รายงานวิจัยฉบับสมบูรณ์

โครงการ :

"GEOMETRY OF BANACH SPACES IN HYPERCONVEX FIXED POINT THEORY"

โดย ศ. ดร. สมพงษ์ ธรรมพงษา และคณะ

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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จากการที่ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) โดยเฉพาะศาสตราจารย์ ดร.วิชัย บุญแสง เล็งเห็นถึงความสำคัญของงานวิจัยองค์ความรู้ใหม่ (Basic Research) ในทุกสาขา และให้การสนับสนุนอย่างต่อเนื่องตลอดมา งานวิจัยทางคณิตศาสตร์จึงได้เจริญก้าวหน้าอย่างไม่เคยปรากฏมาก่อน การแสดงออกของพวกเราถึงความชื่นชมและขอบคุณต่อ สกว.จึงมีปรากฏในทุกสถานที่ โครงงานวิจัยเรื่อง Geometry of Banach Spaces in Hyperconvex Fixed Point Theory นี้เป็นอีกโครงงานหนึ่งที่ได้รับการสนับสนุนด้วยดีจาก สกว.

ผลงานวิจัยที่ได้รับมีทั้งหมด 18 เรื่อง ได้รับการตีพิมพ์ในวารสารนานาชาติที่มีคุณภาพ 17 เรื่อง และอยู่ระหว่างการพิจารณาของ referee 1 เรื่อง โดยส่วนตัวของหัวหน้าคณะวิจัยจะพยายามสร้างผลงานที่คุณภาพเพื่อการ citation ที่สูง สิ่งเหล่านี้พิสูจน์ได้จากผลงานในโครงการวิจัยที่ได้รับการสนับสนุนจาก สกว. ก่อนหน้านี้ ดังนั้นการพิจารณา citation ของ journal ในเทอมของ impact factor จึงไม่ใช่ตัววัดที่สำคัญของเรา การเน้นที่ citation ของตัว paper หรือตัว author เป็นเป้าหมายหลักที่จะนำไปสู่ชื่อเสียงที่แท้จริงของผลงานวิจัย จากผลงานวิจัยที่ผ่านมาคณะวิจัยได้รับเกียรติและได้รับความไว้วางใจให้เป็นเจ้าภาพจัดการประชุม The 8th International Conference on Fixed Point Theory and Its Application (ICFPTA 2007) ขึ้นที่ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ระหว่างวันที่ 16-22 เดือน กรกฎาคม 2550

โครงงานวิจัยนี้มีส่วนประกอบ 3 ส่วนตามแผนงานที่ระบุไว้ในสัญญา คือ

1. Fixed point theory in Banach spaces: a nonstandard analysis approach [1,2,3,5,11,12,13,14,16]

2. Topology of convergence sets [9,10]

3. Fixed point theory in hyperconvex spaces [4,8,9,15,17,18]

ผลสำเร็จของโครงการวิจัย ในด้านคุณภาพมีแนวโน้มว่าจะมี citation ที่ดีซึ่งปรากฏมีขึ้นแล้ว แต่จักต้องใช้ความพยายามมากขึ้นที่จะ publish ผลงานใน journal ในระดับที่สูงขึ้นกว่านี้

ขอขอบพระคุณ สำนักงานกองทุนสนับสนุนการวิจัย เป็นอย่างสูง ไว้ ณ ที่นี้ และขอขอบพระคุณภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ที่ให้ความสะดวกทางด้านอุปกรณ์คอมพิวเตอร์และการพิมพ์

คณะผู้วิจัย
ภาควิชาคณิตศาสตร์
มหาวิทยาลัยเชียงใหม่
30 พฤษภาคม พ.ศ. 2550

บทคัดย่อ

งานวิจัยนี้สร้างทฤษฎีจุดตรึงในปริภูมิบานาคภายใต้เงื่อนไขต่างๆทางเรขาคณิตของปริภูมิ การส่งที่ศึกษาส่วนใหญ่จะเป็นการส่งเพื่อจุดมุ่งหมายนี้ได้มีการสร้างอสมการต่างๆที่หลากหลายที่เกี่ยวข้องกับคุณสมบัติทางเรขาคณิต มีการค้นพบคุณสมบัติทางเรขาคณิตใหม่ๆที่พิสูจน์ได้ในเวลาต่อมาว่าเป็นเครื่องมือใหม่ชิ้นหนึ่งที่สำคัญในทฤษฎีจุดตรึง เครื่องมือสำคัญชิ้นหนึ่งในวิชาการวิเคราะห์นอนแสตนดาร์ดที่ถูกนำมาใช้ตลอดเวลาคือเทคนิค-อุลทราเพาเวอร์

ทฤษฎีบทในปริภูมิบานาคได้ถูกนำมาขยายผลในปริภูมิไฮเพอร์คอนเวกซ์ โดยเฉพาะอย่างยิ่งปริภูมิ CAT(0) การวิจัยอีกส่วนหนึ่งของโครงการคือการศึกษาโทโพโลยีของเซทของการลู่เข้าของการส่ง

คำหลัก: คุณสมบัติทางเรขาคณิต ปริภูมิบานาค เทคนิค-อุลทราเพาเวอร์ ปริภูมิไฮเพอร์คอนเวกซ์ เซทของการลู่เข้า

ABSTRACT

We investigate Fixed Point Theory in Banach spaces under various conditions of their geometry. Most of the mappings of interest are multivalued. To achieve this goal, several inequalities concerning important geometric properties are derived. Some new geometric properties under this research project are introduced. It is proved that many new results can be developed in the direction of these new properties. One of the main ingredients in the study is a technique from nonstandard analysis, namely the ultrapower technique.

The theory is then extended to the class of hyperconvex spaces. The most accomplishment is on the class of CAT(0) spaces, its important subclass. Parts of the research are devoted to the study of the topology of the convergence sets of mappings.

Keywords: geometric property, Banach space, ultrapower technique, hyperconvex space, convergence set

CONTENTS

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1. Project Title

GEOMETRY OF BANACH SPACES IN HYPERCONVEX FIXED POINT THEORY

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3. Research Field

FUNCTIONAL ANALYSIS AND TOPOLOGY

4. Problem statement and importance

The fixed point property (FPP) has been studied since J.Brouwer and S.Banach leading to several celebrated theorems such as Brouwer's Fixed Point Theorem and the principle of Banach's Contraction Mapping. The theory of fixed point property is one of the most important subject in pure and applied Mathematics. It contributes to a variety of applications in many fields of mathematics such as the theory of operators, control theory, approximation theory, and theory of equations. Using geometric property to study FPP has been developed since W.K.Kirk who proved in 1965 that a Banach space with a normal structure has weak fixed point property .

The fixed point property is still proven to be the most important research problem in Mathematics and is continuing to be of interest to mathematicians worldwide. The study of nonexpansive mappings has been substantially motivated by the study of monotone and accretive operators, two classes of operators which arise naturally in the theory of differential equations. As an example, Kato (1967) has obtained the following basic result : For a Banach space X, a subset D of X, and a map T:D ---->X, T is accretive if and only if for each x, y $\in$ D and $\lambda \geq 0$, $\| x-y \| \leq \| x-y+\lambda(Tx-Ty) \|$. Thus a mapping T:D --- > X is accretive if and only if the mapping $J_\lambda = (I-\lambda T)^{-1}$ (called the resolvent of T) is nonexpansive on its domain.

Other examples showing that the notion of nonexpansive mappings and their sets of fixed points play a crucial role in optimization theory (see [5, 39, 41, 42, 48, 50, 56, 57, 58]). In these studies, some forms of convergence theorems for nonexpansive mappings are considered.

The study of nonexpansive mappings and their fixed points could be extended to metric spaces. It is well-known that every nonexpansive mapping on a bounded hyperconvex metric space has a fixed point (Baillon 1988 [3]). The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi [2] 1956. Espinola and Kirk [15] obtained fixed point theorems in $\mathbb{R}$ -trees, whose class forms a subclass of hyperconvex spaces, and applied them to the Graph Theory.

Besides the existence of a fixed point, the first co-investigator is also interested in some topological properties of the fixed point set of a continuous mapping. He recently establishes the notion of the convergence set of a continuous mapping and shows that there is a nice relationship between the convergence set and the fixed point set certain kinds of mappings (including nonexpansive mappings). This relationship opens a new door to study topological properties of the fixed point set using those of the convergence set.

In view of the importance of this subject, the Principal Investigator (PI) and his group intend to continue their work on the mathematics itself, rather than on its applications. One part of the project then aims at establishing new concepts and results for fixed point theory both in Banach spacesand in hyperconvex metric spaces. Two of the main tools, among others, that the PI plans to employ and develop further are " the generalized Jordan- von Neumann constants" proposed by Dhompongsa and others [14], and "the generalized James constants" proposed by Dhompongsa and others [13]. For the second part, the project will concentrate on the topological property of the fixed point set of a given continuous mapping by means of its convergence set, the new notion established earlier by the first co-investigator.

This project will be undertaken for three years in collaboration with the PI's colleague and their students at Chulalongkorn University and Chiang Mai University. The PI has created a group of students working on Geometric Property, in particular, on the Fixed Point Property in Banach spaces. The impact of this project will not only contribute to advance the knowledge of the mathematical society itself, but will also strengthen the research groups, especially students as its members, in Thailand.

5. Objective of Research

5.1 To investigate the significance of the constants $C_{NJ}(a,X)$ and $J(a,X)$. For examples, the preservation of the property $C_{NJ}(a,X)<2$ and $C_{NJ}(a,Y)<2$ by their direct sum $X\oplus Y$ under various norms. (See [12, 13, 14]).

5.2 To investigate the convexity property of direct sums $X\oplus Y$ (See [12, 45]).

5.3 To apply nonstandard analysis to the fixed point theory (See [4, 43, 51, 52, 53, 54, 55]).

5.4 To establish fixed point theorems in the hyperconvexity setting (See [6, 7, 8, 15, 16, 33, 34, 35, 36, 37, 38]).

5.5 To investigate some topological properties of the convergence set.

5.6 To investigate the relationship between the convergence set and the fixed point set and use this relationship to study some topological properties of the fixed point set.

6. Methodology of Research

6.1 Solve problems and conjectures raised in [14] and [13].

6.2 Invent sufficient conditions for the fixed point property of direct sums $X \oplus Y$

6.3 Apply a new tool, namely, the nonstandard analysis technique to the fixed point property in Banach spaces.

6.4 Modify standard proofs formulated in a Banach space setting to a metric space one.

6.5 Invent new topological properties of the convergence set.

6.6 Establish new topological properties of the fixed point set using (6.5).

7. Plan of Research

7.1 Collecting know results concerning the FPP and related concepts in Banach spaces and hyperconvex spaces.

7.2 Preparing specific topics for each member in the research group to concentrate on his/her own research work.

7.3 Sharing results obtained in (7.2), discussing, and preparing manuscripts.

Plan for each 6 month period :

January 1, 2004 - June 30, 2004 : Writing a paper on "Convexity property of direct sums $X \oplus Y$ ".

July 1, 2004 - December 31, 2004 : Writing a paper on "Constants $C_{NJ}(a, X)$, $J(a, X)$ and other related ones" and "Topology of convergence sets".

January 1, 2005 - June 30, 2005 : Writing a paper on "Nonstandard analysis for fixed point property in Banach spaces" and "Topology of convergence sets".

July 1, 2005 - December 31, 2005 : Writing a paper on "Nonstandard analysis for fixed point property in Banach spaces" and "Topology of convergence sets".

January 1, 2006 - June 30, 2006 : Writing a paper on "Fixed point property in hyperconvex spaces".

July 1, 2006 - December 31, 2006 : Writing a paper on "Transition of geometric property in Banach spaces to hyperconvex spaces ".

8. Expected output

We expect to publish at least 2 papers a year.

Tentative titles and journals :

(1) Title : Geometry of Banach spaces in hyperconvex space I, II; Topology of convergence sets.

Journal : Proceedings Amer. Math. Soc (impact factor = 0.369) or London J. Math. Soc. (impact factor = 0.441) or Pacific J. Math. (impact factor = 0.395) or Studia Math. (impact factor = 0.399).

(2) Title : Fixed point theorems: A nonstandard analysis approach I, II; Topology of convergence sets.

Journal : J. Functional Anal. (impact factor = 0.879) or Bull. Austral.. Math. Soc (impact factor = 0.236) or J. Austral. Math. Soc. (impact factor = 0.282) or J. Math. Anal. Appl. (impact factor = 0.444).

(3) Title : Convexity of ψ -direct sums.

Journal : Nonlinear Anal. (impact factor = 0.406)

9. Selected published research papers related to research project matter.

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OUTPUT

We divide the output into 2 categories.

I. Published papers:

- Uniform smoothness and U-convexity of ψ- direct sums, J. Nonlinear and Convex Analysis, 6 (2) (2005), 327-338. Appendix 1

- Fixed point property of direct sums, Nonlinear Anal. 63 (2005), e2177-e2188. Appendix 2

- A note on properties that implies the weak fixed point property, Abst. Appl. Anal. V. 2006, Article ID 34959, Pages 1-12. Appendix 3

- Lim's theorems for multivalued mappings in CAT(0) spaces, J. Math. Anal. Appl. 312 (2005), 478-487. Appendix 4

- The Dominguez-Lorenzo condition and multivalued nonexpansive mappings, Nonlinear Anal. 64 (2006), 958-970. Appendix 5

- Jordan-von Neumann constant and fixed points for multivalued nonexpansive Mappings, J. Math. Anal. Appl. 320 (2006), 916-927. Appendix 6

- Fixed point theorems for multivalued mappings in modular function spaces, Scien. Math. Japon. 63 (2) (2006), 161-169. Appendix 7

- Fixed points of uniformly Lipschitzian mappings, Nonlinear Anal. 65 (2006), 762-772. Appendix 8

- A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006), 983-987. Appendix 9

- Virtually nonexpansive maps and their convergence sets, J. Math. Anal. Appl. 326 (2007), 390-397. Appendix 10

- The James constant of normalized norms on R^2, J. Inequalities Appl. V. 2006, Article ID 26265, Pages 1-12. Appendix 11

- On the modulus of W*-convexity, J. Math. Anal. Appl. 320 (2006), 543-548. Appendix 12

- On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property, J. Math. Anal. Appl. 323 (2006), 1018-1024. Appendix 13

- A new trees-step fixed point iteration scheme for asymptotically nonexpansive Mappings, Appl. Math. Comp. 181 (2006), 1026-1034. Appendix 14

- Nonexpansive set-valued mappings in metric and Banach spaces, Journal of Nonlinear and Convex Analysis, 8 (1) (2007), 35-45. Appendix 15

- An inequality concerning the James constant and the weakly convergent sequence coefficient, Journal of Nonlinear and Convex Analysis (to appear). Appendix 16

- Diametrically contractive multivalued mappings. Appendix 17

I. Submitted paper:

- Common fixed points of a nonexpansive semigroup and a strong convergence theorem for Mann iterations in geodesic metric spaces. Appendix 18

Appendix

Appendix 1: Uniform smoothness and U-convexity of ψ- direct sums,

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UNIFORM SMOOTHNESS AND U-CONVEXITY OF ψ-DIRECT SUMS

SOMPONG DHOMPONGSA, ATTAPOL KAEWKHAO, AND SATIT SAEJUNG

ABSTRACT. We study the ψ-direct sum, introduced by K.-S. Saito and M. Kato, of U-spaces, introduced by K. S. Lau. For Banach spaces X and Y and a continuous convex function ψ on the unit interval $[0,1]$ satisfying certain conditions, let $X \oplus_\psi Y$ be the ψ-direct sum of X and Y equipped with the norm associated with ψ. We first show that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is isometric to the space $X^* \oplus_\varphi Y^*$ for some continuous convex function φ satisfying the same conditions as of ψ. We introduce the so-called u-spaces and show that: (1) $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function. We also show that (2) $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function. As consequences, using the notion of ultrapower, we obtain : (3) $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function, and (4) $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

1. Introduction

For every continuous convex function ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$, there corresponds a unique absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ (see Bonsall and Duncan [3], also [19]). Recently, in [16] the authors introduced the ψ-direct sums $X \oplus_\psi Y$ of Banach spaces X and Y equipped with the norm associated with ψ, and proved that $X \oplus_\psi Y$ is uniformly convex if and only if X, Y are uniformly convex and ψ is strictly convex. We write $X \simeq Y$ to indicate that X and Y are isometric (or Banach isomorphism, see [12]).

The purposes of this paper are to characterize uniform smoothness and U-convexity of $X \oplus_\psi Y$. In Section 2 we shall recall some fundamental facts on the ψ-direct sums of Banach spaces and introduce the dual function φ of ψ so that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is $X^* \oplus_\varphi Y^*$. In Section 3 we shall show that the ultrapower of $X \oplus_\psi Y$ is the ψ-direct sum of the ultrapowers of X and of Y. In Section 4 we shall prove that $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function, and by using the ultrapower technique we obtain that $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function. In Section 5 we introduce new spaces, namely u-spaces, and prove that $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function, and again by using the ultrapowers we have $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

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2. The ψ-direct sums

Let X be a Banach space. Throughout this paper, let X^* be the dual space of X, $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \leq 1\}$, and for $x \neq 0$, $\nabla_x = \{f \in S_{X^*} : f(x) = \|x\|\}$. In this section we shall recall the definition of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y. A norm on $\mathbb{C}^2$ is called *absolute* if $\|(z, w)\| = \|(|z|, |w|)\|$ for all $(z, w) \in \mathbb{C}^2$ and *normalized* if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbb{C}^2$ is denoted by N_a. The l_p-norms $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ on $\mathbb{C}^2$ are examples of such norms, and for any norm $\|\cdot\| \in N_a$,

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$

Let Ψ be the set of all continuous convex functions ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$. N_a and Ψ are in one-to-one correspondence under the following equations. For each $\|\cdot\| \in N_a$, the function ψ defined by $\psi(t) = \|(1-t, t)\|$ $(0 \leq t \leq 1)$ belongs to Ψ. Conversely, for each $\psi \in \Psi$, let $\|(0,0)\|_\psi = 0$, and $\|(z,w)\|_\psi = (|z| + |w|)\psi(\frac{|w|}{|w|+|z|})$ for $(z, w) \neq (0, 0)$ and this norm belongs to N_a (see [3] and [19]). For Banach spaces X and Y, we denote by $X \oplus_\psi Y$ the direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\| = \|(\|x\|, \|y\|)\|_\psi \text{ for } (x, y) \in X \oplus Y.$$

Thus, under this norm, $X \oplus_\psi Y$, which will be called the ψ-direct sum of X and Y, is a Banach space and for all $(x, y) \in X \oplus Y$ we also have (see [16])

$$\|(x, y)\|_\infty \leq \|(x, y)\|_\psi \leq \|(x, y)\|_1.$$

Saito et al. [16] extended the concept to absolute normalized norm on $\mathbb{R}^n$. The corresponding set of all continuous convex functions on the $(n-1)$−simplex $\{(s_1, ..., s_{n-1}) \in \mathbb{R}_+^{n-1} : s_1 + ... + s_{n-1} \leq 1\}$ will be denoted by Ψ_n.

Now we show that the dual space of this ψ-direct sum is a direct sum $X \oplus_\varphi Y$ of the same kind for some $\varphi \in \Psi$. We first define

$$\varphi_\psi(s) = \varphi(s) := \sup_{t \in [0,1]} \frac{st + (1-s)(1-t)}{\psi(t)}$$

for $s \in [0, 1]$. We show that $\varphi \in \Psi$ and call it *the dual function* of ψ.

Proposition 1. *The above function φ is continuous, convex on $[0, 1]$ and satisfies $\varphi(s) \geq \max\{s, 1-s\}$ for all $s \in [0, 1]$.*

Proof. It is easy to see that $\varphi(\cdot)$ is continuous. To show that φ is convex, we let $s_1, s_2 \in [0, 1]$ and consider

$$\begin{aligned}
\varphi\Big(\frac{s_1 + s_2}{2}\Big) &= \sup_{t \in [0,1]} \frac{\frac{s_1+s_2}{2}t + (1 - \frac{s_1+s_2}{2})(1-t)}{\psi(t)} \\
&= \sup_{t \in [0,1]} \frac{1}{2} \frac{s_1 t + s_2 t + (1-s_1)(1-t) + (1-s_2)(1-t)}{\psi(t)} \\
&\leq \frac{1}{2}(\varphi(s_1) + \varphi(s_2)),
\end{aligned}$$

which verifies the convexity of $\varphi(\cdot)$. Next we prove the last assertion. Since $\psi(t) \leq 1$ for all $t \in [0,1]$,

$$\varphi(s) \geq \sup_{t\in[0,1]} \{st + (1-s)(1-t)\} \geq \max\{s, 1-s\}$$

for all $s \in [0,1]$, and the proof is complete. □

Theorem 2. *The dual space $(X \oplus_\psi Y)^*$ is isometric to $X^* \oplus_\varphi Y^*$, where φ is the dual function of ψ. Moreover, each bounded linear functional F in $(X \oplus_\psi Y)^*$ can be (uniquely) represented by (f,g) where $f \in X^*$ and $g \in Y^*$ and*

$$F(x,y) = f(x) + g(y)$$

for all $(x,y) \in X \oplus_\psi Y$. In this case, $\|F\| \leq \|(f,g)\|_\varphi \|(x,y)\|_\psi$.

Proof. Define $T : X^* \oplus_\varphi Y^* \to (X \oplus_\psi Y)^*$ by

$$T(f,g)(x,y) = f(x) + g(y)$$

where $f \in X^*$, $g \in Y^*$, $x \in X$, and $y \in Y$. It is easy to see that T is linear. Moreover, by the definition of φ, we have, recalling that the norm of each nonzero element (f,g) of the φ-direct sum $X^* \oplus_\varphi Y^*$ is defined by

$$\|(f,g)\|_\varphi = (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big),$$

$$\begin{aligned}
|T(f,g)(x,y)| &\leq \|f\|\|x\| + \|g\|\|y\| \\
&= (\|f\| + \|g\|)(\|x\| + \|y\|)\frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\| + \|g\|)(\|x\| + \|y\|)} \\
&\leq (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big)(\|x\| + \|y\|)\psi\Big(\frac{\|y\|}{\|x\| + \|y\|}\Big) \\
&= \|(f,g)\|_\varphi \|(x,y)\|_\psi,
\end{aligned}$$

for all nonzero (f,g). Thus, $T(f,g)$ is actually an element of $(X \oplus_\psi Y)^*$. For each $F \in (X \oplus_\psi Y)^*$, $F(\cdot,0)$ and $F(0,\cdot)$ are bounded linear functionals on X and Y, respectively. Put $f(x) = F(x,0)$ and $g(y) = F(0,y)$, then $T(f,g) = F$ and the surjectivity of T is proved.

Finally we prove that T is an isometry, i.e., $\|T(f,g)\| = \|(f,g)\|_\varphi$. From the above calculation, we always have $\|T(f,g)\| \leq \|(f,g)\|_\varphi$. Now we prove the reverse inequality. We choose sequences $\{t_n\} \subset [0,1]$, $\{x_n\} \subset S_X$, and $\{y_n\} \subset S_Y$ so that

$$\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)\|f\|}{\|f\| + \|g\|} + \frac{t_n\|g\|}{\|f\| + \|g\|}\Big) \to \varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big),$$

$$f(x_n) \to \|f\|, \quad \text{and} \quad g(y_n) \to \|g\| \quad \text{as } n \to \infty.$$

Therefore, since $\frac{1}{\psi(t_n)}((1-t_n)x_n, t_n y_n) \in S_{X\oplus_\psi Y}$,

$$\begin{aligned}
\|T(f,g)\| &\geq \frac{1}{\psi(t_n)}\Big(f((1-t_n)x_n) + g(t_n y_n)\Big) \\
&= (\|f\| + \|g\|)\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)f(x_n)}{\|f\| + \|g\|} + \frac{t_n g(y_n)}{\|f\| + \|g\|}\Big).
\end{aligned}$$

The last expression tends to $\|(f,g)\|_\varphi$ as $n \to \infty$, proving that $\|T(f,g)\| \geq \|(f,g)\|_\varphi$ and this completes the proof. □

Our first application of Theorem 2 is to show that reflexivity is preserved under the ψ-direct sums.

Corollary 3. *For each $\psi \in \Psi$, $X \oplus_\psi Y$ is reflexive if and only if X and Y are reflexive.*

Proof. We only proof the sufficiency. We first show, without using reflexivity, that $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\psi Y^{**}$, i.e., they are isometric. For this, we let φ and then θ be the dual functions of ψ and of φ, respectively. Thus $(X \oplus_\psi Y)^* \simeq X^* \oplus_\varphi Y^*$ by the isometry T where $TF = (F_1, F_2)$, $F_1 = F(\cdot, 0)$ and $F_2 = F(0, \cdot)$; and $(X^* \oplus_\varphi Y^*)^* \simeq X^{**} \oplus_\theta Y^{**}$ by the isometry U where $UG = (G_1, G_2)$, $G_1 = G(\cdot, 0)$ and $G_2 = G(0, \cdot)$. Hence $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\theta Y^{**}$ via the isometry which maps $L \in (X \oplus_\psi Y)^{**}$ to $ULT^{-1} = (LT^{-1}(\cdot, 0), LT^{-1}(0, \cdot)) \in X^{**} \oplus_\theta Y^{**}$ so that $ULT^{-1}(x^*, y^*) = (LT^{-1}(x^*, 0), LT^{-1}(0, y^*)) = (L(x^*, 0), L(0, y^*)) = (L_1(x^*), L_2(y^*))$. In particular, when $L = L_{(x,y)}$, the evaluation map at (x, y), i.e., $L_{(x,y)}(F) = F(x, y) = F_1(x) + F_2(y)$ for $F \in (X \oplus_\psi Y)^*$, $UL_{(x,y)}T^{-1}(x^*, y^*) = x^*(x) + y^*(y) = L_x(x^*) + L_y(y^*) = (L_x, L_y)(x^*, y^*)$. This shows that $\|(x, y)\|_\psi = \|L_{(x,y)}\| = \|(L_x, L_y)\|_\theta$ for $(x, y) \in X \oplus Y$. Therefore, $\psi(\frac{\|y\|}{\|x\|+\|y\|}) = \theta(\frac{\|L_y\|}{\|L_x\|+\|L_y\|}) = \theta(\frac{\|y\|}{\|x\|+\|y\|})$ for $\|x\| + \|y\| \neq 0$. From this we can easily see that $\psi = \theta$.

