called *switching operation.* It is easy to see that the graph obtained from G by a switching will have the same degree sequence as G. The following theorem has been shown by Havel and Hakimi.

Theorem 2.2 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.* □

As a consequence of Theorem 2.2, we can define *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$, the vertices of which are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus we have shown the following corollary.

Corollary 2.3 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.* □

We proved in [7] the following results (Theorem 2.4 - Theorem 2.11).

Theorem 2.4 *Let* $\mathcal{R}(\mathbf{d})$ *be the set of realizations of* $\mathbf{d}$. *If* G *and* G' *are adjacent in the graph* $\mathcal{R}(\mathbf{d})$, *then* $|\chi(G) - \chi(G')| \leq 1$. □

Theorem 2.5 *For any graphic degree sequence* $\mathbf{d}$, *there exist integers* a *and* b *such that* $\mathbf{d}$ *has a realization with chromatic number* c *if and only if* c *is an integer satisfying* $a \leq c \leq b$. □

Let $\mathbf{d}$ be a graphic degree sequence. The *chromatic range* of $\mathbf{d}$ can be defined as the interval of integers specified by Theorem 2.5, i.e.,

$$\chi(\mathbf{d}) = [a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}.$$

Naturally, we can call a *the minimum chromatic number* for $\mathbf{d}$ and we write $\min\chi(\mathbf{d})$. Similarly, $b = \max\chi(\mathbf{d})$, the *maximum chromatic number* for $\mathbf{d}$. We write $\mathbf{d} = r^n$ for the sequence $(r, r, \ldots, r)$ of length n, where r is a non-negative integer and n a positive integer. It is easy to see that $\min\chi(0^n) = \max\chi(0^n) = 1$ and $\min\chi(1^n) = \max\chi(1^n) = 2$. Because of this fact, from now on we will consider $r \geq 2$ and $n \geq r+1$.

Theorem 2.6 *If* $r \geq 2$ *and* $n \geq 2r$, *then*

$$\min\chi(r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 2.7 *If* $r \geq 2$, *then* $\min\chi(r^{r+1}) = \max\chi(r^{r+1}) = r+1$, *and* $\min\chi(r^{r+2}) = \max\chi(r^{r+2}) = (r+2)/2$ *(in this case* r *must be even).* □

Theorem 2.8 *For any* $r \geq 4$ *and odd integer* s *such that* $3 \leq s \leq r$, *let* q *and* t *be integers satisfying* $r + s = sq + t, 0 \leq t < s$. *Then*

$$\min\chi(r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \le t \le s-2, \\ q+2 & \text{if } t = s-1. \end{cases}$$

□

Theorem 2.9 *For any even integer $r \ge 6$ and any even number s such that $4 \le s \le r$, let q and t be integers satisfying $r+s = sq+t$, $0 \le t < s$. Then*

$$\min\chi(r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \ge 2. \end{cases}$$

□

Theorem 2.10 *Let $r \ge 2$. Then*
(1) $\max\chi(r^{2r}) = r$,
(2) $\max\chi(r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \ge 4, \end{cases}$
(3) $\max\chi(r^n) = r+1$ for $n \ge 2r+2$. □

It should be noted that the result of Theorem 2.10 (3) satisfies for a realization $G = K_{r+1} \cup H$ in $\mathcal{R}(\mathbf{d})$, where H is an r-regular graph of order $n-r-1$. By using a suitable switching, we have

$$\max\{\chi(G) \ : \ G \text{ is a connected realization of } r^n\} = r.$$

Theorem 2.11 *For any r ad s such that $3 \le s \le r-1$, we have*
(1) $\max\chi(r^{r+s}) \ge (r+s)/2$ *if $r+s$ is even, and*
(2) $\max\chi(r^{r+s}) \ge (r+s-1)/2$ *if $r+s$ is odd.* □

3. A formula for D(n, k)

We now have enough tools for giving a proof of Theorem 1. For this purpose, results stated in the previous section will play a crucial role. To make the application easier, we state the Theorem 2.5 in a special case as in the following Theorem.

Theorem 3.1 *Let n, k, d be integers such that $3 \le k \le d \le n-2$. Then $\chi(n,k,d) \ne \emptyset$ if and only if* $\min\chi(d^n) \le k \le \max\chi(d^n)$. □

The following Lemma and its corollary can be found in [3]. It will be useful in finding a formula for $D(n,k)$.

Lemma 3.2 *Let n be an integer and l, t the remainder and quotient of the division of n by k, $k \ge 3$. If $t \ge 2$ and $\chi(n,k,d) \ne \emptyset$, then*
(a) $d \le n-t$ *if* $l = 0$,
(b) $d \le n-t-1$ *if* $0 < l < k-1$,

(c) $d \le n - t - 2$ *if* $l = k - 1$. □

Corollary 3.3 *Let* $\epsilon(n,t) = 1$ *or 0 according to the fact that* nt *is odd or even. Suppose* $k \ge 3$ *and* $\chi(n,k,d) \ne \emptyset$. *Then*
(a) $d \le n - t$ *if* $l = 0$,
(b) $d \le n - t - 1 - \epsilon(n,t)$ *if* $0 < l < k - 1$,
(c) $d \le n - t - 2 - \epsilon(n,t+1)$ *if* $l = k - 1$. □

In [3], the authors used the graph G with $\overline{G} = kK_t$ to show that $\chi(kt,k,(k-1)t) \ne \emptyset$ and hence $D(kt,k) = (k-1)t$. In case of $0 < l \le k-1$, they used a method of graph construction to prove that
$D(n,k) = n - t - 1 - \epsilon(n,t)$ for $0 < l < k-1$, and
$D(n,k) = n - t - 2 - \epsilon(n,t+1)$ for $r = k-1$,
and in both cases, the construction could work only when $t \ge k$. We are ready to go to the proof.

Proof of Theorem 1. Since $\chi(n,k,d) = \emptyset$ if nr is odd, we only consider the case when nd is even.

(a) Let $n = kt$, $r = (k-1)t$ and $s = t$. By Theorem 2.8 or 2.9, we have

$$\min\chi(r^{r+s}) = \min\chi(((k-1)t)^{(k-1)t+t}) = k.$$

This shows that $\chi(n,k,r) \ne \emptyset$ and hence $D(kt,k) \ge (k-1)t$, and by Corollary 3.3 (a), we have $D(kt,k) = (k-1)t$.

(b) We consider 3 cases:

Case 1. If both t and n are even, then $\epsilon(n,t+1) = 0$. In this case, let $n = kt + k - 1$, $r = (k-1)t + k - 3$ and $s = t + 2$. We want to show that $\chi(n,k,r) \ne \emptyset$. We first consider a special case when $k = 3$ and $t = 2$, and hence $n = 8, r = 4$ and $s = 4$. By Theorem 2.6 and Theorem 2.10, we have $\min\chi(4^8) = 2$ and $\max\chi(4^8) = 4$, respectively. Hence by Theorem 3.1, we have $\chi(8,3,4) \ne \emptyset$. We now consider $k > 3$ and $t \ge 2$, and then $r \ge 6$ and $s = t + 2 \ge 4$. By assumption on t and n, we can apply Theorem 2.9 we have

$$\min\chi(r^{r+s}) = \left\lceil \frac{r+s}{s} \right\rceil = \left\lceil \frac{kt+k-1}{t+2} \right\rceil = \left\lceil k - \frac{k+1}{t+2} \right\rceil \le k,$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. Similarly, by applying Theorem 2.11, we have

$$\max\chi(r^{r+s}) \ge \frac{r+s}{2} = \frac{kt+k-1}{2} \ge k.$$

Again by Theorem 3.1, we have $\chi(n,k,r) \ne \emptyset$.

Case 2. If t is even and n is odd, then $\epsilon(n,t+1) = 1$. In this case, let $n = kt + k - 1$, $r = (k-1)t + k - 4$ and $s = t + 3$. Since s is odd and r is even, this forces k to be even. Thus $k \geq 4$. Since $r+s = kt+k-1 = (t+3)(k-1)+t-2(k+1)$, $\min\chi(r^{r+s}) \leq k$ by Theorem 2.8. Moreover by Theorem 2.11, we have $\max\chi(r^{r+s}) \geq \frac{r+s-1}{2} \geq k$. Hence by Theorem 3.1, we have $\chi(n,k,r) \neq \emptyset$.

Case 3. If t is odd, then $\epsilon(n,t+1) = 0$. In this case, put $n = kt+k-1$, $r = (k-1)t+k-2$ and $s = t+2$. Since $r+s = kt+k-1 = (t+2)(k-1)+t-(k-1)$, then by Theorem 2.8, $\min\chi(r^{r+s}) \leq k$. From Theorem 2.11, we see that $\max\chi(r^{r+s}) \geq \frac{r+s-1}{2} \geq k$. By applying Theorem 3.1 we have $\chi(n,k,r) \neq \emptyset$.

(c) We consider separately every case, depending on the parity of t and n.

Case 1. If t is even, then $\epsilon(n,t) = 0$. In this case we can put $n = kt+l$, $r = (k-1)t+l-1$ and $s = t+1$. We can write $r+s$ in the form $r+s = kt+l = (t+1)(k-1)+t-(k-l-1)$. This implies $k-l-1 \geq 1$. By Theorem 2.8, we have $\min\chi(r^{r+s}) \leq k$. From Theorems 2.11 and 3.1, we can conclude that $\chi(n,k,r) \neq \emptyset$.

Case 2. If t is odd and n is even, then $\epsilon(n,t) = 0$. Put $n = kt+l$, $r = (k-1)t+l-1$ and $s = t+1$. In this case we can show that $\chi(n,k,r) \neq \emptyset$ by writing $r+s = kt+l = (t+1)(k-1)+t-(k-l-1)$ and applying Theorems 2.9, 2.11 and 3.1.

Case 3. We now consider the case when both t and n are odd. Thus $\epsilon(n,t) = 1$. Now we put $n = kt+l$, $r = (k-1)t+l-2$ and $s = t+2$. We express $r+s$ in the form $r+s = kt+l = (t+2)(k-1)+(t+2)-(2k-l)$ and observe that $2k-l \geq 2$. With an easy application of Theorems 2.8, 2,11 and 3.1, we can conclude that $\chi(n,k,r) \neq \emptyset$. □

4. The values of d for which $\chi(n,k,d) \neq \emptyset$

In the preceding section, we have determined a formula for $D(n,k)$, where $D(n,k) = \max\{d : \chi(n,k,d) \neq \emptyset\}$. In this section we give a proof for Theorem 2 by using the following lemmas under the conditions given in Theorem 1. It should be noted once again that we consider only n,k,d in which nd is even and $k \leq d \leq n/2$. The problem becomes easy when $k \in \{1,2\}$.

Lemma 4.1 *If* $3 \leq k \leq d \leq n/2$, *then* $\chi(n,k,d) \neq \emptyset$.

Proof. First we consider $n = 2k$. Thus $d = k = n/2$ and therefore $\max\chi(k^{2k}) = k$ by Theorem 2.10. This shows that $\chi(2k,k,k) \neq \emptyset$. We now consider $n > 2k$ and $k \leq d \leq n/2$. By Theorem 2.6, we have $\min\chi(d^n) = 2$ or 3 according to n is even or odd. By Theorem 2.10, we have $\max\chi(d^n) = d \geq k$. Thus by Theorem 3.1, we can conclude that $\chi(n,k,d) \neq \emptyset$, proving our lemma. □.

Lemma 4.2 *If* $n/2 < d < D(n,k)$, *then* $\chi(n,k,d) \neq \emptyset$.

Proof. By Theorem 2.11, we have

$$\max\chi(d^n) \geq \begin{cases} (n-1)/2 & \text{if n is odd,} \\ n/2 & \text{if n is even.} \end{cases}$$

Thus max$\chi(d^n) \geq k$. For the sake of convenience in calculating min$\chi(d^n)$, we restate the two Theorems 2.8 and 2.9 in the following form.

If $n/2 < d < D(n,k)$, then

$$\min\chi(d^n) = \begin{cases} q & \text{if } p = 0 \\ q+1 & \text{if } 1 \leq p \leq (n-d-2), \\ q+2 & \text{if } p = n-d-1, \end{cases}$$

where $n = (n-d)q + p,\ 0 \leq p \leq n-d-1$.

Recall that min$\chi(D^n) \leq k$, where $D = D(n,k)$, as we have shown in the preceding section. In order to apply Theorem 3.1, we must show that min$\chi(d^n) \leq k$. Since $t \leq n-D < n-d,\ n = (n-d)q+p,\ 0 \leq p \leq n-d-1$ and $n = kt+l,\ 0 \leq l \leq k-1$, we have $q \leq k-1$. If $q \leq k-2$, then min$\chi(d^n) \leq k$. If $q = k-1$ and $1 \leq p \leq (n-d-2)$ then min$\chi(d^n) \leq k$. Thus there is the only case to be considered when $q = k-1$ and $p = n-d-1$. We want to show that this is impossible by considering the following 3 cases.

Case 1. $n = kt$.

In this case $D(n,k) = (k-1)t$. Since $n/2 < d < (k-1)t$ by assumption, we have $n-d = t+i$ for some $i \geq 1$. Thus $kt = (t+i)(k-1) + t + i - 1 = kt + ik - 1$, a contradiction.

Case 2. $n = kt + k - 1$.

This would imply $D(n,k) = (k-1)t + k - 3 - \epsilon(n, t+1)$. If t is even and n odd, then $D(n,k) = (k-1)t + k - 4$. Since $n/2 < d < D(n,k)$, we have $n-d = t+3+i$ for some $i \geq 1$. Thus $kt+k-1 = (t+3+i)(k-1)+t+3+i-1 = kt+3k+ik-1$. This is also a contradiction. If t is odd, then $D(n,k) = (k-1)t + k - 3$. Thus there exists an $i \geq 1$ such that $n-d = t+2+i$. It would imply $kt+k-1 = (t+2+i)(k-1) + (t+2+i-1) = kt+2k+ik-1$, again a contradiction.

Case 3. $n = kt + l,\ 1 \leq l \leq k-2$.

In this case, $D(n,k) = (k-1)t + l - 1 - \epsilon(n,t)$. We first consider when t is even. Then $D(n,k) = (k-1)t+l-1$. Hence, there exists an $i \geq 1$ such that $n-d = t+1+i$ and $kt+l = (t+1+i)(k-1)+t+1+i-1 = kt+k+ik-1$, so it is impossible. Finally, if both t and n are odd, then $D(n,k) = (k-1)t + l - 2$. Hence there exists an $i \geq 1$ such that $n-d = t+2+i$ and $kt+l = (t+2+i)(k-1)+t+2+i-1 = kt+2k+ik-1$. This contradiction together with the above cases show that the condition $q = k-1$ and $p = n-d-1$ is impossible.

Thus $\chi(n,k,d) \neq \emptyset$. This completes the proof of this lemma. $\square$

From the two Lemmas 4.1 and 4.2, we can see easily that the Theorem 2 is satisfied.

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The matching number of regular graphs*

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Abstract

Let $\alpha_1(G)$ be the matching number of a nonempty graph G. We prove that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, then the values $\alpha_1(G)$ completely cover a line segment $[a, b]$ of positive integers. For an arbitrary graphic degree sequence $\mathbf{d}$, we define $\min(\alpha_1, \mathbf{d})$ and $\max(\alpha_1, \mathbf{d})$ as follows: $\min(\alpha_1, \mathbf{d}) := \min\{\alpha_1(G) : G \in \mathcal{R}(\mathbf{d})\}$ and $\max(\alpha_1, \mathbf{d}) := \max\{\alpha_1(G) : G \in \mathcal{R}(\mathbf{d})\}$, where $\mathcal{R}(\mathbf{d})$ is the set of realizations of $\mathbf{d}$.
The two invariants $a := \min(\alpha_1, \mathbf{d})$ and $b := \max(\alpha_1, \mathbf{d})$ naturally arise. For a graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$ where r is the vertex degree and n is the number of vertices, the exact values of a and b are found in all situations.

Keywords: degree sequence, graph parameter, matching number, interpolation graph parameter.
AMS Subject Classification (2000): 05C07, 05C70

1. Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part, our notation and terminology follows that of Bondy and Murty [3].

As usual, we use $|S|$ to denote the cardinality of a set S. Let $G = (V, E)$ be a graph. The *order* of G is the cardinality of V. To simplify writing, we write $e = uv$ for the edge e that connects the vertices u and v. The *degree* of a vertex v of a graph G is defined as $d(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. A graph G is said to be *r-regular* if all of its vertices have the same degree r. For two disjoint graphs G and H (i.e. $V(G) \cap V(H) = \emptyset$), we denote $G \cup H$ their union, and define pG the union of p copies of G. We denote by C_n and K_n for the cycle and the complete graph of

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order n respectively. The complete bipartite graph with its partite set of cardinality p and q is denoted by $K_{p,q}$. For $r \geq 1$, the notation $B(r; X, Y)$ represents an r-regular bipartite graph with partite sets X and Y. It is easy to see that $B(r; X, Y)$ exists if and only if $|X| = |Y|$ and $1 \leq r \leq |X|$. Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G. Moreover, a graph H of order n is said to have the same degree sequence as G if there is a bijection ϕ from $V(G)$ to $V(H)$ such that $d(v_i) = d(\phi(v_i))$ for all $i = 1, 2, \ldots, n$. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G and in this case, G is called a *realization* of $\mathbf{d}$.

Havel [6] and Hakimi [5] obtained independently a mechanism for determining whether or not a given sequence of non-negative integers is graphic. We state their results in the following theorem.

Theorem 1.1 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.

Let G be a graph, ab and cd are independent edges in G, such that ac and bd are not edges in G. Define $G^{\sigma(a,b;c,d)}$ to be the graph obtained from G by deleting the edges ab and cd and inserting the edges ac and bd. The operation $\sigma(a, b; c, d)$ is called *switching operation.* It is easy to see that the graph obtained from G by a switching will have the same degree sequence as G. The following theorem has been shown by Havel and Hakimi.

Theorem 1.2 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. If G_1 and G_2 are any two realizations of $\mathbf{d}$, then G_2 can be obtained from G_1 by a finite sequence of switchings.*

As a consequence of Theorem 1.2, we can define *the graph $\mathcal{R}(\mathbf{d})$* of realizations of $\mathbf{d}$, the vertices of which are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus we have shown the following corollary.

