By (2.15) and (2.16), we have

$$\|S_nx - Sx\| = \|U_{n,N}x - U_Nx\|$$
$$\le |\alpha_1^{n,1} - \alpha_1^1|\|T_1x - x\| + \sum_{j=2}^{N} |\alpha_1^{n,j} - \alpha_1^j|(\|T_jU_{j-1}x\| + \|U_{N-j}x\|) + \sum_{j=2}^{N} |\alpha_3^{n,j} - \alpha_3^j|(\|U_{j-1}x\| + \|x\|).$$

This together with our assumption, we can conclude that

$$\lim_{n\longrightarrow\infty} \|S_nx - Sx\| = 0. \quad \square$$

3. Main result

In this section, we prove a strong convergence theorem of the iterative scheme (3.1) to a common element of EP and $\bigcap_{i=1}^N F(T_i)$ under some control conditions.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let A be an α-inverse strongly monotone mapping of C into H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \bigcap EP \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \le \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \le 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N$, $\{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \le \theta_3 < 1$. Let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \beta_nx_n + (1 - \beta_n)S_n(a_nu + (1 - a_n)z_n), & \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{a_n\} \in [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions:

(i) $0 < a \le \lambda_n \le b < 2\alpha$, $0 < c \le \beta_n \le d < 1$;

(ii) $\lim_{n\to\infty} \dfrac{\lambda_n}{\lambda_{n+1}} = 1$;

(iii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$;

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$, *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.

Then $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^N F(T_i) \bigcap EP$, where $z = P_{\bigcap_{i=1}^N F(T_i)\bigcap EP}u$.

Proof. First, we show that $(I - \lambda_nA)$ is nonexpansive. Let $x, y \in C$. Since A is α-strongly monotone and $\lambda_n < 2\alpha$ $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - \lambda_nA)x - (I - \lambda_nA)y\|^2 &= \|x - y - \lambda_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 - 2\alpha\lambda_n\|Ax - Ay\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\ &\le \|x - y\|^2. \end{aligned} \tag{3.2}$$

Thus $(I - \lambda_nA)$ is nonexpansive.

Since

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C,$$

we obtain

$$F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (I - \lambda_nA)x_n\rangle \ge 0, \quad \forall y \in C.$$

By Lemma 2.6, we have $z_n = T_{\lambda_n}(x_n - \lambda_nAx_n)$ $\forall n \in \mathbb{N}$.

Let $z \in \bigcap_{i=1}^N F(T_i) \bigcap EP$. Then $F(z, y) + \langle y - z, Az\rangle \ge 0$, $\forall y \in C$.

So $F(z, y) + \frac{1}{\lambda_n}\langle y - z, z - z + \lambda_n Az\rangle \geq 0, \quad \forall y \in C.$

Again by Lemma 2.6, we have $z = T_{\lambda_n}(z - \lambda_n Az)$. Since $I - \lambda_n A$ and T_{λ_n} are nonexpansive, we have

$$\begin{aligned}\|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2\\ &\leq \|x_n - z\|^2,\end{aligned} \tag{3.3}$$

ence $\|z_n - z\| \leq \|x_n - z\|$.

Putting $y_n = a_n u + (1 - a_n)z_n$. Then we have

$$\begin{aligned}\|y_n - z\| &= \|a_n(u - z) + (1 - a_n)(z_n - z)\|\\ &\leq a_n\|u - z\| + (1 - a_n)\|x_n - z\|.\end{aligned} \tag{3.4}$$

is implies that

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1 - \beta_n)(S_n y_n - z)\|\\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)\|y_n - z\|\\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)(a_n\|u - z\| + (1 - a_n)\|x_n - z\|).\end{aligned} \tag{3.5}$$

tting $K = \max\{\|x_1 - z\|, \|u - z\|\}$. By (3.5), we can show by induction that $\|x_n - z\| \leq K, \ \forall n \in \mathbb{N}$. This implies that $\{x_n\}$ bounded. Hence $\{Ax_n\}$, $\{y_n\}$, $\{S_n y_n\}$, $\{z_n\}$ are bounded.

