algebras และได้ขยายการศึกษาใน quasivarieties ด้วยนั่นคือผู้วิจัยศึกษา algebraic properties ของ quasivarieties ที่ generated โดย order - primal algebras และได้เงื่อนไขจำเป็นและเพียงพอสำหรับ $V(\overline{A})$ ซึ่งก่อกำเนิดโดย order - primal algebra ที่จะเป็น minimal variety หรือ minimal quasivariety ผลการศึกษาข้างต้นทั้งหมดได้ถูกเรียบเรียงเป็นบทความทางวิชาการชื่อ "On constantive simple and order-primal algebras" ซึ่งได้รับการตอบรับให้ตีพิมพ์ในวารสารระดับแนวหน้าชื่อ "Order"

K. Denecke ได้แสดงเซต Σ ของ identities ซึ่ง separate ทุก ๆ maximal clones จาก O_A และผู้วิจัยได้แสดงไว้ในโครงการ FGMC แล้วว่าทุก ๆ monotone clones $Pol_p(\leq)$ จะเป็น subclones ของ maximal clones $Pol_p(r)$ เฉพาะ r ที่เป็น central relations หรือ regularly generation relations เท่านั้น สำหรับในโครงการนี้ผู้วิจัยต้องการศึกษาเพื่อหา identities ที่จะ separate $Pol_p(\leq)$ จาก O_A และจาก maximal clones ซึ่งเป็น superset ของ $Pol_p(\leq)$ ซึ่งจะเป็นการตอบคำถาม "Separation of clones by identities" ในระดับชั้นของ monotone clones และผลการศึกษาปรากฏเป็นการตอบวัตถุประสงค์ที่ 3 เพราะผู้วิจัยสามารถจำแนกเซตอันดับทั้งหมดซึ่ง term clone operations ของ order - primal algebras ที่สมนัยกันสอดคล้อง unary hyperidentities $\varphi^{n-2}(x) \approx \varphi^{n-2+K(n)}(x)$ โดยอาศัยฟังก์ชันที่ผู้วิจัยให้ชื่อไว้ว่า long-tail function เป็นเครื่องมือและได้ผลที่สวยงามตามมาคือสามารถพิสูจน์ได้ว่า non-trivial order - primal algebras ที่ไม่สอดคล้องสมการ $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+K(n^2)}(x_1, x_2)$ เป็น primal algebras ซึ่งแสดงว่าผู้วิจัยสามารถตอบคำถาม separation of clones by identities ได้ในชั้นของ unbounded monotone clones ใน $L(O_A)$ เป็นผลสำเร็จ ผู้วิจัยจึงได้รวบรวมผลของการศึกษาทั้งหมดเขียนเป็นบทความชื่อ "Hyperidentities in Order – primal Algebras" และได้รับการตีพิมพ์แล้วในวารสารนานาชาติชื่อ "Journal of Applied Algebra and Discrete Structure"

จาก long– tail functions ยังทำให้ผู้วิจัยขยายงานเพิ่มขึ้นจากวัตถุประสงค์ที่ได้ให้ไว้ นั่นคือการค้นพบกลุ่มของฟังก์ชันที่เป็นตัวจักรกลสำคัญของการจำแนก congruences ทั้งหมดของ finite algebras เพราะ congruence lattice ของ algebra $\overline{A}$ ถูกกำหนดได้โดย unary polynomial operations ของ $\overline{A}$ และถ้า ทุก ๆ unary polynomial operation f ของ $\overline{A}$ มีสมบัติว่า $|Im f| = |A|$ หรือ $|Im f| = 1$ แล้ว $\overline{A}$ จะคือ permutable algebra และนั่นแสดงว่าผู้วิจัยจะได้วิธีการหนึ่งสำหรับจำแนก finite algebras ผู้วิจัยจึงศึกษาปัญหาดังกล่าวและจำแนก equivalence relations บนเซต A ทั้งหมดซึ่ง invariant ภายใต้ unary function f ที่มี pre - period $\lambda(f)$ เป็น n – 1 และ n – 2 และรวบรวมผลงานเขียน manuscript สำหรับบทความทางวิชาการชื่อ "Unary Operations with Long Pre-periods" เพื่อลงพิมพ์ในวารวารระดับนานาชาติต่อไป

เนื่องจากการประยุกต์ของ universal algebras มีมากมายในทาง theoretical computer sciences โดยใช้ heterogeneous หรือ multi-based algebras ผู้วิจัยและผู้ช่วยวิจัยจึงร่วมกันศึกษาเพื่อพัฒนางานในด้านนี้โดยหวังให้เกิดทิศทางของการบูรณาการองค์ความรู้ใหม่ที่ผสมผสานกับแขนงวิชาอื่น ๆ และเพื่อให้เป็นการตอบตามวัตถุประสงค์ข้อที่ 4 ด้วย ผู้วิจัยและผู้ช่วยวิจัยเริ่มต้นวางนัยทั่วไปมโนคติของ essential variables สัมพัทธ์กับ algebra หรือ variety ด้วยการนิยาม essential operation symbols in term สัมพัทธ์กับ algebra หรือ variety โดยเฉพาะกับ menger algebras พร้อมทั้งขยายนิยามเพื่อให้ใช้ได้กับ monoid ของ hypersubstitution และรวบรวมผลของการศึกษาเขียนเป็น manuscript สำหรับบทความทางวิชาการในชื่อ "Essential Operation Symbols in Terms" เพื่อจะลงพิมพ์ในวารวารระดับนานาชาติต่อไป

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Hyperidentities in Order-primal Algebras[1]

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Abstract. In this paper we determine all partial order relations on the finite set A such that an order-primal algebra with the universe A satisfies the unary hyperidentity $\varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$. As a consequence we prove that a non-trivial order-primal algebra, which does not satisfy the equation $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$ as a hyperidentity, is primal.

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Keywords: Order-primal algebra, monotone function, hyperidentity

1. Introduction

A *clone* C on a set A is a set of operations defined on A which is closed under composition and contains all projections. For an algebra $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$, the clone $T(\underline{A})$ of all term operations of $\underline{A}$ is the clone which is generated by the set $\{f_i^{\underline{A}} | i \in I\}$ of all fundamental operations of $\underline{A}$. Let $\leq$ be a partial order relation defined on A. Then the finite algebra $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$ is called *order-primal* if $T(\underline{A})$ is the set $Pol(\leq)$ of all operations defined on A which preserve the partial order $\leq$. Order-primal algebras have received quite a bit of attention recently, see [1], [4], [5]. An order relation $\leq$ on A is called *bounded* if it has the least and the greatest element and *connected* if for any pair (a, b) there exist a natural number n and elements $a = a_0, a_1, \ldots, a_n = b$ such that $a_0 \leq a_1 \geq a_2 \leq \ldots \geq a_n(\leq a_n)$ (or $a_0 \geq a_1 \leq a_2 \geq \ldots \leq a_n(\geq a_n))$. In this case we say that a and b are connected to each other. Clearly, if the partially ordered set $(A; \leq)$ has the least element or the greatest element, then $(A; \leq)$ is connected. The connectedness of elements of an ordered set $(A; \leq)$ is an equivalence

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relation on A. We denote by P_C the corresponding partition of A. If the partition P_C consists of exactly one block A_t with more than one element and all other blocks are one-element, the corresponding equivalence relation is denoted by θ_{A_t}.

Throughout this paper $\leq^*$ denotes the usual order on the set $\mathbf{N}$ of natural numbers. For a partial order relation $\leq$ by $<$ we denote the irreflexive order relation defined by $\leq$. If for all $x, y \in A$ from $x \leq y$ follows $x = y$, then the partially ordered set $(A; \leq)$ is called an *antichain*. If the element y covers the element x, i.e. if $x \leq y$, but there is no z with $x < z < y$, then we will write $x \prec y$.

Let $\underline{A} = (A; (f_i^A)_{i \in I})$ be a finite algebra. An identity $s \approx t$ is called a *hyperidentity* in $\underline{A}$ if whenever the operation symbols occurring in s and in t, respectively, are replaced by any terms of V of the appropriate arity, the identity which results, holds in V. In this case we say also that the clone $T(\underline{A})$ satisfies this hyperidentity. If the type contains unary operation symbols, then we are also interested in unary hyperidentities; i.e., where s and t are terms built up by unary operation symbols and one variable. Let $T^{(1)}(\underline{A})$ be the set of all unary term operations of $\underline{A} = (A; (f_i^A)_{i \in I})$. Then $\underline{A}$ satisfies a unary hyperidentity, $\underline{A} \models_{hyp} s \approx t$, if and only if $s \approx t$ is an identity in the monoid $\mathbf{T}^{(1)}(\underline{A}) = (T^{(1)}(\underline{A}); \circ, id_A)$ where $\circ$ is the composition of unary functions and where id_A is the identity mapping. For a fixed finite set A, let $n := |A| \geq 2$ denote the cardinality of A and let $\kappa(n) :=$ l.c.m. $\{1, 2, \ldots, n\}$ denote the least common multiple of $1, 2, \ldots, n$. We denote by H_A the set of all unary operations defined on A and by S_A the set of all permutations defined on A. The order *ord(g)* of a permutation $g \in S_A$ is the least natural number $m \in \mathbf{N}$ with $g^m = id_A$, where id_A is the identity operation. Here g^m is the m-fold power of g. For $f \in H_A$, let $Imf := \{f(a) | a \in A\}$ be the image of f and let $\lambda(f)$ be the least non-negative integer m such that $Imf^m = Imf^{m+1}$. The number $\lambda(f)$ is called the *pre-period* of f. For $f \in H_A$,the order $ord(f)$ is defined as $ord(f) := ord(f|Imf^{\lambda(f)})$.Then the following properties are satisfied(see [2] or [3]):

(i) $ord(f)$ divides $\kappa(n)$. The restricted function $g := f|Imf^{\lambda(f)}$ is a permutation on $Imf^{\lambda(f)}$ with $f^m(x) = g^{m-\lambda(f)}(f^{\lambda(f)}(x))$ for $m \geq^* \lambda(f)$,

(ii) $0 \leq^* \lambda(f) \leq^* |Imf|$ and $\lambda(f) \leq^* n-1$,

(iii) $\lambda(f) = 0 \Leftrightarrow f \in S_A$,

(iv) $\lambda(f) = n-1$ if and only if there exists an element $d \in A$ such that

$$A = \{d, f(d), f^2(d), \ldots, f^{n-1}(d)\}, f^n(d) = f^{n-1}(d),$$

(v) If $m' \geq^* m$ and p divides p', then $f^m = f^{m+p}$ implies $f^{m'} = f^{m'+p'}$,

(vi) $f^m = f^{m'} \Leftrightarrow m, m' \geq^* \lambda(f)$ and $m \equiv m' \ mod \ ord(f)$

(vii) $f^{\lambda(f)} = f^{\lambda(f)+ord(f)} = f^{\lambda(f)+\kappa(n)}$ (this follows from (iv), (v) and (i)).

A mono-unary hyperidentity in an algebra $\underline{A}$ is a hyperidentity with a single unary operation symbol, say φ. It has the form $\varphi^m(x) \approx^{m'} (x)$ for some $m, m' \in \{0, 1, 2, \ldots\}$. Let $O_A := \bigcup_{n \geq 1} O_A^{(n)}$ be the set of all operations defined on A, i.e., the union of all sets $O_A^{(n)}$ of n-ary operations defined on A. To test whether $\underline{A}$ satisfies the mono-unary hyperidentity $\varphi^m(x) \approx \varphi^{m'}(x)$, we have to check whether the monoid $\mathrm{T}^{(1)}(\underline{A})$ satifies the identity $\varphi^m(x) \approx \varphi^{m'}(x)$. The following results are well-known:

Proposition 1.1.([2])

(i) S_A satisfies the identity $\varphi^m(x) \approx \varphi^{m'}(x)$ if and only if $m \equiv m'$ mod $\kappa(n)$.

(ii) H_A satisfies the identity $\varphi^m(x) \approx varphi^{m'}(x)$ if and only if $m, m' \geq n-1$ and $m \equiv m' \ mod \ \kappa(n)$; in particular, H_A satisfies the identity $\varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n)}(x)$, but H_A does not satisfy $\varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$ and H_A does not satisfy the identity $\varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n-1)}(x)$ if $\kappa(n-1) \neq \kappa(n)$.

(iii) $H_A \setminus S_A$ satisfies the identity $\varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n-1)}(x)$.

In [2] for maximal subclones $C \subset O_A$, mono-unary hyperidentities were determined which are satisfied in C but not in O_A. This answers the question of whether the variety of monoids generated by $C^{(1)}$ is a proper subvariety of the variety generated by $O_A^{(1)} = H_A$. In this paper, we are interested in mono-unary hyperidentities in order-primal algebras; i.e., identities of the monoid $Pol^{(1)}(\leq)$ for arbitrary order relations $\leq$. The main result is the characterization of all order relations $\leq$ such that $Pol^{(1)}(\leq)$ satisfies the identity $\varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$. This identity is important to separate maximal clones from the clone O_A. Using this result, we are able to prove that every order-primal algebra satisfies the binary hyperidentity $\varphi^{n^2-2}(x_1, x_2) \approx \varphi^{n^2-1+\kappa(n^2)}(x_1, x_2)$. This result can be applied to the functional completeness problem of multiple-valued logic.

2. Long-tailed Functions

Clearly, the monoid $(Pol^{(1)}(\leq); \circ, id)$ satisfies the identity $\varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}$ if and only if a function f with $f^{n-1}(x) = f^{n-1+\kappa(n)}$ does not belong to $Pol^{(1)}(\leq)$. In this case, we have $\lambda(f) = n-1$. We will describe some properties of a partial order which is invariant under a unary function f with $\lambda(f) = n-1$. We call a function $f : A \to A$, $|A| = n$ with $\lambda(f) = n-1$ a *long-tailed function*. By (iv) in the introduction, a long-tailed function is characterized by the existence of an element $d \in A$ such that $A = \{d, f(d), f^2(d), \ldots, f^{n-1}(d)\}$. If $A = \{0, 1, \ldots, n-1\}$ then the function defined by $f(x) = x - 1$ if $x \neq 0$ and $f(0) = 0$ is a long-tailed function. In this case, we have $d = n-1$. We will denote a long-tailed function for short by f_{LT}.

Lemma 2.1. *Let $\leq$ be an order relation on A with $|A| = n \geq^* 2$ which is not an antichain and suppose that $f_{LT} \in Pol(\leq)$. Let $d \in A$ be the element such that $A = \{d, f_{LT}(d), \ldots, f_{LT}^{n-2}(d), f_{LT}^{n-1}(d)\}$. Then*

(i) *$f_{LT}^{n-1}(d)$ and $f_{LT}^{n-2}(d)$ are comparable with respect to $\leq$.*

(ii) *$f_{LT}^{n-1}(d)$ is either minimal or maximal.*

(iii) *d is either minimal or maximal.*

(iv) *d is maximal if and only if $f_{LT}^{n-1}(d)$ is minimal.*

(v) *The partition P_C contains exactly one set A_t with more than one element and all other sets of P_C are one-element. Moreover, $\{f_{LT}^{n-2}(d), f_{LT}^{n-1}(d)\} \subseteq A_t$ and $f_{LT}^{n-1}(d)$ is the least or the greatest element of A_t with respect to the restricted order $\leq |A_t$.*

(vi) *If there is a natural number k with $0 \leq^* k \leq^* n-1$ such that $d > f_{LT}^{k}(d)$, then $f_{LT}^{n-1}(d)$ is the least element with respect to $\leq$ and therefore $\leq$ is connected. The dual proposition is also true.*

(vii) *If $f_{LT}^{n-k}(d)$ and $f_{LT}^{n-k-1}(d)$ are comparable then either $f_{LT}^{n-k}(d) \prec f_{LT}^{n-k-1}(d)$ or $f_{LT}^{n-k-1}(d) \prec f_{LT}^{n-k}(d)$ for all $1 \leq^* k \leq^* n-1$.*

Proof. (i) Suppose that $f_{LT}^{n-1}(d)$ and $f_{LT}^{n-2}(d)$ are incomparable. If n = 2 , then $\leq$ is an antichain which contradicts the presumption. Therefore, we may assume that $n >^* 2$ and then $f_{LT}^{n-3}(d) \in A$. If $f_{LT}^{n-3}(d)$ is comparable with $f_{LT}^{n-2}(d)$ or with $f_{LT}^{n-1}(d)$, then $f_{LT}^{n-1}(d)$ and $f_{LT}^{n-2}(d)$ are comparable since f_{LT} preserves the order; a contradiction. Therefore, $\{f_{LT}^{n-1}(d), f_{LT}^{n-2}(d), f_{LT}^{n-3}(d)\}$ is an antichain. So, there is an element k with $3 \leq^* k \leq^* n-1$ such that $\{f_{LT}^{n-1}(d), \ldots, f_{LT}^{n-k}(d)\}$ is an antichain with respect to $\leq$. Suppose that $k <^* n$ is maximal with this property. Then there is a number j with $1 \leq^* j \leq^* k$ such that $f_{LT}^{n-(k+1)}(d)$ is comparable with $f_{LT}^{n-j}(d)$. Since f_{LT} is order-preserving, we obtain that $f_{LT}^{n-k}(d))$ is comparable with $f_{LT}^{n-j+1}(d)$ which contradicts our choice of k.

