สกว

รายงานวิจัยฉบับสมบูรณ์

โครงการ กราฟ domination critical

โดย

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยครั้งนี้สำเร็จลุล่วงได้ด้วยดี โดยการสนับสนุนของสำนักงานกองทุนสนับสนุนการวิจัย ภายใต้โครงการทุนวิจัยองค์ความรู้ใหม่ที่เป็นพื้นฐานต่อการพัฒนา (1 กันยายน 2546 – 28 กุมภาพันธ์ 2549) รหัส BRG4680019

บทคัดย่อ

ให้ $S \subseteq V(G)$ ในกราฟ $G = (V(G), E(G))$ เราจะเรียก S ว่า dominating set ของ G เมื่อสมาชิกแต่ละ v ใน $V(G)$ เป็นสมาชิกใน S หรือประชิดกับจุดใน S และ domination number ของกราฟ G เขียนแทนด้วย $\gamma(G)$ คือจำนวนสมาชิกที่น้อยที่สุดในบรรดา dominating set ของ กราฟ G เราจะกล่าวว่า S เป็น connected dominating set ของ G เมื่ออินดิวซ์สับกราฟ $G[S]$ เป็นกราฟไม่ขาดตอน และ connected domination number ของกราฟ G เขียนแทนด้วย $\gamma_c(G)$ คือจำนวนสมาชิกที่น้อยที่สุดในบรรดา connected dominating set ของ G กราฟ G จะเป็นกราฟ γ–vertex–critical เมื่อ $\gamma(G - v) < \gamma(G)$ สำหรับทุก ๆ จุด v ใน G กราฟ G จะเป็นกราฟ k–γ–critical เมื่อ $\gamma(G) = k$ แต่ $\gamma(G + e) < k$ สำหรับแต่ละเส้น $e \notin E(G)$ ในทำนองเดียวกันเรากล่าวว่า G จะเป็นกราฟ k–γ_c–critical เมื่อ $\gamma_c(G) = k$ แต่ $\gamma_c(G + e) < k$ สำหรับแต่ละเส้น $e \notin E(G)$

สำหรับจำนวนเต็มบวก k และ t ซึ่ง $t \geq 2$ เรากล่าวว่า G เป็นกราฟ k–(γ, t)–critical เมื่อ $\gamma(G) = k$ และสำหรับแต่ละคู่ของจุด u และ v ที่ไม่ประชิดกันใน G ซึ่ง $d(u, v) \leq t$ แล้ว $\gamma(G + e) < k$ ในทำนองเดียวกันเรากล่าวว่า G เป็นกราฟ k–(γ_c, t)—critical เมื่อ $\gamma_c(G) = k$ และสำหรับแต่ละคู่ของจุด u และ v ที่ไม่ประชิดกันใน G ซึ่ง $d(u, v) \leq t$ แล้ว $\gamma_c(G + e) < k$

สำหรับจำนวนเต็มบวก p และ k ที่มีภาวะคู่เสมอกัน กราฟ G ที่มีอันดับ p จะเป็นกราฟ k–factor–critical เมื่อกำจัดเซตของจุดจำนวน k จุดใด ๆ ใน G แล้วกราฟที่เหลือมีการจับคู่สมบูรณ์ เรากล่าวว่า G เป็นกราฟ maximal non–k–factor–critical เมื่อ G ไม่เป็นกราฟ k–factor–critical แต่ $G + e$ เป็นกราฟ k–factor–critical สำหรับเส้นแต่ละเส้น $e \notin E(G)$

ในรายงานการวิจัยฉบับนี้เราศึกษาเงื่อนไขเพียงพอที่จะทำให้กราฟ 3-vertex–critical มีการจับคู่สมบูรณ์และมีการจับคู่ที่ใกล้สมบูรณ์ รวมทั้งเงื่อนไขเพียงพอที่จะทำให้กราฟ 3-vertex–critical เป็นกราฟ k–factor–critical เมื่อ $1 \leq k \leq 3$ สำหรับกราฟ k–γ_c–critical เราศึกษาคุณสมบัติของกราฟนี้ที่มีจุดตัด ซึ่งทำให้ทราบว่ากราฟ 3–γ_c–critical จะมีจุดตัดได้ไม่เกิน 1 จุด ซึ่งนำไปสู่การศึกษาลักษณะเฉพาะเจาะจงของกราฟ 3–γ_c–critical ที่มีจุดตัด 1 จุด เรายังได้ศึกษาเงื่อนไขเพียงพอที่จะทำให้กราฟ 3–γ_c–critical เป็นกราฟ k–factor–critical เมื่อ $1 \leq k \leq 3$ สำหรับงานวิจัยที่เกี่ยวกับกราฟ 3–(γ, t)–critical และกราฟ 3–(γ_c, t)–critical จะเป็นเรื่องของ diameter และความสัมพันธ์ระหว่างกราฟเหล่านี้กับกราฟ 3–γ–critical และ กราฟ 3–γ_c–critical ตามลำดับ เราจบรายงานการวิจัยนี้ด้วยการศึกษาลักษณะเฉพาะเจาะจงของกราฟ maximal non–k–factor–critical

Abstract

A set $S \subseteq V(G)$ is a (vertex) dominating set for G if every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set for G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S for G is a connected dominating set if it induces a connected subgraph of G. The minimum cardinality of a connected dominating set for G is called the connected domination number of G and is denoted by $\gamma_c(G)$. A graph G is said to be γ–vertex–critical if $\gamma(G - v) < \gamma(G)$, for every vertex v in G. Graph G is said to be k–γ–critical if $\gamma(G) = k$ but $\gamma(G + e) < k$ for each edge $e \notin E(G)$. Similarly, G is said to be k–γ_c–critical if $\gamma_c(G) = k$ but $\gamma_c(G + e) < k$ for each edge $e \notin E(G)$.

For positive integers k, t with $t \geq 2$, we say that G is k–(γ, t)–critical if $\gamma(G) = k$ and for every pair of non–adjacent vertices u and v of G with $d(u, v) \leq t$, $\gamma(G + e) < k$. Similarly, G is said to be k–(γ_c, t)–critical if $\gamma_c(G) = k$ and for every pair of non–adjacent vertices u and v of G with $d(u, v) \leq t$, $\gamma_c(G + e) < k$.

A graph G of order p is k–factor–critical , where p and k are positive integers with the same parity, if the deletion of any set of k vertices results in a graph with a perfect matching. G is called maximal non–k–factor–critical if G is not k–factor–critical but $G + e$ is k–factor–critical for every missing edge $e \notin E(G)$.

In this report, we establish sufficient conditions for 3–vertex–critical graphs to contain a perfect matching and a near perfect matching. We also present sufficient conditions for 3–vertex–critical graphs to be k–factor–critical for $1 \leq k \leq 3$. For k –γ_c–critical graphs, we investigate these graphs with cutvertices. It turns out that 3 –γ_c–critical graphs can contain at most one cutvertex which leads to a characterization of 3 –γ_c–critical graphs with a cutvertex. We also establish sufficient conditions for 3 –γ_c–critical graphs to be k–factor–critical for $1 \leq k \leq 3$. Most of the results about 3–(γ, t)–critical and 3–(γ_c, t)–critical graphs concern their diameter and the relationship between these graphs and 3–γ–critical and 3–γ_c–critical graphs respectively. We conclude our report with a characterization of maximal non–k–factor–critical graphs.

หน้าสรุปโครงการ (Executive Summary)

ชื่อโครงการวิจัย กราฟ domination–critical

On domination–critical graphs

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สาขาวิชาที่ทำวิจัย ทฤษฎีกราฟ (ทฤษฎีการจับคู่) ในสาขาคณิตศาสตร์

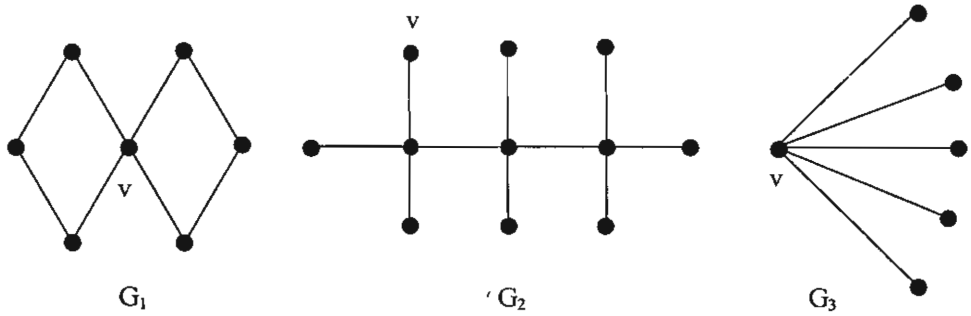
ระยะเวลาดำเนินการ 2 ปี 6 เดือน (1 กันยายน 2546 – 28 กุมภาพันธ์ 2549)

ปัญหาที่ทำวิจัย และความสำคัญของปัญหา

ให้ $S \subseteq V(G)$ ในกราฟ $G = (V(G), E(G))$ เรากล่าวว่า S dominate กราฟ G เมื่อสมาชิกแต่ละ v ใน V(G) เป็นสมาชิกใน S หรือประชิดกับจุดใน S เราเรียก S ว่า dominating set ของ G และ domination number ของกราฟ G เขียนแทนด้วย $\gamma(G)$ คือจำนวนสมาชิกที่น้อยที่สุดในบรรดา dominating set ของ กราฟ G

ให้ $G = (V(G), E(G))$ เป็นกราฟใด ๆ และ $v \in V(G)$ แล้ว domination number ของ $G - v$ อาจยังคงเท่าเดิม หรือเปลี่ยนแปลงไปก็ได้เมื่อเปรียบเทียบกับ domination number ของ G นั่นคือ

อาจเกิดกรณีใดกรณีหนึ่งต่อไปนี้ $\gamma(G - v) < \gamma(G)$ หรือ $\gamma(G - v) = \gamma(G)$ หรือ $\gamma(G - v) > \gamma(G)$ กราฟ G_1, G_2 และ G_3 ในรูป 1 เป็นตัวอย่างของกราฟซึ่ง $\gamma(G - v) < \gamma(G)$, $\gamma(G - v) = \gamma(G)$ และ $\gamma(G - v) > \gamma(G)$ ตามลำดับ



$\gamma(G_1) = 3$ $\quad$ $\gamma(G_2) = 3$ $\quad$ $\gamma(G_3) = 1$

$\gamma(G_1 - v) = 2$ $\quad$ $\gamma(G_2 - v) = 3$ $\quad$ $\gamma(G_3 - v) = 5$

รูป 1

อย่างไรก็ตามในกรณีที่ $\gamma(G - v) < \gamma(G)$ นั้นจะเห็นได้โดยง่ายว่า $\gamma(G - v) = \gamma(G) - 1$ ซึ่งส่วนแรกของงานวิจัยชิ้นนี้จะศึกษากราฟที่มีคุณสมบัติดังกล่าว โดยเราจะให้บทนิยามที่ชัดเจนต่อไปนี้

เรากล่าวว่า G เป็นกราฟ k-vertex-critical เมื่อ $\gamma(G) = k$ และ $\gamma(G - v) = k - 1$ สำหรับแต่ละ $v \in V(G)$ Brigham et. al. [BCD2] ได้ศึกษากราฟ k-vertex-critical และแสดงว่ากราฟ 1-vertex-critical คือกราฟ K_1 เท่านั้น และ G เป็นกราฟ 2 -vertex-critical ก็ต่อเมื่อ $G \cong K_{2n} - F$ โดยที่ F เป็นการจับคู่สมบูรณ์ใน K_{2n} สำหรับกรณีที่ $k \geq 3$ ยังไม่มีผลงานวิจัยใด ๆ ที่จะบ่งบอกลักษณะเฉพาะเจาะจงของกราฟ k-vertex-critical ความจริงแล้วการบ่งบอกลักษณะเฉพาะเจาะจงของกราฟ k-vertex-critical เมื่อ $k \geq 3$ แทบเป็นไปไม่ได้เลยเนื่องจากผลของทฤษฎีบท 1 ต่อไปนี้ซึ่งพิสูจน์โดย Brigham et. al. [BCD2]

Theorem 1: For any graph G there is a k-vertex-critical graph H sucn that G is an induced subgraph of H.

มโนมติที่คล้ายคลึงกันมากกับกราฟ k-vertex-critical คือกราฟ k-edge-critical และกราฟ locally-edge-domination-critical ก่อนที่เราจะให้บทนิยามของกราฟดังกล่าวเราจะให้ข้อสังเกตว่า โดยทั่วไปแล้ว $\gamma(G + e) \leq \gamma(G) - 1$ เมื่อ $e \notin E(G)$ ในที่นี้เราจะกล่าวถึงกลุ่มของกราฟ G ที่มีคุณสมบัติว่า $\gamma(G + e) = \gamma(G) - 1$ เมื่อ $e \notin E(G)$ โดยจะให้บทนิยามที่ชัดเจนดังนี้

กราฟ G = (V(G), E(G)) เป็นกราฟ k-edge-critical เมื่อ $\gamma(G) = k$ และ $\gamma(G + uv) = k - 1$ สำหรับแต่ละ $uv \notin E(G)$ และเรียกกราฟ G ว่ากราฟ k-(γ, t)-critical หรือกราฟ locally-edge-domination-critical เมื่อ $\gamma(G) = k$ และ $\gamma(G + uv) = k - 1$ สำหรับทุก ๆ คู่ของจุด u และ v ใด ๆ ซึ่ง $uv \notin E(G)$ และ ระยะทางระหว่าง u และ v น้อยกว่าหรือเท่ากับ t

Sumner and Blitch [SB] ได้ศึกษากราฟ k-edge-critical และพบว่ากราฟ 1-edge-critical มีแต่เพียงกราฟบริบูรณ์เท่านั้น นอกจากนั้นยังบ่งบอกลักษณะเฉพาะเจาะจงของกราฟ 2-edge-critical และกราฟขาดตอนที่เป็นกราฟ 3-edge-critical ดังนี้

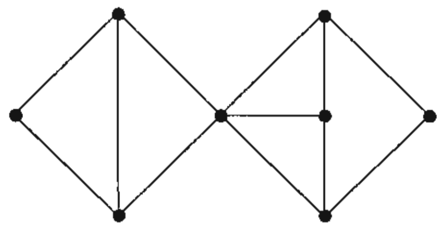
Theorem 2: G is a 2-edge-critical if and only if $\overline{G} = \bigcup_{i=1}^{n} K_{1,n_i}$ $(n \geq 1)$.

Theorem 3: A disconnected graph G is 3-edge-critical if and only if $G = A \cup B$ where either A is trivial and B is 2-edge-critical or A is a complete graph and B is a complete graph minus a perfect matching.

แต่ในกรณีของกราฟไม่ขาดตอนที่เป็นกราฟ 3-edge-critical รวมทั้งกราฟ k-edge-critical เมื่อ $k \geq 4$ ยังไม่มีงานวิจัยชิ้นใดบ่งบอกลักษณะเฉพาะเจาะจงของกราฟดังกล่าว ถึงแม้ว่าจะมีการศึกษาคุณสมบัติต่าง ๆ ของกราฟเหล่านี้บ้างก็ตาม โดยทั่วไปแล้วการศึกษากราฟ k-edge-critical ยุ่งยากซับซ้อนเมื่อ k = 3 และยากมากยิ่งขึ้นเมื่อ $k \geq 4$ (Sumner and Wojcicka. [SW]) ดังนั้นนักวิจัยทั้งหลายจึงทุ่มเทความสนใจให้กับการศึกษากราฟ 3-edge-critical คุณสมบัติที่น่าสนใจประการหนึ่งของกราฟ 3-edge-critical คือการมีการจับคู่ซึ่งเป็นผลงานการพิสูจน์โดย Sumner and Blitch [SB]

Theorem 4: Every even-order, connected 3-edge-critical graph has a perfect matching.

Henning et.al. [HOS] ได้ศึกษากราฟ locally-edge-domination-critical และให้ข้อสังเกตว่าทุก ๆ กราฟ k-edge-critical เป็นกราฟ k-(γ, t)-critical ในขณะที่บทกลับไม่จริง กราฟในรูป 2 เป็นตัวอย่างของกราฟ 3-(γ, 2)-critical แต่ไม่เป็นกราฟ 3-edge-critical



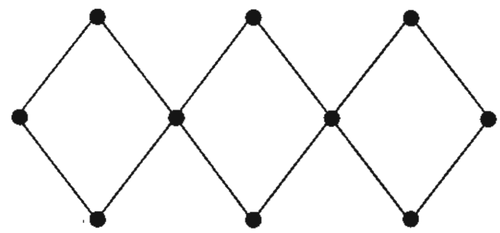
รูป 2

ท่านยังได้ให้ลักษณะเฉพาะเจาะจงของกราฟ 2-(γ, 2)-critical ดังทฤษฎีบท 5 โดยที่นิยามกราฟ double star ว่าเป็นกราฟที่ได้จากกราฟ $K_{1,r}$ และ $K_{1,s}$ โดยการเชื่อมจุดที่มีดีกรี r ของกราฟ $K_{1,r}$ กับจุดที่มีดีกรี s ของกราฟ $K_{1,s}$ ด้วยเส้น

Theorem 5 : A graph G is 2-(γ, 2)-critical if and only if it is the complement of a disjoint union of stars or the complement of a double star.

