บทความเรื่อง

# A characterization of maximal non *k*-factor-critical graphs

โดย

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A *matching* $M$ in $G$ is a subset of $E(G)$ in which no two edges have a vertex in common. A vertex $v$ is saturated by $M$ if some edge of $M$ is incident to $v$; otherwise $v$ is said to be unsaturated. A matching $G$ is *perfect* if it saturates every vertex of $G$. For simplicity we let $V(M)$ denote the vertex set of the subgraph $G[M]$ induced by $M$. A graph $G$ of order $p$ is *$k$-factor-critical*, where $p$ and $k$ are positive integers with the same parity, if the deletion of any set of $k$ vertices results in a graph with a perfect matching. $G$ is called *maximal non-$k$-factor-critical* if $G$ is not $k$-factor-critical but $G + e$ is $k$-factor-critical for every missing edge $e \notin E(G)$. The concept of $k$-factor-critical is a generalization of the concepts of factor critical and bicritical. $k$-factor critical graphs have been studied by many authors including Favaron [3, 4] Favaron and Shi [6, 7] and Favaron et. al. [5].

A closely related concept to $k$-factor-critical is that of *$k$-extendable*. For $1 \leq k \leq n-1$, a connected graph $G$ of order $2n$ with a perfect matching is $k$-extendable if for every matching $M$ of size $k$ in $G$ there is a perfect matching in $G$ containing all of edges of $M$. For convenience, a graph $G$ with a perfect matching is said to be 0-extendable. $G$ is called *maximal non-$k$-extendable* if $G$ is not $k$-extendable but $G + e$ is $k$-extendable for every missing edge $e \notin E(G)$. A connected bipartite graph $G$ with a bipartitioning set $(X, Y)$ such that $|X| = |Y| = n$ is *maximal non-$k$-extendable bipartite* if $G$ is not $k$-extendable but $G + xy$ is $k$-extendable for any edge $xy \notin E(G)$ with $x \in X$ and $y \in Y$. Extendable graphs have been studied by many authors including Plummer [9], Ananchuen and Caccetta [1], Kawarabashi et. al. [8], Ryjáček [12] and Yu [14]. Excellent surveys are the papers of Plummer [10, 11]. In this paper, we introduce the concepts of maximal non-$k$-factor-critical, maximal non $k$-extendable and maximal non $k$-extendable bipartite graphs.

A $2k$-factor-critical graph is obviously $k$-extendable but the converse need not be true since a complete bipartite graph $K_{n,n}$ is $k$-extendable for $0 \leq k \leq n-1$ but is not $2k$-factor-critical. Further, the graph $G$ formed by joining two $K_{2k}$'s with a perfect matching is $k$-extendable non-bipartite but is not $2k$-factor-critical. On the other hand, the graphs $G_1$ and $G_2$, shown in Figure 1.1, are both maximal non-2-extendable graphs and maximal non-4-factor-critical graphs whilst the graphs $G_3$ and $G_4$, shown in Figure 1.2, are both maximal non-2-extendable bipartite graphs since the edge $u_1v_1$ together with the edge $u_2v_2$ cannot extend to a perfect matching in each $G_i$ for $1 \leq i \leq 4$. Note that these graphs are 1-extendable. This is not coincidence as it is true in general, a fact we establish later. However, definitions of maximal non-$k$-factor-critical, maximal-non-$k$-extendable and maximal-non-$k$-extendable bipartite graphs have no suggestion on this property.