(ii) Suppose that $f_{LT}^{n-1}(d)$ is neither maximal nor minimal. Then there are numbers m and k with $0 \leq^* m, k <^* n$ and $m \neq k$ such that $f_{LT}^{m}(d) < f_{LT}^{n-1}(d) < f_{LT}^{k}(d)$ and if $m >^* k$ then we get $f_{LT}^{n-1}(d) = f_{LT}^{m-k}(f_{LT}^{n-1}(d)) \leq f_{LT}^{m-k}(f_{LT}^{k}(d)) = f_{LT}^{m}(d)$; a contradiction. If $m <^* k$, we conclude in a similar way.

(iii) This part can be proved by using similar argument as in (ii), we hence omit the details.

(iv) Suppose that d is not maximal. Then there exists a number k with $n >^* k >^* 0$ such that $d < f_{LT}^{k}(d)$. Since f_{LT} preserves the order, we have $f_{LT}^{n-1-k}(d) < f_{LT}^{n-1-k}(f_{LT}^{k}(d)) = f_{LT}^{n-1}(d)$; hence, $f_{LT}^{n-1}(d)$ is not minimal. The opposite direction can be proved in a similar way.

(v) Clearly, there is a block A_t of P_c with $\{f_{LT}^{n-1}(d), f_{LT}^{n-2}(d)\} \subseteq A_t$ since by (i) $f_{LT}^{n-1}(d)$ and $f_{LT}^{n-2}(d)$ are comparable. If $f_{LT}^{i}(d)$ is comparable with $f_{LT}^{n-1}(d)$ or with $f_{LT}^{n-2}(d)$ for all $i = 0, \ldots, n-3$, then $\leq$ is connected; i.e., $A = A_t$. Otherwise there is a number j such that $f_{LT}^{j}(d)$ is incomparable with $f_{LT}^{n-1}(d)$ and with $f_{LT}^{n-2}(d)$. Let $j \geq^* 2$ be the greatest integer such that $f_{LT}^{n-j}(d)$ is comparable with $f_{LT}^{n-1}(d)$ or with $f_{LT}^{n-2}(d)$. Then $A_t = \{f_{LT}^{n-j}(d), f_{LT}^{n-j+1}(d), \ldots, f_{LT}^{n-2}(d), f_{LT}^{n-1}(d)\}$. Since all other elements have the form $f_{LT}^{n-k}(d)$ with $k >^* j$, the elements $f_{LT}^{n-k}(d)$ cannot belong to A_t. If $f_{LT}^{n-k}(d)$ is comparable with $f_{LT}^{n-k_1}(d)$ where $k_1 >^* j$ and $k_1 <^* k$, then $f_{LT}^{k-j}(f_{LT}^{n-k}(d)) = f_{LT}^{n-j}(d)$ is comparable with $f_{LT}^{k-j}(f_{LT}^{n-k_1}(d)) = f_{LT}^{n+k-(k_1+j)}(d) = f_{LT}^{n+k-k_1-j}(d)$ and $n+k-k_1-j >^* n-k_1$. This contradicts the choice of j; i.e., all other elements are pairwise incomparable. If $f_{LT}^{n-j}(d) \leq f_{LT}^{n-1}(d)$, then for every $1 \leq^* k <^* j$ we have $f_{LT}^{k}(f_{LT}^{n-j}(d)) = f_{LT}^{n-j+k}(d) \leq f_{LT}^{k}(f_{LT}^{n-1}(d)) = f_{LT}^{n-1+k}(d) = f_{LT}^{n-1}(d)$ and $f_{LT}^{n-1}(d)$ is the greatest element of A_t with respect to $\leq$ and if $f_{LT}^{n-j}(d) \geq f_{LT}^{n-1}(d)$, then $f_{LT}^{n-1}(d)$ is the least element.

(vi) can be proved in a similar way as the last proposition of (v).

(vii) Assume that there is an element j with $1 \leq^* j \leq^* n-1$ such that $f_{LT}^{n-1}(d) \leq f_{LT}^{j}(d) \leq f_{LT}^{n-2}(d)$. Then $f_{LT}^{n-1}(d) = f_{LT}(f_{LT}^{n-1}(d)) \leq f_{LT}^{j+1}(d) \leq f_{LT}^{n-1}(d)$ and then $j+1 \geq^* n-1$ because of $\lambda(f_{LT}) = n-1$. But then $n-2 \leq^* j \leq^* n-1$ and thus $f_{LT}^{j}(d) = f_{LT}^{n-1}(d)$ or $f_{LT}^{j}(d) = f_{LT}^{n-2}(d)$. □

3. Connected Orders

Lemma 2.1 shows that if $A_t = A$; i.e., if $\leq$ is connected, then $f_{LT}^{n-1}(d)$ is the least or the greatest element of A with respect to $\leq$. Without restriction of the generality we may assume that $f_{LT}^{n-1}(d)$ is the least element. Then we have: (see also [2])

Lemma 3.1. *Let $\leq$ be a bounded order relation on A and suppose that $f_{LT} \in Pol(\leq)$. Then $\leq$ is a linear order.*

Proof. We repeat the proof given in [Den-P; 88]. Let $f_{LT}^{n-1}(d)$ be the least element. Suppose that 1 is the greatest element. If $1 \neq f_{LT}(1)$, then $1 > f_{LT}(1) \geq f_{LT}(d)$ since $1 > d$ and then we get $1 > f_{LT}(1) \geq f_{LT}^{i}(d)$ for all $i = 1, 2, \ldots, n-1$. Therefore 1 is different from all $f_{LT}^{i}(d)$, $i = 1, \ldots, n-1$

and consequently, $1 = \mathrm{d}$. From $1 > f_{LT}(1)$, we obtain $f_{LT}^{i}(1) \geq f_{LT}^{i+1}(1), i = 1, 2, \ldots, n-2$; i.e., we have a linear order $d > f_{LT}(d) > \ldots > f_{LT}^{n-1}(d)$ on A. Otherwise, if $1 = f_{LT}(1)$ then $f_{LT}(f_{LT}^{n-1}(d)) \neq f_{LT}^{n-1}(d)$ because of $\lambda(f_{LT}) = n-1$ and analogously we obtain the linear order $f_{LT}^{n-1}(d) > f_{LT}^{n-2}(d) > \ldots > d$. □

Now we consider the case that $\leq$ is unbounded and connected. Then, by Lemma 2.1, there exists the least element and suppose that $f_{LT}^{n-1}(d)$ is the least element.

Lemma 3.2. *Let $\leq$ be a connected unbounded order relation on A with the least or the greatest element and suppose that $\leq$ admits a long-tailed function f_{LT} Further, we suppose that $f_{LT}^{n-1}(d)$ is the least (or greatest) element with respect to $\leq$. Then the order relation $\leq$ satisfies the following properties.*

(C1) *There is a longest chain of consecutive powers*

$$f_{LT}^{n-1}(d) \prec f_{LT}^{n-2}(d) \prec \ldots \prec f_{LT}^{n-(k+1)}(d) \prec f_{LT}^{n-(k+2)}(d).$$

(C2) *If $0 \leq^* l \leq^* j$, then $f_{LT}^{n-(j+1)}(d) \not\leq f_{LT}^{n-(l+1)}(d)$.*

(C3) *For each $x \in \{k+2, \ldots, n-1\}$ there is a maximum integer l_x such that $0 \leq^* l_x \leq^* x-1$ and $f_{LT}^{n-(x+1)}(d)$ is incomparable with $f_{LT}^{n-(m+1)}(d)$ for all $l_x <^* m <^* x$ and if $f_{LT}^{n-(m+1)}(d) \leq f_{LT}^{n-(x+1)}(d)$, then $f_{LT}^{n-m}(d) \leq f_{LT}^{n-x}(d)$ for all $m \leq^* l_x$.*

Proof. (i) Without restriction of the generality we may assume that $f_{LT}^{n-1}(d)$ is the least element with respect to $\leq$. Then, by Lemma 2.1 (i) and (vii), we have $f_{LT}^{n-1}(d) \prec f_{LT}^{n-2}(d)$. Since $A = \{d, f_{LT}(d), \ldots, f_{LT}^{n-1}(d)\}$, there exists a maximum number k with $f_{LT}^{n-1}(d) \prec f_{LT}^{n-2}(d) \ldots \prec f_{LT}^{n-(k+1)}(d) \prec f_{LT}^{n-(k+2)}(d)$ and we obtain a longest chain of consecutive powers of $f_{LT}(d)$.

(ii) If $j \leq^* k-1$, then $l <^* j$ implies $f_{LT}^{n-(l+1)}(d) \leq f_{LT}^{n-(j+1)}(d)$ by (i) and thus $f_{LT}^{n-(j+1)}(d) \not\leq f_{LT}^{n-(l+1)}(d)$.

Now we prove the following claim:

Claim. Suppose that $n-1 \geq^* j \geq^* k+1$. We set $l := j-(k+1)$ and prove by induction on l that $j >^* k+1$ implies $f_{LT}^{n-(j+1)}(d) \not\leq f_{LT}^{n-(k+2)}(d)$.

For $l = 1$, we have j = k+2; and from $f_{LT}^{n-(k+3)}(d) < f_{LT}^{n-(k+2)}(d)$ we would obtain $f_{LT}^{n-(k+2)}(d) \leq f_{LT}^{n-(k+1)}(d)$ and by Lemma 2.1 (vii), $f_{LT}^{n-(k+2)}(d) \prec f_{LT}^{n-(k+1)}(d)$, a contradiction to (i). Inductively, suppose that $f_{LT}^{n-(k+2+l)}(d) \not\leq f_{LT}^{n-(k+2)}(d)$ for $l \geq 1$. Then $f_{LT}^{n-(k+2+l+1)}(d) \leq f_{LT}^{n-(k+2)}(d)$ implies $f_{LT}(f_{LT}^{n-(k+2+l+1)}(d)) = f_{LT}^{n-(k+2+l)}(d) \leq f_{LT}(f_{LT}^{n-(k+2)}(d)) = f_{LT}^{n-(k+1)}(d) \leq f_{LT}^{n-(k+2)}(d)$ which contradicts our hypothesis. Therefore, $f_{LT}^{n-(k+2+l+1)}(d) \not\leq f_{LT}^{n-(k+2)}(d)$.

Now, we use the claim to finish the proof of (ii). Assume that $j >^* l \geq^* k-1$. Put $t := l-(k-1)$. If $t = 1$, then $l = k$ and thus $j \geq^* k+1$ and by the claim we have $f_{LT}^{n-(j+1)}(d) \not< f_{LT}^{n-(k+1)}(d)$. Inductively, we suppose that $j >^* k+t$ implies $f_{LT}^{n-(j+1)}(d) \not< f_{LT}^{n-(k+t+1)}(d)$ for $t \geq 1$. Assume that $j >^* l = k+(t+1)$ and $f_{LT}^{n-(j+1)}(d) < f_{LT}^{n-(k+t+2)}(d)$. Then $j-1 >^* k+t$ and by hypothesis $f_{LT}^{n-j}(d) \not< f^{n-(k+t+1)}(d)$ and this contradicts our assumption. Hence, if $j >^* l = k+(t+1)$, then $f_{LT}^{n-(j+1)}(d) \not< f^{n-(k+t+1)}(d)$.

(iii) Let $f_{LT}^{n-x}(d) \in \{f_{LT}^{n-n)}(d) = f_{LT}^{0}(d) = d, f_{LT}(d), \ldots, f_{LT}^{n-(k+3)}(d)\}$; i.e., $x \in \{k+3, k+4, \ldots, n-1, n\}$. Suppose that $f_{LT}^{n-x}(d)$ and $f_{LT}^{n-(x+1)}(d)$ are comparable. Then, by (ii), we have $f_{LT}^{n-x}(d) \leq f_{LT}^{n-(x+1)}(d)$. Applying f_{LT}^{x-k-2} on both sides, we get $f_{LT}^{x-k-2}(f_{LT}^{n-x}(d)) = f_{LT}^{n-(k+2)}(d) \leq f_{LT}^{n-(k+3)}(d) = f_{LT}^{x-k-2}(f_{LT}^{n-(k+1)}(d))$; and by Lemma 2.1, we have $f_{LT}^{n-(k+2)}(d) \prec f_{LT}^{n-(k+3)}(d)$. This is a contradiction to the choice of k in (i). Therefore $f_{LT}^{(n-x)}(d)$ and $f_{LT}^{n-(x+1)}(d)$ are incomparable. Since $f_{LT}^{n-1}(d)$ is the least element, there is a number j with $0 \leq^* j <^* x-1$ such that $f_{LT}^{n-j}(d) < f_{LT}^{n-(x+1)}(d)$. Let l_x be the greatest natural number with this property. Then $f_{LT}^{n-(m+1)}(d)$ and $f_{LT}^{n-(x+1)}(d)$ are incomparable for all $l_x <^* m <^* x-1$. If $m \leq^* l_{x_1}$, then $f_{LT}^{n-(m+1)}(d) \leq f_{LT}^{n-(x+1)}(d)$ implies $f_{LT}^{n-m}(d) \leq f_{LT}^{n-x}(d)$.

If $f_{LT}^{n-1}(d)$ is the greatest element with respect to $\leq$, then we conclude in a similar way. □

Now we prove that an unbounded connected order relation having the least (or the greatest) element and satisfying conditions corresponding to (C1), (C2), (C3) admit a long-tailed function.

Lemma 3.3. *Let $A = \{0, 1, \ldots, n-1\}, n \geq^* 2$ and let $\leq$ be an unbounded* *connected order defined on A . If 0 is the least (or the greatest) element with respect to $\leq$ and if $\leq$ satisfies the following three conditions (or the dual conditions):*

(C1) *There is a longest chain of consecutive integers*

$$0 \prec 1 \prec 2 \prec 3 \ldots \prec k \prec k+1,\ k \geq 1.$$

(C2) *If $0 \leq^* l <^* j$, then $j \not< l$.*

(C3) *For each $x \in \{k+2, k+3, \ldots, n-1\}$, there is a maximum integer $0 \leq^* l_x <^* x-1$ such that x is incomparable with m for all $l_x <^* m <^* x$ and from $t < x$ follows $t-1 < x-1$ for all $t \leq^* l_x$.*

Then the unary function $g : A \to A$, which is defined by $g(x) = x - 1$ if $0 \neq x \in A$and $g(0) = 0$, preserves the order $\leq$.

Proof. It is easy to see that g is a long-tailed function with $A = \{g^0(n-1) = n-1,\ g^1(n-1),\ g^2(n-1),\ \ldots, g^{n-1}(n-1) = 0\}$. Let $x, y \in A$ with $x \leq y$. By (C2) , if $y <^* x$, then $x \not< y$ and thus $x \leq^* y$. We consider the following two cases:

Case 1. If $y \in \{0, 1, \ldots, k+1\}$ then $x \in \{0, 1, \ldots, k+1\}$ because of $x \leq y$ and then $g(x) = x-1 \leq^* y-1 = g(y)$ for all $x \neq 0$ and then $g(x) = x-1 \leq y-1 = g(y)$ by (C1). If $x = 0$, then $g(x) = 0 \leq g(y)$ since 0 is the least element with respect to $\leq$. Therefore g preserves $\leq$.

Case 2. If $y \in \{k+2, \ldots, n-1\}$ then by (C3) there is a maximum integer $l_y, 0 \leq^* l_y <^* y-1$ such that y is incomparable with t for all $l_y <^* t <^* y$; hence, $x \leq^* l_y$ since x is comparable with y . Because of the last sentence of (iii), we have $g(x) = x - 1 < y - 1 = g(y)$ and g preserves $\leq$. □

Lemma 3.2 and Lemma 3.3 characterize all connected unbounded order relations with a least (or greatest) element which admit a long-tailed function f_{LT}.

Theorem 3.4. *Let $\leq$ be a connected unbounded order relation with the least (or the greatest) element. Then a long-tailed function f_{LT} preserves*

$\leq$ if and only if the order $\leq$ satisfies the conditions (C1),(C2),(C3) from Lemma 3.2.

This characterization can be applied to the restriction of a disconnected order to the nontrivial order-component from Lemma 2.1(v).

Corollary 3.5. *Let $\leq$ be a disconnected order relation on A which is not an antichain. Suppose that a long-tailed function f_{LT} preserves $\leq$. Then the relation $\leq|_{A_t}$ is a linear order or is an unbounded order with the least or the greatest element satisfying conditions (C1), (C2), (C3) from Lemma 3.2.*

Proof. Since the order $\leq$ is disconnected, we may assume that there is an integer $j \geq 2$ with

$$A_t = \{f_{LT}^{n-j}(d), f_{LT}^{n-j+1}(d), \ldots, f^{n-2}(d), f^{n-1}(d)\}.$$

We set $d' = f_{LT}^{n-j}(d)$ and consider the unary function $g := f_{LT}|_{A_t}$. Then $A_t = \{d', g(d'), g^2(d'), \ldots, g^{j-1}(d')\}$ and therefore g is a long-tailed function on A_t and then by Lemma 2.1(v), Lemma 3.1 and Lemma 3.2, $\leq$ is a linear order or an unbounded order with the least (or the greatest) element satisfying the conditions (C1),(C2),(C3) from Lemma 3.2 □.