จากการศึกษาเอกสารที่เกี่ยวข้องกับกราฟ k-vertex-critical และกราฟ k-edge-critical เราพบว่ามีบางสิ่งที่กราฟทั้งสองนี้มีคุณสมบัติเหมือนกันเช่น diameter ของกราฟ k-vertex-critical และกราฟ k-edge-critical มีค่าน้อยกว่าหรือเท่ากับ 2k – 2 แต่สิ่งที่แตกต่างกันมากประการหนึ่งคือคุณสมบัติการมีการจับคู่ กราฟ k-vertex-critical ที่มีจุดเป็นจำนวนคู่ไม่จำเป็นต้องมีการจับคู่สมบูรณ์ ตัวอย่างเช่นกราฟ G ในรูป 3 เป็นกราฟ 4-vertex-critical ที่ไม่มีการจับคู่สมบูรณ์ และไม่มีงานวิจัยใด ๆ ที่จะยืนยันคุณสมบัติการมีการจับคู่สมบูรณ์ของกราฟ k-vertex-critical ไม่ว่า k จะเป็นจำนวนเต็มบวกใด ๆ ในขณะที่เราทราบว่ากราฟ k-edge-critical ที่มีจุดเป็นจำนวนคู่มีการจับคู่สมบูรณ์ เมื่อ k = 3 (ดูทฤษฎีบท 4) ทั้งที่เงื่อนไขของการเป็นกราฟ k-vertex-critical ค่อนข้างจะเข้ม และมีมโนมติที่คล้ายคลึงกันมากกับกราฟ k-edge-critical คำถามที่ตามมาก็คือว่าจะต้องมีเงื่อนไขอะไรเพิ่มเติมจึงจะทำให้กราฟ k-vertex-critical มีการจับคู่สมบูรณ์ และจะต้องมีเงื่อนไขอะไรเพิ่มเติมจึงจะทำให้กราฟ k-vertex-critical เป็นกราฟ n-factor-critical การศึกษาเพื่อที่จะหาเงื่อนไขดังกล่าวอาจทำให้

เราค้นพบคุณสมบัติต่าง ๆ ของกราฟ k-vertex-critical เพิ่มเติม ทั้งนี้เพราะว่าการศึกษาคุณสมบัติของกราฟ k-vertex-critical ยังมีน้อยมากเมื่อเทียบกับการศึกษาคุณสมบัติของกราฟ k-edge-critical



รูป 3

จากบทนิยามของกราฟ k-edge-critical และกราฟ locally-edge-domination-critical เราจะสังเกตเห็นว่าบทนิยามของกราฟ locally-edge-domination-critical เป็นบทนิยามที่อ่อนกว่าบทนิยามของกราฟ k-edge-critical คำถามที่ตามมาก็คือว่าคุณสมบัติต่าง ๆ ที่เป็นจริงสำหรับกราฟ k-edge-critical นอกจากเรื่อง diameter จะยังคงเป็นจริงสำหรับกราฟ k-(γ, t)-critical ด้วยหรือไม่ ซึ่งยังไม่มีผู้ใดศึกษาเพิ่มเติม และผลของการศึกษาคุณสมบัติต่าง ๆ ของกราฟ k-(γ, t)-critical อาจนำเราไปสู่การบ่งบอกลักษณะเฉพาะเจาะจงของกราฟ 3-(γ, 2)-critical ที่มี diameter เท่ากับ 3 ซึ่งการบ่งบอกลักษณะเฉพาะเจาะจงของกราฟนี้เป็นปัญหาที่ยากมากปัญหาหนึ่ง (ดู Henning et.al. [HOS])

วัตถุประสงค์

(1) เพื่อที่จะศึกษาเงื่อนไขเพียงพอที่ทำให้กราฟ k-vertex-critical มีการจับคู่สมบูรณ์

(2) เพื่อที่จะศึกษาเงื่อนไขเพียงพอที่ทำให้กราฟ k-vertex-critical เป็นกราฟ n-factor-critical

(3) เพื่อที่จะศึกษาคุณสมบัติต่าง ๆ ของกราฟ locally-edge-domination-critical ในแง่ของพารามิเตอร์ของกราฟเช่น toughness และการจับคู่เป็นต้น

(4) เพื่อที่จะขยายผลงานวิจัยเรื่องกราฟ k-vertex-critical ที่มีอยู่ในปัจจุบัน

(5) เพื่อที่จะขยายผลงานวิจัยเรื่องกราฟ locally-edge-domination-critical ที่มีอยู่ในปัจจุบัน

ระเบียบวิธีการวิจัย

(1) รวบรวมและศึกษาเอกสารที่เกี่ยวข้องจากแหล่งต่าง ๆ

(2) คิดค้นวิธีการในการแก้ปัญหาโดยนำความรู้ที่ได้จากการศึกษาในข้อ (1) มาประยุกต์ใช้

(3) อภิปรายแลกเปลี่ยนความคิดเห็นและขอคำแนะนำจากผู้เชี่ยวชาญ

(4) รวบรวมสิ่งที่ค้นพบ และได้พิสูจน์ นำมาเรียบเรียงเขียนเป็นบทความวิจัย

ผลที่ได้รับ

ผลจากการศึกษาวิจัยทำให้สามารถแบ่งเรื่องที่ศึกษาได้ 3 เรื่องใหญ่ ๆคือ

(1) **คุณสมบัติของกราฟ** 3–vertex–critical **ในแง่ของการมีการจับคู่ และการเป็นกราฟ** k–factor–critical **ซึ่งผลของการศึกษาในเรื่องนี้อยู่ในรายงานการวิจัยบทที่** 1 **และบทที่** 2 **และสามารถนำไปเขียนเรียบเรียงบทความทางวิชาการได้** 3 **บทความดังนี้**

Matchings in 3–vertex–critical graphs: the even case (**ร่วมกับ** Prof.M.D.Plummer)

Matchings in 3–vertex–critical graphs: the odd case (**ร่วมกับ** Prof.M.D.Plummer)

On the connectivity and matchings in 3–vertex–critical claw–free graphs: (**ร่วมกับ** Prof.M.D.Plummer)

(2) **คุณสมบัติของกราฟ** k–γ_c–critical **ที่มี** cutvertex **และคุณสมบัติของกราฟ** 3–γ_c–critical **ที่เป็นกราฟ** k–factor–critical **ซึ่งผลของการศึกษาในเรื่องนี้อยู่ในรายงานการวิจัยบทที่** 3 **และสามารถนำไปเขียนเรียบเรียงบทความทางวิชาการได้** 2 **บทความดังนี้**

On domination critical graphs with cutvertices having connected domination number 3

Matching properties in connected domination critical graphs (**ร่วมกับรศ.ดร.วัชรพงษ์ อนันต์ชื่น และ** Prof.M.D.Plummer)

(3) **คุณสมบัติของกราฟ** k-(γ, t)-critical **และกราฟ** k-(γ_c, t)-critical **ในแง่ของ** diameter **และความสัมพันธ์กับกราฟ** k–γ–critical **และกราฟ** k–γ_c–critical **ตามลำดับ ซึ่งผลของการศึกษาในเรื่องนี้อยู่ในรายงานการวิจัยบทที่** 4 **และอยู่ระหว่างการเรียบเรียงเพื่อเขียนบทความทางวิชาการซึ่งจะให้ชื่อว่า** On local edge connected domination critical graphs

(4) การศึกษาลักษณะเฉพาะเจาะจงของกราฟ maximal non–k–factor–critical ซึ่งผลของการศึกษาในเรื่องนี้อยู่ในรายงานการวิจัยบทที่ 5 และสามารถนำไปเขียนเรียบเรียงบทความทางวิชาการได้ 1 บทความดังนี้

A characterization of maximal non–k–factor–critical graphs (ร่วมกับ รศ.ดร.วัชรพงษ์ อนันต์ชื่น และ Prof. Dr. Louis Caccetta)

หมายเหตุ: นอกจากการวิจัยตามวัตถุประสงค์ของงานวิจัยที่วางไว้ (ดูรายงานการวิจัยบทที่ 1 บทที่ 2 และ หัวข้อ 4.2) ผู้วิจัยยังได้ดำเนินการเพิ่มเติมจากวัตถุประสงค์ที่วางไว้ ทั้งนี้เพราะว่าในขณะที่ผู้วิจัยได้ดำเนินการศึกษาวิจัยอยู่นั้น ได้ปรากฏผลงานวิจัยตีพิมพ์โดย Chen.et.al. [CSM] ในปี คศ. 2004 ซึ่งมีส่วนที่เกี่ยวข้องกับงานวิจัยที่กำลังดำเนินอยู่ จึงทำให้ผู้วิจัยได้เกิดแนวคิดในการที่จะขยายขอบเขตของการศึกษาออกไป และปรากฏผลของการศึกษาตามที่ได้เห็นในรายงานการวิจัยในบทที่ 3 และ ในหัวข้อ 4.3 สำหรับงานวิจัยในบทที่ 5 นั้นเป็นผลมาจากการพบปะพูดคุยกับนักวิจัยในสาขาเดียวกันและเป็นงานที่ผู้วิจัยได้ร่วมวิจัยในขณะที่รับทุนโครงการนี้อยู่ด้วย จึงเห็นสมควรใส่ไว้ในรายงานการวิจัย

Contents

Research paper 1:

N.Ananchuen and M.D.Plummer, Matchings in 3–vertex–critical graphs: the even case, Networks, 45(2005) 210–213. (Impact factor (2004) = 0.571)

Research paper 2:

N.Ananchuen, L.Caccetta and W.Ananchuen, A characterization of maximal non–k–factor–critical graphs, Discrete Math. (to appear). (Impact factor (2004) = 0.374)

Research paper 3:

N.Ananchuen and M.D.Plummer, On the connectivity and matchings in 3–vertex–critical claw–free graphs, Utilitas Mathematica, (to appear). (Impact factor (2004) = 0.169)

Research paper 4:

N.Ananchuen and M.D.Plummer, Matchings in 3–vertex–critical graphs: the odd case (submitted)

Research paper 5:

N.Ananchuen, On domination critical graphs with cutvertices having connected domination number 3 (submitted).

Research paper 6:

N.Ananchuen, W.Ananchuen and M.D.Plummer, Matching properties in connected domination critical graphs (submitted).

Chapter 1

Results on 3-Vertex-Critical Graphs

1.1 Introduction

All graphs considered in this report are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [BM]. Thus G is a graph with vertex set $V(G)$, edge set $E(G)$ and minimum degree $\delta(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly, $G[E']$ denotes the subgraph induced by the edge set E' of G. A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. A vertex v is saturated by M if some edge of M is incident to v; otherwise v is said to be unsaturated. A matching G is *perfect* if it saturates every vertex of G and is *near perfect* if it saturates all but exactly one of the vertices of G. If $|V(G)| \equiv k(\bmod 2)$, graph G is said to be *k-factor-critical* if $G - S$ has a perfect matching for every $S \subseteq V(G)$ with $|S| = k$. (The special cases when $k = 1$ and 2, respectively, have received the most attention in the literature and in these cases the graphs are called *factor-critical* and *bicritical* respectively.) If G is any graph and $S \subseteq V(G)$, then denote by $c(G - S)$ (respectively $c_o(G - S)$) the number of components (respectively odd components) of $G - S$.

A set $S \subseteq V(G)$ is a *(vertex) dominating set* for G if every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in graph G is called the *(vertex) domination number* (or simply the *domination number*) of G and is denoted by $\gamma(G)$. Graph G is said to be *γ-vertex-critical* if $\gamma(G - v) < \gamma(G)$, for every vertex in G. (Clearly, then, $\gamma(G - v) = \gamma(G) - 1$, for every vertex v in G.) The structure of such graphs remains relatively unexplored, even in the case $\gamma = 3$.

The concept of γ-vertex-critical graphs seems to have been first introduced by Sumner [S1]. Clearly, the only 1-vertex-critical graph is K_1 (a single vertex). Sumner pointed out that the 2-vertex-critical graphs are precisely the family of graphs obtained from the complete graphs K_{2n} by deleting a perfect matching. For $\gamma > 2$, however, an understanding of the structure of γ-vertex-critical graphs is far from complete.

The related, yet different, concept of edge criticality with respect to domination number has received more attention. A graph G is called *γ-edge-critical* if $\gamma(G + e) < \gamma(G)$ for every edge $e = uv \notin E(G)$ and $u, v \in V(G)$. (Here again it is clear that in this case $\gamma(G + e) = \gamma(G) - 1$.) It should be immediately pointed out, however, that the two concepts of domination criticality are independent in that there are graphs which are γ-edge-critical, but not γ-vertex-critical, graphs which are γ-vertex-critical, but not γ-edge-critical, graphs which are critical in neither sense and graphs which are critical in both senses. On the other hand, it should also be noted that one can always add edges, if necessary, to a γ-vertex-critical graph so as to produce a graph which is both γ-edge-critical and γ-vertex-critical.

For results about γ-edge-critical graphs, the reader is directed to [S1,SB, HHS, M,B,G] and to the further references that they contain. In particular, in [S1, SB] it is shown that any connected 3-edge-critical graph of even order must contain a perfect matching and this result was the motivation for the present paper. In contrast to their result, we show, by exhibiting an infinite class of examples, that a connected 3-vertex-critical graph of even order need not contain a perfect matching.

For a general reference on matchings in graphs, see [LP].

In [BCD1, BCD2, F, FHM, HHS], the first structural properties of 3-vertex-critical graphs are presented. We now list several of these which shall prove useful to us. We denote by $N(v)$ the neighborhood of vertex v (i.e., the set of all vertices adjacent to v) and by $N[v]$ the closed neighborhood of vertex v; i.e., the set $N(v) \cup \{v\}$. If $S \subseteq V(G)$, then $N_S(v)$ denotes the set $N(v) \cap S$.

Lemma 1.1.1: [F3] If v is a vertex in graph G and if all vertices in $N[v]$ are critical, then there is no vertex $v' \in V(G)$, $v' \neq v$, such that $N[v'] \subseteq N[v]$. ∎

In the next two lemmas, we shall take the phrase "vertex-critical" to mean γ-vertex-critical for some value of γ.

Lemma 1.1.2: [BCD1, BCD2] A graph G is vertex-critical if and only if each block of G is vertex-critical. ∎

Lemma 1.1.3: [BCD1, BCD2] If G is vertex-critical with blocks G_1, $\ldots, G_n$, then

$$\gamma(G) = \left[\sum_{i=1}^{n} \gamma(G_i)\right] - n + 1.$$

∎

In addition to the above results, we shall also make use of the following.

Lemma 1.1.4: If G is 3-vertex-critical and of even order, then G is 2-connected.

Proof: If G is disconnected, then either G consists of two components, one of which is 2-critical and the other 1-critical or else G consists of three components each of which is 1-critical. But in the former case, G must be consist of one component isomorphic to a K_{2n} with a perfect matching deleted and the other component K_1, while in the latter case G must consist of three isolated vertices. Hence in either case, G has odd order, a contradiction.

Thus assume that G is connected, but with cutvertices. Let the blocks of G be $G_1, \ldots, G_n$, where $n \geq 2$. Then by Lemma 1.1.3 we have $3 = \gamma(G) = [\sum_{i=1}^{n} \gamma(G_i)] - n + 1$.

Moreover, by Lemma 1.1.2, each block G_i is vertex-critical and since G is not isomorphic to K_1, no block of G can be a single vertex. So $\gamma(G_i) \geq 2$, for each block G_i. Thus $n = 1$ or 2. But we have assumed that $n \geq 2$ and so $n = 2$ and $\gamma(G_i) = 2$, for $i = 1, 2$. That is, G must consist of two blocks G_1 and G_2 sharing a single cutvertex v. Since $\gamma(G - v) = 2$, it follows that $\gamma(G_1 - v) = \gamma(G_2 - v) = 1$. But then $G_1 - v \cong G_2 - v \cong K_1$ and $|V(G)| = 3$, contradicting the fact that G has even order.

∎

If $v \in V(G)$ we shall denote by G_v the graph $G - v$ and by D_v, a minimum dominating set of $G - v$. The following remarks about D_v are trivial to verify, but as we will appeal to them repeatedly, we list them separately.

Remarks: If G is 3-vertex-critical, then the following hold:

1. For every vertex of G, $|D_v| = 2$.

2. If $D_v = \{x, y\}$, then x and y are not adjacent to v.

3. For every pair of distinct vertices v and w, $D_v \neq D_w$.

1.2 A Result on Perfect Matchings

Tutte's classical theorem on perfect matchings says that if a graph G has no perfect matching, then there is a set $S \subseteq V(G)$ such that the number of components of $G - S$ having odd order is greater than the size of S. We shall call any such set S for which $G - S$ has more than $|S|$ odd components a *Tutte set.* (An alternate name is *antifactor set*; see Sumner [S2].) We shall denote by $c_o(G - S)$ the number of components of $G - S$ having odd order. A graph will be called $K_{1,5}$*-free* if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,5}$.

Our main result will be the culmination of the next three lemmas.

Lemma 1.2.1: Suppose G is 3-vertex-critical of even order and $K_{1,5}$-free, but suppose that G contains no perfect matching. Then if S is any Tutte set in G with $|S| \geq 5$, for every vertex $v \in V(G)$, if D_v is a minimum dominating set for $G - v$, $D_v \subseteq S$.

Furthermore, if $v \in S$, then $|N_S(v)| \geq 2$.

Proof: Let $C_1, \ldots, C_t$ denote the odd components of $G - S$. Since $|S| \geq 5$, and G is of even order, $c_o(G - S) \geq 7$. Suppose to the contrary that there is a vertex $x \in V(G)$ such that $D_x \nsubseteq S$. Clearly, $D_x \nsubseteq \cup_{i=1}^{t} V(C_i)$, since $t \geq 7$ and $|D_x| = 2$. Suppose $D_x = \{y, z\}$. Then without loss of generality, we may suppose that $y \in S$ and $z \notin S$. It follows that y must dominate at least $|S| \geq 5$ odd components which contradicts the fact that G is $K_{1,5}$-free. This completes the proof of the first part of the lemma.

The second part of the lemma follows immediately from the fact that $D_v \subseteq S$, for all $v \in V(G)$ and the fact that v is not adjacent to any vertex of D_v.

∎

Lemma 1.2.2: Suppose G is 3-vertex-critical of even order and is $K_{1,5}$-free, but suppose G contains no perfect matching. Then if S is any Tutte set in G, $2 \leq |S| \leq 4$.

Proof: The fact that $|S| \geq 2$ follows immediately from Lemma 1.1.4.

Suppose to the contrary that S is a Tutte set with $|S| = k \geq 5$. We first show that $k = 6$ and each component of $G - S$ is a singleton.

Since for each $x \in V(G)$, $D_x \subseteq S$ by Lemma 1.2.1, it follows that for every $x \in V(G)$ there is a pair of vertices in $S - \{x\}$, a and b say, such that $D_x = \{a, b\}$. Since there are at most $\binom{k}{2} = k(k-1)/2$ pairs of vertices of S and at least $k + (k+2) = 2k+2$ vertices in G, by Remark 3 it follows that $2k + 2 \leq k(k-1)/2$ and hence $k \geq 6$.

On the other hand, $k + 2 \leq c_o(G - S) \leq 8$ because G is $K_{1,5}$-free and $D_x \subseteq S$ for each $x \in V(G)$. Hence $k = 6$ and $c_o(G - S) = 8$.

Thus there are exactly $\binom{6}{2} = 15$ pairs of vertices in S and hence G has at most 15 vertices. This implies that $G - S$ has no even components and every odd component of $G - S$ must be a singleton as required since there are exactly 8 odd components. So G has exactly 14 vertices and thus at least 14 pairs of vertices in S are realized as a D_x for each $x \in V(G)$.

Let C be the set of vertices which together comprise the eight singleton odd components of $G - S$. Denote the set of odd components of $G - S$ which are adjacent to $v \in S$ by C_v. Clearly, $C_v \subseteq C$. Now let H be a simple graph with $V(H) = S$ and $E(H) = \{xy | D_v = \{x, y\}\}$. For $xy \in E(H)$, we have that $C_x \cup C_y = C$. So, since G is $K_{1,5}$-free, $|C_x| = |C_y| = 4$ and $\{C_x, C_y\}$ partitions C. It follows that H must be bipartite with $|V(H)| = |E(H)| = 6$. Then H must contain a path of length 3 say, u_1, v_1, u_2, v_2, as a subgraph. Therefore, $C_{u_1} = C_{u_2}$ and $C_{v_1} = C_{v_2}$. Then $\{u_1, u_2\}$ and $\{v_1, v_2\}$ cannot be realized as a D_v for any $v \in V(G)$. Hence, there are at most 13 pairs of vertices in S which can be realized as a D_v for some $v \in V(G)$. Since G has exactly 14 vertices, $D_x = D_y$ for some $x \neq y$. But this contradicts Remark 3 and hence completes the proof of our lemma.

∎

Lemma 1.2.3: Suppose G is $K_{1,5}$-free 3-vertex-critical of even order, but suppose G contains no perfect matching. Then if S is any Tutte set in G, $|S| = 4$.

Proof: Suppose, by way of contradiction, that $|S| \neq 4$. Let S be any Tutte set in G. By Lemma 1.2.2, we may suppose that $|S| = 2$ or $|S| = 3$.

Claim. If $v \in S$, and D_v is a minimum dominating set for $G - v$, then $D_v \subseteq S$.

Suppose to the contrary that $D_v \nsubseteq S$ for some $v \in S$. Let $D_v = \{a, b\}$. Then a and b are not adjacent to v by Remark 2. Since $c_o(G - S) \geq 4$, $\{a, b\} \cap S \neq \emptyset$. Let the components of $G - S$ be denoted $C_1, \ldots, C_t$. Without loss of generality, then, we may suppose that $a \in V(C_1)$ and $b \in S$. Then b must be adjacent to every vertex of $C_2 \cup \cdots \cup C_t$. Since G is $K_{1,5}$-free, it follows that $t \leq 5$. We distinguish two cases according to $|S|$.

Case 1. First suppose that $|S| = 2$.

Thus $t = 4$. Consider G_b. D_b must be of the form $\{v, a'\}$ where a' is not adjacent to b. Then $a' \in V(C_1)$. So v is adjacent to every vertex of $V(C_2) \cup V(C_3) \cup V(C_4)$. Choose $c \in V(C_2)$ and consider G_c. Since both v and b are adjacent to c, we must have $D_c \cap \{v, b\} = \emptyset$, a contradiction for then there is at least one of the C_i which D_c cannot dominate. This completes the proof in Case 1.

