Consider $g - f = \sum_{x \in A \cup B} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$

$$\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}.$$

Since, in the product $\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$ and either A or B can be empty, then we can estimate $g - f$ as

$$g - f \leq (d-1)nd^{n-1}.$$

Hence $\quad f \geq h - (d-1)nd^{n-1}$

$$\geq (d-1)^B q - (n2^{n-1} - 2^n + 1)(d-1)^n \sqrt{q} - (d-1)nd^{n-1}.$$

Now, if $q > n^2 d^{3n-2}$, then $f > 0$ as required. ❑

Section 7. Generalized Paley digraphs with the properties $Q(n, k)$ and $Q(m, n, k)$

In this section, our graphs are directed. Note that for q and d positive integers which q a prime power and $d > 1$ is even and $(q-1)/d$ is odd, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a) = -\alpha(a)$ for all $a \in \mathbb{F}_q$. Further more, if a and b are any vertices of $D_q^{(d)}$, then

$$\alpha(a-b) = \begin{cases} 1, & \text{if } a \text{ is dominates } b, \\ 0, & \text{if } a = b, \\ \omega, & \text{otherwise;} \end{cases}$$

where $\omega \in \{e^{\frac{2k\pi}{d}} \mid k = 1, \ldots, d-1\}$.

In this section, we will show that the generalized Paley digraphs having properties $Q(n, k)$ and $Q(m, n, k)$.

Theorem 7.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1}, \tag{7.1}$$

then $D_q^{(d)}$ has property $Q(n, k)$. In particular, for $k = 1$ the graphs $P_q^{(d)}$ has property $Q(n, k)$ whenever $q > n^2 d^{2n}$.

Proof: Let A subset of $V(P_q^{(d)})$ with $|A| = n$. Then there is a vertex $u \notin A$ that dominates every vertex of A if and only if

$$f = \sum_{\substack{x \in \Gamma_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$> (k-1)d^n.$$

Let h be defined similarly as f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.4, we have

$$h \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q}.$$

Consider

$$h - f = \sum_{x \in A} \prod_{i=1}^{n} \{ 1 + \alpha(x-a_i) + \alpha^2(x-a_i) + \ldots + \alpha^{d-1}(x-a_i)\}, \tag{7.2}$$

where $A = \{a_1, a_2, \ldots, a_n\}$.

If $h - f \neq 0$ then for some x_0 the product

$$\prod_{i=1}^{n} \{ 1 + \alpha(x_0 - a_i) + \alpha^2(x_0 - a_i) + \ldots + \alpha^{d-1}(x_0 - a_i)\} \neq 0 \tag{7.3}$$

With out any loss of generality suppose $x_0 = a_k$. For (7.3) to hold we must have $\alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) \neq -1$ for all i. This means that for $i \neq k$ $\alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) \neq d - 1$. Therefore, a_k is unique $h - f = d^{n-1}$. Then, since $h - f$ could be 0 we conclude that

$$h - f \leq d^{n-1}.$$

So

$$f \geq h - d^{n-1}$$

$$f \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q} - d^{n-1}.$$

Now, if inequality (5.1) holds, then $f > (k-1)d^n$ as required. ❑

For the property $Q(m, n, k)$, we have the following result.

Theorem 7.2. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1}, \tag{7.4}$$

then $D_q^{(d)}$ has property $Q(m, n, k)$.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(d)}$ with | A | = m and | B | = n. Then, there are at least k vertices, each of which is dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$

$$\prod_{b \in B} \{ (d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}$$

$$> (k-1)d^{n}.$$

Now, using the method of proof of the theorem 5.1 and 7.1 we have the result. ❑

Section 8. Generalized Paley digraphs with the n-e.c. property

In this section, our graphs are directed. Recalled that a digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Theorem 8.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > n^2 d^{3n-2},$$

then $D_q^{(d)}$ has n-e.c. property.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(d)}$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in F_q \\ x \notin A \cup B}} \prod_{a \in A} \{ 1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a) \}$$

$$\prod_{b \in B} \{ (d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b) \}$$

$$> 0.$$

Now using the method of proof of the Theorem 4.1 we get $f > 0$ when

$$q > n^2 d^{3n-2}.$$

Hence, the result. ❑

References

[1] W. Ananchuen, *On the adjacency property of generalized Paley graphs*, Australas. J. Combin. 6 (2001) 129–147.

[2] W. Ananchuen, *A note on constructing digraphs with prescribed properties*, Australas. J. Combin. 6 (2001) 149–155.

[3] W. Ananchuen and L. Caccetta, *A note on graphs with a prescribed adjacency property*, Bull. Austral. Math. Soc. 51 (1995) 5–15.

[4] W. Ananchuen and L. Caccetta, *On constructing graphs with a prescribed adjacency property*, The Australasian Journal of Combinatorics 10 (1994), 75–83.

[5] W. Ananchuen and L. Caccetta, *On the adjacency properties of Paley graphs*, Networks, 23 (1993) 227–236.

[6] W. Ananchuen and L. Caccetta, *On tournaments with a prescribed property*, ARS Combin. 36 (1993) 89–96.

[7] A. Blass, G. Exoo, and F. Harary, *Paley graphs satisfy all first-order adjacency axioms*, J. Graph Theory 5 (1981) 435–439.

[8] B. Bollobás, **Random Graphs** (Academic Press, London 1985).

[9] A. Bonato, W.H. Holzmann, and H. Kharaghani, *Hadamard matrics and strongly regular graphs with the 3-e.c. adjacency property*, Electron. J. Combin. 8 (2001) 1–9.

[10] J.A. Bondy and U.S.R. Murty, **Graph Theory with Applications**. The MacMillan Press, London (1976).

[11] L. Caccetta, P. Erdös, and K. Vijayan, *A property of random graphs*, ARS Combin. 19A (1985) 287–294.

[12] P.J. Cameron and D. Stark, *A prolific construction of strongly regular graphs with the n-e.c. property*, Electron. J. Combin. 9 (2002) 1–12.

[13] G. Exoo, *On an adjacency property of graphs*, J. Graph Theory 5 (1981) 371–378.

[14] R.L. Graham and J.H. Spencer, *A constructive solution to a tournament problem*, Canad. Math. Bull. 14(1971), pp. 45-48.

[15] W.M. Schmidt, **Equations over Finite Fields; An Elementary Approach**. Lecture Notes in Mathematics, Vol. 536. (Springer-Verlag, Berlin 1976).

