth at a and b whenever ψ is affine on $[a,b]$.

Theorem 14 *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the Banach space $X \oplus_\psi Y$ is a u-space if and only if X and Y are u-spaces and ψ is a u-function.*

Proof. *Necessity* . Suppose there exist a and $b \in [0,1]$ such that ψ is affine on $[a,b]$ but $\psi'_-(a) < \psi'_+(a) = \psi'_-(b)$. Fix $x_0 \in S_X$, $f_0 \in \nabla_{x_0}$, $y_0 \in S_Y$, and $g_0 \in \nabla_{y_0}$. Consider $w = \frac{1}{\psi(a)}((1-a)x_0, ay_0)$ and $z = \frac{1}{\psi(b)}((1-b)x_0, by_0)$. We have $w, z \in S_{X\oplus_\psi Y}$ and $\|w+z\|_\psi = 2$. Indeed,

$$\begin{aligned}
\|w+z\|_\psi &= \left\| \left(\frac{1-a}{\psi(a)}x_0 + \frac{1-b}{\psi(b)}x_0, \frac{a}{\psi(a)}y_0 + \frac{b}{\psi(b)}y_0 \right) \right\|_\psi \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(\frac{\frac{a}{\psi(a)} + \frac{b}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(a\frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} + b\frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \left(\frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}}\psi(a) + \frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}}\psi(b) \right) \\
&= 2.
\end{aligned}$$

To obtain a contradiction, it remains to show that $\nabla_z \neq \nabla_w$. Now, for $\gamma \in [\psi'_-(b), \psi'_+(b)]$, we have

$$\psi(b) - b\gamma \leq \psi(b) - b\psi'_-(b) = \psi(a) - a\psi'_+(a) < \psi(a) - a\psi'_-(a).$$

Thus, $((\psi(a) - a\psi'_-(a))f_0, (\psi(a)+(1-a)\psi'_-(a))g_0) \in \nabla_w \setminus \nabla_z$, that is $\nabla_z \neq \nabla_w$.

Sufficiency. Let us prove that $X \oplus_\psi Y$ is a u-space. Let w and z be elements in the unit sphere of $X \oplus_\psi Y$ such that $\|w+z\|_\psi = 2$. Put $w = (x_1, y_1)$ and $z = (x_2, y_2)$. We have $\|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi = 2$ since $2 = \|w+z\|_\psi = \|(x_1 + x_2, y_1 + y_2)\|_\psi \le \|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi \le \|(\|x_1\|, \|x_2\|)\|_\psi + \|(\|y_1\|, \|y_2\|)\|_\psi = 2$. By the convexity of ψ, it follows that

$$\begin{aligned}
2 &= (\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|)\psi\left(\frac{\|y_1\| + \|y_2\|}{\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|}\right) \\
&\le (\|x_1\| + \|y_1\|)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right) + (\|x_2\| + \|y_2\|)\psi\left(\frac{\|y_2\|}{\|x_2\| + \|y_2\|}\right) \\
&= 2.
\end{aligned}$$

Thus, ψ is affine on $[a \wedge b, a \vee b]$, where $a = \frac{\|y_1\|}{\|x_1\| + \|y_1\|}$ and $b = \frac{\|y_2\|}{\|x_2\| + \|y_2\|}$. Since $\|w+z\| = 2$, there exists $F = (f_1, g_1) \in X^* \oplus_\varphi Y^*$ such that $F \in \nabla_w \cap \nabla_z$. Hence,

$$\begin{aligned}
F(w) &= f_1(x_1) + g_1(y_1) \\
&\le \|f_1\|\|x_1\| + \|g_1\|\|y_1\| \\
&= \frac{\|f_1\|\|x_1\| + \|g_1\|\|y_1\|}{(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|)}(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\
&\le \varphi\left(\frac{\|g_1\|}{\|f_1\| + \|g_1\|}\right)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right)(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\
&= \|F\|_\varphi \|w\|_\psi = 1.
\end{aligned}$$

Thus, we have

$$(\alpha) \qquad f_1(x_1) = \|f_1\|\|x_1\| \text{ and } g_1(y_1) = \|g_1\|\|y_1\|.$$

In the same way, we also have

$$(\beta) \qquad f_1(x_2) = \|f_1\|\|x_2\| \text{ and } g_1(y_2) = \|g_1\|\|y_2\|.$$

Now we show that $\nabla_w = \nabla_z$. We consider first the case when all $\|x_1\|, \|y_1\|, \|x_2\|, \|y_2\|$ are positive. In this case, we can assume that $0 < a \le b < 1$. (α) and (β) give $\frac{f_1}{\|f_1\|} \in \nabla_{\frac{x_1}{\|x_1\|}} \cap \nabla_{\frac{x_2}{\|x_2\|}}$ and $\frac{g_1}{\|g_1\|} \in \nabla_{\frac{y_1}{\|y_1\|}} \cap \nabla_{\frac{y_2}{\|y_2\|}}$. It follows that $\|\frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|}\| = 2$ and $\|\frac{y_1}{\|y_1\|} + \frac{y_2}{\|y_2\|}\| = 2$. Thus, $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$ since both X and Y are u-spaces.

If $a < b$, then, since ψ is affine on $[a, b]$, a and b must be smooth points of ψ. Consequently,

$$(\gamma) \qquad \psi(a) - a\gamma = \psi(b) - b\gamma \text{ and } \psi(a) + (1-a)\gamma = \psi(b) + (1-b)\gamma,$$

where $\gamma = \psi'(a) = \psi'(b)$.

By using (γ) together with Lemma 10 and the equations $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$, we have $\nabla_z = \nabla_w$.

If $a = b$, then, by Lemma 10, we have

$$\begin{aligned}
\nabla_{(x_1,y_1)} &= \{((\psi(a) - a\gamma)f, (\psi(a) + (1-a)\gamma)g) : \gamma \in G(a), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \nabla_{(x_2,y_2)}.
\end{aligned}$$

Thus $\nabla_z = \nabla_w$ as well.

Now we consider the case when exactly one of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ is equal to 0. We assume that $\|y_1\| = 0$, thus $a = 0 < b$ and 0 and b are smooth points. By (α), (β), and by the assumption that X is a u-space, we have $\nabla_{x_1} = \nabla_{\frac{x_2}{\|x_2\|}}$. Since 0 is a smooth point, we have $F = (f_1, 0)$. This in turn implies that $\psi(b) - b\psi'(b) = 1$ and $\psi(b) + (1-b)\psi'(b) = 0$ since $F \in \nabla_w \cap \nabla_z$. Thus, by Lemma 10,

$$\begin{aligned}
\nabla_{(x_2,y_2)} &= \{((\psi(b) - b\psi'(b))f, (\psi(b) + (1-b)\psi'(b))g) : f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_2/\|x_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_1}\} \\
&= \nabla_{(x_1,y_1)}.
\end{aligned}$$

Finally, suppose two of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ are equal to 0. We can assume that $\|y_1\| = \|y_2\| = 0$, thus $a = b = 0$. The proof of the equality $\nabla_z = \nabla_w$ is similar to the one of the case when $a = b$. □

Corollary 15 *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following statements are equivalent:*

(1) $X \oplus_\psi Y$ is a U-space;

(2) $X^* \oplus_\varphi Y^*$ is a U-space;

(3) X and Y are U-spaces and ψ is a u-function;

(4) X and Y are U-spaces and φ is a u-function, where φ is the dual function of ψ.

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Appendix 7: Some convexity properties in Musielak–Orlicz sequence spaces endowed with the Luxemburg norm, (submitted)

SOME CONVEXITY PROPERTIES IN MUSIELAK-ORLICZ SEQUENCE SPACES ENDOWED WITH THE LUXEMBURG NORM

SATIT SAEJUNG

ABSTRACT. Criteria for k-strict convexity, uniform convexity in every direction, property (K), property (H), and property (G) in Musielak-Orlicz sequence spaces and their subspaces endowed with the Luxemburg norm are presented. In particular, we obtain a characterization of such properties of Nakano sequence spaces.
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1. INTRODUCTION

Convexity properties of Banach spaces play essential role in the theory of approximation and optimization. The property of k-strict convexity ensures, for example, that the dimension of the set $P_M(x)$, the Chebyshev map or the best approximation operator, is not greater than k and vice versa (see [13]).

Now we introduce the basic notions and definitions. A convex function $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ is called an *Orlicz function* if it vanishes at zero and is even on the whole line $\mathbb{R}$ and is not identically equal to zero. Denote by l the space of all real sequences $x = (x(i))$. For a given *Musielak-Orlicz function* Φ, i.e. a sequence (φ_i) of Orlicz functions, we define a *convex modular* $I_\Phi : l \to [0, \infty]$ by the formula

$$I_\Phi(x) = \sum_{i=1}^{\infty} \varphi_i(x(i)).$$

The *Musielak-Orlicz sequence space* l_Φ is the space

$$l_\Phi := \{x \in l : I_\Phi(cx) < \infty \text{ for some } c > 0\}.$$

We consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf\{k > 0 : I_\Phi(x/k) \le 1\}.$$

To simplify notation, we put $l_\Phi := (l_\Phi, \|\cdot\|)$. It is known that l_Φ is a Banach space (see [10]).