Next we will show that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.6}$$

tting $u_n = x_n - \lambda_n Ax_n$. Then, we have $z_{n+1} = T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) = T_{\lambda_{n+1}}u_{n+1}$, $z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n) = T_{\lambda_n}u_n$. So we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|a_{n+1}u + (1 - a_{n+1})z_{n+1} - a_n u - (1 - a_n)z_n\|\\ &= \|a_{n+1}u + (1 - a_{n+1})T_{\lambda_{n+1}}u_{n+1} - a_n u - (1 - a_n)T_{\lambda_n}u_n\|\\ &= \|(a_{n+1} - a_n)u + (1 - a_{n+1})(T_{\lambda_{n+1}}u_{n+1} - T_{\lambda_{n+1}}u_n + T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n + T_{\lambda_n}u_n) - (1 - a_n)T_{\lambda_n}u_n\|\\ &= \|(a_{n+1} - a_n)u + (1 - a_{n+1})(T_{\lambda_{n+1}}u_{n+1} - T_{\lambda_{n+1}}u_n)\\ &\quad + (1 - a_{n+1})(T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n) + (1 - a_{n+1})T_{\lambda_n}u_n - (1 - a_n)T_{\lambda_n}u_n\|\\ &\leq |a_{n+1} - a_n|\|u\| + (1 - a_{n+1})\|u_{n+1} - u_n\|\\ &\quad + (1 - a_{n+1})\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\|.\end{aligned} \tag{3.7}$$

nce $I - \lambda_{n+1}A$ is nonexpansive, we have

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|x_{n+1} - \lambda_{n+1}Ax_{n+1} - x_n + \lambda_n Ax_n\|\\ &= \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n + (\lambda_n - \lambda_{n+1})Ax_n\|\\ &\leq \|x_{n+1} - x_n + |\lambda_n - \lambda_{n+1}|Ax_n\|.\end{aligned} \tag{3.8}$$

Lemma 2.6, we have

$$F(T_{\lambda_n}u_n, y) + \frac{1}{\lambda_n}\langle y - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0, \quad \forall y \in C$$

d

$$F(T_{\lambda_{n+1}}u_n, y) + \frac{1}{\lambda_{n+1}}\langle y - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle \geq 0, \quad \forall y \in C.$$

particular, we have

$$F(T_{\lambda_n}u_n, T_{\lambda_{n+1}}u_n) + \frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0, \tag{3.9}$$

d

$$F(T_{\lambda_{n+1}}u_n, T_{\lambda_n}u_n) + \frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle \geq 0. \tag{3.10}$$

mming up (3.9) and (3.10) and using (A2), we obtain

$$\frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle + \frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0, \quad \forall y \in C.$$

It then follows that

$$\left\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, \frac{T_{\lambda_{n+1}}u_n - u_n}{\lambda_{n+1}} - \frac{T_{\lambda_n}u_n - u_n}{\lambda_n}\right\rangle \geq 0.$$

This implies

$$\begin{aligned} 0 &\leq \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n - \frac{\lambda_n}{\lambda_{n+1}}(T_{\lambda_{n+1}}u_n - u_n)\right\rangle \\ &= \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(T_{\lambda_{n+1}}u_n - u_n)\right\rangle. \end{aligned}$$

It follows that

$$\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \left\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\right\| (\|T_{\lambda_{n+1}}u_n\| + \|u_n\|).$$

Hence, we obtain

$$\left\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\right\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L, \tag{3.11}$$

where $L = \sup\{\|u_n\| + \|T_{\lambda_{n+1}}u_n\| : n \in \mathbb{N}\}$.

By (3.7), (3.8) and (3.11), we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq |a_{n+1} - a_n|\|u\| + (1 - a_{n+1})\|u_{n+1} - u_n\| + (1 - a_{n+1})\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + (1 - a_{n+1})(\|x_{n+1} - x_n + |\lambda_n - \lambda_{n+1}|Ax_n\|) \\ &\quad + (1 - a_{n+1})\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + \left\|x_{n+1} - x_n + \lambda_{n+1}\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|Ax_n\right\| \\ &\quad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + \left\|x_{n+1} - x_n + b\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|Ax_n\right\| \\ &\quad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\|. \end{aligned} \tag{3.12}$$

We can rewrite x_{n+1} by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n, \tag{3.13}$$

where $y_n = a_n u + (1 - a_n)z_n$.