Now suppose that X and Y are reflexive. Thus elements in X^{**} and Y^{**} are of the form L_x and L_y for some $x \in X$ and $y \in Y$. To show that $(X \oplus_\psi Y)^{**}$ is reflexive, let $L \in (X \oplus_\psi Y)^{**}$ and consider, for each $F \in (X \oplus Y)^*$, $L(F) = L(F_1, 0) + L(0, F_2) = L_x(F_1) + L_y(F_2) = F_1(x) + F_2(y) = L_{(x,y)}(F)$, for some $x \in X$ and $y \in Y$. That is $L = L_{(x,y)}$ showing that $X \oplus_\psi Y$ is reflexive and the proof is complete. □

We observe that $X \oplus_\psi Y$ is super-reflexive when (and only when) X and Y are super-reflexive. By Henson and Moore [7], this is equivalent to showing that the ultrapower $\widetilde{X \oplus_\psi Y}$ is reflexive. But this follows from Remark 5 below and Corollary 3.

3. Ultrapowers of the ψ-direct sums

The ultrapower of a Banach space is proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. In this section we prove that every ultrapower of a ψ-direct sum is isometric to the ψ-direct sum of their ultrapowers. First we recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i \in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i \in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I}X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from remark (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [17]).

Following T. Landes [11], a normed space Z is a *substitution* space (with index $I \neq \emptyset$ with any cardinality) whenever Z has a (Shauder) basis $(e_i)_{i\in I}$ (unconditional if I is uncountable) and the norm of Z is *monotone*, i.e., $\|z\| \leq \|z'\|$ whenever $0 \leq z_i \leq z'_i$ for all $i \in I$ $(z, z' \in Z)$, where we write $z = \sum_{i\in I} z_i e_i$ for $z \in Z$. Given a family $(X_i)_{i\in I}$ of normed spaces, then the Z *direct sum* $(\bigoplus_{i\in I} X_i)_Z$ of the family (X_i) is defined to be the space $\{x = (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\| e_i \in Z\}$ endowed with the norm $\| \sum_{i\in I} \|x_i\| e_i \|_Z$. $\psi-$ direct sums are examples of $Z-$direct sums.

A property P defined for normed spaces is said to be preserved under the Z-direct-sum-operation, if the Z-direct sums of a family $(X_i)_{i\in I}$ of normed spaces satisfies P whenever all X_i do so.

The following proposition shows that, under some conditions, "normal structure" is preserved under the Z-direct-sum-operation. This result improves the first permanence result for normal structure obtained by Belluce, Kirk, and Steiner [2].

Proposition 4. [11, Theorem 2, Corollary 3 and Corollary 4] *Let Z be a substitution space with index set $I = \{1, \ldots, N\}$ such that*

$$\|z + z'\| < 2 \textit{ whenever } \|z\| = \|z'\| = 1, z_i \geq 0, z'_i \geq 0 \textit{ for all } i \in I,$$

$$\textit{and } z_i = z'_i \textit{ only for these } i \in I \textit{ for which } z_i = z'_i = 0.$$

Thus, normal structure is preserved under the Z-direct-sum-operation. In particular, if Z is strictly convex or $Z = l_p^N$ for any p with $1 < p \leq \infty$.

In case $I = \{1, ..., N\}$ and ψ is strictly convex, it follows from [9] that the norm $\|\cdot\|_\psi$ is monotone and strictly convex on $\mathbb{C}^N$. We note in passing that this result actually holds for $Z-$direct sum : The $Z-$ direct sums $(\bigoplus_i X_i)_Z$ is uniformly convex if and only if Z and each of the Banach space X_i are uniformly convex with a common modulus of convexity (see Dowling [5]).

Remark 5. It is easy to see that the ultrapower of $Z-$direct sum $(\bigoplus_i X_i)_Z$ is isometric to the $Z-$direct sum $(\bigoplus_i \widetilde{X}_i)_Z$ of ultrapowers. Thus in particular,

$\widetilde{(X_1 \oplus \cdots \oplus X_N)_\psi} \simeq (\widetilde{X_1} \oplus \cdots \oplus \widetilde{X_N})_\psi$. This follows from the fact that the $Z-$norm is monotone and from the continuity of norms.

It is known that X is uniformly convex if and only if $\widetilde{X}$ is strictly convex (see [17]). Combining these results and Remark 5 gives

Corollary 6. [9] *Let $X_1, ..., X_N$ be Banach spaces and $\psi \in \Psi_N$. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly convex if and only if $X_1, ...X_N$ are uniformly convex and ψ is strictly convex.*

Thus, in the light of super-reflexivity, we can extend "normal structure" to "uniform normal structure" for ψ-direct sums whenever ψ is strictly convex.

Corollary 7. *Let $X_1, ..., X_N$ be super-reflexive Banach spaces and Z be uniformly convex. Then, the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has uniform normal structure if and only if $X_1, ..., X_N$ have uniform normal structure.*

Proof. Note that, by Khamsi [10], it suffices to show that the ultrapower $\widetilde{(X_1 \oplus \cdots \oplus X_N)_Z}$ has normal structure. But this is an immediate consequence of Remark 5 together with Proposition 4. □

It is well-known that every uniformly nonsquare space is super-reflexive (see [8]). Thus, Corollary 7 and [4, Corollary 3.7] give

Corollary 8. *Let $X_1, ..., X_N$ be Banach spaces and Z be uniformly convex. Then, if $C_{\mathrm{NJ}}(1, X_i) < 2$ for $i = 1, 2, ..., N$, the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has uniform normal structure.*

It is interesting to see if we can conclude that $C_{\mathrm{NJ}}(1, (X_1 \oplus \cdots \oplus X_N)_Z) < 2$ in Corollary 8.

4. Smoothness of the ψ-direct sums

A Banach space X is said to be *smooth* if for any $x \in S_X$, ∇_x is a singleton. We recall that a continuous convex function ψ has left and right derivatives ψ'_L, ψ'_R. Let G be defined on $[0,1]$ by

$$G(0) = [-1, \psi'_R(0)],\ G(1) = [\psi'_L(1), 1],$$
$$G(t) = [\psi'_L(t), \psi'_R(t)]\ (0 < t < 1).$$

Given $\psi \in \Psi$, $t \in [0,1]$, let

$$x(t) = \frac{1}{\psi(t)}(1-t, t)$$

so that $\|x(t)\|_\psi = 1$. In [3], the authors identified the dual of $(\mathbb{C}^2, \|.\|_\psi)$ with $\mathbb{C}^2$ and used this fact to provide a proof of the following lemma.

Lemma 9. [3, Lemma 4] *For ψ, G, and x defined above,*

(1) $\nabla_{x(t)} = \{(\psi(t) - t\gamma, \psi(t) + (1-t)\gamma) : \gamma \in G(t)\}$ *for* $0 < t < 1$,
(2) $\nabla_{x(0)} = \{(1, z(1+\gamma)) : \gamma \in G(0), |z| = 1\}$, *and*
(3) $\nabla_{x(1)} = \{(z(1-\gamma), 1) : \gamma \in G(1), |z| = 1\}$.

In general, using Theorem 2 and Lemma 9, we have the following:

Lemma 10. *Let* $(x,y) \in S_{X\oplus_\psi Y}$ *and* $t = \frac{\|y\|}{\|x\|+\|y\|}$. *Thus*

(1) $\nabla_{(x,y)} = \{((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|}$ *and* $g \in \nabla_{y/\|y\|}\}$ *for* $0 < t < 1$,

(2) $\nabla_{(x,y)} = \{(f, (1+\gamma)g) : \gamma \in G(0), f \in \nabla_x$ *and* $g \in S_{Y^*}\}$ *for* $t = 0$, *and*

(3) $\nabla_{(x,y)} = \{((1-\gamma)f, g) : \gamma \in G(1), g \in \nabla_y$ *and* $f \in S_{X^*}\}$ *for* $t = 1$.

Proof. We prove (1). Let $F = (f,g) \in \nabla_{(x,y)}$, then

$$\begin{aligned}
F((x,y)) &= f(x) + g(y) \\
&\leq \|f\|\|x\| + \|g\|\|y\| \\
&= \frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\| + \|g\|)(\|x\| + \|y\|)}(\|f\| + \|g\|)(\|x\| + \|y\|) \\
&\leq \varphi\left(\frac{\|g\|}{\|f\| + \|g\|}\right)\psi\left(\frac{\|y\|}{\|x\| + \|y\|}\right)(\|f\| + \|g\|)(\|x\| + \|y\|) \\
&= \|F\|_\varphi \|(x,y)\|_\psi = 1.
\end{aligned}$$

Thus, we have $\|f\|\|x\| + \|g\|\|y\| = 1$ and $f(x) = \|f\|\|x\|$ $g(y) = \|g\|\|y\|$, hence $(\|f\|, \|g\|) \in \nabla_{(\|x\|,\|y\|)}$ and $\frac{f}{\|f\|} \in \nabla_{\frac{x}{\|x\|}}$, $\frac{g}{\|g\|} \in \nabla_{\frac{y}{\|y\|}}$. We observe that $(\|x\|, \|y\|) = \frac{1}{\psi(t)}(1-t, t)$, thus it follows from Lemma 9 that

$$\|f\| = \psi(t) - t\gamma \text{ and } \|g\| = \psi(t) + (1-t)\gamma, \text{ for some } \gamma \in G(t).$$

Consequently, we have $(f,g) = (\|f\|\frac{f}{\|f\|}, \|g\|\frac{g}{\|g\|}) = ((\psi(t) - t\gamma)\frac{f}{\|f\|}, (\psi(t) + (1-t))\gamma\frac{g}{\|g\|})$. Thus, we have proved that $\nabla_{(x,y)} \subset \{((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}\}$. On the other hand, let $F = ((\psi(t) - t\gamma)f, (\psi(t) + (1-t)\gamma)g)$ where $\gamma \in G(t), f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}$. Consider, by using Lemma 9,

$$\begin{aligned}
F((x,y)) &= (\psi(t) - t\gamma)f(x) + (\psi(t) + (1-t)\gamma)g(y) \\
&= (\psi(t) - t\gamma)\|x\| + (\psi(t) + (1-t)\gamma)\|y\| \\
&= (\|x\| + \|y\|)\big((\psi(t) - t\gamma)(1-t) + (\psi(t) + (1-t)\gamma)t\big) \\
&= \frac{1}{\psi(t)}\big((\psi(t) - t\gamma)(1-t) + (\psi(t) + (1-t)\gamma)t\big) \\
&= 1.
\end{aligned}$$

Hence, (1) has been proved. The proof of (2) and (3) can be proceeded similarly. □

We say that a function ψ is *smooth* if the following conditions hold:

(1) ψ is *smooth* at every $t \in (0,1)$, i.e., the derivative of ψ exists at t,

(2) the right derivative of ψ at 0 is -1, and

(3) the left derivative of ψ at 1 is 1.

Theorem 11. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is smooth if and only if X and Y are smooth and ψ is smooth.*

Proof. Necessity. Assume that $X \oplus_\psi Y$ is smooth. Because X is isometric to $X \oplus_\psi \{0\}$ which is a subspace of $X \oplus_\psi Y$, then X and similarly Y must be smooth. It remains to prove that ψ is smooth, but by Lemma 10, if ψ is not smooth, there exists $(x, y) \in S_{X\oplus_\psi Y}$ such that $\nabla_{(x,y)}$ contains more than one point which can not happen, and the smoothness of ψ is proved

Sufficiency. This follows from Lemma 10. □

Again, since, for every Banach space X, X is uniformly smooth if and only of $\widetilde{X}$ is smooth, we obtain

Corollary 12. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is uniformly smooth if and only if X and Y are uniformly smooth and ψ is smooth.*

5. U-SPACES AND u-SPACES

A Banach space X is called a *U-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in S_X$, we have $\|x + y\| \leq 2(1 - \delta)$ whenever $f(y) < 1 - \varepsilon$ for some $f \in \nabla_x$ (see [13]). A Banach space X is called a *u-space* if for any $x, y \in S_X$ with $\|x + y\| = 2$, we have $\nabla_x = \nabla_y$. Obviously, every U-space is a u-space.

Remark 13. Let us collect together some properties of u-spaces and U-spaces:

(1) If X^* is a u-space, then X is a u-space. The converse holds whenever X is reflexive.
(2) If X is a U-space, then X is a u-space. The converse holds whenever $\dim X < \infty$.
(3) $\widetilde{X}$ is a u-space if and only if X is a U-space.

Proof. (1) Let $x, y \in S_X$ be such that $\|x + y\| = 2$. We prove that $\nabla_x = \nabla_y$. Let $f \in \nabla_x$, and $h \in \nabla_{x+y}$. It follows that $h(x) = h(y) = 1$ and $\|f + h\| = 2$. By the assumption that X^* is a u-space and $h(y) = 1$, we have $f(y) = 1$. This implies that $\nabla_x \subset \nabla_y$, and then $\nabla_x = \nabla_y$ as required.

(2) The first assertion is obvious and the latter one follows from the compactness of the unit ball.

(3) It is known that $\widetilde{X}$ is a U-space if and only if X is a U-space (see [6] or [15]). In virtue of (2), it suffices to prove that X is a U-space whenever $\widetilde{X}$ is a u-space. Suppose that X is not a U-space. Then there exist an $\epsilon_0 > 0$ and sequences $\{x_n\}, \{y_n\} \subset S_X$, and $\{f_n\} \subset S_{X^*}$ such that $f_n(x_n) = 1$ and $f_n(x_n - y_n) \geq \epsilon_0$ for all $n \in \mathbb{N}$, and $\|x_n + y_n\| \to 2$ as $n \to \infty$. We put $\widetilde{x} = (x_n)_\mathcal{U}$, $\widetilde{y} = (y_n)_\mathcal{U}$ and $\widetilde{f} = (f_n)_\mathcal{U}$. Thus $\|\widetilde{x} + \widetilde{y}\| = 2$, $\widetilde{f}(\widetilde{x}) = 1$ and $\widetilde{f}(\widetilde{y}) \leq 1 - \epsilon_0 < 1$. This means that $\nabla_{\widetilde{x}} \neq \nabla_{\widetilde{y}}$ which implies that $\widetilde{X}$ is not a u-space. □

U-spaces can be considered as the "uniform" version of u-spaces. The following diagram explains this claim as well as it shows how the u-spaces are well-placed (see [1], [4], [6], [14], and [15]):

$$X \text{ is UC} \Leftrightarrow \widetilde{X} \text{ is UC} \Leftrightarrow \widetilde{X} \text{ is SC}$$

$$X \text{ is US} \Leftrightarrow \widetilde{X} \text{ is US} \Leftrightarrow \widetilde{X} \text{ is S}$$

$$X \text{ is UNC} \quad \Leftrightarrow \quad \widetilde{X} \text{ is UNC} \quad \Leftrightarrow \quad \widetilde{X} \text{ is NC}$$

$$X \text{ is a U-space} \quad \Leftrightarrow \quad \widetilde{X} \text{ is a U-space} \quad \Leftrightarrow \quad \widetilde{X} \text{ is a u-space}$$

$$\begin{array}{ccccccccccc}
C_{\mathrm{NJ}}(1,X) < 2 & \Rightarrow & \text{UNS} & & & & & & & & \\
& & \Uparrow & & & & & & & & \\
\text{UC} & \Rightarrow & U & \Rightarrow & \text{UNSQ} & & \text{US} & \Rightarrow & U & \Rightarrow & \text{UNSQ} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{SC} & \Rightarrow & u & \Rightarrow & \text{NSQ} & & \text{S} & \Rightarrow & u & \Rightarrow & \text{NSQ}
\end{array}$$

UC $\equiv$ Uniformly Convex, SC $\equiv$ Strictly Convex, US $\equiv$ Uniformly Smooth, S $\equiv$ Smooth, UNC $\equiv$ Uniformly Noncreasy , NC $\equiv$ Noncreasy, $C_{\mathrm{NJ}}(\cdot) \equiv$ a generalized Jordan-von Neumann constant, UNS $\equiv$ Uniform Normal Structure, UNSQ $\equiv$ Uniformly Nonsquare, NSQ $\equiv$ Nonsquare, U $\equiv$ a U-space, $u \equiv$ a u-space

Examples of u-spaces which are not U-spaces can be obtained from the direct product spaces $(\mathbb{R}^2_{p_1} \oplus \mathbb{R}^2_{p_2} \oplus \mathbb{R}^2_{p_3} \oplus \cdots)_2$ where (p_n) is a sequence of positive numbers strictly decreasing to 1, and $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ where each l_n is the l_n-space. Actually, both spaces are strictly convex, but with the James constant and the Jordan-von Neumann constant are both equal to 2, i.e., the spaces are not uniformly nonsquare, and hence can not be U-spaces. Sims and Smith [18] have shown that the space $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ has asymptotic property (P) but not property (P).

Examples of infinite dimensional u-spaces that are not strictly convex or smooth are easily established.

Let $\psi \in \Psi$. We say that ψ is a *u-function*, if for any interval $[a,b] \subset (0,1)$, we have ψ is smooth at a and b whenever ψ is affine on $[a,b]$.

Theorem 14. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the Banach space $X \oplus_\psi Y$ is a u-space if and only if X and Y are u-spaces and ψ is a u-function.*

Proof. Necessity . Suppose there exist a and $b \in [0,1]$ such that ψ is affine on $[a,b]$ but $\psi'_-(a) < \psi'_+(a) = \psi'_-(b)$. Fix $x_0 \in S_X$, $f_0 \in \nabla_{x_0}$, $y_0 \in S_Y$, and $g_0 \in \nabla_{y_0}$. Consider $w = \frac{1}{\psi(a)}((1-a)x_0, ay_0)$ and $z = \frac{1}{\psi(b)}((1-b)x_0, by_0)$. We have $w, z \in S_{X\oplus_\psi Y}$ and $\|w+z\|_\psi = 2$. Indeed,

$$\begin{aligned}
\|w+z\|_\psi &= \left\| \left(\frac{1-a}{\psi(a)}x_0 + \frac{1-b}{\psi(b)}x_0, \frac{a}{\psi(a)}y_0 + \frac{b}{\psi(b)}y_0 \right) \right\|_\psi \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(\frac{\frac{a}{\psi(a)} + \frac{b}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(a \frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} + b \frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \left(\frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \psi(a) + \frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \psi(b) \right) \\
&= 2.
\end{aligned}$$

To obtain a contradiction, it remains to show that $\nabla_z \neq \nabla_w$. Now, for $\gamma \in [\psi'_-(b), \psi'_+(b)]$, we have

$$\psi(b) - b\gamma \leq \psi(b) - b\psi'_-(b) = \psi(a) - a\psi'_+(a) < \psi(a) - a\psi'_-(a).$$

Thus, $((\psi(a) - a\psi'_-(a))f_0, (\psi(a) + (1-a)\psi'_-(a))g_0) \in \nabla_w \setminus \nabla_z$, that is $\nabla_z \neq \nabla_w$.

Sufficiency. Let us prove that $X \oplus_\psi Y$ is a u-space. Let w and z be elements in the unit sphere of $X \oplus_\psi Y$ such that $\|w+z\|_\psi = 2$. Put $w = (x_1, y_1)$ and $z = (x_2, y_2)$. We have $\|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi = 2$ since $2 = \|w+z\|_\psi = \|(x_1 + x_2, y_1 + y_2)\|_\psi \leq \|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi \leq \|(\|x_1\|, \|x_2\|)\|_\psi + \|(\|y_1\|, \|y_2\|)\|_\varphi = 2$. By the convexity of ψ, it follows that

$$\begin{aligned} 2 &= (\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|)\psi\left(\frac{\|y_1\| + \|y_2\|}{\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|}\right) \\ &\leq (\|x_1\| + \|y_1\|)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right) + (\|x_2\| + \|y_2\|)\psi\left(\frac{\|y_2\|}{\|x_2\| + \|y_2\|}\right) \\ &= 2. \end{aligned}$$

Thus, ψ is affine on $[a \wedge b, a \vee b]$, where $a = \frac{\|y_1\|}{\|x_1\| + \|y_1\|}$ and $b = \frac{\|y_2\|}{\|x_2\| + \|y_2\|}$. Since $\|w + z\| = 2$, there exists $F = (f_1, g_1) \in X^* \oplus_\varphi Y^*$ such that $F \in \nabla_w \cap \nabla_z$. Hence,

$$\begin{aligned} F(w) &= f_1(x_1) + g_1(y_1) \\ &\leq \|f_1\|\|x_1\| + \|g_1\|\|y_1\| \\ &= \frac{\|f_1\|\|x_1\| + \|g_1\|\|y_1\|}{(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|)}(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\ &\leq \varphi\left(\frac{\|g_1\|}{\|f_1\| + \|g_1\|}\right)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right)(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\ &= \|F\|_\varphi \|w\|_\psi = 1. \end{aligned}$$

Thus, we have

$$(\alpha) \qquad f_1(x_1) = \|f_1\|\|x_1\| \text{ and } g_1(y_1) = \|g_1\|\|y_1\|.$$

In the same way, we also have

$$(\beta) \qquad f_1(x_2) = \|f_1\|\|x_2\| \text{ and } g_1(y_2) = \|g_1\|\|y_2\|.$$

Now we show that $\nabla_w = \nabla_z$. We consider first the case when all $\|x_1\|$, $\|y_1\|$, $\|x_2\|$, $\|y_2\|$ are positive. In this case, we can assume that $0 < a \leq b < 1$. (α) and (β) give $\frac{f_1}{\|f_1\|} \in \nabla_{\frac{x_1}{\|x_1\|}} \cap \nabla_{\frac{x_2}{\|x_2\|}}$ and $\frac{g_1}{\|g_1\|} \in \nabla_{\frac{y_1}{\|y_1\|}} \cap \nabla_{\frac{y_2}{\|y_2\|}}$. It follows that $\|\frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|}\| = 2$ and $\|\frac{y_1}{\|y_1\|} + \frac{y_2}{\|y_2\|}\| = 2$. Thus, $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$ since both X and Y are u-spaces.

If $a < b$, then, since ψ is affine on $[a, b]$, a and b must be smooth points of ψ. Consequently,

$$(\gamma) \qquad \psi(a) - a\gamma = \psi(b) - b\gamma \text{ and } \psi(a) + (1-a)\gamma = \psi(b) + (1-b)\gamma,$$

where $\gamma = \psi'(a) = \psi'(b)$.

By using (γ) together with Lemma 10 and the equations $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$, we have $\nabla_z = \nabla_w$.

If $a = b$, then, by Lemma 10, we have

$$\begin{aligned}
&\nabla_{(x_1,y_1)} \\
&= \{((\psi(a) - a\gamma)f, (\psi(a) + (1-a)\gamma)g) : \gamma \in G(a), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \nabla_{(x_2,y_2)}.
\end{aligned}$$

Thus $\nabla_z = \nabla_w$ as well.

Now we consider the case when exactly one of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ is equal to 0. We assume that $\|y_1\| = 0$, thus $a = 0 < b$ and 0 and b are smooth points. By (α), (β), and by the assumption that X is a u-space, we have $\nabla_{x_1} = \nabla_{\frac{x_2}{\|x_2\|}}$. Since 0 is a smooth point, we have $F = (f_1, 0)$. This in turn implies that $\psi(b) - b\psi'(b) = 1$ and $\psi(b) + (1-b)\psi'(b) = 0$ since $F \in \nabla_w \cap \nabla_z$. Thus, by Lemma 10,

$$\begin{aligned}
&\nabla_{(x_2,y_2)} \\
&= \{((\psi(b) - b\psi'(b))f, (\psi(b) + (1-b)\psi'(b))g) : f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_2/\|x_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_1}\} \\
&= \nabla_{(x_1,y_1)}.
\end{aligned}$$

Finally, suppose two of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ are equal to 0. We can assume that $\|y_1\| = \|y_2\| = 0$, thus $a = b = 0$. The proof of the equality $\nabla_z = \nabla_w$ is similar to the one of the case when $a = b$. □

Corollary 15. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following statements are equivalent:*

1. *$X \oplus_\psi Y$ is a U-space;*
2. *$X^* \oplus_\varphi Y^*$ is a U-space;*
3. *X and Y are U-spaces and ψ is a u-function;*
4. *X and Y are U-spaces and φ is a u-function, where φ is the dual function of ψ.*

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Fixed point property of direct sums☆

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Abstract

For a uniformly convex space Z, we show that Z-direct sums $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \ldots, X_N$ with $R(a, X_i) < 1 + a$ for some $a \in (0, 1]$ have the fixed point property for nonexpansive mappings. As a direct consequence, the result holds for all ψ-direct sums with ψ being strictly convex. The same result is extended to all ψ-direct sums $X \oplus_\psi Y$ of spaces X and Y with property (M), whenever $\psi \neq \psi_1$. The permanence of properties that are sufficient for the fixed point property are obtained for Z-direct sums (and then for ψ-direct sums). Such properties include the properties $R(X) < 2$, WNUS, $C_{NJ}(a, X) < 2$, UKK, and NUC.

MSC: 46B20; 46B08

Keywords: Z-direct sums; ψ-direct sums; The fixed point property

1. Introduction

Let X be a Banach space. A self-mapping T of a closed convex subset C of X is said to be a nonexpansive mapping if $\|Tx - Ty\| \leqslant \|x - y\|$ for all x, y in C.

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We will say that X has the weak fixed point property (fpp) if every nonexpansive mapping defined on a nonempty weakly compact convex subset of X has a fixed point.

One open problem in metric fixed point theory is the permanence of fpp under direct sums. Here we consider the problem in two aspects. One of these is to study the permanence of properties that guarantee the fixed point property, the other is to study directly the fixed point property of the direct sum under conditions given on its component spaces. Obviously, the answer to these questions depend on the norm of the product space. We shall be interested in the so-called Z-direct sums and ψ-direct sums the concepts of which will be defined later.

Shortly after Kirk proved in [16] his celebrated fixed point theorem that asserts that every Banach space with normal structure has the fpp, the first permanence result for normal structure was given by Belluce et al. [2]. The result has been improved by Landes [18] whose result in turn has been improved to uniform normal structure under ψ-direct sums by Dhompongsa et al. in [4]. For weak normal structure, the positive answer for $X\oplus_p Y$ is due to Belluce et al. [2]. Landes [18,19] in 1984 and 1986 showed that weak normal structure (WNS) is preserved in $X\oplus_p Y$ for $1 < p \leqslant \infty$, but not for $p = 1$. Some additional conditions on the spaces X and Y that are sufficient for the direct sum $X\oplus_1 Y$ to have WNS are considered in [11,21].

