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รายงานวิจัยฉบับสมบูรณ์

โครงการ :

"GEOMETRY OF BANACH SPACES IN HYPERCONVEX FIXED POINT THEORY"

โดย ศ. ดร. สมพงษ์ ธรรมพงษา และคณะ

31 พฤษภาคม พ.ศ.2550

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

ACKNOWLEDGEMENT

จากการที่ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) โดยเฉพาะศาสตราจารย์ ดร.วิชัย บุญแสง เล็งเห็นถึงความสำคัญของงานวิจัยองค์ความรู้ใหม่ (Basic Research) ในทุกสาขา และให้การสนับสนุนอย่างต่อเนื่องตลอดมา งานวิจัยทางคณิตศาสตร์จึงได้เจริญก้าวหน้าอย่างไม่เคยปรากฏมาก่อน การแสดงออกของพวกเราถึงความชื่นชมและขอบคุณต่อ สกว.จึงมีปรากฏในทุกสถานที่ โครงงานวิจัยเรื่อง Geometry of Banach Spaces in Hyperconvex Fixed Point Theory นี้เป็นอีกโครงงานหนึ่งที่ได้รับการสนับสนุนด้วยดีจาก สกว.

ผลงานวิจัยที่ได้รับมีทั้งหมด 18 เรื่อง ได้รับการตีพิมพ์ในวารสารนานาชาติที่มีคุณภาพ 17 เรื่อง และอยู่ระหว่างการพิจารณาของ referee 1 เรื่อง โดยส่วนตัวของหัวหน้าคณะวิจัยจะพยายามสร้างผลงานที่คุณภาพเพื่อการ citation ที่สูง สิ่งเหล่านี้พิสูจน์ได้จากผลงานในโครงการวิจัยที่ได้รับการสนับสนุนจาก สกว. ก่อนหน้านี้ ดังนั้นการพิจารณา citation ของ journal ในเทอมของ impact factor จึงไม่ใช่ตัววัดที่สำคัญของเรา การเน้นที่ citation ของตัว paper หรือตัว author เป็นเป้าหมายหลักที่จะนำไปสู่ชื่อเสียงที่แท้จริงของผลงานวิจัย จากผลงานวิจัยที่ผ่านมาคณะวิจัยได้รับเกียรติและได้รับความไว้วางใจให้เป็นเจ้าภาพจัดการประชุม The 8th International Conference on Fixed Point Theory and Its Application (ICFPTA 2007) ขึ้นที่ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ระหว่างวันที่ 16-22 เดือน กรกฎาคม 2550

โครงงานวิจัยนี้มีส่วนประกอบ 3 ส่วนตามแผนงานที่ระบุไว้ในสัญญา คือ

1. Fixed point theory in Banach spaces: a nonstandard analysis approach [1,2,3,5,11,12,13,14,16]

2. Topology of convergence sets [9,10]

3. Fixed point theory in hyperconvex spaces [4,8,9,15,17,18]

ผลสำเร็จของโครงการวิจัย ในด้านคุณภาพมีแนวโน้มว่าจะมี citation ที่ดีซึ่งปรากฏมีขึ้นแล้ว แต่จักต้องใช้ความพยายามมากขึ้นที่จะ publish ผลงานใน journal ในระดับที่สูงขึ้นกว่านี้

ขอขอบพระคุณ สำนักงานกองทุนสนับสนุนการวิจัย เป็นอย่างสูง ไว้ ณ ที่นี้ และขอขอบพระคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ที่ให้ความสะดวกทางด้านอุปกรณ์คอมพิวเตอร์และการพิมพ์

คณะผู้วิจัย

ภาควิชาคณิตศาสตร์

มหาวิทยาลัยเชียงใหม่

30 พฤษภาคม พ.ศ. 2550

บทคัดย่อ

งานวิจัยนี้สร้างทฤษฎีจุดตรึงในปริภูมิบานาคภายใต้เงื่อนไขต่างๆทางเรขาคณิตของปริภูมิ การส่งที่ศึกษาส่วนใหญ่จะเป็นการส่งเพื่อจุดมุ่งหมายนี้ได้มีการสร้างอสมการต่างๆที่หลากหลายที่เกี่ยวข้องกับคุณสมบัติทางเรขาคณิต มีการค้นพบคุณสมบัติทางเรขาคณิตใหม่ๆที่พิสูจน์ได้ในเวลาต่อมาว่าเป็นเครื่องมือใหม่ชิ้นหนึ่งที่สำคัญในทฤษฎีจุดตรึง เครื่องมือสำคัญชิ้นหนึ่งในวิชาการวิเคราะห์นอนแสตนดาร์ดที่ถูกนำมาใช้ตลอดเวลาคือเทคนิค-อุลทราเพาเวอร์

ทฤษฎีบทในปริภูมิบานาคได้ถูกนำมาขยายผลในปริภูมิไฮเพอร์คอนเวกซ์ โดยเฉพาะอย่างยิ่งปริภูมิ CAT(0) การวิจัยอีกส่วนหนึ่งของโครงการคือการศึกษาโทโพโลยีของเซทของการลู่เข้าของการส่ง

คำหลัก: คุณสมบัติทางเรขาคณิต ปริภูมิบานาค เทคนิค-อุลทราเพาเวอร์ ปริภูมิไฮเพอร์คอนเวกซ์ เซทของการลู่เข้า

ABSTRACT

We investigate Fixed Point Theory in Banach spaces under various conditions of their geometry. Most of the mappings of interest are multivalued. To achieve this goal, several inequalities concerning important geometric properties are derived. Some new geometric properties under this research project are introduced. It is proved that many new results can be developed in the direction of these new properties. One of the main ingredients in the study is a technique from nonstandard analysis, namely the ultrapower technique.

The theory is then extended to the class of hyperconvex spaces. The most accomplishment is on the class of CAT(0) spaces, its important subclass. Parts of the research are devoted to the study of the topology of the convergence sets of mappings.

Keywords: geometric property, Banach space, ultrapower technique, hyperconvex space, convergence set

CONTENTS

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1. Project Title

GEOMETRY OF BANACH SPACES IN HYPERCONVEX FIXED POINT THEORY

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3. Research Field

FUNCTIONAL ANALYSIS AND TOPOLOGY

4. Problem statement and importance

The fixed point property (FPP) has been studied since J.Brouwer and S.Banach leading to several celebrated theorems such as Brouwer's Fixed Point Theorem and the principle of Banach's Contraction Mapping. The theory of fixed point property is one of the most important subject in pure and applied Mathematics. It contributes to a variety of applications in many fields of mathematics such as the theory of operators, control theory, approximation theory, and theory of equations. Using geometric property to study FPP has been developed since W.K.Kirk who proved in 1965 that a Banach space with a normal structure has weak fixed point property .

The fixed point property is still proven to be the most important research problem in Mathematics and is continuing to be of interest to mathematicians worldwide. The study of nonexpansive mappings has been substantially motivated by the study of monotone and accretive operators, two classes of operators which arise naturally in the theory of differential equations. As an example, Kato (1967) has obtained the following basic result : For a Banach space X, a subset D of X, and a map T:D ---->X, T is accretive if and only if for each x, y $\in$ D and $\lambda \geq 0$, $\| x-y \| \leq \| x-y+\lambda(Tx-Ty) \|$. Thus a mapping T:D --- > X is accretive if and only if the mapping $J_{\lambda} = (I-\lambda T)^{-1}$ (called the resolvent of T) is nonexpansive on its domain.

Other examples showing that the notion of nonexpansive mappings and their sets of fixed points play a crucial role in optimization theory (see [5, 39, 41, 42, 48, 50, 56, 57, 58]). In these studies, some forms of convergence theorems for nonexpansive mappings are considered.

The study of nonexpansive mappings and their fixed points could be extended to metric spaces. It is well-known that every nonexpansive mapping on a bounded hyperconvex metric space has a fixed point (Baillon 1988 [3]). The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi [2] 1956. Espinola and Kirk [15] obtained fixed point theorems in $\mathbb{R}$ -trees, whose class forms a subclass of hyperconvex spaces, and applied them to the Graph Theory.

Besides the existence of a fixed point, the first co-investigator is also interested in some topological properties of the fixed point set of a continuous mapping. He recently establishes the notion of the convergence set of a continuous mapping and shows that there is a nice relationship between the convergence set and the fixed point set certain kinds of mappings (including nonexpansive mappings). This relationship opens a new door to study topological properties of the fixed point set using those of the convergence set.

In view of the importance of this subject, the Principal Investigator (PI) and his group intend to continue their work on the mathematics itself, rather than on its applications. One part of the project then aims at establishing new concepts and results for fixed point theory both in Banach spacesand in hyperconvex metric spaces. Two of the main tools, among others, that the PI plans to employ and develop further are " the generalized Jordan- von Neumann constants" proposed by Dhompongsa and others [14], and "the generalized James constants" proposed by Dhompongsa and others [13]. For the second part, the project will concentrate on the topological property of the fixed point set of a given continuous mapping by means of its convergence set, the new notion established earlier by the first co-investigator.

This project will be undertaken for three years in collaboration with the PI's colleague and their students at Chulalongkorn University and Chiang Mai University. The PI has created a group of students working on Geometric Property, in particular, on the Fixed Point Property in Banach spaces. The impact of this project will not only contribute to advance the knowledge of the mathematical society itself, but will also strengthen the research groups, especially students as its members, in Thailand.

5. Objective of Research

5.1 To investigate the significance of the constants $C_{NJ}(a,X)$ and $J(a,X)$. For examples, the preservation of the property $C_{NJ}(a,X)<2$ and $C_{NJ}(a,Y)<2$ by their direct sum $X\oplus Y$ under various norms. (See [12, 13, 14]).

5.2 To investigate the convexity property of direct sums $X\oplus Y$ (See [12, 45]).

5.3 To apply nonstandard analysis to the fixed point theory (See [4, 43, 51, 52, 53, 54, 55]).

5.4 To establish fixed point theorems in the hyperconvexity setting (See [6, 7, 8, 15, 16, 33, 34, 35, 36, 37, 38]).

5.5 To investigate some topological properties of the convergence set.

5.6 To investigate the relationship between the convergence set and the fixed point set and use this relationship to study some topological properties of the fixed point set.

6. Methodology of Research

6.1 Solve problems and conjectures raised in [14] and [13].

6.2 Invent sufficient conditions for the fixed point property of direct sums $X \oplus Y$

6.3 Apply a new tool, namely, the nonstandard analysis technique to the fixed point property in Banach spaces.

6.4 Modify standard proofs formulated in a Banach space setting to a metric space one.

6.5 Invent new topological properties of the convergence set.

6.6 Establish new topological properties of the fixed point set using (6.5).

7. Plan of Research

7.1 Collecting know results concerning the FPP and related concepts in Banach spaces and hyperconvex spaces.

7.2 Preparing specific topics for each member in the research group to concentrate on his/her own research work.

7.3 Sharing results obtained in (7.2), discussing, and preparing manuscripts.

Plan for each 6 month period :

January 1, 2004 - June 30, 2004 : Writing a paper on "Convexity property of direct sums $X \oplus Y$ ".

July 1, 2004 - December 31, 2004 : Writing a paper on "Constants $C_{NJ}(a, X)$, $J(a, X)$ and other related ones" and "Topology of convergence sets".

January 1, 2005 - June 30, 2005 : Writing a paper on "Nonstandard analysis for fixed point property in Banach spaces" and "Topology of convergence sets".

July 1, 2005 - December 31, 2005 : Writing a paper on "Nonstandard analysis for fixed point property in Banach spaces" and "Topology of convergence sets".

January 1, 2006 - June 30, 2006 : Writing a paper on "Fixed point property in hyperconvex spaces".

July 1, 2006 - December 31, 2006 : Writing a paper on "Transition of geometric property in Banach spaces to hyperconvex spaces ".

8. Expected output

We expect to publish at least 2 papers a year.

Tentative titles and journals :

(1) Title : Geometry of Banach spaces in hyperconvex space I, II; Topology of convergence sets.

Journal : Proceedings Amer. Math. Soc (impact factor = 0.369) or London J. Math. Soc. (impact factor = 0.441) or Pacific J. Math. (impact factor = 0.395) or Studia Math. (impact factor = 0.399).

(2) Title : Fixed point theorems: A nonstandard analysis approach I, II; Topology of convergence sets.

Journal : J. Functional Anal. (impact factor = 0.879) or Bull. Austral.. Math. Soc (impact factor = 0.236) or J. Austral. Math. Soc. (impact factor = 0.282) or J. Math. Anal. Appl. (impact factor = 0.444).

(3) Title : Convexity of ψ -direct sums.

Journal : Nonlinear Anal. (impact factor = 0.406)

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OUTPUT

We divide the output into 2 categories.

I. Published papers:

- Uniform smoothness and U-convexity of ψ- direct sums, J. Nonlinear and Convex Analysis, 6 (2) (2005), 327-338. Appendix 1

- Fixed point property of direct sums, Nonlinear Anal. 63 (2005), e2177-e2188. Appendix 2

- A note on properties that implies the weak fixed point property, Abst. Appl. Anal. V. 2006, Article ID 34959, Pages 1-12. Appendix 3

- Lim's theorems for multivalued mappings in CAT(0) spaces, J. Math. Anal. Appl. 312 (2005), 478-487. Appendix 4

- The Dominguez-Lorenzo condition and multivalued nonexpansive mappings, Nonlinear Anal. 64 (2006), 958-970. Appendix 5

- Jordan-von Neumann constant and fixed points for multivalued nonexpansive Mappings, J. Math. Anal. Appl. 320 (2006), 916-927. Appendix 6

- Fixed point theorems for multivalued mappings in modular function spaces, Scien. Math. Japon. 63 (2) (2006), 161-169. Appendix 7

- Fixed points of uniformly Lipschitzian mappings, Nonlinear Anal. 65 (2006), 762-772. Appendix 8

- A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006), 983-987. Appendix 9

- Virtually nonexpansive maps and their convergence sets, J. Math. Anal. Appl. 326 (2007), 390-397. Appendix 10

- The James constant of normalized norms on R^2, J. Inequalities Appl. V. 2006, Article ID 26265, Pages 1-12. Appendix 11

- On the modulus of W*-convexity, J. Math. Anal. Appl. 320 (2006), 543-548. Appendix 12

- On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property, J. Math. Anal. Appl. 323 (2006), 1018-1024. Appendix 13

- A new trees-step fixed point iteration scheme for asymptotically nonexpansive Mappings, Appl. Math. Comp. 181 (2006), 1026-1034. Appendix 14

- Nonexpansive set-valued mappings in metric and Banach spaces, Journal of Nonlinear and Convex Analysis, 8 (1) (2007), 35-45. Appendix 15

- An inequality concerning the James constant and the weakly convergent sequence coefficient, Journal of Nonlinear and Convex Analysis (to appear). Appendix 16

- Diametrically contractive multivalued mappings. Appendix 17

I. Submitted paper:

- Common fixed points of a nonexpansive semigroup and a strong convergence theorem for Mann iterations in geodesic metric spaces. Appendix 18

Appendix

Appendix 1: Uniform smoothness and U-convexity of ψ - direct sums, J. Nonlinear and Convex Analysis, 6 (2) (2005), 327-338.

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UNIFORM SMOOTHNESS AND U-CONVEXITY OF ψ-DIRECT SUMS

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ABSTRACT. We study the ψ-direct sum, introduced by K.-S. Saito and M. Kato, of U-spaces, introduced by K. S. Lau. For Banach spaces X and Y and a continuous convex function ψ on the unit interval $[0,1]$ satisfying certain conditions, let $X \oplus_\psi Y$ be the ψ-direct sum of X and Y equipped with the norm associated with ψ. We first show that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is isometric to the space $X^* \oplus_\varphi Y^*$ for some continuous convex function φ satisfying the same conditions as of ψ. We introduce the so-called u-spaces and show that: (1) $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function. We also show that (2) $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function. As consequences, using the notion of ultrapower, we obtain : (3) $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function, and (4) $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

1. INTRODUCTION

For every continuous convex function ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$, there corresponds a unique absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ (see Bonsall and Duncan [3], also [19]). Recently, in [16] the authors introduced the ψ-direct sums $X \oplus_\psi Y$ of Banach spaces X and Y equipped with the norm associated with ψ, and proved that $X \oplus_\psi Y$ is uniformly convex if and only if X, Y are uniformly convex and ψ is strictly convex. We write $X \simeq Y$ to indicate that X and Y are isometric (or Banach isomorphism, see [12]).

The purposes of this paper are to characterize uniform smoothness and U-convexity of $X \oplus_\psi Y$. In Section 2 we shall recall some fundamental facts on the ψ-direct sums of Banach spaces and introduce the dual function φ of ψ so that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is $X^* \oplus_\varphi Y^*$. In Section 3 we shall show that the ultrapower of $X \oplus_\psi Y$ is the ψ-direct sum of the ultrapowers of X and of Y. In Section 4 we shall prove that $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function, and by using the ultrapower technique we obtain that $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function. In Section 5 we introduce new spaces, namely u-spaces, and prove that $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function, and again by using the ultrapowers we have $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

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2. The ψ-direct sums

Let X be a Banach space. Throughout this paper, let X^* be the dual space of X, $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \le 1\}$, and for $x \neq 0$, $\nabla_x = \{f \in S_{X^*} : f(x) = \|x\|\}$. In this section we shall recall the definition of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y. A norm on $\mathbb{C}^2$ is called *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $(z,w) \in \mathbb{C}^2$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{C}^2$ is denoted by N_a. The l_p-norms $\|\cdot\|_p$ $(1 \le p \le \infty)$ on $\mathbb{C}^2$ are examples of such norms, and for any norm $\|\cdot\| \in N_a$,

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1.$$

Let Ψ be the set of all continuous convex functions ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t,t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. N_a and Ψ are in one-to-one correspondence under the following equations. For each $\|\cdot\| \in N_a$, the function ψ defined by $\psi(t) = \|(1-t,t)\|$ $(0 \le t \le 1)$ belongs to Ψ. Conversely, for each $\psi \in \Psi$, let $\|(0,0)\|_\psi = 0$, and $\|(z,w)\|_\psi = (|z|+|w|)\psi(\frac{|w|}{|w|+|z|})$ for $(z,w) \neq (0,0)$ and this norm belongs to N_a (see [3] and [19]). For Banach spaces X and Y, we denote by $X \oplus_\psi Y$ the direct sum $X \oplus Y$ equipped with the norm

$$\|(x,y)\| = \|(\|x\|,\|y\|)\|_\psi \text{ for } (x,y) \in X \oplus Y.$$

Thus, under this norm, $X \oplus_\psi Y$, which will be called the ψ-direct sum of X and Y, is a Banach space and for all $(x,y) \in X \oplus Y$ we also have (see [16])

$$\|(x,y)\|_\infty \le \|(x,y)\|_\psi \le \|(x,y)\|_1.$$

Saito et al. [16] extended the concept to absolute normalized norm on $\mathbb{R}^n$. The corresponding set of all continuous convex functions on the $(n-1)$–simplex $\{(s_1,...,s_{n-1}) \in \mathbb{R}_+^{n-1} : s_1 + ... + s_{n-1} \le 1\}$ will be denoted by Ψ_n.

Now we show that the dual space of this ψ-direct sum is a direct sum $X \oplus_\varphi Y$ of the same kind for some $\varphi \in \Psi$. We first define

$$\varphi_\psi(s) = \varphi(s) := \sup_{t\in[0,1]} \frac{st + (1-s)(1-t)}{\psi(t)}$$

for $s \in [0,1]$. We show that $\varphi \in \Psi$ and call it *the dual function* of ψ.

Proposition 1. *The above function φ is continuous, convex on $[0,1]$ and satisfies $\varphi(s) \ge \max\{s, 1-s\}$ for all $s \in [0,1]$.*

Proof. It is easy to see that $\varphi(\cdot)$ is continuous. To show that φ is convex, we let $s_1, s_2 \in [0,1]$ and consider

$$\begin{aligned}\varphi\Big(\frac{s_1+s_2}{2}\Big) &= \sup_{t\in[0,1]} \frac{\frac{s_1+s_2}{2}t + (1-\frac{s_1+s_2}{2})(1-t)}{\psi(t)} \\ &= \sup_{t\in[0,1]} \frac{1}{2}\frac{s_1t + s_2t + (1-s_1)(1-t) + (1-s_2)(1-t)}{\psi(t)} \\ &\le \frac{1}{2}(\varphi(s_1) + \varphi(s_2)),\end{aligned}$$

which verifies the convexity of $\varphi(\cdot)$. Next we prove the last assertion. Since $\psi(t) \leq 1$ for all $t \in [0,1]$,

$$\varphi(s) \geq \sup_{t\in[0,1]} \{st + (1-s)(1-t)\} \geq \max\{s, 1-s\}$$

for all $s \in [0,1]$, and the proof is complete. □

Theorem 2. *The dual space $(X \oplus_\psi Y)^*$ is isometric to $X^* \oplus_\varphi Y^*$, where φ is the dual function of ψ. Moreover, each bounded linear functional F in $(X \oplus_\psi Y)^*$ can be (uniquely) represented by (f,g) where $f \in X^*$ and $g \in Y^*$ and*

$$F(x,y) = f(x) + g(y)$$

for all $(x,y) \in X \oplus_\psi Y$. In this case, $\|F\| \leq \|(f,g)\|_\varphi \|(x,y)\|_\psi$.

Proof. Define $T : X^* \oplus_\varphi Y^* \to (X \oplus_\psi Y)^*$ by

$$T(f,g)(x,y) = f(x) + g(y)$$

where $f \in X^*$, $g \in Y^*$, $x \in X$, and $y \in Y$. It is easy to see that T is linear. Moreover, by the definition of φ, we have, recalling that the norm of each nonzero element (f,g) of the φ-direct sum $X^* \oplus_\varphi Y^*$ is defined by

$$\|(f,g)\|_\varphi = (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\|+\|g\|}\Big),$$

$$\begin{aligned}
|T(f,g)(x,y)| &\leq \|f\|\|x\| + \|g\|\|y\| \\
&= (\|f\| + \|g\|)(\|x\| + \|y\|)\frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\|+\|g\|)(\|x\|+\|y\|)} \\
&\leq (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\|+\|g\|}\Big)(\|x\| + \|y\|)\psi\Big(\frac{\|y\|}{\|x\|+\|y\|}\Big) \\
&= \|(f,g)\|_\varphi \|(x,y)\|_\psi,
\end{aligned}$$

for all nonzero (f,g). Thus, $T(f,g)$ is actually an element of $(X \oplus_\psi Y)^*$. For each $F \in (X \oplus_\psi Y)^*$, $F(\cdot,0)$ and $F(0,\cdot)$ are bounded linear functionals on X and Y, respectively. Put $f(x) = F(x,0)$ and $g(y) = F(0,y)$, then $T(f,g) = F$ and the surjectivity of T is proved.

Finally we prove that T is an isometry, i.e., $\|T(f,g)\| = \|(f,g)\|_\varphi$. From the above calculation, we always have $\|T(f,g)\| \leq \|(f,g)\|_\varphi$. Now we prove the reverse inequality. We choose sequences $\{t_n\} \subset [0,1]$, $\{x_n\} \subset S_X$, and $\{y_n\} \subset S_Y$ so that

$$\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)\|f\|}{\|f\|+\|g\|} + \frac{t_n\|g\|}{\|f\|+\|g\|}\Big) \to \varphi\Big(\frac{\|g\|}{\|f\|+\|g\|}\Big),$$

$$f(x_n) \to \|f\|, \quad \text{and} \quad g(y_n) \to \|g\| \quad \text{as } n \to \infty.$$

Therefore, since $\frac{1}{\psi(t_n)}((1-t_n)x_n, t_n y_n) \in S_{X\oplus_\psi Y}$,

$$\begin{aligned}
\|T(f,g)\| &\geq \frac{1}{\psi(t_n)}\Big(f((1-t_n)x_n) + g(t_n y_n)\Big) \\
&= (\|f\| + \|g\|)\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)f(x_n)}{\|f\|+\|g\|} + \frac{t_n g(y_n)}{\|f\|+\|g\|}\Big).
\end{aligned}$$

The last expression tends to $\|(f,g)\|_\varphi$ as $n \to \infty$, proving that $\|T(f,g)\| \geq \|(f,g)\|_\varphi$ and this completes the proof. □

Our first application of Theorem 2 is to show that reflexivity is preserved under the ψ-direct sums.

Corollary 3. *For each $\psi \in \Psi$, $X \oplus_\psi Y$ is reflexive if and only if X and Y are reflexive.*

Proof. We only proof the sufficiency. We first show, without using reflexivity, that $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\psi Y^{**}$, i.e., they are isometric. For this, we let φ and then θ be the dual functions of ψ and of φ, respectively. Thus $(X \oplus_\psi Y)^* \simeq X^* \oplus_\varphi Y^*$ by the isometry T where $TF = (F_1, F_2)$, $F_1 = F(\cdot, 0)$ and $F_2 = F(0, \cdot)$; and $(X^* \oplus_\varphi Y^*)^* \simeq X^{**} \oplus_\theta Y^{**}$ by the isometry U where $UG = (G_1, G_2)$, $G_1 = G(\cdot, 0)$ and $G_2 = G(0, \cdot)$. Hence $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\theta Y^{**}$ via the isometry which maps $L \in (X \oplus_\psi Y)^{**}$ to $ULT^{-1} = (LT^{-1}(\cdot, 0), LT^{-1}(0, \cdot)) \in X^{**} \oplus_\theta Y^{**}$ so that $ULT^{-1}(x^*, y^*) = (LT^{-1}(x^*, 0), LT^{-1}(0, y^*)) = (L(x^*, 0), L(0, y^*)) = (L_1(x^*), L_2(y^*))$. In particular, when $L = L_{(x,y)}$, the evaluation map at (x, y), i.e., $L_{(x,y)}(F) = F(x, y) = F_1(x) + F_2(y)$ for $F \in (X \oplus_\psi Y)^*$, $UL_{(x,y)}T^{-1}(x^*, y^*) = x^*(x) + y^*(y) = L_x(x^*) + L_y(y^*) = (L_x, L_y)(x^*, y^*)$. This shows that $\|(x, y)\|_\psi = \|L_{(x,y)}\| = \|(L_x, L_y)\|_\theta$ for $(x, y) \in X \oplus Y$. Therefore, $\psi(\frac{\|y\|}{\|x\|+\|y\|}) = \theta(\frac{\|L_y\|}{\|L_x\|+\|L_y\|}) = \theta(\frac{\|y\|}{\|x\|+\|y\|})$ for $\|x\| + \|y\| \neq 0$. From this we can easily see that $\psi = \theta$.

Now suppose that X and Y are reflexive. Thus elements in X^{**} and Y^{**} are of the form L_x and L_y for some $x \in X$ and $y \in Y$. To show that $(X \oplus_\psi Y)^{**}$ is reflexive, let $L \in (X \oplus_\psi Y)^{**}$ and consider, for each $F \in (X \oplus Y)^*$, $L(F) = L(F_1, 0) + L(0, F_2) = L_x(F_1) + L_y(F_2) = F_1(x) + F_2(y) = L_{(x,y)}(F)$, for some $x \in X$ and $y \in Y$. That is $L = L_{(x,y)}$ showing that $X \oplus_\psi Y$ is reflexive and the proof is complete. □

We observe that $X \oplus_\psi Y$ is super-reflexive when (and only when) X and Y are super-reflexive. By Henson and Moore [7], this is equivalent to showing that the ultrapower $\widetilde{X \oplus_\psi Y}$ is reflexive. But this follows from Remark 5 below and Corollary 3.

3. Ultrapowers of the ψ-direct sums

The ultrapower of a Banach space is proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. In this section we prove that every ultrapower of a ψ-direct sum is isometric to the ψ-direct sum of their ultrapowers. First we recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i \in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i \in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I}X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I}\|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from remark (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [17]).

Following T. Landes [11], a normed space Z is a *substitution* space (with index $I \neq \emptyset$ with any cardinality) whenever Z has a (Shauder) basis $(e_i)_{i\in I}$ (unconditional if I is uncountable) and the norm of Z is *monotone*, i.e., $\|z\| \leq \|z'\|$ whenever $0 \leq z_i \leq z'_i$ for all $i \in I$ $(z, z' \in Z)$, where we write $z = \sum_{i\in I} z_i e_i$ for $z \in Z$. Given a family $(X_i)_{i\in I}$ of normed spaces, then the *Z direct sum* $(\bigoplus_{i\in I} X_i)_Z$ of the family (X_i) is defined to be the space $\{x := (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\| e_i \in Z\}$ endowed with the norm $\| \sum_{i\in I} \|x_i\| e_i \|_Z$. $\psi-$ direct sums are examples of $Z-$direct sums.

A property P defined for normed spaces is said to be preserved under the Z-direct-sum-operation, if the Z-direct sums of a family $(X_i)_{i\in I}$ of normed spaces satisfies P whenever all X_i do so.

The following proposition shows that, under some conditions, "normal structure" is preserved under the Z-direct-sum-operation. This result improves the first permanence result for normal structure obtained by Belluce, Kirk, and Steiner [2].

Proposition 4. [11, Theorem 2, Corollary 3 and Corollary 4] *Let Z be a substitution space with index set $I = \{1, \ldots, N\}$ such that*

$$\|z + z'\| < 2 \text{ whenever } \|z\| = \|z'\| = 1, z_i \geq 0, z'_i \geq 0 \text{ for all } i \in I,$$

$$\text{and } z_i = z'_i \text{ only for these } i \in I \text{ for which } z_i = z'_i = 0.$$

Thus, normal structure is preserved under the Z-direct-sum-operation. In particular, if Z is strictly convex or $Z = l_p^N$ for any p with $1 < p \leq \infty$.

In case $I = \{1, ..., N\}$ and ψ is strictly convex, it follows from [9] that the norm $\|\cdot\|_\psi$ is monotone and strictly convex on $\mathbb{C}^N$. We note in passing that this result actually holds for $Z-$direct sum : The $Z-$ direct sums $(\bigoplus_i X_i)_Z$ is uniformly convex if and only if Z and each of the Banach space X_i are uniformly convex with a common modulus of convexity (see Dowling [5]).

Remark 5. It is easy to see that the ultrapower of $Z-$direct sum $(\bigoplus_i X_i)_Z$ is isometric to the $Z-$direct sum $(\bigoplus_i \widetilde{X}_i)_Z$ of ultrapowers. Thus in particular,

$\widetilde{(X_1 \oplus \cdots \oplus X_N)_\psi} \simeq (\widetilde{X_1} \oplus \cdots \oplus \widetilde{X_N})_\psi$. This follows from the fact that the $Z-$norm is monotone and from the continuity of norms.

It is known that X is uniformly convex if and only if $\widetilde{X}$ is strictly convex (see [17]). Combining these results and Remark 5 gives

Corollary 6. [9] *Let $X_1, ..., X_N$ be Banach spaces and $\psi \in \Psi_N$. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly convex if and only if $X_1, ... X_N$ are uniformly convex and ψ is strictly convex.*

Thus, in the light of super-reflexivity, we can extend "normal structure" to "uniform normal structure" for ψ-direct sums whenever ψ is strictly convex.

Corollary 7. *Let $X_1, ..., X_N$ be super-reflexive Banach spaces and Z be uniformly convex. Then, the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has uniform normal structure if and only if $X_1, ..., X_N$ have uniform normal structure.*

Proof. Note that, by Khamsi [10], it suffices to show that the ultrapower $\widetilde{(X_1 \oplus \cdots \oplus X_N)_Z}$ has normal structure. But this is an immediate consequence of Remark 5 together with Proposition 4. □

It is well-known that every uniformly nonsquare space is super-reflexive (see [8]). Thus, Corollary 7 and [4, Corollary 3.7] give

Corollary 8. *Let $X_1, ..., X_N$ be Banach spaces and Z be uniformly convex. Then, if $C_{NJ}(1, X_i) < 2$ for $i = 1, 2, ..., N$, the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has uniform normal structure.*

It is interesting to see if we can conclude that $C_{NJ}(1, (X_1 \oplus \cdots \oplus X_N)_Z) < 2$ in Corollary 8.

4. Smoothness of the ψ-direct sums

A Banach space X is said to be *smooth* if for any $x \in S_X$, ∇_x is a singleton. We recall that a continuous convex function ψ has left and right derivatives ψ'_L, ψ'_R. Let G be defined on $[0,1]$ by

$$G(0) = [-1, \psi'_R(0)],\ G(1) = [\psi'_L(1), 1],$$
$$G(t) = [\psi'_L(t), \psi'_R(t)]\ (0 < t < 1).$$

Given $\psi \in \Psi$, $t \in [0,1]$, let

$$x(t) = \frac{1}{\psi(t)}(1-t, t)$$

so that $\|x(t)\|_\psi = 1$. In [3], the authors identified the dual of $(\mathbb{C}^2, \|.\|_\psi)$ with $\mathbb{C}^2$ and used this fact to provide a proof of the following lemma.

Lemma 9. [3, Lemma 4] *For ψ, G, and x defined above,*

(1) $\nabla_{x(t)} = \{(\psi(t) - t\gamma, \psi(t) + (1-t)\gamma) : \gamma \in G(t)\}$ *for* $0 < t < 1$,
(2) $\nabla_{x(0)} = \{(1, z(1+\gamma)) : \gamma \in G(0), |z| = 1\}$, *and*
(3) $\nabla_{x(1)} = \{(z(1-\gamma), 1) : \gamma \in G(1), |z| = 1\}$.

In general, using Theorem 2 and Lemma 9, we have the following:

Lemma 10. *Let* $(x,y) \in S_{X\oplus_\psi Y}$ *and* $t = \frac{\|y\|}{\|x\|+\|y\|}$. *Thus*

(1) $\nabla_{(x,y)} = \{((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|}$ *and* $g \in \nabla_{y/\|y\|}\}$ *for* $0 < t < 1$,

(2) $\nabla_{(x,y)} = \{(f, (1+\gamma)g) : \gamma \in G(0), f \in \nabla_x$ *and* $g \in S_{Y^*}\}$ *for* $t = 0$, *and*

(3) $\nabla_{(x,y)} = \{((1-\gamma)f, g) : \gamma \in G(1), g \in \nabla_y$ *and* $f \in S_{X^*}\}$ *for* $t = 1$.

Proof. We prove (1). Let $F = (f,g) \in \nabla_{(x,y)}$, then

$$\begin{aligned} F((x,y)) &= f(x) + g(y) \\ &\le \|f\|\|x\| + \|g\|\|y\| \\ &= \frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\| + \|g\|)(\|x\| + \|y\|)}(\|f\| + \|g\|)(\|x\| + \|y\|) \\ &\le \varphi\left(\frac{\|g\|}{\|f\| + \|g\|}\right)\psi\left(\frac{\|y\|}{\|x\| + \|y\|}\right)(\|f\| + \|g\|)(\|x\| + \|y\|) \\ &= \|F\|_\varphi \|(x,y)\|_\psi = 1. \end{aligned}$$

Thus, we have $\|f\|\|x\| + \|g\|\|y\| = 1$ and $f(x) = \|f\|\|x\|$ $g(y) = \|g\|\|y\|$, hence $(\|f\|, \|g\|) \in \nabla_{(\|x\|,\|y\|)}$ and $\frac{f}{\|f\|} \in \nabla_{\frac{x}{\|x\|}}$, $\frac{g}{\|g\|} \in \nabla_{\frac{y}{\|y\|}}$. We observe that $(\|x\|, \|y\|) = \frac{1}{\psi(t)}(1-t, t)$, thus it follows from Lemma 9 that

$$\|f\| = \psi(t) - t\gamma \text{ and } \|g\| = \psi(t) + (1-t)\gamma, \text{ for some } \gamma \in G(t).$$

Consequently, we have $(f,g) = (\|f\|\frac{f}{\|f\|}, \|g\|\frac{g}{\|g\|}) = ((\psi(t) - t\gamma)\frac{f}{\|f\|}, (\psi(t) + (1-t))\gamma\frac{g}{\|g\|})$. Thus, we have proved that $\nabla_{(x,y)} \subset \{((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}\}$. On the other hand, let $F = ((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g)$ where $\gamma \in G(t)$, $f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}$. Consider, by using Lemma 9,

$$\begin{aligned} F((x,y)) &= (\psi(t) - t\gamma)f(x) + (\psi(t) + (1-t)\gamma)g(y) \\ &= (\psi(t) - t\gamma)\|x\| + (\psi(t) + (1-t)\gamma)\|y\| \\ &= (\|x\| + \|y\|)\big((\psi(t) - t\gamma)(1-t) + (\psi(t) + (1-t)\gamma)t\big) \\ &= \frac{1}{\psi(t)}\big((\psi(t) - t\gamma)(1-t) + (\psi(t) + (1-t)\gamma)t\big) \\ &= 1. \end{aligned}$$

Hence, (1) has been proved. The proof of (2) and (3) can be proceeded similarly. □

We say that a function ψ is *smooth* if the following conditions hold:

(1) ψ is *smooth* at every $t \in (0,1)$, i.e., the derivative of ψ exists at t,
(2) the right derivative of ψ at 0 is -1, and
(3) the left derivative of ψ at 1 is 1.

Theorem 11. *Let* X *and* Y *be Banach spaces and* $\psi \in \Psi$. *Then* $X \oplus_\psi Y$ *is smooth if and only if* X *and* Y *are smooth and* ψ *is smooth.*

Proof. Necessity. Assume that $X \oplus_\psi Y$ is smooth. Because X is isometric to $X \oplus_\psi \{0\}$ which is a subspace of $X \oplus_\psi Y$, then X and similarly Y must be smooth. It remains to prove that ψ is smooth, but by Lemma 10, if ψ is not smooth, there exists $(x, y) \in S_{X\oplus_\psi Y}$ such that $\nabla_{(x,y)}$ contains more than one point which can not happen, and the smoothness of ψ is proved

Sufficiency. This follows from Lemma 10. □

Again, since, for every Banach space X, X is uniformly smooth if and only of $\widetilde{X}$ is smooth, we obtain

Corollary 12. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is uniformly smooth if and only if X and Y are uniformly smooth and ψ is smooth.*

5. U-SPACES AND u-SPACES

A Banach space X is called a *U-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in S_X$, we have $\|x + y\| \leq 2(1 - \delta)$ whenever $f(y) < 1 - \varepsilon$ for some $f \in \nabla_x$ (see [13]). A Banach space X is called a *u-space* if for any $x, y \in S_X$ with $\|x + y\| = 2$, we have $\nabla_x = \nabla_y$. Obviously, every U-space is a u-space.

Remark 13. Let us collect together some properties of u-spaces and U-spaces:

(1) If X^* is a u-space, then X is a u-space. The converse holds whenever X is reflexive.
(2) If X is a U-space, then X is a u-space. The converse holds whenever $\dim X < \infty$.
(3) $\widetilde{X}$ is a u-space if and only if X is a U-space.

Proof. (1) Let $x, y \in S_X$ be such that $\|x + y\| = 2$. We prove that $\nabla_x = \nabla_y$. Let $f \in \nabla_x$, and $h \in \nabla_{x+y}$. It follows that $h(x) = h(y) = 1$ and $\|f + h\| = 2$. By the assumption that X^* is a u-space and $h(y) = 1$, we have $f(y) = 1$. This implies that $\nabla_x \subset \nabla_y$, and then $\nabla_x = \nabla_y$ as required.

(2) The first assertion is obvious and the latter one follows from the compactness of the unit ball.

(3) It is known that $\widetilde{X}$ is a U-space if and only if X is a U-space (see [6] or [15]). In virtue of (2), it suffices to prove that X is a U-space whenever $\widetilde{X}$ is a u-space. Suppose that X is not a U-space. Then there exist an $\epsilon_0 > 0$ and sequences $\{x_n\}, \{y_n\} \subset S_X$, and $\{f_n\} \subset S_{X^*}$ such that $f_n(x_n) = 1$ and $f_n(x_n - y_n) \geq \epsilon_0$ for all $n \in \mathbb{N}$, and $\|x_n + y_n\| \to 2$ as $n \to \infty$. We put $\widetilde{x} = (x_n)_\mathcal{U}$, $\widetilde{y} = (y_n)_\mathcal{U}$ and $\widetilde{f} = (f_n)_\mathcal{U}$. Thus $\|\widetilde{x} + \widetilde{y}\| = 2$, $\widetilde{f}(\widetilde{x}) = 1$ and $\widetilde{f}(\widetilde{y}) \leq 1 - \epsilon_0 < 1$. This means that $\nabla_{\widetilde{x}} \neq \nabla_{\widetilde{y}}$ which implies that $\widetilde{X}$ is not a u-space. □

U-spaces can be considered as the "uniform" version of u-spaces. The following diagram explains this claim as well as it shows how the u-spaces are well-placed (see [1], [4], [6], [14], and [15]):

$$X \text{ is UC } \Leftrightarrow \widetilde{X} \text{ is UC } \Leftrightarrow \widetilde{X} \text{ is SC}$$

$$X \text{ is US } \Leftrightarrow \widetilde{X} \text{ is US } \Leftrightarrow \widetilde{X} \text{ is S}$$

$$X \text{ is UNC} \;\Leftrightarrow\; \widetilde{X} \text{ is UNC} \;\Leftrightarrow\; \widetilde{X} \text{ is NC}$$

$$X \text{ is a U-space} \;\Leftrightarrow\; \widetilde{X} \text{ is a U-space} \;\Leftrightarrow\; \widetilde{X} \text{ is a u-space}$$

$$\begin{array}{ccccc} C_{\mathrm{NJ}}(1,X) < 2 & \Rightarrow & \text{UNS} & & \\ & & \Uparrow & & \\ \text{UC} & \Rightarrow & U & \Rightarrow & \text{UNSQ} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{SC} & \Rightarrow & u & \Rightarrow & \text{NSQ} \end{array} \qquad \begin{array}{ccccc} \text{US} & \Rightarrow & U & \Rightarrow & \text{UNSQ} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{S} & \Rightarrow & u & \Rightarrow & \text{NSQ} \end{array}$$

UC ≡ Uniformly Convex, SC ≡ Strictly Convex, US ≡ Uniformly Smooth, S ≡ Smooth, UNC ≡ Uniformly Noncreasy , NC ≡ Noncreasy, $C_{\mathrm{NJ}}(\cdot)$ ≡ a generalized Jordan-von Neumann constant, UNS ≡ Uniform Normal Structure, UNSQ ≡ Uniformly Nonsquare, NSQ ≡ Nonsquare, U ≡ a U-space, u ≡ a u-space

Examples of u-spaces which are not U-spaces can be obtained from the direct product spaces $(\mathbb{R}^2_{p_1} \oplus \mathbb{R}^2_{p_2} \oplus \mathbb{R}^2_{p_3} \oplus \cdots)_2$ where (p_n) is a sequence of positive numbers strictly decreasing to 1, and $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ where each l_n is the l_n-space. Actually, both spaces are strictly convex, but with the James constant and the Jordan-von Neumann constant are both equal to 2, i.e., the spaces are not uniformly nonsquare, and hence can not be U-spaces. Sims and Smith [18] have shown that the space $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ has asymptotic property (P) but not property (P).

Examples of infinite dimensional u-spaces that are not strictly convex or smooth are easily established.

Let $\psi \in \Psi$. We say that ψ is a *u-function*, if for any interval $[a,b] \subset (0,1)$, we have ψ is smooth at a and b whenever ψ is affine on $[a,b]$.

Theorem 14. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the Banach space $X \oplus_\psi Y$ is a u-space if and only if X and Y are u-spaces and ψ is a u-function.*

Proof. Necessity . Suppose there exist a and $b \in [0,1]$ such that ψ is affine on $[a,b]$ but $\psi'_-(a) < \psi'_+(a) = \psi'_-(b)$. Fix $x_0 \in S_X$, $f_0 \in \nabla_{x_0}$, $y_0 \in S_Y$, and $g_0 \in \nabla_{y_0}$. Consider $w = \frac{1}{\psi(a)}((1-a)x_0, ay_0)$ and $z = \frac{1}{\psi(b)}((1-b)x_0, by_0)$. We have $w, z \in S_{X\oplus_\psi Y}$ and $\|w+z\|_\psi = 2$. Indeed,

$$\begin{aligned} \|w+z\|_\psi &= \left\| \left(\frac{1-a}{\psi(a)}x_0 + \frac{1-b}{\psi(b)}x_0, \frac{a}{\psi(a)}y_0 + \frac{b}{\psi(b)}y_0 \right) \right\|_\psi \\ &= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(\frac{\frac{a}{\psi(a)} + \frac{b}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\ &= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(a\frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} + b\frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\ &= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \left(\frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}}\psi(a) + \frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}}\psi(b) \right) \\ &= 2. \end{aligned}$$

To obtain a contradiction, it remains to show that $\nabla_z \neq \nabla_w$. Now, for $\gamma \in [\psi'_-(b), \psi'_+(b)]$, we have

$$\psi(b) - b\gamma \leq \psi(b) - b\psi'_-(b) = \psi(a) - a\psi'_+(a) < \psi(a) - a\psi'_-(a).$$

Thus, $((\psi(a) - a\psi'_-(a))f_0, (\psi(a) + (1-a)\psi'_-(a))g_0) \in \nabla_w \setminus \nabla_z$, that is $\nabla_z \neq \nabla_w$.

Sufficiency. Let us prove that $X \oplus_\psi Y$ is a u-space. Let w and z be elements in the unit sphere of $X \oplus_\psi Y$ such that $\|w+z\|_\psi = 2$. Put $w = (x_1, y_1)$ and $z = (x_2, y_2)$. We have $\|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi = 2$ since $2 = \|w+z\|_\psi = \|(x_1+x_2, y_1+y_2)\|_\psi \leq \|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi \leq \|(\|x_1\|, \|x_2\|)\|_\psi + \|(\|y_1\|, \|y_2\|)\|_\varphi = 2$. By the convexity of ψ, it follows that

$$\begin{aligned}
2 &= (\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|)\psi\left(\frac{\|y_1\| + \|y_2\|}{\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|}\right) \\
&\leq (\|x_1\| + \|y_1\|)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right) + (\|x_2\| + \|y_2\|)\psi\left(\frac{\|y_2\|}{\|x_2\| + \|y_2\|}\right) \\
&= 2.
\end{aligned}$$

Thus, ψ is affine on $[a \wedge b, a \vee b]$, where $a = \frac{\|y_1\|}{\|x_1\|+\|y_1\|}$ and $b = \frac{\|y_2\|}{\|x_2\|+\|y_2\|}$. Since $\|w+z\| = 2$, there exists $F = (f_1, g_1) \in X^* \oplus_\varphi Y^*$ such that $F \in \nabla_w \cap \nabla_z$. Hence,

$$\begin{aligned}
F(w) &= f_1(x_1) + g_1(y_1) \\
&\leq \|f_1\|\|x_1\| + \|g_1\|\|y_1\| \\
&= \frac{\|f_1\|\|x_1\| + \|g_1\|\|y_1\|}{(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|)}(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\
&\leq \varphi\left(\frac{\|g_1\|}{\|f_1\| + \|g_1\|}\right)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right)(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\
&= \|F\|_\varphi \|w\|_\psi = 1.
\end{aligned}$$

Thus, we have

$$(\alpha) \qquad f_1(x_1) = \|f_1\|\|x_1\| \text{ and } g_1(y_1) = \|g_1\|\|y_1\|.$$

In the same way, we also have

$$(\beta) \qquad f_1(x_2) = \|f_1\|\|x_2\| \text{ and } g_1(y_2) = \|g_1\|\|y_2\|.$$

Now we show that $\nabla_w = \nabla_z$. We consider first the case when all $\|x_1\|$, $\|y_1\|$, $\|x_2\|$, $\|y_2\|$ are positive. In this case, we can assume that $0 < a \leq b < 1$. (α) and (β) give $\frac{f_1}{\|f_1\|} \in \nabla_{\frac{x_1}{\|x_1\|}} \cap \nabla_{\frac{x_2}{\|x_2\|}}$ and $\frac{g_1}{\|g_1\|} \in \nabla_{\frac{y_1}{\|y_1\|}} \cap \nabla_{\frac{y_2}{\|y_2\|}}$. It follows that $\|\frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|}\| = 2$ and $\|\frac{y_1}{\|y_1\|} + \frac{y_2}{\|y_2\|}\| = 2$. Thus, $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$ since both X and Y are u-spaces.

If $a < b$, then, since ψ is affine on $[a, b]$, a and b must be smooth points of ψ. Consequently,

$$(\gamma) \qquad \psi(a) - a\gamma = \psi(b) - b\gamma \text{ and } \psi(a) + (1-a)\gamma = \psi(b) + (1-b)\gamma,$$

where $\gamma = \psi'(a) = \psi'(b)$.

By using (γ) together with Lemma 10 and the equations $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$, we have $\nabla_z = \nabla_w$.

If $a = b$, then, by Lemma 10, we have

$$\begin{aligned}
&\nabla_{(x_1,y_1)} \\
&= \{((\psi(a) - a\gamma)f, (\psi(a) + (1-a)\gamma)g) : \gamma \in G(a), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \nabla_{(x_2,y_2)}.
\end{aligned}$$

Thus $\nabla_z = \nabla_w$ as well.

Now we consider the case when exactly one of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ is equal to 0. We assume that $\|y_1\| = 0$, thus $a = 0 < b$ and 0 and b are smooth points. By (α), (β), and by the assumption that X is a u-space, we have $\nabla_{x_1} = \nabla_{\frac{x_2}{\|x_2\|}}$. Since 0 is a smooth point, we have $F = (f_1, 0)$. This in turn implies that $\psi(b) - b\psi'(b) = 1$ and $\psi(b) + (1-b)\psi'(b) = 0$ since $F \in \nabla_w \cap \nabla_z$. Thus, by Lemma 10,

$$\begin{aligned}
&\nabla_{(x_2,y_2)} \\
&= \{((\psi(b) - b\psi'(b))f, (\psi(b) + (1-b)\psi'(b))g) : f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_2/\|x_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_1}\} \\
&= \nabla_{(x_1,y_1)}.
\end{aligned}$$

Finally, suppose two of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ are equal to 0. We can assume that $\|y_1\| = \|y_2\| = 0$, thus $a = b = 0$. The proof of the equality $\nabla_z = \nabla_w$ is similar to the one of the case when $a = b$. □

Corollary 15. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following statements are equivalent:*

(1) *$X \oplus_\psi Y$ is a U-space;*
(2) *$X^* \oplus_\varphi Y^*$ is a U-space;*
(3) *X and Y are U-spaces and ψ is a u-function;*
(4) *X and Y are U-spaces and φ is a u-function, where φ is the dual function of ψ.*

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Fixed point property of direct sums☆

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Abstract

For a uniformly convex space Z, we show that Z-direct sums $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \ldots, X_N$ with $R(a, X_i) < 1+a$ for some $a \in (0, 1]$ have the fixed point property for nonexpansive mappings. As a direct consequence, the result holds for all ψ-direct sums with ψ being strictly convex. The same result is extended to all ψ-direct sums $X \oplus_\psi Y$ of spaces X and Y with property (M), whenever $\psi \neq \psi_1$. The permanence of properties that are sufficient for the fixed point property are obtained for Z-direct sums (and then for ψ-direct sums). Such properties include the properties $R(X) < 2$, WNUS, $C_{NJ}(a, X) < 2$, UKK, and NUC.

MSC: 46B20; 46B08

Keywords: Z-direct sums; ψ-direct sums; The fixed point property

1. Introduction

Let X be a Banach space. A self-mapping T of a closed convex subset C of X is said to be a nonexpansive mapping if $\|Tx - Ty\| \leqslant \|x - y\|$ for all x, y in C.

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We will say that X has the weak fixed point property (fpp) if every nonexpansive mapping defined on a nonempty weakly compact convex subset of X has a fixed point.

One open problem in metric fixed point theory is the permanence of fpp under direct sums. Here we consider the problem in two aspects. One of these is to study the permanence of properties that guarantee the fixed point property, the other is to study directly the fixed point property of the direct sum under conditions given on its component spaces. Obviously, the answer to these questions depend on the norm of the product space. We shall be interested in the so-called Z-direct sums and ψ-direct sums the concepts of which will be defined later.

Shortly after Kirk proved in [16] his celebrated fixed point theorem that asserts that every Banach space with normal structure has the fpp, the first permanence result for normal structure was given by Belluce et al. [2]. The result has been improved by Landes [18] whose result in turn has been improved to uniform normal structure under ψ-direct sums by Dhompongsa et al. in [4]. For weak normal structure, the positive answer for $X \oplus_p Y$ is due to Belluce et al. [2]. Landes [18,19] in 1984 and 1986 showed that weak normal structure (WNS) is preserved in $X \oplus_p Y$ for $1 < p \leqslant \infty$, but not for $p = 1$. Some additional conditions on the spaces X and Y that are sufficient for the direct sum $X \oplus_1 Y$ to have WNS are considered in [11,21].

Sims and Smyth [28], however, considered the problem, known as a *3-space problem*: For a Banach space X and a finite-dimensional Banach space Y, if X has asymptotic (P) so does $X \oplus Y$. Moreover, if X has property (P) and the projection onto X has norm 1, then $X \oplus Y$ has (P). This later result strengthens Theorem 2.3 of [29]. Later Sims and Smyth [28] showed that property (P), asymptotic (P), and some others are inherited from the component spaces to the direct sums. They, as well as Kutzarova and Landesalso in [17], also considered infinite product results. Property (P), introduced by Tan and Xu [29], is sufficient for WNS. In 1997, Prus [24] introduced a class of super-reflexive Banach spaces with fpp the so-called the *uniformly noncreasy* (UNC) spaces. These spaces do not have to have normal structure. For a strictly monotone norm, Wisnicki [30] proved that $X \oplus Y$ has fpp, whenever both X and Y are UNC or have property (P).

A norm on $\mathbb{R}^2$ is called *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $(z,w) \in \mathbb{R}^2$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by N_a. The l_p-norms $\|\cdot\|_p$ for $1 \leqslant p \leqslant \infty$ are such examples and for any $\|\cdot\| \in N_a$, $\|\cdot\|_\infty \leqslant \|\cdot\| \leqslant \|\cdot\|_1$. Let Ψ be the set of all continuous convex functions ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leqslant \psi(t) \leqslant 1$ $(0 \leqslant t \leqslant 1)$. N_a and Ψ are in one-to-one correspondence via the following equations. For each $\|\cdot\| \in N_a$, the function ψ defined by $\psi(t) = \|(1-t,t)\|$ $(0 \leqslant t \leqslant 1)$ belongs to Ψ. Conversely, for each $\psi \in \Psi$ let $\|(0,0)\| = 0$, and $\|(z,w)\| = (|z| + |w|)\psi(|w|/|w| + |z|)$ for $(z,w) \neq (0,0)$. Let ψ_p be a function defined by $\psi_p(t) = ((1-t)^p + t^p)^{1/p}$ if $1 \leqslant p < \infty$; $\psi_p(t) = \max\{1-t, t\}$, if $p = \infty$. It is simple to see that such function ψ_p corresponds to the l_p-norm. We denote by $X \oplus_\psi Y$ the direct sum $X \oplus Y$ equipped with the norm $\|(x,y)\| = \|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$ for $(x,y) \in X \oplus Y$. $X \oplus_\psi Y$ is called the *ψ-direct sum* of X and Y and it is a Banach space satisfying, for all $(x,y) \in X \oplus Y$, $\|(x,y)\|_\infty \leqslant \|(x,y)\|_\psi \leqslant \|(x,y)\|_1$ (see [26]). Saito et al. [25] extended the concept to absolute normalized norm on $\mathbb{R}^N$. The corresponding set of all continuous convex functions on the $(N-1)$-simplex $\{(s_1,\ldots,s_{N-1}) \in \mathbb{R}_+^{N-1} : s_1 + \cdots + s_{N-1} \leqslant 1\}$ will be denoted by Ψ_N. Kato et al. [15] proved that the space $(\mathbb{R}^N, \|\cdot\|_\psi)$ is uniformly convex if and only if ψ is strictly convex. Dowling [8] pointed out that this result fell into a larger framework developed by Day [3]: Let Z be a Banach space with a normalized 1-unconditional basis $(e_i)_{i\in I}$. For Banach spaces $X_i\,(i \in I)$, define the Z-direct sum

$$\left(\bigoplus_i X_i\right)_Z = \left\{(x_i) \in \prod_i X_i : \sum_i \|x_i\| e_i \text{ converges in } Z\right\}$$

and put $\|(x_i)\| = \|\sum_i \|x_i\| e_i\|_Z$ for the norm on $(\bigoplus_i X_i)_Z$. In the language of Day, $(\bigoplus_i X_i)_Z$ is a substitution space of (X_i) in Z. ψ-direct sums are examples of Z-direct sums, simply letting $Z = \mathbb{C}^N$. Dowling showed that $(\bigoplus_i X_i)_Z$ is uniformly convex if and only if Z and each of the Banach space X_i are uniformly convex with a common modulus of convexity. Thus, for $X_i = \mathbb{R}$ for each i, the Z-direct sum $(\bigoplus_i X_i)_Z$ is uniformly convex if and only if Z is uniformly convex.