Output ที่ได้

จากผลการวิจัยที่กล่าวมาข้างต้น สามารถนำมาเรียบเรียงเขียนเป็นบทความทางวิชาการได้ 2 บทความ

เนื่องจากงานวิจัยในครั้งนี้ได้มีการศึกษาการมีสมบัติ n-e.e. ของกราฟและกราฟทิศทางที่สร้างขึ้นมา เพิ่มเติมจากปัญหาและวัตถุประสงค์ที่วางไว้ เดิมจึงได้เปลี่ยนชื่อบทความให้เหมาะสมกับเนื้อหาดังนี้

บทความที่ 1 ชื่อ "Cubic and quadruple Paley graphs with the *n*-e.c. property" โดย W. Ananchuen และ L. Caccetta บทความนี้ได้เสนอเพื่อตีพิมพ์ในวารสาร Discrete Mathematics ซึ่งมี impact factor เท่ากับ 0.395

บทความที่ 2 ชื่อ "Adjacency Properties of Generalized Paley Graphs" โดย W. Ananchuen และ L. Caccetta บทความนี้ได้เสนอเพื่อตีพิมพ์ในวารสาร Journal of Graph Theory ซึ่งมี impact factor เท่ากับ 0.377

หมายเหตุ ค่าเฉลี่ยของ impact factor ของวารสารที่ตีพิมพ์บทความทางทฤษฎีกราฟ มีค่าประมาณ 0.358

การนำไปใช้ประโยชน์

1. เชิงสาธารณะ

- มีเครือข่ายความร่วมมือกับ Prof. Dr. Louis Caccetta, Department of Mathematics and statistics, Curtin University of Technology, GPO Box U 1987, Perth, WA 6001 AUSTRALIA E-mail: L.Caccetta@exchange.curtin.edu.au
- เนื่องจากการสร้างกราฟที่มีสมบัติ P(m, n, k) และ/หรือสมบัติ n-e.c. เป็นเรื่องที่ค่อนข้างยาก ดังนั้นผลลัพธ์ที่ได้จากงานวิจัยนี้อาจช่วยกระตุ้นความสนใจในวงวิชาการในวงกว้างได้

2. เชิงวิชาการ

- สมบัติ P(m, n, k), สมบัติ n-e.c. ของกราฟ และการสร้างกราฟที่มีสมบัติ P(m, n, k) และ/หรือ สมบัติ n-e.c. ได้รับการบรรจุอยู่ในตำราทางทฤษฎีกราฟขั้นสูง เช่น ตำราเรื่อง **Random Graphs** ซึ่งเขียนโดย B. Bollobás (Academic Press, London 1985) งานวิจัยที่กล่าวมาข้างต้นได้ผลลัพธ์ที่ดีกว่าเดิมและยังเป็นการขยายไปยังกรณีทั่วไป ดังนั้นผลการวิจัยที่ค้นพบจะเป็นประโยชน์ต่อการพัฒนาการเรียนการสอน โดยเฉพาะการปรับปรุงตำราทางทฤษฎีกราฟขั้นสูงต่อไปในอนาคต

ภาคผนวก

(Manusrcipts)

บทความเรื่อง

Cubic and quadruple Paley graphs with the *n*-e.c. property

Cubic and quadruple Paley graphs with the n-e.c. property

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Abstract

A graph G is n-existentially closed or n-e.c. if for any two disjoint subsets A and B of vertices of G with $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. It is well-know that almost all graphs are n-e.c. However, few classes of n-e.c. graphs have been constructed. A good construction is the Paley graphs which are defined as follows. Let $q \equiv 1 (\text{mod } 4)$ be a prime power. The vertices of Paley graphs are the elements of the finite field F_q. Two vertices a and b are adjacent if and only if their difference is a quadratic residue. Previous results established that Paley graphs are n-e.c. for sufficiently large q. By using higher order residues on finite fields we can generate other classes of graphs which we called cubic and quadruple Paley graphs. We show that cubic Paley graphs are n-e.c. whenever $q > n^2 3^{3n-2}$ and quadruple Paley graphs are n-e.c. whenever $q > n^2 4^{3n-2}$. A similar result for quadruple Paley digraphs is also obtained.

Keywords : adjacency property, n-e.c. property, Paley graph, Paley digraph

1. Introduction

For a fixed integer $n \geq 1$. A graph G is called *n-existentially closed* or *n-e.c.* if for any two subsets A and B of vertices of G with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. Observe that if a graph G has property n-e.c., then $\overline{G}$, the complement of G, also has property n-e.c. It is well-known that almost all graphs are n-e.c. However, the problem of constructing graphs with the n-e.c. property seems difficult, especially for $n \geq 4$.

The n-e.c. property was first studied by Caccetta et al. [9], where they were called graphs with property $P(n)$. The authors established, using probabilistic argument, the existence of n-e.c. graphs for a range of n. In particular, they determined the largest integer $f(\nu)$ for which there exists a graph on ν vertices having property $P(f(\nu))$ for a given integer ν. They proved that $\log \nu - (2 + o(1))\log \log \nu < f(\nu)\log 2 < \log \nu$. In addition, a class of 2-e.c. graphs was given for all orders ≥ 9.

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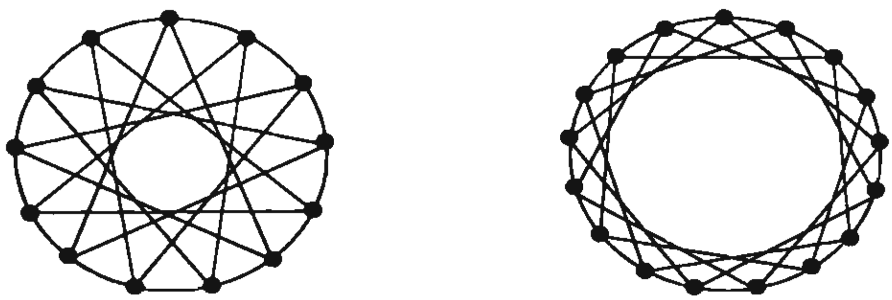
Bonato at al. [8] constructed a new class of 3-e.c. graphs, based on Hadamard matrices. They showed that Bush-type Hadamard matrics of order $16m^2$ give rise to strongly regular 3-e.c. graphs, for each odd m for which $4m$ is the order of a Hadamard matrix. By taking certain affine designs to be Hadamard designs obtained from Paley tournaments, Cameron and Stark [10] have use probabilistic methods to showed that many non-isomorphic strongly regular n-e.c. graphs of order $(q + 1)^2$ exist whenever $q \geq 16n^2 2^{2n}$ is a prime power such that $q \equiv 3 \pmod 4$.

An important graph in the study of the n-e.c. property is the so-called **Paley graph** P_q defined as follows. Let $q \equiv 1 \pmod 4$ be a prime power. The vertices of P_q are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathrm{F}_q$. The n-e.c. property of Paley graphs have been studied by a number of authors [3, 5, 7]; a good discussion is given in the book of Bollobás [7]. With respect to the n-e.c. property, we proved in [3] that if $q \equiv 1 \pmod 4$ is a prime power with $q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + \{(n+1)2^{n-1} - 1\}$, then P_q has the n-e.c. property.

By using higher order residues on finite fields we can generate other classes of graphs. More specifically, for $q \equiv 1 \pmod 3$ a prime power we define the **cubic Paley graph**, $P_q^{(3)}$ as follows. The vertices of $P_q^{(3)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^3$ for some $y \in \mathbf{F}_q$. Since $q \equiv 1 \pmod 3$ is a prime power, -1 is a cubic in $\mathbf{F}_q$. The condition -1 is a cubic in $\mathbf{F}_q$ is needed to ensure that ab is defined to be an edge whenever ba is defined to be an edge. Consequently, $P_q^{(3)}$ is well-defined. Figure 1(a) gives an example.

For $q \equiv 1 \pmod 8$ a prime power, define the **quadruple Paley graph**, $P_q^{(4)}$ as follows. The vertices of $P_q^{(4)}$ are the elements of the finite field $\mathbf{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^4$ for some $y \in \mathbf{F}_q$. Since $q \equiv 1 \pmod 8$ is a prime power, -1 is a quadruple in $\mathbf{F}_q$. Therefore, $P_q^{(4)}$ is well-defined. Figure 1(b) gives an example. The cubic Paley graph and the quadruple Paley graph were first defined in [1].



