BRG4580015 รศ.ดร. วัชรพงษ์ อนันต์ชื่น

สกว
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รายงานวิจัยฉบับสมบูรณ์

การสร้างกราฟและกราฟทิศทาง
ที่สอดคล้องกับสมบัติที่กำหนด

วัชรพงษ์ อนันต์ชื่น

สาขาวิชาศิลปศาสตร์
มหาวิทยาลัยสุโขทัยธรรมาธิราช
นนทบุรี

สิงหาคม พ.ศ. 2547

สัญญาเลขที่ BRG/15/2545

รายงานวิจัยฉบับสมบูรณ์

การสร้างกราฟและกราฟทิศทางที่สอดคล้องกับสมบัติที่กำหนด

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สนับสนุนโดย สำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ Prof. Dr. Louis Caccetta, Department of Mathematics and Statistics, Curtin University of Technology, Perth, Western Australia ที่ได้อนุเคราะห์ให้ข้อเสนอแนะและความคิดเห็นที่เป็นประโยชน์อย่างยิ่งต่อการพัฒนาและปรับปรุงงานวิจัยชิ้นนี้

งานวิจัยครั้งนี้สำเร็จลุล่วงได้ด้วยดี โดยการสนับสนุนของ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ภายใต้โครงการทุนวิจัยองค์ความรู้ใหม่ที่เป็นพื้นฐานต่อการพัฒนา รหัส BRG/15/2545 ผู้วิจัยขอขอบพระคุณเป็นอย่างสูงมา ณ โอกาสนี้

บทคัดย่อ

การสร้างกราฟและกราฟทิศทางที่สอดคล้องกับสมบัติที่กำหนด

ให้ m และ n เป็นจำนวนเต็มบวกหรือศูนย์ และ k เป็นจำนวนเต็มบวกใด ๆ เรากล่าวว่ากราฟ G มีสมบัติ P(m, n, k) ก็ต่อเมื่อ สำหรับทุก ๆ สับเซต A และ B ที่เป็นเซตต่างสมาชิกกันของจุดของ G โดยที่ |A| = m และ |B| = n จะมีอีกอย่างน้อย k จุด ซึ่งแต่ละจุดต่างประชิดกับจุดทุกจุดใน A แต่ไม่ประชิดกับจุดใด ๆ ใน B เลย ยิ่งไปกว่านั้น เรากล่าวว่ากราฟ G มีสมบัติ n-existentially closed หรือกล่าวว่าเป็นกราฟ n-e.c. ก็ต่อเมื่อ สำหรับทุก ๆ สับเซต A และ B ของจุดของ G ซึ่ง $A \cap B = \varnothing$ และ $|A \cup B| = n$ จะมีจุด $u \notin A \cup B$ ซึ่งประชิดกับจุดทุกจุดใน A แต่ไม่ประชิดกับจุดใด ๆ ใน B เลย เป็นที่ทราบกันดีว่ากราฟส่วนใหญ่มีสมบัติ P(m, n, k) และ n-e.c. แต่อย่างไรก็ตามปัญหาการสร้างกราฟที่มีสมบัติ P(m, n, k) และ n-e.c. เป็นปัญหาที่ค่อนข้างยาก ในงานวิจัยนี้เราจะแสดงว่านัยทั่วไปของกราฟพาเลย์ที่สร้างโดยการใช้ส่วนตกค้างกำลังสูงกว่าบนสนามจำกัด ที่มีจำนวนจุดมากพอมีสมบัติ P(m, n, k) และ n-e.c.

ทฤษฎีบทที่คล้ายกันสำหรับนัยทั่วไปของกราฟทิศทางพาเลย์ก็ได้รับการนำเสนอ กล่าวคือกราฟทิศทาง D มีสมบัติ n-e.c. ก็ต่อเมื่อ สำหรับทุก ๆ สับเซต A และ B ของจุดของ D ซึ่ง $A \cap B = \varnothing$ และ $|A \cup B| = n$ จะมีจุด $u \notin A \cup B$ ซึ่งครอบครองจุดทุกจุดใน A และถูกครอบครองด้วยจุดทุกจุดใน B ในงานวิจัยนี้เราจะแสดงว่านัยทั่วไปของกราฟทิศทางพาเลย์ที่สร้างโดยการใช้ส่วนตกค้างกำลังสูงกว่าบนสนามจำกัด ที่มีจำนวนจุดมากพอมีสมบัติ n-e.c.

Keywords: adjacency property, *n*-e.e. property, Paley graph, Paley digraph

2000 Mathematics Subject Classification: 05C75; 05C20

Abstract

On constructing graphs and digraphs with prescribed properties

Let m and n be non-negative integers and k a positive integer. A graph G is said to have property $P(m, n, k)$ if for any disjoint subsets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. Furthermore, a graph G is called *n-existentially closed* or n-e.c. if for any two subsets A and B of vertices of G with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. It is well-known that almost all graphs satisfy the $P(m, n, k)$ property and the n-e.c. property. However, the problem of constructing graphs with the $P(m, n, k)$ property and the n-e.c. property seems difficult. In this report, we show that all sufficiently large generalized Paley graphs defined by using higher order residues on finite fields satisfy the $P(m, n, k)$ property and the n-e.c. property.

Similar results for generalized Paley digraphs are also obtained. More specifically, a digraph D is *n-e.c.* if for any two subsets A and B of vertices of D with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B. In this report, we show that the all sufficiently large generalized Paley digraphs defined by using higher order residues on finite fields are n-e.c.

Keywords: adjacency property, n-e.e. property, Paley graph, Paley digraph

2000 Mathematics Subject Classification: 05C75; 05C20

สารบัญ

หน้าสรุปโครงการ (Executive Summary)

ชื่อโครงการวิจัย การสร้างกราฟและกราฟทิศทางที่สอดคล้องกับสมบัติที่กำหนด
On constructing graphs and digraphs with prescribed properties.

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สาขาวิชาที่ทำวิจัย Adjacency Properties, n-e.c. property, ทฤษฎีกราฟ (สาขาคณิตศาสตร์)

ระยะเวลาดำเนินการ 2 ปี (15 สิงหาคม พ.ศ. 2545 ถึง 14 สิงหาคม พ.ศ. 2547)

ปัญหาที่ทำวิจัย และความสำคัญของปัญหา

การวิจัยนี้เป็นการวิจัยเชิงทฤษฎี เพื่อสร้างองค์ความรู้ใหม่ โดยจะแบ่งงานวิจัยออกเป็นสองส่วน คือ ส่วนที่เกี่ยวกับกราฟอย่างง่าย ซึ่งต่อไปจะเรียกสั้น ๆ ว่า กราฟ (graph) และส่วนที่เกี่ยวกราฟทิศทาง (digraph)

เรากล่าวว่า กราฟ G มีสมบัติ P(m, n, k) ก็ต่อเมื่อ สำหรับทุก ๆ เซต A และ B ที่เป็นเซตต่างสมาชิกกัน (disjoin set) ของจุดของ G ซึ่ง $|A| = m$ และ $|B| = n$ จะมีอีกอย่างน้อย k จุด ที่แต่ละจุด

ต่างประชิดกับจุดทุกจุดใน A แต่ไม่ประชิดกับจุดใด ๆ ใน B เลย กลุ่มของกราฟที่มีสมบัติ P(m, n, k) จะเขียนแทนด้วย $\mathcal{G}$(m, n, k)

สำหรับกรณีที่ m, n $\geq$ 2 ปัญหาการสร้างกลุ่มของกราฟที่มีสมบัติ P(m, n, k) เป็นปัญหาที่ค่อนข้างยาก นับจนถึงปัจจุบันนี้กลุ่มของกราฟที่มีสมบัติดังกล่าวที่เรารู้จักมีเพียงกลุ่มเดียวคือ กลุ่มของกราฟที่สร้างโดยใช้วิธีการของพาเลย์ (Paley construction)

ยิ่งไปกว่านั้น เรากล่าวว่ากราฟ G มีสมบัติ n-existentially closed หรือกล่าวว่าเป็นกราฟ n-e.c. ก็ต่อเมื่อ สำหรับทุก ๆ สับเซต A และ B ของจุดของ G ซึ่ง $A \cap B = \varnothing$ และ $|A \cup B| = n$ จะมีจุด $u \notin A \cup B$ ซึ่งประชิดกับจุดทุกจุดใน A แต่ไม่ประชิดกับจุดใด ๆ ใน B เลย

ในงานวิจัยนี้ เราได้วางนัยทั่วไปของกราฟพาเลย์ โดยใช้ส่วนตกค้าง (residue) กำลังใด ๆ บนสนามจำกัด สิ่งที่เราศึกษาต่อมาคือ เงื่อนไขที่ทำให้การนิยามข้างต้นเป็นการนิยามที่ดี (well-define) และการพิสูจน์ว่ากราฟดังกล่าวที่มีจำนวนจุดมากพอมีสมบัติ P(m, n, k) และ n-e.c. สำหรับจำนวนเต็มบวก m, n และ k ใด ๆ

เรากล่าวว่า กราฟทิศทาง D สอดคล้องกับสมบัติ Q(n, k) ถ้าสำหรับทุก ๆ เซต A ซึ่งเป็นเซตของจุด n จุดของ D จะมีอีกอย่างน้อย k จุด ที่แต่ละจุดครอบครอง (dominate) จุดทุกจุดใน A

เราอาจขยายสมบัติ Q(n, k) ไปเป็น Q(m, n, k) ได้ดังนี้ จะกล่าวว่ากราฟทิศทาง D มีสมบัติ Q(m, n, k) ก็ต่อเมื่อ สำหรับทุก ๆ เซต A และ B ที่เป็นเซตต่างสมาชิกกันของจุดของ D ซึ่ง $|A| = m$ และ $|B| = n$ จะมีอีกอย่างน้อย k จุด ที่แต่ละจุดต่างครอบครองจุดทุกจุดใน A และถูกครอบครองโดยจุดทุกจุดใน B

นอกจากกลุ่มของกราฟทิศทางที่สร้างโดยใช้วิธีการของพาเลย์แล้ว เรายังไม่รู้จักกลุ่มของกราฟทิศทางกลุ่มอื่น ๆ ที่มีสมบัติ Q(n, k) และ Q(m, n, k) อีกเลย

ยิ่งไปกว่านั้น เรากล่าวว่ากราฟทิศทาง D มีสมบัติ n-e.c. ก็ต่อเมื่อ สำหรับทุก ๆ สับเซต A และ B ของจุดของ D ซึ่ง $A \cap B = \varnothing$ และ $|A \cup B| = n$ จะมีจุด $u \notin A \cup B$ ซึ่งครอบครองจุดทุกจุดใน A และถูกครอบครองด้วยจุดทุกจุดใน B

ในงานวิจัยนี้ เราได้วางนัยทั่วไปของกราฟทิศทางพาเลย์ โดยใช้ส่วนตกค้างกำลังใด ๆ บนสนามจำกัด ปัญหาที่เราศึกษาต่อมาคือ เงื่อนไขที่ทำให้การนิยามข้างต้นเป็นการนิยามที่ดี และการพิสูจน์ว่ากราฟทิศทางดังกล่าวที่มีจำนวนจุดมากพอมีสมบัติ Q(n, k), Q(m, n, k) และ n-e.c. สำหรับจำนวนเต็มบวก m, n และ k ใด ๆ

วัตถุประสงค์

1. วางนัยทั่วไปของกราฟพาเลย์ โดยใช้ส่วนตกค้างกำลังใด ๆ บนสนามจำกัด
2. ศึกษาเงื่อนไขที่ทำให้การนิยามข้างต้นเป็นการนิยามที่ดี

3. ศึกษาเงื่อนไขและพิสูจน์ว่ากราฟดังกล่าวมีสมบัติ P(m, n, k) สำหรับจำนวนเต็มบวก m, n และ k ใด ๆ
4. วางนัยทั่วไปของกราฟทิศทางพาเลย์ โดยใช้ส่วนตกค้างกำลังใด ๆ บนสนามจำกัด
5. ศึกษาเงื่อนไขที่ทำให้การนิยามข้างต้นเป็นการนิยามที่ดี
6. ศึกษาเงื่อนไขและพิสูจน์ว่ากราฟทิศทางดังกล่าวมีสมบัติ Q(n, k) และ Q(m, n, k) สำหรับจำนวนเต็มบวก m, n และ k ใด ๆ
7. ศึกษาการมีสมบัติ n-e.c. ของนัยทั่วไปของกราฟพาเลย์และนัยทั่วไปของกราฟทิศทางพาเลย์

<u>หมายเหตุ</u> การศึกษาการมีสมบัติ n-e.c. ของนัยทั่วไปของกราฟพาเลย์และนัยทั่วไปของกราฟทิศทางพาเลย์ เป็นปัญหาและวัตถุประสงค์เพิ่มเติมจากข้อเสนอโครงการวิจัยเดิม และได้ผลลัพธ์ที่น่าพอใจ