In this paper, we will study only finite Z-direct sums. Thus, we may write $(a_1, \ldots, a_N) = a_1e_1 + \cdots + a_Ne_N$ as elements of Z, for $a_1, \ldots, a_N$ in $\mathbb{R}$. The paper is organized as follows. In Section 2, we consider Banach spaces X with $R(a, X) < 1 + a$ for some $a \in (0, 1]$. It is known that spaces with this property have the fpp. Let us call this property 'Dominguez's condition'. The main result of this section is to prove that the finite Z-direct sum $(\bigoplus_i X_i)_Z$ has the fixed point property whenever the spaces (X_i) satisfy Dominguez's condition. We continue the investigation for Banach spaces with property (M) of Kalton in Section 3. The second part of the paper considers the permanence of properties that are sufficient for the fixed point property. The conditions $R(X) < 2$, $C_{NJ}(a, X) < 2$, and the property NUC among others are considered in Sections 5, 6, and 7, respectively.

One tool that seems to be common now in studying the fixed point property is the ultrapower technique. We recall some of its formulation. Let $\mathcal{U}$ be a free ultrafilter on the set of natural numbers. Consider the closed linear subspace of $l_\infty(X)$, $\mathcal{N} = \{(x_n) \in l_\infty(X) : \lim_{n\to\mathcal{U}} \|x_n\| = 0\}$. The ultrapower $\widetilde{X}$ of the space X is defined as the quotient space $l_\infty(X)/\mathcal{N}$. Given an element $x = (x_n) \in l_\infty(X)$, $\widetilde{x}$ stands for the equivalence class of x. The quotient norm in $\widetilde{X}$ satisfies $\|\widetilde{x}\| = \lim_{n\to\mathcal{U}} \|x_n\|$. If $f = (x_n^*)$ is a bounded sequence of functionals in X^*, the expression $\widetilde{f}(\widetilde{x}) = \lim_{n\to\mathcal{U}} x_n^*(x_n)$ for $x = (x_n) \in l_\infty(X)$ defines an element in the dual space of $\widetilde{X}$ with $\|\widetilde{f}\| = \lim_{n\to\mathcal{U}} \|x_n^*\|$. (For more details about the construction of an ultrapower of a Banach space X see, for examples, [1,27].) It is shown in [4] that $\widetilde{X\oplus_\psi Y} = \widetilde{X}\oplus_\psi\widetilde{Y}$.

2. The coefficient $R(a, X)$

Definition 1 (*Garcia-Falset [10]*). Let X be a Banach space, then

$$R(X) := \sup\left\{\liminf_n \|x_n + x\|\right\},$$

where the supremum is taken over all weakly null sequences (x_n) in the unit ball and over all points x of the unit ball.

Garcia-Falset [10] showed that if a Banach space X satisfies $R(X) < 2$, then X has the fpp. Dominguez [7] generalized this result by introducing the coefficient, for $a \geqslant 0$,

$$R(a, X) := \sup \left\{ \liminf_n \|x_n + x\| \right\},$$

where the supremum is taken over all weakly null sequences (x_n) in the unit ball with $D(x_n) \leqslant 1$ and over all points x with $\|x\| \leqslant a$. Here $D(x_n) := \limsup_n \limsup_m \|x_n - x_m\|$. Clearly $R(X) \geqslant R(1, X)$. It was proved in [7] that X has fpp if $R(a, X) < 1 + a$ for some $a \geqslant 0$. It has been observed that $R(l_{2,1}) = 2$, while $R(a, l_{2,1}) < 1 + a$ for some $a \geqslant 0$. Thus this result is a strict improvement of a result in [10]. Moreover, because of this result, Mazcunan Navarro [22] is able to prove a well-known open problem which states that "every uniformly nonsquare Banach space has the fpp".

Theorem 2. *Let $X_1, \ldots, X_N$ be Banach spaces with, for each $i = 1, \ldots, N$, $R(a, X_i) < 1 + a$ for some $a \in (0, 1]$. If Z is uniformly convex, then the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has the fixed point property.*

Proof. The main ingredient of the proof is taken from Dominguez [7]. Suppose that $(X_1 \oplus \cdots \oplus X_N)_Z$ does not have the fpp. Thus, we can find a weakly compact and convex subset K of $(X_1 \oplus \cdots \oplus X_N)_Z$ such that $0 \in K$, $\operatorname{diam} K = 1$ and K is minimal invariant for a nonexpansive mapping T, and we can also find a weakly null approximated fixed point sequence (afps) (z_n) of T in K. We consider the set $W = \{(\widetilde{w_n}) \in \widetilde{K} : \|(\widetilde{w_n}) - (\widetilde{z_n})\| \leqslant 1 - t$ and $D((\widetilde{w_n})) \leqslant t\}$, where $t = 1/(1 + a)$. It is easy to check that W is a closed, convex, and $\widetilde{T}$-invariant set. Furthermore, W is nonempty because it contains $t(\widetilde{z_n})$. Therefore, from Lin's Theorem [20], we know that $\sup\{\|(\widetilde{w_n})\| : (\widetilde{w_n}) \in W\} = 1$ since $0 \in K$. Some parameters will be needed and we define them here. In what follows, $i \in I = \{1, \ldots, N\}$. First choose $M > 1$ so that

$$\frac{1 + a}{M} > \max_i R(a, X_i) \tag{2.1}$$

and then choose $0 < \varepsilon < \eta$ so small that $\eta < M - 1$, $1 + \varepsilon/\eta^2(1 - t) + (1 + t/(1 - t - \varepsilon))\eta < M$,

$$\left(\frac{\eta}{t} + a \left(1 + \frac{\varepsilon}{\eta^2(1 - t)} + \left(1 + \frac{t}{1 - t - \varepsilon} \right) \eta \right) \right) \frac{t}{1 - t - \varepsilon} < M,$$

$$\varepsilon < \frac{1 - N\eta(1 + \eta)}{1 + \eta} \eta,$$

and finally, by uniform convexity of Z and by monotonicity of $\| \cdot \|_Z$, we have

$$\text{if } \|u + v\| > 2(1 - 2\varepsilon) \text{ for } u = (u_1, \ldots, u_N),\ v = (v_1, \ldots, v_N) \in S_{(X_1 \oplus \cdots \oplus X_N)_Z}, \text{ then } |\|u_i\| - \|v_i\|| < \eta^3 \text{ for each } i, \text{ and} \tag{2.2}$$

$$\text{if } \|(c_i)\| - \|(p_i)\| < \varepsilon \text{ for } (c_i),\ (p_i) \in B_{(X_1 \oplus \cdots \oplus X_N)_Z} \text{ with } 0 \leqslant p_i \leqslant c_i \text{ for each } i, \text{ then } c_i - p_i < \eta^3 \text{ for each } i. \tag{2.3}$$

We can find an element $\widetilde{w} = (\widetilde{w_n})$ in W with

$$\|\widetilde{w}\| > 1 - \varepsilon \tag{2.4}$$

and $w_n \xrightarrow{w} w$ for some $w \in (X_1 \oplus \cdots \oplus X_N)_Z$.

Write $w_n = (x_{n1}, \dots, x_{nN})$ and $w = (x_1, \dots, x_N)$. By passing through subsequences, we can assume that all limits mentioned below exist since all sequences under consideration are bounded. For example we assume now that $\|x_{ni}\| \to a(i)$ for some $a(i)$ for each $i \in I$. Let us define the vectors $z_n = w_n - w$ and let $\|z_n\| \to b$. Clearly, from $D(\widetilde{w}) \leqslant t$ and by the weak lower semi-continuity of the norm, we have

$$\|\widetilde{w} - w\| \leqslant t, \quad \|w\| \leqslant 1 - t. \tag{2.5}$$

By (2.4), we must have both $b \geqslant t - \varepsilon > 0$ and $\|w\| \geqslant 1 - t - \varepsilon > 0$. Suppose first that $b \leqslant \|w\|$. For each n, put $s_n = (z_n + w)/2$, $t_n = (\|w\|/\|z_n\|)z_n + w$. Thus, $2s_n = (\|z_n\|/\|w\|)t_n + [(\|w\| - \|z_n\|)/\|w\|]w$, and so $1 - \varepsilon \leqslant \lim_n 2\|s_n\| \leqslant (b/\|w\|)\lim_n \|t_n\| + \|w\| - b$ that implies $\lim_n \|t_n\| \geqslant (1 - \varepsilon + b - \|w\|)/b\|w\| > 2(1 - 2\varepsilon)\|w\|$. Next, suppose $\|w\| < b$. In this case we redefine $t_n = z_n + (\|z_n\|/\|w\|)w$. Thus, $2s_n = (\|w\|/\|z_n\|)t_n + [(\|z_n\| - \|w\|)/\|z_n\|]z_n$, and therefore $1 - \varepsilon \leqslant \lim_n 2\|s_n\| \leqslant (\|w\|/b)\lim_n \|t_n\| + b - \|w\|$, and in consequence $\lim_n \|t_n\| \geqslant [(1 - \varepsilon - b + \|w\|)/\|w\|]b > 2(1 - 2\varepsilon)b$. By applying (2.2), it follows that, for all large n,

$$\left| \frac{\|w\|}{\|z_n\|}\|x_{ni} - x_i\| - \|x_i\| \right| < \eta^3 \tag{2.6}$$

for all i. Let $J = \{i \in I : a(i) > \eta^2\}$. Clearly $J \neq \phi$. Now observe that for some subsequence (n_k) of (n), it is the case that, for some $i \in J$, $D(x_{n_k i}) \leqslant ta(i)(1 + \eta)$. Otherwise, by extracting a subsequence from another we have, for all $i \in J$, $\|x_{n_k i} - x_{n_l i}\| > ta(i)(1 + \eta)$ for all k and l with $k < l$. This leads us to a conclusion, by (2.4), that $\|w_{n_k} - w_{n_l}\| \geqslant \|(\sum_{i \in J} \|x_{n_k i} - x_{n_l i}\| e_i)\| \geqslant t(1+\eta)\|\sum_{i \in J}(a(i)e_i)\| > t(1+\eta)(1 - \varepsilon - N\eta^2)$ for $k < l$. Hence $t \geqslant D(w_n) \geqslant D(w_{n_k}) \geqslant t(1+\eta)(1 - \varepsilon - N\eta^2)$ which, by (2.1), is impossible. Thus, we assume that $a(1) > \eta^2$ and

$$D(x_{n_1}) \leqslant ta(1)(1 + \eta). \tag{2.7}$$

Put $u_{ni} = x_{ni} - x_i$. We claim that

$$\|x_1\| \leqslant taa(1)\left(1 + \frac{\varepsilon}{\eta^2(1-t)} + \left(1 + \frac{b}{\|w\|}\right)\eta\right) \tag{2.8}$$

and

$$\limsup_n \|u_{n_1} + x_1\| = \limsup_n \|x_{n_1}\| > t(1 + a - \gamma)a(1) \tag{2.9}$$

for all $\gamma > 0$ where, recall that, $b = \lim_n \|z_n\|$. By (2.3) we get

$$\|x_{ni} - x_i\| + \|x_i\| - \|x_{ni}\| < \eta^3 \tag{2.10}$$

for all large n. Eq. (2.6) then implies that $(1+\|z_n\|/\|w\|)\|x_i\| \leqslant \|x_{ni}\| + (1+\|z_n\|/\|w\|)\eta^3$, for all large n. Let $n \to \infty$ to obtain $(1+1/a-\varepsilon/(1-t))\|x_1\| \leqslant (1+(t-\varepsilon)/(1-t))\|x_1\| \leqslant (1+b/\|w\|)\|x_1\| \leqslant a(1)+(1+b/\|w\|)\eta^3 \leqslant a(1)(1+(1+b/\|w\|))\eta$, and so $(1+a)/a\|x_1\| \leqslant a(1)(1+[\varepsilon/(\eta^2(1-t))]+(1+b/\|w\|)\eta)$ that provides an estimate in (2.8). Eq. (2.9) is easily obtained. Taking (2.7)–(2.9) into account, we see that $u_{n1}/ta(1)M \in B_{X_1}$, $D(u_{n1}/ta(1)M) \leqslant 1$, and $x_1/ta(1)M \in aB_{X_1}$. This together with (2.1) and (2.9) imply that $R(a, X_1) > \max_i R(a, X_i)$, a contradiction, and the proof is complete. □

Corollary 3. *Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N$ be strictly convex. If for all $i = 1, \ldots, N$, $R(a, X_i) < 1 + a$ for some $a \in (0, 1]$, then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has the fixed point property.*

3. Property (M)

Kalton [14] introduced *property* (M): For $x_n \xrightarrow{w} 0$, the weakly null type $\psi_{(x_n)} := \limsup_n \|x - x_n\|$ is a function of $\|x\|$ only. In [12], Garcia-Falset and Sims proved that if a Banach space X has property (M), then X has the fixed point property. A preliminary result which they used in their proof of the main result in [12] and which we need here is

Lemma A *(Garcia-Falset and Sims [12, Lemma 3.1]). If X has property (M) and (x_n) is a weakly null sequence with* $\limsup_n \|x_n\| = 1$, *then* $D(x_n) = \sup\{\limsup_n \|x_n - x\| : x \in B_X\}$.

We now consider a ψ-direct sum $X \oplus_\psi Y$ when both X and Y have property (M). We first observe that, for each $\psi \in \Psi \backslash \{\psi_1\}$, there exists $t \in (0, \frac{1}{2})$ with $\psi(t) \vee \psi(1-t) < 1$. The following lemma is a consequence of Lemma A.

Lemma 4. *Let X be a Banach space having property* (M). *For every weakly convergence sequence $x_n \xrightarrow{w} x$, we have*

$$\limsup_n \|x_n\| \leqslant D(x_n) + \left(\|x\| \vee \limsup_n \|x_n - x\| - \limsup_n \|x_n - x\|\right).$$

Examples 1 and 2 in [21] show that property (M) is not preserved under the sum $X \oplus_p Y$ for $p \in \{1, \infty\}$. This observation leads us to consider and then obtain the following result.

Theorem 5. *Let Banach spaces X and Y have property* (M) *and $\psi \in \Psi \backslash \{\psi_1\}$. Then the ψ-direct sum $X \oplus_\psi Y$ of X and Y has the weak fixed point property.*

Proof. Let us assume that $X \oplus_\psi Y$ does not have the weak fixed point property. Then we have a weakly compact convex subset K of $X \oplus_\psi Y$ that is minimal for a fixed point free nonexpansive mapping $T : K \to K$. Moreover, we can assume that $\operatorname{diam} K = 1$, and K contains an approximate fixed point sequence (z_n) with $z_n \xrightarrow{w} 0$ and, by the Goebel-Karlovitz Lemma, $\lim_n \|z_n - z\| = 1$ for all $z \in K$. We will consider the following subset of $\widetilde{X}\oplus_\psi\widetilde{Y}$. Let $W := \{\widetilde{(w_n)} \in \widetilde{K}: \|\widetilde{(w_n)} - \widetilde{(z_n)}\| \leqslant t$ and $D(\widetilde{w_n}) \leqslant 1 - t\}$, where $t \in (0, \frac{1}{2})$ is chosen so that $\psi(t) \vee \psi(1-t) < 1$. Observe that $D(w_n) = D(y_n)$ if $\widetilde{(w_n)} = \widetilde{(y_n)}$. Then W is $\widetilde{T}$-invariant, closed, convex and nonempty as $(1-t)\widetilde{(z_n)} \in W$. For $\widetilde{(w_n)} \in W$ we can assume that $w_n \overset{w}{\to} w_o$, $\lim_n \|w_n\| = \|\widetilde{(w_n)}\|$, $\lim_n \|w_n\| = \|(\lim_n \|x_n\|, \lim_n \|y_n\|)\|$and $\lim_n \|w_n - w_o\| = \|(\lim_n \|x_n - x_o\|, \lim_n \|y_n - y_o\|)\|$. Fix $\varepsilon > 0$ such that $0 < \varepsilon < \{(1 - \psi(t)) \wedge (1 - \psi(1-t)) \wedge t\}$.

Case 1 ($\lim_n \|w_n - w_o\| \leqslant \|w_o\|$): We have $\|\widetilde{(w_n)}\| \leqslant \|\widetilde{w_n - w_o}\| + \|w_o\| \leqslant 2\|w_o\| \leqslant 2\|\widetilde{w_n - z_n}\| \leqslant 2t$.

Case 2 ($\lim_n \|w_n - w_o\| > \|w_o\|$): This means that $\lim_n \|x_n - x_0\| > \|x_o\|$ or $\lim_n \|y_n - y_0\| > \|y_o\|$. So we can assume that $\lim_n \|x_n - x_o\| > \|x_o\|$. If $\lim_n \|y_n - y_o\| < \varepsilon$ we have, by using Lemma 5, $\|\widetilde{(w_n)}\| = \lim_n \|w_n\| = \|(\lim_n \|x_n\|, \lim_n \|y_n\|)\| \leqslant \|(D(x_n), \lim_n \|y_n - y_o\| + \|y_o\|)\| \leqslant \|(1-t, t)\| + \varepsilon \leqslant \psi(t) + \varepsilon$. If $\lim_n \|y_n - y_o\| \geqslant \varepsilon$, we have $\|\widetilde{(w_n)}\| = \|(D(x_n), D(y_n) + (\|y_o\| \vee \lim_n \|y_n - y_o\| - \lim_n \|y_n - y_o\|))\| \leqslant \|(D(x_n), D(y_n))\| + (\|y_o\| \vee \lim_n \|y_n - y_o\| - \lim_n \|y_n - y_o\|) = D(w_n) + (\|y_o\| \vee \lim_n \|y_n - y_o\| - \lim_n \|y_n - y_o\|) \leqslant (1-t) + t - \varepsilon = 1 - \varepsilon$. Combining these, we have that elements of W have their norms uniformly bounded away from one. More precisely, we have $\|\widetilde{(w_n)}\| \leqslant \max\{2t, \psi(t) + \varepsilon, \psi(1-t) + \varepsilon, 1 - \varepsilon\} < 1$. This, however, contradicts Lin's Theorem which ensures that W contains elements of norms arbitrarily closed to one.

Corollary 6 (*Marino et al. [21, Proposition 5]*). *Let Banach spaces X and Y have property* (M). *Then the direct sum* $X\oplus_\infty Y$ *of X and Y has the fixed point property.*

In the second part of the paper, we turn to the study of permanence properties. Recently, Dhompongsa et al. [4] considered the permanence of smoothness, uniform smoothness, u-convexity, and U-convexity of Banach spaces. Two of these properties, namely uniform smoothness and U-convexity are known to imply fpp.

4. The coefficient $R(X)$

Since the condition "$R(X) < 2$" is more strict than the condition "$R(1, X) < 2$", we have a stronger result than might be expected from Theorem 2.

Theorem 7. *Let* $X_1, \ldots, X_N$ *be Banach spaces with* $R(X_i) < 2$ *for all* $i = 1, \ldots, N$. *If Z is uniformly convex, then* $R(X_1 \oplus \cdots \oplus X_N)_Z < 2$.

Proof. Since Z is uniformly convex, for $\varepsilon > 0$ which satisfies $\max_i R(X_i)(1 + N\varepsilon) < 2$, there exists $\delta > 0$ such that $\max_i R(X_i)(1 + N\varepsilon) < 2 - \delta$ and for every $z_1, z_2 \in Z$, if $\|z_1 - z_2\| \geqslant \varepsilon$, then $\|z_1 + z_2\| \leqslant 2 - \delta$. Now, let (x_n) be a weakly null sequence in $B_{(X_1 \oplus \cdots \oplus X_N)}$ and $x \in B_{(X_1 \oplus \cdots \oplus X_N)}$. Write $x_n = (x_{n1}, \ldots, x_{nN})$. We want to show that $\liminf_n \|x_n + x\| \leqslant 2 - \delta$. For this end, we can consider subsequences of (x_n) in order to obtain estimates in the argument to follow.

Case 1 ($|\|x_{n1}\| - \|x_1\|| \geqslant \varepsilon$ *for all large* n): We have, for all large n, $\|x_n + x\| = \|\sum_{i \in I} \|x_{ni} + x_i\| e_i\| \leqslant \|\sum_{i \in I} \|x_{ni}\| e_i + \sum_{i \in I} \|x_i\| e_i\| \leqslant 2 - \delta$.

Case 2 (*for all large n*, $|\|x_{ni}\| - \|x_i\|| < \varepsilon$ *for all i*): Let $J = \{i \in I : x_i = 0\}$. Now estimate

$$\begin{aligned}&\|(\|x_{n1} + x_1\|, \ldots, \|x_{nN} + x_N\|)\| \\ &\leqslant \left\|\sum_{i\in J}\|x_{ni}\|e_i\right\| + \left\|\sum_{i\in I\setminus J}(\|x_{ni}\| \vee \|x_i\|)\right\| \frac{x_{ni} + x_i}{\|x_{ni}\| \vee \|x_i\|}\|e_i\| \\ &\leqslant \varepsilon\left\|\sum_{i\in J}e_i\right\| + \left\|\sum_{i\in I\setminus J}(\|x_i\| + \varepsilon)\right\| \frac{x_{ni} + x_i}{\|x_{ni}\| \vee \|x_i\|}\|e_i\|.\end{aligned}$$

Take $\liminf_n$ and get $\liminf_n\|(\|x_{n1} + x_1\|, \ldots, \|x_{nN} + x_N\|)\| \leqslant \varepsilon\|\sum_{i\in J}e_i\| + \|\sum_{i\in I\setminus J}(\|x_i\|+\varepsilon)\max_i R(X_i)e_i\| \leqslant (1+N\varepsilon)\max_i R(X_i)$. Thus, $R(X_1\oplus\cdots\oplus X_N)_Z \leqslant 2-\delta$ as desired. □

Corollary 8. *Let $X_1, \ldots, X_N$ be Banach spaces and $\psi\in\Psi_N$ be strictly convex. If $R(X_i)<2$ for all $i = 1, \ldots, N$, then $R(X_1 \oplus \cdots \oplus X_N)_\psi < 2$.*

In [23], Prus introduced Banach spaces called *nearly uniform smooth* (NUS) *spaces*. These spaces are the dual of NUC spaces. Falset [10] then proved that every NUS space has the fixed point property answering a longstanding open question. Actually, he proved that WNUS, and hence NUS, Banach spaces have the fpp. The notion of WNUS is a natural generalization of the property NUS. A characterization of WNUS spaces is: X is WNUS if and only if X is reflexive and $R(X) < 2$, (see [9]). Thus we immediately have, by Dhompongsa et al. [4, Corollary 3],

Corollary 9. *Let $X_1, \ldots, X_N$ be Banach spaces. If $\psi \in \Psi_N$ is strictly convex, then each of X_i is WNUS if and only if $(X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS.*

5. The $C_{NJ}(a, X)$ constants

In [6], Dhompongsa et al. introduced a generalized Jordan–von Neumann constant $C_{NJ}(a, X)$ for $a \geqslant 0$ defined by

$$C_{NJ}(a, X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2}\right\},$$

where the supremum is taken over all $x, y, z \in B_X$ of which at least one belongs to S_X and $\|y - z\| \leqslant a\|x\|$.

Theorem 10. *Let $X_1, \ldots, X_N$ be Banach spaces with $C_{NJ}(a, X_i) < 2$ for all $i = 1, \ldots, N$. If Z is uniformly convex, then $C_{NJ}(a, (X_1 \oplus \cdots \oplus X_N)_Z) < 2$.*

Proof. Suppose $C_{NJ}(a, (X_1 \oplus \cdots \oplus X_N)_Z) = 2$. Thus, by Dhompongsa et al. [5, Lemma 3.2] there exist sequences $x_n, y_n, z_n \in S_{(X_1\oplus\cdots\oplus X_N)_Z}$ such that $\|x_n + y_n\|, \|x_n - z_n\| \to 2$ and $\|y_n - z_n\| \leqslant a$ for all n. By passing through subsequences, we may assume that all the following sequences converge: $\|x_{ni} + y_{ni}\| \to A_i$, $\|x_{ni} - z_{ni}\| \to B_i$, $\|x_{ni}\| \to C_i$, $\|y_{ni}\| \to C_i$, and $\|z_{ni}\| \to C_i$ for each i. By the assumption that Z is uniformly convex, we obtain $A_i = 2C_i$ and $B_i = 2C_i$ for each i. Now, for each n and for each i, we have $\|y_{ni} - z_{ni}\| = \|\|y_{ni} - z_{ni}\|e_i\| \leqslant \|\sum_{i\in I}(\|y_{ni} - z_{ni}\|)e_i\| \leqslant a$. Finally, we obtain the limits of the following sequences: $\|x_{ni} + y_{ni}\|^2 + \|x_{ni} - z_{ni}\|^2 \to A_i^2 + B_i^2 = 8C_i^2$, $2\|x_{ni}\|^2 + \|y_{ni}\|^2 + \|z_{ni}\|^2 \to 4C_i^2$. Clearly, for some i, $C_i \neq 0$. And for such i, we have $C_{NJ}(a, X_i) = 2$, a contradiction. □

Corollary 11. *Let $X_1, \ldots, X_N$ be Banach spaces and $\psi \in \Psi_N$ be strictly convex. If $C_{NJ}(a, X_i) < 2$, for all $i = 1, \ldots, N$, then $C_{NJ}(a, (X_1 \oplus \cdots \oplus X_N)_\psi) < 2$.*

Corollary 12. *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly nonsquare, whenever $X_1, \ldots, X_N$ are uniformly nonsquare and $\psi \in \Psi_N$ is strictly convex.*

Thus, in this case, $(X_1 \oplus \cdots \oplus X_N)_\psi$ has the fixed point property by Mazcunan Navarro [22, Corollary 4.2.4]. From [5], we know that "$C_{NJ}(a, X) < 2$ if and only if $J(a, X) < 2$". Therefore, the results in this section can be applied to the generalized James constant $J(a, X)$ as well.

6. The uniform Kadec–Klee property (UKK)

A Banach space X is said to have the *uniform Kadec–Klee property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_n \in B_X$, $x_n \rightharpoonup x$ weakly and $\text{sep}(x_n) \geqslant \varepsilon$ imply $\|x\| \leqslant 1 - \delta$. Finally, X is *nearly uniformly convex* (NUC) if for any $\varepsilon \geqslant 0$, there exists $\delta < 1$ such that $x_n \in B_X$ and $\text{sep}(x_n) \geqslant \varepsilon$ imply $\text{co}(x_n) \cap \delta B_X \neq \emptyset$. Here $\text{sep}(x_n) = \inf_{n\neq m}\|x_n - x_m\|$. It is known that UC ⇒ NUC ⇒ UKK ⇒ property-H. It is also well-known by Huff [13] that

$$\text{NUC} \Leftrightarrow \text{UKK} + \text{Reflexive}. \tag{6.1}$$

By strict monotonicity of an element ψ in Ψ we mean its corresponding norm $\|\cdot\|$ is strictly monotone. That is $\|(a, b)\| < \|(a, c)\|$ and $\|(b, a)\| < \|(c, a)\|$ for all $0 \leqslant a, 0 \leqslant b < c$. All strictly convex $\psi \in \Psi$ and all $\psi_p (1 \leqslant p < \infty)$ are strictly monotone. For $p = 1$, this is obvious. Now let $\psi \in \Psi$ be strictly convex. If, for some a, b, c in $[0, \infty)$ with $b < c$, we have $\|(a, b)\| = \|(a, c)\|$, then, by monotonicity [26], $\|(a, b)\| = \|(a, (b+c)/2)\| = \|(a, c)\|$. We put $\Delta = 2a + b + c$. Observe that $a \neq 0$ (by monotonicity of $\|\cdot\|$) and thus $b/(a + b) < c/(a + c)$. Strict convexity of ψ implies that $\psi((b + c)/\Delta) < [(a + b)/\Delta]\psi(b/(a + b)) + [(a + c)/\Delta]\psi(c/(a + c)) = 2/\Delta(a + b)\psi((b/(a + b)) = (2/\Delta)(\Delta/2)\psi((b + c)/\Delta)$, a contradiction, and this proves our claim.

Theorem 13. *For strictly monotone $\psi \in \Psi$, $X \oplus_\psi Y$ has the UKK property if and only if X and Y have the UKK property.*

It is immediate from [4, Corollary 3], (6.1), and Theorem 13 that

Corollary 14. *For strictly monotone $\psi \in \Psi$, $X \oplus_\psi Y$ is NUC if and only if X and Y are NUC.*

Remark 15. (1) Theorem 13 does not hold for the l_∞-norm. Consider any two Banach spaces X and Y with the UKK property. Now assume there are sequences (x_n) and (y_n) in B_X and B_Y, respectively, and $x \in S_X$, $y \in B_Y$ such that $x_n \to x$ and $y_n \xrightarrow{w} y$, $\operatorname{sep}(y_n) \geqslant \frac{1}{2}$, and $|y| < 1-\delta$ for some $\delta > 0$. Clearly, $\|(x, y)\|_\infty = 1 > 1-\delta$. On the other hand, by [4, Theorem 2], $(x_n, y_n) \xrightarrow{w} (x, y)$ and $\operatorname{sep}(x_n, y_n) \geqslant \operatorname{sep}(y_n) \geqslant \frac{1}{2}$. This shows that $X \oplus_\infty Y$ does not have the UKK property.

(2) If $\psi \in \Psi$ is strictly monotone, then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(a+\varepsilon, b)\| \wedge \|(a, b+\varepsilon)\| \geqslant \|(a, b)\| + \delta$ for all $a \geqslant 0, b \geqslant 0$. This follows from the continuity of the norm $\|\cdot\|$ on a compact set.

(3) With the same proof, Theorem 13 can be extended to strictly monotone Z-norms.

Proof of Theorem 13. Suppose on the contrary that $X \bigoplus_\psi Y$ does not have the UKK property. Thus, for some $\varepsilon_0 > 0$, we have for each $k \geqslant 1$, sequences (x_n^k, y_n^k) and (x^k, y^k) in $B_{X \oplus Y}$ such that

$$\operatorname{sep}(x_n^k, y_n^k) \geqslant \varepsilon_0, \quad (x_n^k, y_n^k) \xrightarrow{w} (x^k, y^k) \quad \text{and} \quad \|(x^k, y^k)\| \geqslant 1 - \frac{1}{k}. \tag{6.2}$$

Since $\|a\| + \|b\| \geqslant \|(a, b)\|$, by applying Ramsey's Theorem, we may assume, by passing through subsequences that for some $\varepsilon_1 \in (0, \varepsilon_0)$,

$$\operatorname{sep}(x_n^k) \geqslant \varepsilon_1 \quad \text{for all } k \geqslant 1. \tag{6.3}$$

In the proof below we shall consider two cases, passing through subsequences when necessary. Choose $\delta > 0$ from the definition of the UKK property of X corresponding to $\varepsilon_1/2$. Then choose $\varepsilon_2 > 0$ so small that $\varepsilon_2 < \delta\varepsilon_1/4(1-\delta) \wedge \varepsilon_1/4$.

Case 1: *There exists* $\varepsilon_3 > 0$ *such that* $\|x_n^k\| \geqslant \|x^k\| + \varepsilon_3$ *for all* n, k.

Case 2: $\|x_n^k\| \leqslant \|x^k\| + \varepsilon_2$ *for all* n, k.

For *Case* 1, it follows from Remark 15(2) that for some $\delta_1 \in (0, 1)$, $\|(x_n^k, y_n^k)\| \geqslant \|(\|x^k\| + \varepsilon_3, \|y_n^k\|)\| > \|(x^k, y_n^k)\| + \delta_1$ for all n, k. By (6.2) we see that $(x^k, y_n^k) \xrightarrow{w} (x^k, y^k)$. Thus, $1 \geqslant \lim \|(x_n^k, y_n^k)\| \geqslant \lim \inf_{n\to\infty} \|(x^k, y_n^k)\| + \delta_1 \geqslant \|(x^k, y^k)\| + \delta_1$, which is impossible since $\lim_k \|(x^k, y^k)\| = 1$ by (6.2).

For *Case* 2, we first observe from (6.3) that

$$\|x^k\| \geqslant \varepsilon_1/4 \quad \text{for all large } k. \tag{6.4}$$

As $\lim_n \|x_n^k\| \geqslant \|x^k\|$ we can assume $\|x_n^k\| > 0$ for all n, k. Choose k sufficiently large so that

$$\|x^k\| - \varepsilon_2 \leqslant \|x_n^k\| \leqslant \|x^k\| + \varepsilon_2 \quad \text{for all large } n. \tag{6.5}$$

Thus by (6.2), we have for $n \neq m$,

$$\begin{aligned} \left\| \frac{x_n^k}{\|x_n^k\|} - \frac{x_m^k}{\|x_m^k\|} \right\| &\geqslant \frac{\|x_n^k - x_m^k\|}{\|x^k\|} - \left| \frac{\|x_n^k\| - \|x^k\|}{\|x^k\|} \right| - \left| \frac{\|x_m^k\| - \|x^k\|}{\|x^k\|} \right| \\ &\geqslant (\varepsilon_1 - \varepsilon_2 - \varepsilon_2)/\|x^k\| \geqslant \varepsilon_1/2, \end{aligned}$$

i.e., $\mathrm{sep}(x_n^k/\|x_n^k\|) \geqslant \varepsilon_1/2$. Assume without loss of generality that $\|x_n^k\| \to A$ for some A. Thus, by (6.5) and (6.2), $A \leqslant \|x^k\| + \varepsilon_2$ and $x_n^k/\|x_n^k\| \xrightarrow{w} x^k/A$. But, by (6.4), $\|x^k\|/A \geqslant \|x^k\| /(\|x^k\| + \varepsilon_2) = 1 - \varepsilon_2/(\|x^k\| + \varepsilon_2) > 1 - \delta$, contradicting the UKK property of X. □

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A NOTE ON PROPERTIES THAT IMPLY THE FIXED POINT PROPERTY

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We give relationships between some Banach-space geometric properties that guarantee the weak fixed point property. The results extend some known results of Dalby and Xu.

1. Introduction

A Banach space X is said to satisfy the weak fixed point property (fpp) if every nonempty weakly compact convex subset C, and every nonexpansive mapping $T : C \to C$ (i.e., $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$) has a fixed point, that is, there exists $x \in C$ such that $T(x) = x$. Many properties have been shown to imply fpp. The most recent one is the uniform nonsquareness which is proved by Mazcuñán [20] solving a long stand open problem. Other well known properties include Opial property (Opial [21]), weak normal structure (Kirk [17]), property (M) (García-Falset and Sims [12]), $R(X) < 2$ (García-Falset [10]), and UCED (Garkavi [13]). Connection between these properties were investigated in Dalby [3] and Xu et al. [27]. We aim to continue the study in this direction. In contrast to [3], we do not assume that all Banach spaces are separable.

2. Preliminaries

Let X be a Banach space. For a sequence (x_n) in X, $x_n \xrightarrow{w} x$ denotes the weak convergence of (x_n) to $x \in X$. When $x_n \xrightarrow{w} 0$, we say that (x_n) is a weakly null sequence. $B(X)$ and $S(X)$ stand for the unit ball and the unit sphere of X, respectively. It becomes a common ingredient that when working with a weak null sequence (x_n), we consider the type function $\limsup_{n\to\infty} \|x_n - x\|$ for all $x \in X$. As for a starting point, we recall Opial property.

Opial property [21] states that

$$\text{if } x_n \xrightarrow{w} 0, \quad \text{then } \limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\| \ \forall x \in X,\ x \neq 0. \tag{2.1}$$

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Appendix 3: A note on properties that implies the weak fixed point property, Abst. Appl. Anal. V. 2006, Article ID 34959, Pages 1-12.

If the strict inequality becomes $\leq$, this condition becomes a nonstrict Opial property. On the other hand, if for every $\epsilon > 0$, for each $x_n \xrightarrow{w} 0$ with $\|x_n\| \to 1$, there is an $r > 0$ such that

$$1 + r \leq \limsup_{n\to\infty} \|x_n + x\| \tag{2.2}$$

for each $x \in X$ with $\|x\| \geq \epsilon$, then we have the locally uniformly Opial property (see [27]).

The coefficient $R(X)$, introduced in García-Falset [9], is defined as

$$R(X) := \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0,\ \|x_n\| \leq 1\ \forall n,\ \|x\| \leq 1\right\}. \tag{2.3}$$

So $1 \leq R(X) \leq 2$ and it is not hard to see that in the definition of $R(X)$, "lim inf" can be replaced by "lim sup." Some values of $R(X)$ are $R(c_0) = 1$ and $R(l_p) = 2^{1/p}, 1 < p < \infty$.

A Banach space X has property (M) if whenever $x_n \xrightarrow{w} 0$, then $\limsup_{n\to\infty} \|x_n - x\|$ is a function of $\|x\|$ only. Property (M) which is introduced by Kalton [15] is equivalent to:

$$\text{if } x_n \xrightarrow{w} 0, \quad \|u\| \leq \|v\|, \text{ then } \limsup_{n\to\infty} \|x_n + u\| \leq \limsup_{n\to\infty} \|x_n + v\|. \tag{2.4}$$

Sims [23] introduced a property called weak orthogonality (WORTH) for Banach spaces. A Banach space X is said to have property WORTH if,

$$\text{for every } x_n \xrightarrow{w} 0,\ x \in X, \quad \limsup_{n\to\infty} \|x_n + x\| = \limsup_{n\to\infty} \|x_n - x\|. \tag{2.5}$$

It remains unknown if property WORTH implies fpp. In many situations, the fixed point property can be easily obtained when we assume, in addition, that the spaces being considered have the property WORTH. For examples, WORTH and ε_0-inquadrate for some $\epsilon_0 < 2$ ([24]), WORTH and 2-UNC ([11]) imply fpp.

The following results will be used in Section 3.

Proposition 2.1 [12, Proposition 2.1]. *For the following conditions on a Banach space X, we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv).*

(i) *X has property (M).*

(ii) *X has property WORTH.*

(iii) *If $x_n \xrightarrow{w} 0$, then for each $x \in X$ we have $\limsup_{n\to\infty} \|x_n - tx\|$ is an increasing function of t on $[0, \infty)$.*

(iv) *X satisfies the nonstrict Opial property.*

Property (M) implies the nonstrict Opial property but not weak normal structure. c_0 has property (M) but does not have weak normal structure. In [3, 25] it had been shown that $R(X) = 1$ implies X has property (M).

A generalization of uniform convexity of Banach spaces which is due to Sullivan [26] is now recalled. Let $k \geq 1$ be an integer. Then a Banach space X is said to be k-UR (k-uniformly rotund) if given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\{x_1, \ldots, x_{k+1}\} \subset B(X)$

satisfying $V(x_1,\dots,x_{k+1}) \geq \varepsilon$, then

$$\left\|\frac{\sum_{i=1}^{k+1} x_i}{k+1}\right\| \leq \delta(\varepsilon). \tag{2.6}$$

Here, $V(x_1,\dots,x_{k+1})$ is the volume enclosed by the set $\{x_1,\dots,x_{k+1}\}$, that is,

$$V(x_1,\dots,x_{k+1}) = \sup\left\{\left|\begin{matrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \ddots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{matrix}\right|\right\}, \tag{2.7}$$

where the supremum is taken over all $f_1,\dots,f_k \in B(X^*)$.

Let K be a weakly compact convex subset of a Banach space X and (x_n) a bounded sequence in X. Define a function f on X by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in X. \tag{2.8}$$

Let

$$\begin{aligned} r &\equiv r_K((x_n)) := \inf\{f(x) : x \in K\}, \\ A &\equiv A_K((x_n)) := \{x \in K : f(x) = r\}. \end{aligned} \tag{2.9}$$

Recall that r and A are, respectively, called the asymptotic radius and center of (x_n) relative to K. As K is weakly compact convex, we see that A is nonempty, weakly compact and convex (see [14]). In [18], Kirk proved that the asymptotic center of a bounded sequence w.r.t a bounded closed convex subset of a k-uniformly convex spaces X is compact. This fact will be used in proving Theorem 3.8.

Being k-UR and Opial property are related in the following way.

Theorem 2.2 [19, Theorem 3.5]. *If X is k-UR and satisfies the Opial property, then X satisfies locally uniform Opial property.*

One last concept we need to mention is ultrapowers of Banach spaces. Ultrapowers of a Banach space are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \left\{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\right\}. \tag{2.10}$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.11}$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [1] or [22]).

3. Main results

Recall that a Banach space X is said to have Schur's property if

$$\text{for every sequence } (x_n), \quad x_n \xrightarrow{w} 0 \text{ implies } x_n \longrightarrow 0. \tag{3.1}$$

An element $x \in X$ is said to be an H-point if

$$x_n \xrightarrow{w} x, \quad \|x_n\| \longrightarrow \|x\| \text{ imply } x_n \longrightarrow x. \tag{3.2}$$

X has property (H) if every element of X is an H-point. These concepts are related, in conjunction with the condition $R(X) = 1$, as follow.

THEOREM 3.1. *A Banach space X has Schur's property if and only if $R(X) = 1$ and X has at least one H-point.*

Proof. "⇒" It is well known that Schur's property implies property (H). From the definition of $R(X)$ and Schur's property, we have

$$\begin{aligned} R(X) &= \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0, \|x_n\| \le 1 \ \forall n, \|x\| \le 1\right\} \\ &= \sup\{\|x\| : \|x\| \le 1\} = 1. \end{aligned} \tag{3.3}$$

"⇐" Suppose that there exists a sequence (x_n) converges weakly to 0 but $\|x_n\| \nrightarrow 0$. By passing through a subsequence if necessary, we can assume that $\|x_n\| \to a \neq 0$. Put $y_n = x_n/a$. Clearly $y_n \xrightarrow{w} 0$ and $\|y_n\| \to 1$. Let x_0 be an H-point. If $x_0 = 0$, we are done. We assume now that $x_0 \neq 0$ and in fact we assume that $x_0 \in S(X)$. Thus, as $R(X) = 1$ and the weak lower semicontinuity of the norm,

$$(x_0 - y_n) \xrightarrow{w} x_0, \quad \liminf_{n\to\infty} \|x_0 - y_n\| = 1. \tag{3.4}$$

Choose a subsequence $(y_{n'})$ of (y_n) such that

$$\lim_{n'\to\infty} \|x_0 - y_{n'}\| = 1. \tag{3.5}$$

We see that $(x_0 - y_n') \to x_0$ and $y_n' \to 0$. Thus $\|y_n'\| \to 0$ and $0 = a$, a contradiction. □

A Banach space X has property m_p (resp., m_∞) (cf. [27]) if for all $x \in X$, whenever $x_n \xrightarrow{w} 0$,

$$\limsup_{n\to\infty} \|x + x_n\|^p = \|x\|^p + \limsup_{n\to\infty} \|x_n\|^p$$
$$\left(\text{resp., } \limsup_{n\to\infty} \|x + x_n\| = \max\left\{\|x\|, \limsup_{n\to\infty} \|x_n\|\right\}\right). \tag{3.6}$$

Clearly the above properties imply property (M) and property m_1 implies Opial property.

Property m_1 implies property (H). For, if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$ for some sequence (x_n) and $x \in X$, we have, by m_1,

$$\|x\| = \limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|(x_n - x) + x\| = \|x\| + \limsup_{n\to\infty} \|x_n - x\|. \tag{3.7}$$

This implies that $\limsup_{n\to\infty} \|x_n - x\| = 0$ and thus $x_n \to x$.

It also turns out that property m_∞ and the condition $R(X) = 1$ coincide as the following result shows.

Theorem 3.2. *A Banach space X has property m_∞ if and only if $R(X) = 1$.*

Proof. "⇒" Suppose that X has property m_∞. Thus,

$$\begin{aligned} R(X) &= \sup\left\{\limsup_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\} \\ &= \sup\left\{\max\left\{\|x\|, \limsup_{n\to\infty} \|x_n\|\right\} : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\} = 1. \end{aligned} \tag{3.8}$$

"⇐" To show that X has property m_∞. Given $x_n \xrightarrow{w} 0$ and $x \in X - \{0\}$. Put $a = \max\{\|x\|, \limsup_{n\to\infty} \|x_n\|\}$. Clearly, $\limsup_{n\to\infty}(\|x_n\|/a) \le 1$ and $\|x\|/a \in B(X)$. We note here that $R(X) = 1$ implies property (M) and it in turn implies the nonstrict Opial property. By the weak lower semicontinuity of $\|\cdot\|$ and the nonstrict Opial property, we see that $\|x\| \le \limsup_{n\to\infty} \|x_n - x\|$ and $\limsup_{n\to\infty} \|x_n\| \le \limsup_{n\to\infty} \|x_n - x\|$. Thus $a \le \limsup_{n\to\infty} \|x_n - x\|$. On the other hand, as $R(X) = 1$, we can show that $\limsup_{n\to\infty} \|x_n/a - x/a\| \le 1$. So we can conclude that,

$$\limsup_{n\to\infty} \left\|\frac{x_n}{a} - \frac{x}{a}\right\| = 1, \tag{3.9}$$

and thus $\limsup_{n\to\infty} \|x_n - x\| = a = \max\{\|x\|, \limsup_{n\to\infty} \|x_n\|\}$ and the proof is complete. □

For $p < \infty$, we have the following proposition.

Proposition 3.3. *If X has property $m_p (1 \le p < \infty)$, then $R(X) \le 2^{1/p}$. Moreover, if in addition X does not have Schur's property, then $R(X) = 2^{1/p}$.*

Proof. Define

$$R_p(X) := \sup\Big\{\limsup_{n\to\infty} \|x_n - x\|^p : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\Big\}. \tag{3.10}$$

By property m_p, we have

$$R_p(X) = \sup\Big\{\|x\|^p + \limsup_{n\to\infty} \|x_n\|^p : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\Big\}. \tag{3.11}$$

Thus, $R_p(X) \le 2$ which implies $R(X) \le 2^{1/p}$. On the other hand, if, in addition, X does not have Schur's property, then there exists a weakly null sequence (x_n) such that $x_n \nrightarrow 0$. From this we can construct a weakly null sequence (y_n) in the unit sphere. We can now see that $R_p(X) \ge 2$ and hence $R(X) \ge 2^{1/p}$. Therefore $R(X) = 2^{1/p}$. □

Example 3.4. In l_p $(1 < p < \infty)$, we have $e_n \in S(X)$ and $e_n \xrightarrow{w} 0$, where (e_n) is the standard basis. Clearly

$$\|e_n - e_1\| \xrightarrow{n\to\infty} 2^{1/p}, \tag{3.12}$$

thus $R(l_p) = 2^{1/p}$. Note that l_p has property m_p (cf. [27]).

Some properties are equivalent in a space X with $R(X) = 1$.

THEOREM 3.5. *Let X be a Banach space with $R(X) = 1$. The following conditions are equivalent:*
(i) *X has property m_1;*
(ii) *X satisfies Opial property;*
(iii) *X has Schur's property.*

Proof. (i)⇒(ii) and (iii)⇒(i) are clear. It needs to prove (ii) ⇒ (iii).

Let $x_n \xrightarrow{w} 0$. To show $x_n \to 0$, let $0 \ne x \in X$. By Opial property together with property m_∞, we have

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n + x\| = \max\Big\{\|x\|, \limsup_{n\to\infty} \|x_n\|\Big\}. \tag{3.13}$$

Thus

$$\limsup_{n\to\infty} \|x_n\| < \|x\|, \tag{3.14}$$

for all $x \in X - \{0\}$. This means that $\limsup_{n\to\infty} \|x_n\| = 0$ and thus $\lim_{n\to\infty} \|x_n\| = 0$. Consequently, $x_n \to 0$, and therefore X has Shur's property. □

The Jordan-von Neumann constant $C_{NJ}(X)$ of X is defined by

$$\begin{aligned} C_{NJ}(X) &= \sup\left\{ \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y \in X \text{ not both zero} \right\} \quad ([2]) \\ &= \sup\left\{ \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x \in S(X),\ y \in B(X) \right\} \quad ([16]). \end{aligned} \tag{3.15}$$

Another important constant which is closely related to $C_{NJ}(X)$ is the James constant $J(X)$ defined by Gao and Lau [7] as:

$$\begin{aligned} J(X) &= \sup\{\|x+y\| \wedge \|x-y\| : x,y \in S(X)\} \\ &= \sup\{\|x+y\| \wedge \|x-y\| : x,y \in B(X)\}. \end{aligned} \tag{3.16}$$

In general we have

$$\frac{1}{2}J(X)^2 \le C_{NJ}(X) \le \frac{J(X)^2}{(J(X)-1)^2+1} \quad ([16]). \tag{3.17}$$

With or without having WORTH, Mazcuñán [20] showed that $R(1,X) < 2$ whenever $C_{NJ}(X) < 2$. In gerneral, $R(1,X) \le R(X)$. The constant $R(a,X)$ is introduced by Dominguez [6] as: for a given real number a

$$R(a,X) := \sup\left\{ \liminf_{n\to\infty} \|x+x_n\| \right\}, \tag{3.18}$$

where the supremum is taken over all $x \in X$ with $\|x\| \le a$ and all weakly null sequences (x_n) in the unit ball of X such that

$$\limsup_{n\to\infty}\left(\limsup_{n\to\infty} \|x_n - x_m\| \right) \le 1. \tag{3.19}$$

Replacing $R(1,X)$ in [20] by $R(X)$ we obtain the following theorem.

Theorem 3.6. *If X has property WORTH and $C_{NJ}(X) < 2$, then $R(X) < 2$.*

Proof. Suppose on the contrary that $R(X) = 2$. Thus there exist sequences $(x_n^m), (x^m) \in B(X)$ such that for each m, $x_n^m \xrightarrow{w} 0$ as $n \to \infty$ and

$$\liminf_{n\to\infty} \|x_n^m - x^m\| > 2 - \frac{1}{m} \tag{3.20}$$

for all $m \in N$. Now, by WORTH, we have, for each m,

$$\frac{\|x_n^m + x^m\|^2 + \|x_n^m - x^m\|^2}{2(\|x_n^m\|^2 + \|x^m\|^2)} > \frac{2(2-1/m)^2}{4} = 2 - \frac{2}{m} + \frac{1}{2m^2} \tag{3.21}$$

for all large n. This implies $C_{NJ}(X) = 2$, a contradiction, and therefore $R(X) < 2$ as desired. □

Remark 3.7. Theorem 3.6 says that every Banach space X with property WORTH has fpp or $C_{NJ}(X) = 2 = R(X)$.

THEOREM 3.8. *If X is k-UR and satisfies property (M), then X satisfies Opial property.*

Proof. Suppose that there exist $x_n \overset{w}{\to} 0$ and $0 \neq x_0 \in X$ such that

$$\limsup_{n\to\infty} \|x_n\| \geq \limsup_{n\to\infty} \|x_n - x_0\|. \tag{3.22}$$

Observe that X is therefore not finite dimensional. By the nonstrict Opial property (see Proposition 2.1) we have

$$\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|x_n - x_0\| = \alpha \neq 0. \tag{3.23}$$

We may assume that $\|x_0\| = 1$. Define the type function by

$$f(u) = \limsup_{n\to\infty} \|x_n - u\|. \tag{3.24}$$

Then f is a function of $\|u\|$ and is also nondecreasing in $\|u\|$. Now since $f(0) = f(x_0) = \alpha$ and since $\|x_0\| = 1$, it follows that $f(u) \equiv \alpha$ for all $u \in B(X)$. This implies that $A_{B(X)}(x_n) = B(X)$. Since X is k-UR, Kirk [18] implies that $A_{B(X)}(x_n)$ and so $B(X)$ is compact, that is, X is finite dimensional, a contradiction. □

COROLLARY 3.9. *If X is k-UR and has property (M), then X has the locally uniform Opial property. In particular, properties UR and (M) imply the locally uniform Opial property.*

Proof. This follows from Theorem 2.2 and Theorem 3.8. □

Definition 3.10. Let X be a Banach space.

(i) We say that X has property strict (M) [27, Definition 2.2] if, for each weakly null sequence (x_n), for $u, v \in X$ such that $\|u\| < \|v\|$, $\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \|x_n - v\|$.

(ii) We say that X has property strict (W) if, for each weakly null sequence (x_n), for $x \in X$ we have $\limsup_{n\to\infty} \|x_n - tx\|$ is an strictly increasing function of t on $[0, \infty)$.

It is easy to see that

$$\text{property strict } (M) \Longrightarrow \text{property strict } (W) \Longrightarrow \text{Opial property}. \tag{3.25}$$

PROPOSITION 3.11. *Let X be a Banach space, then X has property strict (M) if and only if it has both properties (M) and strict (W).*

Proof. "⇒" Clear.

"⇐" Suppose X has properies (M) and strict (W). Let (x_n) be a weakly null sequence, $u, v \in X$ with $\|u\| < \|v\|$. By property strict (W) we have

$$\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \left\| x_n - \frac{\|v\|}{\|u\|} u \right\|. \tag{3.26}$$

Since $\|(\|v\|/\|u\|)u\| = \|v\|$, so by property (M) we have $\limsup_{n\to\infty}\|x_n - (\|v\|/\|u\|)u\| = \limsup_{n\to\infty}\|x_n - v\|$. Hence

$$\limsup_{n\to\infty}\|x_n - u\| < \limsup_{n\to\infty}\|x_n - v\|. \tag{3.27}$$

This shows that X has property strict (M). □

PROPOSITION 3.12. *Let X be a Banach space which satisfies Opial property and has property (M). Then X satisfies the locally uniform Opial property.*

Proof. Let (x_n) be a weakly null sequence in X satisfying $\|x_n\| \to 1$ and $c > 0$. Set $r = \limsup_{n\to\infty}\|x_n - (c/\|x\|)x\| - 1$, where $x \in X - \{0\}$. Since X satisfies Opial property, we have $r > 0$. Hence, for $u \in X$ such that $\|u\| \geq c$, we have

$$\limsup_{n\to\infty}\|x_n - u\| \geq \limsup_{n\to\infty}\left\|x_n - \frac{c}{\|u\|}u\right\| = \limsup_{n\to\infty}\left\|x_n - \frac{c}{\|x\|}x\right\| = 1 + r. \tag{3.28}$$

Thus, X satisfies the locally uniform Opial property. □

COROLLARY 3.13 [27, Theorem 2.1]. *Let X be a Banach space which has property strict (M). Then X satisfies the locally uniform Opial property.*

Recall that a Banach space X is uniformly convex in every direction (UCED) Day et al. [4] if, for each $z \in X$ such that $\|z\| = 1$ and $\epsilon > 0$, we have

$$\delta_z(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ x - y = tz,\ |t| \geq \epsilon\right\} > 0. \tag{3.29}$$

THEOREM 3.14. *Suppose that a Banach space X has property WORTH and is also UCED. Then X has the property strict (W).*

Proof. Suppose X fails to have the property strict (W), then there exist a weakly null sequence (x_n), $x \in S(X)$, $t_1, t_2 \in [0,\infty)$, where $t_1 < t_2$, with

$$\limsup_{n\to\infty}\|x_n + t_1x\| \geq \limsup_{n\to\infty}\|x_n + t_2x\|. \tag{3.30}$$

By property WORTH we must have equality. Put $a = \limsup_{n\to\infty}\|x_n + t_1x\|$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\left\|x_n + \frac{t_1+t_2}{2}x\right\| &= \limsup_{n\to\infty}\left\|\frac{x_n + t_1x + x_n + t_2x}{2}\right\| \\ &\leq a\left[1 - \delta_x\left(\frac{t_2 - t_1}{a}\right)\right] < a = \limsup_{n\to\infty}\|x_n + t_1x\|\end{aligned} \tag{3.31}$$

contradicting to having WORTH of X. □

From Proposition 3.11 and Theorem 3.14 we have the following corollary.

COROLLARY 3.15. *Suppose that a Banach space X has property (M) and is also UCED. Then X has property strict (M).*

Finally, we improve the latest upper bound of the Jordan-von Neumann constant $C_{NJ}(X)$ at $(3+\sqrt{5})/4$ for X to have uniform normal structure which is proved in [5].

THEOREM 3.16. *If* $C_{NJ}(X) < (1+\sqrt{3})/2$, *then* X *has uniform normal structure.*

Proof. Since $C_{NJ}(X) < 2$, X is uniformly nonsquare, and consequently, X is reflexive. Thus, normal structure and weak normal structure coincide. By [8, Theorem 5.2], it suffices to prove that X has weak normal structure.

