



โครงการ การควบคุมแบบผ่อนคลายเป็นขั้นของสมการวิวัฒนาการแบบอิมพัลส์
ไม่เชิงเส้น

โดย รองศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม และคณะ

กรกฎาคม 2550

รายงานวิจัยฉบับสมบูรณ์

โครงการ การควบคุมแบบต่อเนื่องของชั้นของสมการวิวัฒนาการแบบอิมพลัส
ไม่เชิงเส้น

คณะผู้วิจัย

1. รองศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม
2. ผศ.ดร.อภิชัย เหมะรุธิน

สังกัด

มหาวิทยาลัยเทคโนโลยีสุรนารี

มหาวิทยาลัยราชภัฏ นครราชสีมา

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย

สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

งานวิจัยนี้ทำการศึกษาระบบควบคุมแบบฟัซซี่ที่มีอิมพลัสประเภทกึ่งเชิงเส้นและไม่เชิงเส้นอย่างเข้ม ได้เริ่มต้นโดยการศึกษาขอบเขตของผลเฉลยของระบบควบคุมดังกล่าว และได้มีการศึกษาแนววิถีแบบฟัซซี่เทียบกับแนววิถีเดิมด้วย ในลำดับต่อมาได้มีการศึกษาถึงการมีอยู่ของการควบคุมที่เหมาะสมที่สุดแบบฟัซซี่ในทั้งสองกรณีและได้มีพิกจน์เกี่ยวกับสมบัติความหนาแน่นของแนววิถีเดิมในระบบฟัซซี่ด้วย

ได้ทำการประยุกต์ผลลัพธ์ข้างต้นเข้ากับปัญหาการควบคุมที่เหมาะสมที่สุดแบบลากรองจ์ โดยได้มีการพิจารณาสมการคล้ายเชิงเส้นพาราโบลิกในปริภูมิยูคลิดเชียน และได้มีการประยุกต์ใช้ทฤษฎีบทที่ได้พิกจน์ไว้แล้วในตอนแรก เพื่อยืนยันถึงการมีระบบควบคุมแบบฟัซซี่ที่เหมาะสมที่สุดของระบบตัวอย่าง

คำสำคัญ สมการวิวัฒนาการแบบอิมพลัสไม่เชิงเส้นอย่างเข้ม การควบคุมที่เหมาะสมที่สุด การฟัซซี่ กลังรูป และวิวัฒนาการสามลำดับ

Abstract

Relaxed control for a class of semilinear and strongly nonlinear impulsive evolution equations are investigated. Boundedness of solutions of semilinear and strongly nonlinear are proved. Properties of original and relaxed trajectories are discussed. The existence of optimal relaxed control are also presented in both semilinear and strongly nonlinear cases. Density property of the original trajectories in the set of relaxed trajectories is proved.

These results can be applied to Lagrange optimal control problem. For illustration, an example of a quasi-linear impulsive parabolic partial differential equation and the corresponding optimal relaxed control are presented.

Keywords: Nonlinear impulsive evolution equations, optimal control, relaxation, semigroup, evolution triple.

Executive Summary

โครงการ โครงการ การควบคุมแบบผ่อนคลายเป็นขั้นของสมการวิวัฒนาการแบบอิมพัลส์ไม่เชิงเส้น

ผู้เสนอ : นายไพโรจน์ สัตยธรรม
หน่วยงาน : สาขาวิชาคณิตศาสตร์
สำนักวิชาวิทยาศาสตร์ มหาวิทยาลัยเทคโนโลยีสุรนารี
อำเภอเมือง จังหวัดนครราชสีมา 30000
โทรศัพท์/FAX 044 224315 Mobile: 089 5849868
e-mail : pairote@sut.ac.th
ระยะเวลาดำเนินงาน : 2 ปี

ปัญหาที่ทำการวิจัย

Recently, the differential equations with impulsive conditions have been studied quite extensively. In which case, the traditional initial value problem

$$x(0) = x_0,$$

is replaced by the impulsive conditions

$$\begin{aligned}x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i))\end{aligned}$$

where $0 < t_1 < t_2 < \dots < t_n < T$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $i = 1, 2, \dots, n$, and F_i 's are some operators. The impulsive conditions are the combinations of the traditional initial value problems and the short-term perturbations where duration can be negligible in comparison with the duration of the process. They have advantage over the traditional initial value problems because they can be used to model other phenomena that cannot be modeled by the traditional initial value problems, such as the dynamics of portfolio strategy subjected to abrupt changes due to terrorists or outbreak of diseases. In these cases an

impulsive control approach, where the investor can change his portfolio only finitely often in a finite time intervals, proved to be useful .

For equations in finite dimensional spaces or equations in general Banach spaces but with linear and continuous operators, the impulsive problems have been examined by many authors where the existence and uniqueness of solutions and stability and property are investigated.

For a **semilinear impulsive evolution equations** with unbounded linear operator A of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + f(x(t)), t > 0, t \neq t_i, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad 0 < t_1 < t_2 < \dots < t_n < T\end{aligned}\tag{1}$$

has been studied by Ahmed. He discussed the problem of existence and uniqueness of mild solution and optimal control but **relaxations** for optimality were not addressed. We wish to present just that and this is the first problem for investigation in this project. Secondly, in a recent paper by the author, the existence of optimal control for a class of a **strongly non linear evolution equation** of the form

$$\begin{aligned}\dot{x}(t) + A(t, x(t)) &= g(t, x(t)), t \neq t_i, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad 0 < t_1 < t_2 < \dots < t_n < T\end{aligned}\tag{2}$$

where A is a nonlinear monotone operator, was proved. Again the **relaxations** for optimality were not addressed. This is the second problem for our investigation in this project.

วัตถุประสงค์

The purpose of this project is to find sufficient conditions for the **relaxations** of optimality for the semilinear evolution equation (1) and the strongly nonlinear evolution equation (2).

ระเบียบวิธีวิจัย

- 1. For the construction of a classical solution of the impulsive system, we propose to use the Schauder fixed point theorem on a suitable Banach space.
- 2. For the construction of a optimal control pair, we propose to use Balder’s result about strong-weak lower semicontinuity of integral functionals.
- 3. For the construction of relaxed systems, we use measure control as a new admissible space.

แผนการดำเนินงานวิจัย

- 1. Investigate the relationship of the original trajectories and the relaxed trajectories
- 2. Investigate the relationship of the optimal value of the original system and the relaxed system
- 3. Given some examples for illustration.

Operation Plan

Activities	Duration/Month			
	1-6	7-12	12-18	18-24
1. Try to prove that $\bar{X} = X_r$	↔			
2. Try to prove that $m = m_r$		↔		
3. Give some examples			↔	
4. Writing a paper				↔

We can prove the following three main theorems

Theorem 1. (Existence of relaxed optimal control pairs for strongly nonlinear evolution equations)

Under some suitable conditions there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$ (See, Theorem 5.1 in [1] for the detail of the proof).

Theorem 2. (Existence of relaxed optimal control pairs for semilinear evolution equations)

Under some suitable conditions there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$ (See, Theorem 5.1 in [2] for the detail of the proof).

Theorem 3. Under some suitable conditions there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$ (See, Theorem 5.1 in [3] for the detail of the proof).

ข้อสรุป:

By using our impulsive model, we can continue to investigate relaxation of the impulsive system and periodic impulsive system which are also important models in finance and nanoelectronic.

End of Executive Summary

เนื้อหางานวิจัย

We consider the following strongly nonlinear impulsive evolution

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = g(t, x(t)), \\ x(0) = x_0 \in H, \\ \Delta x(t_i) = F_i(x(t_i)), i = 1, 2, \dots, n, \end{cases} \quad (1)$$

where A is a nonlinear monotone operator. By using the technique of evolution triple, monotone operator, and Schauder's fixed point theorem, we proved that the (1) has a solution.

Next, we consider the following impulsive controlled system

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = g(t, x(t)) + B(t)u(t), \\ x(0) = x_0 \in H, \\ \Delta x(t_i) = F_i(x(t_i)), i = 1, 2, \dots, n. \end{cases} \quad (2)$$

($0 < t_1 < t_2 < \dots < t_n < T$) where the control function $u(t)$ is an element of space U_{ad} . We proved that, for each u , one can find a trajectory x such that the admissible pair (x, u) is a solution of (2).

Now, let us define a Lagrange cost function

$$J(x, u) = \int_0^T L(t, x(t), u(t)) dt$$

where L is only a lower semicontinuous function (not a convex function) in the variable u . By using technique of convexifying the orientor field (or relaxation), we can prove that there is an admissible pair (x_0, u_0) in the convexified field such that

$$J(x_0, u_0) = \inf_{(x, u) \in A_{ad}} J(x, u)$$

This proves the existence of an optimal control pair. Moreover, we also proved the density property, i.e., there is an original admissible controlled sequence $\{u_n\}$ such that $\{u_n\}$ converges to u as $n \rightarrow \infty$.

We apply our model to a quasi-linear partial differential equation in R^n and proved that such a model in R^n together with Lagrange cost function has an optimal pair in the convexified field.

Now let us consider semilinear impulsive evolution equations of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + f(x(t)), t > 0, t \neq t_i, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad 0 < t_1 < t_2 < \dots, < t_n < T\end{aligned}\tag{3}$$

where A is a linear and unbounded operator. Again, we consider a relaxation problem as above and we also proved the main results such as existence of optimal control pair in the convexified field and density property of the original controls in the set of relaxation controls.

Out put ที่ได้จากโครงการ

We published three papers as the followings:

1. **P. Sattayatham**, Relaxed control for a class of strongly nonlinear impulsive evolution equations, Computer and Mathematics with Applications 52(2006), 779-790.
2. P. Pongchalee, **P. Sattayatham**, and X. Xiang, Relaxation of nonlinear impulsive controlled systems on Banach spaces, Nonlinear Analysis (Article in press 2007).
3. **P. Sattayatham**, Relaxed control for a class of semilinear impulsive evolution equations. (Accepted to published in Pacific journal of Pure and Applied Mathematics 2007).

ภาคผนวก
(Appendix)



Relaxed Control for a Class of Strongly Nonlinear Impulsive Evolution Equations

PAIROTE SATTAYATHAM

School of Mathematics, Suranaree University of Technology
Nakhon Ratchasima 30000, Thailand
sattayatham@yahoo.com

Abstract—Relaxed control for a class of strongly nonlinear impulsive evolution equations is investigated. Existence of solutions of strongly nonlinear impulsive evolution equations is proved and properties of original and relaxed trajectories are discussed. The existence of optimal relaxed control and relaxation results are also presented. For illustration, one example is given. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Impulsive system, Banach space, Nonlinear monotone operator, Evolution triple, Relaxation.

1. INTRODUCTION

In this paper, we present sufficient conditions of optimality for optimal relaxed control problems arising in systems governed by strongly nonlinear impulsive evolution equations on Banach spaces. The general descriptions of such systems were proposed in [1] as given below.

$$\begin{aligned} \dot{x}(t) + A(t, x(t)) &= g(t, x(t), u(t)), & t \in I \setminus D, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), & i = 1, 2, \dots, n, \end{aligned} \tag{1}$$

where $I \equiv (0, T)$ is a bounded open interval of the real line and let the set $D \equiv \{t_1, t_2, \dots, t_n\}$ be a partition on $(0, T)$ such that $0 < t_1 < t_2 < \dots < t_n < T$. In general, the operator A is a nonlinear monotone operator, g is a nonlinear nonmonotone perturbation, $\Delta x(t) \equiv x(t_i^+) - x(t_i^-) \equiv x(t_i^+) - x(t_i)$, $i = 1, 2, \dots, n$, and F_i s are nonlinear operators. This model includes all the standard models used by many authors in the field (see [2,3]). The objective functional is given by $J(x, u) = \int_0^T L(t, x(t), u(t)) dt$.

In a recent paper by the author [1], the existence of optimal control was proved, but sufficient conditions of relaxation for optimality were not addressed. We wish to present just that. Before we can consider such problems, we need some preparation. The rest of the paper is organized as follows. In Section 2, some basic notations and terminologies are presented. Section 3 contains

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some preparatory results. Relaxed impulsive systems are presented in Section 4. Sufficient conditions of relaxation for optimality are discussed in Section 5. In Section 6, we presented an example for illustration.

2. SYSTEM DESCRIPTION

Let V be a real reflexive Banach space with topological dual V^* and H be a real separable Hilbert space. Let $V \hookrightarrow H \hookrightarrow V^*$ be an evolution triple and the embedding $V \hookrightarrow H$ be compact.

The system model considered here is based on this evolution triple (see [4, Chapter 23]).

Let $\langle x, y \rangle$ denote the pairing of an element $x \in V^*$ and an element $y \in V$. If $x, y \in H$, then $\langle x, y \rangle = (x, y)$, where (x, y) is the scalar product on H . The norm in any Banach space X will be denoted by $\|\cdot\|_X$.

Let $p, q \geq 1$ be such that $2 \leq p < +\infty$ and $1/p + 1/q = 1$ and let $I \equiv (0, T)$. For p, q satisfying the preceding conditions, it follows from reflexivity of V that both $L_p(I, V)$ and $L_q(I, V^*)$ are reflexive Banach spaces. The pairing between $L_p(I, V)$ and $L_q(I, V^*)$ will be denoted by $\langle \cdot, \cdot \rangle$.

Define

$$W_{pq}(I) = \{x : x \in L_p(I, V), \dot{x} \in L_q(I, V^*)\},$$

with the norm

$$\|x\|_{W_{pq}(I)} = \|x\|_{L_p(I, V)} + \|\dot{x}\|_{L_q(I, V^*)},$$

where $\dot{x}$ denoted the derivative of x in the generalized sense. The space $(W_{pq}(I), \|\cdot\|_{W_{pq}(I)})$ becomes a Banach space which is clearly reflexive and separable and the embedding $W_{pq}(I) \hookrightarrow C(\bar{I}, H)$ is continuous. If the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq}(I) \hookrightarrow L_p(I, H)$ is also compact (see [4, Problem 23.13(b)]). Sometimes we write $W_{pq}(0, T)$ instead of $W_{pq}(I)$. Similarly, we can define $W_{pq}(s, t)$ for $0 \leq s < t \leq T$ and the space $(W_{pq}(s, t), \|\cdot\|_{W_{pq}(s, t)})$ is also a separable reflexive Banach space. Moreover, the embedding $W_{pq}(s, t) \hookrightarrow C([s, t], H)$ is continuous and the embedding $W_{pq}(s, t) \hookrightarrow L_p((s, t), H)$ is also compact. We define the set $PW_{pq}(0, T) = \{x : x|_{(t_i, t_{i+1})} \in W_{pq}(t_i, t_{i+1}); i = 0, 1, 2, \dots, n, t_0 = 0, t_{n+1} = T\}$. For each $x \in PW_{pq}(0, T)$, we define $\|x\|_{PW_{pq}(0, T)} = \sum_{i=0}^n \|x\|_{W_{pq}(t_i, t_{i+1})}$. As a result, the space $(PW_{pq}(0, T), \|\cdot\|_{PW_{pq}(0, T)})$ becomes a Banach space. Let $PC([0, T], H) = \{x : x \text{ is a map from } [0, T] \text{ into } H \text{ such that } x \text{ is continuous at every point } t \neq t_i, \text{ left continuous at } t = t_i \text{ and right-hand limit } x(t_i^+) \text{ exists for } i = 1, 2, \dots, n\}$. Equipped with the supremum norm topology, it is a Banach space. Consider the following impulsive evolution equation:

$$\begin{aligned} \dot{x}(t) + A(t, x(t)) &= g(t, x(t)), & t \in I \setminus D, \\ x(0) &= x_0 \in H, \\ \Delta x(t_i) &= F_i(x(t_i)), & i = 1, 2, \dots, n \text{ and } 0 < t_1 < t_2 < \dots < t_n < T. \end{aligned} \quad (2)$$

By a solution x of problem (2), we mean a function $x \in PW_{pq}(0, T) \cap PC([0, T], H)$ such that $x(0) = x_0$ and $\Delta x(t_i) = F_i(x(t_i))$, $i = 1, 2, \dots, n$, and satisfies

$$\langle \dot{x}(t), v \rangle + \langle A(t, x(t)), v \rangle = \langle g(t, x(t)), v \rangle$$

for all $v \in V$ and μ -a.e. on I , where μ is the Lebesgue measure on I .

3. PREPARATORY RESULTS

For the study of relaxation for optimality it is essential to guarantee the existence and uniqueness of solutions of the impulsive evolution equation and certain other related equations. Here in this section, for convenience of the reader, we quote some results from the recent work of the author [1]. First, we recall some hypotheses on the data of problem (2).

HYPOTHESIS (A). $A : I \times V \rightarrow V^*$ is an operator such that

- (1) $t \mapsto A(t, x)$ is weakly measurable, i.e., the functions $t \mapsto \langle A(t, x), v \rangle$ are μ -measurable on I , for all $x, v \in V$.
- (2) For each $t \in I$, the operator $A(t) : V \rightarrow V^*$ is uniformly monotone and hemicontinuous, that is, there is a constant $c_1 \geq 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geq c_1 \|x_1 - x_2\|_V^p$$

for all $x_1, x_2 \in V$, and the map $s \mapsto \langle A(t, x + sz), y \rangle$ is continuous on $[0, 1]$ for all $x, y, z \in V$.

- (3) Growth condition: there exists a constant $c_2 > 0$ and a nonnegative function $a_1(\cdot) \in L_q(I)$ such that

$$\|A(t, x)\|_{V^*} \leq a_1(t) + c_2 \|x\|_V^{p-1},$$

for all $x \in V$, for all $t \in I$.