Case 2. So suppose that $|S| = 3$.

Thus $t = 5$. Furthermore, by Case 1, we may also suppose that S is a minimal Tutte set. Now G is $K_{1,5}$-free, so b is adjacent to no vertex of C_1. Thus a dominates all vertices of component C_1.

Now let c denote the third vertex in S. Since S is a minimal Tutte set, vertices v and c are adjacent to at least two components C_i, $1 \leq i \leq 5$. Let $u \in V(C_2) \cup V(C_3) \cup V(C_4) \cup V(C_5)$ be a vertex adjacent to c. Now $D_u = \{v, v'\}$. Since $av \notin E(G)$, $v' \in V(C_1)$. Thus v must dominate each vertex of at least 3 components among $C_2, \cdots, C_5$. Now let $w \in V(C_2) \cup V(C_3) \cup V(C_4) \cup V(C_5)$ be a vertex adjacent to v. Thus w is adjacent to both v and b. Now $D_w = \{c, c'\}$, where $c' \notin S$. This means that c dominates each vertex of at least 2 components among $C_2, \cdots, C_5$. So there is at least one component among $C_2, \cdots, C_5$ such that v, b and c dominate all of its vertices. Let z be a vertex in such a component. Then $D_z \cap S = \phi$, a contradiction for then D_z fails to dominate at least two of the C_i's. This completes the proof in Case 2 and hence the Claim is proved.

It follows immediately from the Claim that $|S| = 3$. Let $S = \{a, b, c\}$. Then for each vertex $v \notin S$, $|N_S(v)| \geq 2$ because if v is not adjacent to say, a and b, then $D_c = \{a, b\}$ would not dominate v. In fact, $|N_S(v)| = 2$ because if $|N_S(v)| = 3$, then $D_v \cap S = \phi$ and thus D_v would not dominate some C_i. This observation together with the fact that each vertex of S is adjacent to at most 4 odd components of $G - S$ implies that $G - S$ has exactly 5 odd components. For each vertex x of S, there exists a vertex $v \notin S$ not adjacent to x but v dominates $S - \{x\}$. So $D_v \cap S = \{x\}$ and x dominates at least

3 odd components of $G - S$. If every vertex of S dominates exactly 3 odd components of $G - S$, then there must exist an odd component of $G - S$ the vertices of which are adjacent to at most one vertex of S, a contradiction of Lemma 1.1.4. Hence there is a vertex of S, say c which dominates exactly 4 odd components of $G - S$. Let $C_1, C_2, \cdots, C_5$ be the odd components of $G - S$. Without any loss of generality, we may assume that a dominates C_1, C_2, C_3, b dominates C_1, C_4, C_5 and c dominates C_2, C_3, C_4 and C_5. Now for each $v \in V(C_1)$, $D_v = \{c, c'\}$. Suppose $c' \notin V(C_1)$. Then $|V(C_1)| = 1$. But then $\{v, c\}$ dominates G, a contradiction. Hence, $c' \in V(C_1)$.

Now if $\gamma(C_1) = 1$, $\{v, c\}$ dominates G, a contradiction. So $\gamma(C_1) \geq 2$. But then C_1 is 2-vertex-critical, and hence of even order by [2,3], a contradiction. Therefore, $|S| = 4$ as required and hence the lemma is proved.

∎

We are now prepared to state and prove our main result.

Theorem 1.2.4: If G is 3-vertex-critical of even order and $K_{1,5}$-free, then G has a perfect matching.

Proof: Suppose to the contrary that G contains no perfect matching and that S is a Tutte set in G.

First we claim that if $|S| \geq 4$, then for all $v \in S$, $D_v \subseteq S$.

If $|S| \geq 5$, then the claim is true by Lemma 1.2.1. So suppose $|S| = 4$. Suppose, to the contrary, that for some vertex $v \in S$, $D_v = \{a, b\}$, where $a \in S$ and $b \in V(G) - S$. Since $c_o(G - S) \geq 6$, vertex a must dominate at least five of the odd components and hence G contains an induced $K_{1,5}$, a contradiction. This completes the proof of the claim.

Next we claim that, in fact, $|S| \neq 4$.

Suppose to the contrary that $|S| = 4$. Choose $x \in S$. Then $D_x \subseteq S$. Suppose $D_x = \{y, z\}$. Without loss of generality, we may then suppose that if w is the fourth vertex of S, then w is adjacent to z. Then D_w must be $\{x, y\}$ and so w is adjacent to neither x nor y. Also since x is not adjacent to z, y must be adjacent to z. But then $D_z \cap \{y, w\} = \emptyset$. So D_z consists of vertex $x \in S$ and a second vertex in $G - S$. But this contradicts the claim verified at the beginning of this proof.

So $|S| \neq 4$ and this contradicts Lemma 1.2.3.

∎

1.3 A New Family of 3-Vertex-Critical Graphs

In the first paper on the subject of 3-vertex-critical graphs [BCD2], the authors present a family of graphs which they denote by $\{G_{m,n}\}$ and claim that these graphs are n-vertex-critical. However, in the case of $n = 3$, this is true only when m is even.

In this section, we present a construction which yields an infinite family of new 3-vertex-critical graphs.

Let k be any positive integer with $k \geq 5$. We proceed to outline the construction of a graph which we will call $H_{k,\binom{k}{2}-k}$. The vertex set will consist of two disjoint subsets of vertices called *central* and *peripheral*, respectively. Let $\{v_1, \ldots, v_k\}$ denote the set of central vertices. The subgraph induced by these central vertices will be the complete graph K_k with the Hamiltonian cycle $v_1 v_2 \cdots v_k v_1$ deleted. The peripheral vertices will be $\binom{k}{2} - k$ in number and will be denoted by the symbol $\sim \{i, j\}$ where the (unordered) pair $\{i, j\}$ $(i \neq j)$ ranges over all the $\binom{k}{2} - k$ subsets of size 2 of the set $1, \ldots, k$, *except* those having $j = i + 2$ where $i + 2$ is read modulo k. The neighbor set of peripheral vertex $\sim \{i, j\}$ will be precisely the set of all central vertices, except i and j. There are no edges joining pairs of peripheral vertices.

Figure 1.3.1 shows as an example the graph $H_{6,9}$.

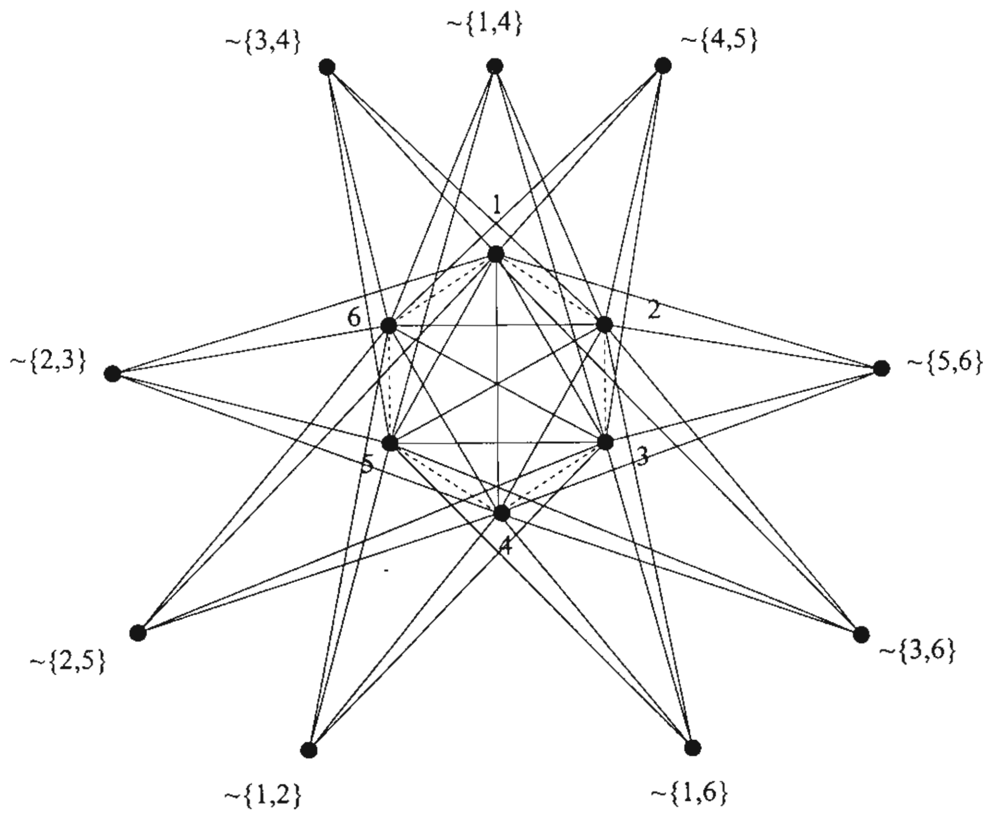


Figure 1.3.1

Each graph $H_{k,\binom{k}{2}-k}$ can, in turn, be used to create a large number of additional 3-vertex-critical graphs as follows. Partition the set of peripheral vertices into $r \geq 3$ subsets $P_1, P_2, P_3, \ldots, P_r$ and add e_i edges to P_i for each $i = 1, \ldots, r$. Here e_i can be any integer such that $0 \leq e_i \leq \binom{|P_i|}{2}$. All such resulting graphs will be 3-vertex-critical.

It should be noted that Sumner proved the following theorem.

Theorem 1.3.1: [S2] If $n \geq 1$ and G is an n-connected $K_{1,n+1}$-free graph of even order, then G contains a perfect matching.

∎

However, there are many 3-vertex-critical $K_{1,5}$-free graphs of even order that are not 4-connected. We show two examples in Figure 1.3.2.

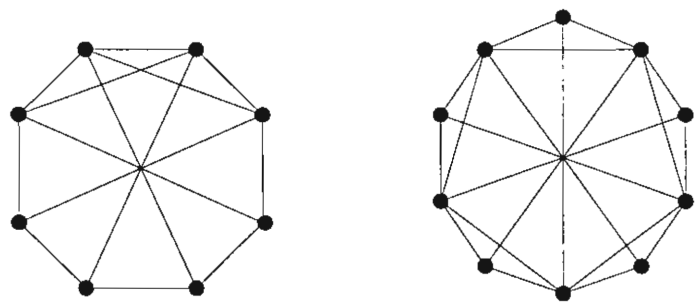


Figure 1.3.2

Sumner and Blitch [S1, SB] showed that every connected 3-edge-critical graph of even order contains a perfect matching. In contrast, it is not true that a connected 3-vertex-critical graph of even order must contain a perfect matching. For each of the infinitely many values of $k \geq 8$ such that $\binom{k}{2}$ is even, the graph $H_{k,\binom{k}{2}-k}$ defined above is such a 3-vertex-critical graph.

The reader will note that we have made considerable use of the additional hypothesis that G be $K_{1,5}$-free in several of our proofs in Section 1.2. Indeed, it would be interesting to know if this extra hypothesis can be weakened. For example, we know of no counterexample to the following.

Conjecture. If G is a 3-vertex-critical graph of even order and $K_{1,7}$-free, then G contains a perfect matching.

1.4 A Result on Near-perfect Matchings

Lemma 1.4.1: Suppose G is a 3-vertex-critical graph which is disconnected. Then either G is isomorphic to three independent vertices or else G is isomorphic to the disjoint union of an even complete graph K_{2n} with a perfect matching removed and one isolated vertex.

Proof: Since $\gamma(G) = 3$, either G consists of three components each having $\gamma = 1$ or else of two components, one of which has $\gamma = 2$ and the other has $\gamma = 1$. But in the former case, each of the three components must be K_1, since each is 1-vertex-critical and in the second case, one component must be 2-vertex-critical and the other 1-vertex-critical. But by an observation first found in [BCD1, BCD2], the 2-vertex-critical component must be an even complete graph with a perfect matching removed and the 1-vertex-critical component must be K_1.

∎

Corollary 1.4.2: If G is a 3-vertex-critical graph with minimum degree greater than 0, then G is connected. ∎

Lemma 1.4.3: If G is 3-vertex-critical and S is a Tutte set in G such that $c_o(G-S) \geq 4$, then each vertex of $G-S$ is not adjacent to at least one vertex of S.

Proof: Suppose $w \in V(G) - S$ such that w is adjacent to every vertex of S. Then $D_w \cap S = \emptyset$ and so $D_w \subseteq V(G) - S$. But this is impossible since the set D_w has size 2 and it must dominate at least three odd components.

∎

Lemma 1.4.4: Suppose G is a $K_{1,5}$-free 3-vertex-critical graph of odd order with $\delta(G) > 0$. Further, suppose that S is a Tutte set for G with $c_o(G-S) \geq |S|+3$. Then $|S| \geq 3$.

Proof: By Corollary 1.4.2, G is connected and hence $|S| \geq 1$.

Suppose first that $|S| = 1$. Say, $S = \{u\}$. Let $C_1, \ldots, C_t$ be the odd components of $G - u$. So $t \geq 4$. Now u is adjacent to vertices in each C_i since G is connected. So since G is $K_{1,5}$-free, $t = 4$ and there are no even components of $G - S$. But $\gamma(G-u) = 2$ implies that $t \leq 2$, a contradiction.

Next, suppose $|S| = 2$. Let $S = \{u, v\}$. Then $c_o(G-S) \geq 5$. Consider G_u. Clearly, $v \in D_u$ and so D_u is composed of the vertex v together with one other vertex, say w, from $V(G-S)$. By Remark 2, $vu \notin E(G)$ and $wu \notin E(G)$. Furthermore, since G is $K_{1,5}$-free, w must lie in one of the odd components of $G-S$. Suppose, without loss of generality, that $w \in V(C_1)$. Then vertex v must be adjacent to all vertices of $C_2 \cup \cdots \cup C_t$ and to all vertices in even components of $G-S$. But again by the fact that G is $K_{1,5}$-free, $t = 5$ and there are no even components of $G-S$. Moreover, by Lemma 1.4.3, none of the vertices of $C_2 \cup \cdots \cup C_5$ is adjacent to u. But v is not adjacent to any vertex of C_1 since G is $K_{1,5}$-free, and so G is disconnected, a contradiction of Corollary 1.4.2.

∎

Lemma 1.4.5: Suppose G is a $K_{1,5}$-free 3-vertex-critical graph of odd order at least 11 with $\delta(G) > 0$. Suppose further that S is a Tutte set in G such that $c_o(G-S) \geq |S|+3$. Then for every vertex $v \in V(G)$, $D_v \subseteq S$.

Proof: Suppose by way of contradiction that there is a vertex v such that $D_v \not\subseteq S$. Since $|S| \geq 3$ by Lemma 1.4.4, $c_o(G-S) \geq 6$. Hence $D_v \cap S \neq \emptyset$. So we may suppose that $D_v = \{u, w\}$, with $u \in S$ and $w \in V(G-S)$. If w were in an even component of $G - S$, then u would have to be adjacent to all vertices in the odd components of $G - S$ and thus u would have to be the center of a $K_{1,5}$ in G, a contradiction. So w must lie in some odd component of $G - S$, say, without loss of generality, that $w \in V(C_1)$. Then u must be adjacent to each vertex of at least four odd components of $G - S$. Thus we may assume that there are exactly six odd components of $G - S$, that $\{v\} = V(C_2)$, that u is adjacent to each vertex of $C_3 \cup \cdots \cup C_6$ and that each of $C_3, \ldots, C_6$ is a complete graph. Moreover, then $|S| = 3$ and G has no even components.

By Lemma 1.4.3 there must exist a vertex $y \in S - \{u\}$ and two vertices lying in two different odd components among $C_3, \ldots, C_6$ such that y is not adjacent to either of these two vertices. More specificly, we may suppose that there are vertices $c_3 \in V(C_3)$ and $c_4 \in V(C_4)$ such that y is adjacent to neither c_3 nor c_4. Since $|S| = 3$, let $S = \{u, y, z\}$.

Claim 1: Vertex z is adjacent to no vertex of $C_5 \cup C_6$.

Suppose to the contrary that z is adjacent to $c_5 \in V(C_5)$. Consider G_{c_5}. Clearly, $D_{c_5} \cap S \neq \emptyset$, but $D_{c_5} \cap (\{z, u\} \cup V(C_5)) = \emptyset$. So $y \in D_{c_5}$ and $|D_{c_5} \cap V(G - S)| = 1$.

Let $D_{c_5} = \{y, w'\}$. Since y is not adjacent to c_3 or c_4, w' is adjacent to both c_3 and c_4. But this is impossible since c_3 and c_4 lie in different odd components. This proves that z is not adjacent to vertex of C_5. By a similar argument, z is not adjacent to vertex of C_6. This proves Claim 1.

Claim 2: Vertex y is adjacent to no vertex of C_5.

Suppose to the contrary that y is adjacent to $a \in V(C_5)$. Consider G_a. Clearly, $D_a \cap S \neq \emptyset$, but $D_a \cap (\{u, y\} \cup V(C_5)) = \emptyset$, since C_5 is complete. So $D_a = \{z, x\}$, where $x \in V(C_6)$ by Claim 1. Thus z is adjacent to every vertex of $C_1 \cup \cdots \cup C_4$ and $V(C_5) = \{a\}$ by Claim 1. Hence, C_i is complete for $1 \leq i \leq 4$.

Now consider G_{c_3}. Clearly, $D_{c_3} \cap S \neq \emptyset$ but $D_{c_3} \cap (\{u, z\} \cup V(C_3)) = \emptyset$ since C_3 is complete and c_3 is adjacent to both u and z. Thus $y \in D_{c_3}$. Because y is not adjacent to c_4, $D_{c_3} = \{y, y'\}$ where $y' \in V(C_4)$. Consequently, y is adjacent to every vertex of $C_1 \cup C_2 \cup C_5 \cup C_6$ and $V(C_3) = \{c_3\}$.

By a similar argument as above, $|V(C_4)| = 1$ and $|V(C_6)| = 1$. Since $|V(G)| \geq 11$, $|V(C_1)| \geq 3$. Let $c_1 \in V(C_1)$. Now consider G_{c_1}. Clearly, $D_{c_1} \cap S \neq \emptyset$, but $D_{c_1} \cap (\{y, z\} \cup V(C_1)) = \emptyset$ since C_1 is complete and c_1 is adjacent to both y and z. Thus $u \in D_{c_1}$ and $|D_{c_1} \cap V(G - S)| = 1$. Let $D_{c_1} = \{u, u'\}$ where $u' \in V(G - S)$. Since G is $K_{1,5}$-free, u is not adjacent to any vertex of $C_1 \cup C_2$ and hence u' is adjacent to v and every vertex of $V(C_1) - \{c_1\}$. But this is impossible since $v \in V(C_2)$. This proves Claim 2.

Now consider G_u. Clearly, $D_u \cap (\cup_{i=3}^{6} V(C_i)) = \emptyset$ since u is adjacent to every vertex of $\cup_{i=3}^{6} V(C_i)$. Thus $D_u \subseteq \{y, z\} \cup V(C_1) \cup V(C_2)$. But then, by Claims 1 and 2, no vertex of D_u is adjacent to any vertex of C_5, a contradiction. This completes the proof of our lemma.

∎

Theorem 1.4.6: Suppose G is a $K_{1,5}$-free 3-vertex-critical graph of odd order at least 11 with $\delta(G) > 0$. Then G contains a near-perfect matching.

Proof: Suppose G does not contain a near-perfect matching. Form a new graph G' from G by adding a new vertex x such that every vertex of G is adjacent to x. Then G' does not contain a perfect matching. So by Tutte's 1-factor theorem and parity, there is a Tutte set S' in G' such that $c_o(G' - S') \geq |S'| + 2$. Since x is adjacent to every vertex of G, it follows that $x \in S'$. Let $S = S' - \{x\}$. Then $c_o(G - S) = c_o(G' - S') \geq |S'| + 2 = |S| + 3$. So by Lemma 1.4.5, $D_v \subseteq S$, for all $v \in V(G)$.