Suppose on the contrary that X does not have weak normal structure. Thus, there exists a weak null sequence (x_n) in $S(X)$ such that for $C := \overline{co}\{x_n : n \geq 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam} C = 1 \quad \forall x \tag{3.32}$$

(cf. [24]). Let $\alpha = \sqrt{1+\sqrt{3}}$. We choose first an $x \in C$ with $\|x\| = 1$. We will consider, without loss of generality

$$\begin{aligned}\lim_{n\to\infty} \|x_n + x\| &\leq R(1,X) \leq J(X) \quad ([20]) \\ &\leq \sqrt{2C_{NJ}(X)} \quad ([16]) < \alpha.\end{aligned} \tag{3.33}$$

By Hanh-Banach theorem there exist $f_n, g \in S(X^*)$ satisfying $f_n(x_n - (1/2)x) = \|x_n - (1/2)x\|$, $\forall n \in \mathbb{N}$ and $g(x) = 1$. Set $\tilde{f} = \widetilde{(f_n)}$. Then $\tilde{f}, \dot{g} \in S(\tilde{X}^*)$ and satisfy

$$\tilde{f}\left(\widetilde{(x_n)}\right) = 1, \qquad \tilde{f}(\dot{x}) = 0, \qquad \dot{g}\left(\widetilde{(x_n)}\right) = 0, \qquad \dot{g}(\dot{x}) = 1. \tag{3.34}$$

Now consider

$$\begin{aligned}\|\tilde{f} - \dot{g}\| &\geq (\tilde{f} - \dot{g})\left(\widetilde{(x_n)} - \dot{x}\right) \\ &= \tilde{f}\left(\widetilde{(x_n)}\right) - \tilde{f}(\dot{x}) - \dot{g}\left(\widetilde{(x_n)}\right) + \dot{g}(\dot{x}) \\ &= 1 + 0 - 0 + 1 \geq 2.\end{aligned} \tag{3.35}$$

On the other hand,

$$\begin{aligned}\|\tilde{f} + \dot{g}\| &\geq (\tilde{f} + \dot{g})\left(\frac{1}{\alpha}\left(\widetilde{(x_n)} + \dot{x}\right)\right) \\ &= \tilde{f}\left(\frac{1}{\alpha}\widetilde{(x_n)}\right) + \tilde{f}\left(\frac{1}{\alpha}\dot{x}\right) - \dot{g}\left(\frac{1}{\alpha}\widetilde{(x_n)}\right) + \dot{g}\left(\frac{1}{\alpha}\dot{x}\right) \\ &= \frac{1}{\alpha} + 0 - 0 + \frac{1}{\alpha} = \frac{2}{\alpha}.\end{aligned} \tag{3.36}$$

Thus we have

$$C_{NJ}(\tilde{X}^*) \geq \frac{\|\tilde{f} + \dot{g}\|^2 + \|\tilde{f} - \dot{g}\|^2}{2(\|\tilde{f}\|^2 + \|\dot{g}\|^2)} \geq \frac{4 + 4/\alpha^2}{4} = 1 + \frac{1}{\alpha^2}. \tag{3.37}$$

Since the Jordan-von Neumann constants of X^*, X, $\tilde{X}$, and $\tilde{X}^*$ are all equal, we must have $C_{NJ}(X) \geq 1 + 1/\alpha^2$, that is,

$$C_{NJ}(X) \geq \frac{1+\sqrt{3}}{2}, \tag{3.38}$$

a contradiction. □

The following corollary is a consequence of the proof of Theorem 3.16.

Corollary 3.17. *If $C_{NJ}(X) < 1 + 1/J(X)^2$, then X has uniform normal structure.*

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Appendix 4: Lim's theorems for multivalued mappings in CAT(0) spaces,

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Lim's theorems for multivalued mappings in CAT(0) spaces ☆

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Lim's theorems for multivalued mappings in CAT(0) spaces ☆

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Abstract

Let X be a complete CAT(0) space. We prove that, if E is a nonempty bounded closed convex subset of X and $T : E \to K(X)$ a nonexpansive mapping satisfying the weakly inward condition, i.e., there exists $p \in E$ such that $\alpha p \oplus (1-\alpha)Tx \subset \overline{I_E(x)}$ $\forall x \in E$, $\forall \alpha \in [0,1]$, then T has a fixed point. In Banach spaces, this is a result of Lim [On asymptotic centers and fixed points of nonexpansive mappings, Canad. J. Math. 32 (1980) 421–430]. The related result for unbounded $\mathbb{R}$-trees is given.

Keywords: Multivalued mappings; Fixed points; CAT(0) spaces; $\mathbb{R}$-trees

1. Introduction

In 1980 [8] and 2001 [9], Lim and, respectively, Xu had proved differently the same result concerning the existence of a fixed point for a nonself nonexpansive compact valued mapping defining on a bounded closed convex subset of a uniformly convex space and satisfying the weak inward condition. While Lim used the method of asymptotic radius,

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Xu used his characterization of uniform convexity. Recently in 2003, Bae [1] considered a closed valued mapping defined on a closed subset of a complete metric space. It was shown that if the mapping is weakly contractive and is metrically inward, then it has a fixed point.

Having all these results, we are interested in extending the Lim–Xu's result to a special kind of metric spaces, namely, CAT(0) spaces. Our proofs follow the ideas of the proofs in Lim [8], Bae [1], and Xu [9].

In Section 2, we give some basic notions and in Sections 3 and 4 we prove our results.

2. Preliminaries

In the course of our proof of the main result, we use an ultrapower of a metric space as an ingredient. Following Khamsi [5], let (X,d) be a bounded metric space and $\mathcal{U}$ a nontrivial ultrafilter on the natural numbers. Consider the countable Cartesian product X^∞ of X and define the equivalence relation $\sim$ on X^∞ by

$$(x_n) \sim (y_n) \quad \text{if } \lim_{\mathcal{U}} d(x_n, y_n) = 0.$$

The limit over $\mathcal{U}$ exists since X is bounded. On the quotient space $\tilde{X}$ of X^∞ over $\sim$, which will be called an ultrapower of X, define the metric $\tilde{d}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} d(x_n, y_n),$$

where $\tilde{x} = \widetilde{(x_n)}$ and $\tilde{y} = \widetilde{(y_n)}$ are elements of $\tilde{X}$. It is easy to see that $\tilde{X}$ is complete whenever X is. For each subset E of X put

$$\dot{E} = \{\widetilde{(x_n)}:\ x_n = x \in E \text{ for any } n \geqslant 1\}.$$

Clearly, X and $\dot{X}$ are isometric.

We present now a brief discussion on CAT(0) spaces (see Kirk [6,7] and Bridson and Haefliger [2]). Although CAT(κ) spaces are defined for all real numbers κ, we restrict ourselves to the case that $\kappa = 0$.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. Obviously, c is an isometry and $d(x,y) = l$. The image of c is called a geodesic segment joining x and y and, when unique, denoted $[x,y]$. A metric space is said to be a geodesic space if any two of its points are joined by a geodesic segment. If there is exactly one geodesic segment joining x to y for all $x, y \in X$, we say that (X,d) is uniquely geodesic.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X,d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edge of Δ). A comparison triangle for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$.

(X,d) is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) comparison axiom:

For every geodesic triangle Δ in X and its comparison triangle $\bar{\Delta}$ in $\mathbb{R}^2$, if $x, y \in \Delta$, and $\bar{x}, \bar{y}$ are their comparison points in $\bar{\Delta}$, respectively, then

$$d(x, y) \leqslant d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let X be a CAT(0) space, and let E be a nonempty closed convex subset of X. The following facts will be needed:

(i) (X, d) is uniquely geodesic.
(ii) $(\bar{X}, \bar{d})$ is a CAT(0) space.
(iii) (X, d) satisfies the (CN) inequality

$$d(x, y_0)^2 \leqslant \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2$$

for all $x, y_1, y_2 \in X$ and y_0 the midpoint of the segment $[y_1, y_2]$. Note that the converse is also true. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies (CN) inequality (cf. [7]).

(iv) Let p, x, y be points in X, let $\alpha \in (0, 1)$, and m_1 and m_2 denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying

$$d(p, m_1) = \alpha d(p, x) \quad \text{and} \quad d(p, m_2) = \alpha d(p, y).$$

Then

$$d(m_1, m_2) \leqslant \alpha d(x, y).$$

(v) For every $x \in X$, there exists a unique point $p(x) \in E$ such that

$$d\big(x, p(x)\big) = \operatorname{dist}(x, E),$$

where $\operatorname{dist}(x, E) := \inf\{d(x, y):\ y \in E\}$.

With the same E and $p(x)$, if $x \notin E$, $y \in E$, and $y \neq p(x)$, then $\angle_{p(x)}(x, y) \geqslant \frac{\pi}{2}$, where $\angle_z(x, y)$ is the Alexandrov angle between the geodesic segments $[z, x]$ and $[z, y]$ for all $x, y, z \in X$ (see [2, p. 176]).

Let (X, d) be a metric space and E a nonempty subset of X. A closed valued mapping $T : \to 2^X \backslash \emptyset$ is said to be metrically inward if for each $x \in E$,

$$Tx \subset MI_E(x),$$

where $MI_E(x)$ is the metrically inward set of E at x defined by

$$MI_E(x) = \big\{z \in X:\ z = x \text{ or there exists } y \in E \text{ such that } y \neq x \text{ and } d(x, z) = d(x, y) + d(y, z)\big\}.$$

In case X is a Banach space, the inward set of E at x is defined by

$$I_E(x) = \big\{x + \lambda(y - x):\ y \in E,\ \lambda \geqslant 1\big\}.$$

In general, $I_E(x) \subset MI_E(x)$ for each $x \in E$, and the equality may not be true.

From now on, X stands for a complete CAT(0) space. Let E be a nonempty bounded subset of X. We shall denote by $F(E)$ the family of nonempty closed subsets of E, by $FC(E)$ the family of nonempty closed convex subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $F(X)$, i.e.,

$$H(A,B)=\max\Big\{\sup_{a\in A}\operatorname{dist}(a,B),\sup_{b\in B}\operatorname{dist}(b,A)\Big\},\quad A,B\in F(X).$$

Definition 2.1. A multivalued mapping $T:E\to F(X)$ is said to be a contraction if there exists a constant $k\in[0,1)$ such that

$$H(Tx,Ty)\leqslant kd(x,y),\quad x,y\in E. \tag{2.1}$$

In this case T is said to be k-contractive. If (2.1) is valid when $k=1$, then T is called nonexpansive.

We use the notation $(1-\alpha)u\oplus\alpha v$, $\alpha\in[0,1]$, to denote the points of the segment $[u,v]$ with distance $\alpha d(u,v)$ from u. For $E\subset X$ and a fixed element $p\in E$, $(1-\alpha)p\oplus\alpha E:=\{(1-\alpha)p\oplus\alpha v\colon v\in E\}$. E is said to be convex if for each pair of points $x,y\in E$, we have $[x,y]\subset E$.

For a nonempty subset E of a CAT(0) space X, it is easy to see that the (metrically) inward set $MI_E(x)$ becomes

$$MI_E(x)=\Big(\bigcup\{z\colon (x,z]\cap E\neq\emptyset\}\Big)\cup\{x\}:=I_E(x).$$

Definition 2.2. A multivalued mapping $T:E\to F(X)$ is said to be inward on E if for some $p\in E$,

$$\alpha p\oplus(1-\alpha)Tx\subset I_E(x)\quad\forall x\in E,\ \forall\alpha\in[0,1],$$

and weakly inward on E if

$$\alpha p\oplus(1-\alpha)Tx\subset\overline{I_E(x)}\quad\forall x\in E,\ \forall\alpha\in[0,1], \tag{2.2}$$

where $\bar{A}$ denotes the closure of a subset A of X.

When E is convex, it is easy to see that

$$I_E(x)=\Big(\bigcup\{[x,y]\colon (x,y]\cap E\neq\emptyset\}\Big)\cup\{x\}.$$

Note that in a normed space setting, the inward (respectively, weakly inward) condition is equivalent to saying that $Tx\subset I_E(x)$ (respectively, $Tx\subset\overline{I_E(x)}$) since in this case, $I_E(x)$ is convex. This is also true for $\mathbb{R}$-trees.

3. Lim's theorems

The following simple result is needed.

Proposition 3.1. *Let E be a nonempty closed convex subset of X, $x \in X$, and $p(x)$ the unique nearest point of x in E. Then*

$$d(x, p(x)) < d(x, y) \quad \forall y \in \overline{I_E(p(x))} \setminus \{p(x)\}.$$

Proof. Let $y \in \overline{I_E(p(x))} \setminus \{p(x)\}$, there is a sequence (y_n) in $I_E(p(x))$ and $y_n \to y$. For all large n we can find $z_n \in (p(x), y_n] \cap E$. Since $z_n \in E$ and $z_n \neq p(x)$, $\angle_{p(x)}(x, z_n) \geqslant \frac{\pi}{2}$ (see [2, p. 176]). Thus in the comparison triangle $\bar{\Delta}(p(x), x, y_n)$, the angle at $\overline{p(x)}$ is also greater than or equal to $\frac{\pi}{2}$ (see [2, p. 161]). By the law of cosines,

$$d(x, p(x))^2 + d(y_n, p(x))^2 \leqslant d(x, y_n)^2.$$

Taking $n \to \infty$, we obtain

$$d(x, p(x)) < d(x, y). \qquad \square$$

One of powerful tools for fixed point theory is the following result.

Theorem 3.2 (J. Caristi [3]). *Assume (M, d) is a complete metric space and $g : M \to M$ is a mapping. If there exists a lower semicontinuous function $\psi : M \to [0, \infty)$ such that*

$$d(x, g(x)) \leqslant \psi(x) - \psi(g(x)) \quad \text{for any } x \in M,$$

then g has a fixed point.

We can now state our main theorem.

Theorem 3.3. *Let E be a nonempty bounded closed convex subset of X and $T : E \to K(X)$ a nonexpansive mapping. Assume T is weakly inward on E. Then T has a fixed point.*

By combining the idea of the proofs in [1,8,9], we thus first establish the following lemma. However, in applying the lemma, we choose to use the ultrapower technique which seems to be alternative.

Lemma 3.4. *Let E be a nonempty closed subset of X and $T : E \to F(X)$ k-contractive for some $k \in [0, 1)$. Assume T satisfies, for all $x \in E$,*

$$Tx \subset \overline{I_E(x)}. \tag{3.1}$$

Then T has a fixed point.

Proof. Let $M = \{(x, z)\colon z \in Tx,\ x \in E\}$ be the graph of T. Give a metric ρ on M by $\rho((x, z), (u, v)) = \max\{d(x, u), d(z, v)\}$. It is easily seen that (M, ρ) is a complete metric space. Choose $\varepsilon > 0$ so that $\varepsilon + (k + 2\varepsilon)(1 + \varepsilon) < 1$.

Now define $\psi : M \to [0, \infty)$ by $\psi(x, z) = \frac{d(x,z)}{\varepsilon}$. Then ψ is continuous on M. Suppose that T has no fixed points, i.e., $\operatorname{dist}(x, Tx) > 0$ for all $x \in E$. Let $(x, z) \in M$. By (3.1), we can find $z' \in I_E(x)$ satisfying $d(z, z') < \varepsilon \operatorname{dist}(x, Tx)$. Now choose $u \in (x, z'] \cap E$ and write $u = (1-\delta)x \oplus \delta z'$ for some $0 < \delta \leqslant 1$. Note that the number δ varies as a function of x. However, for any such δ, we always have

$$\delta\varepsilon + (1-\delta) + (k+2\varepsilon)\delta(1+\varepsilon) < 1. \tag{3.2}$$

Since T is k-contractive and $d(x,u) > 0$, we can find $v \in Tu$ satisfying

$$d(z,v) \leqslant H(Tx,Tu) + \varepsilon d(x,u) \leqslant (k+\varepsilon)d(x,u).$$

Now we define a mapping $g : M \to M$ by $g(x,z) = (u,v)$ $\forall (x,z) \in M$. We claim that g satisfies

$$\rho\big((x,z), g(x,z)\big) < \psi(x,z) - \psi\big(g(x,z)\big) \quad \forall (x,z) \in M. \tag{3.3}$$

Caristi's theorem then implies that g has a fixed point, which contradicts to the strict inequality (3.3) and the proof is complete.

So it remains to prove (3.3). In fact, it is enough to show that

$$\rho\big((x,z),(u,v)\big) < \frac{1}{\varepsilon}\big(d(x,z) - d(u,v)\big).$$

But $d(z,v) \leqslant d(x,u)$, and we only need to prove that $d(x,u) < \frac{1}{\varepsilon}(d(x,z) - d(u,v))$.

Now,

$$\begin{aligned} d(x,u) &= \delta d(x,z') \leqslant \delta\big(d(x,z) + d(z,z')\big) \leqslant \delta\big(d(x,z) + \varepsilon \operatorname{dist}(x,Tx)\big) \\ &\leqslant \delta\big(d(x,z) + \varepsilon d(x,z)\big) \leqslant \delta(1+\varepsilon)d(x,z). \end{aligned}$$

Therefore

$$d(x,u) \leqslant \delta(1+\varepsilon)d(x,z). \tag{3.4}$$

It follows that

$$d(z,v) \leqslant (k+\varepsilon)d(x,u) \leqslant (k+\varepsilon)\delta(1+\varepsilon)d(x,z).$$

Now we let $y = (1-\delta)x \oplus \delta z$, then

$$\begin{aligned} d(u,v) &\leqslant d(u,y) + d(y,z) + d(z,v) \\ &\leqslant \delta d(z,z') + (1-\delta)d(x,z) + (k+\varepsilon)\delta(1+\varepsilon)d(x,z) \\ &\leqslant \delta\varepsilon \operatorname{dist}(x,Tx) + \big((1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z) \\ &\leqslant \delta\varepsilon d(x,z) + \big((1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z) \\ &\leqslant \big(\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z). \end{aligned}$$

Thus

$$d(u,v) \leqslant \big(\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z). \tag{3.5}$$

Inequalities (3.4), (3.5), and (3.2) imply that

$$\begin{aligned} \varepsilon d(x,u) + d(u,v) &\leqslant \varepsilon\delta(1+\varepsilon)d(x,z) + (\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon))d(x,z) \\ &= (\delta\varepsilon + (1-\delta) + (k+2\varepsilon)\delta(1+\varepsilon))d(x,z) < d(x,z). \end{aligned}$$

Therefore $d(x,u) < \frac{1}{\varepsilon}(d(x,z) - d(u,v))$ as desired. □

We are now ready to present the proof of Theorem 3.3.

Proof of Theorem 3.3. For each integer $n \geqslant 1$, the contraction $T_n : E \to K(X)$ is defined by

$$T_n(x) := \frac{1}{n} p \oplus \left(1 - \frac{1}{n}\right) Tx, \quad x \in E,$$

where $p \in E$ is the existing point satisfying the weakly inward condition (2.2). Weak inwardness of T implies that such T_n satisfies the condition (3.1) in Lemma 3.4 and in turn it guarantees that T_n has a fixed point $x_n \in E$. Clearly,

$$\operatorname{dist}(x_n, Tx_n) \leqslant \frac{1}{n-1} \operatorname{diam}(E) \to 0.$$

Let $\tilde{X}$ be a metric space ultrapower of X and

$$\dot{E} = \{\dot{x} = \widetilde{(x_n)}: x_n \equiv x \in E\}.$$

Then $\dot{E}$ is a nonempty closed convex subset of $\tilde{X}$. Since T is compact-valued, we can take $y_n \in Tx_n$ such that

$$d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n), \quad n \geqslant 1.$$

This implies $\widetilde{(x_n)} = \widetilde{(y_n)}$. Since $\dot{E}$ is a closed convex subset of a complete CAT(0) space $\tilde{X}$, $\widetilde{(x_n)}$ has a unique nearest point $\dot{v} \in \dot{E}$, i.e., $\tilde{d}(\widetilde{(x_n)}, \dot{v}) = \operatorname{dist}(\widetilde{(x_n)}, \dot{E})$. As Tv is compact, we can find $v_n \in Tv$ satisfying

$$d(y_n, v_n) = \operatorname{dist}(y_n, Tv) \leqslant H(Tx_n, Tv).$$

It follows from the nonexpansiveness of T that

$$d(y_n, v_n) \leqslant d(x_n, v).$$

This means

$$\tilde{d}\big(\widetilde{(y_n)}, \widetilde{(v_n)}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{v}\big).$$

Since $\widetilde{(x_n)} = \widetilde{(y_n)}$, we have

$$\tilde{d}\big(\widetilde{(x_n)}, \widetilde{(v_n)}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{v}\big). \tag{3.6}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathcal{U}} v_n$. It follows that $\widetilde{(v_n)} = \dot{w}$. This fact and (3.6) imply

$$\tilde{d}\big(\widetilde{(x_n)}, \dot{w}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{v}\big). \tag{3.7}$$

Since $\dot{w} \in \overline{I_{\dot{E}}(\dot{v})}$ as $w \in \overline{I_E(v)}$, (3.7), and Proposition 3.1 then imply that $\dot{w} = \dot{v}$. So $v = w \in Tv$ which then completes the proof. □

As an immediate consequence of Theorem 3.3, we obtain

Corollary 3.5. *Let E be a nonempty bounded closed convex subset of X and $T : E \to K(E)$ a nonexpansive mapping. Then T has a fixed point.*

As we have observed at the end of Definition 2.2, we can restate Theorem 3.3 for $\mathbb{R}$-trees as follows.

Corollary 3.6. *Let X be a complete $\mathbb{R}$-tree, E a nonempty bounded closed convex subset of X, and $T : E \to K(X)$ a nonexpansive mapping. Assume that $Tx \subset \overline{I_E(x)}$ $\forall x \in E$. Then T has a fixed point.*

Finally, as a consequence of Kirk [4, Theorem 4.3] and the idea given in the proof of Theorem 3.3, we can relax the boundedness condition and the compactness of the values of a multivalued self mapping T for $\mathbb{R}$-trees.

Corollary 3.7. *Let (X, d) be a complete $\mathbb{R}$-tree, and suppose E is a closed convex subset of X which does not contain a geodesic ray, and $T : E \to FC(E)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. By [2, p. 176], for each $x \in E$, there exists a unique point $p(x) \in Tx$ such that

$$d(x, p(x)) = \operatorname{dist}(x, Tx).$$

So we have defined a mapping $p : E \to E$. The nonexpansiveness of T and the convexity of Tx imply that p is a nonexpansive mapping. By [4, Theorem 4.3], there exists $z \in E$ such that $z = p(z) \in Tz$ which then completes the proof. □

4. A common fixed point theorem

We consider in this section a common fixed point of nonexpansive mappings. Let $t : E \to E$ and $T : E \to 2^X \setminus \emptyset$. t and T are said to be commuting if $ty \in Ttx$ $\forall y \in Tx$, $\forall x \in E$. If E is a nonempty bounded closed convex subset of X and t is nonexpansive, we know that $\operatorname{Fix}(t)$ is a nonempty bounded closed convex subset of E (see [7, Theorem 12]).

Theorem 4.1. *Let E be a nonempty bounded closed convex subset of X, and let $t : E \to E$ and $T : E \to KC(X)$ be nonexpansive. Assume that for some $p \in \operatorname{Fix}(t)$,*

$$\alpha p \oplus (1 - \alpha)Tx \text{ convex } \forall x \in E,\ \forall \alpha \in [0, 1]. \tag{4.1}$$

If t and T are commuting, then there exists a point $z \in E$ such that $tz = z \in Tz$.

Proof. Let $A = \operatorname{Fix}(t)$. Since $ty \in Ttx = Tx$ for each $x \in A$ and $y \in Tx$, Tx is invariant under t for each $x \in A$, and again by [7, Theorem 12], $Tx \cap A \neq \emptyset$.

Let $\tilde{X}$ be an ultrapower of X and let $p \in A$ satisfying (4.1). As before we define for each $n \geqslant 1$ the contraction $T_n : A \to KC(X)$ by

$$T_n(x) := \frac{1}{n} p \oplus \left(1 - \frac{1}{n}\right) Tx, \quad x \in A.$$

Convexity of A implies $T_n(x) \cap A \neq \emptyset$. Lemma 4.2 below shows that T_n has a fixed point $x_n \in A$. Let y_n be the unique point in Tx_n such that $d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n)$. Thus $\widetilde{(x_n)} = \widetilde{(y_n)}$ since $\operatorname{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$.

Now,

$$d(x_n, ty_n) = d(tx_n, ty_n) \leqslant d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n).$$

Since $y_n \in Tx_n$, we have $ty_n \in Ttx_n = Tx_n$ and thus the uniqueness of y_n implies that $ty_n = y_n$. So $y_n \in Tx_n \cap A$. Since $\dot{A}$ is a closed convex subset of the complete CAT(0) space $\tilde{X}$, there exists a unique point $\dot{z} \in \dot{A}$ such that

$$\tilde{d}\big(\widetilde{(x_n)}, \dot{z}\big) = \operatorname{dist}\big(\widetilde{(x_n)}, \dot{A}\big).$$

For each n there exists a unique point $z_n \in Tz$ such that

$$d(y_n, z_n) = \operatorname{dist}(y_n, Tz).$$

As before we see that $z_n \in Tz \cap A$. By the compactness of $Tz \cap A$, we can find $w \in Tz \cap A$ such that $\lim_{\mathcal{U}} z_n = w$. It follows that $\widetilde{(z_n)} = \dot{w}$.

Observe that

$$d(y_n, z_n) = \operatorname{dist}(y_n, Tz) \leqslant H(Tx_n, Tz) \leqslant d(x_n, z).$$

Therefore $\tilde{d}(\widetilde{(y_n)}, \widetilde{(z_n)}) \leqslant \tilde{d}(\widetilde{(x_n)}, \dot{z})$. Since $\widetilde{(y_n)} = \widetilde{(x_n)}$ and $\widetilde{(z_n)} = \dot{w}$,

$$\tilde{d}\big(\widetilde{(x_n)}, \dot{w}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{z}\big) = \operatorname{dist}\big(\widetilde{(x_n)}, \dot{A}\big).$$

The uniqueness of $\dot{z}$ implies that $\dot{w} = \dot{z}$. Therefore $tz = z = w \in Tz$ as desired. □

It remains to prove our lemma.

Lemma 4.2. *Let A be as above and $T : A \to FC(X)$ be k-contractive for some $k \in [0, 1)$. Assume that T satisfies, for all $x \in A$,*

$$Tx \cap A \neq \emptyset.$$

Then T has a fixed point.

Proof. The proof is similar to the proof of Lemma 3.4. Let $M = \{(x, z): z \in Tx \cap A, x \in A\}$ and define a metric ρ on M by $\rho((x, z), (u, v)) = \max\{d(x, u), d(z, v)\}$. Again (M, ρ) is a complete metric space. Choose $\varepsilon > 0$ so that $\varepsilon + k < 1$.

Define $\psi : M \to [0, \infty)$ by $\psi(x, z) = \frac{d(x,z)}{\varepsilon}$. Suppose that $x \neq z$ for all $(x, z) \in M$. Since Tz is a closed convex subset of X, there exists a unique point $v \in Tz$ such that

$$d(z, v) = \operatorname{dist}(z, Tz).$$

Bearing in mind that $A = \operatorname{Fix}(t)$, thus by the commuting assumption and the uniqueness of v, we have $v \in Tz \cap A$.

Now we define a mapping $g : M \to M$ by $g(x, z) = (z, v)$ for each $(x, z) \in M$. We claim that g satisfies

$$\rho\big((x, z), g(x, z)\big) < \psi(x, z) - \psi\big(g(x, z)\big) \quad \forall (x, z) \in M. \tag{4.2}$$

Again by applying the Caristi's theorem we obtain a contradiction. Thus T has a fixed point.

So it remains to prove (4.2). From the fact that $d(z,v) = \operatorname{dist}(z,Tz) \leqslant H(Tx,Tz) \leqslant kd(x,z)$, we have

$$\varepsilon d(x,z) + d(z,v) \leqslant \varepsilon d(x,z) + kd(x,z) = (\varepsilon + k)d(x,z) < d(x,z).$$

Therefore $\rho((x,z),(z,v)) < \frac{1}{\varepsilon}(d(x,z) - d(z,v))$, and (4.2) is verified. □

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Appendix 5: The Dominguez-Lorenzo condition and multivalued nonexpansive mappings, Nonlinear Anal. 64 (2006), 958-970.

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The Domínguez–Lorenzo condition and multivalued nonexpansive mappings☆

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Abstract

Let E be a nonempty bounded closed convex separable subset of a reflexive Banach space X which satisfies the Domínguez–Lorenzo condition, i.e., an inequality concerning the asymptotic radius of a sequence and the Chebyshev radius of its asymptotic center. We prove that a multivalued nonexpansive mapping $T : E \to 2^X$ which is compact convex valued and such that $T(E)$ is bounded and satisfies an inwardness condition has a fixed point. As a consequence, we obtain a fixed-point theorem for multivalued nonexpansive mappings in uniformly nonsquare Banach spaces which satisfy the property WORTH, extending a known result for the case of nonexpansive single-valued mappings. We also prove a common fixed point theorem for two nonexpansive commuting mappings $t : E \to E$ and $T : E \to KC(E)$ (where $KC(E)$ denotes the class of all compact convex subsets of E) when X is a uniformly convex Banach space.

Keywords: Multivalued nonexpansive mapping; Inwardness condition; Uniform convexity; Non-strict opial condition; Property WORTH; James constant; Uniform nonsquareness

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1. Introduction

One of the most celebrated results about multivalued mappings was given by T.C. Lim [17] in 1974. By using Edelstein's method of asymptotic centers, he proved that every multivalued nonexpansive self-mapping $T : E \to K(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. In 1990, W.A. Kirk and S. Massa [16] proved that if a nonempty bounded closed convex subset E of a Banach space X has a property that the asymptotic center in E of each bounded sequence of X is nonempty and compact, then every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point. In 2001, H.K. Xu [23] extended Kirk and Massa's theorem to a non-self-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

Recently, Domínguez and Lorenzo [10] proved that every nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they give an affirmative answer to problem 6 in [22] which states that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X. Furthermore, they [9] proved that if $T : E \to KC(X)$ is a nonexpansive and $1 - \chi$-contractive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition where E is a nonempty bounded closed convex separable subset of a Banach space X with $\varepsilon_\beta(X) < 1$, then T has a fixed point.

By investigating the proofs in [9] and [10], we observe that the main tool that is used in their proofs is a relationship between the Chebyshev radius of the asymptotic center of a bounded sequence in E and the modulus of noncompact convexity of a Banach space associated with the measure of noncompactness. In this paper, we define the Domínguez–Lorenzo condition and prove that every reflexive Banach space X satisfying the Domínguez–Lorenzo condition and every nonempty bounded closed convex separable subset E of X, every nonexpansive and $1 - \chi$-contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set, and which satisfies the inwardness condition has a fixed point. The main idea of the proof comes from the proofs of Theorems 3.4 and 3.6 in [9]. We also prove that a uniformly nonsquare Banach space X satisfying property WORTH is one of the examples of Banach spaces that satisfy the Domínguez–Lorenzo condition. Moreover, we show that every Banach space which satisfies the Domínguez–Lorenzo condition has a weak normal structure.

Finally, we use a theorem of Deimling [7] to obtain a common fixed point for nonexpansive commuting mappings $t : E \to E$ and $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space.

2. Preliminaries

Let X be a Banach space and E a nonempty subset of X. We shall denote the family of nonempty bounded closed subsets of E by $FB(E)$, the family of nonempty compact subsets of E by $K(E)$, the family of nonempty closed convex subsets of E by $FC(E)$ and the family of nonempty compact convex subsets of E by $KC(E)$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\}, \quad A, B \in FB(X),$$

where $\operatorname{dist}(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B. A multivalued mapping $T : E \to F(X)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leqslant k\|x - y\|, \quad x, y \in E. \tag{1}$$

In this case, we also say that T is k-contractive.

If (1) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$.

Recall that the Kuratowski, separation, and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are, respectively, defined as the numbers:

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameters } \leqslant d\},$$

$$\beta(B) = \sup\{\varepsilon > 0 : \text{ there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \geqslant \varepsilon\},$$

where $sep(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\}$,

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radii } \leqslant d\}.$$

A multivalued mapping $T : E \to 2^X$ is called ϕ-condensing (resp. $1 - \phi$-contractive) where ϕ is a measure of noncompactness if, for each bounded subset B of E with $\phi(B) > 0$, there holds the inequality

$$\phi(T(B)) < \phi(B) (\text{resp. } \phi(T(B)) \leqslant \phi(B)).$$

Here $T(B) = \cup_{x\in B} Tx$.

Before stating our main theorem we need the following results.

Definition 2.1. Let X be a Banach space and $\phi = \alpha$, β or χ. The modulus of noncompact convexity associated with ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf\{1 - \operatorname{dist}(0, A) : A \subset B_X \text{ is convex}, \ \phi(A) \geqslant \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geqslant 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The relationships among the different moduli are

$$\Delta_{X,\alpha}(\varepsilon) \leqslant \Delta_{X,\beta}(\varepsilon) \leqslant \Delta_{X,\chi}(\varepsilon),$$

and consequently,

$$\varepsilon_\alpha(X) \geqslant \varepsilon_\beta(X) \geqslant \varepsilon_\chi(X).$$

See [2] for all these and more details.

Definition 2.2. (a) X is said to satisfy property WORTH [20] if for any $x \in X$ and any weakly null sequence $\{x_n\}$ in X,

$$\limsup_{n\to\infty} \|x_n + x\| = \limsup_{n\to\infty} \|x_n - x\|.$$

(b) X is said to satisfy the Opial condition [18] if, whenever a sequence $\{x_n\}$ in X converges weakly to x, then for $y \neq x$,

$$\limsup_{n} \|x_n - x\| < \limsup_{n} \|x_n - y\|.$$

If the inequality is non-strict, we say that X satisfies the non-strict Opial condition.

It is known that if X satisfies property WORTH, then X satisfies the non-strict Opial condition [12].

Definition 2.3. Let E be a nonempty closed subset of a Banach space X. The inward set of E at $x \in E$ is given by

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geqslant 1,\ y \in E\}.$$

In case E is a nonempty closed convex subset of a Banach space X, we have

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geqslant 0,\ y \in E\}.$$

A multivalued mapping $T : E \to 2^X$ is said to be inward (resp. weakly inward) on E if

$$Tx \subset I_E(x) \text{ (resp. } Tx \subset \overline{I_E(x)}) \text{ for all } x \in E.$$

Our proofs heavily rely on the following result.

Theorem 2.4 (*Deimling [7]*). *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \to FC(X)$ an upper semicontinuous χ-condensing mapping. Assume $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$. Then T has a fixed point.*

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded closed convex subset of X and $\{x_n\}$ a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r(E, \{x_n\}) = \inf \left\{ \limsup_n \|x_n - x\| : x \in E \right\},$$

$$A(E, \{x_n\}) = \left\{ x \in E : \limsup_n \|x_n - x\| = r(E, \{x_n\}) \right\}.$$

It is known that $A(E, \{x_n\})$ is a nonempty weakly compact convex set as E is [14].

Definition 2.5. Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; further, $\{x_n\}$ is called asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.6 (*Goebel [13], Lim [17]*). *Let $\{x_n\}$ and E be as above. Then*

(i) *there always exists a subsequence of $\{x_n\}$ which is regular relative to E; and*
(ii) *if E is seperable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E.*

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\{r_x(C) : x \in E\},$$

where $r_x(C) = \sup\{\|x - y\| : y \in C\}$.

Theorem 2.7 (*Domínguez [8]*). *Let E be a closed convex subset of a reflexive Banach space X and let $\{x_n\}$ be a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E, \{x_n\})) \leqslant (1 - \Delta_{X,\beta}(1^-)) r(E, \{x_n\}).$$

Moreover, if X satisfies the non-strict Opial condition, then

$$r_E(A(E, \{x_n\})) \leqslant (1 - \Delta_{X,\chi}(1^-)) r(E, \{x_n\}).$$

Using Theorem 2.7 as the main tool, Domínguez and Lorenzo [9] proved the following theorem:

Theorem 2.8 (*Dominguez [9, Theorem 3.6]*). *Let X be a Banach space with $\varepsilon_\beta(X) < 1$. Assume that $T : E \to KC(X)$ is a nonexpansive and $1 - \chi$-contractive mapping such that $T(E)$ is a bounded set, and which satisfies the inwardness condition where E is a nonempty bounded closed convex separable subset of X. Then T has a fixed point.*

Moreover, they [10] used the same tool to solve the open problem in [22] on the existence of a fixed point of a multivalued nonexpansive self-mapping $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X.

We now present a formulation of an ultrapower of Banach spaces.

Let $\mathscr{U}$ be a free ultrafilter on the set of natural numbers. Consider the closed linear subspace of $l_\infty(X)$:

$$\mathcal{N} = \left\{ \{x_n\} \in l_\infty(X) : \lim_{\mathscr{U}} \|x_n\| = 0 \right\}.$$

The ultrapower $\widetilde{X}$ of the space X is defined as the quotient space $l_\infty(X)/\mathcal{N}$. Given an element $x = \{x_n\} \in l_\infty(X)$, $\widetilde{x}$ stands for the equivalence class of x. The quotient norm in $\widetilde{X}$ satisfies $\|\widetilde{x}\| = \lim_{\mathscr{U}} \|x_n\|$. For more details on the construction of an ultrapower of a Banach space X, see [1] and [19].

3. Fixed-point theorems

Definition 3.1. A Banach space X is said to satisfy the Domínguez–Lorenzo condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E(A(E, \{x_n\})) \leqslant \lambda r(E, \{x_n\}). \tag{2}$$

We are going to prove that every Banach space satisfying the Domínguez–Lorenzo condition possesses a weak normal structure. A Banach space X is said to have a weak normal structure if any weakly compact convex subset E of X for which $\operatorname{diam}(E) > 0$ contains a point x_0 for which

$$r_{x_0}(E) < \operatorname{diam}(E).$$

Theorem 3.2. *Let X be a Banach space satisfying the Domínguez–Lorenzo condition. Then X has a weak normal structure.*

Proof. Suppose on the contrary that X does not have a weak normal structure. Thus, there exists a weakly null sequence $\{x_n\}$ in B_X such that for $C := \overline{co}\{x_n : n \geqslant 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam}(C) = 1 \text{ for all } x \in C$$

(cf. [21]). By passing through a subsequence, we may assume that $\{x_n\}$ is regular. It is easy to see that $r(C, \{x_n\}) = 1$, $A(C, \{x_n\}) = C$, and $r_C(A(C, \{x_n\})) = r_C(C) = 1$. Since X satisfies the Domínguez–Lorenzo condition with a corresponding $\lambda \in [0, 1)$, it must be the case that

$$1 = r_C(C) \leqslant \lambda r(C, \{x_n\}) < 1.$$

This leads to a contradiction. □

In view of the above theorem and the well-known Kirk's fixed-point theorem [15], we can conclude that every Banach space X which satisfies the Domínguez–Lorenzo condition has a fixed-point property, i.e., for every weakly compact convex subset E of X, every nonexpansive mapping $T : E \to E$ has a fixed point. Moreover, the next theorem shows that every reflexive Banach space that satisfies the Domínguez–Lorenzo condition has a fixed-point property for certain multivalued nonexpansive mappings.

Theorem 3.3. *Let X be a reflexive Banach space satisfying the Domínguez–Lorenzo condition and let E be a bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi$-contractive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition:*

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then T has a fixed point.

Proof. The main idea of the proof follows from the proofs of Theorems 3.4 and 3.6 in [9]. So here we only give a sketch of the proof. First we obtain a sequence of approximate fixed points $\{x_n\}$ of T in E. By the boundedness of $\{x_n\}$, we can assume that $\{x_n\}$ is regular relative to E. Since X satisfies the Domínguez–Lorenzo condition, we obtain

$$r_E(A) \leqslant \lambda r(E, \{x_n\})$$

for some $\lambda \in [0, 1)$, where $A = A(E, \{x_n\})$.

We can show that the mapping $T : A \to KC(X)$ is nonexpansive, $1-\chi$-contractive, and satisfies the condition

$$Tx \cap I_A(x) \neq \emptyset \quad \text{for all } x \in A.$$

Fix $x_0 \in A$, define $T_n : A \to KC(X)$ by

$$T_n x = \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) Tx, \quad x \in A.$$

It is easy to see that T_n is χ-condensing and

$$T_n x \cap I_A(x) \neq \emptyset \quad \text{for all } x \in A.$$

Hence by Theorem 2.4, T_n has a fixed point. Consequently, we obtain a sequence $\{x_n^1\}$ in A satisfying $\lim_n \operatorname{dist}(x_n^1, Tx_n^1) = 0$. Now we can proceed with the proof as in the proofs of Theorems 3.4 and 3.6 in [9] to obtain a fixed point. □

From Theorem 2.7 it can be seen that every Banach space X with $\varepsilon_\beta(X) < 1$ satisfies the Domínguez–Lorenzo condition. We now present some other Banach spaces which satisfy the Domínguez–Lorenzo condition. Here we consider the James constant or the nonsquare constant $J(X)$.

For a Banach space X, the James constant, or the nonsquare constant is defined by Gao and Lau [11] as

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}.$$

Clearly, X is uniformly nonsquare if and only if $J(X) < 2$.

Theorem 3.4. *Let X be a Banach space satisfying property WORTH and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E,\{x_n\})) \leqslant \frac{J(X)}{2} r(E,\{x_n\}).$$

Proof. Denote $r = r(E,\{x_n\})$ and $A = A(E,\{x_n\})$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can assume, by passing through a subsequence if necessary, that x_n converges weakly to some element in E, say x. It should be noted that passing through a subsequence of $\{x_n\}$ does not have any effect on the asymptotic radius of the whole sequence $\{x_n\}$ because $\{x_n\}$ is regular. Let us observe here that for any subsequence $\{y_n\}$ of $\{x_n\}$, $r_E(A(E,\{x_n\})) \leqslant r_E(A(E,\{y_n\}))$. This observation will be needed at the end of the proof. Since X satisfies property WORTH, it satisfies the non-strict Opial condition, and thus it must be the case that $x \in A$, that is,

$$\limsup_{n\to\infty} \|x_n - x\| = r. \tag{3}$$

Now let $z \in A$. Thus $\limsup_{n\to\infty}\|x_n - z\| = r$. By regularity of $\{x_n\}$, we can choose a subsequence $\{x_{n'}\}$ of $\{x_n\}$ so that $\lim_{n'\to\infty}\|x_{n'} - x\| = r = \lim_{n'\to\infty}\|x_{n'} - z\|$. Property WORTH and the fact that $x_{n'} - x \overset{w}{\to} 0$ yield the following:

$$\begin{aligned} r &= \lim_{n'\to\infty} \|x_{n'} - z\| \\ &= \lim_{n'\to\infty} \|(x_{n'} - x) + (x - z)\| \\ &= \lim_{n'\to\infty} \|(x_{n'} - x) - (x - z)\| \\ &= \lim_{n'\to\infty} \|x_{n'} - 2x + z\|. \end{aligned}$$

Thus we have

$$\lim_{n'\to\infty} \left\|\frac{x_{n'} - z}{r}\right\| = 1 = \lim_{n'\to\infty} \left\|\frac{x_{n'} - 2x + z}{r}\right\|. \tag{4}$$

Let us consider an ultrapower $\widetilde{X}$ of X. Put

$$\tilde{u} = \frac{1}{r}\{x_{n'} - z\}_{\mathcal{U}} \quad \text{and} \quad \tilde{v} = \frac{1}{r}\{x_{n'} - 2x + z\}_{\mathcal{U}}.$$

By (4) we know that $\tilde{u}, \tilde{v} \in S_{\widetilde{X}}$. We see that

$$\begin{aligned} \|\tilde{u} + \tilde{v}\| &= \lim_{\mathcal{U}} \left\|\frac{1}{r}(x_{n'} - z) + \frac{1}{r}(x_{n'} - 2x + z)\right\| \\ &= \lim_{\mathcal{U}} \left\|\frac{2}{r}(x_{n'} - x)\right\| \\ &= \frac{2}{r} \lim_{\mathcal{U}} \|(x_{n'} - x)\| \\ &= \frac{2}{r}(r) = 2. \end{aligned}$$

On the other hand,

$$\|\tilde{u} - \tilde{v}\| = \lim_{\mathcal{U}} \left\| \frac{1}{r}(x_{n'} - z) - \frac{1}{r}(x_{n'} - 2x + z) \right\|$$
$$= \frac{2}{r}\|x - z\|.$$

Thus

$$J(\widetilde{X}) \geqslant \|\tilde{u} + \tilde{v}\| \wedge \|\tilde{u} - \tilde{v}\|$$
$$= 2 \wedge \frac{2}{r}\|x - z\|$$
$$= \frac{2}{r}\|x - z\|.$$

Since the James constants of X and of $\widetilde{X}$ are the same, we obtain

$$J(X) \geqslant \frac{2}{r}\|x - z\|.$$

This holds for arbitrary $z \in A$. Hence we have

$$r_x(A) \leqslant \frac{J(X)}{2} r,$$

and therefore, by the previous observation, $r_E(A) \leqslant \frac{J(X)}{2} r$. □

From the above theorem we immediately have

Corollary 3.5. *Let X be a uniformly nonsquare Banach space satisfying property WORTH. Then X satisfies the Domínguez–Lorenzo condition.*

Proof. Uniform nonsquareness of X is equivalent to $J(X) < 2$. Put $\lambda = \frac{J(X)}{2}$. Then $\lambda < 1$ and by Theorem 3.4 the result follows. □

Theorem 3.3 and Corollary 3.5 give

Corollary 3.6. *Let X be a uniformly nonsquare Banach space satisfying property WORTH and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition:*

$$Tx \subset I_E(x) \quad \textit{for all } x \in E,$$

then T has a fixed point.

Proof. By Corollary 3.5, X satisfies the Domínguez–Lorenzo condition. It is known that uniform nonsquareness implies reflexivity of X. Since X has a non-strict opial condition, we can conclude that the nonexpansive mapping $T : E \to K(X)$ with a bounded range is $1 - \chi$-contractive (see [9]). Now Theorem 3.3 can be applied to obtain a fixed point. □

Questions. (1) It has been shown in [4, Theorem 3.1] that a Banach space X has a uniform normal structure whenever $J(X) < \frac{1+\sqrt{5}}{2}$. It is natural to ask if the condition of being uniformly nonsquare and property WORTH can be replaced by the condition "$J(X) < \frac{1+\sqrt{5}}{2}$" or some other upper bounds.

(2) A similar question about the Jordan–von Neumann constants can be asked in the sense of (1). Here we ask if we can replace the condition of being uniform nonsquareness and having property WORTH by the condition $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ or some other upper bounds. Note that it has been shown in [5, Theorem 3.16] that a Banach space X has a uniform normal structure whenever $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$.

4. The common fixed points in uniformly convex Banach spaces

In this section, we extend a result of common fixed points for CAT(0) spaces [6, Theorem 4.1] to uniformly convex Banach spaces.

Definition 4.1. Let E be a nonempty bounded closed convex subset of a Banach space X, $t : E \to X$, and $T : E \to FB(X)$. Then t and T are said to be commuting if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds

$$tx \in Tty.$$

Theorem 4.2. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $t : E \to E$, $T : E \to KC(E)$ a single-valued and a multi-valued nonexpansive mapping, respectively. Assume that t and T are commuting. Then t and T have a common fixed point, i.e., there exists a point x in E such that $x = tx \in Tx$.*

Proof. It is known that the fixed point set of t, denoted by $\mathrm{Fix}(t)$, is nonempty, closed, and convex. Let $x \in \mathrm{Fix}(t)$. Since t and T are commuting, we have $ty \in Tx$ for each $y \in Tx$. We see that, for $x \in \mathrm{Fix}(t)$, $Tx \cap \mathrm{Fix}(t) \neq \emptyset$. For a fixed element $x_0 \in \mathrm{Fix}(t)$, define a contraction $T_n : \mathrm{Fix}(t) \to KC(E)$ by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)T(x), \quad x \in \mathrm{Fix}(t).$$

It is easy to see that for each $x \in \mathrm{Fix}(t)$, $T_n x \cap \mathrm{Fix}(t) \neq \emptyset$ as T does.

Since $\mathrm{Fix}(t)$ is a nonexpansive retract of E [3], we can show that $T_n : \mathrm{Fix}(t) \to KC(E)$ is χ-condensing. Indeed, let B be a bounded subset of $\mathrm{Fix}(t)$ and $\chi(B) > 0$. Given $d > 0$ be such that

$$B \subset \bigcup_{i=1}^{n} B(x_i, d), \quad x_i \in E.$$

Let R be a nonexpansive retraction of E onto $\mathrm{Fix}(t)$.

For each $a \in B(x_i, d) \cap B$, we have

$$\|Rx_i - a\| = \|Rx_i - Ra\| \leqslant \|x_i - a\| \leqslant d.$$

Therefore $B(x_i, d) \cap B \subset B(Rx_i, d)$ for each $i \in \{1, \ldots, n\}$, and hence

$$B \subset \bigcup_{i=1}^{n} B(Rx_i, d).$$

Since T_n is $(1 - \frac{1}{n})$-contractive,

$$T_n(B) \subset \bigcup_{i=1}^{n} \left(T_n Rx_i + \left(1 - \frac{1}{n}\right) d B(0, 1) \right).$$

Thus

$$\chi(T_n(B)) \leqslant \left(1 - \frac{1}{n}\right) \chi(B) < \chi(B),$$

and T_n is χ-condensing. Now we can apply Theorem 2.4 to conclude that T_n has a fixed point, say x_n. Moreover, we can show that

$$\operatorname{dist}(x_n, Tx_n) \to 0.$$

Let $\widetilde{X}$ be a Banach space ultrapower of X and

$$\dot{\operatorname{Fix}}(t) = \{\dot{x} = \widetilde{(x_n)} : x_n \equiv x \in \operatorname{Fix}(t)\}.$$

Then $\dot{\operatorname{Fix}}(t)$ is a nonempty closed convex subset of $\widetilde{X}$. Now, for each $n \in \mathbb{N}$, let y_n be the unique nearest point of x_n in Tx_n, i.e., $\|x_n - y_n\| = \operatorname{dist}(x_n, Tx_n)$. Consequently, $\widetilde{(x_n)} = \widetilde{(y_n)}$. Since t is nonexpansive and $x_n \in \operatorname{Fix}(t)$, we have

$$\|x_n - ty_n\| = \|tx_n - ty_n\| \leqslant \|x_n - y_n\|$$

for each $n \in \mathbb{N}$. Since $ty_n \in Tx_n$, we have $y_n = ty_n \in \operatorname{Fix}(t)$ for each $n \in \mathbb{N}$. Since $\dot{\operatorname{Fix}}(t)$ is a closed convex subset of a uniformly convex Banach space $\tilde{X}$, $\widetilde{(x_n)}$ has a unique nearest point $\dot{v} \in \dot{\operatorname{Fix}}(t)$, i.e., $\|\widetilde{(x_n)} - \dot{v}\| = \operatorname{dist}(\widetilde{(x_n)}, \dot{\operatorname{Fix}}(t))$. As Tv is closed and convex, we can find $v_n \in Tv$ satisfying

$$\|y_n - v_n\| = \operatorname{dist}(y_n, Tv) \leqslant H(Tx_n, Tv).$$

We note here that $v_n \in \operatorname{Fix}(t)$ for each n. It follows from the nonexpansiveness of T that

$$\|y_n - v_n\| \leqslant \|x_n - v\|.$$

This means

$$\|\widetilde{(y_n)} - \widetilde{(v_n)}\| \leqslant \|\widetilde{(x_n)} - \dot{v}\|.$$

Since $\widetilde{(x_n)} = \widetilde{(y_n)}$, we have

$$\|\widetilde{(x_n)} - \widetilde{(v_n)}\| \leqslant \|\widetilde{(x_n)} - \dot{v}\|. \tag{5}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathcal{U}} v_n$. It follows that $\widetilde{(v_n)} = \dot{w}$. This fact and (5) imply that

$$\|\widetilde{(x_n)} - \dot{w}\| \leqslant \|\widetilde{(x_n)} - \dot{v}\|. \tag{6}$$

Moreover, $w \in \mathrm{Fix}(t)$ and then $\dot{w} \in \dot{\mathrm{Fix}}(t)$. Hence $\dot{w} = \dot{v}$ and so $v = w \in Tv$ which then completes the proof. ☐

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Appendix 6: Jordan-von Neumann constant and fixed points for multivalued nonexpansive Mappings, J. Math. Anal. Appl. 320 (2006), 916-927.

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The Jordan–von Neumann constants and fixed points for multivalued nonexpansive mappings ☆

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Abstract

The purpose of this paper is to study the existence of fixed points for nonexpansive multivalued mappings in a particular class of Banach spaces. Furthermore, we demonstrate a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ of a Banach space X. Using this fact, we prove that if $C_{NJ}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty weakly compact convex subset of a Banach space X, and $KC(E)$ is the class of all nonempty compact convex subsets of E.

Keywords: Multivalued nonexpansive mapping; Weakly convergent sequence coefficient; Jordan–von Neumann constant; Normal structure; Regular asymptotically uniform sequence

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1. Introduction

In 1969, Nadler [18] established the multivalued version of Banach's contraction principle. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. In 1974, Lim [17], using Edelstein's method of asymptotic centers, proved the existence of a fixed point for a multivalued nonexpansive self-mapping $T : E \to K(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space. In 1990, Kirk and Massa [15] extended Lim's theorem. They proved that every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X for which the asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, Xu [22] extended Kirk–Massa's theorem to nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

In 2004, Domínguez and Lorenzo [10] proved that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they can give an affirmative answer of a problem in [21] proving that every nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space. Recently, Dhompongsa et al. [5], gave an existence of a fixed point for a multivalued nonexpansive and $(1 - \chi)$-contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set and which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of a reflexive Banach space which satisfies the Domínguez–Lorenzo condition, i.e., an inequality concerning the asymptotic radius and the Chebyshev radius of the asymptotic center for some types of sequences. Consequently, they could show that if X is a uniformly nonsquare Banach space satisfying property WORTH and $T : E \to KC(X)$ is a nonexpansive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of X, then T has a fixed point. Furthermore, they also ask: Does $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ imply the existence of a fixed point for multivalued nonexpansive mappings?

In this paper, we organize as follows. We define a property for Banach spaces which we call property (D) (see definition in Section 3), which is weaker than the Domínguez–Lorenzo condition and stronger than weak normal structure and we prove that if X is a Banach space satisfying property (D) and E is a nonempty weakly compact convex subset of X, then every nonexpansive mapping $T : E \to KC(E)$ has a fixed point. Then we state a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ of a Banach space X. Finally, using this fact, we prove that if $C_{NJ}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point. In particular, we give a partial answer to the question which has been asked in [5].

2. Preliminaries

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) := \max\Big\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\Big\}, \quad A,B\in FB(X),$$

where $\operatorname{dist}(a,B) := \inf\{\|a-b\|: b\in B\}$ is the distance from the point a to the subset B. A multivalued mapping $T: E\to FB(X)$ is said to be a contraction if there exists a constant $k\in[0,1)$ such that

$$H(Tx,Ty)\leqslant k\|x-y\|, \quad x,y\in E. \tag{1}$$

If (1) is valid when $k=1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x\in Tx$.

Throughout the paper we let X^* stand for the dual space of a Banach space X. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded subset of X. The number $r(A) := \inf\{\sup_{y\in A}\|x-y\|: x\in A\}$ is called the Chebyshev radius of A. The number $\operatorname{diam}(A) := \sup\{\|x-y\|: x,y\in A\}$ is called the diameter of A. A Banach space X has normal structure (respectively weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

for every bounded closed (respectively weakly compact) convex subset A of X with $\operatorname{diam}(A)>0$. X is said to have uniform normal structure (respectively weak uniform normal structure) if

$$\inf\left\{\frac{\operatorname{diam}(A)}{r(A)}\right\} > 1,$$

where the infimum is taken over all bounded closed (respectively weakly compact) convex subsets A of X with $\operatorname{diam}(A)>0$. The weakly convergent sequence coefficient $WCS(X)$ [3] of X is the number

$$WCS(X) := \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) := \limsup_n\{\|x_i-x_j\|: i,j\geqslant n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) := \inf\{\limsup_n\|x_n-y\|: y\in\overline{\operatorname{co}}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$.