- (4) Coerciveness: there exists a constant $c_3 > 0$ and $c_4 \geq 0$ such that

$$\langle A(t, x), x \rangle \geq c_3 \|x\|_V^p - c_4, \quad \text{for all } x \in V, \quad \text{for all } t \in I.$$

HYPOTHESIS (G). $g : I \times H \rightarrow V^*$ is an operator such that

- (1) g is measurable in the first variable and continuous in the second variable.
- (2) g is locally Lipschitz with respect to x , i.e., for any $\rho > 0$, there exists a positive constant $L(\rho)$ such that

$$\|g(t, x_1) - g(t, x_2)\|_{V^*} \leq L(\rho) \|x_1 - x_2\|_H,$$

for all $t \in I$ and all $x_1, x_2 \in H$ with $\|x_1\| \leq \rho, \|x_2\| \leq \rho$.

- (3) There exist $c_5 > 0$, $2 \leq k < p$, and a nonnegative function $h_1(\cdot) \in L_q(I)$ such that

$$\|g(t, x)\|_{V^*} \leq h_1(t) + c_5 \|x\|_H^{k-1}$$

for all $x \in V, t \in I$.

HYPOTHESIS (F). $F_i : H \rightarrow H$ is locally Lipschitz continuous on H , i.e., for any $\rho > 0$, there exists a constant $L_i(\rho)$ such that

$$\|F_i(x_1) - F_i(x_2)\|_H \leq L_i(\rho) \|x_1 - x_2\|_H$$

for all $x_1, x_2 \in H$ with $\|x_1\|_H \leq \rho, \|x_2\|_H \leq \rho$ ($i = 1, 2, \dots, n$).

It is sometimes convenient to rewrite system (2) into an operator equation. To do this, we set $X = L_p(I, V)$ and hence $X^* = L_q(I, V^*)$. Moreover, we set

$$\begin{aligned} A(x)(t) &= A(t, x(t)), \\ G(x)(t) &= g(t, x(t)), \end{aligned} \tag{3}$$

for all $x \in X$ and for all $t \in (0, T)$. Then the original equation (2) is equivalent to the following operator equation (see [4, Theorem 30.A]):

$$\begin{aligned} \dot{x} + Ax &= G(x), \\ x(0) &= x_0 \in H, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad i = 1, 2, \dots, n \quad \text{and} \quad 0 < t_1 < t_2 < \dots < t_n < T. \end{aligned} \tag{4}$$

It follows from Theorem 30.A of Zeidler [4] that equation (4) defines an operator $A : X \rightarrow X^*$ such that A is uniformly monotone, hemicontinuous, coercive, and bounded. Moreover, by using Hypothesis (G)(3) and using the same technique as in Theorem 30.A, one can show that the operator $G : L_p(I, H) \rightarrow X^*$ is also bounded and satisfies

$$\|G(v)\|_{X^*} \leq M_1 + M_2 \|v\|_{L_p(I, H)}^{k-1}, \quad \text{for all } v \in L_p(I, H).$$

We state the following results from [1, Theorem B].

THEOREM 3.1. *Under Assumptions (A), (F), and (G), system (2) has a unique solution $x \in PW_{pq}(0, T) \cap PC([0, T], H)$ and there is a constant $M > 0$ such that $\|x\|_{PW_{pq}(0, T)} \leq M$ and $\|x\|_{PC([0, T], H)} \leq M$.*

PROOF. See [1, Theorem B] for the proof of existence and boundedness. The uniqueness follows from Assumption (G)(2). To see this, suppose that system (2) has two solutions x_1, x_2 . Then it follows from integration by part formula and monotonicity of $A(t, x)$ that

$$\begin{aligned} \|x_1(t) - x_2(t)\|_H^2 &= 2 \int_0^t \left\langle \dot{x}_1(s) - \dot{x}_2(s), x_1(s) - x_2(s) \right\rangle_{V^* - V} ds \\ &= -2 \int_0^t \langle A(s, x_1(s)) - A(s, x_2(s)), x_1(s) - x_2(s) \rangle_{V^* - V} ds \\ &\quad + 2 \int_0^t \langle g(s, x_1(s)) - g(s, x_2(s)), x_1(s) - x_2(s) \rangle_{V^* - V} ds \\ &\leq 2 \int_0^t \langle g(s, x_1(s)) - g(s, x_2(s)), x_1(s) - x_2(s) \rangle_{V^* - V} ds \\ &\leq 2 \int_0^t \|g(s, x_1(s)) - g(s, x_2(s))\|_{V^*} \|x_1(s) - x_2(s)\|_V ds \\ &\leq 2L(\rho) \int_0^t \|x_1(s) - x_2(s)\|_H \|x_1(s) - x_2(s)\|_V ds \\ &\leq 2L(\rho)C_1 \int_0^t \|x_1(s) - x_2(s)\|_H^2 ds, \end{aligned}$$

for some positive constant C_1 . By Gronwall's lemma, we get $x_1(t) = x_2(t)$ for all $t \in [0, T]$. Hence $x_1 = x_2$ and this prove the uniqueness of the solution of system (2).

Now, let us consider the corresponding control system. We model the control space by a separable complete metric space Z (i.e., a Polish space). By P_f (P_{fc}), we denote a class of nonempty closed (closed and convex) subset of Z . Let $I = (0, T)$. Recall that a multifunction $\Gamma : I \rightarrow P_f(Z)$ is said to be measurable if for each $F \in P_f Z$, $\Gamma^{-1}(F)$ is Lebesgue measurable in I . We defined S_Γ to be the set of all measurable selections of $\Gamma(\cdot)$, i.e.,

$$S_\Gamma = \{u : I \rightarrow Z \mid u(t) \text{ is measurable and } u(t) \in \Gamma(t), \mu\text{-a.e. } t \in I\},$$

where μ is the Lebesgue measure on I . Note that the set $S_\Gamma \neq \emptyset$ if $\Gamma : I \rightarrow P_f(Z)$ is measurable (see [5, Theorem 2.23, p. 100]). Consider the following control systems:

$$\begin{aligned} \dot{x}(t) + A(t, x(t)) &= g(t, x(t), u(t)), \quad t \in I, \\ x(0) &= x_0 \in H, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad i = 1, 2, \dots, n, \quad 0 < t_1 < t_2 < \dots < t_n < T. \end{aligned} \tag{5}$$

Here, we require the operator A and F_i s of (5) to satisfy Hypotheses (A), and (F), respectively, as in Section 3. We now give some new hypotheses for the remaining data.

HYPOTHESIS (U). $U : I \rightarrow P_{fc}(Z)$ is a measurable multifunction satisfying $U(\cdot) \subset F$, where F is a compact subset of Z . For the admissible controls, we choose the set $U_{ad} = S_U$.

HYPOTHESIS (G1). $g : I \times H \times Z \rightarrow V^*$ is an operator such that

- (1) $t \mapsto g(t, x, z)$ is measurable, and the map $(x, z) \mapsto g(t, x, z)$ is continuous on $H \times Z$.
- (2) For each fixed z , $g(t, x, z)$ is locally Lipschitz continuous with respect to x and uniformly in t .
- (3) There exist constants $a, b > 0$ such that

$$\|g(t, x, z)\|_{V^*} \leq a + b\|x\|_H^{k-1},$$

for all $x \in H, t \in (0, T)$, and $z \in Z$, where $2 \leq k < p$.

By Assumption (U), the control set S_U is nonempty and is called the class of original control. Now, let us define

- $X_0 = \{x \in PW_{pq}(I) \cap PC(\bar{I}, H) \mid x \text{ is a solution of (5) corresponding to } u\}$.
- X_0 is called the class of original trajectories.
- $A_0 = \{(x, u) \in PW_{pq}(I) \cap PC(\bar{I}, H) \times S_U \mid x \text{ is a solution of (5) corresponding to } u\}$.
- A_0 is called the class of admissible state-control pairs.

The following theorem guarantees that $X_0 \neq \emptyset$. Its proof follows immediately from Theorem 3.1 by defining the function $g_u(t, x) = g(t, x, u)$ and noting that g_u satisfies all hypotheses of Theorem 3.1.

THEOREM 3.2. Assume that Hypotheses (A), (F), (G1), and (U) hold. For every $u \in S_U$, equation (5) has a unique solution $x(u) \in PW_{pq}(I) \cap PC(\bar{I}, H)$. Moreover the set X_0 is bounded in $PW_{pq}(I) \cap PC(\bar{I}, H)$, i.e., $\|x(u)\|_{PW_{pq}(0, T)} \leq M$ and $\|x(u)\|_{PC([0, T], H)} \leq M$ for all $u \in S_U$.

4. RELAXED IMPULSIVE SYSTEMS

We consider the following optimal control problem:

$$\inf \left\{ J(x, u) = \int_0^T L(t, x(t), u(t)) dt \right\} \quad (P)$$

subject to equation (5).

It is well known that, to solve the optimization problem involving (P) and obtain an optimal state-control pair, we need some kind of convexity hypothesis on the orientor field $L(t, x(t), u(t))$. If the convexity hypothesis is no longer satisfied, in order to get an optimal admissible pair, we need to pass to a larger system with measure control (or know as "relaxed control") in which the orientor field have been convexified. For this purpose, we introduce the relaxed control and the corresponding relaxed systems.

Let Z be a separable complete metric space (i.e., a Polish space) and $B(Z)$ be its Borel σ -field. Let (Ω, Σ, μ) be a measure space. We will denote the space of probability measures on the measurable space $(Z, B(Z))$ by $M_+^1(Z)$.

A Caratheodory integrand on $\Omega \times Z$ is a function $f : \Omega \times Z \rightarrow R$ such that $f(\cdot, x)$ is Σ -measurable on Ω , $f(\omega, \cdot)$ is continuous on Z for all $\omega \in \Omega$, and $\sup\{|f(\omega, z)| : z \in Z\} \leq \alpha(\omega)$, a.e., for some functions $\alpha(\cdot) \in L_1(\Omega)$. We denote the set of all Caratheodory integrands on $\Omega \times Z$ by $\text{Car}(\Omega, Z)$. By a transition probability, we mean a function $\lambda : \Omega \times B(Z) \rightarrow [0, 1]$ such that for every $A \in B(Z)$, $\lambda(\cdot, A)$ is Σ -measurable and for every $\omega \in \Omega$, $\lambda(\omega, \cdot) \in M_+^1(Z)$. We use $R(\Omega, Z)$ to denote the set of all transition probability from (Ω, Σ) into $(Z, B(Z))$. Following Balder [6], we can define a topology on $R(\Omega, Z)$ as follows: let $f \in \text{Car}(\Omega, Z)$ and define

$$I_f(\lambda) = \int_{\Omega} \int_Z f(\omega, z) \lambda(\omega) (dz) d\mu(\omega). \quad (6)$$

The weak topology on $R(\Omega, Z)$ is defined as the weakest topology for which all functionals $I_f : R(\Omega, Z) \rightarrow R$, $f \in \text{Car}(\Omega, Z)$, are continuous.

Suppose $\Omega = I = [0, T]$ and Z is a compact Polish space, then the space $\text{Car}(I, Z)$ can be identified with the separable Banach space $L_1(I, C(Z))$ where $C(Z)$ is the space of all real-valued continuous functions on Z . To see this, we associate to each Caratheodory integrand $\phi(\cdot, \cdot)$ the map $t \mapsto \phi(t, \cdot) \in C(Z)$. Let $M(Z)$ be the space of all regular bounded countably additive measure defined on $B(Z)$. We note that $M(Z)$ is a Banach space under the total variation norm, i.e., $\|\lambda\|_{M(Z)} = |\lambda|(Z)$. Then by Riesz representation theorem, the dual $[C(Z)]^*$ can be identified algebraically and metrically with $M(Z)$. The duality pair between $M(Z)$ and $C(Z)$ is given by

$$\langle \lambda, f \rangle = \int_Z f(z) \lambda(dz). \quad (7)$$

So $M(Z)$ is a separable (see [7, p. 265]) dual Banach space and hence has a Radon-Nikodym property. This observation, combined with Theorem 1 of Diestel and Uhr [8, p. 98], tells us that

$$L_1(I, C(Z))^* = L_\infty(I, M(Z)).$$

So the weak topology on $R(I, Z)$ coincides with the relative $w^*(L_\infty(I, M(Z)), L_1(I, C(Z)))$ topology. The duality pair between $L_\infty(I, M(Z))$ and $L_1(I, C(Z))$ is given by

$$\begin{aligned} \langle \lambda, f \rangle &= \int_0^T \langle \lambda(t), f(t) \rangle dt \\ &= \int_0^T \int_Z f(t)(z) \lambda(t)(dz) dt \\ &= \int_0^T \int_Z f(t, z) \lambda(t)(dz) dt, \end{aligned} \quad (8)$$

which is the same formula as in (6) with $f(t, z) \equiv f(t)(z)$. This fact will be useful in the study of the relaxed control system where the control functions are transition probability.

Now we introduce some assumptions imposed on the class of relaxed control which will be denoted by S_Σ .

ASSUMPTION (U1). Z is a compact Polish space, $U : I \rightarrow P_{fc}(Z)$ is a measurable multifunction.

Define $\Sigma(t) = \{\lambda \in M_+^1(Z), \lambda(U(t)) = 1\}$ and let $S_\Sigma \subseteq R(I, Z)$ be the set of transition probabilities on $I \times B(Z)$ that are measurable selections of $\Sigma(\cdot)$. For any $u \in S_U$, we define the relaxation $\delta_u \in S_\Sigma$ of u by $\delta_u(t) \equiv \text{Dirac probability measure at } u(t)$. Then we can identify $S_U \subseteq S_\Sigma$. From now on, we shall consider S_U and S_Σ as a subspace of the topological space $R(I, Z)$ with the weak topology defined above.

We list two lemmas which will be useful in discussing relaxation problem. The proofs can be found in [7, Theorem IV 2.1] and [6, Corollary 3], respectively.

LEMMA 4.1. Suppose Z is a compact Polish space. Then S_Σ is convex, compact, and sequentially compact.

LEMMA 4.2. S_U is dense in S_Σ .

THEOREM 4.3. Let $h : I \times H \times Z \rightarrow R$ be such that

1. $t \mapsto (t, x, z)$ is measurable and $(x, z) \mapsto h(t, x, z)$ is continuous.
2. $|h(t, x, z)| \leq \psi(t) \in L_1(I)$ for all $(x, z) \in H \times Z$.

If $x_k \rightarrow x \in C([0, T], H)$ then

$$\bar{h}_k(\cdot, \cdot) \rightarrow \bar{h}(\cdot, \cdot), \quad \text{in } L_1(I, C(Z)),$$

as $k \rightarrow \infty$, where $\bar{h}_k(t, z) = h(t, x_k(t), z)$ and $\bar{h}(t, z) = h(t, x(t), z)$.

PROOF. The proof is similar to Lemma 3.3 in [3].

Next, let us consider this new larger system know as "relaxed impulsive system"

$$\begin{aligned} \dot{x}(t) + A(t, x(t)) &= \int_Z g(t, x(t), z) \lambda(t)(dz), \quad 0 \leq t \leq T, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad i = 1, 2, \dots, n. \end{aligned} \quad (9)$$

We will denote the set of trajectories of (9) by X_r , i.e., $X_r = \{x \in PW_{pq}(I) \cap PC(\bar{I}, H) \mid x \text{ is a solution of (9) corresponding to } \lambda \in S_\Sigma\}$. Moreover, the set of admissible state-control pairs

of (9) will be denoted by $A_r = \{(x, \lambda) \in PW_{pq}(I) \cap PC(\bar{I}, H) \times S_\Sigma \mid x \text{ is a solution of (9) corresponding to } \lambda \in S_\Sigma\}$.

Note that $X_0 \subseteq X_r$, since $S_U \subseteq S_\Sigma$ and if the hypotheses of Theorem 3.2 are satisfied, $X_0 \neq \emptyset \Rightarrow X_r \neq \emptyset$. To see this, given any relaxed control $\lambda \in S_\Sigma$, if we set $\bar{g}(t, x(t), \lambda(t)) = \int_Z g(t, x(t), z) \lambda(t)(dz)$ then, working as in the proof of Theorem 3.2, one can show that there exists a relaxed admissible trajectory $x(\lambda)$ corresponding to λ . We summarize the above discussion into a theorem.

THEOREM 4.4. Assume that Hypotheses (A), (F), (G1), and (U1) hold. For every $\lambda \in S_\Sigma$, equation (9) has a unique solution $x(\lambda) \in PW_{pq}(I) \cap PC(\bar{I}, H)$. Moreover the set X_r is bounded in $PW_{pq}(I) \cap PC(\bar{I}, H)$, i.e., $\|x(\lambda)\|_{PW_{pq}(0,T)} \leq M$ and $\|x(\lambda)\|_{PC([0,T],H)} \leq M$ for all $\lambda \in S_\Sigma$.

The next theorem gives us a useful relation between X_0 and X_r .

THEOREM 4.5. If Assumptions (A), (F), (G1), and (U1) hold, then $X_r = \bar{X}_0$ (closure is taken in $PC(\bar{I}, H)$).

Before proving this theorem, we need a lemma.

LEMMA 4.6. If Assumptions (A), (F), (G1), and (U1) hold and $\lambda_k \rightarrow \lambda$ in $R(I, Z)$, suppose that $\{x_k, x\}$ is the solution of (9) corresponding to $\{\lambda_k, \lambda\}$, by working with a subsequence if necessary, $x_k \rightarrow x$ in $PC(\bar{I}, H)$ as $k \rightarrow \infty$.