Corollary 1.3 *The graph $\mathcal{R}(\mathbf{d})$ is connected.*

2. Interpolation theorem

Let $\mathcal{G}$ be the class of all simple graphs, a function $f : \mathcal{G} \to \mathbb{Z}$ is called a *graph parameter* if $f(G) = f(H)$, whenever $G \cong H$. If f is a graph parameter and $\mathcal{J} \subseteq \mathcal{G}$, f is called an *interpolation graph parameter with respect to $\mathcal{J}$* if there exist integers a and b such that $\{f(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \leq k \leq b\}$. We have shown in [7], [8] and [9] that the chromatic number χ, the clique number ω, and the order of maximum induced forests I are interpolation graph parameters with respect to $\mathcal{J} = \mathcal{R}(\mathbf{d})$.

If f is an interpolation graph parameter with respect to $\mathcal{J}$, it is natural to write $\min(f, \mathcal{J}) = \min\{f(G) : G \in \mathcal{J}\}$ and likewise $\max(f, \mathcal{J}) = \max\{f(G) : G \in \mathcal{J}\}$.

In the case where $\mathcal{J} = \mathcal{R}(\mathbf{d})$, we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$ respectively.

Definition 2.1 A subset M of the edge set E of a graph $G = (V, E)$ is an *independent edge set* or *matching* in G if no two distinct edges in M have common vertices. A matching M is *maximum* in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of a maximum matching of G, written $\alpha_1(G)$, is called the *matching number* of G.

It is clear that α_1 is a graph parameter. In this section we show that the matching number, α_1, is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$, the set of all graphs with a degree sequence $\mathbf{d}$. We then find $\min(\alpha_1, \mathbf{d})$ and $\max(\alpha_1, \mathbf{d})$ for all graphic degree sequences $\mathbf{d} = r^n := (r, r, \ldots, r)$ of r-regular graphs of order n.

Theorem 2.2 *If G is a graph with $\alpha_1(G) = \alpha_1$ and $\sigma(a, b; c, d)$ is a switching on G, then $\alpha_1(G^{\sigma(a,b;c,d)}) \geq \alpha_1 - 1$.*

Proof. Let M be an independent set of edges in E with $|M| = \alpha_1(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on G. If $\{ab, cd\} \cap M = \emptyset$, then $|M| = |M^\sigma|$. If $\{ab, cd\} \subseteq M$, then $|M| = |M^\sigma|$. Finally, if M contains exactly one edge from the set $\{ab, cd\}$, then $|M^\sigma| = |M| - 1$. Therefore $\alpha_1(G^\sigma) \geq \alpha_1 - 1$.

Corollary 2.3 *If σ is a switching on G, then $|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$.*

Proof. Since a switching is an involution, we may assume that $\alpha_1(G) \geq \alpha_1(G^\sigma)$. By Theorem 2.2, $\alpha_1(G^\sigma)$ is either $\alpha_1(G) - 1$ or $\alpha_1(G)$. In both cases we have $|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$.

Theorem 2.4 *For any graphic degree sequence $\mathbf{d}$, there exist integers a and b such that there exists a graph G with degree sequence $\mathbf{d}$ and $\alpha_1(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Proof. The proof follows directly from Corollaries 1.3 and 2.3.

3. Matching number of regular graphs

Let $\mathbf{d}$ be a graphic degree sequence. We have already defined the graph $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$. *The range of matching numbers of* $\mathcal{R}(\mathbf{d})$ can be defined as the interval of integers specified by Theorem 2.4, i.e.,

$$\alpha_1(\mathbf{d}) = \{\alpha_1(G) \ : \ G \in \mathcal{R}(\mathbf{d})\} = [a, b] = \{c \in \mathbb{Z} \ : \ a \leq c \leq b\}.$$

Naturally, we can call a the *minimum matching number* of $\mathcal{R}(\mathbf{d})$, thus $a := \min(\alpha_1, \mathbf{d})$. Similarly, $b := \max(\alpha_1, \mathbf{d})$, the *maximum matching number* of $\mathcal{R}(\mathbf{d})$. We write

$\mathbf{d} = r^n$ for the sequence $(r, r, \ldots, r)$ of length n, where r is a non-negative integer and n is a positive integer. By the definition of graphic degree sequence, $\mathbf{d} = r^n$ is graphic if and only if $rn \equiv 0 (\mathrm{mod}\, 2)$ and $n \geq r + 1$. It is easy to see that $\min(\alpha_1, 0^n) = \max(\alpha_1, 0^n) = 0$ and $\min(\alpha_1, 1^{2n}) = \max(\alpha_1, 1^{2n}) = n$. Because of this fact, from now on we will consider $r \geq 2$ and $n \geq r + 1$.

We first investigate the value of $\max(\alpha_1, r^n)$ by the following theorem.

Theorem 3.1 *For $r \geq 2, n \geq r+1$ and $nr \equiv 0 (\mathrm{mod}\, 2)$, there exists an r-regular hamiltonian graph of order n. In particular,* $\max(\alpha_1, r^n) = \lfloor \frac{n}{2} \rfloor$.

Proof. Since there is a well known result by Dirac [4] that an r-regular graph of order n with $r \geq \frac{n}{2}$ is hamiltonian, we need to consider only when $r < \frac{n}{2}$. It is easy to construct a hamiltonian graph with 10 or fewer vertices. It is also easy to construct such a graph with $r = 2$ or 3. Let $X = \{x_0, x_1, \ldots, x_{t-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{t-1}\}$ where $t \geq 5$. For an integer r with $1 \leq r \leq t$, take the edge set

$$E = \{x_i y_{i+j} : \ i = 0, 1, 2, \ldots, t-1, j = 0, 1, 2, \ldots, r-1\},$$

where all subscripts are taken mod t. It is clear that the graph $B(r; X, Y) = (X \cup Y, E)$ is an r-regular bipartite graph and it is hamiltonian if $r \geq 2$. Suppose $n = 2t+1$ and r is an even integer with $4 \leq r \leq t$, the graph $G = (X \cup Y \cup \{u\}, E')$, where

$$E' = [E(B(r-2; X, Y)) \setminus \{x_i y_i : \ i = 0, 1, 2, \ldots, \tfrac{r}{2}\}] \cup \{x_i x_{i+1}, y_i y_{i+1} : \ i = 0, 1, 2, \ldots, t-1\} \cup \{ux_i, uy_i : \ i = 0, 1, 2, \ldots, \tfrac{r}{2}\},$$

is an r-regular hamiltonian graph. □

The rest of this section we will use the generalized result of Tutte and the result of Wallis [13] to obtain all values of $\min(\alpha_1, r^n)$. A 1-*factor* of a graph G is a 1-regular spanning subgraph of G. A 1-*factorization* of G is a set of pairwise edge-disjoint 1-factors which together contain each edge of G. It is well known that K_{2n} and $K_{n,n}$ have 1-factorizations for all positive integers n. The question of which graphs contain 1-factors is one that has attracted considerable attention. For a comprehensive review we refer to the survey of Akiyama and Kano [1] and to Wallis [14].

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte [12]. A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $O(G)$ the number of odd components of G. The following theorem is due to Tutte [12].

Theorem 3.2 *G has a perfect matching if and only if*

$$O(G \setminus S) \leq |S| \text{ for all } S \subseteq V.$$

Berge [2] generalized Tutte's result and it makes easier for application.

Theorem 3.3 *The number of edges in a maximum matching of a graph G is $\frac{1}{2}(|V(G)| - d)$, where $d = \max_{S \subseteq V(G)}\{O(G \setminus S) - |S|\}$.*

Let $F(r,d)$ be the minimum order of an r-regular graph G with $\alpha_1(G) = \frac{1}{2}(|V(G)|-d)$. It is clear that $|V(G)| \equiv d (\mathrm{mod}\ 2)$. Wallis [13] found $F(r,2)$ for all $r \geq 3$. In other words, he proved the following theorem.

Theorem 3.4 *Let G be an r-regular graph with no 1-factor and no odd component. Then*

$$|V(G)| \geq \begin{cases} 3r+7 & \text{if } r \text{ is odd, } r \geq 3, \\ 3r+4 & \text{if } r \text{ is even, } r \geq 6, \\ 22 & \text{if } r = 4. \end{cases}$$

Furthermore, no such graphs exists for $r = 1$ or 2.

Suppose G is an r-regular graph with $\alpha_1(G) = \frac{1}{2}(|V(G)| - d)$. By Theorem 3.3 there exists a k-subset K of $V(G)$ such that $O(G \setminus K) = k + d$. If $k = 0$, then r is even, G contains d odd components, and each component of G has order at least $r+1$. Suppose $k \geq 1$ and $G \setminus K$ has an odd component with p vertices, where p less than or equal to r. The number of edges within the component is at most $\frac{1}{2}p(p-1)$. This means that the sum of degree of these p vertices in $G \setminus K$ is at most $p(p-1)$. But G is an r-regular graph, so the sum of these p vertices in G is pr. The number of edges joining to the component K must be at least $pr - p(p-1)$. For a fixed r and for integer p satisfying $1 \leq p \leq r$ the function $f(p) = pr - p(p-1),\ 1 \leq p \leq r$ has minimum value $f(1) = f(r) = r$. So any odd component with r or less vertices is joined to K by r or more edges.

Suppose there are O_+ odd components of $G \setminus K$ with more than r vertices and O_- odd components with less than or equal to r vertices. Thus

$$O_+ + O_- = k + d, \tag{1}$$

$$O_+ + rO_- \leq kr. \tag{2}$$

From these 2 relations, we have
$O_+ \geq \lceil \frac{rd}{r-1} \rceil = d + \lceil \frac{d}{r-1} \rceil$ and $k \geq \lceil \frac{d}{r-1} \rceil$. Thus we have the following theorem.

Theorem 3.5 *Let r be an even integer, $r \geq 2$. Then $F(r,d) = d(r+1)$.*

Proof. Since $\alpha_1(dK_{r+1}) = \frac{1}{2}(|V(G')| - d)$, $F(r,d) \leq d(r+1)$. On the other hand let G be an r-regular graph of order n such that $\alpha_1(G) = \frac{1}{2}(n-d)$. There exists a subset K of $V(G)$ such that $O(G \setminus K) = d + k$. We have already shown that $O_+(G \setminus K) \geq d$. Thus $n \geq d(r+1)$. It follows that $F(r,d) = d(r+1)$.

□

Corollary 3.6 *Let r be an even integer, $r \geq 2$. If $n = (r+1)d + e, 0 \leq e \leq r$, then* $\min(\alpha_1, r^n) = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$.

Proof. Let $G' = (d-1)K_{r+1} \cup K$, where K is an r-regular graph of order $r+1+e$. Then $\alpha_1(G') = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$. Let G be an r-regular graph of order $n = (r+1)d+e, 0 \geq e \geq r$. By Theorem 3.5 we have $\alpha_1(G) \geq \lfloor \frac{1}{2}(n-d-1) \rfloor + 1 = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$. Thus $\min(\alpha_1, r^n) = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$.

□

Suppose r is odd and $r \geq 3$. Let G be an r-regular graph of order n such that $\alpha_1(G) = \frac{1}{2}(n-d)$. Then d must be even. Put $d = 2q$. By Theorem 3.3 there exists a nonempty subset K of $V(G)$ of cardinality k such that $O(G \setminus K) = k + 2q$. By (1) and (2), we have

$$\begin{aligned} n \geq k + (r+2)O_+ &\geq \lceil \tfrac{2q}{r-1} \rceil + (r+2)(2q + \lceil \tfrac{2q}{r-1} \rceil) \\ &= \lceil \tfrac{2q}{r-1} \rceil (r+3) + 2q(r+2). \end{aligned} \tag{3}$$

Wallis [13] defined $G(x, y)$ to be a graph with $x + y$ vertices, x being of degree $x + y - 3$ and y of degree $x + y - 2$. $G(x, y)$ exists if and only if y is even and $y \geq 2$. It is noted that for any graph $G(x, y)$, it has y vertices of degree the same degree, r say, and x vertices of degree $r - 1$. Let $x_i, y_i, i = 1, 2, \ldots, m$ be integers such that $G(x_i, y_i)$ exists for all $i = 1, 2, \ldots, m$. We construct the graph

$$G(x_1, y_1) * G(x_2, y_2) * \ldots * G(x_m, y_m)$$

from disjoint copies of the graphs by inserting a new vertex, u say, and joining u to all vertices of $G(x_i, y_i)$ which have the lesser degree, for $i = 1, 2, \ldots, m$.

Using this notation, we see that for an odd integer $r \geq 3$ and $q = 1, 2, \ldots, \frac{r-1}{2}$, for any odd positive integers $a_i, i = 1, 2, \ldots, 1 + 2q$ whose sum is r,

$$G_q = G(a_1, r + 2 - a_1) * G(a_2, r + 2 - a_2) * \ldots * G(a_{1+2q}, r + 2 - a_{1+2q})$$

is an r-regular graph on $(r + 2)(1 + 2q) + 1$ vertices with $\alpha_1(G_q) = \frac{1}{2}(|V(G_q)| - 2q)$. We have the following theorem.

Theorem 3.6 *For an odd integer $r \geq 3$. Then*
1. *$F(r, 2q) = (r + 2)(1 + 2q) + 1$, for $q = 1, 2, \ldots, \frac{r-1}{2}$,*
2. *if $q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2}$, then $F(r, 2q) = sF(r, r - 1) + F(r, 2t)$, where $F(r, 0) = 0$.*

Proof. For an odd integer $r \geq 3$ and let G be an r-regular graph of order n with $\alpha_1(G) = \frac{1}{2}(n - 2q)$.

(1) By (3) and for $q = 1, 2, \ldots, \frac{r-1}{2}$, we have
$n \geq \lceil \frac{2q}{r-1} \rceil (r + 3) + 2q(r + 2) = (r + 2)(1 + 2q) + 1$.
The graph G_q constructed above has the property that $\alpha_1(G_q) = \frac{1}{2}(|V(G_q)| - 2q)$ and $|G_q| = (r+2)(1+2q)+1$. Thus $F(r, 2q) = (r+2)(1+2q)+1$, for all $q = 1, 2, \ldots, \frac{r-1}{2}$.

(2) If $q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2}$, we have $n \geq \lceil \frac{2q}{r-1} \rceil (r + 3) + 2q(r + 2)$. Thus

$$n \geq \begin{cases} s[(r + 2)r + 1] & \text{if } t = 0, \\ s[(r + 2)r + 1] + (r + 2)(1 + 2t) + 1 & \text{if } 0 < t < \frac{r-1}{2}. \end{cases}$$

Thus if $t = 0$, the graph $sG_{\frac{r-1}{2}}$ has order $s[(r+2)r+1]$ and $\alpha_1(sG_{\frac{r-1}{2}}) = \frac{1}{2}(s[(r+2)r+1]-2q)$. If $t > 0$, the graph $sG_{\frac{r-1}{2}} \cup G_t$ has order $s[(r+2)r+1]+(r+2)(1+2t)+1$

and $\alpha_1(sG_{\frac{r-1}{2}} \cup G_t) = \frac{1}{2}(s[(r+2)r+1] + (r+2)(1+2t) + 1 - 2q)$. Therefore $F(r,2q) = sF(r,r-1) + F(r,2t)$, where $F(r,0) = 0$.

□

Corollary 3.7 *Let r be an odd integer, $r \geq 3$. If $F(r,2q) \leq n < F(r,2(q+1)$, then* $\min(\alpha_1, r^n) = \frac{1}{2}(n-2q)$.

Proof. Since r is odd, n is an even integer satisfying $F(r,2q) \leq n < F(r,2(q+1)$, and by the definition of $F(r,2(q+1))$, $\min(\alpha_1, r^n) \geq \frac{1}{2}(n-2(q+1)+1 = \frac{1}{2}(n-2q)$. It is easy to construct an r-regular graph G of order n having $\min(\alpha_1, r^n) = \frac{1}{2}(n-2q)$. Thus $\min(\alpha_1, r^n) = \frac{1}{2}(n-2q)$.

□

4. Connected realizations

A study of interpolation theorems on graph parameters may be considered into two parts. First is to consider whether or not a given graph parameter f is interpolated over a subclass $\mathcal{J}$ of $\mathcal{G}$. If it is, we will continue for the second part, namely, finding the values of $\min(f, \mathcal{J})$ and $\max(f, \mathcal{J})$. For the graph parameter α_1, we have answered two parts of the interpolation theorems completely for the class $\mathcal{R}(r^n)$. We have been suggested by Prof. Mikio Kano as a personal communication to consider the interpolation theorems on graph parameters to the class of connected realizations of a given graphic degree sequence $\mathbf{d}$. With the constraint on connected realizations of a given degree sequence, we will see later that the first part of the interpolation theorem is followed immediately but the second part of the interpolation theorem seems to be more difficult. Let P be a property which a graph may possess. Denote by $\mathcal{R}(\mathbf{d},\ P)$ the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by those vertices which correspond to graphs with property P. Properties for which $\mathcal{R}(\mathbf{d},\ P)$ is connected are called *complete*. If a property P is complete we may find all the graphs of a given degree sequence with the property by switching constrained to graphs with the property.

Colbourn (cited from [11]) showed that the property of being a tree is complete and Syslo [10] extended this to the property of being unicyclic. Taylor (cited from [11]) generalized these results by showing that the property of being connected is complete. The property of being 2-connected was shown to be complete also by Taylor [11]. Thus by Corollary 2.3, we have the following theorems.

Theorem 3.7 *For any graphic degree sequence* $\mathbf{d}$, *there exist integers a and b such that there is a connected graph G with degree sequence* $\mathbf{d}$ *and $\alpha_1(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Theorem 3.8 *For any graphic degree sequence* $\mathbf{d}$, *there exist integers a and b such that there is a 2-connected graph G with degree sequence* $\mathbf{d}$ *and $\alpha_1(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

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Graph of realizations and chromatic numbers

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Abstract

We consider the problem of determining the structure of induced subgraphs of the graph of realizations of a degree sequence **d** with prescribe chromatic number. We prove in this paper some significant results when **d** is a regular degree sequence.

1. Introduction

All graphs considered in this paper are finite and simple. We use notation and terminology as in the textbook of Parthasarathy [5]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use $|S|$ to denote the cardinality of a set S. We define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. We simply write $e = uv$ for the edge e that joins the vertices u and v. The maximum degree of a graph G is denoted by $\Delta(G)$. A graph G is said to be r*-regular* if all of its vertices have the same degree r. For two disjoint graphs G and H (i.e. $V(G) \cap V(H) = \emptyset$), we denote $G \cup H$ their union, and define pG the union of p copies of G. For a graph $G = (V, E)$ and $v \in V$, we denote by $\mathcal{N}_G(v)$ to be $\{u \in V : vu \in E\}$.

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G, where $d(v_i)$ is the degree of vertex v_i. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case,

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G is called a *realization* of $\mathbf{d}$. We denote $\mathcal{R}(\mathbf{d})$ for the set of all graphs with degree sequence $\mathbf{d}$.