(a) $P_{13}^{(3)}$ (b) $P_{17}^{(4)}$

Figure 2.1. Graphs $P_{13}^{(3)}$ and $P_{17}^{(4)}$.

Paley constructions have played an important role in constructing classes of graphs with the n-e.c. property, especially for $n \geq 4$, see [3, 7, 10]. In addition to directly providing graphs with interesting adjacency properties, Paley designs played an important role in the construction of strongly regular n-e.c. graphs given in [10]. In the same paper it was noted that the case of affine geometries in place of Paley designs can provide n-e.c. graphs only for $n \leq 3$. In Section 3, we show that the cubic Paley graph $P_q^{(3)}$ has the n-e.c. property whenever $q > n^2 3^{3n-2}$, and the quadruple Paley graph $P_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$.

Another version of adjacency property that has been studied is the following. Let m and n be non-negative integers and k a positive integer. A graph G is said to have *property* $P(m, n, k)$ if for any disjoint sets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property $P(m, n, k)$ is denoted by $\mathcal{G}(m, n, k)$. The class $\mathcal{G}(m, n, k)$ has been studied by Ananchuen [1], Ananchuen and Caccetta [3, 5], Blass et. al. [6] and Exoo [11]. In [1] we proved that the cubic and quadruple Paley graphs are n-e.c. for sufficially large q.

The concept of n-e.c. property of graphs can be extended to digraphs as follows. If (i, j) is an arc in a digraph D, then we say vertex i *dominates* vertex j. A digraph D is *n-e.c.* if for any two subsets A and B of vertices of D with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Let $q \equiv 5 \pmod 8$ be a prime power. Define the **quadruple Paley digraph**, $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadruple in $\mathbf{F}_q$; that is $a - b = y^4$ for some $y \in \mathbf{F}_q$. The n-e.c. property of Paley digraphs have been studied by [4, 7].

In Section 4, we prove that $D_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$.

2. Preliminaries

We make use of the following basic notation and terminology. Let $\mathbf{F}_q$ be a finite field of order q where q is a prime power and let $\mathbf{F}_q[x]$ be a polynomial ring over $\mathbf{F}_q$.

A **character** χ of $\mathbf{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbf{F}_q$, is a map from $\mathbf{F}_q^*$ to the multiplicative group of complex numbers with $|\chi(x)| = 1$ for all $x \in \mathbf{F}_q^*$ and with $\chi(xy) = \chi(x)\chi(y)$ for any $x, y \in \mathbf{F}_q^*$. Among the character of $\mathbf{F}_q^*$, we have the **trivial** character χ_o defined by $\chi_o(x) = 1$ for all $x \in \mathbf{F}_q^*$; all other character of $\mathbf{F}_q^*$ are called **nontrivial**. A character χ is of **order** d if $\chi^d = \chi_o$ and d is the smallest positive integer with this property.

It is customary to extent the definition of nontrivial character χ to the whole $\mathbf{F}_q$ by defining $\chi(0) = 0$. For χ_o we define $\chi_o(0) = 1$.

Observe that

$$\chi^t(a) = \chi(a^t), \tag{2.1}$$

for any $a \in \mathbf{F}_q$ and t a positive integer.

The following lemma, due to Schmidt [12], is very useful to our work.

Lemma 2.1. *Let χ be a nontrivial character of order d of $\mathbf{F}_q$. Suppose $f(x) \in \mathbf{F}_q[x]$ has precisely s distinct zero and it is not a d^{th} power; that is $f(x)$ is not the form $c\{g(x)\}^d$, where $c \in \mathbf{F}_q$ and $g(x) \in \mathbf{F}_q[x]$. Then*

$$\left| \sum_{x \in \mathbf{F}_q} \chi(f(x)) \right| \leq (s-1)\sqrt{q}. \qquad \square$$

Let g be a fixed primitive element of the finite field $\mathbf{F}_q$; that is g is a generator of the cyclic group $\mathbf{F}_q^*$. Define a function α by

$$\alpha(g^t) = e^{\frac{2\pi i t}{3}},$$

where $i^2 = -1$. Therefore, α is a cubic character, character of order 3, of $\mathbf{F}_q$. The values of α are the elements of the set $\{1, \omega, \omega^2\}$ where $\omega = e^{\frac{2\pi i}{3}}$. Note that α^2 is also a cubic character. Moreover, if a is not a cubic of an element of $\mathbf{F}_q^*$, then $\alpha(a) + \alpha^2(a) = -1$. This fact is very important in our methodology.

Further, define a function β by

$$\beta(g^t) = i^t.$$

Therefore, β is the quadruple character, character of order 4, of $\mathbf{F}_q$. The values of β are in the set $\{1, -1, i, -i\}$. Observe that β^3 is also a quadruple character while β^2 is a quadratic character. Moreover, if a is not a quadruple of an element of $\mathbf{F}_q^*$, then $\beta(a) + \beta^2(a) + \beta^3(a) = -1$. This fact is very important in our methodology.

The following lemmas were proved in [1].

Lemma 2.2. *Let α be a cubic character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$ with $|A \cup B| = n$. Put*

$$g = \sum_{x \in \mathbf{F}_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Then

$$g \geq 2^{|B|} q - (n2^{n-1} - 2^n + 1)2^n \sqrt{q}. \qquad \square$$

Lemma 2.3. *Let β be a quadruple character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$ with $|A \cup B| = n$. Put*

$$h = \sum_{x \in \mathbf{F}_q} \prod_{a \in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Then

$$h \geq 3^{|B|} q - (n2^{n-1} - 2^n + 1)3^n \sqrt{q}. \qquad \square$$

3. The cubic and quadruple Paley graphs

For $q \equiv 1 \pmod 3$ a prime power, there exists a cubic character α of $\mathbf{F}_q$ and $\alpha(-a) = \alpha(a)$ for all $a \in \mathbf{F}_q$. Further, for $q \equiv 1 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and $\beta(-a) = \beta(a)$ for all $a \in \mathbf{F}_q$.

Observe that if a and b are any vertices of $P_q^{(3)}$, then for $t = 1$ and 2

$$\alpha^t(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ \omega \text{ or } \omega^2, & \text{otherwise.} \end{cases}$$

Also, if a and b are any vertices of $P_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b, \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise.} \end{cases}$$

Note that β^2 is a quadratic character; that is

$$\beta^2(a-b) = \begin{cases} 1, & \text{if } a-b \text{ is a quadratic ressidue}, \\ 0, & \text{if } a = b, \\ -1, & \text{otherwise.} \end{cases}$$

Our first result concerns cubic Paley graph having property n-e.c. for any fixed integer $n \geq 1$.

Theorem 3.1. *Let $q \equiv 1 \pmod 3$ be a prime power. If*

$$q > n^2 3^{3n-2},$$

then $P_q^{(3)}$ has the n-e.c. property. Furthermore, for $n > 1$ the graph $P_q^{(3)}$ is n-e.c whenever $q > n^2 3^{3n-4}$.