The subspace h_Φ, called *the space of finite* (or *order continuous*) *elements*, is defined by

$$h_\Phi := \{x \in l_\Phi : I_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

We say a Musielak-Orlicz function $\Phi = (\varphi_i)$ *satisfies the δ_2-condition* ($\Phi \in \delta_2$) if there exist constants $K \ge 2$, $u_0 > 0$ and a sequence (c_i) of positive numbers such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\varphi_i(2u) \le K\varphi_i(u) + c_i$$

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holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\varphi_i(u) \leq u_0$.

It is well known that $h_\Phi = l_\Phi$ if and only if $\Phi \in \delta_2$ (see [5]).

We also say a Musielak-Orlicz function $\Phi = (\varphi_i)$ *satisfies the* $(*)$*-condition* if for any $\varepsilon \in (0,1)$ there exists a $\delta > 0$ such that, for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, $\varphi_i((1+\delta)u) \leq 1$ whenever $\varphi_i(u) \leq 1 - \varepsilon$ (see [9]).

2. k-Strict Convexity

Let X be a Banach space. Denoted by $S(X)$ and $B(X)$ the unit sphere and the unit ball of X, respectively. A point $x \in S(X)$ is called an *extreme point* if for any two elements x_1 and x_2 in $B(X)$ satisfying $x = \frac{x_1+x_2}{2}$ implies that $x_1 = x_2$.

A point $x \in S(X)$ is called a *k-extreme point*, for $k \in \mathbb{N}$, if for any $k+1$ elements $x_1, x_2, \ldots, x_{k+1}$ in $B(X)$ satisfying $x = \frac{x_1+x_2+\cdots+x_{k+1}}{k+1}$ implies that the set $\{x_1, x_2, \ldots, x_{k+1}\}$ is linearly dependent. It is clear from the definition that if the dimension of a Banach space X is less than or equal to k, then every point in $S(X)$ is always a k-extreme point.

Remark 2.1.

(1) A point $x \in S(X)$ is a 1-extreme point if and only if it is an extreme point.
(2) If a point x in $S(X)$ is a k-extreme point, then it is also a $(k+1)$-extreme point.

An interval $[a, b]$ is called a *structural affine interval* (SAI) of an Orlicz function φ if φ is affine on $[a, b]$, i.e.,

$$\varphi(\lambda a + (1-\lambda)b) = \lambda\varphi(a) + (1-\lambda)\varphi(b)$$

for all $\lambda \in [0,1]$, but not affine either on $[a-\varepsilon, b]$ or $[a, b+\varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_n, b_n]\}_n$ be the set of all SAIs of φ. Put

$$SC_\varphi = \mathbb{R} \setminus \cup_n (a_n, b_n).$$

Let $a_\varphi = \sup\{u \in \mathbb{R} : \varphi(u) = 0\}$.

Theorem 2.2. *A point $x = (x(i)) \in S(l_\Phi)$ is a k-extreme point if and only if the following conditions are satisfied:*

(i) $I_\Phi(x) = 1$,
(ii) $\#\{i \in \mathbb{N} : |x(i)| \in [0, a_{\varphi_i})\} \leq k-1$ *and*
(iii) $\#\{i \in \mathbb{N} : x(i) \notin SC_{\varphi_i}\} \leq k$.

In particular, a point $x = (x(i)) \in S(l_\Phi)$ is an extreme point if and only if $I_\Phi(x) = 1$, $\#\{i \in \mathbb{N} : |x(i)| \in [0, a_{\varphi_i})\} = 0$, and $\#\{i \in \mathbb{N} : x(i) \notin SC_{\varphi_i}\} \leq 1$. Here $\#A$ denotes the cardinality of the set A.

Proof. Necessity. We note that if $x = (x(i)) \in S(l_\Phi)$ is a k-extreme point, then $(|x(i)|)$ is also a k-extreme point. So we may assume that $x(i) \geq 0$ for all $i \in \mathbb{N}$. Suppose that (i) does not hold, i.e., $I_\Phi(x) < 1$. By the continuity of each φ_i, there exists $\varepsilon > 0$ such that

$$\varphi_i(x(i) \pm \varepsilon) \leq \varphi_i(x(i)) + \frac{1 - I_\Phi(x)}{2},$$

for all $i = 1, \ldots, k+1$. Without loss of generality, we assume in addition that $x(1) > 0$. We define

$$x_1 = (x(1) + \varepsilon)e_1 + (x(k+1) - \varepsilon)e_{k+1} + \sum_{i \neq 1, k+1} x(i)e_i$$

and, for each $j \in \{2, \dots, k+1\}$, we also define

$$x_j = (x(j-1) - \varepsilon)e_{j-1} + (x(j) + \varepsilon)e_j + \sum_{i \neq j-1, j} x(i)e_i.$$

Here e_i is the sequence which has the i-th term 1 and all other terms 0. It is easy to see that $x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}$ and $\{x_1, \dots, x_{k+1}\} \subset B(l_\Phi)$. To prove that $x_1, \dots, x_{k+1}$ are linearly independent, let $a_1, \dots, a_{k+1} \in \mathbb{R}$ be such that $a_1 x_1 + \cdots + a_{k+1} x_{k+1} = 0$. In particular, we have

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} a_i\right) x(1) + (a_1 - a_2)\varepsilon &= 0, \\
\left(\sum_{i=1}^{k+1} a_i\right) x(2) + (a_2 - a_3)\varepsilon &= 0, \\
&\vdots \\
\left(\sum_{i=1}^{k+1} a_i\right) x(k+1) + (a_{k+1} - a_1)\varepsilon &= 0.
\end{aligned}$$

Clearly, $(\sum_{i=1}^{k+1} a_i)(\sum_{i=1}^{k+1} x(i)) = 0$. Knowing that $\sum_{i=1}^{k+1} x(i) \geq x(1) > 0$, thus $\sum_{i=1}^{k+1} a_i = 0$ and this gives $(a_1 - a_2)\varepsilon = (a_2 - a_3)\varepsilon = \cdots = (a_{k+1} - a_1)\varepsilon = 0$. Hence $a_1 = \cdots = a_{k+1} = 0$ and $x_1, \dots, x_{k+1}$ are proved to be linearly independent. This implies that x is not a k-extreme point, a contradiction.

Suppose (iii) does not hold. Without loss of generality, we assume $x(i) \notin SC_{\varphi_i}$ for all $i \in N$ where $N = \{1, \dots, k+1\}$. Hence $x(i) \in (b_i, c_i)$ where $[b_i, c_i]$ is a structural affine interval of φ_i. Let $\varphi_i(u) = \alpha_i u + \beta_i$ for $u \in (b_i, c_i)$ for some constants $\alpha_i \geq 0$ and $\beta_i \in \mathbb{R}$ for each $i \in N$. We note that $\beta_i = 0$ whenever $\alpha_i = 0$. Let $A = \{i \in N : \alpha_i > 0\}$. Consider the following cases:

Case 1: $\#(A) \geq 2$. For simplicity, we assume that $A = \{1, \dots, m\}$ where $2 \leq m \leq k+1$. Choose $\varepsilon_1, \dots, \varepsilon_{k+1} > 0$ such that

$$\alpha_1 \varepsilon_1 = \cdots = \alpha_m \varepsilon_m \quad \text{and} \quad x(i) \pm \varepsilon_i \in (a_i, b_i) \quad \text{for all } i \in N.$$

We define

$$\begin{aligned}
x_1 &= (x(1) + \varepsilon_1)e_1 + (x(m) - \varepsilon_m)e_m + \sum_{i \neq 1, m} x(i)e_i, \\
x_j &= (x(j-1) - \varepsilon_{j-1})e_{j-1} + (x(j) + \varepsilon_j)e_j + \sum_{i \neq j-1, j} x(i)e_i \quad \text{for } j = 2, \dots, m-1, \\
x_m &= (x(m-1) - \varepsilon_{m-1})e_{m-1} + (x(m) + \varepsilon_m)e_m + (x(m+1) + \varepsilon_{m+1})e_{m+1}, \\
&\quad + \sum_{i \neq m-1, m, m+1} x(i)e_i \\
x_j &= (x(j+1) + \varepsilon_{j+1})e_{j+1} + \sum_{i \neq j+1} x(i)e_i \quad \text{for } j = m+1, \dots, k, \text{ and finally,} \\
x_{k+1} &= \sum_{i=m+1}^{k+1} (x(i) - \varepsilon_i)e_i + \sum_{i \neq m+1, \dots, k+1} x(i)e_i.
\end{aligned}$$

It is easy to see that $x = \frac{x_1+\cdots+x_{k+1}}{k+1}$. Moreover, $I_\Phi(x_1) = \cdots = I_\Phi(x_{k+1}) = 1$. Indeed, by the fact that $\alpha_1\varepsilon_1 = \alpha_m\varepsilon_m$, we have

$$\begin{aligned} I_\Phi(x_1) &= \varphi_1(x(1)+\varepsilon_1) + \varphi_m(x(m)-\varepsilon_m) + \sum_{i\neq 1,m} \varphi_i(x(i)) \\ &= \alpha_1 x(1) + \alpha_1\varepsilon_1 + \beta_1 + \alpha_m x(m) - \alpha_m\varepsilon_m + \beta_m + \sum_{i\neq 1,m} \varphi_i(x(i)) \\ &= \varphi_1(x(1)) + \varphi_m(x(m)) + \sum_{i\neq 1,m} \varphi_i(x(i)) = I_\Phi(x) = 1. \end{aligned}$$