ระเบียบวิธีวิจัย

1. รวบรวมเอกสารทางวิชาการที่เกี่ยวข้อง ซึ่งมีทั้งตำราและบทความ แล้วศึกษาหาความรู้จากเอกสารเหล่านั้น
2. คิดค้นวิธีการในการแก้ปัญหา โดยนำความรู้ที่ได้รับมาประยุกต์ใช้ในการสร้างกลุ่มของกราฟและกลุ่มของกราฟทิศทางที่มีสมบัติตามที่ต้องการ และที่สำคัญสำหรับการทำวิจัยในครั้งนี้คือการพิสูจน์ (โดยใช้วิธีการทางคณิตศาสตร์) ว่า กลุ่มของกราฟและกลุ่มของกราฟทิศทางดังกล่าวมีสมบัติตามที่ต้องการ
3. ขอคำปรึกษาจาก Prof. Dr. Louis Caccetta, School of Mathematics and Statistics, Curtin University of Technology ประเทศออสเตรเลีย ซึ่งเป็นผู้เชี่ยวชาญทางด้านนี้โดยตรง
4. รวบรวมสิ่งที่ค้นพบ และได้พิสูจน์ นำมาเรียบเรียงเขียนเป็นบทความวิจัย

ผลที่ได้รับ

ผลการวิจัยครั้งนี้ได้ผลลัพธ์ตามวัตถุประสงค์ที่วางไว้ทุกประการ ยิ่งไปกว่านั้นจากการศึกษาการมีสมบัติ n-e.c. ของกราฟที่สร้างจากการวางนัยทั่วไปของกราฟพาเลย์และกราฟทิศทางพาเลย์ โดยใช้ส่วนตกค้างกำลังสูงกว่าบนสนามจำกัด เราได้ค้นพบทฤษฎีบทใหม่ ๆ ที่น่าสนใจ ดังรายงานสรุปผลการวิจัยอย่างย่อต่อไปนี้

1. นิยาม cubic Paley graphs, $P_q^{(3)}$ ดังนี้ จุดของ $P_q^{(3)}$ คือสมาชิกของ สนามจำกัด $\mathbb{F}_q$ จุด a และ b ใด ๆ ประชิดกันก็ต่อเมื่อผลต่างของ a และ b เป็นส่วนตกค้างกำลังสาม (cubic residue) เราพบมาก่อนหน้านี้แล้วว่าการนิยามข้างต้นจะเป็นการนิยามที่ดี เมื่อ $q \equiv 1 \pmod 3$ และได้พิสูจน์ทฤษฎีบทต่อไปนี้

Theorem 1. *Let $q \equiv 1 \pmod 3$ be a prime power. If*

$$q > n^2 3^{3n-2},$$

then $P_q^{(3)}$ has the n-e.c. property. Furthermore, for $n > 1$ the graph $P_q^{(3)}$ is n-e.c. whenever $q > n^2 3^{3n-4}$.

2. นิยาม quadruple Paley graphs, $P_q^{(4)}$ ดังนี้ จุดของ $P_q^{(4)}$ คือสมาชิกของ สนามจำกัด $\mathbb{F}_q$ จุด a และ b ใด ๆ ประชิดกันก็ต่อเมื่อผลต่างของ a และ b เป็นส่วนตกค้างกำลัง สี่ (quadruple residue) เราพบมาก่อนหน้านี้แล้วว่าการนิยามข้างต้นจะเป็นการนิยามที่ดี เมื่อ $q \equiv 1 \pmod 8$ และได้พิสูจน์ทฤษฎีบทต่อไปนี้

Theorem 2. *Let $q \equiv 1 \pmod 8$ be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $P_q^{(4)}$ has the n-e.c. property.

3. นิยาม quadruple Paley digraphs, $D_q^{(4)}$ ดังนี้ จุดของ $D_q^{(4)}$ คือสมาชิกของสนามจำกัด $\mathbb{F}_q$ จุด u ครอบครองจุด v ก็ต่อเมื่อผลต่างของ u และ v เป็นส่วนตกค้างกำลัง สี่ (quadruple residue) เราพบมาก่อนหน้านี้แล้วว่าการนิยามข้างต้นจะเป็นการนิยามที่ดี เมื่อ $q \equiv 5 \pmod 8$ และได้พิสูจน์ทฤษฎีบทต่อไปนี้

Theorem 3. *Let $q \equiv 5 \pmod 8$ be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $D_q^{(4)}$ has n-e.c. property.

4. ได้วางนัยทั่วไปของกราฟพาเลย์ โดยใช้ส่วนตกค้างกำลังสูงกว่าบนสนามจำกัด กล่าวคือ ให้ q และ d เป็นจำนวนเต็มบวก โดยที่ q เป็นจำนวนเฉพาะกำลัง และ

d > 1 เป็นจำนวนคี่ หรือ (q – 1)/d เป็นจำนวนคู่

นิยาม generalized Paley graphs, $P_q^{(d)}$ ดังนี้ จุดของกราฟ $P_q^{(d)}$ คือสมาชิกของสนามจำกัด $\mathbb{F}_q$ จุด a และ b ใด ๆ ของ $G_q^{(d)}$ ประชิดกัน ก็ต่อเมื่อ ผลต่างของ a และ b เป็นส่วนตกค้างกำลัง d ใน $\mathbb{F}_q$

เราพบว่าการนิยามข้างต้นเป็นการนิยามที่ดี และได้พิสูจน์ว่า generalized Paley graphs $G_q^{(d)}$ มีสมบัติ ต่าง ๆ ดังต่อไปนี้

Theorem 4. *Let q and d be positive integers such that q is a prime power and d > 1 is odd or (q − 1)/d is even. If*

$$q > (t2^{t-1} - 2^t + 1)(d-1)^m \sqrt{q} + [m + (d-1)n + (k-1)d]\,(d-1)^{-n} d^{t-1},$$

then $P_q^{(d)} \in \mathcal{A}(m, n, k)$ *for all m, n with* $m + n \le t$.

Theorem 5. *Let q and d be positive integers such that q is a prime power and d > 1 is odd or (q − 1)/d is even. If*

$$q > (n2^{2n} - 2^n + 1)(d-1)^n \sqrt{q} + [(d-1)n + (k-1)](d-1)^{-n} d^{2n-1},$$

then $P_q^{(d)}$ *has property P(n, n, k). In particular, for k = 1 the graphs* $P_q^{(d)}$ *has property P(n, n, 1) whenever* $q > n^2 d^{4n}$.

Theorem 6. *Let q and d be positive integers such that q is a prime power and d > 1 is odd or (q − 1)/d is even. If*

$$q > n^2 d^{3n-2},$$

then $P_q^{(d)}$ *has the n-e.c. property.*

5. ได้วางนัยทั่วไปของกราฟทิศทางพาเลย์ โดยใช้ส่วนตกค้างกำลังสูงกว่าบนสนามจำกัด กล่าวคือ

ให้ q และ d เป็นจำนวนเต็มบวก โดยที่ q เป็นจำนวนเฉพาะกำลัง และ

d > 1 เป็นจำนวนคู่ และ (q − 1)/d เป็นจำนวนคี่

นิยาม generalized Paley digraphs, $D_q^{(d)}$ ดังนี้ จุดของกราฟ $D_q^{(d)}$ คือสมาชิกของสนามจำกัด $\mathbb{F}_q$ จุด u ครอบครองจุด b ก็ต่อเมื่อ ผลต่างของ a และ b เป็นส่วนตกค้างกำลัง d ใน $\mathbb{F}_q$

เราพบว่าการนิยามข้างต้นเป็นการนิยามที่ดี และได้พิสูจน์ว่า generalized Paley digraphs $D_q^{(d)}$ มีสมบัติ ต่าง ๆ ดังต่อไปนี้

Theorem 7. *Let q and d be positive integers such that q is a prime power and d > 1 is even and (q − 1)/d is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1},$$

then $D_q^{(d)}$ *has property* $Q(n, k)$. *In particular, for* $k = 1$ *the graphs* $P_q^{(d)}$ *has property* $Q(n, k)$ *whenever* $q > n^2 d^{2n}$.

Theorem 8. *Let q and d be positive integers such that q is a prime power and* $d > 1$ *is even and* $(q - 1)/d$ *is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1},$$

then $D_q^{(d)}$ *has property* $Q(m, n, k)$.

Theorem 9. *Let q and d be positive integers such that q is a prime power and* $d > 1$ *is even and* $(q - 1)/d$ *is odd. If*

$$q > n^2 d^{3n-2},$$

then $D_q^{(d)}$ *has n-e.c. property.*

บทความที่ได้จากงานวิจัย

จากผลงานวิจัย สามารถนำมาเรียบเรียงเขียนเป็นบทความทางวิชาการได้ 2 บทความ

เนื่องจากการวิจัยในครั้งนี้ได้มีการศึกษาการมีสมบัติ n-e.c. ของกราฟและกราฟทิศทางที่สร้างขึ้นมา เพิ่มเติมจากปัญหาและวัตถุประสงค์ที่วางไว้เดิม จึงได้เปลี่ยนชื่อบทความให้เหมาะสมกับเนื้อหาดังนี้

บทความที่ 1 ชื่อ "Cubic and quadruple Paley graphs with the *n*-e.c. property" โดย W. Ananchuen และ L. Caccetta บทความนี้ได้เสนอเพื่อตีพิมพ์ในวารสาร Discrete Mathematics ซึ่งมี impact factor เท่ากับ 0.395

บทความที่ 2 ชื่อ "Adjacency Properties of Generalized Paley Graphs" โดย W. Ananchuen และ L. Caccetta บทความนี้ได้เสนอเพื่อตีพิมพ์ในวารสาร Journal of Graph Theory ซึ่งมี impact factor เท่ากับ 0.377

<u>**หมายเหตุ**</u> ค่าเฉลี่ยของ impact factor ของวารสารที่ตีพิมพ์บทความทางทฤษฎีกราฟ มีค่าประมาณ 0.358

เนื้อหางานวิจัย

Section 1. Introduction

A *graph* G consists of a non-empty set of elements, called *vertices*, and a list of unordered pair of these elements, called *edges*. The set of vertices of the graph G is called *vertex set* of G, and the list of edges is called *edge set* of G. If a and b are vertices of a graph G, then an edge of the form ab or ba is said to *join* a and b. We also say that a and b are *adjacent*. A *loop* is an edge of a graph joining a vertex to itself. Two or more edges joining the same pair of vertices are called *multiple edges*. All graphs considered in this paper are finite, loopless and have no multiple edges. A *complete* graph is one with every pair of vertices adjacent. For the most part, our notation and terminology follows that of Bondy and Murty [10]. Thus G is a graph with vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices and $\varepsilon(G)$ edges. However, we denote the complement of G by $\bar{G}$.

If we think of the edge between two vertices as an order pair, a natural direction from first vertex to the second vertex can be associated with the edge. Such an edge will be called an *arc* (to maintain the historical and terminology), and a graph in which each edge has such a direction will be called a *directed graph* or *digraph*. An orientation of a complete graph is called a *tournament*.

For a fixed integer $n \geq 1$. A graph G is called *n-existentially closed* or *n-e.c.* if for any two subsets A and B of vertices of G with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. Observe that if a graph G has property n-e.c., then $\bar{G}$, the complement of G, also has property n-e.c. It is well-known that almost all graphs are n-e.c. However, the problem of constructing graphs with the n-e.c. property seems difficult, especially for $n \geq 4$.

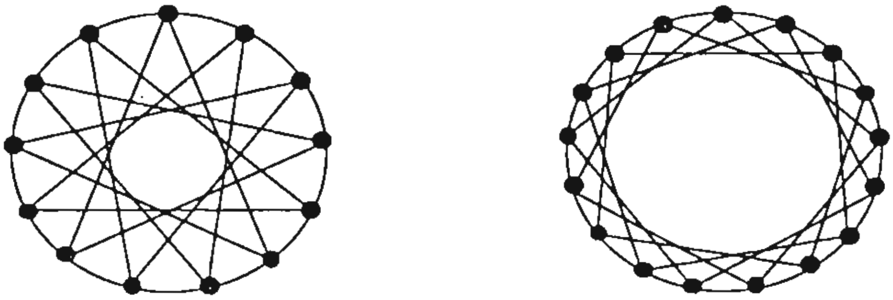
The n-e.c. property was first studied by Caccetta et al. [11], where they were called graphs with property $P(n)$. The authors established, using probabilistic argument, the existence of n-e.c. graphs for a range of n. In particular, they determined the largest integer $f(\nu)$ for which there exists a graph on ν vertices having property $P(f(\nu))$ for a given integer ν. They proved that $\log \nu - (2 + o(1))\log \log \nu < f(\nu)\log 2 < \log \nu$. In addition, a class of 2-e.c. graphs was given for all orders ≥ 9.