Next, we show that

$$\lim_{n\to\infty} \|S_n y_n - x_n\| = 0. \tag{3.14}$$

For $k \in \{2, 3, \ldots, N\}$, we have

$$\begin{aligned} \|U_{n+1,k}y_n - U_{n,k}y_n\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}y_n + \alpha_2^{n+1,k}U_{n+1,k-1}y_n + \alpha_3^{n+1,k}y_n \\ &\quad - \alpha_1^{n,k}T_kU_{n,k-1}y_n - \alpha_2^{n,k}U_{n,k-1}y_n - \alpha_3^{n,k}y_n\| \\ &= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}y_n - T_kU_{n,k-1}y_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}y_n \\ &\quad + (\alpha_3^{n+1,k} - \alpha_3^{n,k})y_n + \alpha_2^{n+1,k}(U_{n+1,k-1}y_n - U_{n,k-1}y_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}y_n\| \\ &\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\| \\ &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|y_n\| + \alpha_2^{n+1,k}\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}y_n\| \\ &= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\| \\ &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|y_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}y_n\| \\ &\leq \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\| \end{aligned}$$

$$
\begin{aligned}
&+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|\|y_n\|+|(\alpha_1^{n,k}-\alpha_1^{n+1,k})+(\alpha_3^{n,k}-\alpha_3^{n+1,k})|\|U_{n,k-1}y_n\|\\
&\le \|U_{n+1,k-1}y_n-U_{n,k-1}y_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\|\\
&+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|\|y_n\|+|\alpha_1^{n,k}-\alpha_1^{n+1,k}|\|U_{n,k-1}y_n\|+|\alpha_3^{n,k}-\alpha_3^{n+1,k}|\|U_{n,k-1}y_n\|\\
&=\|U_{n+1,k-1}y_n-U_{n,k-1}y_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|(\|T_kU_{n,k-1}y_n\|+\|U_{n,k-1}y_n\|)\\
&+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|(\|y_n\|+\|U_{n,k-1}y_n\|).
\end{aligned}
\tag{3.15}
$$

By (3.15), we obtain that for each $n\in\mathbb{N}$,

$$
\begin{aligned}
\|S_{n+1}y_n-S_ny_n\| &= \|U_{n+1,N}y_n-U_{n,N}y_n\|\\
&\le \|U_{n+1,1}y_n-U_{n,1}y_n\|+\sum_{j=2}^{N}|\alpha_1^{n+1,j}-\alpha_1^{n,j}|(\|T_jU_{n,j-1}y_n\|+\|U_{n,j-1}y_n\|)\\
&+\sum_{j=2}^{N}|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|y_n\|+\|U_{n,j-1}y_n\|)\\
&=|\alpha_1^{n+1,1}-\alpha_1^{n,1}|\|T_1y_n-y_n\|+\sum_{j=2}^{N}|\alpha_1^{n+1,j}-\alpha_1^{n,j}|(\|T_jU_{n,j-1}y_n\|+\|U_{n,j-1}y_n\|)\\
&+\sum_{j=2}^{N}|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|y_n\|+\|U_{n,j-1}y_n\|).
\end{aligned}
$$

This together with condition (iv), we obtain

$$
\lim_{n\to\infty}\|S_{n+1}y_n-S_ny_n\|=0.
\tag{3.16}
$$

By (3.12), we have

$$
\begin{aligned}
\|S_{n+1}y_{n+1}-S_ny_n\| &\le \|y_{n+1}-y_n\|+\|S_{n+1}y_n-S_ny_n\|\\
&\le |a_{n+1}-a_n|\|u\|+\|x_{n+1}-x_n\|+b\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\|Ax_n\|\\
&+\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|L+|a_{n+1}-a_n|\|T_{\lambda_n}u_n\|+\|S_{n+1}y_n-S_ny_n\|.
\end{aligned}
\tag{3.17}
$$

This together with (3.16) and conditions (ii) and (iii), we obtain

$$
\limsup_{n\to\infty}(\|S_{n+1}y_{n+1}-S_ny_n\|-\|x_{n+1}-x_n\|)\le 0.
\tag{3.18}
$$