Sims and Smyth [28], however, considered the problem, known as a *3-space problem*: For a Banach space X and a finite-dimensional Banach space Y, if X has asymptotic (P) so does $X \oplus Y$. Moreover, if X has property (P) and the projection onto X has norm 1, then $X \oplus Y$ has (P). This later result strengthens Theorem 2.3 of [29]. Later Sims and Smyth [28] showed that property (P), asymptotic (P), and some others are inherited from the component spaces to the direct sums. They, as well as Kutzarova and Landesalso in [17], also considered infinite product results. Property (P), introduced by Tan and Xu [29], is sufficient for WNS. In 1997, Prus [24] introduced a class of super-reflexive Banach spaces with fpp the so-called the *uniformly noncreasy* (UNC) spaces. These spaces do not have to have normal structure. For a strictly monotone norm, Wisnicki [30] proved that $X \oplus Y$ has fpp, whenever both X and Y are UNC or have property (P).

A norm on $\mathbb{R}^2$ is called *absolute* if $\|(z, w)\| = \|(|z|, |w|)\|$ for all $(z, w) \in \mathbb{R}^2$ and *normalized* if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by N_a. The l_p-norms $\|\cdot\|_p$ for $1 \leqslant p \leqslant \infty$ are such examples and for any $\|\cdot\| \in N_a$, $\|\cdot\|_\infty \leqslant \|\cdot\| \leqslant \|\cdot\|_1$. Let Ψ be the set of all continuous convex functions ψ on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leqslant \psi(t) \leqslant 1$ $(0 \leqslant t \leqslant 1)$. N_a and Ψ are in one-to-one correspondence via the following equations. For each $\|\cdot\| \in N_a$, the function ψ defined by $\psi(t) = \|(1-t, t)\|$ $(0 \leqslant t \leqslant 1)$ belongs to Ψ. Conversely, for each $\psi \in \Psi$ let $\|(0, 0)\| = 0$, and $\|(z, w)\| = (|z| + |w|)\psi(|w|/|w| + |z|)$ for $(z, w) \neq (0, 0)$. Let ψ_p be a function defined by $\psi_p(t) = ((1-t)^p + t^p)^{1/p}$ if $1 \leqslant p < \infty$; $\psi_p(t) = \max\{1-t, t\}$, if $p = \infty$. It is simple to see that such function ψ_p corresponds to the l_p-norm. We denote by $X\oplus_\psi Y$ the direct sum $X \oplus Y$ equipped with the norm $\|(x, y)\| = \|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$ for $(x, y) \in X \oplus Y$. $X\oplus_\psi Y$ is called the *ψ-direct sum* of X and Y and it is a Banach space satisfying, for all $(x, y) \in X \oplus Y$, $\|(x, y)\|_\infty \leqslant \|(x, y)\|_\psi \leqslant \|(x, y)\|_1$ (see [26]). Saito et al. [25] extended the concept to absolute normalized norm on $\mathbb{R}^N$. The corresponding set of all continuous convex functions on the $(N-1)$-simplex $\{(s_1, \ldots, s_{N-1}) \in \mathbb{R}_+^{N-1}: s_1 + \cdots + s_{N-1} \leqslant 1\}$ will be denoted by Ψ_N. Kato et al. [15] proved that the space $(\mathbb{R}^N, \|\cdot\|_\psi)$ is uniformly convex if and only if ψ is strictly convex. Dowling [8] pointed out that this result fell into a larger framework developed by Day [3]: Let Z be a Banach space with a normalized 1-unconditional basis $(e_i)_{i\in I}$. For Banach spaces $X_i\,(i \in I)$, define the Z-direct sum

$$\left(\bigoplus_i X_i\right)_Z = \left\{(x_i) \in \prod_i X_i : \sum_i \|x_i\| e_i \text{ converges in } Z\right\}$$

and put $\|(x_i)\| = \|\sum_i \|x_i\| e_i\|_Z$ for the norm on $(\bigoplus_i X_i)_Z$. In the language of Day, $(\bigoplus_i X_i)_Z$ is a substitution space of (X_i) in Z. ψ-direct sums are examples of Z-direct sums, simply letting $Z = \mathbb{C}^N$. Dowling showed that $(\bigoplus_i X_i)_Z$ is uniformly convex if and only if Z and each of the Banach space X_i are uniformly convex with a common modulus of convexity. Thus, for $X_i = \mathbb{R}$ for each i, the Z-direct sum $(\bigoplus_i X_i)_Z$ is uniformly convex if and only if Z is uniformly convex.

In this paper, we will study only finite Z-direct sums. Thus, we may write $(a_1, \ldots, a_N) = a_1 e_1 + \cdots + a_N e_N$ as elements of Z, for $a_1, \ldots, a_N$ in $\mathbb{R}$. The paper is organized as follows. In Section 2, we consider Banach spaces X with $R(a, X) < 1 + a$ for some $a \in (0, 1]$. It is known that spaces with this property have the fpp. Let us call this property 'Dominguez's condition'. The main result of this section is to prove that the finite Z-direct sum $(\bigoplus_i X_i)_Z$ has the fixed point property whenever the spaces (X_i) satisfy Dominguez's condition. We continue the investigation for Banach spaces with property (M) of Kalton in Section 3. The second part of the paper considers the permanence of properties that are sufficient for the fixed point property. The conditions $R(X) < 2$, $C_{NJ}(a, X) < 2$, and the property NUC among others are considered in Sections 5, 6, and 7, respectively.

One tool that seems to be common now in studying the fixed point property is the ultrapower technique. We recall some of its formulation. Let $\mathcal{U}$ be a free ultrafilter on the set of natural numbers. Consider the closed linear subspace of $l_\infty(X)$, $\mathcal{N} = \{(x_n) \in l_\infty(X) : \lim_{n\to\mathcal{U}} \|x_n\| = 0\}$. The ultrapower $\widetilde{X}$ of the space X is defined as the quotient space $l_\infty(X)/\mathcal{N}$. Given an element $x = (x_n) \in l_\infty(X)$, $\widetilde{x}$ stands for the equivalence class of x. The quotient norm in $\widetilde{X}$ satisfies $\|\widetilde{x}\| = \lim_{n\to\mathcal{U}} \|x_n\|$. If $f = (x_n^*)$ is a bounded sequence of functionals in X^*, the expression $\widetilde{f}(\widetilde{x}) = \lim_{n\to\mathcal{U}} x_n^*(x_n)$ for $x = (x_n) \in l_\infty(X)$ defines an element in the dual space of $\widetilde{X}$ with $\|\widetilde{f}\| = \lim_{n\to\mathcal{U}} \|x_n^*\|$. (For more details about the construction of an ultrapower of a Banach space X see, for examples, [1,27].) It is shown in [4] that $\widetilde{X \oplus_\psi Y} = \widetilde{X} \oplus_\psi \widetilde{Y}$.

2. The coefficient $R(a, X)$

Definition 1 (*Garcia-Falset [10]*). Let X be a Banach space, then

$$R(X) := \sup\left\{\liminf_n \|x_n + x\|\right\},$$

where the supremum is taken over all weakly null sequences (x_n) in the unit ball and over all points x of the unit ball.

Garcia-Falset [10] showed that if a Banach space X satisfies $R(X) < 2$, then X has the fpp. Dominguez [7] generalized this result by introducing the coefficient, for $a \geqslant 0$,

$$R(a, X) := \sup \left\{ \liminf_n \|x_n + x\| \right\},$$

where the supremum is taken over all weakly null sequences (x_n) in the unit ball with $D(x_n) \leqslant 1$ and over all points x with $\|x\| \leqslant a$. Here $D(x_n) := \limsup_n \limsup_m \|x_n - x_m\|$. Clearly $R(X) \geqslant R(1, X)$. It was proved in [7] that X has fpp if $R(a, X) < 1 + a$ for some $a \geqslant 0$. It has been observed that $R(l_{2,1}) = 2$, while $R(a, l_{2,1}) < 1 + a$ for some $a \geqslant 0$. Thus this result is a strict improvement of a result in [10]. Moreover, because of this result, Mazcunan Navarro [22] is able to prove a well-known open problem which states that "every uniformly nonsquare Banach space has the fpp".

Theorem 2. *Let $X_1, \ldots, X_N$ be Banach spaces with, for each $i = 1, \ldots, N$, $R(a, X_i) < 1 + a$ for some $a \in (0, 1]$. If Z is uniformly convex, then the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has the fixed point property.*

Proof. The main ingredient of the proof is taken from Dominguez [7]. Suppose that $(X_1 \oplus \cdots \oplus X_N)_Z$ does not have the fpp. Thus, we can find a weakly compact and convex subset K of $(X_1 \oplus \cdots \oplus X_N)_Z$ such that $0 \in K$, $\operatorname{diam} K = 1$ and K is minimal invariant for a nonexpansive mapping T, and we can also find a weakly null approximated fixed point sequence (afps) (z_n) of T in K. We consider the set $W = \{(\widetilde{w_n}) \in \widetilde{K} : \|(\widetilde{w_n}) - (\widetilde{z_n})\| \leqslant 1 - t$ and $D((\widetilde{w_n})) \leqslant t\}$, where $t = 1/(1 + a)$. It is easy to check that W is a closed, convex, and $\widetilde{T}$-invariant set. Furthermore, W is nonempty because it contains $t(\widetilde{z_n})$. Therefore, from Lin's Theorem [20], we know that $\sup\{\|(\widetilde{w_n})\| : (\widetilde{w_n}) \in W\} = 1$ since $0 \in K$. Some parameters will be needed and we define them here. In what follows, $i \in I = \{1, \ldots, N\}$. First choose $M > 1$ so that

$$\frac{1 + a}{M} > \max_i R(a, X_i) \tag{2.1}$$

and then choose $0 < \varepsilon < \eta$ so small that $\eta < M - 1$, $1 + \varepsilon/\eta^2(1 - t) + (1 + t/(1 - t - \varepsilon))\eta < M$,

$$\left(\frac{\eta}{t} + a \left(1 + \frac{\varepsilon}{\eta^2(1 - t)} + \left(1 + \frac{t}{1 - t - \varepsilon} \right) \eta \right) \right) \frac{t}{1 - t - \varepsilon} < M,$$

$$\varepsilon < \frac{1 - N\eta(1 + \eta)}{1 + \eta} \eta,$$

and finally, by uniform convexity of Z and by monotonicity of $\| \cdot \|_Z$, we have

$$\begin{aligned} &\text{if } \|u + v\| > 2(1 - 2\varepsilon) \text{ for } u = (u_1, \ldots, u_N),\ v = (v_1, \ldots, v_N) \in S_{(X_1 \oplus \cdots \oplus X_N)_Z}, \\ &\text{then } |\|u_i\| - \|v_i\|| < \eta^3 \text{ for each } i, \text{ and} \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\text{if } \|(c_i)\| - \|(p_i)\| < \varepsilon \text{ for } (c_i),\ (p_i) \in B_{(X_1 \oplus \cdots \oplus X_N)_Z} \text{ with } 0 \leqslant p_i \leqslant c_i \\ &\text{for each } i, \text{ then } c_i - p_i < \eta^3 \text{ for each } i. \end{aligned} \tag{2.3}$$

We can find an element $\widetilde{w} = (\widetilde{w_n})$ in W with

$$\|\widetilde{w}\| > 1 - \varepsilon \tag{2.4}$$

and $w_n \xrightarrow{w} w$ for some $w \in (X_1 \oplus \cdots \oplus X_N)_Z$.

Write $w_n = (x_{n1}, \ldots, x_{nN})$ and $w = (x_1, \ldots, x_N)$. By passing through subsequences, we can assume that all limits mentioned below exist since all sequences under consideration are bounded. For example we assume now that $\|x_{ni}\| \to a(i)$ for some $a(i)$ for each $i \in I$. Let us define the vectors $z_n = w_n - w$ and let $\|z_n\| \to b$. Clearly, from $D(\widetilde{w}) \leqslant t$ and by the weak lower semi-continuity of the norm, we have

$$\|\widetilde{w} - w\| \leqslant t, \quad \|w\| \leqslant 1 - t. \tag{2.5}$$

By (2.4), we must have both $b \geqslant t - \varepsilon > 0$ and $\|w\| \geqslant 1 - t - \varepsilon > 0$. Suppose first that $b \leqslant \|w\|$. For each n, put $s_n = (z_n + w)/2$, $t_n = (\|w\|/\|z_n\|)z_n + w$. Thus, $2s_n = (\|z_n\|/\|w\|)t_n + [(\|w\| - \|z_n\|)/\|w\|]w$, and so $1 - \varepsilon \leqslant \lim_n 2\|s_n\| \leqslant (b/\|w\|)\lim_n \|t_n\| + \|w\| - b$ that implies $\lim_n \|t_n\| \geqslant (1 - \varepsilon + b - \|w\|)/b\|w\| > 2(1 - 2\varepsilon)\|w\|$. Next, suppose $\|w\| < b$. In this case we redefine $t_n = z_n + (\|z_n\|/\|w\|)w$. Thus, $2s_n = (\|w\|/\|z_n\|)t_n + [(\|z_n\| - \|w\|)/\|z_n\|]z_n$, and therefore $1 - \varepsilon \leqslant \lim_n 2\|s_n\| \leqslant (\|w\|/b)\lim_n \|t_n\| + b - \|w\|$, and in consequence $\lim_n \|t_n\| \geqslant [(1 - \varepsilon - b + \|w\|)/\|w\|]b > 2(1 - 2\varepsilon)b$. By applying (2.2), it follows that, for all large n,

$$\left| \frac{\|w\|}{\|z_n\|} \|x_{ni} - x_i\| - \|x_i\| \right| < \eta^3 \tag{2.6}$$

for all i. Let $J = \{i \in I : a(i) > \eta^2\}$. Clearly $J \neq \phi$. Now observe that for some subsequence (n_k) of (n), it is the case that, for some $i \in J$, $D(x_{n_k i}) \leqslant ta(i)(1 + \eta)$. Otherwise, by extracting a subsequence from another we have, for all $i \in J$, $\|x_{n_k i} - x_{n_l i}\| > ta(i)(1 + \eta)$ for all k and l with $k < l$. This leads us to a conclusion, by (2.4), that $\|w_{n_k} - w_{n_l}\| \geqslant \|(\sum_{i \in J} \|x_{n_k i} - x_{n_l i}\| e_i)\| \geqslant t(1+\eta)\|\sum_{i \in J}(a(i)e_i)\| > t(1+\eta)(1 - \varepsilon - N\eta^2)$ for $k < l$. Hence $t \geqslant D(w_n) \geqslant D(w_{n_k}) \geqslant t(1+\eta)(1 - \varepsilon - N\eta^2)$ which, by (2.1), is impossible. Thus, we assume that $a(1) > \eta^2$ and

$$D(x_{n_1}) \leqslant ta(1)(1 + \eta). \tag{2.7}$$

Put $u_{ni} = x_{ni} - x_i$. We claim that

$$\|x_1\| \leqslant taa(1)\left(1 + \frac{\varepsilon}{\eta^2(1 - t)} + \left(1 + \frac{b}{\|w\|}\right)\eta\right) \tag{2.8}$$

and

$$\limsup_n \|u_{n_1} + x_1\| = \limsup_n \|x_{n_1}\| > t(1 + a - \gamma)a(1) \tag{2.9}$$

for all $\gamma > 0$ where, recall that, $b = \lim_n \|z_n\|$. By (2.3) we get

$$\|x_{ni} - x_i\| + \|x_i\| - \|x_{ni}\| < \eta^3 \tag{2.10}$$

for all large n. Eq. (2.6) then implies that $(1+\|z_n\|/\|w\|)\|x_i\| \leqslant \|x_{ni}\| + (1+\|z_n\|/\|w\|)\eta^3$, for all large n. Let $n \to \infty$ to obtain $(1+1/a-\varepsilon/(1-t))\|x_1\| \leqslant (1+(t-\varepsilon)/(1-t))\|x_1\| \leqslant (1+b/\|w\|)\|x_1\| \leqslant a(1)+(1+b/\|w\|)\eta^3 \leqslant a(1)(1+(1+b/\|w\|))\eta$, and so $(1+a)/a\|x_1\| \leqslant a(1)(1+[\varepsilon/(\eta^2(1-t))]+(1+b/\|w\|)\eta)$ that provides an estimate in (2.8). Eq. (2.9) is easily obtained. Taking (2.7)–(2.9) into account, we see that $u_{n1}/ta(1)M \in B_{X_1}$, $D(u_{n1}/ta(1)M) \leqslant 1$, and $x_1/ta(1)M \in aB_{X_1}$. This together with (2.1) and (2.9) imply that $R(a, X_1) > \max_i R(a, X_i)$, a contradiction, and the proof is complete. □

Corollary 3. *Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N$ be strictly convex. If for all $i = 1, \ldots, N$, $R(a, X_i) < 1 + a$ for some $a \in (0, 1]$, then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has the fixed point property.*

3. Property (M)

Kalton [14] introduced *property* (M): For $x_n \xrightarrow{w} 0$, the weakly null type $\psi_{(x_n)} := \limsup_n \|x - x_n\|$ is a function of $\|x\|$ only. In [12], Garcia-Falset and Sims proved that if a Banach space X has property (M), then X has the fixed point property. A preliminary result which they used in their proof of the main result in [12] and which we need here is

Lemma A *(Garcia-Falset and Sims [12, Lemma 3.1]). If X has property (M) and (x_n) is a weakly null sequence with* $\limsup_n \|x_n\| = 1$, *then* $D(x_n) = \sup\{\limsup_n \|x_n - x\| : x \in B_X\}$.

We now consider a ψ-direct sum $X \oplus_\psi Y$ when both X and Y have property (M). We first observe that, for each $\psi \in \Psi \backslash \{\psi_1\}$, there exists $t \in (0, \frac{1}{2})$ with $\psi(t) \vee \psi(1-t) < 1$. The following lemma is a consequence of Lemma A.

Lemma 4. *Let X be a Banach space having property* (M). *For every weakly convergence sequence $x_n \xrightarrow{w} x$, we have*

$$\limsup_n \|x_n\| \leqslant D(x_n) + \left(\|x\| \vee \limsup_n \|x_n - x\| - \limsup_n \|x_n - x\| \right).$$

Examples 1 and 2 in [21] show that property (M) is not preserved under the sum $X \oplus_p Y$ for $p \in \{1, \infty\}$. This observation leads us to consider and then obtain the following result.

Theorem 5. *Let Banach spaces X and Y have property* (M) *and $\psi \in \Psi \backslash \{\psi_1\}$. Then the ψ-direct sum $X \oplus_\psi Y$ of X and Y has the weak fixed point property.*

Proof. Let us assume that $X \oplus_\psi Y$ does not have the weak fixed point property. Then we have a weakly compact convex subset K of $X \oplus_\psi Y$ that is minimal for a fixed point free nonexpansive mapping $T : K \to K$. Moreover, we can assume that $\operatorname{diam} K = 1$, and K contains an approximate fixed point sequence (z_n) with $z_n \xrightarrow{w} 0$ and, by the Goebel-Karlovitz Lemma, $\lim_n \|z_n - z\| = 1$ for all $z \in K$. We will consider the following subset of $\widetilde{X}\oplus_\psi\widetilde{Y}$. Let $W := \{\widetilde{(w_n)} \in \widetilde{K} : \|\widetilde{(w_n)} - \widetilde{(z_n)}\| \leqslant t$ and $D(\widetilde{w_n}) \leqslant 1-t\}$, where $t \in (0, \frac{1}{2})$ is chosen so that $\psi(t)\vee\psi(1-t) < 1$. Observe that $D(w_n) = D(y_n)$ if $\widetilde{(w_n)} = \widetilde{(y_n)}$. Then W is $\widetilde{T}$-invariant, closed, convex and nonempty as $(1-t)\widetilde{(z_n)} \in W$. For $\widetilde{(w_n)} \in W$ we can assume that $w_n \xrightarrow{w} w_o$, $\lim_n\|w_n\| = \|\widetilde{(w_n)}\|$, $\lim_n\|w_n\| = \|(\lim_n\|x_n\|, \lim_n\|y_n\|)\|$and $\lim_n\|w_n - w_o\| = \|(\lim_n\|x_n - x_o\|, \lim_n\|y_n - y_o\|)\|$. Fix $\varepsilon > 0$ such that $0<\varepsilon<\{(1-\psi(t))\wedge(1-\psi(1-t))\wedge t\}$.

Case 1 ($\lim_n\|w_n - w_o\| \leqslant \|w_o\|$): We have $\|\widetilde{(w_n)}\| \leqslant \|\widetilde{w_n - w_o}\| + \|w_o\| \leqslant 2\|w_o\| \leqslant 2\|\widetilde{w_n - z_n}\| \leqslant 2t$.

Case 2 ($\lim_n\|w_n - w_o\| > \|w_o\|$): This means that $\lim_n\|x_n - x_0\| > \|x_o\|$ or $\lim_n\|y_n - y_0\| > \|y_o\|$. So we can assume that $\lim_n\|x_n - x_o\| > \|x_o\|$. If $\lim_n\|y_n - y_o\| < \varepsilon$ we have, by using Lemma 5, $\|\widetilde{(w_n)}\| = \lim_n\|w_n\| = \|(\lim_n\|x_n\|, \lim_n\|y_n\|)\| \leqslant \|(D(x_n), \lim_n\|y_n - y_o\| + \|y_o\|)\| \leqslant \|(1-t, t)\| + \varepsilon \leqslant \psi(t)+\varepsilon$. If $\lim_n\|y_n - y_o\| \geqslant \varepsilon$, we have $\|\widetilde{(w_n)}\| = \|(D(x_n), D(y_n) + (\|y_o\|\vee\lim_n\|y_n-y_o\| - \lim_n\|y_n - y_o\|))\| \leqslant \|(D(x_n), D(y_n))\| + (\|y_o\|\vee\lim_n\|y_n - y_o\| - \lim_n\|y_n-y_o\|) = D(w_n) + (\|y_o\|\vee\lim_n\|y_n-y_o\| - \lim_n\|y_n - y_o\|) \leqslant (1-t)+t-\varepsilon = 1-\varepsilon$. Combining these, we have that elements of W have their norms uniformly bounded away from one. More precisely, we have $\|\widetilde{(w_n)}\| \leqslant \max\{2t, \psi(t)+\varepsilon, \psi(1-t)+\varepsilon, 1-\varepsilon\} < 1$. This, however, contradicts Lin's Theorem which ensures that W contains elements of norms arbitrarily closed to one.

Corollary 6 (*Marino et al. [21, Proposition 5]*). *Let Banach spaces X and Y have property* (M). *Then the direct sum $X\oplus_\infty Y$ of X and Y has the fixed point property.*

In the second part of the paper, we turn to the study of permanence properties. Recently, Dhompongsa et al. [4] considered the permanence of smoothness, uniform smoothness, u-convexity, and U-convexity of Banach spaces. Two of these properties, namely uniform smoothness and U-convexity are known to imply fpp.

4. The coefficient $R(X)$

Since the condition "$R(X) < 2$" is more strict than the condition "$R(1, X) < 2$", we have a stronger result than might be expected from Theorem 2.

Theorem 7. *Let $X_1, \ldots, X_N$ be Banach spaces with $R(X_i) < 2$ for all $i = 1, \ldots, N$. If Z is uniformly convex, then $R(X_1 \oplus \cdots \oplus X_N)_Z < 2$.*

Proof. Since Z is uniformly convex, for $\varepsilon > 0$ which satisfies $\max_i R(X_i)(1+N\varepsilon) < 2$, there exists $\delta > 0$ such that $\max_i R(X_i)(1+N\varepsilon) < 2-\delta$ and for every $z_1, z_2 \in Z$, if $\|z_1 - z_2\| \geqslant \varepsilon$, then $\|z_1 + z_2\| \leqslant 2-\delta$. Now, let (x_n) be a weakly null sequence in $B_{(X_1\oplus\cdots\oplus X_N)}$ and $x \in B_{(X_1\oplus\cdots\oplus X_N)}$. Write $x_n = (x_{n1}, \ldots, x_{nN})$. We want to show that $\liminf_n\|x_n + x\| \leqslant 2-\delta$. For this end, we can consider subsequences of (x_n) in order to obtain estimates in the argument to follow.

Case 1 ($\|\|x_{n1}\| - \|x_1\|\| \geqslant \varepsilon$ *for all large n*): We have, for all large n, $\|x_n + x\| = \|\sum_{i\in I}\|x_{ni} + x_i\|e_i\| \leqslant \|\sum_{i\in I}\|x_{ni}\|e_i + \sum_{i\in I}\|x_i\|e_i\| \leqslant 2-\delta$.

Case 2 (*for all large n*, $|\|x_{ni}\| - \|x_i\|| < \varepsilon$ *for all i*): Let $J = \{i \in I : x_i = 0\}$. Now estimate

$$\begin{aligned}
&\|(\|x_{n1} + x_1\|, \ldots, \|x_{nN} + x_N\|)\| \\
&\leqslant \left\|\sum_{i\in J} \|x_{ni}\| e_i\right\| + \left\|\sum_{i\in I\setminus J} (\|x_{ni}\| \vee \|x_i\|) \left\| \frac{x_{ni} + x_i}{\|x_{ni}\| \vee \|x_i\|} \right\| e_i\right\| \\
&\leqslant \varepsilon \left\|\sum_{i\in J} e_i\right\| + \left\|\sum_{i\in I\setminus J} (\|x_i\| + \varepsilon) \left\| \frac{x_{ni} + x_i}{\|x_{ni}\| \vee \|x_i\|} \right\| e_i\right\|.
\end{aligned}$$

Take $\liminf_n$ and get $\liminf_n \|(\|x_{n1} + x_1\|, \ldots, \|x_{nN} + x_N\|)\| \leqslant \varepsilon \|\sum_{i\in J} e_i\| + \|\sum_{i\in I\setminus J} (\|x_i\| + \varepsilon) \max_i R(X_i) e_i\| \leqslant (1 + N\varepsilon) \max_i R(X_i)$. Thus, $R(X_1 \oplus \cdots \oplus X_N)_Z \leqslant 2 - \delta$ as desired. □

Corollary 8. *Let* $X_1, \ldots, X_N$ *be Banach spaces and* $\psi \in \Psi_N$ *be strictly convex. If* $R(X_i) < 2$ *for all* $i = 1, \ldots, N$, *then* $R(X_1 \oplus \cdots \oplus X_N)_\psi < 2$.