Bonato et al. [9] constructed a new class of 3-e.c. graphs, based on Hadamard matrices. They showed that Bush-type Hadamard matrics of order $16m^2$ give rise to strongly regular 3-e.c. graphs, for each odd m for which $4m$ is the order of a Hadamard matrix. By taking certain affine designs to be Hadamard designs obtained from Paley tournaments, Cameron and Stark [12] have use probabilistic methods to showed that many non-isomorphic strongly regular n-e.c. graphs of order $(q + 1)^2$ exist whenever $q \geq 16n^2 2^{2n}$ is a prime power such that $q \equiv 3 (\mathrm{mod}\ 4)$.

An important graph in the study of the n-e.c. property is the so-called *Paley graph* P_q defined as follows. Let $q \equiv 1 (\mathrm{mod}\ 4)$ be a prime power. The vertices of P_q are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbb{F}_q$. The n-e.c. property of Paley graphs have been studied by a number of authors [3, 5, 8]; a good discussion is given in the book of Bollobás [8]. With respect to the n-e.c. property, we proved in [3] that if $q \equiv 1 (\mathrm{mod}\ 4)$ is a prime power with $q > \{(n - 3)2^{n-1} + 2\}\sqrt{q} + \{(n + 1)2^{n-1} - 1\}$, then P_q has the n-e.c. property.

For $q \equiv 1 (\mathrm{mod}\ 3)$ a prime power we define the *cubic Paley graph*, $P_q^{(3)}$ as follows. The vertices of $P_q^{(3)}$ are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^3$ for some $y \in \mathbb{F}_q$. Since $q \equiv 1 (\mathrm{mod}\ 3)$ is a prime power, -1 is a cubic in $\mathbb{F}_q$. The condition -1 is a cubic in $\mathbb{F}_q$ is needed to ensure that ab is defined to be an edge whenever ba is defined to be an edge. Consequently, $P_q^{(3)}$ is well-defined. Figure 1.1(a) gives an example.

For $q \equiv 1 (\mathrm{mod}\ 8)$ a prime power, define the *quadruple Paley graph*, $P_q^{(4)}$ as follows. The vertices of $P_q^{(4)}$ are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^4$ for some $y \in \mathbb{F}_q$. Since $q \equiv 1 (\mathrm{mod}\ 8)$ is a prime power, -1 is a quadruple in $\mathbb{F}_q$. Therefore, $P_q^{(4)}$ is well-defined. Figure 1.1(b) gives an example. The cubic Paley graph and the quadruple Paley graph were first defined in [1].



(a) $P_{13}^{(3)}$ (b) $P_{17}^{(4)}$

Figure 1.1. Graphs $P_{13}^{(3)}$ and $P_{17}^{(4)}$.

Paley constructions have played an important role in constructing classes of graphs with the n-e.c. property, especially for $n \geq 4$, see [3, 8, 12]. In addition to directly providing graphs with interesting adjacency properties, Paley designs played an important role in the construction of strongly regular n-e.c. graphs given in [12]. In the same paper it was noted that the case of affine geometries in place of Paley designs can provide n-e.c. graphs only for $n \leq 3$. In Section 3, we show that the cubic Paley graph $P_q^{(3)}$ has the n-e.c. property whenever $q > n^2 3^{3n-2}$, and the quadruple Paley graph $P_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$.

The concept of n-e.c. property of graphs can be extended to digraphs as follows. If (i, j) is an arc in a digraph D, then we say vertex i *dominates* vertex j. A digraph D is *n-e.c.* if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Let $q \equiv 5 (\mathrm{mod}\ 8)$ be a prime power. Define the *quadruple Paley digraph*, $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbb{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadruple in $\mathbb{F}_q$; that is $a - b = y^4$ for some $y \in \mathbb{F}_q$. The n-e.c. property of Paley digraphs have been studied by [6, 8].

In Section 4, we prove that $D_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$.

We now turn our attention to the property $P(m, n, k)$

Let m and n be non-negative integers and k a positive integer. A graph G is said to have *property* $P(m, n, k)$ if for any disjoint subsets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property $P(m, n, k)$ is denoted by $\mathcal{G}(m, n, k)$. Observe that if a graph G has property $P(m, n, k)$, then $\overline{G}$, the complement of G, has property $P(n, m, k)$. It is well-known that almost all graphs have property $P(m, n, k)$. Despite this result, few graphs have been constructed which exhibit the property $P(m, n, k)$; some constructions for the class $\mathcal{G}(1, n, k)$ were given in [4]. The class $\mathcal{G}(m, n, k)$ has been studied by many authors including: Ananchuen [1]; Ananchuen and Cacetta [3, 5]; Blass et al. [7]; Bollobás [8]; and Exoo[13].

An important graph in the study of the property $P(m, n, k)$ is the so-called *Paley graph* P_q defined as follows. Let $q \equiv 1 (\text{mod } 4)$ be a prime power. The vertices of P_q are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbb{F}_q$. The adjacency properties of Paley graphs have been studied by a number of authors [3, 5, 7, 11]; a good discussion is given in the book of Bollobás [8]. With respect to the property $P(n, n, 1)$ we proved in [5] that if $q \equiv 1 (\text{mod } 4)$ is a prime power with $q > ((2n - 3)2^{2n-1} + 4)^2$, then $P_q \in \mathcal{G}(n, n, 1)$.

By using higher order residues on finite fields we can generate other classes of graphs. More specifically, let q and d be positive integers such that q is a prime power and

$$d > 1 \text{ is odd or } (q - 1)/d \text{ is even.}$$

We define the *generalized Paley graph*, $P_q^{(d)}$ as follows. The vertices of $P_q^{(d)}$ are the elements of finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^d$ for some $y \in \mathbb{F}_q$. Since q is a prime power and $d > 1$ is odd or $(q - 1)/d$ is even, $-1 = y^d$ for some $y \in \mathbb{F}_q$. The condition -1 is a d^{th} power of an element of $\mathbb{F}_q$ is needed to ensure that ba is defined to be an edge precisely whenever ab is defined to be an edge. Consequently, $P_q^{(d)}$ is well-defined. It has been proved that all sufficiently large the cubic and quadruple Paley graphs satisfy the $P(m, n, k)$ property.

In Section 5, we will show that the generalized Paley graphs satisfy the property $P(n, n, 1)$ whenever $q > n^2d^{4n}$.

In section 6, we prove that the generalzied Paley graph has the n-e.c. property whenever $q > n^2d^{3n-2}$.

The concept of adjacency property of graphs can be extended to digraphs as follows. If (i, j) is an arc in a digraph D, then we say vertex i *dominates* vertex j. A digraph D is said to have *property* $Q(n, k)$ if every subset of n vertices of D is dominated by at least k other vertices. Graham and Spencer [14] defined the following digraph. Let $p \equiv 3 \pmod 4$ be a prime. The vertices of digraph D_p are $\{0, 1, \dots, p-1\}$ and D_p contains the arc (a, b) if and only if $a - b$ is a quadratic residue modulo p. The digraph D_p is sometimes referred to as the *Paley tournament*. Graham and Spencer [14] proved that D_p has property $Q(n, 1)$ whenever $p > n^2 2^{2n-2}$. Bollobás [8] extended these results to prime powers. More specifically, if $q \equiv 3 \pmod 4$ is a prime power, the Paley tournament D_q is defined as follows. The vertex set of D_q are the elements of the finite field $\mathbb{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadratic residue in $\mathbb{F}_q$. Bollobás [8] noted that D_q has property $Q(n, 1)$ whenever $q > \{(n-2)2^{n-1} + 1\}\sqrt{q} + n2^{n-1}$. Ananchuen and Caccetta [5] proved that D_q has property $Q(n, k)$ whenever
$q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + k2^{n-1}$.

Let q and d be positive integers such that q is a prime power and

$$d > 1 \text{ is even and } (q-1)/d \text{ is odd.}$$

We define the *generalized Paley digraph*, $D_q^{(d)}$ as follows. The vertices of $D_q^{(d)}$ are the elements of the finite field $\mathbb{F}_q$. A vertex a joins to vertex b by and arc if and only if $a - b = y^d$ for some $y \in \mathbb{F}_q$. Since $d > 1$ is even and $(q-1)/d$ is odd, -1 is not a d^{th} power of any element of $\mathbb{F}_q$. The condition -1 is not a d^{th} power of any element of $\mathbb{F}_q$ is needed to ensure that (b, a) is not defined to be an arc whenever (a, b) is defined to be an arc. Consequently, $D_q^{(d)}$ is well-defined.

In Section 7, we show that the generalized Paley digraph $D_q^{(d)}$ has the property $Q(n, 1)$ whenever $q > n^2d^{2n}$.

In Section 8, we show that the generalized Paley digraph $D_q^{(d)}$ is n-e.c. whenever $q > n^2d^{3n-2}$.

Section 2. Preliminaries

We make use of the following basic notation and terminology. Let $\mathbb{F}_q$ be a finite field of order q where q is a prime power. A *character* χ on $\mathbb{F}_q^*$, the multiplicative group of the non–zero elements of $\mathbb{F}_q$, is a homomorphism from $\mathbb{F}_q^*$ to the multiplicative group of complex number with $|\chi(x)| = 1$ for all x. Among the character of $\mathbb{F}_q^*$, we have the *trivial character* χ_o defined by $\chi_o(x) = 1$ for all $x \in \mathbb{F}_q^*$; all other character of $\mathbb{F}_q^*$ are called *nontrivial*. A character χ is of *order d* if $\chi^d = \chi_o$ and d is the smallest positive integer with this property.

It is customary to extent the definition of character χ to the whole $\mathbb{F}_q$ by putting $\chi(0) = 0$ and $\chi_o(0) = 1$.

Observe that (see[15])

$$\sum_{\substack{\chi \text{ of order dividing } d \\ \chi \neq \chi_o}} \chi(x) = \begin{cases} d-1, & \text{if } x = y^d \text{ for some } y \in \mathbb{F}_q^*, \\ 0, & \text{if } x = 0, \\ -1, & \text{otherwise.} \end{cases} \tag{2.1}$$

This fact is very important in our methodology. Moreover,

$$\chi(a^r) = \chi^r(a) \tag{2.2}$$

for any $a \in \mathbb{F}_q$ and r is a positive integer.

The following lemma, due to Schmidt [15], is very useful to our work.

Lemma 2.1. *Let χ be a nontrivial character of $\mathbb{F}_q$ of order diving $d > 1$. If a polynomial $f(x)$ has precisely s distinct zeros and it is not a d^{th} power, then*

$$\left| \sum_{x \in \mathbb{F}_q} \chi(f(x)) \right| \leq (s-1)\sqrt{q}\,. \qquad \square$$

For g a fixed primitive element of the finite field $\mathbb{F}_q$; that is g is a generator of the cyclic group $\mathbb{F}_q^*$. Defined a function α by

$$\alpha(g^k) = e^{\frac{2k\pi i}{d}},$$

where $i^2 = -1$. Hence, α is a character of order dividing d and the value of α are the elements of the set $\{e^{\frac{2k\pi i}{d}} \mid k = 0, 1, \ldots, d-1\}$. It is not too difficult to verify that $\alpha, \alpha^2, \ldots, \alpha^{d-1}$ are characters of order dividing d and are all difference.

For $d = 3$, α is a cubic character, character of order 3, of $\mathbb{F}_q$. The values of α are the elements of the set $\{1, \omega, \omega^2\}$ where $\omega = e^{\frac{2\pi i}{3}}$. Note that α^2 is also a cubic character. Moreover, if a is not a cubic of an element of $\mathbb{F}_q^*$, then $\alpha(a) + \alpha^2(a) = -1$. This fact is very important in our methodology.

For $d = 4$, α is the quadruple character, character of order 4, of $\mathbb{F}_q$. The values of α are in the set $\{1, -1, i, -i\}$. Observe that α^3 is also a quadruple character while α^2 is a quadratic character. Moreover, if a is not a quadruple of an element of $\mathbb{F}_q^*$, then $\alpha(a) + \alpha^2(a) + \alpha^3(a) = -1$. This fact is very important in our methodology.

The following lemmas were proved in [1].

Lemma 2.2. *Let α be a cubic character of $\mathbb{F}_q$ and let A and B be disjoint subsets of $\mathbb{F}_q$ with $|A \cup B| = n$. Put*

$$g = \sum_{x\in\mathbb{F}_q} \prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b\in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Then

$$g \geq 2^{|B|} q - (n2^{n-1} - 2^n + 1)2^n \sqrt{q}.$$ ❑

Lemma 2.3. *Let β be a quadruple character of $\mathbb{F}_q$ and let A and B be disjoint subsets of $\mathbb{F}_q$ with $|A \cup B| = n$. Put*

$$h = \sum_{x\in\mathbb{F}_q} \prod_{a\in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b\in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Then

$$h \geq 3^{|B|} q - (n2^{n-1} - 2^n + 1)3^n \sqrt{q}.$$ ❑

The following two lammas are extensively used in establishing our results.

Lemma 2.3. *Let α be a character of order d of $\mathbb{F}_q$ and let A and B be disjoint subsets. Put*

$$g = \sum_{x\in \mathbb{F}_q} \prod_{a\in A}\{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$\prod_{b\in B}\{ (d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}.$$

Then

$$g \geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q},$$

where $|A| = m$, $|B| = n$ *and* $m + n = t$.