4. Characterization of Order Relations Admitting a Long-tailed Function

Theorem 4.1. *Let $\leq$ be an order relation on the finite set $A = \{0, \ldots, n-1\}$ with $n \geq^* 2$. Then $\leq$ admits a long-tailed function f_{LT} if and only if $\leq$ satisfies one of the following conditions:*

(i) *$(A; \leq)$ is an antichain,*

(ii) *$(A; \leq)$ is a chain,*

(iii) *$\leq$ is unbounded and connected with the least (the greatest) element and satisfies conditions (C1),(C2),(C3) from Lemma 3.2,*

(iv) *$\leq$ is disconnected and only one connected component A_t with respect to $\leq$ contains more than one element and the restricted order $\leq|_{A_t}$ is* *connected with the least (the greatest) element and satisfies conditions (C1),(C2),(C3) from Lemma 3.2.*

Proof. We prove at first that in all four cases $Pol(\leq)$ contains a long-tailed function f_{LT}. In the first case, this is trivial since $Pol(\leq) = O_A$. Let $0 \prec 1 \prec \ldots \prec n-1$ be a chain. Then f_{LT} defined by $f_{LT}(x) = x - 1$ if $x \neq 0$ and $f_{LT}(0) = 0$ is a long-tailed function since $\{0, 1, \ldots, n-1\} = \{f_{LT}^{n-1}(n-1), \ldots, f_{LT}(n-1), n-1\}$ and f_{LT} preserves the order. In the third case, Lemma 3.3 shows that $Pol(\leq)$ contains a long-tailed function. In the fourth case, if the restricted order $\leq /A_t$ is unbounded, then by Lemma 3.3, there is a long-tailed function $g : A_t \to A_t$ which preserves the restricted order $\leq |_{A_t}$. This means that there is an element $d \in A_t$ such that $A_t = \{d, g(d), \ldots, g^{n-m}(d)\}$. Now we define a function $f : A \to A$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in A_t \\ d & \text{if } x = a_{t-1} \\ a_{i+1} & \text{if } x = a_i \text{ and } 1 \leq i < t-1 \end{cases}$$

Clearly,

$$A = \{a_1, f(a_1), \ldots, f^{t-1}(a_1) = d, f^t(a_1) = f(d) = g(d), \ldots, f^{n-1}(a_1)\}$$

and this shows that f is a long-tailed function. Then f preserves the order since for $x \leq y$, $x, y \in A_t$ and $f/_{A_t} = g$ is order-preserving. If $\leq /A_t$ is bounded, then it is a linear order and we can find an order-preserving LT-function in a similar way.

If conversely $f_{LT'}$ is order preserving; i.e., $f_{LT} \in Pol(\leq)$ and $(A; \leq)$ is not an antichain, then $\leq$ can be connected or disconnected. If it is connected and bounded, it is a chain by Lemma 3.1. If $\leq$ is connected and unbounded, then by Lemma 2.1 it has the least (or the greatest) element and by Lemma 3.3, conditions (C1), (C2), (C3) from Lemma 3.2 are satisfied. If $\leq$ is disconnected, then by Lemma 2.1 (v) and Corollary 3.5, the order has the properties given in (iv). □

5. Mono-unary Hyperidentities in Order-primal Algebras

Now we apply Theorem 4.1 to determine mono-unary hyperidentities which are valid in order-primal algebras.

Corollary 5.1 *Let $\underline{A} = (A; (f_i^A)_{i\in I})$ be an order-primal algebra with respect to an order relation $\leq$ on A. Then $\underline{A} \models_{hyp} \varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$ for a unary operation symbol φ if and only if $\leq$ is different from one of the orders described in Theorem 4.1 (i)-(iv).* □

A finite algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ is called *primal* if $T(\underline{A}) = O_A$. Primal algebras can be characterized by maximal subclones of O_A in the following way : $\underline{A}$ is primal if and only if $T(\underline{A})$ is not contained in one of the maximal subclones of O_A which are described by I. Rosenberg [6]. For his description Rosenberg used the following six classes of relations:

Class (1): is the class of all bounded order relations on A.

Class (2): is the class of all binary relations $\{(a, s(a)) | a \in A\}$ where s is a permutation on A without invariant elements with all cycles of the same prime length.

Class (3): is the class of all quaternary relations α defined by $(a, b, c, d) \in \alpha$ if and only if $a + b = c + d$ where $+$ is the addition of an elementary abelian p-group (p prime) on A.

Class (4): is the class of all non-trivial equivalence relations on A.

Class(5): is the class of all central; i.e., totally reflexive and totally symmetric h-ary relations r with $1 \leq h \leq |A|$, having a nontrivial center. A relation is called totally reflexive if it contains all h-tuples with a repetition of elements. An h-ary relation is said to be totally symmetric if $(x_0, \ldots, x_{h-1}) \in r$ implies $(x_{s(0)}, \ldots, x_{s(h-1)}) \in r$ for any permutation s of $\{0, \ldots, n-1\}$. The center of a totally symmetric and reflexive h-ary relation r is the set of all elements $c \in A$ such that $(c, x_1, \ldots, x_{h-1}) \in r$ for all $x_1, \ldots, x_{h-1} \in A$. For $h = 1$, the relation r is simply a subset of A.

Class(6): is the class of all h-regularly generated relations which are defined for $3 \leq h \leq |A|$ by the following steps: for $m \geq 1, m \in \mathbf{N}$ a set $\theta_0, \ldots, \theta_{m-1}$

of equivalence relations on A is called h-regular if each $\theta_i, 0 \leq i \leq m-1$ defines exactly h-equivalence classes and if the intersection $\cap_{i=0}^{m-1}\varepsilon_i$ of arbitrary m equivalence calsses ε_i of θ_i is nonempty. An h-ary relation ϱ is said to be h-regularly generated associated with $\theta_0, \ldots, \theta_{m-1}$ if $(a_1, \ldots, a_h) \in \rho$ if and only if for each $0 \leq i \leq m-1$ at least two of the elements $a_1, \ldots, a_h$ are equivalent modulo θ_i .

We discuss briefly in which maximal clones the clones $Pol(\leq)$ for the order relations from Theorem 4.1 are contained:

If $(A; \leq)$ is an antichain, then $Pol(\leq) = O_A$, so the order-primal algebra $\underline{A}$ is primal. Since every chain on a finite set is bounded, in the second case $Pol(\leq)$ is a maximal clone of Class (1). In the third case, the connectedness of two elements defines a binary central relation with a one-element center, consisting of the least (greatest) element and thus $Pol(\leq)$ contained in one of the maximal classes of types(5).

In the fourth case, $Pol(\leq)$ is not a subclone of a maximal clone of operations preserving a relation on A of one of the classes (1), (2), (3). In this case the connectedness of two elements defines a congruence relation with exactly one non-singleton block A_t and $Pol(\leq) \subseteq Pol(\theta_{A_t})$.

6. Binary Hyperindentities in Order-primal Algebras

In [2], special hyperidentities with binary operation symbols are considered. The basic idea is that unary operations $h : A^2 \to A^2$ can be expressed in the form

$$h(a_1, a_2) = (f \otimes f')(a_1, a_2) = (f(a_1, a_2), f'(a_1, a_2))$$

with binary operations $f, f' \in O_A^{(2)}$. The correspondence $h \mapsto (f \otimes f')$ defines a bijection from $O_{A^2}^{(1)}$ onto $O_A^{(2)} \times O_A^{(2)}$. For binary operation symbols $\varphi_i, \varphi'_i, i \in I$, we define $\psi_i(x_1, x_2) := (\varphi_i(x_1, x_2), \varphi'_i(x_1, x_2))$. The composition of ψ_1 and ψ_2 is defined by

$$\psi_1\psi_2(x_1, x_2) = (\varphi_1(\varphi_2(x_1, x_2), \varphi'_2(x_1, x_2)), \varphi'_1(\varphi_2(x_1, x_2), \varphi'_2(x_1, x_2))).$$

We define binary terms $Pr\psi_i, Pr'\psi_i$ inductively by

(i) $Pr\psi_i = \varphi_i, Pr'\psi_i := \varphi'_i$ for all $i \in I$,

(ii) For $\xi = \psi_{i_1} \dots \psi_{i_k}\psi_{i_{k+1}}$, put $\psi := \psi_{i_2} \dots \psi_{i_{k+1}}$ and define

$$Pr\xi := \varphi_{i_1}(Pr\psi, Pr'\psi), Pr'\xi := \varphi'_{i_1}(Pr\psi, Pr'\psi).$$

For $\eta := \psi_{i_1} \dots \psi_{i_k}$ and $\tau := \psi_{i_1} \dots \psi_{i_l}$, we say that the algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ satisfies the hyperidentity $\eta(x_1, x_2) \approx \tau(x_1, x_2)$ if $\underline{A} \models_{hyp} (Pr\eta)(x_1, x_2) \approx (Pr\tau)(x_1, x_2)$. If $\rho \subseteq A^h$ is an h-ary relation on A, then $\rho\otimes\rho$ is an h-ary relation on A^2 defined by $\rho\otimes\rho := \{((a_1, b_1), \dots, (a_h, b_h)) \in (A^2)^h \mid (a_1, \dots, a_h) \in \rho$ and $(b_1, \dots, b_h) \in \rho\}$. Further we say that $f \otimes f'$ defined by $(f \otimes f')(x_1, x_2) := (f(x_1, x_2), f'(x_1, x_2)), f, f' \in O_A^{(2)}$ preserves $\rho \otimes \rho$ if $\{f, f'\} \subseteq Pol\rho$.

For a primal algebra $\underline{A}$, we have $\underline{A} \models \psi^m(x_1, x_2) \approx \psi^{m'}(x_1, x_2)$ if and only if $m, m' \geq n^2 - 1$ and $m \equiv m' \bmod \kappa(n^2)(n = |A|)$; i.e., $\underline{A}$ satisfies in particular $\psi^{n^2-1}(x_1, x_2) \approx \varphi^{n^2-1+\kappa(n^2)}(x_1, x_2)$ but not$\varphi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$. Our aim is to prove the following theorem:

Theorem 6.1. *Every order-primal algebra $\underline{A} = (A; (f_i^A)_{i\in I}), |A| \geq 2$, for an order relation different from an antichain satisfies the hyperidentity $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$.*

Proof. For an order relation $\leq$ on A, we define $\lambda_{\leq} := max\{\lambda(f) | f \in Pol_A^{(1)}(\leq)\}$. Then we have: if $\lambda_{\leq} \leq^* n - 2$, then $\lambda_{\leq\otimes\leq} = max\{\lambda(h) | h \in Pol_{A^2}^{(1)}(\leq \otimes \leq)\} \leq^* n^2 - 2$. Therefore, for every order-primal algebra for an order $\leq$ different from the orders in Theorem 4.1, we have $\underline{A} \models \psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$.

We consider the order relations listed in Theorem 4.1. If $(A; \leq)$ is a chain, then $\leq \otimes \leq$ is a bounded order on A^2 which is not a chain since there are incomparable elements (for instance (0, 1) and (1, 0) for the least element 0 and the greatest element 1). Therefore, $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$ is satisfied. □

In [2], it was proved that for an equivalence relation θ on A which has at least two blocks with more than one element the semigroup $\underline{Pol^{(1)}\theta}$ satisfies the identity $\psi^{n-2}(x_1, x_2) \approx \psi^{n-2+\kappa(n)}(x_1, x_2)$. If $\leq$ is disconnected and only one connected component A_t with respect to $\leq$ contains more than one element, then $Pol(\leq) \subseteq Pol(\theta_{A_t})$, but $\theta_{A_t} \otimes \theta_{A_t}$ is an equivalence relation on A^2 with more than one block of cardinality greater than 1. Therefore, in this case, $\underline{A}$ satisfies also $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$. That means, only the case where $\leq$ is unbounded and connected with the least(the greatest) element satisfying Conditions (i), (ii), (iii) from Lemma 3.2 is open. As we have already mentioned, in this case $Pol(\leq)$ is contained in a maximal clone $Pol(\varrho)$ for a binary central relation ϱ. But in [Den-P; 88] was proved that $\underline{Pol^{(1)}(\varrho)}$ satisfies $\varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n)}(x)$ for every central relation. The following lemma fills in the gap and finishes the proof of Theorem 6.1.

Lemma 6.2. *Let $(A; \leq)$ be an unbounded connected ordered set with the least (or dually, with the greatest) element. Then $Pol_{A^2}(\leq \otimes \leq)$ cannot admit a long-tailed function.*

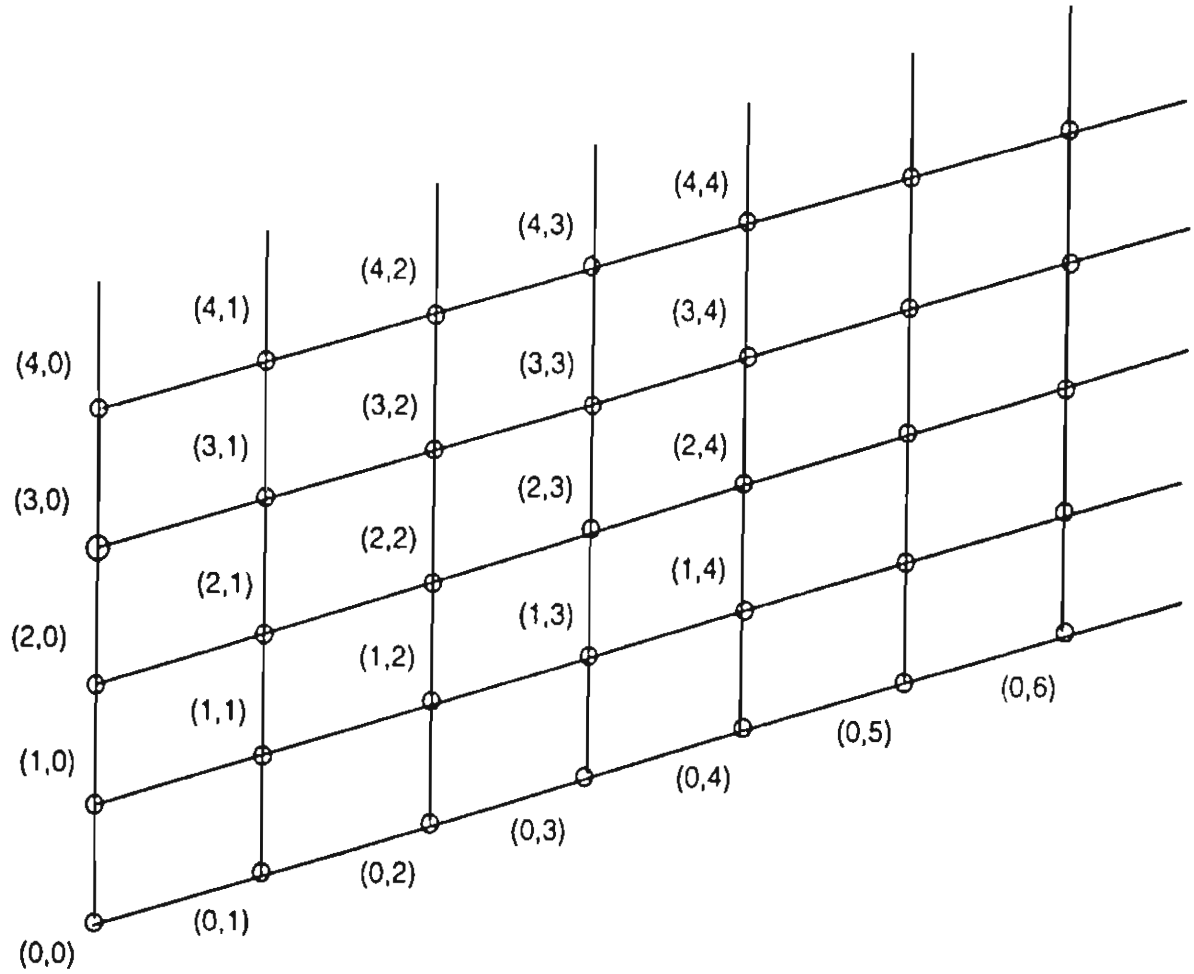
Proof. Since $(A; \leq)$ is unbounded, there are elements a and c in A which are incomparable. Let 0 be the least element of A with respect to $\leq$ and denote by 1 an element of A which covers 0; i.e., $0 \prec 1$. If there are elements of A which are incomparable with 1, then we will denote one of such elements by a. If 1 is comparable with all elements of A, we will denote an element which covers 1 by 2. Since A is finite, there are a least integer k and an element $a \in A$ with

$$0 \prec 1 \prec 2 \ldots \prec k \prec k+1 \text{ and } k \prec a$$

such that $\{a, k+1\}$ is an anti-chain. Let $\{a_1, a_2, \ldots, a_t\}$ be the set of all elements in A such that $a_i \succ k$ for all $1 \leq i \leq t$ and such that $\{k+1, a_1, \ldots, a_t\}$ is an anti-chain. Then $(A^2; \leq \otimes \leq)$ is connected since $(0,0)$ is the least element, but also unbounded. Suppose that $(A^2; \leq \otimes \leq)$ is invariant under an order-preserving LT-function g such that

$A^2 = \{d, g(d), \ldots, g^{n^2-1}(d) = (0,0)\}$ for some $d \in A^2$. We set $\bar{0} :=$ $(0,0)$ and $\overline{j-1} := g^{n^2-j}(d)$ for $1 \leq^* j \leq^* n^2$. We enumerate the pairs of A^2 in the following way :(0,0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2),(3, 0), (2, 1), (1, 2), (0, 3), (4, 0), (3, 1), (2, 2), (1, 3), (0, 4), (5, 0),(4, 1), (3, 2), (2, 3), (1, 4), (0, 5).