Now let $|S| = k$. There are $\binom{k}{2}$ different pairs of vertices of S and at least $k + 3 + k = 2k + 3$ vertices in G. So by Remark 3, $2k + 3 \leq \binom{k}{2}$ and so $k \geq 6$.

On the other hand, choose any vertex $w \in S$. Then $D_w \subseteq S$ by Lemma 1.4.5. But then, since G is $K_{1,5}$-free, $c_o(G - S) \leq 8$. So we have $k + 3 \leq c_o(G - S) \leq 8$, or $k \leq 5$, a contradiction.

∎

Note that the assumption that $|V(G)| \geq 11$ is necessary in both Lemma 1.4.5 and Theorem 1.4.6, for the graph G shown in Figure 1.4.1 has odd order 9 and $\delta(G) > 0$, is $K_{1,5}$-free and 3-vertex-critical, but, if we let $S = \{u, y, z\}$, then $D_{c_i} \not\subseteq S$, for $i = 1, \ldots, 6$ and G has no near-perfect matching.

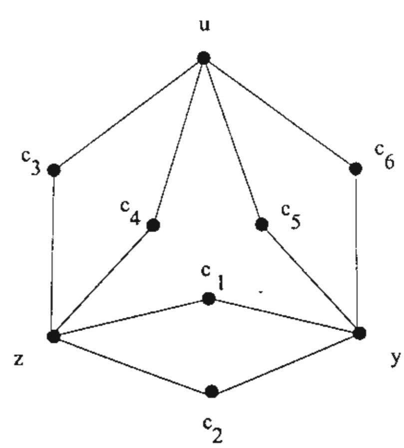


Figure 1.4.1

1.5. A Factor-critical Result

Lemma 1.5.1: Suppose G is a $K_{1,4}$-free 3-vertex-critical graph of odd order with minimum degree at least 3. If G_v has no perfect matching for some $v \in V(G)$ and $S_v \subseteq V(G_v)$ is a Tutte set for G_v with $c_o(G_v - S_v) \geq |S_v| + 2$, then $|S_v| \geq 2$.

Proof: First, note that G is connected by Corollary 1.4.2. Suppose that $v \in V(G), S_v \subseteq V(G)$ and $c_o(G_v - S_v) \geq |S_v| + 2$. Let S be $S_v \cup \{v\}$. Then $c_o(G - S) = c_o(G_v - S_v) \geq |S_v| + 2 = |S| + 1$. Let $C_1, C_2, ..., C_t$ be odd components of $G - S$. We first show that $S_v \neq \emptyset$. Suppose to the contradiction that $S_v = \emptyset$. Thus v is a cut vertex of G and $S = \{v\}$. Since G is connected, it follows that v is adjacent to at least one vertex of each component of $G - S$. Hence, $2 \leq c_o(G - S) < 4$ because of $K_{1,4}$-freedom of G. Clearly, $c_o(G - S) \neq 3$ since G has odd order. Then $c_o(G - S) = 2$. Since $\delta(G)$ is at least 3, $|V(C_i)| \geq 3$ for $i = 1, 2$. Hence, $D_v \cap V(C_i) \neq \emptyset$ for $i = 1, 2$. Let $D_v = \{a_1, a_2\}$ where $a_i \in V(C_i)$ for $i = 1, 2$. Then a_i is adjacent to every vertex of $V(C_i)$, but a_i is not adjacent to v. Since v is a cut vertex of G and G is connected, there exist vertices b_1 of C_1 and b_2 of C_2 say, such that $b_1v \in E(G)$ and $b_2v \in E(G)$. Clearly, $b_i \neq a_i$ for $i = 1, 2$. Now consider G_{b_1}. Then $D_{b_1} \cap \{v, a_1\} = \emptyset$. Thus $D_{b_1} \cap V(C_i) \neq \emptyset$ for $i = 1, 2$. Let $D_{b_1} = \{c_1, c_2\}$, where $c_i \in V(C_i)$ for $i = 1, 2$. Note that $b_1c_1 \notin E(G)$. Now c_1 is adjacent to every vertex of $V(C_1) - \{b_1\}$ and c_2 is adjacent to every vertex of $V(C_2)$. If $c_2v \in E(G)$, then $\{c_2, a_1\}$ dominates G. This contradicts the fact that $\gamma(G) = 3$. Thus $c_2v \notin E(G)$ and then $c_1v \in E(G)$.

Now consider G_{b_2}. Clearly, $D_{b_2} \cap \{v, a_2\} = \emptyset$. Then $D_{b_2} \cap V(C_i) \neq \emptyset$ for $i = 1, 2$. Let $D_{b_2} = \{d_1, d_2\}$ where $d_i \in V(C_i)$ for $i = 1, 2$. Note that $b_2d_2 \notin E(G)$. Then $d_2 \neq a_2$ and $d_2 \neq c_2$. Now d_1 is adjacent to every vertex of $V(C_1)$ and d_2 is adjacent to every vertex of $V(C_2) - \{b_2\}$. If $d_2v \in E(G)$, then $G[\{v; b_1, c_1, b_2, d_2\}]$ becomes a $K_{1,4}$ centered at v, a contradiction. Hence $d_2v \notin E(G)$ and so $d_1v \in E(G)$. But now $\{d_1, c_2\}$ dominates G, again a contradiction. Therefore, $S_v \neq \emptyset$.

So $|S_v| \geq 1$. Suppose $|S_v| = 1$. Let $S_v = \{u\}$. Then $S = S_v \cup \{v\} = \{u, v\}$. Since G is $K_{1,4}$-free and connected, $2 \leq c_o(G - S) \leq 6$. In fact, $c_o(G - S) = 3$ or $c_o(G - S) = 5$ because of the odd order of G. Suppose $c_o(G - S) = 5$. Now consider G_u. Clearly, $v \in D_u$ and $|D_u \cap V(C_i)| = 1$ for some $i, 1 \leq i \leq 5$. Then v becomes the center of a $K_{1,4}$, a contradiction. Hence, $c_o(G - S) \neq 5$. Therefore, $c_o(G - S) = 3$. Since $\gamma(G) = 3$ and $|S| = 2$, it follows that there is a vertex of $\bigcup_{i=1}^{3} V(C_i)$, x_1 say, such that $x_1u \notin E(G)$ and $x_1v \notin E(G)$. Without loss of generality, we may assume that $x_1 \in V(C_1)$. Since $\delta(G) \geq 3$, $|V(C_1)| \geq 5$ and $|V(C_i)| \geq 3$ for $i = 2, 3$. Let $y_1 \in V(C_1) - \{x_1\}$. Now consider G_{y_1}. Clearly, $G_{y_1} \neq \{u, v\}$, since $x_1u \notin E(G)$ and $x_1v \notin E(G)$. However, $D_{y_1} \cap \{u, v\} \neq \emptyset$ since $c_o(G - S) = 3$ and $|V(C_1) - \{y_1\}| \geq 4$. Without loss of generality, we may assume that $u \in D_{y_1}$. Thus $v \notin D_{y_1}$. Since $x_1u \notin E(G), D_{y_1} - \{u\} \subseteq V(C_1) - \{y_1\}$. Thus u is adjacent to every vertex of $V(C_2) \cup V(C_3)$.

Now consider G_u. Since $c_o(G - S) = 3, v \in D_u$. Because $x_1v \notin E(G), D_u - \{v\} \subseteq V(C_1)$. Thus v is adjacent to every vertex of $V(C_2) \cup V(C_3)$. Let $x_2 \in V(C_2)$. Note that $x_2u \in E(G)$ and $x_2v \in E(G)$. Then $D_{x_2} \cap \{u, v\} = \emptyset$. Thus $D_{x_2} \subseteq \bigcup_{i=1}^{3} V(C_i)$. Since $|D_{x_2}| = 2$, $D_{x_2} \cap V(C_i) = \emptyset$ for some i=1, 2, 3. But this contradicts to the fact that D_{x_2} is a dominating set of G_{x_2}

since $|V(C_i)| \geq 3$ for all $i, 1 \leq i \leq 3$. Hence, $|S_v| \neq 1$. Therefore $|S_v| \geq 2$ as required.

∎

Theorem 1.5.2: Suppose G is a $K_{1,4}$-free 3-vertex-critical graph of odd order with minimum degree at least 3. Further, suppose that G_v has no perfect matching for some $v \in V(G)$ and S_v is a Tutte set of $V(G_v)$ with $c_o(G_v - S_v) \geq |S_v| + 2$. Then for every vertex x of $V(G)$, $D_x \subseteq S_v \cup \{v\}$.

Proof: Let $S = S_v \cup \{v\}$. Thus, by Lemma 1.5.1, $|S| \geq 3$. Further, $c_o(G - S) = c_o(G_v - S_v) \geq |S_v| + 2 = |S| + 1 \geq 4$. Now let $C_1, C_2, ..., C_t$ be the odd components of $G - S$ and let $E_1, E_2, ..., E_n$ be the even components of $G - S$. Suppose to the contrary that there is a vertex x of $V(G)$ such that $D_x \nsubseteq S$. However, $D_x \cap S \neq \emptyset$ since $c_o(G - S) \geq 4$ and $|D_x| = 2$. Let $D_x \cap S = \{u\}$ and $D_x - S = \{y\}$. That is $D_x = \{u, y\}$. Clearly, $ux \notin E(G)$ and $yx \notin E(G)$. Suppose $G - S$ has an even component E_1 and suppose $y \in V(E_1)$. Then $t = 4$, or else u is the center of an induced $K_{1,4}$. So $|S| = 3$. Now vertex u is adjacent to all the vertices in at least three of the C_i's, say, without loss of generality, that u is adjacent to all vertices of $V(C_2) \cup V(C_3) \cup V(C_4)$. Then u is adjacent to no vertex of C_1, again because u is not the center of any induced $K_{1,4}$. But then $\{x\} = V(C_1)$ and $\deg_G(x) \leq 2$, a contradiction.

Thus $y \in \bigcup_{i=1}^{t} V(C_i)$. Without loss of generality, we may assume that $y \in V(C_1)$. Since G is $K_{1,4}$-free, the number of components of $G - S$ is at most 5 as otherwise u becomes a center of $K_{1,4}$. Thus $3 \leq |S| \leq 4$. Further, if $|S| = 4$, $G - S$ has no even components and if $|S| = 3$, then $G - S$ has at most one even component.

Claim 1: Each vertex of $G - S$ is not adjacent to at least one vertex of S.

Suppose to the contrary that there exists a vertex w of $G - S$ such that w is adjacent to every vertex of S. Now consider G_w. Clearly, $D_w \cap S = \emptyset$. Thus $D_w \subseteq V(G - S)$. But this is not possible since D_w is a dominating set of G_w of size 2 and $c_o(G - S) \geq 4$. Hence, our claim is proved.

Now we distinguish three cases according to the location of x.

Case 1: Suppose $x \in V(C_1)$.

Since $y \in V(C_1)$, u is adjacent to every vertex of $\bigcup_{i=2}^{t} V(C_i)$ and every vertex of $\bigcup_{i=1}^{n} V(E_i)$. It follows that $t = 4$ and $G - S$ has no even components because of $K_{1,4}$-freedom in G. Thus $|S| = 3$. Further, for $2 \leq i \leq 4$, C_i is complete and u is not adjacent to any vertex of $V(C_1)$, again by $K_{1,4}$-freedom in G. Then y is adjacent to every vertex of $V(C_1) - \{x\}$. It follows from Claim 1 that there is a vertex of $S - \{u\}$, say w, such that w is not adjacent to at least two vertices of $G - S$ lying in two different components of $C_2 \cup C_3 \cup C_4$.

Without loss of generality, we may assume that w is not adjacent to c_2 and c_3, where $c_2 \in V(C_2)$ and $c_3 \in V(C_3)$, respectively. Let $z \in S - \{u, w\}$. Then $S = \{u, w, z\}$.

We first show that z is not adjacent to any vertex of $V(C_4)$. Suppose to the contradiction that $zc_4 \in E(G)$ for some $c_4 \in V(C_4)$. Now consider G_{c_4}. Clearly, $D_{c_4} \cap S \neq \emptyset$, but $D_{c_4} \cap (\{z, u\} \cup V(C_4)) = \emptyset$. Thus $w \in D_{c_4}$ and $|D_{c_4} \cap V(G - S)| = 1$. Let $D_{c_4} \cap V(G - S) = \{w'\}$. Since $wc_2 \notin E(G)$ and $wc_3 \notin E(G)$, it follows that $w'c_2 \in E(G)$ and $w'c_3 \in E(G)$. But this is impossible since w' can be in only one odd component of $G - S$. Hence, z is not adjacent to any vertex of $V(C_4)$. Consequently, $|V(C_4)| \geq 3$ since $\delta(G) \geq 3$.

We next show that w is not adjacent to any vertex of $V(C_4)$. Suppose by way of contradiction that $wa \in E(G)$ for some $a \in V(C_4)$. Now consider G_a. Clearly, $D_a \cap S \neq \emptyset$, but $D_a \cap (\{u, w\} \cup V(C_4)) = \emptyset$. Thus $z \in D_a \cap S$. Since $|V(C_4) - \{a\}| \geq 2$ and z is not adjacent to any vertex of $V(C_4)$, it follows that $D_a - \{z\} \subseteq V(C_4)$. But this is impossible since $D_a \cap V(C_4) = \emptyset$. Hence, w is not adjacent to any vertex of $V(C_4)$.

Now let $b \in V(C_4)$ and consider G_b. Since C_4 is complete and $ub \in E(G)$, it follows that $D_b \cap (\{u\} \cup V(C_4)) = \emptyset$. Then $D_b \subseteq (S - \{u\}) \cup \bigcup_{i=1}^{3} V(C_i)$. Since z and w are not adjacent to any vertex of $V(C_4)$, no vertex of D_b is adjacent to any vertex of $V(C_4) - \{b\}$. This contradicts the fact that D_b is a dominating set of G_b since $|V(C_4) - \{b\}| \geq 2$. Hence, $x \notin V(C_1)$.

Case 2: Next, suppose $x \in V(G - S) - V(C_1)$.

If x belongs to some even component E_1 of $G - S$, then $V(E_1) - \{x\} \neq \emptyset$; say $z \in V(E_1) - \{x\}$. But then u is adjacent to z and to every vertex in $V(C_2) \cup V(C_3) \cup V(C_4)$. It then follows that u is the center of an induced $K_{1,4}$ and we have a contradiction.

Hence, without loss of generality, we may assume that $x \in V(C_2)$. We distinguish two cases according to $|S|$.

Case 2.1: Suppose $|S| = 3$. Since G is $K_{1,4}$-free, $c_o(G - S) \leq 5$. Thus $c_o(G - S) = 4$ since G has odd order. Since $ux \notin E(G)$ and $\delta(G) \geq 3$, it follows that $|V(C_2)| \geq 3$. Then u is adjacent to every vertex of $\bigcup_{i=2}^{4} V(C_i) - \{x\}$. Since G is $K_{1,4}$-free, $G - S$ has no even components and $C_2 - x$, C_3 and C_4 are complete.

Let $z \in S - \{u\}$. We next show that z is not adjacent to any vertex of $V(C_4)$. Suppose to the contrary that $za_4 \in E(G)$ for some $a_4 \in V(C_4)$. Then $D_{a_4} \cap (\{u, z\} \cup V(C_4)) = \emptyset$ since u is adjacent to every vertex of $V(C_4)$ and $V(C_4)$ is complete. Let $S - \{u, z\} = \{w\}$. Clearly, $w \in D_{a_4}$. Then $wa_4 \notin E(G)$ and w dominates $V(C_4) - \{a_4\}$. Now $|V(C_4)| \geq 3$ because $\delta(G) \geq 3$. Let $b_4 \in V(C_4) - \{a_4\}$. Then $b_4u \in E(G)$ and $b_4w \in E(G)$. Consequently, $D_{b_4} \cap (\{u, w\} \cup V(C_4)) = \emptyset$.

Since $c_o(G - S) = 4, z \in D_{b_4}$. So $zb_4 \notin E(G)$, but z dominates $V(C_4) - \{b_4\}$. Now if $c_4 \in V(C_4) - \{a_4, b_4\}$, c_4 is adjacent to every vertex of S. This contradicts Claim 1. Hence, z is not adjacent to any vertex of $V(C_4)$ for every $z \in S - \{u\}$.

Because $\delta(G) \geq 3$, $|V(C_4)| \geq 3$. Suppose $c \in V(C_4)$. Since u is adjacent to every vertex of $V(C_4)$, $cu \in E(G)$. Thus $D_c \cap (\{u\} \cup V(C_4)) = \emptyset$ since C_4 is complete. Then $D_c \subseteq (S - \{u\}) \cup \bigcup_{i=1}^{3} V(C_i)$. Since $V(C_4) - \{c\} \neq \emptyset$, every vertex of $V(C_4) - \{c\}$ is adjacent to at least one vertex of D_c. But this is impossible since $D_c \subseteq ((S - \{u\}) \cup \bigcup_{i=1}^{3} V(C_i))$ and none of the vertices of $(S - \{u\}) \cup \bigcup_{i=1}^{3} V(C_i)$ is adjacent to any vertex of $V(C_4)$. This complete the proof of Case 2.1.

Case 2.2: Suppose $|S| = 4$. Thus $c_o(G - S) = 5$ and $G - S$ has no even components. If $V(C_2) - \{x\} \neq \emptyset$, then u dominates $\bigcup_{i=2}^{5} V(C_i) - \{x\}$. This contradicts the fact that G is $K_{1,4}$-free. Thus $V(C_2) - \{x\} = \emptyset$. Since $ux \notin E(G)$ and $\delta(G) \geq 3$, it follows that x is adjacent to every vertex of $S - \{u\}$. Because G is $K_{1,4}$-free and u dominates $\bigcup_{i=3}^{5} V(C_i)$, each odd component C_i is complete for all i, $3 \leq i \leq 5$.

Claim 2: For each $a \in S$, $|D_a \cap S| = 2$.

Clearly, $D_a \cap S \neq \emptyset$ since $c_o(G - S) = 5$. If $|D_a \cap S| = 1$, then G contains a $K_{1,4}$ centered at the vertex of $D_a \cap S$, a contradiction. Hence, Claim 2 is proved.

As a consequence of Claim 2 and Remark 2, $|N_G(a) \cap (S - \{a\})| \leq 1$ for each $a \in S$. Now let $S - \{u\} = \{w, z, v\}$. Without any loss of generality, we may assume that $uw \notin E(G)$ and $uz \notin E(G)$. Since $D_x = \{u, y\}$, y is adjacent to both w and z. Now consider D_u. We next show that $v \in D_u$. Suppose to contrary that $v \notin D_u$. By Claim 2, $D_u = \{w, z\}$. Since $wy \in E(G)$ and $wx \in E(G)$, it follows that w can dominate vertices in at most one component among C_3, C_4 and C_5 because of $K_{1,4}$-freedom of G. Without any loss of generality, then, we may assume that w is adjacent to no vertex in $C_4 \cup C_5$. Then z must dominate $C_4 \cup C_5$. But then z becomes a center of $K_{1,4}$ since $zy \in E(G)$ and $zx \in E(G)$. This contradiction proves that $v \in D_u$. Hence, $vu \notin E(G)$. Because $D_x = \{u, y\}$, it follows that $yv \in E(G)$. Now every vertex of $S - \{u\}$ is adjacent to both x and y. Since G is $K_{1,4}$-free, v can dominate vertices in at most one component of $C_3 \cup C_4 \cup C_5$. Thus the vertex of $D_u - \{v\}$ must be in S by Claim 2 and hence must be a center of an induced $K_{1,4}$, again a contradiction. This completes the proof of Case 2.2.