Similarly, we also have $I_\Phi(x_j) = 1$ for all $j = 2,\ldots,k+1$. Now we prove that $x_1,\ldots,x_{k+1}$ are linearly independent and as a consequence we obtain a contradiction. Let $a_1,\ldots,a_{k+1} \in \mathbb{R}$ be such that $a_1x_1 + \cdots + a_{k+1}x_{k+1} = 0$. Hence

$$a_1x_1(i) + \cdots + a_{k+1}x_{k+1}(i) = 0$$

for all $i \in \mathbb{N}$. In particular, we have

$$\begin{aligned} \left(\sum_{i=1}^{k+1} a_i\right) x(1) + (a_1 - a_2)\varepsilon_1 &= 0, \\ \left(\sum_{i=1}^{k+1} a_i\right) x(2) + (a_2 - a_3)\varepsilon_2 &= 0, \\ &\vdots \\ \left(\sum_{i=1}^{k+1} a_i\right) x(m) + (a_m - a_1)\varepsilon_m &= 0. \end{aligned}$$

Combining all these we have

$$\left(\sum_{i=1}^{k+1} a_i\right)\left(\frac{x(1)}{\varepsilon_1} + \cdots + \frac{x(m)}{\varepsilon_m}\right) = 0.$$

Since $\frac{x(1)}{\varepsilon_1} + \cdots + \frac{x(m)}{\varepsilon_m} \neq 0$, $\sum_{i=1}^{k+1} a_i = 0$. Therefore $a_1 = \cdots = a_m$. Furthermore, for all $j = m,\ldots,k+1$, we have

$$0 = \left(\sum_{i=1}^{k+1} a_i\right) x(j) + (a_j - a_{k+1})\varepsilon_j = (a_j - a_{k+1})\varepsilon_j.$$

Again we obtain $a_m = \cdots = a_{k+1}$ and so $a_1 = \cdots = a_{k+1} = 0$.

Case 2: $\#(A) = 1$. We assume that $A = \{1\}$. Choose $\varepsilon_2,\ldots,\varepsilon_{k+1} > 0$ such that

$$x(i) \pm \varepsilon_i \in (a_i, b_i) \text{ for } i = 2,\ldots,k+1.$$

We define, for $j = 1,\ldots,k$,

$$\begin{aligned} x_j &= (x(j+1) + \varepsilon_{j+1})e_{j+1} + \sum_{i\neq j+1} x(i)e_i \text{ and} \\ x_{k+1} &= \sum_{j=2}^{k+1}(x(j) - \varepsilon_j)e_j + \sum_{i\neq 2,\ldots,k+1} x(i)e_i. \end{aligned}$$

Clearly, $x = \frac{x_1+\cdots+x_{k+1}}{k+1}$, and $I_\Phi(x_1) = \cdots = I_\Phi(x_{k+1}) = 1$. To prove the linear independence of $x_1,\ldots,x_{k+1}$, let $a_1,\ldots,a_{k+1} \in \mathbb{R}$ be such that $a_1x_1 + \cdots +$

$a_{k+1}x_{k+1} = 0$. Hence $a_1x(1) + \cdots + a_{k+1}x(1) = 0$. Note that $x(1) > 0$. Otherwise, $x(1) \in SC_{\varphi_1}$ which contradicts to our assumption. Thus, $\sum_{i=1}^{k+1} a_i = 0$ and so

$$0 = \left(\sum_{i=1}^{k+1} a_i\right) x(j+1) + (a_j - a_{k+1})\varepsilon_{j+1} = (a_j - a_{k+1})\varepsilon_{j+1}$$

for all $j = 1, \ldots, k$. Therefore $a_1 = \cdots = a_{k+1} = 0$.

Case 3: $\#(A) = 0$. Since $I_\Phi(x) = 1$, there exists $i_0 \notin N$ such that $\varphi_{i_0}(x(i_0)) > 0$. Let us consider the following subcases.

Subcase 3.1: $x(i_0) \notin SC_{\varphi_{i_0}}$. If we put $A' = (A \setminus \{1\}) \cup \{i_0\}$ and repeat the proof of Case 2, then we obtain a contradiction.

Subcase 3.2: $x(i_0) \in SC_{\varphi_{i_0}}$. Choose $\varepsilon_1, \ldots, \varepsilon_{k+1} > 0$ such that

$$x(i) \pm \varepsilon_i \in (b_i, c_i) \text{ for } i = 1, \ldots, k+1.$$

Define

$$x_1 = (x(1) + \varepsilon_1)e_1 + (x(k+1) - \varepsilon_{k+1})e_{k+1} + \sum_{i \neq 1, k+1} x(i)e_i$$

and, for each $j \in \{2, \ldots, k+1\}$, we also define

$$x_j = (x(j-1) - \varepsilon_{j-1})e_{j-1} + (x(j) + \varepsilon_j)e_j + \sum_{i \neq j-1, j} x(i)e_i.$$

Again, we have $x_1, \ldots, x_{k+1} \in S(l_\Phi)$ and $x = \frac{x_1 + \cdots + x_{k+1}}{k+1}$. Since $x(i_0) \in SC_{\varphi_{i_0}}$, we have $\sum_{i=1}^{k+1} a_i = 0$ and repeat the proof of the necessity of (i), we get $a_1 = \cdots = a_{k+1} = 0$.

In all cases we encounter with contradictions since x is a k-extreme point and thus the necessity of (iii) is established.

To prove (ii). Suppose that $x(i) \in [0, a_{\varphi_i})$ for all $i = 1, \ldots, k$. Choose $\varepsilon > 0$ so that $x(i) \pm \varepsilon \in (-a_{\varphi_i}, a_{\varphi_i})$ for all $i = 1, \ldots, k$. For $j = 1, \ldots, k$, we define

$$x_j = (x(j) - \varepsilon)e_j + \sum_{i \neq j} x(i)e_i$$

and

$$x_{k+1} = \sum_{i=1}^{k}(x(i) + \varepsilon)e_i + \sum_{i=k+1}^{\infty} x(i)e_i.$$

Obviously $x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}$ and $\{x_1, \ldots, x_{k+1}\} \subset S(l_\Phi)$. Now we prove the linear independence of these elements. If $a_1x_1 + \cdots + a_{k+1}x_{k+1} = 0$, then $a_1x_1(i) + \cdots + a_{k+1}x_{k+1}(i) = 0$ for all $i \in \mathbb{N}$. Since $I_\Phi(x) = 1$, there exists an index $i_0 \geq k$ such that $\varphi_{i_0}(x(i_0)) > 0$. This implies that $x(i_0) \neq 0$. It follows from (iii) that $x(i_0) \in SC_{\varphi_{i_0}}$. Then $a_1 + \cdots + a_{k+1} = 0$. Moreover, we have

$$\begin{aligned} 0 &= a_1x(1) - a_1\varepsilon + a_2x(1) + \cdots + a_ix(1) + a_{k+1}x(1) - a_{k+1}\varepsilon \\ &= a_1x(1) + a_2x(1) + \cdots + a_kx(1) + a_{k+1}x(1) + a_{k+1}\varepsilon - a_1\varepsilon \\ &= a_{k+1}\varepsilon - a_1\varepsilon. \end{aligned}$$

This gives $a_1 = a_{k+1}$. Similarly, we have $a_j = a_{k+1}$ for $j = 2, \ldots, k$. Hence $a_1 = \cdots = a_{k+1} = 0$. Therefore x cannot be a k-extreme point.

Sufficiency. Let $x \in S(l_\Phi)$ be such that the conditions (i)-(iii) hold. Given elements $x_1, \dots, x_{k+1}$ in the unit sphere of l_Φ with

$$x = \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1}.$$

By the condition (i) and the convexity of the modular, we obtain $I_\Phi(x_1) = \cdots = I_\Phi(x_{k+1}) = 1$. Furthermore, for each $i \in \mathbb{N}$, $\{x_j(i) : j = 1, \dots, k+1\}$ is either a singleton or a set contained in the same SAI of φ_i. To prove that $\{x_1, \dots, x_{k+1}\}$ is linearly dependent, we shall find $a_1, \dots, a_{k+1} \in \mathbb{R}$ such that $a_1 x_1 + \cdots + a_{k+1} x_{k+1} = 0$ where a_i's are not all zero. It follows by the condition (iii) that, for all but k coordinates, $\{x_j(i) : j = 1, \dots, k+1\}$ is a singleton. For the sake of convenience we assume that $\{x_j(i) : j = 1, \dots, k+1\}$ is a singleton for all $i \geq k+1$. Then

$$(\star) \qquad I_\Phi\left(\sum_{i=1}^{k} x_1(i) e_i\right) = \cdots = I_\Phi\left(\sum_{i=1}^{k} x_{k+1}(i) e_i\right).$$