Proof: Let A and B be disjoint subsets of $V(P_q^{(3)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbf{F}_q \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}$$

$$> 0.$$

Let

$$g = \sum_{x \in \mathbf{F}_q} \prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B} \{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Now, by Lemma 2.2 we have

$$g \geq 2^n q - (n2^{n-1} - 2^n + 1)2^n\sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} \prod_{a \in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\} \prod_{b \in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}.$$

Since, in the product $\prod_{a \in A}\{1 + \alpha(x-a) + \alpha^2(x-a)\}$ each factor is at most 3 and one factor is 1 and in the product $\prod_{b \in B}\{2 - \alpha(x-b) - \alpha^2(x-b)\}$ each factor is at most 3 and one factor is 2 we have

$$\begin{aligned} g - f &\leq 3^{n-1}|A| + 3^{n-1}2|B| \\ &= (|A| + 2|B|)3^{n-1} \\ &\leq 2n3^{n-1}. \end{aligned}$$

Consequently,

$$f \geq 2^n q - (n2^{n-1} - 2^n + 1)2^n\sqrt{q} - 2n3^{n-1}.$$

Now, if $q > n^2 3^{3n-2}$, then $f > 0$ as required.

It is easily cheeked that $f > 0$ when $q > n^2 3^{3n-4}$ for $n > 1$. ❑

Remark 3.1. The bound for q in Theorem 3.1 can be improved to $n^2 3^{2.5n}$ for $1 \leq n \leq 55$.

We now turn our attention to the adjacent property of the quadruple Paley graph $P_q^{(4)}$.

Theorem 3.2 *Let $q \equiv 1 \pmod 8$ be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $P_q^{(4)}$ has the n-e.c property.

Proof: Let A and B be disjoint subsets of $V(P_q^{(4)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in F \\ x \notin A \cup B}} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

$$> 0.$$

Let

$$h = \sum_{x \in F} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

Now, by Lemma 2.3, we have

$$h \geq 3^n q - (n2^{n-1} - 2^n + 1)3^n\sqrt{q}.$$

Consider

$$h - f = \sum_{x \in A \cup B} \prod_{a \in A}\{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B}\{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

Since, in the product $\prod_{a \in A} \{ 1 + \beta(x - a) + \beta^2(x - a) + \beta^3(x - a)\}$ each factor is at most 4 and one factor is 1 and in the product $\prod_{b \in B} \{ 3 - \beta(x - b) - \beta^2(x - b) - \beta^3(x - b)\}$ each factor is at most 4 and one factor is 3 we have

$$h - f \leq |A| 4^{n-1} + 3|B| 4^{n-1}$$
$$\leq 3n4^{n-1}.$$

Consequently,

$$f \geq 3^n q - (n2^{n-1} - 2^n + 1)3^n \sqrt{q} - 3n4^{n-1}.$$

Now, if $q > n^2 4^{3n-2}$, then $f > 0$ as required. ❑

Remark 3.2. The bound for q in Theorem 3.2 can be improved to $q > n^2 4^{3n-3}$ for $n > 1$ or $n^2 4^{2.5n}$ for $1 \leq n \leq 14$.

4. Quadruple Paley digraphs

In this section, our graphs are directed. Recalled that, digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A \cap B = \emptyset$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B. For $q \equiv 5 (\mathrm{mod}\ 8)$ be a prime power. Define the quadruple Paley digraph $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadruple in $\mathbf{F}_q$. Since $q \equiv 5(\mathrm{mod}\ 8)$ is a prime power, -1 is not a quadruple in $\mathbf{F}_q$. The condition -1 is not a quadruple in $\mathbf{F}_q$ is needed to ensure that (b, a) is not defined to be an arc when (a, b) is defined to be an arc. Consequently, $D_q^{(4)}$ is well-defined. However, $D_q^{(4)}$ is not a tournament. Figure 4.1 displays the digraph $D_{13}^{(4)}$. The quadruple Paley digraph was first defined in [2].

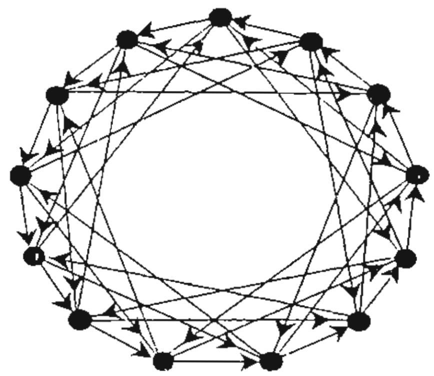


Figure 4.1. Paley digraph $D_{13}^{(4)}$.

For $q \equiv 5 (\mathrm{mod}\ 8)$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and noting that if a and b are any vertices of $D_q^{(4)}$, then for $t = 1$ and 3

$$\beta^t(a-b) = \begin{cases} 1, & \text{if } a \text{ dominates } b, \\ 0, & \text{if } a = b, \\ -1, i \text{ or } -i, & \text{otherwise}. \end{cases}$$

Note that β^2 is a quadratic character. Further, $\beta(-a) = -\beta(a)$ for any $a \in \mathbf{F}_q$.

Theorem 4.1. *Let $q \equiv 5 (\mathrm{mod}\ 8)$ be a prime power. If*

$$q > n^2 4^{3n-2},$$

then $D_q^{(4)}$ has n-e.c. property.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(4)}$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in F \\ x \notin A \cup B}} \prod_{a \in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}$$

$$> 0.$$

Now using the method of proof of the Theorem 3.2 we get $f > 0$ when

$$q > n^2 4^{3n-2}$$

Hence, the result. ❑

Remark 4.1. The bound for q in Theorem 4.1 can be improved to $n^2 4^{2.5n}$ for $1 \le n \le 14$.

Reference

[1] W. Ananchuen, On the adjacency property of generalized Paley graphs, Australas. J. Combin. 6 (2001) 129–147.

[2] W. Ananchuen, A note on constructing digraphs with prescribed properties, Australas. J. Combin. 6 (2001) 149–155.

[3] W. Ananchuen and L. Caccetta, On the adjacency properties of Paley graphs, Networks, 23 (1993) 227–236.

[4] W. Ananchuen and L. Caccetta, On tournaments with a prescribed property, ARS Combin. 36 (1993) 89–96.

[5] W. Ananchuen and L. Caccetta, A note on graphs with a prescribed adjacency property, Bull. Austral. Math. Soc. 51 (1995) 5–15.

[6] A. Blass, G. Exoo and F. Harary, Paley graphs satisfy all first-order adjacency axioms, J Graph Theory 5 (1981) 435–439.

[7] B. Bollobás, Random Graphs (Academic Press, London 1985).

[8] A. Bonato, W.H. Holzmann and H. Kharaghani, Hadamard matrics and strongly regular graphs with the 3-e.c. adjacency property, Electron. J. Combin. 8 (2001) 1–9.

[9] L. Caccetta, P. Erdös and K. Vijayan, A property of random graphs, ARS Combin. 19A (1985) 287 - 294.

[10] P.J. Cameron and D. Stark, A prolific construction of strongly regular graphs with the n-e.c property, Electron. J. Combin. 9 (2002) 1–12
[11] G. Exoo, On an adjacency property of graphs, J. Graph Theory 5 (1981) 371–378.
[12] W.M. Schmidt, Equations over Finite Fields, An Elementary Approach, Lecture Notes in Mathematics, Vol. 536, (Springer-Verlag, Berlin 1976).