This can be demonstrated by the following picture:



That means with $d = m_0 := 0, m_i = m_{i-1} + i + 1$ for $1 \leq^* i \leq^* k$ we set $\overline{m_i + l} := (i - l + 2, l - 1)$ for $1 \leq^* l \leq^* m_{i+1} - m_i$. For the proof we need the following claim.

claim: $\overline{m_k + 1} \in \{(k+1, 0), (a_1, 0), \ldots, (a_t, 0), (k, 1)\}$ and $\overline{m_k + 2} = (k, 1)$.

Clearly, $\overline{m_k + 2} = (k, 1) \succ (k, 0) = \overline{m_{k-1} + 1}$ and this implies $\overline{m_k + 1} \succ \overline{m_{k-1}} = (0, k)$ which is impossible since $\{(0, k), (k+1, 0), (a_1, 0), \ldots, (a_t, 0), (k, 1)\}$ is an anti-chain. □

In [2], it was shown that a finite algebra $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$ with $\{f_i^{\underline{A}}) \mid i \in I\} = Pol(r)$ for a central relation r does not satisfy the binary hyperidentity $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-n^2)}(x_1, x_2)$. On the other hand side, if $\leq$ is a connected unbounded order with the least (the greatest) element, then $Pol(\leq)$ is contained in $Pol(r)$ for a central relation r. Therefore one can separate the corresponding clones by a hyperidentity. Further we have

Corollary 6.4. *Let $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$ be a finite nontrivial order-primal algebra and assume that $\psi^{n^2-2}(x_1, x_2) \approx \psi^{n^2-2+\kappa(n^2)}(x_1, x_2)$ is not a hyperidentity in $\underline{A}$. Then $\underline{A}$ is primal (and the order is an anti-chain).*

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Unary Operations with Long Pre-periods*

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Abstract

It is well-known that the congruence lattice $Con\underline{A}$ of an algebra $\underline{A}$ is uniquely determined by the unary polynomial operations of $\underline{A}$. If $|A| = n$ and if $|Imf| = |A|$ or $|Imf| = 1$ for every unary polynomial operation f of $\underline{A}$, then $\underline{A}$ is called a permutation algebra. Permutation algebras play an important role in tame congruence theory. If $f : A \to A$ is not a permutation then $|A| > |Imf|$ and there is a least natural number $\lambda(f)$ with $Imf^{\lambda(f)} = Imf^{\lambda(f)+1}$. We consider unary operations with $\lambda(f) = n - 1$ and $\lambda(f) = n - 2$ and ask for equivalence relations on A which are invariant under such unary operations
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1 Some Facts on Unary operations

Let $f : A \to A$ be a unary operation defined on a finite set A with $n := |A|$, $n \geq 2$. Let $H_A = \{f | f : A \to A\}$ be the set of all unary operations (transformation) defined on A and let S_A be the set of all permutations on A. If id_A is the identity operation on A and $g \in S_A$, then the order $Ord(g)$ of g is the least natural number m with $g^m = id_A$.

Let $f \in H_A$. Then we set $Imf := \{f(a) | a \in A\}$ and let $\lambda(f)$ be the least natural number m such that $Imf^m = Imf^{m+1}$. The number $\lambda(f)$

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is called the *pre-period* of f. For $f \in H_A$ we put $o(f) := Ord(f|Imf^{\lambda(f)})$ where f^m is defined as m-fold composition of f (with $f^0 := id_A$). Then

Remark 1.1:

(i) $0 \leq \lambda(f) \leq |Imf|$ and $\lambda(f) \leq n-1$,

(ii) $\lambda(f) = 0 \Leftrightarrow f \in S_A$,

(iii) $\lambda(f) = n-1$ if and only if there exists an element $d \in A$ such that $A = \{d, f(d), f^2(d), \dots, f^{n-1}(d)\} = f^n(d)$. (see e.g. [Den-P, 88])

A unary operation $f : A \to A$ with $\lambda(f) = n-1, |A| = n$ was called a LT-function (long-tailed function) in [Den-R; 04]. Remark 1.1 (iii) characterizes LT-functions. In the next section we consider unary operations f with $\lambda(f) = n-2, n \geq 3$.

2 $LT_1 - functions$

Now we want to characterize unary operations $f : A \to A$ with $|A| \geq 3$ and $\lambda(f) = n-2$.

Defintion 2.1: Let A be a finite set with $|A| \geq 3$. Then a unary operation $f : A \to A$ with $\lambda(f) = n-2$ is said to be a $LT_1 - function$.

We will prove some simple properties of $LT_1 - functions$:

Lemma 2.2 : Let $f : A \to A$ be a unary operation and assume that $|A| = n \geq 3$ and $\lambda(f) = n-2$. Then the following propositions are satisfied:

(i) $A \supset Imf \supset Imf^2 \supset \dots \supset Imf^{n-2}$,

(ii) $|Imf^{n-2}| = 1$ or $|Imf^{n-2}| = 2$,

(iii) If $|Imf^{n-2}| = 2$ then $|A| = |Imf| + 1$,

(iv) If $|Imf^{n-2}| = 1$ then $|A| = |Imf| + 2$,

(v) $|Imf^k| = |Imf^{k+1}| + 1$ for $k = 1, \dots, n-3$.

Proof: (i) In any cases we have $A \supseteq Imf \supseteq Imf^2 \supseteq \ldots \supseteq Imf^{n-2}$. Proper inclusions follow from the definition of $\lambda(f)$ and from $\lambda(f) = n-2$.

(ii) Because of (i) we have $|Imf^{n-2}| \leq 2$ and thus $|Imf^{n-2}| = 1$ or $|Imf^{n-2}| = 2$.

(iii) This is also a consequence of (i).

(iv) By (i) and $|Imf^{n-2}| = 1$ there is a k with $1 \leq k < n-1$ such that $|Imf^k| = |Imf^{k+1}| + 2$. Assume that $k < n-2$. Then there are elements $a \neq b$ and $c \neq d$ in Imf^k such that $f(a) = f(b)$ and $f(c) = f(d)$ in Imf^{k+1}. Therefore, there are elements $a, b, c, d \in A$ with $a \neq b, c \neq d$ such that $f(a) = f(b)$ and $f(c) = f(d)$ and this means, $|A| = |Imf| + 2$.

(v) If $|Imf^{n-2}| = 2$, then from (i) we obtain $|Imf^k| = |Imf^{k+1}| + 1$ for $k = 0, \ldots, n-3$ and if $|Imf^{n-2}| = 1$, then by (iv), $|A| = |Imf| + 2$ and then by (i), $|Imf^k| = Imf^{k+1} + 1$ for $k = 1, \ldots, n-3$. □

Our aim is to characterize unary operations on A with $\lambda(f) = n-2$. At first we will prove some technical lemmas which we need for our characterization theorem. If $|Imf^{n-2}| = 1$, then by Lemma 2.2 (iv), $|A| = |Imf| + 2$. Therefore, there are elements $a, b, c, d, s, t \in A, a \neq b, c \neq d$ such that $f(a) = f(b) = s, f(c) = f(d) = t$. Clearly, if $|A| = 3$, then f is constant.

Remark 2.3: If $|A| \geq 3$ and $|A| = |Imf| + 2$, then there are different elements $u, v \in A$ with $u \neq v$ such that for all $t' \in A$ we have $f(t') \notin \{u, v\}$. Moreover, the function $f|_{A\setminus\{a,b,c,d\}} : A\setminus\{a, b, c, d\} \to A\setminus\{s, t, u, v\}$ is a bijection.

Lemma 2.4: Assume that $f : A \to A$ is a unary operation with $f(a) = f(b) = s, f(c) = f(d) = t$ and $|A| = |Imf| + 2$. If $s, t \notin \{a, b, c, d\}$ and $f(s) \notin \{a, b, c, d\}$ or $f(t) \notin \{a, b, c, d\}$, then $|Imf^k| \geq 2$ for all $k \geq 1$.

Proof: $s, t \notin \{a, b, c, d\}$ implies $f(s) \notin \{s, t, u, v\}$ since $f|_{A\setminus\{a,b,c,d\}} : A\setminus\{a, b, c, d\} \to A\setminus\{s, t, u, v\}$ is bijective. Therefore $f(s) \neq s$ and the set $\{s = f^0(s), f(s)\}$ is a two-element subset of Imf. Inductively, assume that $\{f^{k-1}(s), f^k(s)\}$ is a two-element subset of Imf^k. We consider the following cases:

(a) $f^k(s) \notin \{a, b, c, d\}$ and $f^{k-1}(s) \notin \{a, b, c, d\}$. Then the injectivity of $f|_{A\setminus\{a,b,c,d\}}$ implies that $f^k(s) \neq f^{k+1}(s)$.

(b) $f^k(s) \notin \{a, b, c, d\}$ and $f^{k-1}(s) \in \{a, b, c, d\}$. Then $f^k(s) = f(f^{k-1}(s)) \in \{s, t\}$ and $f^{k+1}(s) = f(f^k(s)) \notin \{s, t\}$, so $f^{k+i}(s) \neq f^k(s)$.

(c) $f^k(s) \notin \{a,b,c,d\}$. Then $f^{k+1}(s) \notin \{s,t\}$ where $s,t \in \{a,b,c,d\}$ and then $f^k(s) \neq f^{k+1}(s)$.

Then there follows that $\{f^k(s), f^{k+1}(s)\} \subseteq Imf^k$ for all $k \geq 1$. This means $|Imf^k| \geq 2$ for all $k \geq 1$. $\square$

Lemma 2.5: Assume that $\lambda(f) = n-2, |Imf^{n-2}| = 1, f(a) = f(b) = s$ and $f(c) = f(d) = t$. Then

(i) if $s,t \notin \{a,b,c,d\}$, then $f(s) \notin \{a,b,c,d\}$ or $f(t) \notin \{a,b,c,d\}$,

(ii) $s \in \{a,b,c,d\}$ or $t \in \{a,b,c,d\}$,

(iii) if $s \in \{a,b\}$ and $s \neq t$, there exists an m such that $f^m(c) \in \{a,b\}\backslash\{s\}$ and $\{c,d\} \cap \{u.v\} \neq \phi$ where $u,v \in A$ are the elements which do not occur in Imf (by Remark 2.3),

(iv) if $|A| \geq 4$ and $s = t$, then $\{u,v\} \neq \{b,c\}$ and $\{u.v\} \cap \{b,c\} \neq \phi$.

Proof: (i) If $f(s) = s$ and $f(t) = t$ or $f(s) = t$ and $f(t) = s$, then $s,t \in Imf^k$ for all k. This contradicts $|Imf^{n-1}| = 1$ since $s \neq t$ because of $s,t \notin \{a,b,c,d\}$. Therefore $f(s), f(t) \notin \{s,t\}$. If $f(s) \in \{a,b\}$ or $f(t) \in \{c,d\}$, then $f^2(s) = s$ or $f^2(t) = t$. This means $s \in Imf^{n-2}$ or $t \in Imf^{n-2}$ and $s \in Imf^{n-3}$ or $t \in Imf^{n-3}$ and then $f(s) \neq s \in Imf^{n-2}$ or $f(t) \neq t \in Imf^{n-2}$ and $|Imf^{n-2}| \geq 2$, a contradiction. Thus $f(s) \notin \{a,b\}$ and $f(t) \notin \{c,d\}$. If $f(s) \in \{c,d\}$ and $f(t) \in \{a,b\}$, then $f^2(s) = t$ and $f^2(t) = s$ and then $f(f^2(a)) = f^2(c)$ and $f(f^2(c)) = f^2(a)$. Therefore $f^2(a)$ and $f^2(c)$ are two elements of A which are mapped to each other and thus they belong to Imf^k for all $k \geq 1$. Since $f(s) \neq f(t)$, we have $|Imf^{n-2}| \geq 2$, a contradiction. This means $f(s) \notin \{c,d,\}$ or $f(t) \notin \{a,b\}$. Therefore, $f(s) \notin \{a,b,c,d\}$ or $f(t) \notin \{a,b,c,d\}$.

(ii) If $s,t \notin \{a,b,c,d\}$, then by (i) and Lemma 2.4 we get $|Imf^k| \geq 2$ for all $k \geq 1$ and then $|Imf^{n-2}| \geq 2$, a contradiction.

(iii) Clearly, $f^{n-2}(c) \in Imf^{n-2}$ and $f^{n-2}(c) = s$. Let r be the least positive integer such that $f^r(c) = s$. Then $1 < r \leq n-2$ and $1 \leq r-1 < n-2$ and $f^{r-1}(c) \in \{a,b\}$. Indeed, from $f^{r-1}(c) \notin \{a,b\}$ would follow $|Imf| \leq |A| - 3$, a contradiction. By the choice of r we have $f^{r-1}(c) \in \{a,b\}\backslash\{s\}$ with $r-1 \geq 1$. We choose $m = r-1$.

Next, suppose that $\{c,d\} \cap \{u,v\} = \phi$. Then $\{c,d\} \subseteq Imf$. Therefore, there are $p,q \in A$ with $f(p) = c$ and $f(q) = d$. Now $a,b,c,d \in Imf$

with $f(a) = f(b)$ and $f(c) = f(d)$ implies that $|Imf^2| \leq |Imf| - 2$, a contradiction.

(iv) Suppose that $\{u, v\} = \{b, c\}$ or $\{u, v\} \cap \{b, c\} = \phi$.

(a) $\{u, v\} = \{b, c\}$. Then $A \backslash \{s, u, v\} = A \backslash \{a, b, c\}$ and $f|_{A \backslash \{a,b,c\}}$ is a permutation on $A \backslash \{a, b, c\}$. Since $|A| = n \geq 4$, we have $|A \backslash \{a, b, c\}| \geq 1$ and $\phi \neq A \backslash \{a, b, c\} \subseteq Imf^k$ for all $k \geq 1$. Since $a \in Imf^{n-2}$, we have $|Imf^{n-2}| \geq 2$, a contradiction.

(b) $\{u, v\} \cap \{b, c\} = \phi$. Then $\{b, c\} \subseteq Imf$. Then there are elements $p, q \in A$ such that $f(p) = b \neq c = f(q)$ which implies that $\{f(a) = a, f(p) = b, f(q) = c\}$ is a subset of Imf. But $f^2(q) = f^2(p) = f^2(a) = a \in Imf^2$, which shows that $|Imf^2| \leq |Imf| - 2 < |Imf| - 1$, a contradiction. □

From Lemma 2.2(ii) it follows that we have to consider the two cases $|Imf^{n-2}| = 1$ and $|Imf^{n-2}| = 2$. Now, we will give a characterization of $LT_1 - functions$ with $|Imf^{n-2}| = 1$ which corresponds to the characterization of $LT - functions$ given in Remark 1.1.

Theorem2.6: Let A be a finite set with $|A| = n \geq 3$ and let $f : A \to A$ be an operation. Then $\lambda(f) = n - 2$ and $|Imf^{n-2}| = 1$ if and only if there are distinct elements $u, v \in A$ such that $A = \{u, v, f(v), \ldots, f^{n-2}(v)\}$ and such that there is an exponent k with $0 \leq k \leq n - 2$ with $f(u) = f^{k+1}(v)$ and a number m with $m + k = n - 2$ such that $f^{m+1} = f^m(u)$.

Proof: Assume that $\lambda(f) = n - 2$ and $|Imf^{n-2}| = 1$. Then by Lemma 2.2(iv) we get $|A| = |Imf| + 2$ and there are two elements u, v with $u \neq v$ which do not occur in Imf. Because of Lemma 2.2(v) the restriction g of f on the (n-2)-element set Imf is a LT-function. Then by Remark 1.1(iii) there is an element $d \in Imf$ such that $Imf = \{d, g(d), \ldots, g^{n-3}(d)\} = \{d, f|_{Imf}(d), \ldots, f^{n-3}|_{Imf}(d)\}$ and $g^{n-3}(d) = g^{n-2}(d)$. Since $d \in Imf$, there is an element $v \in A$ such that $d = f(v)$ and the element v cannot belong to $Imf = \{d, g(d), \ldots, g^{n-3}(d)\}$. Further, the second element u which does not belong to Imf cannot be mapped to u or to v. Therefore, it is mapped to one of the elements $d, g(d), \ldots, g^{n-3}(d)$; let's say to $g^k(d) = f^{k+1}(v) = f(u)$. Then $A = \{v, f(v), \ldots, f^k(v)\} \cup \{u, f(u), \ldots, f^m(u)\}$ with $m + k = n - 2$ and because of $|Imf^{n-2}| = 1$, we have $f^m(u) = f^{m+1}(u)$.

Conversely, we assume that there are different elements $u, v \in A$ such that $A = \{u, v, f(v), \ldots, f^{n-2}(v)\}$ and such that there is an exponent k with $0 \leq k \leq n-2$ with $f(u) = f^{k+1}(v)$ and a number m with $m+k = n-2$ and such that $f^{m+1}(u) = f^m(u)$. Clearly, all elements are pairwise distinct and we have $a := f^m(u) \neq f^{m+1}(u) =: b$ and $c := f^k(v) \neq u$. Then $f(a) = f^{n+1}(u) = f^m(u) = f(b)$ and $f(c) = f^{k+1}(v) = f(u)$. Hence, $|A| = |Imf| + 2$. Next, we show that $\lambda(f) = n - 2$. Because of $|A| = |Imf| + 2$, it is enough to show that $A \supset Imf \supset Imf^2 \supset \ldots \supset Imf^{n-2}$.