In [23], Prus introduced Banach spaces called *nearly uniform smooth* (NUS) *spaces*. These spaces are the dual of NUC spaces. Falset [10] then proved that every NUS space has the fixed point property answering a longstanding open question. Actually, he proved that WNUS, and hence NUS, Banach spaces have the fpp. The notion of WNUS is a natural generalization of the property NUS. A characterization of WNUS spaces is: X is WNUS if and only if X is reflexive and $R(X) < 2$, (see [9]). Thus we immediately have, by Dhompongsa et al. [4, Corollary 3],

Corollary 9. *Let* $X_1, \ldots, X_N$ *be Banach spaces. If* $\psi \in \Psi_N$ *is strictly convex, then each of* X_i *is WNUS if and only if* $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is WNUS.*

5. The $C_{NJ}(a, X)$ constants

In [6], Dhompongsa et al. introduced a generalized Jordan–von Neumann constant $C_{NJ}(a, X)$ for $a \geqslant 0$ defined by

$$C_{NJ}(a, X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \right\},$$

where the supremum is taken over all $x, y, z \in B_X$ of which at least one belongs to S_X and $\|y - z\| \leqslant a\|x\|$.

Theorem 10. *Let* $X_1, \ldots, X_N$ *be Banach spaces with* $C_{NJ}(a, X_i) < 2$ *for all* $i = 1, \ldots, N$. *If Z is uniformly convex, then* $C_{NJ}(a, (X_1 \oplus \cdots \oplus X_N)_Z) < 2$.

Proof. Suppose $C_{NJ}(a, (X_1 \oplus \cdots \oplus X_N)_Z) = 2$. Thus, by Dhompongsa et al. [5, Lemma 3.2] there exist sequences $x_n, y_n, z_n \in S_{(X_1 \oplus \cdots \oplus X_N)_Z}$ such that $\|x_n + y_n\|, \|x_n - z_n\| \to 2$ and $\|y_n - z_n\| \leqslant a$ for all n. By passing through subsequences, we may assume that all the following sequences converge: $\|x_{ni} + y_{ni}\| \to A_i$, $\|x_{ni} - z_{ni}\| \to B_i$, $\|x_{ni}\| \to C_i$, $\|y_{ni}\| \to C_i$, and $\|z_{ni}\| \to C_i$ for each i. By the assumption that Z is uniformly convex, we obtain $A_i = 2C_i$ and $B_i = 2C_i$ for each i. Now, for each n and for each i, we have $\|y_{ni} - z_{ni}\| = \|\|y_{ni} - z_{ni}\|e_i\| \leqslant \|\sum_{i\in I}(\|y_{ni} - z_{ni}\|)e_i\| \leqslant a$. Finally, we obtain the limits of the following sequences: $\|x_{ni} + y_{ni}\|^2 + \|x_{ni} - z_{ni}\|^2 \to A_i^2 + B_i^2 = 8C_i^2$, $2\|x_{ni}\|^2 + \|y_{ni}\|^2 + \|z_{ni}\|^2 \to 4C_i^2$. Clearly, for some i, $C_i \neq 0$. And for such i, we have $C_{NJ}(a, X_i) = 2$, a contradiction. □

Corollary 11. *Let $X_1, \ldots, X_N$ be Banach spaces and $\psi \in \Psi_N$ be strictly convex. If $C_{NJ}(a, X_i) < 2$, for all $i = 1, \ldots, N$, then $C_{NJ}(a, (X_1 \oplus \cdots \oplus X_N)_\psi) < 2$.*

Corollary 12. *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly nonsquare, whenever $X_1, \ldots, X_N$ are uniformly nonsquare and $\psi \in \Psi_N$ is strictly convex.*

Thus, in this case, $(X_1 \oplus \cdots \oplus X_N)_\psi$ has the fixed point property by Mazcunan Navarro [22, Corollary 4.2.4]. From [5], we know that "$C_{NJ}(a, X) < 2$ if and only if $J(a, X) < 2$". Therefore, the results in this section can be applied to the generalized James constant $J(a, X)$ as well.

6. The uniform Kadec–Klee property (UKK)

A Banach space X is said to have the *uniform Kadec–Klee property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_n \in B_X$, $x_n \rightharpoonup x$ weakly and $\mathrm{sep}(x_n) \geqslant \varepsilon$ imply $\|x\| \leqslant 1 - \delta$. Finally, X is *nearly uniformly convex* (NUC) if for any $\varepsilon \geqslant 0$, there exists $\delta < 1$ such that $x_n \in B_X$ and $\mathrm{sep}(x_n) \geqslant \varepsilon$ imply $\mathrm{co}(x_n) \cap \delta B_X \neq \emptyset$. Here $\mathrm{sep}(x_n) = \inf_{n\neq m}\|x_n - x_m\|$. It is known that UC ⇒ NUC ⇒ UKK ⇒ property-H. It is also well-known by Huff [13] that

$$\text{NUC} \Leftrightarrow \text{UKK} + \text{Reflexive}. \tag{6.1}$$

By strict monotonicity of an element ψ in Ψ we mean its corresponding norm $\|\cdot\|$ is strictly monotone. That is $\|(a, b)\| < \|(a, c)\|$ and $\|(b, a)\| < \|(c, a)\|$ for all $0 \leqslant a, 0 \leqslant b < c$. All strictly convex $\psi \in \Psi$ and all $\psi_p (1 \leqslant p < \infty)$ are strictly monotone. For $p = 1$, this is obvious. Now let $\psi \in \Psi$ be strictly convex. If, for some a, b, c in $[0, \infty)$ with $b < c$, we have $\|(a, b)\| = \|(a, c)\|$, then, by monotonicity [26], $\|(a, b)\| = \|(a, (b+c)/2)\| = \|(a, c)\|$. We put $\Delta = 2a + b + c$. Observe that $a \neq 0$ (by monotonicity of $\|\cdot\|$) and thus $b/(a + b) < c/(a + c)$. Strict convexity of ψ implies that $\psi((b + c)/\Delta) < [(a + b)/\Delta]\psi(b/(a + b)) + [(a + c)/\Delta]\psi(c/(a + c)) = 2/\Delta(a + b)\psi((b/(a + b)) = (2/\Delta)(\Delta/2)\psi((b + c)/\Delta)$, a contradiction, and this proves our claim.

Theorem 13. *For strictly monotone $\psi \in \Psi$, $X \oplus_\psi Y$ has the UKK property if and only if X and Y have the UKK property.*

It is immediate from [4, Corollary 3], (6.1), and Theorem 13 that

Corollary 14. *For strictly monotone $\psi \in \Psi$, $X \oplus_\psi Y$ is NUC if and only if X and Y are NUC.*

Remark 15. (1) Theorem 13 does not hold for the l_∞-norm. Consider any two Banach spaces X and Y with the UKK property. Now assume there are sequences (x_n) and (y_n) in B_X and B_Y, respectively, and $x \in S_X$, $y \in B_Y$ such that $x_n \rightarrow x$ and $y_n \xrightarrow{w} y$, $\text{sep}(y_n) \geqslant \frac{1}{2}$, and $|y| < 1 - \delta$ for some $\delta > 0$. Clearly, $\|(x, y)\|_\infty = 1 > 1 - \delta$. On the other hand, by [4, Theorem 2], $(x_n, y_n) \xrightarrow{w} (x, y)$ and $\text{sep}(x_n, y_n) \geqslant \text{sep}(y_n) \geqslant \frac{1}{2}$. This shows that $X \oplus_\infty Y$ does not have the UKK property.

(2) If $\psi \in \Psi$ is strictly monotone, then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(a+\varepsilon, b)\| \wedge \|(a, b+\varepsilon)\| \geqslant \|(a, b)\| + \delta$ for all $a \geqslant 0$, $b \geqslant 0$. This follows from the continuity of the norm $\|\cdot\|$ on a compact set.

(3) With the same proof, Theorem 13 can be extended to strictly monotone Z-norms.

Proof of Theorem 13. Suppose on the contrary that $X \bigoplus_\psi Y$ does not have the UKK property. Thus, for some $\varepsilon_0 > 0$, we have for each $k \geqslant 1$, sequences (x_n^k, y_n^k) and (x^k, y^k) in $B_{X \oplus Y}$ such that

$$\text{sep}(x_n^k, y_n^k) \geqslant \varepsilon_0, \quad (x_n^k, y_n^k) \xrightarrow{w} (x^k, y^k) \quad \text{and} \quad \|(x^k, y^k)\| \geqslant 1 - \frac{1}{k}. \tag{6.2}$$

Since $\|a\| + \|b\| \geqslant \|(a, b)\|$, by applying Ramsey's Theorem, we may assume, by passing through subsequences that for some $\varepsilon_1 \in (0, \varepsilon_0)$,

$$\text{sep}(x_n^k) \geqslant \varepsilon_1 \quad \text{for all } k \geqslant 1. \tag{6.3}$$

In the proof below we shall consider two cases, passing through subsequences when necessary. Choose $\delta > 0$ from the definition of the UKK property of X corresponding to $\varepsilon_1/2$. Then choose $\varepsilon_2 > 0$ so small that $\varepsilon_2 < \delta\varepsilon_1/4(1 - \delta) \wedge \varepsilon_1/4$.

Case 1: *There exists* $\varepsilon_3 > 0$ *such that* $\|x_n^k\| \geqslant \|x^k\| + \varepsilon_3$ *for all* n, k.

Case 2: $\|x_n^k\| \leqslant \|x^k\| + \varepsilon_2$ *for all* n, k.

For *Case* 1, it follows from Remark 15(2) that for some $\delta_1 \in (0, 1)$, $\|(x_n^k, y_n^k)\| \geqslant \|(\|x^k\| + \varepsilon_3, \|y_n^k\|)\| > \|(x^k, y_n^k)\| + \delta_1$ for all n, k. By (6.2) we see that $(x^k, y_n^k) \xrightarrow{w} (x^k, y^k)$. Thus, $1 \geqslant \lim \|(x_n^k, y_n^k)\| \geqslant \lim \inf_{n\to\infty} \|(x^k, y_n^k)\| + \delta_1 \geqslant \|(x^k, y^k)\| + \delta_1$, which is impossible since $\lim_k \|(x^k, y^k)\| = 1$ by (6.2).

For *Case* 2, we first observe from (6.3) that

$$\|x^k\| \geqslant \varepsilon_1/4 \quad \text{for all large } k. \tag{6.4}$$

As $\lim_n \|x_n^k\| \geqslant \|x^k\|$ we can assume $\|x_n^k\| > 0$ for all n, k. Choose k sufficiently large so that

$$\|x^k\| - \varepsilon_2 \leqslant \|x_n^k\| \leqslant \|x^k\| + \varepsilon_2 \quad \text{for all large } n. \tag{6.5}$$

Thus by (6.2), we have for $n \neq m$,

$$\begin{aligned} \left\| \frac{x_n^k}{\|x_n^k\|} - \frac{x_m^k}{\|x_m^k\|} \right\| &\geqslant \frac{\|x_n^k - x_m^k\|}{\|x^k\|} - \left| \frac{\|x_n^k\| - \|x^k\|}{\|x^k\|} \right| - \left| \frac{\|x_m^k\| - \|x^k\|}{\|x^k\|} \right| \\ &\geqslant (\varepsilon_1 - \varepsilon_2 - \varepsilon_2)/\|x^k\| \geqslant \varepsilon_1/2, \end{aligned}$$

i.e., $\mathrm{sep}(x_n^k/\|x_n^k\|) \geqslant \varepsilon_1/2$. Assume without loss of generality that $\|x_n^k\| \to A$ for some A. Thus, by (6.5) and (6.2), $A \leqslant \|x^k\| + \varepsilon_2$ and $x_n^k/\|x_n^k\| \xrightarrow{w} x^k/A$. But, by (6.4), $\|x^k\|/A \geqslant \|x^k\| /(\|x^k\| + \varepsilon_2) = 1 - \varepsilon_2/(\|x^k\| + \varepsilon_2) > 1 - \delta$, contradicting the UKK property of X. □

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A NOTE ON PROPERTIES THAT IMPLY THE FIXED POINT PROPERTY

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We give relationships between some Banach-space geometric properties that guarantee the weak fixed point property. The results extend some known results of Dalby and Xu.

1. Introduction

A Banach space X is said to satisfy the weak fixed point property (fpp) if every nonempty weakly compact convex subset C, and every nonexpansive mapping $T : C \to C$ (i.e., $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$) has a fixed point, that is, there exists $x \in C$ such that $T(x) = x$. Many properties have been shown to imply fpp. The most recent one is the uniform nonsquareness which is proved by Mazcuñán [20] solving a long stand open problem. Other well known properties include Opial property (Opial [21]), weak normal structure (Kirk [17]), property (M) (García-Falset and Sims [12]), $R(X) < 2$ (García-Falset [10]), and UCED (Garkavi [13]). Connection between these properties were investigated in Dalby [3] and Xu et al. [27]. We aim to continue the study in this direction. In contrast to [3], we do not assume that all Banach spaces are separable.

2. Preliminaries

Let X be a Banach space. For a sequence (x_n) in X, $x_n \xrightarrow{w} x$ denotes the weak convergence of (x_n) to $x \in X$. When $x_n \xrightarrow{w} 0$, we say that (x_n) is a weakly null sequence. $B(X)$ and $S(X)$ stand for the unit ball and the unit sphere of X, respectively. It becomes a common ingredient that when working with a weak null sequence (x_n), we consider the type function $\limsup_{n\to\infty} \|x_n - x\|$ for all $x \in X$. As for a starting point, we recall Opial property.

Opial property [21] states that

$$\text{if } x_n \xrightarrow{w} 0, \quad \text{then } \limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\| \ \forall x \in X,\ x \neq 0. \tag{2.1}$$

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Appendix 3: A note on properties that implies the weak fixed point property, Abst. Appl. Anal. V. 2006, Article ID 34959, Pages 1-12.

If the strict inequality becomes $\leq$, this condition becomes a nonstrict Opial property. On the other hand, if for every $\epsilon > 0$, for each $x_n \xrightarrow{w} 0$ with $\|x_n\| \to 1$, there is an $r > 0$ such that

$$1 + r \leq \limsup_{n\to\infty} \|x_n + x\| \tag{2.2}$$

for each $x \in X$ with $\|x\| \geq \epsilon$, then we have the locally uniformly Opial property (see [27]).

The coefficient $R(X)$, introduced in García-Falset [9], is defined as

$$R(X) := \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0,\ \|x_n\| \leq 1\ \forall n,\ \|x\| \leq 1\right\}. \tag{2.3}$$

So $1 \leq R(X) \leq 2$ and it is not hard to see that in the definition of $R(X)$, "liminf" can be replaced by "limsup." Some values of $R(X)$ are $R(c_0) = 1$ and $R(l_p) = 2^{1/p}, 1 < p < \infty$.

A Banach space X has property (M) if whenever $x_n \xrightarrow{w} 0$, then $\limsup_{n\to\infty} \|x_n - x\|$ is a function of $\|x\|$ only. Property (M) which is introduced by Kalton [15] is equivalent to:

$$\text{if } x_n \xrightarrow{w} 0, \quad \|u\| \leq \|v\|, \text{ then } \limsup_{n\to\infty} \|x_n + u\| \leq \limsup_{n\to\infty} \|x_n + v\|. \tag{2.4}$$

Sims [23] introduced a property called weak orthogonality (WORTH) for Banach spaces. A Banach space X is said to have property WORTH if,

$$\text{for every } x_n \xrightarrow{w} 0,\ x \in X, \quad \limsup_{n\to\infty} \|x_n + x\| = \limsup_{n\to\infty} \|x_n - x\|. \tag{2.5}$$

It remains unknown if property WORTH implies fpp. In many situations, the fixed point property can be easily obtained when we assume, in addition, that the spaces being considered have the property WORTH. For examples, WORTH and ε_0-inquadrate for some $\epsilon_0 < 2$ ([24]), WORTH and 2-UNC ([11]) imply fpp.

The following results will be used in Section 3.

Proposition 2.1 [12, Proposition 2.1]. *For the following conditions on a Banach space X, we have (i)⇒(ii)⇒(iii)⇒(iv).*

(i) *X has property (M).*

(ii) *X has property WORTH.*

(iii) *If $x_n \xrightarrow{w} 0$, then for each $x \in X$ we have $\limsup_{n\to\infty} \|x_n - tx\|$ is an increasing function of t on $[0, \infty)$.*

(iv) *X satisfies the nonstrict Opial property.*

Property (M) implies the nonstrict Opial property but not weak normal structure. c_0 has property (M) but does not have weak normal structure. In [3, 25] it had been shown that $R(X) = 1$ implies X has property (M).

A generalization of uniform convexity of Banach spaces which is due to Sullivan [26] is now recalled. Let $k \geq 1$ be an integer. Then a Banach space X is said to be k-UR (k-uniformly rotund) if given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\{x_1, \dots, x_{k+1}\} \subset B(X)$

satisfying $V(x_1,\dots,x_{k+1}) \geq \varepsilon$, then

$$\left\| \frac{\sum_{i=1}^{k+1} x_i}{k+1} \right\| \leq \delta(\varepsilon). \tag{2.6}$$

Here, $V(x_1,\dots,x_{k+1})$ is the volume enclosed by the set $\{x_1,\dots,x_{k+1}\}$, that is,

$$V(x_1,\dots,x_{k+1}) = \sup\left\{ \left| \begin{matrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \ddots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{matrix} \right| \right\}, \tag{2.7}$$

where the supremum is taken over all $f_1,\dots,f_k \in B(X^*)$.

Let K be a weakly compact convex subset of a Banach space X and (x_n) a bounded sequence in X. Define a function f on X by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in X. \tag{2.8}$$

Let

$$\begin{aligned} r \equiv r_K((x_n)) &:= \inf\{f(x) : x \in K\}, \\ A \equiv A_K((x_n)) &:= \{x \in K : f(x) = r\}. \end{aligned} \tag{2.9}$$

Recall that r and A are, respectively, called the asymptotic radius and center of (x_n) relative to K. As K is weakly compact convex, we see that A is nonempty, weakly compact and convex (see [14]). In [18], Kirk proved that the asymptotic center of a bounded sequence w.r.t a bounded closed convex subset of a k-uniformly convex spaces X is compact. This fact will be used in proving Theorem 3.8.

Being k-UR and Opial property are related in the following way.

Theorem 2.2 [19, Theorem 3.5]. *If X is k-UR and satisfies the Opial property, then X satisfies locally uniform Opial property.*

One last concept we need to mention is ultrapowers of Banach spaces. Ultrapowers of a Banach space are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I,X_i)$ denote the subspace of the product space $\Pi_{i\in I}X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \left\{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\right\}. \tag{2.10}$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.11}$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [1] or [22]).

3. Main results

Recall that a Banach space X is said to have Schur's property if

$$\text{for every sequence } (x_n), \quad x_n \xrightarrow{w} 0 \text{ implies } x_n \longrightarrow 0. \tag{3.1}$$

An element $x \in X$ is said to be an H-point if

$$x_n \xrightarrow{w} x, \quad \|x_n\| \longrightarrow \|x\| \text{ imply } x_n \longrightarrow x. \tag{3.2}$$

X has property (H) if every element of X is an H-point. These concepts are related, in conjunction with the condition $R(X) = 1$, as follow.

THEOREM 3.1. *A Banach space X has Schur's property if and only if $R(X) = 1$ and X has at least one H-point.*

Proof. "$\Rightarrow$" It is well known that Schur's property implies property (H). From the definition of $R(X)$ and Schur's property, we have

$$\begin{aligned} R(X) &= \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0, \|x_n\| \le 1 \ \forall n, \|x\| \le 1\right\} \\ &= \sup\{\|x\| : \|x\| \le 1\} = 1. \end{aligned} \tag{3.3}$$

"$\Leftarrow$" Suppose that there exists a sequence (x_n) converges weakly to 0 but $\|x_n\| \nrightarrow 0$. By passing through a subsequence if necessary, we can assume that $\|x_n\| \to a \neq 0$. Put $y_n = x_n/a$. Clearly $y_n \xrightarrow{w} 0$ and $\|y_n\| \to 1$. Let x_0 be an H-point. If $x_0 = 0$, we are done. We assume now that $x_0 \neq 0$ and in fact we assume that $x_0 \in S(X)$. Thus, as $R(X) = 1$ and the weak lower semicontinuity of the norm,

$$(x_0 - y_n) \xrightarrow{w} x_0, \quad \liminf_{n\to\infty} \|x_0 - y_n\| = 1. \tag{3.4}$$

Choose a subsequence $(y_{n'})$ of (y_n) such that

$$\lim_{n'\to\infty} \|x_0 - y_{n'}\| = 1. \tag{3.5}$$

We see that $(x_0 - y_n') \to x_0$ and $y_n' \to 0$. Thus $\|y_n'\| \to 0$ and $0 = a$, a contradiction. □

A Banach space X has property m_p (resp., m_∞) (cf. [27]) if for all $x \in X$, whenever $x_n \overset{w}{\to} 0$,

$$\begin{gathered}\limsup_{n\to\infty}\|x+x_n\|^p = \|x\|^p + \limsup_{n\to\infty}\|x_n\|^p \\ \left(\text{resp., } \limsup_{n\to\infty}\|x+x_n\| = \max\left\{\|x\|, \limsup_{n\to\infty}\|x_n\|\right\}\right).\end{gathered} \tag{3.6}$$

Clearly the above properties imply property (M) and property m_1 implies Opial property.

Property m_1 implies property (H). For, if $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$ for some sequence (x_n) and $x \in X$, we have, by m_1,

$$\|x\| = \limsup_{n\to\infty}\|x_n\| = \limsup_{n\to\infty}\|(x_n - x) + x\| = \|x\| + \limsup_{n\to\infty}\|x_n - x\|. \tag{3.7}$$

This implies that $\limsup_{n\to\infty}\|x_n - x\| = 0$ and thus $x_n \to x$.

It also turns out that property m_∞ and the condition $R(X) = 1$ coincide as the following result shows.

Theorem 3.2. *A Banach space X has property m_∞ if and only if $R(X) = 1$.*

Proof. "⇒" Suppose that X has property m_∞. Thus,

$$\begin{aligned} R(X) &= \sup\left\{\limsup_{n\to\infty}\|x_n - x\| : x_n \overset{w}{\longrightarrow} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\} \\ &= \sup\left\{\max\left\{\|x\|, \limsup_{n\to\infty}\|x_n\|\right\} : x_n \overset{w}{\longrightarrow} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\} = 1. \end{aligned} \tag{3.8}$$

"⇐" To show that X has property m_∞. Given $x_n \overset{w}{\to} 0$ and $x \in X - \{0\}$. Put $a = \max\{\|x\|, \limsup_{n\to\infty}\|x_n\|\}$. Clearly, $\limsup_{n\to\infty}(\|x_n\|/a) \le 1$ and $\|x\|/a \in B(X)$. We note here that $R(X) = 1$ implies property (M) and it in turn implies the nonstrict Opial property. By the weak lower semicontinuity of $\|\cdot\|$ and the nonstrict Opial property, we see that $\|x\| \le \limsup_{n\to\infty}\|x_n - x\|$ and $\limsup_{n\to\infty}\|x_n\| \le \limsup_{n\to\infty}\|x_n - x\|$. Thus $a \le \limsup_{n\to\infty}\|x_n - x\|$. On the other hand, as $R(X) = 1$, we can show that $\limsup_{n\to\infty}\|x_n/a - x/a\| \le 1$. So we can conclude that,

$$\limsup_{n\to\infty}\left\|\frac{x_n}{a} - \frac{x}{a}\right\| = 1, \tag{3.9}$$

and thus $\limsup_{n\to\infty}\|x_n - x\| = a = \max\{\|x\|, \limsup_{n\to\infty}\|x_n\|\}$ and the proof is complete. □

For $p < \infty$, we have the following proposition.

Proposition 3.3. *If X has property $m_p (1 \le p < \infty)$, then $R(X) \le 2^{1/p}$. Moreover, if in addition X does not have Schur's property, then $R(X) = 2^{1/p}$.*

Proof. Define

$$R_p(X) := \sup\Big\{\limsup_{n\to\infty} \|x_n - x\|^p : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\Big\}. \tag{3.10}$$

By property m_p, we have

$$R_p(X) = \sup\Big\{\|x\|^p + \limsup_{n\to\infty} \|x_n\|^p : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\Big\}. \tag{3.11}$$

Thus, $R_p(X) \le 2$ which implies $R(X) \le 2^{1/p}$. On the other hand, if, in addition, X does not have Schur's property, then there exists a weakly null sequence (x_n) such that $x_n \nrightarrow 0$. From this we can construct a weakly null sequence (y_n) in the unit sphere. We can now see that $R_p(X) \ge 2$ and hence $R(X) \ge 2^{1/p}$. Therefore $R(X) = 2^{1/p}$. □

Example 3.4. In $l_p\,(1 < p < \infty)$, we have $e_n \in S(X)$ and $e_n \xrightarrow{w} 0$, where (e_n) is the standard basis. Clearly

$$\|e_n - e_1\| \xrightarrow{n\to\infty} 2^{1/p}, \tag{3.12}$$

thus $R(l_p) = 2^{1/p}$. Note that l_p has property m_p (cf. [27]).

Some properties are equivalent in a space X with $R(X) = 1$.

THEOREM 3.5. *Let X be a Banach space with $R(X) = 1$. The following conditions are equivalent:*

(i) *X has property m_1;*
(ii) *X satisfies Opial property;*
(iii) *X has Schur's property.*

Proof. (i)⇒(ii) and (iii)⇒(i) are clear. It needs to prove (ii) ⇒ (iii).

Let $x_n \xrightarrow{w} 0$. To show $x_n \to 0$, let $0 \ne x \in X$. By Opial property together with property m_∞, we have

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n + x\| = \max\Big\{\|x\|, \limsup_{n\to\infty} \|x_n\|\Big\}. \tag{3.13}$$

Thus

$$\limsup_{n\to\infty} \|x_n\| < \|x\|, \tag{3.14}$$

for all $x \in X - \{0\}$. This means that $\limsup_{n\to\infty} \|x_n\| = 0$ and thus $\lim_{n\to\infty} \|x_n\| = 0$. Consequently, $x_n \to 0$, and therefore X has Shur's property. □

The Jordan-von Neumann constant $C_{NJ}(X)$ of X is defined by

$$\begin{aligned} C_{NJ}(X) &= \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y\in X \text{ not both zero}\right\} \quad ([2]) \\ &= \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x\in S(X),\ y\in B(X)\right\} \quad ([16]). \end{aligned} \tag{3.15}$$

Another important constant which is closely related to $C_{NJ}(X)$ is the James constant $J(X)$ defined by Gao and Lau [7] as:

$$\begin{aligned} J(X) &= \sup\{\|x+y\| \wedge \|x-y\| : x,y\in S(X)\} \\ &= \sup\{\|x+y\| \wedge \|x-y\| : x,y\in B(X)\}. \end{aligned} \tag{3.16}$$

In general we have

$$\frac{1}{2}J(X)^2 \le C_{NJ}(X) \le \frac{J(X)^2}{(J(X)-1)^2+1} \quad ([16]). \tag{3.17}$$

With or without having WORTH, Mazcuñán [20] showed that $R(1,X) < 2$ whenever $C_{NJ}(X) < 2$. In gerneral, $R(1,X) \le R(X)$. The constant $R(a,X)$ is introduced by Dominguez [6] as: for a given real number a

$$R(a,X) := \sup\left\{\liminf_{n\to\infty}\|x+x_n\|\right\}, \tag{3.18}$$

where the supremum is taken over all $x\in X$ with $\|x\|\le a$ and all weakly null sequences (x_n) in the unit ball of X such that

$$\limsup_{n\to\infty}\left(\limsup_{n\to\infty}\|x_n-x_m\|\right)\le 1. \tag{3.19}$$

Replacing $R(1,X)$ in [20] by $R(X)$ we obtain the following theorem.