PROOF. Suppose that $\lambda_k \rightarrow \lambda$ in $R(I, Z)$ as $k \rightarrow \infty$ and $\{x_k, x\}$ is the solution of (9) corresponding to $\{\lambda_k, \lambda\}$. Since $(x_k, \lambda_k) \in A_r$ for each positive integer k , then (x_k, u_k) must satisfy the equation

$$\begin{aligned} \dot{x}_k(t) + A(t, x_k(t)) &= \int_Z g(t, x_k(t), z) \lambda_k(t)(dz), \\ x_k(0) &= x_0 \in H, \\ \Delta x_k(t_i) &= F_i(x_k(t_i)), \quad i = 1, 2, \dots, n, \quad 0 < t_1 < t_2 < \dots < t_n < T. \end{aligned} \quad (10)$$

To finish the proof, we try to choose $y \in X_r$ such that y is a solution of (9) corresponding to this λ and $x_k \rightarrow y$ in $PC(\bar{I}, H)$ as $k \rightarrow \infty$. The uniqueness property of the solution of (9) implies $x = y$ and hence $x_k \rightarrow x$ in $PC(\bar{I}, H)$. This proves that $x \in \bar{X}_0$ and we are done. We shall do this by considering each case separately.

CASE 1. Find y on the interval $(0, t_1)$. For notational convenience, we let $I_1 = (0, t_1)$, $X_1 = L_p(I_1, V)$, and $X_1^* = L_q(I_1, V^*)$. We note that $X_1 = L_p(I_1, V)$ can be considered as a closed subspace of $X = L_p(I, V)$. Let x_k^1 and λ_k^1 be the restriction of the functions x_k, λ_k on the interval I_1 ($k = 1, 2, \dots$). Hence, by Theorem 4.4, $\{x_k^1\}$ is bounded in $W_{pq}(I_1)$. By reflexivity of $W_{pq}(I_1)$ there is a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$x_k^1 \xrightarrow{w} x^1, \quad \text{in } W_{pq}(I_1), \quad \text{as } k \rightarrow \infty. \quad (11)$$

Since the embedding $W_{pq}(I_1) \hookrightarrow X_1$ is continuous, the embedding $W_{pq}(I_1) \hookrightarrow L_p(I_1, H)$ is compact and the operator $A: X_1 \rightarrow X_1^*$ maps bounded sets to bounded sets, it follows from (11) that there exists a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$\begin{aligned} x_k^1 &\xrightarrow{w} x^1 \quad \text{in } X_1, & x_k^1 &\xrightarrow{w} \dot{x}^1, \quad \text{in } X_1^*, \\ Ax_k^1 &\xrightarrow{w} \xi, \quad \text{in } X_1^*, \\ x_k^1 &\xrightarrow{s} x^1, \quad \text{in } L_p(I_1, H), \end{aligned} \quad (12)$$

and, by [3, Theorem 3.B], $x_k^1 \xrightarrow{s} x^1$ in $C([0, t_1], H)$, as $k \rightarrow \infty$ and for some $\xi \in X_1^*$. Consider the following equation:

$$\begin{aligned} \dot{x}_k^1(t) + A(t, x_k^1(t)) &= \int_Z g(t, x_k^1(t), z) \lambda_k^1(t)(dz), \quad 0 \leq t < t_1, \\ x_k^1(0) &= x_0. \end{aligned} \quad (13)$$

Define an operator $G_k : I_1 \rightarrow V^*$ and $G : I_1 \rightarrow V^*$ as follows:

$$G_k(t) = \int_Z g(t, x_k^1(t), z) \lambda_k^1(t)(dz), \quad k = 1, 2, 3, \dots,$$

$$G(t) = \int_Z g(t, x^1(t), z) \lambda^1(t)(dz).$$

It follows from Assumption (G1) that G and $G_k \in L_q(I_1, V^*)$.

With this new notation, equation (13) can be rewritten into an equivalent operator equation of the form

$$\begin{aligned} \dot{x}_k^1 + A(x_k^1) &= G_k; & 0 < t < t_1, \\ x_k^1(0) &= x_0. \end{aligned} \quad (14)$$

For each fixed $v \in V$, define

$$\begin{aligned} \bar{g}_k(t, z) &= \langle g(t, x_k^1(t), z), v \rangle_{V^* \cdot V}, \\ \bar{g}(t, z) &= \langle g(t, x^1(t), z), v \rangle_{V^* \cdot V}. \end{aligned}$$

It follows from Assumption (G1) that, for each fixed $t \in I_1$,

$$\bar{g}_k(t, \cdot), \text{ and } \bar{g}(t, \cdot) \in C(Z),$$

and furthermore

$$\bar{g}_k(\cdot, \cdot), \text{ and } \bar{g}(\cdot, \cdot) \in L_1(I_1, C(Z)).$$

Since $x_k^1 \xrightarrow{s} x^1$ in $C([0, t_1], H)$ (see equation (12)), then Theorem 4.3 gives

$$\bar{g}_k(\cdot, \cdot) \rightarrow \bar{g}(\cdot, \cdot), \quad \text{in } L_1(I_1, C(Z)), \quad \text{as } k \rightarrow \infty.$$

Since $\lambda_k^1 \rightarrow \lambda^1$ in $R(I_1, Z)$, by equation (7), we have $\lambda_k^1 \xrightarrow{w} \lambda^1$ in $(L_1(I_1, C(Z)))^*$ as $k \rightarrow \infty$. Hence, it follows from Proposition 21.6(e) of Zeidler [4, p. 216] that

$$\langle\langle \lambda_k^1, \bar{g}_k \rangle\rangle \rightarrow \langle\langle \lambda^1, \bar{g} \rangle\rangle, \quad \text{as } k \rightarrow \infty.$$

This means that

$$\int_0^{t_1} \int_Z \langle g(t, x_k^1(t), z), v \rangle \lambda_k^1(t)(dz) dt \rightarrow \int_0^{t_1} \int_Z \langle g(t, x^1(t), z), v \rangle \lambda^1(t)(dz) dt \quad (15)$$

as $k \rightarrow \infty$. The convergence in (15) is true for all $v \in V$ then we get

$$G_k \xrightarrow{w} G, \quad \text{as } k \rightarrow \infty, \quad \text{in } L_q(I_1, V^*).$$

By equation (12), $x_k^1 \xrightarrow{s} x^1$ in $C([0, t_1], H)$ and this implies $x_k^1(0) \rightarrow x^1(0)$ in H as $k \rightarrow \infty$. Referring to the initial condition, we have $x_k^1(0) = x_0 \in H$ for all $k = 1, 2, 3, \dots$. Thus, $x^1(0) = x_0$.

Up to this point, we can conclude that x^1 satisfies the following equation:

$$\begin{aligned} \dot{x}^1(t) + \xi(t) &= \int_Z g(t, x^1(t), z) \lambda(t)(dz), \\ x^1(0) &= x_0 \in H. \end{aligned}$$

Next we aim to prove that $\xi = Ax^1$ in X_1^* .

To prove this we note from equation (14) that

$$\begin{aligned} \langle\langle A(x_k^1), x_k^1 \rangle\rangle_{X_1^* - X_1} &= \langle\langle A(x_k^1), x^1 \rangle\rangle_{X_1^* - X_1} \\ &- \langle\langle \dot{x}_k^1, x_k^1 - x^1 \rangle\rangle_{X_1^* - X_1} + \langle\langle G_k, x_k^1 - x^1 \rangle\rangle_{X_1^* - X_1}. \end{aligned} \quad (16)$$

From integration by part formula, we have

$$\begin{aligned} \langle\langle \dot{x}_k^1, x_k^1 - x^1 \rangle\rangle_{X_1^* - X_1} &= \langle\langle \dot{x}^1, x_k^1 - x^1 \rangle\rangle_{X_1^* - X_1} \\ &+ \frac{1}{2} \left(\|x_k^1(t_1) - x^1(t_1)\|_H^2 - \|x_k^1(0) - x^1(0)\|_H^2 \right). \end{aligned} \quad (17)$$

Substitute (17) into (16) and noting that the second term on the right-hand side of (17) is always nonnegative, then we get

$$\begin{aligned} \langle\langle A(x_k^1), x_k^1 \rangle\rangle_{X_1^* - X_1} &\leq \langle\langle A(x_k^1), x^1 \rangle\rangle_{X_1^* - X_1} - \langle\langle \dot{x}^1, x_k^1 - x^1 \rangle\rangle_{X_1^* - X_1} \\ &\quad + \|x_k^1(0) - x^1(0)\|_H^2 + \langle\langle G_k, x_k^1 - x^1 \rangle\rangle_{X_1^* - X_1}. \end{aligned}$$

Therefore,

$$\lim_{k \rightarrow \infty} \langle\langle A(x_k^1), x_k^1 \rangle\rangle_{X_1^* - X_1} \leq \langle\langle \xi, x^1 \rangle\rangle_{X_1^* - X_1},$$

and hence A satisfies condition (M) (see [4, p. 474]). Then we have

$$A(x^1) = \xi.$$

Now we can say that x^1 is the solution of the following equation:

$$\begin{aligned} \dot{x}^1(t) + A(t, x^1(t)) &= \int_Z g(t, x^1(t), z) \lambda(t)(dz), \\ x^1(0) &= x_0 \in H. \end{aligned}$$

This proves that x^1 satisfies (9) on the interval $(0, t_1)$ and x^1 is the required y on $(0, t_1)$.

CASE 2. Find y on the interval (t_1, t_2) . The proof is similar to Case 1. Here, let $I_2 = (t_1, t_2)$, $X_2 = L_p(I, V)$, and $X_2^* = L_q(I, V^*)$. Let x_k^2, u_k^2 be the restriction of the functions x_k and u_k on the interval I_2 , respectively ($k = 1, 2, \dots$). It follows from equation (10) that (x_k^2, u_k^2) satisfies the operator equation

$$\begin{aligned} \dot{x}_k^2 + A(x_k^2) &= G_k; \quad t_1 < t < t_2, \\ x_k^2(t_1^+) &= x_k^2(t_1^-) + F_1(x_k^2(t_1)), \end{aligned} \quad (18)$$

where $x_k^2(t_1^-) = x_k^2(t_1) = x_k^1(t_1)$ ($k = 1, 2, 3, \dots$). By using the same proof as in Case 1, we get that

$$x_k^2 \xrightarrow{w} x^2, \quad \text{in } W_{pq}(t_1, t_2), \quad \text{and} \quad x_k^2 \xrightarrow{s} x^2, \quad \text{in } C([t_1, t_2], H), \quad \text{as } k \rightarrow \infty,$$

which implies that $x_k^2(t_1^+) \rightarrow x^2(t_1^+)$ in H as $k \rightarrow \infty$ and, moreover, x^2 is also satisfied the operator equation

$$\dot{x}^2 + A(x^2) = G; \quad t_1 < t < t_2.$$

We are left to verify the initial condition at t_1 . To see this, we note that the expression on the right-hand side of (18) converges to $x^1(t_1) + F_1(x^1(t_1))$ as $k \rightarrow \infty$ (see Hypothesis (F)). On the other hand, the left-hand side $x_k^2(t_1^+) \rightarrow x^2(t_1^+)$ in H as $k \rightarrow \infty$. Hence, $x^2(t_1^+) = x^1(t_1) + F_1(x^1(t_1)) \equiv x^2(t_1^-) + F_1(x^2(t_1))$. This proves that x^2 satisfies (9) on the interval (t_1, t_2) and x^2 is the required y on (t_1, t_2) . Continuing this process we can find y on the interval (t_j, t_{j+1}) , $j = 0, 1, \dots, n$. By piecing them together from $j = 1, 2, \dots, n$ and taking into account the impact

of jump, we obtain y which is the solution of (9) corresponding to the relaxed control λ satisfying $x_k \rightarrow y$ in $PC(\bar{I}, H)$ as $k \rightarrow \infty$. Since $x = y$, $x_k \rightarrow x$ in $PC(\bar{I}, H)$ as $k \rightarrow \infty$. The proof of Lemma 4.6 is now complete.

PROOF OF THEOREM 4.5. First, we shall show that $X_r \subseteq \bar{X}_0$. Letting $x \in X_r$, then there exists $\lambda \in S_\Sigma$ such that $(x, \lambda) \in A_r$. By virtue of density result Lemma 4.2, there exists a sequence $\{u_k\} \in S_U$ such that $\delta_{u_k} \rightarrow \lambda$ in $R(I, Z)$. Let x_k be the solution of (9) corresponding to u_k . Then we have a sequence $\{(x_k, u_k)\} \subset A_0$. Since $(x_k, u_k) \in A_0$ for each positive integer k , then (x_k, u_k) must satisfy the equation

$$\begin{aligned}\dot{x}_k(t) + A(t, x_k(t)) &= \int_Z g(t, x_k(t), z) \delta_{u_k}(t)(dz), \\ x_k(0) &= x_0 \in H, \\ \Delta x_k(t_i) &= F_i(x_k(t_i)), \quad i = 1, 2, \dots, n, \quad k = 1, 2, 3, \dots, \\ 0 &< t_1 < t_2 < \dots < t_n < T.\end{aligned}$$

Applying Lemma 4.6, we get $x_k \rightarrow x$ in $PC(\bar{I}, H)$. This proves that $x \in \bar{X}_0$ and hence $X_r \subseteq \bar{X}_0$. Finally, we will show that X_r is closed in $PC(\bar{I}, H)$. Let $\{x_k\}$ be a sequence of points in X_r such that $x_k \rightarrow x$ in $PC(\bar{I}, H)$ as $k \rightarrow \infty$. By definition of X_r , there is a sequence $\{\lambda_k\}$ of points in S_Σ such that $(x_k, \lambda_k) \in A_r$, $k = 1, 2, 3, \dots$. By Lemma 4.1, S_Σ is compact in $R(I, Z)$ under the weak topology. Moreover, $R(I, Z)$ -topology coincides with the relative $w^*(L_\infty(I, M(Z)), L_1(I, C(Z)))$ topology which is metrizable (see [2, p. 276]). Then, by passing to a subsequence if necessary, we may assume that $\lambda_k \rightarrow \lambda$ in $R(I, Z)$. Applying Lemma 4.6, there is $x \in X_r$ such that $x_k \rightarrow x$ in $PC(\bar{I}, H)$ as $k \rightarrow \infty$. Hence X_r is closed in $PC(\bar{I}, H)$ and, consequently, $\bar{X}_0 \subseteq \bar{X}_r = X_r$. The proof of Theorem 4.5 is now complete.

The following corollary is an immediate consequence of Lemma 4.6.

COROLLARY 4.7. Under assumption of Theorem 4.5, the function $\lambda \mapsto x(\lambda)$ is continuous from $S_\Sigma \subseteq R(I, Z)$ into $PC(\bar{I}, H)$.

5. EXISTENCE OF OPTIMAL CONTROLS

Consider the following Lagrange optimal control problem (P_r) : find a control policy $\bar{\lambda} \in S_\Sigma$, such that it imparts a minimum to the cost functional J given by

$$J(\lambda) \equiv J(x^\lambda, \lambda) \equiv \int_I \int_Z l(t, x^\lambda(t), z) \lambda(t)(dz) dt, \quad (P_r)$$

where x^λ is the solution of system (9) corresponding to the control $\lambda \in S_\Sigma$. We make the following hypothesis concerning the integrand $l(\cdot, \cdot, \cdot)$.

HYPOTHESIS (L). $l : I \times H \times Z \rightarrow R \cup \{+\infty\}$ is Borel measurable satisfying the following conditions:

- (1) $(\xi, z) \mapsto l(t, \xi, z)$ is lower semicontinuous on $H \times Z$ for each fixed t .
- (2) $\psi(t) \leq l(t, \xi, z)$ almost everywhere with $\psi(t) \in L_1(I)$.

Let $m_r = \inf\{J(\lambda) : \lambda \in S_\Sigma\}$. We have the following theorem on the existence of optimal impulsive control.

THEOREM 5.1. Suppose Assumptions (A), (F), (G1), (U1), (L) hold and Z is compact Polish space, then there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$.