Some equivalent definitions of the weakly convergent sequence coefficient can be found in [2, p. 120] as follows:

$$WCS(X) = \inf\left\{\frac{\lim_{n,m:\,n\neq m}\|x_n-x_m\|}{\lim_{n\to\infty}\|x_n\|}: \{x_n\} \text{ converges weakly to zero}, \lim_{n\to\infty}\|x_n\| \text{ and } \lim_{n,m;\,n\neq m}\|x_n-x_m\| \text{ exist}\right\}$$

and

$$WCS(X) = \inf\left\{ \lim_{n,m:\, n\neq m} \|x_n - x_m\|:\ \{x_n\} \text{ converges weakly to zero, } \|x_n\| = 1 \text{ and } \lim_{n,m:\, n\neq m} \|x_n - x_m\| \text{ exists} \right\}.$$

It is easy to see, from the definition of $WCS(X)$, that $1 \leqslant WCS(X) \leqslant 2$, and it is known that $WCS(X) > 1$ implies X has weak uniform normal structure [3].

For a Banach space X, the Jordan–von Neumann constant $C_{NJ}(X)$ of X, introduced by Clarkson [4], is defined by

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2}:\ x, y \in X \text{ not both zero} \right\}.$$

The constant $R(a, X)$, which is a generalized Garcia-Falset coefficient [12], is introduced by Domínguez [7]: For a given nonnegative real number a,

$$R(a, X) := \sup\left\{ \liminf_n \|x + x_n\| \right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leqslant a$ and all weakly null sequences $\{x_n\}$ in the unit ball of X such that $\lim_{n,m:\, n\neq m} \|x_n - x_m\| \leqslant 1$.

A relationship between the constant $R(1, X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ can be found in [6]:

$$R(1, X) \leqslant \sqrt{2C_{NJ}(X)}.$$

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded closed subset of X and $\{x_n\}$ a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r(E, \{x_n\}) = \inf\left\{ \limsup_{n\to\infty} \|x_n - x\|:\ x \in E \right\},$$

$$A(E, \{x_n\}) = \left\{ x \in E:\ \limsup_{n\to\infty} \|x_n - x\| = r(E, \{x_n\}) \right\}.$$

It is known that $A(E, \{x_n\})$ is a nonempty weakly compact convex set whenever E is [14].

Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{x_n\}$ is called asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform relative to E if $\{x_n\}$ is regular and asymptotically uniform relative to E.

Lemma 2.1. (Goebel [13], Lim [17]) *Let $\{x_n\}$ and E be as above. Then*

(i) *there always exists a subsequence of $\{x_n\}$ which is regular relative to E;*
(ii) *if E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E.*

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\{r_x(C)\colon x \in E\},$$

where $r_x(C) = \sup\{\|x - y\|\colon y \in C\}$.

A last concept which we need to mention is the concept of ultrapowers of Banach spaces. Ultrapowers are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I\colon x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A\colon A \subset I,\ i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the following facts:

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and
(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\prod_{i\in I} X_i$ equipped with the norm $\|\{x_i\}\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \Big\{\{x_i\} \in l_\infty(I, X_i)\colon \lim_{\mathcal{U}} \|x_i\| = 0\Big\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $\{x_i\}_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) and the definition of the quotient norm that

$$\big\|\{x_i\}_{\mathcal{U}}\big\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an ultrapower of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see Aksoy and Khamsi [1] or Sims [19]).

3. Main results

Definition 3.1. A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform relative to E we have

$$r\big(E, \{y_n\}\big) \leqslant \lambda r\big(E, \{x_n\}\big). \tag{2}$$

The Domínguez–Lorenzo condition introduced in [5] is defined as follows:

Definition 3.2. A Banach space X is said to satisfy the Domínguez–Lorenzo condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leqslant \lambda r\big(E, \{x_n\}\big).$$

It is clear from the definition that property (D) is weaker than the Domínguez–Lorenzo condition. In fact, property (D) is strictly weaker than the Domínguez–Lorenzo condition as shown in [8]. The next result shows that property (D) is stronger than weak normal structure.

Theorem 3.3. *Let X be a Banach space satisfying property* (D). *Then X has weak normal structure.*

Proof. Suppose on the contrary that there exists a weakly null sequence $\{x_n\} \subset B_X$ such that $\lim_{n\to\infty} \|x_n - x\| = 1$ for all $x \in C = \overline{\text{co}}(\{x_n\})$ (see [20]). By passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to C. We see that $r(C, \{x_n\}) = 1$ and $A(C, \{x_n\}) = C$. Moreover, $\{x_n\}$ is asymptotically uniform relative to C. Indeed, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$; we have

$$A\big(C, \{x_{n_k}\}\big) = \Big\{x \in C\colon \limsup_{k\to\infty} \|x_{n_k} - x\| = r\big(C, \{x_{n_k}\}\big) = 1\Big\} = C.$$

Since $\{x_n\} \subset C = A(C, \{x_n\})$ and X satisfies property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r\big(C, \{x_n\}\big) \leqslant \lambda r\big(C, \{x_n\}\big)$$

which leads to a contradiction. □

The following results will be very useful in order to prove our main theorem.

Theorem 3.4. (Domínguez and Lorenzo [9]) *Let E be a nonempty weakly compact separable subset of a Banach space X, $T : E \to K(E)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in E such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \quad \forall x \in A := A\big(E, \{z_n\}\big).$$

Theorem 3.5. (Domínguez and Lorenzo [10]) *Let E be a nonempty weakly compact convex separable subset of a Banach space X. Assume that $T : E \to KC(E)$ is a contraction. If A is a closed convex subset of E such that $Tx \cap A \neq \emptyset$ for all $x \in A$, then T has a fixed point in A.*

We can now state our main theorem.

Theorem 3.6. *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies property* (D). *Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.*

Proof. The first part of the proof is similar to the proof of Theorem 4.2 in [9]. Therefore, we only sketch this part of the proof. From [16] we can assume that E is separable. Fix $z_0 \in E$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n}z_0 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E.$$

By Nadler's theorem [18], for any $n \in \mathbb{N}$, T_n has a fixed point, say x_n^1. It is easy to prove that $\lim_{n\to\infty} \operatorname{dist}(x_n^1, Tx_n^1) = 0$. By Lemma 2.1, we can assume that sequence $\{x_n^1\} \subset E$ is a regular asymptotically uniform relative to E. Denote $A_1 = A(E, \{x_n^1\})$. By Theorem 3.4 we can assume that $Tx \cap A_1 \neq \emptyset$ for all $x \in A_1$. Fix $z_1 \in A_1$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n}z_1 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E.$$

Convexity of A_1 implies $T_n(x) \cap A_1 \neq \emptyset$ for all $x \in A_1$. By Theorem 3.5, T_n has a fixed point in A_1, say x_n^2. Consequently, we can get a sequence $\{x_n^2\} \subset A_1$ which is regular asymptotically uniform relative to E and $\lim_{n\to\infty} \operatorname{dist}(x_n^2, Tx_n^2) = 0$. Since X satisfies the property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r(E, \{x_n^2\}) \leqslant \lambda r(E, \{x_n^1\}).$$

By induction, we can find a sequence $\{x_n^k\} \subset A_{k-1} = A(E, \{x_n^{k-1}\})$ which is regular asymptotically uniform relative to E,

$$\lim_{n\to\infty} \operatorname{dist}(x_n^k, Tx_n^k) = 0,$$

and

$$r(E, \{x_n^k\}) \leqslant \lambda r(E, \{x_n^{k-1}\}) \quad \text{for all } k \in \mathbb{N}.$$

Consequently,

$$r(E, \{x_n^k\}) \leqslant \lambda r(E, \{x_n^{k-1}\}) \leqslant \cdots \leqslant \lambda^{k-1} r(E, \{x_n^1\}).$$

We now begin the second part of the proof. In view of [2, p. 48], we may assume that for each $k \in \mathbb{N}$,

$$\lim_{n,m;\, n\neq m} \|x_n^k - x_m^k\| \quad \text{exists},$$

and in addition $\|x_n^k - x_m^k\| < \lim_{n,m;\, n\neq m} \|x_n^k - x_m^k\| + \frac{1}{2^k}$ for all $n, m \in \mathbb{N}$ and $n \neq m$. Let $\{y_n\}$ be the diagonal sequence $\{x_n^n\}$. We claim that $\{y_n\}$ is a Cauchy sequence. For each $n \geqslant 1$, we have for any positive number m,

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leqslant \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - y_{n-1}\| \\
&= \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - x_{n-1}^{n-1}\| \\
&\leqslant \|y_n - x_m^{n-1}\| + \lim_{i,j:\, i\neq j} \|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}}.
\end{aligned}$$

Taking upper limit as $m \to \infty$,

$$\begin{aligned}\|y_n - y_{n-1}\| &\leqslant \limsup_{m\to\infty}\|y_n - x_m^{n-1}\| + \lim_{i,j:\,i\neq j}\|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}} \\ &\leqslant r\big(E,\{x_n^{n-1}\}\big) + \limsup_i\|x_i^{n-1} - y_n\| + \limsup_j\|x_j^{n-1} - y_n\| + \frac{1}{2^{n-1}} \\ &= 3r\big(E,\{x_n^{n-1}\}\big) + \frac{1}{2^{n-1}} \\ &\leqslant 3\lambda^{n-2}r\big(E,\{x_n^1\}\big) + \frac{1}{2^{n-1}}.\end{aligned}$$

Since $\lambda < 1$, we conclude that there exists $y \in E$ such that y_n converges to y. Consequently,

$$\operatorname{dist}(y, Ty) \leqslant \|y - y_n\| + \operatorname{dist}(y_n, Ty_n) + H(Ty_n, Ty) \to 0 \quad \text{as } n \to \infty.$$

Hence y is a fixed point of T. □

Theorem 3.7. *Let E be a nonempty weakly compact convex subset of a Banach space X with*

$$C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

Proof. We will prove that X satisfies property (D). Since $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$, we choose $\lambda = \frac{2\sqrt{C_{\mathrm{NJ}}(X)-1}}{WCS(X)} < 1$. Let D be a nonempty weakly compact convex subset of X, $\{x_n\} \subset D$, and $\{y_n\} \subset A(D, \{x_n\})$ be regular asymptotically uniform sequences relative to D. We will show that (2) is satisfied. By choosing a subsequence, if necessary, we can assume that $\{y_n\}$ converges weakly to $y \in D$ and

$$\lim_{k,j:\,k\neq j}\|y_k - y_j\| = l \quad \text{for some } l \geqslant 0. \tag{3}$$

Let $r = r(D, \{x_n\})$. The condition (2) easily follows when $r = 0$ or $l = 0$. We assume now that $r > 0$ and $l > 0$. Let $\varepsilon > 0$ so small that $0 < \varepsilon < l \wedge r$. From (3) we assume that

$$\big|\|y_k - y_j\| - l\big| < \varepsilon \quad \text{for all } k \neq j. \tag{4}$$

Fix $k \neq j$. Since $y_k, y_j \in A(D, \{x_n\})$ and using the convexity of $A(D, \{x_n\})$, we can assume, passing through a subsequence, that

$$\|x_n - y_k\| < r + \varepsilon, \qquad \|x_n - y_j\| < r + \varepsilon, \tag{5}$$

and

$$\left\|x_n - \frac{y_k + y_j}{2}\right\| > r - \varepsilon \quad \text{for all large } n. \tag{6}$$

By the definition of $C_{\mathrm{NJ}}(X)$, by (4)–(6) we have for n large enough,

$$C_{\mathrm{NJ}}(X) \geqslant \frac{\|2x_n - (y_k + y_j)\|^2 + \|y_k - y_j\|^2}{2\|x_n - y_k\|^2 + 2\|x_n - y_j\|^2} \geqslant \frac{4(r-\varepsilon)^2 + (l-\varepsilon)^2}{4(r+\varepsilon)^2}.$$

Since ε is arbitrarily small, it follows that

$$C_{\mathrm{NJ}}(X) \geqslant \frac{4r^2 + l^2}{4r^2}.$$

Since

$$WCS(X) = \inf\left\{ \frac{\lim_{j,k;\, j\neq k} \|u_j - u_k\|}{\limsup_j \|u_j\|} : u_j \xrightarrow{w} 0,\ \lim_{j,k;\, j\neq k} \|u_j - u_k\| \text{ exists} \right\},$$

we can deduce that

$$C_{\mathrm{NJ}}(X) \geqslant 1 + \frac{WCS(X)^2 (\limsup_n \|y_n - y\|)^2}{4r^2} \geqslant 1 + \frac{WCS(X)^2 r(D, \{y_n\})^2}{4r^2}.$$

Consequently,

$$r\big(D, \{y_n\}\big) \leqslant \frac{2\sqrt{C_{\mathrm{NJ}}(X) - 1}}{WCS(X)} r = \lambda r\big(D, \{x_n\}\big)$$

as desired. □

In order to prove our next result, we need the following theorem which states a relationship between the weakly convergent sequence coefficient and the Jordan–von Neumann constant of a Banach space X.

Theorem 3.8. *For a Banach space X,*

$$\big[WCS(X)\big]^2 \geqslant \frac{2C_{\mathrm{NJ}}(X) + 1}{2[C_{\mathrm{NJ}}(X)]^2}.$$

Proof. Since $C_{\mathrm{NJ}}(X) \leqslant 2$ and the result is obvious if $C_{\mathrm{NJ}}(X) = 2$, we can assume that $C_{\mathrm{NJ}}(X) < 2$. It is known that $C_{\mathrm{NJ}}(X) < 2$ implies X and X^* are reflexive. Put $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$. Let $\{x_n\}$ be a normalized weakly null sequence in X and $d := \lim_{n,m;\, n\neq m} \|x_n - x_m\|$. Consider a sequence $\{f_n\}$ of norm one functionals for which $f_n(x_n) = 1$. Since X^* is reflexive we can assume that $\{f_n\}$ converges weakly to some f in X^*. Let ε be an arbitrary positive number and choose $K \in \mathbb{N}$ large enough so that $|f(x_n)| < \varepsilon$ and $d - \varepsilon \leqslant \|x_n - x_m\| \leqslant d + \varepsilon$ for any $m \neq n$; $m, n \geqslant K$. Then we have

$$\lim_n (f_n - f)(x_K) = 0 \quad \text{and} \quad \lim_n f_K(x_n) = 0.$$

Since $\lim_{n,m;\, n\neq m} \|\frac{x_n - x_m}{d+\varepsilon}\| < 1$ and $\|\frac{x_K}{d+\varepsilon}\| \leqslant 1$, we have, by the definition of $R(1, X)$,

$$\limsup_n \|x_n + x_K\| \leqslant (d + \varepsilon) R(1, X) \leqslant (d + \varepsilon)\sqrt{2C_{\mathrm{NJ}}(X)} = (d + \varepsilon)\alpha.$$

We construct elements of $\widetilde{X}$ and $\widetilde{X}^*$:

$$\tilde{x} = \left\{ \frac{x_n - x_K}{d + \varepsilon} \right\}_{\mathcal{U}}, \qquad \tilde{y} = \left\{ \frac{x_n + x_K}{(d + \varepsilon)\alpha} \right\}_{\mathcal{U}}, \qquad \tilde{f} = \{f_n\}_{\mathcal{U}} \quad \text{and} \quad \tilde{g} = \dot{f}_K.$$

Here $\dot{h}$ denotes an equivalence class of the sequence $\{h_n\}$ such that $h_n \equiv h$ for all $n \in \mathbb{N}$. Clearly $\tilde{x}, \tilde{y} \in B_{\widetilde{X}}$ and $\tilde{f}, \tilde{g} \in S_{\widetilde{X}^*}$. Moreover,

$$\tilde{f}\big(\{x_n\}_{\mathcal{U}}\big) = 1 \quad \text{and} \quad \big|\tilde{f}(\dot{x}_K)\big| = \big|\dot{f}(\dot{x}_K)\big| < \varepsilon.$$

On the other hand,

$$\tilde{g}\big(\{x_n\}_{\mathcal{U}}\big)=0 \quad \text{and} \quad \tilde{g}(x_K^*)=1.$$

Let consider

$$\begin{aligned}\|\tilde{f}-\tilde{g}\| &\geqslant (\tilde{f}-\tilde{g})(\tilde{x})=\tilde{f}(\tilde{x})-\tilde{g}(\tilde{x})\\ &=\frac{1}{d+\varepsilon}\big(\tilde{f}\big(\{x_n\}_{\mathcal{U}}\big)-\tilde{f}(x_K^*)-\big[\tilde{g}\big(\{x_n\}_{\mathcal{U}}\big)-\tilde{g}(x_K^*)\big]\big)\\ &\geqslant \frac{1}{d+\varepsilon}(1-\varepsilon-0+1)=\frac{2-\varepsilon}{d+\varepsilon}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\tilde{f}+\tilde{g}\| &\geqslant (\tilde{f}+\tilde{g})(\tilde{y})=\tilde{f}(\tilde{y})+\tilde{g}(\tilde{y})\\ &=\frac{1}{(d+\varepsilon)\alpha}\big(\tilde{f}\big(\{x_n\}_{\mathcal{U}}\big)+\tilde{f}(x_K^*)+\tilde{g}\big(\{x_n\}_{\mathcal{U}}\big)+\tilde{g}(x_K^*)\big)\\ &\geqslant \frac{1}{(d+\varepsilon)\alpha}(1-\varepsilon+0+1)=\frac{2-\varepsilon}{(d+\varepsilon)\alpha}.\end{aligned}$$

Thus we have

$$\begin{aligned}C_{\mathrm{NJ}}(\widetilde{X}^*) &\geqslant \frac{\|\tilde{f}+\tilde{g}\|^2+\|\tilde{f}-\tilde{g}\|^2}{2\|\tilde{f}\|^2+2\|\tilde{g}\|^2}\\ &\geqslant \frac{\left(\frac{2-\varepsilon}{d+\varepsilon}\right)^2+\left(\frac{2-\varepsilon}{(d+\varepsilon)\alpha}\right)^2}{4}\\ &=\left(\frac{1}{d+\varepsilon}\right)^2\left(\frac{(2-\varepsilon)^2}{4}+\frac{(2-\varepsilon)^2}{4\alpha^2}\right).\end{aligned}$$

Since ε is arbitrary and the Jordan–von Neumann constants of X^*, X, $\widetilde{X}$ and $\widetilde{X}^*$ are all equal, we obtain

$$C_{\mathrm{NJ}}(X)\geqslant\left(\frac{1}{d^2}\right)\left(1+\frac{1}{2C_{\mathrm{NJ}}(X)}\right).$$

Thus

$$\big[WCS(X)\big]^2\geqslant\frac{2C_{\mathrm{NJ}}(X)+1}{2[C_{\mathrm{NJ}}(X)]^2}. \qquad \square$$

Using Theorem 3.8, we obtain the following corollary.

Corollary 3.9. [6, Theorem 3.16] *Let X be a Banach space. If $C_{\mathrm{NJ}}(X)<\frac{1+\sqrt{3}}{2}$, then X and X^* has uniform normal structure.*

Proof. Let $\widetilde{X}$ be a Banach space ultrapower of X. Since $C_{\mathrm{NJ}}(\widetilde{X})=C_{\mathrm{NJ}}(X)$, Theorem 3.8 can be applied to $\widetilde{X}$. The inequality in Theorem 3.8 implies $WCS(\widetilde{X})>1$ if $C_{\mathrm{NJ}}(\widetilde{X})<\frac{1+\sqrt{3}}{2}$. Since $WCS(\widetilde{X})>1$ implies $\widetilde{X}$ has weak normal structure [3] and since $\widetilde{X}$

is reflexive, it must be the case that $\widetilde{X}$ has normal structure. By [11, Theorem 5.2], X has uniform normal structure as desired. □

Using the inequality appearing in Theorem 3.8, and numerical calculus, it is not difficult to see that $C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}$ if $C_{NJ}(X) < c_0 = 1.273\ldots$. Thus we can state:

Corollary 3.10. *Let E be a nonempty bounded closed convex subset of a Banach space X with*

$$C_{NJ}(X) < c_0 = 1.273\ldots.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

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Appendix 7: Fixed point theorems for multivalued mappings in modular function spaces, Scien. Math. Japon. 63 (2) (2006), 161-169.

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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN MODULAR FUNCTION SPACES*

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ABSTRACT. The purpose of this paper is to study the existence of fixed points for multivalued nonexpansive mappings in modular function spaces. We apply our main result to obtain fixed point theorems for multivalued mappings in the Banach spaces L_1 and l_1.

1. Introduction

The theory of modular spaces was initiated by Nakano [15] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see, for instance, [2, 3, 7, 8]). In particular, some fixed point theorems for (singlevalued) nonexpansive mappings in modular function spaces are given in [8]. In 1969, Nadler [14] established the multivalued version of Banach's contraction principle in metric spaces. Since then the metric fixed point theory for multivalued mappings has been rapidly developed and many of papers have appeared proving the existence of fixed points for multivalued nonexpansive mappings in special classes of Banach spaces (see, for instance, [4, 5, 9, 11]). In this paper, we study similar problems in the setting of modular function spaces. Namely, we prove that every ρ−contraction $T : C \to F_\rho(C)$ has a fixed point where ρ is a convex function modular satisfying the Δ_2−type condition and C is a nonempty ρ−bounded ρ−closed subset of L_ρ. By using this result, we can assert the existence of fixed points for multivalued ρ−nonexpansive mappings. Finally, we apply our main result to obtain fixed point theorems in the Banach space L_1 (resp. l_1) for multivalued mappings whose domains are compact in the topology of the convergence locally in measure (resp. w^*−topology).

2. Preliminaries

We start by recalling some basic concepts about modular function spaces. For more details the reader is referred to [10, 12].

Let Ω be a nonempty set and Σ be a nontrivial σ−algebra of subsets of Ω. Let $\mathcal{P}$ be a δ−ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ−finite measure space). By $\mathcal{E}$ we denote the

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linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}, |g_n| \le |f|$, and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Let us recall that a set function $\mu : \Sigma \to [0, \infty]$ is called a σ-subadditive measure if $\mu(\emptyset) = 0$, $\mu(A) \le \mu(B)$ for any $A \subset B$ and $\mu(\bigcup A_n) \le \sum \mu(A_n)$ for any sequence of sets $\{A_n\} \subset \Sigma$. By 1_A we denote the characteristic function of the set A.

Definition 2.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if :

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,
(P_2) $\rho(f, E) \le \rho(g, E)$ whenever $|f(\omega)| \le |g(\omega)|$ for any $\omega \in \Omega, f, g \in \mathcal{E}$, and $E \in \Sigma$,
(P_3) $\rho(f, .) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,
(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,
(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,
(P_6) for any $\alpha > 0$, $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \subset \mathcal{P}$ and decreases to $\emptyset$.

A σ-subadditive measure ρ is said to be additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$ whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $f \in \mathcal{M}$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup \{\rho(g, E) : g \in \mathcal{E}, |g(\omega)| \le |f(\omega)| \text{ for every } \omega \in \Omega\}.$$

Definition 2.2. A set E is said to be ρ-null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. For example, we will say frequently $f_n \to f$ ρ-a.e.

Note that a countable union of ρ-null sets is still ρ-null. In the sequel we will identify sets A and B whose symmetric difference $A \Delta B$ is ρ-null, similarly we will identify measurable functions which differ only on a ρ-null set.

In the above condition, we define the function $\rho : \mathcal{M} \to [0, \infty]$ by $\rho(f) = \rho(f, \Omega)$. We know from [10] that ρ satisfies the following properties :

(i) $\rho(f) = 0$ if and only if $f = 0$ ρ-a.e.
(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.
(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$.
In addition, if the following property is satisfied
(iii)' $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$,
we say that ρ is a convex modular.

A function modular ρ is called σ-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(1_{K_n}) < \infty$ and $\Omega = \bigcup K_n$.

The modular ρ defines a corresponding modular space L_ρ, which is given by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an F-norm. Recall that a functional $\|\cdot\| : X \to [0, \infty]$ defines an F-norm on a linear space X if and only if

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|\alpha x\| = \|x\|$ whenever $|\alpha| = 1$,
(3) $\|x + y\| \le \|x\| + \|y\|$,
(4) $\|\alpha_n x_n - \alpha x\| \to 0$ if $\alpha_n \to \alpha$ and $\|x_n - x\| \to 0$.

The modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf \left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

We know from [10] that the linear space $(L_\rho, \|\cdot\|_\rho)$ is a complete metric space. If ρ is convex, the formula

$$\|f\|_l = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

defines a norm which is frequently called the Luxemburg norm. The formula

$$\|f\|_a = \inf\left\{\frac{1}{k}(1 + \rho(kf)) : k > 0\right\}$$

defines a different norm which is called the Amemiya norm. Moreover, $\|\cdot\|_l$ and $\|\cdot\|_a$ are equivalent norms. We can also consider the space

$$E_\rho = \{f \in \mathcal{M} : \rho(\alpha f, \cdot) \text{ is order continuous for all } \alpha > 0\}.$$

Definition 2.3. A function modular ρ is said to satisfy the Δ_2-condition if

$$\sup_{n\geq 1} \rho(2f_n, D_k) \to 0 \text{ as } k \to \infty \text{ whenever } \{f_n\} \subset \mathcal{M}, D_k \in \Sigma$$

$$\text{decreases to } \emptyset \text{ and } \sup_{n\geq 1} \rho(f_n, D_k) \to 0 \text{ as } k \to \infty.$$

It is known that the Δ_2-condition is equivalent to $E_\rho = L_\rho$.

Definition 2.4. A function modular ρ is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \leq K\rho(f)$.

In general, the Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that the Δ_2- type condition implies the Δ_2-condition.

Definition 2.5. Let L_ρ be a modular space.

(1) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.
(2) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega : f_n(\omega) \nrightarrow f(\omega)\}$ is ρ-null.
(3) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.
(4) A subset C of L_ρ is called ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of C always belongs to C.
(5) A subset C of L_ρ is called ρ-compact if every sequence in C has a ρ-convergent subsequence in C.
(6) A subset C of L_ρ is called ρ-a.e. compact if every sequence in C has a ρ-a.e. convergent subsequence in C.
(7) A subset C of L_ρ is called ρ-bounded if

$$\text{diam}_\rho(C) = \sup\{\rho(f - g) : f, g \in C\} < \infty.$$

We know by [10] that under the Δ_2-condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2-type condition.

Definition 2.6. Let ρ be as above. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)} : f \in L_\rho, 0 < \rho(f) < \infty\right\} \text{ for all } 0 \leq t < \infty.$$

The following properties of the growth function can be found in [3].

Lemma 2.7. *Let ρ be as above. Then the growth function ω has the following properties :*

(1) $\omega(t) < \infty$, $\forall t \in [0, \infty)$.
(2) $\omega : [0,\infty) \to [0,\infty)$ *is a convex, strictly increasing function. So, it is continuous.*
(3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$.
(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0,\infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 2.8 (T. Domínguez Benavides et al. [3]). *Let ρ be as above. Then*

$$\|f\|_\rho \leq \frac{1}{\omega^{-1}(1/\rho(f))} \text{ whenever } f \in L_\rho\backslash\{0\}.$$

The following lemma is a technical lemma which will be need because of lack of the triangular inequality.

Lemma 2.9 (T. Domínguez Benavides et al. [3]). *Let ρ be as above, $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then*

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \limsup_{n\to\infty} \rho(f_n + g_n) = \limsup_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \liminf_{n\to\infty} \rho(f_n + g_n) = \liminf_{n\to\infty} \rho(f_n).$$

In the same way as the Hausdorff distance defined on the family of bounded closed subsets of a metric space, we can define the analogue to the Hausdorff distance for modular function spaces. We will call $\rho-$Hausdorff distance even though it is not a metric.

Definition 2.10. Let C be a nonempty subset of L_ρ. We shall denote by $F_\rho(C)$ the family of nonempty $\rho-$closed subsets of C and by $K_\rho(C)$ the family of nonempty $\rho-$compact subsets of C. Let $H_\rho(\cdot,\cdot)$ be the $\rho-$Hausdorff distance on $F_\rho(L_\rho)$, i.e.,

$$H_\rho(A, B) = \max\left\{\sup_{f\in A} \operatorname{dist}_\rho(f, B), \sup_{g\in B} \operatorname{dist}_\rho(g, A)\right\}, \quad A, B \in F_\rho(L_\rho),$$

where $\operatorname{dist}_\rho(f, B) = \inf\{\rho(f-g) : g \in B\}$ is the $\rho-$distance between f and B. A multivalued mapping $T : C \to F_\rho(L_\rho)$ is said to be a $\rho-$contraction if there exists a constant $k \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \leq k\rho(f - g), \quad f, g \in C. \tag{2.1}$$

If (2.1) is valid when $k = 1$, then T is called $\rho-$nonexpansive. A function $f \in C$ is called a fixed point for a multivalued mapping T if $f \in Tf$.

3. Main results

We begin stating the Banach Contraction Principle for multivalued mappings in modular function spaces.

Theorem 3.1. *Let ρ be a convex function modular satisfying the Δ_2-type condition, C a nonempty $\rho-$bounded $\rho-$closed subset of L_ρ, and $T : C \to F_\rho(C)$ a $\rho-$contraction mapping, i.e., there exists a constant $k \in [0, 1)$ such that*

$$H_\rho(Tf, Tg) \leq k\rho(f - g), \quad f, g \in C. \tag{3.1}$$

Then T has a fixed point.

Proof. Let $f_0 \in C$ and $\alpha \in (k,1)$. Since Tf_0 is nonempty, there exists $f_1 \in Tf_0$ such that $\rho(f_0 - f_1) > 0$ (otherwise f_0 is a fixed point of T). In view of (3.1), we have

$$\text{dist}_\rho(f_1, Tf_1) \le H_\rho(Tf_0, Tf_1) \le k\rho(f_0 - f_1) < \alpha\rho(f_0 - f_1).$$

Since $\text{dist}_\rho(f_1, Tf_1) = \inf\{\rho(f_1 - g) : g \in Tf_1\}$, it follows that there exists $f_2 \in Tf_1$ such that

$$\rho(f_1 - f_2) < \alpha\rho(f_0 - f_1).$$

Similarly, there exists $f_3 \in Tf_2$ such that

$$\rho(f_2 - f_3) < \alpha\rho(f_1 - f_2).$$

Continuing in this way, there exists a sequence $\{f_n\}$ in C satisfying $f_{n+1} \in Tf_n$ and

$$\begin{aligned}\rho(f_n - f_{n+1}) &< \alpha\rho(f_{n-1} - f_n)\\ &< \alpha^2(\rho(f_{n-2} - f_{n-1}))\\ &< \dots\\ &< \alpha^{n-1}(\rho(f_1 - f_2))\\ &< \alpha^n(\rho(f_0 - f_1))\\ &\le \alpha^n \text{diam}_\rho(C).\end{aligned}$$

Let $M = \text{diam}_\rho(C)$, then

$$\frac{1}{\alpha^n M} < \frac{1}{\rho(f_n - f_{n+1})}.$$

By Lemma 2.7, we have

$$\left(\omega^{-1}\left(\frac{1}{\alpha}\right)\right)^n \omega^{-1}\left(\frac{1}{M}\right) < \omega^{-1}\left(\frac{1}{\rho(f_n - f_{n+1})}\right).$$

It follows that

$$\frac{1}{\omega^{-1}\left(\frac{1}{\rho(f_n - f_{n+1})}\right)} < \frac{1}{\left(\omega^{-1}\left(\frac{1}{\alpha}\right)\right)^n \omega^{-1}\left(\frac{1}{M}\right)}.$$

By Lemma 2.8, we obtain

$$\|f_n - f_{n+1}\|_\rho < \left(\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)}\right)^n \cdot \frac{1}{\omega^{-1}\left(\frac{1}{M}\right)}.$$

Since ω^{-1} is strictly increasing, we have $\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)} < 1$. This implies that $\{f_n\}$ is a Cauchy sequence in $(L_\rho, \|\cdot\|_\rho)$. Since $(L_\rho, \|\cdot\|_\rho)$ is a complete metric space, there exists $f \in L_\rho$ such that $\{f_n\}$ is $\|\cdot\|_\rho$-convergent to f. Since under the Δ_2-type condition, norm convergence and modular convergence are identical, $\{f_n\}$ is ρ-convergent to f and $f \in C$ because C is ρ-closed. Since $f_n \in Tf_{n-1}$, we have

$$\text{dist}_\rho(f_n, Tf) \le H_\rho(Tf_{n-1}, Tf) \le k\rho(f_{n-1} - f) \longrightarrow 0. \tag{3.2}$$

We observe that, for each n, there exists $g_n \in Tf$ such that

$$\rho(f_n - g_n) \le \text{dist}_\rho(f_n, Tf) + \frac{1}{n}. \tag{3.3}$$

Thus, (3.2) and (3.3) imply that $\lim_{n\to\infty} \rho(f_n - g_n) = 0$. By Lemma 2.9,

$$\limsup_{n\to\infty} \rho(g_n - f) = \limsup_{n\to\infty} \rho(g_n - f_n + f_n - f) = \limsup_{n\to\infty} \rho(f_n - f) = 0.$$

Since Tf is ρ-closed, we can conclude that $f \in Tf$. □

The following results will be very useful in the proof of our main theorem.

Theorem 3.2 (M. A. Khamsi [7]). *Let $\{f_n\} \subset L_\rho$ be ρ–a.e. convergent to 0. Assume there exists $k > 1$ such that*

$$\sup_{n \geq 1} \rho(kf_n) = M < \infty.$$

Let $g \in E_\rho$, then we have

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g).$$

The following lemma guarantees that every nonempty ρ–compact subset of L_ρ attains a nearest point.

Lemma 3.3. *Let ρ be a convex function modular satisfying the Δ_2–type condition, $f \in L_\rho$, and K a nonempty ρ–compact subset of L_ρ. Then there exists $g_0 \in K$ such that*

$$\rho(f - g_0) = dist_\rho(f, K).$$

Proof. Let $m = \text{dist}_\rho(f, K)$. For each $n \in \mathbb{N}$, there exists $g_n \in K$ such that

$$m - \frac{1}{n} \leq \rho(f - g_n) \leq m + \frac{1}{n}.$$

By the ρ–compactness of K, we can assume, by passing through a subsequence, that $g_n \xrightarrow{\rho} g_0 \in K$. By Lemma 2.9, we obtain

$$\begin{aligned} m = \limsup_{n\to\infty} \rho(g_n - f) &= \limsup_{n\to\infty} \rho(g_n - g_0 + g_0 - f) \\ &= \limsup_{n\to\infty} \rho(g_0 - f) \\ &= \rho(g_0 - f). \end{aligned}$$

□

We can now state our main theorem.

Theorem 3.4. *Let ρ be a convex function modular satisfying the Δ_2–type condition, C a nonempty ρ–a.e. compact ρ–bounded convex subset of L_ρ, and $T : C \to K_\rho(C)$ a ρ–nonexpansive mapping. Then T has a fixed point.*

Proof. Fix $f_0 \in C$. For each $n \in \mathbb{N}$, the ρ–contraction $T_n : C \to F_\rho(C)$ is defined by

$$T_n(f) = \frac{1}{n} f_0 + (1 - \frac{1}{n})Tf, \quad f \in C.$$

By Theorem 3.1, we can conclude that T_n has a fixed point, say f_n. It is easy to see that

$$\text{dist}_\rho(f_n, Tf_n) \leq \frac{1}{n}\text{diam}_\rho(C) \longrightarrow 0.$$

Because of ρ–a.e. compactness of C, we can assume, by passing through a subsequence, that $f_n \xrightarrow{\rho-a.e.} f$ for some $f \in C$. By Lemma 3.3, for each $n \in \mathbb{N}$, there exists $g_n \in Tf_n$ and $h_n \in Tf$ such that

$$\rho(f_n - g_n) = \text{dist}_\rho(f_n, Tf_n)$$

and

$$\rho(g_n - h_n) = \text{dist}_\rho(g_n, Tf) \leq H_\rho(Tf_n, Tf) \leq \rho(f_n - f).$$

Because of ρ–compactness of Tf, we can assume, by passing through a subsequence, that $h_n \xrightarrow{\rho} h \in Tf$. Since ρ satisfies the Δ_2–type condition, there exists $K > 0$ such that

$\rho(2(f_n - f)) \le K\rho(f_n - f)$ for all $n \in \mathbb{N}$.
This implies that

$$\sup_{n\ge 1} \rho(2(f_n - f)) \le K \sup_{n\ge 1} \rho(f_n - f) < \infty.$$

By Theorem 3.2 and Lemma 2.9, we obtain

$$\begin{aligned}\liminf_{n\to\infty} \rho(f_n - f) + \rho(f - h) &= \liminf_{n\to\infty} \rho(f_n - f + f - h)\\ &= \liminf_{n\to\infty} \rho(f_n - h)\\ &= \liminf_{n\to\infty} \rho(f_n - g_n + g_n - h_n + h_n - h)\\ &= \liminf_{n\to\infty} \rho(g_n - h_n)\\ &\le \liminf_{n\to\infty} \rho(f_n - f).\end{aligned}$$

It follows that $\rho(f - h) = 0$ and then we have $f = h \in Tf$. □

Consider the space $L_p(\Omega, \mu)$ for a σ-finite measure μ with the usual norm. Let C be a bounded closed convex subset of L_p for $1 < p < \infty$ and $T : C \to K(C)$ a multivalued nonexpansive mapping. Because of uniform convexity of L_p, it is known that T has a fixed point. For $p = 1$, T can fail to have a fixed point even in the singlevalued case for a weakly compact convex set C (see [1]). However, since L_1 is a modular space where $\rho(f) = \int_\Omega |f| d\mu = \|f\|$ for all $f \in L_1$, Theorem 3.4 implies the existence of a fixed point when we define mappings on a ρ-a.e. compact ρ-bounded convex subset of L_1. Thus we can state :

Corollary 3.5. *Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a nonempty bounded convex set which is compact for the topology of the convergence locally in measure, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. Under the above hypothesis, ρ-a.e. compact sets and compact sets in the topology of the convergence locally in measure are identical (see [2]). Consequently, Theorem 3.4 can be applied to obtain a fixed point for T. □

In the case of the space l_1 we also can obtain a bounded closed convex set C and a nonexpansive mapping $T : C \to C$ which is fixed point free. Indeed, consider the following easy and well known example :
Let

$$C = \Big\{ \{x_n\} \in l_1 : 0 \le x_n \le 1 \text{ and } \sum_{n=1}^{\infty} x_n = 1 \Big\}.$$

Define a nonexpansive mapping $T : C \to C$ by

$$T(x) = (0, x_1, x_2, x_3, \ldots) \text{ where } x = \{x_n\}.$$

Then T is a fixed point free. However, if we consider $L_\rho = l_1$ where $\rho(x) = \|x\|$, $\forall x \in l_1$. Then ρ-a.e. convergence and w^*-convergence are identical on bounded subsets of l_1 (see [3]). This fact leads us to obtain the following corollary :

Corollary 3.6. *Let C be a nonempty w^*-compact convex subset of l_1 and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. By the above argument, we know that ρ-a.e. compact bounded sets and w^*-compact sets are identical. Then we can apply Theorem 3.4 to assert the existence of a fixed point of T. □

In fact Corollary 3.5 and 3.6 are consequences of a general result: Assume that X is a linear normed space and τ is a Hausdorff topology on X. We say that X satisfies the strict τ-Opial property if

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for each sequence $\{x_n\}$ in X which converges to x for the topology τ and each $y \neq x$. Following the same argument as in [11] it is easy to prove the following theorem:

Theorem 3.7. *Let X be a Banach space, C a convex bounded sequentially τ-compact subset of X, and $T : C \to K(C)$ a nonexpansive mapping. If X satisfies the strict τ-Opial property, then T has a fixed point.*

When X is a modular function space equipped with either Luxemburg or Amemiya norm, we can consider the topology τ of convergence ρ-a.e. In this case, Theorem 3.7 yields to the following:

Theorem 3.8. *Let ρ be a convex additive $\sigma-$finite function modular satisfying the Δ_2-type condition. Assume that L_ρ is equipped either with Luxemburg or Amemiya norm. Let C be a nonempty $\rho-$a.e. compact $\rho-$bounded convex subset of L_ρ, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. From [6] (Theorem 4.1 and 4.3), X satisfies the uniform Opial property with respect to the topology of ρ-a.e. convergence. Since ρ-a.e. compact sets and ρ-a.e. sequentially compact sets are identical for this topology (see [2]), we can deduce the result from Theorem 3.7 $\square$

Remark. In the case of the space $L^1(\Omega)$ we have

$$\rho(f) = \int_\Omega |f| d\mu = \|f\|_l = \|f\|_a$$

and we can deduce Corollary 3.5 and 3.6 from Theorem 3.8.

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Fixed points of uniformly lipschitzian mappings

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Abstract

Two fixed point theorems for uniformly lipschitzian mappings in metric spaces, due respectively to E. Lifšic and to T.-C. Lim and H.-K. Xu, are compared within the framework of the so-called CAT(0) spaces. It is shown that both results apply in this setting, and that Lifšic's theorem gives a sharper result. Also, a new property is introduced that yields a fixed point theorem for uniformly lipschitzian mappings in a class of hyperconvex spaces, a class which includes those possessing property (P) of Lim and Xu.

MSC: primary 54H25, 54E40; secondary 47H09

Keywords: Uniformly lipschitzian mappings; Fixed points; CAT(0) spaces; Hyperconvex spaces

1. Introduction

A mapping $T : M \to M$ of a metric space (M, d) is said to be uniformly lipschitzian if there exists a constant k such that $d(x, y) \leq kd(T^n x, T^n y)$, for all $x, y \in M$ and $n \in \mathbb{N}$. This class of mappings was introduced by Goebel and Kirk in [5], where it was shown that if C is a bounded closed convex subset of a uniformly convex Banach space X, then there exists a constant $k > 1$, depending on the modulus of convexity of X, such that every uniformly lipschitzian mapping $T : C \to C$ with constant k has a fixed point. Since then there have been a number of extensions of this result, typically in a Banach space setting (see, e.g., the discussion in [6]). However two results in a metric setting are noteworthy. The first is a result of Lifšic [11] and the second is

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due to Lim and Xu [12]. Here we compare these results, taking as an underlying framework the so-called CAT(0) spaces. We show in particular that within this framework both the Lifšic and the Lim–Xu theorems apply, and that Lifšic's theorem yields the sharper conclusion. This is an important feature of the paper because it provides a class of spaces which are **not** Banach spaces, but for which the Lifšic characteristic can be calculated, and which satisfy all of the assumption of the Lim–Xu theorem. This appears to be the first example of such a class of spaces.

We also introduce a new property that yields a fixed point theorem for uniformly lipschitzian mappings in certain hyperconvex spaces. The precise relationship of this new property to ones previously studied is not yet clear. However the proof is a departure from the usual methods, and the result yields the Lim–Xu theorem in a hyperconvex setting as a corollary.

We begin with some basic definitions and notation that will be needed later. Let (X, d) be a bounded metric space. For a nonempty subset D of X, set

$$\begin{aligned}
&r_x(D) = \sup\{d(x, y) : y \in D\}, \qquad x \in X;\\
&r(D) = \inf\{r_x(D) : x \in X\};\\
&C(D) = \{x \in X : r_x(D) = r(D)\};\\
&\delta(D) = \sup\{d(x, y) : x, y \in D\};\\
&\mathrm{cov}(D) = \cap\{B : B \text{ is a closed ball and } D \subset B\}.
\end{aligned}$$

The number $r(D)$ is called the *Chebyshev radius* of D (in X) and $C(D)$ is called the *Chebyshev center* of D.

A subset A of X is said to be admissible if $\mathrm{cov}(A) = A$. The number

$$\tilde{N}(X) := \sup\left\{\frac{r(A)}{\delta(A)}\right\},$$

where the supremum is taken over all nonempty bounded admissible subsets A of X for which $\delta(A) > 0$ is called the *normal structure coefficient* of X. If $\tilde{N}(X) \le c$ for some constant $c < 1$, then X is said to have *uniform normal structure*. (For some authors, $\tilde{N}(X)$ would be the inverse of the normal structure coefficient.)

The metric space (X, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha\in\Gamma} B(x_\alpha; r_\alpha) \neq \emptyset$$

for any collection of points $\{x_\alpha\}_{\alpha\in\Gamma}$ in X and positive numbers $\{r_\alpha\}_{\alpha\in\Gamma}$ such that $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$ for any α, β in Γ. The classical spaces ℓ_∞ and L_∞ are examples of hyperconvex Banach spaces. Two facts are pertinent to what follows: $\tilde{N}(X) = 1/\sqrt{2}$ if X is a Hilbert space and $\tilde{N}(X) = 1/2$ if X is hyperconvex.

We now turn to the definition of the Lifšic characteristic of a metric space X. Balls in X are said to be *c-regular* if the following holds: for each $k < c$ there exist $\mu, \alpha \in (0, 1)$ such that for each $x, y \in X$ and $r > 0$ with $d(x, y) \ge (1 - \mu)r$, there exists $z \in X$ such that

$$B(x; (1 + \mu)r) \bigcap B(y; k(1 + \mu)r) \subset B(z; \alpha r). \tag{1.1}$$

The *Lifšic characteristic* $\kappa(X)$ of X is defined as follows:

$$\kappa(X) = \sup\{c \ge 1 : \text{balls in } X \text{ are } c\text{-regular}\}.$$

Theorem 1 *(Lifšic [11]). Let (X, d) be a bounded complete metric space. Then every uniformly k-lipschitzian mapping $T : X \to X$ with $k < \kappa(X)$ has a fixed point.*

In [12], Lim and Xu introduced the so-called property (P) for metric spaces. A metric space (X, d) is said to have *property* (P) if given two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, there exists $z \in \bigcap_{n\geq 1} \operatorname{cov}(\{z_j : j \geq n\})$ such that

$$\limsup_n d(z, x_n) \leq \limsup_j \limsup_n d(z_j, x_n).$$

The following theorem is the main result of [12].

Theorem 2 *([12, Theorem 7]). Let (X, d) be a complete bounded metric space with both property (P) and uniform normal structure. Then every uniformly k-lipschitzian mapping $T : X \to X$ with $k < \tilde{N}(X)^{-\frac{1}{2}}$ has a fixed point.*

It is known that the Lifšic characteristic of a Hilbert space is $\sqrt{2}$, and in Section 3 we show that the Lifšic characteristic of an $\mathbb{R}$-tree is 2. Therefore in these spaces Lifšic's theorem yields the sharper result. We also show that the same is true in the CAT(0) spaces, a class of spaces that includes these two spaces as extreme cases.

2. CAT(κ) spaces

Let (X, d) be a geodesic metric space in which each two points $x, y \in X$ are joined by a unique geodesic (metric) segment denoted $[x, y]$. A subset $Y \subset X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

Denote by M_κ^2 the following classical metric spaces:

(1) if $\kappa = 0$ then M_0^2 is the Euclidean plane $\mathbb{E}^2$;
(2) if $\kappa < 0$ then M_κ^2 is obtained from the classical hyperbolic plane $\mathbb{H}^2$ by multiplying the hyperbolic distance by $1/\sqrt{-\kappa}$.

A metric space X is said to be a CAT(κ) *space* (the term is due to M. Gromov — see, e.g., [1, p. 159]) if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in M_κ^2. We make this precise below. For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [1] or Burago et al. [2].

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that $d_{M_\kappa^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. The triangle inequality assures that comparison triangles always exists. If a point x is on an edge $[x_i, x_j]$ of Δ, then $\bar{x} \in \bar{\Delta}$ is called a comparison point of x if

$$d(x_i, x) = d_{M_\kappa^2}(\bar{x}_i, \bar{x}) \quad \text{and} \quad d(x_j, x) = d_{M_\kappa^2}(\bar{x}_j, \bar{x}).$$

A geodesic metric space is said to be a CAT(κ) space if all geodesic triangles of appropriate size satisfy the following CAT(κ) comparison axiom.

CAT(κ): Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset M_\kappa^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(κ) *inequality* if for all $x, y \in \Delta$,

$$d(x, y) \leq d_{M_\kappa^2}(\bar{x}, \bar{y}). \tag{2.1}$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the respective comparison points of x, y.

Of particular interest are the complete CAT(0) spaces, sometimes called *Hadamard spaces*. These spaces are uniquely geodesic and they include, as a very special case, the following class of spaces.

Definition 3. An $\mathbb{R}$-*tree* is a metric space T such that:

(i) there is a unique geodesic segment (denoted by $[x, y]$) joining each pair of points $x, y \in T$;
(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Proposition 4 (*[1, Chapter II.1]*). *The following relations hold:*

(1) *If X is a* CAT(κ) *space, then it is a* CAT(κ') *space for every $\kappa' \geq \kappa$.*
(2) *X is a* CAT(κ) *space for all $\kappa < 0$ if and only if X is an $\mathbb{R}$-tree.*

One consequence of (1) and (2) is that any result proved for CAT(0) spaces automatically carries over to any CAT(κ) spaces for $\kappa < 0$, and, in particular, to $\mathbb{R}$-trees.

Another fundamental property of CAT(0) spaces that we will need in the following is the so-called CN inequality. In fact a geodesic space is a CAT(0) space if and only if this inequality holds (see [1, p. 163]).

The CN inequality: for all $p, q, r \in X$ and all m with $d(q, m) = d(r, m) = d(q, r)/2$, one has

$$d(p, q)^2 + d(p, r)^2 \geq 2d(m, p)^2 + \frac{1}{2}d(q, r)^2. \tag{2.2}$$

All CAT(κ) spaces for $\kappa \leq 0$ have uniform normal structure with normal structure coefficient $c \leq 1/\sqrt{2}$. The precise values of c depend on κ. (See [10]; also the discussion in [8].)

3. The Lifšic characteristic of CAT(0) spaces

Theorem 5. *If (X, d) is a complete* CAT(0) *space, then $\kappa(X) \geq \sqrt{2}$. Moreover, if X is an $\mathbb{R}$-tree, $\kappa(X) = 2$.*

First a preliminary observation. Every bounded closed convex subset of a CAT(0) space has a unique Chebyshev center which is a singleton. Since closed convex subsets of a CAT(0) space are nonexpansive retracts of the space [1, p. 176], the unique minimal ball containing such a set must be centered at a point of the set. In other words, every bounded closed convex set contains its Chebyshev center. Thus in the definition of the Lifšic characteristic of such space, the inclusion (1.1) may be replaced with:

$$r\left(B(x; (1+\mu)r) \bigcap B(y; k(1+\mu)r)\right) \leq \alpha r, \tag{3.1}$$

where $r(\cdot)$ denotes the Chebyshev radius.

Proof of Theorem 1. We first show that in general $\kappa(X) \geq \sqrt{2}$. Let $r > 0$, choose $x, y \in X$ with $d(x, y) = r$ and let $\bar{x}, \bar{y} \in \mathbb{R}^2$ be any two points with $\|\bar{x} - \bar{y}\| = d(x, y)$.

Suppose $k = \kappa(X) < \sqrt{2}$. Then

$$r\left(B(x; r) \bigcap B(y; kr)\right) \leq \xi r$$

for some $\xi < 1$. (This is because the Lifšic characteristic of $\mathbb{R}^2$ is $\sqrt{2}$.) Now choose $\alpha \in (\xi, 1)$. Then for $\mu \in (0, 1)$ sufficiently near 0 and $\alpha \in (0, 1)$ sufficiently near 1,

$$r\left(B(\bar{x}; (1+\mu)r) \bigcap B(\bar{y}; k(1+\mu)r)\right) \le \alpha r,$$

and we may assume in addition only that $d(x, y) \ge (1-\mu)r$. Let

$$\bar{S} := B(\bar{x}; (1+\mu)r) \bigcap B(\bar{y}; k(1+\mu))r$$

and

$$S := B(x; (1+\mu)r) \bigcap B(y; k(1+\mu)r).$$

The Chebyshev center $\bar{c}$ of $\bar{S}$ lies on the segment $[\bar{x}, \bar{y}]$. Also if $u \in S$ and if $\Delta(\bar{y}, \bar{x}, \bar{u})$ is a comparison triangle for $\Delta(y, x, u)$ in $\mathbb{R}^2$, then $\bar{u} \in \bar{S}$. Therefore $\|\bar{u} - \bar{c}\| \le \alpha r$. If c is the point of the segment $[x, y]$ for which $d(y, c) = \|\bar{y} - \bar{c}\|$, then (using the CAT(0) inequality)

$$d(u, c) \le \|\bar{u} - \bar{c}\| \le \alpha r.$$

Since this is true for any $u \in S$ it follows that $r(S) \le \alpha r$, and since $k < \sqrt{2}$ was arbitrary, we have $\kappa(X) \ge \sqrt{2}$.

We now suppose X is an $\mathbb{R}$-tree, and we show that $\kappa(X) = 2$ by a direct calculation. Let $x, y \in X$ with $d(x, y) = r$, and let $k < 2$. Set

$$S := B(x; r) \bigcap B(y; kr).$$

We show that $\operatorname{diam}(S) \le 2(k-1)r$. Let $u, v \in S$. There exist points $p, q \in [x, y]$ such that $d(x, v) = d(x, p) + d(p, v)$ and $d(x, u) = d(x, q) + d(q, u)$. Similarly, $(y, v) = d(y, p) + d(p, v)$ and $d(y, u) = d(y, q) + d(q, u)$. Without loss of generality we may assume $d(x, p) = d(x, q) + d(q, p)$. Therefore

$$d(u, v) = d(u, q) + d(q, p) + d(p, v).$$

Since $u, v \in B(y; kr)$ we now have

$$\begin{aligned} 2kr &\ge d(u, y) + d(y, v) \\ &= d(y, q) + d(q, u) + d(y, p) + d(p, v) \\ &= r - d(x, p) + d(p, v) + r - d(x.q) + d(q, v) \\ &= 2r + d(u, v). \end{aligned}$$

This implies $d(u, v) \le 2(k-1)r$. Therefore, for $\mu \in (0, 1)$ sufficiently small and $\alpha \in (0, 1)$ sufficiently near 1,

$$\operatorname{diam}\left(B(x; (1+\mu)r) \bigcap B(x; k(1+\mu)r)\right) \le 2\alpha r$$

when $d(x, y) \ge (1-\mu)r$. Since X is hyperconvex (thus $\tilde{N}(X) = 1/2$) this in turn implies

$$r\left(B(x; (1+\mu)r) \bigcap B(x; k(1+\mu)r)\right) \le \alpha r. \quad \square$$

In view of the Lifšic theorem we have the following result.

Theorem 6. *Let (X, d) be a bounded complete* CAT(0) *space. Then every uniformly k-lipschitzian mapping $T : X \to X$ with $k < \sqrt{2}$ has a fixed point.*

The case when X is an $\mathbb{R}$-tree is moot because every bounded (indeed every geodesically bounded) complete $\mathbb{R}$-tree has the fixed point property for continuous maps. This fact is a consequence of results of G.S. Young [13, cf. Theorem 16]. For a direct proof, see [9].

Remark 1. It seems reasonable to conjecture that the Lifšic characteristic of a CAT(κ) space for $\kappa < 0$ is a continuous increasing function of κ which takes values in the interval $(\sqrt{2}, 2)$.

Remark 2. If $T : X \to X$ is uniformly k-lipschitzian, then T is nonexpansive relative to a metric r on X that satisfies

$$d(x, y) \leq r(x, y) \leq kd(x, y).$$

Also, if $T : X \to X$ is nonexpansive relative to a metric s on X with

$$\alpha d(x, y) \leq s(x, y) \leq \beta d(x, y),$$

then T is uniformly $\frac{\beta}{\alpha}$-lipschitzian on (X, d). (For the details, see [5].) While these observations might seem interesting, their usefulness in this context is mitigated by the fact that the CAT(κ) inequality is not necessarily preserved under small perturbations of the metric.

4. CAT(0) spaces and property (P)

In this section we show that every complete CAT(0) space has property (P). Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and let K be a closed and convex subset of X. Define $\varphi : X \to \mathbb{R}$ by setting $\varphi(x) = \limsup_{n\to\infty} d(x, x_n)$, $x \in X$.