Further, the above examples suggest that there may be a relationship between maximal non-$k$-factor-critical graphs and maximal non-$k$-extendable graphs. In this paper, we establish the strong connection between these two classes of graphs. More precisely, we establish that for a connected graph $G$ on $2n$ vertices with a perfect matching, $G$ is maximal non-$k$-extendable if and only if $G$ is maximal non-$2k$-factor-critical for $1 \leq k \leq n-1$. We also provide a characterization of maximal non-$k$-factor-critical graphs, maximal non-$k$-extendable graphs and

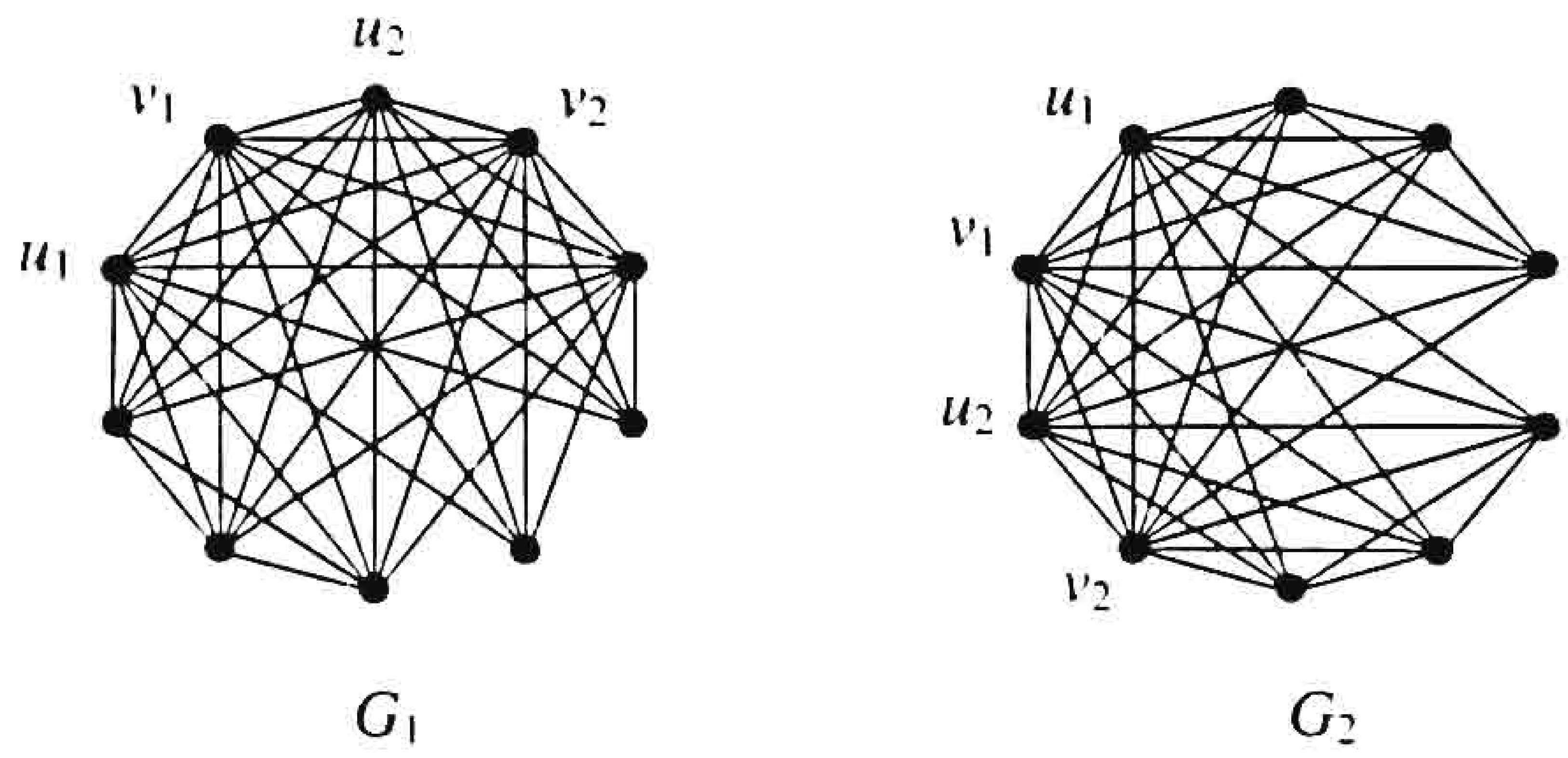


**Figure 1.1**

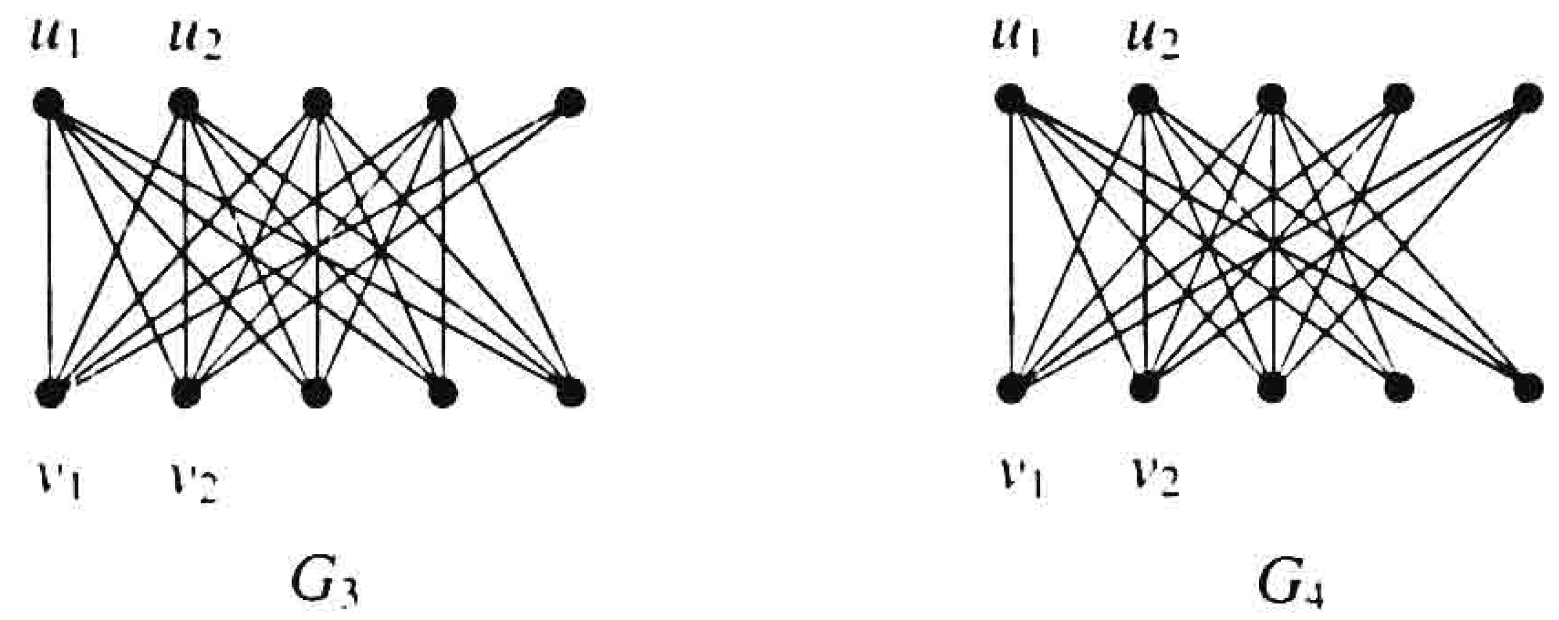


**Figure 1.2**

maximal non-$k$-extendable bipartite graphs.

## 2. Maximal non-$k$-factor-critical graphs

In this section, we establish a characterization of maximal non-$k$-factor-critical graphs. We begin with the following lemma.

**Lemma 2.1:** *For positive integers $p$ and $k$ having the same parity, and non-negative integers $s, t_1, t_2, \dots, t_{s+2}$ with $0 \le s \le \frac{1}{2}(p-k) - 1$ and $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p-k) - s - 1$, the graph*

$$G = K_{k+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1}$$

*is maximal non-$k$-factor-critical of order $p$.*

**Proof:** Let $H = K_{k+s}$ and $G_i = K_{2t_i+1}$ for $1 \le i \le s+2$. Then $G = H \vee \bigcup_{i=1}^{s+2} G_i$. Let $T$ be a subset of $V(H)$ with $|T| = k$. Clearly, $G - T = K_s \vee \bigcup_{i=1}^{s+2} G_i$ has no perfect matching. Thus $G$ is not $k$-factor-critical.