Proof: Let $A \cup B = \{c_1, c_2, \ldots, c_t\}$. Expanding g and noting that $\sum_{x\in \mathbb{F}_q}(d-1)^n = (d-1)^n q$,

we can write

$$|g-(d-1)^n q| \leq |\sum_{x\in \mathbb{F}_q} \sum_{\chi\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}} \sum_{i=1}^{t} (d-1)^{t-1}\chi(x-c_i)| + |\sum_{x\in \mathbb{F}_q} \sum_{\chi_i\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}} \sum_{i_1<i_2}$$

$$\{(d-1)^{t-2}\chi_1(x-c_{i_1})\chi_2(x-c_{i_2})\}| + \ldots + |\sum_{x\in \mathbb{F}_q} \sum_{\chi_i\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}} \sum_{i_1<i_2<\ldots<i_s}$$

$$\{(d-1)^{t-s}\chi_1(x-c_{i_1})\chi_2(x-c_{i_2})\ldots\chi_s(x-c_{i_s})\}| + \ldots +$$

$$|\sum_{x\in \mathbb{F}_q} \sum_{\chi_i\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}} \{\chi_1(x-c_1)\chi_2(x-c_2)\ldots\chi_t(x-c_t)\}|.$$

Now, by (2.1), (2.2) and Lemma 2.1 we have

$$|g-(d-1)^n q| \leq \sum_{s=1}^{t} (d-1)^s(d-1)^{t-s}\binom{t}{s}(s-1)\sqrt{q}$$

$$= (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q}.$$

Therefore, $g \geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q}$ as required. ❑

Lemma 2.4. *Let α be a character of order d of $\mathbb{F}_q$ and A be a subset of n vertices of $\mathbb{F}_q$. Put*

$$h = \sum_{x \in \mathbb{F}_q} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a) \}.$$

Then

$$h \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q}\ .$$

Proof: Let $A = \{a_1, a_2, \ldots, a_n\}$. We can write

$$h = \sum_{x \in \mathbb{F}_q} 1 + \sum_{\chi \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i=1}^{n} \chi(x - a_i) + \sum_{x \in \mathbb{F}_q} \sum_{\chi \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i_1 < i_2} \chi_1(x - a_{i_1})\chi_2(x - a_{i_2}) + \ldots + \sum_{x \in \mathbb{F}_q} \sum_{\chi \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i_1 < i_2 < \ldots < i_s} \chi_1(x - a_{i_1})\chi_2(x - a_{i_2})\ldots\chi_s(x - a_{i_s}) + \ldots$$

$$+ \sum_{x \in \mathbb{F}_q} \sum_{\chi \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \chi_1(x - a_1)\chi_2(x - a_2)\ldots\chi_n(x - a_n).$$

Now, by (2.1), (2.2) and Lemma 2.1 we have

$$|h - q| \leq \sum_{s=1}^{n} (d-1)^s \binom{n}{s} (s-1)\sqrt{q}$$

$$= [1 + (nd - n - d)d^{n-1}]\sqrt{q}\ .$$

Therefore, $h \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q}$ as required. ❑

Section 3. The cubic and quadruple Paley graphs

For $q \equiv 1 \pmod 3$ a prime power, there exists a cubic character α of $\mathbb{F}_q$ and $\alpha(-a) = \alpha(a)$ for all $a \in \mathbb{F}_q$. Further, for $q \equiv 1 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbb{F}_q$ and $\beta(-a) = \beta(a)$ for all $a \in \mathbb{F}_q$.

Observe that if a and b are any vertices of $P_q^{(3)}$, then for $t = 1$ and 2

$$\alpha^t(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ \omega \text{ or } \omega^2, & \text{otherwise.} \end{cases}$$

Also, if a and b are any vertices of $P_q^{(4)}$, then for $t = 1$ and 3

$$\beta'(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character; that is

$$\beta^2(a-b) = \begin{cases} 1, & \text{if } a-b \text{ is a quadratic ressidue,} \\ 0, & \text{if } a = b, \\ -1, & \text{otherwise.} \end{cases}$$

Our first result concerns cubic Paley graph having property n-e.c. for any fixed integer $n \geq 1$.

Theorem 3.1. *Let $q \equiv 1 (\text{mod } 3)$ be a prime power. If*

$$q > n^2 3^{3n-2},$$

then $P_q^{(3)}$ has the n-e.c. property. Furthermore, for $n > 1$ the graph $P_q^{(3)}$ is n-e.c. whenever $q > n^2 3^{3n-4}$.

Proof: Let A and B be disjoint subsets of $V(P_q^{(3)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}$$

$$> 0.$$

Let

$$g = \sum_{x \in \mathbb{F}_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Now, by Lemma 2.2 we have

$$g \geq 2^{|B|} q - (n2^{n-1} - 2^n + 1) 2^n \sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Since, in the product $\prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\}$ each factor is at most 3 and one factor is 1 and in the product $\prod_{b\in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}$ each factor is at most 3 and one factor is 2 we have

$$\begin{aligned} g - f &\le 3^{n-1}|A| + 3^{n-1}2|B| \\ &= (|A| + 2|B|)3^{n-1} \\ &\le 2n3^{n-1}. \end{aligned}$$

Consequently,

$$f \ge 2^{|B|}q - (n2^{n-1} - 2^n + 1)2^n\sqrt{q} - 2n3^{n-1}.$$

Now, if $q > n^2 3^{3n-2}$, then $f > 0$ as required.

It is easily cheeked that $f > 0$ when $q > n^2 3^{3n-4}$ for $n > 1$. ❑

Remark 3.1. The bound for q in Theorem 3.1 can be improved to $n^2 3^{2.5n}$ for $1 \le n \le 55$.

We now turn our attention to the adjacent property of the quadruple Paley graph $P_q^{(4)}$.

Theorem 3.2 *Let* $q \equiv 1 (\text{mod } 8)$ *be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $P_q^{(4)}$ *has the n-e.c. property.*

Proof: Let A and B be disjoint subsets of $V(P_q^{(4)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x\in F_q \\ x\notin A\cup B}} \prod_{a\in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b\in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

$$> 0.$$

Let

$$h = \sum_{x \in \mathbb{F}_q} \prod_{a \in A} \{ 1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{ 3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Now, by Lemma 2.3, we have

$$h \geq 3^{|B|} q - (n2^{n-1} - 2^n + 1)3^n \sqrt{q}.$$

Consider

$$h - f = \sum_{x \in A \cup B} \prod_{a \in A} \{ 1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{ 3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Since, in the product $\prod_{a \in A} \{ 1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}$ each factor is at most 4 and one factor is 1 and in the product $\prod_{b \in B} \{ 3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$ each factor is at most 4 and one factor is 3 we have

$$\begin{aligned} h - f &\leq |A| 4^{n-1} + 3|B| 4^{n-1} \\ &\leq 3n4^{n-1}. \end{aligned}$$

Consequently,

$$f \geq 3^{|B|} q - (n2^{n-1} - 2^n + 1)3^n \sqrt{q} - 3n4^{n-1}.$$

Now, if $q > n^2 4^{3n-2}$, then $f > 0$ as required. ❑

Remark 3.2. The bound for q in Theorem 3.2 can be improved to $q > n^2 4^{3n-3}$ for $n > 1$ or $n^2 4^{2.5n}$ for $1 \leq n \leq 14$.

Section 4. The Quadruple Paley digraphs

In this section, our graphs are directed. Recalled that, digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B. For $q \equiv 5 \pmod 8$ be a prime power. Define the quadruple Paley digraph $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbb{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadruple in $\mathbb{F}_q$. Since $q \equiv 5 \pmod 8$ is a prime power, -1 is not a quadruple in $\mathbb{F}_q$. The condition -1 is not a quadruple in $\mathbb{F}_q$ is needed to ensure that (b, a) is not defined to be an arc when (a, b) is defined to be an arc. Consequently, $D_q^{(4)}$ is well-defined. However, $D_q^{(4)}$ is not a tournament. Figure 4.1 displays the digraph $D_{13}^{(4)}$. The quadruple Paley digraph was first defined in [2].

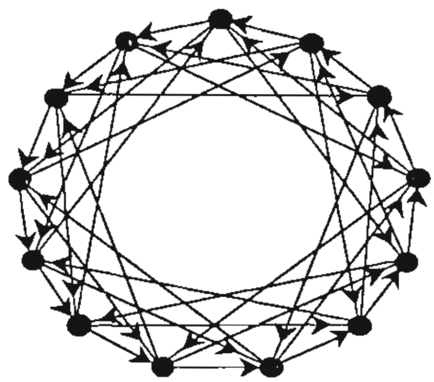


Figure 4.1. Paley digraph $D_{13}^{(4)}$.

For $q \equiv 5 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbb{F}_q$ and noting that if a and b are any vertices of $D_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(a-b) = \begin{cases} 1, & \text{if } a \text{ dominates } b, \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character. Further, $\beta(-a) = -\beta(a)$ for any $a \in \mathbb{F}_q$.

Theorem 4.1. *Let* $q \equiv 5 \pmod 8$ *be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $D_q^{(4)}$ *has* n*-e.c. property.*

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(4)}$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

$$> 0.$$

Now using the method of proof of the Theorem 3.2 we get $f > 0$ when

$$q > n^2 4^{3n-2}$$

Hence, the result. ❑

Remark 4.1. The bound for q in Theorem 4.1 can be improved to $n^2 4^{2.5n}$ for $1 \le n \le 14$.

Section 5. Generalized Paley graphs with the $P(m, n, k)$ property

Note that, for q and d positive integers such that q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a) = \alpha(a)$ for all $a \in \mathbb{F}_q$. Further more, if α is a character of order d of $\mathbb{F}_q$ and a and b are vertices of $P_q^{(d)}$, then

$$\alpha(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ \omega, & \text{otherwise;} \end{cases}$$

where $\omega \in \{ e^{\frac{2k\pi i}{d}} \mid k = 1, \ldots, d-1 \}$.

Our first result for this section concerns the generalized Paley graphs having property $P(m, n, k)$.

Theorem 5.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even. If*

$$q > (t2^{t-1} - 2^t + 1)(d-1)^m \sqrt{q} + [m + (d-1)n + (k-1)d]\,(d-1)^{-n} d^{t-1}, \quad (5.1)$$

then $P_q^{(d)} \in \mathscr{A}(m, n, k)$ for all m, n with $m + n \le t$.

Proof: It clearly suffices to establish the result for $m + n = t$. Let A and B be disjoint subsets of $V(P_q^{(d)})$ with $|A| = m$ and $|B| = n$. Then there are at least k other vertices, each of which adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a) \}$$

$$\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-\ldots-\alpha^{d-1}(x-b)\}$$

$\geq kd^t$.

To show that $f \geq kd^t$, it is clearly sufficient to establish that $f > (k-1)d^t$.

Let g be defined similarly as f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.3 we have

$$g \geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q}\,.$$

Consider

$$g - f = \sum_{x\in A\cup B}\prod_{a\in A}\{1+\alpha(x-a)+\alpha^2(x-a)+\ldots+\alpha^{d-1}(x-a)\}\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-\alpha^{d-1}(x-b)\}$$

$$\leq d^{t-1}m + (d-1)d^{t-1}n$$

$$= [m + (d-1)n]d^{t-1}$$

since, in the product $\prod_{a\in A}\{1+\alpha(x-a)+\alpha^2(x-a)+\ldots+\alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-\ldots-\alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$. Therefore,

$$f \geq g - t(d-1)d^{t-1}$$

$$\geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q} - [m + (d-1)n]d^{t-1}.$$

Now, if inequality (5.1) holds, then $f > (k-1)d^t$ as required. ❑

For the case $m = n$, we have the following shaper result.

Theorem 5.2. *Let q and d be positive integers such that q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even. If*

$$q > (n2^{2n} - 2^n + 1)(d-1)^n\sqrt{q} + [(d-1)n + (k-1)](d-1)^{-n}d^{2n-1}, \qquad (5.2)$$

then $P_q^{(d)}$ has property $P(n, n, k)$. In particular, for $k = 1$ the graphs $P_q^{(d)}$ has property $P(n, n, 1)$ whenever $q > n^2d^{4n}$.