Since $|A| = |Imf| + 2$ and $Imf \supseteq \{f(v), \ldots, f^{n-2}(v)\}$ we get $Imf = \{f(v), \ldots, f^{n-2}(v)\}$. Now, we consider the cases m =1 and $m > 1$.

If m = 1, then $A = \{u, f(u)\} \cup \{v, f(v), \ldots, f^k(v)\}$. If k = 0, then $A = \{v, u, f(u)\}$ and $Imf = \{f(u)\}, n = 3, \lambda(f) = 1 = 3 - 2, |Imf| = 1$, i.e. f is a $LT_1 - function$. If $k > 0$, then $f^k(v) \in Imf^t$ for all $1 \leq t \leq k$ with $n - 2 = k + 1$; and in this case we have $f(u) \neq f^k(v) \in Imf^t$, but $f(u) = f^{k+1}(v) \notin Imf^{t+1}$ for all $1 \leq t \leq k$ which implies $Imf^t \supset Imf^{t+1}$ for all $1 \leq t \leq k < n - 2$ and f is a $LT_1 - function$.

If $m > 1$, Then $f^{k+1}(v) = f(u)$ implies $f^{m-1} = f^{m-2}(f(u)) = f^{m-2}(f^{k+1}(v)) = f^{m+k-1}(v) = f^{n-2}(v) \in Imf^{n-3}$. Since $Imf^{n-3} \supset Imf^t$ for all $t \leq n-3$, we have $f^{m-1}(u) \in Imf^t$ for all $1 \leq t < n - 2$. Now $f^{m-1}(u) \neq f^m(u)$ in Imf^t, whereas $f(f^{m-1}(u)) = f(f^m(u))$ implies that $Imf^t \supset Imf^{t+1}$ for all $1 \leq t < n - 2$. This shows $A \supset Imf \supset Imf^2 \ldots \supset Imf^{n-2}$ and together with $|A| = |Imf| + 2$, this means $|imf^{n-2}| = 1$ and $\lambda(f) = n - 2$. This finishes the proof. □

Now, we consider the case that $|Imf^{n-2}| = 2$. Then $|A| = |Imf| + 1$ and $|Imf^k| = |Imf^{k+1}| + 1$ for $k = 1, \ldots, n-3$. The restriction $f^{n-2}|Imf$ can be the identity function or a permutation on a two-element set.

Theorem 2.7: Let A be a set with $|A| = n \geq 3$ and let $f : A \to A$ be a unary operation. Then $\lambda(f) = n - 2$ and $|Imf^{n-2}| = 2$ if and only if there are different elements $u, v \in A$ such that either

(i) $A = \{v, u, f(u), \ldots, f^{n-2}(u)\}$ with $v = f(v)$ and $f^{n-1}(u) = f^{n-2}(u)$, or

(ii) $A = \{u, f(u), f^2(u) \ldots, v = f^{n-2}(u), f^{n-1}(u)\}$ where $v = f^n(u) = f^{n-2}(u)$.

Proof: By Lemma 2.2(iii) we have $|A| = |Imf| + 1 > |Imf|$. Therefore, there are exactly two elements $a, b \in A$ such that $f(a) = f(b) = s$ and there is an element $y \in A$ with $y \neq f(t)$ for all $t \in A$. Then the restriction of f to $A \backslash \{a, b\}$ is a bijective mapping $f|_{A \backslash \{a,b\}} \to A \backslash \{s, y\}$. We consider the two cases $s \in \{a, b\}$ and $s \notin \{a, b\}$.

Case1: $s \in \{a, b\}$. Without loss of generality we may assume that $s = a$. Then $f(s) = s \in Imf^{n-2}$. Now we consider two subcases. $y \in \{a, b\}$ and $y \notin \{a, b\}$.

Case1.1: $y \in \{a, b\}$. Then $\{a, b\} = \{s, y\}$ and $f|_{A\setminus\{a,b\}}$ is a permutation. Since $|A| \geq 3$ and $|Imf^{n-2}| = 2$, we have $|A| = 3$ because of $s \in Imf^{n-2}$. Then $A = \{a, y, f(y)\}$ with $f(a) = a$ and $f^2(y) = y$. This corresponds to (i).

Case1.2: $y \notin \{a, b\}$. Because of the bijectivity of $f|_{A\setminus\{a,b\}} : A\setminus\{a, b\} \to A\setminus\{s, y\}$, we can choose elements $x_1, x_2, \ldots, x_{q-1}$ from $A\setminus\{a, b, y\}$ and $x_q = y$ such that $f(x_1) = b$ and $f(x_i) = x_{i-1}$ for $1 \leq i \leq q$. So, $X = \{x_q, f(x_q), \ldots, f^q(x_q) = b, f^{q+1}(x_q) = a\} \subseteq A$. Since $|Imf^{n-2}| = 2$, we have $A\setminus X \neq \phi$ and $f|_{A\setminus X}$ is a permutation. Thus $|A\setminus X| = 1$ and $|A| = q+3$. It follows that $A = \{c, y, f(y), \ldots, f^{q+1}(y)\}$ with $c \in A\setminus X, f(c) = c$ and $f^{q+2}(y) = f^{q+1}(y)$. This corresponds to (i).

Case2: $s \notin \{a, b\}$. Then s is not a fix point with respect to f since otherwise $|A| \geq |Imf| - 2$, a contradiction. We consider the two subcases $f(s) \notin \{a, b\}$ and $f(s) \in \{a, b\}$.

Case2.1: $f(s) \notin \{a, b\}$. The bijectivity of $f|_{A\setminus\{a,b\}}$ onto $A\setminus\{s, y\}$ implies $s \neq f(s)$ and $\{s, f(s)\} \subseteq A\setminus\{a, b\}$. Now assume that there is a $k \geq 1$ such that $\{s, f(s), \ldots, f^k(s)\}$ is a subset of $A\setminus\{a, b\}$ consisting of pairwise distinct elements. From the injectivity of $f|_{A\setminus\{a,b\}}$ and $f^{k-1}(s) \neq f^k(s)$ we obtain $f^k(s) \neq f^{k+1}(s)$. If $f^{k+t}(s) = f^t(s)$ for some $1 \leq t \leq k$, then $1 \leq t - 1 \leq k$ and $f(f^k(s) = f(f^{t-1}(s))$ implies $f^k(s) = f^{t-1}(s)$ which contradicts our assumption. If $f^{k+1}(s) = s$, then by Lemma 2.2 we get another contradiction. This means, $X = \{s, f(s), \ldots, f^k(s), \ldots\}$ is an infinite set, a contradiction.

Case2.2: $f(s) \in \{a, b\}$. Assume that $f(s) = a$. From Lemma 2.2 we obtain $a, s \in Imf^{n-2}$. So, $f(t) \notin \{a, s\}$ for all $t \notin \{a, b, s\}$ since $|A| = |Imf| + 1$. If $y = b$, then $A\setminus\{a, b, s\} = A\setminus\{a, s, y\}$. Hence $f|_{A\setminus\{a,b,s\}}$ is a permutation. There follows that $A = \{a, b, s\} = \{b, f(b), f^2(b)\}$ where $f^3(b) = f(b)$. This corresponds to (ii). The other subcase of 2.2 is that $y \neq b$. Then $b \in A\setminus\{s, y\}$. By surjectivity of $f|_{A\setminus\{a,b\}}$ onto $A\setminus\{s, y\}$ and because of the finiteness of $|A\setminus\{a, b\}|$ we may choose q-1 pairwise distinct elements $x_1, x_2, \ldots, x_{q-1}$ which are different from a and from b such that $f(x_i) = f(x_{i-1})$ for $1 \leq i \leq q$ and with $x_0 = b, x_q = y$. Therefore, $X = \{x_q, f(x_q), \ldots, f^q(x_q) = b, f^{q+1}(x_q) = s, f^{q+2}(x_q) = a\} \subseteq A$ and $f|_{A\setminus X}$ is a permutation. Assume that $A\setminus X \neq \phi$, then $|Imf^{n-2}| \geq 3$, a contradiction. Thus $A = X$ and $q + 2 = n - 1$, i.e. $q = n - 3$ with

$u = x_q$ we have $A = \{u, f(u), \ldots, f^{n-1}(u)\}$ with $f^n(u) = f^{n-2}(u)$ and this corresponds to (ii).

Conversely, let A be a finite set with $|A| \geq 3$ and let $f : A \to A$ be a function satisfying (i) or (ii). Then we have $f^{n-1}(u) \neq f^{n-2}(u)$ (in case (a)) or $f^{n-1}(u) \neq f^{n-3}(u)$ (in case (b)) but $f(f^{n-1}(u)) = f^n(u) = f^{n-1}(u) = f(f^{n-2}(u))$ (in case (a)) or $f(f^{n-1}(u)) = f^n(u) = f^{n-2}(u) = f(f^{n-3}(u))$ (in case (b)). In either cases, we have $A \supset Imf$.

If $n = 3$, then $A = \{a, b, f(b)\}$ where $f(a) = a$ and $f^2(b) = f(b)$. Thus $\lambda(f) = 1 = 3 - 2$ and $|Imf| = 2$, i.e. f is a $LT_1 - function$. Therefore we may assume that $n \geq 4$ we show that $A \supset Imf \supset Imf^2 \supset \ldots \supset Imf^{n-2} = Imf^{n-1}$. Since $u \notin Imf$, we have $A \supset Imf \supset Imf^2 \supset \ldots \supset Imf^{n-2}$. In case (i) the elements $f^{n-3}(u)$ and $f^{n-2}(u)$ are distinct in Imf^t which have the same image in Imf^{t+1} for all $1 \leq t \leq n - 3$. Similarly, in case (b), we have $f^{n-3}(u)$ and $f^{n-1}(u)$ being distinct elements in Imf^t having the same image in Imf^{t+1} for all $1 \leq t \leq n-3$. This shows that $|Imf^t| \geq |Imf^{t+1}| + 1$ which implies that $Imf^t \supset Imf^{t+1}$ for all $1 \leq t \leq n - 3$. Therefore we have $\lambda(f) = n - 2$. We have to show that $|Imf^{n-2}| = 2$. In case (i), f has two different fix points, v and $f^{n-2}(u)$. Both are elements of Imf^{n-2}. Therefore $|Imf^{n-2}| = 2$. In case (ii), $f^{n-1}(u)$ and $f^{n-2}(u)$ are different elements from Imf^{n-2} and therefore, using $\lambda(f) = n - 2$, we have $|Imf| = 2$. □

3 Invariant Equivalence Relations

Let $\theta \subseteq A \times A$ be an equivalence relation on the finite set A, $|A| \geq 2$ and let $f : A \to A$ be an arbitary unary operation defined on A. Then we say, f preserves θ, or θ is invariant under f if the following is satisfied

$$\forall a, b \in A((a, b) \in \theta \Rightarrow (f(a), f(b)) \in \theta)$$

Let $Pol^{(1)}\theta$ be the set of all functions defined on A which preserve θ. Then we ask the flowing question: which equivalence relations are invariant under LT or $LT_1 - function$. For $LT - functions$ the answer is given by the following theorem.

Theorem 3.1 : Let A be a finite set with $|A| \geq 2$ and let θ be a non-trivial equivalence relation defined on A. Then $f \in Pol^{(1)}\theta$ is a LT-function if and only if there is only one block with respect to θ which has more than one element.

Proof: Since $\lambda(f) = n-1$, there is an element $d \in A$ such that $A = \{d, f(d), \ldots, f^{n-1}(d)\}$ and $f^{n-1}(d) = f^n(d)$. Since θ is non-trivial, there is a block B with respect to θ containing more than one element. Then there exists a least integer $i \geq 0$ and an integer j with $i < j \leq n-1$ such that $(f^i(d), f^j(d)) \in \theta$. From this we obtain $(f^s(d), f^{n-1}(d)) \in \theta$ for all $s \geq i$, This means, the elements $f^i(d), f^{i+1}(d), \ldots, f^{n-1}(d)$ belong to B; and by the choice of i, all other elements form singleton blocks.

Conversely, let θ be a non-trivial equivalence relation on A having only one block with more than one element. We denote the elements of A by $a_0, \ldots, a_{n-1}$ and may assume that $\{a_i, \ldots, a_{n-1}\}$ is the only non-trivial block with respect to θ. Then the operation $f : A \to A$ defined by $f(a_i) = a_{i+1}$ for $0 \leq i < n-1$ and $f(a_{n-1}) = a_{n-1}$ preserves θ and is obviously a LT-function. □

We will answer to the same question for $LT_1 - functions$, i.e. if $\lambda(f) = n-2$. Here we have again to distinguish the cases $|Imf^{n-2}| = 1$ and $|Imf^{n-2}| = 2$.

Preposition 3.2: Let A be a finite set with $|A| = n \geq 3$ and $\theta \subseteq A \times A$ be a non-trivial equivalence relation on A. Then there is a unary operation f with $\lambda(f) = n-2$ and $|Imf^{n-2}| = 1$ which preserves θ if and only if either

(i) there exists only one block with respect to θ with more than one element, or

(ii) there are exactly two blocks with respect to θ with more than one element and one of them consists exactly two elements.

Proof: Assune that $f : A \to A$ with $\lambda(f) = n-2$ and $|Imf^{n-2}| = 1$. Then there are distinct elements $u, v \in A$ and intergers $m \geq 1$ and $k \geq 0$ such that $m + k = n-2$ and $A = \{u, v, f(v), \ldots, f^{n-2}(v)\} = \{u, f(u), \ldots, f^m(u)\} \cup \{v, f(v), \ldots, f^k(v)\}$ where $f^{m+1}(u) = f^m(u) = f^{m+k}(v) = f^{m+k+1}(v)$ and $f(u) = f^{k+1}(v)$. Let θ be a nontrivial equivalence relation on A which is invariant under f and assume that $X := \{v, f(v), \ldots, f^{k-m}(v)\}$ and $Y := \{u, f(u), \ldots, f^m(u)\}$. Then $|X| \geq 3$ and $|Y| \geq 2$ and $f|_X$ and $f|_Y$ are LT-functions. Moreover, $f|_X$ preserved $\bar{\theta} := \theta|_{X \times X}$ and $f|_Y$ preserves $\bar{\bar{\theta}} := \theta|_{Y \times Y}$. By theorem 3.1, there is exactly one block with respect to $\bar{\theta}$ and $\bar{\bar{\theta}}$; respectively, which has more than one element. We consider the following cases:

Case 1: The block of u with respect to θ consists only one element. Then $\theta = \theta|_{X \times X} \cup \{(u, u)\} = \bar{\theta} \cup \{(u, u)\}$. Hence, there exists only one element of $A|\theta$ having cardinality greater than one.

Case 2: $(u, f^t(v)) \in \theta$ for some $0 \leq t < k$. If t = 0, then $(u, v) \in \theta$ which implies $(f^{k+1}(v) = f(u), f(v)) \in \theta$, so $\{f(v), \ldots, f^{m+k}(v)\} = X \backslash \{v\}$ is a subset of the block C of f(v) with respect to $\bar{\theta} = \theta|_{X \times X}$ and also with respect to θ; hence $\theta = A \times A$ if $u \in C$, a contradiction since θ is non-trivial. This shows $u \notin C$ and $B = \{u, v\}$ and $C = X \backslash \{v\}$ are the only elements of A/θ. Since $k > 0$, this gives (ii).

If $t > 0$, then also $k > 0$ and $f(u) \neq f^k(v)$. Then $\{f^{t+1}(v), \ldots, f^{k+1}(v) = f(u), \ldots, f^{m+k}(v)\}$ is a subset of the block C of f(u) with respect to $\bar{\theta}$ (and also with respect to θ) containing $f^k(v)$ and $f(u)$; hence $|C| > 1$.

If $u \in C$, then C is the only block with respect to θ with cardinality greater than 1 and if $u \notin C$, then $\{u, f^t(u)\}$ and C are the only block with respect to θ having cardinalities greater than 1 and $|\{u, f^t(v)\}| = 2$.

Case 3: $(u, f^t(u)) \in \theta$ for some $1 \leq t \leq m$ and $(u, f^s(v)) \notin \theta$ for all $0 \leq s < k$. Then Y is a block with respect to θ and the block of each $f^s(v)$ for $0 \leq s < k$ is singleton. Therefore, Y is the only block with respect to θ with $|Y| \geq 2$.

Case 4: $(u, f^k(v)) \in \theta$. If $(c, d) \notin \theta$ for all $c \neq d$ in $A - \{u, f^k(v)\}$ then $\{u, f^k(v)\}$ is the only block with respect to θ having more than one element. We consider that there are $c \neq d$ in $A - \{u, f^k(v)\}$ such that $(c, d) \in \theta$. If c or d belong to $X - Y$ then $\{c, d, f^k(v)\}$ is a subset of the only block C with respect to $\bar{\theta}$ (hence with respect to θ) with $|C| > 1$; and so, $C \cup \{u\}$ is the only block with respect to θ which has more than one element. But, if c and d are both in $Y - \{u\}$ then they are in the only block C with respect to $\bar{\theta}$ (hence with respect to θ) with $|C| > 1$; so, in this case, C and $\{u, f^k(v)\}$ are the only blocks having more than one element and one of them has cardinality 2.