We next show that $G$ is maximal. Let $u$ and $v$ be non-adjacent vertices in $G$ and let us consider $G' = G + uv$. Clearly, $u$ and $v$ are vertices of $G_i$ and $G_j$ for some $i \neq j$, respectively. Without any loss of generality, we may assume that $u \in V(G_1)$ and $v \in V(G_2)$. Now let $T'$ be a subset of $V(G')$ with $|T'| = k$. Further, let $r = |V(H) \cap T'|$ and $r_i = |V(G_i) \cap T'|$ for $1 \leq i \leq s+2$. Clearly, $r + \sum_{i=1}^{s+2} r_i = k$. Then $0 \leq r \leq k$. We now distinguish 3 cases according to $r$.

**Case 1:** $r = k$.

Then $G' - T' \cong K_s \vee (D \cup \bigcup_{i=3}^{s+2} G_i)$ where $V(D) = V(G_1) \cup V(G_2)$ and $E(D) = E(G_1) \cup E(G_2) \cup \{uv\}$. Clearly, $G' - T'$ has a perfect matching.

**Case 2:** $r = k - 1$.

Then $r_j = 1$ for some $j, 1 \leq j \leq s+2$. Thus $G' - T'$ contains at most $s+1$ odd components and the equality holds for $r_1 = 1$ or $r_2 = 1$. It is not difficult to show that $G' - T'$ has a perfect matching.

**Case 3:** $r \leq k - 2$.

Then $H - T' = K_{r'}$ where $r' = k + s - r \geq s + 2$ and $G' - (V(H) \cup T')$ contains at most $s+2$ odd components. The edge $uv$ is in $E(G' - T')$ if $T' \cap \{u, v\} = \phi$. It is not difficult to show that $G' - T'$ has a perfect matching.

Therefore, in all cases, $G' = G + uv$ is $k$-factor-critical and hence $G$ is maximal non-$k$-factor-critical. □

Before we establish a characterization of maximal non-$k$-factor-critical graphs we recall Tutte's Theorem which we make use of in our proof.

**Theorem 2.2:** Tutte's Theorem (see Bondy and Murty [2] p.76)
*A graph $G$ has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subset V(G)$.* □

Now we are ready for our main theorem in this section.

**Theorem 2.3:** *Let $G$ be a connected graph on $p$ vertices and $k$ a positive integer having the same parity with $p$. $G$ is maximal non-$k$-factor-critical if and only if*

$$G \cong K_{k+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1}$$

*where $s$ and $t_i$ are non-negative integers with $0 \leq s \leq \frac{1}{2}(p-k) - 1$ and $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p-k) - s - 1$.*

**Proof:** The sufficiency follows from Lemma 2.1. Now we prove the necessity. Since $G$ is maximal non-$k$-factor-critical, there is a subset $T$ of $V(G)$ of size $k$ such that $G' = G - T$ has no perfect matching. Then, by Tutte's Theorem, there is a subset $S'$ of $V(G')$ such that $o(G' - S') > |S'|$. Put $s = |S'|$. Because $G'$ is of even order, it follows that $s$ and $o(G' - S')$ must have the same parity. Thus $o(G' - S') \geq s + 2$.

Let $C_1, C_2, ..., C_r$ be odd components of $G' - S'$. We first show that $r = s+2$. Suppose to the contrary that $r \geq s + 3$. Then $r \geq s + 4$. Let $c_i \in V(C_i)$ for $i = 1, 2$ and let us consider $G + c_1c_2$. Clearly, $(G + c_1c_2) - (T \cup S')$ contains at least $s + 2$ odd components. Thus $G + c_1c_2$ is not $k$-factor-critical. This contradicts the fact that $G$ is maximal non-$k$-factor-critical. Hence, $r = s + 2$ as required.