Proof: Let A and B be disjoint subset of $V(P_q^{(d)})$ with $|A| = |B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$\prod_{b \in B} \{ (d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$$

$$> (k-1)d^{2n}$$

Let h be defined similarly as f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, -by Lemma 2.3, we have

$$h \geq (d-1)^n q - (n2^{2n} - 2^{2n} + 1)(d-1)^{2n}\sqrt{q}$$

Consider

$$h - f = \sum_{x \in A \cup B} \prod_{i=1}^{n} \{ 1 + \alpha(x-a_i) + \alpha^2(x-a_i) + \ldots + \alpha^{d-1}(x-a_i)\}$$

$$\{(d-1) - \alpha(x-b_i) - \alpha^2(x-b_i) - \ldots - \alpha^{d-1}(x-b_i)\}, \qquad (5.3)$$

where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

If $h - f \neq 0$ then for some x_0 the product

$$\prod_{i=1}^{n} \{ 1 + \alpha(x_0-a_i) + \alpha^2(x_0-a_i) + \ldots + \alpha^{d-1}(x_0-a_i)\}$$

$$\{(d-1) - \alpha(x_0-b_i) - \alpha^2(x_0-b_i) - \ldots - \alpha^{d-1}(x_0-b_i)\} \neq 0 \qquad (5.4)$$

With out any loss of generality suppose $x_0 = a_k$. For (3.1) to hold we must have

$$\alpha(a_k-a_i) + \alpha^2(a_k-a_i) + \ldots + \alpha^{d-1}(a_k-a_i) \neq -1 \text{ and}$$

$$\alpha(a_k-b_i) + \alpha^2(a_k-b_i) + \ldots + \alpha^{d-1}(a_k-b_i) \neq d-1 \text{ for all } i.$$

This means that

$$\alpha(a_k-a_i) + \alpha^2(a_k-a_i) + \ldots + \alpha^{d-1}(a_k-a_i) \neq d-1 \text{ for } i \neq k \text{ and}$$

$$\alpha(a_k-b_i) + \alpha^2(a_k-b_i) + \ldots + \alpha^{d-1}(a_k-b_i) \neq -1 \text{ for all } i.$$

Hence, the term in (5.4) with $x = b_i$ for all i contributes zero to the sum. Thus we can write (5.3) as

$$h - f = \sum_{x \in A} \prod_{i=1}^{n} \{ 1 + \alpha(x-a_i) + \alpha^2(x-a_i) + \ldots + \alpha^{d-1}(x-a_i)\}$$

$$\{(d-1) - \alpha(x-b_i) - \alpha^2(x-b_i) - \ldots - \alpha^{d-1}(x-b_i)\}$$

$\leq n(d-1)d^{2n-1}$,

since in the product $\prod_{a\in A}\{1+\alpha(x-a)+\alpha^2(x-a)+\ldots+\alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-\ldots-\alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$. Therfore,

$$f\leq h-n(d-1)d^{2n-1}$$

$$f\geq (d-1)^n q-(n2^{2n}-2^{2n}+1)(d-1)^{2n}\sqrt{q}\ -n(d-1)d^{2n-1}.$$

Now, if inequality (5.2) holds, then $f>(k-1)d^{2n}$ as required. ❑

Section 6. Generalized Paley graphs with the n-e.c. property

In this section, we will show that the generalized Paley graphs having property n-e.c.

Theorem 6.1. *Let q and d be positive integers such that q is a prime power and $d>1$ is odd or $(q-1)/d$ is even. If*

$$q>n^2d^{3n-2},$$

then $P_q^{(d)}$ has the n-e.c. property.

Proof: Let A and B be disjoint subset of $V(P_q^{(d)})$ with $|A\cup B|=n$. Then there is a vertex $u\notin A\cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f=\sum_{\substack{x\in\mathbb{F}_q\\ x\notin A\cup B}}\prod_{a\in A}\{1+\alpha(x-a)+\alpha^2(x-a)+\ldots+\alpha^{d-1}(x-a)\}$$

$$\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-\ldots-\alpha^{d-1}(x-b)\}$$

$$>0.$$

Let g be defined similarly as f except that the sum is taken over all $x\in\mathbb{F}_q$. Now, by Lemma 2.3, we have

$$g\geq (d-1)^B q-(n2^{n-1}-2^n+1)(d-1)^n\sqrt{q}$$

Consider $g - f = \sum_{x \in A \cup B} \prod_{a \in A} \{1 + \alpha(x - a) + \alpha^2(x - a) + \ldots + \alpha^{d-1}(x - a)\}$

$$\prod_{b \in B} \{(d - 1) - \alpha(x - b) - \alpha^2(x - b) - \ldots - \alpha^{d-1}(x - b)\}.$$

Since, in the product $\prod_{a \in A} \{1 + \alpha(x - a) + \alpha^2(x - a) + \ldots + \alpha^{d-1}(x - a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b \in B} \{(d - 1) - \alpha(x - b) - \alpha^2(x - b) - \ldots - \alpha^{d-1}(x - b)\}$ each factor is at most d and one factor is $d - 1$ and either A or B can be empty, then we can estimate $g - f$ as

$$g - f \leq (d - 1)md^{n-1}.$$

Hence $\quad f \geq h - (d - 1)md^{n-1}$

$$\geq (d-1)^B q - (n2^{n-1} - 2^n + 1)(d - 1)^n \sqrt{q} - (d - 1)md^{n-1}.$$

Now, if $q > n^2 d^{3n-2}$, then $f > 0$ as required. ❑

Section 7. Generalized Paley digraphs with the properties $Q(n, k)$ and $Q(m, n, k)$

In this section, our graphs are directed. Note that for q and d positive integers which q a prime power and $d > 1$ is even and $(q - 1)/d$ is odd, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a) = -\alpha(a)$ for all $a \in \mathbb{F}_q$. Further more, if a and b are any vertices of $D_q^{(d)}$, then

$$\alpha(a - b) = \begin{cases} 1, & \text{if } a \text{ is dominates } b, \\ 0, & \text{if } a = b, \\ \omega, & \text{otherwise;} \end{cases}$$

where $\omega \in \{ e^{\frac{2k\pi}{d}} \mid k = 1, \ldots, d - 1\}$.

In this section, we will show that the generalized Paley digraphs having properties $Q(n, k)$ and $Q(m, n, k)$.

Theorem 7.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1}, \tag{7.1}$$

then $D_q^{(d)}$ has property $Q(n, k)$. In particular, for $k = 1$ the graphs $P_q^{(d)}$ has property $Q(n, k)$ whenever $q > n^2 d^{2n}$.

Proof: Let A subset of $V(P_q^{(d)})$ with $|A| = n$. Then there is a vertex $u \notin A$ that dominates every vertex of A if and only if

$$f = \sum_{\substack{x \in \Gamma_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$> (k-1)d^n.$$

Let h be defined similarly as f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.4, we have

$$h \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q}.$$

Consider

$$h - f = \sum_{x \in A} \prod_{i=1}^{n} \{ 1 + \alpha(x - a_i) + \alpha^2(x - a_i) + \ldots + \alpha^{d-1}(x - a_i)\}, \tag{7.2}$$

where $A = \{a_1, a_2, \ldots, a_n\}$.

If $h - f \neq 0$ then for some x_0 the product

$$\prod_{i=1}^{n} \{ 1 + \alpha(x_0 - a_i) + \alpha^2(x_0 - a_i) + \ldots + \alpha^{d-1}(x_0 - a_i)\} \neq 0 \tag{7.3}$$

With out any loss of generality suppose $x_0 = a_k$. For (7.3) to hold we must have $\alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) \neq -1$ for all i. This means that for $i \neq k$ $\alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) \neq d - 1$. Therefore, a_k is unique $h - f = d^{n-1}$. Then, since $h - f$ could be 0 we conclude that

$$h - f \leq d^{n-1}.$$

So

$$f \geq h - d^{n-1}$$

$$f \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q} - d^{n-1}.$$

Now, if inequality (5.1) holds, then $f > (k-1)d^n$ as required. ❑

For the property $Q(m, n, k)$, we have the following result.

Theorem 7.2. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1}, \tag{7.4}$$

then $D_q^{(d)}$ has property $Q(m, n, k)$.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(d)}$ with | A | = m and | B | = n. Then, there are at least k vertices, each of which is dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$\prod_{b \in B} \{ (d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$$

$$> (k-1)d^n.$$

Now, using the method of proof of the theorem 5.1 and 7.1 we have the result. ❑

Section 8. Generalized Paley digraphs with the n-e.c. property

In this section, our graphs are directed. Recalled that a digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Theorem 8.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > n^2 d^{3n-2},$$

then $D_q^{(d)}$ has n-e.c. property.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(d)}$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$\prod_{b \in B} \{ (d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$$

$$> 0.$$

Now using the method of proof of the Theorem 4.1 we get $f > 0$ when

$$q > n^2 d^{3n-2}.$$

Hence, the result. ❑

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Output ที่ได้

จากผลการวิจัยที่กล่าวมาข้างต้น สามารถนำมาเรียบเรียงเขียนเป็นบทความทางวิชาการได้ 2 บทความ

เนื่องจากงานวิจัยในครั้งนี้ได้มีการศึกษาการมีสมบัติ n-e.e. ของกราฟและกราฟทิศทางที่สร้างขึ้นมา เพิ่มเติมจากปัญหาและวัตถุประสงค์ที่วางไว้ เดิมจึงได้เปลี่ยนชื่อบทความให้เหมาะสมกับเนื้อหาดังนี้

บทความที่ 1 ชื่อ "Cubic and quadruple Paley graphs with the *n*-e.c. property" โดย W. Ananchuen และ L. Caccetta บทความนี้ได้เสนอเพื่อตีพิมพ์ในวารสาร Discrete Mathematics ซึ่งมี impact factor เท่ากับ 0.395

บทความที่ 2 ชื่อ "Adjacency Properties of Generalized Paley Graphs" โดย W. Ananchuen และ L. Caccetta บทความนี้ได้เสนอเพื่อตีพิมพ์ในวารสาร Journal of Graph Theory ซึ่งมี impact factor เท่ากับ 0.377

หมายเหตุ ค่าเฉลี่ยของ impact factor ของวารสารที่ตีพิมพ์บทความทางทฤษฎีกราฟ มีค่าประมาณ 0.358

การนำไปใช้ประโยชน์

1. เชิงสาธารณะ

- มีเครือข่ายความร่วมมือกับ Prof. Dr. Louis Caccetta, Department of Mathematics and statistics, Curtin University of Technology, GPO Box U 1987, Perth, WA 6001 AUSTRALIA E-mail: L.Caccetta@exchange.curtin.edu.au
- เนื่องจากการสร้างกราฟที่มีสมบัติ P(m, n, k) และ/หรือสมบัติ n-e.c. เป็นเรื่องที่ค่อนข้างยาก ดังนั้นผลลัพธ์ที่ได้จากงานวิจัยนี้อาจช่วยกระตุ้นความสนใจในวงวิชาการในวงกว้างได้

2. เชิงวิชาการ

- สมบัติ P(m, n, k), สมบัติ n-e.c. ของกราฟ และการสร้างกราฟที่มีสมบัติ P(m, n, k) และ/หรือสมบัติ n-e.c. ได้รับการบรรจุอยู่ในตำราทางทฤษฎีกราฟขั้นสูง เช่น ตำราเรื่อง **Random Graphs** ซึ่งเขียนโดย B. Bollobás (Academic Press, London 1985) งานวิจัยที่กล่าวมาข้างต้นได้ผลลัพธ์ที่ดีกว่าเดิมและยังเป็นการขยายไปยังกรณีทั่วไป ดังนั้นผลการวิจัยที่ค้นพบจะเป็นประโยชน์ต่อการพัฒนาการเรียนการสอน โดยเฉพาะการปรับปรุงตำราทางทฤษฎีกราฟขั้นสูงต่อไปในอนาคต

ภาคผนวก

(Manusrcipts)

บทความเรื่อง

Cubic and quadruple Paley graphs with the *n*-e.c. property

Cubic and quadruple Paley graphs with the n-e.c. property

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Abstract

A graph G is n-existentially closed or n-e.c. if for any two disjoint subsets A and B of vertices of G with $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. It is well-know that almost all graphs are n-e.c. However, few classes of n-e.c. graphs have been constructed. A good construction is the Paley graphs which are defined as follows. Let $q \equiv 1 (\text{mod } 4)$ be a prime power. The vertices of Paley graphs are the elements of the finite field F_q. Two vertices a and b are adjacent if and only if their difference is a quadratic residue. Previous results established that Paley graphs are n-e.c. for sufficiently large q. By using higher order residues on finite fields we can generate other classes of graphs which we called cubic and quadruple Paley graphs. We show that cubic Paley graphs are n-e.c. whenever $q > n^2 3^{3n-2}$ and quadruple Paley graphs are n-e.c. whenever $q > n^2 4^{3n-2}$. A similar result for quadruple Paley digraphs is also obtained.

Keywords : adjacency property, n-e.c. property, Paley graph, Paley digraph

1. Introduction

For a fixed integer $n \geq 1$. A graph G is called *n-existentially closed* or *n-e.c.* if for any two subsets A and B of vertices of G with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. Observe that if a graph G has property n-e.c., then $\overline{G}$, the complement of G, also has property n-e.c. It is well-known that almost all graphs are n-e.c. However, the problem of constructing graphs with the n-e.c. property seems difficult, especially for $n \geq 4$.

The n-e.c. property was first studied by Caccetta et al. [9], where they were called graphs with property $P(n)$. The authors established, using probabilistic argument, the existence of n-e.c. graphs for a range of n. In particular, they determined the largest integer $f(\nu)$ for which there exists a graph on ν vertices having property $P(f(\nu))$ for a given integer ν. They proved that $\log \nu - (2 + o(1))\log\log \nu < f(\nu)\log 2 < \log \nu$. In addition, a class of 2-e.c. graphs was given for all orders ≥ 9.