It follows from (3.13) and (3.17) and Lemma 2.4, $\lim_{n\to\infty}\|S_ny_n-x_n\|=0$.
This implies that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|S_ny_n-x_n\|=0.
\tag{3.19}
$$

Next, we show that

$$
\lim_{n\to\infty}\|x_n-z_n\|=0.
\tag{3.20}
$$

By monotonicity of A and nonexpansiveness of T_{λ_n}, we have

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(S_ny_n-z)\|^2\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|y_n-z\|^2\\
&=\beta_n\|x_n-z\|^2+(1-\beta_n)\|a_n(u-z)+(1-a_n)(z_n-z)\|^2\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)(a_n\|u-z\|^2+(1-a_n)\|z_n-z\|^2) \qquad (3.21)\\
&=\beta_n\|x_n-z\|^2+(1-\beta_n)(a_n\|u-z\|^2+(1-a_n)\|T_{\lambda_n}(x_n-\lambda_nAx_n)-T_{\lambda_n}(z-\lambda_nAz)\|^2)\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)(a_n\|u-z\|^2+(1-a_n)\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)\|^2)\\
&=\beta_n\|x_n-z\|^2+(1-\beta_n)(a_n\|u-z\|^2+(1-a_n)\|(x_n-z)-\lambda_n(Ax_n-Az)\|^2)\\
&=\beta_n\|x_n-z\|^2+(1-\beta_n)(a_n\|u-z\|^2+(1-a_n)(\|x_n-z\|^2\\
&\quad-2\lambda_n\langle x_n-z,Ax_n-Az\rangle+\lambda_n^2\|Ax_n-Az\|^2))
\end{aligned}
$$

$$\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)(a_n\|u-z\|^2 + (1-a_n)(\|x_n - z\|^2 - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2))$$
$$= \beta_n\|x_n - z\|^2 + (1-\beta_n)(a_n\|u-z\|^2 + (1-a_n)(\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2))$$
$$\leq \|x_n - z\|^2 + (1-\beta_n)a_n\|u-z\|^2 + (1-a_n)(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2. \quad (3.22)$$

By (3.22), we have

$$(1-a_n)(1-\beta_n)\lambda_n(2\alpha - \lambda_n)\|Ax_n - Az\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\beta_n)a_n\|u-z\|^2. \quad (3.23)$$

Since $0 < a \leq \lambda_n \leq b < 2\alpha$ and $0 < c \leq \beta_n \leq d < 1$, we have

$$(1-a_n)(1-d)a(2\alpha - \lambda_n)\|Ax_n - Az\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\beta_n)a_n\|u-z\|^2$$
$$\leq \|x_{n+1} - x_n\|(\|x_n - z\| + \|x_{n+1} - z\|) + (1-\beta_n)a_n\|u-z\|^2. \quad (3.24)$$

This implies, by (3.19) and condition *(iii)*, that

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \quad (3.25)$$

Since T_{λ_n} is a firmly nonexpansive, we have

$$\|z_n - z\|^2 = \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2$$
$$\leq \langle (x_n - \lambda_n Ax_n) - (z - \lambda_n Az), z_n - z\rangle$$
$$= \frac{1}{2}(\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 + \|z_n - z\|^2 - \|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az) - (z_n - z)\|^2)$$
$$\leq \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Az)\|^2)$$
$$= \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Az\rangle - \lambda_N^2\|Ax_n - Az\|^2). \quad (3.26)$$

It follows that

$$\|z_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|. \quad (3.27)$$

By (3.21) and (3.27), we have

$$\|x_{n+1} - z\|^2 \leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[a_n\|u-z\|^2 + (1-a_n)\|z_n - z\|^2]$$
$$\leq \beta_n\|x_n - z\|^2 + a_n\|u-z\|^2 + (1-\beta_n)\|z_n - z\|^2$$
$$\leq \beta_n\|x_n - z\|^2 + a_n\|u-z\|^2 + (1-\beta_n)(\|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|)$$
$$\leq \|x_n - z\|^2 + a_n\|u-z\|^2 - (1-\beta_n)\|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|. \quad (3.28)$$

This implies

$$(1-\beta_n)\|x_n - z_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + a_n\|u-z\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|.$$

Hence

$$(1-d)\|x_n - z_n\|^2 \leq \|x_{n+1} - x_n\|(\|x_n - z\| + \|x_{n+1} - z\|) + a_n\|u-z\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|.$$

By (3.19) and (3.25), we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \quad (3.29)$$

Since $y_n = a_n u + (1-a_n)z_n$, we have $\|y_n - z_n\| = a_n\|u - z_n\|$.