บทความเรื่อง

Adjacency Properties of Generalized Paley Graphs

Adjacency Properties of Generalized Paley Graphs

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Abstract

Let m and n be non-negative integers and k a positive integer. A graph G is said to have property $P(m,n,k)$ if for any disjoint subsets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. Furthermore, a graph G is called n-existentially closed or n-e.c. if for any two subsets A and B of vertices of G with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. It is well-known that almost all graphs satisfy the $P(m,n,k)$ property and the n-e.c. property. However, the problem of constructing graphs with the $P(m,n,k)$ property and the n-e.c. property seems difficult. In this paper, we show that all sufficiently large generalized Paley graphs defined by using higher order residues on finite fields satisfy the $P(m,n,k)$ property and the n-e.c. property. Similar results for generalized Paley digraphs are also obtained.

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1. Introduction

Let m and n be non-negative integers and k a positive integer. A graph G is said to have *property* $P(m,n,k)$ if for any disjoint subsets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property $P(m,n,k)$ is denoted by $\mathcal{G}(m,n,k)$. Observe that if a graph G has property $P(m,n,k)$, then $\overline{G}$, the complement of G, has property $P(n,m,k)$. It is well-known [6] that almost all graphs have property $P(m,n,k)$. Despite this result, few graphs have been constructed which exhibit the property $P(m,n,k)$. The class $\mathcal{G}(m,n,k)$ has been studied by many authors including:

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Ananchuen [1]; Ananchuen and Caccetta [2, 4]; Blass et al. [6]; Bollobás [7]; and Exoo [11].

An important graph in the study of the property $P(m, n, k)$ is the so-called Paley graph P_q defined as follows. Let $q \equiv 1 (\text{mod } 4)$ be a prime power. The vertices of P_q are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbb{F}q$. The adjacency properties of Paley graphs have been studied by a number of authors [2, 4, 6, 7]; a good discussion is given in the book of Bollobás [7]. With respect to the property $P(n, n, 1)$ we proved in [2] that if $q \equiv 1(\text{mod } 4)$ is a prime power with $q > ((2n-3)2^{2n-1}+4)^2$, then $P_q \in \mathcal{G}(n, n, 1)$.

By using higher order residues on finite fields we can generate other classes of graphs. More specifically, let q and d be positive integers such that q is a prime power and

$$d > 1 \text{ is odd or } (q-1)/d \text{ is even.}$$

We define the *generalized Paley graph*, $P_q^{(d)}$ as follows. The vertices of $P_q^{(d)}$ are the elements of finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if $a - b = y^d$ for some $y \in \mathbb{F}_q$. Since q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even, $-1 = y^d$ for some $y \in \mathbb{F}_q$. The condition -1 is a d^{th} power of an element of $\mathbb{F}_q$ is needed to ensure that ba is defined to be an edge precisely whenever ab is defined to be an edge. Consequently, $P_q^{(d)}$ is well-defined. Clearly, $P_q^{(2)}$ is the Paley graph. $P_q^{(3)}$ is called the cubic Paley graph and $P_q^{(4)}$ the quadruple Paley graph in [1]. It has been proved [1] that all sufficiently large cubic and quadruple Paley graphs satisfy the $P(m, n, k)$ property.

In Section 3, we will show that the generalized Paley graphs satisfy the property $P(n, n, 1)$ whenever $q > n^2d^{4n}$.

Another version of adjacency property that has been stuaied is the following. For a fixed integer $n \geq 1$. A graph G is called *n-existentially closed* or *n-e.c.* if for any two subsets A and B of vertices of G with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ that is adjacent to every vertex of A but not adjacent to any vertex of B. Observe that if a graph G has property n-e.c., then $\overline{G}$, the complement of G, also has property n-e.c. It is well-known that for any fixed n, almost all graphs are n-e.c. However, the problem of constructing graphs with the n-e.c. property seems difficult, especially for $n \geq 4$.

The n-e.c. property was first studied by Caccetta et al. [9], where they were called graphs with property $P(n)$. The authors established, using a probabilistic argument, the existence of n-e.c. graphs for a range of n. In particular, they determined the largest integer $f(n)$ for which there exists a graph on n vertices having property $P(f(n))$ for a given integer n. They proved that $\log n - (2 + o(1)) \log \log n < f(n) \log 2 < \log n$. In addition, a class of 2-e.c. graphs was given for all orders ≥ 9.

Bonato et al. [8] constructed a new class of 3-e.c. graphs, based on Hadamard matrices. They showed that Bush-type Hadamard matrics of order $16m^2$ gives rise to strongly regular 3-e.c. graphs, for each odd m for which $4m$ is the order of a Hadamard matrix. By taking certain affine designs to be Hadamard designs obtained from Paley tournaments, Cameron and Stark [10] have use probabilistic methods to show that many non-isomorphic strongly regular n-e.c. graphs of order $(q+1)^2$ exist whenever $q \geq 16n^2 2^{2n}$ is a prime power such that $q \equiv 3 (\text{mod } 4)$. Ananchuen and Caccetta [5] show that the cubic Paley graph $P_q^{(3)}$ has the n-e.c. property whenever $q > n^2 3^{3n-2}$, and the quadruple Paley graph $P_q^{(4)}$ has the n-e.c. property whenever $q > n^2 4^{3n-2}$. In section 4, we prove that the generalized Paley graph has the n-e.c. property whenever $q > n^2 d^{3n-2}$.

The concept of adjacency property of graphs can be extended to digraphs as follows. If (i, j) is an arc in a digraph D, then we say vertex i *dominates* vertex j. A digraph D is said to have property $Q(n, k)$ if every subset of n vertices of D is dominated by at least k other vertices. Graham and Spencer [12] defined the following digraph. Let $p \equiv 3 (\text{mod } 4)$ be a prime. The vertices of digraph D_p are $\{0, 1, ..., p-1\}$ and D_p contains the arc (a, b) if and only if $a - b$ is a quadratic residue modulo p. The digraph D_p is sometimes referred to as the Paley tournament. Graham and Spencer [12] proved that D_p has property $Q(n, 1)$ whenever $p > n^2 2^{2n-2}$. Bollobás [7] extended these results to prime powers. More specifically, if $q \equiv 3 (\text{mod } 4)$ is a prime power, the Paley tournament D_q is defined as follows. The vertex set of D_q are the elements of the finite field $\mathbb{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadratic residue in $\mathbb{F}_q$. Bollobás [7] noted that D_q has property $Q(n, 1)$ whenever $q > \{(n-2)2^{n-1} + 1\} + n2^{n-1}$. Ananchuen and Caccetta [3] proved that D_q has property $Q(n, k)$ whenever $q > \{(n-3)2^{n-1} + 2\} + k2^{n-1}$.

Let q and d be positive integers such that q is a prime power and

$$d > 1 \text{ is even and } (q-1)/d \text{ is odd.}$$

We define the *generalized Paley digraph*, $D_q^{(d)}$ as follows. The vertices of $D_q^{(d)}$ are the elements of the finite field $\mathbb{F}_q$. A vertex a joins to vertex b by an arc if and only if $a - b = y^d$ for some $y \in \mathbb{F}_q$. Since $d > 1$ is even and $(q-1)/d$ is odd, -1 is not a d^{th} power of any element of $\mathbb{F}_q$. The condition -1 is not a d^{th} power of any element of $\mathbb{F}_q$ is needed to ensure that (b, a) is not defined to be an arc whenever (a, b) is defined to be an arc. Consequently, $D_q^{(d)}$ is well-defined.