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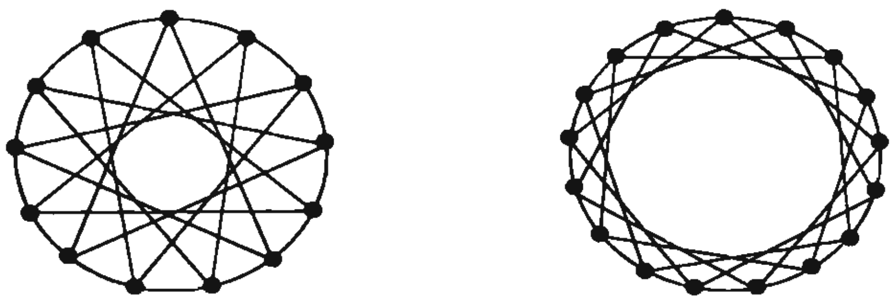
Bonato at al. [8] constructed a new class of 3-e.c. graphs, based on Hadamard matrices. They showed that Bush-type Hadamard matrics of order $16m^2$ give rise to strongly regular 3-e.c. graphs, for each odd m for which $4m$ is the order of a Hadamard matrix. By taking certain affine designs to be Hadamard designs obtained from Paley tournaments, Cameron and Stark [10] have use probabilistic methods to showed that many non-isomorphic strongly regular n-e.c. graphs of order $(q + 1)^2$ exist whenever $q \geq 16n^2 2^{2n}$ is a prime power such that $q \equiv 3 \pmod 4$.

An important graph in the study of the n-e.c. property is the so-called **Paley graph** P_q defined as follows. Let $q \equiv 1 \pmod 4$ be a prime power. The vertices of P_q are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathrm{F}_q$. The n-e.c. property of Paley graphs have been studied by a number of authors [3, 5, 7]; a good discussion is given in the book of Bollobás [7]. With respect to the n-e.c. property, we proved in [3] that if $q \equiv 1 \pmod 4$ is a prime power with $q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + \{(n+1)2^{n-1} - 1\}$, then P_q has the n-e.c. property.

By using higher order residues on finite fields we can generate other classes of graphs. More specifically, for $q \equiv 1 \pmod 3$ a prime power we define the **cubic Paley graph**, $P_q^{(3)}$ as follows. The vertices of $P_q^{(3)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^3$ for some $y \in \mathbf{F}_q$. Since $q \equiv 1 \pmod 3$ is a prime power, -1 is a cubic in $\mathbf{F}_q$. The condition -1 is a cubic in $\mathbf{F}_q$ is needed to ensure that ab is defined to be an edge whenever ba is defined to be an edge. Consequently, $P_q^{(3)}$ is well-defined. Figure 1(a) gives an example.

For $q \equiv 1 \pmod 8$ a prime power, define the **quadruple Paley graph**, $P_q^{(4)}$ as follows. The vertices of $P_q^{(4)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^4$ for some $y \in \mathbf{F}_q$. Since $q \equiv 1 \pmod 8$ is a prime power, -1 is a quadruple in $\mathbf{F}_q$. Therefore, $P_q^{(4)}$ is well-defined. Figure 1(b) gives an example. The cubic Paley graph and the quadruple Paley graph were first defined in [1].



(a) $P_{13}^{(3)}$ (b) $P_{17}^{(4)}$

Figure 2.1. Graphs $P_{13}^{(3)}$ and $P_{17}^{(4)}$.

Paley constructions have played an important role in constructing classes of graphs with the n-e.c. property, especially for $n \geq 4$, see [3, 7, 10]. In addition to directly providing graphs with interesting adjacency properties, Paley designs played an important role in the construction of strongly regular n-e.c. graphs given in [10]. In the same paper it was noted that the case of affine geometries in place of Paley designs can provide n-e.c. graphs only for $n \leq 3$. In Section 3, we show that the cubic Paley graph $P_q^{(3)}$ has the n-e.c. property whenever $q > n^2 3^{3n-2}$, and the quadruple Paley graph $P_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$.

Another version of adjacency property that has been studied is the following. Let m and n be non-negative integers and k a positive integer. A graph G is said to have *property* $P(m, n, k)$ if for any disjoint sets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property $P(m, n, k)$ is denoted by $\mathcal{G}(m, n, k)$. The class $\mathcal{G}(m, n, k)$ has been studied by Ananchuen [1], Ananchuen and Caccetta [3, 5], Blass et. al. [6] and Exoo [11]. In [1] we proved that the cubic and quadruple Paley graphs are n-e.c. for sufficially large q.

The concept of n-e.c. property of graphs can be extended to digraphs as follows. If (i, j) is an arc in a digraph D, then we say vertex i *dominates* vertex j. A digraph D is *n-e.c.* if for any two subsets A and B of vertices of D with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Let $q \equiv 5 (\mathrm{mod}\ 8)$ be a prime power. Define the **quadruple Paley digraph**, $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadruple in $\mathbf{F}_q$; that is $a - b = y^4$ for some $y \in \mathbf{F}_q$. The n-e.c. property of Paley digraphs have been studied by [4, 7].

In Section 4, we prove that $D_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$.

2. Preliminaries

We make use of the following basic notation and terminology. Let $\mathbf{F}_q$ be a finite field of order q where q is a prime power and let $\mathbf{F}_q[x]$ be a polynomial ring over $\mathbf{F}_q$.

A **character** χ of $\mathbf{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbf{F}_q$, is a map from $\mathbf{F}_q^*$ to the multiplicative group of complex numbers with $|\chi(x)| = 1$ for all $x \in \mathbf{F}_q^*$ and with $\chi(xy) = \chi(x)\chi(y)$ for any $x, y \in \mathbf{F}_q^*$. Among the character of $\mathbf{F}_q^*$, we have the **trivial** character χ_o defined by $\chi_o(x) = 1$ for all $x \in \mathbf{F}_q^*$; all other character of $\mathbf{F}_q^*$ are called **nontrivial**. A character χ is of **order** d if $\chi^d = \chi_o$ and d is the smallest positive integer with this property.

It is customary to extent the definition of nontrivial character χ to the whole $\mathbf{F}_q$ by defining $\chi(0) = 0$. For χ_o we define $\chi_o(0) = 1$.

Observe that

$$\chi^t(a) = \chi(a^t), \tag{2.1}$$

for any $a \in \mathbf{F}_q$ and t a positive integer.

The following lemma, due to Schmidt [12], is very useful to our work.

Lemma 2.1. *Let χ be a nontrivial character of order d of $\mathbf{F}_q$. Suppose $f(x) \in \mathbf{F}_q[x]$ has precisely s distinct zero and it is not a d^{th} power; that is $f(x)$ is not the form $c\{g(x)\}^d$, where $c \in \mathbf{F}_q$ and $g(x) \in \mathbf{F}_q[x]$. Then*

$$\left|\sum_{x\in\mathbf{F}_q}\chi(f(x))\right| \le (s-1)\sqrt{q}. \qquad \square$$

Let g be a fixed primitive element of the finite field $\mathbf{F}_q$; that is g is a generator of the cyclic group $\mathbf{F}_q^*$. Define a function α by

$$\alpha(g^t) = e^{\frac{2\pi i t}{3}},$$

where $i^2 = -1$. Therefore, α is a cubic character, character of order 3, of $\mathbf{F}_q$. The values of α are the elements of the set $\{1, \omega, \omega^2\}$ where $\omega = e^{\frac{2\pi i}{3}}$. Note that α^2 is also a cubic character. Moreover, if a is not a cubic of an element of $\mathbf{F}_q^*$, then $\alpha(a) + \alpha^2(a) = -1$. This fact is very important in our methodology.

Further, define a function β by

$$\beta(g^t) = i^t.$$

Therefore, β is the quadruple character, character of order 4, of $\mathbf{F}_q$. The values of β are in the set $\{1, -1, i, -i\}$. Observe that β^3 is also a quadruple character while β^2 is a quadratic character. Moreover, if a is not a quadruple of an element of $\mathbf{F}_q^*$, then $\beta(a) + \beta^2(a) + \beta^3(a) = -1$. This fact is very important in our methodology.

The following lemmas were proved in [1].

Lemma 2.2. *Let α be a cubic character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$ with $|A \cup B| = n$. Put*

$$g = \sum_{x\in\mathbf{F}_q}\prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\}\prod_{b\in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Then

$$g \ge 2^{|B|}q - (n2^{n-1} - 2^n + 1)2^n\sqrt{q}. \qquad \square$$

Lemma 2.3. *Let β be a quadruple character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$ with $|A \cup B| = n$. Put*

$$h = \sum_{x\in\mathbf{F}_q}\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}\prod_{b\in B}\{3-\beta(x-b)-\beta^2(x-b)-\beta^3(x-b)\}.$$

Then

$$h \ge 3^{|B|}q - (n2^{n-1} - 2^n + 1)3^n\sqrt{q}. \qquad \square$$

3. The cubic and quadruple Paley graphs

For $q \equiv 1 \pmod 3$ a prime power, there exists a cubic character α of $\mathbf{F}_q$ and $\alpha(-a) = \alpha(a)$ for all $a \in \mathbf{F}_q$. Further, for $q \equiv 1 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and $\beta(-a) = \beta(a)$ for all $a \in \mathbf{F}_q$.

Observe that if a and b are any vertices of $P_q^{(3)}$, then for $t = 1$ and 2

$$\alpha^t(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ \omega \text{ or } \omega^2, & \text{otherwise.} \end{cases}$$

Also, if a and b are any vertices of $P_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character: that is

$$\beta^2(a-b) = \begin{cases} 1, & \text{if } a-b \text{ is a quadratic ressidue}, \\ 0, & \text{if } a = b, \\ -1, & \text{otherwise.} \end{cases}$$

Our first result concerns cubic Paley graph having property n-e.c. for any fixed integer $n \geq 1$.

Theorem 3.1. *Let $q \equiv 1 \pmod 3$ be a prime power. If*

$$q > n^2 3^{3n-2},$$

then $P_q^{(3)}$ has the n-e.c. property. Furthermore, for $n > 1$ the graph $P_q^{(3)}$ is n-e.c whenever $q > n^2 3^{3n-4}$.

Proof: Let A and B be disjoint subsets of $V(P_q^{(3)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbf{F}_q \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}$$
$$> 0.$$

Let

$$g = \sum_{x \in \mathbf{F}_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Now, by Lemma 2.2 we have

$$g \geq 2^n q - (n2^{n-1} - 2^n + 1)2^n\sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} \prod_{a \in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\}\prod_{b \in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Since, in the product $\prod_{a \in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\}$ each factor is at most 3 and one factor is 1 and in the product $\prod_{b \in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}$ each factor is at most 3 and one factor is 2 we have

$$\begin{aligned} g - f &\leq 3^{n-1}|A| + 3^{n-1}2|B| \\ &= (|A| + 2|B|)3^{n-1} \\ &\leq 2n3^{n-1}. \end{aligned}$$

Consequently,

$$f \geq 2^n q - (n2^{n-1} - 2^n + 1)2^n\sqrt{q} - 2n3^{n-1}.$$

Now, if $q > n^2 3^{3n-2}$, then $f > 0$ as required.

It is easily cheeked that $f > 0$ when $q > n^2 3^{3n-4}$ for $n > 1$. ❑

Remark 3.1. The bound for q in Theorem 3.1 can be improved to $n^2 3^{2.5n}$ for $1 \leq n \leq 55$.

We now turn our attention to the adjacent property of the quadruple Paley graph $P_q^{(4)}$.

Theorem 3.2 *Let $q \equiv 1 \pmod 8$ be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $P_q^{(4)}$ has the n-e.c property.

Proof: Let A and B be disjoint subsets of $V(P_q^{(4)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in F \\ x \notin A \cup B}} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}\prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$
$$> 0.$$

Let

$$h = \sum_{x \in F} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}\prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

Now, by Lemma 2.3, we have

$$h \geq 3^n q - (n2^{n-1} - 2^n + 1)3^n\sqrt{q}.$$

Consider

$$h - f = \sum_{x \in A \cup B} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}\prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Since, in the product $\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}$ each factor is at most 4 and one factor is 1 and in the product $\prod_{b\in B}\{3-\beta(x-b)-\beta^2(x-b)-\beta^3(x-b)\}$ each factor is at most 4 and one factor is 3 we have

$$h-f\le |A|4^{n-1}+3|B|4^{n-1}$$
$$\le 3n4^{n-1}.$$

Consequently,

$$f\ge 3^n q-(n2^{n-1}-2^n+1)3^n\sqrt{q}-3n4^{n-1}.$$

Now, if $q>n^24^{3n-2}$, then $f>0$ as required. ❑

Remark 3.2. The bound for q in Theorem 3.2 can be improved to $q>n^24^{3n-3}$ for $n>1$ or $n^24^{2.5n}$ for $1\le n\le 14$.