Theorem 3.6. *If X has property WORTH and $C_{NJ}(X) < 2$, then $R(X) < 2$.*

Proof. Suppose on the contrary that $R(X) = 2$. Thus there exist sequences $(x_n^m),(x^m)\in B(X)$ such that for each m, $x_n^m \xrightarrow{w} 0$ as $n\to\infty$ and

$$\liminf_{n\to\infty}\|x_n^m - x^m\| > 2-\frac{1}{m} \tag{3.20}$$

for all $m\in N$. Now, by WORTH, we have, for each m,

$$\frac{\|x_n^m+x^m\|^2+\|x_n^m-x^m\|^2}{2(\|x_n^m\|^2+\|x^m\|^2)} > \frac{2(2-1/m)^2}{4} = 2-\frac{2}{m}+\frac{1}{2m^2} \tag{3.21}$$

for all large n. This implies $C_{NJ}(X) = 2$, a contradiction, and therefore $R(X) < 2$ as desired. □

Remark 3.7. Theorem 3.6 says that every Banach space X with property WORTH has fpp or $C_{NJ}(X) = 2 = R(X)$.

THEOREM 3.8. *If X is k-UR and satisfies property (M), then X satisfies Opial property.*

Proof. Suppose that there exist $x_n \xrightarrow{w} 0$ and $0 \neq x_0 \in X$ such that

$$\limsup_{n\to\infty} \|x_n\| \geq \limsup_{n\to\infty} \|x_n - x_0\|. \tag{3.22}$$

Observe that X is therefore not finite dimensional. By the nonstrict Opial property (see Proposition 2.1) we have

$$\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|x_n - x_0\| = \alpha \neq 0. \tag{3.23}$$

We may assume that $\|x_0\| = 1$. Define the type function by

$$f(u) = \limsup_{n\to\infty} \|x_n - u\|. \tag{3.24}$$

Then f is a function of $\|u\|$ and is also nondecreasing in $\|u\|$. Now since $f(0) = f(x_0) = \alpha$ and since $\|x_0\| = 1$, it follows that $f(u) \equiv \alpha$ for all $u \in B(X)$. This implies that $A_{B(X)}(x_n) = B(X)$. Since X is k-UR, Kirk [18] implies that $A_{B(X)}(x_n)$ and so $B(X)$ is compact, that is, X is finite dimensional, a contradiction. □

COROLLARY 3.9. *If X is k-UR and has property (M), then X has the locally uniform Opial property. In particular, properties UR and (M) imply the locally uniform Opial property.*

Proof. This follows from Theorem 2.2 and Theorem 3.8. □

Definition 3.10. Let X be a Banach space.

(i) We say that X has property strict (M) [27, Definition 2.2] if, for each weakly null sequence (x_n), for $u, v \in X$ such that $\|u\| < \|v\|$, $\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \|x_n - v\|$.

(ii) We say that X has property strict (W) if, for each weakly null sequence (x_n), for $x \in X$ we have $\limsup_{n\to\infty} \|x_n - tx\|$ is an strictly increasing function of t on $[0, \infty)$.

It is easy to see that

$$\text{property strict } (M) \Longrightarrow \text{property strict } (W) \Longrightarrow \text{Opial property}. \tag{3.25}$$

PROPOSITION 3.11. *Let X be a Banach space, then X has property strict (M) if and only if it has both properties (M) and strict (W).*

Proof. "⇒" Clear.

"⇐" Suppose X has properies (M) and strict (W). Let (x_n) be a weakly null sequence, $u, v \in X$ with $\|u\| < \|v\|$. By property strict (W) we have

$$\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \left\| x_n - \frac{\|v\|}{\|u\|} u \right\|. \tag{3.26}$$

Since $\|(\|v\|/\|u\|)u\| = \|v\|$, so by property (M) we have $\limsup_{n\to\infty} \|x_n - (\|v\|/\|u\|)u\| = \limsup_{n\to\infty} \|x_n - v\|$. Hence

$$\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \|x_n - v\|. \tag{3.27}$$

This shows that X has property strict (M). □

PROPOSITION 3.12. *Let X be a Banach space which satisfies Opial property and has property (M). Then X satisfies the locally uniform Opial property.*

Proof. Let (x_n) be a weakly null sequence in X satisfying $\|x_n\| \to 1$ and $c > 0$. Set $r = \limsup_{n\to\infty} \|x_n - (c/\|x\|)x\| - 1$, where $x \in X - \{0\}$. Since X satisfies Opial property, we have $r > 0$. Hence, for $u \in X$ such that $\|u\| \geq c$, we have

$$\limsup_{n\to\infty} \|x_n - u\| \geq \limsup_{n\to\infty} \left\| x_n - \frac{c}{\|u\|}u \right\| = \limsup_{n\to\infty} \left\| x_n - \frac{c}{\|x\|}x \right\| = 1 + r. \tag{3.28}$$

Thus, X satisfies the locally uniform Opial property. □

COROLLARY 3.13 [27, Theorem 2.1]. *Let X be a Banach space which has property strict (M). Then X satisfies the locally uniform Opial property.*

Recall that a Banach space X is uniformly convex in every direction (UCED) Day et al. [4] if, for each $z \in X$ such that $\|z\| = 1$ and $\epsilon > 0$, we have

$$\delta_z(\epsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ x - y = tz,\ |t| \geq \epsilon \right\} > 0. \tag{3.29}$$

THEOREM 3.14. *Suppose that a Banach space X has property WORTH and is also UCED. Then X has the property strict (W).*

Proof. Suppose X fails to have the property strict (W), then there exist a weakly null sequence (x_n), $x \in S(X)$, $t_1, t_2 \in [0,\infty)$, where $t_1 < t_2$, with

$$\limsup_{n\to\infty} \|x_n + t_1 x\| \geq \limsup_{n\to\infty} \|x_n + t_2 x\|. \tag{3.30}$$

By property WORTH we must have equality. Put $a = \limsup_{n\to\infty} \|x_n + t_1 x\|$, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \left\| x_n + \frac{t_1 + t_2}{2} x \right\| &= \limsup_{n\to\infty} \left\| \frac{x_n + t_1 x + x_n + t_2 x}{2} \right\| \\ &\leq a\left[1 - \delta_x\left(\frac{t_2 - t_1}{a} \right) \right] < a = \limsup_{n\to\infty} \|x_n + t_1 x\| \end{aligned} \tag{3.31}$$

contradicting to having WORTH of X. □

From Proposition 3.11 and Theorem 3.14 we have the following corollary.

COROLLARY 3.15. *Suppose that a Banach space X has property (M) and is also UCED. Then X has property strict (M).*

Finally, we improve the latest upper bound of the Jordan-von Neumann constant $C_{NJ}(X)$ at $(3+\sqrt{5})/4$ for X to have uniform normal structure which is proved in [5].

THEOREM 3.16. *If* $C_{NJ}(X) < (1+\sqrt{3})/2$, *then* X *has uniform normal structure.*

Proof. Since $C_{NJ}(X) < 2$, X is uniformly nonsquare, and consequently, X is reflexive. Thus, normal structure and weak normal structure coincide. By [8, Theorem 5.2], it suffices to prove that X has weak normal structure.

Suppose on the contrary that X does not have weak normal structure. Thus, there exists a weak null sequence (x_n) in $S(X)$ such that for $C := \overline{co}\{x_n : n \geq 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam} C = 1 \quad \forall x \tag{3.32}$$

(cf. [24]). Let $\alpha = \sqrt{1+\sqrt{3}}$. We choose first an $x \in C$ with $\|x\| = 1$. We will consider, without loss of generality

$$\begin{aligned} \lim_{n\to\infty} \|x_n + x\| &\leq R(1,X) \leq J(X) \quad ([20]) \\ &\leq \sqrt{2C_{NJ}(X)} \quad ([16]) < \alpha. \end{aligned} \tag{3.33}$$

By Hanh-Banach theorem there exist $f_n, g \in S(X^*)$ satisfying $f_n(x_n - (1/2)x) = \|x_n - (1/2)x\|$, $\forall n \in \mathbb{N}$ and $g(x) = 1$. Set $\tilde{f} = \widetilde{(f_n)}$. Then $\tilde{f}, \dot{g} \in S(\tilde{X}^*)$ and satisfy

$$\tilde{f}\left(\widetilde{(x_n)}\right) = 1, \qquad \tilde{f}(\dot{x}) = 0, \qquad \dot{g}\left(\widetilde{(x_n)}\right) = 0, \qquad \dot{g}(\dot{x}) = 1. \tag{3.34}$$

Now consider

$$\begin{aligned} \|\tilde{f} - \dot{g}\| &\geq (\tilde{f} - \dot{g})\left(\widetilde{(x_n)} - \dot{x}\right) \\ &= \tilde{f}\left(\widetilde{(x_n)}\right) - \tilde{f}(\dot{x}) - \dot{g}\left(\widetilde{(x_n)}\right) + \dot{g}(\dot{x}) \\ &= 1 + 0 - 0 + 1 \geq 2. \end{aligned} \tag{3.35}$$

On the other hand,

$$\begin{aligned} \|\tilde{f} + \dot{g}\| &\geq (\tilde{f} + \dot{g})\left(\frac{1}{\alpha}\left(\widetilde{(x_n)} + \dot{x}\right)\right) \\ &= \tilde{f}\left(\frac{1}{\alpha}\widetilde{(x_n)}\right) + \tilde{f}\left(\frac{1}{\alpha}\dot{x}\right) - \dot{g}\left(\frac{1}{\alpha}\widetilde{(x_n)}\right) + \dot{g}\left(\frac{1}{\alpha}\dot{x}\right) \\ &= \frac{1}{\alpha} + 0 - 0 + \frac{1}{\alpha} = \frac{2}{\alpha}. \end{aligned} \tag{3.36}$$

Thus we have

$$C_{NJ}(\tilde{X}^*) \geq \frac{\|\tilde{f} + \dot{g}\|^2 + \|\tilde{f} - \dot{g}\|^2}{2(\|\tilde{f}\|^2 + \|\dot{g}\|^2)} \geq \frac{4 + 4/\alpha^2}{4} = 1 + \frac{1}{\alpha^2}. \tag{3.37}$$

Since the Jordan-von Neumann constants of X^*, X, $\widetilde{X}$, and $\widetilde{X}^*$ are all equal, we must have $C_{NJ}(X) \geq 1 + 1/\alpha^2$, that is,

$$C_{NJ}(X) \geq \frac{1+\sqrt{3}}{2}, \tag{3.38}$$

a contradiction. □

The following corollary is a consequence of the proof of Theorem 3.16.

Corollary 3.17. *If $C_{NJ}(X) < 1 + 1/J(X)^2$, then X has uniform normal structure.*

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Appendix 4: Lim's theorems for multivalued mappings in CAT(0) spaces,

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Abstract

Let X be a complete CAT(0) space. We prove that, if E is a nonempty bounded closed convex subset of X and $T : E \to K(X)$ a nonexpansive mapping satisfying the weakly inward condition, i.e., there exists $p \in E$ such that $\alpha p \oplus (1-\alpha)Tx \subset \overline{I_E(x)}$ $\forall x \in E, \forall \alpha \in [0,1]$, then T has a fixed point. In Banach spaces, this is a result of Lim [On asymptotic centers and fixed points of nonexpansive mappings, Canad. J. Math. 32 (1980) 421–430]. The related result for unbounded $\mathbb{R}$-trees is given.

Keywords: Multivalued mappings; Fixed points; CAT(0) spaces; $\mathbb{R}$-trees

1. Introduction

In 1980 [8] and 2001 [9], Lim and, respectively, Xu had proved differently the same result concerning the existence of a fixed point for a nonself nonexpansive compact valued mapping defining on a bounded closed convex subset of a uniformly convex space and satisfying the weak inward condition. While Lim used the method of asymptotic radius,

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Xu used his characterization of uniform convexity. Recently in 2003, Bae [1] considered a closed valued mapping defined on a closed subset of a complete metric space. It was shown that if the mapping is weakly contractive and is metrically inward, then it has a fixed point.

Having all these results, we are interested in extending the Lim–Xu's result to a special kind of metric spaces, namely, CAT(0) spaces. Our proofs follow the ideas of the proofs in Lim [8], Bae [1], and Xu [9].

In Section 2, we give some basic notions and in Sections 3 and 4 we prove our results.

2. Preliminaries

In the course of our proof of the main result, we use an ultrapower of a metric space as an ingredient. Following Khamsi [5], let (X,d) be a bounded metric space and $\mathcal{U}$ a nontrivial ultrafilter on the natural numbers. Consider the countable Cartesian product X^∞ of X and define the equivalence relation $\sim$ on X^∞ by

$$(x_n) \sim (y_n) \quad \text{if } \lim_{\mathcal{U}} d(x_n, y_n) = 0.$$

The limit over $\mathcal{U}$ exists since X is bounded. On the quotient space $\tilde{X}$ of X^∞ over $\sim$, which will be called an ultrapower of X, define the metric $\tilde{d}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} d(x_n, y_n),$$

where $\tilde{x} = \widetilde{(x_n)}$ and $\tilde{y} = \widetilde{(y_n)}$ are elements of $\tilde{X}$. It is easy to see that $\tilde{X}$ is complete whenever X is. For each subset E of X put

$$\dot{E} = \{\widetilde{(x_n)}:\ x_n = x \in E \text{ for any } n \geqslant 1\}.$$

Clearly, X and $\dot{X}$ are isometric.

We present now a brief discussion on CAT(0) spaces (see Kirk [6,7] and Bridson and Haefliger [2]). Although CAT(κ) spaces are defined for all real numbers κ, we restrict ourselves to the case that $\kappa = 0$.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. Obviously, c is an isometry and $d(x,y) = l$. The image of c is called a geodesic segment joining x and y and, when unique, denoted $[x,y]$. A metric space is said to be a geodesic space if any two of its points are joined by a geodesic segment. If there is exactly one geodesic segment joining x to y for all $x, y \in X$, we say that (X,d) is uniquely geodesic.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X,d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edge of Δ). A comparison triangle for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$.

(X,d) is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) comparison axiom:

For every geodesic triangle Δ in X and its comparison triangle $\bar{\Delta}$ in $\mathbb{R}^2$, if $x, y \in \Delta$, and $\bar{x}, \bar{y}$ are their comparison points in $\bar{\Delta}$, respectively, then

$$d(x, y) \leqslant d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let X be a CAT(0) space, and let E be a nonempty closed convex subset of X. The following facts will be needed:

(i) (X, d) is uniquely geodesic.
(ii) $(\bar{X}, \bar{d})$ is a CAT(0) space.
(iii) (X, d) satisfies the (CN) inequality

$$d(x, y_0)^2 \leqslant \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2$$

for all $x, y_1, y_2 \in X$ and y_0 the midpoint of the segment $[y_1, y_2]$. Note that the converse is also true. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies (CN) inequality (cf. [7]).

(iv) Let p, x, y be points in X, let $\alpha \in (0, 1)$, and m_1 and m_2 denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying

$$d(p, m_1) = \alpha d(p, x) \quad \text{and} \quad d(p, m_2) = \alpha d(p, y).$$

Then

$$d(m_1, m_2) \leqslant \alpha d(x, y).$$

(v) For every $x \in X$, there exists a unique point $p(x) \in E$ such that

$$d\big(x, p(x)\big) = \operatorname{dist}(x, E),$$

where $\operatorname{dist}(x, E) := \inf\{d(x, y):\ y \in E\}$.

With the same E and $p(x)$, if $x \notin E$, $y \in E$, and $y \neq p(x)$, then $\angle_{p(x)}(x, y) \geqslant \frac{\pi}{2}$, where $\angle_z(x, y)$ is the Alexandrov angle between the geodesic segments $[z, x]$ and $[z, y]$ for all $x, y, z \in X$ (see [2, p. 176]).

Let (X, d) be a metric space and E a nonempty subset of X. A closed valued mapping $T : \to 2^X \backslash \emptyset$ is said to be metrically inward if for each $x \in E$,

$$Tx \subset MI_E(x),$$

where $MI_E(x)$ is the metrically inward set of E at x defined by

$$MI_E(x) = \big\{z \in X:\ z = x \text{ or there exists } y \in E \text{ such that } y \neq x \text{ and } d(x, z) = d(x, y) + d(y, z)\big\}.$$

In case X is a Banach space, the inward set of E at x is defined by

$$I_E(x) = \big\{x + \lambda(y - x):\ y \in E,\ \lambda \geqslant 1\big\}.$$

In general, $I_E(x) \subset MI_E(x)$ for each $x \in E$, and the equality may not be true.

From now on, X stands for a complete CAT(0) space. Let E be a nonempty bounded subset of X. We shall denote by $F(E)$ the family of nonempty closed subsets of E, by $FC(E)$ the family of nonempty closed convex subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $F(X)$, i.e.,

$$H(A,B)=\max\Big\{\sup_{a\in A}\operatorname{dist}(a,B),\ \sup_{b\in B}\operatorname{dist}(b,A)\Big\},\quad A,B\in F(X).$$

Definition 2.1. A multivalued mapping $T:E\to F(X)$ is said to be a contraction if there exists a constant $k\in[0,1)$ such that

$$H(Tx,Ty)\leqslant kd(x,y),\quad x,y\in E. \tag{2.1}$$

In this case T is said to be k-contractive. If (2.1) is valid when $k=1$, then T is called nonexpansive.

We use the notation $(1-\alpha)u\oplus\alpha v$, $\alpha\in[0,1]$, to denote the points of the segment $[u,v]$ with distance $\alpha d(u,v)$ from u. For $E\subset X$ and a fixed element $p\in E$, $(1-\alpha)p\oplus\alpha E:=\{(1-\alpha)p\oplus\alpha v\colon v\in E\}$. E is said to be convex if for each pair of points $x,y\in E$, we have $[x,y]\subset E$.

For a nonempty subset E of a CAT(0) space X, it is easy to see that the (metrically) inward set $MI_E(x)$ becomes

$$MI_E(x)=\Big(\bigcup\{z\colon (x,z]\cap E\neq\emptyset\}\Big)\cup\{x\}:=I_E(x).$$

Definition 2.2. A multivalued mapping $T:E\to F(X)$ is said to be inward on E if for some $p\in E$,

$$\alpha p\oplus(1-\alpha)Tx\subset I_E(x)\quad\forall x\in E,\ \forall\alpha\in[0,1],$$

and weakly inward on E if

$$\alpha p\oplus(1-\alpha)Tx\subset\overline{I_E(x)}\quad\forall x\in E,\ \forall\alpha\in[0,1], \tag{2.2}$$

where $\bar{A}$ denotes the closure of a subset A of X.

When E is convex, it is easy to see that

$$I_E(x)=\Big(\bigcup\{[x,y]\colon (x,y]\cap E\neq\emptyset\}\Big)\cup\{x\}.$$

Note that in a normed space setting, the inward (respectively, weakly inward) condition is equivalent to saying that $Tx\subset I_E(x)$ (respectively, $Tx\subset\overline{I_E(x)}$) since in this case, $I_E(x)$ is convex. This is also true for $\mathbb{R}$-trees.

3. Lim's theorems

The following simple result is needed.

Proposition 3.1. *Let E be a nonempty closed convex subset of X, $x \in X$, and $p(x)$ the unique nearest point of x in E. Then*

$$d(x, p(x)) < d(x, y) \quad \forall y \in \overline{I_E(p(x))} \setminus \{p(x)\}.$$

Proof. Let $y \in \overline{I_E(p(x))} \setminus \{p(x)\}$, there is a sequence (y_n) in $I_E(p(x))$ and $y_n \to y$. For all large n we can find $z_n \in (p(x), y_n] \cap E$. Since $z_n \in E$ and $z_n \neq p(x)$, $\angle_{p(x)}(x, z_n) \geqslant \frac{\pi}{2}$ (see [2, p. 176]). Thus in the comparison triangle $\bar{\Delta}(p(x), x, y_n)$, the angle at $\overline{p(x)}$ is also greater than or equal to $\frac{\pi}{2}$ (see [2, p. 161]). By the law of cosines,

$$d(x, p(x))^2 + d(y_n, p(x))^2 \leqslant d(x, y_n)^2.$$

Taking $n \to \infty$, we obtain

$$d(x, p(x)) < d(x, y). \qquad \square$$

One of powerful tools for fixed point theory is the following result.

Theorem 3.2 (J. Caristi [3]). *Assume (M, d) is a complete metric space and $g : M \to M$ is a mapping. If there exists a lower semicontinuous function $\psi : M \to [0, \infty)$ such that*

$$d(x, g(x)) \leqslant \psi(x) - \psi(g(x)) \quad \text{for any } x \in M,$$

then g has a fixed point.

We can now state our main theorem.

Theorem 3.3. *Let E be a nonempty bounded closed convex subset of X and $T : E \to K(X)$ a nonexpansive mapping. Assume T is weakly inward on E. Then T has a fixed point.*

By combining the idea of the proofs in [1,8,9], we thus first establish the following lemma. However, in applying the lemma, we choose to use the ultrapower technique which seems to be alternative.

Lemma 3.4. *Let E be a nonempty closed subset of X and $T : E \to F(X)$ k-contractive for some $k \in [0, 1)$. Assume T satisfies, for all $x \in E$,*

$$Tx \subset \overline{I_E(x)}. \tag{3.1}$$

Then T has a fixed point.

Proof. Let $M = \{(x, z)\colon z \in Tx,\ x \in E\}$ be the graph of T. Give a metric ρ on M by $\rho((x, z), (u, v)) = \max\{d(x, u), d(z, v)\}$. It is easily seen that (M, ρ) is a complete metric space. Choose $\varepsilon > 0$ so that $\varepsilon + (k + 2\varepsilon)(1 + \varepsilon) < 1$.

Now define $\psi : M \to [0, \infty)$ by $\psi(x, z) = \frac{d(x,z)}{\varepsilon}$. Then ψ is continuous on M. Suppose that T has no fixed points, i.e., $\operatorname{dist}(x, Tx) > 0$ for all $x \in E$. Let $(x, z) \in M$. By (3.1), we can find $z' \in I_E(x)$ satisfying $d(z, z') < \varepsilon \operatorname{dist}(x, Tx)$. Now choose $u \in (x, z'] \cap E$ and write $u = (1-\delta)x \oplus \delta z'$ for some $0 < \delta \leqslant 1$. Note that the number δ varies as a function of x. However, for any such δ, we always have

$$\delta\varepsilon + (1-\delta) + (k+2\varepsilon)\delta(1+\varepsilon) < 1. \tag{3.2}$$

Since T is k-contractive and $d(x,u) > 0$, we can find $v \in Tu$ satisfying

$$d(z,v) \leqslant H(Tx, Tu) + \varepsilon d(x,u) \leqslant (k+\varepsilon)d(x,u).$$

Now we define a mapping $g : M \to M$ by $g(x,z) = (u,v)$ $\forall (x,z) \in M$. We claim that g satisfies

$$\rho\big((x,z), g(x,z)\big) < \psi(x,z) - \psi\big(g(x,z)\big) \quad \forall (x,z) \in M. \tag{3.3}$$

Caristi's theorem then implies that g has a fixed point, which contradicts to the strict inequality (3.3) and the proof is complete.

So it remains to prove (3.3). In fact, it is enough to show that

$$\rho\big((x,z),(u,v)\big) < \frac{1}{\varepsilon}\big(d(x,z) - d(u,v)\big).$$

But $d(z,v) \leqslant d(x,u)$, and we only need to prove that $d(x,u) < \frac{1}{\varepsilon}(d(x,z) - d(u,v))$.

Now,

$$\begin{aligned} d(x,u) &= \delta d(x,z') \leqslant \delta\big(d(x,z) + d(z,z')\big) \leqslant \delta\big(d(x,z) + \varepsilon \operatorname{dist}(x,Tx)\big) \\ &\leqslant \delta\big(d(x,z) + \varepsilon d(x,z)\big) \leqslant \delta(1+\varepsilon)d(x,z). \end{aligned}$$

Therefore

$$d(x,u) \leqslant \delta(1+\varepsilon)d(x,z). \tag{3.4}$$

It follows that

$$d(z,v) \leqslant (k+\varepsilon)d(x,u) \leqslant (k+\varepsilon)\delta(1+\varepsilon)d(x,z).$$

Now we let $y = (1-\delta)x \oplus \delta z$, then

$$\begin{aligned} d(u,v) &\leqslant d(u,y) + d(y,z) + d(z,v) \\ &\leqslant \delta d(z,z') + (1-\delta)d(x,z) + (k+\varepsilon)\delta(1+\varepsilon)d(x,z) \\ &\leqslant \delta\varepsilon \operatorname{dist}(x,Tx) + \big((1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z) \\ &\leqslant \delta\varepsilon d(x,z) + \big((1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z) \\ &\leqslant \big(\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z). \end{aligned}$$

Thus

$$d(u,v) \leqslant \big(\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z). \tag{3.5}$$

Inequalities (3.4), (3.5), and (3.2) imply that

$$\begin{aligned} \varepsilon d(x,u) + d(u,v) &\leqslant \varepsilon\delta(1+\varepsilon)d(x,z) + (\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon))d(x,z) \\ &= (\delta\varepsilon + (1-\delta) + (k+2\varepsilon)\delta(1+\varepsilon))d(x,z) < d(x,z). \end{aligned}$$

Therefore $d(x,u) < \frac{1}{\varepsilon}(d(x,z) - d(u,v))$ as desired. □

We are now ready to present the proof of Theorem 3.3.