We next show that $G' - S'$ has no even components. Suppose to the contrary that $G' - S'$ contains $D$ as an even component. Let $d \in D$ and $c_1 \in V(C_1)$. Now consider $G + dc_1$. Clearly, $(G + dc_1) - (T \cup S')$ contains exactly $s + 2$ odd components since the components $D$ and $C_1$ together with the edge $dc_1$ forms an odd component of $G + dc_1$. Thus $G + dc_1$ is not $k$-factor-critical, a contradiction. This proves that $G' - S'$ has no even components.

Now we claim that $G[T \cup S']$ is complete. Suppose it is not the case. Then there exist vertices $x$ and $y$ in $T \cup S'$ such that $xy \notin E(G)$. Now consider $G + xy$. Since $(G + xy) - (T \cup S')$ contains exactly $s + 2$ odd components, $G + xy$ is not $k$-factor-critical. This contradiction proves that $G[T \cup S']$ is complete. By a similar argument, it is easy to establish that each $C_i$ is complete for $1 \leq i \leq s+2$. Further, for $1 \leq i \leq s+2$, each vertex of $C_i$ is adjacent to every vertex of $T \cup S'$.

Now, for $1 \leq i \leq s + 2$, let $|V(C_i)| = 2t_i + 1$ for some non-negative integer $t_i$. Then $p = |V(G)| = k + s + \sum_{i=1}^{s+2} |V(C_i)| = k + 2s + 2 + 2\sum_{i=1}^{s+2} t_i \geq k + 2s + 2$. Hence, $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p - k) - s - 1$ and $0 \leq s \leq \frac{1}{2}(p - k) - 1$ as required. This completes the proof of our theorem. □

As a corollary we have:

**Corollary 2.4:** *If $G$ is a maximal non-$k$-factor-critical graph on $p$ vertices where $k$ is a positive integer greater than 1 having the same parity with $p$, then $G$ is $(k - 2)$-factor-critical.* □

## 3. Maximal non-$k$-extendable graphs

In this section, we turn our attention to a closely related concept to maximal non-$k$-factor-critical, namely that of maximal non-$k$-extendable graphs. We establish a characterization of maximal non-$k$-extendable graphs and it turns out that these classes of graphs have very closed relationship.

**Theorem 3.1:** *Let $G$ be a connected graph with a perfect matching on $2n$ vertices. For $1 \leq k \leq n - 1$, $G$ is maximal non-$k$-extendable if and only if*

$$G \cong K_{2k+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1}$$

*where $s$ and $t_i$ are non-negative integers with $0 \le s \le n-k-1$ and $\sum_{i=1}^{s+2} t_i = n-k-s-1$.*

**Proof:** The sufficiency follows from Lemma 2.1 and the definitions of factor-critical graphs and $k$-extendable graphs. For the necessity, the proof is almost identical with the proof in Theorem 2.3 so we omit it. □

As a corollary we have:

**Corollary 3.2:** *Let $G$ be a maximal non-$k$-extendable graph on $2n$ vertices for $1 \le k \le n-1$. Then $G$ is $(k-1)$-extendable.* □

**Corollary 3.3:** *Let $G$ be a maximal non-$k$-extendable graph on $2n$ vertices for $1 \le k \le n-1$. If $E' \subseteq E(K_{2n}) \backslash E(G)$ with $|E'| \ge 1$, then $G + E'$ is $k$-extendable.*

**Proof:** The result follows by applying a similar argument as in the proof of Lemma 2.1 to the graph $G + E'$. □

**Remark 3.1:** (1) A connected graph with a perfect matching which is not $k$-extendable need not be $(k-1)$-extendable. For example, a cycle on $2n \ge 8$ vertices is not 3-extendable and it is not 2-extendable. In the case of a maximal non-$k$-extendable graph $G$, $G$ is not $k$-extendable but it is $(k-1)$-extendable. Although immediately obvious, one can simply prove from the definition that a maximal non-$k$-extendable graph is $(k-1)$-extendable.