4. Quadruple Paley digraphs

In this section, our graphs are directed. Recalled that, digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A\cap B=\varnothing$ and $|A\cup B|=n$, there is a vertex $u\notin A\cup B$ such that u dominates every vertex of A and dominated by every vertex of B. For $q\equiv 5 \pmod 8$ be a prime power. Define the quadruple Paley digraph $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex a joins to vertex b by an arc if and only if $a-b$ is a quadruple in $\mathbf{F}_q$. Since $q\equiv 5 \pmod 8$ is a prime power, -1 is not a quadruple in $\mathbf{F}_q$. The condition -1 is not a quadruple in $\mathbf{F}_q$ is needed to ensure that (b, a) is not defined to be an arc when (a, b) is defined to be an arc. Consequently, $D_q^{(4)}$ is well-defined. However, $D_q^{(4)}$ is not a tournament. Figure 4.1 displays the digraph $D_{13}^{(4)}$. The quadruple Paley digraph was first defined in [2].

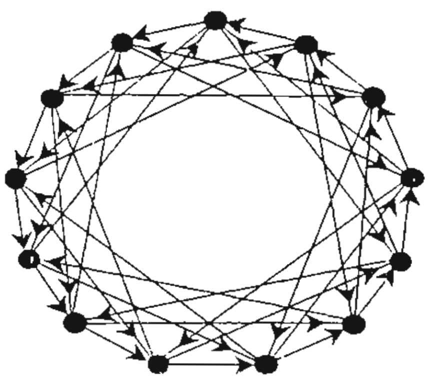


Figure 4.1. Paley digraph $D_{13}^{(4)}$.

For $q \equiv 5 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and noting that if a and b are any vertices of $D_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(a-b) = \begin{cases} 1, & \text{if } a \text{ dominates } b, \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character. Further, $\beta(-a) = -\beta(a)$ for any $a \in \mathbf{F}_q$.

Theorem 4.1. *Let $q \equiv 5 \pmod 8$ be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $D_q^{(4)}$ has n-e.c. property.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(4)}$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in F \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

$$> 0.$$

Now using the method of proof of the Theorem 3.2 we get $f > 0$ when

$$q > n^2 4^{3n-2}$$

Hence, the result. ❑

Remark 4.1. The bound for q in Theorem 4.1 can be improved to $n^2 4^{2.5n}$ for $1 \leq n \leq 14$.

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บทความเรื่อง

Adjacency Properties of Generalized Paley Graphs

Adjacency Properties of Generalized Paley Graphs

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Abstract

Let m and n be non-negative integers and k a positive integer. A graph G is said to have property $P(m,n,k)$ if for any disjoint subsets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. Furthermore, a graph G is called n-existentially closed or n-e.c. if for any two subsets A and B of vertices of G with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. It is well-known that almost all graphs satisfy the $P(m,n,k)$ property and the n-e.c. property. However, the problem of constructing graphs with the $P(m,n,k)$ property and the n-e.c. property seems difficult. In this paper, we show that all sufficiently large generalized Paley graphs defined by using higher order residues on finite fields satisfy the $P(m,n,k)$ property and the n-e.c. property. Similar results for generalized Paley digraphs are also obtained.

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1. Introduction

Let m and n be non-negative integers and k a positive integer. A graph G is said to have *property* $P(m,n,k)$ if for any disjoint subsets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property $P(m,n,k)$ is denoted by $\mathcal{G}(m,n,k)$. Observe that if a graph G has property $P(m,n,k)$, then $\overline{G}$, the complement of G, has property $P(n,m,k)$. It is well-known [6] that almost all graphs have property $P(m,n,k)$. Despite this result, few graphs have been constructed which exhibit the property $P(m,n,k)$. The class $\mathcal{G}(m,n,k)$ has been studied by many authors including:

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Ananchuen [1]; Ananchuen and Caccetta [2, 4]; Blass et al. [6]; Bollobás [7]; and Exoo [11].

An important graph in the study of the property $P(m, n, k)$ is the so-called Paley graph P_q defined as follows. Let $q \equiv 1 (\text{mod } 4)$ be a prime power. The vertices of P_q are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbb{F}q$. The adjacency properties of Paley graphs have been studied by a number of authors [2, 4, 6, 7]; a good discussion is given in the book of Bollobás [7]. With respect to the property $P(n, n, 1)$ we proved in [2] that if $q \equiv 1(\text{mod } 4)$ is a prime power with $q > ((2n - 3)2^{2n-1} + 4)^2$, then $P_q \in \mathcal{G}(n, n, 1)$.

By using higher order residues on finite fields we can generate other classes of graphs. More specifically, let q and d be positive integers such that q is a prime power and

$$d > 1 \text{ is odd or } (q - 1)/d \text{ is even.}$$

We define the *generalized Paley graph*, $P_q^{(d)}$ as follows. The vertices of $P_q^{(d)}$ are the elements of finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^d$ for some $y \in \mathbb{F}_q$. Since q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even, $-1 = y^d$ for some $y \in \mathbb{F}_q$. The condition -1 is a d^{th} power of an element of $\mathbb{F}_q$ is needed to ensure that ba is defined to be an edge precisely whenever ab is defined to be an edge. Consequently, $P_q^{(d)}$ is well-defined. Clearly, $P_q^{(2)}$ is the Paley graph. $P_q^{(3)}$ is called the cubic Paley graph and $P_q^{(4)}$ the quadruple Paley graph in [1]. It has been proved [1] that all sufficiently large cubic and quadruple Paley graphs satisfy the $P(m, n, k)$ property.

In Section 3, we will show that the generalized Paley graphs satisfy the property $P(n, n, 1)$ whenever $q > n^2 d^{4n}$.

Another version of adjacency property that has been stuaied is the following. For a fixed integer $n \geq 1$. A graph G is called *n-existentially closed* or *n-e.c.* if for any two subsets A and B of vertices of G with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. Observe that if a graph G has property n-e.c., then $\overline{G}$, the complement of G, also has property n-e.c. It is well-known that for any fixed n, almost all graphs are n-e.c. However, the problem of constructing graphs with the n-e.c. property seems difficult, especially for $n \geq 4$.

The n-e.c. property was first studied by Caccetta et al. [9], where they were called graphs with property $P(n)$. The authors established, using a probabilistic argument, the existence of n-e.c. graphs for a range of n. In particular, they determined the largest integer $f(n)$ for which there exists a graph on n vertices having property $P(f(n))$ for a given integer n. They proved that $\log n - (2 + o(1)) \log\log n < f(n) \log 2 < \log n$. In addition, a class of 2-e.c. graphs was given for all orders ≥ 9.

Bonato et al. [8] constructed a new class of 3-e.c. graphs, based on Hadamard matrices. They showed that Bush-type Hadamard matrics of order $16m^2$ gives rise to strongly regular 3-e.c. graphs, for each odd m for which $4m$ is the order of a Hadamard matrix. By taking certain affine designs to be Hadamard designs obtained from Paley tournaments, Cameron and Stark [10] have use probabilistic methods to show that many non-isomorphic strongly regular n-e.c. graphs of order $(q+1)^2$ exist whenever $q \geq 16n^2 2^{2n}$ is a prime power such that $q \equiv 3 (\text{mod } 4)$. Ananchuen and Caccetta [5] show that the cubic Paley graph $P_q^{(3)}$ has the n-e.c. property whenever $q > n^2 3^{3n-2}$, and the quadruple Paley graph $P_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$. In section 4, we prove that the generalized Paley graph has the n-e.c. property whenever $q > n^2 d^{3n-2}$.

The concept of adjacency property of graphs can be extended to digraphs as follows. If (i,j) is an arc in a digraph D, then we say vertex i *dominates* vertex j. A digraph D is said to have property $Q(n,k)$ if every subset of n vertices of D is dominated by at least k other vertices. Graham and Spencer [12] defined the following digraph. Let $p \equiv 3 (\text{mod } 4)$ be a prime. The vertices of digraph D_p are $\{0, 1, ..., p-1\}$ and D_p contains the arc (a, b) if and only if $a - b$ is a quadratic residue modulo p. The digraph D_p is sometimes referred to as the Paley tournament. Graham and Spencer [12] proved that D_p has property $Q(n,1)$ whenever $p > n^2 2^{2n-2}$. Bollobás [7] extended these results to prime powers. More specifically, if $q \equiv 3 (\text{mod } 4)$ is a prime power, the Paley tournament D_q is defined as follows. The vertex set of D_q are the elements of the finite field $\mathbb{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadratic residue in $\mathbb{F}_q$. Bollobás [7] noted that D_q has property $Q(n,1)$ whenever $q > \{(n-2)2^{n-1} + 1\} + n2^{n-1}$. Ananchuen and Caccetta [3] proved that D_q has property $Q(n,k)$ whenever $q > \{(n-3)2^{n-1} + 2\} + k2^{n-1}$.

Let q and d be positive integers such that q is a prime power and

$d > 1$ is even and $(q-1)/d$ is odd.

We define the *generalized Paley digraph*, $D_q^{(d)}$ as follows. The vertices of $D_q^{(d)}$ are the elements of the finite field $\mathbb{F}_q$. A vertex a joins to vertex b by an arc if and only if $a - b = y^d$ for some $y \in \mathbb{F}_q$. Since $d > 1$ is even and $(q-1)/d$ is odd, -1 is not a d^{th} power of any element of $\mathbb{F}_q$. The condition -1 is not a d^{th} power of any element of $\mathbb{F}_q$ is needed to ensure that (b,a) is not defined to be an arc whenever (a,b) is defined to be an arc. Consequently, $D_q^{(d)}$ is well-defined.

In Section 5, we show that the generalized Paley digraph $D_q^{(d)}$ has the property $Q(n,1)$ whenever $q > n^2 d^{2n}$.

A digraph D is n-*e.c.* if for any two subsets A and B of vertices of D with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

In Section 6, we show that the generalized Paley digraph $D_q^{(d)}$ is n-e.c. whenever $q > n^2 d^{3n-2}$.

2. Preliminaries

We make use of the following basic notation and terminology. Let $\mathbb{F}_q$ be a finite field of order q where q is a prime power. A *character* χ on $\mathbb{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbb{F}_q$, is a homomorphism from $\mathbb{F}_q^*$ to the multiplicative group of complex number with $|\chi(x)| = 1$ for all x. Among the characters of $\mathbb{F}_q^*$, we have the *trivial character* χ_o defined by $\chi_o(x) = 1$ for all $x \in \mathbb{F}_q^*$; all other characters of $\mathbb{F}_q^*$ are called *nontrivial*. A character χ is of *order* d if $\chi^d = \chi_o$ and d is the smallest positive integer with this property.

It is customary to extent the definition of character χ to the whole $\mathbb{F}_q$ by putting $\chi(0) = 0$ and $\chi_o(0) = 1$.

Observe that (see[13])

$$\sum_{\substack{\chi \text{ of order dividing } d \\ \chi \neq \chi_o}} \chi(x) = \begin{cases} d-1, & \text{if } x = y^d \text{ for some } y \in \mathbb{F}_q^*, \\ 0, & \text{if } x = 0, \\ -1, & \text{otherwise.} \end{cases} \quad (2.1)$$

This fact is very important in our methodology. Moreover.

$$\chi(a^r) = \chi^r(a), \quad (2.2)$$

for any $a \in \mathbb{F}_q$ and r is a positive integer.

The following lemma, due to Schmidt [13], is very useful to our work.

Lemma 2.1. *Let χ be a nontrivial character of order d of $\mathbb{F}_q$. Suppose $f(x) \in \mathbb{F}_q[x]$ has precisely s distinct zero and it is not a d^{th} power; that is $f(x)$ is not the form $c\{g(x)\}^d$, where $c \in \mathbb{F}_q$ and $g(x) \in \mathbb{F}_q[x]$. Then*

$$|\sum_{x \in \mathbb{F}_q} \chi(f(x)| \leq (s-1)\sqrt{q}. \quad \square$$

For g a fixed primitive element of the finite field $\mathbb{F}_q$; that is g is a generator of the cyclic group $\mathbb{F}_q^*$. Define a function α by

$$\alpha(g^k) = e^{\frac{2k\pi i}{d}},$$

where $i^2 = -1$. Therefore, α is a character of order dividing d and the value of α are the elements of the set $\{e^{\frac{2k\pi i}{d}} | k = 0, 1, \ldots, d-1\}$. It is not too difficult to verify that $\alpha, \alpha^2, \ldots, \alpha^{d-1}$ are characters of order dividing d and are all different.

The following two lemmas are extensively used in establishing our results.

Lemma 2.2. *Let α be a character of order d of F_q and let A and B be disjoint subsets of F_q. Put*

$$g = \sum_{x \in \mathbb{F}_q} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$
$$\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}].$$

Then

$$g \geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t \sqrt{q},$$

where $|A| = m$, $|B| = n$ and $m + n = t$.