Proposition 7. *There exists a unique point $u \in K$ such that*

$$\varphi(u) = \inf_{x\in K} \varphi(x).$$

Proof. Let $r = \inf_{x\in K} \varphi(x)$ and let $\varepsilon > 0$. Then by assumption there exists $x \in K$ such that $\varphi(x) < r + \varepsilon$; thus for n sufficiently large $d(x, x_n) < r + \varepsilon$, i.e., for n sufficiently large $x \in B(x_n; r + \varepsilon)$. Thus

$$C_\varepsilon := \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} B(x_i; r + \varepsilon) \cap K \right) \neq \emptyset.$$

As the ascending union of convex sets, clearly C_ε is convex. Also the closure $\overline{C_\varepsilon}$ of C_ε is also convex (see [1, Proposition 1.4(1)]). Therefore

$$C := \bigcap_{\varepsilon>0} \overline{C_\varepsilon} \neq \emptyset.$$

Clearly for $u \in C$, $\varphi(u) \leq r$. Uniqueness of such a u follows from the CN inequality (2.2). Specifically, suppose $u, v \in C$ with $u \neq v$. Then if m is the midpoint of the geodesic joining u and v,

$$d(m, x_n)^2 \leq \frac{d(u, x_n)^2 + d(v, x_n)^2}{2} - \frac{1}{4} d(u, v)^2.$$

This implies $\varphi(m)^2 \leq r^2 - \frac{1}{4} d(u, v)^2$ — a contradiction. □

In view of the above, X has property (P) if given two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, there exists $z \in \bigcap_{n=1}^{\infty} \operatorname{cov}\{z_j : j \geq n\}$ such that

$$\varphi(z) \leq \limsup_{j \to \infty} \varphi(z_j),$$

where φ is defined as above.

Theorem 8. *A complete* CAT(0) *space* (X, d) *has property* (P).

Proof. Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in X and define $\varphi(x) = \limsup_{n \to \infty} d(x, x_n)$, $x \in X$. For each n, let

$$C_n := \operatorname{cov}\{z_j : j \geq n\}.$$

By Proposition 7 there exists a unique point $u_n \in C_n$ such that

$$\varphi(u_n) = \inf_{x \in C_n} \varphi(x).$$

Moreover, since $z_j \in C_n$ for $j \geq n$, $\varphi(u_n) \leq \varphi(z_j)$ for all $j \geq n$. Thus $\varphi(u_n) \leq \limsup_{j \to \infty} \varphi(z_j)$ for all n. We assert that $\{u_n\}$ is a Cauchy sequence. To see this, suppose not. Then there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there exist $i, j \geq N$ such that $d(u_i, u_j) \geq \varepsilon$. Also, since the sets $\{C_n\}$ are descending, the sequence $\{\varphi(u_n)\}$ is increasing. Let $d := \lim_{n \to \infty} \varphi(u_n)$. Choose $\xi > 0$ so small that $\frac{2d\xi + \xi^2}{2} < \varepsilon/8$, and choose N so large that $|\varphi(u_i) - \varphi(u_j)| \leq \xi$ if $i, j \geq N$. Now choose $i > j \geq N$ so that $d(u_i, u_j) \geq \varepsilon$, let m_j denote the midpoint of the geodesic joining u_i and u_j, and let $n \in \mathbb{N}$. Then by the (CN) inequality

$$d(m_j, x_n)^2 \leq \frac{d(u_i, x_n)^2 + d(u_j, x_n)^2}{2} - \frac{\varepsilon}{4}.$$

This implies

$$\begin{aligned}
\varphi(m_j)^2 &\leq \frac{\varphi(u_i)^2 + \varphi(u_j)^2}{2} - \frac{\varepsilon}{4} \\
&\leq \frac{(\varphi(u_j) + \xi)^2 + \varphi(u_j)^2}{2} - \frac{\varepsilon}{4} \\
&= \varphi(u_j)^2 + \frac{2\varphi(u_j)\xi + \xi^2}{2} - \frac{\varepsilon}{4} \\
&< \varphi(u_j)^2 - \frac{\varepsilon}{8}.
\end{aligned}$$

Since $m_j \in C_j$, this contradicts the definition of u_j.

This proves that $\{u_n\}$ is a Cauchy sequence. Consequently there exists $z \in \bigcap_{n=1}^{\infty} C_n$ such that $\lim_{n \to \infty} u_n = z$ and, since φ is continuous, $\lim_{n \to \infty} \varphi(u_n) = \varphi(z)$. Since $\varphi(u_n) \leq \limsup_{j \to \infty} \varphi(z_j)$ for all n, we conclude that

$$\varphi(z) \leq \limsup_{j \to \infty} \varphi(z_j). \quad \square$$

5. Hyperconvex spaces

Since hyperconvex metric spaces have uniform normal structure, it is a consequence of Theorem 2 that if M is a bounded hyperconvex metric space with property (P), then every uniformly k-lipschitzian $T : M \to M$ has a fixed point for $k < \sqrt{2}$. Here, by embedding the problem in a larger space, we show that uniformly lipschitzian mappings have fixed points under an assumption that appears to be weaker than property (P). Consequently we recover the Lim and Xu result in a hyperconvex setting.

Every metric space (X, d) can be embedded isometrically into a hyperconvex space. To see this let

$$\ell_\infty(X) = \left\{ \{m_x\}_{x \in X} : m_x \in \mathbb{R} \text{ for all } x \text{ and } \sup_{x \in X} |m_x| < \infty \right\}.$$

Define the distance d_∞ on $\ell_\infty(X)$ by $d_\infty(\{m_x\}, \{n_x\}) = \sup_{x \in X} |m_x - n_x|$. Thus the metric space $(\ell_\infty(X), d_\infty)$ is hyperconvex. Fix $a \in X$ and consider the map $I : X \to \ell_\infty(X)$ defined by $I(b) = \{d(b, x) - d(a, x)\}_{x \in X}$. It is easy to see that I is an isometry.

For a nonempty subset D of a bounded hyperconvex metric space (X, d), it is known that $r(D) = \frac{1}{2}\delta(D)$ and

$$C(D) = C(\text{cov}(D)) = \bigcap_{x \in \text{cov}(D)} B(x; r(D)) \neq \emptyset$$

(see [4] for details).

Let $H = (H, d)$ be a bounded hyperconvex space. Embed H into $\ell_\infty(H)$ isometrically via the mapping $h \mapsto \{d(h, p) - d(a, p)\}_{p \in H}$, where a is a fixed element of H. Write $h_p = d(h, p) - d(a, p)$ for each $h, p \in H$. Let $R : \ell_\infty(H) \to H$ be a nonexpansive retraction.

Now let $\{e_n\}$ be the standard basis in the classical ℓ_∞ space, i.e., $\{e_n\} = \{\delta_{nj}\}_{j \in \mathbb{N}}$ where δ_{nj} is the Kronecker delta. Observe that $\bigcap_{n \geq 1} \text{cov}(\{e_j : j \geq n\}) = \{0\}$,

$$\limsup_n e_{nj} - \liminf_n e_{nj} < \varepsilon \text{ for all } j,$$

and

$$\limsup_n \sup_j |e_{nj} - 0| = 1 > 1 - \varepsilon \qquad \text{for all } \varepsilon \in (0, 1).$$

Following this observation, we say that a sequence $\{x_n\}$ in H is a copy of $\{e_n\}$ if for each $\varepsilon \in (0, 1)$, there exists a sequence $\{p_n\}$ in H with $\limsup_n x_{np_j} - \liminf_n x_{np_j} \leq \varepsilon\delta(\{x_n\})$ for all j, and $\limsup_n \sup_j |x_{np_j} - z_{p_j}| > (1 - \varepsilon)\delta(\{x_n\})$ for all $z \in \bigcap_{n \geq 1} \text{cov}(\{x_j : j \geq n\})$.

Thus, if H does not contain a copy of $\{e_n\}$, given $\{x_n\}$ in H there exists $\varepsilon \in (0, 1)$, depending on $\{x_n\}$, such that if $\{p_n\}$ is any sequence in H for which

$$\limsup_n x_{np_j} - \liminf_n x_{np_j} < \varepsilon\delta(\{x_n\}) \text{ for all } j,$$

then there exists $z \in \bigcap_{n \geq 1} \text{cov}(\{x_j : j \geq n\})$ for which

$$\limsup_n \sup_j |x_{np_j} - z_{p_j}| \leq (1 - \varepsilon)\delta(\{x_n\}).$$

This prompts the following definition. We say that H has *property* (P_ε) if H does not contain a copy of $\{e_n\}$ in the following uniform sense: there exists an $\varepsilon \in (0, 1)$ such that, for a sequence $\{x_n\}$ and a collection $\{p_\lambda\}_{\lambda \in \Lambda}$ of points in H with $\limsup_n x_{np_\lambda} - \liminf_n x_{np_\lambda} < \varepsilon\delta(\{x_n\})$ for all λ, there exists $z \in \bigcap_{n \geq 1} \text{cov}(\{x_j : j \geq n\})$ satisfying

$$\limsup_n \sup_\lambda |x_{np_\lambda} - z_{p_\lambda}| \leq (1 - \varepsilon)\delta(\{x_n\}). \tag{5.1}$$

In proving the theorem of [12], Lim and Xu use property (P) to construct a sequence $\{x_n\}$ in X satisfying for each integer $j \geq 0$,

$$\limsup_n d(x_{j+1}, T^n x_j) \leq \tilde{c}\delta(\{T^n x_j\}), \tag{5.2}$$

$$d(x_{j+1}, y) \leq \limsup_n d(T^n x_j, y), \qquad \text{for all } y \in X. \tag{5.3}$$

In the proof to follow, we shall construct such a sequence using property (P_ε).

We first observe that if a hyperconvex space H has property (P), then it has property $(P_{1/2})$. To see this we let $\{x_n\}$ and $\{p_n\}$ be any sequences in H. Let $z_n \in C(\{x_j : j \geq n\})$. By property (P), we can take a point $z \in \bigcap_{n\geq 1} \operatorname{cov}(\{z_j : j \geq n\})$ such that

$$\limsup_n d(z, x_n) \leq \limsup_j \limsup_n d(z_j, x_n) \leq \frac{1}{2}\delta(\{x_n\}). \tag{5.4}$$

Clearly,

$$\limsup_n \sup_j |x_{np_j} - z_{p_j}| \leq \limsup_n d(x_n, z) \leq \frac{1}{2}\delta(\{x_n\}).$$

We do not know whether the converse is true, that is, whether property $(P_{1/2})$ implies property (P). Indeed, if $\varepsilon' < \varepsilon$ then $P_\varepsilon \Rightarrow P_{\varepsilon'}$, but we see no reason why the reverse implication should be true.

Theorem 9. *Let H be a bounded hyperconvex metric space which has property (P_ε) for $\varepsilon \in (0, \frac{1}{2}]$. Then every uniformly k-lipschitzian mapping $T : H \to H$ with $k < \sqrt{\frac{1}{1-\varepsilon}}$ has a fixed point.*

Proof. We embed H in $\ell_\infty(X)$ as described above. Choose $\varepsilon' \in (0, \varepsilon)$ so that $k < \sqrt{\frac{1}{1-\varepsilon'}}$. Fix $x_0 \in H$ and consider the sequence $\{T^n x_0\}$. Define sets $A_n = \{T^j x_0 : j \geq n\}$ and $\mathcal{C} = \{p \in H : \limsup_n \{T^n x_0\}_p - \liminf_n \{T^n x_0\}_p < \varepsilon\delta(A_1)\}$. By (5.1) we see that with some $z \in \bigcap_{n\geq 1} \operatorname{cov}(A_n)$,

$$\limsup_n \sup_{p\in\mathcal{C}} |\{T^n x_0\}_p - z_p| \leq (1-\varepsilon)\delta(A_1). \tag{5.5}$$

Thus, since $0 < \varepsilon' < \epsilon$, for all large n, we have

$$\sup_{p\in\mathcal{C}} |z_p - \{T^n x_0\}_p| < (1-\varepsilon')\delta(A_1). \tag{5.6}$$

Let $p \in \mathcal{C}'$, where $\mathcal{C}'$ is the complement of $\mathcal{C}$. Write

$$a = \inf\{T^n x_0\}_p; \qquad b = \liminf_n \{T^n x_0\}_p;$$
$$c = \limsup_n \{T^n x_0\}_p; \qquad d = \sup\{T^n x_0\}_p.$$

Observe that we either have

$$a \leq d - \varepsilon\delta(A_1) \leq a + \varepsilon\delta(A_1) \leq d \qquad \text{or} \qquad a + \varepsilon\delta(A_1) < d - \varepsilon\delta(A_1),$$

from which we respectively have

$$[b, c] \cap [d - \varepsilon\delta(A_1), a + \varepsilon\delta(A_1)] \neq \emptyset \qquad \text{or} \qquad [b, c] \cap [a + \varepsilon\delta(A_1), d - \varepsilon\delta(A_1)] \neq \emptyset.$$

In either case, we can find a point $w_p \in [b, c]$ such that

$$|a - w_p| \leq \varepsilon\delta(A_1) \quad \text{and} \quad |d - w_p| \leq \varepsilon\delta(A_1) \quad \text{or}$$
$$|a - w_p| \leq (1-\varepsilon)\delta(A_1) \quad \text{and} \quad |d - w_p| \leq (1-\varepsilon)\delta(A_1).$$

Let $w = (w_p)_p$ where $w_p = z_p$ for $p \in C$. Finally, let $x_1 = R(w)$, the image of w under the retraction R.

As $\varepsilon \leq \frac{1}{2}$, we have $\varepsilon \leq (1-\varepsilon)$. Thus

$$|\sup_n \{T^n x_0\}_p - w_p| \leq (1-\varepsilon)\delta(A_1)$$
$$|w_p - \inf_n \{T^n x_0\}_p| \leq (1-\varepsilon)\delta(A_1).$$

So, for each j,

$$|w_p - \{T^j x_0\}_p| \leq \max\left\{\sup_n \{T^n x_0\}_p - w_p, w_p - \inf_n \{T^n x_0\}_p\right\} \leq (1-\varepsilon)\delta(A_1),$$

and thus

$$\sup_{p \in C'} |w_p - \{T^n x_0\}_p| \leq (1-\varepsilon)\delta(A_1) \quad \text{for each } n. \tag{5.7}$$

Finally, (5.6) and (5.7) imply for $j \geq n$ where n is sufficiently large,

$$d(x_1, T^j x_0) = \sup_{p \in H} |x_{1p} - \{T^j x_0\}_p| \leq \sup_{p \in H} |w_p - \{T^j x_0\}_p|$$
$$\leq (1-\varepsilon')\delta(A_1).$$

Consequently,

$$\limsup_n d(x_1, T^n x_0) \leq (1-\varepsilon')\delta(A_1). \tag{5.8}$$

Consider a ball $B(y, r_y(A_n))$ in H. Since $|w_p - y_p| \leq r_y(A_n)$ for all p, bearing in mind that $z \in \mathrm{cov}(A_n)$, we thus have

$$d(x_1, y) = d(R(w), R(y)) \leq d(w, y) \leq r(y, A_n).$$

This inequality holds for each y, therefore $x_1 \in \bigcap \mathrm{cov}(A_n)$, and so

$$d(x_1, y) \leq \limsup_n d(T^n x_0, y) \quad \text{for all } y \in H. \tag{5.9}$$

By induction we can obtain a sequence $\{x_n\}$ in H satisfying (5.2) and (5.3). The proof now can be completed as in the proof of Theorem 7 in [12]. □

Since hyperconvex spaces satisfy property $(P_{1/2})$, we have the following.

Corollary 10. *Let H be a bounded hyperconvex space which has property (P). Then every uniformly k-lipschitzian mapping $T : H \to H$ with $k < \sqrt{2}$ has a fixed point.*

Corollary 11. *Let $T : H \to H$ be a uniformly k-lipschitzian mapping. Suppose that each orbit $\{T^n x\}_n$ of T is not a copy of $\{e_n\}$, i.e., all orbits $\{\{T^n x\}_n : x \in H\}$ satisfy (5.1) for some $\varepsilon \in (0, 1)$. If $k < \sqrt{\frac{1}{1-\varepsilon}}$, then T has a fixed point.*

Remark. In [7] it is shown that if M is bounded, hyperconvex, and satisfies property (P), then every left reversible totally ordered uniformly k-lipschitzian semigroup of self-mapping of M has a common fixed point for $k < \sqrt{2}$. Extensions of the results of Lifšic and Lim–Xu to lipschitzian semigroups are given in [3].

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Appendix 9: A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006), 983-987.

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Note

A note on fixed point sets in CAT(0) spaces

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Abstract

We show that the fixed point set of a quasi-nonexpansive selfmap of a nonempty convex subset of a CAT(0) space is always closed, convex and contractible. Moreover, we give a construction of a continuous selfmap of a CAT(0) space whose fixed point set is prescribed.

Keywords: Fixed point set; CAT(0) spaces; Nonexpansive maps

Introduction

Geometric and topological properties of the fixed point set have been studied to some degree for nonexpansive maps of certain kinds of metric spaces. For example, it is known that the fixed point set of a nonexpansive selfmap of a bounded hyperconvex space X is also hyperconvex and hence a nonexpansive retract of X (see [3]). Recently, in [4] and [5], W.A. Kirk developed the fixed point theory for CAT(0) spaces and proved an interesting fact about the fixed point set:

Theorem. [5] *Suppose X is a nonempty bounded closed convex subset of a complete* CAT(0) *space, and suppose $f : X \to X$ is nonexpansive. Then the fixed point set of f is nonempty, closed and convex.*

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Motivated by the theorem above in the situation where the existence of a fixed point is assumed, we try to explore the connection between fixed point sets and closed convex subsets in CAT(0) spaces in greater generality.

In Section 1, we show that the fixed point set (if nonempty) of a quasi-nonexpansive selfmap (which may not even be continuous, see below) of a convex subset of a CAT(0) space is always closed, convex and hence contractible. Therefore, we have a complete homotopical/homological description of the fixed point set of a quasi-nonexpansive selfmap of a nonempty convex subset of a CAT(0) space.

In Section 2, we try to do something opposite. It is also known that, for a nonempty complete convex subset K of a CAT(0) space X, we can always find a nonexpansive selfmap of X (for example, the nearest point projection in [2]) whose fixed point set is precisely K. Hence, for a complete CAT(0) space X, there is a nonexpansive map whose fixed point set is any prescribed nonempty closed convex subset of X. However, the convexity assumption here is very crucial because there is no nonexpansive selfmap of $\mathbb{C}$ whose fixed point set is exactly the unit circle (by considering where 0 should be mapped). Interestingly, if we only require the continuity of the map, the convexity assumption can be dropped. In fact, we will give an explicit construction a continuous selfmap of a CAT(0) space X whose fixed point set is any prescribed nonempty closed subset of X.

Throughout this paper, for a space X, the fixed point set of $f : X \to X$ will be denoted by $F(f)$. Since we are interested in properties of $F(f)$, we will always assume that $F(f) \neq \emptyset$. According to [1], the map f will be called *quasi-nonexpansive* if $d(f(x), p) \leqslant d(x, p)$ for each $x \in X$ and $p \in F(f)$. Clearly, a nonexpansive map is quasi-nonexpansive, but not vice versa. In fact, it is easy to see that a quasi-nonexpansive map needs not be continuous (e.g., $f(x) = x^2$ for all $x \in [0, 1)$ and $f(1) = 0$).

A CAT(0) space is simply a geodesic metric space whose each geodesic triangle is at least as thin as its comparison triangle in the euclidean plane (see [2,4,5] for precise definitions and properties). Since a CAT(0) space (X, d) is always uniquely geodesic, we will use $[x, y]$ to denote the geodesic segment joining x and y, and $(1 - t)x \oplus ty$ to denote the unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ for any $t \in [0, 1]$. With this notation, we immediately obtain the following facts: for any $x, y, z \in (X, d)$ and $s, t \in [0, 1]$,

(1) $d((1 - t)x \oplus ty, (1 - s)x \oplus sy) = |t - s| d(x, y)$,
(2) $d((1 - t)x \oplus ty, (1 - t)x \oplus tz) \leqslant d(y, z)$.

One nice thing about a CAT(0) space is that it is always contractible. Moreover, one can easily prove that a subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space if and only if it is convex. Therefore, a convex subset of a CAT(0) space is always contractible.

1. Fixed point sets of quasi-nonexpansive maps

Lemma 1.1. *The fixed point set of a quasi-nonexpansive selfmap of a metric space is always closed.*

Proof. Suppose $f:(X,d)\to(X,d)$ is a quasi-nonexpansive map and (x_n) a sequence in $F(f)$ converging to x. Then, for each $\varepsilon>0$, there is $N\in\mathbb{N}$ such that $d(x,x_N)<\varepsilon/2$ and hence $d(x,f(x))\leqslant d(x,x_N)+d(x_N,f(x))\leqslant 2d(x,x_N)<\varepsilon$. Since ε is arbitrary, we must have $d(x,f(x))=0$; i.e., $x\in F(f)$. Therefore, $F(f)$ is closed. □

Lemma 1.2. *Let (X,d) be a* CAT(0) *space and $x,y,z\in X$. If*

$$d(x,z)+d(y,z)=d(x,y),$$

then $z\in[x,y]$.

Proof. Let $\Delta(\bar{x},\bar{y},\bar{z})$ be the comparison triangle (up to isomorphism) in $\mathbb{R}^2$ of the geodesic triangle $\Delta(x,y,z)$, and $w\in[x,y]$ be such that $d(x,w)=d(x,z)$. By the above assumption, it follows that $[\bar{x},\bar{y}]$ is simply a straight line with the point $\bar{z}$ in between; i.e., $\bar{z}\in[\bar{x},\bar{y}]$. Moreover, since $d(x,w)=d(x,z)$, we must have $d(\bar{x},\bar{w})=d(\bar{x},\bar{z})$ and hence $\bar{w}=\bar{z}$. Then, by the CAT(0) inequality, we have

$$d(w,z)\leqslant d(\bar{w},\bar{z})=0,$$

which implies $z=w\in[x,y]$. □

Theorem 1.3. *Let (X,d) be a convex subset of a* CAT(0) *space and $f:X\to X$ a quasi-nonexpansive map whose fixed point set is nonempty. Then $F(f)$ is closed, convex and hence contractible.*

Proof. Since (X,d) is convex in a CAT(0) space, it is also a CAT(0) space. Let $x,y\in F(f)$ and $z\in[x,y]$. Since f is quasi-nonexpansive, we have $d(x,f(z))\leqslant d(x,z)$ and $d(y,f(z))\leqslant d(y,z)$. Hence, we obtain the inequalities

$$d(x,y)\leqslant d\big(x,f(z)\big)+d\big(f(z),y\big)\leqslant d(x,z)+d(z,y)=d(x,y),$$

which implies $d(x,f(z))+d(f(z),y)=d(x,y)$. In fact, it is not difficult to see that $d(x,z)=d(x,f(z))$ and $d(z,y)=d(f(z),y)$. For if $d(x,z)<d(x,f(z))$ or $d(y,z)<d(y,f(z))$, the above inequalities will give

$$d(x,y)\leqslant d\big(x,f(z)\big)+d\big(f(z),y\big)<d(x,z)+d(z,y)\leqslant d(x,y),$$

which is a contradiction. Now, by the previous lemma, we have $f(z)\in[x,y]$, and hence $z=f(z)$ because z is the only point in $[x,y]$ satisfying $d(x,z)=d(x,f(z))$. Therefore, $[x,y]\subseteq F(f)$; i.e., $F(f)$ is convex and hence contractible. □

2. Maps with prescribed fixed point sets

Theorem 2.1. *Let A be a nonempty subset of a* CAT(0) *space (X,d). Then there exists a continuous map $f:X\to X$ such that $F(f)=\bar{A}$.*

Proof. For each $x\in X$, let $k_x=\frac{d(x,A)}{1+d(x,A)}\in[0,1]$. First, note that for each $x,y\in X$,

$$\begin{aligned}
|k_x - k_y| &= \left|\frac{d(x,A)}{1+d(x,A)} - \frac{d(y,A)}{1+d(y,A)}\right| \\
&= \left|1 - \frac{1}{1+d(x,A)} - 1 + \frac{1}{1+d(y,A)}\right| \\
&= \left|\frac{1}{1+d(y,A)} - \frac{1}{1+d(x,A)}\right| \\
&= \left|\frac{d(x,A) - d(y,A)}{(1+d(y,A))(1+d(x,A))}\right| \\
&\leqslant \left|d(x,A) - d(y,A)\right|.
\end{aligned}$$

Now, fix $x_0 \in A$ and define $f : X \to X$ by

$$f(x) = (1-k_x)x \oplus k_x x_0 \quad \text{for all } x \in X.$$

To see that f is continuous, we let $x \in X$, $\epsilon > 0$ and $\delta = \frac{\epsilon}{d(x,x_0)+1} > 0$. Then, for each $y \in B_d(x,\delta)$, we have

$$\begin{aligned}
d\big(f(x), f(y)\big) &= d\big((1-k_x)x \oplus k_x x_0, (1-k_y)y \oplus k_y x_0\big) \\
&\leqslant d\big((1-k_x)x \oplus k_x x_0, (1-k_y)x \oplus k_y x_0\big) \\
&\quad + d\big((1-k_y)x \oplus k_y x_0, (1-k_y)y \oplus k_y x_0\big) \\
&\leqslant |k_x - k_y| d(x, x_0) + d(x,y) \\
&\leqslant \left|d(x,A) - d(y,A)\right| d(x_0, x) + d(x,y) \\
&\leqslant d(x,y)\big(d(x_0,x) + 1\big) \\
&< \delta\big(d(x_0,x) + 1\big) \\
&= \epsilon.
\end{aligned}$$

Finally, it is easy to see that $f(x) = x$ if and only if $(1-k_x)x \oplus k_x x_0 = x$ if and only if $k_x = 0$ if and only if $\frac{d(x,A)}{1+d(x,A)} = 0$ if and only if $d(x,A) = 0$ if and only if $x \in \overline{A}$. Therefore, $F(f) = \overline{A}$ as desired. □

Corollary 2.2. *Let K be a nonempty closed subset of a* CAT(0) *space (X,d). Then there exists a continuous map $f : X \to X$ whose fixed point set is precisely K.*

Example 2.3. Let $X = [0,1]$, $K = \{0, 1, 1/2, 1/3, \ldots\}$ a closed subset of X and $x_0 = 0$. It is now easy to verify that the map $f : X \to X$ from the previous theorem is of the form $f(x) = \frac{x}{1+d(x,K)}$ for each $x \in X$. Notice that if we let $f_0(0) = 0$ and, for each $n \in \mathbb{N}$, $f_n : [\frac{1}{n+1}, \frac{1}{n}] \to [\frac{1}{n+1}, \frac{1}{n}]$ be defined by

$$f_n(x) = \begin{cases} \frac{x}{1+\frac{1}{n}-x}, & \frac{2n+1}{2n(n+1)} \leqslant x \leqslant \frac{1}{n}, \\ \frac{x}{1-\frac{1}{n+1}+x}, & \frac{1}{n+1} \leqslant x \leqslant \frac{2n+1}{2n(n+1)}, \end{cases}$$

then $f = \bigcup_{n=0}^{\infty} f_n$ which is clearly continuous. Moreover, since $F(f_0) = \{0\}$ and $F(f_n) = \{\frac{1}{n+1}, \frac{1}{n}\}$, it follows that $F(f) = \bigcup_{n=0}^{\infty} F(f_n) = K$.

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Appendix 10: Virtually nonexpansive maps and their convergence sets,

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Virtually nonexpansive maps and their convergence sets

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Abstract

We introduce the notion of a virtually nonexpansive selfmap of a metric space and show that the fixed point set of such a map is generally a retract of its convergence set. We also show that the class of virtually nonexpansive maps properly contains the class of (continuous) asymptotically quasi-nonexpansive maps. When the domain is complete and the fixed point set is totally bounded, we give another description of the convergence set of a virtually nonexpansive map and use it to show that the convergence set is always a G_δ-set. We also discuss some criteria to obtain an explicit retraction from the domain onto the fixed point set in Banach space settings.

Keywords: Convergence set; Fixed point set; Nonexpansive retract

0. Introduction

It is well know that, in many situations, the fixed point set of a nonexpansive map $f:(X,d)\to(X,d)$ is a nonexpansive retract of X. For example, when X is a locally weakly compact convex subset of a Banach space and satisfies the conditional fixed point property [2]. More recently, this result has been extended to an asymptotically nonexpansive map [4]. From homotopical point of view, all those results imply that the fixed point set cannot be homotopically more complicated than X. However, in those situations, X is already contractible, we immediately obtain the fact that the fixed point set is always contractible. Although this fact is not technically new in the sense that it can be proved directly in many situations [3,7], it really initiates the study of the

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homotopical structure of the fixed point set via a retraction in a more general settings. Therefore, in this paper, we try to form a basis of such a study by introducing a new kind of selfmap of a metric space, called a virtually nonexpansive map, which enjoys many nice properties as far as a retraction onto the fixed point set is concerned.

The paper is organized as follows. In Section 1, we prove that a virtually nonexpansive map is more general than an asymptotically quasi-nonexpansive map. Moreover, for a metric space (X,d) and a virtually nonexpansive map $f: X \to X$ whose fixed point set is nonempty, we naturally obtain a retraction from a certain subset of X (instead of X itself) onto the fixed point set. We call such a subset the convergence set of f and it consists of every point x whose Picard iteration $(f^n(x))$ converges. Then, in Section 2, we further explore the convergence set of a virtually nonexpansive map topologically by showing that the convergence set is always a G_δ-set in X when the domain X is complete and the fixed point set is totally bounded. Finally, in Section 3, we discuss situations where we can obtain a retraction from the domain X onto the fixed point set. Unlike the usual retraction arising from Zorn's lemma, we can apply the results on strong convergence of various iteration schemes approximating a fixed point [5,6,8,9] to obtain such a retraction explicitly.

Throughout this paper, (X,d) always denotes a metric space and $f: X \to X$ a continuous map. The fixed point set of f will be denoted by $F(f)$. The continuity of f guarantees that $F(f)$ is a closed subset of X. Since we are only interested in the structure of $F(f)$, we will always assume that $F(f) \neq \emptyset$.

Recall that a map $f: X \to X$ is called

- *a nonexpansive map* (*NX*) if $d(f(x), f(y)) \leqslant d(x,y)$ for any $x, y \in X$;
- *a quasi-nonexpansive map* (*QNX*) if $d(f(x), p) \leqslant d(x,p)$ for any $x \in X$ and $p \in F(f)$;
- *an asymptotically nonexpansive map* (*ANX*) if there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), f^n(y)) \leqslant k_n d(x,y)$ for any $x, y \in X$ and $n \in \mathbb{N}$;
- *an asymptotically quasi-nonexpansive map* (*AQNX*) if there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), p) \leqslant k_n d(x,p)$ for any $x \in X$, $p \in F(f)$ and $n \in \mathbb{N}$.

Notice also that a QNX may not be continuous in general, but in our settings, the continuity is always assumed. Moreover, we always have the following implications:

$$\begin{array}{ccc} \text{NX} & \Rightarrow & \text{QNX} \\ \Downarrow & & \Downarrow \\ \text{ANX} & \Rightarrow & \text{AQNX}. \end{array}$$

1. Virtually nonexpansive maps

We define the convergence set of f to be

$$C(f) = \{x \in X \mid \text{the sequence } (f^n(x)) \text{ converges}\}.$$

Notice that we naturally obtain the map $f^\infty: C(f) \to F(f)$ is defined by

$$f^\infty(x) = \lim_{n\to\infty} f^n(x)$$

which may not be continuous in general. Since we always have $f^\infty(x) = x$ for any $x \in F(f)$, the map f^∞ will be a retraction whenever it is continuous. Moreover, when f^∞ is continuous, any retraction from a superset of $C(f)$ (e.g., the domain X itself) onto $F(f)$ that satisfies a certain condition (which is weaker than being f-ergodic in the sense of [4]) is simply a continuous extension of f^∞ by the following theorem.

Theorem 1.1. *Suppose f^∞ is continuous and $R : C(f) \to F(f)$ is any retraction. If $R \circ f = R$, then $R = f^\infty$.*

Proof. For each $x \in C(f)$, we clearly have

$$f^\infty(x) = R\big(f^\infty(x)\big) = R\Big(\lim_{n\to\infty} f^n(x)\Big) = \lim_{n\to\infty} R\big(f^n(x)\big) = \lim_{n\to\infty} R(x) = R(x). \qquad \square$$

The next theorem is our criterion for the continuity of f^∞ that will be used throughout. Let

$$E(f) = \big\{x \in X \mid \text{the family of iterates } \{f^n\} \text{ is equicontinuous at } x\big\}.$$

Theorem 1.2. *If $C(f) \subseteq E(f)$, then f^∞ is continuous.*

Proof. Let $x \in C(f)$ and $\varepsilon > 0$.

By equicontinuity, there exists $\delta > 0$ such that for any $y \in X$ and $n \in \mathbb{N}$,

$$d\big(f^n(x), f^n(y)\big) < \frac{\varepsilon}{3}$$

whenever $d(x, y) < \delta$.

Now, let $y \in C(f)$ with $d(x, y) < \delta$. By convergence, there exists $N \in \mathbb{N}$ such that $d(f^\infty(x), f^N(x)) < \frac{\varepsilon}{3}$ and $d(f^\infty(y), f^N(y)) < \frac{\varepsilon}{3}$. Hence, it follows that

$$d\big(f^\infty(x), f^\infty(y)\big) \leqslant d\big(f^\infty(x), f^N(x)\big) + d\big(f^N(x), f^N(y)\big) + d\big(f^N(y), f^\infty(y)\big) < \varepsilon$$

which proves the continuity of f^∞ at x as desired. $\square$

Definition 1.3. A continuous map f is said to be a *virtually nonexpansive map* (*or VNX*) if $C(f) \subseteq E(f)$.

From the definition above, we can restate the previous theorem as:

Theorem 1.4. *If f is a VNX, then f^∞ is continuous and hence a retraction.*

Example 1.5. If $f : X \to X$ is an isometry, then f is clearly a VNX since we have $F(f) = C(f) \subseteq E(f) = X$.

Example 1.6. Let D^2 and S^1 denote the unit disk and the unit circle in $\mathbb{C}$, respectively. Also, let $S^1_{\mathbb{Q}} = \{e^{i\theta} \mid \theta \in \mathbb{Q}\} \subseteq S^1$ and $X = D^2 - S^1_{\mathbb{Q}}$. Define $f : X \to X$ by $f(z) = z^2$. It is easy to see that $F(f) = \{0\}$ and $C(f) = \operatorname{Int}(D^2) = E(f)$. Hence, f is a VNX.

The converse of the previous theorem is not always true as we will see in the next example.

Example 1.7. Let $f : S^1 \to S^1$ be defined by $f(z) = z^2$. Then we have $F(f) = \{1\}$, $C(f) \neq \emptyset$ and $E(f) = \emptyset$. However, f^∞ is clearly continuous.

The next theorem surprisingly gives us another classification of a VNX in terms of the fixed point set.

Theorem 1.8. *$F(f) \subseteq E(f)$ iff $C(f) \subseteq E(f)$.*

Proof. ($\Leftarrow$) Follows directly from $F(f) \subseteq C(f)$.

($\Rightarrow$) Suppose $F(f) \subseteq E(f)$. Let $x \in C(f)$, $x_0 = f^{\infty}(x) \in F(f)$ and $\varepsilon > 0$.

(1) Since $x_0 \in F(f) \subseteq E(F)$, there exists $r > 0$ such that, for each $n \in \mathbb{N}$ and each $y \in X$, if $d(x_0, y) < r$ then $d(x_0, f^n(y)) < \frac{\varepsilon}{2}$. With out loss of generality, we may assume that $r < \frac{\varepsilon}{2}$.
(2) Since the sequence $(f^n(x))$ converges to x_0, there exists $N \in \mathbb{N}$ such that $d(x_0, f^n(x)) < r < \frac{\varepsilon}{2}$ for all $n \geqslant N$.
(3) By continuity of $f, f^2, \ldots, f^N$ at x, there exist $\delta_1, \delta_2, \ldots, \delta_N > 0$ such that, for each $n = 1, 2, \ldots, N$ and $y \in X$, if $d(x, y) < \delta_n$ then $d(f^n(x), f^n(y)) < \varepsilon$.

Now, let $\delta > 0$ be such that $\delta \leqslant \min\{\delta_1, \delta_2, \ldots, \delta_N\}$ and $B(x, \delta) \subseteq f^{-N}(B(x_0, r))$. This can be done because f^N is continuous and $B(x_0, r)$ is open.

Then, for each $n \in \mathbb{N}$ and $y \in B(x, \delta)$, we consider the following 2 cases:

Case $n \leqslant N$: $d(f^n(x), f^n(y)) < \varepsilon$ by (3).
Case $n > N$: Suppose $n = N + i$ for some $i \in \mathbb{N}$. By (1) and (2), we clearly have

$$d\big(f^n(x), f^n(y)\big) = d\big(f^{N+i}(x), f^{N+i}(y)\big) \leqslant d\big(f^{N+i}(x), x_0\big) + d\big(x_0, f^{N+i}(y)\big) < r + \frac{\varepsilon}{2} < \varepsilon.$$

Hence, $x \in E(f)$ as desired. □

The next theorem shows that the set of all VNXs contains the set of all AQNXs.

Theorem 1.9. *Every AQNX is a VNX.*

Proof. Let f be an AQNX. Then there exists a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), p) \leqslant k_n d(x, p)$ for any $x \in X$, $p \in F(f)$ and $n \in \mathbb{N}$. By the previous theorem, it suffices to show that $F(f) \subseteq E(f)$.

Let $p \in F(f)$ and $\varepsilon > 0$. Since $(k_n) \to 1$, we can find $K > 0$ such that $k_n \leqslant K$ for all $n \in \mathbb{N}$. We now set $\delta = \frac{\varepsilon}{K}$. Then for any $x \in X$ with $d(x, p) < \delta$, we clearly have

$$d\big(f^n(x), p\big) \leqslant k_n d(x, p) < k_n \delta = k_n \frac{\varepsilon}{K} \leqslant \varepsilon$$

for all $n \in \mathbb{N}$. Therefore, $p \in E(f)$ as desired. □

The next example ensures that the set of all VNXs is strictly larger that the set of all AQNXs.

Example 1.10. Let $X = D^2 - \{0\}$ and $f : X \to X$ be defined by $f(z) = \frac{z}{|z|}$. Clearly, $F(f) = S^1$, $C(f) = X$ and $E(f) = X$. Hence, f is a VNX. Now suppose f is an AQNX. Then, there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $|f^n(z) - p| \leqslant k_n |z - p|$ for any $z \in X$, $p \in S^1$ and $n \in \mathbb{N}$. Pick a point $z \in D^2 - \{0\}$ with $|z| < 1$ and let $q = f(z) \in S^1$ and $p \in S^1$

the opposite point of q. Then it follows that $2 = |q - p| = |f^n(z) - p| \leqslant k_n|z - p|$ and hence $2 \leqslant \lim_{n\to\infty} k_n|z - p| = |z - p|$. This clearly contradicts the fact that $|z - p| < 2$. Therefore, f is not an AQNX.

2. Topology of $C(f)$

Unlike $F(f)$ which is always closed, $C(f)$ looks more complicated even for a nice space (see Example 2.1). However, for a VNX, it turns out that we can describe $C(f)$ in some situations. Therefore, in this section, we will start by giving another description of $C(f)$ that generalizes Theorem 2.1 in [1] to a VNX when its fixed point set is totally bounded. Then we will show that, under some circumstances, $C(f)$ is always a G_δ-set in X.

Example 2.1. Convergence sets of various VNXs:

(1) If $f : [-1, 1] \to [-1, 1]$ is given by $f(x) = -x^3$, then $C(f) = (-1, 1)$ which is open.
(2) If $f : D^2 \to D^2$ is given by $f(z) = e^{i\pi/4}z$, then $C(f) = \{0\}$ which is closed.

Theorem 2.2. *Let (X, d) be a complete metric space and let $f : X \to X$ be a VNX. If $F(f)$ is totally bounded, then*

$$C(f) = \{x \in X \mid d(O(f, x), F(f)) = 0\},$$

where $O(f, x)$ is the set of iterates of x under f; i.e.,

$$O(f, x) = \{f^n(x) \mid n \in \mathbb{N} \cup \{0\}\}.$$

Proof. Clearly, we always have $C(f) \subseteq \{x \in X \mid d(O(f, x), F(f)) = 0\}$.

Conversely, let $x \in X$ be such that $d(O(f, x), F(f)) = 0$. Since (X, d) is assumed to be complete, it suffices to show that the sequence $(f^n(x))$ is Cauchy. Let $\varepsilon > 0$. Since $F(f)$ is totally bounded by assumption and complete (being a closed subspace of a complete space), it is compact. Then, there exists $p \in \overline{O(f, x)} \cap F(f)$.

Since f is a VNX, we have $p \in E(f)$. Then, there exists $\delta > 0$ such that, for each $n \in \mathbb{N}$ and each $y \in X$, if $d(p, y) < \delta$, then $d(p, f^n(y)) < \frac{\varepsilon}{2}$.

Since $p \in \overline{O(f, x)}$, we must have $B(p, \delta) \cap O(f, x) \neq \emptyset$. Then, there exists $N \in \mathbb{N}$ such that $d(p, f^N(x)) < \delta$.

It follows that $d(p, f^{N+n}(x)) = d(p, f^n(f^N(x))) < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$.

Hence, for $m, n \in \mathbb{N}$, we have

$$d(f^{N+m}(x), f^{N+n}(x)) \leqslant d(f^{N+m}(x), p) + d(p, f^{N+n}(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is $(f^n(x))$ is Cauchy as desired. □

The following example shows that when $F(f)$ is not totally bounded, the previous theorem may not be true.

Example 2.3. Let B be the closed unit ball in ℓ_∞ and $X = B \cup \{x_n \mid n = 1, 2, \ldots\}$ where

$$x_n = \Big(0, \ldots, 0, \underbrace{1 + \frac{1}{n}}_{n\text{th-position}}, 0, \ldots\Big).$$

Let $f : X \to X$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in B, \\ x_{n+1} & \text{if } x = x_n. \end{cases}$$

Clearly $F(f) = B = C(f)$, so $F(f)$ is not totally bounded. Moreover, it is not difficult to verify that the set $\{x_n \mid n = 1, 2, \ldots\}$ is discrete and hence $\{f^n\}$ is clearly equicontinuous on X. So we have $C(f) \subseteq E(f) = X$; i.e., f is a VNX. Since $d(x_n, F(f)) = d(x_n, B) = \frac{1}{n}$, we have $d(O(f, x_1), F(f)) = 0$. However, $x_1 \notin C(f)$.

Theorem 2.4. *Let (X, d) be a complete metric space and let $f : X \to X$ be a VNX. If $F(f)$ is totally bounded, then $C(f)$ is a G_δ-set in X.*

Proof. We first note that, for each $m \in \mathbb{N}$, the set $V_{1/m} = \{x \in X \mid d(x, F(f)) < \frac{1}{m}\}$ is open in X. Then, for each $n \in \mathbb{N}$, the continuity of f^n implies that $f^{-n}(V_{1/m})$ is open in X. Therefore, by the previous theorem, it suffices to show that

$$C(f) = \{x \in X \mid d(O(f, x), F(f)) = 0\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} f^{-n}(V_{\frac{1}{m}})$$

but this is easily obtained from the series of equivalences:

$$\begin{aligned} d(O(f, x), F(f)) = 0 \quad &\Leftrightarrow \quad \forall m\ \exists n \left(d(f^n(x), F(f)) < \frac{1}{m}\right) \\ &\Leftrightarrow \quad \forall m\ \exists n \left(f^n(x) \in V_{\frac{1}{m}}\right) \\ &\Leftrightarrow \quad \forall m\ \exists n \left(x \in f^{-n}(V_{\frac{1}{m}})\right) \\ &\Leftrightarrow \quad x \in \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} f^{-n}(V_{\frac{1}{m}}). \qquad \square \end{aligned}$$

3. Retraction via iteration schemes

In this section, we will use results on strong convergence of well-known iteration schemes approximating a fixed point in Banach spaces [5,6,8,9] to obtain an explicit retraction from the domain X onto $F(f)$. Recall that, from our results in Section 1, f^∞ is always continuous whenever f is nonexpansive or quasi-nonexpansive. In fact, it follows readily from the definitions that f^∞ is (quasi-) nonexpansive whenever f is.

Theorem 3.1. *Let X be a closed subset of a Banach space. If $T : X \to X$ is continuous, quasi-nonexpansive and asymptotically regular map with $F(T) \neq \emptyset$ and*

$$\liminf_{n\to\infty} d(y_n, F(T)) = 0$$

for any sequence (y_n) in X with $\lim_{n\to\infty} \|(I - T)y_n\| = 0$, then T^∞ is a quasi-nonexpansive retraction from X onto $F(T)$.

Proof. By [9], the conditions in this theorem imply that $(T^n x_0)$ converges for each $x_0 \in X$; i.e., $C(T) = X$. Hence, by the previous observation, T^∞ is a quasi-nonexpansive retraction from X onto $F(T)$. $\square$

Now following [5], for a convex subset X of a normed linear space, a continuous map $T : X \to X$ and $\lambda, \mu \in [0, 1]$, we let

$$T_{\lambda,\mu} = (1-\lambda)I + \lambda T T_{\mu},$$

where $T_{\mu} = (1-\mu)I + \mu T$. Notice that $T_{1,0} = T$ and $T_{\mu} = T_{\mu,0}$. Moreover, $(T^n_{1,0})$, (T^n_{μ}) and $(T^n_{\lambda,\mu})$ are simply Picard, Mann and Ishikawa iteration schemes, respectively.

Lemma 3.2. *With X and T as above, and $\lambda, \mu \in [0, 1]$ we have the following:*

(1) *$F(T) \subseteq F(T_{\lambda,\mu})$.*
(2) *If T is nonexpansive and $(\lambda, \mu) \neq (1, 1)$, then $F(T) = F(T_{\lambda,\mu})$.*
(3) *If T is nonexpansive, then $T_{\lambda,\mu}$ is nonexpansive.*
(4) *If T is quasi-nonexpansive, then $T_{\lambda,\mu}$ is quasi-nonexpansive for $p \in F(T)$.*

Proof. The proof is rather straightforward and can be found in [5] for $\mu \in [0, 1)$ and $\lambda \in (0, 1)$. However, since $T_{0,0} = T_{0,1} = I$, $T_{1,0} = T$ and $T_{1,1} = T^2$, the theorem is also true for any $\lambda, \mu \in \{0, 1\}$. □

Theorem 3.3. *Let X be a bounded closed convex subset of a uniformly convex Banach space B, $\mu \in [0, 1)$ and $\lambda \in (0, 1)$. If $T : X \to X$ is a nonexpansive map satisfying one of the following conditions:*

(1) *$(I - TT_{\mu})$ maps closed sets in X into closed sets in X,*
(2) *TT_{μ} is demicompact at θ,*
(3) *there exists $k > 0$ such that $\|(T - TT_{\mu})x\| \geqslant kd(x, F(T))$ for each $x \in X$,*

then $T^{\infty}_{\lambda,\mu}$ is a nonexpansive retraction from X onto $F(T)$.

Proof. Since T is a nonexpansive map, we have $F(T_{\lambda,\mu}) = F(T)$ and $T_{\lambda,\mu}$ is nonexpansive by Lemma 3.2. Hence, the map $T^{\infty}_{\lambda,\mu} : C(T_{\lambda,\mu}) \to F(T_{\lambda,\mu})$ is continuous and nonexpansive. By [5], since T satisfies one of the above conditions, the sequence $(T^n_{\lambda,\mu}x_0)$ converges to a fixed point of T for any $x_0 \in X$; i.e., $C(T_{\lambda,\mu}) = X$. Therefore, $T^{\infty}_{\lambda,\mu}$ is a nonexpansive retraction from X onto $F(T)$. □

From the above idea, we obtain the following two general theorems.

Theorem 3.4. *Let X be a convex subset of a Banach space and $T : X \to X$ a nonexpansive map. Suppose further that X and T satisfy some conditions that ensure the existence of $\lambda, \mu \in [0, 1]$ for which $F(T) \neq \emptyset$ and $C(T_{\lambda,\mu}) = X$ (and $F(T) = F(T^2)$ if $\lambda = 1 = \mu$). Then $T^{\infty}_{\lambda,\mu}$ is a nonexpansive retraction from X onto $F(T)$.*

Proof. By the assumption and by Lemma 3.2, we have $F(T) = F(T_{\lambda,\mu})$ and $T_{\lambda,\mu}$ is also nonexpansive. Hence, $T^{\infty}_{\lambda,\mu}$ is the desired nonexpansive retraction from $C(T_{\lambda,\mu}) = X$ onto $F(T_{\lambda,\mu}) = F(T)$. □

Theorem 3.5. *Let X be a convex subset of a Banach space and $T : X \to X$ a continuous quasi-nonexpansive map. Suppose further that X and T satisfy some conditions that ensure* *the existence of* $\lambda, \mu \in [0, 1]$ *for which* $\emptyset \neq F(T) = F(T_{\lambda,\mu})$ *and* $C(T_{\lambda,\mu}) = X$. *Then* $T^{\infty}_{\lambda,\mu}$ *is a quasi-nonexpansive retraction from* X *onto* $F(T)$.

Proof. By the assumption $F(T) = F(T_{\lambda,\mu})$ and by Lemma 3.2, the map $T_{\lambda,\mu}$ is clearly quasi-nonexpansive. Hence, $T^{\infty}_{\lambda,\mu}$ is the desired quasi-nonexpansive retraction from $C(T_{\lambda,\mu}) = X$ onto $F(T_{\lambda,\mu}) = F(T)$. □

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Appendix 11: The James constant of normalized norms on R^2, J. Inequalities Appl. V. 2006, Article ID 26265, Pages 1-12.

THE JAMES CONSTANT OF NORMALIZED NORMS ON $\mathbb{R}^2$

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We introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms. We also give a criterion for deciding whether a given norm in this class is uniformly nonsquare. Moreover, an estimate for the James constant is presented and the exact value of some certain norms is computed. This gives a partial answer to the question raised by Kato et al.

1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$) is said to be *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z,w \in \mathbb{C}$ (resp., $\mathbb{R}$), and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(z,w)\|_p = \begin{cases} \left(|z|^p + |w|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|z|,|w|\} & \text{if } p = \infty. \end{cases} \tag{1.1}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$), and Ψ_2 the family of all continuous convex functions ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t,\, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. According to Bonsall and Duncan [1], AN_2 and Ψ_2 are in a one-to-one correspondence under the equation

$$\psi(t) = \|(1-t,t)\| \quad (0 \le t \le 1). \tag{1.2}$$

Indeed, for all $\psi \in \Psi_2$, let

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \ne (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{1.3}$$

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Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1.2). From this result, we can consider many non-ℓ_p-type norms easily. Now let

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases} \tag{1.4}$$

Then $\psi_p(t) \in \Psi_2$ and, as is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with ψ_p.

If X is a Banach space, then X is *uniformly nonsquare* if there exists $\delta \in (0,1)$ such that for any $x, y \in S_X$,

$$\text{either } \|x+y\| \le 2(1-\delta) \qquad \text{or} \qquad \|x-y\| \le 2(1-\delta), \tag{1.5}$$

where $S_X = \{x \in X : \|x\| = 1\}$. The *James constant* $J(X)$ is defined by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x,\ y \in S_X\}. \tag{1.6}$$

The *modulus of convexity of* X, $\delta_X : [0,2] \to [0,1]$ is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x,\ y \in S_X,\ \|x-y\| \ge \varepsilon\right\}. \tag{1.7}$$

The preceding parameters have been recently studied by several authors (cf. [4–6, 8, 9]). We collect together some known results.

Proposition 1.1. *Let X be a nontrivial Banach space, then*

(i) $\sqrt{2} \le J(X) \le 2$ *(Gao and Lau [5]),*

(ii) *if X is a Hilbert space, then $J(X) = \sqrt{2}$; the converse is not true (Gao and Lau [5]),*

(iii) *X is uniformly nonsquare if and only if $J(X) < 2$ (Gao and Lau [5]),*

(iv) $2J(X) - 2 \le J(X^*) \le J(X)/2 + 1$, $J(X^{**}) = J(X)$, *and there exists a Banach space X such that $J(X^*) \ne J(X)$ (Kato et al. [8]),*

(v) *if $2 \le p \le \infty$, then $\delta_{\ell_p}(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$ (Hanner [6]),*

(vi) $J(X) = \sup\{\varepsilon \in (0,2) : \delta_X(\varepsilon) \le 1 - \varepsilon/2\}$ *(Gao and Lau [5]).*

The paper is organized as follows. In Section 2 we introduce a new class of normalized norms on $\mathbb{R}^2$. This class properly contains all absolute normalized norms of Bonsall and Duncan [1]. The so-called generalized Day-James space, ℓ_ψ-ℓ_φ, where $\psi, \varphi \in \Psi_2$, is introduced and studied. More precisely, we prove that $(\ell_\psi\text{-}\ell_\varphi)^* = \ell_{\psi^*}\text{-}\ell_{\varphi^*}$ where ψ^* and φ^* are the dual functions of ψ and φ, respectively. In Section 3, the upper bound of the James constant of the generalized Day-James space is given. Furthermore, we compute $J(\ell_\psi\text{-}\ell_\infty)$ and deduce that every generalized Day-James space except ℓ_1-ℓ_1 and ℓ_∞-ℓ_∞ is uniformly nonsquare. This result strengthens Corollary 3 of Saito et al. [10].

2. Generalized Day-James spaces

In this section, we introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms of Bonsall and Duncan [1]. Moreover, we introduce a two-dimensional normed space which is a generalization of Day-James ℓ_p-ℓ_q spaces.

Lemma 2.1. *Let $\psi \in \Psi_2$ and let $\|\cdot\|_{\psi,\psi_\infty}$ be a function on $\mathbb{R}^2$ defined by, for all $(z,w) \in \mathbb{R}^2$,*

$$\begin{aligned}\|(z,w)\|_{\psi,\psi_\infty} &:= \max\{\|(z^+,w^+)\|_\psi, \|(z^-,w^-)\|_\psi\},\\ &= \begin{cases}\|(z,w)\|_\psi & \text{if } zw \geq 0,\\ \|(z,w)\|_\infty & \text{if } zw \leq 0,\end{cases}\end{aligned} \tag{2.1}$$

where x^+ and x^- are positive and negative parts of $x \in \mathbb{R}$, that is, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$. Then $\|\cdot\|_{\psi,\psi_\infty}$ is a norm on $\mathbb{R}^2$.

For convenience, we put $\mathcal{B}_{\psi_1,\psi_2} := \{(z,w) \in \mathbb{R}^2 : \|(z,w)\|_{\psi_1,\psi_2} \leq 1\}$.

Theorem 2.2. *Let $\psi, \varphi \in \Psi_2$ and*

$$\|(z,w)\|_{\psi,\varphi} := \begin{cases}\|(z,w)\|_\psi & \text{if } zw \geq 0,\\ \|(z,w)\|_\varphi & \text{if } zw \leq 0\end{cases} \tag{2.2}$$

for all $(z,w) \in \mathbb{R}^2$. Then $\|\cdot\|_{\psi,\varphi}$ is a norm on $\mathbb{R}^2$. Denote by N_2 the family of all such preceding norms.

Proof. Let $\psi, \varphi \in \Psi_2$, we only show $\|\cdot\|_{\psi,\varphi}$ satisfies the triangle inequality. To this end, it suffices to prove that $\mathcal{B}_{\psi,\varphi}$ is convex. By Lemma 2.1, we have that $\mathcal{B}_{\psi,\psi_\infty}$ and $\mathcal{B}_{\varphi,\psi_\infty}$ are closed unit balls of $\|\cdot\|_{\psi,\psi_\infty}$ and $\|\cdot\|_{\varphi,\psi_\infty}$, respectively, and so $\mathcal{B}_{\psi,\psi_\infty}$ and $\mathcal{B}_{\varphi,\psi_\infty}$ are convex sets. We define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T((z,w)) = (-z,w) \quad \forall (z,w) \in \mathbb{R}^2. \tag{2.3}$$

Then T is a linear operator and $T(\mathcal{B}_{\varphi,\psi_\infty}) = \mathcal{B}_{\psi_\infty,\varphi}$, which implies that $\mathcal{B}_{\psi_\infty,\varphi}$ is convex and so $\mathcal{B}_{\psi,\varphi} = \mathcal{B}_{\psi_\infty,\varphi} \cap \mathcal{B}_{\psi,\psi_\infty}$ is convex. □

Taking $\psi = \psi_p$ and $\varphi = \psi_q$ $(1 \leq p,q \leq \infty)$ in Theorem 2.2, we obtain the following.