(2) In [14] Yu proved that if $G$ is a $k$-extendable graph on $2n$ vertices with $1 \le k \le n-1$, then $G + e$ is $(k-1)$-extendable for any edge $e \notin E(G)$. Hence, adding a new edge into a $k$-extendable graph $G$ might destroy the $k$-extendability property of $G$. However for a maximal non-$k$-extendable graph, this is not so, no matter how many edges in $E(K_{2n}) \backslash E(G)$ are added into $G$. The resulting graph is still $k$-extendable providing that the number of edges is at least 1.

By Theorems 2.3 and 3.1, we have the theorem.

**Theorem 3.3:** *Let $G$ be a connected graph on $2n$ vertices with a perfect matching. For $1 \le k \le n-1$, $G$ is maximal non-$k$-extendable if and only if $G$ is maximal non-$2k$-factor-critical.* □

**Remark 3.2:** As we mention in the Introduction that $k$-extendable graphs need not be $2k$-factor-critical but for a maximal non $k$-extendable graph $G$, $G + e$ is both $k$-extendable and $2k$-factor-critical for any edge $e \notin E(G)$.

**Remark 3.3:** A variation of $k$-extendability is that of induced matching extendability or IM-extendability for short which was introduced by Yuan [15].

A matching $M$ of $G$ is induced if $E([V(M)]) = M$. A graph $G$ is IM-extendable if every induced matching of $G$ is included in a perfect matching of $G$. Notice that an IM-extendable graph is 1-extendable. Further, a $k$-extendable graph with no induced matching of size greater than $k$ is IM-extendable. Wang and Yuan [13] introduced a concept of maximal IM-unextendable graphs. A graph $G$ is called maximal IM-unextendable if it is not IM-extendable but $G+xy$ is IM-extendable for every two non-adjacent vertices $x$ and $y$ of $G$. They established that the only maximal IM-unextendable graph is $M_k \vee (K_s \vee (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_{s+2}}))$ where $M_k$ is an induced matching of size $k \geq 1, s$ is a non-negative integer and each $n_i$ is odd. Observe that the class of maximal IM-unextendable graphs coincides with the class of maximal non-$k$-extendable graphs only for $k = 1$.

## 4. Maximal non-$k$-extendable bipartite graphs

In this section, we extend our idea on maximal non-$k$-extendable graphs to the case of bipartite graphs as follows. Let $G$ be a connected bipartite graph on $2n$ vertices with a bipartitioning set $(X, Y)$ such that $|X| = |Y| = n$. For non-negative integers $k$ and $n$ with $0 \leq k \leq n-1$, $G$ is *maximal non-$k$-extendable bipartite* if G is not $k$-extendable but $G + e$ is $k$-extendable for any edge $e = xy \notin E(G)$ where $x \in X, y \in Y$. Thus we are interested in adding a missing edge $e \notin E(G)$ which has one of its end vertices in $X$ and the other in $Y$. We also establish a characterization of maximal non-$k$-extendable bipartite graphs. We first recall Hall's Theorem.

**Theorem 4.1:** Hall's Theorem (see Bondy and Murty [2] p.72)
*Let $G$ be a bipartite graph with bipartitioning $(X, Y)$. Then $G$ contains a matching that saturates every vertex in $X$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.* □

**Lemma 4.2:** *For any non-negative integers $n, k$ and $s$ with $1 \leq s \leq n-1$ and $2 \leq k+s \leq n$, let $(X, Y)$ be a bipartitioning set of $K_{n,n}$ and let $S \subseteq X, T \subseteq Y$ with $|S| = s$ and $|T| = n-k-s+1$. Then*

$$G = K_{n,n} - \{xy \mid x \in S, y \in T\}$$

*is a maximal non-$k$-extendable bipartite graph on $2n$ vertices.*

**Proof:** The result is obvious for $k = 0$. We have to consider only $k \geq 1$. Let $M$ be a matching of size $k$ in $G$ which each $e_i = u_i v_i \in M, u_i \in X \backslash S, v_i \in Y \backslash T$ for $1 \leq i \leq k$. Then $S \subseteq X \backslash V(M)$ with $|N_{G-V(M)}(S)| = s-1 < s = |S|$. Thus $G - V(M)$ has no perfect matching by Hall's Theorem. Hence, $G$ is not $k$-extendable.