A *switching* $\sigma(a, b; c, d)$ on a graph G is a replacement of any two independent edges ab and cd of G by the edges ac and bd, where ac and bd are not edges in G. (See **Fig. 1** below.) The graph obtained from G and a switching is denoted by G^σ.

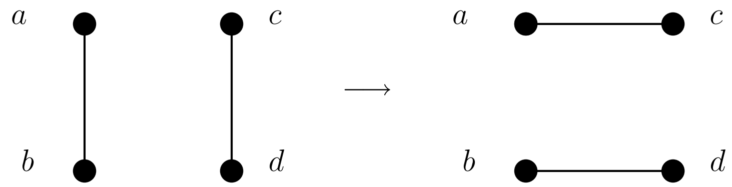

Fig.1

It was shown by Havel [4] and Hakimi [3] that if G_1 and G_2 are any two realizations of $\mathbf{d}$, then G_2 can be obtained from G_1 by a finite sequence of switchings. Thus The graph $\mathcal{GR}(\mathbf{d})$ of realizations of $\mathbf{d}$ can be defined as the graph whose vertex set is $\mathcal{R}(\mathbf{d})$; two vertices being adjacent in $\mathcal{GR}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus the graph $\mathcal{GR}(\mathbf{d})$ is connected.

Let P be a property which a graph may possess. Denote by $\mathcal{R}(\mathbf{d}, P)$ the subgraph of the graph $\mathcal{GR}(\mathbf{d})$ induced by those vertices which correspond to graphs with property P. Properties for which $\mathcal{R}(\mathbf{d}, P)$ is connected are called *complete.* If a property P is complete we may find all the graphs of a given degree sequence with the property by switching constrained to graphs with the property.

Colbourn [2] showed that the property of being a tree is complete and Syslo [9] extended this to the property of being unicyclic. Taylor (cited from [10]) generalized these results by showing that the property of being connected is complete. The property of being 2-connected was shown to be complete also by Taylor [10]. We denote by $\mathcal{CR}(\mathbf{d})$ for the set of all connected realizations of $\mathbf{d}$.

Let $\mathcal{G}$ be the set of all simple graphs, a function $f : \mathcal{G} \to \mathbb{Z}$ is called a graph parameter if $f(G) = f(H)$, whenever $G \cong H$. If f is a graph parameter and $\mathcal{J} \subseteq \mathcal{G}$, f is called an *interpolation graph parameter with respect to* $\mathcal{J}$ if there exist integers a and b such that $\{f(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \leq k \leq b\}$.

Lemma 1.1 *Let f be a graph parameter. For a graph G and σ is a switching on G. If f has a property that $|f(G) - f(G^\sigma)| \leq 1$, then f is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$.*

$\square$

Lemma 1.2 *Let f be a graph parameter. For a graph G and σ is a switching on G. If f has a property that $|f(G) - f(G^\sigma)| \leq 1$, then f is an interpolation graph parameter with respect to $\mathcal{CR}(\mathbf{d})$.*

□

Let G be a graph. The problem of determining the minimum number of colors required to color the vertices of G so that every pair of adjacent vertices receive different colors is one of the most difficult problem in graph theory. For a graph G, this minimum is known as *the chromatic number* of G, and denoted by $\chi(G)$. However, the set of those graphs G in which $\chi(G) \leq 2$ consists of all bipartite graphs. For a nonbipartite graph G, it is difficult to tell whether $\chi(G) = 3$ or not. Another graph parameter which closely related to the chromatic number is *the clique number* and is defined by the maximum order of the complete subgraphs of a graph G and is denoted by $\omega(G)$. It is clear that for a graph G, $\omega(G) \leq \chi(G)$. We respectively proved in [6], [7] that if G is a graph and σ is a switching on G, then $|\chi(G) - \chi(G^\sigma)| \leq 1$ and $|\omega(G) - \omega(G^\sigma)| \leq 1$. Hence, χ and ω are interpolation graph parameters with respect to $\mathcal{R}(\mathbf{d})$ and hence with respect to $\mathcal{CR}(\mathbf{d})$.

2. Preliminaries

Brooks [1] observed that every graph G may be colored by $\Delta(G) + 1$ colors and he characterized the graphs for which $\Delta(G)$ colors are not enough.

Theorem 2.1 *Any graph G satisfies $\chi(G) \leq 1 + \Delta(G)$, with equality holds if and only if either of the following holds:*
(1) *some component of G is the complete graph $K_{\Delta+1}$, where $\Delta = \Delta(G)$;*
(2) *some component of G is an odd cycle, and $\Delta(G) = 2$.*

□

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. The *chromatic range* of $\mathbf{d}$ can be defined as the interval of integers as

$$\chi(\mathbf{d}) := [a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}.$$

Naturally, we can call a the *minimum chromatic number* of $\mathbf{d}$, and we write $a := \min(\chi, \mathbf{d})$. Similarly, $b := \max(\chi, \mathbf{d})$, the *maximum chromatic number* of $\mathbf{d}$. Note that Brooks' theorem implies $b \leq 1 + \Delta$, where $\Delta = \max\{d_1, d_2, \ldots, d_n\}$. Thus for a regular degree sequence r^n, we have $b \leq 1 + r$. We found in [6] the corresponding value of b for all values of r and n except the cases when r and n are even and $r + 4 \leq n \leq 2r - 2$. In general the Brooks' bound can be very far from the actual value. Reed [8] observed and made a series of conjectures for upper bounds of the chromatic number of a graph G in terms of $\Delta(G)$ and $\omega(G)$.

For each $c \in \chi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; c)$ denote the subgraph of $\mathcal{GR}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number c. Similarly for any $c \in \chi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; p \leq c)$ denote the subgraph of $\mathcal{GR}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number $p \leq c$. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d}; c)$ and $\mathcal{R}(\mathbf{d}; p \leq c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d}; c)$ and of $\mathcal{R}(\mathbf{d}; p \leq c)$? In particular, are these graphs connected?

If $\mathcal{R}(\mathbf{d};\ c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with chromatic number c by beginning with one such realization and applying a suitable sequence if switchings producing only graphs with chromatic number c. Similarly if $\mathcal{R}(\mathbf{d};\ p \leq c)$ is connected.

Note that $\chi(0^n) = \{1\}$, $\chi(1^n) = \{2\}$ (n is even), $\chi(2^4) = \{2\}$ $\chi(2^n) = \{2,3\}$ if n is even and $n \geq 6$, and $\chi(2^n) = \{3\}$ if n is odd and $n \geq 3$. For $\chi(3^n)$, we have $\chi(3^4) = \{4\}$, $\chi(3^6) = \{2,3\}$ and $\chi(3^n) = \{2,3,4\}$ if n is even and $n \geq 8$.

3. Main results

Let G be a graph with $\chi(G) = k$ and σ a switching on G. σ is called a *k-safe switching* if $\chi(G^\sigma) = k$. and a sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a sequence of *k-safe switchings* if for each $i, i = 1, 2, \ldots, t$, $\chi(G^{\sigma_1,\sigma_2,\ldots,\sigma_i}) = k$.

Theorem 3.1 *If* $r \geq 3$ *and* $\max(\chi, r^n) = r + 1$, *then the graphs* $\mathcal{R}(r^n; r+1)$ *and* $\mathcal{R}(r^n; p \leq r)$ *are connected.*

Proof. Note that if $r \geq 3$, then $\max(\chi, r^n) = r + 1$ if and only if $n = r + 1$ or $n \geq 2r + 2$. Moreover, if G is a realization of r^n and $\chi(G) = r + 1$ if and only if G has K_{r+1} as a component. The theorem is true for $n = r + 1$. For $n \geq 2r + 2$, let G_1 and G_2 be any two realizations of r^n such that $\chi(G_1) = \chi(G_2) = r + 1$. Thus $G_1 = K_{r+1} \cup H_1$ and $G_2 = K_{r+1} \cup H_2$. Since H_1 and H_2 are r-regular graphs of order $n - r - 1$, H_2 can be obtained from H_1 by a finite number of switchings. Thus G_2 can be as well as obtained by those switchings. Furthermore, it is easy to observe that those switchings are $(r + 1)$-safe switchings.

Let G be a realization of r^n such that $\chi(G) \leq r$. If G is disconnected, then G does not contain K_{r+1} as its component. Thus there exists a suitable sequence of switchings which transform G to a connected realization of r^n such that each resulting graph obtained in this transformation will have chromatic number less than or equal to r. By using the result by Taylor [10], the proof is complete.

□

Let C_m be the cycle of order m. Thus C_m exists for all integer $m \geq 3$, we call C_m *odd cycle* or *even cycle* according to m is odd or even. A realization of 2^n can be written as $\cup_{i=1}^t C_{n_i}$ and $\sum_{i=1}^t n_i = n$. It is well known that if $G = \cup_{i=1}^t C_{n_i}$, then $\chi(G) = 2$ if and only if for all i, n_i is even. It is clear that any two even cycles C_r and C_s, there is a 2-safe switching which transforms these two cycles to C_{r+s}. Thus $\mathcal{GR}(2^n;\ 2)$ is connected. The corresponding result can be obtained for the graph $\mathcal{R}(2^n;\ 3)$ with only one exceptional.

Theorem 3.2 *If* $3 \in \chi(2^n)$, *then the graph* $\mathcal{R}(2^n;\ 3)$ *is connected if and only if* $n \neq 10$.

Proof. Observe that if n is odd and $n \geq 3$, then $\chi(2^n) = \{3\}$. Thus the graph $\mathcal{R}(2^n;\ 3)$ is connected. If $n = 10$, we can not transform $2C_5$ to $2C_3 \cup C_4$ without passing C_{10}. Thus the graph $\mathcal{R}(2^{10};\ 3)$ is not connected. For $n \geq 12$, let $\cup_{i=1}^t C_{n_i}$ be a realization of 2^n and suppose that n_1 is odd. Then there is a sequence of 3-safe switchings which transforms $\cup_{i=1}^{t} C_{n_i}$ to $C_{n_1} \cup C_{n-n_1}$. Since $n \geq 12$, $C_5 \cup C_{n-5}$ can be transformed to $C_5 \cup C_3 \cup C_{n-8}$ and to $C_3 \cup C_{n-3}$. If m is odd and $m \geq 7$, then $C_m \cup C_{n-m}$ can be transformed to $C_3 \cup C_{m-3} \cup C_{n-m}$ then to $C_3 \cup C_{n-3}$. The proof is complete.

□

Lemma 3.3 *Let G and H be bipartite graphs with bipartitioning sets X and Y. If $d_G(v) = d_H(v)$ for all $v \in X \cup Y$, then there is a sequence of 2-safe switchings that transforms G into H.*

Proof. We prove by induction on $|X|$. It is clear that if $|X| = 1$, then $G = H$. Thus the theorem is true for $|X| = 1$. Suppose $|X| > 1$. Let G be a bipartite graph with bipartitioning sets X and Y. Let v be a vertex of maximum degree in X and suppose that $d_G(x) = k$. Let $u_1, u_2, \ldots, u_k$ be k vertices of highest degree in Y and $u_1 \geq u_2 \geq \ldots \geq u_k$. If $u_1 \notin \mathcal{N}_G(v)$, then there exists $u \in \mathcal{N}_G(v) \setminus \{u_1, u_2, \ldots, u_k\}$ and $v_i \in \mathcal{N}_G(u_1)$ such that $uv_i \notin E(G)$. It is easy to choose a 2-safe switching σ so that the resulting graph G^σ will have the bipartitioning sets X and Y and $u_1 \in \mathcal{N}_{G^\sigma}(v)$. Thus we may assume that there exists i such that $1 < i < k$, with $vu_1, vu_2, \ldots, vu_{i-1} \in \mathcal{N}_G(v)$ but $vu_i \notin \mathcal{N}_G(v)$. Thus there exists $u \in \mathcal{N}_G(v) \setminus \{u_1, u_2, \ldots, u_k\}$ and $v_j \in \mathcal{N}_G(u_i)$ such that $uv_j \notin E(G)$. It is easy to choose a 2-safe switching σ so that the resulting graph G^σ will have the bipartitioning sets X and Y and $u_1, u_2, \ldots, u_i \in \mathcal{N}_{G^\sigma}(v)$. By continuing this process, we get a sequence of 2-safe switchings which transforms G to G' with $\mathcal{N}_{G'} = \{u_1, u_2, \ldots, u_k\}$. The same result can be obtained a graph H' from the graph H. Thus the graphs $G' \setminus \{v\}$ and $H' \setminus \{v\}$ are bipartite graphs with bipartitioning sets $X \setminus \{v\}$ and Y. By induction, there exists a sequence of 2-safe switchings which transforms $G' \setminus \{v\}$ to $H' \setminus \{v\}$. The proof is complete.

□

The following theorem is a consequence of the above lemma.

Theorem 3.4 *If $2 \in \chi(r^n)$, then the graph $\mathcal{R}(r^n\,; 2)$ is connected.*

□

Theorem 3.5 *If $c \in \chi(3^n)$, then the graph $\mathcal{R}(3^n;\ c)$ is connected.*

Proof. We have already proved when $c = 2, 4$. Note that $G \in \mathcal{R}(3^n)$ with $\chi(G) = 3$ if and only if G does not contain K_4 as its component and G has an odd cycle as its subgraph. It is easy to check that the theorem is true if $n \leq 8$. Let G be a cubic graph of order $n \geq 10$ with $\chi(G) = 3$. We first suppose that G has a triangle T with $V(T) = \{x, y, z\}$ and $E(T) = \{xy, yz, xz\}$. If x and y have a common neighbour $v \in G \setminus T$, then there exists an edge ab in $G \setminus T$ independent to the edge xv. Thus there is a 3-safe switching σ such that G^σ contains a triangle T such that any two vertices of T have no common neighbour in $G \setminus T$.

Now suppose that G contains an odd cycle C of smallest order $k \geq 5$. Let x, y be two adjacency vertices in the cycle. Thus x and y have no common neighbour in G. Let z be the neighbour of y in $G \setminus C$. Then there exists a 3-safe switching

σ such that G^σ contains a triangle $T = \{x, y, z\}$. Thus for a graph $G \in \mathcal{R}(3^n)$ and $\chi(G) = 3$, there is a sequence of 3-safe switchings which transforms G to G' such that G' contains a triangle $T' = \{x, y, z\}$ and $G' \setminus T'$ is a graph having exactly 3 vertices of degree 2. Since G is an arbitrary graph in $\mathcal{R}(3^n)$, the proof is complete. $\square$

Conjecture: *If $c \in \chi(r^n)$, then the graph $\mathcal{R}(r^n;\ c)$ is connected.*

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Decycling regular graphs

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Abstract

For a graph G and $S \subseteq V(G)$, if $G - S$ is acyclic, then S is said to be a *decycling set* of G. The cardinality of smallest decycling set of G is called the *decycling number* of G and is denoted by $\phi(G)$. We prove in this paper that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, then the values $\phi(G)$ completely cover a line segment $[a, b]$ of positive integers. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$, where r is the vertex degree and n is the order of the graph, the exact value of $\min(\phi, r^n)$ and $\max(\phi, r^n)$ are found in all situations. As an application, we can find all cubic graphs of order $2n$ having the smallest decycling number.

Keywords: degree sequence, graph parameter, decycling number, interpolation graph parameter, cubic graph.

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1. Introduction

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is difficult even for some simply defined graphs. For a graph G, this minimum number is known as the *decycling number* of G, and is denoted by $\phi(G)$. The class of those graphs G of which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n-2$ and $K_{p,\,q} = p-1$ if $p \leq q$, where K_n denotes the complete graph of order n and $K_{p,\,q}$ denotes the complete bipartite graph with partite sets of cardinality p and q. The exact values of decycling numbers for many classes of graphs were obtained and cited in [1]. In the same paper, they posed the following problems:
Problem 1. Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?
Problem 2. Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

Problem 1 has been answered in [2] by proving that for a random cubic graph G of order n, $\phi(G) = \lceil \frac{n}{2} + \frac{1}{2} \rceil$ holds asymptotically almost surely, but no answer for the second problem yet.

We prove in this paper that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, the values $\phi(G)$ completely cover a line segment $[a, b]$ of positive integers. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$ where r is the vertex degree and n is the number of graph vertices, the exact values of $\min(\phi, r^n)$ and $\max(\phi, r^n)$ are found in all situations. Finally we shall answer Problem 1.

In this paper we only consider finite simple graphs. For the most part, our notation and terminology follows that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Since we only deal with finite and simple graphs, we will use the following notation and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. As usual, we use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that joins the vertex u to the vertex v. The *degree* of a vertex v of a graph G is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. Let S and T be disjoint subsets of $V(G)$ of a graph G. We denote by $e(S, T)$ the number of edges in G that connect S to T. If $S \subseteq V(G)$, the graph $G|_S$ is the subgraph induced by S in G and denotes

$e(S)$ the number of edges in the graph $G|_S$. A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called *cubic graph.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of G, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underline graph G is clear in the context. A graph H of order n is said to have the same degree sequence as G if there is a bijection f from $V(G)$ to $V(H)$ such that $d_G(v_i) = d_H(f(v_i))$ for all $i = 1, 2, \ldots, n$. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by [5] and [4]. We state their results in the following theorem.

Theorem 1.1 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

$\square$

Let G be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Let

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) \cup \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from G by a switching has the same degree sequence as G. The following theorem has been shown by [5] and [4].

Theorem 1.2 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.*

$\square$

As a consequence of Theorem 1.2, we can define *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus we obtain the following theorem.

Theorem 1.3 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

$\square$

2. Interpolation theorem

Let $\mathcal{G}$ be the class of all simple graphs, a function $f : \mathcal{G} \to \mathbb{Z}$ is called a *graph parameter* if $f(G) = f(H)$, whenever $G \cong H$. If f is a graph parameter and $\mathcal{J} \subseteq \mathcal{G}$,

f is called an *interpolation graph parameter with respect to* $\mathcal{J}$ if there exist integers a and b such that

$$\{f(G) : G \in \mathcal{J}\} = [a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

We have shown in [7, 8, 9] that the chromatic number χ, the clique number ω, and the matching number α_1 are interpolation graph parameters with respect to $\mathcal{J} = \mathcal{R}(\mathbf{d})$.

If f is an interpolation graph parameter with respect to $\mathcal{J}$, it is natural to write $\min(f, \mathcal{J}) = \min\{f(G) : G \in \mathcal{J}\}$ and $\max(f, \mathcal{J}) = \max\{f(G) : G \in \mathcal{J}\}$.

In the case where $\mathcal{J} = \mathcal{R}(\mathbf{d})$ we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$ respectively.

Theorem 2.1 *If G is a graph and $\sigma(a, b; c, d)$ is a switching on G, then* $\phi(G^{\sigma(a,b;c,d)}) \leq \phi(G) + 1$.