In Section 5, we show that the generalized Paley digraph $D_q^{(d)}$ has the property $Q(n, 1)$ whenever $q > n^2 d^{2n}$.

A digraph D is *n-e.c.* if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

In Section 6, we show that the generalized Paley digraph $D_q^{(d)}$ is n-e.c. whenever $q > n^2 d^{3n-2}$.

2. Preliminaries

We make use of the following basic notation and terminology. Let $\mathbb{F}_q$ be a finite field of order q where q is a prime power. A *character* χ on $\mathbb{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbb{F}_q$, is a homomorphism from $\mathbb{F}_q^*$ to the multiplicative group of complex number with $|\chi(x)| = 1$ for all x. Among the characters of $\mathbb{F}_q^*$, we have the *trivial character* χ_o defined by $\chi_o(x) = 1$ for all $x \in \mathbb{F}_q^*$; all other characters of $\mathbb{F}_q^*$ are called *nontrivial*. A character χ is of *order* d if $\chi^d = \chi_o$ and d is the smallest positive integer with this property.

It is customary to extent the definition of character χ to the whole $\mathbb{F}_q$ by putting $\chi(0) = 0$ and $\chi_o(0) = 1$.

Observe that (see[13])

$$\sum_{\substack{\chi \text{ of order dividing } d \\ \chi \neq \chi_o}} \chi(x) = \begin{cases} d-1, & \text{if } x = y^d \text{ for some } y \in \mathbb{F}_q^*, \\ 0, & \text{if } x = 0, \\ -1, & \text{otherwise.} \end{cases} \quad (2.1)$$

This fact is very important in our methodology. Moreover.

$$\chi(a^r) = \chi^r(a), \quad (2.2)$$

for any $a \in \mathbb{F}_q$ and r is a positive integer.

The following lemma, due to Schmidt [13], is very useful to our work.

Lemma 2.1. *Let χ be a nontrivial character of order d of $\mathbb{F}_q$. Suppose $f(x) \in \mathbb{F}_q[x]$ has precisely s distinct zero and it is not a d^{th} power; that is $f(x)$ is not the form $c\{g(x)\}^d$, where $c \in \mathbb{F}_q$ and $g(x) \in \mathbb{F}_q[x]$. Then*

$$|\sum_{x \in \mathbb{F}_q} \chi(f(x)| \leq (s-1)\sqrt{q}. \quad \square$$

For g a fixed primitive element of the finite field $\mathbb{F}_q$; that is g is a generator of the cyclic group $\mathbb{F}_q^*$. Define a function α by

$$\alpha(g^k) = e^{\frac{2k\pi i}{d}},$$

where $i^2 = -1$. Therefore, α is a character of order dividing d and the value of α are the elements of the set $\{e^{\frac{2k\pi i}{d}} | k = 0, 1, \ldots, d-1\}$. It is not too difficult to verify that $\alpha, \alpha^2, \ldots, \alpha^{d-1}$ are characters of order dividing d and are all different.

The following two lemmas are extensively used in establishing our results.

Lemma 2.2. *Let α be a character of order d of F_q and let A and B be disjoint subsets of F_q. Put*

$$g = \sum_{x \in \mathbb{F}_q} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$
$$\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\}].$$

Then

$$g \geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t \sqrt{q},$$

where $|A| = m$, $|B| = n$ and $m + n = t$.

Proof: Let $A \cup B = \{c_1, c_2, \ldots, c_t\}$. Expanding g and noting that $\sum_{x \in \mathbb{F}_q} (d-1)^n = (d-1)^n q$, we can write

$$|g - (d-1)^n q| \leq |\sum_{x \in \mathbb{F}_q} \sum_{\chi \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i=1}^{t} (d-1)^{t-1} \chi(x - c_i)| +$$
$$|\sum_{x \in \mathbb{F}_q} \sum_{\chi_j \in \{\alpha, \alpha^2, \ldots, \alpha^{d-1}\}} \sum_{i_1 < i_2} (d-1)^{t-2} \chi_1(x - c_{i_1})\chi_2(x - c_{i_2})| + \ldots +$$

$$|\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\dots,\alpha^{d-1}\}}\sum_{i_1<i_2\dots<i_s}(d-1)^{t-s}\chi_1(x-c_{i_1})\chi_2(x-c_{i_2})\dots\chi_s(x-c_{i_s})|$$
$$+\dots+|\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\dots,\alpha^{d-1}\}}(d-1)^{t-s}\chi_1(x-c_1)\chi_2(x-c_2)\dots\chi_t(x-c_t)|.$$

Now, by (2.1), (2.2) and Lemma 2.1 we have

$$|g-(d-1)^n q| \le \sum_{s=1}^{t}(d-1)^s(d-1)^{t-s}\binom{t}{s}(s-1)\sqrt{q}$$
$$= (t2^{t-1}-2^t+1)(d-1)^t.$$

Therefore, $g \ge (d-1)^n q - (t2^{t-1}-2^t+1)(d-1)^t\sqrt{q}$ as required. □

Lemma 2.3. *Let α be a character of order d of F_q and let A be a subsets of n vertices of F_q. Put*

$$h = \sum_{x\in\mathbb{F}_q}\prod_{a\in A}\{1+\alpha(x-a)+\alpha^2(x-a)+\dots+\alpha^{d-1}(x-a)\}.$$

Then

$$h \ge q - [1+(nd-n-d)d^{n-1}]\sqrt{q}.$$

Proof: Let $A=\{a_1,a_2,\dots,a_n\}$. We can write

$$h = \sum_{x\in\mathbb{F}_q}1 + \sum_{\chi\in\{\alpha,\alpha^2,\dots,\alpha^{d-1}\}}\sum_{i=1}^{n}\chi(x-a_i)+$$
$$\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\dots,\alpha^{d-1}\}}\sum_{i_1<i_2}\chi_1(x-a_{i_1})\chi_2(x-a_{i_2})+\dots+$$
$$\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\dots,\alpha^{d-1}\}}\sum_{i_1<i_2\dots<i_s}\chi_1(x-a_{i_1})\chi_2(x-a_{i_2})\dots\chi_s(x-a_{i_s})+\dots+$$
$$\sum_{x\in\mathbb{F}_q}\sum_{\chi_j\in\{\alpha,\alpha^2,\dots,\alpha^{d-1}\}}\chi_1(x-a_1)\chi_2(x-a_2)\dots\chi_n(x-a_n).$$

Now, by (2.1), (2.2) and Lemma 2.1 we have

$$|h-q| \le \sum_{s=1}^{n}(d-1)^s\binom{n}{s}(s-1)\sqrt{q}$$
$$= [1+(nd-n-d)d^{n-1}]\sqrt{q}.$$

Therefore, $h \ge q - [1+(nd-n-d)d^{n-1}]\sqrt{q}$ as required. □

3. The property $P(m,n,k)$

Note that, for q and d positive integers such that q is a prime power and $d>1$ is odd or $(q-1)/d$ is even, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a)=\alpha(a)$ for all $a\in\mathbb{F}_q$. Furthermore, if α is a character of order d of $\mathbb{F}_q$ and a and b are vertices of $P_q^{(d)}$, then

$$\alpha(a-b) = \begin{cases} 1, & \text{if } a \text{ is adjacent to } b,\\ 0, & \text{if } a=b,\\ \omega, & \text{otherwise;}\end{cases}$$

where $\omega \in \{e^{\frac{2k\pi i}{d}} \mid k=1,\dots,d-1\}$.