This implies $\lim_{n\to\infty}\|y_n - z_n\| = 0$.

By (3.14) and (3.29), we have

$$\|S_n y_n - y_n\| \leq \|S_n y_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\| \to 0 \quad \text{as } n\to\infty. \quad (3.30)$$

Next, putting $z_0 = P_{\bigcap_{i=1}^N F(T_i)\bigcap EP}u$, we shall show that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \leq 0. \quad (3.31)$$

To show this inequality, take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle = \limsup_{k\to\infty}\langle u - z_0, y_{n_k} - z_0\rangle. \quad (3.32)$$

Without loss of generality, we may assume that $y_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. We first show $\omega \in EP$. We have $z_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Since $z_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$, we obtain

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), we have $\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq F(y, z_n)$. Then

$$\langle Ax_{n_k}, y - z_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle y - z_{n_k}, z_{n_i} - x_{n_k}\rangle \geq F(y, z_{n_k}), \quad \forall y \in C. \tag{3.33}$$

Put $z_t = ty + (1-t)\omega$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.33) we have

$$\begin{aligned}\langle z_t - z_{n_k}, Az_t\rangle &\geq \langle z_t - z_{n_k}, Az_t\rangle - \langle z_t - z_{n_k}, Ax_{n_i}\rangle - \left\langle z_t - z_{n_k}, \frac{z_{n_i} - x_{n_k}}{\lambda_{n_k}}\right\rangle + F(z_t, z_{n_k})\\ &= \langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle - \left\langle z_t - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\lambda_{n_k}}\right\rangle + F(z_t, z_{n_k}).\end{aligned}$$

Since $\|z_{n_k} - x_{n_k}\| \to 0$, we have $\|Az_{n_k} - Ax_{n_k}\| \to 0$. Further, from the monotonicity of A, we have $\langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle \geq 0$. So, from (A4) we have

$$\langle z_t - \omega, Az_t\rangle \geq F(z_t, \omega) \quad \text{as } k \to \infty. \tag{3.34}$$

From (A1), (A4) and (3.34), we also have

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, \omega)\\ &\leq tF(z_t, y) + (1-t)\langle z_t - \omega, Az_t\rangle\\ &= tF(z_t, y) + (1-t)t\langle y - \omega, Az_t\rangle,\end{aligned}$$

hence

$$0 \leq F(z_t, y) + (1-t)\langle y - \omega, Az_t\rangle.$$

Letting $t \to 0$, we have

$$0 \leq F(\omega, y) + \langle y - \omega, A\omega\rangle \quad \forall y \in C. \tag{3.35}$$

Therefore $\omega \in EP$.

Next, we show that $\omega \in \bigcap_{i=1}^N F(T_i)$. We can assume that

$$\alpha_1^{n_k, j} \to \alpha_1^j \in (0, 1) \quad \text{and} \quad \alpha_1^{n_k, N} \to \alpha_1^N \in (0, 1] \quad \text{as } k \to \infty \text{ for } j = 1, 2, \ldots, N-1 \tag{3.36}$$

and

$$\alpha_3^{n_k, j} \to \alpha_3^j \in [0, 1) \quad \text{as } k \to \infty \text{ for } j = 1, 2, \ldots, N. \tag{3.37}$$

Let S be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ where $\beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, for $j = 1, 2, \ldots, N$. By Lemma 2.9, we have

$$\lim_{k\to\infty} \|S_{n_k}x - Sx\| = 0 \tag{3.38}$$

for all $x \in C$.