Proof. Let S be a decycling set of G with $|S| = \phi(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on G. We claim that $(G - S)^\sigma$ contains at most one cycle. If $ab, cd \in E(G - S)$, then $(G - S)^\sigma$ is well defined. Since $G - S$ is a forest, there is at most one path in $G - S$ from a to c. If there is a path from a to c in $G - S$, then the path can be modified as the unique path from b to d. Thus the claim is true. If one or both of ab, cd are not edges in $G - S$, then the claim is also true. Therefore $\phi(G^\sigma) \leq \phi(G) + 1$.

□

Corollary 2.2 *If σ is a switching on G, then* $|\phi(G) - \phi(G^\sigma)| \leq 1$.

Proof. Since a switching is symmetric, we may assume that $\phi(G) \leq \phi(G^\sigma)$. By Theorem 2.1, $\phi(G^\sigma)$ is either $\phi(G) + 1$ or $\phi(G)$. In both cases we have $|\phi(G) - \phi(G^\sigma)| \leq 1$.

□

Theorem 2.3 *For a given graphic degree sequence* $\mathbf{d}$*, there exist integers a and b such that there is a graph G with degree sequence* $\mathbf{d}$ *and $\phi(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Proof. The proof follows directly from Theorem 1.3 and Corollary 2.2.

□

Let $\mathbf{d}$ be a graphic degree sequence. We have already defined the graph $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$. The *range of decycling numbers* of $\mathcal{R}(\mathbf{d})$ can be defined as the interval of integers specified by Theorem 2.3, i.e.,

$$\phi(\mathbf{d}) = \{\phi(G) \ : \ G \in \mathcal{R}(\mathbf{d})\} = [a, b] = \{c \in \mathbb{Z} \ : \ a \leq c \leq b\}.$$

Naturally, we can call a the *minimum decycling number* for $\mathbf{d}$, thus $a := \min(\phi, \mathbf{d})$ and $b := \max(\phi, \mathbf{d})$, the *maximum decycling number* for $\mathbf{d}$. We write $\mathbf{d} = r^n$ for the sequence $(r, r, \ldots, r)$ of length n, where r is a non-negative integer and n is a positive integer. By the definition of graphic degree sequence, $\mathbf{d} = r^n$ is graphic if and only if $rn \equiv 0 (\mathrm{mod}\ 2)$ and $n \geq r + 1$. Moreover, $\mathcal{R}(r^n)$ contains a disconnected graph if and only if $n \geq 2r+2$. It is easy to see that $\min(\phi, 0^n) = \max(\phi, 0^n) = 0$ and $\min(\phi, 1^{2n}) = \max(\phi, 1^{2n}) = 0$. When $r = 2$, we have $\min(\phi, 2^n) = 1$ and $\max(\phi, 2^n) = \lfloor \frac{n}{3} \rfloor$. Because of this fact, from now on we will consider the cases when $r \geq 3$ and $n \geq r+1$.

3. $\min(\phi, r^n)$

It was a remark in [1] that if G is a connected graph with maximum degree Δ, then $\phi(G) \geq \frac{|E(G)|-|V(G)|+1}{\Delta-1}$. Thus if G is a connected r-regular graph of order n, then $\phi(G) \geq \frac{nr-2n+2}{2(r-1)}$.

In order to obtain the exact values of $\min(\phi, r^n)$ we first state some useful facts arising from elementary arithmetic as follows:
1. Let n and r be integers, $n > r \geq 3$. Then $r - 1 \leq \frac{nr-2n+2}{2(r-1)}$ if and only if $n \geq 2r$.
2. Let n and r be integers, $n \geq 2r$ and $nr \equiv 0 \pmod 2$. Then $\frac{nr-2n+2}{2(r-1)}$ is an integer if and only if n is even and $n = 2 + 2(r-1)q$ for some positive integers q, or n is odd and $n = r + 1 + 2(r-1)q$ for some positive integers q.

Theorem 3.1

$$\min(\phi, r^n) = \begin{cases} r-1 & \text{if } r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & \text{if } n \geq 2r. \end{cases}$$

The proof of Theorem 3.1 follows from Lemmas 3.2, 3.3, 3.4, and 3.5.

□

If S is a decycling set of an r-regular graph G with $E(G-S) = \emptyset$, then for any $v \in S$, $S - \{v\}$ is also a decycling set of G. Thus for a minimum decycling set S of an r-regular graph G, there exists $v \in V(G-S)$ such that $d_{(G-S)}(v) = 1$. It follows that $\min(\phi, r^n) \geq r-1$. With this observation, we will see that the lower bound is precise when $n \leq 2r$. Moreover, the lower bound can be improved to $\lceil \frac{nr-2n+2}{2(r-1)} \rceil$, otherwise.

Lemma 3.2 $\min(\phi, r^n) = r-1, \text{if } r+1 \leq n \leq 2r-1.$

Proof. In order to achieve the exact the value of $\min(\phi, r^n)$, we now construct an r-regular graph G of order n such that $\phi(G) = r-1$. Put $n = r+j, 1 \leq j \leq r-1$. It should be noted that an r-regular graph of order $n = r+j$ exists if and only if j is odd or both r and j are even.

Let $X = \{s_1, s_2, \ldots, s_{r-2}\}$ and $Y = \{t_1, t_2, \ldots, t_{j+2}\}$, where $r \leq j+2$. Let G be a graph with $V(G) = X \cup Y$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{t_1t_2, t_2t_3, \ldots, t_{j+1}t_{j+2}, t_{j+2}t_1\}, E_2 = \{s_pt_q : 1 \leq p \leq r-2, 1 \leq q \leq j+2\}$, and $E_3 = E(H)$, where H is an $(r-j-2)$-regular graph on X. Note that an $(r-j-2)$-regular graph of order $r-2$ exists if and only if an r-regular graph of order $r+j$ exists. It is clear that G is an r-regular graph of order n with a decycling set $S = X \cup \{t_1\}$ of cardinality $r-1$. Hence S is minimum.

Suppose $r = j + 1$. In this particular case, we see that an r-regular graph of order $n = r + j = 2r - 1$ exists if and only if j is odd and r is even. Let $X = \{s_1, s_2, \ldots, s_{r-2}\}$ and $Y = \{t_1, t_2, \ldots, t_{r+1}\}$. Let G be a graph with $V(G) = X \cup Y$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{t_1t_2, t_2t_3, \ldots, t_rt_{r+1}, t_{r+1}t_1\}$,
$E_2 = \{s_pt_q : 1 \leq p \leq r-2, 1 \leq q \leq r\} - \{s_1t_2, s_2t_3, \ldots, s_{\frac{r-2}{2}}t_{\frac{r}{2}}\}$, and
$E_3 = \{s_1t_{r+1}, t_2t_{r+1}, s_2t_{r+1}, t_3t_{r+1}, \ldots, t_{\frac{r}{2}}t_{r+1}\}$. Thus G is an r-regular graph of order $2r-1$ with a decycling set $S = X \cup \{t_{r+1}\}$ of cardinality $r-1$. Hence S is minimum. The proof is complete. □

Lemma 3.3 $\min(\phi, r^{2r}) = r - 1$, *for all* $r \geq 3$, *and* $\min(\phi, r^{2r+1}) = r$, *for all even integers* $r \geq 4$.

Proof. Let X and Y be disjoint sets, $|X| = |Y| = r$. The complete bipartite graph G on the partite sets X and Y is an r-regular graph of order $2r$. For each $x \in X$, put $S = X - \{x\}$ and $T = Y \cup \{x\}$ we see that S is a decycling set of G, $E(G|_S) = \emptyset$ and $G|_T$ is a tree. Since $|S| = r-1$, S is minimum. Thus $\min(\phi, r^{2r}) = r - 1$.

For an even integer $r \geq 4$, let $X = \{s_1, s_2, \ldots, s_r\}$ and $Y = \{t_1, t_2, \ldots, t_r\}$ be disjoint sets. An r-regular graph G whose $V(G) = X \cup Y \cup \{v\}$, where v is a new vertex not in $X \cup Y$, and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{s_pt_q : 1 \leq p \leq r, 1 \leq q \leq r, p \neq q\}$, $E_2 = \{t_1t_2, t_3t_4, \ldots, t_{r-1}t_r\}$, and $E_3 = \{vs_p : 1 \leq p \leq r\}$. Since $\lceil \frac{(2r+1)r-2(2r+1)+2}{2(r-1)} \rceil = r, \min(\phi, r^{2r+1}) = r$. □

Lemma 3.4 *If* $n \geq 2r$ *and* $\frac{nr-2n+2}{2(r-1)}$ *is an integer, then* $\min(\phi, r^n) = \frac{nr-2n+2}{2(r-1)}$.

Proof. *Case 1.* Suppose n is even and $n \geq 2r$. Since $\frac{nr-2n+2}{2(r-1)} = \frac{n}{2} - \frac{n-2}{2(r-1)}$, we write $n = 2(r-1)q + 2, q \geq 1$. By induction on q, it is true when $q = 1$, by Lemma 3.3. Moreover, the graph G constructed in Lemma 3.3 has the property that $V(G) = S \cup T, E(G|_S) = \emptyset$, $|S| = r - 1$, and $G|_T$ is a tree on $r + 1$ vertices.

Suppose there exists an r-regular graph G_0 on $n = 2(r-1)q + 2$ vertices with $V(G_0) = S_0 \cup T_0, S_0 \cap T_0 = \emptyset, S_0 = \{s_1, s_2, \ldots, s_a\}$ and $T_0 = \{t_1, t_2, \ldots, t_b\}$ where $a = (r-2)q + 1$ and $b = rq + 1$. We suppose further that $E(G_0|_{S_0}) = \emptyset$, and $G_0|_{T_0}$ is a tree with $d(t_b) = 1$. Let $S_1 = \{u_1, u_2, \ldots, u_{r-2}\}$ and $T_1 = \{v_1, v_2, \ldots, v_r\}$. Now let G be a graph with $V(G) = S \cup T$, where $S = S_0 \cup S_1, T = T_0 \cup T_1$, and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(G_0) - \{s_at_b\}, E_2 = \{u_pv_q : 1 \leq p \leq r-2, 1 \leq q \leq r\}, E_3 = \{t_bv_1, v_1v_2, \ldots, v_{r-1}v_r, v_rs_a\}$. It is clear that G is an r-regular graph on $2(r-1)(q+1) + 2$ vertices, $E(G|_S) = \emptyset$, and $\phi(G) = (r-2)(q+1) + 1$, where $S = S_0 \cup S_1$. Moreover $G|_T$ is a tree and $d_{G|_T}(t_b) = 1$.

Case 2. Suppose n is odd and r is even. Write $n = 2(r-1)q + r + 1, q \geq 1$, and consider when $q = 1$. Let $S = \{s_1, s_2, \ldots, s_a\}$ and $T = \{t_1, t_2, \ldots, t_b\}$, where $a = \frac{3}{2}r - 2$ and $b = \frac{3}{2}r + 1$. Let H be an $(r-2)$-regular bipartite graph with partite sets S and $T' = \{t_1, t_2, \ldots, t_a\}$. Since $|S - \{s_a\}| = \frac{3}{2}r - 3 = 3(\frac{r}{2} - 1)$, $S - \{s_a\}$ can be partitioned into $\frac{r}{2} - 1$ sets each of which contains 3 elements. Let $P = \{S_1, S_2, \ldots, S_{\frac{r}{2}-1}\}$ be such a partition. Let K be a bipartite graph with partite sets $S-\{s_a\}$ and $\{t_{b-2},t_{b-1},t_b\}$ such that $E(K)$ is the union of all edges in 2-regular bipartite graphs with partite sets $\{t_{b-2},t_{b-1},t_b\}$ and S_i, for all $i=1,2,\ldots,\frac{r}{2}-1$. Finally let G be a graph with $V(G)=S\cup T$ and $E(G)=E(H)\cup E(K)\cup E_1$, where $E_1=\{s_at_1,s_at_b,t_1t_2,t_2t_3,\ldots,t_{a-1}t_a\}$. Therefore G is an r-regular graph on $3r-1$ vertices with $\phi(G)=|S|=\frac{3}{2}r-2$. Moreover $G|_T$ is a path with $d_{G|_T}(t_b)=1$ and $E(G|_S)=\emptyset$.

Suppose there exists an r-regular graph G_0 on $2(r-1)q+r+1, q\geq 1$, vertices with $V(G_0)=S_0\cup T_0, E(G_0|_{S_0})=\emptyset$, $|S_0|=a=\frac{3}{2}r-2+(q-1)(r-2)$, and $G_0|_{T_0}$ is a path $t_1t_2\ldots t_b$ of length $b=\frac{3}{2}r+1+(q-1)r$. Let $S_1=\{u_1,u_2,\ldots,u_{r-2}\}$ and $T_1=\{v_1,v_2,\ldots,v_r\}$. We may assume that $s_at_b\in G_0$. Now let G be a graph with $V(G)=S_0\cup S_1\cup T_0\cup T_1$ and $E(G)=E_1\cup E_2\cup E_3$, where $E_1=E(G_0)-\{s_at_b\}, E_2=\{u_pv_q: 1\leq p\leq r-2, 1\leq q\leq r\}$, and $E_3=\{s_av_r,t_bv_1,v_1v_2,v_2v_3,\ldots,v_{r-1}v_r\}$. It is clear that G is an r-regular graph on $n=2(r-1)(q+1)+r+1$ vertices, $E(G|_S)=\emptyset$, and$|S|=\phi(G)=\frac{3}{2}r-2+q(r-2)$, where $S=S_0\cup S_1$. Also note that $G-S=G|_T=t_1t_2\ldots t_bv_1v_2\ldots v_r$ is a path and $d_{G|_T}(t_1)=d_{G|_T}(v_r)=1$.

□

Let $f(n,r)=\lceil\frac{nr-2n+2}{2(r-1)}\rceil$, $r\geq 3$ and $n\geq 2r$. If $\frac{nr-2n+2}{2(r-1)}$ is an integer, then it is easy to show that

$$f(n+2i,r)=\begin{cases} f(n,r)+i & \text{if } 1\leq i\leq r-2,\\ f(n+2(r-1),r) & \text{if } i=r-2.\end{cases}$$

Lemma 3.5 *If* $n\geq 2r$, *then* $\min(\phi,r^{n+2})\leq\min(\phi,r^n)+1$.

Proof. For $n\geq 2r$ we have constructed an r-regular graph G on n vertices having minimum decycling set when $n=2r$ and $n=2r+1$ in Lemma 3.3. We have also constructed an r-regular graph G on n vertices when $\frac{nr-2n+2}{2(r-1)}$ is an integer and $\phi(G)=\frac{nr-2n+2}{2(r-1)}$ in Lemma 3.4. Moreover, such constructions give us decycling sets S with $E(G|_S)=\emptyset$. Let G_0 be an r-regular graph on n vertices with $\phi(G_0)=\min(\phi,r^n)$. We may further assume that $V(G_0)=S_0\cup T_0$ where $E(G_0|_{S_0})=\emptyset$, and $G_0|_{T_0}$ is a forest. Since G_0 is r-regular and $|S_0|\geq r-1$, by Hall's Theorem (See, e.g. [6] page 227), there exists a set of $r-1$ independent edges M_0 joining S_0 to T_0. Let G be a graph with $V(G)=V(G_0)\cup\{x,y\}$ and $E(G)=E_1\cup E_2\cup E_3$, where $E_1=E(G_0)-M_0, E_2=\{xt_1,xt_2,\ldots,xt_{r-1},s_1y,s_2y,\ldots,s_{r-1}y\}$, and $E_3=\{xy\}$, where $M_0=\{s_1t_1,s_2t_2,\ldots,s_{r-1}t_{r-1}\}$, with $s_i\in S_0$ and $t_i\in T_0$. Thus G is an r-regular graph on $n+2$ vertices with a decycling set $S_0\cup\{x\}$. Therefore $\phi(G)\leq\phi(G_0)+1$.

□

It is interesting to observe that the proof of Theorem 3.1 is now complete.

4. $\max(\phi, r^n)$

The problem of determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and the sum of the two numbers equals the order of the graph. Let $F \subseteq V(G)$ of a graph G. F is called an *induced forest* of G, if $G|_F$ contains no cycle. For a graph G, we define, $I(G)$ as:

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

Observe that for a minimum decycling set S of a graph G, if $v \in S$, then there exists a connected component C of $G - S$ such that v is adjacent to at least two vertices of C. Thus $\Delta(G|_S) \leq \Delta(G) - 2$. With this observation, we find that if G is an r-regular graph and S is a minimum decycling set of G, the graph $G|_S$ may not be an $(r-2)$-regular graph. This causes a difficulty in finding $\max(\phi, r^n)$ if we consider only the class of regular graphs. It is reasonable to enlarge the class of regular graphs into the following class of graphs. Let Δ be a nonnegative integer and n be a positive integer such that $n \geq \Delta + 1$. Let $\mathcal{G}(\Delta, n)$ be the class of all graphs of order n and of maximum degree Δ. *The* (Δ, n)*-graph* is a graph having $\mathcal{G}(\Delta, n)$ as its vertex set and two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge.

Lemma 4.1 *The* (Δ, n)*-graph is connected.*

Proof. For any graph $G \in \mathcal{G}(\Delta, n)$, if $F = K_{1,\Delta} \cup (n - \Delta - 1)K_1$ and $G \neq F$, G can be obtained from F by a finite sequence of adding edges. The proof is complete. □

Lemma 4.2 *If* G_1 *and* G_2 *are adjacent in the* (Δ, n)*-graph, then* $|\phi(G_1) - \phi(G_2)| \leq 1$.

Proof. Without loss of generality we may assume that G_2 is a graph obtained from G_1 by adding an edge e. Thus $\phi(G_1) \leq \phi(G_2)$. On the other hand, if S is a minimum decycling set of G_1, then $G_2 - S$ contains at most one cycle. Thus $\phi(G_2) \leq |S| + 1$. Therefore $\phi(G_1) \leq \phi(G_2) \leq \phi(G_1) + 1$. The proof is complete. □

As a consequence of Theorems 4.1 and 4.2, we have the following corollary.

Corollary 4.3 *For any class of graphs* $\mathcal{G}(\Delta, n)$*, there exist integers* a *and* b *such that there is a graph* $G \in \mathcal{G}(\Delta, n)$ *with* $\phi(G) = c$ *if and only if* c *is an integer satisfying* $a \leq c \leq b$. □

Note that the result of Corollary 4.3 is an interpolation theorem of ϕ with respect to $\mathcal{G}(\Delta, n)$, and it is easy to see that $\min(\phi, \mathcal{G}(\Delta, n)) = 0$, the graph F in the proof of Lemma 4.1 is such a graph. In order to investigate the exact values of $\max(\phi, \mathcal{G}(\Delta, n))$, we first give its upper bound.