Corollary 2.3 (Day-James ℓ_p-ℓ_q spaces). *For $1 \leq p$, $q \leq \infty$, denote by ℓ_p-ℓ_q the Day-James space, that is, $\mathbb{R}^2$ with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,*

$$\|(z,w)\|_{p,q} = \begin{cases}\|(z,w)\|_p & \text{if } zw \geq 0,\\ \|(z,w)\|_q & \text{if } zw \leq 0.\end{cases} \tag{2.4}$$

James [7] considered the ℓ_p-$\ell_{p'}$ space as an example of a Banach space which is isometric to its dual but which is not given by a Hilbert norm when $p \neq 2$. Day [2] considered even more general spaces, namely, if $(X, \|\cdot\|)$ is a two-dimensional Banach space and $(X^*, \|\cdot\|^*)$ its dual, then the X-X^* space is the space X with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,

$$\|(z,w)\|_{X,X^*} = \begin{cases}\|(z,w)\| & \text{if } zw \geq 0,\\ \|(z,w)\|^* & \text{if } zw \leq 0.\end{cases} \tag{2.5}$$

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For $\psi, \varphi \in \Psi_2$, denote by ℓ_ψ-ℓ_φ the *generalized Day-James space*, that is, $\mathbb{R}^2$ with the norm $\|\cdot\|_{\psi,\varphi}$ defined by (2.2). For ψ_p defined by (1.4), we write ℓ_ψ-ℓ_p for ℓ_ψ-ℓ_{ψ_p}. For example, if $1 \le p, q \le \infty$, ℓ_p-ℓ_q means ℓ_{ψ_p}-ℓ_{ψ_q}.

It is worthwhile to mention that there is a normalized norm which is not absolute.

Proposition 2.4. *There is $\psi \in \Psi_2$ such that ℓ_ψ-ℓ_∞ is not isometrically isomorphic to ℓ_φ-ℓ_φ for all $\varphi \in \Psi_2$.*

Proof. Let

$$\psi(t) := \begin{cases} 1 - t & \text{if } 0 \le t \le \dfrac{1}{8}, \\ \dfrac{11 - 4t}{12} & \text{if } \dfrac{1}{8} \le t \le \dfrac{1}{2}, \\ \dfrac{1+t}{2} & \text{if } \dfrac{1}{2} \le t \le 1. \end{cases} \tag{2.6}$$

We observe that the sphere of ℓ_ψ-ℓ_∞ is the octagon whose right half consists of 4 segments of different lengths. Suppose that there are $\varphi \in \Psi_2$ and an isometric isomorphism from ℓ_ψ-ℓ_∞ onto ℓ_φ-ℓ_φ. Since the image of each segment in ℓ_ψ-ℓ_∞ is again a segment of the same length in ℓ_φ-ℓ_φ, the sphere of ℓ_φ-ℓ_φ must be the octagon whose each corresponding side has the same length (measured by $\|\cdot\|_\varphi$). We show that this cannot happen. Consider $(1,0) \in S_{\ell_\varphi\text{-}\ell_\varphi}$. If $(1,0)$ is an extreme point of $B_{\ell_\varphi\text{-}\ell_\varphi}$, then $S_{\ell_\varphi\text{-}\ell_\varphi}$ contains 4 segments of same lengths since $\|\cdot\|_\varphi$ is absolute. On the other hand, if $(1,0)$ is an not extreme point of $B_{\ell_\varphi\text{-}\ell_\varphi}$, again $S_{\ell_\varphi\text{-}\ell_\varphi}$ contains 4 segments of same lengths. □

Next, we prove that the dual of a generalized Day-James space is again a generalized Day-James space. Recall that, for $\psi \in \Psi_2$, the *dual function* ψ^* of ψ is defined by

$$\psi^*(s) = \max_{0 \le t \le 1} \frac{(1-s)(1-t) + st}{\psi(t)} \tag{2.7}$$

for all $s \in [0,1]$. It was proved that $\psi^* \in \Psi_2$ and $(\ell_\psi\text{-}\ell_\psi)^* = \ell_{\psi^*}\text{-}\ell_{\psi^*}$ (see [3, Proposition 1 and Theorem 2]). We generalize this result to our spaces as follows.

Theorem 2.5. *For $\psi, \varphi \in \Psi_2$, there is an isometric isomorphism that identifies $(\ell_\psi\text{-}\ell_\varphi)^*$ with $\ell_{\psi^*}\text{-}\ell_{\varphi^*}$ such that if $f \in (\ell_\psi\text{-}\ell_\varphi)^*$ is identified with the element $(z,w) \in \ell_{\psi^*}\text{-}\ell_{\varphi^*}$, then*

$$f(u,v) = zu + wv \tag{2.8}$$

for all $(u,v) \in \mathbb{R}^2$.

Proof. We can prove analogous to [3, Theorem 2]. □

3. The James constant and uniform nonsquareness

The next lemmas are crucial for proving the main theorems.

Lemma 3.1. *Let $\psi, \varphi \in \Psi_2$. Then*

(i) $\|\cdot\|_\infty \le \|\cdot\|_{\psi,\varphi} \le \|\cdot\|_1$,

(ii) $(1/M_{\psi,\varphi})\|\cdot\|_{\varphi} \le \|\cdot\|_{\psi,\varphi} \le M_{\varphi,\psi}\|\cdot\|_{\psi}$,
(iii) $(1/M_{\varphi,\psi})\|\cdot\|_{\varphi} \le \|\cdot\|_{\psi,\varphi} \le M_{\psi,\varphi}\|\cdot\|_{\varphi}$,
where $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$ *and* $M_{\psi,\varphi} = \max_{0\le t\le 1}\psi(t)/\varphi(t)$.

LEMMA 3.2. *Let* $\psi,\varphi \in \Psi_2$ *and let* Q_i $(i = 1,\dots,4)$ *denote the ith quadrant in* $\mathbb{R}^2$. *Suppose that* $x, y \in S_{\ell_\psi\text{-}\ell_\varphi}$, *then the following statements are true.*
(i) *If* $x, y \in Q_1$, *then* $x + y \in Q_1$ *and* $x - y \in Q_2 \cup Q_4$.
(ii) *If* $x, y \in Q_2$, *then* $x + y \in Q_2$ *and* $x - y \in Q_1 \cup Q_3$.
(iii) *If* $\psi(t) \le \varphi(t)$ *for all* $t \in [0,1]$ *and* $x - y \in Q_2^\circ \cup Q_4^\circ$, *where* Q_2° *and* Q_4° *are the interiors of* Q_2 *and* Q_4, *respectively, then* $x + y \in Q_1 \cup Q_3$.

We will estimate the James constant of ℓ_ψ-ℓ_φ.

THEOREM 3.3. *Let* $\psi,\varphi \in \Psi_2$ *with* $\psi(t) \le \varphi(t)$ *for all* $t \in [0,1]$, *let* $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$, *and let* $\delta_\psi(\cdot)$ *be the modulus of convexity of* ℓ_ψ-ℓ_ψ. *Then for* $\varepsilon \in [0,2]$,

$$\delta_{\psi,\varphi}(\varepsilon) \ge \min\left\{1 - M_{\varphi,\psi}(1 - \delta_\psi(\varepsilon)),\ \delta_\psi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right\}, \tag{3.1}$$

where $\delta_{\psi,\varphi}(\cdot)$ *is the modulus of convexity of* ℓ_ψ-ℓ_φ. *Consequently,*

$$J(\ell_\psi\text{-}\ell_\varphi) \le \sup\left\{\varepsilon \in (0,2) : \varepsilon \le 2M_{\varphi,\psi}(1 - \delta_\psi(\varepsilon)) \text{ or } \varepsilon \le 2\left(1 - \delta_\psi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right)\right\}. \tag{3.2}$$

Proof. By Lemma 3.1(ii), we have

$$\|\cdot\|_\psi \le \|\cdot\|_{\psi,\varphi} \le M_{\varphi,\psi}\|\cdot\|_\psi. \tag{3.3}$$

We now evaluate the modulus of convexity $\delta_{\psi,\varphi}$ for ℓ_ψ-ℓ_φ. We consider two cases.

Case 1. Take $\|x\|_{\psi,\varphi} = \|y\|_{\psi,\varphi} = 1$ with $\|x - y\|_{\psi,\varphi} \ge \varepsilon$, where $x - y \in Q_1 \cup Q_3$. Thus $\|x\|_\psi \le 1$, $\|y\|_\psi \le 1$, and $\|x - y\|_\psi \ge \varepsilon$, which implies that

$$\frac{1}{2}\|x + y\|_\psi \le 1 - \delta_\psi(\varepsilon). \tag{3.4}$$

This in turn implies

$$\frac{1}{2}\|x + y\|_{\psi,\varphi} \le \frac{1}{2}M_{\varphi,\psi}\|x + y\|_\psi \le M_{\varphi,\psi}(1 - \delta_\psi(\varepsilon)), \tag{3.5}$$

thus

$$1 - \frac{1}{2}\|x + y\|_{\psi,\varphi} \ge 1 - M_{\varphi,\psi}(1 - \delta_\psi(\varepsilon)). \tag{3.6}$$

Case 2. Now take x, y as above, but with $x - y \in Q_2^\circ \cup Q_4^\circ$. By Lemma 3.2(iii), $x + y \in Q_1 \cup Q_3$. Since $\|x - y\|_{\psi,\varphi} \ge \varepsilon$,

$$\|x - y\|_\psi \ge \frac{\|x - y\|_{\psi,\varphi}}{M_{\varphi,\psi}} \ge \frac{\varepsilon}{M_{\varphi,\psi}}. \tag{3.7}$$

Then

$$\frac{1}{2}\|x+y\|_{\psi,\varphi} = \frac{1}{2}\|x+y\|_{\psi} \le 1-\delta_{\psi}\Big(\frac{\varepsilon}{M_{\varphi,\psi}}\Big), \tag{3.8}$$

and so

$$1-\frac{1}{2}\|x+y\|_{\psi,\varphi} \ge \delta_{\psi}\Big(\frac{\varepsilon}{M_{\varphi,\psi}}\Big). \tag{3.9}$$

Hence we obtain (3.1). By Proposition 1.1(vi), (3.2) follows. □

The following corollary shows that we can have equality in (3.2).

Corollary 3.4 [4, 8]. *If $1 \le q \le p < \infty$ and $p \ge 2$, then*

$$J(\ell_p\text{-}\ell_q) \le 2\Big(\frac{2^{p/q}}{2^{p/q}+2}\Big)^{1/p}. \tag{3.10}$$

In particular, if $p=2$ and $q=1$, then $J(\ell_2\text{-}\ell_1) = \sqrt{8/3}$.

Proof. It follows that since

$$M_{\psi_q,\psi_p} = 2^{1/q-1/p}, \qquad \delta_{\ell_p\text{-}\ell_p}(\varepsilon) = 1-\bigg(1-\Big(\frac{\varepsilon}{2}\Big)^p\bigg)^{1/p}. \tag{3.11}$$

Moreover, if $p=2$ and $q=1$, then $J(\ell_2\text{-}\ell_1) \le \sqrt{8/3}$. Now we put

$$x_0 = \Big(\frac{2+\sqrt{2}}{2\sqrt{3}}, \frac{2-\sqrt{2}}{2\sqrt{3}}\Big), \qquad y_0 = \Big(\frac{2-\sqrt{2}}{2\sqrt{3}}, \frac{2+\sqrt{2}}{2\sqrt{3}}\Big). \tag{3.12}$$

Then

$$\|x_0\|_{2,1} = \|y_0\|_{2,1} = 1, \qquad \|x_0 \pm y_0\|_{2,1} = \sqrt{\frac{8}{3}}. \tag{3.13}$$

□

Theorem 3.5. *Let $\psi,\varphi \in \Psi_2$ with $\psi(t) \le \varphi(t)$ for all $t\in[0,1]$, let $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$, and let $\delta_{\varphi}(\cdot)$ be the modulus of convexity of $\ell_{\varphi}\text{-}\ell_{\varphi}$. Then for $\varepsilon\in[0,2]$,*

$$\delta_{\psi,\varphi}(\varepsilon) \ge 1-M_{\varphi,\psi}\bigg(1-\delta_{\varphi}\Big(\frac{\varepsilon}{M_{\varphi,\psi}}\Big)\bigg), \tag{3.14}$$

where $\delta_{\psi,\varphi}(\cdot)$ is the modulus of convexity of $\ell_{\psi}\text{-}\ell_{\varphi}$. Consequently,

$$J(\ell_{\psi}\text{-}\ell_{\varphi}) \le \sup\bigg\{\varepsilon\in(0,2) : \varepsilon \le 2M_{\varphi,\psi}\bigg(1-\delta_{\varphi}\Big(\frac{\varepsilon}{M_{\varphi,\psi}}\Big)\bigg)\bigg\}. \tag{3.15}$$

Proof. By Lemma 3.1(iii), we have

$$\frac{1}{M_{\varphi,\psi}}\|\cdot\|_{\varphi} \le \|\cdot\|_{\psi,\varphi} \le \|\cdot\|_{\varphi}. \tag{3.16}$$

We now evaluate the modulus of convexity $\delta_{\psi,\varphi}$ for ℓ_ψ-ℓ_φ. Let

$$\|x\|_{\psi,\varphi} = \|y\|_{\psi,\varphi} = 1 \quad \text{with } \|x-y\|_{\psi,\varphi} \geq \varepsilon. \tag{3.17}$$

Then

$$\begin{gathered} \frac{1}{M_{\varphi,\psi}}\|x\|_\varphi \leq 1, \qquad \frac{1}{M_{\varphi,\psi}}\|y\|_\varphi \leq 1, \\ \frac{1}{M_{\varphi,\psi}}\|x-y\|_\varphi \geq \frac{1}{M_{\varphi,\psi}}\|x-y\|_{\psi,\varphi} \geq \frac{\varepsilon}{M_{\varphi,\psi}}, \end{gathered} \tag{3.18}$$

which implies that

$$\frac{1}{2M_{\varphi,\psi}}\|x+y\|_\varphi \leq 1 - \delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right). \tag{3.19}$$

This in turn implies that

$$\frac{1}{2M_{\varphi,\psi}}\|x+y\|_{\psi,\varphi} \leq \frac{1}{2M_{\varphi,\psi}}\|x+y\|_\varphi \leq 1 - \delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right), \tag{3.20}$$

thus

$$1 - \frac{1}{2}\|x+y\|_{\psi,\varphi} \geq 1 - M_{\varphi,\psi}\left(1 - \delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right). \tag{3.21}$$

Hence we obtain (3.14). By Proposition 1.1(vi), (3.15) follows. □

COROLLARY 3.6. *If $2 \leq q \leq p < \infty$, then*

$$J(\ell_p\text{-}\ell_q) \leq 2^{1-1/p}. \tag{3.22}$$

It is easy to see that the estimate (3.22) is better than one obtained in [4, Example 2.4(3)].

For some generalized Day-James spaces, [8, Corollary 4] of Kato et al. gives only rough result for the estimate of the James constant, that is, for $\psi \in \Psi_2$,

$$\frac{2}{M} \leq J(\ell_\psi\text{-}\ell_\infty) \leq 2M, \tag{3.23}$$

where $M = \max_{0\leq t\leq 1} \psi_\infty(t)/\psi(t)$.

However, the following theorem gives the exact value of the James constant of these spaces.

THEOREM 3.7. *Let $\psi \in \Psi_2$. Then*

$$J(\ell_\psi\text{-}\ell_\infty) = 1 + \frac{1/2}{\psi(1/2)}. \tag{3.24}$$

Proof. For our convenience, we write $\|\cdot\|$ instead of $\|\cdot\|_{\psi,\psi_\infty}$. Let $x, y \in S_{\ell_\psi\text{-}\ell_\infty}$. We prove that

$$\text{either } \|x+y\| \leq 1 + \frac{1/2}{\psi(1/2)} \qquad \text{or} \qquad \|x-y\| \leq 1 + \frac{1/2}{\psi(1/2)}. \tag{3.25}$$

Let us consider the following cases.

Case 1. $x, y \in Q_1$. Let $x = (a,b)$ and $y = (c,d)$ where $a,b,c,d \in [0,1]$. By Lemma 3.2(i), we have $x - y \in Q_2 \cup Q_4$. Then

$$\|x-y\| = \max\{|a-c|, |b-d|\} \leq 1 \leq 1 + \frac{1/2}{\psi(1/2)}. \tag{3.26}$$

Case 2. $x, y \in Q_2$. If x, y lies in the same segment, then $\|x-y\| \leq 1$. We now suppose that $x = (-1, a)$ and $y = (-c, 1)$ where $a, c \in [0,1]$.

Subcase 2.1. $a \leq (1/2)/\psi(1/2)$ and $c \leq (1/2)/\psi(1/2)$. Then

$$\|x+y\| = \|(-1-c, 1+a)\|_\infty = \max\{1+c,\ 1+a\} \leq 1 + \frac{1/2}{\psi(1/2)}. \tag{3.27}$$

Subcase 2.2. $a \geq (1/2)/\psi(1/2)$ or $c \geq (1/2)/\psi(1/2)$. Put $z = (-1, 1)$, then

$$\|x-y\| \leq \|x-z\| + \|z-y\| = 1 - a + 1 - c \leq 1 + 1 - \frac{1/2}{\psi(1/2)} \leq 1 + \frac{1/2}{\psi(1/2)}. \tag{3.28}$$

From now on, we may assume without loss of generality that there is $\beta \in [1/2, 1]$ such that $\psi(\beta) \leq \psi(t)$ for all $t \in [0,1]$. Indeed, $J(\ell_\psi\text{-}\ell_\infty) = J(\ell_{\tilde\psi}\text{-}\ell_\infty)$ where $\tilde\psi(t) = \psi(1-t)$ for all $t \in [0,1]$.

Case 3. $x \in Q_1$ and $y \in Q_2$. Let $x = (a,b)$, $y = (-c, 1)$ where $a,b,c \in [0,1]$. We consider three subcases.

Subcase 3.1. $a \leq (1/2)/\psi(1/2)$ or $c \leq (1/2)/\psi(1/2)$. Then

$$\|x-y\| = \|(a+c, b-1)\|_\infty = \max\{a+c,\ 1-b\} \leq 1 + \frac{1/2}{\psi(1/2)}. \tag{3.29}$$

Subcase 3.2. $(1/2)/\psi(1/2) \leq a \leq c$. Then $b \leq (1/2)/\psi(1/2)$ and

$$\|x+y\| = \|(a-c, b+1)\|_\infty = \max\{c-a,\ 1+b\} \leq 1 + \frac{1/2}{\psi(1/2)}. \tag{3.30}$$

Subcase 3.3. $(1/2)/\psi(1/2) < c \leq a$. We write $a = (1-t_0)/\psi(t_0)$, $b = t_0/\psi(t_0)$ where $t_0 = b/(a+b)$ and $0 \leq t_0 \leq 1/2$. By the convexity of ψ and $\psi(t) \geq \psi(\beta)$ for all $0 \leq t \leq 1$, we have $\psi(t_0) \geq \psi(1/2)$ and so $1/\psi(t_0) \leq 1/\psi(1/2)$. By Lemma 3.1(i),

$$\begin{aligned}\|x+y\| &= \|(a,b)+(-c,1)\| \leq \|(a-c,b+1)\|_1 \\ &= a-c+b+1 = \frac{1}{\psi(t_0)}+1-c \\ &\leq \frac{1}{\psi(1/2)}+1-\frac{1/2}{\psi(1/2)} = 1+\frac{1/2}{\psi(1/2)}.\end{aligned} \tag{3.31}$$

Case 4. $x \in Q_1$ and $y \in Q_2$. Let $x=(a,b)$, $y=(-1,c)$ where $a,b,c \in [0,1]$. We consider three subcases.

Subcase 4.1. $b \leq (1/2)/\psi(1/2)$ or $c \leq (1/2)/\psi(1/2)$. Then

$$\|x+y\| = \|(a-1,b+c)\|_\infty = \max\{1-a,\ b+c\} \leq 1+\frac{1/2}{\psi(1/2)}. \tag{3.32}$$

Subcase 4.2. $(1/2)/\psi(1/2) < b \leq c$. Then $a \leq (1/2)/\psi(1/2)$ and

$$\|x-y\| = \|(1+a,b-c)\|_\infty = \max\{1+a,c-b\} \leq 1+\frac{1/2}{\psi(1/2)}. \tag{3.33}$$

Subcase 4.3. $(1/2)/\psi(1/2) < c \leq b$. We write $a=(1-t_0)/\psi(t_0)$, $b=t_0/\psi(t_0)$, where $t_0 = b/(a+b)$ and $1/2 \leq t_0 \leq 1$. We choose $\alpha = b/(a+2b-1)$, then

$$\frac{1}{2} \leq \alpha \leq 1, \qquad a = \frac{1-2\alpha}{\alpha}b+1. \tag{3.34}$$

Since $b-c \leq 1+a$ and $b \leq 1$,

$$\frac{b-c}{1+a+b-c} \leq \frac{1}{2} \leq t_0 \leq \alpha. \tag{3.35}$$

Let

$$\psi_\alpha(t) = \begin{cases} \dfrac{\alpha-1}{\alpha}t+1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases} \tag{3.36}$$

We see that $\psi_\alpha(t_0) = \psi(t_0)$. By the convexity of ψ, we have

$$\psi(t) \leq \psi_\alpha(t) \quad \forall t \leq t_0. \tag{3.37}$$

Therefore,

$$\begin{aligned}
\|x-y\| &= \|(a+1,b-c)\|_\psi = (1+a+b-c)\psi\left(\frac{b-c}{1+a+b-c}\right) \\
&\le (1+a+b-c)\psi_\alpha\left(\frac{b-c}{1+a+b-c}\right) = \frac{\alpha-1}{\alpha}(b-c)+1+a+b-c \\
&= 1+a+\frac{2\alpha-1}{\alpha}b-\frac{2\alpha-1}{\alpha}c = 1+1-\frac{2\alpha-1}{\alpha}c \qquad (3.38)\\
&< 1+1-\frac{2\alpha-1}{\alpha}\frac{1/2}{\psi(1/2)} = 1+\frac{1/2}{\psi(1/2)}+1-\frac{3\alpha-1}{2\alpha}\frac{1}{\psi(1/2)} \\
&= 1+\frac{1/2}{\psi(1/2)}+1-\frac{\psi_\alpha(1/2)}{\psi(1/2)} \le 1+\frac{1/2}{\psi(1/2)}.
\end{aligned}$$

Finally, we conclude that

$$J(\ell_\psi\text{-}\ell_\infty) \le 1+\frac{1/2}{\psi(1/2)}. \qquad (3.39)$$

Now, we put $x_0 = ((1/2)/\psi(1/2),(1/2)/\psi(1/2))$ and $y_0 = (-1,1)$, then

$$\|x_0\| = \|y_0\| = 1, \qquad \|x_0 \pm y_0\| = 1+\frac{1/2}{\psi(1/2)}. \qquad (3.40)$$

Thus,

$$J(\ell_\psi\text{-}\ell_\infty) \ge \min\{\|x_0-y_0\|, \|x_0+y_0\|\} = 1+\frac{1/2}{\psi(1/2)}. \qquad (3.41)$$

This together with (3.39) completes the proof. □

Corollary 3.8 [4, Example 2.4(2)]. *Let $1 \le p \le \infty$, then*

$$J(\ell_p\text{-}\ell_\infty) = 1+\left(\frac{1}{2}\right)^{1/p}. \qquad (3.42)$$

Indeed, $\psi_p(1/2) = 2^{1/p-1}$.

We now obtain the bounds for $J(\ell_\psi\text{-}\ell_1)$.

Corollary 3.9. *Let $\psi \in \Psi_2$. Then*

$$2\min_{0\le t\le 1}\psi(t) \le J(\ell_\psi\text{-}\ell_1) \le \frac{3}{2}+\frac{1}{2}\min_{0\le t\le 1}\psi(t). \qquad (3.43)$$

Proof. Note that $\psi^*(1/2) = \max_{0\le t\le 1}(1/2)/\psi(t) = 1/2\min_{0\le t\le 1}\psi(t)$. By Theorem 3.7, we have $J(\ell_{\psi^*}\text{-}\ell_\infty) = 1+\min_{0\le t\le 1}\psi(t)$. Applying Proposition 1.1(iv), the assertion is obtained. □

We now improve the upper bound for $J(\ell_p\text{-}\ell_1)$ (see also Corollary 3.4).

COROLLARY 3.10. *Let $1 \le p < \infty$. Then*

$$J(\ell_p\text{-}\ell_1) \le \frac{3}{2} + \left(\frac{1}{2}\right)^{2-1/p}. \tag{3.44}$$

In particular, if $p \ge 2$, then

$$J(\ell_p\text{-}\ell_1) \le \min\left\{\frac{4}{(2^p+2)^{1/p}}, \frac{3}{2} + \left(\frac{1}{2}\right)^{2-1/p}\right\}. \tag{3.45}$$

The following corollary follows by Theorem 3.7 and Corollary 3.9.

COROLLARY 3.11. *Let $\psi \in \Psi_2$. Then*
(i) *ℓ_ψ-ℓ_∞ is uniformly nonsquare if and only if $\psi \ne \psi_\infty$,*
(ii) *ℓ_ψ-ℓ_1 is uniformly nonsquare if and only if $\psi \ne \psi_1$.*

We can say more about the uniform nonsquareness of ℓ_ψ-ℓ_φ.

THEOREM 3.12. *Let $\psi, \varphi \in \Psi_2$. Then all ℓ_ψ-ℓ_φ except ℓ_1-ℓ_1 and ℓ_∞-ℓ_∞ are uniformly nonsquare.*

Proof. If $\psi = \varphi$, we are done by [10, Corollary 3]. Assume that $\psi \ne \varphi$. We prove that ℓ_ψ-ℓ_φ is uniformly nonsquare. Suppose not, that is, there are $x, y \in S_{\ell_\psi\text{-}\ell_\varphi}$ such that $\|x \pm y\|_{\psi,\varphi} = 2$. We consider three cases.

Case 1. $x, y \in Q_1$. Then

$$\begin{aligned} \|x\|_{\psi,1} = \|x\|_\psi = \|x\|_{\psi,\varphi} = 1, \\ \|y\|_{\psi,1} = \|y\|_\psi = \|y\|_{\psi,\varphi} = 1. \end{aligned} \tag{3.46}$$

It follows by Lemma 3.2(i) that $x + y \in Q_1$ and $x - y \in Q_2 \cup Q_4$. Therefore

$$\begin{aligned} \|x + y\|_{\psi,1} = \|x + y\|_{\psi,\varphi} = 2, \\ 2 = \|x - y\|_{\psi,\varphi} \le \|x - y\|_1 = \|x - y\|_{\psi,1} \le 2. \end{aligned} \tag{3.47}$$

Hence $\|x \pm y\|_{\psi,1} = 2$ and this implies that ℓ_ψ-ℓ_1 is not uniformly nonsquare. By Corollary 3.11(ii), we have $\psi = \psi_1$. Again, since ℓ_ψ-$\ell_\varphi = \ell_1$-ℓ_φ is not uniformly nonsquare, $\varphi = \psi_1 = \psi$; a contradiction.

Case 2. $x, y \in Q_2$. It is similar to Case 1, so we omit the proof.

Case 3. $x := (a, b) \in Q_1$ and $y := (-c, d) \in Q_2$ where $a, b, c, d \in [0, 1]$. Since $\|x + y\|_{\psi,\varphi} = 2$, the line segment joining x and y must lie in the sphere. In particular, there is $\alpha \in [0, 1]$ such that

$$(0, 1) = \alpha x + (1 - \alpha) y. \tag{3.48}$$

It follows that $b = 1$ since $b, d \le 1$. Similarly consider x and $-y$ instead of x and y, we can also conclude that $a = 1$. Hence $\|(1, 1)\|_\psi = \|(1, 1)\|_{\psi,\varphi} = 1$, that is, $\psi(1/2) = 1/2$. Then $\psi = \psi_\infty$ and so ℓ_ψ-$\ell_\varphi = \ell_\infty$-$\ell_\varphi$ is not uniformly nonsquare. By Corollary 3.11(i), we have $\varphi = \psi_\infty = \psi$; a contradiction. □

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On the modulus of W^*-convexity ☆

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Abstract

In this paper, we prove that the moduli of W^*-convexity, introduced by Ji Gao [J. Gao, The W^*-convexity and normal structure in Banach spaces, Appl. Math. Lett. 17 (2004) 1381–1386], of a Banach space X and of the ultrapower $\widetilde{X}$ of X itself coincide whenever X is super-reflexive. Moreover, we improve a sufficient condition for uniform normal structure of the space and its dual. This generalizes and strengthens the main results of [J. Gao, The W^*-convexity and normal structure in Banach spaces, Appl. Math. Lett. 17 (2004) 1381–1386].

Keywords: Modulus of W^*-convexity; Uniform normal structure; Super-reflexive space

1. Introduction

Let X be a Banach space, and let $B_X := \{x \in X\colon \|x\| \leqslant 1\}$ and $S_X := \{x \in X\colon \|x\| \leqslant 1\}$ denote the unit ball and unit sphere of X, respectively.

In [4], Gao introduced the modulus of W^*-convexity of a Banach space X, as follows:

$$W_X^*(\varepsilon) = \inf\left\{\frac{1}{2} f(x-y)\colon x, y \in S_X,\ \|x-y\| \geqslant \varepsilon,\ f \in \nabla_x\right\}.$$

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Appendix 12: On the modulus of W*-convexity, J. Math. Anal. Appl. 320 (2006), 543-548.

Here $\nabla_x := \{f \in S_{X^*}\colon f(x) = \|x\|\}$ is the set of norm 1 supporting functionals at x. This modulus is the same as one introduced by Bynum in [2] and by Prus–Szczepanik in [10]. More precisely,

$$W_X^*(\cdot) = \frac{1}{2}\beta_X(\cdot) = V_X(\cdot).$$

This paper is organized as follows: In Section 2 we give some equivalent formulations of the modulus of W^*-convexity and prove that if a Banach space X is super-reflexive, then the moduli of W^*-convexity of the ultrapower $\widetilde{X}$ of X and of X itself coincide. This result improves Gao's result without assuming the uniform convexity of the space. Using ultrapower methods, in Section 3, we show that a Banach space X and its dual X^* have uniform normal structure whenever $W_X^*(\varepsilon) > \frac{1}{2}\max\{0, \varepsilon - 1\}$ for some $\varepsilon \in (0, 2)$.

2. The modulus of W^*-convexity

First we recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}$ in X *converges to x with respect to $\mathcal{F}$*, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N}\colon x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an *ultrafilter* if it is maximal with respect to set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A\colon A \subset \mathbb{N}, i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called *nontrivial*. Let $l_\infty(X)$ denote the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \Big\{(x_n) \in l_\infty(X)\colon \lim_{\mathcal{U}} \|x_n\| = 0\Big\}.$$

The *ultrapower* of X, denoted by $\widetilde{X}$, is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\|(x_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|.$$

Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically. For more details see [11].

Lemma 1. (Bishop–Phelps–Bollobás [1]) *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $h(z) > 1 - \varepsilon^2/4$, then there exists $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Lemma 2. (Megginson [8]) *Suppose that $x, y \in S_X$ and $\|x - y\| = \varepsilon$ where $0 < \varepsilon < 2$. Then there exist sequences $\{x_n\}, \{y_n\} \subset S_X$ such that $\|x_n - y_n\| > \varepsilon$ for all $n \in \mathbb{N}$, $x_n \to x$, and $y_n \to y$.*

We first give equivalent formulations of the modulus $W_X^*(\varepsilon)$.

Theorem 3. *For a Banach space X and* $0 \leqslant \varepsilon < 2$,

$$\begin{aligned} W_X^*(\varepsilon) &= \inf\left\{\frac{1}{2}f(x-y)\colon x \in S_X,\ y \in B_X,\ \|x-y\| \geqslant \varepsilon,\ f \in \nabla_x\right\} \\ &= \inf\left\{\frac{1}{2}f(x-y)\colon x, y \in S_X,\ \|x-y\| > \varepsilon,\ f \in \nabla_x\right\} \\ &= \inf\left\{\frac{1}{2}f(x-y)\colon x, y \in S_X,\ \|x-y\| = \varepsilon,\ f \in \nabla_x\right\}. \end{aligned}$$

Proof. For convenience, let $W_1(\varepsilon)$, $W_2(\varepsilon)$ and $W_3(\varepsilon)$ denote the infima in the proposition in the order in which they occur. We first prove that $W_X^*(\varepsilon) \leqslant W_1(\varepsilon)$. Let $x \in S_X$, $y \in B_X$ and $f \in \nabla_x$ be such that $\|x-y\| \geqslant \varepsilon$. Suppose that $\|y\| < 1$. Then $y = \lambda z + (1-\lambda)z'$ for some $z, z' \in S_X$ and $\lambda \in (0,1)$ such that $f(z) = f(z') = f(y)$. By the triangle inequality, we have

$$\varepsilon \leqslant \|x-y\| \leqslant \lambda\|x-z\| + (1-\lambda)\|x-z'\|.$$

This implies either $\|x-z\| \geqslant \varepsilon$ or $\|x-z'\| \geqslant \varepsilon$, so $\frac{1}{2}f(x-z) \geqslant W_X^*(\varepsilon)$ or $\frac{1}{2}f(x-z') \geqslant W_X^*(\varepsilon)$. Hence $\frac{1}{2}f(x-y) \geqslant W_X^*(\varepsilon)$.

Now, we prove that $W_2(\varepsilon) \leqslant W_3(\varepsilon)$. Let $x, y \in S_X$ and $f \in \nabla_x$ be such that $\|x-y\| = \varepsilon$. Then there exist sequences $\{x_n\}, \{y_n\} \subset S_X$ such that $\|x_n - y_n\| > \varepsilon$ for all $n \in \mathbb{N}$, $x_n \to x$, and $y_n \to y$. Hence $f(x_n) \to f(x) = 1$. By the Bishop–Phelps–Bollobás Theorem, there are sequences $\{x_n'\} \subset S_X$ and $\{f_n'\} \subset S_{X^*}$ such that $f_n' \in \nabla_{x_n'}$ for all $n \in \mathbb{N}$, $f_n' - f \to 0$ and $x_n' - x_n \to 0$. Passing to subsequences we may assume that $\|x_n' - y_n\| > \varepsilon$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} W_2(\varepsilon) &\leqslant \frac{1}{2}f_n'(x_n' - y_n) = \frac{1}{2}(f_n' - f)(x_n' - y_n) + \frac{1}{2}f_n(x_n' - x_n) + \frac{1}{2}f(x_n - y_n) \\ &\leqslant \|f_n' - f\| + \frac{1}{2}\|x_n' - x_n\| + \frac{1}{2}f(x_n - y_n) \\ &\to \frac{1}{2}f(x-y). \end{aligned}$$

Finally, since

$$W_1(\varepsilon) \leqslant W_2(\varepsilon) \quad \text{and} \quad W_3(\varepsilon) \leqslant W_X^*(\varepsilon),$$

the proof is finished. □

The following is our main result.

Theorem 4. *Suppose that X is super-reflexive. Then* $W_X^*(\cdot) = W_{\tilde{X}}^*(\cdot)$ *on* $[0,2)$. *In particular, if* $W_X^*(\varepsilon) > 0$ *for some* $\varepsilon \in (0,2)$, *then* $W_X^*(\varepsilon) = W_{\tilde{X}}^*(\varepsilon)$.

Proof. Clearly, $W_X^*(\varepsilon) \geqslant W_{\tilde{X}}^*(\varepsilon)$ for all $\varepsilon \in [0,2)$. We now prove the reverse inequality. Let $\tilde{x}, \tilde{y} \in S_{\tilde{X}}$ and $\tilde{f} \in \nabla_{\tilde{x}}$ be such that $\|\tilde{x} - \tilde{y}\| > \varepsilon$ where $\varepsilon \in [0,2)$. We write $\tilde{x} = (x_n)_{\mathcal{U}}$

and $\tilde{y} = (y_n)_{\mathcal{U}}$, where $x_n, y_n \in X$ for all $n \in \mathbb{N}$. By the super-reflexivity of X, we can write $\tilde{f} = (f_n)_{\mathcal{U}}$ where $f_n \in X^*$ for all $n \in \mathbb{N}$. Then, we have

$$\lim_{\mathcal{U}} \|x_n\| = \lim_{\mathcal{U}} \|y_n\| = \lim_{\mathcal{U}} \|f_n\| = \lim_{\mathcal{U}} f_n(x_n) = 1, \qquad \lim_{\mathcal{U}} \|x_n - y_n\| > \varepsilon.$$

Put $x_n' = x_n/\|x_n\|$, $y_n' = y_n/\|y_n\|$, and $f_n' = f_n/\|f_n\|$ for all $n \in \mathbb{N}$. By the Bishop–Phelps–Bollobás Theorem, there are sequences $\{x_n''\} \subset S_X$ and $\{f_n''\} \subset S_{X^*}$ such that $f_n'' \in \nabla_{x_n''}$ for all $n \in \mathbb{N}$, and $\lim_{\mathcal{U}} \|f_n'' - f_n'\| = \lim_{\mathcal{U}} \|x_n'' - x_n'\| = 0$. Then

$$\begin{aligned}
\frac{1}{2}\tilde{f}(\tilde{x} - \tilde{y}) &= \lim_{\mathcal{U}} \frac{1}{2} f_n(x_n - y_n) = \lim_{\mathcal{U}} \frac{1}{2} f_n'(x_n' - y_n') \\
&= \lim_{\mathcal{U}} \frac{1}{2}\big(f_n'(x_n' - x_n'') + (f_n' - f_n'')(x_n'') + f_n''(x_n'' - y_n')\big) \\
&= \lim_{\mathcal{U}} \frac{1}{2} f_n''(x_n'' - y_n') \\
&\geqslant W_X^*(\varepsilon).
\end{aligned}$$

This means that $W_{\tilde{X}}^*(\varepsilon) \geqslant W_X^*(\varepsilon)$ and the proof of the fist part is finished.

Finally, if $W_X^*(\varepsilon) > 0$ for some $\varepsilon \in (0, 2)$, then X is uniformly nonsquare (see [4]) and hence super-reflexive (see [6]). This completes the proof. □

It is worth noting that [4, Theorem 3] is true with the weaker assumption.

3. Uniform normal structure

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* provided the inequality

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for every $x, y \in C$. Now, a Banach space X is said to have the *fixed point property* if every nonexpansive mapping $T : C \to C$, where C is a nonempty bounded closed convex subset of a Banach space X, has a fixed point.

Recall that a bounded convex subset K of a Banach space X is said to have *normal structure* if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\big\{\|x_0 - y\|: y \in H\big\} < \sup\big\{\|x - y\|: x, y \in H\big\}.$$

A Banach space X is said to have *weak normal structure* if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. A Banach space X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\big\{\|x_0 - y\|: y \in K\big\} < c \sup\big\{\|x - y\|: x, y \in K\big\}.$$

It was proved by W.A. Kirk that every reflexive Banach space with normal structure has the fixed point property (see [7]).

The following extends [4, Theorem 2] and [10, Corollary 3.18].

Theorem 5. *If $W_X^*(\varepsilon) > \frac{1}{2}\max\{0, \varepsilon - 1\}$ for some $\varepsilon \in (0,2)$, then X and X^* have uniform normal structure.*

Proof. It suffices to prove that X has weak normal structure whenever $W_X^*(\varepsilon) > \frac{1}{2}\max\{0, \varepsilon - 1\}$ for some $\varepsilon \in (0,2)$, since $W_X^*(\varepsilon) > 0$ implies that X is super-reflexive, and then $W_{\tilde{X}}^*(\varepsilon) = W_X^*(\varepsilon)$. Now suppose that X fails to have weak normal structure. Then, by the classical argument (see [5]), there exists a weakly null sequence $\{x_n\}_{n=1}^\infty$ such that

$$\lim_n \|x - x_n\| = 1 \quad \text{for all } x \in \operatorname{co}\{x_n\}_{n=1}^\infty.$$

Let $\{f_n\} \subset S_{X^*}$ be such that $f_n \in \nabla_{x_n}$ for all $n \in \mathbb{N}$. By the reflexivity of X^*, we may assume that $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$. We now choose a subsequence of $\{x_n\}_{n=1}^\infty$, denoted again by $\{x_n\}_{n=1}^\infty$, such that

$$\lim_n \|x_n - x_{n+1}\| = 1, \qquad \big|(f_{n+1} - f)(x_n)\big| < \frac{1}{n}, \quad \text{and} \quad \big|f_n(x_{n+1})\big| < \frac{1}{n}$$

for all $n \in \mathbb{N}$. It follows that $\lim_n f_{n+1}(x_n) = \lim_n (f_{n+1} - f)(x_n) + f(x_n) = 0$.

Put $\tilde{x} = (x_n - x_{n+1})_{\mathcal{U}}$, $\tilde{y} = ((1-\lambda)x_n + \lambda x_{n+1})_{\mathcal{U}}$, and $\tilde{f} = (f_n)_{\mathcal{U}}$ where $\lambda \in [0,1)$. Then

$$\tilde{f}(\tilde{x}) = 1, \qquad \|\tilde{x} - \tilde{y}\| \geqslant 1 + \lambda, \quad \text{and} \quad \frac{1}{2}\tilde{f}(\tilde{x} - \tilde{y}) = \frac{\lambda}{2}.$$

But, this implies $W_X^*(\varepsilon) = W_{\tilde{X}}^*(\varepsilon) \leqslant \frac{1}{2}\max\{0, \varepsilon - 1\}$ for all $\varepsilon \in [0,2)$.

Similarly, put $\tilde{x} = (f_n)_{\mathcal{U}}$, $\tilde{y} = (\lambda f_{n+1} - (1-\lambda)f_{n+2})_{\mathcal{U}}$, and $\tilde{f} = (x_n - x_{n+2})_{\mathcal{U}}$ where $\lambda \in [0,1)$. Hence

$$\tilde{f}(\tilde{x}) = 1, \qquad \|\tilde{x} - \tilde{y}\| \geqslant 1 + \lambda, \quad \text{and} \quad \frac{1}{2}\tilde{f}(\tilde{x} - \tilde{y}) = \frac{\lambda}{2}.$$

But, this implies $W_{X^*}^*(\varepsilon) = W_{\tilde{X}^*}^*(\varepsilon) = W_{(\tilde{X})^*}^*(\varepsilon) \leqslant \frac{1}{2}\max\{0, \varepsilon - 1\}$ for all $\varepsilon \in [0,2)$. □

Since $W_X^*(\varepsilon) \geqslant \delta_X(\varepsilon)$, the modulus of convexity defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\|\colon x, y \in B_X,\ \|x - y\| \geqslant \varepsilon\right\},$$

we have the following corollary which strengthens Theorem 8 of Gao [3] and Corollary 3 of Prus [9].

Corollary 6. *If $\delta_X(\varepsilon) > \max\{\frac{\varepsilon-1}{2}, 0\}$ for some $\varepsilon \in (0,2)$, then X and X^* have uniform normal structure.*

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On James and von Neumann–Jordan constants and sufficient conditions for the fixed point property

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Abstract

In this paper, we prove that a Banach space X and its dual space X^* have uniform normal structure if $C_{\mathrm{NJ}}(X) < (1+\sqrt{3})/2$. The García-Falset coefficient $R(X)$ is estimated by the $C_{\mathrm{NJ}}(X)$-constant and the weak orthogonality coefficient introduced by B. Sims. Finally, we present an affirmative answer to a conjecture by L. Maligranda concerning the relation between the James and $C_{\mathrm{NJ}}(X)$-constants for a Banach space.

Keywords: Von Neumann–Jordan constant; James constant; García-Falset coefficient; Weak orthogonality coefficient; Uniform normal structure

1. Introduction

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T: C \to C$ is said to be *nonexpansive* provided the inequality

$$\|Tx - Ty\| \leqslant \|x - y\|$$

holds for every $x, y \in C$. A Banach space X is said to have the *fixed point property* if every nonexpansive mapping $T: C \to C$, where C is a nonempty bounded closed convex subset of X, has a fixed point.

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Many mathematicians have established that, under various geometric properties of the Banach space X often measured by different constants, the fixed point property of X is guaranteed.

How the classical modulus of convexity $\delta_X(\cdot)$ of a Banach space X, introduced by J.A. Clarkson in 1936 [3], relates to the fixed point property has been widely studied. It is well known [9, Theorem 5.12, p. 122] that if $\delta_X(1) > 0$ then X and X^* have the fixed point property. Recently, J. García-Falset proved that every weakly nearly uniformly smooth space has the fixed point property. To prove this, he introduced the following coefficient, the so-called *García-Falset coefficient*:

$$R(X) := \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\},$$

where the supremum is taken over all weakly null sequences (x_n) in B_X $(:= \{x \in X\colon \|x\| \leqslant 1\})$ and all $x \in S_X$ $(:= \{x \in X\colon \|x\| = 1\})$. He proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed point property [8].

2. Uniform normal structure

A Banach space X is said to have (*weak*) *normal structure* (see [1]) if for every (weakly compact) closed bounded convex subset K in X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\left\{\|x_0 - y\|\colon y \in K\right\} < \sup\left\{\|x - y\|\colon x, y \in K\right\}.$$

In reflexive spaces, normal structure and weak normal structure are the same. It is well known (see [9]) that if X fails to have weak normal structure, then there exist a weakly compact convex subset $C \subset X$ and a sequence $(x_n) \subset C$ such that $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^n) \to \operatorname{diam} C = 1$. A Banach space X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\left\{\|x_0 - y\|\colon y \in K\right\} < c\sup\left\{\|x - y\|\colon x, y \in K\right\}.$$

It was proved by W.A. Kirk that every reflexive Banach space with normal structure has the fixed point property (see [13]).

In a recent paper, M. Kato, L. Maligranda and Y. Takahashi [12] gave a sufficient condition for uniform normal structure in terms of the *von Neumann–Jordan constant* $C_{NJ}(X)$, which was defined in 1937 by J.A. Clarkson as

$$C_{NJ}(X) := \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)}\colon x, y \in X \text{ and } \|x\| + \|y\| \neq 0\right\}.$$

This result has been recently improved by S. Dhompongsa, P. Piraisangjun and S. Saejung in [5], where among other things, the authors show that X and X^* have uniform normal structure whenever $C_{NJ}(X) < (3 + \sqrt{5})/4$. To improve this result, we start with the following lemma.

Lemma 1. *Let X be a Banach space for which B_{X^*} is w^*-sequentially compact (for example, X is reflexive or separable, or has an equivalent smooth norm). Suppose that X fails to have weak normal structure. Then, for any $\varepsilon > 0$, there exist $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ such that the following conditions are satisfied*:

(a) $|\|z_i - z_j\| - 1| < \varepsilon$ *and* $|g_i(z_j)| < \varepsilon$ *for all* $i \neq j$,

(b) $g_i(z_i) = 1$ *for* $i = 1, 2, 3$, *and*
(c) $\|z_3 - (z_2 + z_1)\| \geqslant \|z_2 + z_1\| - \varepsilon$.

Proof. By the assumptions, there exist sequences $(x_n) \subset X$ and $(f_n) \subset S_{X^*}$ such that

(1) $x_n \xrightarrow{w} 0$,
(2) $\operatorname{diam}\{x_n\}_{n=1}^{\infty} = 1 = \lim_{n\to\infty} \|x_n - x\|$ for all $x \in \overline{\mathrm{co}}\{x_n\}_{n=1}^{\infty}$,
(3) $f_n(x_n) = \|x_n\|$ for all $n \in \mathbb{N}$, and
(4) $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$.

Observe that 0 is in the weakly closed convex hull of $\{x_n\}_{n=1}^{\infty}$ which equals the norm closed convex hull $\overline{\mathrm{co}}\{x_n\}_{n=1}^{\infty}$. This implies that $\lim_{n\to\infty} \|x_n\| = 1$.

Let $\varepsilon \in (0, 1)$ be given. Pick $\eta = \varepsilon/2$. We first choose a natural number n_1 so that

$$\left|f(x_{n_1})\right| < \frac{\eta}{2} \quad \text{and} \quad 1 - \eta \leqslant \|x_{n_1}\| \leqslant 1.$$

Next, we choose $n_2 > n_1$ so that

$$1 - \eta \leqslant \|x_{n_2}\| \leqslant 1, \qquad 1 - \eta \leqslant \|x_{n_2} - x_{n_1}\| \leqslant 1,$$
$$\left|f_{n_1}(x_{n_2})\right| < \eta, \qquad \left|f(x_{n_2})\right| < \frac{\eta}{2}, \quad \text{and} \quad \left|(f_{n_2} - f)(x_{n_1})\right| < \frac{\eta}{2}.$$

This implies that

$$\left|f_{n_2}(x_{n_1})\right| \leqslant \left|(f_{n_2} - f)(x_{n_1})\right| + \left|f(x_{n_1})\right| < \eta.$$

Finally, we choose $n_3 > n_2$ so that

$$1 - \eta \leqslant \|x_{n_3}\| \leqslant 1, \qquad 1 - \eta \leqslant \|x_{n_3} - x_{n_1}\| \leqslant 1, \qquad 1 - \eta \leqslant \|x_{n_3} - x_{n_2}\| \leqslant 1,$$
$$\left|f_{n_1}(x_{n_3})\right| < \eta, \qquad \left|f_{n_2}(x_{n_3})\right| < \eta, \qquad \left|(f_{n_3} - f)(x_{n_1})\right| < \frac{\eta}{2},$$
$$\left|(f_{n_3} - f)(x_{n_2})\right| < \frac{\eta}{2}, \quad \text{and}$$
$$\left\|x_{n_3} - \left(\frac{x_{n_2}}{\|x_{n_2}\|} + \frac{x_{n_1}}{\|x_{n_1}\|}\right)\right\| \geqslant \left\|\frac{x_{n_2}}{\|x_{n_2}\|} + \frac{x_{n_1}}{\|x_{n_1}\|}\right\| - \eta.$$

Similarly, we also obtain that

$$\left|f_{n_3}(x_{n_1})\right| < \eta \quad \text{and} \quad \left|f_{n_3}(x_{n_2})\right| < \eta.$$

Set

$$z_1 := \frac{x_{n_1}}{\|x_{n_1}\|}, \qquad z_2 := \frac{x_{n_2}}{\|x_{n_2}\|}, \qquad z_3 := \frac{x_{n_3}}{\|x_{n_3}\|},$$
$$g_1 := f_{n_1}, \qquad g_2 := f_{n_2}, \qquad g_3 := f_{n_3}.$$

We now prove that (a)–(c) are satisfied. Clearly, (b) holds. Moreover, for $i \neq j$,

$$\left|g_i(z_j)\right| = \frac{|f_{n_i}(x_{n_j})|}{\|x_{n_j}\|} < \frac{\eta}{1 - \eta} < 2\eta = \varepsilon.$$

Next, we observe that

$$\begin{aligned}\|z_i - z_j\| &= \left\| \frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x_{n_j}}{\|x_{n_j}\|} \right\| \\ &\leqslant \left\| \frac{x_{n_i}}{\|x_{n_i}\|} - x_{n_i} \right\| + \|x_{n_i} - x_{n_j}\| + \left\| x_{n_j} - \frac{x_{n_j}}{\|x_{n_j}\|} \right\| \\ &= \left|1 - \|x_{n_i}\|\right| + \|x_{n_i} - x_{n_j}\| + \left|1 - \|x_{n_j}\|\right| \\ &< 1 + 2\eta < 1 + \varepsilon,\end{aligned}$$

and

$$\|z_i - z_j\| \geqslant g_i(z_i - z_j) = g_i(z_i) - g_i(z_j) \geqslant 1 - \eta > 1 - \varepsilon$$

for all $i \neq j$, that is (a) is satisfied. Finally, (c) is satisfied, since

$$\begin{aligned}\left\|z_3 - (z_2 + z_1)\right\| &\geqslant \left\|x_{n_3} - (z_2 + z_1)\right\| - \|x_{n_3} - z_3\| \\ &\geqslant \|z_2 + z_1\| - \eta - \left|1 - \|x_{n_3}\|\right| \\ &> \|z_2 + z_1\| - \varepsilon.\end{aligned}$$

This completes the proof. □

Theorem 2. *A Banach space X and its dual space X^* have uniform normal structure if $C_{\mathrm{NJ}}(X) < (1 + \sqrt{3})/2$.*

Proof. Since $C_{\mathrm{NJ}}(X^*) = C_{\mathrm{NJ}}(X)$ [12], it suffices to prove only that X has weak normal structure if $C_{\mathrm{NJ}}(X) < (1 + \sqrt{3})/2$. Suppose that X fails to have weak normal structure. Let $\varepsilon > 0$ and choose $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the conditions in Lemma 1. We put $\alpha^2 = 1 + \sqrt{3}$ and consider the following cases:

Case 1. $\|z_2 + z_1\| \leqslant \alpha$. Then we have

$$\begin{aligned}\frac{\|g_2 + g_1\|^2 + \|g_2 - g_1\|^2}{2(\|g_2\|^2 + \|g_1\|^2)} &\geqslant \frac{\left((g_2 + g_1)\left(\frac{z_2 + z_1}{\alpha}\right)\right)^2 + \left((g_2 - g_1)\left(\frac{z_2 - z_1}{\|z_2 - z_1\|}\right)\right)^2}{4} \\ &\geqslant \frac{\left(\frac{2 - 2\varepsilon}{\alpha}\right)^2 + \left(\frac{2 - 2\varepsilon}{1 + \varepsilon}\right)^2}{4}.\end{aligned}$$

Case 2. $\|z_2 + z_1\| > \alpha$.

Case 2.1. $\|z_3 - z_2 + z_1\| \leqslant \alpha$. In this case, we have

$$\begin{aligned}\frac{\|g_3 + g_1\|^2 + \|g_3 - g_1\|^2}{2(\|g_3\|^2 + \|g_1\|^2)} &\geqslant \frac{\left((g_3 + g_1)\left(\frac{z_3 - z_2 + z_1}{\alpha}\right)\right)^2 + \left((g_3 - g_1)\left(\frac{z_3 - z_1}{\|z_3 - z_1\|}\right)\right)^2}{4} \\ &\geqslant \frac{\left(\frac{2 - 4\varepsilon}{\alpha}\right)^2 + \left(\frac{2 - 2\varepsilon}{1 + \varepsilon}\right)^2}{4}.\end{aligned}$$

Case 2.2. $\|z_3 - z_2 + z_1\| > \alpha$. In this case, we have

$$\frac{\|(z_3 - z_2) + z_1\|^2 + \|(z_3 - z_2) - z_1\|^2}{2(\|z_3 - z_2\|^2 + \|z_1\|^2)} \geqslant \frac{\alpha^2 + (\|z_2 + z_1\| - \varepsilon)^2}{2((1+\varepsilon)^2 + 1)} \geqslant \frac{\alpha^2 + (\alpha - \varepsilon)^2}{2((1+\varepsilon)^2 + 1)}.$$

By the arbitrariness of ε and the fact that $C_{NJ}(X^*) = C_{NJ}(X)$, we conclude that

$$C_{NJ}(X) \geqslant \min\left\{\frac{1}{\alpha^2} + 1, \frac{\alpha^2}{2}\right\} = \frac{1+\sqrt{3}}{2}. \qquad \square$$

3. The García-Falset coefficient

The WORTH-property was introduced by B. Sims in [15] as follows: a Banach space X has the *WORTH-property* if

$$\lim_{n\to\infty} \big| \|x_n + x\| - \|x_n - x\| \big| = 0$$

for all $x \in X$ and all weakly null sequences (x_n). In [16], the author defined the *coefficient of weak orthogonality*, which measures the "degree of WORTHwhileness," by

$$w(X) := \sup\big\{\lambda\colon \lambda \liminf_{n\to\infty} \|x_n + x\| \leqslant \liminf_{n\to\infty} \|x_n - x\|\big\},$$

where the supremum is taken over all $x \in X$ and all weakly null sequences (x_n). It is known that a Banach space has the WORTH-property if and only if $w(X) = 1$.

Relation between the coefficient of weak orthogonality, the García-Falset coefficient, and the James and von Neumann–Jordan constant is given in the following theorem. Recall that the *James constant* $J(X)$ is defined by

$$J(X) := \sup\big\{\min\{\|x + y\|, \|x - y\|\}\colon x, y \in S_X\big\}.$$

Theorem 3. *Let X be a Banach space. Then*

(1) $R(X)w(X) \leqslant J(X)$, *and*
(2) $(R(X))^2(1 + (w(X))^2) \leqslant 4C_{NJ}(X)$.