Case 3: So suppose $x \in S$.

Clearly, since G is $K_{1,4}$-free, $c_o(G - S) = 4$ and $|S| = 3$. Then u dominates $\bigcup_{i=2}^{4} V(C_i)$. Thus C_i is complete for $2 \leq i \leq 4$. By an argument similar to that used in the proof in Case 2.1, each vertex of $S - \{u\}$ is not adjacent to any vertex of $V(C_4)$. Further, for $c \in V(C_4)$, $D_c \subseteq (S - \{u\}) \cup \bigcup_{i=1}^{3} V(C_i)$. But this is also impossible since no vertex of $(S - \{u\}) \cup \bigcup_{i=1}^{3} V(C_i)$ is adjacent to any vertex of $V(C_4)$. This completes the proof of Case 3 and hence the proof of our theorem.

∎

Theorem 1.5.3: If G is a $K_{1,4}$-free 3-vertex-critical graph of odd order with minimum degree at least 3, then G is factor-critical.

Proof: Suppose to the contrary that G is not factor-critical. Then there is a vertex v of G such that $G - v = G_v$ has no perfect matching. By Tutte's 1-factor theorem and the fact that G_v has even order, there exists a Tutte set $S_v \subseteq V(G_v)$ such that $c_o(G_v - S_v) \geq |S_v| + 2$. Then, by Lemma 1.5.1, $|S_v| \geq 2$. Let S be $S_v \cup \{v\}$. Then S is a Tutte set in G and $c_o(G - S) \geq |S| + 1 \geq 4$. Now let $|S| = k$. Since for each $x \in V(G), D_x \subseteq S$ by Theorem 1.5.2, it follows that for every vertex x of G there is a pair of vertices in $S - \{x\}$, say a and b, such that $D_x = \{a, b\}$. Since there are $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs of vertices of S and at least $k + (k + 1) = 2k + 1$ vertices in G, by Remark 3 it follows that $2k + 1 \leq \frac{k(k-1)}{2}$ and hence $k \geq 6$.

On the other hand, $k + 1 \leq c_o(G - S) \leq 6$ because G is $K_{1,4}$-free and $D_x \subseteq S$ for each $x \in V(G)$. Hence, $k \leq 5$, a contradiction. This completes the proof our theorem.

∎

Our bound on the minimum degree in Theorem 1.5.3 is best possible since the graph G in Figure 1.5.1 is $K_{1,4}$-free 3-vertex-critical connected of odd order with minimum degree 2, but is not factor-critical since $G - v$ has no perfect matching.

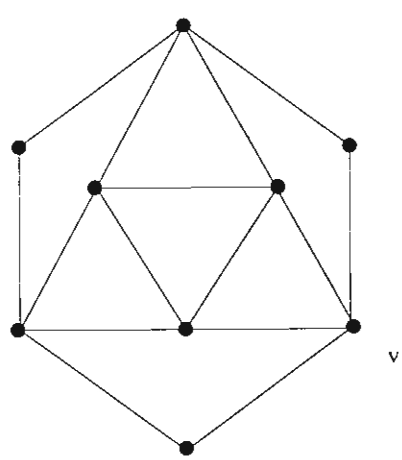


Figure 1.5.1

Note that there are infinitely many 3-vertex-critical connected graphs of odd order containing $K_{1,4}$, for the graphs shown in Figure 1.5.2 all belong to this family.

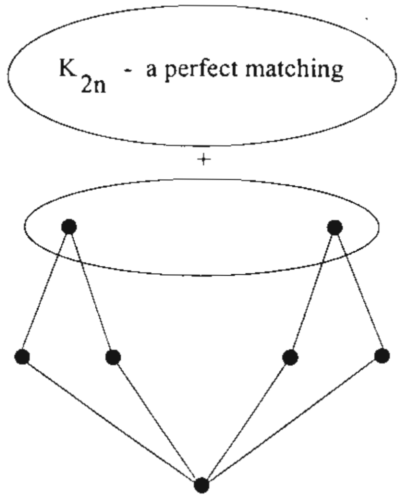


Figure 1.5.2

Moreover, there are also infinitely many $K_{1,4}$-free 3-vertex-critical connected graphs of odd order with minimum degree at least 3. The graph $G_{2k,3}$ for any positive integer k, introduced by Brigham, Chin and Dutton [BCD1, BCD2], is such a graph where $V(G_{2k,3}) = \{v_0, v_1, ..., v_{4k+2}\}$ and $E(G_{2k,3}) = \{v_iv_j | 1 \leq (i-j) \bmod (4k+3) \leq k\}$. Figure 1.5.3 shows $G_{6,3}$ and $G_{8,3}$.

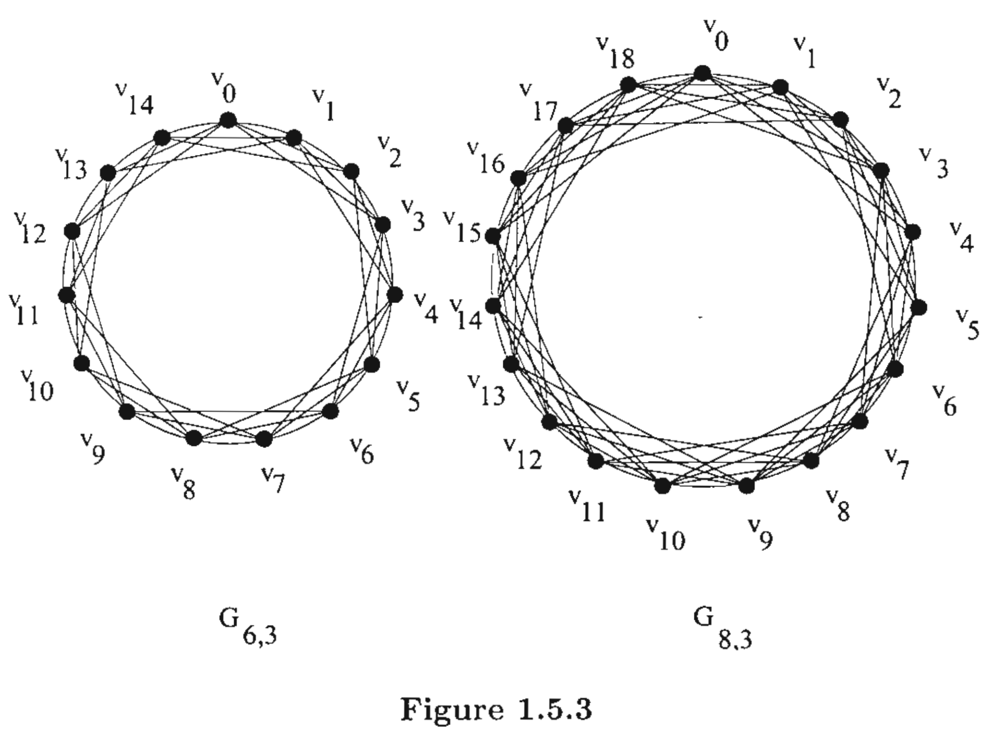


Figure 1.5.3

Theorem 1.2.4 states that if G is a $K_{1,5}$-free 3-vertex-critical connected graph of even order, then G has a perfect matching. One might expect that the hypothesis that the graph be $K_{1,4}$-free in Theorem 1.5.3 can also be weakened to say that the graph be $K_{1,5}$-free. But this is not the case since the graphs in Figure 1.5.4 (with $r, s \geq 3$) are $K_{1,5}$-free 3-vertex-critical connected graphs of odd order with minimum degree at least 3, (in fact, with minimum degree at least 4), but are not factor-critical.

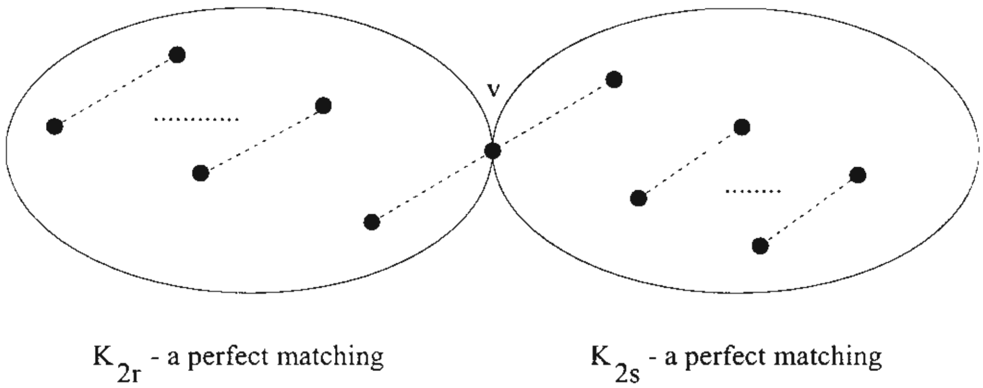


Figure 1.5.4

Note that $G - v$ has no perfect matching. Further, G contains $K_{1,4}$ as a subgraph. If we increase the connectivity of the graphs involved, however, we believe that one can relax the property of $K_{1,4}$-free to $K_{1,5}$-free. So we conclude this section with the following conjecture.

Conjecture: If G is a $K_{1,5}$-free 3-vertex-critical 2-connected graph of odd order with minimum degree at least 3, then G is factor-critical.

Chapter 2

Results on 3-Vertex-Critical Claw-Free Graphs

2.1 Introduction

A graph is called *claw-free* if it has no induced subgraph isomorphic to the bipartite graph $K_{1,3}$. In this chapter, three new theorems about the connectivity of 3-vertex-critical graphs which are also claw-free are presented, together with three corollaries about their k-factor-criticality.

Recall that G_v denotes the graph $G-v$, D_v any minimum dominating set of the graph $G-v$. Further, $N(v)$ denotes the set of all vertices adjacent to vertex v and $N[v]$ the closed neighborhood of v.

The following remarks about D_v are easily verified, but since we will appeal to them repeatedly, we list them separately.

Remarks: If G is 3-vertex-critical, then the following hold:

1. For every vertex v of G, $|D_v| = 2$.
2. If $D_v = \{x, y\}$, then x and y are not adjacent to v.
3. For every pair of distinct vertices v and w, $D_v \neq D_w$.

We shall need the following four lemmas in establishing our results.

Lemma 2.1.1: ([BCD1]) A connected graph G is 2-vertex-critical if and only if G is isomorphic to K_{2n} with a perfect matching removed.

∎

Lemma 2.1.2: ([FHM; Theorem 2]) The diameter d of a $\gamma-$vertex-critical graph G satisfies d $\leq 2(\gamma - 1)$ for $\gamma \geq 2$.

∎

Lemma 2.1.3: ([FHM; Theorem 6]) A connected graph G with diameter 4 is 3-vertex-critical if and only if it has two blocks each of which is 2-vertex-critical.

∎

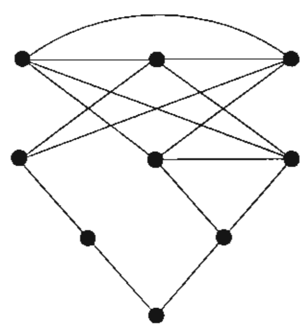
Lemma 2.1.4: ([FHM; Lemma 5]) If there exist vertices u and v such that $N_G[u] \subseteq N_G[v]$, then G is not γ-vertex-critical for any γ.

∎

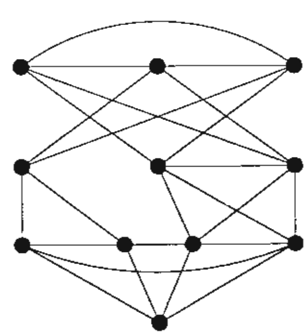
We now present a construction which yields a new infinite family of claw-free 3-vertex-critical graphs.

For positive integers t, r and s, we construct the graph $G(t, r, s)$ as follows. Let $X = \{x_1, x_2, ..., x_t\}$, $Y = \{y_1, y_2, ..., y_r\}$, $T = \{u_1, u_2, ..., u_t,$

$v_1, v_2, ..., v_r\}$ and $S = \{z_1, z_2, ..., z_s, w_1, w_2, ..., w_s\}$. Then set $V(G(t,r,s)) = X \cup Y \cup T \cup S \cup \{a\}$, thus yielding a set of $2t+2r+2s+1$ distinct vertices. Join vertex a to each vertex of S. Form complete graphs on each of X, Y and T and form a complete graph on S, except for the perfect matching $\{z_i w_i | 1 \leq i \leq s\}$. Finally, join each x_i to each vertex of $(T - \{u_i\}) \cup \{z_1, z_2, ..., z_s\}$ and join each y_i to each vertex of $(T - \{v_i\}) \cup \{w_1, w_2, ..., w_s\}$. It is not difficult to show that $G(t,r,s)$ is a claw-free 3-vertex-critical graph. Figure 2.1.1 shows the graphs $G(1,2,1)$ and $G(1,2,2)$. Note that these graphs are 2-connected and 3-connected, respectively. Our theorems in the next section guarantee certain connectivity for claw-free 3-vertex-critical graphs, given sufficient minimum degree. The graphs $G(1,2,1)$ and $G(1,2,2)$ show these assumptions on minimum degree to be best possible.



G(1,2,1)



G(1,2,2)

Figure 2.1.1

Lemma 2.1.5: If G is a claw-free 3-vertex-critical connected graph, then G has diameter at most 3.

Proof: Let d be the diameter of G. Then, by Lemma 2.1.2, $d \leq 4$. Suppose, to the contrary, that $d = 4$. Then, by Lemma 2.1.3, G has two blocks, each of which is 2-vertex-critical. Then each block of G must be a complete graph of even order without one perfect matching by Lemma 2.1.1. Since G is connected, each of these blocks has at least four vertices. Further, these two blocks must overlap in one vertex, u say. But then u becomes a center of $K_{1,3}$, a contradiction. This completes the proof of the lemma.

∎

To see that the above upper bound on the diameter is best possible, the reader is again directed to the infinite family described after Lemma 2.1.4 above.

We shall also make use of the following theorem on factor-critical graphs. (See [FFR, LY].)

Theorem 2.1.6: If G is $(k+1)$-connected, claw-free and of order n, and if $n-k$ is even, then G is k-factor-critical.

∎

Finally, the next two lemmas will be used repeatedly to obtain our main results.

Lemma 2.1.7: Let G be a k-connected claw-free graph and suppose $k \geq 1$. Suppose S is cutset of $V(G)$ with $|S| = k$. Then

1. For any component C of $G - S$, $N_G(x) \cap C \neq \emptyset$ for every $x \in S$,

2. $G - S$ has exactly two components.

Proof: Part (1) follows immediately from the fact that S is a minimum cutset. Part (2) then follows by claw-freedom.

∎

Lemma 2.1.8: Suppose G and S are defined as in Lemma 2.1.7. In addition, suppose G is also 3-vertex-critical. Let C_1 and C_2 be the two components of $G - S$. Further, let $A = V(C_1) - \bigcup\limits_{x \in S} N_G(x)$ and $B = V(C_2) - \bigcup\limits_{x \in S} N_G(x)$. Then

1. For each $i = 1, 2$, $G[N_{C_i}(x)]$ is complete for every $x \in S$,

2. $A = \emptyset$ or $B = \emptyset$; further, if $k = 2$, then $A \neq \emptyset$ or $B \neq \emptyset$,

3. if $A = \emptyset$ and $|V(C_1)| \geq 2$, then $\bigcap\limits_{x \in S} N_{C_1}(x) = \emptyset$.

Proof: Statement (1) follows immediately from Lemma 2.1.7(1) and the fact that G is claw-free.

(2) If $A \neq \emptyset$ and $B \neq \emptyset$, then the diameter of G must be at least 4. This contradicts Lemma 2.1.5. Hence, $A = \emptyset$ or $B = \emptyset$. Now suppose $k = 2$ and suppose further that $A = \emptyset$ and $B = \emptyset$. Then $\gamma(G) = 2$, a contradiction. Hence, $A \neq \emptyset$ or $B \neq \emptyset$. This proves (2).

(3) Suppose $A = \emptyset$, but $\bigcap_{x \in S} N_{C_1}(x) \neq \emptyset$. Let $w \in \bigcap_{x \in S} N_{C_1}(x)$. It follows by (1), and the fact that $A = \emptyset$, that w must be adjacent to every vertex of $V(C_1) - \{w\}$. Thus $N_G[w] = V(C_1) \cup S$. Since $|V(C_1) - \{w\}| \geq 1$, there exists a vertex $z \in V(C_1) - \{w\}$. Clearly, $N_G[z] \subseteq N_G[w]$. But this contradicts Lemma 2.1.4 and completes the proof of our lemma.

∎

2.2 Main Results

Theorem 2.2.1: Let G be a connected claw-free 3-vertex-critical graph. Then G is 2-connected.

Proof: Suppose that G is not 2-connected. Then there exists a cutvertex $v \in V(G)$. Moreover, $G - v$ contains exactly two components by claw-freedom. Let these two components be C_1 and C_2. Let A and B be as given in Lemma 2.1.8 and suppose $A = \emptyset$. Then v dominates $V(C_1)$. Thus $N[u] \subseteq N[v]$ for each vertex u in $V(C_1)$. But this contradicts Lemma 2.1.4 and completes the proof of our theorem.

∎

Theorem 2.2.2: Let G be a connected claw-free 3-vertex-critical graph. Then if G is of even order or if $\delta(G) \geq 3$, then G is 3-connected.

Proof: Suppose, to the contrary, that G is not 3-connected. By Theorem 2.2.1, G is 2-connected, so G must have a (minimum) cutset $S = \{u, v\}$. By Lemma 2.1.7(2), there must be exactly two components in $G - S$. Denote these components by C_1 and C_2. Let A and B be as given in Lemma 2.1.8. By Lemma 2.1.8(2), we may suppose that $A = \emptyset$, but $B \neq \emptyset$. We now distinguish three cases according to $|V(C_1)|$.

Case 1: $|V(C_1)| = 1$.

Let $\{z\} = V(C_1)$. Then z is adjacent to both u and v. Thus $\delta(G) = 2$ and hence G is of even order by our hypothesis. By Lemma 2.1.4, $uv \notin E(G)$ otherwise $N_G[z] \subseteq N_G[u]$.

Claim: For each $w \in V(C_2) - B$, $D_w = \{z, w'\}$ where $w' \in B$.

Suppose without any loss of generality that $w \in N_{C_2}(u)$. Then $D_w \cap (N_{C_2}(u) \cup \{u\}) = \emptyset$ by claw-freedom in G. We first show that $z \in D_w$. Suppose to the contrary that $z \notin D_w$. Clearly, $v \in D_w$. Since $uv \notin E(G)$, the single vertex - call it a - of $D_w - \{v\}$ must be adjacent to every vertex of $B \cup \{u\}$. This implies that vertex a is in $N_{C_2}(u)$, a contradiction. Hence, $z \in D_w$.

Since $B \neq \emptyset$, $D_w - \{z\} \subseteq V(C_2)$. Let $D_w - \{z\} = \{w'\}$. Then w' dominates $V(C_2) - \{w\}$. If $w' \in N_{C_2}(v)$, then $\{u, w'\}$ dominates G, a contradiction. Hence, $w' \notin N_{C_2}(v)$. Since $D_w \cap (N_G(u) \cup \{u\}) = \emptyset$, $w' \in B$. This proves our claim.

Now let $V(C_2) - B = \{w_1, w_2, \ldots, w_t\}$ where $t \geq 1$. By Claim 1, there exists a set of vertices $\{w'_1, w'_2, \ldots, w'_t\} \subseteq B$ such that $D_{w_i} = \{z, w'_i\}$ for $1 \leq i \leq t$. Clearly, $w'_i \neq w'_j$ for $i \neq j$ by Remark 3. Therefore, $|V(C_2) - B| \leq |B|$. Further, $|B| \geq 2$ because of connectedness of C_2. Since $z \in V(C_1)$, for each i, w'_i dominates $V(C_2) - \{w_i\}$. Remark 2 together with the fact that $|B| \geq 2$ implies that $w_i \in D_{w'_i}$. Then w_i dominates $B - \{w'_i\}$. Thus B is complete by claw-freedom of G and the fact that w'_i dominates $V(C_2) - \{w_i\}$.