We also assume in the worst case that $\{i \in \mathbb{N} : x(i) \notin SC_{\varphi_i}\} = \{1, \dots, k\}$. Let $\{i \in \mathbb{N} : |x(i)| \in [0, a_{\varphi_i})\} = \{1, \dots, m\}$ where $m \leq k-1$ and let $K = \{1, \dots, k\} \setminus \{1, \dots, m\}$. If $x(i) = 0$ for all $i \geq k+1$, the following system of equations

$$\begin{aligned} a_1 x_1(1) + a_1 x_2(1) + \cdots + a_{k+1} x_{k+1}(1) &= 0, \\ a_1 x_1(2) + a_1 x_2(2) + \cdots + a_{k+1} x_{k+1}(2) &= 0, \\ &\vdots \\ a_1 x_1(k) + a_1 x_2(k) + \cdots + a_{k+1} x_{k+1}(k) &= 0. \end{aligned}$$

always has a nontrivial solution. On the other hand, if there exists a coordinate $i \geq k+1$ such that $x(i) \neq 0$, then

$$a_1 + a_2 + \cdots + a_{k+1} = 0.$$

Consider the matrix

$$\begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_{k+1}(1) \\ x_1(2) & x_2(2) & \cdots & x_{k+1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(k) & x_2(k) & \cdots & x_{k+1}(k) \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

For $k \in K$, let $\varphi_k(u) = \alpha_k u + \beta_k$ when $u \in [b_k, c_k]$, where $[b_k, c_k]$ is a structural affine interval of φ_k containing $x(k)$, $\alpha_k > 0$ and $\beta_k \in \mathbb{R}$. By $(\star)$, we have

$$\sum_{k \in K} (\alpha_k x_1(k) + \beta_k) = \sum_{k \in K} (\alpha_k x_2(k) + \beta_k) = \cdots = \sum_{k \in K} (\alpha_k x_{k+1}(k) + \beta_k).$$

This implies that the above matrix is equivalent to this following matrix

$$\begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_{k+1}(1) \\ x_1(2) & x_2(2) & \cdots & x_{k+1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(m) & x_2(m) & \cdots & x_{k+1}(m) \\ \varphi_{m+1}(x_1(m+1)) & \varphi_{m+1}(x_2(m+1)) & \cdots & \varphi_{m+1}(x_{k+1}(m+1)) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_k(x_1(k)) & \varphi_k(x_2(k)) & \cdots & \varphi_k(x_{k+1}(k)) \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then there exists a nontrivial solution $\{a_i : i = 1, \dots, k+1\}$ for the above system. This implies the linear dependence of $\{x_1, \dots, x_{k+1}\}$. □

A Banach space X is said to be *k-strictly convex* if each point in its unit sphere is a k-extreme point (see [13]). Also, a Banach space X is said to be *strictly convex* if each point in its unit sphere is an extreme point.

Let $\sigma_i = \sup\{u \geq 0 : \varphi_i$ is strictly convex on $[0, u]$ and $\varphi_i(u) \leq 1\}$. By the previous theorem, we obtain the following characterizations.

Corollary 2.3. *The Musielak-Orlicz sequence space l_Φ is k-strictly convex if and only if the following conditions are satisfied*

(1) *$\Phi = (\varphi_i)$ satisfies the δ_2-condition,*

(2) *each φ_i vanishes only at zero for all but $k-1$ indices i's and*

(3) *$\varphi_{i_1}(\sigma_{i_1}) + \varphi_{i_2}(\sigma_{i_2}) + \cdots + \varphi_{i_k}(\sigma_{i_k}) \geq 1$ for all k distinct indices $i_1, i_2, \dots, i_k$.*

In particular, h_Φ is k-strictly convex if and only if the conditions (2) and (3) are satisfied.

3. Uniform Convexity in Every Direction

A Banach space X is said to be *uniformly convex in every direction* (UCED) if for any nonzero $z \in X$ there exists $\delta > 0$ such that if $x \in X$, $\|x\| = 1$ and $\|x + z\| \leq 1$ then $\|x + \frac{z}{2}\| \leq 1 - \delta$. Equivalently, if $x_n, z \in X$, $\|x_n\|, \|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ imply $z = 0$. It is easy to see that every UCED space is strictly convex.

Lemma 3.1. [6] *Let $v_i \in \mathbb{R}$, $i = 1, \dots, 4$ and $v_1 < v_2 < v_3 < v_4$. If φ is strictly convex on $[v_2, v_3]$, then there exists $p \in (0,1)$ such that*

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1-p}{2}(\varphi(u) + \varphi(v))$$

for all $u \in [v_1, v_2]$ and $v \in [v_3, v_4]$.

Lemma 3.2. [7] *Let φ be strictly convex on $[-a, a]$. Then for each $\varepsilon > 0$, $d_1, d_2 \in (0, a]$, $d_1 < d_2$, there exists $p \in (0,1)$ such that*

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1-p}{2}(\varphi(u) + \varphi(v))$$

if all $|u - v| \geq \varepsilon \max(|u|, |v|)$ and $\max(|u|, |v|) \in [d_1, d_2]$.

Lemma 3.3. [9] *If the Musielak-Orlicz function Φ satisfies the δ_2-condition and the $(*)$-condition, then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| \leq 1 - \delta$ whenever $I_\Phi(x) \leq 1 - \varepsilon$.*

Theorem 3.4. *The following statements are equivalent:*

(1) l_Φ *is* UCED;
(2) h_Φ *is* UCED;
(3) *the following conditions are satisfied:*
  (a) Φ *satisfies the* δ_2*-condition and the* $(*)$*-condition.*
  (b) *each* φ_i *vanishes only at zero.*
  (c) $\varphi_i(\sigma_i) + \varphi_j(\sigma_j) \geq 1$ *for all* $i \neq j$.

Proof. (1)$\Rightarrow$(2) is trivial. To prove (2)$\Rightarrow$(3), it suffices to prove only the necessity of (a). Suppose first that $\Phi \notin \delta_2$, then there exists $x = (x(i)) \in S(l_\Phi)$ such that $I_\Phi(x) \leq \varepsilon_0 < 1$ and $I_\Phi(\lambda x) = \infty$ for all $\lambda > 1$. We can find a strictly increasing sequence $\{i_n\}$ of natural numbers so that

$$\left\| \sum_{i=i_n+1}^{i_{n+1}} x(i)e_i \right\| \geq \frac{n}{n+1}$$

for all $n \in \mathbb{N}$. Define $x_n = \sum_{i=i_n+1}^{i_{n+1}} x(i)e_i$. Then $\|x_n\| \to 1$. We may assume that $x(1) \neq 0$. Put $z = x(1)e_1 \neq 0$. Then we have $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$. This is a contradiction.

We next prove that Φ satisfies the $(*)$-condition. For an arbitrary $\varepsilon \in (0,1)$ and $i \neq 1$, let $u \in \mathbb{R}$ be such that $\varphi_i(u) \leq 1 - \varepsilon$. Put $z = 2ae_1$ where $\varphi_1(a) = \varepsilon$. Since h_Φ is UCED, there exists $\delta' > 0$ such that $\|x + \frac{z}{2}\| \leq 1 - \delta'$ for any $x \in h_\Phi$ with $\|x\|, \|x + z\| \leq 1$. If we put $x_n = ue_n - ae_1$, then $\|x_n\|, \|x_n + z\| = 1$. Hence $\|ue_n\| = \|x + \frac{z}{2}\| \leq 1 - \delta'$. This implies that $\varphi_i(\frac{u}{1-\delta'}) \leq 1$ for all $i \neq 1$. By the continuity of φ_1, if $\varphi_1(u) \leq 1 - \epsilon$, there exists $\delta'' > 0$ such that $\varphi_1((1 + \delta'')u) \leq 1$. Put $\delta = \min\left\{\frac{\delta'}{1-\delta'}, \delta''\right\}$. Then the necessity of the $(*)$-condition is proved.

(3)$\Rightarrow$(1) Let $z = (z(i)) \in l_\Phi$ be a nonzero element. Consider the set

$$A = \{x = (x(i)) : I_\Phi(x) = 1 \text{ and } I_\Phi(x + z) \leq 1\}.$$

We first consider these two following cases:

I : There exists an index k such that $\sigma_k > 0$, $z(k) \neq 0$, $|x(k)| \leq \sigma_k$ and $|x(k) + z(k)| \leq \sigma_k$

II : There exist an index k and numbers $t_1, t_2 \in (-\varphi_k^{-1}(1), \varphi_k^{-1}(1))$, $t_1 < t_2$, $\sigma_k > 0$ such that

(i) φ_k is strictly convex on $[t_1, t_2]$,
(ii) $x(k) \leq t_1$ and $x(k) + z(k) \geq t_2$ or $x(k) + z(k) \leq t_1$ and $x(k) \geq t_2$, and
(iii) $\varphi_k(x(k)) \geq \varphi_k(\sigma_k)$ or $\varphi_k(x(k) + z(k)) \geq \varphi_k(\sigma_k)$.

We will estimate the value of $I_\Phi\left(x + \frac{z}{2}\right)$.