PROOF. If $J(\lambda) = +\infty$ for all $\lambda \in S_\Sigma$, then every control is admissible. Assume $\inf\{J(\lambda) : \lambda \in S_\Sigma\} = m_r < +\infty$. By Assumption (L), we have $m_r > -\infty$. Hence m_r is finite. Let $\{\lambda_k\}$ be a minimizing sequence so that $\lim_{k \rightarrow \infty} J(\lambda_k) = m_r$. By Lemma 4.1, S_Σ is compact in the topology

$R(I, Z)$. Hence, by passing to a subsequence if necessary, we may assume that $\lambda_k \rightarrow \bar{\lambda}$ in $R(I, Z)$ as $k \rightarrow \infty$. This means that $\lambda_k \xrightarrow{w^*} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \rightarrow \infty$. Let $\{x_k, \bar{x}\}$ be the solution of (9) corresponding to $\{\lambda_k, \bar{\lambda}\}$. By Lemma 4.6, we get $x_k \rightarrow \bar{x}$ in $PC(I, H)$ and $(\bar{x}, \bar{\lambda}) \in A_r$. Next, we shall prove that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

As before, we identify the space of Caratheodory integrand $\text{Car}(I, Z)$ with the separable Banach space $L_1(I, C(Z))$. We note that every semicontinuous measurable integrand $l : I \times H \times Z \rightarrow R \cup \{+\infty\}$ is the limit of an increasing sequence of Caratheodory integrand $\{l_j\} \in L_1(I, C(Z))$ for each fixed $h \in H$. Thus, there exists an increasing sequence of Caratheodory integrand $\{l_j\} \in L_1(I, C(Z))$ such that

$$l_j(t, \bar{x}(t), z) \uparrow l(t, \bar{x}(t), z), \quad \text{as } j \rightarrow \infty, \quad \text{for all } t \in I, \quad z \in Z.$$

Since $x_k \rightarrow \bar{x}$ in $PC(I, H)$, by applying Theorem 4.3 on each subinterval of $[0, T]$, $l_j(t, x_k(t), z) \rightarrow l_j(t, \bar{x}(t), z)$ as $k \rightarrow \infty$ for almost all $t \in I$ and all $z \in Z$. We note that since $\lambda_k \xrightarrow{w^*} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \rightarrow \infty$, then

$$\begin{aligned} J(\bar{x}, \bar{\lambda}) &= \langle \bar{\lambda}, l \rangle = \int_I \int_Z l(t, \bar{x}(t), z) \bar{\lambda}(t)(dz) dt \\ &= \lim_{j \rightarrow \infty} \int_I \int_Z l_j(t, \bar{x}(t), z) \bar{\lambda}(t)(dz) dt \\ &= \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_I \int_Z l_j(t, x_k(t), z) \lambda_k(t)(dz) dt \\ &\leq \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_I \int_Z l_j(t, x_k(t), z) \lambda_k(t)(dz) dt = m_r. \end{aligned}$$

However, by definition of m_r , it is obvious that $J(\bar{x}, \bar{\lambda}) \geq m_r$. Hence $J(\bar{x}, \bar{\lambda}) = m_r$. This implies that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

REMARK. If $J_0(x, u) = \int_I l(t, x(t), u(t)) dt$ is the cost functional for the original problem and $m = \inf\{J_0(x, u) : u \in U_{ad}\}$, in general we have $m_r \leq m$. It is desirable that $m_r = m$, i.e., our relaxation is reasonable. With some stronger conditions on l , i.e., the map $(\xi, \eta, z) \mapsto l(t, \xi, z)$ is continuous and $||l(t, \xi, z)| \leq \theta_R(t)$ for all most all $t \in I$ and $\theta_R \in L_1(I)$, one can show that $m_r = m$. The proof is similar to Theorem 4.B. in [3].

6. EXAMPLE

In this section we present an example of a strongly nonlinear impulsive system for which our general theory can be applied. Let $I = (0, T)$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary $\partial\Omega$. For $p \geq 2$ and $\theta \geq 0$, we consider the following quasi-linear parabolic control problem:

$$\begin{aligned} \frac{\partial}{\partial t} x(t, z) - \sum_{i=1}^N D_i \left(|D_i x(t, z)|^{p-2} D_i x(t, z) \right) &= g(t, z, x(t, z), u), & \text{on } I \times \Omega, \\ x|_{I \times \partial\Omega} &= 0, & x(0, z) = x_0(z), \\ \Delta x(t_i, z) &= F_i(x(t_i, z)), & i = 1, 2, \dots, n, \end{aligned} \quad (19)$$

where $(0 < t_1 < t_2 < \dots < t_n < T)$.

Here the operator $D_i = \frac{\partial}{\partial x_i}$ ($i = 1, 2, \dots, N$). We need the following hypotheses on the data of (19).

HYPOTHESIS (G'). $f : I \times \Omega \times R \times R^N \rightarrow R$ is an operator such that

- (1) $(t, z) \mapsto f(t, z, \xi, u)$ is measurable $I \times \Omega$; and the map $(\xi, u) \mapsto f(t, z, \xi, u)$ is continuous on $R \times R^N$.

(2) There exist constants $b_1 > 0$, $b_2(t, z) \in L_2(I \times \Omega)$ such that

$$|f(t, z, \xi, u)| \leq b_1|\xi| + b_2(t, z).$$

HYPOTHESIS (F'). $F_i : L_2(\Omega) \rightarrow L_2(\Omega)$ ($i = 1, 2, \dots, n$) are operators such that for any $\rho > 0$ there exists a constant $L_i(\rho) > 0$ such that

$$\|F_i(h_1) - F_i(h_2)\|_{L_2(\Omega)} \leq L_i(\rho) \|h_1 - h_2\|_{L_2(\Omega)}$$

for all $h_1, h_2 \in L_2(\Omega)$ with $\|h_1\|_{L_2(\Omega)} \leq \rho$, $\|h_2\|_{L_2(\Omega)} \leq \rho$ ($i = 1, 2, \dots, n$).

THEOREM 6.1. If Hypotheses (G') and (F') hold and letting $x_0(\cdot) \in L_2(\Omega)$, $u \in U_{ad}$ (defined below), then system (19) has a solution $x \in L_p(I, PW^{1,p}(\Omega) \cap PC(\bar{I}, L_2(\Omega)))$ such that $\frac{\partial x}{\partial t} \in L_q(I, W^{-1,q}(\Omega))$.

PROOF. In this system, the evolution triple is $V = W_0^{1,p}(\Omega)$, $H = L_2(\Omega)$, and $V^* = W^{-1,q}(\Omega)$. All embeddings are compact (Sobolev embedding theorem). Define an operator $A : I \times V \rightarrow V^*$ by

$$\langle A(t, x), y \rangle_V = \int_{\Omega} \sum_{i=1}^N |D_i(x)|^{p-2} (D_i x)(D_i y) dz. \quad (20)$$

One can easily check that $A(t, x)$ satisfies Hypothesis (A) in Section 3. The uniform monotonicity of $A(t, \cdot)$ is a consequence of the result of Zeidler [4, p. 783].

In the sequel, we suppose that $\beta_{1i}(\cdot)$, $\beta_{2i}(\cdot)$ ($1 \leq i \leq N$) are continuous functions from $[0, T]$ to R and satisfy $\beta_{1i}(\cdot) \leq \beta_{2i}(\cdot)$ ($1 \leq i \leq N$) for all $t \in [0, T]$, $1 \leq i \leq N$. There exists a constant $a > 0$ such that $-a \leq \beta_{1i}(t) \leq \beta_{2i}(t) \leq a$ for all $t \in [0, T]$, $1 \leq i \leq N$. Set $Z = [-a, a]^N \subset R^N$. Then Z is a compact Polish space. Define

$$U(t) = \{(w_i(t)) \in R^N : \beta_{1i}(t) \leq w_i(t) \leq \beta_{2i}(t), 1 \leq i \leq N\}.$$

It is clear that $U : I \rightarrow P_{fc}(Z)$ is measurable. The set of admissible controls U_{ad} is chosen as $U_{ad} \equiv S_U \equiv \{u : I \rightarrow R^N \mid u \text{ is measurable and } u(t) \in U(t) \text{ a.e. } t \in [0, T]\}$. Hence the multifunction U satisfies (U1).

Next, for $t \in I$, $\phi \in H$, $w \in Z$ define a function $b^w : I \times H \times V \rightarrow R$ by $b^w(t, \phi, \psi) = \int_{\Omega} f(t, z, \phi, w) \psi(z) dz$. Then, the map $\psi \mapsto b^w(t, \phi, \psi)$ is bounded on V and hence is a continuous linear form on V . Thus there exists an operator $g : I \times H \times Z \rightarrow V^*$ such that

$$b^w(t, \phi, \psi) = \langle g(t, \phi, w), \psi \rangle_{V^* - V}. \quad (21)$$

By using Hypothesis (G'), we obtain that g satisfies Hypothesis (G1) of Section 3.

Using the operator A and g as defined in equation (20) and (21), one can rewrite system (19) in an abstract form as in (9). So, applying Theorem 4.4, system (19) has a solution.

REFERENCES

1. P. Sattayatham, Strongly nonlinear impulsive evolution equations and optimal control, *Nonlinear Analysis* **52**, 1005–1020, (2004).
2. N.S. Papageorgiou, Properties of the relaxed trajectories of evolution equations and optimal control, *SIAM J. Control Optim.* **27**, 267–288, (1989).
3. X. Xiang, P. Sattayatham and W. Wei, Relaxed controls for a class of strongly nonlinear delay evolution equations, *Nonlinear Analysis* **52**, 703–723, (2003).
4. E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Vol. I, II, Springer, New York, (1990).
5. X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, Boston, MA, (1995).
6. E. Balder, A general denseness result for relaxed control theory, *Bull. Austral. Math. Soc.* **20**, 463–475, (1984).
7. J. Warga, *Optimal Control of Differential and Functional Equations*, Academic Press, New York, (1972).
8. J. Diestel and J.J. Uhr, *Vector Measures*, American Mathematical Society, (1997).



Relaxation of nonlinear impulsive controlled systems on Banach spaces[☆]

P. Pongchalee^a, P. Sattayatham^{a,*}, X. Xiang^b

^a School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand

^b Department of Mathematics, Guizhou University, Guiyang Guizhou 550025, People's Republic of China

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Abstract

Relaxation control for a class of semilinear impulsive controlled systems is investigated. Existence of mild solutions for semilinear impulsive controlled systems is proved. By introducing a regular countably additive measure, we convexify the original control systems and obtain the corresponding relaxed control systems. The existence of optimal relaxed controls and relaxation results is also proved.

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1. Introduction

Let $I \equiv [0, T]$ be a closed and bounded interval of the real line. Let $D \equiv \{t_1, t_2, \dots, t_n\}$ be a partition on $(0, T)$ such that $0 < t_1 < t_2 < \dots < t_n < T$. A semilinear impulsive controlled system can be described by the following evolution equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)) & t \in (0, T) \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), & i = 1, 2, \dots, n, \end{cases} \quad (1.1)$$

where A is the infinitesimal generator of a C_0 -semigroup $\{T(t), t \geq 0\}$ in a Banach space X , the functions $f, J_i, i = 1, 2, \dots$, are continuous nonlinear operators from X to X , and $\Delta x(t_i) \equiv x(t_i + 0) - x(t_i - 0) = x(t_i + 0) - x(t_i)$. This system contains the jump in the state x at time t_i with J_i determining the size of the jump at t_i . In this paper, we aim to prove the existence of state-control pairs of the system (1.1). Moreover, by defining the objective functional $J(x, u) = \int_0^T L(t, x(t), u(t))dt$, we shall find sufficient conditions to guarantee the existence of optimal state-control pairs when convexity conditions on a certain orientor field are not assumed. This is the relaxation

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* Corresponding author. Tel.: +66 2 044224315; fax: +66 2 044224185.

E-mail addresses: ppongchalee@hotmail.com (P. Pongchalee), pairote@sut.ac.th (P. Sattayatham).

problem. By introducing regular countable additive measures, we convexify the original control systems and obtain the corresponding relaxed control systems. Under some reasonable assumptions, we prove that the set of original trajectories is dense in the set of relaxed trajectories in an appropriate space. The existence of optimal relaxed controls is obtained under some regularity hypotheses concerning the cost functional.

In recent years, relaxed systems have attracted much attention since some orientor fields do not satisfy the convexity condition. See, for instance, [1,6,7]. Ahmed [1] dealt with this problem and introduced measure-valued controls in which the control space is compact and values of relaxed control are countable additive measures, while Papageorgiou [6] and other authors including us continue to discuss this problem in another direction. However, to our knowledge, there are few authors who have studied the problem of relaxed controls of systems governed by impulsive evolution equations, particularly, relaxation on semilinear impulsive evolution equations. We organize the paper as follows. In Section 2, we describe the original control systems and the corresponding relaxed control systems. The properties of relaxed trajectories are given in Section 3. Section 4 is devoted to the existence of relaxed optimal controls and relaxation theorems.

2. Original and relaxed controlled systems

In what follows, let the Banach space $(X, \|\cdot\|_X)$ be the state space, $I \equiv [0, T]$ be a closed and bounded interval of the real line, $C(I, X)$ denote the space of continuous functions, and $C^1(I, X)$ denote the space of first-order continuous differentiable functions. Let $L(X, Y)$ denote the space of bounded linear operators from X to Y and $L(X)$ denote the space of bounded linear operators from X to X .

We denote the ball $\{x \in X : \|x\| \leq r\}$ by B_r . Define $PC(I, X) \equiv \{x : I \rightarrow X : x(t) \text{ is continuous at } t \neq t_i, \text{ left continuous at } t = t_i, \text{ and the right hand limit } x(t_i^+) \text{ exists}\}$. Equipped with the supremum norm topology, it is a Banach space.

We introduce the following assumptions.

[A]: The operator A is the infinitesimal generator of a C_0 -semigroup $\{T(t), t \geq 0\}$ on X .

[F]: $f : I \times X \rightarrow X$ is an operator such that

- (1) $t \rightarrow f(t, \xi)$ is measurable and locally Lipschitz continuous with respect to the last variable, i.e., for any finite number $\rho > 0$, there exists a constant $L_1(\rho) > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_X \leq L_1(\rho)\|x_1 - x_2\|_X,$$

$$\forall x_1, x_2 \in B_\rho.$$

- (2) There exists a constant $k > 0$ such that $\|f(t, x)\|_X \leq k(1 + \|x\|_X)$.

[J]: $J_i : X \rightarrow X$ is an operator such that

- (1) J_i maps a bounded set to a bounded set.
- (2) There exist constants $h_i > 0, i = 1, 2, \dots, n$, such that

$$\|J_i(x) - J_i(y)\| \leq h_i\|x - y\|, \quad x, y \in X.$$

Consider the following impulsive systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) & t \in [0, T] \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), & i = 1, 2, \dots, n. \end{cases} \quad (2.1)$$

By a mild solution of (2.1), we shall mean that a function $x \in PC(I, X)$ satisfies the following integral equation:

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + \sum_{0 < t_i < t} T(t-t_i)J_i(x(t_i)).$$

Theorem 1. Suppose the assumptions [A], [F], and [J] hold; then for every $x_0 \in X$ the system (2.1) has a unique mild solution $x \in PC(I, X)$ and the mild solution depends continuously on the initial conditions—that is, if $x_0, y_0 \in X$ and if $x(t), y(t)$ are mild solutions of Eq. (2.1) which satisfy $x(0) = x_0$ and $y(0) = y_0$. Then there exists a constant $C > 0$ s.t.

$$\sup_{t \in [0, T]} \|x(t) - y(t)\| \leq C\|x_0 - y_0\|_X.$$

Proof. Firstly, we consider the following general differential equation without impulse:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) & t > 0, \\ x(0) = x_0. \end{cases} \quad (2.1.1)$$

Define a closed ball $\bar{B}(x_0, 1)$ as follows:

$$\bar{B}(x_0, 1) = \{x \in C([0, T_1], X), \|x(t) - x_0\| \leq 1, 0 \leq t \leq T_1\},$$

where T_1 will be chosen later. Define a map P on $\bar{B}(x_0, 1)$ by

$$(Px)(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau$$

and let $M \equiv \sup_{t \in [0, T]} \|T(t)\|$. Using assumption [F], one can verify that P maps $\bar{B}(x_0, 1)$ to $\bar{B}(x_0, 1)$. To prove this, we note that

$$\begin{aligned} \|(Px)(t) - x_0\| &\leq \|T(t)x_0 - x_0\| + \int_0^t \|T(t-\tau)\| \|f(\tau, x(\tau))\| d\tau \\ &\leq Mk(1+\rho)t + \|T(t)x_0 - x_0\|. \end{aligned}$$

Since $T(t)$ is the strongly continuous C_0 -semigroup, there exists $T_{11} > 0$ such that for all $t \in [0, T_{11}]$, $\|T(t)x_0 - x_0\| \leq \frac{1}{2}$. Now, let $0 < T_{22} < \frac{1}{2Mk(1+\rho)}$. Set $T'_1 = \min\{T_{11}, T_{22}\}$; hence for all $t \in [0, T'_1]$ we have $\|(Px)(t) - x_0\| \leq 1$. Hence $P : \bar{B}(x_0, 1) \rightarrow \bar{B}(x_0, 1)$.

Let $x_1, x_2 \in \bar{B}(x_0, 1)$. By assumption [F](1), we have

$$\begin{aligned} \|(Px_1)(t) - (Px_2)(t)\| &\leq \int_0^t \|T(t-\tau)\| \|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\| d\tau \\ &\leq MtL_1(\rho)\|x_1 - x_2\|. \end{aligned}$$

Now, let $0 < T''_1 = \frac{1}{2MtL_1(\rho)}$; then $\|(Px_1)(t) - (Px_2)(t)\| \leq \frac{1}{2}\|x_1 - x_2\|$. Hence, we shall choose $T_1 = \min\{T'_1, T''_1\}$ to guarantee that P is a contraction map on $\bar{B}(x_0, 1)$. This implies that (2.1.1) has a unique mild solution on $[0, T_1]$. Again, using the assumption [F], we can obtain the a priori estimate of mild solutions of Eq. (2.1.1). To see this, we note that

$$\begin{aligned} \|x(t)\| &\leq \|T(t)x_0\| + \int_0^t \|T(t-\tau)\| \|f(\tau, x(\tau))\| d\tau \\ &\leq M\|x_0\| + MkT + Mk \int_0^t \|x(\tau)\| d\tau. \end{aligned}$$

By the Gronwall inequality, we obtain

$$\begin{aligned} \|x(t)\| &\leq (M\|x_0\| + MkT)e^{Mk \int_0^t d\tau} \\ &\leq (M\|x_0\| + MkT)e^{MkT} \equiv \bar{M}. \end{aligned}$$

That is, there exists a constant $\bar{M} = (M\|x_0\| + MkT)e^{MkT} > 0$ such that for $t \in [0, T]$ we have $\|x(t)\| \leq \bar{M}$. Then we can prove the global existence of the mild solution of system (2.1.1) on $[0, T]$.