Now we establish that $G$ is maximal. Let $e = xy \notin E(G)$ where $x \in X$ and $y \in Y$. Clearly, $x \in S$ and $y \in T$.

Consider $G' = G + xy$. Let $M'$ be a matching of size $k$ in $G'$ and

$$k_1 = |(X\backslash S) \cap V(M')|, \qquad k_2 = |S \cap V(M')|,$$

$$k_3 = |(Y\backslash T) \cap V(M')|, \qquad \text{and} \qquad k_4 = |T \cap V(M')|.$$

Then $k_1 + k_2 = k = k_3 + k_4$, $|X\backslash(S \cup V(M'))| = n - k_1 - s$ and $|Y\backslash(T \cup V(M'))| = k + s - 1 - k_3$. We distinguish two cases according to $k_1$.

**Case 1:** $k_1 = k$.

Clearly, $k_2 = 0$ and $|S\backslash V(M')| = s$.

**Subcase 1.1.** $k_4 = 0$. Then $k_3 = k$ and $xy \in E(G' - V(M'))$. There must be a matching $M_1'$ of $G' - V(M')$ of size $s - 1$ joining vertices of $S\backslash\{x\}$ to vertices of $Y\backslash(T \cup V(M'))$ and a matching $M_2'$ of $G' - V(M')$ of size $n - k - s$ joining vertices of $T\backslash\{y\}$ to vertices of $X\backslash(S \cup V(M'))$. Now $G' - V(M')$ contains $M_1' \cup M_2' \cup \{xy\}$ as a perfect matching as required.

**Subcase 1.2.** $k_4 \geq 1$. Then $k_3 \leq k - 1$. Thus $s \leq k + s - 1 - k_3$. Now let $M_1''$ be a matching of $G' - V(M')$ of size $s$ joining vertices of $S$ to vertices of $Y\backslash(T \cup V(M'))$. Further, let $M_2''$ be a matching of $G' - V(M')$ of size $n - k - s + 1 - k_4$ joining vertices of $T\backslash V(M')$ to vertices of $X\backslash(S \cup V(M'))$. Now $G - V(M' \cup M_1'' \cup M_2'') \cong K_{m,m}$, where $m = k_4 - 1$, contains a perfect matching $M_3''$. Hence, $M_1'' \cup M_2'' \cup M_3''$ forms a perfect matching of $G' - V(M')$.

**Case 2:** $k_1 \leq k - 1$.

Then $k_2 \geq 1$. Further, $n-k-s+1 \leq n-k_1-s$ and $s-k_2 \leq s-1 \leq k-k_3+s-1$. Now let $M_1'''$ be a matching of $G' - V(M')$ of size $s - k_2$ joining vertices of $S\backslash V(M')$ to vertices of $Y\backslash(T \cup V(M'))$. Further, let $M_2'''$ be a matching of $G' - V(M')$ of size $n - k - s + 1 - k_4$ joining vertices of $T\backslash V(M')$ to vertices of $X\backslash(S \cup V(M'))$. Now $G - V(M' \cup M_1''' \cup M_2''') \cong K_{m,m}$, where $m = k_2 + k_4 - 1$, contains a perfect matching $M_3'''$. Hence, $M_1''' \cup M_2''' \cup M_3'''$ is a perfect matching of $G' - V(M')$. Therefore, $G' = G + xy$ is $k$-extendable as required. This completes the proof of our lemma. □

Now we establish the main result of this section.