I : Let $n \in \mathbb{N}$ be such that $\varphi_k(z(k)) \leq n$. Then

$$|z(k)| \geq \frac{\varphi_k(z(k))}{n} \max(|x(k)|, |x(k) + z(k)|).$$

Otherwise, since φ_k vanishes only at zero,

$$\varphi_k(z(k)) < \frac{\varphi_k(z(k))}{n} \max(\varphi_k(x(k)), \varphi_k(x(k)+z(k))) \leq \frac{\varphi_k(z(k))}{n}\varphi_k(\sigma_k) \leq \frac{\varphi_k(z(k))}{n}$$

which is impossible. Moreover we also have

$$\frac{|z(k)|}{2} \leq \max(|x(k)|, |x(k) + z(k)|) \leq \sigma_k.$$

Now we apply Lemma 3.2 with $\frac{\varphi_k(z(k))}{n}$, $\frac{|z(k)|}{2}$, σ_k in place of ε, d_1, d_2, respectively. Then there exists $p_k \in (0,1)$ such that

$$\varphi_k\left(x(k)+\frac{z(k)}{2}\right) \le \frac{1-p_k}{2}(\varphi_k(x(k))+\varphi_k(x(k)+z(k))).$$

This implies

$$\begin{aligned} I_\Phi\left(x+\frac{z}{2}\right) &\le 1-\frac{p_k}{2}(\varphi_k(x(k))+\varphi_k(x(k)+z(k))) \\ &\le 1-\frac{p_k}{2}\max(\varphi_k(x(k)),\varphi_k(x(k)+z(k))) \\ &\le 1-\frac{p_k}{2}\varphi_k\left(\frac{z(k)}{2}\right). \end{aligned}$$

II : Applying Lemma 3.1 with $-\varphi_k^{-1}(1), t_1, t_2, \varphi_k^{-1}(1)$ in place of v_i, respectively, we obtain $p_k \in (0,1)$ such that

$$\varphi_k\left(x(k)+\frac{z(k)}{2}\right) \le \frac{1-p_k}{2}(\varphi_k(x(k))+\varphi_k(x(k)+z(k))).$$

This implies

$$\begin{aligned} I_\Phi\left(x+\frac{z}{2}\right) &\le 1-\frac{p_k}{2}(\varphi_k(x(k))+\varphi_k(x(k)+z(k))) \\ &\le 1-\frac{p_k}{2}\varphi_k(\sigma_k). \end{aligned}$$

Without loss of generality, we assume that

$$\varphi_1(z(1)) = \max\{\varphi_i(z(i)) : i \in \mathbb{N}\},$$
$$\varphi_2(z(2)) = \max\{\varphi_i(z(i)) : i \in \mathbb{N}, i \neq 1\},$$

and define the following sets

$$\begin{aligned} A_1 &= \{x \in A : |x(1)| \le \sigma_1 \text{ and } |x(1)+z(1)| \le \sigma_1\}, \\ A_2 &= \{x \in A : |x(1)| > \sigma_1 \text{ and } |x(1)+z(1)| > \sigma_1\}, \\ A_3 &= \{x \in A : |x(1)| \le \sigma_1 \text{ and } |x(1)+z(1)| > \sigma_1\}, \text{ and} \\ A_4 &= \{x \in A : |x(1)| > \sigma_1 \text{ and } |x(1)+z(1)| \le \sigma_1\}. \end{aligned}$$

It is evident that $A = \cup_{i=1}^4 A_i$.

We note that if $z(i) = 0$ for all $i \ge 2$, then

$$\begin{aligned} I_\Phi\left(x+\frac{z}{2}\right) &= \varphi_1\left(x(1)+\frac{z(1)}{2}\right)+\sum_{i=2}^{\infty}\varphi_i(x(i)) \\ &= \varphi_1\left(x(1)+\frac{z(1)}{2}\right)+1-\varphi_1(x(1)) \\ &\le 1-\varphi_1\left(\frac{z(1)}{2}\right). \end{aligned}$$

From now on, we may assume that $z(2) \neq 0$.

First, if $\sigma_1 = 0$, then by (c) we have $\varphi_i(\sigma_i) = 1$ for all $i \ge 2$. We apply I with $k = 2$.

Secondly, if $\varphi_1(\sigma_1) = 1$, then we shall apply the case I with $k = 1$.

We now assume that $\sigma_i > 0$ for all $i \in \mathbb{N}$. In the virtue of I it is enough to consider only the sets A_2, A_3 and A_4.

Suppose that the numbers $x(1)$ and $x(1) + z(1)$ are of the different sign. Then, for such x from $A_2 \cup A_3 \cup A_4$, it falls in the case II when $k = 1$ by putting $t_1, t_2 \in \{\pm\sigma_1, 0\}$.

Now assume that $x(1)$ and $x(1) + z(1)$ are of the same sign. If $x \in A_2$ then $\varphi_2(x(2)) \le \varphi_2(\sigma_2)$ and $\varphi_2(x(2) + z(2)) \le \varphi_2(\sigma_2)$ since $\varphi_1(\sigma_1) + \varphi_2(\sigma_2) \ge 1$. Therefore, case I is applicable for $k = 2$. Now let $x \in A_3$. Note that the signs of $x(1)$ and $z(1)$ must be the same. Let $m \in \mathbb{N}$ such that $\sigma_1 - \frac{|z(1)|}{m} > 0$ and let

$$B_3 = \left\{x \in A_3 : |x(1)| \le \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

Putting t_1, t_2 as $\pm(\sigma_1 - \frac{|z(1)|}{m})$, $\pm\sigma_1$, so elements in B_3 satisfy the assumption of II. Denoted by $\widetilde{B_3}$ the complement of B_3 in A_3 i.e.

$$\widetilde{B_3} = \left\{x \in A_3 : |x(1)| > \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

Then $|x(1) + z(1)| = |x(1)| + |z(1)| > \sigma_1 - \frac{|z(1)|}{m} + |z(1)| = \sigma_1 + \frac{(m-1)|z(1)|}{m}$. Therefore, $\varphi_1(x(1) + z(1)) > \varphi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)$, which implies $\varphi_2(x(2) + z(2)) \le 1 - \varphi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right) < \varphi_2(\sigma_2)$. If $|x(2)| \le \sigma_2$ then we are in case I with $k = 2$. If $|x(2)| > \sigma_2$ then we are in case II for $k = 2$ with

$$t_1, t_2 \text{ are chosen respectively from } \pm\left|\varphi_2^{-1}\left(1 - \varphi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)\right)\right|, \pm\sigma_2.$$

For $x \in A_4$ we also make analogous considerations. Note that $x(1)$ and $z(1)$ must have the different signs. However $x(1)$ and $x(1) + z(1)$ have the same sign. Thus $|x(1)| > |z(1)|$ and $|x(1) + z(1)| = |x(1)| - |z(1)|$. Let

$$B_4 = \left\{x \in A_4 : |x(1) + z(1)| \le \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

If $x \in B_4$ then the conditions of Case II are satisfied. Put

$$\widetilde{B_4} = \left\{x \in A_4 : |x(1) + z(1)| > \sigma_1 - \frac{|z(1)|}{m}\right\}.$$

Therefore $|x(1)| - |z(1)| = |x(1) + z(1)| > \sigma_1 - \frac{|z(1)|}{m}$ which implies $|x(1)| > \sigma_1 + \frac{(m-1)|z(1)|}{m}$. Hence $\varphi_2(x(2)) \le 1 - \varphi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right) < \varphi_2(\sigma_2)$. If $|x(2) + z(2)| \le \sigma_2$ then we are in case I with $k = 2$. If $|x(2) + z(2)| > \sigma_2$, then we are in case II for $k = 2$ with

$$t_1, t_2 \text{ are chosen respectively from } \pm\left|\varphi_2^{-1}\left(1 - \varphi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)\right)\right|, \pm\sigma_2.$$

Thus, for all $x \in A$, we have that $I_\Phi\left(x + \frac{z}{2}\right) \le \frac{1-p}{2}$ for some $p > 0$. The number p depends only on z. Indeed, p depends on the numbers

$$\left\{\pm\sigma_1, \pm\sigma_2, 0, \pm\varphi_2^{-1}\left(1 - \varphi_1\left(\sigma_1 + \frac{(m-1)|z(1)|}{m}\right)\right), \pm\left(\sigma_1 - \frac{|z(1)|}{m}\right)\right\}.$$

Hence by the $(*)$-condition and the δ_2-condition there exists $\delta > 0$ such that $\|x + \frac{z}{2}\| \le 1 - \delta$ for all $x \in S(l_\Phi)$ with $\|x + z\| \le 1$. The proof is now complete. $\square$

4. Property (K), Property (H) and Property (G)

A point $x \in S(X)$ is called an *H-point* if $x_n \to x$ whenever $(x_n) \subset X$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$. A point $x \in S(X)$ is called a *PC-point* if the identity map $id : B(X) \to B(X)$ is weak-to-norm continuous at x. Equivalently, for any $\varepsilon > 0$ there exist $\delta > 0$ and finitely many linear functionals $x_1^*, x_2^*, \dots, x_n^* \in X^*$ such that

$$\|y - x\| < \varepsilon$$

whenever $\|y\| \leq 1$ and $|x_i^*(y - x)| < \delta$ for all $i = 1, 2, \dots, n$.