Proof of Theorem 3.3. For each integer $n \geqslant 1$, the contraction $T_n : E \to K(X)$ is defined by

$$T_n(x) := \frac{1}{n} p \oplus \left(1 - \frac{1}{n}\right) Tx, \quad x \in E,$$

where $p \in E$ is the existing point satisfying the weakly inward condition (2.2). Weak inwardness of T implies that such T_n satisfies the condition (3.1) in Lemma 3.4 and in turn it guarantees that T_n has a fixed point $x_n \in E$. Clearly,

$$\operatorname{dist}(x_n, Tx_n) \leqslant \frac{1}{n-1} \operatorname{diam}(E) \to 0.$$

Let $\tilde{X}$ be a metric space ultrapower of X and

$$\dot{E} = \{\dot{x} = \widetilde{(x_n)}:\ x_n \equiv x \in E\}.$$

Then $\dot{E}$ is a nonempty closed convex subset of $\tilde{X}$. Since T is compact-valued, we can take $y_n \in Tx_n$ such that

$$d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n), \quad n \geqslant 1.$$

This implies $\widetilde{(x_n)} = \widetilde{(y_n)}$. Since $\dot{E}$ is a closed convex subset of a complete CAT(0) space $\tilde{X}$, $\widetilde{(x_n)}$ has a unique nearest point $\dot{v} \in \dot{E}$, i.e., $\tilde{d}(\widetilde{(x_n)}, \dot{v}) = \operatorname{dist}(\widetilde{(x_n)}, \dot{E})$. As Tv is compact, we can find $v_n \in Tv$ satisfying

$$d(y_n, v_n) = \operatorname{dist}(y_n, Tv) \leqslant H(Tx_n, Tv).$$

It follows from the nonexpansiveness of T that

$$d(y_n, v_n) \leqslant d(x_n, v).$$

This means

$$\tilde{d}\big(\widetilde{(y_n)}, \widetilde{(v_n)}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{v}\big).$$

Since $\widetilde{(x_n)} = \widetilde{(y_n)}$, we have

$$\tilde{d}\big(\widetilde{(x_n)}, \widetilde{(v_n)}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{v}\big). \tag{3.6}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathcal{U}} v_n$. It follows that $\widetilde{(v_n)} = \dot{w}$. This fact and (3.6) imply

$$\tilde{d}\big(\widetilde{(x_n)}, \dot{w}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{v}\big). \tag{3.7}$$

Since $\dot{w} \in \overline{I_{\dot{E}}(\dot{v})}$ as $w \in \overline{I_E(v)}$, (3.7), and Proposition 3.1 then imply that $\dot{w} = \dot{v}$. So $v = w \in Tv$ which then completes the proof. □

As an immediate consequence of Theorem 3.3, we obtain

Corollary 3.5. *Let E be a nonempty bounded closed convex subset of X and $T : E \to K(E)$ a nonexpansive mapping. Then T has a fixed point.*

As we have observed at the end of Definition 2.2, we can restate Theorem 3.3 for $\mathbb{R}$-trees as follows.

Corollary 3.6. *Let X be a complete $\mathbb{R}$-tree, E a nonempty bounded closed convex subset of X, and $T : E \to K(X)$ a nonexpansive mapping. Assume that $Tx \subset \overline{I_E(x)}$ $\forall x \in E$. Then T has a fixed point.*

Finally, as a consequence of Kirk [4, Theorem 4.3] and the idea given in the proof of Theorem 3.3, we can relax the boundedness condition and the compactness of the values of a multivalued self mapping T for $\mathbb{R}$-trees.

Corollary 3.7. *Let (X, d) be a complete $\mathbb{R}$-tree, and suppose E is a closed convex subset of X which does not contain a geodesic ray, and $T : E \to FC(E)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. By [2, p. 176], for each $x \in E$, there exists a unique point $p(x) \in Tx$ such that

$$d\big(x, p(x)\big) = \mathrm{dist}(x, Tx).$$

So we have defined a mapping $p : E \to E$. The nonexpansiveness of T and the convexity of Tx imply that p is a nonexpansive mapping. By [4, Theorem 4.3], there exists $z \in E$ such that $z = p(z) \in Tz$ which then completes the proof. □

4. A common fixed point theorem

We consider in this section a common fixed point of nonexpansive mappings. Let $t : E \to E$ and $T : E \to 2^X \setminus \emptyset$. t and T are said to be commuting if $ty \in Ttx$ $\forall y \in Tx$, $\forall x \in E$. If E is a nonempty bounded closed convex subset of X and t is nonexpansive, we know that $\mathrm{Fix}(t)$ is a nonempty bounded closed convex subset of E (see [7, Theorem 12]).

Theorem 4.1. *Let E be a nonempty bounded closed convex subset of X, and let $t : E \to E$ and $T : E \to KC(X)$ be nonexpansive. Assume that for some $p \in \mathrm{Fix}(t)$,*

$$\alpha p \oplus (1 - \alpha)Tx \text{ convex } \forall x \in E,\ \forall \alpha \in [0, 1]. \tag{4.1}$$

If t and T are commuting, then there exists a point $z \in E$ such that $tz = z \in Tz$.

Proof. Let $A = \mathrm{Fix}(t)$. Since $ty \in Ttx = Tx$ for each $x \in A$ and $y \in Tx$, Tx is invariant under t for each $x \in A$, and again by [7, Theorem 12], $Tx \cap A \neq \emptyset$.

Let $\tilde{X}$ be an ultrapower of X and let $p \in A$ satisfying (4.1). As before we define for each $n \geqslant 1$ the contraction $T_n : A \to KC(X)$ by

$$T_n(x) := \frac{1}{n} p \oplus \left(1 - \frac{1}{n}\right) Tx, \quad x \in A.$$

Convexity of A implies $T_n(x) \cap A \neq \emptyset$. Lemma 4.2 below shows that T_n has a fixed point $x_n \in A$. Let y_n be the unique point in Tx_n such that $d(x_n, y_n) = \mathrm{dist}(x_n, Tx_n)$. Thus $\widetilde{(x_n)} = \widetilde{(y_n)}$ since $\mathrm{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$.

Now,

$$d(x_n, ty_n) = d(tx_n, ty_n) \leqslant d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n).$$

Since $y_n \in Tx_n$, we have $ty_n \in Ttx_n = Tx_n$ and thus the uniqueness of y_n implies that $ty_n = y_n$. So $y_n \in Tx_n \cap A$. Since $\dot{A}$ is a closed convex subset of the complete CAT(0) space $\tilde{X}$, there exists a unique point $\dot{z} \in \dot{A}$ such that

$$\tilde{d}\big(\widetilde{(x_n)}, \dot{z}\big) = \operatorname{dist}\big(\widetilde{(x_n)}, \dot{A}\big).$$

For each n there exists a unique point $z_n \in Tz$ such that

$$d(y_n, z_n) = \operatorname{dist}(y_n, Tz).$$

As before we see that $z_n \in Tz \cap A$. By the compactness of $Tz \cap A$, we can find $w \in Tz \cap A$ such that $\lim_{\mathcal{U}} z_n = w$. It follows that $\widetilde{(z_n)} = \dot{w}$.

Observe that

$$d(y_n, z_n) = \operatorname{dist}(y_n, Tz) \leqslant H(Tx_n, Tz) \leqslant d(x_n, z).$$

Therefore $\tilde{d}(\widetilde{(y_n)}, \widetilde{(z_n)}) \leqslant \tilde{d}(\widetilde{(x_n)}, \dot{z})$. Since $\widetilde{(y_n)} = \widetilde{(x_n)}$ and $\widetilde{(z_n)} = \dot{w}$,

$$\tilde{d}\big(\widetilde{(x_n)}, \dot{w}\big) \leqslant \tilde{d}\big(\widetilde{(x_n)}, \dot{z}\big) = \operatorname{dist}\big(\widetilde{(x_n)}, \dot{A}\big).$$

The uniqueness of $\dot{z}$ implies that $\dot{w} = \dot{z}$. Therefore $tz = z = w \in Tz$ as desired. □

It remains to prove our lemma.

Lemma 4.2. *Let A be as above and $T : A \to FC(X)$ be k-contractive for some $k \in [0, 1)$. Assume that T satisfies, for all $x \in A$,*

$$Tx \cap A \neq \emptyset.$$

Then T has a fixed point.

Proof. The proof is similar to the proof of Lemma 3.4. Let $M = \{(x, z) : z \in Tx \cap A, x \in A\}$ and define a metric ρ on M by $\rho((x, z), (u, v)) = \max\{d(x, u), d(z, v)\}$. Again (M, ρ) is a complete metric space. Choose $\varepsilon > 0$ so that $\varepsilon + k < 1$.

Define $\psi : M \to [0, \infty)$ by $\psi(x, z) = \frac{d(x,z)}{\varepsilon}$. Suppose that $x \neq z$ for all $(x, z) \in M$. Since Tz is a closed convex subset of X, there exists a unique point $v \in Tz$ such that

$$d(z, v) = \operatorname{dist}(z, Tz).$$

Bearing in mind that $A = \operatorname{Fix}(t)$, thus by the commuting assumption and the uniqueness of v, we have $v \in Tz \cap A$.

Now we define a mapping $g : M \to M$ by $g(x, z) = (z, v)$ for each $(x, z) \in M$. We claim that g satisfies

$$\rho\big((x, z), g(x, z)\big) < \psi(x, z) - \psi\big(g(x, z)\big) \quad \forall (x, z) \in M. \tag{4.2}$$

Again by applying the Caristi's theorem we obtain a contradiction. Thus T has a fixed point.

So it remains to prove (4.2). From the fact that $d(z, v) = \mathrm{dist}(z, Tz) \leqslant H(Tx, Tz) \leqslant kd(x, z)$, we have

$$\varepsilon d(x, z) + d(z, v) \leqslant \varepsilon d(x, z) + kd(x, z) = (\varepsilon + k)d(x, z) < d(x, z).$$

Therefore $\rho((x, z), (z, v)) < \frac{1}{\varepsilon}(d(x, z) - d(z, v))$, and (4.2) is verified. □

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The Domínguez–Lorenzo condition and multivalued nonexpansive mappings☆

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Abstract

Let E be a nonempty bounded closed convex separable subset of a reflexive Banach space X which satisfies the Domínguez–Lorenzo condition, i.e., an inequality concerning the asymptotic radius of a sequence and the Chebyshev radius of its asymptotic center. We prove that a multivalued nonexpansive mapping $T : E \to 2^X$ which is compact convex valued and such that $T(E)$ is bounded and satisfies an inwardness condition has a fixed point. As a consequence, we obtain a fixed-point theorem for multivalued nonexpansive mappings in uniformly nonsquare Banach spaces which satisfy the property WORTH, extending a known result for the case of nonexpansive single-valued mappings. We also prove a common fixed point theorem for two nonexpansive commuting mappings $t : E \to E$ and $T : E \to KC(E)$ (where $KC(E)$ denotes the class of all compact convex subsets of E) when X is a uniformly convex Banach space.

Keywords: Multivalued nonexpansive mapping; Inwardness condition; Uniform convexity; Non-strict opial condition; Property WORTH; James constant; Uniform nonsquareness

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1. Introduction

One of the most celebrated results about multivalued mappings was given by T.C. Lim [17] in 1974. By using Edelstein's method of asymptotic centers, he proved that every multivalued nonexpansive self-mapping $T : E \to K(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. In 1990, W.A. Kirk and S. Massa [16] proved that if a nonempty bounded closed convex subset E of a Banach space X has a property that the asymptotic center in E of each bounded sequence of X is nonempty and compact, then every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point. In 2001, H.K. Xu [23] extended Kirk and Massa's theorem to a non-self-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

Recently, Domínguez and Lorenzo [10] proved that every nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they give an affirmative answer to problem 6 in [22] which states that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X. Furthermore, they [9] proved that if $T : E \to KC(X)$ is a nonexpansive and $1 - \chi$-contractive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition where E is a nonempty bounded closed convex separable subset of a Banach space X with $\varepsilon_\beta(X) < 1$, then T has a fixed point.

By investigating the proofs in [9] and [10], we observe that the main tool that is used in their proofs is a relationship between the Chebyshev radius of the asymptotic center of a bounded sequence in E and the modulus of noncompact convexity of a Banach space associated with the measure of noncompactness. In this paper, we define the Domínguez–Lorenzo condition and prove that every reflexive Banach space X satisfying the Domínguez–Lorenzo condition and every nonempty bounded closed convex separable subset E of X, every nonexpansive and $1 - \chi$-contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set, and which satisfies the inwardness condition has a fixed point. The main idea of the proof comes from the proofs of Theorems 3.4 and 3.6 in [9]. We also prove that a uniformly nonsquare Banach space X satisfying property WORTH is one of the examples of Banach spaces that satisfy the Domínguez–Lorenzo condition. Moreover, we show that every Banach space which satisfies the Domínguez–Lorenzo condition has a weak normal structure.

Finally, we use a theorem of Deimling [7] to obtain a common fixed point for nonexpansive commuting mappings $t : E \to E$ and $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space.

2. Preliminaries

Let X be a Banach space and E a nonempty subset of X. We shall denote the family of nonempty bounded closed subsets of E by $FB(E)$, the family of nonempty compact subsets of E by $K(E)$, the family of nonempty closed convex subsets of E by $FC(E)$ and the family of nonempty compact convex subsets of E by $KC(E)$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\}, \quad A, B \in FB(X),$$

where $\operatorname{dist}(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B. A multivalued mapping $T : E \to F(X)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leqslant k\|x - y\|, \quad x, y \in E. \tag{1}$$

In this case, we also say that T is k-contractive.

If (1) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$.

Recall that the Kuratowski, separation, and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are, respectively, defined as the numbers:

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameters } \leqslant d\},$$

$$\beta(B) = \sup\{\varepsilon > 0 : \text{ there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \geqslant \varepsilon\},$$

where $sep(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\}$,

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radii } \leqslant d\}.$$

A multivalued mapping $T : E \to 2^X$ is called ϕ-condensing (resp. $1 - \phi$-contractive) where ϕ is a measure of noncompactness if, for each bounded subset B of E with $\phi(B) > 0$, there holds the inequality

$$\phi(T(B)) < \phi(B) (\text{resp. } \phi(T(B)) \leqslant \phi(B)).$$

Here $T(B) = \cup_{x\in B} Tx$.

Before stating our main theorem we need the following results.

Definition 2.1. Let X be a Banach space and $\phi = \alpha$, β or χ. The modulus of noncompact convexity associated with ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = \inf\{1 - \operatorname{dist}(0, A) : A \subset B_X \text{ is convex}, \ \phi(A) \geqslant \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geqslant 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The relationships among the different moduli are

$$\Delta_{X,\alpha}(\varepsilon) \leqslant \Delta_{X,\beta}(\varepsilon) \leqslant \Delta_{X,\chi}(\varepsilon),$$

and consequently,

$$\varepsilon_\alpha(X) \geqslant \varepsilon_\beta(X) \geqslant \varepsilon_\chi(X).$$

See [2] for all these and more details.

Definition 2.2. (a) X is said to satisfy property WORTH [20] if for any $x \in X$ and any weakly null sequence $\{x_n\}$ in X,

$$\limsup_{n\to\infty} \|x_n + x\| = \limsup_{n\to\infty} \|x_n - x\|.$$

(b) X is said to satisfy the Opial condition [18] if, whenever a sequence $\{x_n\}$ in X converges weakly to x, then for $y \neq x$,

$$\limsup_{n} \|x_n - x\| < \limsup_{n} \|x_n - y\|.$$

If the inequality is non-strict, we say that X satisfies the non-strict Opial condition.

It is known that if X satisfies property WORTH, then X satisfies the non-strict Opial condition [12].

Definition 2.3. Let E be a nonempty closed subset of a Banach space X. The inward set of E at $x \in E$ is given by

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geqslant 1,\ y \in E\}.$$

In case E is a nonempty closed convex subset of a Banach space X, we have

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geqslant 0,\ y \in E\}.$$

A multivalued mapping $T : E \to 2^X$ is said to be inward (resp. weakly inward) on E if

$$Tx \subset I_E(x) \text{ (resp. } Tx \subset \overline{I_E(x)}) \text{ for all } x \in E.$$

Our proofs heavily rely on the following result.

Theorem 2.4 (*Deimling [7]*). *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \to FC(X)$ an upper semicontinuous χ-condensing mapping. Assume $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$. Then T has a fixed point.*

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded closed convex subset of X and $\{x_n\}$ a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r(E,\{x_n\}) = \inf\left\{\limsup_n \|x_n - x\| : x \in E\right\},$$

$$A(E,\{x_n\}) = \left\{x \in E : \limsup_n \|x_n - x\| = r(E,\{x_n\})\right\}.$$

It is known that $A(E,\{x_n\})$ is a nonempty weakly compact convex set as E is [14].

Definition 2.5. Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E,\{x_n\}) = r(E,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; further, $\{x_n\}$ is called asymptotically uniform relative to E if $A(E,\{x_n\}) = A(E,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.6 (*Goebel [13], Lim [17]*). *Let $\{x_n\}$ and E be as above. Then*

(i) *there always exists a subsequence of $\{x_n\}$ which is regular relative to E; and*
(ii) *if E is seperable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E.*

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\{r_x(C) : x \in E\},$$

where $r_x(C) = \sup\{\|x - y\| : y \in C\}$.

Theorem 2.7 (*Domínguez [8]*). *Let E be a closed convex subset of a reflexive Banach space X and let $\{x_n\}$ be a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E,\{x_n\})) \leqslant (1 - \Delta_{X,\beta}(1^-))r(E,\{x_n\}).$$

Moreover, if X satisfies the non-strict Opial condition, then

$$r_E(A(E,\{x_n\})) \leqslant (1 - \Delta_{X,\chi}(1^-))r(E,\{x_n\}).$$

Using Theorem 2.7 as the main tool, Domínguez and Lorenzo [9] proved the following theorem:

Theorem 2.8 (*Dominguez [9, Theorem 3.6]*). *Let X be a Banach space with $\varepsilon_\beta(X) < 1$. Assume that $T : E \to KC(X)$ is a nonexpansive and $1-\chi$-contractive mapping such that $T(E)$ is a bounded set, and which satisfies the inwardness condition where E is a nonempty bounded closed convex separable subset of X. Then T has a fixed point.*

Moreover, they [10] used the same tool to solve the open problem in [22] on the existence of a fixed point of a multivalued nonexpansive self-mapping $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X.

We now present a formulation of an ultrapower of Banach spaces.

Let $\mathscr{U}$ be a free ultrafilter on the set of natural numbers. Consider the closed linear subspace of $l_\infty(X)$:

$$\mathscr{N} = \left\{ \{x_n\} \in l_\infty(X) : \lim_{\mathscr{U}} \|x_n\| = 0 \right\}.$$

The ultrapower $\widetilde{X}$ of the space X is defined as the quotient space $l_\infty(X)/\mathscr{N}$. Given an element $x = \{x_n\} \in l_\infty(X)$, $\widetilde{x}$ stands for the equivalence class of x. The quotient norm in $\widetilde{X}$ satisfies $\|\widetilde{x}\| = \lim_{\mathscr{U}} \|x_n\|$. For more details on the construction of an ultrapower of a Banach space X, see [1] and [19].

3. Fixed-point theorems

Definition 3.1. A Banach space X is said to satisfy the Domínguez–Lorenzo condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E(A(E, \{x_n\})) \leqslant \lambda r(E, \{x_n\}). \tag{2}$$

We are going to prove that every Banach space satisfying the Domínguez–Lorenzo condition possesses a weak normal structure. A Banach space X is said to have a weak normal structure if any weakly compact convex subset E of X for which $\operatorname{diam}(E) > 0$ contains a point x_0 for which

$$r_{x_0}(E) < \operatorname{diam}(E).$$

Theorem 3.2. *Let X be a Banach space satisfying the Domínguez–Lorenzo condition. Then X has a weak normal structure.*

Proof. Suppose on the contrary that X does not have a weak normal structure. Thus, there exists a weakly null sequence $\{x_n\}$ in B_X such that for $C := \overline{co}\{x_n : n \geqslant 1\}$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam}(C) = 1 \text{ for all } x \in C$$

(cf. [21]). By passing through a subsequence, we may assume that $\{x_n\}$ is regular. It is easy to see that $r(C, \{x_n\}) = 1$, $A(C, \{x_n\}) = C$, and $r_C(A(C, \{x_n\})) = r_C(C) = 1$. Since X satisfies the Domínguez–Lorenzo condition with a corresponding $\lambda \in [0, 1)$, it must be the case that

$$1 = r_C(C) \leqslant \lambda r(C, \{x_n\}) < 1.$$

This leads to a contradiction. □

In view of the above theorem and the well-known Kirk's fixed-point theorem [15], we can conclude that every Banach space X which satisfies the Domínguez–Lorenzo condition has a fixed-point property, i.e., for every weakly compact convex subset E of X, every nonexpansive mapping $T : E \to E$ has a fixed point. Moreover, the next theorem shows that every reflexive Banach space that satisfies the Domínguez–Lorenzo condition has a fixed-point property for certain multivalued nonexpansive mappings.

Theorem 3.3. *Let X be a reflexive Banach space satisfying the Domínguez–Lorenzo condition and let E be a bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi$-contractive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition*:

$$Tx \subset I_E(x) \quad \textit{for all } x \in E,$$

then T has a fixed point.

Proof. The main idea of the proof follows from the proofs of Theorems 3.4 and 3.6 in [9]. So here we only give a sketch of the proof. First we obtain a sequence of approximate fixed points $\{x_n\}$ of T in E. By the boundedness of $\{x_n\}$, we can assume that $\{x_n\}$ is regular relative to E. Since X satisfies the Domínguez–Lorenzo condition, we obtain

$$r_E(A) \leqslant \lambda r(E, \{x_n\})$$

for some $\lambda \in [0, 1)$, where $A = A(E, \{x_n\})$.

We can show that the mapping $T : A \to KC(X)$ is nonexpansive, $1-\chi$-contractive, and satisfies the condition

$$Tx \cap I_A(x) \neq \emptyset \quad \text{for all } x \in A.$$

Fix $x_0 \in A$, define $T_n : A \to KC(X)$ by

$$T_n x = \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) Tx, \quad x \in A.$$

It is easy to see that T_n is χ-condensing and

$$T_n x \cap I_A(x) \neq \emptyset \quad \text{for all } x \in A.$$

Hence by Theorem 2.4, T_n has a fixed point. Consequently, we obtain a sequence $\{x_n^1\}$ in A satisfying $\lim_n \operatorname{dist}(x_n^1, Tx_n^1) = 0$. Now we can proceed with the proof as in the proofs of Theorems 3.4 and 3.6 in [9] to obtain a fixed point. □

From Theorem 2.7 it can be seen that every Banach space X with $\varepsilon_\beta(X) < 1$ satisfies the Domínguez–Lorenzo condition. We now present some other Banach spaces which satisfy the Domínguez–Lorenzo condition. Here we consider the James constant or the nonsquare constant $J(X)$.

For a Banach space X, the James constant, or the nonsquare constant is defined by Gao and Lau [11] as

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}.$$

Clearly, X is uniformly nonsquare if and only if $J(X) < 2$.

Theorem 3.4. *Let X be a Banach space satisfying property WORTH and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E,\{x_n\})) \leqslant \frac{J(X)}{2} r(E,\{x_n\}).$$

Proof. Denote $r = r(E,\{x_n\})$ and $A = A(E,\{x_n\})$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can assume, by passing through a subsequence if necessary, that x_n converges weakly to some element in E, say x. It should be noted that passing through a subsequence of $\{x_n\}$ does not have any effect on the asymptotic radius of the whole sequence $\{x_n\}$ because $\{x_n\}$ is regular. Let us observe here that for any subsequence $\{y_n\}$ of $\{x_n\}$, $r_E(A(E,\{x_n\})) \leqslant r_E(A(E,\{y_n\}))$. This observation will be needed at the end of the proof. Since X satisfies property WORTH, it satisfies the non-strict Opial condition, and thus it must be the case that $x \in A$, that is,

$$\limsup_{n\to\infty} \|x_n - x\| = r. \tag{3}$$

Now let $z \in A$. Thus $\limsup_{n\to\infty}\|x_n - z\| = r$. By regularity of $\{x_n\}$, we can choose a subsequence $\{x_{n'}\}$ of $\{x_n\}$ so that $\lim_{n'\to\infty}\|x_{n'} - x\| = r = \lim_{n'\to\infty}\|x_{n'} - z\|$. Property WORTH and the fact that $x_{n'} - x \xrightarrow{w} 0$ yield the following:

$$\begin{aligned} r &= \lim_{n'\to\infty} \|x_{n'} - z\| \\ &= \lim_{n'\to\infty} \|(x_{n'} - x) + (x - z)\| \\ &= \lim_{n'\to\infty} \|(x_{n'} - x) - (x - z)\| \\ &= \lim_{n'\to\infty} \|x_{n'} - 2x + z\|. \end{aligned}$$

Thus we have

$$\lim_{n'\to\infty} \left\|\frac{x_{n'} - z}{r}\right\| = 1 = \lim_{n'\to\infty} \left\|\frac{x_{n'} - 2x + z}{r}\right\|. \tag{4}$$

Let us consider an ultrapower $\widetilde{X}$ of X. Put

$$\tilde{u} = \frac{1}{r}\{x_{n'} - z\}_{\mathscr{U}} \quad \text{and} \quad \tilde{v} = \frac{1}{r}\{x_{n'} - 2x + z\}_{\mathscr{U}}.$$

By (4) we know that $\tilde{u}, \tilde{v} \in S_{\widetilde{X}}$. We see that

$$\begin{aligned} \|\tilde{u} + \tilde{v}\| &= \lim_{\mathscr{U}} \left\|\frac{1}{r}(x_{n'} - z) + \frac{1}{r}(x_{n'} - 2x + z)\right\| \\ &= \lim_{\mathscr{U}} \left\|\frac{2}{r}(x_{n'} - x)\right\| \\ &= \frac{2}{r} \lim_{\mathscr{U}} \|(x_{n'} - x)\| \\ &= \frac{2}{r}(r) = 2. \end{aligned}$$

On the other hand,

$$\begin{aligned}\|\tilde{u}-\tilde{v}\| &= \lim_{\mathcal{U}} \left\| \frac{1}{r}(x_{n'}-z) - \frac{1}{r}(x_{n'}-2x+z) \right\| \\ &= \frac{2}{r}\|x-z\|.\end{aligned}$$

Thus

$$\begin{aligned}J(\widetilde{X}) &\geqslant \|\tilde{u}+\tilde{v}\| \wedge \|\tilde{u}-\tilde{v}\| \\ &= 2 \wedge \frac{2}{r}\|x-z\| \\ &= \frac{2}{r}\|x-z\|.\end{aligned}$$

Since the James constants of X and of $\widetilde{X}$ are the same, we obtain

$$J(X) \geqslant \frac{2}{r}\|x-z\|.$$

This holds for arbitrary $z \in A$. Hence we have

$$r_x(A) \leqslant \frac{J(X)}{2} r,$$

and therefore, by the previous observation, $r_E(A) \leqslant \frac{J(X)}{2} r$. □

From the above theorem we immediately have

Corollary 3.5. *Let X be a uniformly nonsquare Banach space satisfying property WORTH. Then X satisfies the Domínguez–Lorenzo condition.*

Proof. Uniform nonsquareness of X is equivalent to $J(X) < 2$. Put $\lambda = \frac{J(X)}{2}$. Then $\lambda < 1$ and by Theorem 3.4 the result follows. □

Theorem 3.3 and Corollary 3.5 give

Corollary 3.6. *Let X be a uniformly nonsquare Banach space satisfying property WORTH and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition:*

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then T has a fixed point.