Conversely, let A be a set with $|A| = n \geq 3$ and let $\theta \subseteq A \times A$ satisfy either case (i) or case(ii). We may assume that $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and either $B = \{a_i, a_{i+1}, \ldots, a_{n-1}\}$ for some $0 < i < n-1$ in case (i) or $B = \{a_0, a_i\}$ and $C = \{a_{i+1}, \ldots, a_{n-1}\}$ for some $0 < i < n-1$ in case (ii) are the blocks with respect to θ. In either cases, we define $f : A \to A$ by $f(a_i) = a_{i+1}$ if $i \notin \{0, n-1\}$, $f(a_0) = a_{i+1}$ and $f(a_{n-1}) = a_{n-1}$. In both cases f preserves θ. Further, we have $f(a_{i-1}) = f(a_0)$ and $f(a_{n-2}) = a_{n-1} = f(a_{n-1})$ for $a_{i-1} \neq a_0$ and $a_{n-2} \neq a_{n-1}$ and there are no other elements $c \neq d$ in A such that f(c) = f(d). Thus $|Imf| = |A| - 2$. Since $a_{n-1} \neq a_{n-2}$ and $a_{n-1}, a_{n-2} \in Imf^k$ for all $1 \leq k < n-2$ and $f(a_{n-1}) = f(a_{n-2})$, we have $Imf^k \supset Imf^{k+1}$ for all $1 \leq k < n-2$. Together with $A = |Imf| + 2$

we get $|Imf^{n-2}| \leq 1$. But, $a_{n-1} \in Imf^{n-2}$ implies $|Imf^{n-2}| = 1$ and $Imf^{n-2} = Imf^{n-3}$ and therefore $\lambda(f) = n-2$. $\square$

Now we will consider the case $|Imf^{n-2}| = 2$.

Proposition 3.3: Let A be a finite set with $n \geq 3$ and let $\theta \subseteq A \times A$ be a nontrivial equivalence relation. Then there is a unary operation $f : A \to A$ with $\lambda(f) = n-2$ and $|Imf^{n-2}| = 2$ such that θ is invariant under f if and only if either

(i) there is only one block B with respect to θ which has more than one element; or

(ii) there are only two blocks B and C with respect to θ which have more than one element and $|B| - |C| \leq 1$.

Proof: By Theorem 2.7. there are different elements $u, v \in A$ such that either

(i) $A = \{v, u, f(u), \ldots, f^{n-2}(u)\}$ with $v = f(v)$ and $f^{n-1}(u) = f^{n-2}(u)$. or

(ii) $A = \{u, f(u), \ldots, v = f^{n-2}(u), f^{n-1}(u)\}$ where $v = f^n(u) = f^{n-2}(u)$.

We consider at first case(i).

(i) Let $X := A \backslash \{v\}$. Then $f|X$ is a LT-function and $\theta|_{X \times X}$ is invariant under$f|X$. Therefore, there is only one block with respect to $\theta|_{X \times X}$ which has more than one element. If $v \notin B$, then $\theta = \theta|_{X \times X} \cup \{(v, v)\}$ and therefore B is the only block with respect to θ with more than one element. If $v \in B$, then $(v, f^t(u)) \in \theta$ for some $0 \leq t \leq n-2$ and thus $(v, f^s(u)) \in \theta$ for all s with $t \leq s \leq n-2$ and then also $(v, f^{n-2}(u)) \in \theta$. This means that every block B with respect to θ with $|B| > 1$ and $v \in B$ contain also $f^{n-2}(u)$.

Now, $B = \{v, f^{n-2}(v)\}$ implies that the block of each $f^t(u)$ for $0 \leq t < n-2$ is singleton, hence B is the only block with respect to θ which has more than one element. But, if $\{v, f^{n-2}(u)\}$ is a proper subset of B, then $B \backslash \{v\}$ is the only block with respect to $\theta|_{X \times X}$ with $|B \backslash \{v\}| > 1$ which implies that the block of each $f^t(u) \notin B$ with respect to $\theta|_{X \times X}$ (and also with respect to θ) is singleton; hence, B is the only block with respect to θ with $|B| > 1$.

Now we consider the second case.

(ii) From $v = f^n(u) = f^{n-2}(u)$ we obtain $f^{2q}(f^{n-1}(u)) = f^{n-1}(u), f^{2q}(f^{n-2}(u)) = f^{n-2}(u), f^{2q+1}(f^{n-2}(u)) = f^{n-1}(u)$ and

$f^{2q+1}(f^{n-1}(u)) = f^{n-2}(u)$ for all $q \geq 1$. Since θ is nontrivial, there are numbers i, j with $0 \leq i < j \leq n-1$ such that $(f^i(u), f^j(u)) \in \theta$. From this we obtain easily $(f^i(u), f^{n-1}(u)) \in \theta$ or $(f^i(u), f^{n-2}(u)) \in \theta$. If $(f^{n-1}(u), f^{n-2}(u)) \in \theta$, there is only one block B with respect to θ such that $|B| \geq 2$. We assune that $(f^{n-1}(u), f^{n-2}(u)) \notin \theta$ then we get $(f^i(u), f^j(u)) \in \theta, (f^j(u), f^{n-1}(u)) \in \theta$ or $(f^i(u), f^j(u)) \in \theta$ or $(f^j(u), f^{n-2}(u)) \in \theta$. Without restriction of the generality we assume that $(f^i(u), f^{n-1}(u)) \in \theta$. Let $f^{n-1}(u) \in B \in A|\theta$ and let $f^{n-2}(u)) \in C \in A|\theta$. Then $|B| \geq 2, |C| \geq 1$ and $B \cap C = \phi$.

Now let D be a block of θ with more than one element. Then $(f^s(u), f^t(u)) \in \theta$ for some $0 \leq s < t < n-1$. Then from $(f^s(u), f^{n-1}(u)) \in \theta$ or $(f^s(u), f^{n-2}(u)) \in \theta$ there follows that either D = B or D = C. In either cases, if $|C| = 1$, there is only one $B \in A|\theta$ such that $|B| \geq 2$. If $|C| \geq 2$, then B and C are the only two elements of $A|\theta$ containing more than one element. For the proof of the last statement in (ii) we have only to consider the case that $(f^{n-1}(u), f^{n-2}(u)) \notin \theta$. Let i with $0 \leq i < n-1$ be the least integer such that $(f^i(u), f^j(u)) \in \theta$ for some $j > i$. If i and j have different parity, it can be easily checked that $(f^{n-1}(u), f^{n-2}(u)) \in \theta$. Therefore $(f^i(u), f^{i+1}(u)) \notin \theta$. Let $f^i(u) \in B \in A|\theta$ and $f^{i+1}(u) \in C \in A|\theta$. Then $|B| \geq 2$ since $f^i(u) \neq f^j(u)$ are in B. Then we have either $f^{n-1}(u) \in B$ or $f^{n-2}(u) \in B$. In the first case we get $f^{n-2}(u) \in C$ and if $f^s(u), f^t(u) \in B$ or $f^s(u), f^t(u) \in C$, we get s - t = 2q for some $q \geq 1$. By the choice of i, we can write $B = \{f^i(u), f^{i+2}(u), \ldots, f^{n-3}(u), f^{n-1}(u)\}$ and $C = \{f^{i+1}(u), f^{i+3}(u), \ldots, f^{n-1}(u), f^{n-2}(u)\}$. So, the function $\alpha : B \to C$, defined by $\alpha(f^t(u)) = f^{t+1}(u)$ if $t \neq n-1$ and $\alpha(f^{n-1}(u)) = f^{n-2}(u)$ is injective on $B \setminus \{f^{n-1}(u)\}$. Together with $\alpha(f^{n-3}(u)) = f^{n-2}(u) = \alpha(f^{n-1}(u))$ this gives $|B| = |C| + 1$.

In the case $f^{n-2}(u) \in B$ we have $f^{n-1}(u) \in C$ and $B = \{f^i(u), f^{i+1}(u), \ldots, f^{n-1}(u), f^{n-2}(u)\}$ and $C = \{f^{i+1}(u), f^{i+3}(u), \ldots, f^{n-3}(u), f^{n-1}(u)\}$. In this case, $\alpha : B \to C$ defined by $\alpha(f^t(u)) = f^{t+1}(u)(1 \leq t \leq n-2)$ is a bijection and $|B| = |C|$.

Conversely, let A be a set with $|A| = n \geq 3$ and let $\theta \subseteq A \times A$ satisfy either case (i) or case (ii). We may assune that $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and either $B = \{a_i, a_{i+1}, \ldots, a_{n-1}\}$ for some $0 < i < n-1$ in case (i) or $B = \{a_i, a_{i+2}, a_{i+4}, \ldots, a_{n-1}\}$ and $C = \{a_{i+1}, a_{i+3}, \ldots, a_{n-2}\}$ for some i with $0 < i < n-1$ such that i and n-1 have the same parity. In case (i) we define $f : A \to A$ by $f(a_i) = a_{i+1}$ if $a_i \notin \{a_{n-2}, a_{n-1}\}, f(a_i) = a_i$ if $a_i \in \{a_{n-2}, a_{n-1}\}$. In case (ii), let define $f : A \to A$ by $f(a_i) = a_{i+1}$ if

$a_i \neq a_{n-1}$ and $f(a_{n-1}) = a_{n-2}$. In either cases, it is clear that that θ is invariant under f.

Since $f(a_{n-3}) = f(a_{n-2}) = a_{n-2}$ in case (i) and $f(a_{n-3}) = a_{n-2} = f(a_{n-1})$ in case (ii) we have $Imf \supset A$ in both cases.

Since $f|_{A\setminus\{a_{n-1}\}}$ is a LT-function and $f(a_{n-1}) = a_{n-1}$ in case (i) we have $\lambda(f) = n-2$ in case (i). In case(ii), a_{n-3} and a_{n-2} are in Imf^k for all k with $1 \leq k < n-2$ which implies that

$$A \supset Imf \supset \ldots \supset Imf^{n-2} \text{ and } Imf^{n-2} = \{a_{n-1}, a_{n-2}\};$$

hence $Imf^{n-2} = Imf^{n-1}$. So $|Imf^{n-2}| = 2$ and $\lambda(f) = n-2$. □

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ภาคผนวก (ง)

ESSENTIAL OPERATION SYMBOLS IN TERMS *

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Abstract

Generalizating the concept of an essential variable in a term with respect to an algebra or variety we define essential operation symbols in a term with respect to an algebra or with respect to a variety of algebras. After proving some elementary propositions, we extend our definition to essential operation symbols in hypersubsttutions and determine some monoids of hypersubstitutions which contain the same essential operation symbols.

Using the concept of a unitary Menger algebra of rank n we define so-called operator term and prove that essential operation symbols in terms with respect to an algebra correspond to essential veriables in operator terms with respect to the same algebra. Using the isomorphism between the monoid of all clone endomorphisms at the monoid of hypersubstitution we get the equivalence between essential variables in hyperterm with respect to clone V and essential operation symbols in hypersubstitutions with respect to the variety V.

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1 Introduction

In [Sht-D; 98] the concept of an essential variable in a term with respect to an algebra or with respect to a variety was introduced. Let $\tau = (n_i)_{i \in I}$ be an arbitrary type and let $W_\tau(X_n)$ be the set of all $n-ary$ terms of type τ built up by the $n_i - ary$ operation symbols $f_i, i \in I$, and by variables from an alphabet $X_n = \{x_1, \ldots, x_n\}$ and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ where $X = \{x_1, \ldots, x_n, \ldots\}$ is an arbitrary countably infinite alphabet. Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ with the sequence

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$(f_i^{\mathcal{A}})_{i\in I}$ of fundamental operations where $(f_i^{\mathcal{A}})$ is n_i-ary. Let $:= (W_\tau(X); (\overline{f}_i)_{i\in I})$ with $\overline{f}_i(t_1,\ldots,t_{n_i}) := f_i(t_1,\ldots,t_{n_i})$ be the absolutely free albebra. On the set $W_\tau(X_n)$ of all $n-ary$ terms of type τ we may define an $(n+1)-ary$operation $S^n : W_\tau(X_n)^{n+1} \to W_\tau(X_n)$ by

$S^n(x_i, t_i, \ldots, t_n) := t_i$,

$S^n(f_i(s_1,\ldots,s_{n_i}), t_1,\ldots,t_n) := f_i(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_{n_i},t_1,\ldots,t_n))$.

Adding the variables $x_1,\ldots,x_n$ asnullary operations we get an algebra $n-clone\tau := (W_\tau(X_n); S^n, x_1,\ldots,x_n)$ which is called a uniary Menger algebra of rank n. An algebra of the same type can be constructed if we define a superposition operation $S^{n,A}$ on the set $O^{(n)}(A)$ of all $n-ary$ operations $f : A^n \to A$ defined on A.

$S^{n,A} : O^{(n)}(A)^{n+1} \to O^{(n)}(A)$ is defined by $S^{n,A}(f_0,(f_1,\ldots,f_n)(a_1,\ldots,a_n) := f_0(f_1(a_1,\ldots,a_n),\ldots,f_n(a_1,\ldots,a_n))$ for all $a_1,\ldots,a_n \in A^n$. Then $\mathcal{O}^{(n)}(A) := (O^{(n)}(A); S^{n,A}, e_1^{n,A},\ldots,e_n^{n,A})$ is also a unitary Menger algebra of rank n since it satisfies the identities which can be used to define $n-clone\tau$ and which will be given later on. $\mathcal{F}(\mathcal{A}) :=< \{f_i^A \mid i \in I\} >$ denotes the subalgebra of $\mathcal{O}^{(n)}(A)$ defined by the fundamental operations of $\mathcal{A}$.

Let $\beta : x \to W_\tau(X)$ be a substitution. By the freeness of $\mathcal{F}_\tau(X)$ each such mapping can be uniquely extended to an endomorphism $\overline{\beta} : \mathcal{F}_\tau(X) \to \mathcal{F}_\tau(X)$.

Definition 1.1([Sht-D; 98]) *Let t be an $n-ary$ term of type τ. A variable x_i is essential in t with respect to an algebra $\mathcal{A}$ of typed τ iff there is a substitution $\beta : X_n \to W_\tau(X_{n+1})$ with $\beta(x_j) = x_j$ for all $j \neq i$ and $\beta(x_i) = x_{n+1}$ such that $\overline{\beta}(t) \approx t$ is not an identy in $\mathcal{A}$.*

By $Ess(t,\mathcal{A})$ we denote the set of all variables which are essential in t with respect to $\mathcal{A}$. If $s \approx t$ is satisfied as an identity in the algebra $\mathcal{A}$ we will write $\mathcal{A} \models s \approx t$. To every term $t \in W_\tau(X_n)$ and every algebra $\mathcal{A}$ of type τ there belong an induced $n-ary$ term operation which is inductively defined by

$x_i^{\mathcal{A}} := e_i^{n,A}$ where $e_i^{n,A} : A^n \to A$ and $e_i^{n,A}(a_1,\ldots,a_n) = a_i$ is the $n-ary$ projection n th $i-$th component and $1 \leq i \leq n$,

$(f_i(s_1,\ldots,s_{n_i}))^{\mathcal{A}} := S^{n,A}(f_i^{\mathcal{A}}, s_1^{\mathcal{A}},\ldots,s_{n_i}^{\mathcal{A}})$.

Then $\mathcal{A} \models s \approx t$ means simply $s^{\mathcal{A}} = t^{\mathcal{A}}$. If V is a variety of algebra of type τ, then $V \models s \approx t$ means $\mathcal{A} \models s \approx t \ \ \forall \mathcal{A} \in V$.

The following properties of the $Ess(t,\mathcal{A})$ are quite clear.

1. $x_i \in Ess(x_i,\mathcal{A}) \Longleftrightarrow |A| > 1$,

2. $x_i \notin vat(t) \Longrightarrow x_i \notin Ess(t,\mathcal{A})$
(here $var(t)$ denote the of all varibles occurring in the term t),

3. $\mathcal{A} \models s \approx t \Longrightarrow Ess(s, \mathcal{A}) = Ess(t, \mathcal{A})$.

It is very natural to define variables which are essential in the term t with respect to a variety V of the same type.

Definition 1.2 *Let V be a variety of type $\tau, t \in W_\tau(X_n)$. Then a variable $x_i|inX_n$ is called essentialin t with respect to a varity V if it is essential in t with respect to the free algebra $\mathcal{F}_V(X)$ with $X = \{x_1, \ldots, x_n, \ldots\}$. The set of all variables in t which are essential with respect to the variety V is denoted by $Ess(t, V)$.*

Then for variety V and W of type τ we have

4. $V \subseteq W \Longrightarrow Ess(t, V) \subseteq Ess(t, W)$.

Hypersubstitutions of type τ are mapping which assign to each $n_i - ary$ operation symbol of type τ an $n_i - ary$ term of the same type. If $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ is a hypersubstitution of type τ then its extension $\hat{\sigma} : W_\tau(X_n) \to W_\tau(X_n)$ is defined inductively by the following steps:

(i) If $t = x_i$ for some $1 \leq i \leq n$, then $\hat{\sigma}[t] = x_i$,

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ for the $n_i - ary$ operation symbol f_i and some $n_i - ary$ term, then $\hat{\sigma}[t] = S_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]$.

Let $Hyp(\tau)$ be the set of all hypersubstitutions of typr τ. Together with the identityhypersubstitution σ_{id} mapping f_j to $f_j(x_1, \ldots, x_{n_j})$ for all $j \in I$ we get a monoid.

Here the superposition operation $S_n^{n_i} : W_\tau(X_{n_i}) \times (W_\tau(X_n))^{n_i} \to W_\tau(X_n)$ is defined in a similar way as we defined S^n.