It is easy to see that every PC-point is an H-point. Moreover, if X is reflexive, both notions are the same ([1]).

Lemma 4.1. [9] *If the Musielak-Orlicz function $\Phi = (\varphi_i)$ satisfies the δ_2-condition and the $(*)$-condition and each φ_i vanishes only at zero, then for each $\varepsilon > 0$, there exist $\delta > 0$ such that $|I_\Phi(x) - I_\Phi(y)| < \varepsilon$ whenever $I_\Phi(x) \leq 1$, $I_\Phi(y) \leq 1$ and $I_\Phi(x - y) \leq \delta$.*

Lemma 4.2. [12] *If Φ does not satisfy the δ_2-condition, then $S(l_\Phi)$ contains no H-points*

Theorem 4.3. *Suppose that a Musielak-Orlicz function Φ satisfies the $(*)$-condition and each φ_i vanishes only at zero. Then the following statements are equivalent:*

(1) *$x \in S(l_\Phi)$ is a PC-point;*
(2) *x is an H-point;*
(3) *$\Phi \in \delta_2$.*

Proof. (1)$\Rightarrow$(2) is obvious. See [12] for a proof of the implication (2)$\Rightarrow$(3).

(3)$\Rightarrow$(1) Suppose $\Phi \in \delta_2$. Given $\varepsilon > 0$. There exists $\delta \in (0, \varepsilon)$ such that

$$\|y\| < \varepsilon \text{ whenever } I_\Phi(y) \leq 2\delta$$

and there exists $\delta' \in (0, \delta)$ such that

$$|I_\Phi(y) - I_\Phi(z)| < \delta \text{ whenever } I_\Phi(y - z) \leq \delta', I_\Phi(y) \leq 1, I_\Phi(z) \leq 1.$$

Choose $i_0 \in \mathbb{N}$ so that $\sum_{i=i_0+1}^{\infty} \varphi_i(x(i)) < \delta$. Note that $\alpha := \min_{i=1,\dots,i_0} \Phi_i^{-1}(\frac{\delta'}{i_0}) > 0$. Put $A_\delta = \{y \in B(l_\Phi) : |\langle y - x, e_i\rangle| = |y(i) - x(i)| < \alpha \text{ for all } i = 1, \dots, i_0\}$. For any $x \in A_\delta$, we have

$$\sum_{i=1}^{i_0} \varphi_i(y(i) - x(i)) < \sum_{i=1}^{i_0} \varphi_i(\alpha) \leq \delta'.$$

Moreover, we also have

$$\begin{aligned} \sum_{i=i_0+1}^{\infty} \varphi_i(y(i)) &\leq 1 - \sum_{i=1}^{i_0} \varphi_i(y(i)) \\ &= \sum_{i=1}^{i_0} \varphi_i(x(i)) - \sum_{i=1}^{i_0} \varphi_i(y(i)) + \sum_{i=i_0+1}^{\infty} \varphi_i(x(i)) \leq 2\delta. \end{aligned}$$

These yield

$$\begin{aligned} I_\Phi\left(\frac{y-x}{2}\right) &\leq \sum_{i=1}^{i_0} \varphi_i\left(\frac{y(i) - x(i)}{2}\right) + \frac{1}{2}\left(\sum_{i=i_0+1}^{\infty} \varphi_i(y(i)) + \varphi_i(x(i))\right) \\ &\leq 2\delta. \end{aligned}$$

Hence $\|y - x\| < \varepsilon$, i.e. $A_\delta \subset x + \varepsilon B(l_\Phi)$. Therefore x is a PC-point. □

A Banach space X is said to have *property* (H) *(property* (K), *resp.)* if each point in its unit sphere is an H-point (PC-point, resp.).

Corollary 4.4. *Suppose that a Musielak-Orlicz function Φ satisfies the $(*)$-condition and each φ_i vanishes only at zero. Then the following statements are equivalent:*

(1) *l_Φ has property* (K);
(2) *h_Φ has property* (K);
(3) *l_Φ has property* (H);
(4) *h_Φ has property* (H);
(5) $\Phi \in \delta_2$.

A point $x \in S(X)$ is called a *denting point* if for any $\varepsilon > 0$, $x \notin \overline{\text{co}}\{B(X)\backslash(x + \varepsilon B(X))\}$. Recall that $\overline{\text{co}}(A)$ denotes the closed convex hull of A. If each point in $S(X)$ is a denting point, we say that X has *property* (G).

Recently, B.-L. Lin, et al. ([11]) proved that $x \in S(X)$ is a denting point if and only if it is a PC-point and an extreme point (see [11]). This gives the following characterizations:

Theorem 4.5. *Suppose that a Musielak-Orlicz function Φ satisfies the $(*)$-condition and each φ_i vanishes only at zero. Then $x = (x(i)) \in S(l_\Phi)$ is a denting point if and only if $\Phi \in \delta_2$ and $\#\{i \in \mathbb{N} : x(i) \notin SC_{\varphi_i}\} \le 1$.*

In particular, the following statements are equivalent:

(1) *l_Φ has property* (G);
(2) *h_Φ has property* (G);
(3) *l_Φ is strictly convex.*

5. Convexity Properties in Nakano Sequence Spaces and Orlicz sequence spaces

In this section, we give the characterizations of properties in the previous sections for Nakano sequence spaces. Recall that a *Nakano sequence space* $l^{\{p_i\}}$ is a Musielak-Orlicz sequence space with

$$\varphi_i(u) = |u|^{p_i}$$

where $1 \le p_i < \infty$. An *Orlicz sequence spaces* l_M is just the Musielak-Orlicz sequence space l_Φ such that $\varphi_i = M$ for all $i \in \mathbb{N}$.

Theorem 5.1. *For the Nakano sequence space $l^{\{p_i\}}$, we have*

(1) ([4, Theorem 3]) *$l^{\{p_i\}}$ is k-strictly convex if and only if* $\limsup_{i\to\infty} p_i < \infty$ *and* $\#\{i \in \mathbb{N} : p_i = 1\} \le k$,
(2) ([4, Theorem 22 and Final remark]) *$l^{\{p_i\}}$ is* UCED *if and only if $l^{\{p_i\}}$ has property* (G)*; if and only if* $\limsup_{i\to\infty} p_i < \infty$ *and* $\#\{i \in \mathbb{N} : p_i = 1\} \le 1$, *and*
(3) ([4, Theorem 6 and Final remark]) *$l^{\{p_i\}}$ has property* (K) *if and only if $l^{\{p_i\}}$ has property* (H)*; if and only if* $\limsup_{i\to\infty} p_i < \infty$.

Theorem 5.2. *For the Orlicz sequence space l_M, we have*

(1) ([3, Theorem 2.11]) *l_M is k-strictly convex if and only if $M \in \delta_2$ and M is strictly convex on* $[0, M^{-1}(\frac{1}{k})]$.

(2) ([14, Theorem 2] and [8]) l_M *is* UCED *if and only if* l_M *has property* (G)*; if and only if* $M \in \delta_2$ *and* M *is strictly convex on* $[0, M^{-1}(\frac{1}{2})]$, *and*
(3) *if, in addition,* M *vanishes only at zero,* ([14, Theorem 1.1]) l_M *has property* (K) *if and only if* l_M *has property* (H)*; if and only if* $M \in \delta_2$.

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Appendix 8: On the modulus of U–convexity, (submitted)

ON THE MODULUS OF U-CONVEXITY

SATIT SAEJUNG

ABSTRACT. In this paper, we prove that the moduli of U-convexity, introduced by Gao (1995), of the ultrapower $\tilde{X}$ of a Banach space X and of X itself coincide whenever X is super-reflexive. As consequences, some known results have been proved and improved. More precisely, we prove that $u_X(1) > 0$ implies that both X and the dual space X^* of X have normal structure and hence the property WORTH in Corollary 7 of María Mazcuñán-Navarro (2003) can be discarded.

1. INTRODUCTION

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* provided the inequality

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for every $x, y \in C$. If there exists $k < 1$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leqslant k\|x - y\|,$$

then by the Banach Contraction Principle, T has a unique fixed point in C, that is, there exists one and only one $x \in C$ such that $Tx = x$. Perhaps the most obvious question raised by the study of the Banach Contraction Principle is: What happens when $k = 1$? The simple example $Tx = x + 1$ for $x \in \mathbb{R}$ shows that the counterpart of Banach's Theorem fails to hold. Furthermore, the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = 1 + \ln(1 + e^x)$ provides an example of a fixed point free mapping which satisfies the inequality

$$\|Tx - Ty\| < \|x - y\|$$

for every $x, y \in \mathbb{R}$.

Now, a Banach space X is said to have the *fixed point property* if every nonexpansive mapping $T : C \to C$, where C is a nonempty bounded closed convex subset of a Banach space X, has a fixed point.

Many mathematicians have established that, under various geometric properties of the Banach space X often measured by different moduli of convexity, the fixed point property of X is guaranteed.