Lemma 4.4 *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq 1$, then $\phi(G) \leq \frac{n(\Delta-1)}{\Delta+1}$.*

Proof. The theorem is trivial if $\Delta \leq 2$. If $\Delta = 3$ and S is a minimum decycling set of G, then for each $v \in S, d_{G|_S}(v) \leq 1$. This means that both $G|_S$ and $G - S$ are decycling sets of G and hence $|S| \leq \frac{n}{2} = \frac{n(\Delta-1)}{\Delta+1}$. Suppose $\Delta \geq 4$ and S is a minimum decycling set of G. Thus $\Delta(G|_S) \leq \Delta(G) - 2$. Since $I(G) \geq I(G|_S)$, it follows that $n - \phi(G) \geq |S| - \phi(S) \geq |S| - \frac{|S|(\Delta-3)}{\Delta-1}$. Therefore $\phi(G) \leq \frac{n(\Delta-1)}{\Delta+1}$.

□

Theorem 4.5 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then*
(1) $\max(\phi, \mathbf{d}) = n - 2$ if and only if $\mathcal{R}(\mathbf{d}) = \{K_n\}$ and
(2) if $K_n \notin \mathcal{R}(\mathbf{d})$, then $\max(\phi, \mathbf{d}) = n - 3$ if and only if there exists a union of stars as a realization of $\overline{\mathbf{d}}$, where $\overline{\mathbf{d}} = (n - d_n, n - d_{n-1}, \ldots, n - d_1)$.

Proof. (1) By Lemma 4.4, we have $\max(\phi, \mathbf{d}) \leq n - 2$ and $\max(\phi, \mathbf{d}) = n - 2$ if and only if any induced subgraph of 3 vertices of $G \in \mathcal{R}(\mathbf{d})$ forms a triangle. Thus $\max(\phi, \mathbf{d}) = n - 2$ if and only if $\mathcal{R}(\mathbf{d}) = \{K_n\}$
(2) By (1), we have $\max(\phi, \mathbf{d}) \leq n - 3$. If $\mathcal{R}(\overline{\mathbf{d}})$ does not contain a union of stars as its realization, then for every realization $\overline{G}$ of $\overline{\mathbf{d}}$, G must contain a C_3 or P_4 as an induced subgraph. Thus $I(G) \geq 4$ which is equivalent to $\max(\phi, \mathbf{d}) \leq n - 4$. Conversely, if $\overline{G} \in \mathcal{R}(\overline{\mathbf{d}})$ and $\overline{G}$ is a union of stars, then any induced subgraph of 4 vertices of G must contain a triangle or C_4. Therefore $\max(\phi, \mathbf{d}) = n - 3$.

□

Theorem 4.6 *Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. Then*
(1) $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q$, if $t = 0$,
(2) $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q - 1$, if $t = 1$, and
(3) $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q - 2$, if $2 \leq t \leq \Delta$.

Proof. (1) By Lemma 4.4, $\max(\phi, \mathcal{G}(\Delta, (\Delta + 1)q)) \leq n - 2q$. It is easy to see that the graph $qK_{\Delta+1} \in \mathcal{G}(\Delta, (\Delta + 1)q)$ and $\phi(qK_{\Delta+1}) = n - 2q$. Therefore $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q$, if $t = 0$.
(2) By Lemma 4.4, $\max(\phi, \mathcal{G}(\Delta, (\Delta+1)q+1)) \leq n-2q-1$. It is easy to see that the graph $qK_{\Delta+1} \cup K_1 \in \mathcal{G}(\Delta, (\Delta+1)q+1)$ and $\phi(qK_{\Delta+1} \cup K_1) = n-2q-1$. Therefore $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q - 1$, if $t = 1$.
(3) Let $n = (\Delta+1)q+t, 2 \leq t \leq \Delta$. Since $qK_{\Delta+1} \cup K_t \in \mathcal{G}(\Delta, n)$ and $\phi(qK_{\Delta+1} \cup K_t) = n - 2q - 2$, $\max(\phi, \mathcal{G}(\Delta, n)) \geq n - 2q - 2$. We first consider the case $q = 1$. Since $n = (\Delta+1)q+t, 2 \leq t \leq \Delta$, the minimum degree of $\overline{G}$ is t for all graphs $G \in \mathcal{G}(\Delta, n)$. Thus $\overline{G}$ is not a union of stars and therefore $\max(\phi, \mathcal{G}(\Delta, n)) = n - 4$ for $q = 1$.

Suppose $q \geq 2$ and q is the smallest integer such that $\max(\phi, \mathcal{G}(\Delta, (\Delta+1)q+1)) = n - 2q - 1$. Let v be a vertex of G of degree Δ and let H be the graph obtained from G by deleting the vertex v and its neighbors. It is clear that H has order $(\Delta + 1)(q - 1) + t$ and by minimality of q, there exists a minimum decycling set S of H of order at most $n - 2(q - 1) - 2 = n - 2q$. Equivalently, $|H - S| \geq 2q$. Since

$(H-S)\cup\{v\}$ is an induced forest of G and $|(H-S)\cup\{v\}|\geq 2q+1$, we have $|(H-S)\cup\{v\}|=2q+1$. Put $F=(V(H)-S)\cup\{v\}$ and $D=G-F$, thus D is a minimum decycling set of G with $|D|=n-2q-1$. Since $v\in V(G|_F)$ and $d_{G|_F}(v)=0$, $G|_F$ contains at most $2q$ vertices of degree at least 1. Let T the subgraph of $G|_F$ consisting of all nontrivial components. By being maximality of $G|_F$, $V(T)\neq\emptyset$. Thus $1\leq|V(T)|\leq 2q$. Since $|D|=(\Delta-1)q+t-1$, $e(D,T)\geq 2(\Delta-1)q+2(t-1)$. But $\frac{2(\Delta-1)q+2(t-1)}{2q}>\Delta-1$, there exists a vertex $f\in T$ such that $d_G(f)>\Delta$. This is a contradiction. The proof is complete.

□

Theorem 4.7 *For $r\geq 3$, and $r+1\leq n\leq 2r+1$,*
(1) $\max(\phi,r^n)=n-2$, *if and only if* $n=r+1$,
(2) $\max(\phi,r^n)=n-3$, *if and only if* $n=r+2$,
(3) $\max(\phi,r^n)=n-4$, *for all even integers* n, $r+3\leq n$,
(4) $\max(\phi,r^n)=n-4$, *for all odd integers* n, $r+3\leq n$ *and* $n\geq f(j)$,
(5) $\max(\phi,r^n)=n-5$, *for all odd integers* n, $r+3\leq n$ *and* $n<f(j)$,
where $f(j)=\frac{5}{2}(j-1)$ *if* $j\equiv 3(\mathrm{mod}\ 4)$, *and*
$f(j)=1+\frac{5}{2}(j-1)$ *if* $j\equiv 1(\mathrm{mod}\ 4)$.

Proof. (1) and (2) follow directly from Theorem 4.5.
(3) From (1) and (2), we have $\max(\phi,r^n)\leq n-4$ for all integers $n=r+j$, $3\leq j\leq r+1$. Suppose n is even and let G be an r-regular graph of order n such that $\overline{G}$ is bipartite. Thus the induced subgraph of any 5 vertices of G must contain a cycle in G. Thus $\phi(G)=n-4$. Therefore $\max(\phi,r^n)=n-4$,
(4) In [7], we have shown that if j is odd and $j\geq 3$, then there exists a $(j-1)$-regular triangle-free graph of odd order n if and only if $n\geq f(j)$, where $f(j)=\frac{5}{2}(j-1)$ if $j\equiv 3(\mathrm{mod}\ 4)$, and $f(j)=1+\frac{5}{2}(j-1)$ if $j\equiv 1(\mathrm{mod}\ 4)$. Note that if H is a triangle-free graph of order at least 5 and for any subset K of H with $|K|=5$, then $E(H|_K)\leq 5$. Thus if $n\geq f(j)$ and H is a $(j-1)$-regular triangle-free graph on n vertices, then any induced subgraph of 5 vertices of H contains at most 5 edges in H. Therefore $\phi(\overline{H})\geq n-4$. Hence $\max(\phi,r^n)=n-4$.
(5) If $n<f(j)$ and $j\geq 5$, then any $(j-1)$-regular graph H on n vertices must contain a triangle. Let $T=\{u,v,w\}$ be a set of 3 vertices of H which induces a triangle. Since $n\leq\frac{5}{2}(j-1)$ and H is a $(j-1)$-regular graph, there exist $x,y\in T$ such that either $N_H(x)\cap N_H(y)\neq\emptyset$ or there exist $a\in N_H(x)$ and $b\in N_H(y)$ such that $ab\in E(H)$. In either case there are at least 5 vertices of $V(\overline{H})$ induced a forest in $\overline{H}$. Thus $I(\overline{H})\geq 5$ and it is equivalent to $\phi(\overline{H})\leq n-5$.

Finally, let $X=\{x_1,x_2,\ldots,x_t\}$ and $Y=\{y_1,y_2,\ldots,y_t\},t=\frac{n-1}{2}$. Since $n\leq\frac{5}{2}(j-1),n$ is odd, and $j\geq 5$, there exists a $(j-1)$-regular bipartite graph K with the partite sets X and Y and $\{x_iy_i:i=1,2,\ldots,t\}\subseteq E(K)$. Now choose H to be a $(j-1)$-regular graph with $V(H)=V(K)\cup\{v\}$ and $E(H)=E_1\cup E_2$, where $E_1=E(K)-\{x_iy_i:i=1,2,\ldots,\frac{j-1}{2}\}$ and $E_2=\{vx_i,vy_i:i=1,2,\ldots,\frac{j-1}{2}\}$. It is clear that H is a $(j-1)$-regular graph on n vertices such that $\phi(\overline{H})=n-5$.

□

Theorem 4.8 *For $n \geq 2r+2$ and $r \geq 3$, write $n = (r+1)q + t, q \geq 2$ and $0 \leq t \leq r$. Then*
(1) $\max(\phi, r^n) = n - 2q$ *if* $t = 0$,
(2) $\max(\phi, r^n) = n - 2q - 1$ *if* $t = 1$,
(3) $\max(\phi, r^n) = n - 2q - 2$ *if* $2 \leq t \leq r-1$,
(4) $\max(\phi, r^n) = n - 2q - 3$ *if* $t = r$.

Proof. The proof of (1), (2) and (3) follows directly from Theorems 4.6 and 4.7. (4) Note that $\max(\phi, \mathcal{G}(2, 3q+2)) = q$ and a graph $G \in \mathcal{G}(2, 3q+2)$ with $\phi(G) = q$ if and only if $G = (q-1)C_3 \cup C_5$ or $(q-2)C_3 \cup 2C_4$. It should be also noted that an r-regular graph of order $(r+1)q + r$ exists if and only if r is even. By Theorem 4.6, we have $\max(\phi, r^{(r+1)q+r}) \leq (r-1)q + (r-2)$. We first consider the case $r = 4$. Suppose there is a 4-regular graph G of order $5q+4$ and $\phi(G) = 3q+2$. Let S be a minimum decycling set of G such that $|S| = 3q+2$. Put $F = V(G) - S$. Since for each vertex $v \in S$ we have $e(\{v\}, F) \geq 2$, $e(S, F) \geq 2(3q+2)$. If $e(S, F) = 2(3q+2)$ holds for every minimum decycling set S of G, then S is a 2-regular graph of order $3q+2$. In this case $G|_S$ is either $(q-1)C_3 \cup C_5$ or $(q-2)C_3 \cup 2C_4$.

Let F_1 be an induced forest of S of order $2q+2$. Then F_1 is also a maximum induced forest of G and $G - F_1$ is a 2-regular graph of order $3q+2$. It is clear that F_1 can be obtained from S by removing one vertex on each cycle. Since F_1 contains an induced path of at least 3 vertices and there exists a vertex $v \in F$ adjacent to exactly two vertices in the path, we can choose a maximum induced forest F_2 of S of order $2q+2$ in such a way that $e(\{v\}, F_2) = 1$. Thus $F_2 \cup \{v\}$ is an induced forest of G of order $2q+3$, this is a contradiction. If $e(S, F) > 2(3q+2)$, then $S \in \mathcal{G}(2, 3q+2)$ and S is not regular. Therefore $I(G) \geq I(S) \geq 2q+3$. A graph $(q-1)K_5 \cup H$, where H is a 4-regular graph of order 9 satisfying the condition in Theorem 4.7(5), has the property that $\phi((q-1)K_5 \cup H) = 3(q-1) + 4 = 3q+1$. Thus $\max(\phi, 4^{5q+4}) = 3q+1$.

Now suppose $r \geq 6$ and let G be an r-regular graph of order $n = (r+1)q + r$ and $\phi(G) = (r-1)q + (r-2)$. Let S be a minimum decycling set of G such that $|S| = (r-1)q + (r-2)$. Put $F_1 = V(G) - S$. If $e(S, F_1) = 2((r-1)q + (r-2))$, then S is an $(r-2)$-regular graph of order $(r-1)q + (r-2)$. By induction on r, $I(S) \geq 2q+3$. Since $I(G) \geq I(S) \geq 2q+3$, we get a contradiction. Thus $e(S, F_1) > 2(n - 2q - 2)$. Since r is even and $e(\{v\}, F_1) \geq 2$ for all $v \in S$, there exists $v \in S$ such that $e(\{v\}, F_1) \geq 4$ or there exist two vertices $u, v \in S$ such that $e(\{u\}, F_1) \geq 3$ and $e(\{v\}, F_1) \geq 3$. Thus $e(S, F_1) \geq 2(r-1)q + 2(r-2) + 2$. On the other hand by counting edges from S to F_1, we find that $e(S, F_1) = 2(r-1)q + 2(r-2) + 2$ and $F_1 = (q+1)K_2$. Put $G_1 = G - F_1$ and $G_i = G_{i-1} - F_i$, for $2 \leq i \leq \frac{r-2}{2}$ and F_i is a maximum induced forest of G_{i-1}. If $2q+2 = I(G) = I(G_1) = I(G_2) = \ldots = I(G_{\frac{r-4}{2}})$, then $G_{\frac{r-2}{2}}$ has order $3q+2$, maximum degree 2 and not regular. Thus $I(G_{\frac{r-2}{2}}) \geq 2q+3$. This is a contradiction.

Therefore $\min(I, r^{(r+1)q+r}) \geq 2q+3$. A graph $(q-1)K_{r+1} \cup H$, where H is an r-regular graph of order $2r+1$ satisfying the condition in Theorem 4.7(5), has the property that $\phi((q-1)K_{r+1} \cup H) = (r-1)(q-1) + (2r+1-5) = (r-1)q + r - 3 = n - 2q - 3$. Therefore $\max(\phi, r^{(r+1)q+r}) = n - 2q - 3$.

□

5. Decycling number of cubic graphs

Let $\mathcal{R}(3^{2n})$ be the class of cubic graphs of order $2n$. As a consequence of the previous sections we have the following result concerning the class of cubic graphs.

Theorem 5.1 *For any integer $n \geq 2$, we have*

$$\min(\phi, 3^{2n}) = \lceil \frac{n+1}{2} \rceil, \textit{ and}$$

$$\max(\phi, 3^{2n}) = \begin{cases} n & \textit{if } n \textit{ is even} \\ n-1 & \textit{if } n \textit{ is odd.} \end{cases}$$

□

Thus, by Theorem 5.1, the range of decycling can be obtained, namely

$$\phi(3^{2n}) = \{\phi(G) \ : \ G \in \mathcal{R}(3^{2n})\} = [\min(\phi, 3^{2n}), \max(\phi, 3^{2n})].$$

For each $c \in \phi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d};\, c)$ denote the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number c. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d};\, c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d};\, c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d};\, c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number c by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with decycling number c. In this section, we prove that the induced subgraph $\mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$ is connected.

Let $G \in \mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$ and S be a minimum decycling set of G. The structure of $\mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$, for $n = 2, 3$, can be easily verified. From now on we will consider when $2n \geq 8$. Put $F = V(G) - S$. Thus, by counting edges in the graph G, we get $e(S) + e(S,F) + e(F) = 3n$, $2|S| \leq e(S,F) \leq 3|S|$ and $e(S,F) = 3|S| - 2e(S)$. Since $|E(G|_F)| \leq |F| - 1$, $\frac{3n-1}{2} - 1 \geq e(F) = 3n - 3\lceil \frac{n+1}{2} \rceil + e(S)$. The following Lemma is easily obtained.

Lemma 5.2 *Let $G \in \mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$ and S be a minimum decycling set of G. Put $F = V(G) - S$. Then*
(1) $e(S) = 0$, *if n is odd and $e(S) \leq 1$, if n is even,*
(2) *if n is odd, then $G|_F$ is a tree,*
(3) *if n is even and $e(S) = 1$, then $G|_F$ is a tree,*
(4) *if n is even and $e(S) = 0$, then $G|_F$ has 2 connected components.*

□

Let $\mathcal{F}(n_1, n_2, n_3)$ be the class of trees T having n_i vertices of degree $i, i = 1, 2, 3$ and $\Delta(T) \leq 3$. Thus for any $T \in \mathcal{F}(n_1, n_2, n_3)$ we have $|V(T)| = n_1 + n_2 + n_3$, $n_1 = n_3 + 2$, and $n_1 \geq 2$.

Lemma 5.3 *Let n be an odd integer with $n \geq 5$ and $N = \frac{3n-1}{2}$. Then there exists $G \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ with a minimum decycling set S such that $G - S \in \mathcal{F}(n_1, n_2, n_3)$ if and only if there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3(n+1)}{2}$, and $n_1 \geq 2$.*

Proof. Suppose there is a cubic graph G of order $2n$ with $\phi(G) = \frac{n+1}{2}$. Then by Lemma 5.1, there exists a minimum decycling set S such that $|S| = \frac{n+1}{2}$, $e(S) = 0$, and $G - S$ is a tree of order N. Put $F = V(G) - S$. Since $\Delta(G|_F) \leq 3$, we define $n_i, i = 1, 2, 3$ as the number of vertices of $G|_F$ of degree i. It is clear that $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{n+1}{2}$ and $n_1 \geq 2$.