Proof. By Corollary 3.5, X satisfies the Domínguez–Lorenzo condition. It is known that uniform nonsquareness implies reflexivity of X. Since X has a non-strict opial condition, we can conclude that the nonexpansive mapping $T : E \to K(X)$ with a bounded range is $1-\chi$-contractive (see [9]). Now Theorem 3.3 can be applied to obtain a fixed point. □

Questions. (1) It has been shown in [4, Theorem 3.1] that a Banach space X has a uniform normal structure whenever $J(X) < \frac{1+\sqrt{5}}{2}$. It is natural to ask if the condition of being uniformly nonsquare and property WORTH can be replaced by the condition "$J(X) < \frac{1+\sqrt{5}}{2}$" or some other upper bounds.

(2) A similar question about the Jordan–von Neumann constants can be asked in the sense of (1). Here we ask if we can replace the condition of being uniform nonsquareness and having property WORTH by the condition $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ or some other upper bounds. Note that it has been shown in [5, Theorem 3.16] that a Banach space X has a uniform normal structure whenever $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$.

4. The common fixed points in uniformly convex Banach spaces

In this section, we extend a result of common fixed points for CAT(0) spaces [6, Theorem 4.1] to uniformly convex Banach spaces.

Definition 4.1. Let E be a nonempty bounded closed convex subset of a Banach space X, $t : E \to X$, and $T : E \to FB(X)$. Then t and T are said to be commuting if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds

$$tx \in Tty.$$

Theorem 4.2. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $t : E \to E$, $T : E \to KC(E)$ a single-valued and a multi-valued nonexpansive mapping, respectively. Assume that t and T are commuting. Then t and T have a common fixed point, i.e., there exists a point x in E such that $x = tx \in Tx$.*

Proof. It is known that the fixed point set of t, denoted by $\mathrm{Fix}(t)$, is nonempty, closed, and convex. Let $x \in \mathrm{Fix}(t)$. Since t and T are commuting, we have $ty \in Tx$ for each $y \in Tx$. We see that, for $x \in \mathrm{Fix}(t)$, $Tx \cap \mathrm{Fix}(t) \neq \emptyset$. For a fixed element $x_0 \in \mathrm{Fix}(t)$, define a contraction $T_n : \mathrm{Fix}(t) \to KC(E)$ by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)T(x), \quad x \in \mathrm{Fix}(t).$$

It is easy to see that for each $x \in \mathrm{Fix}(t)$, $T_n x \cap \mathrm{Fix}(t) \neq \emptyset$ as T does.

Since $\mathrm{Fix}(t)$ is a nonexpansive retract of E [3], we can show that $T_n : \mathrm{Fix}(t) \to KC(E)$ is χ-condensing. Indeed, let B be a bounded subset of $\mathrm{Fix}(t)$ and $\chi(B) > 0$. Given $d > 0$ be such that

$$B \subset \bigcup_{i=1}^{n} B(x_i, d), \quad x_i \in E.$$

Let R be a nonexpansive retraction of E onto $\mathrm{Fix}(t)$.

For each $a \in B(x_i, d) \cap B$, we have

$$\|Rx_i - a\| = \|Rx_i - Ra\| \leqslant \|x_i - a\| \leqslant d.$$

Therefore $B(x_i, d) \cap B \subset B(Rx_i, d)$ for each $i \in \{1, \dots, n\}$, and hence

$$B \subset \bigcup_{i=1}^{n} B(Rx_i, d).$$

Since T_n is $(1 - \frac{1}{n})$-contractive,

$$T_n(B) \subset \bigcup_{i=1}^{n} \left(T_n Rx_i + \left(1 - \frac{1}{n}\right) dB(0, 1)\right).$$

Thus

$$\chi(T_n(B)) \leqslant \left(1 - \frac{1}{n}\right) \chi(B) < \chi(B),$$

and T_n is χ-condensing. Now we can apply Theorem 2.4 to conclude that T_n has a fixed point, say x_n. Moreover, we can show that

$$\mathrm{dist}(x_n, Tx_n) \to 0.$$

Let $\widetilde{X}$ be a Banach space ultrapower of X and

$$\dot{\mathrm{Fix}}(t) = \{\dot{x} = \widetilde{(x_n)} : x_n \equiv x \in \mathrm{Fix}(t)\}.$$

Then $\dot{\mathrm{Fix}}(t)$ is a nonempty closed convex subset of $\widetilde{X}$. Now, for each $n \in \mathbb{N}$, let y_n be the unique nearest point of x_n in Tx_n, i.e., $\|x_n - y_n\| = \mathrm{dist}(x_n, Tx_n)$. Consequently, $\widetilde{(x_n)} = \widetilde{(y_n)}$. Since t is nonexpansive and $x_n \in \mathrm{Fix}(t)$, we have

$$\|x_n - ty_n\| = \|tx_n - ty_n\| \leqslant \|x_n - y_n\|$$

for each $n \in \mathbb{N}$. Since $ty_n \in Tx_n$, we have $y_n = ty_n \in \mathrm{Fix}(t)$ for each $n \in \mathbb{N}$. Since $\dot{\mathrm{Fix}}(t)$ is a closed convex subset of a uniformly convex Banach space $\tilde{X}$, $\widetilde{(x_n)}$ has a unique nearest point $\dot{v} \in \dot{\mathrm{Fix}}(t)$, i.e., $\|\widetilde{(x_n)} - \dot{v}\| = \mathrm{dist}(\widetilde{(x_n)}, \dot{\mathrm{Fix}}(t))$. As Tv is closed and convex, we can find $v_n \in Tv$ satisfying

$$\|y_n - v_n\| = \mathrm{dist}(y_n, Tv) \leqslant H(Tx_n, Tv).$$

We note here that $v_n \in \mathrm{Fix}(t)$ for each n. It follows from the nonexpansiveness of T that

$$\|y_n - v_n\| \leqslant \|x_n - v\|.$$

This means

$$\|\widetilde{(y_n)} - \widetilde{(v_n)}\| \leqslant \|\widetilde{(x_n)} - \dot{v}\|.$$

Since $\widetilde{(x_n)} = \widetilde{(y_n)}$, we have

$$\|\widetilde{(x_n)} - \widetilde{(v_n)}\| \leqslant \|\widetilde{(x_n)} - \dot{v}\|. \tag{5}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathscr{U}} v_n$. It follows that $\widetilde{(v_n)} = \dot{w}$. This fact and (5) imply that

$$\|\widetilde{(x_n)} - \dot{w}\| \leqslant \|\widetilde{(x_n)} - \dot{v}\|. \tag{6}$$

Moreover, $w \in \mathrm{Fix}(t)$ and then $\dot{w} \in \dot{\mathrm{Fix}}(t)$. Hence $\dot{w} = \dot{v}$ and so $v = w \in Tv$ which then completes the proof. □

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Appendix 6: Jordan-von Neumann constant and fixed points for multivalued nonexpansive Mappings, J. Math. Anal. Appl. 320 (2006), 916-927.

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The Jordan–von Neumann constants and fixed points for multivalued nonexpansive mappings ☆

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Abstract

The purpose of this paper is to study the existence of fixed points for nonexpansive multivalued mappings in a particular class of Banach spaces. Furthermore, we demonstrate a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ of a Banach space X. Using this fact, we prove that if $C_{NJ}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty weakly compact convex subset of a Banach space X, and $KC(E)$ is the class of all nonempty compact convex subsets of E.

Keywords: Multivalued nonexpansive mapping; Weakly convergent sequence coefficient; Jordan–von Neumann constant; Normal structure; Regular asymptotically uniform sequence

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1. Introduction

In 1969, Nadler [18] established the multivalued version of Banach's contraction principle. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. In 1974, Lim [17], using Edelstein's method of asymptotic centers, proved the existence of a fixed point for a multivalued nonexpansive self-mapping $T: E \to K(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space. In 1990, Kirk and Massa [15] extended Lim's theorem. They proved that every multivalued nonexpansive self-mapping $T: E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X for which the asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, Xu [22] extended Kirk–Massa's theorem to nonself-mapping $T: E \to KC(X)$ which satisfies the inwardness condition.

In 2004, Domínguez and Lorenzo [10] proved that every multivalued nonexpansive mapping $T: E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they can give an affirmative answer of a problem in [21] proving that every nonexpansive self-mapping $T: E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space. Recently, Dhompongsa et al. [5], gave an existence of a fixed point for a multivalued nonexpansive and $(1-\chi)$-contractive mapping $T: E \to KC(X)$ such that $T(E)$ is a bounded set and which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of a reflexive Banach space which satisfies the Domínguez–Lorenzo condition, i.e., an inequality concerning the asymptotic radius and the Chebyshev radius of the asymptotic center for some types of sequences. Consequently, they could show that if X is a uniformly nonsquare Banach space satisfying property WORTH and $T: E \to KC(X)$ is a nonexpansive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of X, then T has a fixed point. Furthermore, they also ask: Does $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ imply the existence of a fixed point for multivalued nonexpansive mappings?

In this paper, we organize as follows. We define a property for Banach spaces which we call property (D) (see definition in Section 3), which is weaker than the Domínguez–Lorenzo condition and stronger than weak normal structure and we prove that if X is a Banach space satisfying property (D) and E is a nonempty weakly compact convex subset of X, then every nonexpansive mapping $T: E \to KC(E)$ has a fixed point. Then we state a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ of a Banach space X. Finally, using this fact, we prove that if $C_{NJ}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T: E \to KC(E)$ has a fixed point. In particular, we give a partial answer to the question which has been asked in [5].

2. Preliminaries

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) := \max\Big\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\Big\}, \quad A, B \in FB(X),$$

where $\operatorname{dist}(a,B) := \inf\{\|a-b\|: b \in B\}$ is the distance from the point a to the subset B. A multivalued mapping $T: E \to FB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \leqslant k\|x-y\|, \quad x, y \in E. \tag{1}$$

If (1) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$.

Throughout the paper we let X^* stand for the dual space of a Banach space X. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded subset of X. The number $r(A) := \inf\{\sup_{y\in A}\|x-y\|: x \in A\}$ is called the Chebyshev radius of A. The number $\operatorname{diam}(A) := \sup\{\|x-y\|: x, y \in A\}$ is called the diameter of A. A Banach space X has normal structure (respectively weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

for every bounded closed (respectively weakly compact) convex subset A of X with $\operatorname{diam}(A) > 0$. X is said to have uniform normal structure (respectively weak uniform normal structure) if

$$\inf\left\{\frac{\operatorname{diam}(A)}{r(A)}\right\} > 1,$$

where the infimum is taken over all bounded closed (respectively weakly compact) convex subsets A of X with $\operatorname{diam}(A) > 0$. The weakly convergent sequence coefficient $WCS(X)$ [3] of X is the number

$$WCS(X) := \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) := \limsup_n\{\|x_i - x_j\|: i, j \geqslant n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) := \inf\{\limsup_n \|x_n - y\|: y \in \overline{\operatorname{co}}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$.

Some equivalent definitions of the weakly convergent sequence coefficient can be found in [2, p. 120] as follows:

$$WCS(X) = \inf\left\{\frac{\lim_{n,m:\,n\neq m}\|x_n - x_m\|}{\lim_{n\to\infty}\|x_n\|}: \{x_n\} \text{ converges weakly to zero, } \lim_{n\to\infty}\|x_n\| \text{ and } \lim_{n,m:\,n\neq m}\|x_n - x_m\| \text{ exist}\right\}$$

and

$$WCS(X) = \inf\Big\{ \lim_{n,m:\, n\neq m} \|x_n - x_m\|:\ \{x_n\} \text{ converges weakly to zero}, \\ \|x_n\| = 1 \text{ and } \lim_{n,m:\, n\neq m} \|x_n - x_m\| \text{ exists}\Big\}.$$

It is easy to see, from the definition of $WCS(X)$, that $1 \leqslant WCS(X) \leqslant 2$, and it is known that $WCS(X) > 1$ implies X has weak uniform normal structure [3].

For a Banach space X, the Jordan–von Neumann constant $C_{NJ}(X)$ of X, introduced by Clarkson [4], is defined by

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2}:\ x, y \in X \text{ not both zero}\right\}.$$

The constant $R(a, X)$, which is a generalized Garcia-Falset coefficient [12], is introduced by Domínguez [7]: For a given nonnegative real number a,

$$R(a, X) := \sup\Big\{ \liminf_n \|x + x_n\| \Big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leqslant a$ and all weakly null sequences $\{x_n\}$ in the unit ball of X such that $\lim_{n,m:\, n\neq m} \|x_n - x_m\| \leqslant 1$.

A relationship between the constant $R(1, X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ can be found in [6]:

$$R(1, X) \leqslant \sqrt{2C_{NJ}(X)}.$$

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded closed subset of X and $\{x_n\}$ a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r\big(E, \{x_n\}\big) = \inf\Big\{ \limsup_{n\to\infty} \|x_n - x\|:\ x \in E\Big\},$$

$$A\big(E, \{x_n\}\big) = \Big\{ x \in E:\ \limsup_{n\to\infty} \|x_n - x\| = r\big(E, \{x_n\}\big)\Big\}.$$

It is known that $A(E, \{x_n\})$ is a nonempty weakly compact convex set whenever E is [14].

Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{x_n\}$ is called asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform relative to E if $\{x_n\}$ is regular and asymptotically uniform relative to E.

Lemma 2.1. (Goebel [13], Lim [17]) *Let $\{x_n\}$ and E be as above. Then*

(i) *there always exists a subsequence of $\{x_n\}$ which is regular relative to E;*
(ii) *if E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E.*

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\{r_x(C) \colon x \in E\},$$

where $r_x(C) = \sup\{\|x - y\| \colon y \in C\}$.

A last concept which we need to mention is the concept of ultrapowers of Banach spaces. Ultrapowers are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I \colon x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A \colon A \subset I,\ i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the following facts:

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and
(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\prod_{i\in I} X_i$ equipped with the norm $\|\{x_i\}\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \Big\{\{x_i\} \in l_\infty(I, X_i) \colon \lim_{\mathcal{U}} \|x_i\| = 0\Big\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $\{x_i\}_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) and the definition of the quotient norm that

$$\big\|\{x_i\}_{\mathcal{U}}\big\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an ultrapower of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see Aksoy and Khamsi [1] or Sims [19]).

3. Main results

Definition 3.1. A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform relative to E we have

$$r\big(E, \{y_n\}\big) \leqslant \lambda r\big(E, \{x_n\}\big). \tag{2}$$

The Domínguez–Lorenzo condition introduced in [5] is defined as follows:

Definition 3.2. A Banach space X is said to satisfy the Domínguez–Lorenzo condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leqslant \lambda r\big(E, \{x_n\}\big).$$

It is clear from the definition that property (D) is weaker than the Domínguez–Lorenzo condition. In fact, property (D) is strictly weaker than the Domínguez–Lorenzo condition as shown in [8]. The next result shows that property (D) is stronger than weak normal structure.

Theorem 3.3. *Let X be a Banach space satisfying property* (D). *Then X has weak normal structure.*

Proof. Suppose on the contrary that there exists a weakly null sequence $\{x_n\} \subset B_X$ such that $\lim_{n\to\infty} \|x_n - x\| = 1$ for all $x \in C = \overline{\mathrm{co}}(\{x_n\})$ (see [20]). By passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to C. We see that $r(C, \{x_n\}) = 1$ and $A(C, \{x_n\}) = C$. Moreover, $\{x_n\}$ is asymptotically uniform relative to C. Indeed, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$; we have

$$A\big(C, \{x_{n_k}\}\big) = \Big\{x \in C\colon \limsup_{k\to\infty} \|x_{n_k} - x\| = r\big(C, \{x_{n_k}\}\big) = 1\Big\} = C.$$

Since $\{x_n\} \subset C = A(C, \{x_n\})$ and X satisfies property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r\big(C, \{x_n\}\big) \leqslant \lambda r\big(C, \{x_n\}\big)$$

which leads to a contradiction. □

The following results will be very useful in order to prove our main theorem.

Theorem 3.4. (Domínguez and Lorenzo [9]) *Let E be a nonempty weakly compact separable subset of a Banach space X, $T : E \to K(E)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in E such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \quad \forall x \in A := A\big(E, \{z_n\}\big).$$

Theorem 3.5. (Domínguez and Lorenzo [10]) *Let E be a nonempty weakly compact convex separable subset of a Banach space X. Assume that $T : E \to KC(E)$ is a contraction. If A is a closed convex subset of E such that $Tx \cap A \neq \emptyset$ for all $x \in A$, then T has a fixed point in A.*

We can now state our main theorem.

Theorem 3.6. *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies property* (D). *Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.*

Proof. The first part of the proof is similar to the proof of Theorem 4.2 in [9]. Therefore, we only sketch this part of the proof. From [16] we can assume that E is separable. Fix $z_0 \in E$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n}z_0 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E.$$

By Nadler's theorem [18], for any $n \in \mathbb{N}$, T_n has a fixed point, say x_n^1. It is easy to prove that $\lim_{n\to\infty} \operatorname{dist}(x_n^1, Tx_n^1) = 0$. By Lemma 2.1, we can assume that sequence $\{x_n^1\} \subset E$ is a regular asymptotically uniform relative to E. Denote $A_1 = A(E, \{x_n^1\})$. By Theorem 3.4 we can assume that $Tx \cap A_1 \neq \emptyset$ for all $x \in A_1$. Fix $z_1 \in A_1$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n}z_1 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E.$$

Convexity of A_1 implies $T_n(x) \cap A_1 \neq \emptyset$ for all $x \in A_1$. By Theorem 3.5, T_n has a fixed point in A_1, say x_n^2. Consequently, we can get a sequence $\{x_n^2\} \subset A_1$ which is regular asymptotically uniform relative to E and $\lim_{n\to\infty} \operatorname{dist}(x_n^2, Tx_n^2) = 0$. Since X satisfies the property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r(E, \{x_n^2\}) \leqslant \lambda r(E, \{x_n^1\}).$$

By induction, we can find a sequence $\{x_n^k\} \subset A_{k-1} = A(E, \{x_n^{k-1}\})$ which is regular asymptotically uniform relative to E,

$$\lim_{n\to\infty} \operatorname{dist}(x_n^k, Tx_n^k) = 0,$$

and

$$r(E, \{x_n^k\}) \leqslant \lambda r(E, \{x_n^{k-1}\}) \quad \text{for all } k \in \mathbb{N}.$$

Consequently,

$$r(E, \{x_n^k\}) \leqslant \lambda r(E, \{x_n^{k-1}\}) \leqslant \cdots \leqslant \lambda^{k-1} r(E, \{x_n^1\}).$$

We now begin the second part of the proof. In view of [2, p. 48], we may assume that for each $k \in \mathbb{N}$,

$$\lim_{n,m;\, n\neq m} \|x_n^k - x_m^k\| \quad \text{exists},$$

and in addition $\|x_n^k - x_m^k\| < \lim_{n,m;\, n\neq m} \|x_n^k - x_m^k\| + \frac{1}{2^k}$ for all $n, m \in \mathbb{N}$ and $n \neq m$. Let $\{y_n\}$ be the diagonal sequence $\{x_n^n\}$. We claim that $\{y_n\}$ is a Cauchy sequence. For each $n \geqslant 1$, we have for any positive number m,

$$\begin{aligned}\|y_n - y_{n-1}\| &\leqslant \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - y_{n-1}\| \\ &= \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - x_{n-1}^{n-1}\| \\ &\leqslant \|y_n - x_m^{n-1}\| + \lim_{i,j:\, i\neq j} \|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}}.\end{aligned}$$

Taking upper limit as $m \to \infty$,

$$\begin{aligned}\|y_n - y_{n-1}\| &\leqslant \limsup_{m\to\infty}\|y_n - x_m^{n-1}\| + \lim_{i,j:\,i\neq j}\|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}}\\ &\leqslant r\big(E,\{x_n^{n-1}\}\big) + \limsup_i\|x_i^{n-1} - y_n\| + \limsup_j\|x_j^{n-1} - y_n\| + \frac{1}{2^{n-1}}\\ &= 3r\big(E,\{x_n^{n-1}\}\big) + \frac{1}{2^{n-1}}\\ &\leqslant 3\lambda^{n-2}r\big(E,\{x_n^1\}\big) + \frac{1}{2^{n-1}}.\end{aligned}$$

Since $\lambda < 1$, we conclude that there exists $y \in E$ such that y_n converges to y. Consequently,

$$\operatorname{dist}(y, Ty) \leqslant \|y - y_n\| + \operatorname{dist}(y_n, Ty_n) + H(Ty_n, Ty) \to 0 \quad \text{as } n \to \infty.$$

Hence y is a fixed point of T. □

Theorem 3.7. *Let E be a nonempty weakly compact convex subset of a Banach space X with*

$$C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

Proof. We will prove that X satisfies property (D). Since $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$, we choose $\lambda = \frac{2\sqrt{C_{\mathrm{NJ}}(X)-1}}{WCS(X)} < 1$. Let D be a nonempty weakly compact convex subset of X, $\{x_n\} \subset D$, and $\{y_n\} \subset A(D, \{x_n\})$ be regular asymptotically uniform sequences relative to D. We will show that (2) is satisfied. By choosing a subsequence, if necessary, we can assume that $\{y_n\}$ converges weakly to $y \in D$ and

$$\lim_{k,j:\,k\neq j}\|y_k - y_j\| = l \quad \text{for some } l \geqslant 0. \tag{3}$$

Let $r = r(D, \{x_n\})$. The condition (2) easily follows when $r = 0$ or $l = 0$. We assume now that $r > 0$ and $l > 0$. Let $\varepsilon > 0$ so small that $0 < \varepsilon < l \wedge r$. From (3) we assume that

$$\big|\|y_k - y_j\| - l\big| < \varepsilon \quad \text{for all } k \neq j. \tag{4}$$

Fix $k \neq j$. Since $y_k, y_j \in A(D, \{x_n\})$ and using the convexity of $A(D, \{x_n\})$, we can assume, passing through a subsequence, that

$$\|x_n - y_k\| < r + \varepsilon, \qquad \|x_n - y_j\| < r + \varepsilon, \tag{5}$$

and

$$\left\|x_n - \frac{y_k + y_j}{2}\right\| > r - \varepsilon \quad \text{for all large } n. \tag{6}$$

By the definition of $C_{\mathrm{NJ}}(X)$, by (4)–(6) we have for n large enough,

$$C_{\mathrm{NJ}}(X) \geqslant \frac{\|2x_n - (y_k + y_j)\|^2 + \|y_k - y_j\|^2}{2\|x_n - y_k\|^2 + 2\|x_n - y_j\|^2} \geqslant \frac{4(r-\varepsilon)^2 + (l-\varepsilon)^2}{4(r+\varepsilon)^2}.$$

Since ε is arbitrarily small, it follows that

$$C_{NJ}(X) \geqslant \frac{4r^2 + l^2}{4r^2}.$$

Since

$$WCS(X) = \inf\left\{\frac{\lim_{j,k;\,j\neq k}\|u_j - u_k\|}{\limsup_j \|u_j\|} : u_j \xrightarrow{w} 0,\ \lim_{j,k;\,j\neq k}\|u_j - u_k\| \text{ exists}\right\},$$

we can deduce that

$$C_{NJ}(X) \geqslant 1 + \frac{WCS(X)^2(\limsup_n \|y_n - y\|)^2}{4r^2} \geqslant 1 + \frac{WCS(X)^2 r(D,\{y_n\})^2}{4r^2}.$$

Consequently,

$$r(D,\{y_n\}) \leqslant \frac{2\sqrt{C_{NJ}(X) - 1}}{WCS(X)} r = \lambda r(D,\{x_n\})$$

as desired. □

In order to prove our next result, we need the following theorem which states a relationship between the weakly convergent sequence coefficient and the Jordan–von Neumann constant of a Banach space X.