Now, we are ready to construct a mild solution for the impulsive system (2.1). For $t \in [0, t_1]$, the above result implies that $x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau$ is the unique mild solution of the system (2.1) on $[0, t_1]$. Clearly the solution is continuous on $[0, t_1)$ and since $T(t)$ is a continuous semigroup, then $x(t)$ can be extended continuously until the point of time t_1 which is denoted by $x(t_1)$. It is easy to see that $x(t_1) \in X$. Since J_1 maps bounded sets to bounded subsets of X , the jump is uniquely determined by the expression

$$x(t_1 + 0) = x(t_1 - 0) + J_1(x(t_1 - 0)) \equiv x(t_1) + J_1(x(t_1)) \equiv x_1.$$

Consider the time $t \in (t_1, t_2)$. We have

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + T(t-t_1)J_1(x(t_1)).$$

Again, $x \in C((t_1, t_2), X)$ and can be extended continuously until the point of time t_2 which is denoted by $x(t_2) \in X$. By the previous result, $x(\cdot)$ is a mild solution of Eq. (2.1) on $(t_1, t_2]$. Because J_2 maps bounded sets to bounded sets, the jump is uniquely determined by

$$x(t_2 + 0) = x(t_2 - 0) + J_2(x(t_2 - 0)) \equiv x(t_2) + J_2(x(t_2)) \equiv x_2.$$

This procedure can be repeated on $t \in (t_2, t_3], (t_3, t_4], \dots, (t_n, T]$. Thus we obtain a unique mild solution of problem (2.1) on $[0, T]$ and it is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + \sum_{0 < t_i < t} T(t-t_i)J_i(x(t_i)), \quad 0 \leq t \leq T.$$

For the proof of continuous dependence on the initial value, one can use the Gronwall inequality to find a constant C such that $\|x(t) - y(t)\| \leq C\|x_0 - y_0\|_X$ for all $t \in [0, T]$. The proof is now complete. $\square$

Now, we introduce the admissible controls space U_{ad} .

Let Γ be a compact Polish space (i.e., a separable complete metric space).

We define

$$U_{ad} = \{u: [0, T] \rightarrow \Gamma | u \text{ is strongly measurable}\}.$$

By the measurable selection theorem, $U_{ad} \neq \emptyset$ (see [3]). We make the following assumptions for our control systems.

Assumptions

[F1] $f: I \times X \times \Gamma \rightarrow X$ is an operator such that

- (1) $t \mapsto f(t, \xi, \eta)$ is measurable, and $(\xi, \eta) \mapsto f(t, \xi, \eta)$ is continuous on $X \times \Gamma$.
- (2) For any finite number $\rho > 0$, there exists a constant $L(\rho) > 0$ such that

$$\|f(t, x_1, \sigma) - f(t, x_2, \sigma)\|_X \leq L(\rho)\|x_1 - x_2\|_X,$$

for all $\|x_1\| < \rho$, $\|x_2\| < \rho$, and $t \in I$, $\sigma \in \Gamma$.

- (3) There exists a constant $k_F > 0$ such that

$$\|f(t, x, \sigma)\|_X \leq k_F(1 + \|x\|_X) \quad (t \in I, \sigma \in \Gamma).$$

Consider the following original control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{ad}. \end{cases} \quad (2.2)$$

Theorem 2. Suppose the assumptions [A], [J], and [F1] hold. Then for every $x_0 \in X$ and $u \in U_{ad}$, the system (2.2) has a unique mild solution $x \in PC(I, X)$ which satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau), u(\tau))d\tau + \sum_{0 < t_i < t} T(t-t_i)J_i(x(t_i)).$$

Proof. Let $u \in U_{ad}$ and define $g_u(t, x) = f(t, x, u)$. Since f is measurable, then $g_u: I \times X \rightarrow X$ is measurable on $[0, T]$ for each fixed $x \in X$. Hence g_u satisfies the assumption [F]. By Theorem 1, the system (2.2) has a unique mild solution $x \in PC(I, X)$.

In order to introduce the relaxed control system corresponding to (2.2), we need some preparations which are drawn from ([4], p. 618–650). Let Γ be a compact Polish space, and $C(\Gamma)$ consist of all continuous real-valued functions. Endowed with the supremum norm, $C(\Gamma)$ is a Banach space. Let $\Phi(C)$ be a σ -field generated by the collection C of all closed sets of Γ and let $\Sigma_{rca}(\Gamma)$ be the space of all regular countably additive measures on the measurable space $(\Gamma, \Phi(C))$. For $\mu \in \Sigma_{rca}(\Gamma)$, $|\mu|$ denotes the total variation of μ . $\square$

Lemma 3. The dual space $C(\Gamma)^*$ can be identified algebraically and metrically with $\Sigma_{\text{rca}}(\Gamma)$ with the norm

$$\|\mu\|_{\Sigma_{\text{rca}}(\Gamma)} = |\mu|(\Gamma).$$

The duality pairing of $C(\Gamma)$ and $\Sigma_{\text{rca}}(\Gamma)$ is given by

$$\langle f, \mu \rangle = \int_{\Gamma} f(\sigma) \mu(d\sigma)$$

for $f \in C(\Gamma)$, $\mu \in \Sigma_{\text{rca}}(\Gamma)$.

Let $L^1(I, C(\Gamma))$ be the space of all (the equivalence class of) strongly measurable $C(\Gamma)$ -valued functions $u(\cdot)$ defined on I such that

$$\|u\| = \int_I \|u(t)\| dt < +\infty.$$

$L^1(I, C(\Gamma))$ is a Banach space. $L_w^\infty(I, C(\Gamma)^*)$ is the space of all $C(\Gamma)^*$ -valued $C(\Gamma)$ -weakly measurable functions $g(\cdot)$ such that there exists $C > 0$ with

$$|\langle g(t), y \rangle| \leq C \|y\|_{C(\Gamma)} \quad \text{a.e. in } 0 \leq t \leq T, \quad (2.2.1)$$

for each $y \in C(\Gamma)$ (the null set where (2.2.1) fails to hold may depend on y). Two functions $g(\cdot), h(\cdot)$ are said to be equivalent in $L_w^\infty(I, C(\Gamma)^*)$ (in symbols, $g \approx h$) if $\langle g(t), y \rangle = \langle h(t), y \rangle$ a.e. in $0 \leq t \leq T$ for each $y \in C(\Gamma)$.

Lemma 4. The dual $L^1(I, C(\Gamma))^*$ is isometrically isomorphic to $L_w^\infty(I, C(\Gamma)^*)$. The duality pairing between the two spaces is given by

$$\langle g, f \rangle = \int_0^T \langle g(t), f(t) \rangle dt,$$

where $g \in L_w^\infty(I, C(\Gamma)^*)$ and $f \in L^1(I, C(\Gamma))$.

Since Γ is a compact metric space, $C(\Gamma)^*$ is a separable Banach space (see [8], p. 265) and hence has the Radon–Nikodym property which tells us that $L^1(I, C(\Gamma))^* = L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$.

Definition 1. The space $R(I, \Gamma)$ of relaxed controls consists of all $\mu(\cdot)$ in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma)) = L^1(I, C(\Gamma))^*$ that satisfy

(i) if $f(\cdot, \cdot) \in L^1(I, C(\Gamma))$ is such that $f(t, \sigma) \geq 0$ for $\sigma \in \Gamma$ a.e. in $0 \leq t \leq T$ then

$$\int_0^T \int_{\Gamma} f(t, \sigma) \mu(t, d\sigma) dt \geq 0,$$

(ii) if $\chi(t)$ is the characteristic function of a measurable set $e \subseteq [0, T]$, and $\mathbf{1} \in C(\Gamma)$ is the function $\mathbf{1}(\sigma) = 1$, then

$$\int_0^T \int_{\Gamma} (\chi(t) \otimes \mathbf{1}(\sigma)) \mu(t, d\sigma) dt = |e|.$$

Note that $\chi(\cdot) \otimes \mathbf{1}(\cdot) \in L^1(I, C(\Gamma))$.

We note that (ii) can be generalized to

$$\int_0^T \int_{\Gamma} (\phi(t) \otimes \mathbf{1}(\sigma)) \mu(t, d\sigma) dt = \int_0^T \phi(t) dt$$

for any $\phi(\cdot) \in L^1(I)$.

In fact, for $\mu(\cdot) \in R(I, \Gamma)$, we have

$$\|\mu\|_{L^\infty(I, \Sigma_{\text{rca}}(\Gamma))} \leq 1, \quad \mu(t) \geq 0, \text{ and } \mu(t, \Gamma) = 1 \quad \text{a.e. in } 0 \leq t \leq T.$$

In particular,

$$\|\mu(t)\|_{\Sigma_{\text{rca}}(\Gamma)} = 1 \quad \text{a.e. in } 0 \leq t \leq T.$$

Lemma 5. Let $\{\mu_n(\cdot)\}$ be a sequence in $R(I, \Gamma)$. Then there exists a subsequence which is $L^1(I, C(\Gamma))$ -weakly convergent in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ to $\mu(\cdot) \in R(I, \Gamma)$.

Sometimes, using another equivalent definition of $R(I, \Gamma)$ is more convenient. We denote by $\Pi_{\text{rca}}(\Gamma)$ the set of all probability measures μ in $\Sigma_{\text{rca}}(\Gamma)$. We denote the Dirac measure with mass at u using the functional notation $\delta(\cdot - u)$ or δ_u . The set $D = \{\delta_u : u \in \Gamma\}$ of all Dirac measures is a subset of $\Pi_{\text{rca}}(\Gamma)$.

Lemma 6. $\Pi_{\text{rca}}(\Gamma)$ is $C(\Gamma)$ -weakly compact, also $C(\Gamma)$ -weakly closed in $\Sigma_{\text{rca}}(\Gamma)$.

Let $\overline{\text{conv}}$ denote the closed convex hull (closure taken in the weak $C(\Gamma)$ -topology). Then

$$\Pi_{\text{rca}}(\Gamma) = \overline{\text{conv}}(D).$$

Since $C(\Gamma)$ is separable, the equivalent relation in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ is equality almost everywhere. Let us define the set

$$R(I, \Pi_{\text{rca}}(\Gamma)) = \{u \in L^\infty(I, \Sigma_{\text{rca}}), \exists v \text{ s.t. } v \approx u \text{ and } v(t) \in \Pi_{\text{rca}}(\Gamma) \text{ a.e. in } 0 \leq t \leq T\}.$$

If $u(\cdot) \in U_{\text{ad}}$ then one can check that the Dirac delta with mass at $u(t)$ (written as $\delta(\cdot - u(t))$) is an element of $R(I, \Pi_{\text{rca}}(\Gamma))$. Hence we can identify U_{ad} as a subset of $R(I, \Pi_{\text{rca}}(\Gamma))$. We note further that $R(I, \Pi_{\text{rca}}(\Gamma)) = R(I, \Gamma)$ (see [4], Theorem 12.6.7).

Now, let us consider the new larger system known as the “relaxed impulsive system”:

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \quad (2.3)$$

The admissible control space is $U_r = R(I, \Pi_{\text{rca}}(\Gamma))$. The function $F : I \times X \times \Sigma_{\text{rca}}(\Gamma) \rightarrow X$ is defined by

$$F(t, x)\mu = \int_{\Gamma} f(t, x, \sigma)\mu(d\sigma).$$

The following theorem is an immediate consequence of Theorem 2.

Theorem 7. Assume that assumptions [A], [J] and [F1] hold. For every $\mu(\cdot) \in U_r$, the relaxed control system (2.3) has a unique solution.

3. Properties of relaxed trajectories

In this section, we will denote the set of original trajectories and relaxed trajectories of the system (2.2) by X_0 and the system (2.3) by X_r , i.e.,

$$X_0 = \{x \in PC([0, T]; X) \mid x \text{ is a solution of (2.2) corresponding to } u(\cdot) \in U_{\text{ad}}\}$$

and

$$X_r = \{x \in PC([0, T]; X) \mid x \text{ is a solution of (2.3) corresponding to } \mu(\cdot) \in U_r\}.$$

Theorems 2 and 7 show that $X_0 \neq \emptyset$ implies $X_r \neq \emptyset$. Moreover, since $U_{\text{ad}} \subseteq U_r$ we have $X_0 \subseteq X_r$.

Next, we introduce one more hypothesis concerning the operator A .

[A1] An operator A is the infinitesimal generator of a compact C_0 -semigroup $\{T(t), t \geq 0\}$.

Lemma 8. Let A satisfy assumption [A1] on Banach space X . Let $1 < p$ and define

$$S(g(\cdot)) = \int_0^\cdot T(\cdot - s)g(s)ds \quad \forall g(\cdot) \in L^p(I, X).$$

Then $S : L^p(I, X) \rightarrow C(I, X)$ is compact.

Proof. See Lemma 3.2 in [5]. $\square$

Lemma 9. Let X be reflexive and separable. Suppose the assumptions [A1] and [F1] hold. If $\{\mu^n(\cdot)\}$ is a sequence in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ with $\mu^n(\cdot) \rightarrow \mu(\cdot)$ $L^1(I, C(\Gamma))$ -weakly as $n \rightarrow \infty$ then

$$\rho_n(\cdot) = \int_0^\cdot T(\cdot - \tau) \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma) \mathrm{d}\tau \rightarrow 0 \quad \text{in } C(I, X) \text{ as } n \rightarrow \infty,$$

where $x \in C([0, T], X)$.

Proof. Due to reflexivity of X , $\{T^*(t), t \geq 0\}$ is a C_0 -semigroup in Banach space X^* (see [2], p. 47). Define $g_n(\tau) = \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)$; then

$$\begin{aligned} \|g_n(\tau)\| &\leq \int_\Gamma \|f(\tau, x(\tau), \sigma)\|(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma) \\ &\leq k_F(1 + \|x(\tau)\|)\|\mu^n(\tau) - \mu(\tau)\|_{\Sigma_{\text{rca}}(\Gamma)} \\ &\leq 2k_F(1 + \|x(\tau)\|). \end{aligned}$$

Since $x(t)$ is the solution of (2.3), then it is bounded by $\bar{M}$. This implies that $\{g_n(\cdot)\}$ is bounded in $L^p(I, X)$, $1 < p < +\infty$. Hence there exists a subsequence (denoted with the same symbol) with $g_n(\cdot) \xrightarrow{w} g(\cdot)$ in $L^p(I, X)$.

By Lemma 8, we have

$$\rho_n(\cdot) = \int_0^\cdot T(\cdot - \tau)g_n(\tau)\mathrm{d}\tau \xrightarrow{s} \int_0^\cdot T(\cdot - \tau)g(\tau)\mathrm{d}\tau \equiv \rho(\cdot) \quad \text{in } C(I, X).$$

For fixed $0 \leq t \leq T$, $h^* \in X^*$, we have

$$\begin{aligned} \langle \rho_n(t), h^* \rangle &= \int_0^t \langle T(t - \tau)g_n(\tau), h^* \rangle \mathrm{d}\tau \\ &= \int_0^t \langle g_n(\tau), T^*(t - \tau)h^* \rangle \mathrm{d}\tau \\ &= \int_0^t \int_\Gamma \langle f(\tau, x(\tau), \sigma), T^*(t - \tau)h^* \rangle (\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma) \mathrm{d}\tau \\ &= \int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma) \mathrm{d}\tau \end{aligned}$$

where $\xi(\tau, \sigma) = \langle f(\tau, x(\tau), \sigma), T^*(t - \tau)h^* \rangle$.

By assumption [F1], for τ fixed, the map $\sigma \mapsto \xi(\tau, \sigma)$ is continuous. This implies that $\xi(\tau, \sigma) \in C(\Gamma)$ and

$$|\xi(\tau, \sigma)| \leq k(1 + \|x(\tau)\|).$$

Hence $\xi(\cdot, \cdot) \in L^1(I, C(\Gamma))$.

Since $\mu^n(\cdot) \rightarrow \mu(\cdot)$ $L^1(I, C(\Gamma))$ -weakly in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$, then

$$\int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma) \mathrm{d}\tau \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies that, for fixed $t \in I$,

$$\langle \rho_n(t), h^* \rangle \rightarrow 0 \quad \forall h^* \in X^*.$$

Hence $\rho_n(t) \xrightarrow{w} 0$ as $n \rightarrow \infty$. Thus $\rho(t) \equiv 0$. This means that $\rho_n(\cdot) \rightarrow 0$ as $n \rightarrow \infty$ in $C(I, X)$. $\square$

Remark. Using the same proof, one can see that the result of Lemma 9 is also true when $x \in PC([0, T], X)$.