Proof. For $\eta > 0$, there are $x \in S_X$ and (x_n) in B_X such that

$$\liminf_{n\to\infty} \|x_n + x\| \geqslant R(X) - \eta.$$

We may extract a subsequence, still denoted by (x_n), such that

$$\lim_{n\to\infty} \|x_n + x\| \geqslant R(X) - \eta \quad \text{and} \quad \lim_{n\to\infty} \|x_n - x\|$$

exist. Now, we have

$$\begin{aligned} J(X) &\geqslant \min\left\{\lim_{n\to\infty} \|x_n + x\|, \lim_{n\to\infty} \|x_n - x\|\right\} \\ &\geqslant w(X) \lim_{n\to\infty} \|x_n + x\| \\ &\geqslant w(X)\big(R(X) - \eta\big). \end{aligned}$$

Similarly,

$$\begin{aligned}4C_{\mathrm{NJ}}(X) &\geqslant \lim_{n\to\infty} \|x_n + x\|^2 + \|x_n - x\|^2 \\ &\geqslant \left(1 + (w(X))^2\right) \lim_{n\to\infty} \|x_n + x\|^2 \\ &\geqslant \left(1 + (w(X))^2\right)\left(R(X) - \eta\right)^2.\end{aligned}$$

Letting $\eta \to 0$ gives the results. □

Remark 4. The preceding estimates are tight when $X = l_p$ where $1 < p \leqslant 2$. In fact, $R(l_p) = J(l_p) = 2^{1/p}$, $C_{\mathrm{NJ}}(l_p) = 2^{2/p-1}$ and $w(l_p) = 1$ (see [4,6,7,16], respectively).

As a consequence of the preceding theorem, we have the following.

Corollary 5. *Let X be a Banach space.*

(1) [7, Proposition 3.6] *If X has the WORTH-property, then $R(X) \leqslant J(X)$.*
(2) *If $C_{\mathrm{NJ}}(X) < 1 + (w(X))^2$, then $R(X) < 2$ and $R(X^*) < 2$. Furthermore, this result is sharp.*

Proof. (1) follows since $w(X) = 1$.

To prove (2), we first observe that X is reflexive. The assertion follows immediately from the fact that $C_{\mathrm{NJ}}(X^*) = C_{\mathrm{NJ}}(X)$ and $w(X^*) = w(X)$ [11, Theorem 5]. Finally, let us consider the space $X = l_{2,1}$, which is l_2 renormed according to $\|x\|_{2,1} := \|x^+\|_2 + \|x^-\|_2$, where x^+ and x^- are the positive and the negative part or x, respectively, defined as $x^+ = (\max\{x(i), 0\})$ and $x^- = x^+ - x$ (see [2]). It was proved in [11, Theorem 8] that $C_{\mathrm{NJ}}(l_{2,1}) = 3/2$ and it is easy to see that $w(l_{2,1}) = 1/\sqrt{2}$, and $R(l_{2,1}) = 2$. □

Finally, we give an affirmative answer to the following conjecture proposed by L. Maligranda [14]. This result improves the connection between $C_{\mathrm{NJ}}(X)$ and $J(X)$ [12, Theorem 3].

Theorem 6. *For any Banach space X,*

$$C_{\mathrm{NJ}}(X) \leqslant 1 + \frac{(J(X))^2}{4}.$$

Proof. It is easy to see that $C_{\mathrm{NJ}}(X) = \sup\{C_{\mathrm{NJ}}(t, X)\colon t \in [0, 1]\}$ where

$$C_{\mathrm{NJ}}(t, X) := \sup\left\{\frac{\|x + ty\|^2 + \|x - ty\|^2}{2(1 + t^2)}\colon x, y \in S_X\right\}.$$

We now prove that, for all $t \in [0, 1]$,

$$C_{\mathrm{NJ}}(t, X) \leqslant 1 + \frac{t^2(J(X))^2 + 2t(1 - t)J(X)}{2(1 + t^2)}. \tag{1}$$

First we observe that

$$C_{\mathrm{NJ}}(t, X) \leqslant \frac{(J(t, X))^2 + (1 + t)^2}{2(1 + t^2)}, \tag{2}$$

where

$$J(t, X) := \sup\left\{\min\left\{\|x + ty\|, \|x - ty\|\right\}\colon x, y \in S_X\right\}.$$

Indeed, $\|x+ty\|^2 + \|x-ty\|^2 \leqslant (\min(\|x+ty\|, \|x-ty\|))^2 + (1+t)^2$ for all $x, y \in S_X$, and we obtain (2). Moreover,

$$J(t, X) \leqslant tJ(X) + 1 - t. \tag{3}$$

This follows since $\|x \pm ty\| = \|t(x \pm y) + (1-t)x\| \leqslant t\|x \pm y\| + 1 - t$. By (2) and (3), we have

$$\begin{aligned} C_{\mathrm{NJ}}(t, X) &\leqslant \frac{(J(t,X))^2 + (1+t)^2}{2(1+t^2)} \\ &\leqslant \frac{(tJ(X) + (1-t))^2 + (1+t)^2}{2(1+t^2)} \\ &= \frac{t^2(J(X))^2 + 2t(1-t)J(X) + 2(1+t^2)}{2(1+t^2)} \\ &= 1 + \frac{t^2(J(X))^2 + 2t(1-t)J(X)}{2(1+t^2)}. \end{aligned}$$

Hence, by an elementary calculation, we obtain that

$$C_{\mathrm{NJ}}(X) \leqslant \sup\left\{1 + \frac{t^2(J(X))^2 + 2t(1-t)J(X)}{2(1+t^2)} : t \in [0,1]\right\} = 1 + \frac{(J(X))^2}{4}. \qquad \square$$

Remark 7. If X is not uniformly nonsquare [10] (for example, $X = l_1, l_\infty$ or c_0), then $C_{\mathrm{NJ}}(X) = 1 + \frac{1}{4}(J(X))^2$.

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A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings ☆

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Abstract

In the present paper, we define and study a new three-step iterative scheme inspired by Suantai [J. Math. Anal. Appl. 311 (2005) 506–517]. This scheme includes many well-known iterations, for examples, modified Mann-type, modified Ishikawa-type iterative schemes, and the three-step iterative scheme of Xu and Noor. Several convergence theorems of this scheme are established for asymptotically nonexpansive mappings. Our results extend and improve the recent ones announced by Schu [J. Math. Anal. Appl. 158 (1991) 407–413; Bull. Aust. Math. Soc. 43 (1991) 153–159], Xu and Noor [J. Math. Anal. Appl. 267 (2002) 444–453], Suantai [J. Math. Anal. Appl. 311 (2005) 506–517], and many others. A misleading conclusion of Theorem 2.6 in Suantai's paper is also corrected.

Keywords: Asymptotically nonexpansive mapping; Uniformly convex Banach space; Mann-type iteration; Ishikawa-type iteration; Three-step iteration

1. Introduction

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T: C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geqslant 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geqslant 1$. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leqslant L\|x - y\|,$$

for all $x, y \in C$ and all $n \geqslant 1$. It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geqslant 1\}$.

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The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as an important generalization of the class of nonexpansive mappings (i.e., mappings $T: C \to C$ such that $\|Tx - Ty\| \leqslant \|x - y\|$, $\forall x, y \in C$). They proved that if C is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of C, then T has a fixed point. Iterative methods for approximating fixed points of certain mappings have been studied by various authors, using the Mann iterative scheme (see, e.g., [6]) or the Ishikawa iterative scheme (see, e.g., [7]). In 1991, Schu [2,3] introduced a modified Mann iterative scheme to approximate fixed points of asymptotically nonexpansive mappings in a Hilbert space.

In 2002, Xu and Noor [4] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in a Banach space. Glowinski and Le Tallec [8] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [8] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [9] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [8] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences.

Recently, Suantai [1] defined a new three-step iteration which is an extension of Xu and Noor [4] iterations and gave some weak and strong convergence theorems of the iterations for asymptotically nonexpansive mappings in a uniformly convex Banach space. Inspired by the preceding iteration scheme, we define a new iteration scheme as follows.

Let C be a nonempty convex subset of a real Banach space X and $T: C \to C$ be a mapping.

Algorithm 1. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + \gamma_n T^n x_n + (1 - \alpha_n - \beta_n - \gamma_n) x_n, \quad n \geqslant 1, \end{aligned} \tag{1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ are appropriate sequences in $[0, 1]$. The iterative scheme (1) is called the *three-step mean value iterative scheme*.

If $\gamma_n \equiv 0$, then Algorithm 1 reduces to

Algorithm 2. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \quad n \geqslant 1, \end{aligned} \tag{2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\alpha_n + \beta_n\}$ are appropriate sequences in $[0, 1]$. The iterative scheme (2) is called the *modified Noor iterative scheme*, defined by Suantai [1].

If $c_n = \beta_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to

Algorithm 3. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geqslant 1, \end{aligned} \tag{3}$$

where $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$. The iterative scheme (3) is called the *Noor iterative scheme*, defined by Xu and Noor [4].

If $a_n = c_n = \beta_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to

Algorithm 4. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= b_n T^n x_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{4}$$

where $\{b_n\}$ and $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

Similarly, if $b_n = c_n = \alpha_n = \gamma_n \equiv 0$ then Algorithm 1 reduces to

Algorithm 4′. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n, \\ x_{n+1} &= \beta_n T^n z_n + (1-\beta_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{5}$$

where $\{a_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0,1]$.

Let us note that schemes (4) and (5) are called the *modified Ishikawa iterative scheme*.
If $a_n = b_n = c_n = \beta_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to

Algorithm 5. For a given $x_1 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1-\alpha_n)x_n, \quad n \geqslant 1, \tag{6}$$

where $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$. The iterative scheme (6) is called the *modified Mann iterative scheme*.

2. Auxiliary lemmas

For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$. In the sequel, we shall need the following lemmas.

Lemma 1 [10, Lemma 1]. *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1+\delta_n)a_n + b_n, \quad n \geqslant 1.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 2. *Let X be a real Banach space and C be a nonempty convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ (i.e., $F(T) := \{x \in C: x = Tx\} \neq \emptyset$) and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1, then we have the following conclusions:*

(i) *$\lim_n \|x_n - p\|$ exists for any $p \in F(T)$.*
(ii) *$\lim_n d(x_n, F(T))$ exists, where $d(x, F(T))$ denotes the distance from x to the fixed-point set $F(T)$.*

Proof. Let $p \in F(T)$ and $L := \sup\{k_n : n \geqslant 1\}$. Using (1), we have

$$\begin{aligned} \|z_n - p\| &\leqslant a_n\|T^n x_n - p\| + (1-a_n)\|x_n - p\| \leqslant (1 + a_n(k_n - 1))\|x_n - p\| \leqslant k_n\|x_n - p\|, \\ \|y_n - p\| &\leqslant b_n\|T^n z_n - p\| + c_n\|T^n x_n - p\| + (1 - b_n - c_n)\|x_n - p\| \leqslant b_n k_n\|z_n - p\| + c_n k_n\|x_n - p\| + (1 - b_n - c_n)\|x_n - p\| \\ &\leqslant 1 + (b_n k_n + b_n + c_n)(k_n - 1))\|x_n - p\| \leqslant (1 + (L+2)(k_n - 1))\|x_n - p\|, \end{aligned}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant \alpha_n\|T^n y_n - p\| + \beta_n\|T^n z_n - p\| + \gamma_n\|T^n x_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| \\ &\leqslant \alpha_n k_n\|y_n - p\| + \beta_n k_n\|z_n - p\| + \gamma_n k_n\|x_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| \\ &\leqslant (1 + (\alpha_n + \alpha_n k_n(L+2) + \beta_n(k_n + 1) + \gamma_n)(k_n - 1))\|x_n - p\| \\ &\leqslant (1 + (L^2 + 3L + 3)(k_n - 1))\|x_n - p\| = (1 + K(k_n - 1))\|x_n - p\|, \end{aligned} \tag{7}$$

where $K := L^2 + 3L + 3$ is some positive constant. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, the conclusions of the lemma follow from Lemma 1. This completes the proof. □

By Xu's inequality [11, Theorem 2], we have the following lemma.

Lemma 3 [12, Lemma 1.4]. *Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leqslant r\}$, $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \xi, \vartheta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

Interchanging the roles of vectors x, y and z in Lemma 3, and summing together we immediately obtain the following.

Lemma 4. *Let X be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w\|^2$$
$$\leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 - \tfrac{1}{3}\vartheta(\lambda g(\|x - w\|) + \mu g(\|y - w\|) + \xi g(\|z - w\|)),$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \xi, \vartheta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

The following lemmas are the important ingredients for proving our main results in the next section.

Lemma 5. *Let X be a uniformly convex Banach space and C be a nonempty convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1, then we have the following assertions:*

(i) *If $0 < \liminf_n \alpha_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, then $\lim_n\|T^n y_n - x_n\| = 0$.*
(ii) *If $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, then $\lim_n\|T^n z_n - x_n\| = 0$.*
(iii) *If $0 < \liminf_n \gamma_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, then $\lim_n\|T^n x_n - x_n\| = 0$.*
(iv) *If $0 < \liminf_n \alpha_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n(b_n + c_n) < 1$, then $\lim_n\|T^n x_n - x_n\| = 0$.*
(v) *If $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n a_n < 1$, then $\lim_n\|T^n x_n - x_n\| = 0$.*

Proof. By Lemma 2, we know that $\lim_n\|x_n - p\|$ exists for any $p \in F(T)$. It follows that $\{x_n - p\}$, $\{T^n x_n - p\}$, $\{y_n - p\}$, $\{T^n y_n - p\}$, $\{z_n - p\}$ and $\{T^n z_n - p\}$ are all bounded. We may assume that such sequences belong to B_r where $r > 0$. By Lemma 4, we have

$$\|z_n - p\|^2 \leqslant a_n\|T^n x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 \leqslant (1 + a_n(k_n^2 - 1))\|x_n - p\|^2 \leqslant (1 + (k_n^2 - 1))\|x_n - p\|^2,$$

$$\begin{aligned}\|y_n - p\|^2 &\leqslant b_n\|T^n z_n - p\|^2 + c_n\|T^n x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 \\ &\quad - \tfrac{1}{3}(1 - b_n - c_n)(b_n g(\|T^n z_n - x_n\|) + c_n g(\|T^n x_n - x_n\|)) \\ &\leqslant b_n k_n^2\|z_n - p\|^2 + (c_n k_n^2 + (1 - b_n - c_n))\|x_n - p\|^2 - \tfrac{1}{3}b_n(1 - b_n - c_n)g(\|T^n z_n - x_n\|) \\ &\leqslant (1 + (b_n + b_n k_n^2 + c_n)(k_n^2 - 1))\|x_n - p\|^2 - \tfrac{1}{3}b_n(1 - b_n - c_n)g(\|T^n z_n - x_n\|) \\ &\leqslant (1 + (L^2 + 2)(k_n^2 - 1))\|x_n - p\|^2 - \tfrac{1}{3}b_n(1 - b_n - c_n)g(\|T^n z_n - x_n\|),\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\leqslant \alpha_n\|T^n y_n - p\|^2 + \beta_n\|T^n z_n - p\|^2 + \gamma_n\|T^n x_n - p\|^2 + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\|^2 \\
&\quad -\tfrac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)(\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)) \\
&\leqslant \alpha_n k_n^2\|y_n - p\|^2 + \beta_n k_n^2\|z_n - p\|^2 + \gamma_n k_n^2\|x_n - p\|^2 + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\|^2 \\
&\quad -\tfrac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)(\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)) \\
&\leqslant (1 + (\alpha_n + \alpha_n k_n^2(L^2+2) + \beta_n + \beta_n k_n^2 + \gamma_n)(k_n^2 - 1))\|x_n - p\|^2 - \tfrac{1}{3}\alpha_n k_n^2 b_n(1-b_n-c_n) \\
&\quad g(\|T^n z_n - x_n\|) - \tfrac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)(\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)) \\
&\leqslant \|x_n - p\|^2 + (L^4 + 3L^2 + 3)\|x_n - p\|^2(k_n^2 - 1) - \tfrac{1}{3}\alpha_n b_n(1-b_n-c_n)g(\|T^n z_n - x_n\|) \\
&\quad -\tfrac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)(\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)) \\
&\leqslant \|x_n - p\|^2 + (L^4 + 3L^2 + 3)r^2(k_n^2 - 1) - \tfrac{1}{3}\alpha_n b_n(1-b_n-c_n)g(\|T^n z_n - x_n\|) \\
&\quad -\tfrac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)(\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)).
\end{aligned}$$

Let $\sigma_n = (L^4 + 3L^2 + 3)r^2(k_n^2 - 1)$. Then

$$\alpha_n(1-\alpha_n-\beta_n-\gamma_n)g(\|T^n y_n - x_n\|) \leqslant 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n), \tag{8}$$

$$\beta_n(1-\alpha_n-\beta_n-\gamma_n)g(\|T^n z_n - x_n\|) \leqslant 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n), \tag{9}$$

$$\gamma_n(1-\alpha_n-\beta_n-\gamma_n)g(\|T^n x_n - x_n\|) \leqslant 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n), \tag{10}$$

and

$$\alpha_n b_n(1-b_n-c_n)g(\|T^n z_n - x_n\|) \leqslant 3(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sigma_n). \tag{11}$$

We now prove (i). Since $\lim_n k_n = 1$ and $\lim_n\|x_n - p\|$ exists, it follows from (8) that $\lim_n \alpha_n(1-\alpha_n-\beta_n-\gamma_n)$ $g(\|T^n y_n - x_n\|) = 0$. From g is continuous strictly increasing with $g(0) = 0$ and $0 < \liminf_n \alpha_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, we have $\lim_n\|T^n y_n - x_n\| = 0$.

By using a similar method, together with inequalities (9) and (10), it can be shown that (ii) and (iii) are satisfied, respectively.

(iv) By using (1), we have

$$\begin{aligned}
\|T^n x_n - x_n\| &\leqslant \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \leqslant k_n\|x_n - y_n\| + \|T^n y_n - x_n\| \\
&\leqslant k_n b_n\|T^n z_n - x_n\| + k_n c_n\|T^n x_n - x_n\| + \|T^n y_n - x_n\|.
\end{aligned} \tag{12}$$

To show that

$$\lim_n \|T^n x_n - x_n\| = 0, \tag{13}$$

let $\{m_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k\|T^{n_k}x_{n_k} - x_{n_k}\| = 0$.

If $\liminf_j b_{m_j} > 0$, it follows from (11) that

$$\alpha_{m_j} b_{m_j}(1-b_{m_j}-c_{m_j})g(\|T^{m_j}z_{m_j} - x_{m_j}\|) \leqslant 3(\|x_{m_j} - p\|^2 - \|x_{m_j+1} - p\|^2 + \sigma_{m_j}).$$

From $\lim_n k_n = 1$ and $\lim_n\|x_n - p\|$ exists, we have

$$\lim_j \alpha_{m_j} b_{m_j}(1-b_{m_j}-c_{m_j})g(\|T^{m_j}z_{m_j} - x_{m_j}\|) = 0.$$

Since g is continuous strictly increasing with $g(0) = 0$, $\liminf_j \alpha_{m_j} > 0$ and $0 < \liminf_j b_{m_j} \leqslant \limsup_j(b_{m_j} + c_{m_j}) < 1$, it follows that $\lim_j\|T^{m_j}z_{m_j} - x_{m_j}\| = 0$. This together with (i) and (12) gives

$$\lim_j (1 - k_{m_j}c_{m_j})\|T^{m_j}x_{m_j} - x_{m_j}\| = 0.$$

As $\liminf_n(1-k_nc_n) = 1-\limsup_n c_n > 0$, we have

$$\lim_j \|T^{m_j}x_{m_j} - x_{m_j}\| = 0.$$

On the other hand, if $\liminf_j b_{m_j} = 0$, then we may extract a subsequence $\{b_{n_k}\}$ of $\{b_{m_j}\}$ so that $\lim_k b_{n_k} = 0$. This together with (i) and (12) gives

$$\lim_k (1-k_{n_k}c_{n_k})\|T^{n_k}x_{n_k} - x_{n_k}\| = 0,$$

and so

$$\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

By Double Extract Subsequence Principle, we obtain (13).

(v) If $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n a_n < 1$, then

$$\begin{aligned}\|T^nx_n - x_n\| &\leqslant \|T^nz_n - T^nx_n\| + \|T^nz_n - x_n\| \leqslant k_n\|z_n - x_n\| + \|T^nz_n - x_n\| \\ &= k_na_n\|T^nx_n - x_n\| + \|T^nz_n - x_n\|. \end{aligned} \tag{14}$$

Since $\lim_n k_n = 1$ and $\limsup_n a_n < 1$,

$$\liminf_n(1-a_nk_n) = 1 - \limsup_n a_nk_n = 1 - \limsup_n a_n > 0.$$

This together with (ii) and (14) implies that $\lim_n\|T^nx_n - x_n\| = 0$ and the proof is finished. □

Lemma 6. *Let X be a real Banach space and C be a nonempty convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\lim_n k_n = 1$ and, $\{x_n\}$ be a sequence defined in C by Algorithm 1. If $\lim_n\|T^nx_n - x_n\| = 0$, then $\lim_n\|Tx_n - x_n\| = 0$.*

Proof. Using (1), we have

$$\begin{aligned}\|T^nz_n - x_n\| &\leqslant \|T^nz_n - T^nx_n\| + \|T^nx_n - x_n\| \leqslant k_n\|z_n - x_n\| + \|T^nx_n - x_n\|, \\ &= k_na_n\|T^nx_n - x_n\| + \|T^nx_n - x_n\| \to 0, \\ \|T^ny_n - x_n\| &\leqslant \|T^ny_n - T^nx_n\| + \|T^nx_n - x_n\| \leqslant k_n\|y_n - x_n\| + \|T^nx_n - x_n\|, \\ &\leqslant k_nb_n\|T^nz_n - x_n\| + k_nc_n\|T^nx_n - x_n\| + \|T^nx_n - x_n\| \to 0,\end{aligned}$$

and so

$$\begin{aligned}&\|x_{n+1} - T^nx_{n+1}\| \\ &\quad\leqslant \|x_{n+1} - x_n\| + \|T^nx_{n+1} - T^nx_n\| + \|T^nx_n - x_n\| \leqslant (1+k_n)\|x_{n+1} - x_n\| + \|T^nx_n - x_n\| \\ &\quad\leqslant \alpha_n(1+k_n)\|T^ny_n - x_n\| + \beta_n(1+k_n)\|T^nz_n - x_n\| + \gamma_n(1+k_n)\|T^nx_n - x_n\| + \|T^nx_n - x_n\| \to 0.\end{aligned}$$

Thus

$$\|x_{n+1} - Tx_{n+1}\| \leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^nx_{n+1} - x_{n+1}\| \to 0,$$

which implies $\lim_n\|Tx_n - x_n\| = 0$. This completes the proof. □

3. Main results

In this section, we establish several strong convergence theorems of the three-step mean value iterative scheme for asymptotically nonexpansive mappings. The first result extends and improves [1, Theorem 2.3], (and, of course [4, Theorem 2.1]). As in [1], it is misleading to conclude [1, Theorem 2.6] from [1, Theorem 2.3] and by setting $a_n = b_n = c_n = \beta_n \equiv 0$ in Algorithm 2. Because the requirement of the assumption in [1, Theorem 2.3] is $\liminf_n b_n > 0$. However, we can have [1, Theorem 2.6] which is a consequence of the following theorem.

Theorem 7. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \alpha_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and*
(ii) $\limsup_n(b_n + c_n) < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Let $\{u_n\}$ be a given sequence in C. Recall that a mapping $T: C \to C$ with the nonempty fixed-point set $F(T)$ in C satisfies Condition (A) with respect to the sequence $\{u_n\}$ [13] if there is a nondecreasing function $f:[0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(u_n, F(T))) \leqslant \|u_n - Tu_n\|, \quad \text{for all } n \geqslant 1.$$

Proof. By Lemma 5(iv) and Lemma 6,

$$\lim_n \|Tx_n - x_n\| = 0. \tag{15}$$

Let f be a nondecreasing function corresponding to Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n, F(T))) \leqslant \|Tx_n - x_n\| \to 0$$

and so $\lim_n d(x_n, F(T)) = 0$. By Lemma 2, we know that $\lim_n\|x_n - p\|$ exists for all $p \in F(T)$, it follows that $\{x_n - p\}$ is bounded. Then there is a constant M such that

$$K\|x_n - p\| \leqslant M, \quad \text{for all } n \geqslant 1.$$

This together with (7), shows that

$$\|x_{n+1} - p\| \leqslant \|x_n - p\| + M(k_n - 1).$$

Also

$$\begin{aligned}\|x_{n+m} - p\| &\leqslant \|x_{n+m-1} - p\| + M(k_{n+m-1} - 1) \leqslant \|x_{n+m-2} - p\| + M(k_{n+m-2} - 1) + M(k_{n+m-1} - 1) \\ &\leqslant \|x_n - p\| + M\sum_{j=n}^{n+m-1}(k_j - 1),\end{aligned} \tag{16}$$

for all $n, m \geqslant 1$. We now prove that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$. Since

$$\lim_n d(x_n, F(T)) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}(k_n - 1) < \infty,$$

there exists a positive integer N such that

$$d(x_n, F(T)) < \frac{\varepsilon}{3} \quad \text{and} \quad \sum_{j=n}^{\infty}(k_j - 1) \leqslant \frac{\varepsilon}{6M},$$

for all $n \geqslant N$. In particular,

$$d(x_N, F(T)) < \frac{\varepsilon}{3}.$$

There must exist $q \in F(T)$ such that

$$\|x_N - q\| = d(x_N, q) < \frac{\varepsilon}{3}.$$

From (16) it follows that, for all $n \geqslant N$ and $m \geqslant 1$,

$$\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - q\| + \|x_n - q\| \leqslant 2\|x_n - q\| + M \sum_{j=n}^{n+m-1} (k_j - 1)$$
$$\leqslant 2\|x_N - q\| + 2M \sum_{j=N}^{n-1} (k_j - 1) + M \sum_{j=n}^{n+m-1} (k_j - 1)$$
$$\leqslant 2\|x_N - q\| + 2M \sum_{j=N}^{\infty} (k_j - 1) < 2\frac{\varepsilon}{3} + 2M\frac{\varepsilon}{6M} = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. In virtue of the completeness of C, we may assume that $x_n \to q'$ as $n \to \infty$ where $q' \in C$. By the continuity of T and (15), we have $Tq' = q'$, so q' is a fixed point of T. This completes the proof. □

When $\gamma_n \equiv 0$ in Theorem 7, the present result is obtained without the restrictions $\liminf_n b_n > 0$ and the boundedness of C as were the cases in [1]. Moreover, if T is completely continuous, then T satisfies Condition (A) with respect to $\{x_n\}$ (see also [14, Corollary 2.5]).

Corollary 8. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $0 < \liminf_n \alpha_n \leqslant \limsup_n(\alpha_n + \beta_n) < 1$ *and*
(ii) $\limsup_n(b_n + c_n) < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

When $c_n = \beta_n = \gamma_n \equiv 0$ in Theorem 7, we also have the result which is an improvement of [4, Theorem 2.1]. Furthermore, Theorem 7 includes [4, Theorems 2.2 and 2.3], as special cases.

Corollary 9. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 3 with the following restrictions:*

(i) $0 < \liminf_n \alpha_n \leqslant \limsup_n \alpha_n < 1$ *and*
(ii) $\limsup_n b_n < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Next, as a consequence of Lemma 5(v) and Lemma 6, we have the following theorem.

Theorem 10. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and*
(ii) $\limsup_n a_n < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Finally, as a consequence of Lemma 5(iii) and Lemma 6, we have the following result which does seem to be new and not implied by any known iterative scheme.

Theorem 11. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restriction*:

$$0 < \liminf_n \gamma_n \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1.$$

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 12. By using the same ideas and techniques, we can also discuss the weak convergence for asymptotically nonexpansive mappings and thereby improve the results obtained by Bose [15], Schu [3], Tan and Xu [16] and, Chang [17].

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Appendix 15: Nonexpansive set-valued mappings in metric and Banach spaces, Journal of Nonlinear and Convex Analysis, 8 (1) (2007), 35-45.

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NONEXPANSIVE SET-VALUED MAPPINGS IN METRIC AND BANACH SPACES

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ABSTRACT. We extend recent homotopy results of Sims, Xu, and Yuan for set-valued maps to a CAT(0) setting. We also introduce an ultrapower approach to proving fixed point theorems for nonexpansive set-valued mappings, both in this setting and in Banach spaces. This method provides an efficient way of recovering all of the classical Banach space results.

1. INTRODUCTION

In [24], Sims, Xu and Yuan obtain homotopic invariance theorems for nonexpansive set-valued mappings in Banach spaces having Opial's property. They base their results on the fact that if T is a multivalued nonexpansive mapping having nonempty compact values, then the demiclosedness principle for $I - T$ is valid in such spaces. (If C is a nonempty closed convex subset of a Banach space X and if T maps points of C to nonempty closed subsets of X, then T is said to be *demiclosed* on C if the graph of T is closed in the product topology of $(X, \sigma) \times (X, \|\cdot\|)$ where σ and $\|\cdot\|$ denote the weak and strong topologies, respectively). One objective of this paper is to show that the results of [24] extend to CAT(0) spaces (see below) despite the fact that no weak topology is present. The results we obtain are set-valued analogs of single-valued results found in [14].

We also introduce a new approach to the classical fixed point theorems for nonexpansive mappings in Banach spaces by reformulating the arguments in an ultrapower context. This approach seems to illuminate many underlying ideas.

2. CAT(0) SPACES

A metric space is a CAT(0) *space* (the term is due to M. Gromov — see, e.g., [2], p. 159) if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. For a precise definition and a detailed discussion of the properties of such spaces, see Bridson and Haefliger [2] or Burago, et al. [4]. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include the classical hyperbolic spaces, Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [10]; also inequality (4.3) of [23] and subsequent comments), and many others.

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Let (x_n) be a bounded sequence in a complete CAT(0) space X and for $x \in X$ set

$$r(x,(x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x,(x_n)) : x \in X\}.$$

The *asymptotic center* $A((x_n))$ of (x_n) is the set

$$A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}.$$

Recall that a bounded sequence (x_n) is *regular* if $r(x_n) = r(u_n)$ for every subsequence (u_n) of (x_n). It is easy to see that every bounded sequence in X has a regular subsequence (see, e.g., [9], p. 166).

It is known (see, e.g., [6], Proposition 7) that in a CAT(0) space, $A((x_n))$ consists of exactly one point. We will also need the following important fact about asymptotic centers.

Proposition 2.1. *If K is a closed convex subset of X and if (x_n) is a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Proof. Let $x \in X$ be the asymptotic center of (x_n). It is known that the nearest point projection $P : X \to K$ exists and is nonexpansive ([2], p. 177). If $x \notin K$ then $r(x,(x_n)) < r(P(x),(x_n))$, and we would have a contradiction. □

3. A FIXED POINT THEOREM

Let C be a subset of a complete CAT(0) space X. We use $D(\cdot,\cdot)$ to denote the Hausdorff distance on the set $\mathcal{B}(C)$ of nonempty bounded closed subsets of C. Thus for $A, B \in \mathcal{B}(C)$,

$$D(A,B) = \inf\{\rho > 0 : A \subseteq N_\rho(B) \text{ and } B \subseteq N_\rho(A)\}$$

where $N_\rho(S) = \{x \in C : dist(x, S) \le \rho\}$.

A set-valued mapping $T : C \to 2^X \backslash \emptyset$ satisfying

$$D(T(x), T(y)) \le kd(x,y)$$

is called a *contraction* if $k \in [0,1)$ and *nonexpansive* if $k = 1$.

For convenience and brevity we work in an ultrapower setting. This seems to be a new approach in this context. Let $\mathcal{U}$ be a nontrivial ultrafilter on the natural numbers $\mathbb{N}$. Fix $p \in X$, and let $\tilde{X}$ denote the metric space ultrapower of X over $\mathcal{U}$ relative to p. Thus the elements of $\tilde{X}$ consist of equivalence classes $[(x_i)]_{i\in\mathbb{N}}$ for which

$$\lim_{\mathcal{U}} d(x_i, p) < \infty.$$

with $(u_i) \in [(x_i)]$ if and only if $\lim_{\mathcal{U}} d(x_i, u_i) = 0$. Note that $\tilde{X}$ is also a CAT(0) space ([2], p. 187).

For $C \subseteq X$, let

$$\tilde{C} = \{\tilde{x} = [(x_n)] : x_n \in C \text{ for each } n\}.$$

and for $x \in X$, let $\dot{x} = [(x_n)]$ where $x_n = x$ for each $n \in \mathbb{N}$. Finally, let $\dot{X}$ and $\dot{C}$ denote the respective canonical isometric copy of X and C in $\tilde{X}$.

A nonexpansive set-valued mapping $T : C \to \mathcal{B}(X)$ induces a nonexpansive set-valued mapping $\tilde{T}$ defined on $\tilde{C}$ as follows:

$$\tilde{T}(\tilde{x}) = \left\{\tilde{u} \in \tilde{X} : \exists \text{ a representative } (u_n) \text{ of } \tilde{u} \text{ with } u_n \in T(x_n) \text{ for each } n\right\}.$$

To see that $\tilde{T}$ is nonexpansive (and hence well-defined), let $\tilde{x}, \tilde{y} \in \tilde{C}$, with $\tilde{x} = [(x_n)]$ and $\tilde{y} = [(y_n)]$. Then

$$\begin{aligned} D\left(\tilde{T}(\tilde{x}), \tilde{T}(\tilde{y})\right) &\leq \lim_{\mathcal{U}} D(T(x_n), T(y_n)) \\ &\leq \lim_{\mathcal{U}} d(x_n, y_n) \\ &= d_{\mathcal{U}}(\tilde{x}, \tilde{y}). \end{aligned}$$

The following fact (see, e.g., [11], Proposition 1) will be needed.

(3.1) If $S \subseteq C$ is compact, then $\dot{S} = \tilde{S}$.

We will also need the well-known fact that if $C \subseteq X$ is closed and if $T : C \to \mathcal{B}(C)$ is a set-valued contraction mapping, then T has a fixed point. (This fact holds for all complete metric spaces, see [21]).

Next we have a result that is analogous to Proposition 7 of [16] for Banach spaces satisfying the Opial property. The proof is an adaptation of the one given in [16].

Proposition 3.1. *x is the asymptotic center of a regular sequence $(x_n) \subset X$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$ in the ultrapower $\tilde{X}$.*

Proof. ($\Rightarrow$) Suppose x is the asymptotic center of (x_n), and suppose $d_{\mathcal{U}}(\dot{y}, \tilde{x}) \leq d_{\mathcal{U}}(\dot{x}, \tilde{x})$ for some $y \in X$. Choose a subsequence (u_n) of (x_n) such that

$$\lim_{n\to\infty} d(y, u_n) = \liminf_{n\to\infty} d(y, x_n).$$

Using the fact that (x_n) is regular we have

$$\begin{aligned} \lim_{n\to\infty} d(y, u_n) &\leq \lim_{\mathcal{U}} d(y, x_n) \\ &= d_{\mathcal{U}}(\dot{y}, \tilde{x}) \\ &\leq d_{\mathcal{U}}(\dot{x}, \tilde{x}) \\ &\leq \limsup_{n\to\infty} d(x, x_n) \\ &= r((x_n)) \\ &= \limsup_{n\to\infty} d(x, u_n). \end{aligned}$$

Thus $\lim_{n\to\infty} d(y, u_n) \leq \limsup_{n\to\infty} d(x, u_n)$, and $y = x$ by uniqueness of the asymptotic center.

($\Leftarrow$) Suppose $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$, and suppose y is the asymptotic center of (x_n). Then by the implication ($\Rightarrow$) $\dot{y}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x}$, whence $\dot{x} = \dot{y}$; thus $x = y$. □

With the above observation, we are in a position to prove the fixed point theorem we will need in the next section. Here $\mathcal{K}(X)$ denotes the family of nonempty compact subsets of X, and we use D to denote the usual Hausdorff metric on $\mathcal{K}(X)$.

Theorem 3.2. *Let K be a closed convex subset of a complete CAT(0) space X, and let $T : K \to \mathcal{K}(X)$ be a nonexpansive mapping. Suppose $dist(x_n, T(x_n)) \to 0$ for some bounded sequence $(x_n) \subset K$. Then T has a fixed point.*

Proof. By passing to a subsequence we may suppose (x_n) is regular. Let x be the asymptotic center of (x_n). By Proposition 3.1 $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$. By Proposition 2.1, $x \in K$ and also $\dot{x} \in \dot{K}$. Since $\tilde{x} \in \tilde{T}(\tilde{x})$, $\tilde{x}$ must lie in a ρ-neighborhood of $\tilde{T}(\dot{x})$ for $\rho = D\left(\tilde{T}(\tilde{x}), \tilde{T}(\dot{x})\right)$. Since $\tilde{T}(\dot{x})$ is compact, $dist\left(\tilde{x}, \tilde{T}(\dot{x})\right) = d_{\mathcal{U}}(\tilde{x}, \dot{u})$ for some $\dot{u} \in \tilde{T}(\dot{x})$. But since $\tilde{T}(\dot{x}) \subset \dot{X}$, if $\dot{u} \neq \dot{x}$ we have the contradiction

$$d_{\mathcal{U}}(\tilde{x}, \dot{u}) > d_{\mathcal{U}}(\tilde{x}, \dot{x}) \geq D\left(\tilde{T}(\tilde{x}), \tilde{T}(\dot{x})\right) = \rho.$$

Therefore $\dot{x} = \dot{u} \in \tilde{T}(\dot{x})$. However $\tilde{T}(\dot{x}) = \widetilde{T(x)}$, so by (3.1) this in turn implies $x \in T(x)$. □

Remark 3.3. Convexity of K is needed the preceding argument only to assure that the asymptotic center of (x_n) lies in K. The theorem actually holds under the weaker assumption that K is closed and contains the asymptotic centers of all of its regular sequences.

4. Homotopic invariance

The following is an analog of Theorem 3.1 of [24].

Theorem 4.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, with int$(C) \neq \emptyset$, let $\{T_t\}_{0\leq t\leq 1}$ be a family of λ-contractions from C to $\mathcal{K}(X)$ which is equi-continuous in $t \in [0,1]$ over C. Assume that some T_t has a fixed point in C, and assume every T_t is fixed point free on ∂C. Then T_t has a fixed point in C for each $t \in [0,1]$.*

Proof. Let $V = \{t \in [0,1] : T_t$ has a fixed point in $C\}$. Then V is nonempty by assumption. We show that V is both open and closed in $[0,1]$ and therefore conclude that $V = [0,1]$. The proof that V is open in $[0,1]$ is identical to the one given in the proof of Lemma 3.1 of [24]. To show that V is closed, assume $(t_n) \subset V$ is such that $t_n \to t_0$. Then for each n there exists $x_n \in C$ such that $x_n \in T_{t_n}(x_n)$. By equi-continuity we have

$$dist(x_n, T_{t_0}(x_n)) \leq D(T_{t_n}(x_n), T_{t_0}(x_n)) \to 0.$$

By Theorem 3.2, T_{t_0} has a fixed point in C, so $t_0 \in V$. □

We now turn to an analog of Theorem 4.1 of [24].

Theorem 4.2. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose $T, G : C \to \mathcal{K}(X)$ are two set-valued nonexpansive mappings and suppose there exists a homotopy $H : [0,1] \times C \to \mathcal{K}(X)$ such that*

(1) $H(0,\cdot) = T(\cdot)$ *and* $H(1,\cdot) = G(\cdot)$;

(2) *for each* $t \in [0,1]$, $H(t,\cdot)$ *is a set-valued nonexpansive mapping from* C *to* $\mathcal{K}(X)$;

(3) $H(t,x)$ *is equi-continuous in* $t \in [0,1]$ *over* C;

(4) *for each sequence* (t_n) *in* $[0,1]$ *with*

$$\inf_{x\in C} dist(x, H(t_n, x)) > 0,$$

$\lim_{n\to\infty} t_n = t_0$ *implies* $\inf_{x\in C} dist(x, H(t_0, x)) > 0$.

Then T has a fixed point in C if and only if G has a fixed point in C.

Proof. Assume T has a fixed point in C, and let

$$V = \{t \in [0,1] : \text{ there exists } x \in C \text{ such that } x \in H(t,x)\}.$$

We can show that V is closed as in the proof of Theorem 4.1. Suppose V is not open. Then there exists $t_0 \in V$ and a sequence $(t_n) \subset [0,1] \setminus V$ such that $\lim_{n\to\infty} t_n = t_0$. Since $t_n \notin V$, $\text{dist}(x, H(t_n, x)) > 0$ for all $n \in \mathbb{N}$ and $x \in C$. We claim that

$$\inf_{x\in C} dist(x, H(t_n, x)) > 0 \text{ for all } n \in \mathbb{N}.$$

Otherwise, there exists a sequence $(x_m) \subset C$ such that

$$\lim_{m\to\infty} dist(x_m, H(t_n, x_m)) = 0,$$

and by Theorem 3.2 $H(t_n,\cdot)$ has a fixed point. But this contradicts $t_n \notin V$, so we have the claim. Condition (4) now implies

$$\inf_{x\in C} dist(x, H(t_0, x)) > 0,$$

which in turn implies $t_0 \notin V$ and this is a contradiction. Therefore V is open in $[0,1]$, and hence $V = [0,1]$, from which the conclusion follows. □

The other results of [24], including the alternative principles, carry over the present setting as well.

Remark 4.3. In view of Remark 4.3, in both Theorems 4.1 and 4.2 the assumption of convexity can be replaced by the assumption that C contains the asymptotic center of each of its regular sequences.

5. Banach spaces

As we shall see, the ultrapower approach used in proving Theorem 3.2 also provides a very efficient method for proving the classical Banach space fixed point theorems for nonexpansive set-valued mappings.

Let C be a subset of a Banach space X. We will use 2^C, $\mathcal{B}(C)$, $\mathcal{K}(C)$, and $\mathcal{KC}(C)$ to denote respectively the family of all subsets of C, the family of nonempty bounded closed subsets of C, the family of nonempty compact subsets of C, and the family of nonempty compact convex subsets of C. As before we use $D(\cdot,\cdot)$ to denote the Hausdorff distance on $\mathcal{B}(C)$.

We adopt all the notation and definitions of Sections 2 and 3, but with the distance d replaced with the norm $\|\cdot\|$.

Recall that a Banach space is said to have the *Opial property* if given whenever (x_n) converges weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \text{ for each } y \in X \text{ with } y \neq x.$$

Now let $\mathcal{U}$ be a nontrivial ultrafilter over the natural numbers $\mathbb{N}$ and let $\tilde{X}$ denote the Banach space ultrapower of X over $\mathcal{U}$. We will use the standard notation for this setting, see for example [1] or [12].

As in the CAT(0) case, a nonexpansive set-valued mapping $T : C \to \mathcal{B}(X)$ induces a nonexpansive set-valued mapping $\tilde{T}$ defined on $\tilde{C}$ as follows:

$$\tilde{T}(\tilde{x}) = \left\{\tilde{u} \in \tilde{X} : \exists \text{ a representative } (u_n) \text{ of } \tilde{u} \text{ with } u_n \in T(x_n) \text{ for each } n\right\}.$$

The following simple idea, which is extracted from the proof of Theorem 3.2, is the basis for all of our Banach space results. Recall that a set C is said to be (uniquely) proximinal if each point $x \in X$ has a (unique) nearest point in C.

Lemma 5.1. *Let K be a subset of a Banach space X, suppose $T : K \to 2^X\backslash\emptyset$ is nonexpansive, and suppose there exists $x_0 \in K$ such that $x_0 \in T(x_0)$. Suppose C is a subset of K for which $T : C \to \mathcal{K}(C)$, and suppose C is uniquely proximinal in K. Then T has a fixed point in C. Indeed, the point of C which is nearest to x_0 is a fixed point of T.*

Proof. If $x_0 \in C$ we are finished. Otherwise let x be the unique point of C nearest to x_0. We assert that $x \in T(x)$. Since $x_0 \in T(x_0)$, x_0 must lie in a ρ-neighborhood of $T(x)$ for $\rho = D(T(x_0), T(x))$. Therefore, since $T(x)$ is compact, $dist(x_0, T(x)) = \|x_0 - u\| \le \rho$ for some $u \in T(x)$. But since $T(x) \subset C$, if $u \neq x$,

$$\|x_0 - u\| > \|x_0 - x\| \ge D(T(x), T(x_0)) = \rho,$$

and we have a contradiction. Therefore $u = x \in T(x)$. □

The preceding lemma quickly yields the following result. Notice that boundedness of C is not needed. This observation may be known, but we are not aware of an explicit citation.

Theorem 5.2. *Let X be a uniformly convex Banach space, and let C be a closed convex subset of X. If $T : C \to \mathcal{K}(C)$ is a nonexpansive mapping that satisfies*

$$(5.1) \qquad dist\,(x_n, T\,(x_n)) \to 0 \qquad as\ n \to \infty$$

for some bounded sequence (x_n) in C, then T has a fixed point.

Proof. Let $\tilde{x} = [(x_n)] \in \tilde{C}$. As we have observed, $\tilde{T} : \tilde{C} \to 2^{\tilde{C}} \backslash \emptyset$ is nonexpansive. Also (5.1) implies $\tilde{x} \in \tilde{T}(\tilde{x})$. Since uniform convexity is a super property, $\tilde{X}$ is uniformly convex and then $\tilde{x}$ has a unique nearest point $\dot{x} \in \dot{C}$. Since $\tilde{T} : \dot{C} \to \mathcal{K}\left(\dot{C}\right)$, Lemma 5.1 implies there exists $\dot{x} \in \dot{C}$ such that $\dot{x} \in \tilde{T}(\dot{x})$. However by (3.1) $\tilde{T}(\dot{x}) = \dot{T(x)}$, and this in turn implies that $x \in T(x)$. □

If X has the Opial property, the assumption that $T : C \to \mathcal{K}(C)$ can be weakened to $T : C \to \mathcal{K}(X)$. For this we will make use of the following fact.

Proposition 5.3 ([16]). *Let X be a Banach space that has the Opial property. Then $x \in X$ is the weak limit of a regular sequence $(x_n) \subset X$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$ in the ultrapower $\tilde{X}$.*

Theorem 5.4. *Let X be a Banach space that has the Opial property, and let C be a weakly compact subset of X. If $T : C \to \mathcal{K}(X)$ is a nonexpansive mapping that satisfies*

$$(5.2) \qquad dist\,(x_n, T\,(x_n)) \to 0 \qquad as\ n \to \infty$$

for some bounded sequence (x_n) in C, then T has a fixed point.

Proof. By passing to a subsequence if necessary we may suppose that (x_n) is regular and converges weakly, say to $x \in C$. By Proposition 5.3 $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x}$. The proof is now identical to the proof of Theorem 3.2 upon replacing $d_{\mathcal{U}}$ with $\|\cdot\|_{\mathcal{U}}$. □

As a corollary of the preceding results we have the classical results of both Lim and Lami Dozo.

Theorem 5.5 ([18], [19]). *Suppose X is either a uniformly convex Banach space, or a reflexive Banach space that has the Opial property. Let C be a bounded closed convex subset of X, and suppose $T : C \to \mathcal{K}(C)$ is nonexpansive. Then T has a fixed point.*

Proof. Fix $z \in C$, and for each $n \geq 1$, consider the contraction mapping $T_n : C \to \mathcal{K}(C)$ defined by

$$T_n(x) = \frac{1}{n} z + \left(1 - \frac{1}{n}\right) T(x), \qquad x \in C.$$

Then by Nadler's theorem [21], for each $n \geq 1$ there exists $x_n \in C$ such that $x_n \in T_n(x_n)$. Moreover

$$dist\,(x_n, T\,(x_n)) \leq \frac{1}{n} diam\,(C) \to 0 \text{ as } n \to \infty.$$

□

We now turn to an extension of Lim's theorem to inward mappings. The *inward set* $I_C(x)$ of x relative to C is the set

$$I_C(x) = \{x + c(u - x) : u \in C \text{ and } c \geq 1\}.$$

A mapping $T : C \to B(X)$ is said to be *weakly inward* if $T(x)$ is in the closure $\overline{I_C(x)}$ of $I_C(x)$ for each $x \in C$.

The following two facts will be needed.

Lemma 5.6 ([8], Corollary 2). *Suppose C is a closed convex subset of a Banach space X and suppose $T : C \to \mathcal{K}(X)$ is a weakly inward contraction on C. Then T has a fixed point in C.*

Lemma 5.7. *Let X be a uniformly convex Banach space, let C be a closed convex subset of X, and suppose $x_0 \in X$. Let x be the unique point of C which is nearest to x_0. Then x is the unique point of $\overline{I_C(x)}$ which is nearest to x_0.*

Proof. Suppose not, and let y be the unique point of $\overline{I_C(x)}$ which is nearest to x_0. Then, since $C \subseteq \overline{I_C(x)}$ and $y \in \overline{I_C(x)} \backslash C$, it must be the case that

$$\|y - x_0\| < \|x - x_0\|.$$

By the continuity of $\|\cdot\|$ there exists $z \in I_C(x) \backslash C$ such that $\|z - x_0\| < \|x - x_0\|$. This implies $z = (1 - \alpha) x + \alpha w$ for some $w \in C$ and $\alpha > 1$. Hence

$$\|w - x_0\| \leq \frac{1}{\alpha} \|z - x_0\| + \left(1 - \frac{1}{\alpha}\right) \|x - x_0\| < \|x - x_0\|,$$

a contradiction. □

The following theorem was first proved for inward mappings independently by Downing and Kirk [8] and by Reich [22]. The slightly more general formulation below is due to H. K. Xu (see [25], Theorem 3.4). Our proof is much shorter than the one given in [25] (although it depends on deeper facts).

Theorem 5.8. *Let C be a bounded closed convex subset of a uniformly convex Banach space X, and suppose $T : C \to \mathcal{K}(X)$ is nonexpansive and weakly inward on C. Then T has a fixed point.*

Proof. As in the proof of Theorem 5.5, approximate T with the contraction mappings T_n. Each of the mapping T_n is also weakly inward and by Lemma 5.6 has a fixed point x_n. Since the sequence (x_n) satisfies $dist(x_n, T(x_n)) \to 0$. Let $\tilde{x} = [(x_n)]$, and let $\dot{x}$ be the unique point of $\dot{C}$ which is nearest $\tilde{x}$. Since $\tilde{T}$ is nonexpansive there exists a point $\tilde{y} \in \tilde{T}(\dot{x})$ such that $\|\tilde{y} - \tilde{x}\|_{\mathcal{U}} \leq \|\dot{x} - \tilde{x}\|_{\mathcal{U}}$, and since $\tilde{T}$ is weakly inward on $\dot{C}$, $\tilde{y} \in \overline{I_{\dot{C}}(\dot{x})}$. Lemma 5.7 implies $\tilde{y} = \dot{x}$. Thus $\dot{x} \in \tilde{T}(\dot{x})$ and the conclusion follows. □

Finally we remark that it is possible to use this approach to prove the following theorem of Kirk and Massa ([15]; also see [13]). We omit the details because the ultrapower proof is not appreciably shorter than the one given in [15] (which also uses nonstandard techniques). Indeed, this result has recently been extended to spaces

X for which $\varepsilon_\beta(X) < 1$, where $\varepsilon_\beta(X)$ denotes the characteristic of noncompact convexity for the separation measure of noncompactness (see [7]).

Theorem 5.9. *Suppose C is a nonempty bounded closed convex subset of a Banach space X, and suppose $T : C \to KC(C)$ is nonexpansive. Suppose also that the asymptotic center in C of each bounded sequence in X is nonempty and compact. Then T has a fixed point.*

Remark 5.10. It might be worth noting that Lemma 5.1 holds for mappings taking only closed values if it is assumed that the space is uniformly convex.

Lemma 5.11. *Let K be a subset of a uniformly convex Banach space X, suppose $T : K \to 2^X\backslash\emptyset$ is nonexpansive, and suppose there exists $x_0 \in K$ such that $x_0 \in T(x_0)$. Suppose C is a closed convex subset of K for which $T : C \to B(C)$. Then the point of C which is nearest to x_0 is a fixed point of T.*

Proof. If $x_0 \in C$ we are finished. Otherwise let x be unique the point of C nearest to x_0. We assert that $x \in T(x)$. Suppose not. Since $x_0 \in T(x_0)$, x_0 must lie in a ρ-neighborhood of $T(x)$ for $\rho = D(T(x_0), T(x))$. If $dist(x_0, T(x)) > \|x_0 - x\|$ we have a contradiction as in the proof of Lemma 5.1. On the other hand, if $dist(x_0, T(x)) = \|x - x_0\|$, then there exists a sequence $(u_n) \subset T(x)$ such that $\|x_0 - u_n\| \to \|x_0 - x\|$ as $n \to \infty$. Since $\left\|x_0 - \dfrac{x+u_n}{2}\right\| > \|x_0 - x\|$, the uniform convexity of X yields $\|x - u_n\| \to 0$ as $n \to \infty$. Since $T(x)$ is closed, $x \in T(x)$. □

Remark 5.12. In Theorems 3.1 and 4.1 of [24] the domain of the mappings is assumed to be weakly compact and convex. However weak compactness suffices – the convexity assumption may be dropped. To see this one could either use Theorem 5.4 in lieu of the demiclosedness principal in the proofs of those theorems, or observe that convexity is not needed in the proof of the demiclosedness principal itself (Lemma 2.1 of [24])

Remark 5.13. For an analog of Theorem 5.8 in a CAT(0) space, see [5]. In fact, Theorem 5.8 extends to the uniformly convex hyperbolic metric spaces in the sense of Reich and Shafrir [23]. Such spaces include both the CAT(0) spaces and uniformly convex Banach spaces.

6. Weak convergence in CAT(0) spaces

We conclude with a question. A comparison of Propositions 3.1 and 5.3 clearly suggests that the following would be a reasonable way to define weak convergence in a CAT(0) space, especially since it does indeed coincide with weak convergence in a Hilbert space.

Definition 6.1. *A sequence [net] (x_n) in X is said to* converge weakly *to $x \in X$ if x is the unique asymptotic center of (u_n) for every subsequence [subnet] (u_n) of (x_n).*

This notion of convergence was first introduced in metric spaces by T. C. Lim [20], who called it Δ-convergence. (T. Kuczumow [17] introduced a similar notion of convergence in Banach spaces which he called 'almost convergence'.)

This raises a very fundamental question: For what CAT(0) spaces, aside from Hilbert space, does the notion convergence in Definition 6.1 actually correspond to convergence relative to some topology? Specifically, *when is there a topology* τ *on* X *such that a net* (x_α) *converges to* x *in the sense of Definition 6.1 if and only if* (x_α) *is* τ*-convergent to* x?

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Appendix 16: An inequality concerning the James constant and the weakly convergent sequence coefficient, Journal of Nonlinear and Convex Analysis (to appear).

AN INEQUALITY CONCERNING THE JAMES CONSTANT AND THE WEAKLY CONVERGENT SEQUENCE COEFFICIENT

S. DHOMPONGSA AND A. KAEWKHAO*

1. Introduction

As it is well-known, the notions of normal structure and uniform normal structure play important role in metric fixed point theory (see Goebel and Kirk [20]). Some parameters and constants defined on Banach spaces can be used to verify whether a specific Banach space enjoys uniform normal structure. These constants include the James constants and the Jordan-von Neumann constants, which are introduced by Gao and Lau [16] and Clarkson [7], respectively.

For a Banach space X, we show that the James constant $J(X)$ is related to, as an inequality, the weakly convergent sequence coefficient $WSC(X)$ defined by Bynum [5]. As a consequence, we obtain the the latest upper bound of the James constant $J(X)$ at $\frac{1+\sqrt{5}}{2}$ for X to have uniform normal structure [9, Dhompongsa et. al]. By applying Domínguez and Xu's theorem [15, Theorem 3.2], we also obtain fixed point results for asymptotically regular mappings.