An identity $s \approx t$ of terms of type τ is called a hyperidentity of a variety V if for every substitution of $n_i - ary$ terms of V for the operation symbols f_i in $s \approx t$ the resulting identity holds in $V(i \in I)$, i.e. if $V \models \hat{\sigma}[s] \approx \hat{\sigma}[t]$ for every $\sigma \in Hpy(\tau)$. If $s \approx t$ is a hyperidentity in V we will also write $V \models_{hyp} s \approx t$.
Essential variables in hypersubstitutions of type $\tau = (n)$ can be defined as follows:

Definition 1.3 *Let $t \in W_{(n)}(X_n)$, let V be a variety of type $(n), n \geq 1$, and let $\sigma_t : f \mapsto t$ be a hypersubstitution of type (n). Then x_i is essential in σ_t with respect to the variety V if x_i is essential in t with respect to V.*

Then we can consider the following set of hypersubstitutions:
$M_i(V) = \{\sigma_t \mid x_i \in Ess(t, V), t \in W_{(n)}(X_n)\}$.
In general, $M_i(V)$ is not a submonoid of $Hyp(n)$. In [Den-K; 01] we gave the necessary sufficienced conditions for V to make $M_i(V)$ a monoid.

2 Essential Operation Symbols

We extend the definiton of essential variables to essential operation symbols

Definition 2.1 *Let $t \in W_\tau(X_n)$ be an $n-ary$ term of type τ, let $\mathcal{A}$ be an algebra of type τ and let $op_s(t)$ be the set of all operation symbols occurring in the term t. An operation symbol f_i of arity n_i is essential in t with respect to $\mathcal{A}$ iff there is a hypersubstitution σ of type τ and an $n_i - ary$ operation symbol $g \notin op_s(t)$such that $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ for every $j \neq i, j \in I$ and $\sigma(f_i) = g(x_1, \ldots, x_{n_i})$ with $f_i(x_1, \ldots, x_{n_i}) \approx g(x_1, \ldots, x_{n_i})$ is not an identity in $\mathcal{A}$ and $\mathcal{A} \not\models \hat{\sigma}[t] \approx t$.*

The opposite case, f_i is called fictitious in t wity respect to $\mathcal{A}$.
Let $Hypess(t, \mathcal{A})$ be the set of all essential operation symbols in t with respect to $\mathcal{A}$.

Corollary 2.2 *Let $s, t \in W_\tau(X_n)$ and let $\mathcal{A}$ be an algebra of type τ. Then*

$$\mathcal{A} \models_{hyp} s \approx t \Longrightarrow Hypess(s, \mathcal{A}) = Hypess(t, \mathcal{A}).$$

Proof. Assume that $f_i \in Hypess(t, \mathcal{A})$. Then there exists a hypersubstitution $\sigma \in Hyp(\tau)$ and an $n_i - ary$ operation symbol $g \notin op_s(t)$ such that $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ for every $j \neq i, j \in I$ and $\sigma(f_i) = g(x_1, \ldots, x_{n_i})$ with $\mathcal{A} \not\models \hat{\sigma}[t] \approx t$.

If $\mathcal{A} \models \hat{\sigma}[s] \approx s$ then together with $\mathcal{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t]$ we would have $\mathcal{A} \models \hat{\sigma}[t] \approx t$, a contradiction. Therefore $\mathcal{A} \not\models \hat{\sigma}[s] \approx s$ and $f_i \in Hypess(s, \mathcal{A})$. This shows $Hypess(t, \mathcal{A}) \subseteq Hypess(s, \mathcal{A})$ and similarly, $Hypess(s, \mathcal{A}) \subseteq Hypess(t, \mathcal{A})$. □

For the set of all operation symbols occurring in the term t we have $f_i \notin op_s(t) \Rightarrow f_i \notin Hypess(t, \mathcal{A})$. Further, for variables x_i we have $Hypess(x_i, \mathcal{A}) = \emptyset$.

One more consequence of the definition is:

Proposition 2.3 *Let $\mathcal{A}$ be an algebra of type τ and assume that the type contains one at least binary operation symbol f_i. Then $\mathcal{A}$ is trivial iff for every $t \in W_\tau(X)$ we have $Hypess(t, \mathcal{A}) = \emptyset$.*

Proof. The trivial algebra satisfies $\hat{\sigma}[t] \approx t$ for every hypersubstitution and for every term. Therefore, there is no essential operation symbol in t with respect to

$\mathcal{A}$. Conversely, assume that for every term $t \in W_\tau(X)$ we have $Hypess(x_i, \mathcal{A}) = \emptyset$. That means, for every hypersubstitution $\sigma \in Hyp(\tau)$ and for every $i \in I$ with $\sigma(f_i) = g(x_1, \ldots, x_{n_i}) \neq f_i(x_1, \ldots, x_{n_i}), g \notin op_s(t)$ and $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ if $j \neq i$ we have $\mathcal{A} \models \hat{\sigma}[t] \approx t$. Consider the hypersubstitutions σ_1 with $\sigma_1(f_i) = e_1(x_1, \ldots, x_{n_i})$ and $\sigma_1(f_j) = f_j(x_1, \ldots, x_{n_i})$ if $j \neq i$ and $\sigma_2(f_i) = e_{n_i}(x_1, \ldots, x_{n_i})$ and $\sigma_2(f_j) = f_j(x_1, \ldots, x_{n_i})$ if $j \neq i$ where $e_i^{n_i, \mathcal{A}}$ is the n_i-ary projection on the $i-th$ component and the term $t = f_i(x_1, \ldots, x_{n_i})$ where f_i is at least binary operation symbol. Then $\mathcal{A} \models \hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})] \approx f_i(x_1, \ldots, x_{n_i})$ i.e. $\mathcal{A} \models e_1(x_1, \ldots, x_{n_i}) \approx f_i(x_1, \ldots, x_{n_i})$ and similary $\mathcal{A} \models e_{n_i}(x_1, \ldots, x_{n_i}) \approx f_i(x_1, \ldots, x_{n_i})$. But then $\mathcal{A} \models e_1(x_1, \ldots, x_{n_i}) \approx e_{n_i}(x_1, \ldots, x_{n_i})$ and this means $\mathcal{A}$ is trivial. □

As in the case of essential variable in terms with respect to algebras we may extend our definition to varieties.

Definition 2.4 *Let V be a variety of type τ and assume that $t \in W_\tau(X_n)$. Then f_i is called essential in t with respect to the variety V if f_i is essential in t with respect to the free algebra $F_V(X)$.*

In this case we consider the set $Hypess(t, V)$.
An easy consequence of this definition is

Corollary 2.5 *If V is a subvariety of W, then $Hypess(t, V) \subseteq Hypess(t, W)$.*

Proof. If $V \subseteq W$ we get $IdW \subseteq IdV$ for the sets of all identities satisfied in W and in V respectively. Assume that $f_i \notin Hypess(t, W)$. Then for every $\sigma \in Hyp(\tau)$ with $\sigma(f_i) = g(x_1, \ldots, x_{n_i}) \neq f_i(x_1, \ldots, x_{n_i}), g \notin op_s(t)$ and $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ if $j \neq i$ we have $W \models \hat{\sigma}[t] \approx t$. But then $V \models \hat{\sigma}[t] \approx t$ and $f_i \notin Hypess(t, V)$. □

Essential operation symbols in hypersubstitutions can be defined as follows:

Definition 2.6 *Let V be a variety of type τ and let $\sigma \in Hyp(\tau)$. Then f_i is essential in σ with respect to V iff f_i is essential in the term $\sigma(f_i)$ with respect to V.*

We consider the set of all hypersubstitutions σ such that f_i is essential in σ, i.e.

$$M_i(V) := \{\sigma \mid \sigma \in Hyp(\tau) \text{ and } f_i \text{ is essential in } \sigma\}.$$

Assume now that $\tau = (n), n \geq 2$. In this case we write simply $M(V)$.

In general, $M(V)$ is not a submonoid of $Hyp(n)$. We ask the following question: For which varieties of type(n) do the sets $M(V)$ form submonoids of $Hyp(n)$?

Now for proving the set $M(V)$ with $V \neq V_i = Mod\{f(x_1, \ldots, x_n) \approx x_i\}$ forms a submonoids of $Hyp(n)$, we need one auxiliary result.

Lemma 2.7 *Let $V_i = Mod\{f(x_1, \ldots, x_n) \approx x_i\}$. Then V_i is minimal variety.*

Proof. Let V be a variety such that $V \subseteq V_i$ and $V \neq V_i$. By using the fact that $s \approx t \in IdV_i$ iff there exists x_j which occurs in s and in t. Since $V \subseteq V_i$ and $V \neq V_i$, there is a term $t \in W_{(n)}(X_n)$ with $x_i \notin var(t)$ such that $t \approx x_i \in IdV$.
If $t = x_j, j \neq i$ and $x_j \approx x_i$, then V is trivial.
Let $t = f(t_1, \ldots, t_n)$ where $t_i \in W_{(n)}(X_n)$ and $f(t_1, \ldots, t_n) \approx x_i \in IdV$. Using the substitution rule, we substitute in $f(t_1, \ldots, t_n) \approx x_i$ on both sides for x_i the variable $x_j \neq x_i$. This gives $f(t_1, \ldots, t_n) \approx x_j \in IdV$ and then V is trivial. □

Proposition 2.8 *Let $\tau = (n), n \geq 2$ and let V be a non-trivial variety of type (n) with $V \neq V_i = Mod\{f(x_1, \ldots, x_n) \approx x_i\}$. Then*
$M(V) = \{\sigma \mid f \in hypess(\sigma(f), V)\}$ is a submonoid of $Hyp(n)$.

Proof. Since $V \neq V_i$ and every variety V with $V \subseteq V_i$ is trivial, there is a hypersubstitution $\sigma \in Hyp(n)$ with $\sigma(f) = e_i(x_1, \ldots, x_n)$ where $e_i^{n,V}$ is the $n-ary$ projection on the $i-th$ component such that $V \not\models \hat{\sigma}[f(x_1, \ldots, x_n)] \approx f(x_1, \ldots, x_n)$. That means, f is essential in $f(x_1, \ldots, x_n) = \sigma_{id}(f)$ with respect to V and so we get $\sigma_{id} \in M(V)$.
Let $\sigma_1, \sigma_2 \in M(V)$. Then $f \in hypess(\sigma_1(f), V)$ and $f \in hypess(\sigma_2(f), V)$ with $\sigma_1(f) \neq x_i, \sigma_2(f) \neq x_i$ for all $i = 1, 2, \ldots, n$ and there are hypersubstitutions $\sigma, \sigma' \in Hyp(n)$ with $\sigma(f) = g_1(x_1, \ldots, x_n)$, $\sigma'(f) = g_2(x_1, \ldots, x_n)$ where $g_1 \notin op_s(\sigma_1(f))$ and $g_2 \notin op_s(\sigma_2(f))$ such that $V \not\models \hat{\sigma}[\sigma_1(f)] \approx \sigma_1(f)$ and $V \not\models \hat{\sigma}'[\sigma_2(f)] \approx \sigma_2(f)$. That means, there exist variables $x_k, x_l \in X_n$ such that $x_k \notin var(\sigma_1(f))$ and $x_l \notin var(\sigma_2(f))$.
Let $\sigma_1 \circ_h \sigma_2(f) = \hat{\sigma}_1[\sigma_2(f)] = w$ where w is $n-ary$ term with $w \neq x_i$ for all $i = 1, 2, \ldots, n$. Then by induction on complexity of w, we get $x_l \notin var(w)$. Therefore, there is a hypersubstitution $\sigma'' \in Hyp(n)$ with $\sigma''(f) = e_l(x_1, \ldots, x_n), e_l \notin op_s(w)$ such that $V \not\models \hat{\sigma}''[w] \approx w$. That means, f is essential in $\sigma_1 \circ_h \sigma_2(f)$ with respect to V and so we have $\sigma_1 \circ_h \sigma_2 \in M(V)$. □

Proposition 2.9 *Let $\tau = (n_i)_{i \in I}, n_i \geq 2$ and let V be a non-trivial variety of type τ with $V \neq V_i = Mod\{f_i(x_1, \ldots, x_n) \approx x_j, i \in I\}$. Then*
$M_i(V) = \{\sigma \mid f_i \in hypess(\sigma(f_i), V)\}$ is a submonoid of $Hyp(\tau)$.

Proof. Since $V \neq V_i$ and every variety V with $V \subseteq V_i$ is trivial, there is a hypersubstitution $\sigma \in Hyp(\tau)$ with $\sigma(f_i) = e_i(x_1, \ldots, x_{n_i})$ and $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ where $e_i^{n,V}$ is the $n_i - ary$ projection on the $i - th$ component such that $V \not\models \hat{\sigma}[f_i(x_1, \ldots, x_{n_i})] \approx f_i(x_1, \ldots, x_{n_i})$. That means, f_i is essential in $f_i(x_1, \ldots, x_{n_i}) = \sigma_{id}(f_i)$ with respect to V and so we get $\sigma_{id} \in M_i(V)$.
Let $\sigma_1, \sigma_2 \in M_i(V)$. Then $f_i \in hypess(\sigma_1(f_i), V)$ and $f_i \in hypess(\sigma_2(f_i), V)$ with $\sigma_1(f_i) \neq x_i, \sigma_2(f_i) \neq x_i$ for all $i = 1, 2, \ldots, n_i$ and there are hypersubstitutions $\sigma, \sigma' \in Hyp(\tau)$ with $\sigma(f_i) = g_1(x_1, \ldots, x_{n_i})$, $\sigma'(f_i) = g_2(x_1, \ldots, x_{n_i})$ and $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j}), \sigma'(f_j) = f_j(x_1, \ldots, x_{n_j})$ if $j \neq i$ where $g_1 \notin op_s(\sigma_1(f_i))$ and $g_2 \notin op_s(\sigma_2(f_i))$ such that $V \not\models \hat{\sigma}[\sigma_1(f_i)] \approx \sigma_1(f_i)$ and $V \not\models \hat{\sigma}'[\sigma_2(f_i)] \approx \sigma_2(f_i)$. That means, there exist variables $x_k, x_l \in X_{n_i}$ such that $x_k \notin var(\sigma_1(f_i))$ and $x_l \notin var(\sigma_2(f_i))$.
Let $\sigma_1 \circ_h \sigma_2(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] = w$ where w is $n_i - ary$ term with $w \neq x_i$ for all $i = 1, 2, \ldots, n_i$. Then by induction on complexity of w, we get $x_l \notin var(w)$. Therefore, there is a hypersubstitution $\sigma'' \in Hyp(\tau)$ with $\sigma''(f_i) = e_l(x_1, \ldots, x_{n_i}), e_l \notin op_s(w)$ and $\sigma''(f_j) = f_j(x_1, \ldots, x_{n_j})$ if $j \neq i$ such that $V \not\models \hat{\sigma}''[w] \approx w$. That means, f_i is essential in $\sigma_1 \circ_h \sigma_2(f_i)$ with respect to V and so we have $\sigma_1 \circ_h \sigma_2 \in M_i(V)$. □

3 Essential Operation Symbolsin Terms and Essential Variables in Hyperterms

Is this section we study this correspondence in a restricted setting, that of $n - ary$ type and algebras. We will call a type of algebras $n - ary$ if all the operation symbols of the type are $n - ary$, for some fixed natural number n, and any algebra of such a type will be called an $n - ary$ algebra. Throughout ths section we assume that τ_n is such a fixed $n - ary$ type, with operation symbols $(f_i)_{i \in I}$ indexed by some set I.

By $W_{\tau_n}(X_n)$ we denote the set of all $n - ary$ terms of type τ_n built up from the operation symbols f_i of type τ_n and the alphabet X_n.

Next we want to consider some algebra of another type $\tau = (n + 1, 0, \ldots, 0)$, having n nullary operation and one $(n + 1) - ary$ operation, for a fixed natural number n. The first algebra of this type can be defined using the $n - ary$ term operations of an algebra of our first type τ_n. That is, we let $\mathcal{A} = (A; (f_i^{\mathcal{A}})^{i \in I})$ be any $n - ary$ algebra. Every $n - ary$ term $t \in W_{\tau_n}(X_n)$ induces an $n - ary$ term operation $t^{\mathcal{A}}$ on $\mathcal{A}$ which is inductively defined by the following steps:

(i) For every $1 \leq j \leq n$, the variable $x_j \in X_n$ induces the $n-ary\ j-th$ projection $e_j^{n,\mathcal{A}} : A^n \to A$ defined by $e_j^{n,\mathcal{A}}(a_1, \ldots, a_n) := a_j$;

(ii) If $t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}$ are the $n-ary$ term operations which are induched by the terms $t_1, \ldots, t_n \in W_{\tau_n}(X_n)$, then $(f_i(t_1, \ldots, t_n))^{\mathcal{A}} := f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$ is the $n-ary$ term operation induced by $f_i(t_1, \ldots, t_n)$.

We will use $\mathcal{T}^{(n)}(\mathcal{A})$ for the set of all $n-ary$ term operation of the $n-ary$ algebra $\mathcal{A}$. To define an algebra of type τ on this set, we select the n project operation $e_j^{n,\mathcal{A}}$, for $j = 1, 2, \ldots, n$ as the nullary operations, and we de an $(n+1)-ary$ superposition opration $S^{n,\mathcal{A}}$, inductively defind by

$$S^{n,\mathcal{A}}(e_j^{n,\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) := t_j^{\mathcal{A}}, \text{ for } 1 \leq j \leq n; \quad \text{and}$$

$$S^{n,\mathcal{A}}((f_i(s_1, \ldots, s_n))^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) \\ := f_i^{\mathcal{A}}(S^{n,\mathcal{A}}(s_1^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}), \ldots, S^{n,\mathcal{A}}(s_n^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})).$$

This gives analgebra $\mathcal{T}^{(n)}(\mathcal{A}) = (T^{(n)}(\mathcal{A}); S^{n,\mathcal{A}}, e_1^{n,\mathcal{A}}, \ldots, e_n^{n,\mathcal{A}})$, called the $n-ary$ (term) clone of the $n-ary$ algebra $\mathcal{A}$.