How the classical modulus of convexity $\delta_X(\cdot)$ of a Banach space X, introduced by J. A. Clarkson in 1936, relates to the fixed point property has been widely studied. It is well-known ([9, Theorem 5.12, page 122]) that if $\delta_X(1) > 0$ then X and X^* have the fixed point property. Recently, J. García Falset proved that every weakly nearly uniformly smooth space has the fixed point property. To prove this, he introduced the following coefficient

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\}$$

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where the supremum is taken over all weakly null sequences $\{x_n\}$ in $B_X(:= \{x \in X : \|x\| \leqslant 1\})$ and all $x \in S_X(:= \{x \in X : \|x\| = 1\})$. Indeed, he proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed point property ([8]).

On the other hand, in 1995, Ji Gao defined the following modulus, for $\varepsilon \in [0, 2]$,

$$u_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X \text{ and } f(x - y) \geqslant \varepsilon \text{ for some } f \in \nabla_x\right\}.$$

Here ∇_x denotes the set of all norm 1 supporting functionals f of $x \in S_X$, i.e, $f(x) = \|x\| = 1$. It is easy to see that $u_X(\varepsilon) \geqslant \delta_X(\varepsilon)$ for all $\varepsilon \in [0, 2]$. The inequality may be strict even when X is a Hilbert space. In fact, $u_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon}{2}}$ for $\varepsilon \in [0, 2]$ where H is a Hilbert space. Ji Gao proved that if there exists $\delta > 0$ such that $u_X(\frac{1}{2} - \delta) > 0$, then X has uniform normal structure ([4]).

Mazcuñán-Navarro (2003) proved a relationship between two of the above notions. Namely, if there exists $\delta > 0$ such that $u_X(1 - \delta) > 0$, then $R(X) < 2$ ([11, Theorem 5]).

This paper is organized as follows: In section 2 we prove some inequalities concerning the modulus of U-convexity, introduced by Gao, and other constants. By these inequalities, we immediately obtain some results proved by Gao (1995) and Mazcuñán-Navarro (2003). Finally, in section 3, we prove that if a Banach space X is superreflexive, then the moduli of U-convexity of the ultrapower $\widetilde{X}$ of X and of X itself coincide. Using ultrapower method we show, a Banach space X and its dual X^* have uniform normal structure whenever $u_X(1) > 0$. The paper concludes with an example showing that such a condition is sharp.

2. The Modulus of U-convexity

It was proved in [3] that $u_X(\cdot)$ is continuous on $[0, 2)$. Hence we restate [11, Theorem 5] that

Theorem 1. *Let X be a Banach space with $u_X(1) > 0$. Then $R(X) < 2$.*

Furthermore, the above result follows directly from the following inequality and the continuity of $u_X(\cdot)$.

Proposition 2. *Let X be a Banach space. Then*

$$R(X) \leqslant \inf\{\max\{\varepsilon + 1, 2(1 - u_X(\varepsilon))\} : \varepsilon \in [0, 1]\}.$$

Proof. Suppose the inequality does not hold. Then there exist $\varepsilon \in [0, 1)$, a weakly null sequence $\{x_n\}$ in B_X and $x \in S_X$ such that

$$\liminf_{n\to\infty} \|x + x_n\| > \max\{\varepsilon + 1, 2(1 - u_X(\varepsilon))\}.$$

Take $f \in \nabla_x$. So $f(x_n) \to 0$ as $n \to \infty$. Hence $f(x - x_n) \geqslant \varepsilon$ for all sufficiently large n and we then have $\|x + x_n\| \leqslant 2(1 - u_X(\varepsilon))$, a contradiction. □

The following example shows us that $\varepsilon = 1$ is the largest number such that

"$u_X(\varepsilon) > 0 \Rightarrow R(X) < 2$."

Example 3. For $p \in (1, \infty)$, let us consider the l_p space equipped with the norm

$$\|x\|' = \|x^+\|_p + \|x^-\|_p$$

where x^+ and x^- are positive and negative parts of $x \in l_p$, i.e, $(x^+)_n = \max\{x_n, 0\}$ and $(x^-)_n = \max\{-x_n, 0\}$. We write $l_{p,1}$ to denote the space $(l_p, \|\cdot\|')$. This space was introduced and studied by Bynum (see [2]). It is not difficult to see that $R(l_{p,1}) = 2$ and hence $u_{l_{p,1}}(1) = 0$. Moreover, it is well-known that $u_{l_{p,1}}(\varepsilon) \geqslant \delta_{l_{p,1}}(\varepsilon) > 0$ for all $\varepsilon > 2^{\frac{1}{p}}$ (see [2]).

Now we let $X = (\oplus l_{p_n,1})_{l_2}$ where $\{p_n\} \subset (1, \infty)$ is a sequence tending to infinity. It is easy to see that $R(X) = 2$ and $u_X(\varepsilon) > 0$ for all $\varepsilon > 1$.

In an attempt to simplify Schäffer's notion of girth and perimeter [13], the James constant

$$J(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in B_X\}$$

are studied. It is easy to see that a Banach space X is uniformly nonsquare if and only if $J(X) < 2$.

As we prove Proposition 2, a relationship between the modulus of U-convexity and the James constant is obtained.

Proposition 4. *Let X be a Banach space. Then*

$$J(X) \leqslant \inf\{\max\{\varepsilon + 1, 2(1 - u(\varepsilon))\} : \varepsilon \in [0, 1]\}.$$

In particular, if $u_X(1) > 0$, then X is uniformly nonsquare ([4, Theorem 2]).

In order to extend this result (see Theorem 8), we need the following two Lemmas.

Lemma 5 (Bishop-Phelps-Bollobás [1]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - h(z) < \frac{\varepsilon^2}{4}$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Lemma 6. *Let*

$$\begin{aligned} u'_X(\varepsilon) &= \inf_{\eta > 0} \inf \Big\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X, f(x) > 1 - \eta, \\ &\qquad \text{and } f(x - y) \geqslant \varepsilon \text{ for some } f \in S_{X^*}\Big\}. \end{aligned}$$

Then for each $\varepsilon \in [0, 2)$ and for each $\xi > 0$, there exists $\eta > 0$ such that

$$u'(\varepsilon) + \xi > u(\varepsilon - \eta) - \frac{\eta}{2}.$$

Proof. Let $\xi > 0$. Then there exist $\eta > 0$, $x, y \in S_X$ and $f \in S_{X^*}$ such that

$$1 - \frac{1}{2}\|x + y\| < u'(\varepsilon) + \xi, \ f(x - y) \geqslant \varepsilon, \text{ and } f(x) > 1 - \frac{\eta^2}{4}.$$

By Bishop-Phelps-Bollobás' Theorem, there exist $z \in S_X$ and $g \in \nabla_z$ such that

$$\|g - f\| < \eta \text{ and } \|z - x\| < \eta.$$

Hence $1 - \frac{1}{2}\|x + y\| \geqslant 1 - \frac{1}{2}\|z + y\| - \frac{\eta}{2}$. Furthermore,

$$\begin{aligned} g(z - y) &= 1 - g(y) \\ &= 1 - (g - f)(y) - f(y) \\ &\geqslant 1 - \|g - f\| - 1 + \varepsilon \\ &> \varepsilon - \eta. \end{aligned}$$

Therefore, by the definition of $u(\cdot)$,

$$u'(\varepsilon) + \xi > u(\varepsilon - \eta) - \frac{\eta}{2}.$$

□

Now by the continuity of $u(\cdot)$ and the fact that $u(\cdot) \geqslant u'(\cdot)$ on $[0,2)$, we have

Corollary 7. *$u(\cdot) = u'(\cdot)$ on $[0,2)$.*

Theorem 8. *A Banach space X is uniformly nonsquare if and only if there exists $\delta > 0$ such that $u_X(2-\delta) > 0$.*

Proof. The necessity is trivially true since $u_X(\varepsilon) \geqslant \delta_X(\varepsilon)$ for all $\varepsilon \in [0,2]$. We now prove the sufficiency. Since there exists $\delta > 0$ such that $u_X(2-\delta) > 0$, we choose $\eta > 0$ so that $u_X(2-\eta) > \eta$. Suppose that X is not uniformly nonsquare. Then there exist sequence $\{x_n\}, \{y_n\} \subset S_X$ such that

$$\big|\|x_n + y_n\| - 1\big| < \frac{1}{n} \quad \text{and} \big|\|x_n - y_n\| - 1\big| < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Let $f_n \in \nabla_{x_n}$. Then $f_n(y_n) \to 0$. Indeed, $|f_n(y_n)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we put

$$x'_n = \frac{x_n + y_n}{\|x_n + y_n\|} \quad \text{and} \quad y'_n = \frac{-x_n + y_n}{\|-x_n + y_n\|}.$$

Hence $f_n(x'_n) > \frac{n-1}{n+1}$ and $f_n(x'_n - y'_n) > \frac{2(n-1)}{n+1}$ for all $n \in \mathbb{N}$, and $\|x'_n + y'_n\| \to 2$ as $n \to \infty$. For all sufficiently large n, we have $u_X(2-\eta) \leq \eta$, a contradiction. ☐

Recall that a bounded convex subset K of a Banach space X is said to have *normal structure* if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space X is said to have *weak normal structure* if every weakly compact convex subset of X that contains more than one point has normal structure. A Banach space X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x - y\| : x, y \in K\}.$$

Combining Theorem 8 with the WORTH property, introduced by B. Sims [15], we have

Corollary 9. *If there exists $\delta > 0$ such that $u_X(2-\delta) > 0$ and X has the WORTH property, then X has normal structure.*

In particular, if $u_X(1) > 0$ and X has the WORTH property, then X has normal structure ([11, Corollary 8]). In the next section, we will see that this conclusion still holds regardless of whether or not X has the WORTH property.