Suppose $|B| \geq t + 1$. Choose $b \in B - \{w'_1, w'_2, \ldots, w'_t\}$. Then b dominates $V(C_2)$. Thus $\{b, z\}$ dominates G, a contradiction. Hence, $|B| = t$. This implies that $|V(G)| = 2t+3$ contradicting the fact that G is of even order. This proves Case 1.

Case 2: $|V(C_1)| = 2$.

Let $V(C_1) = \{x, y\}$. Clearly, $xy \in E(G)$. By Lemma 2.1.8(3), we may suppose that u is adjacent to x, but not to y, and v is adjacent to y, but not to x. Thus $\deg x = \deg y = 2$. But then by hypothesis, G is of even order. Now consider G_v. Clearly, $D_v \cap (N_G(v) \cup \{v\}) = \emptyset$. Suppose $u \in D_v$. Since $uy \notin E(G)$, $x \in D_v$. Then u dominates $V(C_2)$. But this is impossible since $B \neq \emptyset$. Hence, $u \notin D_v$. Thus $|D_v \cap V(C_1)| = 1$ and $|D_v \cap V(C_2)| = 1$.

Let $\{w\} = D_v \cap V(C_2)$. Then w dominates $V(C_2)$. If $w \in V(C_2) - B$, then $N_G[b] \subseteq N_G[w]$ for every vertex $b \in B$, a contradiction. Hence, $w \in B$. If there is a vertex $z \in B - \{w\}$, then $N[z] \subseteq N[w]$, again a contradiction, and so $B - \{w\} = \emptyset$. Thus $B = \{w\}$. Now let $a \in N_{C_2}(u)$. Consider G_a. Since $N_{C_2}(u)$ is complete by Lemma 2.1.8(1), $D_a \cap (N_{C_2}(u) \cup \{u, w\}) = \emptyset$. But $D_a \cap V(C_2) \neq \emptyset$ because $B \neq \emptyset$. Thus $v \notin D_a$ otherwise no vertex of D_a is adjacent to x. Hence, $D_a \cap V(C_1) \neq \emptyset$. Let $\{a'\} = D_a \cap V(C_2)$. Clearly, $a' \in N_{C_2}(v)$ and a' dominates $V(C_2) - \{a\}$. Similarly, $a \in D_{a'}$ and a dominates $V(C_2) - \{a'\}$. Hence, $V(C_2) - B$ is isomorphic to a complete graph of even order 2t with a perfect matching deleted. Therefore, $|V(G)| = 2t + 5$, contradicting the fact that G is of even order. This completes the proof of Case 2.

Case 3: $|V(C_1)| \geq 3$.

Then by Lemma 2.1.8(3), sets $N_{C_1}(u)$ and $N_{C_1}(v)$ must partition $V(C_1)$, since $A = \emptyset$. So, without loss of generality, we may suppose $|N_{C_1}(u)| \geq 2$.

Let $x \in N_{C_1}(u)$. Consider G_x. Clearly, $|D_x| = 2$ and $D_x \cap (N_{C_1}(u) \cup \{u\}) = \emptyset$. (Note that $N_{C_1}(u)$ is complete by Lemma 2.1.8(1).) Since $|N_{C_1}(u) - \{x\}| \geq 1$ and v is not adjacent to any vertex of $N_{C_1}(u)$ by Lemma 2.1.8(3), it follows that $D_x \cap N_{C_1}(v) \neq \emptyset$. Let $D_x = \{y, w\}$ where $y \in N_{C_1}(v)$. Again, by Lemma 2.1.8(3), $yu \notin E(G)$. Thus $wu \in E(G)$. Since y is not adjacent to any vertex of $V(C_2)$ and $B \neq \emptyset$, it follows that $w \in N_{C_2}(u)$. Further, w dominates $V(C_2) \cup \{u\}$. Because $B \neq \emptyset$, there is a vertex $z \in B$. Clearly, $N_G[z] \subseteq N_G[w]$. This contradicts Lemma 2.1.4 and completes the proof of the theorem.

∎

Theorem 2.2.3: Let G be a connected claw-free 3-vertex-critical graph. Then if $\delta(G) \geq 5$, G is 4-connected.

Proof: Suppose to the contrary that G is not 4-connected. By Theorem 2.2.2, G is 3-connected, so there exists a cutset consisting of three vertices in G, say $S = \{u, v, w\}$. By Lemma 2.1.7(2), $G - S$ consists of exactly two components, C_1 and C_2 say. Let $A = V(C_1) - (N_G(u) \cup N_G(v) \cup N_G(w))$ and $B = V(C_2) - (N_G(u) \cup N_G(v) \cup N_G(w))$. Then by Lemma 2.1.7(1), $N_G(x) \cap V(C_i) \neq \emptyset$ for every $x \in \{u, v, w\}$ and for $i = 1, 2$. By Lemma 2.1.8(2), $A = \emptyset$ or $B = \emptyset$. Without loss of generality, we may assume that $A = \emptyset$. Note that since $\delta(G) \geq 5$,$|V(C_1)| \geq 4$ by Lemma 2.1.8(3). Further, $|V(C_2)| \geq 3$.

Let $x \in N_{C_2}(u)$. Consider G_x. Clearly, $|D_x| = 2$ and $D_x \cap (N_{C_2}(u) \cup \{u\}) = \emptyset$, since $N_{C_2}(u)$ is complete by Lemma 2.1.8(1). We distinguish two cases according to D_x.

Case 1: $D_x \cap \{v, w\} = \emptyset$.

Since $|V(C_1)| \geq 4$ and $|V(C_2)| \geq 3$, it follows that $D_x \cap V(C_1) \neq \emptyset$ and $D_x \cap V(C_2) \neq \emptyset$. Put $D_x = \{y, z\}$ where $y \in V(C_1)$ and $z \in V(C_2)$. Then y dominates $V(C_1)$ and z dominates $V(C_2) - \{x\}$. Clearly, $yx \notin E(G)$ and $zx \notin E(G)$. By Lemma 2.1.7(1) and the claw-freedom of G, $zu \notin E(G)$. Thus $yu \in E(G)$ since $D_x = \{y, z\}$. Since $|V(C_1)| \geq 4$ and $A = \emptyset$, it follows by Lemma 2.1.8(3) that y is not adjacent to at least one vertex of $\{v, w\}$. Without loss of generality, we may assume that $yv \notin E(G)$. Then $zv \in E(G)$. It follows from Lemma 2.1.7(1) and the claw-freedom of G that $vx \notin E(G)$. We now distinguish two cases according to yw.

Case 1.1: $yw \in E(G)$.

Note that y dominates $V(C_1) \cup \{u, w\}$. Choose $a \in V(C_1) - \{y\}$. If $av \notin E(G)$, then $N_G[a] \subseteq N_G[y]$ contradicting Lemma 2.1.4. Thus $av \in E(G)$ for every $a \in V(C_1) - \{y\}$. Hence, $N_{C_1}(v) = V(C_1) - \{y\}$. By Lemma 2.1.8(1), $G[V(C_1) - \{y\}]$ is complete. Since y dominates $V(C_1) \cup \{u, w\}$, $G[V(C_1)]$ is complete. We next show that $N_{C_1}(u) = \{y\}$.

Suppose to the contrary that there is a vertex $y_1 \in V(C_1) - \{y\}$ such that $y_1u \in E(G)$. Consider G_{y_1}. Clearly, $D_{y_1} \cap (V(C_1) \cup \{v, u\}) = \emptyset$. Then $D_{y_1} \subseteq \{w\} \cup V(C_2)$. Since $|V(C_1)| \geq 4$, $w \in D_{y_1}$. Then w dominates $V(C_1) - \{y_1\}$. Next, choose $y_2 \in V(C_1) - \{y, y_1\}$. Consider G_{y_2}. Clearly, $D_{y_2} \cap (V(C_1) \cup \{v, w\}) = \emptyset$. Then $D_{y_2} \subseteq \{u\} \cup V(C_2)$. Since $|V(C_1)| \geq 4$, $u \in D_{y_2}$. Then u dominates $V(C_1) - \{y_2\}$. Now, if $y_3 \in V(C_1) - \{y, y_1, y_2\}$, then y_3 is adjacent to v, w and u. But this contradicts Lemma 2.1.8(3). Hence, $N_{C_1}(u) = \{y\}$. By applying a similar argument, we have $N_{C_1}(w) = \{y\}$.

Now if $a, b \in V(C_1) - \{y\}$, then $N_G[a] = V(C_1) \cup \{v\} = N_G[b]$. But this contradicts Lemma 2.1.4 and hence completes the proof in this case.

Case 1.2: $yw \notin E(G)$.

Since $D_x = \{y, z\}$, $zw \in E(G)$. Now z dominates $(V(C_2) - \{x\}) \cup \{v, w\}$. By Lemma 2.1.7(1) and the claw-freedom of G, $wx \notin E(G)$. Consider G_z. Clearly, $D_z \cap ((V(C_2) - \{x\}) \cup \{v, w\}) = \emptyset$. Then $D_z \subseteq \{u, x\} \cup V(C_1)$. Since $|V(C_2)| \geq 3$, $D_z \cap \{u, x\} \neq \emptyset$. If $D_z = \{u, x\}$, then $uw \in E(G)$ since $xw \notin E(G)$. But then $G[u; w, x, y]$ becomes a claw centered at u, a contradiction. Hence, $D_z \neq \{u, x\}$. Now we show that $u \notin D_z$. Suppose to the contrary that

$u \in D_z$. Then $x \notin D_z$. Thus u dominates $V(C_2) - \{z\}$. By Lemma 2.1.8(1), $G[V(C_2) - \{z\}]$ is complete. Since z dominates $V(C_2) - \{x\}$, $G[V(C_2)]$ is complete except for the edge xz. Let $x_1 \in V(C_2) - \{x, z\}$. Then $V(C_2) \cup \{u\} \subseteq N_G[x_1]$. Consider G_{x_1}. Clearly, $D_{x_1} \cap (V(C_2) \cup \{u\}) = \emptyset$. Thus $D_{x_1} \subseteq \{v, w\} \cup V(C_1)$. But then no vertex of D_{x_1} is adjacent to x since $x \in V(C_2)$ and v and w are not adjacent to x. This contradiction proves that $u \notin D_z$. Then $x \in D_z$. Let $\{y_1\} = D_z - \{x\}$. Since $x \in V(C_2)$ and $y_1 \neq u$, $y_1 \in V(C_1)$. Because x is not adjacent to any vertex of $V(C_1) \cup \{v, w\}$, y_1 must dominate $V(C_1) \cup \{v, w\}$. Thus $y_1 \neq y$. By Lemma 2.1.8(3), $y_1u \notin E(G)$. Now consider G_{y_1}. Clearly, $D_{y_1} \cap (V(C_1) \cup \{v, w\}) = \emptyset$. Thus $D_{y_1} \subseteq \{u\} \cup V(C_2)$. Since $|V(C_1)| \geq 4$, $u \in D_{y_1}$. Then u dominates $V(C_1) - \{y_1\}$. By Lemma 2.1.8(1), $G[V(C_1) - \{y_1\}]$ is complete. Since y_1 dominates $V(C_1) \cup \{v, w\}$, $G[V(C_1)]$ is complete. Let $y_2 \in V(C_1) - \{y, y_1\}$. Then $V(C_1) \cup \{u\} \subseteq N_G[y_2]$. Consider G_{y_2}. Clearly, $D_{y_2} \cap (V(C_1) \cup \{u\}) = \emptyset$. Then $D_{y_2} \subseteq \{v, w\} \cup V(C_2)$. But then no vertex of D_{y_2} is adjacent to y since $y \in V(C_1)$ and v and w are not adjacent to y, a contradiction. This completes the proof in Case 1.2 and hence in Case 1.

Case 2: $D_x \cap \{v, w\} \neq \emptyset$.

Without any loss of generality, we may assume that $v \in D_x$. We distinguish three cases according to $D_x - \{v\}$.

Case 2.1: $D_x - \{v\} \in V(C_2)$.

Then v dominates $V(C_1)$ and thus $G[V(C_1)]$ is complete by Lemma 2.1.8 (1). Let $y_1 \in N_{C_1}(u)$. Then $V(C_1) \cup \{u, v\} \subseteq N_G[y_1]$. Consider G_{y_1}. Clearly, $D_{y_1} \cap (V(C_1) \cup \{u, v\}) = \emptyset$. Thus $D_{y_1} \subseteq \{w\} \cup V(C_2)$. Since $|V(C_1)| \geq 4$, $w \in D_{y_1}$. Then w dominates $V(C_1) - \{y_1\}$. Next suppose $y_2 \in V(C_1) - \{y_1\}$. Then $V(C_1) \cup \{v, w\} \subseteq N_G[y_2]$. Consider G_{y_2}. By a similar argument, we have $u \in D_{y_2}$ and u dominates $V(C_1) - \{y_2\}$. Now suppose $y_3 \in V(C_1) - \{y_1, y_2\}$. Clearly, y_3 is adjacent to v,w and u. This contradicts Lemma 2.1.8(3) and completes the proof in this case.

Case 2.2: $D_x - \{v\} = \{w\}$.

Then $vx \notin E(G)$ and $wx \notin E(G)$. Further, $V(C_1) = N_{C_1}(v) \cup N_{C_1}(w)$ and $vu \in E(G)$ or $wu \in E(G)$. Without any loss of generality, we may assume that $vu \in E(G)$.

Claim 2.2.1: $N_{C_1}(v) \cap N_{C_1}(w) = \emptyset$.

Suppose to the contrary that $N_{C_1}(v) \cap N_{C_1}(w) \neq \emptyset$. Let $a_1 \in N_{C_1}(v) \cap N_{C_1}(w)$. Then a_1 is adjacent to every vertex of $V(C_1) - \{a_1\}$ by Lemma 2.1.8(1). By Lemma 2.1.8(3), $a_1u \notin E(G)$. Consider G_{a_1}. Clearly, $D_{a_1} \cap (V(C_1) \cup \{v, w\}) = \emptyset$. Thus $D_{a_1} \subseteq \{u\} \cup V(C_2)$. Since $|V(C_1)| \geq 4$, $u \in D_{a_1}$. Then u dominates $V(C_1) - \{a_1\}$. By Lemma 2.1.8(1), $G[V(C_1) - \{a_1\}]$ is complete. Since a_1 is adjacent to every vertex of $V(C_1) - \{a_1\}$, $G[V(C_1)]$ is complete. Suppose $a_2 \in V(C_1) - \{a_1\}$. Since $V(C_1) = N_{C_1}(v) \cup N_{C_1}(w)$, $a_2v \in E(G)$ or $a_2w \in E(G)$. Suppose $a_2v \in E(G)$. Now $V(C_1) \cup \{u, v\} \subseteq N_G[a_2]$. By Lemma 2.1.8(3), $a_2w \notin E(G)$. Consider G_{a_2}. By a similar argument, we have $w \in D_{a_2}$ and w dominates $V(C_1) - \{a_2\}$. Now every vertex of $V(C_1) - \{a_1, a_2\}$

is adjacent to both u and w. Therefore, by Lemma 2.1.8(3), none is adjacent to v. Let $a_3 \in V(C_1) - \{a_1, a_2\}$. Consider G_{a_3}. Clearly, $D_{a_3} \cap (V(C_1) \cup \{u, w\}) = \emptyset$. Thus $D_{a_3} \subseteq \{v\} \cup V(C_2)$. But then no vertex of D_{a_3} is adjacent to a_4 for $a_4 \in V(C_1) - \{a_1, a_2, a_3\}$, a contradiction. Hence, $a_2 v \notin E(G)$. By a similar argument, $a_2 w \notin E(G)$. Thus $a_2 \notin N_{C_1}(v) \cup N_{C_1}(w)$. But this contradicts the fact that $V(C_1) = N_{C_1}(v) \cup N_{C_1}(w)$. Hence, our claim is proved.

Claim 2.2.2: $N_{C_1}(u) \subseteq N_{C_1}(v)$.

Suppose to the contrary that there is a vertex $b \in N_{C_1}(u)$ such that $b \notin N_{C_1}(v)$. Since $ux \in E(G)$ and $uv \in E(G)$, but $vx \notin E(G)$, it follows that $G[u; v, b, x]$ is a claw centered at u. This contradiction proves that $b \in N_{C_1}(v)$ for every $b \in N_{C_1}(u)$. Hence, $N_{C_1}(u) \subseteq N_{C_1}(v)$ as claimed.

Now consider G_v. Clearly, $D_v \cap (N_G(v) \cup \{u, v\}) = \emptyset$. Since $|N_{C_1}(v)| \geq 1$, $D_v \cap N_{C_1}(w) \neq \emptyset$ by Claim 2.2.1. Thus $D_v - N_{C_1}(w) \neq \{w\}$ since $wx \notin E(G)$ and no vertex of $N_{C_1}(w)$ is adjacent to x. Now let $D_v = \{y, z\}$ where $y \in N_{C_1}(w)$. Clearly, $z \in V(C_2)$. Thus y dominates $V(C_1)$. By Claim 2.2.2, $yu \notin E(G)$. Hence, z dominates $V(C_2) \cup \{u\}$. Now consider G_z. Clearly, $D_z \cap (V(C_2) \cup \{u\}) = \emptyset$. Thus $D_z \subseteq \{v, w\} \cup V(C_1)$. But then no vertex of D_z is adjacent to x since $x \in V(C_2)$ and v and w are not adjacent to x. This completes the proof in Case 2.2.

Case 2.3: $D_x - \{v\} \in V(C_1)$.

Then v dominates $V(C_2) - \{x\}$ and $B = \emptyset$. By Lemma 2.1.8(1), $G[V(C_2) - \{x\}]$ is complete. Since $vx \notin E(G)$ and $\delta(G) \geq 5$, $|V(C_2)| \geq 4$.

Claim 2.3.1: $N_{C_2}(u) = \{x\}$.

Suppose to the contrary that u is adjacent to some vertex of $V(C_2) - \{x\}$, x_1 say. Then $xx_1 \in E(G)$ by Lemma 2.1.8(1). Now $V(C_2) \cup \{u, v\} \subseteq N_G[x_1]$. Consider G_{x_1}. Clearly, $D_{x_1} \cap (V(C_2) \cup \{u, v\}) = \emptyset$. Then $D_{x_1} \subseteq \{w\} \cup V(C_1)$. Since $|V(C_2)| \geq 4$, $w \in D_{x_1}$. Further, w dominates $V(C_2) - \{x_1\}$. By Lemma 2.1.8(1), $G[V(C_2) - \{x_1\}]$ is complete. Consequently, $G[V(C_2)]$ is complete since $xx_1 \in E(G)$ and $G[V(C_2) - \{x\}]$ is complete. Next suppose $x_2 \in V(C_2) - \{x, x_1\}$. Then $V(C_2) \cup \{v, w\} \subseteq N_G[x_2]$. Consider G_{x_2}. Clearly, $D_{x_2} \cap (V(C_2) \cup \{v, w\}) = \emptyset$. Then $D_{x_2} \subseteq \{u\} \cup V(C_1)$. Since $|V(C_2)| \geq 4$, $u \in D_{x_2}$. Further, u dominates $V(C_2) - \{x_2\}$. Now for every $z \in V(C_2) - \{x, x_1, x_2\}$, $N_G[z] = V(C_2) \cup \{u, v, w\}$. Then $N_G[x] \subseteq N_G[z]$. This contradicts Lemma 2.1.4. Hence, $N_{C_2}(u) = \{x\}$.

Claim 2.3.2: $N_{C_2}(w) = \{x\}$.