Conversely, suppose there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{n+1}{2}$, and $n_1 \geq 2$. We first consider $n_1 = 2$. Thus $n_2 = \frac{3n-5}{2}$, $n_3 = 0$. Let T is a path of order N with $V(T) = F_1 \cup F_2$, where $F_1 = \{f_1, f_2\}$ and $F_2 = \{f_3, f_4, \ldots, f_N\}$ are the sets of vertices of T of degree 1 and of degree 2 of T, respectively. Let G be a graph with $V(G) = V(T) \cup S$, where $S = \{s_1, s_2, \ldots, s_{\frac{n+1}{2}}\}$, and $E(G) = \{s_1f_1, s_1f_2, s_1f_3, s_2f_1, s_2f_2, s_2f_4\} \cup E_1$, where $E_1 = \bigcup_{i=0}^{\frac{n-5}{2}} \{s_{3+i}f_{3i+5}, s_{3+i}f_{3i+6}, s_{3+i}f_{3i+7}\}$. It is clear that $G \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ and $G - S \in \mathcal{F}(2, n_2, 0)$. Suppose $n_1 \geq 3$. Since $n_1' = n_1 - 1$, $n_2' = n_2 + 2$ and $n_3' = n_3 - 1$ satisfy the conditions of the theorem, by induction on n_1, there exists $G' \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ and a minimum decycling set S such that $G' - S \in \mathcal{F}(n_1', n_2', n_3')$. Let $F_i' = \{f \in G' - S : d_{G'-S}(f) = i\}, i = 1, 2, 3$ be the corresponding vertices in $G' - S$ of degree i. Since $n_2' \geq 2$, there exists $v, w \in F_2'$ and $v \neq w$. There exist $s_1, s_2 \in S$ such that $vs_1, ws_2 \in E(G')$. If $s_1 \neq s_2$, then the graph $G = G'^{\sigma(u,v;w,s_2)}$ contains a maximum induced forest T with n_1 vertices of degree 1. If $s_1 = s_2$ and $us_1 \notin E(G')$, there exists $s \in S - \{s_1\}$ such that $us \in E(G')$. Thus $G'^{\sigma(u,s;v,s_1)}$ is a graph such that v and w have different neighbors in S. Finally, if $s_1 = s_2$ and $us_1 \in E(G')$, there exists $s \in S - \{s_1\}$ such that $us \in E(G')$ and there exists $x \in (G' - S) - \{u, v, w\}$ such that $xs \in E(G')$. Thus $G'^{\sigma(v,s_1;x,s)}$ is a graph such that v and w have different neighbors in S. Thus the proof is complete.

□

Lemma 5.4 *Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} - 1$. Then there exists $G \in \mathcal{R}(3^{2n};\ \frac{n}{2} + 1)$ with a minimum decycling set S such that $G - S \in \mathcal{F}(n_1, n_2, n_3)$ and $e(S) = 1$ if and only if there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3n}{2} + 1$, and $n_1 \geq 2$.*

Proof. The proof follows from Lemma 5.2 and similar argument in Lemma 5.3.

□

Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} - 1$. For a set S of cardinality $\frac{n}{2} + 1$, a graph $G \in \mathcal{R}(3^{2n};\ \frac{n}{2} + 1)$ with the minimum decycling set S, G is called a cubic graph of type 0 or 1 if $e(S) = 0$ or $e(S) = 1$, respectively. Let $G \in$

$\mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ be of type 0. Then there exists a decycling set S of G such that $|S| = \frac{n}{2}+1$, $E(G|_S) = \emptyset$, and $F = G - S$ is a forest containing two connected components. Thus there exist $f_1, f_2 \in V(F)$ such that $d_F(f_1) \leq 1$, $d_F(f_2) \leq 1$, and f_1, f_2 are not in the same components of F. There must also exist $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $s_1f_1, s_2f_2 \in E(G)$. Thus $G^\sigma(s_1, f_1; s_2, f_2) \in \mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ having S a minimum decycling set and $G^\sigma(s_1, f_1; s_2, f_2)$ is of type 1. Thus the graphs of two types can be transformed to each other by a suitable switching.

Lemma 5.5 *Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2}-1$. Then there exists a cubic graph G of order $2n$ with minimum decycling set S such that $|S| = \frac{n}{2}+1$ and $e(S) = 0$ if and only if there exists a switching σ such that $G^\sigma \in \mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ with a minimum decycling set S and $e(S) = 1$.*

□

Let n, N, n_1, n_2, n_3 be integers satisfying conditions in Lemma 5.3. Let $\mathcal{F}(n_1, n_2, n_3)$ be the class of trees T having n_i vertices of degree $i, i = 1, 2, 3$ and $\Delta(T) \leq 3$. We first consider in the case when $n_1 = 2$. Thus $n_3 = 0$ and $\mathcal{F}(2, n_2, 0)$ contains the path of N vertices. Let P_N be the path $f_1f_2\cdots f_N$ and let $S_t = \{s_1, s_2, \ldots, s_t\}$, where $t = \frac{n+1}{2}$. It is clear that there are cubic graphs obtained by joining $3t$ edges from S to vertices in P_N and such graphs are not unique. Now we will construct a cubic graph of order $2n$. When $N = 4$, it is easy to see that there is a unique cubic graph G_4 obtained in this way. That is $V(G_4) = S_2 \cup V(P_4)$ and $E(G_4) = \{s_1f_1, s_1f_2, s_1f_4, s_2f_1, s_2f_3, s_2f_4\}$.

A cubic graph G_7 of order 10 can be constructed by taking $V(G_7) = S_3 \cup V(P_7)$ and $E(G_7) = (E(G_4) - \{s_2f_4\}) \cup \{s_2f_7, s_3f_5, s_3f_6, s_3f_7\}$. Thus G_7 is a cubic graph of order 10 with $\phi(G_7) = 3$. Similarly the graph G_{10} can be obtained from G_7 by extending the path P_7 to P_{10}, removing the edge s_3f_7 and inserting edges $s_3f_{10}, s_4f_8, s_4f_9, s_4f_{10}$. In general, if $t \geq 4$, then $N = 3t - 2$. We can construct the cubic graph G_N obtained from G_{N-3} by extending the path P_{N-3} to P_N, removing the edge $s_{t-1}f_{N-3}$ and inserting edges $s_{t-1}f_N, s_tf_{N-2}, s_tf_{N-1}, s_tf_N$.

Lemma 5.6 *Let G be a cubic graph of order $2n$, n is an odd integer, with $\phi(G) = \frac{n+1}{2}$. If G has a path P_N as a maximum induced forest, where $N = \frac{3n-1}{2}$, then G_N can be obtained from G by a finite sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1\sigma_2\ldots\sigma_i}$ is a cubic graph with P_N as its induced forest.*

Proof. It is easy if $N = 4$. Let G be a cubic graph of order $2n$ with P_N as its induced forest. Put $P_N = f_1f_2\cdots f_N$ and $S_t = \{s_1, s_2, \ldots, s_t\}$ where $t = \frac{n+1}{2}$. If $s_tf_N \notin E(G)$, there are exactly 2 vertices in S which are adjacent to f_N and there are exactly 3 vertices in $V(P_N)$ which are adjacent to s_t. Thus there exist $s_i \in S$ and $f_j \in V(P_N)$ such that $s_if_N, s_tf_j \in E(G)$ and $s_if_j \notin E(G)$. The graph $G^1 = G^{\sigma(s_t, f_j; f_N, s_i)}$ has a common edge s_tf_N with G_N. If $s_tf_{N-1} \notin E(G^1)$, there exists $s \in S_t$ such that $sf_{N-1} \in G^1$. Put $s = s_{t-1}$. Since P_N is a path and $t \geq 3$, $|N(s_{t-1}) \cap N(s_t)| \leq 1$. Thus there is a switching which transforms G^1 to G^2 such that G^2 has s_tf_N, s_tf_{N-1} as common edges with G_N. By continuing in this way, we can transform the graph G by a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ to G_N such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1\sigma_2\ldots\sigma_i}$ is a cubic graph with P_N as its induced forest.

□

Lemma 5.7 *Let G be a cubic graph of order $2n$, n is an odd integer, with $\phi(G) = \frac{n+1}{2}$. If G does not have P_N as its maximum induced forest. Then a cubic graph G_N can be obtained from G by a finite sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1\sigma_2\ldots\sigma_i} \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ and $G^{\sigma_1\sigma_2\ldots\sigma_k} = G_N$.*

Proof. In the proof of Lemma 5.3 and Lemma 5.6 a sequence of suitable switchings can be obtained in order to transform G into G_N.

□

Similar argument can be made to obtain the same result for cubic graphs of order $2n$ and n is even.

Combining the results in this section, we have the following theorem.

Theorem 5.8 *The induced subgraph $\mathcal{R}(3^{2n};\ \lceil\frac{n+1}{2}\rceil)$ is connected.*

□

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The decycling number of cubic graphs

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Abstract

For a graph G, a subset $S \subseteq V(G)$, is said to be a *decycling set* of G if $G - S$ is acyclic. The cardinality of smallest decycling set of G is called the *decycling number* of G and it is denoted by $\phi(G)$.

Bau and Beineke posed the following problems: Which cubic graphs G with $\mid G \mid= 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$? In this paper, we give an answer to this Problem.

Keywords: degree sequence, decycling number, cubic graph.

1. Introduction

We consider in this paper only finite simple graphs. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Since we deal only with finite and simple graphs, we will use the following notations and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. As usual, we use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that connects the vertices u and v. The *degree* of a vertex v of a graph G is defined as $d(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree

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of a graph G is usually denoted by $\Delta(G)$. Let S and T be disjoint subsets of $V(G)$ of a graph G. We denote by $e(S,T)$ the number of edges in G that connect from S and T. If S is a subset of $V(G)$ of a graph G, by the graph S we mean the induced subgraph of S in G and we denote $e(S)$ to be the number of edges in the graph S. A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called a *cubic graph.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G. Moreover, a graph H of order n is said to have the same degree sequence as G if there is a bijection f from $V(G)$ to $V(H)$ such that $d(v_i) = d(f(v_i))$ for all $i = 1, 2, \ldots, n$. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G and in this case, G is called a *realization* of $\mathbf{d}$.

Let G be a graph, and let ab and cd be independent edges in G such that ac and bd are not edges in G. Define $G^{\sigma(a,b;c,d)}$ to be the graph obtained from G by deleting the edges ab and cd and replacing the edges ac and bd. The operation $\sigma(a, b; c, d)$ is called *switching operation.* It is easy to see that the graph obtained from G by a switching will have the same degree sequence as G. The following theorem has been shown by Havel [4] and Hakimi [3].

Theorem 1.1 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.*

□

As a consequence of Theorem 1.1, we can define *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$, the vertices of which are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus, as a direct consequence of Theorem 1.1, we have shown the following theorem.

Theorem 1.2 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

□

Let G be a graph. The problem of determining the minimum number of vertices whose removal eliminates all cycles in the graph G is difficult even for some simply defined graphs as stated in Bau and Beineke [1]. For a graph G, this minimum is known as the *decycling number* of G, and denoted by $\phi(G)$. However, the class of those graphs G in which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n-2$ and $\phi(K_{p,q}) = p-1$ if $p \leq q$, where K_n and $K_{p,q}$ denote the complete graph of order n and the complete bipartite graph with partite sets of cardinality p and q, respectively. The value of $\phi(G)$ for many classes of the graphs were obtained by Bau and Beineke [1] including an upper bound of connected cubic graphs of a given girth. In the same paper, they posed the following problems:

Problem 1. Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?
Problem 2. Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

We shall answer the Problem 1.

We proved in [5] that the graph parameter ϕ has the property that if the graphs G_1 and G_2 are adjacent in the graph $\mathcal{R}(\mathbf{d})$, then $|\phi(G_1) - \phi(G_2)| \leq 1$. Thus for any graphic degree sequence $\mathbf{d}$, there exist integers a and b such that there is a graph G with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$. We proved in the same paper that if $\mathcal{R}(3^{2n})$ is the class of all cubic graphs of order $2n$, then $\min\{\phi(G) : G \in \mathcal{R}(3^{2n})\} = \lceil \frac{n+1}{2} \rceil$. Thus to answer the Problem 1 is equivalence to find all cubic graphs of order $2n$ in $\mathcal{R}(3^{2n})$ having minimum cardinality of decycling set.

2. Main results

For a graphic degree sequence $\mathbf{d}$, let $\phi(\mathbf{d}) = \{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$. Thus there exist integers a and b such that $\phi(\mathbf{d}) = \{k \in \mathbb{Z} : a \leq k \leq b\}$. For each $c \in \phi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; c)$ denote the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number c. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d}; c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d}; c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d}; c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number c by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with decycling number c. In this section, we find all cubic graphs of order $2n$ with decycling number $\lceil \frac{n+1}{2} \rceil$ and prove that the induced subgraph $\mathcal{R}(3^{2n};\ \lceil \frac{n+1}{2} \rceil)$ is connected.

Let G be a cubic graph of order $2n$ with a minimum decycling set S of cardinality $\lceil \frac{n+1}{2} \rceil$. Since there is only one cubic graph of order 4 and there are only two cubic graphs of order 6, it is easy to that those graphs have the decycling number $\lceil \frac{n+1}{2} \rceil$. From now on we will consider when $2n \geq 8$. Put $F = G - S$. Thus $e(S) + e(S, F) + e(F) = 3n$, $2|S| \leq e(S, F) \leq 3|S|$ and $e(S, F) = 3|S| - 2e(S)$. Hence $2n - \lceil \frac{n+1}{2} \rceil - 1 \geq e(F) = 3n - 3\lceil \frac{n+1}{2} \rceil + e(S)$. Thus we have the following Lemma:

Lemma 2.1 *Let G be a cubic graph of order $2n$ with a minimum decycling set S of cardinality $\lceil \frac{n+1}{2} \rceil$. Put $F = G - S$. Then*
(1) $e(S) = 0$, *if n is odd and $e(S) \leq 1$, if n is even,*
(2) *if n is odd, then F is a tree,*
(3) *if n is even and $e(S) = 1$, then F is a tree,*
(4) *if n is even and $e(S) = 0$, then F has 2 connected components.*

□

For an odd integer $n \geq 5$, there exists a cubic graph G with independent decycling set S of G of cardinality $\frac{n+1}{2}$. Furthermore, $F = G - S$ is a tree of order $N = \frac{3n-1}{2}$ and $\Delta(F) \leq 3$. Let $n_i, i = 1, 2, 3$ be the number of vertices of F of degree i. It is clear that $n_1+n_2+n_3=N$, $2n_1+n_2=\frac{3(n+1)}{2}$ and $n_1\geq 2$.

Let $\mathcal{F}(n_1,n_2,n_3)$ be the class of trees F having n_i vertices of degree $i, i=1,2,3$ and $\Delta(F)\leq 3$. Thus for any $F\in\mathcal{F}(n_1,n_2,n_3)$ we have $|V(F)|=n_1+n_2+n_3$, $n_1=n_3+2$, and $n_1\geq 2$.

Lemma 2.2 *Let n be an odd integer with $n\geq 5$ and $N=\frac{3n-1}{2}$. Then $\mathcal{R}(\phi;\ \frac{n+1}{2})\neq\emptyset$ if and only if there exist nonnegative integers $n_i, i=1,2,3$, $n_1+n_2+n_3=N$, $2n_1+n_2=\frac{3(n+1)}{2}$ and $n_1\geq 2$.*

Proof. We have already proved the first part of this lemma. Suppose there exist nonnegative integers $n_i, i=1,2,3$, $n_1+n_2+n_3=N$, $2n_1+n_2=\frac{n+1}{2}$, and $n_1\geq 2$. We first consider when $n_1=2$. Thus $n_2=\frac{3n-5}{2}$, $n_3=0$. Let F is a path of order N with $V(F)=F_1\cup F_2$, where $F_1=\{f_1,f_2\}$ and $F_2=\{f_3,f_4,\ldots,f_N\}$ are the sets of vertices of F of degree 1 and of degree 2 of F, respectively. Let G be a graph with $V(G)=V(F)\cup S$, where $S=\{s_1,s_2,\ldots,s_{\frac{n+1}{2}}\}$, and $E(G)=\{s_1f_1,s_1f_2,s_1f_3,s_2f_1,s_2f_2,s_2f_4\}\cup E_1$, where $E_1=\bigcup_{i=0}^{\frac{n-5}{2}}\{s_{3+i}f_{3i+5},s_{3+i}f_{3i+6},s_{3+i}f_{3i+7}\}$. It is clear that $G\in\mathcal{R}(3^{2n};\ \frac{n+1}{2})$. Suppose $n_1\geq 3$. Since $n_1'=n_1-1$, $n_2'=n_2+2$ and $n_3'=n_3-1$ satisfy the conditions of the theorem, by induction on n_1, there exists $G'\in\mathcal{R}(3^{2n};\ \lceil\frac{n+1}{2}\rceil)$ and a minimum decycling set S such that $F'=G'-S\in\mathcal{F}(n_1',n_2',n_3')$. Let $F_i', i=1,2,3$ be the corresponding vertices in F' of degree i. Since $n_2'\geq 2$, there exists $v,w\in F_2'$ and $v\neq w$. Since v has two neighbors in F', there exists $u\in V(F')$ such that $u\neq w$, $uv\in E(F')$, and $uw\notin E(F')$. Let $s_1,s_2\in S$ such that $vs_1,ws_2\in E(G')$. We will consider into 2 cases.

Case 1. If $s_1\neq s_2$, then the graph $G=G'^{\sigma(u,v;w,s_2)}$ contains a maximum induced forest F with $V(F)=V(F')$ having n_1 vertices of degree 1.

Case 2. If $s_1=s_2$, then there exist $s\in S-\{s_1\}$ and $x\in V(F')-\{v,w\}$ such that $xs\in E(G')$ and $xs_1\notin E(G')$. Thus the graph $G'^{\sigma}(x,s;s_1,v)$ contains a maximum induced forest F with $V(F)=V(F')$, v,w are of degree 2 in F, and v,w have different neighbors in S. By applying a suitable switching in Case 1., we get a cubic graph G with n_1 vertices of degree 1. Thus the proof is complete.

□

Lemma 2.3 *Let n be an even integer with $n\geq 4$ and $N=\frac{3n}{2}-1$. Then there exists a cubic graph G of order $2n$ with minimum decycling set S such that $|S|=\frac{n}{2}+1$ and $e(S)=1$ if and only if there exist nonnegative integers $n_i, i=1,2,3$, $n_1+n_2+n_3=N$, $2n_1+n_2=\frac{3n}{2}+1$ and $n_1\geq 2$.*

Proof. The proof follows from Lemma 2.1 and the similar argument form Lemma 2.2.