Proof: Let $A \cup B = \{c_1, c_2, \ldots, c_t\}$. Expanding g and noting that $\sum_{x \in \mathbb{F}_q} (d-1)^n = (d-1)^n q$, we can write

$$|g - (d-1)^n q| \leq |\sum_{x \in \mathbb{F}_q} \sum_{\chi \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i=1}^{t} (d-1)^{t-1} \chi(x - c_i)| +$$
$$|\sum_{x \in \mathbb{F}_q} \sum_{\chi_j \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i_1 < i_2} (d-1)^{t-2} \chi_1(x - c_{i_1}) \chi_2(x - c_{i_2})| + \ldots +$$

$$|\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}}\sum_{i_1<i_2\ldots<i_s}(d-1)^{t-s}\chi_1(x-c_{i_1})\chi_2(x-c_{i_2})\ldots\chi_s(x-c_{i_s})|$$
$$+\ldots+|\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}}(d-1)^{t-s}\chi_1(x-c_1)\chi_2(x-c_2)\ldots\chi_t(x-c_t)|.$$

Now, by (2.1), (2.2) and Lemma 2.1 we have

$$|g-(d-1)^n q|\le\sum_{s=1}^{t}(d-1)^s(d-1)^{t-s}\binom{t}{s}(s-1)\sqrt{q}$$
$$=(t2^{t-1}-2^t+1)(d-1)^t.$$

Therefore, $g\ge(d-1)^n q-(t2^{t-1}-2^t+1)(d-1)^t\sqrt{q}$ as required. □

Lemma 2.3. *Let α be a character of order d of F_q and let A be a subsets of n vertices of F_q. Put*

$$h=\sum_{x\in\mathbb{F}_q}\prod_{a\in A}\{1+\alpha(x-a)+\alpha^2(x-a)+\ldots+\alpha^{d-1}(x-a)\}.$$

Then

$$h\ge q-[1+(nd-n-d)d^{n-1}]\sqrt{q}.$$

Proof: Let $A=\{a_1,a_2,\ldots,a_n\}$. We can write

$$h=\sum_{x\in\mathbb{F}_q}1+\sum_{\chi\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}}\sum_{i=1}^{n}\chi(x-a_i)+$$
$$\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}}\sum_{i_1<i_2}\chi_1(x-a_{i_1})\chi_2(x-a_{i_2})+\ldots+$$
$$\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}}\sum_{i_1<i_2\ldots<i_s}\chi_1(x-a_{i_1})\chi_2(x-a_{i_2})\ldots\chi_s(x-a_{i_s})+\ldots+$$
$$\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\ldots,\alpha^{d-1}\}}\chi_1(x-a_1)\chi_2(x-a_2)\ldots\chi_n(x-a_n).$$

Now, by (2.1), (2.2) and Lemma 2.1 we have

$$|h-q|\le\sum_{s=1}^{n}(d-1)^s\binom{n}{s}(s-1)\sqrt{q}$$
$$=[1+(nd-n-d)d^{n-1}]\sqrt{q}.$$

Therefore, $h\ge q-[1+(nd-n-d)d^{n-1}]\sqrt{q}$ as required. □

3. The property $P(m,n,k)$

Note that, for q and d positive integers such that q is a prime power and $d>1$ is odd or $(q-1)/d$ is even, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a)=\alpha(a)$ for all $a\in\mathbb{F}_q$. Furthermore, if α is a character of order d of $\mathbb{F}_q$ and a and b are vertices of $P_q^{(d)}$, then

$$\alpha(a-b)=\begin{cases}1, & \text{if } a \text{ is adjacent to } b,\\ 0, & \text{if } a=b,\\ \omega, & \text{otherwise;}\end{cases}$$

where $\omega\in\{e^{\frac{2k\pi i}{d}}\,|\,k=1,\ldots,d-1\}$.

Our first result concerns the generalized Paley graphs having property $P(m,n,k)$.

Theorem 3.1. *Let q and d be positive integers such that q is a prime power and $d>1$ is odd or $(q-1)/d$ is even. If*

$q > (t2^{t-1} - 2^t + 1)(d-1)^m\sqrt{q} + [m + (d-1)n + (k-1)d](d-1)^{-n}d^{t-1}$, (3.1)
then $P_q^{(d)} \in \mathcal{G}(m,n,k)$ *for all* m, n *with* $m + n \le t$.

Proof: It clearly suffices to establish the result for $m + n = t$. Let A and B be disjoint subsets of $V(P_q^{(d)})$ with $|A| = m$ and $|B| = n$. Then there are at least k other vertices, each of which adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}] \ge kd^t.$$

To show that $f \ge kd^t$, it is clearly sufficient to establish that $f > (k-1)d^t$.

Let g be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2 we have

$$g \ge (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}]$$
$$\le d^{t-1}m + (d-1)d^{t-1}n$$
$$= [m + (d-1)n]d^{t-1}$$

since, in the product $\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$. Therefore,

$$f \ge g - t(d-1)d^{t-1}$$
$$\ge (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q} - [m + (d-1)n]d^{t-1}.$$

Now, if inequality (3.1) holds, then $f > (k-1)d^t$ as required. □

For the case $m = n$, we have the following sharper result.

Theorem 3.2. *Let q and d be positive integers such that q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even. If*

$$q > (n2^{2n} - 2^n + 1)(d-1)^n\sqrt{q} + [(d-1)n + (k-1)](d-1)^{-n}d^{2n-1}, \quad (3.2)$$

then $P_q^{(d)}$ has property $P(n,n,k)$. In particular, for $k = 1$ the graphs $P_q^{(d)}$ has property $P(n,n,1)$ whenever $q > n^2 d^{4n}$.

Proof: Let A and B be disjoint subset of $V(P_q^{(d)})$ with $|A| = |B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-...-\alpha^{d-1}(x-b)\}]$$
$$\geq (k-1)d^{2n}.$$

Let h be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2, we have

$$h \geq (d-1)^n q - (n2^{2n} - 2^{2n} + 1)(d-1)^{2n}\sqrt{q}.$$

Consider

$$h - f = \sum_{x\in A\cup B}\prod_{i=1}^{n}[\{1+\alpha(x-a_i)+\alpha^2(x-a_i)+...+\alpha^{d-1}(x-a_i)\}$$
$$\{(d-1)-\alpha(x-b_i)-\alpha^2(x-b_i)-...-\alpha^{d-1}(x-b_i)\}], \quad (3.3)$$

where $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$.

If $h - f \neq 0$, then for some x_o the product

$$\prod_{i=1}^{n}\{1+\alpha(x_o-a_i)+\alpha^2(x_o-a_i)+...+\alpha^{d-1}(x_o-a_i)\}$$
$$\{(d-1)-\alpha(x_o-b_i)-\alpha^2(x_o-b_i)-...-\alpha^{d-1}(x_o-b_i)\} \neq 0. \quad (3.4)$$

With out any loss of generality suppose $x_o = a_k$. For (3.3) to hold we must have

$$\alpha(a_k-a_i)+\alpha^2(a_k-a_i)+...+\alpha^{d-1}(a_k-a_i) \neq -1$$

and

$$\alpha(a_k-b_i)+\alpha^2(a_k-b_i)+...+\alpha^{d-1}(a_k-b_i) \neq d-1 \text{ for all } i.$$

This means that

$$\alpha(a_k-a_i)+\alpha^2(a_k-a_i)+...+\alpha^{d-1}(a_k-a_i) = d-1 \text{ for } i \neq k$$

and

$$\alpha(a_k-b_i)+\alpha^2(a_k-b_i)+...+\alpha^{d-1}(a_k-b_i) = -1 \text{ for all } i.$$

Hence, the term in (3.4) with $x = b_i$ for all i contributes zero to the sum. Thus we can write (3.3) as

$$h - f = \sum_{x\in A}[\prod_{i=1}^{n}\{1+\alpha(x-a_i)+\alpha^2(x-a_i)+...+\alpha^{d-1}(x-a_i)\}$$
$$\{(d-1)-\alpha(x-b_i)-\alpha^2(x-b_i)-...-\alpha^{d-1}(x-b_i)\}]$$
$$\leq n(d-1)d^{2n-1},$$

since in the product $\prod_{i=1}^{n}\{1+\alpha(x-a_i)+\alpha^2(x-a_i)+...+\alpha^{d-1}(x-a_i)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{i=1}^{n}\{(d-1)-\alpha(x-b_i)-\alpha^2(x-b_i)-...-\alpha^{d-1}(x-b_i)\}$ each factor is at most d and one factor is $d-1$. Therefore,

$$f \geq h - n(d-1)d^{2n-1}$$
$$f \geq (d-1)^n q - (n2^{2n} - 2^{2n} + 1)(d-1)^{2n}\sqrt{q} - n(d-1)d^{2n-1}.$$

Now, if inequality (3.2) holds, then $f > (k-1)d^{2n}$ as required. It is easily checked that $f > 0$ whenever $q > n^2 d^{4n}$ for $k = 1$. □

4. The n-e.c. property

In this section, we will show that the generalized Paley graphs having property n-e.c.

Theorem 4.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even. If*

$$q > n^2 d^{3n-2},$$

then $P_q^{(d)}$ has the n-e.c. property.

Proof: Let A and B be disjoint subsets of $V(P_q^{(d)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} \Big[\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\} \Big] > 0.$$

Let g be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2, we have

$$g \geq (d-1)^{|B|} q - (n2^{n-1} - 2^n + 1)(d-1)^n \sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} \Big[\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\} \Big].$$

Since, in the product $\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$ and either A or B can be empty, then we can estimate $g - f$ as

$$g - f \leq (d-1) n d^{n-1}.$$

Hence

$$\begin{aligned} f &\geq h - (d-1) n d^{n-1} \\ &\geq (d-1)^{|B|} q - (n2^{n-1} - 2^n + 1)(d-1)^n \sqrt{q} - (d-1) n d^{n-1}. \end{aligned}$$

Now, if $q > n^2 d^{3n-2}$, then $f > 0$ as required. □

5. The property $Q(n,k)$

Note that for q and d positive integers which q a prime power and $d > 1$ is even and $(q-1)/d$ is odd, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a) = -\alpha(a)$ for all $a \in \mathbb{F}_q$. Further more, if a and b are any vertices of $D_q^{(d)}$, then

$$\alpha(a-b) = \begin{cases} 1, & \text{if } a \text{ dominates } b, \\ 0, & \text{if } a = b, \\ \omega, & \text{otherwise;} \end{cases}$$

where $\omega \in \{e^{\frac{2k\pi i}{d}} \mid k = 1, \ldots, d-1\}$.

In this section, we will show that the generalized Paley digraphs having property $Q(n,k)$.

Theorem 5.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1}, \quad (5.1)$$

then $D_q^{(d)}$ has property $Q(n, k)$. In particular, the graphs $D_q^{(d)}$ has property $Q(n, 1)$ whenever $q > n^2 d^{2n}$.

Proof: Let A subset of $V(D_q^{(d)})$ with $|A| = n$. Then there is a vertex $u \notin A$ that dominates every vertex of A if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A}} \prod_{a \in A} \{1 + \alpha(x - a) + \alpha^2(x - a) + \ldots + \alpha^{d-1}(x - a)\}$$
$$> (k - 1)d^n.$$

Let h be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2, we have

$$h \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q}.$$

Consider

$$h - f = \sum_{x \in A} \prod_{i=1}^{n} \{1 + \alpha(x - a_i) + \alpha^2(x - a_i) + \ldots + \alpha^{d-1}(x - a_i)\}, \quad (5.2)$$

where $A = \{a_1, a_2, \ldots, a_n\}$.

If $h - f \neq 0$, then for some x_o the product

$$\prod_{i=1}^{n} \{1 + \alpha(x_o - a_i) + \alpha^2(x_o - a_i) + \ldots + \alpha^{d-1}(x_o - a_i)\} \neq 0. \quad (5.3)$$

With out any loss of generality suppose $x_o = a_k$. For (5.2) to hold we must have $\alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) \neq -1$ for all i. This means that for $i \neq k, \alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) = d - 1$. Therefore, a_k is unique $h - f = d^{n-1}$. Then, since $h - f$ could be 0 we conclude that

$$h - f \geq d^{n-1}.$$

So

$$f \geq h - d^{n-1}$$
$$\geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q} - d^{n-1}.$$

Now, if inequality (5.1) holds, then $f > (k - 1)d^n$ as required. It is easily checked that $f > 0$ whenever $q > n^2 d^{2n}$ for $k = 1$. □

6. The n-e.c. property for digraphs

Recalled that a digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Theorem 6.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q - 1)/d$ is odd. If*

$$q > n^2 d^{3n-2},$$

then $D_q^{(d)}$ has the n-e.c. property.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(d)}$ with $|A \cap B| = n$. Then there is a vertex $u \in A \cap B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbf{F}_q \\ x \notin A \cup B}} \Big[\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}\Big]$$
$$> 0.$$

Now using the method of proof of the Theorem 4.1 we get $f > 0$ whenever $q > n^2 d^{3n-2}$.

Hence, the result. □

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