3. Normal structures and the modulus of U-convexity

The ultrapower of a Banach space is proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. First we recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and
(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I}X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I}\|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows easily from (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically. For more details see [14].

The main result in this paper is the following

Theorem 10. *Suppose that X is super-reflexive. Then $u_{\widetilde{X}}(\cdot) = u_X(\cdot)$ for all $\varepsilon \in [0, 2)$. In particular, if $u_X(\varepsilon) > 0$ for some $\varepsilon \in (0, 2)$, then $u_{\widetilde{X}}(\varepsilon) = u_X(\varepsilon)$.*

Proof. It is easy to see that $u_{\widetilde{X}}(\varepsilon) \leqslant u_X(\varepsilon)$ for all $\varepsilon \in [0, 2)$. It suffices to prove that $u_{\widetilde{X}}(\varepsilon) \geqslant u'_X(\varepsilon)$ for all $\varepsilon \in [0, 2)$ where $u'_X(\cdot)$ is defined in Lemma 6. Let $\widetilde{x}, \widetilde{y} \in S_{\widetilde{X}}$ and $\widetilde{f} \in \nabla_{\widetilde{x}}$ be such that $\widetilde{f}(\widetilde{x} - \widetilde{y}) \geqslant \varepsilon$. We write $\widetilde{x} = (x_n)_{\mathcal{U}}$ and $\widetilde{y} = (y_n)_{\mathcal{U}}$ where $x_n, y_n \in X$ for all $n \in \mathbb{N}$. By the super-reflexivity of X, we also write $\widetilde{f} = (f_n)_{\mathcal{U}}$ where $f_n \in X^*$ for all $n \in \mathbb{N}$ (see [14]). Then, we have

$$\lim_{\mathcal{U}} \|x_n\| = \lim_{\mathcal{U}} \|y_n\| = \lim_{\mathcal{U}} f_n(x_n) = 1 \text{ and } \lim_{\mathcal{U}} f_n(y_n) \leqslant 1 - \varepsilon.$$

Discarding some terms of the above sequences, we may assume that no x_n, y_n or f_n is 0. Then put $x'_n = \frac{x_n}{\|x_n\|}$, $y'_n = \frac{y_n}{\|y_n\|}$ and $f'_n = \frac{f_n}{\|f_n\|}$. Given $\eta > 0$, we have $\{n \in \mathbb{N} : f'_n(x'_n) > 1 - \eta\} \in \mathcal{U}$ and $\{n \in \mathbb{N} : 1 - \frac{1}{2}\|x'_n + y'_n\| > u'_X(\varepsilon) - \eta\} \in \mathcal{U}$. It follows that

$$1 - \frac{1}{2}\|\widetilde{x} + \widetilde{y}\| = 1 - \frac{1}{2}\lim_{\mathcal{U}} \|x_n + y_n\| \geqslant u'_X(\varepsilon) - \eta.$$

This implies that $u_{\widetilde{X}}(\varepsilon) \geqslant u'_X(\varepsilon)$ and the proof is complete. □

Recall that a Banach space X is said to be a *U-space* if $u_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2)$. In order to prove that being U-space is a super-property, i.e. every Banach space finitely representable in a U-space is a U-space, Gao and Lau used some equivalent forms of U-spaces proved through the properties of Asplund spaces (see [6, Theorem 3.7]). Here we also obtain this through a new approach, as a consequence of Theorem 10.

Corollary 11. [6, Theorem 4.3] *A Banach space X is a U-space if and only if $\widetilde{X}$ is a U-space.*

Proposition 12. *If $u_X(1) > 0$, then X and X^* have uniform normal structure.*

Proof. It suffices to prove that X has weak normal structure whenever $u_{\tilde{X}}(1) > 0$ or $u_{\tilde{X}^*}(1) > 0$. Since $u_X(\varepsilon) > 0$ implies that X is super-reflexive. Then $u_X(1) = u_{\tilde{X}}(1) > 0$. Now suppose that X fails to have weak normal structure. Then, by the classical argument, there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_n \|x - x_n\| = 1 \text{ for all } x \in \operatorname{co}\{x_n\}_{n=1}^{\infty}.$$

We choose a subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted again by $\{x_n\}_{n=1}^{\infty}$, such that

$$\lim_n \|x_n - x_{n+1}\| = 1, \ |f_{n+1}(x_n)| < \frac{1}{n}, \text{ and } |f_n(x_{n+1})| < \frac{1}{n}$$

for all $n \in \mathbb{N}$ where $f_n \in \nabla_{x_n}$. Put $\tilde{x} = (x_n - x_{n+1})$, $\tilde{y} = (x_{n+1})$ and $\tilde{f} = (f_n)$. Then $\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x} - \tilde{y}) = \|\tilde{x}\| = \|\tilde{y}\| = 1$. Furthermore,

$$2 \geqslant \|\tilde{x} + \tilde{y}\| = \lim_{\mathcal{U}} \|2x_{n+1} - x_n\| \geqslant \lim_{\mathcal{U}} f_{n+1}(2x_{n+1} - x_n) = 2.$$

Hence $u_{\tilde{X}}(1) = 0$.

Next, let $\tilde{g} = (-f_{n+1})$. Hence

$$2 \leqslant \|\tilde{f} + \tilde{g}\| \leqslant (\tilde{f} + \tilde{g})(\tilde{x}) \leqslant 2.$$

Moreover, $\tilde{g}(-\tilde{y}) = 1$ and $\tilde{f}(-\tilde{y}) = 0$. This implies that

$$u_{X^*}(1) = u_{\tilde{X}^*}(1) = u_{(\tilde{X})^*}(1) = 0.$$

The proof is finished. □

Theorem 13. *If $u_X(\epsilon) > \max\{0, \frac{\epsilon-1}{2}\}$ for some $\epsilon \in (0,2)$, then X has uniform normal structure. Furthermore, if $u_X(\epsilon) > \max\{0, \epsilon - 1\}$ for some $\epsilon \in (0,2)$, then both X and X^* have uniform normal structure.*

Proof. Let us repeat the proof of Proposition 12. Let $t \in [0,1]$. Now we put $\tilde{x} = (x_n - x_{n+1})$, $\tilde{y} = (-tx_n - (1-t)x_{n+1})$ and $\tilde{f} = (f_n)$. Then $\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \|\tilde{x}\| = \|\tilde{y}\| = 1$. Furthermore, we have $\tilde{f}(\tilde{x} - \tilde{y}) = 1 + t$ and

$$\begin{aligned} \|\tilde{x} + \tilde{y}\| &= \lim_{\mathcal{U}} \|(1-t)x_n - (2-t)x_{n+1}\| \\ &\geqslant \lim_{\mathcal{U}} (-f_{n+1})((1-t)x_n - (2-t)x_{n+1}) \\ &= 2 - t. \end{aligned}$$

Hence $u_{\tilde{X}}(1+t) \leqslant \frac{t}{2}$ and this implies that $u_X(\varepsilon) \leqslant \max\{0, \frac{\varepsilon-1}{2}\}$ for all $\varepsilon \in (0,2)$, a contradiction.

Next, we put $\tilde{g} = (-tf_n - (1-t)f_{n+1})$. It is easy to see that $\tilde{f}(\tilde{z}) = 1$ and $(\tilde{f} - \tilde{g})(\tilde{z}) = 1 + t$ where $\tilde{z} = (x_n)$. Moreover, we have

$$\|\tilde{f} + \tilde{g}\| = \lim_{\mathcal{U}} (1-t)\|f_n - f_{n+1}\| \geqslant \lim_{\mathcal{U}} (1-t)(f_n - f_{n+1})(x_n - x_{n+1}) = 2(1-t).$$

Therefore $u_{X^*}(1+t) \leqslant t$ or $u_{X^*}(\varepsilon) \leqslant \max\{0, \varepsilon - 1\}$ for all $\varepsilon \in (0,2)$. Hence if $u_{X^*}(\varepsilon) > \max\{0, \varepsilon - 1\}$ for some $\varepsilon \in (0,2)$, then X has normal structure. □

Corollary 14. ([5, Theorem 8] and [12, Corollary 3]) *If $\delta_X(\epsilon) > \max\{\frac{\epsilon-1}{2}, 0\}$ for some $\epsilon \in (0,2)$, then X has uniform normal structure.*

Example 15. For $p \in (1, \infty)$, we denoted by $l_{p,\infty}$ the l_p space with the norm

$$\|x\| = \max\{\|x^+\|_p, \|x^-\|_p\}.$$

It is known that $l_{p,\infty}$ is a super-reflexive space that fails normal structure ([2]). Hence $u_{l_{p,\infty}}(1) = 0$ while $u_{l_{p,\infty}}(\varepsilon) \geqslant \delta_{l_{p,\infty}}(\varepsilon) > 0$ for all $\varepsilon > 1$. This example shows that the condition in Proposition 12 is best possible.

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