Suppose to the contrary that w is adjacent to some vertex of $V(C_2) - \{x\}$, y say. Note that $(V(C_2) - \{x\}) \cup \{v, w\} \subseteq N_G[y]$. Consider G_y. Clearly, $D_y \cap (V(C_2) - \{x\}) \cup \{v, w\} = \emptyset$. Then $D_y \subseteq \{u, x\} \cup V(C_1)$. Since $N_{C_2}(u) = \{x\}$ and $|V(C_2)| \geq 4$, it follows that $x \in D_y$. Further, x dominates $V(C_2) - \{y\}$. Since $G[V(C_2) - \{x\}]$ is complete, $G[V(C_2)]$ is complete except for the edge xy. By Lemma 2.1.7(1) and the fact that $wy \in E(G)$ and $xy \notin E(G)$, it follows that $wx \notin E(G)$ as otherwise w becomes a center of claw. Next suppose $y_1 \in V(C_2) - \{x, y\}$. Then $V(C_2) \cup \{v\} \subseteq N_G[y_1]$. Consider G_{y_1}. Clearly, $D_{y_1} \cap (V(C_2) \cup \{v\}) = \emptyset$. Then $D_{y_1} \subseteq \{u, w\} \cup V(C_1)$. Since $N_{C_2}(u) = \{x\}$

and $|V(C_2)| \geq 4$, it follows that $w \in D_{y_1}$. Further, w dominates $V(C_2) - \{x, y_1\}$. Now let $y_2 \in V(C_2) - \{x, y, y_1\}$. Then $V(C_2) \cup \{v, w\} \subseteq N_G[y_2]$. Consider G_{y_2}. Clearly, $D_{y_2} \cap (V(C_2) \cup \{v, w\}) = \emptyset$. Then $D_{y_2} \subseteq \{u\} \cup V(C_1)$. But then no vertex of D_{y_2} is adjacent to any vertex of $V(C_2) - \{x, y_2\}$, a contradiction. Hence, $N_{C_2}(w) \cap (V(C_2) - \{x\}) = \emptyset$. It follows by Lemma 2.1.7(1) that $N_{C_2}(w) = \{x\}$ as claimed.

Now let $z \in V(C_2) - \{x\}$ such that $zx \in E(G)$. Then $N_G[z] = V(C_2) \cup \{v\}$. Consider G_z. Clearly, $D_z \cap (V(C_2) \cup \{v\}) = \emptyset$. Then $D_z \subseteq \{u, w\} \cup V(C_1)$. But then no vertex of D_z is adjacent to any vertex of $V(C_2) - \{x, z\}$, a contradiction. This completes the proof of Case 2.3 and hence the theorem is proved.

∎

We now have the following corollary the proof of which is immediate by Theorems 2.1.6, 2.2.1, 2.2.2 and 2.2.3.

Corollary 2.2.4: 1. Let G be a connected claw-free 3-vertex-critical graph of odd order. Then G is factor-critical.

2. Let G be a connected claw-free 3-vertex-critical graph of even order. Then G is bicritical.

3. Let G be a connected claw-free 3-vertex-critical graph of odd order. Then if $\delta(G) \geq 5$, G is 3-factor-critical.

∎

Note that the members of the infinite family shown in Figure 2.1.1 also satisfy the hypotheses of Corollary 2.2.4(1).

It is known that every 3-factor-critical graph must be 3-connected. (See [F1; Theorem 2.5].) On the other hand, clearly the graph $G(1,2,2)$ shown in Section 2.1 is 3-connected and has minimum degree 4, but is not 3-factor-critical. Thus the bound on minimum degree in Corollary 2.2.4(3) is best possible. Note also that each $G(t,r,s)$ for $t + r \geq 4$ and $s \geq 3$ satisfies the hypotheses of Corollary 2.2.4(3).

Chapter 3

Results on Connected Domination Critical Graphs

3.1 Introduction

Recall that a set $S \subseteq V(G)$ is a *(vertex) dominating set* for G if every vertex of G either belongs to S or is adjacent to a vertex of S and the minimum cardinality of a dominating set for G is called the *domination number* of G and is denoted by $\gamma(G)$. We say that a dominating set S for G is a *connected dominating set* if $G[S]$ is connected. The minimum cardinality of a connected dominating set for G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. Observe that $\gamma(G) \leq \gamma_c(G)$ and if $\gamma(G) = 1$, then $\gamma(G) = \gamma_c(G)$. Further, a graph containing a connected dominating set is connected.

Graph G is said to be $k - \gamma-$*critical* if $\gamma(G) = k$ but $\gamma(G+e) < k$ for each edge $e \notin E(G)$. (Clearly, then $\gamma(G+e) = k-1$, for every edge $e \notin E(G)$). The study of $k - \gamma-$critical graphs was begun by Sumner and Blitch [SB] in 1983. Clearly, the only $1 - \gamma-$critical graphs are K_n for $n \geq 1$. Sumner and Blitch showed that a graph G is $2 - \gamma-$critical if and only if $\overline{G} = \bigcup_{i=1}^{r} K_{1,n_i}$ for $n_i \geq 1$ and $r \geq 1$. Since 1980 $k - \gamma-$critical graphs have attracted considerable attention with many authors contributing results. For summaries of most known results, see [HHS; Chapter 16] as well as [FTWZ] and the references that they contain. Most of these results concern $3 - \gamma-$critical graphs. The structure of $k - \gamma-$critical graphs for $k \geq 4$ is far from completely understood.

The similar concept of edge criticality with respect to the connected domination number just has received attention only recently. Graph G is said to be $k - \gamma_c-$*critical* if $\gamma_c(G) = k$ but $\gamma_c(G+e) < k$ for each edge $e \notin E(G)$. Clearly, the only $1 - \gamma_c-$critical graphs are K_n for $n \geq 1$. Chen et.al. [CSM] were the first to study $k - \gamma_c-$critical graphs. They pointed out that for each edge $e \notin E(G)$, $\gamma_c(G) - 2 \leq \gamma_c(G+e) \leq \gamma_c(G) - 1$. Observe that $\gamma_c(C_n) = n-2$. Clearly, $\gamma_c(C_5 + e) = 2$ for any edge $e \notin E(C_5)$ but $\gamma_c(C_8 + uv) = 4$ if u and v are vertices of C_8 at distance 4.

If S is a connected dominating set for G, we shall denote by $S \succ_c G$. Further, if u and v are non-adjacent vertices of G and $\{u\} \cup S_1 \succ_c G - v$ for some $S_1 \subseteq V(G) \backslash \{u,v\}$, we will follow previously accepted notation and write $[u, S_1] \rightarrow_c v$. If $S_1 = \{z\}$, then we write $[u,z] \rightarrow_c v$ instead of $[u,\{z\}] \rightarrow_c v$.

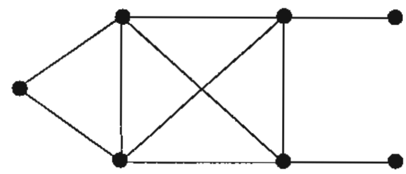


Figure 3.1.1

Chen et.al.[CSM] established the following theorems:

Theorem 3.1.1: A connected graph G is $2-\gamma_c-$critical if and only if $\overline{G}=\bigcup_{i=1}^{r} K_{1,n_i}$ for $n_i \geq 1$ and $r \geq 2$. ■

Theorem 3.1.2: Let G be a connected $3-\gamma_c-$critical graph and S an independent set with $s \geq 3$ vertices. Then the vertices in S may be ordered as $a_1, a_2, \ldots a_s$ in such a way that there exists a path $x_1, x_2, \ldots, x_{s-1}$ in $G-S$ with $[a_i, x_i] \rightarrow_c a_{i+1}$ for $i=1,2,\ldots,s-1$. ■

Theorem 3.1.3: Let G be a connected $3-\gamma_c-$critical graph.
1. If S is a cutset of G, then $c(G-S) \leq |S|+1$.
2. If G has even order, then G contains a perfect matching.
3. The diameter of G is at most 3. ■

Observe that Theorem 3.1.1 is similar to a characterization of $2-\gamma-$critical graphs mentioned above except for the lower bound on r. Further, Theorems 3.1.2 and 3.1.3 are true for $3-\gamma-$critical graphs. One might expect that all results on $3-\gamma-$critical graphs are also valid for $3-\gamma_c-$critical graphs. But this is not the case if we consider $3-\gamma_c-$critical graphs with cutvertices. Ananchuen and Plummer [AP3] showed that a connected $3-\gamma-$critical graph may contain more than one cutvertex. The graph in Figure 3.1.1 is as an example. They also characterized connected $3-\gamma-$critical graphs with more than one cutvertex.

In this chapter, we show that a $3-\gamma_c-$critical graph can contain at most one cutvertex. A characterization of $3-\gamma_c-$critical graphs with a cutvertex is given in Section 3.3. Section 3.2 contains results for $k-\gamma_c-$critical graphs with cutvertices for $k \geq 3$. We conclude this chapter with the results about matchings in $3-\gamma_c-$critical graphs in Section 3.4.

The following remarks are trivial to verify, but as we will appeal to them repeatedly, we list them separately.

Remark: If G is a $3-\gamma_c-$critical graph and u and v are non-adjacent vertices of G, then the following hold:

1. $\gamma_c(G + uv) = 2$,

2. If $N_G[u] \cup N_G[v] \neq V(G)$, then there exists a vertex $z \in V(G) \backslash \{u, v\}$ such that $[u, z] \rightarrow_c v$ or $[v, z] \rightarrow_c u$. Further, if $[u, z] \rightarrow_c v$, then $uz \in E(G)$ but $v \notin N_G(u) \cup N_G(z)$ and if $[v, z] \rightarrow_c u$, then $vz \in E(G)$ but $u \notin N_G(v) \cup N_G(z)$.

3.2. $k - \gamma_c-$ Critical Graphs with Cutvertices.

Lemma 3.2.1: For $k \geq 3$, let G be a $k - \gamma_c-$critical graph with a cutvertex x. Then

1. $G - x$ contains exactly two components,

2. If C_1 and C_2 are the components of $G - x$, then $G\left[N_{C_1}(x)\right]$ and $G\left[N_{C_2}(x)\right]$ are complete.

Proof: Let C_1, C_2, $\cdots$, C_t, $t \geq 2$, be the components of $G - x$.

(1) Suppose to the contrary that $t \geq 3$. Let $c_1 \in N_{C_1}(x)$ and $c_2 \in N_{C_2}(x)$. Consider $G + c_1c_2$. Since G is $k - \gamma_c-$critical, $\gamma_c(G + c_1c_2) < k$. Let S be a minimum connected dominating set for $G + c_1c_2$. Then $|S| \leq k - 1$. Since $t \geq 3$ and $G[S]$ is connected, it follows that $x \in S$. Then S is also a connected dominating set for G because $\{c_1, c_2\} \subseteq N_G(x)$. But this contradicts the fact that $\gamma_c(G) = k$ since $|S| \leq k - 1$. Hence, $t = 2$ as required. This proves (1).

(2) Suppose to the contrary that $G[N_{C_1}(x)]$ is not complete. Then there exist non-adjacent vertices a and b of $N_{C_1}(x)$. Consider $G + ab$. By a similar argument as in the proof of (1), a minimum connected dominating set S_1 for $G + ab$ of size at most $k - 1$ is also a connected dominating set for G. This contradicts the fact that $\gamma_c(G) = k$. Hence, $G[N_{C_1}(x)]$ is complete. Similarly, $G[N_{C_2}(x)]$ is complete. This proves (2) and completes the proof of our lemma. ■

Lemma 3.2.2: For $k \geq 3$, let G be a $k - \gamma_c-$critical graph with a cutvertex x and let C_1 and C_2 be the components of $G - x$. Suppose S is a minimum connected dominating set for G. Then

1. $x \in S$,

2. For $i = 1, 2$; $\gamma_c(C_i) \leq k - 1$,

3. If C is a non-singleton component of $G - x$ with $\gamma_c(C) = k - 1$, then C is $(k - 1) - \gamma_c-$critical.

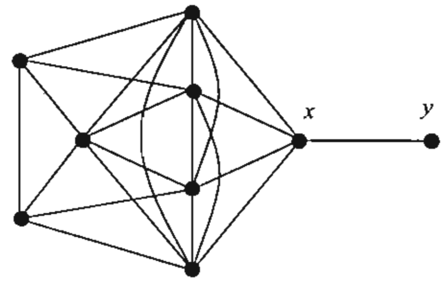


Figure 3.2.1

Proof: (1) follows immediately by the fact that $G[S]$ is connected.

(2) is obvious if $\gamma_c(C_i) \leq 2$ since $k \geq 3$. So we may suppose $\gamma_c(C_i) \geq 3$. If $S \cap V(C_1) = \emptyset$, then, since $x \in S$, $V(C_1) \subseteq N_G(x)$. By Lemma 3.2.1(2), $\gamma_c(C_1) = 1$, a contradiction. Hence, $S \cap V(C_1) \neq \emptyset$. Similarly, $S \cap V(C_2) \neq \emptyset$. Because $G[S]$ is connected and $x \in S$, it follows that $S \cap N_{C_i}(x) \neq \emptyset$ for $i = 1, 2$. By Lemma 3.2.1(2), $S \cap V(C_i) \succ_c C_i$. Hence, $\gamma_c(C_i) \leq |S \cap V\ (C_i)| \leq k - 1$.

(3) Let a and b be non-adjacent vertices of C. By Lemma 3.2.1(2), $\{a, b\} \nsubseteq N_C(x)$. Consider $G' = G + ab$. Since G is $k - \gamma_c$−critical, there exists a connected dominating set S_1 of size at most $k - 1$ for G'. Since $G'[S_1]$ is connected, $x \in S_1$. By a similar argument as in the proof of (2), $S_1 \cap V(C) \succ_c C + ab$. Hence, $\gamma_c(C + ab) \leq k - 2$. Therefore, C is $(k - 1) - \gamma_c$−critical as required. This completes the proof of our lemma. ■

Remark: Suppose $\gamma_c(C) = t < k - 1$ where C is defined as in Lemma 3.2.2. Then C need not be $t - \gamma_c$−critical. The graph G, in Figure 3.2.1, is $3 - \gamma_c$−critical with a cutvertex x. Clearly, $C = G - \{x, y\}$ is a non-singleton component of $G - x$ with $\gamma_c(C) = 1$ and is not $1 - \gamma_c$−critical.

Theorem 3.2.3 : For $k \geq 3$, let G be a $k - \gamma_c$−critical graph with a cutvertex x. Suppose C_1 and C_2 are the components of $G - x$. Let $A = G[V(C_1) \cup \{x\}]$ and $B = G[V(C_2) \cup \{x\}]$ .Then

1. $k - 1 \leq \gamma_c(A) + \gamma_c(B) \leq k$.
2. $\gamma_c(A) + \gamma_c(B) = k$ if and only if exactly one of C_1 and C_2 is singelton.

Proof: Let S be a minimum connected dominating set for G. By Lemma 3.2.2(1), $x \in S$.

(1) We distinguish two cases.

Case 1: $S \cap V(C_1) = \emptyset$ or $S \cap V(C_2) = \emptyset$.

Suppose without any loss of generality that $S \cap V(C_1) = \emptyset$. Then $V(C_1) \subseteq$

$N_G(x)$ and thus $\gamma_c(A) = 1$. Since $\gamma_c(G) \geq 3$, $V(C_2)\backslash N_G(x) \neq \emptyset$. Since $G[S]$ is connected, there exists a vertex $x_1 \in N_{C_2}(x) \cap S$. Then, by Lemma 3.2.1(2), $S - \{x\} \succ_c B$. Hence, $\gamma_c(B) \leq k - 1$. If there exists a connected dominating set S_1 of size at most $k - 2$ for B, then $S_1 \cup \{x\}$ becomes a connected dominating set of size at most $k - 1$ for G, a contradiction. Hence, $\gamma_c(B) = k - 1$. Therefore, $\gamma_c(A) + \gamma_c(B) = k$.

Case 2: $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$.

Because $x \in S$, $|S \cap V(C_1)| + |S \cap V(C_2)| = k - 1$. Since $G[S]$ is connected, there exists $y_i \in S \cap N_{C_i}(x)$ for $i = 1, 2$. By Lemma 3.2.1(2), $S \cap V(C_i) \succ_c V(C_i) \cup \{x\}$. Hence, $\gamma_c(V(C_i) \cup \{x\}) \leq |S \cap V(C_i)|$. We next show that for $i = 1, 2$, $\gamma_c(V(C_i) \cup \{x\}) = |S \cap V(C_i)|$. Suppose to the contrary that $\gamma_c(V(C_1) \cup \{x\}) \leq |S \cap V(C_1)| - 1$. Let S' be a minimum connected dominating set for $V(C_1) \cup \{x\}$. Then $S' \cap N_{C_1}(x) \neq \emptyset$. Thus $S' \cup \{x\} \cup (S \cap V(C_2)) \succ_c G$. But this contradicts the fact that $\gamma_c(G) = k$ since $|S' \cup \{x\} \cup (S \cap V(C_2)| \leq |S \cap V(C_1)| - 1 + 1 + |S \cap V(C_2)| = k - 1$. This proves that $\gamma_c(V(C_1) \cup \{x\}) = |S \cap V(C_1)|$. Similarly, $\gamma_c(V(C_2) \cup \{x\}) = |S \cap V(C_2)|$. Therefore, $\gamma_c(A) + \gamma_c(B) = k - 1$. Hence, (1) is proved.

(2) The sufficiency is immediate. So we need only prove the necessity. Let $\gamma_c(A) + \gamma_c(B) = k$. If $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$, then, by the proof of Case 2, $\gamma_c(A) + \gamma_c(B) = k - 1$, a contradiction. Hence, $S \cap V(C_1) = \emptyset$ or $S \cap V(C_2) = \emptyset$. Suppose without any loss of generality, we may assume that $S \cap V(C_1) = \emptyset$. Then $V(C_1) \subseteq N_G(x)$. Since $\gamma_c(G) \geq 3$, it follows that $V(C_2)\backslash N_G(x) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$. We next show that $|V(C_1)| = 1$.

Suppose to the contrary that $|V(C_1)| \geq 2$. Let $a_1 \in V(C_1) \cap N_G(x)$ and $a_2 \in V(C_2) \cap N_G(x)$. Consider $G + a_1a_2$. Then there exists a set $S_1 \subseteq V(G)\backslash\{a_1, a_2\}$ of size at most $k - 2$ such that $\{a_1, a_2\} \cup S_1 \succ_c G + a_1a_2$ or $[a_1, S_1] \succ_c a_2$ or $[a_2, S_1] \succ_c a_1$. Suppose $\{a_1, a_2\} \cup S_1 \succ_c G + a_1a_2$. Then $|S_1| \leq k - 3$. Thus $(S_1 \cap V(C_2)) \cup \{a_2\} \succ_c C_2$. Then $(S_1 \cap V(C_2)) \cup \{a_2, x\} \succ_c G$. But this contradicts the fact that $\gamma_c(G) = k$ since $|S_1 \cap V(C_2)| + |\{a_2, x\}| \leq k - 1$. Hence, $\{a_1, a_2\} \cup S_1$ does not dominate $G + a_1a_2$. We next suppose that $[a_1, S_1] \succ_c a_2$. Thus $|S_1| \leq k - 2$ and $S_1 \cap N_G(a_2) = \emptyset$. Thus $x \notin S_1$. Since $G[S_1 \cup \{a_1\}]$ is connected, $S_1 \subseteq V(C_1)$. But then no vertex of $S_1 \cup \{a_1\}$ is adjacent to a vertex of $V(C_2)\backslash\{a_2\}$, a contradiction. Hence, $\{a_1\} \cup S_1$ does not dominate $G - a_2$. Therefore, $[a_2, S_1] \succ_c a_1$. By an argument similar to that above, $x \notin S_1$ and $S_1 \subseteq V(C_2)$. But then no vertex of $S_1 \cup \{a_2\}$ is adjacent to a vertex of $V(C_1)\backslash\{a_1\}$, a contradiction. Hence, $|V(C_1)| = 1$ as claimed. Therefore, C_1 is singleton. This completes the proof of our theorem. ■

3.3 A Characterization of $3-\gamma_c-$ Critical Graphs with a Cutvertex.

Our first theorem improves Theorem 3.1.3(1) established by Chen et.al.[CSM] when a cutset is not singleton.