**Theorem 4.3:** *Let $G$ be a connected bipartite graph on $2n$ vertices with a bipartitioning set $(X, Y)$ such that $|X| = |Y|$. For $0 \leq k \leq n - 1$, $G$ is maximal non-$k$-extendable bipartite if and only if there are subsets $S \subseteq X, T \subseteq Y$ with $|S| = s$ and $|T| = n - k - s + 1$ such that*

$$G \cong K_{n,n} - \{xy \mid x \in S, y \in T\}$$

*for an integer $s$ with $1 \leq s \leq n - 1$ and $2 \leq k + s \leq n$.*

**Proof:** The sufficiency follows from Lemma 4.2. So we need only prove the necessity. Since $G$ is maximal non-$k$-extendable bipartite, there is a matching $M$ of size $k$ in $G$ such that $G - V(M)$ has no perfect matching. Let $(X', Y')$ be a bipartitioning set of $G' = G - V(M)$. Clearly, $X' = X\backslash V(M)$ and $Y' = Y\backslash V(M)$. Further, $|X'| = n - k = |Y'|$. Since $G'$ has no perfect matching, by Hall's Theorem, there is a subset $S \subseteq X'$ such that $s = |S| \geq |N_{G'}(S)| + 1 \geq 1$.

Clearly, $s \leq n-k$. We next show that $s = |N_{G'}(S)|+1$. Suppose to the contrary that $s \geq |N_{G'}(S)|+2$. Then $|Y'\backslash N_{G'}(S)| = n-k-|N_{G'}(S)| \geq n-k-s+2 \geq 2$. Let $x \in S$ and $y \in Y'\backslash N_{G'}(S)$. Clearly, $xy \notin E(G)$. But $(G+xy)-V(M) = G'+xy$ contains $S$ as a subset of $X'$ with $s = |S| > (s-2)+1 \geq |N_{G'}(S)|+1 = |N_{G'+xy}(S)|$. Thus $(G+xy)-V(M)$ has no perfect matching. Hence, $G+xy$ is not $k$-extendable. This contradicts the fact that $G$ is maximal non-$k$-extendable bipartite. Therefore, $s = |N_{G'}(S)|+1$.

We next show that each vertex of $S$ is adjacent to every vertex of $(V(M)\cap Y)\cup N_{G'}(S)$. Suppose this is not the case. Then there are vertices $a \in S$ and $b \in (V(M)\cap Y)\cup N_{G'}(S)$ such that $ab \notin E(G)$. Clearly, $(G+ab)-V(M)$ contains $S$ as a subset of $X'$ with $s = |S| = |N_{G'}(S)|+1 = |N_{(G+ab)-V(M)}(S)|+1$. Thus $(G+ab)-V(M)$ has no perfect matching. Hence, $G+ab$ is not $k$-extendable. This contradicts the fact that $G$ is maximal non-$k$-extendable bipartite and proves that each vertex of $S$ is adjacent to every vertex of $(V(M)\cap Y)\cup N_{G'}(S)$. By a similar argument, one can establish that each vertex of $X\backslash S$ is adjacent to every vertex of $Y$. Consequently, each vertex of $(V(M)\cap Y)\cup N_{G'}(S)$ is adjacent to every vertex of $X$ and each vertex of $T = Y\backslash(V(M)\cup N_{G'}(S)) = \overline{N_G}(S)\cap Y$ is adjacent to every vertex of $X\backslash S$. Note that

$$|V(M)\cap X| + |X'\backslash S| = k+(n-k-s) = n-s,$$

$$|V(M)\cap Y| + |N_{G'}(S)| = k+s-1$$

and $$|T| = |\overline{N_G}(S)\cap Y| = n-(k+s-1) = n-k-s+1.$$

Hence, $G \cong K_{n,n} - \{xy \mid x \in S, y \in T\}$. Clearly, if $k+s=1$ or $n-s=0$, then $G$ is disconnected, contradicting the connectedness of $G$. Hence, $k+s \geq 2$ and $n-s \geq 1$. This completes the proof of our theorem. □

**Remark 4.1:** Note that the maximal non-$k$-extendable bipartite graph $G$ in Theorem 4.3 is isomorphic to the graph

$$\overline{K}_s \vee \overline{K}_{k+s-1} \vee \overline{K}_{n-s} \vee \overline{K}_{n-k-s+1}.$$

As a corollary we have:

**Corollary 4.4:** *Let $G$ be a maximal non-$k$-extendable bipartite graph on $2n$ vertices, $1 \leq k \leq n-1$. Then $G$ is $(k-1)$-extendable.* □

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