Theorem 10. Let X be reflexive and separable. Suppose the assumptions [A1], [J], and [F1] hold. If $x(\cdot, \mu)$ is the solution of (2.3) corresponding to μ then, for every $\varepsilon > 0$, there exists $u(\cdot) \in U_{\text{ad}}$ such that $x(\cdot, u)$ is a solution of (2.2) corresponding to u and satisfying

$$\|x(\cdot, \mu) - x(\cdot, u)\|_{PC(I, X)} < \varepsilon, \quad t \in I.$$

Proof. Let $\mu(\cdot) \in U_r$; since $U_{ad} \subseteq U_r$ and U_{ad} is dense in U_r , there thus exists a sequence $\{u_n\} \subseteq U_{ad}$ such that $u_n \xrightarrow{w^*} \mu$. Let $x_n(\cdot) = x(\cdot, u_n)$ be the solution of (2.2) corresponding to u_n and $x(\cdot) = x(\cdot, \mu)$ be the solution of (2.3) corresponding to μ . Since

$$\begin{aligned} x_n(t) &= T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x_n(\tau), u_n(\tau))d\tau + \sum_{0 < t_i < t} T(t-t_i)J_i(x_n(t_i)) \\ &= T(t)x_0 + \int_0^t T(t-\tau) \left[\int_{\Gamma} f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau)(d\sigma) \right] d\tau + \sum_{0 < t_i < t} T(t-t_i)J_i(x_n(t_i)) \end{aligned}$$

and

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau) \left[\int_{\Gamma} f(\tau, x(\tau), \sigma)\mu(\tau)(d\sigma) \right] d\tau + \sum_{0 < t_i < t} T(t-t_i)J_i(x(t_i)),$$

we have

$$\begin{aligned} x_n(t) - x(t) &= \int_0^t T(t-\tau) \left[\int_{\Gamma} (f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau), \sigma)\delta_{u_n}(\tau))(d\sigma) \right] d\tau \\ &\quad + \int_0^t T(t-\tau) \left[\int_{\Gamma} f(\tau, x(\tau), \sigma)(\delta_{u_n}(\tau) - \mu(\tau))(d\sigma) \right] d\tau \\ &\quad + \sum_{0 < t_i < t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))] \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By the Lipschitz condition [F1], we get

$$|I_1| \leq M \int_0^t L(\rho) \|x_n(\tau) - x(\tau)\| d\tau,$$

where $I_1 \equiv \int_0^t T(t-\tau) \left[\int_{\Gamma} (f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau), \sigma)\delta_{u_n}(\tau))(d\sigma) \right] d\tau$, and M is a bound for $\|T(t)\|$ in $0 \leq t \leq T$.

Using assumption [J](2), we have

$$|I_3| \leq \sum_{0 < t_i < t} M h_i \|x_n(t_i) - x(t_i)\|,$$

where $I_3 \equiv \sum_{0 < t_i < t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))]$.

We denote the second integral I_2 by $\rho_n(t)$, i.e.,

$$\rho_n(t) \equiv I_2 \equiv \int_0^t T(t-\tau) \left[\int_{\Gamma} f(\tau, x(\tau), \sigma)(\delta_{u_n}(\tau) - \mu(\tau))(d\sigma) \right] d\tau.$$

Thus

$$\|x_n(t) - x(t)\| \leq M \int_0^t L(\rho) \|x_n(\tau) - x(\tau)\| d\tau + \|\rho_n(t)\| + \sum_{0 < t_i < t} M h_i \|x_n(t_i) - x(t_i)\|.$$

By the impulsive Gronwall inequality, we get

$$\|x_n(t) - x(t)\| \leq C \|\rho_n(t)\|,$$

where $C \equiv \prod_{0 < t_i < t} (1 + M h_i) \exp(ML(\rho)t)$.

By using Lemma 9, we show that $\rho_n(\cdot) \rightarrow 0$ as $n \rightarrow \infty$ in $PC([0, T], X)$. Hence $x_n(\cdot) \rightarrow x(\cdot)$ as $n \rightarrow \infty$ in $PC([0, T], X)$. The proof is complete. $\square$

4. Relaxed optimal controls and relaxation theorems

Consider the following Lagrange optimal control (P_r): Find a control policy $\mu_0 \in U_r$ such that it imparts a minimum to the cost functional J given by

$$J(\mu) \equiv J(x_\mu, \mu) \equiv \int_I \int_\Gamma l(t, x_\mu(t), \sigma) \mu(t)(d\sigma) dt, \quad (P_r)$$

where x_μ is a solution of the system (2.3) corresponding to the control $\mu \in U_r$.

We make the following hypotheses concerning the integrand $l(\cdot, \cdot, \cdot)$.

[L] $l : I \times X \times \Gamma \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an operator such that

- (1) $(t, \xi, \sigma) \mapsto l(t, \xi, \sigma)$ is measurable,
- (2) $(\xi, \sigma) \mapsto l(t, \xi, \sigma)$ is lower semicontinuous,
- (3) $|l(t, \xi, \sigma)| \leq \theta_R(t)$ for almost all $t \in I$ provided that $\|\xi\|_X \leq R$, $\sigma \in \Gamma$ and $\theta_R(t) \in L^1(I)$.

Before proving the existence of the relaxed control, we need a lemma.

Lemma 11. Suppose $h : I \times X \times \Gamma \rightarrow \mathbb{R}$ satisfies

- (1) $t \mapsto h(t, \xi, \sigma)$ is measurable, $(\xi, \sigma) \mapsto h(t, \xi, \sigma)$ is continuous,
- (2) $|h(t, \xi, \sigma)| \leq \psi_R(t) \in L^1(I)$ provided that $\|\xi\|_X \leq R$ and $\sigma \in \Gamma$.

If $x_n \rightarrow x$ in $C(I, X)$ then $h_n(\cdot, \cdot) \rightarrow h(\cdot, \cdot)$ in $L^1(I, C(\Gamma))$ as $n \rightarrow \infty$, where $h_n(t, \sigma) = h(t, x_n(t), \sigma)$ and $h(t, \sigma) = h(t, x(t), \sigma)$.

Proof. It follows immediately from the first hypothesis of this lemma that

$$h_n, h \in L^1(I, C(\Gamma)).$$

For each fixed $t \in I$, we shall show that $h_n(t, \cdot) \rightarrow h(t, \cdot)$ in $C(\Gamma)$ as $n \rightarrow \infty$.

By definition, we have

$$\sup_{\sigma \in \Gamma} |h_n(t, \sigma) - h(t, \sigma)| = \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}.$$

Since Γ is compact, there exists $\sigma_n \in \Gamma$ such that

$$|h_n(t, \sigma_n) - h(t, \sigma_n)| = \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}$$

and we can assume $\sigma_n \rightarrow \sigma^*$ as $n \rightarrow \infty$. We note that

$$\begin{aligned} \sup_{\sigma \in \Gamma} |h_n(t, \sigma) - h(t, \sigma)| &= |h_n(t, \sigma_n) - h(t, \sigma_n)| \\ &\leq |h_n(t, \sigma_n) - h_n(t, \sigma^*)| + |h_n(t, \sigma^*) - h(t, \sigma^*)| + |h(t, \sigma^*) - h(t, \sigma_n)|. \end{aligned}$$

Then, by continuity of h , we have $|h_n(t, \sigma_n) - h(t, \sigma_n)| \rightarrow 0$ as $n \rightarrow \infty$.

This means

$$\|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Assuming that $x_n \rightarrow x$ in $C(I, X)$ as $n \rightarrow \infty$ then there exists R such that $\|x_n(t)\|, \|x(t)\| \leq R$.

Hence, by the second hypothesis of this lemma, we have

$$\|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} \leq \psi_R(t).$$

This implies that

$$\int_I \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We have

$$h_n(\cdot, \cdot) \rightarrow h(\cdot, \cdot) \quad \text{in } L^1(I, C(\Gamma)) \text{ as } n \rightarrow \infty.$$

This proves the lemma. $\square$

Let $m_r = \inf\{J(\mu) : \mu \in U_r\}$. We have the following existence of relaxed optimal control.

Theorem 12. Suppose assumptions [A1], [F1], [J] and [L] hold. Then there exists $\mu^* \in U_r$ such that $J(\mu^*) = m_r$.

Proof. Let $\{\mu_n\}$ be a minimizing sequence so that $\lim_{n \rightarrow \infty} J(\mu_n) = m_r$. Recall that U_r is w^* -compact in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$; by passing to a subsequence if necessary, we may assume $\mu_n \xrightarrow{w^*} \mu^*$ in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ as $n \rightarrow \infty$. Next, we shall prove that (x, μ^*) is an optimal pair, where x is the solution of (2.3) corresponding to μ^* .

Since every lower semicontinuous measurable integrand is the limit of an increasing sequence of Caratheodory integrands, there exists an increasing sequence of Caratheodory integrands $\{l_k\}$ such that

$$l_k(t, \xi, \sigma) \uparrow l(t, \xi, \sigma) \quad \text{as } k \rightarrow \infty \text{ for all } t \in I, \sigma \in \Gamma.$$

Invoking the definition of weak topology and applying Lemma 11 on each subinterval of $[0, T]$, $l_k(t, x_n(t), \sigma) \rightarrow l_k(t, x(t), \sigma)$ as $n \rightarrow \infty$ for almost all $t \in I$ and all $\sigma \in \Gamma$, then

$$\begin{aligned} J(x, \mu^*) &= J(\mu^*) = \int_I \int_\Gamma l(t, x(t), \sigma) \mu^*(t)(d\sigma) dt \\ &= \lim_{k \rightarrow \infty} \int_I \int_\Gamma l_k(t, x(t), \sigma) \mu^*(t)(d\sigma) dt \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_I \int_\Gamma l_k(t, x_n(t), \sigma) \mu_n(t)(d\sigma) dt \\ &\leq \lim_{n \rightarrow \infty} \int_I \int_\Gamma l(t, x_n(t), \sigma) \mu_n(t)(d\sigma) dt \\ &= m_r. \end{aligned}$$

However, by definition of m_r , it is obvious that $J(x, \mu^*) \geq m_r$. Hence $J(x, \mu^*) = m_r$.

This implies that (x, μ^*) is an optimal pair. $\square$

If $J(u) = \int_I l(t, x(t), u(t)) dt$ is the cost function for the original problem, and $J(u_0) = \inf\{J(u), u \in U_{\text{ad}}\} = m_0$, in general, since $U_{\text{ad}} \subseteq U_r$, we have $m_r \leq m_0$. It is desirable that $m_r = m_0$, i.e., our relaxation is reasonable. We have the following relaxation theorem. For this, we need hypotheses on l stronger than [L]:

[L1] $l : I \times X \times \Gamma \rightarrow \mathbb{R}$ is an operator such that

- (1) $(t, \xi, \sigma) \rightarrow l(t, \xi, \sigma)$ is measurable,
- (2) $(\xi, \sigma) \rightarrow l(t, \xi, \sigma)$ is continuous,
- (3) $|l(t, \xi, \sigma)| \leq \theta_R(t)$ for almost all $t \in I$, provided $\|\xi\|_X \leq R, \sigma \in \Gamma$ and $\theta_R \in L^1(I)$.

Theorem 13. If assumptions [A1], [J], [F1], and [L1] hold and Γ is compact then $m_0 = m_r$.

Proof. Let (x, μ^*) be the optimal pair (the existence was guaranteed by the previous theorem); that is $m_r = J(x, \mu^*)$. By Theorem 10, there exists $\{u^n\} \subseteq U_{\text{ad}}$ and $\{x_n\} \subseteq PC(I, X)$ such that

$$\delta_{u_n}(\cdot) \rightarrow \mu^*(\cdot) \text{ } L^1(I, C(\Gamma))\text{-weakly in } L^\infty(I, \Sigma_{\text{rca}}(\Gamma)),$$

and $x_n \rightarrow x$ in $PC(I, X)$ as $n \rightarrow \infty$.

Applying Lemma 11 to each subinterval of $[0, T]$, one can verify that

$$l(\cdot, x_n(\cdot), \cdot) \rightarrow l(\cdot, x(\cdot), \cdot) \quad \text{in } L^1(I, C(\Gamma)).$$

By definition of the weak topology on U_r , we have

$$\begin{aligned} J(u_n) &= J(\delta_{u_n}) = \int_I \int_\Gamma l(t, x_n(t), \sigma) \delta_{u_n}(t)(d\sigma) dt \\ &\rightarrow \int_I \int_\Gamma l(t, x(t), \sigma) \mu^*(t)(d\sigma) dt = J(x, \mu^*) = m_r. \end{aligned}$$

But, by definition of m_0 , $J(u_n) \geq m_0$. Hence $m_r = \lim_{n \rightarrow \infty} J(u_n) \geq m_0$. This implies $m_0 = m_r$. The proof is now complete. $\square$

References

- [1] N.U. Ahmed, Properties of relaxed trajectories for a class of nonlinear evolution equations on a Banach space, *SIAM J. Control Optim.* 21 (1983) 953–967.
- [2] N.U. Ahmed, Semigroup Theory with Applications to System and Control, in: Pitman Research Notes in Maths series, vol. 246, Longman Scientific Technical, New York, 1991.
- [3] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.
- [4] H.O. Fattorini, Infinite Dimensional Optimization and Control Theory, Cambridge University Press, 1999.
- [5] X. Li, J. Yong, Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Boston, 1995.
- [6] N.S. Papageorgious, Properties of the relaxed trajectories of evolutions and optimal control, *SIAM J. Control Optim.* 27 (2) (1989) 267–288.
- [7] X. Xiang, P. Sattayatham, W. Wei, Relaxed controls for a class of strongly nonlinear delay evolution equations, *Nonlinear Anal.* 52 (2003) 703–723.
- [8] J. Warga, Optimal Control of Differential and Functional Differential Equations, Springer, New York, 1996.

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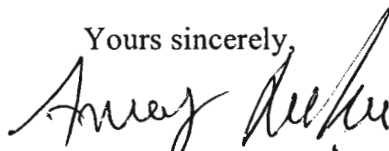
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Dear Prof. Pairote Sattayatham,

It is my pleasure to inform you that your paper "Relaxed control for a class of semilinear impulsive evolution equations" is accepted to be published in PJPAM.

Thank you for your selection of our journal.

Yours sincerely,



Prof. Amnuay Kananthai

The Chief Editor of PJPAM

May 30, 2007

Relaxed control for a class of semilinear impulsive evolution equations.*

P. Sattayatham

School of Mathematics, Suranaree University of Technology,

Nakhon Ratchasima 30000, Thailand

E-mail address: pairote@sut.ac.th

Abstract

Relaxed control for a class of semilinear impulsive evolution equations is investigated. Boundedness of solutions of semilinear impulsive evolution equations is proved and properties of original and relaxed trajectories are discussed. The existence of optimal relaxed control and relaxation results are also presented.

Keywords: Impulsive system, Banach space, semilinear equation, relaxation.

AMS (MOS) subject classification: 34G20

1 Introductions

In this paper, we present sufficient conditions of optimality for optimal relaxed control problems arising in systems governed by semilinear impulsive evolution equations on Banach spaces. The general descriptions of such systems are given below.

$$\frac{d}{dt}x(t) = Ax(t) + f(t, x(t), u(t)), \quad t \in I \setminus D \quad (1a)$$

$$x(0) = x_0 \in X, \quad (1b)$$

$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1, 2, \dots, n, \quad (1c)$$

where $I \equiv [0, T]$ is a bounded closed interval of the real line R , and let the set

$D \equiv \{t_1, t_2, \dots, t_n\}$ be a partition on $[0, T]$ such that $0 < t_1 < t_2 < \dots < t_n < T$. In general, the operator $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$, f is a nonlinear perturbation, $\Delta x(t) \equiv x(t_i^+) - x(t_i^-) \equiv x(t_i^+) - x(t_i)$, $i = 1, 2, \dots, n$, and F_i 's are nonlinear operators. This model includes all the standard models used by many authors in the field (see Sattayatham&Huawu [8], Ahmed [1]). The objective functional is given by $J(x, u) = \int_0^T L(t, x(t), u(t))dt$.

In recent years impulsive evolution equations on infinite dimensional Banach spaces have been considered in several papers by Liu [5] and Ahmed [1]. Liu considers the problem of existence and regularity of the solution while Ahmed considers the optimal impulsive control problem and necessary conditions, but sufficient conditions of relaxation for optimality were not addressed. We wish to present just that. Before we can consider such problems, we need some preparation. The rest of the paper is organized as follows. In section 2, some basic notations

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and terminology are presented. Section 3 contains some preparatory results. Relaxed impulsive systems are presented in section 4. Sufficient conditions of relaxation for optimality are discussed in section 5.

2 System descriptions

Let X be a Banach space. Let $C([0, T], X)$ be the Banach space of all continuous functions from $[0, T]$ into X with the supremum norm, i.e., $\|x\| = \sup\{\|x(t)\|_X : 0 \leq t \leq T\}$. The operator $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$.

Let $PC([0, T], X)$ denote the space of piecewise continuous functions on $[0, T]$ with values in X which are left continuous and possessing right hand limits. Equipped with the supremum norm topology, it is a Banach space. Consider the following evolution systems

$$\frac{d}{dt}x(t) = Ax(t) + g(t, x(t)), \quad t \in I \setminus D, \quad (2a)$$

$$x(0) = x_0 \in X, \quad (2b)$$

$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1, 2, \dots, n. \quad (2c)$$

By a mild solution $x(t, x_0)$ of the system (2a)-(2c) corresponding to the initial state $x_0 \in X$, we mean a function $x \in PC([0, T], X)$ such that $x(0) = x_0$, and satisfy the following integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)g(s, x(s))ds + \sum_{0 < t_i < t} T(t-t_i)F_i(x(t_i)), \quad 0 \leq t \leq T.$$

3 Preparatory results

For the study of relaxation for optimality, it is essential to guarantee the existence and uniqueness of solutions of the impulsive evolution equation and certain other related equations. Here in this section, for the convenience of the reader, we quote some results from the recent work of Liu [4]. But first, we recall some hypotheses on the data of problem (2a)-(2c).