2. Preliminaries and Notations

Throughout the paper we let X and X^* stand for a Banach space and its dual space, respectively. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y\in A}\|x-y\| : x \in A\}$ is called the Chebyshev radius of A. The number $\text{diam}(A) = \sup\{\|x-y\| : x, y \in A\}$ is called the diameter of A. A Banach space X has normal structure (resp. weak normal structure) if

$$r(A) < \text{diam}(A)$$

for every bounded closed (resp. weakly compact) convex subset A of X with $\text{diam}(A) > 0$. The normal structure coefficient $N(X)$ of X [5, Bynum] is the number

$$N(X) = \inf\left\{\frac{\text{diam}(A)}{r(A)}\right\},$$

where the infimum is taken over all bounded closed convex subsets A of X with $\text{diam}(A) > 0$. The weakly convergent sequence coefficient $WCS(X)$ [5] of X is the

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number

$$WCS(X) = \inf \{ \frac{A(\{x_n\})}{r_a(\{x_n\})} \},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) = \limsup_n \{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) = \inf\{\limsup_n \|x_n - y\| : y \in \overline{co}\{x_n\}\}$ is the asymptotic radius of $\{x_n\}$. A space X with $N(X) > 1$ (resp. $WCS(X) > 1$) is said to have uniform (resp. weak uniform) normal structure. For a Banach space X, the James constant, or the nonsquare constant is defined by Gao and Lau [16] as

$$J(X) = \sup \{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}.$$

It is known that $J(X) < 2$ if and only if X is uniformly nonsquare. Dhompongsa et. al [9, Theorem 3.1] showed that if $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure. The Jordan-von Neumann constant $C_{NJ}(X)$ of X, which is introduced by Clarkson [7], is defined by

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero} \right\}.$$

A relation between these two constants is

$$\frac{(J(X))^2}{2} \leq C_{NJ}(X) \leq \frac{(J(X))^2}{(J(X) - 1)^2 + 1} (\text{[23, Kato et. al]}).$$

From this relation, it is easy to conclude that $C_{NJ}(X) < 2$ is equivalent to $J(X) < 2$. Recently, Dhompongsa and Kaewkhao [10, Theorem 3.16] obtained the latest upper bound of the Jordan-von Neumann constant $C_{NJ}(X)$ at $\frac{1+\sqrt{3}}{2}$ for X to have uniform normal structure. However, it is still not clear that if the upper bounds of the James constants and of the Jordan-von Neumann constants are sharp for having uniform normal structure (see a conjecture in [9]). The constant $R(a, X)$, which is a generalized Garcí a-Falset coefficient [18], is introduced by Domínguez [12] : for a given positive real number a

$$R(a, X) := \sup\{\liminf_n \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequence $\{x_n\}$ in the unit ball of X such that

$$D(x_n) = \limsup_n \left(\limsup_m \|x_n - x_m\| \right) \leq 1.$$

Concerning with this coefficient, Domínguez obtained a fixed point theorem which states that if X is a Banach space with $R(a, X) < 1 + a$ for some a, then X has the weak fixed point property (for details about the (weak) fixed point property, the readers are referred to Goebel and Kirk [21]). In [28], Mazcuñán-Navarro showed that

$$R(1, X) \leq J(X),$$

and then combined it with the fixed point theorem of Domínguez to solve a long stand open problem. Indeed, it was proved that the uniform nonsquareness implies the weak fixed point property. A mapping $T : X \to X$ is called asymptotically regular if

$$\lim_n \|T^n x - T^{n+1} x\| = 0 \quad \text{for all } x \in X.$$

The concept of asymptotically regular mappings is due to Browder and Petryshyn [2]. We set

$$s(T) = \liminf_n |T^n|,$$

where $|T^n| = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x-y\|} : x, y \in C,\ x \neq y\right\}$. Fixed point results for asymptotically regular mappings can be found in [3, 4, 15, 13, 14, 22, 25, 26]. Most of these results are related to geometric coefficients in Banach spaces. We state here the one using the weak convergent sequence coefficients.

Theorem 2.1. [15, Theorem 3.2] *Suppose X is a Banach space with $WCS(X) > 1$, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is a uniformly Lipschitzian mapping such that $s(T) < \sqrt{WCS(X)}$. Suppose in addition that T is asymptotically regular on C. Then T has a fixed point.*

One last concept we need to mention is ultrapowers of Banach spaces. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and
(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|\{x_i\}\| := \sup_{i\in I}\|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) above and the definition of the quotient norm that

$$\|\{x_i\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an ultrapower of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see Aksoy and Khamsi [1] or Sims [30]).

3. The James constants

Before we present the first result, we need another equivalent definition of the weakly convergent sequence coefficient $WCS(X)$ of X which is shown in [11, Theorem 1.1] as follows :

Definition 3.1. For a Banach space X,

$$WCS(X) = \inf\{\frac{1}{r_a(x_n)} : x_n \xrightarrow{w} 0, \lim_{n\neq m} \|x_n - x_m\| = 1\}.$$

Now we can state the following.

Theorem 3.2. *For a Banach space X,*

$$WCS(X) \geq \frac{J(X)+1}{[J(X)]^2}.$$

In particular, if $J(X) < \frac{1+\sqrt{5}}{2}$, then $WCS(X) > 1$.

Proof. For a sake of convenience we put $J(X) = \alpha$. Let $\{x_n\}$ be a weakly null sequence in X such that

$$\lim_{n\neq m} \|x_n - x_m\| = 1. \tag{3.1}$$

Put $C = \overline{co}\{x_n\}$ and $r = r_a\{x_n\}$. Since $0 \in C$, we obtain

$$r \leq \limsup_n \|x_n\|. \tag{3.2}$$

Fix $\varepsilon > 0$. By (3.1) there exists $K \in \mathbb{N}$ such that

$$\limsup_n \|x_n - x_m\| \leq 1 + \varepsilon, \quad \forall m \geq K, \tag{3.3}$$

and it follows from the weak lower semicontinuity of the norm that

$$\|x_m\| \leq 1 + \varepsilon, \quad \forall m \geq K.$$

We have, for all $m \geq K$,

$$\limsup_n \|\frac{x_n}{1+\varepsilon} + \frac{x_m}{1+\varepsilon}\| = \limsup_n \|\frac{x_{(n+K)}}{1+\varepsilon} + \frac{x_m}{1+\varepsilon}\| \leq R(1, X) \leq J(X) = \alpha. \tag{3.4}$$

From (3.2), we can find an integer $M \geq K$ such that

$$r(1-\varepsilon) \leq \|x_M\|. \tag{3.5}$$

By definition of r and convexity of C, we must have

$$r \leq \limsup_n \|x_n - (\frac{\alpha-1}{\alpha+1})x_M\|. \tag{3.6}$$

We can assume, by passing through a subsequence if necessary, that "lim sup" in (3.6) can be replaced by "lim". Now let $\widetilde{X}$ be a Banach space ultrapower of X over an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Set

$$\tilde{x} = \{\frac{x_n - x_M}{1+\varepsilon}\}_{\mathcal{U}} \text{ and } \tilde{y} = \{\frac{x_n + x_M}{(1+\varepsilon)\alpha}\}_{\mathcal{U}}.$$

(3.3) and (3.4) guarantee that $\tilde{x}, \tilde{y} \in B_{\widetilde{X}}$. Consider, by using (3.6),

$$\begin{aligned}
\|\tilde{x} + \tilde{y}\| &= \lim_{\mathcal{U}} \|\frac{x_n - x_M}{1+\varepsilon} + \frac{x_n + x_M}{(1+\varepsilon)\alpha}\| \\
&= (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha}) \lim_{\mathcal{U}} \|x_n - (\frac{\alpha-1}{\alpha+1})x_M\| \\
&= (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha}) \lim_n \|x_n - (\frac{\alpha-1}{\alpha+1})x_M\| \\
&\geq (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha})r.
\end{aligned}$$

On the other hand, by using the weak lower semicontinuity of $\|\cdot\|$ and (3.5),

$$\begin{aligned}\|\tilde{x}-\tilde{y}\| &= \lim_{\mathcal{U}}\|\frac{x_n-x_M}{1+\varepsilon}-\frac{x_n+x_M}{(1+\varepsilon)\alpha}\| \\ &= (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha})\lim_{\mathcal{U}}\|(\frac{\alpha-1}{\alpha+1})x_n-x_M\| \\ &\geq (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha})\liminf_n\|(\frac{\alpha-1}{\alpha+1})x_n-x_M\| \\ &\geq (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha})\|x_M\| \\ &\geq (\frac{1-\varepsilon}{1+\varepsilon})(\frac{\alpha+1}{\alpha})r.\end{aligned}$$

It follows from the definition of $J(\widetilde{X})$ that

$$\begin{aligned}J(\widetilde{X}) &\geq \|\tilde{x}+\tilde{y}\|\wedge\|\tilde{x}-\tilde{y}\| \\ &\geq (\frac{1}{1+\varepsilon})(\frac{\alpha+1}{\alpha})r\wedge(\frac{1-\varepsilon}{1+\varepsilon})(\frac{\alpha+1}{\alpha})r \\ &= (\frac{1-\varepsilon}{1+\varepsilon})(\frac{\alpha+1}{\alpha})r.\end{aligned}$$

According to the fact that $J(X)=J(\widetilde{X})$ ([17, Gao and Lau]) and $\alpha=J(X)$, we have

$$J(X)\geq(\frac{1-\varepsilon}{1+\varepsilon})(\frac{J(X)+1}{J(X)})r.$$

Thus,

$$\frac{1}{r}\geq(\frac{1-\varepsilon}{1+\varepsilon})(\frac{J(X)+1}{[J(X)]^2}).$$

Since ε is arbitrary, it follows that

$$\frac{1}{r}\geq\frac{J(X)+1}{[J(X)]^2}.$$

Hence, by Definition 3.1 we conclude that

$$WCS(X)\geq\frac{J(X)+1}{[J(X)]^2}$$

as desired. □

As a consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.3. [9, Theorem 3.1] *Let X be a Banach space. If $J(X)<\frac{1+\sqrt{5}}{2}$, then X has uniform normal structure.*

Proof. Let $\widetilde{X}$ be a Banach space ultrapower of X over an ultrafilter. Since $J(\widetilde{X})=J(X)$, Theorem 3.2 can be applied to $\widetilde{X}$ and then $WCS(\widetilde{X})>1$. Since $WCS(\widetilde{X})>1$, $\widetilde{X}$ has weak normal structure [5] and since $\widetilde{X}$ is reflexive, it must be the case that $\widetilde{X}$ has normal structure. By [17, Theorem 5.2], X has uniform normal structure as desired. □

In view of Theorem 2.1, we obtain a fixed point result about asymptotically regular mappings concerning the James constants.

Corollary 3.4. *Suppose X is a Banach space such that $J(X) < \frac{1+\sqrt{5}}{2}$, C is a nonempty closed bounded convex subset of X, and $T : C \to C$ is a uniformly Lipschitzian mapping such that*

$$s(T) < \frac{\sqrt{J(X)+1}}{J(X)}.$$

Suppose in addition that T is asymptotically regular on C. Then T has a fixed point.

Proof. This follows immediately from Theorem 3.2 and Theorem 2.1. □

4. Conclusions and Open Problems

The objective of this section is to examine what is known, and not known, about fixed point results for several kinds of mappings related to the two constants. In the notion of geometric properties in Banach spaces especially the notions of normal and uniform normal structure, four important kinds of mappings are involved. Let recall their definitions. Let C be a subset of a Banach space X and $T : C \to C$ be a mapping. Firstly, T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers satisfying $\lim_n k_n = 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ $\forall x, y \in C, \forall n \in \mathbb{N}$ [19, Goebel and Kirk]. Secondly, if $k_n \equiv 1, \forall n \in \mathbb{N}$, then T is called a nonexpansive mapping. Thirdly, if there exists a constant k such that $k_n \equiv k, \forall n \in \mathbb{N}$, then T is said to be uniformly Lipschitzian. The final one is an asymptotically regular mapping which has already been defined in Section 2. Now we collect fixed point results for such mappings concerning the two constants.

In the following, let C be a closed bounded convex subset of a Banach space X.

Fact 4.1. [28, Mazcuñán-Navarro] *If $J(X) < 2$, equivalently $C_{NJ}(X) < 2$, then every nonexpansive mapping $T : C \to C$ has a fixed point.*

In [24, Theorem 1], Kim and Xu proved that if a Banach space X has uniform normal structure, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point. By combining this theorem with Corollary 3.3, we obtain the following

Fact 4.2. *If $J(X) < \frac{1+\sqrt{5}}{2}$, or $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point.*

In [6], Casini and Maluta proved the existence of fixed points of a uniformly k−Lipschitzian mapping T with $k < \sqrt{N(X)}$ in a space X with uniform normal structure. (As before, $N(X)$ is the normal structure coefficient of X.) Prus showed in [29] that $N(X) \ge J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}$ (see also Llorens-Fuster [27]). On the other hand, Kato et al. [23] showed that $N(X) \ge \frac{1}{\sqrt{C_{NJ}(X) - \frac{1}{4}}}$ (see also [27]). By using the results just mentioned, we now conclude the following results.

Fact 4.3. (1) *Suppose $J(X) < \frac{3}{2}$, and $T : C \to C$ is a uniformly k−Lipschitzian mapping such that*

$$k < \sqrt{J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}}.$$

Then T has a fixed point.
(2) *Suppose $C_{NJ}(X) < \frac{5}{4}$, and $T : C \to C$ is a uniformly $k-$Lipschitzian mapping such that*

$$k < \frac{1}{\sqrt{\sqrt{C_{NJ}(X) - \frac{1}{4}}}}.$$

Then T has a fixed point.

We end this paper by posing some open questions about these concepts.

Problem 4.4. Are the upper bounds of the James constants and of the Jordan-von Neumann constants sharp for a space to have uniform normal structure ?

Problem 4.5. Does the asymptotically regularity of T in Corollary 3.4 can be dropped ?

Problem 4.6. Can the upper bounds of $J(X)$ and $C_{NJ}(X)$ appearing in Fact 4.3 be improved ?

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Appendix 17: Diametrically contractive multivalued mappings, J. Fixed Point Theory and Applications (to appear).

Diametrically contractive multivalued mappings

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Abstract

Diametrically contractive mappings on a complete metric space are introduced by V.I. Istratescu. We extend and generalize this idea to multivalued mappings. An easy example shows that our fixed point theorem is more applicable than a former one obtained by H.K. Xu. A convergence theorem of Picard iteratives is also provided for multivalued mappings on hyperconvex spaces, thereby extending a Proinov's result.

1 Introduction

Let (X, d) be a complete metric space. A mapping $T : X \to X$ is a *contraction* if for some $\alpha \in (0, 1)$,

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

The mapping T is said to be *contractive* if

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X, x \neq y.$$

By the well-known Banach's contraction principle, every contraction has a unique fixed point x_0 and the Picard iteration $\{T^n x\}$ converges to x_0 for every $x \in X$. Examples in [7, 10] show that a contractive mapping may fail to have a fixed point. However, a question of the existence of a fixed point is of interest. In fact, it has been left open the following question:

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[5] Let M be a weakly compact subset of a Banach space and let $T: M \to M$ be contractive. Does T have a fixed point?

Istratescu [6] introduced a proper subclass of the class of the contractive mappings, whose elements are called the diametrically contractive mappings. Xu [10] proved, in the framework of Banach spaces, the following theorem.

Theorem 1.1 *[10, Theorem 2.3]* Let M be a weakly compact subset of a Banach space X and let $T: M \to M$ be a diametrically contractive mapping. Then T has a fixed point.

The following problems raised in [10] had been answered in the negative way in [2]:

Problem 1 *Can we substitute "weakly compact" subset with "closed convex bounded" one in Theorem 1.1?*

Problem 2 *If T is diametrically contractive and x^* is the fixed point of T, do we have $T^n x \to x^*$ for all (or at least for some) $x \in M$?*

In this paper, we weaken the condition in the definition of diametrically contractive mappings and obtain a corresponding fixed point theorem for nonself multivalued mappings. Moreover, we also apply a Proinov's fixed point theorem to a selection of a multivalued mapping with externally hyperconvex values and obtain its unique fixed point on a hyperconvex metric space.

2 Diametrically Contractive Mappings

In [6], Istratescu introduced a new class of mappings strictly lying between contractions and contractive mappings.

Definition 2.1 A mapping T on a complete metric space (X,d) is said to be *diametrically contractive* if $\delta(TA) < \delta(A)$ for all closed subsets A with $0 < \delta(A) < \infty$.
(Here $\delta(A) := sup\{d(x,y) : x, y \in A\}$ is the diameter of $A \subset X$.)

In the following, we consider a multivalued version of mappings in Theorem 1.1. We also can weaken the condition required in Definition 2.1.

Let $\mathcal{F}(X)$ be the collection of nonempty closed subsets of X and let Fix T denote the set of fixed points of T. Recall that $TA = \cup_{a \in A} Ta$.

Theorem 2.2 Let M be a weakly compact subset of a Banach space X and let $T: M \to \mathcal{F}(X)$, $Tx \cap M \neq \emptyset$ for all $x \in M$ and $\delta(TA \cap A) < \delta(A)$ for all closed sets A with $\delta(A) > 0$. Then T has a unique fixed point.

Proof The uniqueness of the fixed point is obvious. To prove the existence we consider the family $\mathcal{U} := \{A \subset M : A$ is a nonempty weakly compact subset of $M,\ TA \cap M \subset A\}$. Clearly, $\mathcal{U} \neq \emptyset$. Partially order $\mathcal{U}$ by saying that $A_1 \preceq A_2$ if $A_1 \supset A_2$ for $A_1, A_2 \in \mathcal{U}$. Every chain $\mathcal{C}$ in $\mathcal{U}$ has a finite intersection property, thus it has a nonempty intersection. That is $B := \cap_{A \in \mathcal{C}} A \neq \emptyset$. Since $TB \cap M \subset TA \cap M \subset A$ for all $A \in \mathcal{C}$, $TB \cap M \subset B$, i.e., $B \in \mathcal{U}$, and it is an upper bound of $\mathcal{C}$. Thus $\mathcal{U}$ has a maximal element, say A. Fix $x \in A$. As $A \in \mathcal{U}$ we see that $Tx \cap M \subset TA \cap M \subset A$. That is to say $Tx \cap A \neq \emptyset$ for all $x \in A$.

Put $A_0 = \overline{TA \cap A}^w$. Therefore $A_0 = \overline{TA \cap A}^w \subset \overline{TA \cap M}^w \subset \overline{A}^w = A$ and so $A_0 \subset A$. Moreover, we have $TA_0 \cap M \subset TA \cap M \subset A$. Therefore $TA_0 \cap M \subset TA \cap A \subset \overline{TA \cap A}^w = A_0$. Thus $A_0 \in \mathcal{U}$ and by maximality of A, we have $A = A_0 = \overline{TA \cap A}^w$. By lower semicontinuity of the norm of X we see that $\delta(A) = \delta(\overline{TA \cap A}^w) = \delta(TA \cap A)$. Since T is diametrically contractive we must have $\delta(A) = 0$ and A consists of exactly one point, say ξ. By the condition $\emptyset \neq TA \cap M \subset A$ we see that $\xi \in T\xi$, and we have a fixed point. $\square$

The proof given above is a modification of the proof of Theorem 1.1. The following example shows that Theorem 2.2 is strictly stronger than Theorem 1.1.

Example 2.3 $M = [0,5]$, $T : M \to \mathbb{R}$ defined by $Tx = x + 1$ if $x \leq 3$,and $Tx = 4$ if $x > 3$. Now, if A is a closed subset of M, then there will be a, b in M such that $A \subset [a,b]$ and $\delta(A) = b - a$. If $[a,b] \subset [0,3]$, then $TA \subset [a+1, b+1]$ and $TA \cap A \subset [a+1, b]$. Thus $\delta(TA \cap A) \leq b - a - 1 < \delta(A)$. If $a \leq 3 \leq b$, then $TA \subset [a+1, 4]$ and therefore $\delta(TA \cap A) \leq 3 - a < b - a = \delta(A)$. The case when $[a,b] \subset [3,5]$, T clearly satisfies $\delta(TA \cap A) = 0 < \delta(A)$. Thus T has a fixed point by Theorem 2.2. Note that 4 is the unique fixed point of T. We observe that T does not satisfy the condition in Theorem 1.1 because $\delta(T[0,1]) = 1 = \delta([0,1])$.

Example 2.4 Let $Tx = [0, x - \log(x+1)]$ for $x \in [0, 100]$. If A is a bounded closed subset of $[0, 100]$, then for some $a, b > 0$ we have $A \subset [a,b]$, and $\delta(A) = b - a$. Clearly $TA \subset \cup_{x \in A}[0, x - \log(x+1)] \subset [0, b - \log(b+1)]$, and so $TA \cap A \subset [a, b - \log(b+1)]$. Therefore $\delta(TA \cap A) < \delta(A)$. 0 is the unique fixed point of T.

Next we will replace the diameter $\delta(A)$ of a set A by $\alpha(A)$, where α is the Kuratowski measure of noncompactness:

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by finitely many sets with diameters } \leq \epsilon\}.$$

Definition 2.5 Let M be a nonempty subset of a metric space (X, d). A mapping $T : M \to 2^X$ is said to be *a k-set contraction* if, for each $A \subset M$ with A bounded, TA is bounded and $\alpha(TA) \leq k\alpha(A)$. If $\alpha(TA) < \alpha(A)$ for all such A, then T is said to be *α-condensing*.

Suppose that M is a bounded subset of a metric space (X, d). Then:
(i) $co(M) = \bigcap\{B \subset X : B$ is a closed ball in X such that $M \subset B\}$, and
(ii) M is said to be *subadmissible* [1], if for each $A \in \langle M\rangle, co(A) \subset M$, where $\langle M\rangle$ denotes the class of all nonempty finite subsets of M.

For a nonempty subset M of X and a topological space Y, if two set-valued mappings $T, F : M \to 2^Y$ satisfy the condition $T(co(A) \cap M) \subset F(A)$, for any $A \in \langle M\rangle$, then F is called a generalized KKM mapping with respect to T.

Let $T : M \to 2^Y$ be a set-valued mapping such that the family $\{\overline{Fx} : x \in M\}$ has the finite intersection property (where $\overline{Fx}$ denotes the closure of Fx) for each generalized KKM mapping $F : M \to 2^Y$ with respect to T , then we say that T has the KKM property. Denote

$KKM(M,Y) = \{T : M \to 2^Y : T$ has the KKM property$\}$.

Theorem 2.6 *[4, Theorem 1]* Let (X, d) be a complete metric space and M be a nonempty bounded nearly subadmissible subset of X. If $T \in KKM(M, M)$ is a k-set contraction, $0 < k < 1$, and closed with $\overline{TM} \subset M$, then T has a fixed point in M.

The next result shows that we can replace k-set contractions in Theorem 2.6 by α-condensing mappings.

Theorem 2.7 Let (X, d) be a complete metric space and M be a nonempty bounded nearly subadmissible subset of X. If $T \in KKM(M, M)$ is α-condensing, and closed with $\overline{TM} \subset M$, then T has a fixed point in M.

In the course of the proof, we will apply the technique in the proof of the following lemma.

Lemma 2.8 *[9, Lemma 2.2]* Let F be a selfmappping of an arbitrary set Y and let $f : Y \to \mathbb{R}_+$ be a nonnegative valued function defined on Y. Suppose that the following conditions hold:

(i) There exists a function $\varphi \in \Phi_1$ (i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying: for any $\epsilon > 0$, there exists $\delta > \epsilon$ such that $\epsilon < t < \delta$ impiles $\varphi(t) \le \epsilon$) such that $f(Fy) \le \varphi(f(y))$ for all $y \in Y$;
(ii) $f(y) > 0$ implies $f(Fy) < f(y)$ and $f(y) = 0$ implies $f(Fy) = 0$.

Then $\lim f(F^n y) = 0$ for each $y \in Y$.

Proof of Theorem 2.7 We follow the proof of Theorem 2.6. Let $y \in M$ be any point, and let $M_0 = M$. Define $M_1 = co(T(M_0) \cup \{y\}) \cap M$, and $M_{n+1} = co(T(M_n) \cup \{y\}) \cap M$, for each n. Then

$$\alpha(M_{n+1}) \le \alpha(T(M_n)) < \alpha(M_n) \le \dots < \alpha(M), \tag{1}$$

for each n (see[4]).
If we can prove that

$$\lim \alpha(M_n) = 0, \tag{2}$$

then the rest of the proof will follow the same lines as of Theorem 2.6. To achieve (2), we will apply the proof of Lemma 2.8. For each $t \in R_+$, let $A_t = \{M_n : \alpha(M_n) \leq t\}$ and $B_t = \{\alpha(M_{n+1}) : M_n \in A_t\}$. From (1), it is seen that $B_t \neq \emptyset$ and bounded if $A_t \neq \emptyset$. Define $\varphi(t) = \sup B_t$ if $A_t \neq \emptyset$. Otherwise, put $\varphi(t) = 0$. We claim that $\varphi \in \Phi_1$. Consider the set A_ϵ for $\epsilon > 0$. If $\alpha(M_n) \leq \epsilon$ for some n, let n_0 be the smallest such n. If $n_0 = 0$, then by (1) it is seen that $\varphi(t) \leq \epsilon$ for all $t > \epsilon$. Otherwise, let $\delta = \alpha(M_{n_0-1})$. Thus $\delta > \epsilon$, and if $\epsilon < t < \delta$, then by (1) we have $\varphi(t) \leq \alpha(M_{n_0+1}) < \alpha(M_{n_0}) \leq \epsilon$. Therefore $\varphi \in \Phi_1$. We now prove (2).
Clearly,we have

$$\alpha(M_{n+1}) \leq \varphi(\alpha(M_n)) \text{ and } \alpha(M_{n+1}) < \alpha(M_n) \text{ by (1) for each } n. \tag{3}$$

It follows from (1) that $\{\alpha(M_n)\}$ is strictly decreasing, hence it converges to some $\epsilon \geq 0$. Suppose $\epsilon > 0$. Since $\varphi \in \Phi_1$, we have for some $\delta > \epsilon$, $\varphi(t) \leq \epsilon$ for all $t \in (\epsilon, \delta)$. Choose n_0 so that $\epsilon < \alpha(M_{n_0}) < \delta$. Thus $\varphi(\alpha(M_{n_0})) \leq \epsilon$. But then (3) implies $\alpha(M_n) \leq \epsilon$ for all $n > n_0$ which contradicts to (1). Hence (2) follows. □

3 Picard Iteratives for Multivalued Mappings on Hyperconvex Metric Spaces

A metric space (X, d) is *hyperconvex* if for any family of points $\{x_\alpha\}$ in X and any family of positive numbers $\{r_\alpha\}$ satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have $\cap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$ where $B(x, r)$ is the closed ball with center at x and radius r. A subset E of X is said to be *externally hyperconvex* if for any of those families $\{x_\alpha\}, \{r_\alpha\}$ with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $dist(x_\alpha, E) \leq r_\alpha$, we have $\cap_\alpha B(x_\alpha, r_\alpha) \cap E \neq \emptyset$. The class of all externally hyperconvex subsets of X will be denoted by $\mathcal{E}(X)$. Let H be the Hausdorff metric.

Let t be a singlevalued selfmapping on a metric space (X, d). A fixed point of t is said to be *contractive* (cf.[8]) if all Picard iteratives of t converge to this fixed point. A selfmapping t on a metric space (X, d) is said to be *asymptotically regular* (cf.[3]) if $\lim d(t^n(x), t^{n+1}(x)) = 0$ for each x in X. Extend the concept naturally to multivalued mappings with the Hausdorff metric taken into action.

Theorem 3.1 *[9, Theorem 4.1]* Let t be a continuous and asymptotically regular selfmapping on a complete metric space satisfying the following conditions:

(i) There exists $\varphi \in \Phi_1$ such that $d(t(x), t(y)) \leq \varphi(D(x, y))$ for all $x, y \in X$;

(ii) $d(t(x), t(y)) < D(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then t has a contractive fixed point. Here $D(x,y) = d(x,y) + r[d(x,t(x)) + d(y,t(y))], r \geq 0$.

Replacing D by d, we present a multivalued version of Theorem 3.1 on a special setting, namely, on the class of hyperconvex metric spaces.

Theorem 3.2 Let (X, d) be a bounded hyperconvex metric space, $T : X \to \mathcal{E}(X)$ be asymptotically regular satisfying the following conditions:

(i) There exists $\varphi \in \Phi_1$ such that $\varphi(x) \leq x$, $\varphi(x+y) \leq \varphi(x) + \varphi(y)$, $\varphi(x) = 0$ if and only if $x = 0$, and $H(Tx, Ty) \leq \varphi(d(x, y))$ for all x, y in X;

(ii) $H(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then, if $\delta(T^n x) \to 0$ for each $x \in X$, T has a contractive fixed point. That is, there exists a unique point ξ in X such that, for each $x \in X$, $T^n x \to \{\xi\} = FixT$.

Proof The uniqueness of the fixed point is evident. We are going to find a selection $t : X \to X$ so that $t(x) \in Tx$ for all $x \in X$ and t satisfies the conditions in Theorem 3.1. Thus, there is a point ξ in Fix t satisfying $t^n(x) \to \xi$ for all $x \in X$. To find a selection t, we Zornimize the family $\mathcal{F} = \{(A, t) : \emptyset \neq A \subset X, t : A \to A$ asymptotically regular, $t(a) \in Ta$ for all $a \in A$, and t satisfies (i) and (ii) in Theorem 3.1$\}$. Partially order $\mathcal{F}$ by $(A_1, t_1) \preceq (A_2, t_2)$ if $A_1 \subset A_2$ and $t_2|_{A_1} = t_1$.

Suppose $A = \emptyset$ or $(A, t) \in \mathcal{F}$ and $x_0 \in X \setminus A$. We shall define a countable set $\{x_0, x_1, x_2, \ldots\}$, possibly finite, and an extension function t^* of t over $A \cup \{x_0, x_1, x_2, \ldots\}$ so that $(A \cup \{x_0, x_1, x_2, \ldots\}, t^*) \in \mathcal{F}$. Let $x_0, x_1, \ldots, x_n \in X \setminus A$ have been defined for some $n \geq 0$ so that, for $1 \leq k \leq n$, $x_k \in Tx_{k-1}$, and when $n \geq 2$, $d(x_{i+1}, x_{j+1}) \leq \varphi(d(x_i, x_j))$ and $d(x_{i+1}, x_{j+1}) < d(x_i, x_j)$ for $i < j$ in $\{1, \ldots, n-1\}$. Moreover, $d(t(x), x_{i+1}) \leq \varphi(d(x, x_i))$ and $d(t(x), x_{i+1}) < d(x, x_i)$ for $i \in \{1, \ldots, n-1\}$ and all $x \in A$.

Put $r_k = \varphi(d(x_{k-1}, x_n))$, and $r_{t(x)} = \varphi(d(x, x_n))$ for each $1 \leq k \leq n$, and for all $x \in A$. Thus, for $1 \leq k \leq n$, $x \in A$, and for $i < j$ in $\{1, \ldots, n-1\}$,

$dist(x_k, Tx_n) \leq H(Tx_{k-1}, Tx_n) \leq \varphi(d(x_{k-1}, x_n)) = r_k$,

$dist(t(x), Tx_n) \leq H(Tx, Tx_n) \leq r_{t(x)}$,

$d(t(x), x_k) \leq \varphi(d(x, x_{k-1})) \leq \varphi(d(x, x_n)) + \varphi(d(x_{k-1}, x_n)) = r_{t(x)} + r_k$, and

$d(x_i, x_j) \leq \varphi(d(x_{i-1}, x_{j-1})) \leq \varphi(d(x_{i-1}, x_n)) + \varphi(d(x_{j-1}, x_n)) = r_i + r_j$.

Finally, for $x, y \in A$, $d(t(x), t(y)) \leq \varphi(d(x, y)) \leq \varphi(d(x, x_n)) + \varphi(d(y, x_n)) = r_{t(x)} + r_{t(y)}$. Therefore there exists a point $x_{n+1} \in \cap_{x \in A} B(t(x), r_{t(x)}) \bigcap \cap_{k=1}^{n} B(x_k, r_k) \cap Tx_n$. The point x_{n+1} has the following property: For $k \leq n$, $d(x_k, x_{n+1}) \leq r_k = \varphi(d(x_{k-1}, x_n))$, and for each $x \in A, d(t(x), x_{n+1}) \leq r_{t(x)} = \varphi(d(x, x_n))$. Clearly, $d(x_k, x_{n+1}) < d(x_{k-1}, x_n)$ and $d(t(x), x_{n+1}) < d(x, x_n)$.

If $x_{n+1} \in A$, the process terminates. Otherwise, we obtain a subset $\{x_0, x_1, \ldots, x_n, \ldots\}$ of $X \setminus A$ satisfying the conditions (i) and (ii) in Theorem 3.1 where we extend t to t^* by defining $t^*(x_n) = x_{n+1}$ for $n \geq 0$. Thus $(A \cup \{x_0, x_1, \ldots\}, t^*) \in \mathcal{F}$.

In summary the above argument shows that, if $A = \emptyset$, then $(\{x_0, x_1, \ldots\}, t^*) \in \mathcal{F}$, i.e, $\mathcal{F} \neq \emptyset$. On the other hand if (A, t) is a maximal element in $\mathcal{F}$ (by Zorn's Lemma), we must have $A = X$. That is (X, t) belongs to $\mathcal{F}$ for some t. Apply Theorem 3.1, to conclude that there exists a fixed point ξ of t such that $t^n(x) \to \xi$ for each $x \in X$. Consequently, $T^n x \to \{\xi\}$ for each $x \in X$ and $FixT = \{\xi\}$. □

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Appendix 18: Common fixed points of a nonexpansive semigroup and a strong convergence theorem for Mann iterations in geodesic metric spaces (submitted).

Common fixed points of a nonexpansive semigroup and a strong convergence theorem for Mann iterations in geodesic metric spaces*

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Abstract

First, we consider a strongly continuous semigroup of nonexpansive mappings defined on a closed convex subset of a complete CAT(0) space and prove a strong convergence of a Mann iteration to a common fixed point of the mappings. This result is motivated by a result of Kirk (2002) and of Suzuki (2002). Second, we obtain a result on limits of subsequences of Mann iteratives of multivalued nonexpansive mappings on metric spaces of hyperbolic type, which leads to a strong convergence theorem for nonexpansive mappings on these spaces.

Keywords: Geodesic metric spaces, Nonexpansive semigroups, Fixed points.

AMS Subject Classification: 47H09, 54H25.

1 Introduction

In [7] , Kirk and Panyanak introduce a concept of weak convergence in CAT(0) spaces by saying that a net (x_α) converges weakly to x if x is the unique asymptotic center of every subnet of (x_α). One of their main results is that: "Every bounded closed convex set in a complete CAT(0) space is weakly compact in the sense that every net in the set has a weakly convergent subset." In Banach spaces, this holds when the spaces are reflexive. It is observed that every CAT(0)

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space is reflexive. Indeed, if (B_n) is a sequence of bounded closed and convex subsets of a CAT(0) space, then choose a sequence (x_n) with $x_n \in B_n$ for each n. With respect to an ultrapower $(\tilde{X}, \tilde{d})$ of (X, d), there exists a unique element $x \in B_1$ such that $\tilde{d}(\tilde{x}, \dot{x}) = dist(\tilde{x}, \dot{B_1})$ where $\tilde{x} = [(x_n)]$. If $x \notin B_2$, let $\nu \in B_2$ nearest to x. Now $d(x, x_n) \geq d(\nu, x_n)$ for all $n \geq 2$ imply that $\tilde{d}(\tilde{x}, \dot{\nu}) \leq \tilde{d}(\tilde{x}, \dot{x})$, a contradiction. Thus this shows that $x \in \cap_n B_n$.

In the first part of this paper, we extend this idea and consider a general metric space (X, d) and a topology τ on X which is weaker than the topology τ_d induced by the metric d. We obtain a metric version of Kirk's celebrated theorem for Banach spaces. By applying the result in [7] mentioned above we are able to extend a result of [9] on common fixed points of a nonexpansive semigroup on a Hilbert space to a complete CAT(0) space.

In 1983, K. Goebel and W.A. Kirk proved in [2] that: If $(x_n), (y_n)$ are sequences in a space of hyperbolic type, where (x_n) is defined by $x_1 \in M$, x_{n+1} is the point in the segment joining x_n, y_n with $d(x_n, x_{n+1}) = \alpha_n d(x_n, y_n)$, and if $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$, (α_n) is divergent in sum, the set $\{d(x_n, y_{n+i}) | n, i \in \mathbb{N}\}$ is bounded, and there is $b \in (0,1)$ such that $\alpha_n \leq b$ for each $n \in \mathbb{N}$, then $d(x_n, y_n) \to 0$. For a multivalued mapping, we consider in the second part of the paper Mann iterations on metric spaces of hyperbolic type and proof that all limits of convergent subsequences are fixed points by using this result of Goebel and Kirk. As a consequence, a strong convergence theorem on spaces of hyperbolic type is established.

2 Preliminaries

A notion of weak convergence in CAT(0) spaces based on the fact that in Hilbert spaces a bounded weakly convergent sequence always converges to its unique asymptotic center has been studied in [7]. Let (X, d) be a metric space, (x_n) be a bounded sequence in X and E a bounded subset of X. We associate with the number

$$r(E, (x_n)) = \inf\{\limsup_{n\to\infty} d(x, x_n) : x \in E\}$$

and the set

$$A(E, (x_n)) = \{x \in E : r(x, (x_n)) = r(E, (x_n))\},$$

where $r(x, (x_n)) = r(\{x\}, (x_n))$. $r(E, (x_n))$ and $A(E, (x_n))$ are called asymptotic radius and asymptotic center of (x_n) relative to E, respectively.

In a CAT(0) space, the asymptotic center $A(E, (x_n))$ of (x_n) consists of exactly one point whenever E is closed and convex. A sequence (x_n) in a CAT(0) space X is said to *converge weakly* to $x \in X$ if x is the unique asymptotic center of every subsequence of (x_n). Notice that given $(x_n) \subset X$ such that (x_n) converges weakly to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x, x_n) < \limsup_{n\to\infty} d(y, x_n).$$

Thus every CAT(0) space X has the Opial property.

Following Kirk and Panyanak [7], we define the asymptotic radius and asymptotic center for bounded nets analogous to the way they are defined for sequences. A bounded net (x_α) in a CAT(0) space X is said to *converge weakly* to $x \in X$ if x is the asymptotic center of every subnet of (x_α).

Every bounded closed convex set in a complete CAT(0) space is weakly compact. That is, if E is a bounded closed convex subset of a complete CAT(0) space and (x_α) is a net in E, then some subnet of (x_α) converges weakly to a point in E. Define a weak topology τ_w by calling a subset E *weakly closed* if it contains all their weak limit points. It gives a topology on a CAT(0) space that has many properties of the weak topology on reflexive Banach spaces.

Recall that a bounded sequence (x_n) in X is said to be *regular* relative to a bounded subset E of X if $r(E,(x_n)) = r(E,(u_n))$ for every subsequence (u_n) of (x_n). Furthermore, a sequence (x_n) in X which is regular relative to $E \subset X$ is said to be *asymptotically uniform* relative to E if $A(E,(x_n)) = A(E,(u_n))$ for every subsequence (u_n) of (x_n). In Banach spaces, every bounded sequence (x_n) has a regular subsequence, and if E is a separable subset of X, then (x_n) contains an asymptotically uniform subsequence relative to E (see, e.g., [3] p. 166). The proof can be carried over to the present setting without change.

3 Asymptotically weak normal structure

For a Banach space X, it is well-known that, if X does not have weak normal structure, then there must be a weakly null sequence (x_n) in $B(X)$ such that for $C := \bar{co}\{x_n : n \geq 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \text{diam} C = 1 \text{ for all } x \in C$$

(cf. [8]). The converse is obviously true. In terms of asymptotic centers, we can say that X does not have weak normal structure if and only if for some weakly closed convex set K and some sequence (x_n) in K, $A(K,(x_n)) = K$. Thus we extend the concept to metric spaces as follows.

Let (X,d) be a metric space, τ another topology on X which is weaker than τ_d. We say that X has *asymptotically weak normal structure* (with respect to τ) if every τ-closed set K with $diam(K) > 0$ and every sequence (x_n) in X which is regular relative to K, we have $A(K,(x_n))$ is τ-closed and $\emptyset \neq A(K,(x_n)) \neq K$. In practice, we may restrict ourselves to special kinds of sets K. For example, K can be τ−closed and convex when a concept of convexity is available. Thus, in this case, CAT(0) spaces have asymptotically weak normal structure with respect to τ_w.

Theorem 3.1 *Let E be a bounded τ-compact subset of X, $T : E \to E$ a nonexpansive mapping with $\inf_{x \in E} d(x, Tx) = 0$. If X has asymptotically weak normal structure, then T has a fixed point.*

Proof. Let $\mathcal{F}$ denote the family of all nonempty τ-closed subsets of E, each of which is mapped into itself by T. By Zorn's lemma $\mathcal{F}$ has a minimal element with respect to inclusion which we denote by K. We complete the proof by showing that K consists of a single point. Suppose on the contrary that $\operatorname{diam}(K) > 0$. We let (x_n) be a sequence in E such that $d(x_n, Tx_n) \to 0$. By passing through a subsequence we may assume that (x_n) is regular relative to K. Thus, $\emptyset \neq A(K, (x_n)) \neq K$. If $x \in A(K, (x_n))$, then

$$d(Tx, x_n) \leq d(Tx, Tx_n) + d(Tx_n, x_n).$$

Taking limit supremum on both sides we get $\limsup_n d(Tx, x_n) \leq r(K, (x_n)) + 0$ which implies that $Tx \in A(K, (x_n))$. Thus $A(K, (x_n)) \in \mathcal{F}$. Since X has asymptotically weak normal structure, $A(K, (x_n))$ is properly contained in K which contradicts to the minimality of K.

■

4 Common fixed points of nonexpansive semigroups

Let E be a closed convex subset of a Hilbert space H and T a nonexpansive mapping on E. We denote by $\mathcal{F}(T)$ the set of fixed points of T. Fixed $x_0 \in E$. Then for each $\alpha \in (0,1)$ there exists a unique point x_α of E satisfying $x_\alpha = (1-\alpha)Tx + \alpha x$ because the map $x_\alpha \mapsto (1-\alpha)Tx + \alpha x$ is a contraction. In 1967, Browder [1] and Halpern [4] proved independently the following theorem:

Theorem 4.1 (Browder [1], Halpern [4]) *Let E be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on E with a fixed point set $\mathcal{F}(T)$. Let (α_n) be a sequence in $(0,1)$ converging to 0. Fix x_0 and define a sequence (x_n) by*

$$x_n = (1-\alpha_n)Tx_n + \alpha_n x_0$$

for all n. Then (x_n) converges strongly to the element of $\mathcal{F}(T)$ nearest to x_0.

Let $S = \{T_t : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on a closed convex subset E of a Hilbert space H, i.e.,

(1) for each $t \in \mathbb{R}_+$, T_t is a nonexpansive mapping from E into itself;

(2) $T_0 x = x$ for all $x \in E$;

(3) $T_{s+t} = T_s T_t$ for all $s, t \in \mathbb{R}_+$;

(4) for each $x \in E$, the mapping $T(\cdot, x)$ from $\mathbb{R}_+$ into E is continuous.

We put $\mathcal{F}(S) = \bigcap_{t \in \mathbb{R}_+} \mathcal{F}(t)$. In 2002, Suzuki [9] proved a common fixed point theorem for a strongly continuous semigroup of nonexpansive mappings.

Theorem 4.2 (T. Suzuki [9]) *Let E be a closed convex subset of a Hilbert space H. Let $S = \{T_t : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on E such that $\mathcal{F}(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1, t_n > 0$ and $\lim_n t_n = \lim_n \frac{\alpha_n}{t_n} = 0$. Fix $x_0 \in E$ and define a sequence $\{x_n\}$ in E by*

$$x_n = (1 - \alpha_n) T_{t_n} x_n + \alpha_n x_0$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to the element of $\mathcal{F}(S)$ nearest to x_0.

In 2003, Kirk [6] studied a convergence theorem on CAT(0) spaces. He proved the following theorem.

Theorem 4.3 (Kirk [6]) *Let E be a bounded closed convex subset of a complete CAT(0) space X, let T be a nonexpansive mapping on E, let $x_0 \in E$ and for each $t \in (0, 1)$, let x_t be the fixed point of the mapping $x \mapsto tTx \oplus (1 - t)x_0$. Then $\lim_{t \to 1^-} x_t$ converges to the unique fixed point of T which is nearest to x_0.*

Motivated by the above results, we prove the following theorem.

Theorem 4.4 *Let E be a bounded closed convex subset of a complete CAT(0) space X, let $S = \{T_t : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on E. Let (α_n) and (t_n) be sequences of real numbers satisfying $0 < \alpha_n < 1, t_n > 0$ and $\lim_n t_n = \lim_n \frac{\alpha_n}{t_n} = 0$. Let $x_0 \in E$ and for each n let x_n be a fixed point of the mapping $x \mapsto (1 - \alpha_n) T_{t_n} x \oplus (1 - \alpha_n) x_0$. Then $\mathcal{F}(S) \neq \emptyset$ and $\{x_n\}$ converges strongly to the element of $\mathcal{F}(S)$ nearest to x_0.*

Proof. The proof will proceed as follow : (i) We will prove first that some subsequence (u_n) of (x_n) converges weakly to a point x in $\mathcal{F}(S)$. Thus $\mathcal{F}(S) \neq \emptyset$. The main computation in the proof follows the idea in [9]. (ii) We then show that a subsequence of (u_n) converges strongly to x. Finally, we prove that (iii) x is the point in $\mathcal{F}(S)$ nearest to x_0.

(i): Let (w_n) be an arbitrary subsequence of (x_n). Then there exists a subsequence (u_n) of (w_n) converging weakly to an element x of E. We assume without loss of generality that (u_n) is regular. Define the corresponding sequences (s_n) and (β_n) so that

$$u_n = (1 - \beta_n) T_{s_n} u_n + \beta_n x_0 \quad \text{for each } n.$$

Let $t > 0$. Thus

$$\begin{aligned} d(T_t x, u_n) &\le d(T_0 u_n, T_{s_n} u_n) + d(T_{s_n} u_n, T_{2s_n} u_n) + \dots + d(T_{([\frac{t}{s_n}]-1)s_n} u_n, T_{[\frac{t}{s_n}]} u_n) + \\ &\quad d(T_{[\frac{t}{s_n}]s_n} u_n, T_{[\frac{t}{s_n}]s_n} x) + d(T_{[\frac{t}{s_n}]s_n}, T_t x) \\ &\le [\frac{t}{s_n}] d(T_{s_n} u_n, u_n) + d(u_n, x) + d(T_{t-[\frac{t}{s_n}]s_n} x, x) \\ &\le \beta_n [\frac{t}{s_n}] d(T_{s_n} u_n, x_0) + d(u_n, x) + d(T_{t-[\frac{t}{s_n}]s_n} x, x) \\ &\le \frac{t\beta_n}{s_n} d(T_{s_n} u_n, x_0) + d(u_n, x) + \max_{0 \le s \le s_n} d(T_s x, x) \end{aligned}$$

for all $n \ge 1$. This implies $\limsup_n d(T_t x, u_n) \le \limsup_n d(x, u_n)$, and then $T_t x = x$. That is x is a common fixed point.

(ii): Let $\bar{\Delta}(\bar{T}_{s_m} u_m, \bar{T}_{s_n} u_n, \bar{0})$ be a comparison triangle of $\Delta(T_{s_m} u_m, T_{s_n} u_n, x_0)$. If a subsequence of $(\bar{T}_{s_n} u_n)$ converges to $\bar{0}$, then the corresponding subsequence of (u_n) converges. Now suppose, without loss of generality, that the sequence $(||\bar{T}_{s_n} u_n||)$ converges to a positive real number. To show that (u_n) converges, we suppose on the contrary that $r := r(x_0, (u_n)) > 0$. Since $d(T_{s_n} u_n, x) \le d(u_n, x)$, we see that $x \ne x_0$. Choose a subsequence (u_{n_k}) of (u_n) so that $\lim_k d(u_{n_k}, x) = r$ and $\lim_k d(u_{n_k}, x_0) = a > 0$. Let $\bar{\Delta}(\bar{T}_{s_n} u_n, \bar{x}, \bar{0})$ be a comparison triangle of $\Delta(T_{s_n} u_n, x, x_0)$ and let $\bar{u}_{n_k}$ be the corresponding point, in the comparison triangle, of u_{n_k}. We may assume that the limit $\lim_k \|\bar{u}_{n_k} - \bar{x}\|$ exists which is clearly equal to r since $\bar{u}_n - \bar{T}_{s_n} u_n \to 0$. As $\|\bar{u}_{n_k} - \bar{x}\| \ge \|\bar{T}_{s_n} u_n - \bar{x}\|$, the angle $\angle \bar{0}\bar{u}_{n_k}\bar{x}$ is at least $\frac{\pi}{2}$ radians. Consider a right triangle ABC which has $|AB| = ||\bar{x}||$, $|BC| = r$, $|AC| = a$, and has a right angle at C. If $CD \perp AB$ where D lies on the segment AB, there exists $\delta = \delta(r) > 0$ such that $|CD| < r - \delta$. Let z be the point on the segment $[x_0, x]$ joining x_0 and x so that $d(x_0, z) = |AD|$. Let $\bar{z}$ be the corresponding point of z on the segment $[\bar{0}, \bar{x}]$. Finally, we estimate:

$$\begin{aligned} d(u_{n_k}, z) &\le ||\bar{u}_{n_k} - \bar{z}|| \\ &\le ||\bar{u}_{n_k} - \bar{x}|| - \delta_k \\ &\le \beta_{n_k} ||\bar{T}_{s_{n_k}} u_{n_k}|| + ||\bar{T}_{s_{n_k}} u_{n_k} - \bar{x}|| - \delta_k \\ &= \beta_{n_k} d(x_0, T_{s_{n_k} u_{n_k}}) + d(T_{s_{n_k}} u_{n_k}, x) - \delta_k \\ &\le \beta_{n_k} d(x_0, T_{s_{n_k} u_{n_k}}) + d(u_{n_k}, x) - \delta_k. \end{aligned}$$

We can choose the sequence (δ_k) so that $\lim_k \delta_k = \delta$. Then

$$\limsup_{k \to \infty} d(u_{n_k}, z) \le 0 + r - \delta < r,$$

a contradiction.

(iii) : We show that x is the point in $\mathcal{F}(S)$ nearest to x_0. Let $p \in \mathcal{F}(S)$. Consider a comparison triangle $\bar{\Delta}(\bar{T}_{s_{n_k}} u_{n_k}, \bar{p}, \bar{0})$ of $\Delta(T_{s_{n_k}} u_{n_k}, p, x_0)$. The inequalities

$$||\bar{T}_{s_{n_k}} u_{n_k} - \bar{p}|| = d(T_{s_{n_k}} u_{n_k}, T_{s_{n_k}} p) \le d(u_{n_k}, p) \le ||u_{n_k} - \bar{p}||$$

imply that the angle $\angle\bar{0}\bar{u}_{n_k}\bar{p} \geq \frac{\pi}{2}$. Therefore,

$$d^2(x_0,p) = ||\bar{p}||^2 \geq ||\bar{p} - \bar{u}_{n_k}||^2 + ||\bar{u}_{n_k}||^2 \geq d^2(p,u_{n_k}) + d^2(u_{n_k},x_0).$$

Thus, $d^2(x_0,p) = ||\bar{p}||^2 \geq d^2(p,x) + d^2(x,x_0) \geq d^2(x_0,x)$. This shows that x is the point nearest to x_0. ■

Remark. If we assume that $\mathcal{F}(S) \neq \emptyset$, then we can drop the boundedness of E. Indeed, if u is the nearest point of $\mathcal{F}(S)$ from x_0, then

$$\begin{aligned} d(x_n,u) &= d((1-\alpha_n)T_{t_n}x_n \oplus \alpha_n x_0, u) \\ &= (1-\alpha_n)d(T_{t_n}x_n,u) + \alpha_n d(x_0,u) \\ &\leq (1-\alpha_n)d(x_0,u) + \alpha_n d(x_n,u). \end{aligned}$$

Thus $d(x_n,u) \leq d(x_0,u)$ for all n and therefore (x_n) is bounded.

5 A Strong convergence theorem of Mann iterations

Following Kirk [5], we suppose (X,d) is a metric space containing a family $\mathcal{L}$ of metric segments such that (a) each two points x,y in X are endpoints of exactly one member $[x,y]$ of $\mathcal{L}$ and (b) if $p,x,y \in X$ and if $m \in [x,y]$ satisfies $d(x,m) = \alpha d(x,y)$ for $\alpha \in [0,1]$, then

$$d(p,m) \leq (1-\alpha)d(p,x) + \alpha d(p,y). \tag{5.1}$$

Spaces of this type are said to be of hyperbolic type (Takahashi [10] called these spaces convex metric spaces). CAT(0) spaces as well as normed linear spaces and hyperconvex metric spaces are of hyperbolic type. A subset of X is said to be *convex* if every segment joining two points in the set entirely lies in the set.

The following proposition was proved in [2].

Proposition 5.1 ([2]) *Let (x_n), (y_n) be sequences in a space of hyperbolic type (X,d), let $(\alpha_n) \subseteq [0,1]$. If $(x_n),(y_n)$, and (α_n) satisfy the conditions :*
(i) $x_{n+1} \in [x_n,y_n]$ with $d(x_n,x_{n+1}) = \alpha_n d(x_n,y_n)$,
(ii) $d(y_{n+1},y_n) \leq d(x_{n+1},x_n)$,
(iii) $\sup_{i,n} d(y_{i+n},x_i) < \infty$,
(iv) $\alpha_n \leq b < 1$, for each $n \in \mathbb{N}$, and
(v) $\sum \alpha_n = \infty$,
then $d(y_n,x_n) \to 0$.

Let X be a metric space. A subset P of X is called *proximinal* if for each $x \in X$, there exists an element $p \in P$ such that

$$d(x,p) = dist(x,P) = inf\,\{d(x,y) : y \in P\}.$$

Let E be a subset of X. We denote by $\wp(E)$ the family of nonempty proximinal subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $\wp(X)$, i.e.,

$$H(A,B) = max\left\{sup_{a\in A}dist(a,B), sup_{b\in B}dist(b,A)\right\},$$

where $dist(a,B) = inf\left\{d(a,b)|b \in B\right\}$.

A multivalued mapping $T : E \to \wp(E)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq d(x,y),$$

for each x, y in E.

The sequence of Mann iteration (x_n) is defined for $x_1 \in X$ by:

(M) Let y_1 be any point in Tx_1, x_2 be the point in the segment joining x_1, y_1 with $d(x_1,x_2) = \alpha_1 d(x_1,y_1)$. In general, x_{n+1} is the point in the segment joining x_n, y_n with $d(x_n,x_{n+1}) = \alpha_n d(x_n,y_n)$, $\alpha_n \in [0,1], n \geq 1$, where $y_n \in Tx_n$ is such that $d(y_{n-1},y_n) = dist(y_{n-1},Tx_n)$.

Proposition 5.2 *Let E be a convex subset of a space of hyperbolic type X, $T : E \to E$ be a nonexpansive mapping. Let (x_n) be the sequence of Mann iteration defined by (M). Assume that*
(i) $\sum \alpha_n = \infty$,
(ii) $\alpha_n \in [0,b]$, for some $b \in (0,1)$,
(iii) $\sup_{i,n} d(y_{i+n},x_i) < \infty$.
If (x_n) has a convergence subsequence with limit in E, then its limit is a fixed point.

Proof. Since

$$\begin{aligned} d(y_n,y_{n+1}) &= dist(y_n,Tx_{n+1}),\\ &\leq H(Tx_n,Tx_{n+1}),\\ &\leq d(x_n,x_{n+1}). \end{aligned}$$

Therefore, Proposition 5.1 implies $d(x_n,y_n) \to 0$. Let (x_{n_k}) be a subsequence of (x_n) such that $d(x_{n_k},x) \to 0$ for some $x \in E$. We show that x is a fixed point of T. Consider the following inequalities:

$$\begin{aligned} dist(x,Tx) &= \inf_{a\in Tx} d(x,a),\\ &\leq \inf_{a\in Tx}[d(x,y_{n_k}) + d(y_{n_k},a)],\\ &= d(x,y_{n_k}) + \inf_{a\in Tx} d(y_{n_k},a),\\ &= d(x,y_{n_k}) + dist(y_{n_k},Tx),\\ &\leq d(x,y_{n_k}) + H(Tx_{n_k},Tx). \end{aligned}$$

Since $d(x_{n_k}, x) \to 0$ and $d(x_n, y_n) \to 0$, we have $d(y_{n_k}, x) \to 0$. Moreover, the nonexpansiveness of T implies that $H(Tx_{n_k}, Tx) \to 0$. Thus $dist(x, Tx) = 0$ and x is a fixed point of T. ■

When T is a single-valued nonexpansive mapping we see from (5.1) that $d(x_{n+1}, p) \leq d(x_n, p)$ for all fixed points p of T. Thus in this case we have a strong convergence theorem for spaces of hyperbolic type.

Corollary 5.3 *Let E be a compact convex subset of a space of hyperbolic type, $T : E \to E$ be a nonexpansive mapping and (x_n) be the sequence of Mann iteration defined for $x_1 \in E$ by $d(x_n, x_{n+1}) = \alpha_n d(x_n, Tx_n)$ for $n \geq 1$. Assume that*
(i) $\sum \alpha_n = \infty$,
(ii) $\alpha_n \in [0, b]$, *for some* $b \in (0, 1)$,
(iii) $\sup_{i,n} d(y_{i+n}, x_i) < \infty$.
Thus $x_n \to p$, *for some* $p \in FixT$.

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