An algebra of the type $\tau = (n+1, 0, \ldots, 0)$ can also be defined on the set $W_{\tau_n}(X_n)$ of $n-ary$ terms of type τ_n In this case the $(n+1)-ary$ superposition operation S^n is defined inductively by

$$S^n(x_j, t_1, \ldots, t_n) := t_j, \text{ for } 1 \leq j \leq n; \quad \text{and}$$

$$S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) \\ := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)).$$

Selecting the variable terms $x_1, \ldots, x_n$ for the nullary operation, we form the algebra

$$n-clone\tau_n := (W_{\tau_n}(X_n); S^n, x_1, \ldots, x_n)$$

call the $n-ary\ \tau_n-clone$.

Now we want to consider identities of the type $\tau = (n+1, 0, \ldots, 0)$. To do this we use the new language built from an $(n+1)-ary$ operation symbol $\tilde{S}^n$ and n nullary operation symbols $\lambda_1, \ldots, \lambda_n$. We also introduce an alphabet of new variables, $\mathcal{X} := \{X_i \mid i \in I\}$.

Terms of typ $(\tau) = (n+1, 0, \ldots, 0)$ are defined in the following way:

(i) X_i is a term of type τ for all $i \in I$,

(ii) $\lambda_1, \ldots, \lambda_n$ are terms of type τ,

(iii) if $T, T_1, \ldots, T_n$ are terms and if $\tilde{S}^n$ in $(n+1)-ary$, then $\tilde{S}^n(T, T_1, \ldots, T_n)$ is a term of type τ.

By $W_\tau(\mathcal{X}_n)$ we denote the of all $n-ary$ terms of type τ and let $W_\tau(\mathcal{X})$ be the set of all terms of type τ.

Our identities will use terms from the new language $W_\tau(\mathcal{X})$.

Lemma 3.1[Den-J-W; 03] *The algebra $n-clone\tau-n$ satisfies the following identities of type τ:*

(C1) $\tilde{S}^n(X_0, \tilde{S}^n(X_{j_1}, X_2, \ldots, X_{n+1}), \ldots, \tilde{S}^n(X_{j_n}, X_2, \ldots, X_{n+1}))$
$\approx \tilde{S}^n(\tilde{S}^n(X_0, X_{j_1}, \ldots, X_{j_n}), X_2, \ldots, X_{n+1}))$,

(C2) $\tilde{S}^n(\lambda_j, X_1, \ldots, X_n) \approx X_j$, *for* $1 \leq j \leq n$,

(C3) $\tilde{S}^n(X_j, \lambda_1, \ldots, \lambda_n) \approx \lambda_j$, *for* $1 \leq j \leq n$.

Next we will show the fact that there is a bijection with $W_\tau(\mathcal{X})$ and $W_{\tau_n}(X_n)$.

Lemma 3.2 *If $\eta : \{X_i \mid i \in I\} \cup \{\lambda_1, \ldots, \lambda_n\} \rightarrow \{f_i(x_1, \ldots, x_n) \mid i \in I\} \cup \{x_1, \ldots, x_n\}$ is a bijection with $\eta(X_i) = f_i(x_1, \ldots, x_n)$ and $\eta(\lambda_i) = X_i$, then η can be extended to a bijection with $W_\tau(\mathcal{X} = \{X_i \mid i \in I\})$ and $W_{\tau_n}(X_n)$.*

Proof. Let $\bar{\eta} : W_\tau(\mathcal{X}) \rightarrow W_{\tau_n}(X_n)$ defined by

$$\bar{\eta}(X_i) := f_i(x_1, \ldots, x_n) \quad \text{for all } i \in I$$

$$\bar{\eta}(\lambda_i) := X_i \quad \text{for all } i = 1, 2, \ldots, n$$

$$\bar{\eta}(\tilde{S}^n(T_0, T_1, \ldots, T_n) := S^n(\bar{\eta}(T_0), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)).$$

Given $t \in W_{\tau_n}(X_n)$. To prove that there exists a term $T \in W_\tau(\mathcal{X})$ such that $\bar{\eta}(T) = t$ by using induction on the complexity of the term t.
If $t = x_i$ for some $i, i \in \{1, 2, \ldots, n\}$, then there exists $\lambda_i \in W_\tau(\mathcal{X})$ such that $\bar{\eta}(\lambda_i) := X_i$

If $t = f_i(x_1, \ldots, x_n)$ and assume that there exist $T_1, T_2, \ldots, T_n \in W_\tau(\mathcal{X})$ such that $\bar{\eta}(T_1) = t_1, \ldots, \bar{\eta}(T_n) = t_n$, then we have

$$\bar{\eta}(\tilde{S}^n(X_0, T_1, \ldots, T_n) = S^n(\bar{\eta}(X_0), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)) = f_i(x_1, \ldots, x_n).$$

This is, $\bar{\eta}$ is surjective.

Let $T, T' \in W_\tau(\mathcal{X})$ with $\bar{\eta}(T) = \bar{\eta}(T')$. To prove that $T = T'$

If $T = X_i$, then $\bar{\eta}(X_i) = f_i(x_1, \ldots, x_n) = \bar{\eta}(T')$ and so we get $T' = X_i$.

If $T = \lambda_i$, then $\bar{\eta}(\lambda_i) := X_i \bar{\eta}(T')$ and so we get $T' = \lambda_i$

If $T = \tilde{S}^n(T_0, T_1, \ldots, T_n), T' = \tilde{S}^n(T_0', T_1', \ldots, T_n')$ and assume that if $\bar{\eta}(T_j) = \bar{\eta}(T_j')$, then $T_j = T_j'$ for all $j = 0, 1, \ldots n$, we have

$\bar{\eta}(\tilde{S}^n(T_0, T_1, \ldots, T_n) = \bar{\eta}(\tilde{S}^n(T_0', T_1', \ldots, T_n')$ and then

$$S^n(\bar{\eta}(T_0), \bar{\eta}(T_1), \ldots, \bar{\eta}T_n) = S^n(\bar{\eta}(T_0'), \bar{\eta}(T_1'), \ldots, \bar{\eta}T_n').$$

That is, $\bar{\eta}(T_j) = \bar{\eta}(T_j')$ and then for all $j = 0, 1, \ldots n$. Therefore $T = T'$, this means $\bar{\eta}$ is surjective. Altogether, $\bar{\eta}$ is bijectivee. $\Box$

Now we extend the definition of essential variables in terms to essential variables in hyperterms

Definition 3.3 *Let $\mathcal{A}$ be be a non-trivial $n-ary$ algebra of type τ_n and let $T \in W_\tau(\mathcal{X})$. Then $X_i, 1 \leq i \leq m$ is called essential in T with respect to $\mathcal{T}^{(n)}(\mathcal{A})$ iff*

$$\mathcal{T}^{(n)}(\mathcal{A}) \not\models \bar{\beta}(T) = T,$$

where $\beta : \mathcal{X}_m \rightarrow W_\tau(\mathcal{X}_{m-1})$ is a mapping defined by

$\beta(X_j) = X_j$ for all $j \neq i$ and $\beta(X_i) = X_{m-1}$;

and when $\bar{\beta}$ is the extension of β to a mapping defined on terms, i.e.

$$\bar{\beta} : W_\tau(\mathcal{X}_m) \rightarrow W_\tau(\mathcal{X}_{m-1}).$$

For proving the following Theorem, we need one more auxiliary results.

Lemma 3.4 *If $\sigma = \bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha$ with $\alpha : \{f_i \mid i \in I\} \rightarrow \{f_i(x_1, \ldots, x_n) \mid i \in I\}$.*

then $\hat{\sigma} = \bar{\eta} \circ \bar{\beta} \circ \eta^{-1}$.

Proof. Let $t \in W_{\tau_n}(X_n)$. It is easy to show that $\hat{\sigma}[t] = \bar{\eta} \circ \bar{\beta} \circ \eta^{-1}[t]$ by using induction on the complexcity of the term t. ☐

Lemma 3.5 *Let $\mathcal{A}$ be be a non-trivial $n-ary$ algebra of type τ_n and let $T \in W_\tau(\mathcal{X}_m)$and let $f_{m+1}(x_1, \ldots, x_n) \approx f_i(x_1, \ldots, x_n) \notin Id\mathcal{A}$ with $f_{m+1} \notin op_s(\bar{\eta})$. Then*

$$\bar{\beta}(T)^{T^{(n)}(\mathcal{A})} \neq T^{T^{(n)}(\mathcal{A})} \iff \bar{\eta}(\beta(T))^{\mathcal{A}} \neq \bar{\eta}(T)^{\mathcal{A}},$$

where $\beta : \mathcal{X}_m \to W_\tau(\mathcal{X}_{m+1})$ is a mapping defined by

$$\beta(X_j) = X_j \text{ for all } j \neq i \text{ and } \beta(X_i) = X_{m+1}$$

and when $\bar{\beta}$ is the extension of β to a mapping defined on terms, i.e.

$$\bar{\beta} : W_\tau(\mathcal{X}_m) \to W_\tau(\mathcal{X}_{m+1})$$

and when $\bar{\eta} : W_\tau(\mathcal{X}) \to W_{\tau_n}(X_n)$ defined by

$$\bar{\eta}(X_i) := f_i(x_1, \ldots, x_n) \quad \text{for all } i \in I$$

$$\bar{\eta}(\lambda_i) := X_i \quad \text{for all } i = 1, 2, \ldots, n$$

$$\bar{\eta}(\tilde{S}^n(T_0, T_1, \ldots, T_n) := S^n(\bar{\eta}(T_0), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)).$$

Proof. Let $T \in W_\tau(\mathcal{X}_m)$. To show that

$$\bar{\beta}(T)^{T^{(n)}(\mathcal{A})} = T^{T^{(n)}(\mathcal{A})} \iff \bar{\eta}(\beta(T))^{\mathcal{A}} = \bar{\eta}(T)^{\mathcal{A}},$$

by using induction on the complexity on the term T.

If $T = X_i$, then $\bar{\beta}(T)^{T^{(n)}(\mathcal{A})} = \bar{\beta}(X_i)^{T^{(n)}(\mathcal{A})} = X_{m+1}^{T^{(n)}(\mathcal{A})} \neq X_i^{T^{(n)}(\mathcal{A})}$
$\iff \bar{\eta}(\bar{\beta}(T))^{\mathcal{A}} = \bar{\eta}(X_{m+1})^{\mathcal{A}} = f_{m+1}^{\mathcal{A}}(x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}} \neq \bar{\eta}(T)^{\mathcal{A}}$.

If $T = \tilde{S}^n(T_0, T_1, \ldots, T_n)$ and let $\beta(T) = T' = \tilde{S}^n(T_0', T_1', \ldots, T_n')$ and assume that $\bar{\eta}(T_j')^{\mathcal{A}} = \bar{\eta}(T_j)^{\mathcal{A}}$ for all $j = 1, 2, \ldots, n \iff (T_j')^{T^{(n)}(\mathcal{A})}$
$= (T_j)^{T^{(n)}(\mathcal{A})}$ for all $j = 1, 2, \ldots, n$. Then we have $\bar{\eta}(\bar{\beta}(T))^{\mathcal{A}} =$
$S^{n,\mathcal{A}}(\bar{\eta}(T_0')^{\mathcal{A}}, \bar{\eta}(T_1')^{\mathcal{A}}, \ldots, \bar{\eta}(T_n')^{\mathcal{A}}) = \bar{\eta}(T)^{\mathcal{A}} = S^{n,\mathcal{A}}(\bar{\eta}(T_0)^{\mathcal{A}}, \bar{\eta}(T_1)^{\mathcal{A}}, \ldots, \bar{\eta}(T_n)^{\mathcal{A}})$
$\iff \bar{\beta}(T)^{T^{(n)}(\mathcal{A})} = S^{n,\mathcal{A}}((T_0')^{T^{(n)}(\mathcal{A})}, \ldots, (T_n')^{T^{(n)}(\mathcal{A})}) = T^{T^{(n)}(\mathcal{A})}) =$

$S^{n,\mathcal{A}}((T_0)^{\mathcal{T}^{(n)}(\mathcal{A})}, \ldots, (T_n)^{\mathcal{T}^{(n)}(\mathcal{A})}).$ □

Now we want to prove that for essential variables in hyperterms with respect to the $n-ary$ clone of $n-ary$ algebra $\mathcal{A}$ and essential operation symbols in terms with respect to the algebra $\mathcal{A}$ are equivalent.

Theorem 3.6 *Let $\mathcal{A}$ be a non-trivial $n-ary$ algebra of type τ_n and let $\mathcal{T}^{(n)}(\mathcal{A}) = (T^{(n)}(\mathcal{A}); S^{n,\mathcal{A}}, e_1^{n,\mathcal{A}}, \ldots, e_n^{n,\mathcal{A}})$ and also let $T \in W_\tau(\mathcal{X})$. Then*

$$X_i \in Ess(T, \mathcal{T}^{(n)}(\mathcal{A})) \iff f_i \in Hess(\bar{\eta}(T), \mathcal{A})$$

when $\bar{\eta} : W_\tau(\mathcal{X}) \to W_{\tau_n}(X_n)$ defined by

$$\bar{\eta}(X_i) := f_i(x_1, \ldots, x_n) \quad \text{for all } i \in I$$

$$\bar{\eta}(\lambda_i) := X_i \quad \text{for all } i = 1, 2, \ldots, n$$

$$\bar{\eta}(\tilde{S}^n(T_0, T_1, \ldots, T_n) := S^n(\bar{\eta}(T_0), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)).$$

Proof. Let $T \in W_\tau(\mathcal{X})$ and assume thatb$X_i \in Ess(T, \mathcal{T}^{(n)}(\mathcal{A}))$. Then there is a natural number m such that $T \in W_\tau(\mathcal{X}_m)$ and we have $T \neq X_j$ for all $j = 1, 2, \ldots, m, T \neq \lambda_i$ for all $j = 1, 2, \ldots, n$ and there is a mapping β such that

$$\mathcal{T}^{(n)}(\mathcal{A}) \not\models \bar{\beta}(T) = T,$$

where $\beta : \mathcal{X}_m \to W_\tau(\mathcal{X}_{m+1})$ is a mapping defined by

$$\beta(X_j) = X_j \text{ for all } j \neq i \text{ and } \beta(X_i) = X_{m+1}$$

and when $\bar{\beta}$ is the extension of β to a mapping defined on terms, i.e.

$\bar{\beta} : W_\tau(\mathcal{X}_m) \to W_\tau(\mathcal{X}_{m+1})$. To show that $f_i \in Hess(\bar{\eta}(T), \mathcal{A})$.
Since $X_i \in var(T)$, we have $f_i \in op_s(\bar{\eta}(T))$.

Define a hypersubstitution σ of type τ_n with $\sigma = \bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha$. Then by LEmma 3.5 we have

$$\bar{\beta}(T) \approx T \notin Id\mathcal{T}^{(n)}(\mathcal{A}) \Rightarrow \bar{\eta}(\bar{\beta}(T)) \approx \bar{\eta}(T) \notin Id\mathcal{A} \Rightarrow \hat{\sigma}[\bar{\eta}(T)] \approx \bar{\eta}(T) \notin Id\mathcal{A}.$$

This means, $f_i \in Hess(\bar{\eta}(T), \mathcal{A})$.

Conversely, assume that $f_i \in Hess(\bar{\eta}(T), \mathcal{A})$, then there exists σ of type τ_n and an $n-ary$ operation symbol $f_{m+1} \notin op_s(\bar{\eta}(T))$such that $\sigma(f_j) = f_j(x_1, \ldots, x_n)$

for every $j \neq i, j \in I$ and $\sigma(f_i) = f_{m+1}(x_1, \ldots, x_n)$ with $f_i(x_1, \ldots, x_{n_i}) \approx f_{m+1}(x_1, \ldots, x_n)$ is not an identity in $\mathcal{A}$ and $\mathcal{A} \not\models \hat{\sigma}[\bar{\eta}(T)] \approx \bar{\eta}(T)]$ and so we get $\bar{\eta}(T) \neq x_i$ for all $i = 1, 2, \ldots, n$ and $X_i \in var(T)$.
To show that $X_i \in Ess(T, \mathcal{T}^{(n)}(\mathcal{A}))$.

Define a mapping $\beta : \mathcal{X}_m \to W_\tau(\mathcal{X}_{m+1})$ by

$$\beta(X_j) = X_j \text{ for all } j \neq i \text{ and } \beta(X_i) = X_{m+1}.$$

Then by Lemma 3.5 we have

$$\hat{\sigma}[\bar{\eta}(T)] \approx \bar{\eta}(T) \notin Id\mathcal{A} \Rightarrow \bar{\eta}(\bar{\beta}(T)) \approx \bar{\eta}(T) \notin Id\mathcal{A} \Rightarrow \bar{\beta}(T) \approx T \notin Id\mathcal{T}^{(n)}(\mathcal{A}).$$

This means, $X_i \in Ess(T, \mathcal{T}^{(n)}(\mathcal{A}))$. □

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