(G) $g : I \times X \rightarrow X$ is an operator such that

- (1) $t \mapsto g(t, x)$ is measurable, the map $x \mapsto g(t, x)$ is continuous, and there exists a constant $L > 0$ such that

$$\|g(t, x) - g(t, y)\| \leq L \|x - y\|, \quad t \in I, \quad x, y \in X.$$

- (2) There exists a constant $k > 0$ such that $\|g(t, x)\| \leq k(1 + \|x\|)$, $t \in I, x \in X$.

(F) $F_i : X \rightarrow X, i = 1, 2, \dots, n$, are continuous and there exist constants $h_i > 0, i = 1, 2, \dots, n$, such that

$$\|F_i(x) - F_i(y)\| \leq h_i \|x - y\|, \quad x, y \in X.$$

(A) Let $\mathcal{U}(\cdot)$ be the strongly continuous semigroup generated by the unbounded operator A . Let $B(X)$ be the Banach space of all linear and bounded operators on X . Denote

$$M \equiv \max_{t \in [0, T]} \|T(t)\|_{B(X)},$$

and assume that

$$M[LT + \sum_{k=1}^n h_k] < 1.$$

By the uniform boundedness principle $\|T(t)\|_{B(X)}$ is bounded on $[0, T]$, so M in hypothesis (A) is finite. We state the following results which give sufficient conditions for the existence of a mild solution.

Theorem 3.1 Let hypotheses (A), (G), and (F) be satisfied. Then for every $x_0 \in X$, equations (2a)-(2c) has a unique mild solution. Moreover, the set of mild solutions is bounded in $PC([0, T], X)$.

Proof. For the existence and uniqueness of the mild solutions, see Liu ([5], Theorem 2.1). Moreover, one can prove the boundness of the set of mild solutions by using Grownwall inequality. To see this, suppose that $x(\cdot)$ is a mild solution of equation (2a)-(2c). Then we have

$$\begin{aligned} \|x(t)\| &\leq \|T(t)x_0\| + \int_0^t \|T(t-s)\| \|g(s, x(s))\| ds + \sum_{0 < t_i < t} \|T(t-t_i)\| \|F_i(x(t_i))\| \\ &\leq M\|x_0\| + Mk \int_0^t (1 + \|x(s)\|) ds + M \sum_{0 < t_i < t} \|F_i(x(t_i))\| \\ &\leq M\|x_0\| + MkT + Mk \int_0^t \|x(s)\| ds + M_1. \end{aligned}$$

Applying Grownwall inequality on each subinterval for which $x(t)$ is continuous, we obtain

$$\begin{aligned} \|x(t)\| &\leq (M_1 + M\|x_0\| + MkT)e^{Mk \int_0^t d\tau} \\ &\leq (M_1 + M\|x_0\| + MkT)e^{MkT} \equiv M_2, \end{aligned}$$

for some constants M_1 and M_2 . This proves that the set of mild solutions is bounded in $PC([0, T], X)$.

Now, let us consider the corresponding control system. We model the control space by a separable complete metric space Z (i.e., a Polish space). By P_f (P_{fc}), we denote a class of nonempty closed (closed and convex) subsets of Z . Let $I = [0, T]$. Recall that a multifunction $\Gamma : I \rightarrow P_f(Z)$ is said to be measurable if for each $F \in P_f(Z)$, $\Gamma^{-1}(F)$ is Lebesgue measurable in I . We defined S_Γ to be the set of all measurable selections of $\Gamma(\cdot)$, i.e.,

$$S_\Gamma = \{u : I \rightarrow Z \mid u(t) \text{ is measurable and } u(t) \in \Gamma(t), \mu\text{-a.e. } t \in I\},$$

where μ is the Lebesgue measure on I . Note that the set $S_\Gamma \neq \emptyset$ if $\Gamma : I \rightarrow P_f(Z)$ is measurable (see Li&Yong [6], Theorem 2.23, p.100). Consider the following control systems

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + g(t, x(t), u(t)), \quad t \in [0, T] \setminus D, \\ x(0) &= x_0 \in X, \\ \Delta x(t_i) &= F_i(x(t_i)) \quad , \quad i = 1, 2, \dots, n. \end{aligned} \tag{3}$$

Here, we require the operators A , and F_i 's of (3) to satisfy hypothesis (A) and (F) respectively. We now give some new hypotheses for the remaining data.

- (U) $U : I \rightarrow P_{fc}(Z)$ is a measurable multifunction satisfying $U(\cdot) \subset K$, where K is a compact subset of Z . For the admissible controls, we choose the set $U_{ad} = S_U$.
- (G1) $g : I \times X \times Z \rightarrow X$ is an operator such that
 - (1) $t \mapsto g(t, x, z)$ is measurable, the map $(x, z) \mapsto g(t, x, z)$ is continuous on $X \times Z$, and there is a constant $L > 0$ such that

$$\|g(t, x_1, z) - g(t, x_2, z)\| \leq L \|x_1 - x_2\|, \text{ for all } t \in I, x_1, x_2 \in X, \text{ and } z \in Z.$$

- (2) There exists a constant $k > 0$ such that $\|g(t, x, z)\| \leq k(1 + \|x\|)$, $t \in I$, $x \in X$ and $z \in Z$.

By assumption (U), the control set S_U is nonempty and is called the class of original control. Now, let us define

$X_0 = \{x \in PC([0, T], X) \mid x \text{ is a solution of (3) corresponding to } u\}$.

X_0 is called the class of original trajectories.

$A_0 = \{(x, u) \in PC([0, T], X) \times S_U \mid x \text{ is a solution of (3) corresponding to } u\}$.

A_0 is called the class of admissible state-control pairs.

The following theorem guarantees that $X_0 \neq \emptyset$. Its proof follows immediately from Theorem 3.1 by defining the function $g_u(t, x) = g(t, x, u)$ and noting that g_u satisfies all hypotheses of Theorem 3.1.

Theorem 3.2 Assume that hypotheses (A), (F), (G1) and (U) hold. For every $u \in S_U$, equation (3) has a unique mild solution $x(u) \in PC([0, T], X)$. Moreover, the set of mild solutions is bounded in $PC([0, T], X)$.

4 Relaxed impulsive systems

We consider the following optimal control problem

$$(P) \quad \inf \{J(x, u) = \int_0^T L(t, x(t), u(t)) dt\}$$

subject to equation (3).

It is well known that, to solve optimization problems involving (P) and obtain an optimal state-control pair, we need some kind of convexity hypothesis on the orientor field $L(t, x(t), u(t))$. If the convexity hypothesis is no longer satisfied, in order to get an optimal admissible pair, we need to pass to a larger system with measure control (or know as "relaxed control") in which the orientor field has been convexified. For this purpose, we introduce the relaxed control and the corresponding relaxed systems.

Let Z be a separable complete metric space (i.e. a Polish space) and $B(Z)$ be its Borel σ -field. Let (Ω, Σ, μ) be a measure space. We will denote the space of probability measures on the measurable space $(Z, B(Z))$ by $M_+^1(Z)$.

A Caratheodory integrand on $\Omega \times Z$ is a function $f : \Omega \times Z \rightarrow R$ such that $f(\cdot, x)$ is Σ -measurable on Ω , $f(\omega, \cdot)$ is continuous on Z for all $\omega \in \Omega$, and $\sup\{|f(\omega, z)| : z \in Z\} \leq \alpha(\omega)$, a.e., for some functions $\alpha(\cdot) \in L_1(\Omega)$. We denote the set of all Caratheodory integrands on $\Omega \times Z$ by $Car(\Omega, Z)$.

By a transition probability, we mean a function $\lambda : \Omega \times B(Z) \rightarrow [0, 1]$ such that for every $A \in B(Z)$, $\lambda(\cdot, A)$ is Σ -measurable and for every $\omega \in \Omega$, $\lambda(\omega, \cdot) \in M_+^1(Z)$. We use $R(\Omega, Z)$ to

denote the set of all transition probabilities from (Ω, Σ) into $(Z, B(Z))$. Following Balder [2], we can define a topology on $R(\Omega, Z)$ as follows : Let $f \in Car(\Omega, Z)$ and define

$$I_f(\lambda) = \int_{\Omega} \int_Z f(\omega, z) \lambda(\omega)(dz) d\mu(\omega). \quad (4)$$

The weak topology on $R(\Omega, Z)$ is defined as the weakest topology for which all functionals $I_f : R(\Omega, Z) \rightarrow R$, $f \in Car(\Omega, Z)$, are continuous.

Supposing $\Omega = I = [0, T]$ and Z is a compact Polish space, then the space $Car(I, Z)$ can be identified with the separable Banach space $L_1(I, C(Z))$ where $C(Z)$ is the space of all real valued continuous functions on Z . To see this, we associate to each Caratheodory integrand $\phi(\cdot, \cdot)$ the map $t \mapsto \phi(t, \cdot) \in C(Z)$. Let $M(Z)$ be the space of all regular bounded countably additive measures defined on $B(Z)$. We note that $M(Z)$ is a Banach space under the total variation norm, i.e., $\|\lambda\|_{M(Z)} = |\lambda|(Z)$. Then by the Riesz representation theorem, the dual $[C(Z)]^*$ can be identified algebraically and metrically with $M(Z)$. The duality pair between $M(Z)$ and $C(Z)$ is given by

$$\langle \lambda, f \rangle = \int_Z f(z) \lambda(dz).$$

So $M(Z)$ is a separable (see Warga [9], p.265) dual Banach space and hence has a Radon-Nikodym property. This observation combined with Theorem 1 of Diestel and Uhr [3, p. 98], tells us that

$$L_1(I, C(Z))^* = L_{\infty}(I, M(Z)). \quad (5)$$

Hence the weak topology on $R(I, Z)$ coincides with the relative $w^*(L_{\infty}(I, M(Z)), L_1(I, C(Z)))$ -topology.

The duality pair between $L_{\infty}(I, M(Z))$ and $L_1(I, C(Z))$ is given by

$$\begin{aligned} \langle \langle \lambda, f \rangle \rangle &= \int_0^T \langle \lambda(t), f(t) \rangle dt \\ &= \int_0^T \int_Z f(t)(z) \lambda(t)(dz) dt \\ &= \int_0^T \int_Z f(t, z) \lambda(t)(dz) dt, \end{aligned} \quad (6)$$

which is the same formula as in (4) with $f(t, z) \equiv f(t)(z)$. This fact will be useful in the study of the relaxed control system where the control functions are transition probabilities.

Now we introduce some assumptions imposed on the class of relaxed controls which will be denoted by S_{Σ} .

(U1) Z is a compact Polish space, $U : I \rightarrow P_{fc}(Z)$ is a measurable multifunction.

Define $\Sigma(t) = \{\lambda \in M_+^1(Z), \lambda(U(t)) = 1\}$ and let $S_{\Sigma} \subseteq R(I, Z)$ be the set of transition probabilities on $I \times B(Z)$ that are measurable selections of $\Sigma(\cdot)$. For any $u \in S_U$, we define the relaxation $\delta_u \in S_{\Sigma}$ of u by $\delta_u(t) \equiv$ Dirac probability measure at $u(t)$. Then we can identify $S_U \subseteq S_{\Sigma}$. From now on, we shall consider S_U and S_{Σ} as a subspace of the topological space $R(I, Z)$ with the weak topology defined above.

We list two lemmas which will be useful in discussing the relaxation problem. The proofs can be found in Warga [9, Theorem IV 2.1] and Balder [2, Corollary 3] respectively.

Lemma 4.1 Suppose Z is a compact Polish space. Then S_Σ is convex, compact, and sequentially compact.

Lemma 4.2 S_U is dense in S_Σ .

The following theorem is the Arzela-Ascoli Theorem for continuous vector-valued functions. A proof of this result can be found in Carroll [3, Thm. 8.18, p. 34].

Theorem 4.3 (Arzela-Ascoli) A subset $K \subseteq C(I, H)$ is relatively compact if and only if K is equicontinuous and for all $t \in I$, $K(t) = \{f(t) \mid f \in K\}$ is a relatively compact subset of H .

Next, let us consider this new larger system know as "relaxed impulsive system"

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \int_Z g(t, x(t), z) \lambda(t)(dz), \quad 0 \leq t \leq T, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad i = 1, 2, \dots, n. \end{aligned} \tag{7}$$

We will denote the set of trajectories of (7) by X_r , i.e.,

$X_r = \{x \in PC(I, X) \mid x \text{ is a mild solution of (7) corresponding to } \lambda \in S_\Sigma\}$.

Moreover, the set of admissible state-control pairs of (7) will be denoted by

$A_r = \{(x, \lambda) \in PC(I, X) \times S_\Sigma \mid x \text{ is a mild solution of (7) corresponding to } \lambda \in S_\Sigma\}$.

Note that $X_0 \subseteq X_r$, since $S_U \subseteq S_\Sigma$, and, if the hypotheses of Theorem 3.2 are satisfied, $X_0 \neq \emptyset \Rightarrow X_r \neq \emptyset$. To see this, given any relaxed control $\lambda \in S_\Sigma$, if we set $\bar{g}(t, x(t), \lambda(t)) = \int_Z g(t, x(t), z) \lambda(t)(dz)$ then, working as in the proof of Theorem 3.2, one can show that there exists a relaxed admissible trajectory $x(\lambda)$ corresponding to λ . We now summarize the above discussion into a theorem.

Theorem 4.4 Assume that hypotheses (A), (F), (G1) and (U1) hold. For every $\lambda \in S_\Sigma$, equation (7) has a unique mild solution $x(\lambda) \in PC(I, X)$. Moreover the set X_r is bounded in $PC(I, X)$, i.e., $\|x(\lambda)\|_{PC(I, X)} \leq M$ for all $\lambda \in S_\Sigma$.

The next theorem gives us a useful relation between X_0 and X_r .

Theorem 4.5 If assumptions (A), (G), (F), (G1) and (U1) hold, then $X_r = \overline{X_0}$ (closure is taken in $PC(I, X)$).

Before proving this theorem, we need a lemma.

Lemma 4.6 If assumptions (A), (G), (F), (G1) and (U1) hold and $\lambda_k \rightarrow \lambda$ in $R(I, Z)$. Suppose that $\{x_k, x\}$ is the solution of (7) corresponding to $\{\lambda_k, \lambda\}$. Assume further that there exists $y \in PC(I, X)$ such that $x_k \rightarrow y$ as $k \rightarrow \infty$. Then y is a solution of (7) corresponding to the control variable λ .

Proof Since x_k is a solution of (7) corresponding to the control variable λ_k then

$$x_k(t) = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz)ds + \sum_{0 < t_i < t} T(t-t_i) F_i(x_k(t_i)), \quad 0 \leq t \leq T.$$

We aim to prove that y is a solution of (7) corresponding to the control variable λ , i.e., we shall show that

$$y(t) = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, y(s), z) \lambda(s)(dz) ds + \sum_{0 < t_i < t} T(t-t_i) F_i(y(t_i)), 0 \leq t \leq T.$$

For each fixed $0 \leq t \leq T$, and $h^* \in X^*$, we denote the duality pair between X and X^* by $\langle \cdot, \cdot \rangle$ and denote $h_t^*(s, z) \equiv \langle T(t-s)g(s, y(s), z), h^* \rangle$, where $0 \leq s \leq t \leq T$, $z \in Z$. It follows from (G1) that $h_t^*(s, z)$ is a Carathéodory integrand. Then, by the topology on $R(I, Z)$, we have

$$\int_{[0,t]} \int_Z \langle T(t-s)g(s, y(s), z), h^* \rangle \lambda_k(s)(dz) dt \rightarrow \int_{[0,t]} \int_Z \langle T(t-s)g(s, y(s), z), h^* \rangle \lambda(s)(dz) dt \text{ in } R,$$

as $n \rightarrow \infty$. Hence

$$\langle \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) dt, h^* \rangle \rightarrow \langle \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda(s)(dz) dt, h^* \rangle \text{ in } R,$$

as $n \rightarrow \infty$. Since h^* is an arbitrary element in X^* then

$$\int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) dt \rightarrow \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda(s)(dz) dt \text{ in } X, \quad (8)$$

as $n \rightarrow \infty$. Moreover, we note that

$$\begin{aligned} & \left\| \int_0^t T(t-s) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) dt \right. \\ & \quad \left. + \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) dt - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda(s)(dz) dt \right\| \\ & \leq \left\| \int_0^t T(t-s) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) ds \right\| \quad (9) \\ & \quad + \left\| \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda(s)(dz) ds \right\| \end{aligned}$$

It follows from equation (8) that the second expression of inequality (9) converges to zero as $k \rightarrow \infty$. The first expression also converges to zero. To see this, we note that

$$\begin{aligned} & \left\| \int_0^t T(t-s) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z) \lambda_k(s)(dz) ds \right\|_X \\ & \leq \int_{[0,t]} \int_Z \|T(t-s)\|_{B(X)} \|g(s, x_k(s), z) - g(s, y(s), z)\|_X \lambda_k(s)(dz) ds \\ & \leq \int_{[0,t]} \int_Z ML \|x_k(s) - y(s)\|_X \lambda_k(s)(dz) ds \quad (\text{assumption G1}) \\ & \leq \int_{[0,t]} \|x_k(s) - y(s)\|_X \left(\int_Z ML \lambda_k(s)(dz) \right) ds \\ & \leq \int_{[0,t]} \|x_k(s) - y(s)\|_X ML \lambda_k(s)(Z) ds \\ & \leq ML \int_{[0,t]} \|x_k(s) - y(s)\|_X ds \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (\lambda_k(s) \text{ is a probability measure on } Z) \end{aligned}$$

Now we turn to the jump part.

$$\begin{aligned}
& \left\| \sum_{0 < t_i < t} T(t - t_i) F_i(x_k(t_i)) - \sum_{0 < t_i < t} T(t - t_i) F_i(y(t_i)) \right\|_X \\
& \leq \sum_{0 < t_i < t} \|T(t - t_i)\|_{B(X)} \|F_i(x_k(t_i)) - F_i(y(t_i))\|_X \\
& \leq M \sum_{0 < t_i < t} \|F_i(x_k(t_i)) - F_i(y(t_i))\|_X \\
& \leq M \sum_{0 < t_i < t} h_i \|x_k(t_i) - y(t_i)\|_X \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

This prove that

$$\lim_{k \rightarrow \infty} x_k = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, y(s), z) \lambda(s)(dz)ds + \sum_{0 < t_i < t} T(t - t_i) F_i(x_k(t_i)).$$

By the assumption, we get $\lim_{k \rightarrow \infty} x_k = y$. Hence y is the solution (7) as required.