□

Lemma 2.4 *Let n be an even integer with $n\geq 4$ and $N=\frac{3n}{2}-1$. Then there exists a cubic graph G_1 of order $2n$ with minimum decycling set S_1 such that $|S_1|=\frac{n}{2}+1$ and $e(S_1)=0$ if and only if there exist a cubic graph G of order $2n$ with* *a minimum decycling set S of cardinality $\frac{n+1}{2}$, $e(S) = 1$, and a suitable switching.*

$\square$

Let n, N, n_1, n_2, and n_3 be integers satisfying the conditions in Lemma 2.2. We first consider in the case when $n_1 = 2$ in the class $\mathcal{F}(n_1, n_2, n_3)$. Thus $n_3 = 0$ and $\mathcal{F}(2, n_2, 0)$ contains the path of N vertices. Let P_N be the path $f_1 f_2 \cdots f_N$ and let $S_t = \{s_1, s_2, \ldots, s_t\}$ be a set of independent vertices, where $t = \frac{n+1}{2}$. It is clear that there are cubic graphs obtained by joining $3t$ edges from S to vertices in P_N. In particular case when $N = 4$, there is a unique cubic graph G_4 obtained in this way. That is $V(G_4) = V(S_2) \cup V(P_4)$ and $E(G_4) = \{s_1 f_1, s_1 f_2, s_1 f_4, s_2 f_1, s_2 f_3, s_2 f_4\}$. Let G_7 be a cubic graph with $V(G_7) = V(S_3) \cup V(P_7)$ and $E(G_7) = (E(G_4) - \{s_2 f_4\}) \cup \{s_2 f_7, s_3 f_5, s_3 f_6, s_3 f_7\}$. Thus G_7 is a cubic graph of order 10 with $\phi(G_7) = 3$. The graph G_{10} can be obtained from G_7 by extending the path P_7 to P_{10}, removing the edge $s_3 f_7$ and inserting edges $s_3 f_{10}, s_4 f_8, s_4 f_9, s_4 f_{10}$. In general, if $t \geq 4$, then $N = 3t - 2$. We can construct the cubic graph G_N obtained from G_{N-3} by extending the path P_{N-3} to P_N, removing the edge $s_{t-1} f_{N-3}$ and inserting edges $s_{t-1} f_N, s_t f_{N-2}, s_t f_{N-1}, s_t f_N$. The graph G_N is called the *standard cubic graph of order* $2n$.

Lemma 2.5 *Let n is an odd integer and G a cubic graph of order $2n$ with $\phi(G) = \frac{n+1}{2}$. If G has a path P_N as a maximum induced forest, where $N = \frac{3n-1}{2}$, then G_N can be obtained from G and a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for every $i = 1, 2, \ldots t$, $G^{\sigma_1 \sigma_2 \ldots \sigma_i}$ is a cubic graph with P_N as its induced forest.*

Proof. It is easy if $N = 4$. Let G be a cubic graph of order $2n$ with P_N as its induced forest. Put $P_N = f_1 f_2 \cdots f_N$ and $S_t = \{s_1, s_2, \ldots, s_t\}$ where $t = \frac{n+1}{2}$. If $s_t f_N \notin E(G)$, there are exactly 2 vertices in S which are adjacent to f_N and there are exactly 3 vertices in $V(P_N)$ which are adjacent to s_t. Thus there exist $s_i \in S$ and $f_j \in V(P_N)$ such that $s_i f_N, s_t f_j \in E(G)$ and $s_i f_j \notin E(G)$. The graph $G^1 = G^{\sigma_1}$, where $\sigma_1 = \sigma(s_t, f_j; f_N, s_i)$, has a common edge $s_t f_N$ with G_N. If $s_t f_{N-1} \notin E(G^1)$, there exists $s \in S_t$ such that $s f_{N-1} \in G^1$. Put $s = s_{t-1}$. Since P_N is a path and $t \geq 3$, $|N(s_{t-1}) \cap N(s_t)| \leq 1$. Thus there exists $f_j \in V(P_N)$ such that $f_j \neq f_{N-1}, s_t f_j \in E(G^1)$, and $s_{t-1} f_j \notin E(G^1)$. Therefore the graph $G^2 = G^{1\sigma_2}$, where $\sigma_2 = \sigma(s_t, f_j; f_{N-1} s_{t-1}$, has $s_t f_N, s_t f_{N-1}$ as common edges with G_N. By continuing in this way, we can transform the graph G by a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ to G_N such that for every $i = 1, 2, \ldots t$, $G^{\sigma_1 \sigma_2 \ldots \sigma_i}$ is a cubic graph with P_N as its induced forest.

$\square$

Lemma 2.6 *Let n is an odd integer and G a cubic graph of order $2n$ with $\phi(G) = \frac{n+1}{2}$. If G has no P_N as its maximum induced forest. Then G_N can be obtained from G and a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for every $i = 1, 2, \ldots t$, $\phi(G^{\sigma_1 \sigma_2 \ldots \sigma_i}) = \frac{n+1}{2}$.*

Proof. In the proof of Lemma 2.2 and Lemma 2.5 a sequence of suitable switchings can be obtained in order to transform G into G_N.

$\square$

Similar argument can be made to obtain the same result for cubic graphs of order $2n$ and n is even.

We have constructed all cubic graphs of order $2n$ having decycling set of cardinality $\frac{n+1}{2}$. In particular, we have proved the following theorem.

Theorem 2.7 *The induced subgraph* $\mathcal{R}(3^{2n}; \lceil \frac{n+1}{2} \rceil)$ *is connected.*

$\square$

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Decycling Number of Regular Graphs

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Abstract

For a graph G and $S \subset V(G)$, if $G - S$ is acyclic, then S is called a *decycling set* or *feedback set* of G. The *decycling number* of G is defined to be

$$\phi(G) := \min\{|S| : S \subset V(G) \text{ is a decycling set}\}.$$

This paper reviews some recent results and problems on the decycling number of graphs, with an emphasis on the decycling number of regular graphs and random regular graphs. A computational method using differential equations developed by N.C. Wormald has been applied to an algorithm over the probability space of random regular graphs to obtain asymptotic results that are difficult to obtain otherwise. Some completely deterministic interpolation results have also been obtained recently by N. Punnim for the decycling number of regular graphs. These will be reviewed, together with some open problems in this area.

1 Decycling Number

Let G be a graph and let $X \subset E(G)$. Then the minimum $|X|$ such that $G - X$ is acyclic is the dimension of the cycle space of the graph. If G is connected, then the dimension of the cycle space of G is $\|G\| - |G| + 1$, where as in [8], $|G|$ and $\|G\|$ denote respectively the *order* and *size* of G. It was natural to investigate the corresponding problem in terms of vertices, and this was indeed considered by Kirchhoff [12] in his work on spanning trees.

Let G be a graph and $S \subset V(G)$. If $G - S$ is acyclic then S is called a *decycling set* of G. The *decycling number* or the *feedback number* of G is defined to be

$$\phi(G) := \min\{|S| : S \subset V(G) \text{ is a decycling set}\}.$$

The computation of $\phi(G)$, in general, is difficult [11]. A decycling set S with $|S| = \phi(G)$ is said to be a *minimum decycling set* of G. The determination of the decycling numbers of graphs in the following families is shown to be computationally difficult: planar graphs, bipartite graphs, and perfect graphs. The problem is known to be polynomial for the following families of graphs: cubic graphs [13, 21], permutation graphs [14], and interval and comparability graphs (graphs with transitive orientation) [15]. Further investigation of this parameter is therefore well motivated.

The decycling number of the n-cube was determined for $n \in \{1, 2, \cdots, 8\}$ and fairly close bounds were also obtained for $n \in \{9, 10, \cdots, 13\}$ (see [3] for a summary). This also provides a coding theoretic connection. The decycling number of grid graphs (the cartesian product of two paths) was considered in [5]. While these problems are not completely solved, progress has been made on computation of the decycling numbers of these classes of graphs. In [3, 5], a problem has been left open for determination of the decycling numbers of the cartesian products of cycles. For a solution of this problem for the cartesian product of two cycles, see [17]. Several problems had been left open in the summary paper [3], among which was a problem of determination of all cubic graphs with

$$\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$$

where $|G| = 2n$. A hint of an algorithm for determining the graphs for this problem has been given in [18] using the method of switching and direct constructions.

Approximation algorithms have been investigated. In [2], it was shown that a polynomial time algorithm exists for computation of a decycling set of cardinality $2\phi(G)$ in an arbitrary graph G.

Application of the computational method of differential equations is classical and can be found everywhere, in- and outside mathematics. In this note, an application of differential equation method to a simple algorithm over the probability space of random regular graphs will be illustrated. This method was developed by N.C. Wormald [22, 23, 24]. and has been producing sharp asymptotic results which are otherwise difficult to obtain. While determining the decycling number of regular graphs for degree $d \geq 4$ appeared to be difficult, good deterministic interpolation results have been obtained recently by N. Punnim for regular graphs using a simple switching technique. These results will also be reviewed in this paper.

2 Cubic Graphs

Let $S \subset V(G)$ be a decycling set of G with $|S| = \phi(G)$. Denote by $\Delta(G)$ the maximum degree of G and by $\omega(G)$ the number of components of G. Then since $G - S$ is acyclic,

$$\|G - S\| = |G - S| - \omega(G - S) \ \text{ and } \ \omega(G - S) \geq 1.$$

Hence

$$|S| + \omega(G - S) + \|G|_S\| \geq \phi + 1.$$

Then by counting the edges in different classes,

$$\begin{aligned}
\Delta\phi \ &\geq |[S, G - S]| + 2\,\|G|_S\| \\
&= \|G\| - \|G - S\| - \|G|_s\| + 2\|G|_S\| \\
&= \|G\| - \|G - S\| + \omega(G - S) + \|G|_S\| \\
&= \|G\| - |G| + |S| + \omega(G - S) + \|G|_S\| \\
&\geq \|G\| - |G| + 1 + \phi.
\end{aligned}$$

Hence

$$\phi(G) \geq \frac{\|G\| - |G| + 1}{\Delta - 1}.$$

For a graph G regular of degree d, it is natural to ask whether there is a constant c such that

$$\phi(G) \leq \frac{\|G\| - |G| + 1}{d - 1} + c.$$

This is not the case. Let H be a cubic graph and let $|H| = 2n$. Replace each vertex of H with a triangle and denote the resulting graph by G. Then $|G| = 6n$ and $\phi(G) \geq 2n$. Hence

$$\phi(G) - \frac{\|G\| - |G| + 1}{2} \geq 2n - \frac{3n + 1}{2} \geq \frac{n}{2}.$$

This motivated the following problems in [3].

Problem 2.1 Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$?

Problem 2.2 Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$?

It seems that there is an algorithm for determing the graphs asked by Problem 2.1, as shown in [18] using a switching technique and explicit constructions. This will be reviewed in Section 5.

Bounds on the decycling number of cubic graphs have been investigated for some time. For a connected cubic graph G with girth $g(G) = g$, Speckenmeyer proved the following theorem in [20].

Theorem 2.1 *Let G be a connected cubic graph with girth $g(G) = g$. Then*

$$\phi(G) \leq \frac{g+1}{4g-2}|G| + \frac{g-1}{2g-1}.$$

In [25], Zheng and Lu sharpened this for connected cubic graphs of order at least 8 without triangles.

Theorem 2.2 *Let G be a connected cubic graph with $|G| \geq 8$ and without triangles. Then*

$$\phi(G) \leq \left\lceil \frac{|G|}{3} \right\rceil.$$

This settles a conjecture by Bondy, Hopkins and Staton in [6] in the affirmative.

A sharp upper bound for the decycling number of cubic graphs has been obtained in [16] by Liu and Zhao and that for connected graphs with maximum degree 3 has been obtained in [1]. Let $\mathcal{G}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of K_4 with one subdivided edge at every vertex of degree 1.

Theorem 2.3 *Let G be a cubic graph with $g(G) = g$. Then*

$$\phi(G) \leq \frac{g}{4(g-1)}|G| + \frac{g-3}{2g-2}$$

if $G \notin \{K_4, Q_3, W\} \cup \mathcal{G}$ where Q_3 is the 3-cube and W is the Wagner's graph. If $G \in \mathcal{G}$, then

$$\phi(G) = \frac{3}{8}|G| + \frac{1}{4}.$$

Corollary 2.1 *If $g(G) \geq 3$ then $\phi(G) \leq \frac{3}{8}|G|$ for $G \notin \{K_4\} \cup \mathcal{G}$. If G is a connected cubic graph with $g(G) \geq 4$ and $G \neq Q_3$ or W, then $\phi(G) \leq \frac{|G|}{3}$.*

Theorem 2.4 *Let G be a connected graph of maximum degree 3. If $G \neq K_4$ then*

$$\phi(G) \leq \left\lfloor \frac{\|G\|+1}{4} \right\rfloor.$$

The family $\mathcal{G}$ of graphs show the sharpness of this result.

A polynomial time algorithm to decide the decycling number of any cubic graph has been found by Li and Liu in [13].

3 Random Regular Graphs

The computational method of differential equations may be applied when one deals with random variables. This has been applied frequently in classical continuous processes. For the probability space of random regular graphs, an algorithm gives rise to discrete random variables. This makes it possible for applications of the differential equation method.

The method may be summarized as follows. For a random process, some random variables arise from execution of an algorithm, whose behavior may be modelled by differential equations based on the mathematical expectations of their changes over a unit increment of time. Solutions (usually numerical) of these equations provide asymtotic estimates for the variables under a main theorem concering existence and uniqueness of solutions for the differential equations, which is established by a second moment or large deviation inequality.

A *random graph process* is a sequence

$$G_0, G_1, \cdots, G_n, \cdots$$

so that $V(G_0) = V(G_1) = \cdots = V(G_k) = \cdots$ and $G_{i+1} = G_i \cup A_i$ where A_i is a set of edges that is chosen according to a stochastic rule. If for all $i \geq t$, $G_{i+1} = G_i$ then G_t is called the *final graph* (or the *terminal graph*) of the process. The smallest such t is called the *stopping time*. After the stopping time $A_t = \emptyset$.

A sequence $\{A_n\}$ of events $A_n \in \Omega$ *occurs asymptotically almost surely* or *a.a.s.* for brevity, if

$$\lim_{n\to\infty} \mathbf{P}\{A_n\} = 1.$$

Probability and expectation are denoted by $\mathbf{P}$ and $\mathbf{E}$ respectively.

The *uniform* model for random regular graphs (indeed for graphs) is classic. In the uniform model, each graph in the space has equal probability for being sampled– they occur with uniform probability.

There is another model, the *pairing* model. Assume that $dn \equiv 0 \pmod 2$. A set of dn points is partitioned into n vertices containing d points each. A partition of these dn points into $dn/2$ pairs is called a *pairing*. There is a bijection between the set of all such pairings and the set of all d-regular multigraphs of order n. Denote by $\mathcal{G}_{n,d}$ the uniform space of these pairings, and by $\mathcal{H}_n$ the uniform space of random hamiltonian cycles on the same vertex set as $\mathcal{G}_{n,d}$.

Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be two discrete probability spaces over the same underlying set for each $n \geq 1$. Sequences of spaces $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ are said to be *contiguous* if any sequence of events A_n $(n \geq 1)$ occurs a.a.s. in $\{\mathcal{P}_n\}$ if and only if it occurs a.a.s. in $\{\mathcal{Q}_n\}$. For simplicity, we also say that the spaces $\mathcal{P}_n$ and $\mathcal{Q}_n$ are contiguous and write $\mathcal{P}_n \approx \mathcal{Q}_n$.

For two probability spaces $\mathcal{P}$ and $\mathcal{Q}$ of random graphs on the same vertex set, the *sum* $\mathcal{P} + \mathcal{Q}$ is the space whose elements are determined by the random multigraph (called *superposition* of G and H) whose edge set is the edges of G together with the edges of H, where $G \in \mathcal{P}$ and $H \in \mathcal{Q}$ are generated independently. Define the *graph-restricted sum* $\mathcal{P} \oplus \mathcal{Q}$ to be the space which is the restriction of $\mathcal{P} + \mathcal{Q}$ to simple graphs. The operations $+$ and $\oplus$ are commutative and associative.

We now cite an important and deep theorem of Robinson and Wormald (see [22, 23, 24] for details).

Theorem 3.1 *Let $d \geq 3$ and n be even. Then*

$$\mathcal{G}_{n,d} \approx \mathcal{H}_n \oplus (d-2)\mathcal{G}_{n,1}.$$

This theorem has been applied in obtaining an asymtotic value for the decycling number of random cubic graphs in [4].

Theorem 3.2 *Let G be a random cubic graph. Then*

$$\lim_{|G|\to\infty} \mathbf{P}\left\{\phi(G) = \left\lceil \frac{|G|}{4} + \frac{1}{2} \right\rceil\right\} = 1.$$

Also in [4], the pairing model has been used in running a greedy algorithm for random regular graphs of degree $d \geq 4$ to obtain fairly close probabilistic asymtotic bounds for the decycling number.

Table 1: Lower and upper bounds.

d	$b(d)$	$B(d)$
4	1/3	0.3787
5	0.3786	0.4512
6	0.4232	0.5043
7	0.4610	0.5459
8	0.4932	0.5800
9	0.5210	0.6085
10	0.5453	0.6328

Theorem 3.3 *Let $d \geq 4$. For a random d-regular graph G,*

$$\lim_{|G|\to\infty} \mathbf{P}\left\{b(d) \leq \frac{\phi(G)}{|G|} \leq B(d)\right\} = 1$$

where $b(d)$ and $B(d)$ are constants given in Table 1.

The proof of Theorem 3.3 is by an algorithm using the strategy of *deferred decisions* over the space of random cubic graphs that is contiguous with the probability space of direct sum of a random hamiltonian cycle and a random 1-factor (perfect matching) by Theorem 3.1. The algorithm is executed simultaneously with generating the perfect matching M. The main idea of this algorithm is as follows. When a vertex i is being processed by the algorithm, the *direction* of the matching edge incident with i is first revealed by generating this edge at random with correct probability. The direction of an edge of the random matching is *backward* if the other end of the edge is a vertex that is already processed, and *forward* otherwise. Only if it is an edge whose direction is backward, the current vertex i is then chosen for membership in a decycling set S; otherwise the next vertex along the random hamiltonian cycle is processed. Note that if an edge of the random matching is backward, then the other end of this edge is not a vertex that is already decided membership in S.

Suppose that the random hamiltonian cycle is

$$H = (1, 2, \cdots, n, 1).$$

Begin from vertex 1. When the algorithm processes vertex i, the probability that the edge incident with i that is in the random matching is backward is equal to the number of unmatched vertices in $[1, \cdots, i-1]$ divided by $n-i+1$. In this case, the vertex that matches with i is chosen uniformly at random from those available and add this edge to M_{i-1} to obtain M_i. If adding this edge and the edge $\{i-1, i\}$ (and $\{n, 1\}$ if $i = n$) to G_{i-1} creates no cycle, then let G_i be the graph obtained this way and set $S_i \leftarrow S_{i-1}$; otherwise set $G_i \leftarrow G_{i-1}$ and $S_i \leftarrow S_{i-1} \cup \{i\}$. In the case that the edge incident with i that is in the random matching is forward, leave the vertex i unmatched and set $M_i \leftarrow M_{i-1}$, $S_i \leftarrow S_{i-1}$, and $G_i \leftarrow G_{i-1} \cup \{i-1, i\}$. Increment i and repeat this until $i = n$. For $i = n$ return $S := S_n$ and $M := M_n$.