Theorem 3.8. *For a Banach space X,*

$$[WCS(X)]^2 \geqslant \frac{2C_{NJ}(X) + 1}{2[C_{NJ}(X)]^2}.$$

Proof. Since $C_{NJ}(X) \leqslant 2$ and the result is obvious if $C_{NJ}(X) = 2$, we can assume that $C_{NJ}(X) < 2$. It is known that $C_{NJ}(X) < 2$ implies X and X^* are reflexive. Put $\alpha = \sqrt{2C_{NJ}(X)}$. Let $\{x_n\}$ be a normalized weakly null sequence in X and $d := \lim_{n,m;\,n\neq m}\|x_n - x_m\|$. Consider a sequence $\{f_n\}$ of norm one functionals for which $f_n(x_n) = 1$. Since X^* is reflexive we can assume that $\{f_n\}$ converges weakly to some f in X^*. Let ε be an arbitrary positive number and choose $K \in \mathbb{N}$ large enough so that $|f(x_n)| < \varepsilon$ and $d - \varepsilon \leqslant \|x_n - x_m\| \leqslant d + \varepsilon$ for any $m \neq n$; $m, n \geqslant K$. Then we have

$$\lim_n (f_n - f)(x_K) = 0 \quad \text{and} \quad \lim_n f_K(x_n) = 0.$$

Since $\lim_{n,m;\,n\neq m}\|\frac{x_n - x_m}{d+\varepsilon}\| < 1$ and $\|\frac{x_K}{d+\varepsilon}\| \leqslant 1$, we have, by the definition of $R(1,X)$,

$$\limsup_n \|x_n + x_K\| \leqslant (d+\varepsilon)R(1,X) \leqslant (d+\varepsilon)\sqrt{2C_{NJ}(X)} = (d+\varepsilon)\alpha.$$

We construct elements of $\widetilde{X}$ and $\widetilde{X}^*$:

$$\tilde{x} = \left\{\frac{x_n - x_K}{d+\varepsilon}\right\}_{\mathcal{U}}, \qquad \tilde{y} = \left\{\frac{x_n + x_K}{(d+\varepsilon)\alpha}\right\}_{\mathcal{U}}, \qquad \tilde{f} = \{f_n\}_{\mathcal{U}} \quad \text{and} \quad \tilde{g} = \dot{f}_K.$$

Here $\dot{h}$ denotes an equivalence class of the sequence $\{h_n\}$ such that $h_n \equiv h$ for all $n \in \mathbb{N}$. Clearly $\tilde{x}, \tilde{y} \in B_{\widetilde{X}}$ and $\tilde{f}, \tilde{g} \in S_{\widetilde{X}^*}$. Moreover,

$$\tilde{f}(\{x_n\}_{\mathcal{U}}) = 1 \quad \text{and} \quad |\tilde{f}(\dot{x}_K)| = |\dot{f}(\dot{x}_K)| < \varepsilon.$$

On the other hand,

$$\tilde{g}(\{x_n\}_{\mathcal{U}}) = 0 \quad \text{and} \quad \tilde{g}(\dot{x_K}) = 1.$$

Let consider

$$\begin{aligned}\|\tilde{f} - \tilde{g}\| &\geqslant (\tilde{f} - \tilde{g})(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x}) \\ &= \frac{1}{d+\varepsilon}\left(\tilde{f}(\{x_n\}_{\mathcal{U}}) - \tilde{f}(\dot{x_K}) - \left[\tilde{g}(\{x_n\}_{\mathcal{U}}) - \tilde{g}(\dot{x_K})\right]\right) \\ &\geqslant \frac{1}{d+\varepsilon}(1 - \varepsilon - 0 + 1) = \frac{2-\varepsilon}{d+\varepsilon}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\tilde{f} + \tilde{g}\| &\geqslant (\tilde{f} + \tilde{g})(\tilde{y}) = \tilde{f}(\tilde{y}) + \tilde{g}(\tilde{y}) \\ &= \frac{1}{(d+\varepsilon)\alpha}\left(\tilde{f}(\{x_n\}_{\mathcal{U}}) + \tilde{f}(\dot{x_K}) + \tilde{g}(\{x_n\}_{\mathcal{U}}) + \tilde{g}(\dot{x_K})\right) \\ &\geqslant \frac{1}{(d+\varepsilon)\alpha}(1 - \varepsilon + 0 + 1) = \frac{2-\varepsilon}{(d+\varepsilon)\alpha}.\end{aligned}$$

Thus we have

$$\begin{aligned}C_{\mathrm{NJ}}(\widetilde{X}^*) &\geqslant \frac{\|\tilde{f} + \tilde{g}\|^2 + \|\tilde{f} - \tilde{g}\|^2}{2\|\tilde{f}\|^2 + 2\|\tilde{g}\|^2} \\ &\geqslant \frac{\left(\frac{2-\varepsilon}{d+\varepsilon}\right)^2 + \left(\frac{2-\varepsilon}{(d+\varepsilon)\alpha}\right)^2}{4} \\ &= \left(\frac{1}{d+\varepsilon}\right)^2\left(\frac{(2-\varepsilon)^2}{4} + \frac{(2-\varepsilon)^2}{4\alpha^2}\right).\end{aligned}$$

Since ε is arbitrary and the Jordan–von Neumann constants of X^*, X, $\widetilde{X}$ and $\widetilde{X}^*$ are all equal, we obtain

$$C_{\mathrm{NJ}}(X) \geqslant \left(\frac{1}{d^2}\right)\left(1 + \frac{1}{2C_{\mathrm{NJ}}(X)}\right).$$

Thus

$$\left[WCS(X)\right]^2 \geqslant \frac{2C_{\mathrm{NJ}}(X) + 1}{2[C_{\mathrm{NJ}}(X)]^2}. \qquad \square$$

Using Theorem 3.8, we obtain the following corollary.

Corollary 3.9. [6, Theorem 3.16] *Let X be a Banach space. If $C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2}$, then X and X^* has uniform normal structure.*

Proof. Let $\widetilde{X}$ be a Banach space ultrapower of X. Since $C_{\mathrm{NJ}}(\widetilde{X}) = C_{\mathrm{NJ}}(X)$, Theorem 3.8 can be applied to $\widetilde{X}$. The inequality in Theorem 3.8 implies $WCS(\widetilde{X}) > 1$ if $C_{\mathrm{NJ}}(\widetilde{X}) < \frac{1+\sqrt{3}}{2}$. Since $WCS(\widetilde{X}) > 1$ implies $\widetilde{X}$ has weak normal structure [3] and since $\widetilde{X}$

is reflexive, it must be the case that $\widetilde{X}$ has normal structure. By [11, Theorem 5.2], X has uniform normal structure as desired. □

Using the inequality appearing in Theorem 3.8, and numerical calculus, it is not difficult to see that $C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}$ if $C_{NJ}(X) < c_0 = 1.273\ldots$. Thus we can state:

Corollary 3.10. *Let E be a nonempty bounded closed convex subset of a Banach space X with*

$$C_{NJ}(X) < c_0 = 1.273\ldots.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

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Appendix 7: Fixed point theorems for multivalued mappings in modular function spaces, Scien. Math. Japon. 63 (2) (2006), 161-169.

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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN MODULAR FUNCTION SPACES*

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ABSTRACT. The purpose of this paper is to study the existence of fixed points for multivalued nonexpansive mappings in modular function spaces. We apply our main result to obtain fixed point theorems for multivalued mappings in the Banach spaces L_1 and l_1.

1. Introduction

The theory of modular spaces was initiated by Nakano [15] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see, for instance, [2, 3, 7, 8]). In particular, some fixed point theorems for (singlevalued) nonexpansive mappings in modular function spaces are given in [8]. In 1969, Nadler [14] established the multivalued version of Banach's contraction principle in metric spaces. Since then the metric fixed point theory for multivalued mappings has been rapidly developed and many of papers have appeared proving the existence of fixed points for multivalued nonexpansive mappings in special classes of Banach spaces (see, for instance, [4, 5, 9, 11]). In this paper, we study similar problems in the setting of modular function spaces. Namely, we prove that every ρ−contraction $T : C \to F_\rho(C)$ has a fixed point where ρ is a convex function modular satisfying the Δ_2−type condition and C is a nonempty ρ−bounded ρ−closed subset of L_ρ. By using this result, we can assert the existence of fixed points for multivalued ρ−nonexpansive mappings. Finally, we apply our main result to obtain fixed point theorems in the Banach space L_1 (resp. l_1) for multivalued mappings whose domains are compact in the topology of the convergence locally in measure (resp. w^*−topology).

2. Preliminaries

We start by recalling some basic concepts about modular function spaces. For more details the reader is referred to [10, 12].

Let Ω be a nonempty set and Σ be a nontrivial σ−algebra of subsets of Ω. Let $\mathcal{P}$ be a δ−ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ−finite measure space). By $\mathcal{E}$ we denote the

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linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}, |g_n| \le |f|$, and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Let us recall that a set function $\mu : \Sigma \to [0, \infty]$ is called a σ-subadditive measure if $\mu(\emptyset) = 0$, $\mu(A) \le \mu(B)$ for any $A \subset B$ and $\mu(\bigcup A_n) \le \sum \mu(A_n)$ for any sequence of sets $\{A_n\} \subset \Sigma$. By 1_A we denote the characteristic function of the set A.

Definition 2.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if :

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,
(P_2) $\rho(f, E) \le \rho(g, E)$ whenever $|f(\omega)| \le |g(\omega)|$ for any $\omega \in \Omega, f, g \in \mathcal{E}$, and $E \in \Sigma$,
(P_3) $\rho(f, .) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,
(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,
(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,
(P_6) for any $\alpha > 0$, $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \subset \mathcal{P}$ and decreases to $\emptyset$.

A σ-subadditive measure ρ is said to be additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$ whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $f \in \mathcal{M}$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\left\{\rho(g, E) : g \in \mathcal{E}, |g(\omega)| \le |f(\omega)| \text{ for every } \omega \in \Omega\right\}.$$

Definition 2.2. A set E is said to be ρ-null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega) \text{ does not hold}\}$ is ρ-null. For example, we will say frequently $f_n \to f$ ρ-a.e.

Note that a countable union of ρ-null sets is still ρ-null. In the sequel we will identify sets A and B whose symmetric difference $A \Delta B$ is ρ-null, similarly we will identify measurable functions which differ only on a ρ-null set.

In the above condition, we define the function $\rho : \mathcal{M} \to [0, \infty]$ by $\rho(f) = \rho(f, \Omega)$. We know from [10] that ρ satisfies the following properties :

(i) $\rho(f) = 0$ if and only if $f = 0$ ρ-a.e.
(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.
(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$.
In addition, if the following property is satisfied
(iii)' $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$,
we say that ρ is a convex modular.

A function modular ρ is called σ-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(1_{K_n}) < \infty$ and $\Omega = \bigcup K_n$.

The modular ρ defines a corresponding modular space L_ρ, which is given by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an F-norm. Recall that a functional $\|\cdot\| : X \to [0, \infty]$ defines an F-norm on a linear space X if and only if

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|\alpha x\| = \|x\|$ whenever $|\alpha| = 1$,
(3) $\|x + y\| \le \|x\| + \|y\|$,
(4) $\|\alpha_n x_n - \alpha x\| \to 0$ if $\alpha_n \to \alpha$ and $\|x_n - x\| \to 0$.

The modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

We know from [10] that the linear space $(L_\rho, \|\cdot\|_\rho)$ is a complete metric space. If ρ is convex, the formula

$$\|f\|_l = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

defines a norm which is frequently called the Luxemburg norm. The formula

$$\|f\|_a = \inf\left\{\frac{1}{k}(1 + \rho(kf)) : k > 0\right\}$$

defines a different norm which is called the Amemiya norm. Moreover, $\|\cdot\|_l$ and $\|\cdot\|_a$ are equivalent norms. We can also consider the space

$$E_\rho = \{f \in \mathcal{M} : \rho(\alpha f, \cdot) \text{ is order continuous for all } \alpha > 0\}.$$

Definition 2.3. A function modular ρ is said to satisfy the Δ_2–condition if

$$\sup_{n\geq 1} \rho(2f_n, D_k) \to 0 \text{ as } k \to \infty \text{ whenever } \{f_n\} \subset \mathcal{M}, D_k \in \Sigma$$

$$\text{decreases to } \emptyset \text{ and } \sup_{n\geq 1} \rho(f_n, D_k) \to 0 \text{ as } k \to \infty.$$

It is known that the Δ_2–condition is equivalent to $E_\rho = L_\rho$.

Definition 2.4. A function modular ρ is said to satisfy the Δ_2–type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \leq K\rho(f)$.

In general, the Δ_2–type condition and Δ_2–condition are not equivalent, even though it is obvious that the Δ_2– type condition implies the Δ_2–condition.

Definition 2.5. Let L_ρ be a modular space.

(1) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ–convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.
(2) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ–a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega : f_n(\omega) \nrightarrow f(\omega)\}$ is ρ–null.
(3) A subset C of L_ρ is called ρ–closed if the ρ–limit of a ρ–convergent sequence of C always belongs to C.
(4) A subset C of L_ρ is called ρ–a.e. closed if the ρ–a.e. limit of a ρ–a.e. convergent sequence of C always belongs to C.
(5) A subset C of L_ρ is called ρ–compact if every sequence in C has a ρ–convergent subsequence in C.
(6) A subset C of L_ρ is called ρ–a.e. compact if every sequence in C has a ρ–a.e. convergent subsequence in C.
(7) A subset C of L_ρ is called ρ–bounded if

$$\text{diam}_\rho(C) = \sup\{\rho(f - g) : f, g \in C\} < \infty.$$

We know by [10] that under the Δ_2–condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2–type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2–type condition.

Definition 2.6. Let ρ be as above. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)} : f \in L_\rho, 0 < \rho(f) < \infty\right\} \text{ for all } 0 \leq t < \infty.$$

The following properties of the growth function can be found in [3].

Lemma 2.7. *Let ρ be as above. Then the growth function ω has the following properties :*

(1) $\omega(t) < \infty$, $\forall t \in [0, \infty)$.
(2) $\omega : [0, \infty) \to [0, \infty)$ *is a convex, strictly increasing function. So, it is continuous.*
(3) $\omega(\alpha\beta) \le \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$.
(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \le \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0, \infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 2.8 (T. Domínguez Benavides et al. [3]). *Let ρ be as above. Then*

$$\|f\|_\rho \le \frac{1}{\omega^{-1}(1/\rho(f))} \text{ whenever } f \in L_\rho \setminus \{0\}.$$

The following lemma is a technical lemma which will be need because of lack of the triangular inequality.

Lemma 2.9 (T. Domínguez Benavides et al. [3]). *Let ρ be as above, $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then*

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \limsup_{n\to\infty} \rho(f_n + g_n) = \limsup_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \liminf_{n\to\infty} \rho(f_n + g_n) = \liminf_{n\to\infty} \rho(f_n).$$

In the same way as the Hausdorff distance defined on the family of bounded closed subsets of a metric space, we can define the analogue to the Hausdorff distance for modular function spaces. We will call ρ−Hausdorff distance even though it is not a metric.

Definition 2.10. Let C be a nonempty subset of L_ρ. We shall denote by $F_\rho(C)$ the family of nonempty ρ−closed subsets of C and by $K_\rho(C)$ the family of nonempty ρ−compact subsets of C. Let $H_\rho(\cdot,\cdot)$ be the ρ−Hausdorff distance on $F_\rho(L_\rho)$, i.e.,

$$H_\rho(A, B) = \max\left\{\sup_{f\in A} \operatorname{dist}_\rho(f, B), \sup_{g\in B} \operatorname{dist}_\rho(g, A)\right\}, \quad A, B \in F_\rho(L_\rho),$$

where $\operatorname{dist}_\rho(f, B) = \inf\{\rho(f-g) : g \in B\}$ is the ρ−distance between f and B. A multivalued mapping $T : C \to F_\rho(L_\rho)$ is said to be a ρ−contraction if there exists a constant $k \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \le k\rho(f - g), \quad f, g \in C. \tag{2.1}$$

If (2.1) is valid when $k = 1$, then T is called ρ−nonexpansive. A function $f \in C$ is called a fixed point for a multivalued mapping T if $f \in Tf$.

3. Main results

We begin stating the Banach Contraction Principle for multivalued mappings in modular function spaces.

Theorem 3.1. *Let ρ be a convex function modular satisfying the Δ_2−type condition, C a nonempty ρ−bounded ρ−closed subset of L_ρ, and $T : C \to F_\rho(C)$ a ρ−contraction mapping, i.e., there exists a constant $k \in [0, 1)$ such that*

$$H_\rho(Tf, Tg) \le k\rho(f - g), \quad f, g \in C. \tag{3.1}$$

Then T has a fixed point.

Proof. Let $f_0 \in C$ and $\alpha \in (k,1)$. Since Tf_0 is nonempty, there exists $f_1 \in Tf_0$ such that $\rho(f_0 - f_1) > 0$ (otherwise f_0 is a fixed point of T). In view of (3.1), we have

$$\mathrm{dist}_\rho(f_1, Tf_1) \leq H_\rho(Tf_0, Tf_1) \leq k\rho(f_0 - f_1) < \alpha\rho(f_0 - f_1).$$

Since $\mathrm{dist}_\rho(f_1, Tf_1) = \inf\{\rho(f_1 - g) : g \in Tf_1\}$, it follows that there exists $f_2 \in Tf_1$ such that

$$\rho(f_1 - f_2) < \alpha\rho(f_0 - f_1).$$

Similarly, there exists $f_3 \in Tf_2$ such that

$$\rho(f_2 - f_3) < \alpha\rho(f_1 - f_2).$$

Continuing in this way, there exists a sequence $\{f_n\}$ in C satisfying $f_{n+1} \in Tf_n$ and

$$\begin{aligned}\rho(f_n - f_{n+1}) &< \alpha\rho(f_{n-1} - f_n) \\ &< \alpha^2(\rho(f_{n-2} - f_{n-1})) \\ &< \ldots \\ &< \alpha^{n-1}(\rho(f_1 - f_2)) \\ &< \alpha^n(\rho(f_0 - f_1)) \\ &\leq \alpha^n \mathrm{diam}_\rho(C).\end{aligned}$$

Let $M = \mathrm{diam}_\rho(C)$, then

$$\frac{1}{\alpha^n M} < \frac{1}{\rho(f_n - f_{n+1})}.$$

By Lemma 2.7, we have

$$\left(\omega^{-1}\left(\frac{1}{\alpha}\right)\right)^n \omega^{-1}\left(\frac{1}{M}\right) < \omega^{-1}\left(\frac{1}{\rho(f_n - f_{n+1})}\right).$$

It follows that

$$\frac{1}{\omega^{-1}\left(\frac{1}{\rho(f_n - f_{n+1})}\right)} < \frac{1}{\left(\omega^{-1}\left(\frac{1}{\alpha}\right)\right)^n \omega^{-1}\left(\frac{1}{M}\right)}.$$

By Lemma 2.8, we obtain

$$\|f_n - f_{n+1}\|_\rho < \left(\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)}\right)^n \cdot \frac{1}{\omega^{-1}\left(\frac{1}{M}\right)}.$$

Since ω^{-1} is strictly increasing, we have $\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)} < 1$. This implies that $\{f_n\}$ is a Cauchy sequence in $(L_\rho, \|\cdot\|_\rho)$. Since $(L_\rho, \|\cdot\|_\rho)$ is a complete metric space, there exists $f \in L_\rho$ such that $\{f_n\}$ is $\|\cdot\|_\rho$-convergent to f. Since under the Δ_2-type condition, norm convergence and modular convergence are identical, $\{f_n\}$ is ρ-convergent to f and $f \in C$ because C is ρ-closed. Since $f_n \in Tf_{n-1}$, we have

$$\mathrm{dist}_\rho(f_n, Tf) \leq H_\rho(Tf_{n-1}, Tf) \leq k\rho(f_{n-1} - f) \longrightarrow 0. \tag{3.2}$$

We observe that, for each n, there exists $g_n \in Tf$ such that

$$\rho(f_n - g_n) \leq \mathrm{dist}_\rho(f_n, Tf) + \frac{1}{n}. \tag{3.3}$$

Thus, (3.2) and (3.3) imply that $\lim_{n\to\infty} \rho(f_n - g_n) = 0$. By Lemma 2.9,

$$\limsup_{n\to\infty} \rho(g_n - f) = \limsup_{n\to\infty} \rho(g_n - f_n + f_n - f) = \limsup_{n\to\infty} \rho(f_n - f) = 0.$$

Since Tf is ρ-closed, we can conclude that $f \in Tf$. □

The following results will be very useful in the proof of our main theorem.

Theorem 3.2 (M. A. Khamsi [7]). *Let $\{f_n\} \subset L_\rho$ be ρ–a.e. convergent to 0. Assume there exists $k > 1$ such that*

$$\sup_{n\geq 1} \rho(kf_n) = M < \infty.$$

Let $g \in E_\rho$, then we have

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g).$$

The following lemma guarantees that every nonempty ρ–compact subset of L_ρ attains a nearest point.

Lemma 3.3. *Let ρ be a convex function modular satisfying the Δ_2–type condition, $f \in L_\rho$, and K a nonempty ρ–compact subset of L_ρ. Then there exists $g_0 \in K$ such that*

$$\rho(f - g_0) = dist_\rho(f, K).$$

Proof. Let $m = \mathrm{dist}_\rho(f, K)$. For each $n \in \mathbb{N}$, there exists $g_n \in K$ such that

$$m - \frac{1}{n} \leq \rho(f - g_n) \leq m + \frac{1}{n}.$$

By the ρ–compactness of K, we can assume, by passing through a subsequence, that $g_n \xrightarrow{\rho} g_0 \in K$. By Lemma 2.9, we obtain

$$\begin{aligned} m = \limsup_{n\to\infty} \rho(g_n - f) &= \limsup_{n\to\infty} \rho(g_n - g_0 + g_0 - f) \\ &= \limsup_{n\to\infty} \rho(g_0 - f) \\ &= \rho(g_0 - f). \end{aligned}$$

□

We can now state our main theorem.

Theorem 3.4. *Let ρ be a convex function modular satisfying the Δ_2–type condition, C a nonempty ρ–a.e. compact ρ–bounded convex subset of L_ρ, and $T : C \to K_\rho(C)$ a ρ–nonexpansive mapping. Then T has a fixed point.*

Proof. Fix $f_0 \in C$. For each $n \in \mathbb{N}$, the ρ–contraction $T_n : C \to F_\rho(C)$ is defined by

$$T_n(f) = \frac{1}{n} f_0 + (1 - \frac{1}{n})Tf, \quad f \in C.$$

By Theorem 3.1, we can conclude that T_n has a fixed point, say f_n. It is easy to see that

$$\mathrm{dist}_\rho(f_n, Tf_n) \leq \frac{1}{n}\mathrm{diam}_\rho(C) \longrightarrow 0.$$

Because of ρ–a.e. compactness of C, we can assume, by passing through a subsequence, that $f_n \xrightarrow{\rho-a.e.} f$ for some $f \in C$. By Lemma 3.3, for each $n \in \mathbb{N}$, there exists $g_n \in Tf_n$ and $h_n \in Tf$ such that

$$\rho(f_n - g_n) = \mathrm{dist}_\rho(f_n, Tf_n)$$

and

$$\rho(g_n - h_n) = \mathrm{dist}_\rho(g_n, Tf) \leq H_\rho(Tf_n, Tf) \leq \rho(f_n - f).$$

Because of ρ–compactness of Tf, we can assume, by passing through a subsequence, that $h_n \xrightarrow{\rho} h \in Tf$. Since ρ satisfies the Δ_2–type condition, there exists $K > 0$ such that

$\rho(2(f_n - f)) \leq K\rho(f_n - f)$ for all $n \in \mathbb{N}$.
This implies that

$$\sup_{n\geq 1} \rho(2(f_n - f)) \leq K \sup_{n\geq 1} \rho(f_n - f) < \infty.$$

By Theorem 3.2 and Lemma 2.9, we obtain

$$\begin{aligned}\liminf_{n\to\infty} \rho(f_n - f) + \rho(f - h) &= \liminf_{n\to\infty} \rho(f_n - f + f - h) \\ &= \liminf_{n\to\infty} \rho(f_n - h) \\ &= \liminf_{n\to\infty} \rho(f_n - g_n + g_n - h_n + h_n - h) \\ &= \liminf_{n\to\infty} \rho(g_n - h_n) \\ &\leq \liminf_{n\to\infty} \rho(f_n - f).\end{aligned}$$

It follows that $\rho(f - h) = 0$ and then we have $f = h \in Tf$. □

Consider the space $L_p(\Omega, \mu)$ for a σ-finite measure μ with the usual norm. Let C be a bounded closed convex subset of L_p for $1 < p < \infty$ and $T : C \to K(C)$ a multivalued nonexpansive mapping. Because of uniform convexity of L_p, it is known that T has a fixed point. For $p = 1$, T can fail to have a fixed point even in the singlevalued case for a weakly compact convex set C (see [1]). However, since L_1 is a modular space where $\rho(f) = \int_\Omega |f| d\mu = \|f\|$ for all $f \in L_1$, Theorem 3.4 implies the existence of a fixed point when we define mappings on a ρ-a.e. compact ρ-bounded convex subset of L_1. Thus we can state :

Corollary 3.5. *Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a nonempty bounded convex set which is compact for the topology of the convergence locally in measure, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. Under the above hypothesis, ρ-a.e. compact sets and compact sets in the topology of the convergence locally in measure are identical (see [2]). Consequently, Theorem 3.4 can be applied to obtain a fixed point for T. □

In the case of the space l_1 we also can obtain a bounded closed convex set C and a nonexpansive mapping $T : C \to C$ which is fixed point free. Indeed, consider the following easy and well known example :
Let

$$C = \Big\{\{x_n\} \in l_1 : 0 \leq x_n \leq 1 \text{ and } \sum_{n=1}^{\infty} x_n = 1\Big\}.$$

Define a nonexpansive mapping $T : C \to C$ by

$$T(x) = (0, x_1, x_2, x_3, ...) \text{ where } x = \{x_n\}.$$

Then T is a fixed point free. However, if we consider $L_\rho = l_1$ where $\rho(x) = \|x\|$, $\forall x \in l_1$. Then ρ-a.e. convergence and w^*-convergence are identical on bounded subsets of l_1 (see [3]). This fact leads us to obtain the following corollary :

Corollary 3.6. *Let C be a nonempty w^*-compact convex subset of l_1 and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. By the above argument, we know that ρ-a.e. compact bounded sets and w^*-compact sets are identical. Then we can apply Theorem 3.4 to assert the existence of a fixed point of T. □

In fact Corollary 3.5 and 3.6 are consequences of a general result: Assume that X is a linear normed space and τ is a Hausdorff topology on X. We say that X satisfies the strict τ-Opial property if

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for each sequence $\{x_n\}$ in X which converges to x for the topology τ and each $y \neq x$. Following the same argument as in [11] it is easy to prove the following theorem:

Theorem 3.7. *Let X be a Banach space, C a convex bounded sequentially τ-compact subset of X, and $T : C \to K(C)$ a nonexpansive mapping. If X satisfies the strict τ-Opial property, then T has a fixed point.*

When X is a modular function space equipped with either Luxemburg or Amemiya norm, we can consider the topology τ of convergence ρ-a.e. In this case, Theorem 3.7 yields to the following:

Theorem 3.8. *Let ρ be a convex additive $\sigma-$finite function modular satisfying the Δ_2-type condition. Assume that L_ρ is equipped either with Luxemburg or Amemiya norm. Let C be a nonempty $\rho-$a.e. compact $\rho-$bounded convex subset of L_ρ, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. From [6] (Theorem 4.1 and 4.3), X satisfies the uniform Opial property with respect to the topology of ρ-a.e. convergence. Since ρ-a.e. compact sets and ρ-a.e. sequentially compact sets are identical for this topology (see [2]), we can deduce the result from Theorem 3.7 □

Remark. In the case of the space $L^1(\Omega)$ we have

$$\rho(f) = \int_\Omega |f| d\mu = \|f\|_l = \|f\|_a$$

and we can deduce Corollary 3.5 and 3.6 from Theorem 3.8.

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