Lemma 4.7 If assumptions (A), (G), (F), (G1) and (U1) hold, the semigroup $\{T(t)\}$ in the assumption (A) is compact, and $\lambda_k \rightarrow \lambda$ in $R(I, Z)$. Suppose that $\{x_k, x\}$ is the solution of (7) corresponding to $\{\lambda_k, \lambda\}$, by working with a subsequence is necessary, $x_k \rightarrow x$ in $PC(I, X)$ as $k \rightarrow \infty$.

Proof Suppose that $\lambda_k \rightarrow \lambda$ in $R(I, Z)$ as $k \rightarrow \infty$ and $\{x_k, x\}$ is the solution of (7) corresponding to $\{\lambda_k, \lambda\}$. Since $(x_k, \lambda_k) \in A_r$ for each positive integer k , then (x_k, λ_k) must satisfy the equation

$$x_k(t) = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz)ds + \sum_{0 < t_i < t} T(t - t_i) F_i(x_k(t_i)), 0 \leq t \leq T,$$

while (x, λ) satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, x(s), z) \lambda(s)(dz)ds + \sum_{0 < t_i < t} T(t - t_i) F_i(x(t_i)), 0 \leq t \leq T.$$

To finish the proof, we try to choose $y \in X_r$ such that y is a solution of (7) corresponding to this λ and $x_k \rightarrow y$ in $PC(I, H)$ as $k \rightarrow \infty$. The unique property of solution of (7) implies $x = y$ and hence $x_k \rightarrow x$ in $PC(I, X)$ and we are done. Since x_k is a mild solution of (7), then by Theorem 4.4, $\{x_k\}$ is a bounded sequence in $PC(I, X)$. By using the same technique as in [7, p. 193], one can prove that the set $\{x_k\}$ is equicontinuous. To see this, let $\rho, t, t' \in [0, T]$ be such that $0 < \rho < t < t'$. Let $N \equiv \sup\{\|g(s, x_k(s), z)\| : s \in [0, T], z \in Z\}$ which is finite and independent of k since g is continuous on the compact space Z , the solution set $\{x_k\}$ is bounded, and hypothesis G_1 . Then

$$\begin{aligned}
\|x_k(t') - x_k(t)\| &\leq \|T(t')x_0 - T(t)x_0\| + \left\| \int_t^{t'} T(t' - s) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) ds \right\| \\
&\quad + \left\| \left(\int_0^{t-\rho} + \int_{t-\rho}^t \right) (T(t' - s) - T(t - s)) \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) ds \right\| \\
&\quad + \sum_{t < t_i < t'} \|T(t' - t_i)\| \|F_i(x_k(t_i))\| + \sum_{0 < t_i < t'} \|T(t' - t_i)T(t - t_i)\| \|F_i(x_k(t_i))\| \\
&\leq \|T(t')x_0 - T(t)x_0\| + M \int_t^{t'} \left\| \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) \right\| ds \\
&\quad + \int_0^{t-\rho} \|T(t' - s) - T(t - s)\| \left\| \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) \right\| ds \\
&\quad + \int_{t-\rho}^t \|T(t' - s) - T(t - s)\| \left\| \int_Z g(s, x_k(s), z) \lambda_k(s)(dz) \right\| ds \\
&\quad + \sum_{t < t_i < t'} M \|F_i(x_k(t_i))\| + \sum_{0 < t_i < t'} \|T(t' - t_i) - T(t - t_i)\| \|F_i(x_k(t_i))\| \\
&\leq \|T(t')x_0 - T(t)x_0\| + MN(t' - t) + N \int_0^{t-\rho} \|T(t' - s) - T(t - s)\| ds + 2MN\rho \\
&\quad + \sum_{t < t_i < t'} M \|F_i(x_k(t_i))\| + \sum_{0 < t_i < t'} \|T(t' - t_i) - T(t - t_i)\| \|F_i(x_k(t_i))\|. \quad (10)
\end{aligned}$$

Since $t > \rho > 0$ is arbitrary, and since $T(t)$ is continuous in the uniform operator topology for $t \geq \rho > 0$, the first four terms on right-hand of inequality (10) tend to zero as t tends to t' and ρ tends to zero. Moreover, the two jump terms also tend to zero as t tend to t' since there is no jump in the interval (t, t') if length $|t - t'|$ is small enough. This proves that the set $\{x_k\}$ is equicontinuous.

Let $K^1 \equiv \{x_k^1\}$ be the restriction of the sequence $\{x_k\}$ on the interval $[0, t_1]$, i.e., $x_k^1(t) = x_k(t)$ on $[0, t_1]$ and equal to zero elsewhere. Clear $K(0) = \{x_0\}$ is compact in H . For $0 < \epsilon < t \leq t_1$, define

$$K_\epsilon^1(t) = \{T(\epsilon)x_k^1(t - \epsilon) : k = 1, 2, \dots\}$$

For each $t \in [0, t_1]$, $K^1(t)$ is a bounded subset of H and, by our hypothesis, $T(t)$ is a compact operator for $t > 0$, it follows from the above expression that $K_\epsilon^1(t)$ is relative compact for $t \in (\epsilon, t_1]$. Further, by using the same proof as in (10), one can show that

$$\sup\{\|x_k^1(t) - T(\epsilon)x_k^1(t - \epsilon)\| : k = 1, 2, \dots\} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Then the set $K_\epsilon^1(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact set. Hence $K^1(t)$ itself is relatively compact. Applying Arzela-Ascoli Theorem, the sequence $\{x_k^1\}$ is relatively compact in $C([0, t_1], X)$. Then there exists a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$x_k^1 \rightarrow y^1 \text{ in } C([0, t_1], X) \text{ as } k \rightarrow \infty.$$

Now, let $\{x_k^2\}$ be the restriction of the sequence $\{x_k\}$ on the interval $(t_1, t_2]$, i.e., $x_k^2(t) = x_k(t)$ on $(t_1, t_2]$ and equal to zero elsewhere. By using the same proof as above, there exists a subsequence of $\{x_k^2\}$, again denoted by $\{x_k^2\}$, such that

$$x_k^2 \rightarrow y^2 \text{ in } C((t_1, t_2], X) \text{ as } k \rightarrow \infty.$$

It is obvious that $y^2(t^+) = \lim_{k \rightarrow \infty} x_k(t_1^+)$. Hence y^2 possesses a right hand limit. Continue this process until to the interval $(t_{n-1}, t_n]$. Define a function y on $[0, T]$ as follows:

$$y(t) = \begin{cases} x(0) & \text{if } t = 0 \\ y^i(t) & \text{if } t \in (t_{i-1}, t_i), i = 0, 1, 2, \dots, n \end{cases}$$

Then $y \in PC([0, T], H)$ and there is a subsequence of $\{x_k\}$ converges to y . Applying lemma 4.6, we get y is also a solution of (7). By uniqueness of the solution of (7), we get $x = y$. Hence there is a subsequence of $\{x_k\}$ converges to x and we are done.

Proof of Theorem 4.5 Firstly, we shall show that $X_r \subset \overline{X_0}$. Let $x \in X_r$, then there exists $\lambda \in S_\Sigma$ such that $(x, \lambda) \in A_r$. By virtue of the density result as in Lemma 4.2, there exists a sequence $\{u_k\} \subset S_U$ such that $\delta_{u_k} \rightarrow \lambda$ in $R(I, Z)$. Let x_k be the solution of (7) corresponding to u_k . Then we have a sequence $\{(x_k, u_k)\} \subset A_0$. Since for each k , $(x_k, u_k) \in A_0$ then (x_k, u_k) must satisfy the equation

$$\begin{aligned} \dot{x}_k(t) &= Ax_k(t) + \int_Z g(t, x_k(t), z) \delta_{u_k}(t)(dz), \\ x_k(0) &= x_0 \in X, \\ \Delta x_k(t_i) &= F_i(x_k(t_i)), \quad i = 1, 2, \dots, n, \quad k = 1, 2, 3, \dots \\ (0 &< t_1 < t_2 < \dots < t_n) \end{aligned}$$

Applying Lemma 4.7, we get $x_k \rightarrow x$ in $PC(I, X)$. This proves that $x \in \overline{X_0}$, and hence $X_r \subset \overline{X_0}$. Finally, we will show that X_r is closed in $PC(I, X)$. Let $\{x_k\}$ be a sequence of points in X_r such that $x_k \rightarrow x$ in $PC(I, X)$ as $k \rightarrow \infty$. By definition of X_r , there is a sequence $\{\lambda_k\}$ of points in S_Σ such that $(x_k, \lambda_k) \in A_r$, $k = 1, 2, 3, \dots$. Lemma 4.1 implies that S_Σ is compact in $R(I, Z)$ under the weak topology. Moreover, $R(I, Z)$ -topology coincides with the relative $\omega^*(L_\infty(I, M(Z)), L_1(I, C(Z)))$ -topology. Then, by passing to a subsequence if necessary, we may assume that $\lambda_k \rightarrow \lambda$ in $R(I, Z)$ as $k \rightarrow \infty$. Applying Lemma 4.7, there is $x \in X_r$ such that $x_k \rightarrow x$ in $PC(I, X)$ as $k \rightarrow \infty$. Hence X_r is closed in $PC(I, X)$ and consequently, $\overline{X_0} \subset \overline{X_r} = X_r$. The proof of Theorem 4.5 is now complete.

The following corollary is an immediate consequence of lemma 4.7

Corollary 4.8 Under assumption of Theorem 4.5, the function $\lambda \mapsto x(\lambda)$ is continuous from $S_\Sigma \subseteq R(I, Z)$ into $PC(I, X)$.

5 Existence of optimal controls

Consider the following Lagrange optimal control problem (P_r) : Find a control policy $\bar{\lambda} \in S_\Sigma$, such that it imparts a minimum to the cost functional J given by

$$(P_r) \quad J(\lambda) \equiv J(x^\lambda, \lambda) \equiv \int_I \int_Z l(t, x^\lambda(t), z) \lambda(t)(dz) dt,$$

where x^λ is the solution of the system (7) corresponding to the control $\lambda \in S_\Sigma$. We form the following hypothesis concerning the integrand $l(\cdot, \cdot, \cdot)$.

(L) $l : I \times H \times Z \rightarrow R \cup \{+\infty\}$ is Borel measurable satisfying the following conditions

- (1) $(\xi, z) \mapsto l(t, \xi, z)$ is lower semicontinuous on $H \times Z$ for each fixed t .
- (2) There exist $\psi(t) \in L_1(I, R)$ such that $|l(t, \xi, z)| \leq \psi(t)$ for almost all $t \in I$.
- (3) l maps bounded set into bounded set.

Let $m_r = \inf\{J(\lambda) : \lambda \in S_\Sigma\}$. We have the following theorem on the existence of optimal impulsive control.

Theorem 5.1 Suppose assumptions (A),(F),(G1),(U1),(L) hold and Z is a compact Polish space, then there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$.

Before giving the proof of Theorem 5.1, we need a lemma. The proof is similar to Lemma 3.3 in [10].

Lemma 5.2 Let $h : I \times H \times Z \rightarrow R$ be such that

1. $t \mapsto (t, x, z)$ is measurable and $(x, z) \mapsto h(t, x, z)$ is continuous.
2. $|h(t, x, z)| \leq \psi(t) \in L_1(I)$ for all $(x, z) \in H \times Z$.

If $x_k \rightarrow x \in C([0, T], H)$ then

$$\bar{h}_k(\cdot, \cdot) \rightarrow \bar{h}(\cdot, \cdot) \text{ in } L_1(I, C(Z))$$

as $k \rightarrow \infty$, where $\bar{h}_k(t, z) = h(t, x_k(t), z)$ and $\bar{h}(t, z) = h(t, x(t), z)$.

Proof of Theorem 5.1 If $J(\lambda) = +\infty$ for all $\lambda \in S_\Sigma$, then every control is admissible. Assume $\inf\{J(\lambda) : \lambda \in S_\Sigma\} = m_r < +\infty$. By assumption (L), we have $m_r > -\infty$. Hence m_r is finite. Let $\{\lambda_k\}$ be a minimizing sequence so that $\lim_{k \rightarrow \infty} J(\lambda_k) = m_r$. By Lemma 4.1, S_Σ is compact in the topology $R(I, Z)$. Hence, by passing to a subsequence if necessary, we may assume that $\lambda_k \rightarrow \bar{\lambda}$ in $R(I, Z)$ as $k \rightarrow \infty$. This means that $\lambda_k \xrightarrow{w^*} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \rightarrow \infty$. Let $\{x_k, \bar{x}\}$ be the solution of (7) correspond to $\{\lambda_k, \bar{\lambda}\}$. By Lemma 4.5, we get $x_k \rightarrow \bar{x}$ in $PC(I, X)$ and $(\bar{x}, \bar{\lambda}) \in A_r$. Next, we shall prove that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

As before, we identify the space of Caratheodory integrand $Car(I, Z)$ with the separable Banach space $L_1(I, C(Z))$. We note that every lower semicontinuous measurable integrand $l : I \times H \times Z \rightarrow R \cup \{+\infty\}$ is the limit of an increasing sequence of Caratheodory integrand $\{l_j\} \in L_1(I, C(Z))$ for each fixed $h \in H$. Thus, there exists an increasing sequence of Caratheodory integrands $\{l_j\} \in L_1(I, C(Z))$ such that

$$l_j(t, \bar{x}(t), z) \uparrow l(t, \bar{x}(t), z) \text{ as } j \rightarrow \infty \text{ for all } t \in I, z \in Z.$$

Since $x_k \rightarrow \bar{x}$ in $PC(I, X)$, by applying Lemma 5.2 on each subinterval of $[0, T]$, $l_j(t, x_k(t), z) \rightarrow l_j(t, \bar{x}(t), z)$ as $k \rightarrow \infty$ for almost all $t \in I$ and all $z \in Z$. We note that since $\lambda_k \xrightarrow{w^*} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \rightarrow \infty$, then

$$\begin{aligned} J(\bar{x}, \bar{\lambda}) &= \langle \bar{\lambda}, l \rangle = \int_I \int_Z l(t, \bar{x}(t), z) \bar{\lambda}(t)(dz) dt \\ &= \lim_{j \rightarrow \infty} \int_I \int_Z l_j(t, \bar{x}(t), z) \bar{\lambda}(t)(dz) dt \\ &= \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_I \int_Z l_j(t, x_k(t), z) \lambda_k(t)(dz) dt \\ &\leq \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_I \int_Z l_j(t, x_k(t), z) \lambda_k(t)(dz) dt = m_r. \end{aligned}$$

However, by definition of m_r , it is obvious that $J(\bar{x}, \bar{\lambda}) \geq m_r$. Hence $J(\bar{x}, \bar{\lambda}) = m_r$. This implies that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

Remark. If $J_0(x, u) = \int_I l(t, x(t), u(t)) dt$ is the cost functional for the original problem and $m = \inf\{J_0(x, u) : u \in U_{ad}\}$. In general we have $m_r \leq m$. It is desirable that $m_r = m$, i.e., our

relaxation is reasonable. With some stronger conditions on l , i.e., the map $(\xi, \eta, z) \mapsto l(t, \xi, z)$ is continuous and $|l(t, \xi, z)| \leq \theta_R(t)$ for all most all $t \in I$ and $\theta_R \in L_1(I)$, one can show that $m_r = m$. The proof is similar to Theorem 4.B. in [10].

6 References

1. N.U. Ahmed, Optimal impulsive control for impulsive systems in Banach spaces, *Internat. J. Differential Equations Appl.*,1 (2000), 37-52.
2. E. Balder, A general denseness result for relaxed control theory, *Bull. Austral. Math. Soc.* 20 (1984), 463-475.
3. R. Carroll, *Abstract Methods in Partial Differential Equations*, Harper and Row, New York, 1969.
4. J. Diestel and J.J.Uhr, *Vector measures*, American mathematical Society, 1997.
5. J.H. Liu, Nonlinear impulsive evolution equations, *Dynamics of Continuous, Discrete and Impulsive Systems*, 6(1999), 77-85.
6. X. Li and J. Yong, *Optimal control theory for infinite dimensional systems*, Birkhauser, Boston, 1995.
7. A. Parzy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
8. P. Sattayatham and Huawu, Relaxation and optimal controls for a class of infinite dimensional nonlinear evolution suystems, *Journal of Guizhou university*, Vol. 16, No. 4 (1999), 242-250.
9. J. Warga, *Optimal control of differential and functional equations*, Academic press, New York, 1972.
10. X.L. Xiang, P. Sattayatham, and Wei Wei, Relaxation controls for a class of strongly nonlinear delay evolution equations, *Nonlinear Analysis* 52 (2003), 703-723.