Theorem 3.3.1: Let G be a $3-\gamma_c-$critical graph and S a cutset of G with $|S| = s \geq 2$. Then $c(G-S) \leq |S|$. Further, the upper bound on the number of components is best possible.

Proof: Suppose to the contrary that $c(G-S) \geq |S| + 1 = s+1 \geq 3$. By Theorem 3.1.3(1), $c(G-S) = s+1$. Let $C_1, C_2, \ldots, C_{s+1}$ be the components of $G-S$. For $1 \leq i \leq s+1$, let $c_i \in V(C_i)$. Then $A = \{c_1, c_2, \ldots, c_{s+1}\}$ is independent. By Theorem 3.1.2, the vertices in A may be ordered as $a_1, a_2, \ldots, a_{s+1}$ in such a way that there exists a path $x_1, x_2, \ldots, x_s$ in $G-A$ with $[a_i, x_i] \rightarrow_c a_{i+1}$ for $1 \leq i \leq s$. Note that $a_i x_i \in E(G)$ but $x_i a_{i+1} \notin E(G)$. Further, $x_i \in S$. Thus $S = \{x_1, x_2, \ldots, x_s\}$ and a_1 is adjacent to every vertex of S. Observe that

$$\{a_1, x_2\} \cup \left(\bigcup_{i=2}^{s+1} V(C_i)\backslash\{a_2\}\right) \subseteq N_G(x_1),$$

$$\{a_s, x_{s-1}\} \cup \left(\bigcup_{i=1}^{s+1} V(C_i)\backslash\left(V(C_s) \cup \{a_{s+1}\}\right)\right) \subseteq N_G(x_s),$$

and for $2 \leq j \leq s-1$,

$$\{a_j, x_{j-1}, x_{j+1}\} \cup \left(\bigcup_{i=1}^{s+1} V(C_i)\backslash\left(V(C_j) \cup \{a_{j+1}\}\right)\right) \subseteq N_G(x_j).$$

Now consider $G + a_1 a_{s+1}$. Then, by Remark (2) of Section 3.1, there exists a vertex z such that $[a_1, z] \rightarrow_c a_{s+1}$ or $[a_{s+1}, z] \rightarrow_c a_1$. In either case, $z \in S$. Then $\{a_{s+1}, z\}$ does not dominate $G - a_1$ since a_1 is adjacent to every vertex of S. Hence, $[a_1, z] \rightarrow_c a_{s+1}$. Since $[a_i, x_i] \rightarrow_c a_{i+1}$ for $1 \leq i \leq s$ and $z a_{s+1} \notin E(G)$, it follows that $z = x_s$. Then x_s dominates $\bigcup_{i=1}^{s+1} V(C_i)\backslash\{a_{s+1}\}$. If $s = 2$, then $\{x_1, x_2\} \succ_c G$, a contradiction. Hence, $s \geq 3$. For $2 \leq k \leq s-1$, consider $G + a_k a_{s+1}$. Then, by Remark (2), there exists a vertex z_1 such that $[a_k, z_1] \rightarrow_c a_{s+1}$ or $[a_{s+1}, z_1] \rightarrow_c a_k$. We show that in either case $x_s x_{k-1} \in E(G)$. Suppose $[a_k, z_1] \rightarrow_c a_{s+1}$. Then $z_1 = x_s$. Since $a_k x_{k-1} \notin E(G)$, $x_s x_{k-1} \in E(G)$ as claimed. Now suppose $[a_{s+1}, z_1] \rightarrow_c a_k$. Then $z_1 = x_{k-1}$. Since $a_{s+1} x_s \notin E(G)$, $x_{k-1} x_s \in E(G)$ as claimed. Hence, $x_s x_i \in E(G)$, for $1 \leq i \leq s-1$ since $x_{s-1} x_s \in E(G)$. Because $[a_2, x_2] \rightarrow_c a_3$ and $s \geq 3$, it follows that $x_2 a_{s+1} \in E(G)$. But then $\{x_s, x_2\}$ is a connected dominating set for G, a contradiction. Hence, $c(G-S) \leq |S|$ as claimed.■

We next show that the upper bound on the number of components in Theorem 3.3.1 is best possible. For an integer $n \geq 3$, we construct a graph G_n as

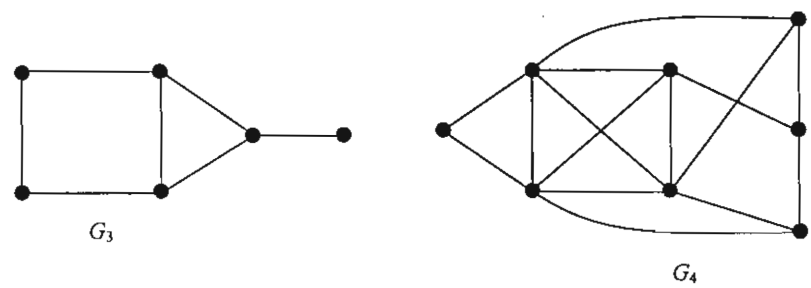


Figure 3.3.1

follows. Let $X = \{x_1, x_2, \ldots, x_{n-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$. Then set $V(G) = X \cup Y \cup \{a, b\}$, thus yielding a set of $2n$ distinct vertices. Form a complete graph on X. Join each x_i to each vertex of $(Y \backslash \{y_i\}) \cup \{a\}$ and finally join b to each vertex of $(Y \backslash \{y_{n-1}\}) \cup \{a\}$. It is not difficult to show that G_n is $3-\gamma_c$-critical. Note that $|X \cup \{b\}| = n$ and $G_n - (X \cup \{b\})$ contains exactly n components. Figure 3.3.1 shows the graphs G_3 and G_4.

Corollary 3.3.2: Let G be a $3-\gamma_c$-critical graph with a cutvertex x. Suppose C_1 and C_2 are the components of $G - x$. Then exactly one of C_1 and C_2 is a singleton.

Proof: Clearly, at most one of C_1 or C_2 is a singleton. If $V(C_1)\backslash N_G(x) \neq \emptyset$ and $V(C_2)\backslash N_G(x) \neq \emptyset$, then the distance from u to v is at least 4 for $u \in V(C_1)\backslash N_G(x)$ and $v \in V(C_2)\backslash N_G(x)$. This contradicts Theorem 3.1.3(3). Hence, $V(C_1)\backslash N_G(x) = \emptyset$ or $V(C_2)\backslash N_G(x) = \emptyset$. Since $\gamma_c(G) = 3$, it follows that $V(C_1)\backslash N_G(x) \neq \emptyset$ or $V(C_2)\backslash N_G(x) \neq \emptyset$. We may assume without any loss of generality that $V(C_2)\backslash N_G(x) = \emptyset$ but $V(C_1)\backslash N_G(x) \neq \emptyset$. Thus $\gamma_c(G[V(C_2) \cup \{x\}]) = 1$. By Theorem 3.2.3(1), $\gamma_c(G[V(C_1) \cup \{x\}]) = 1$ or 2. Suppose first that $\gamma_c(G[V(C_1) \cup \{x\}]) = 1$. Let $\{a\}$ be a minimum connected dominating set for $G[V(C_1) \cup \{x\}]$. Clearly, $a \neq x$ but $ax \in E(G)$. But then $\{a, x\} \succ_c G$, a contradiction. Hence, $\gamma_c(G[V(C_1) \cup \{x\}]) = 2$. By Theorem 3.2.3(2), exactly one of C_1 and C_2 is singleton. Because $\gamma_c(G) = 3, |V(C_1)| \geq 2$. Thus C_2 is singleton. This completes the proof of our corollary. ∎

Corollary 3.3.2 need not be true for $k \geq 4$. The graphs G_1 and G_2 in Figure 3.3.2 are $4-\gamma_c$-critical and $5-\gamma_c$-critical, respectively. Note that none of components of $G_i - x$ is singleton.

The following corollary follows immediately from Theorem 3.2.3(2) and Lemma 3.3.2.

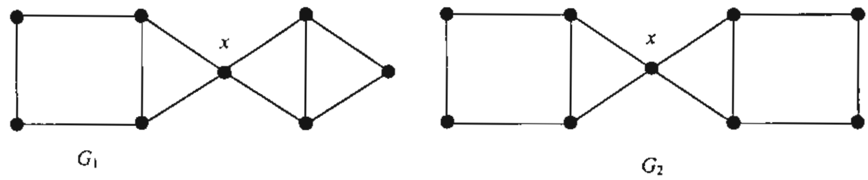


Figure 3.3.2

Corollary 3.3.3: Let G be a $3-\gamma_c$–critical graph with a cutvertex x. Suppose C_1 and C_2 are the components of $G - x$ with C_2 is singleton. Then $\gamma_c(G[V(C_1) \cup \{x\}]) = 2$. ■

Our next result establishes the number of cutvertices in $3-\gamma_c$–critical graphs.

Theorem 3.3.4: If G is a $3 - \gamma_c$–critical graph, then G contains at most one cutvertex.

Proof: Suppose to the contrary that x_1 and x_2 are distinct cutvertices of G. By Lemma 3.2.1(1) and Corollary 3.3.2, $G- x_1$ contains exactly 2 components, say C_1 and C_2, where C_2 is singleton. Let $\{y\} = V(C_2)$. Clearly, $N_G(y) = \{x_1\}$. Now consider $G- x_2$. Again, by Lemma 3.2.1 and Corollary 3.3.2, $G- x_2$ contains exactly 2 components, one of which is a singleton. Let $\{w\}$ be the vertex set of the singleton component of $G- x_2$. Then $w \neq y$ and $N_G(w) = \{x_2\}$. Clearly, $\{w, x_2\} \subseteq V(C_1)$. Since $\gamma_c(G) = 3$, $|V(C_1)| \geq 3$. Thus $G - \{x_1, x_2\}$ contains at least 3 components contradicting Theorem 3.3.1. This proves our theorem. ■

We now present a construction which yields two infinite families of $3 - \gamma_c$–critical graphs with a cutvertex. For positive integers n_i and r with $r \geq 2$, let $H = \bigcup_{i=1}^{r} K_{1,n_i}$. For $1 \leq j \leq r$, let c_j be the center of K_{1,n_j} in H and w_1^j, w_2^j ,..., $w_{n_j}^j$ the end vertices of K_{1,n_j} in H. We now construct the graphs G_{c_1} and G_{c_2} as follows. Set $V(G_{c_1}) = V(H) \cup \{x, y\}$ and $E(G_{c_1}) = E(\overline{H}) \cup \{xy\} \cup \{xw_i^j \mid 1 \leq i \leq n_j \text{ and } 1 \leq j \leq r\}$. Next set $V(G_{c_2}) = V(H) \cup \{x, y\} \cup U$ where $|U| \geq 1$ and $E(G_{c_2}) = E(\overline{H}) \cup \{xy\} \cup \{xw_i^j \mid 1 \leq i \leq n_j \text{ and } 1 \leq j \leq r\} \cup \{uz \mid u \in U \text{ and } z \in V(H) \cup (U \backslash \{u\})\}$. Note that $E(G_{c_2}) = E(G_{c_1}) \cup \{uz \mid u \in U$ and $z \in V(H) \cup (U \backslash \{u\})\}$. It is not difficult to show that G_{c_1} and G_{c_2} are both $3 - \gamma_c$–critical with the single cutvertex x. Note that $\gamma_c(G_{c_1} - \{x, y\}) = 2$ but $\gamma_c(G_{c_2} - \{x, y\}) = 1$. Figure 3.3.3 shows as examples the graphs G_{c_1} and G_{c_2} of order 7 and 8, respectively.

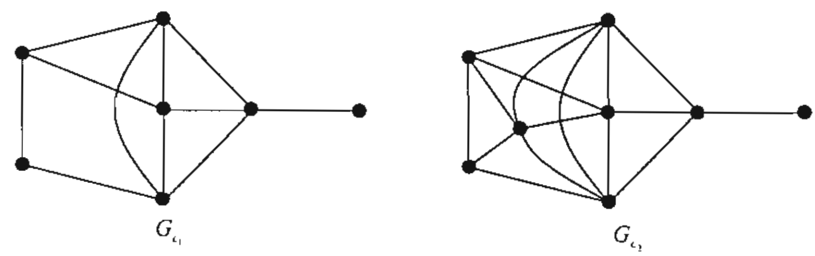


Figure 3.3.3

Theorem 3.3.5: G is a $3-\gamma_c$−critical graph with a cutvertex if and only if $G \in \{G_{c_1}, G_{c_2}\}$.

Proof: The sufficiency follows from our construction. So we only prove the necessity. Let x be a cutvertex of G. By Lemma 3.2.1(1) and Corollary 3.3.2, $G-x$ contains exactly two components, one of them is singleton. Let C_1 and C_2 be the components of $G-x$ with $V(C_2)=\{y\}$. Clearly, $N_G(y)=\{x\}$. By Corollary 3.3.3, $\gamma_c(G[V(C_1)\cup\{x\}])=2$. Let S be a minimum connected dominating set for $G[V(C_1)\cup\{x\}]$.

Claim: $x \notin S$.

Suppose to the contrary that $x \in S$. Let $\{x_1\}=S\backslash\{x\}$. Since $G[S]$ is connected, $xx_1 \in E(G)$. Because $N_G(y)=\{x\}$, $\{x, x_1\} \succ_c G$, a contradiction. This proves our claim.

It follows by our claim that $S \succ_c C_1$ and thus $\gamma_c(C_1)\le 2$. We distinguish two cases.

Case 1: $\gamma_c(C_1)=2$.

By Lemma 3.2.2(3), C_1 is $2-\gamma_c$−critical. Thus $\overline{C_1}=\bigcup_{i=1}^{r} K_{1,n_i}$ for $n_i \ge 1$ and $r \ge 2$ by Theorem 3.1.1. Let c_j be the center of K_{1,n_j} in $\overline{C_1}$ and $w_1^j, w_2^j, \ldots, w_{n_j}^j$ the end vertices of K_{1,n_j} in $\overline{C_1}$. We need to show that $N_{C_1}(x)=\bigcup_{j=1}^{r}\{w_i^j \mid 1\le i\le n_j\}$.

Claim 1.1: For $n_j \ge 1$, if x is adjacent to c_j, then x is not adjacent to any vertex of $\{w_1^j, w_2^j, \ldots, w_{n_j}^j\}$.

This claim follows directly from Lemma 3.2.1(2) and the fact that $c_jw_i^j \notin E(G)$ for $1 \le i \le n_j$.

Claim 1.2: If $n_j \ge 2$, then x is not adjacent to c_j.

Suppose to the contrary that x is adjacent to c_j for some j with $n_j \geq 2$. Then, by Claim 1.1, x is not adjacent to any vertex of $\{w_1^j, w_2^j, \ldots, w_{n_j}^j\}$. Consider $G + c_j w_1^j$. Since $y \notin N_G[c_j] \cup N_G[w_1^j]$, by Remark (2), there exists a vertex $z \in V(G)\backslash\{c_j, w_1^j\}$ such that $[c_j, z] \rightarrow_c w_1^j$ or $[w_1^j, z] \rightarrow_c c_j$. In either case, $z \in \{x, y\}$ since $N_G(y) = \{x\}$. Because $\{c_j, w_1^j, y\}$ is independent, $z \neq y$. Hence, $z = x$. If $[c_j, x] \rightarrow_c w_1^j$, then no vertex of $\{c_j, x\}$ is adjacent to w_2^j, a contradiction. Hence. $\{c_j, x\}$ does not dominate $G - w_1^j$. Therefore, $[w_1^j, x] \rightarrow_c c_j$. But this contradicts the connectedness of $G[\{w_1^j, x\}]$ since $xw_1^j \notin E(G)$. This proves our claim.

Claim 1.3: For $n_j \geq 2$, x is adjacent to every vertex of $\{w_i^j | 1 \leq i \leq n_j\}$.

Suppose to the contrary that there exists a vertex w_t^j, for some $1 \leq t \leq n_j$ and for some j, such that $xw_t^j \notin E(G)$. By Claim 1.2, $xc_j \notin E(G)$. Consider $G + xw_t^j$. Since x and w_t^j are not adjacent to c_j, by Remark (2), there exists a vertex $z \in V(G)\backslash\{x, w_t^j\}$ such that $[x, z] \rightarrow_c w_t^j$ or $[w_t^j, z] \rightarrow_c x$. If $[w_t^j, z] \rightarrow_c x$, then $z \neq y$ since $xy \in E(G)$. But then no vertex of $\{w_t^j, z\}$ is adjacent to y since $N_G(y) = \{x\}$, a contradiction. Hence, $\{w_t^j, z\}$ does not dominate $G - x$. Therefore, $[x, z] \rightarrow_c w_t^j$. Then $xz \in E(G)$ and $zw_t^j \notin E(G)$. Since $N_G(w_t^j) = V(G)\backslash\{x, y, c_j\}$ and $xc_j \notin E(G)$, it follows that $z = y$. But then no vertex of $\{x, z\}$ is adjacent to c_j, a contradiction. This proves our claim.

Claim 1.4: For $n_j = 1$, x is adjacent to exactly one of $\{c_j, w_1^j\}$.

Suppose to the contrary that x is adjacent to neither c_j nor w_1^j. Consider $G + c_j w_1^j$. By Remark (2), there exists a vertex $z \in V(G)\backslash\{c_j, w_1^j\}$ such that $[c_j, z] \rightarrow_c w_1^j$ or $[w_1^j, z] \rightarrow_c c_j$. Suppose $[c_j, z] \rightarrow_c w_1^j$. Since $G[\{c_j, z\}]$ is connected, $z \notin \{x, y\}$ because $(N_G(x) \cup N_G(y)) \cap \{c_j\} = \emptyset$. But then no vertex of $\{c_j, z\}$ is adjacent to y, a contradiction. Hence, $\{c_j, z\}$ does not dominate $G - w_1^j$. By a similar argument, $\{w_1^j, z\}$ does not dominate $G - c_j$. Thus $\gamma_c(G + c_j w_1^j) > 2$, a contradiction. Hence, x is adjacent to c_j or w_1^j. By Claim 1.1, x is adjacent to exactly one of $\{c_j, w_1^j\}$.

Without any loss of generality, we may assume that $xw_1^j \in E(G)$ for each j with $n_j = 1$. Now $N_G(x) = \{y\} \cup \bigcup_{j=1}^{r} \{w_i^j \mid 1 \leq i \leq n_j\}$. Hence, $G \cong G_{c_1}$ as required.

Case 2: $\gamma_c(C_1) = 1$.

Let u be a vertex of C_1 with $\{u\} \succ_c C_1$. If $u \in N_{C_1}(x)$, then $\{u, x\} \succ_c G$, a contradiction. Hence, $u \notin N_{C_1}(x)$ and $N_G[u] = V(C_1)$. Let $U = \{u \mid \{u\} \succ_c C_1\}$. Clearly, $|U| \geq 1$. $C_1\backslash U \neq \emptyset$ and $\gamma_c(C_1 - U) \geq 2$. Further, $N_{C_1}(x) \cap U = \emptyset$.

Claim 2.1: If a and b are non-adjacent vertices of C_1, then $ax \in E(G)$ but $bx \notin E(G)$ or $bx \in E(G)$ but $ax \notin E(G)$. Further, if $ax \in E(G)$, then a dominates $V(C_1)\backslash\{b\}$. Similarly, if $bx \in E(G)$, then b dominates $V(C_1)\backslash\{a\}$.