Our first result concerns the generalized Paley graphs having property $P(m,n,k)$.

Theorem 3.1. *Let q and d be positive integers such that q is a prime power and $d>1$ is odd or $(q-1)/d$ is even. If*

$q > (t2^{t-1} - 2^t + 1)(d-1)^m\sqrt{q} + [m + (d-1)n + (k-1)d](d-1)^{-n}d^{t-1},$ (3.1)
then $P_q^{(d)} \in \mathcal{G}(m,n,k)$ *for all* m, n *with* $m + n \leq t$.

Proof: It clearly suffices to establish the result for $m + n = t$. Let A and B be disjoint subsets of $V(P_q^{(d)})$ with $|A| = m$ and $|B| = n$. Then there are at least k other vertices, each of which adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x\in\mathbb{F}_q \\ x\notin A\cup B}} [\prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a) + ... + \alpha^{d-1}(x-a)\}$$
$$\prod_{b\in B}\{(d-1) - \alpha(x-b) - \alpha^2(x-b) - ... - \alpha^{d-1}(x-b)\}]$$
$$\geq kd^t.$$

To show that $f \geq kd^t$, it is clearly sufficient to establish that $f > (k-1)d^t$.

Let g be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2 we have

$$g \geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q}.$$

Consider

$$g - f = \sum_{x\in A\cup B} [\prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a) + ... + \alpha^{d-1}(x-a)\}$$
$$\prod_{b\in B}\{(d-1) - \alpha(x-b) - \alpha^2(x-b) - ... - \alpha^{d-1}(x-b)\}]$$
$$\leq d^{t-1}m + (d-1)d^{t-1}n$$
$$= [m + (d-1)n]d^{t-1}$$

since, in the product $\prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a) + ... + \alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b\in B}\{(d-1) - \alpha(x-b) - \alpha^2(x-b) - ... - \alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$. Therefore,

$$f \geq g - t(d-1)d^{t-1}$$
$$\geq (d-1)^n q - (t2^{t-1} - 2^t + 1)(d-1)^t\sqrt{q} - [m + (d-1)n]d^{t-1}.$$

Now, if inequality (3.1) holds, then $f > (k-1)d^t$ as required. □

For the case $m = n$, we have the following sharper result.

Theorem 3.2. *Let* q *and* d *be positive integers such that* q *is a prime power and* $d > 1$ *is odd or* $(q-1)/d$ *is even. If*

$q > (n2^{2n} - 2^n + 1)(d-1)^n\sqrt{q} + [(d-1)n + (k-1)](d-1)^{-n}d^{2n-1},$ (3.2)

then $P_q^{(d)}$ *has property* $P(n,n,k)$. *In particular, for* $k = 1$ *the graphs* $P_q^{(d)}$ *has property* $P(n,n,1)$ *whenever* $q > n^2 d^{4n}$.

Proof: Let A and B be disjoint subset of $V(P_q^{(d)})$ with $|A| = |B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x\in\mathbb{F}_q \\ x\notin A\cup B}} [\prod_{a\in A}\{1 + \alpha(x-a) + \alpha^2(x-a) + ... + \alpha^{d-1}(x-a)\}$$

$$\prod_{b\in B}\{(d-1)-\alpha(x-b)-\alpha^2(x-b)-...-\alpha^{d-1}(x-b)\}]$$
$$\geq (k-1)d^{2n}.$$

Let h be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2, we have

$$h \geq (d-1)^n q - (n2^{2n} - 2^{2n} + 1)(d-1)^{2n}\sqrt{q}.$$

Consider

$$h - f = \sum_{x\in A\cup B}\prod_{i=1}^{n}[\{1+\alpha(x-a_i)+\alpha^2(x-a_i)+...+\alpha^{d-1}(x-a_i)\}$$
$$\{(d-1)-\alpha(x-b_i)-\alpha^2(x-b_i)-...-\alpha^{d-1}(x-b_i)\}],\quad(3.3)$$

where $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$.

If $h - f \neq 0$, then for some x_o the product

$$\prod_{i=1}^{n}\{1+\alpha(x_o-a_i)+\alpha^2(x_o-a_i)+...+\alpha^{d-1}(x_o-a_i)\}$$
$$\{(d-1)-\alpha(x_o-b_i)-\alpha^2(x_o-b_i)-...-\alpha^{d-1}(x_o-b_i)\}\neq 0. \quad (3.4)$$

With out any loss of generality suppose $x_o = a_k$. For (3.3) to hold we must have

$$\alpha(a_k-a_i)+\alpha^2(a_k-a_i)+...+\alpha^{d-1}(a_k-a_i)\neq -1$$

and

$$\alpha(a_k-b_i)+\alpha^2(a_k-b_i)+...+\alpha^{d-1}(a_k-b_i)\neq d-1 \text{ for all } i.$$

This means that

$$\alpha(a_k-a_i)+\alpha^2(a_k-a_i)+...+\alpha^{d-1}(a_k-a_i)= d-1 \text{ for } i\neq k$$

and

$$\alpha(a_k-b_i)+\alpha^2(a_k-b_i)+...+\alpha^{d-1}(a_k-b_i)= -1 \text{ for all } i.$$

Hence, the term in (3.4) with $x = b_i$ for all i contributes zero to the sum. Thus we can write (3.3) as

$$h - f = \sum_{x\in A}[\prod_{i=1}^{n}\{1+\alpha(x-a_i)+\alpha^2(x-a_i)+...+\alpha^{d-1}(x-a_i)\}$$
$$\{(d-1)-\alpha(x-b_i)-\alpha^2(x-b_i)-...-\alpha^{d-1}(x-b_i)\}]$$
$$\leq n(d-1)d^{2n-1},$$

since in the product $\prod_{i=1}^{n}\{1+\alpha(x-a_i)+\alpha^2(x-a_i)+...+\alpha^{d-1}(x-a_i)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{i=1}^{n}\{(d-1)-\alpha(x-b_i)-\alpha^2(x-b_i)-...-\alpha^{d-1}(x-b_i)\}$ each factor is at most d and one factor is $d-1$. Therefore,

$$f \geq h - n(d-1)d^{2n-1}$$
$$f \geq (d-1)^n q - (n2^{2n} - 2^{2n} + 1)(d-1)^{2n}\sqrt{q} - n(d-1)d^{2n-1}.$$

Now, if inequality (3.2) holds, then $f > (k-1)d^{2n}$ as required. It is easily checked that $f > 0$ whenever $q > n^2 d^{4n}$ for $k = 1$. □

4. The n-e.c. property

In this section, we will show that the generalized Paley graphs having property n-e.c.

Theorem 4.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is odd or $(q-1)/d$ is even. If*

$$q > n^2 d^{3n-2},$$

then $P_q^{(d)}$ has the n-e.c. property.