It is clear that $G_i = G|_{[1,\cdots,i]\backslash S_i}$. Equivalently, G_i is the subgraph of H induced by $[1, \cdots, i]\backslash S_i$ and the matching edges incident with these vertices. It is also clear from the algorithm that G_i is acyclic, and in particular, G_n is acyclic. Hence S_n is a decycling set of G. It is also easy to see that the only possible edge in $G|_{S_n}$ is the edge $\{n-1, n\}$. Otherwise, $S = S_n$ is an independent set. This means that

$$|[S, G - S]| + \|G|_S\| \geq 3|S| - 1.$$

The main part of the proof of Theorem 3.2 is to show that the subgraph G_n of G is a.a.s. connected (i.e., a tree). This was effected by way of two intermediate results:

(1) *For any fixed constant* $K \geq 3$ *and* $j = \lceil n^{1/3} \rceil$, *the number of vertices in the interval* $[1, \cdots, n-j]$ *that are in* S_i *and the latest component of* G_{i-1} *has at most* K *unmatched vertices is a.a.s.* $O(\log^3 n)$;

(2) *A.a.s., no vertex in the interval* $[n-j+1, \cdots, n]$ *is matched with another vertex in this interval.*

Once it is established that G_n is a.a.s. a tree, the total number of edges of G may be counted

$$\frac{3n}{2} \geq (3|S| - 1) + (n - |S| - 1)$$

and hence

$$|S| \leq \frac{n}{4} + 1$$

and therefore

$$\phi(G) \leq \frac{n}{4} + 1.$$

But as seen in the beginning paragraphs of Section 2,

$$\phi(G) \geq \left\lceil \frac{n}{4} + \frac{1}{2} \right\rceil .$$

Hence Theorem follows since $\lceil n/4 + 1/2 \rceil = \lceil n/4 + 1 \rceil$ for each even integer n.

4 Differential Equations

The idea of proof of Theorem 3.3 will be discussed in this section. Of course, Theorem 3.1 may be applied again, just as in the proof of Theorem 3.2, with a slight modification of the algorithm. Let $H = (1, 2, \cdots, n, 1)$ be a random hamilton cycle. Beginning at 1, the algorithm walks along H, the vertex currently visited is given membership in the decycling set if (1) at least two edges in the matchings are backward, or (2) only one edge in the matchings is backward but it creates a cycle with vertices so far placed in the growing forest, or (3) the vertex before the current vertex which had at least one backward matching edge was not placed in the decycling set.

The standard differential equation method [22] is applied to yield a probabilistic asymtotic upper bound for $\phi(G)/n$. The upper bound provided by this method is weaker than one obtained by using a natural algorithm on the pairing model, idea of which will be reviewed in this section.

This algorithm returns a decycling set and an induced forest (usually a tree) simultaneously with generating a random pairing uniformly. Simultaneous generation of random structures and execution of algorithms has been used many times [21]. Two points that are being paired are called *mates*. The method of deferred decision is again used: when a vertex is placed in the decycling set, points that pairs back to the growing tree are first determined, and their mates are chosen with correct probability, and the mates of the other points deferred (left undetermined). The particular property that the random pairing has a.a.s. happens to hold for the uniformly random d-regular graphs as well.

The algorithm executes as follows.

Recall that in the pairing model, vertices of a random d-regular graph are sets containing d points each. Let S_t and T_t denote the decycling set and the growing forest respectively at time t. Let U_t be the set of unpaired points in the vertices of the growing forest (tree) T_t. Choose a point $x \in U_t$ uniformly at random. Select a mate y of x from the points in the vertices not in $S_t \cup T_t$. The vertices in $V(G) \backslash (S_t \cup T_t)$ are said to be *untreated* and points in these vertices are said to be *untreated* points. Let $u = \{y, z_1, \cdots, z_{d-1}\}$ be the vertex containing y. For each of z_i decide whether the mate of z_i is in U_t. (This is to be done with correct probability, given that the pairing is uniform subject to all of U_t being paired with untreated points. See below for the estimate of this probability.) If no mate of z_i is in U_t, then set $S_{t+1} \leftarrow S_t$ and set T_{t+1} be the forest obtained from T_t by adding u together with the edge joining u

and the vertex containing x. Else, set $S_{t+1} \leftarrow S_t \cup \{u\}$, $T_{t+1} \leftarrow T_t$ and select mates for those z_i with mates in U_t as determined above. These are selected uniformly at random from $U_t \backslash \{x\}$. In the second case, the mates of those points which are untreated are left undetermined. All new pairs that are determined are placed in the pairing P. If $t+1 = dn/2$ then stop and return $S := S_{dn/2}$, $T := T_{dn/2}$ and the pairing P. Otherwise, increment t and repeat.

Let $X(t)$ denote the number of untreated vertices at time t. Hence the number of untreated points in the untreated vertices is $X(t)d$. Let $Z(t) = |T_t|$ and $Y(t) = |U_t|$ (the number of unpaired points in untreated vertices). Let u be the vertex currently being treated. In a general step of the algorithm, i.e., $X(t) > \log n$, the probability that a point in u is paired with an upaired point of T_t is asymptotically

$$\frac{Y(t)}{X(t)d}.$$

This is because the pairing is uniform subject to all $Y(t)$ unpaired points in T_t being paired with the $X(t)d$ points in the untreated vertices. During the processing of the unpaired points in u, the change in this probability is $o(1)$. Hence the probability that u is to be placed in the growing forest T_t is

$$P(t)^{d-1} + o(1) \text{ where } P(t) = 1 - \frac{Y(t)}{X(t)d}. \tag{1}$$

Consequently,

$$\mathbf{E}\left[Z(t+1) - Z(t)\right] = P(t)^{d-1} + o(1).$$

For a suitably small $\epsilon > 0$, three intervals for t are dealt with separately (τ_0 to be determined later): (1) $t \leq \epsilon n$, (2) $\epsilon n < t < (\tau_0 - \epsilon)n$, and (3) $t \geq (\tau_0 - \epsilon)n$. Special consideration has to be given to the end intervals. The first inerval has to be examined separately and carefully. The last interval contains a negligible number of steps and it may be made sure that it does not alter the general analysis. The analysis of the middle interval (the main interval) is to be reviewed here.

After an analysis of the first interval, it may be assumed that $Y(t) > 0$ for $t = \lfloor \epsilon n \rfloor$. Assume that $Y(t) > 0$ at each step in the second interval (see [4]). In the middle interval, asymptotically T_t is a tree. In the vertices of T_t there are $Z(t)d$ points, $2Z(t) + O(1)$ are used by $E(T_t)$ and $Y(t)$ are unpaired. Hence $(d-2)Z(t) - Y(t)$ have been paired with vertices in S_t. Thus S_t has

$$\begin{aligned} W(t) &= d\left[n - Z(t) - X(t)\right] - \left[(d-2)Z(t) - Y(t)\right] \\ &= d\left[n - X(t) - 2Z(t)\right] + Y(t) + 2Z(t) \end{aligned}$$

unpaired points.

At the moment the algorithm upgrades the pairing P, points x counted by $Y(t)$ is used up. For each of the $d-1$ points in u other than y, the probability that its mate is a point counted by $Y(t)$ is $1 - P(t)$. Hence the expected number of such points in u paired to T_t is $(d-1)[1 - P(t)]$.

Event	Probability	Value of Y
$u+ \to S_t$	$1 - P(t)^{d-1}$	$-1 - \dfrac{(d-1)[1-P(t)]}{1-P(t)^{d-1}}$
$u+ \to T_t$	$P(t)^{d-1}$	$-1 + \dfrac{(d-1)[d \cdot X(t) - Y(t)]}{d \cdot X(t) - Y(t) + W(t)}$

Therefore,

$$\begin{aligned} \mathbf{E}\left[Y(t+1) - Y(t)\right] &= (d-1) \cdot \tfrac{d \cdot X(t) - Y(t)}{d \cdot X(t) - Y(t) + W(t)} \cdot P(t)^{d-1} \\ &\quad -(d-1)\left[1 - P(t)\right] - 1. \end{aligned}$$

It is clear that

$$\mathbf{E}\left[X(t+1) - X(t)\right] = -1.$$

Let

$$\tau = \frac{t}{n},\ w(\tau) = \frac{W(t)}{n},\ x(\tau) = \frac{X(t)}{n},\ y(\tau) = \frac{Y(t)}{n},\ z(t) = \frac{Z(t)}{n}.$$

Then

$$p(\tau) = 1 - \frac{y(\tau)}{d \cdot x(\tau)}$$

and

$$w(\tau) = d \cdot [1 - x(\tau) - 2z(\tau)] + y(\tau) + 2z(\tau).$$

Now we have the following system of differential equations

$$\begin{cases} \dfrac{dx}{d\tau} = -1 \\ \dfrac{dy}{d\tau} = (d-1)p^{d-1} \cdot \dfrac{xd - y}{xd - y + w} - (d-1)(1-p) - 1 \\ \dfrac{dz}{d\tau} = p^{d-1}. \end{cases} \tag{2}$$

This becomes an initial value problem with the natural initial conditions:

$$x(0) = 1,\ y(0) = z(0) = 0.$$

This initial value problem has been solved numerically, from which τ_0 is determined to be the first positive τ satisfying $x = 0$, $xd - y + w = 0$.

For all $\epsilon > 0$, $z(\tau_0) - \epsilon$ is a.a.s. a lower bound on the proportion of vertices in the induced forest T (tree in general). Hence a.a.s.,

$$\frac{\phi(G)}{|G|} \leq 1 - z(\tau_0) + \epsilon.$$

This is so because $Z(t) = |T_t|$.

Note that from this algorithm it follows that there is a.a.s. an induced tree of the same asymptotic size as the induced forest found. Note also that no analytic solution or qualitative behavior other than the trend of $z(\tau)$ has been attempted. No general formula for $B(d)$ for all d is given. These may furnish further research in this direction.

A comparison of numerical results produced by the contiguity model and the pairing model was also performed in the work leading to [4].

The lower bound $b(d)$ is obtained by solving an algebraic equation numerically, which provides an estimate of the expected number of trees and forests.

This is a brief sketch of the proof of Theorem 3.3.

The following problem, suggested by Table 1 is open.

Problem 4.1 For $G \in \mathcal{G}_{n,4}$,

$$\lim_{|G|\to\infty} \mathbf{P}\left\{\phi(G) = \left\lceil \frac{|G|}{3} + \frac{1}{3} \right\rceil\right\} = 1?$$

5 Regular Graphs

The result of Theorem is a probabilistic asymptotic result which furnishes a good insight and does not furnish a deterministic solution to Problem 2.1. Existence of an algorithm for deciding which graphs are in the class for which the decycling number is minimum is indicated in [18]. This will be reviewed in this section.

A simple switching technique that was used in [9, 10] has been applied in [18] to prove that the values of $\phi(G)$ cover a segment $[a, b]$ of positive integers, where a and b are defined as follows. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs with degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$ two parameters a and b may be defined, by lettting

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}.$$

For the degree sequence of a d-regular graph of order n

$$\mathbf{d} = d^n := (\underbrace{d, d, \cdots, d}_{n}).$$

A simple and useful switching technique was employed in the proof of the necessity part of the well known theorem of Haval and Hakimi [9, 10].

Theorem 5.1 *Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ be a non increasing sequence of non negative integers and let*

$$\mathbf{d}' = (d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n).$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$*, rearranged in non increasing order, is graphic.*

Let G be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Then the *switching* operation $\sigma(a, b; c, d)$ on G is defined to be

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) \cup \{ac, bd\}.$$

Note that the degree sequence of $G^{\sigma(a,b;c,d)}$ is the same as that of G. The following was also established in [9, 10]. A graph G with degree sequence $\mathbf{d}$ is called a *realization* of $\mathbf{d}$.

Theorem 5.2 *Let* $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ *be a graphic sequence. If* G *and* H *are any two realizations of* $\mathbf{d}$*, then there is a finite sequence*

$$G = G_0, G_1, \cdots, G_s = H$$

such that for each $i \in \{1, \cdots, s\}$*, there is a switching* σ_i *and* $G_i = G_{i-1}^{\sigma_i}$.

Define the *realization* graph $\mathcal{R}(\mathbf{d})$ to be the graph with vertices all realizations of the sequence $\mathbf{d}$, with two vertices adjacent if and only if one may be obtained from the other by a switching. As a consequence of this definition and Theorem 5.2, we have

Theorem 5.3 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

Let $\mathcal{G}$ be the set of all simple graphs and X be a set. A function $f : \mathcal{G} \to X$ is called an *invariant* of graphs if for all $G, H \in \mathcal{G}$, $G \simeq H \Rightarrow f(G) = f(H)$. For example, (1) if X is the set of all integer sequences, then the degree sequence function $f = \mathbf{d}$ is an invariant; (2) if X is the category of all groups then the automorphism group function $f = \mathrm{Aut}$ is an invariant. (3) Each of a and b defined above is an integer invariant for graphs; other integer invariants include the determinant of the adjacency matrix of G.

Let $\mathcal{J} \subseteq \mathcal{G}$, $X = \mathbb{Z}$ and f be an integer valued function defined on $\mathcal{J}$. Then f is called an *interpolation parameter* on $\mathcal{J}$ if there exist $a, b \in \mathbb{Z}$ such that

$$\{f(G) : G \in \mathcal{J}\} = [a, b] := \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

If f is an interpolation parameter, then it is natural to write

$$a := \min(f, \mathcal{J}) = \min\{f(G) : G \in \mathcal{J}\}$$

and

$$b := \max(f, \mathcal{J}) = \max\{f(G) : G \in \mathcal{J}\}.$$

For $\mathcal{J} = \mathcal{R}(\mathbf{d})$, write more simply $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$ respectively.

Now let $S \subset V(G)$ be a decycling set of G with $|S| = \phi(G)$ and let $\sigma = \sigma(a, b; c, d)$ be a switching on G. If $ab, cd \in E(G - S)$, then $(G - S)^{\sigma}$ is defined. Since $G - S$ is a forest, there is at most one path in $G - S$ connecting a and c. If there is a path connecting a and c, then it may be modified to a unique path connecting b and d.

Hence $(G-S)^\sigma$ contains at most one cycle. If one or both of ab, cd are not edges of $G-S$, then $(G-S)^\sigma$ contains at most one cycle. Hence $\phi(G^\sigma) \leq \phi(G)+1$.

Theorem 5.4 *If G is a graph and $\sigma = \sigma(a,b;c,d)$ is a switching on G, then $\phi(G^\sigma) \leq \phi(G)+1$.*

Since the switching operation is symmetric, it may be assumed that $\phi(G) \leq \phi(G^\sigma)$.

Corollary 5.1 *If σ is a switching on G, then $|\phi(G^\sigma) - \phi(G)| \leq 1$.*

From Theorem 5.3 and Corollary 5.1, it was very simple to deduce the following [18]:

Theorem 5.5 *For a given graphic degree sequence $\mathbf{d}$, there exist integers a and b such that there is a graph G with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if $a \leq c \leq b$.*

By this theorem, one may introduce the definition

$$\phi(\mathbf{d}) := \{\phi(G) : G \in \mathcal{R}(\mathbf{d})\} = [a,b] := \{c \in \mathbb{Z} : a \leq c \leq b\}.$$

For a regular graphic sequence $\mathbf{d} = d^n$, a sequence of direct construction has been given in [18] to show the existence of graphs with decycling number a or b.

For each $c \in \phi(\mathbf{d})$, denote by $\mathcal{R}(\mathbf{d};c)$ the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by graphs G with $\phi(G) = c$. The problem of determining the induced subgraph $\mathcal{R}(\mathbf{d};c)$ is open for general $c \in [a,b]$. A particular question is to decide whether $\mathcal{R}(\mathbf{d};c)$ is connected for each $c \in [a,b]$. The following was proved in [18].

Theorem 5.6 *The induced subgraph $\mathcal{R}\left(3^{2n}; \left\lceil \frac{n+1}{2} \right\rceil\right)$ is connected.*

This certainly pointed a direction for an algorithm for determining all cubic graphs attaining the trivial lower bound for the decycling number proposed in the first paragraphs of Section 2. In such an algorithm, one needs to guarantee that the decycling number is held invariant by the switching performed at each step of the algorithm.

Under what condition on the quadruple of vertices on which a switching is performed, the decycling number is invariant under the switching?

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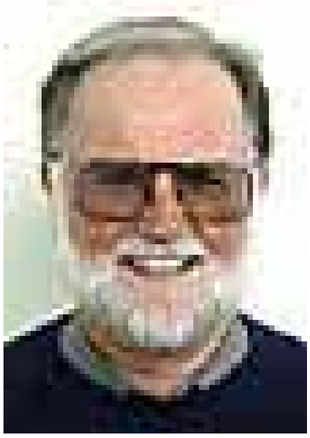

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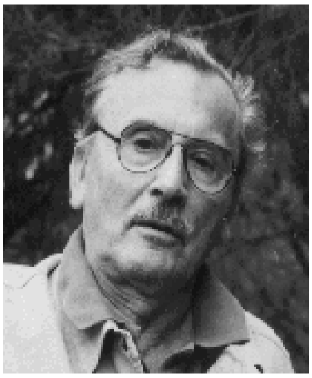

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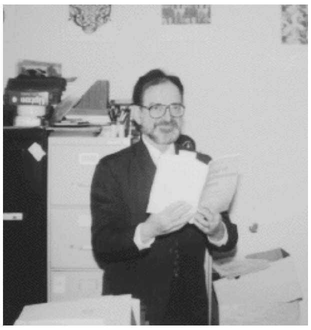

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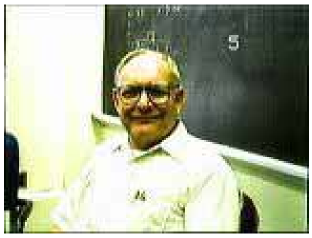

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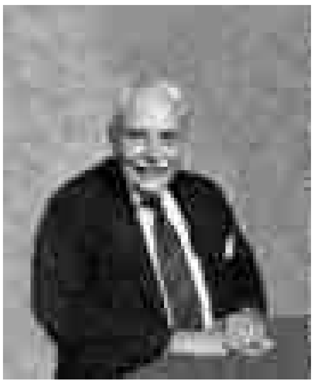

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