Proof: Let A and B be disjoint subsets of $V(P_q^{(d)})$ with $|A \cup B| = n$. Then there is a vertex $u \notin A \cup B$ that adjacent to every vertex of A but not adjacent to any vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbb{F}_q \\ x \notin A \cup B}} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \dots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \dots - \alpha^{d-1}(x-b)\}] > 0.$$

Let g be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2, we have

$$g \geq (d-1)^{|B|} q - (n2^{n-1} - 2^n + 1)(d-1)^n \sqrt{q}.$$

Consider

$$g - f = \sum_{x \in A \cup B} [\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \dots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \dots - \alpha^{d-1}(x-b)\}].$$

Since, in the product $\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \dots + \alpha^{d-1}(x-a)\}$ each factor is at most d and one factor is 1 and in the product $\prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \dots - \alpha^{d-1}(x-b)\}$ each factor is at most d and one factor is $d-1$ and either A or B can be empty, then we can estimate $g - f$ as

$$g - f \leq (d-1) n d^{n-1}.$$

Hence

$$\begin{aligned} f &\geq h - (d-1) n d^{n-1} \\ &\geq (d-1)^{|B|} q - (n2^{n-1} - 2^n + 1)(d-1)^n \sqrt{q} - (d-1) n d^{n-1}. \end{aligned}$$

Now, if $q > n^2 d^{3n-2}$, then $f > 0$ as required. □

5. The property $Q(n, k)$

Note that for q and d positive integers which q a prime power and $d > 1$ is even and $(q-1)/d$ is odd, there exists a character α of order d of $\mathbb{F}_q$ and $\alpha(-a) = -\alpha(a)$ for all $a \in \mathbb{F}_q$. Further more, if a and b are any vertices of $D_q^{(d)}$, then

$$\alpha(a-b) = \begin{cases} 1, & \text{if } a \text{ dominates } b, \\ 0, & \text{if } a = b, \\ \omega, & \text{otherwise;} \end{cases}$$

where $\omega \in \{e^{\frac{2k\pi i}{d}} | k = 1, \dots, d-1\}$.

In this section, we will show that the generalized Paley digraphs having property $Q(n, k)$.

Theorem 5.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > [1 + (nd - n - d)d^{n-1}]\sqrt{q} + (1 + kd - d)d^{n-1}, \qquad (5.1)$$

then $D_q^{(d)}$ *has property* $Q(n,k)$. *In particular, the graphs* $D_q^{(d)}$ *has property* $Q(n,1)$ *whenever* $q > n^2d^{2n}$.

Proof: Let A subset of $V(D_q^{(d)})$ with $|A| = n$. Then there is a vertex $u \notin A$ that dominates every vertex of A if and only if

$$f = \sum_{\substack{x\in\mathbb{F}_q \\ x\notin A}} \prod_{a\in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\}$$
$$> (k-1)d^n.$$

Let h be defined similarly to f except that the sum is taken over all $x \in \mathbb{F}_q$. Now, by Lemma 2.2, we have

$$h \geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q}.$$

Consider

$$h - f = \sum_{x\in A} \prod_{i=1}^{n} \{1 + \alpha(x-a_i) + \alpha^2(x-a_i) + \ldots + \alpha^{d-1}(x-a_i)\}, \qquad (5.2)$$

where $A = \{a_1, a_2, \ldots, a_n\}$.

If $h - f \neq 0$, then for some x_o the product

$$\prod_{i=1}^{n} \{1 + \alpha(x_o - a_i) + \alpha^2(x_o - a_i) + \ldots + \alpha^{d-1}(x_o - a_i)\} \neq 0. \qquad (5.3)$$

With out any loss of generality suppose $x_o = a_k$. For (5.2) to hold we must have $\alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) \neq -1$ for all i. This means that for $i \neq k, \alpha(a_k - a_i) + \alpha^2(a_k - a_i) + \ldots + \alpha^{d-1}(a_k - a_i) = d - 1$. Therefore, a_k is unique $h - f = d^{n-1}$. Then, since $h - f$ could be 0 we conclude that

$$h - f \geq d^{n-1}.$$

So

$$f \geq h - d^{n-1}$$
$$\geq q - [1 + (nd - n - d)d^{n-1}]\sqrt{q} - d^{n-1}.$$

Now, if inequality (5.1) holds, then $f > (k-1)d^n$ as required. It is easily checked that $f > 0$ whenever $q > n^2d^{2n}$ for $k = 1$. □

6. The n-e.c. property for digraphs

Recalled that a digraph D is n-e.c. if for any two subsets A and B of vertices of D with $A \cap B = \varnothing$ and $|A \cup B| = n$, there is a vertex $u \notin A \cup B$ such that u dominates every vertex of A and dominated by every vertex of B.

Theorem 6.1. *Let q and d be positive integers such that q is a prime power and $d > 1$ is even and $(q-1)/d$ is odd. If*

$$q > n^2d^{3n-2},$$

then $D_q^{(d)}$ *has the* n*-e.c. property.*

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(d)}$ with $|A \cap B| = n$. Then there is a vertex $u \in A \cap B$ that dominates every vertex of A but is dominated by every vertex of B if and only if

$$f = \sum_{\substack{x \in \mathbf{F}_q \\ x \notin A \cup B}} \Big[\prod_{a \in A} \{1 + \alpha(x-a) + \alpha^2(x-a) + \ldots + \alpha^{d-1}(x-a)\} \prod_{b \in B} \{(d-1) - \alpha(x-b) - \alpha^2(x-b) - \ldots - \alpha^{d-1}(x-b)\} \Big]$$

> 0.

Now using the method of proof of the Theorem 4.1 we get $f > 0$ whenever $q > n^2 d^{3n-2}$.

Hence, the result. □

References

[1] W. Ananchuen, On the adjacency property of generalized Paley graphs, *Australas. J. Combin.* **6** (2001) 129-147.

[2] W. Ananchuen and L. Caccetta, On the adjacency properties of Paley graphs, *Networks* **23** (1993) 227-236.

[3] W. Ananchuen and L. Caccetta, On tournaments with a prescribed property, *ARS Combin.* **36** (1993) 89-96.

[4] W. Ananchuen and L. Caccetta, A note on graphs with a prescribed adjacency property, *Bull. Austral. Math. Soc.* **51** (1995) 5-15.

[5] W. Ananchuen and L. Caccetta, Cubic and quadruple Paley graphs with n-e.c. property, (submitted)

[6] A. Blass, G. Exoo, and F. Harary, Paley graphs satisfy all first-order adjacency axioms, *J. Graph Theory* **5** (1981) 435-439.

[7] B. Bollobás, *Random Graphs*. Academic Press, London (1985).

[8] A. Bonato, W.H. Holzmann, and H. Kharaghani, Hadamard matrics and strongly regular graphs with the 3-e.c. adjacency property, *Electron. J. Combin.* **8** (2001) 1-9.

[9] L. Caccetta, P. Erdös, and K. Vijayan, A property of random graphs, *ARS Combin.* **19A** (1985) 287-294.

[10] P.J. Cameron and D. Stark, A prolific construction of strongly regular graphs with the n-e.c. property, *Electron. J. Combin.* **9** (2002) 1-12.

[11] G. Exoo, On an adjacency property of graphs, *J. Graph Theory* **5** (1981) 371-378.

[12] R.L. Graham and J.H. Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.* **14** (1971) 45-48.

[13] W.M. Schmidt, *Equations over Finite Fields, An Elementary Approach*. Lecture Notes in Mathematics, Vol. 536. Springer-Verlag, Berlin (1976).