By Lemma 2.8, we have $\bigcap_{i=1}^N F(T_i) = F(S)$. Assume that $S\omega \neq \omega$. By using the Opial property and (3.30) and (3.38), we have

$$\begin{aligned}\liminf_{k\to\infty} \|y_{n_k} - \omega\| &< \liminf_{k\to\infty} \|y_{n_k} - S\omega\|\\ &\leq \liminf_{k\to\infty}(\|y_{n_k} - S_{n_k}y_{n_k}\| + \|S_{n_k}y_{n_k} - S_{n_k}\omega\| + \|S_{n_k}\omega - S\omega\|)\\ &\leq \liminf_{k\to\infty} \|y_{n_k} - \omega\|,\end{aligned}$$

which is a contradiction. Thus $S\omega = \omega$, so $\omega \in F(S) = \bigcap_{i=1}^N F(T_i)$.

Hence $\omega \in \bigcap_{i=1}^N F(T_i) \bigcap EP$.

Since $y_{n_k} \rightharpoonup \omega$ and $\omega \in \bigcap_{i=1}^N F(T_i) \bigcap EP$, we have

$$\limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle = \limsup_{k\to\infty} \langle u - z_0, y_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.39}$$

By using (3.3), we have

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &= \|\beta_n(x_n-z_0)+(1-\beta_n)(S_ny_n-z_0)\|^2\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|y_n-z_0\|^2\\ &= \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|a_nu+(1-a_n)z_n-z_0\|^2\\ &= \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|a_n(u-z_0)+(1-a_n)(z_n-z_0)\|^2\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)((1-a_n)^2\|z_n-z_0\|^2+2a_n\langle u-z_0,y_n-z_0\rangle)\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)(1-a_n)\|z_n-z_0\|^2+2(1-\beta_n)a_n\langle u-z_0,y_n-z_0\rangle\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)(1-a_n)\|x_n-z_0\|^2+2(1-\beta_n)a_n\langle u-z_0,y_n-z_0\rangle\\ &= (1-(1-\beta_n)a_n)\|x_n-z_0\|^2+2(1-\beta_n)a_n\langle u-z_0,y_n-z_0\rangle.\end{aligned}$$

Since $\sum_{i=1}^{\infty}(1-\beta_n)a_n=\infty$ and $\limsup_{n\to\infty}2\langle u-z_0,y_n-z_0\rangle\le 0$, we can conclude from Lemma 2.3 that

$$\lim_{n\to\infty}\|x_n-z_0\|=0. \quad \square$$

4. Applications

Using our main theorem (Theorem 3.1), we obtain the following strong convergence theorems in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space and let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i)\bigcap EP(F)\neq\emptyset$. For $j=1,2,\dots,N$, let $\alpha_j^{(n)}=(\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j})$ be such that $\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j}\in[0,1]$, $\alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j}=1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1}\subset[\eta_1,\theta_1]$ with $0<\eta_1\le\theta_1<1$, $\{\alpha_1^{n,N}\}\subset[\eta_N,1]$ with $0<\eta_N\le 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N$, $\{\alpha_3^{n,j}\}_{j=1}^N\subset[0,\theta_3]$ with $0\le\theta_3<1$. Let S_n be the S-mappings generated by $T_1,T_2,\dots,T_N$ and $\alpha_1^{(n)},\alpha_2^{(n)},\dots,\alpha_N^{(n)}$. Let $u\in C$ and $x_1\in C$ and let $\{z_n\}\subset C$ and $\{x_n\}\subset C$ be sequences generated by*

$$\begin{cases}F(z_n,y)+\dfrac{1}{\lambda_n}\langle y-z_n,z_n-x_n\rangle\ge 0, & \forall y\in C,\\ x_{n+1}=\beta_nx_n+(1-\beta_n)S_n(a_nu+(1-a_n)z_n), & \forall n\in\mathbb{N},\end{cases}\tag{4.1}$$

where $\{a_n\}\in[0,1]$, $\{\beta_n\}\subset[0,1]$ and $\{\lambda_n\}\subset[0,2\alpha]$ satisfy the following conditions:

(i) $0<a\le\lambda_n\le b<2\alpha,\ 0<c\le\beta_n\le d<1$;

(ii) $\lim_{n\to\infty}\dfrac{\lambda_n}{\lambda_{n+1}}=1$;

(iii) $\lim_{n\to\infty}a_n=0,\ \sum_{n=1}^{\infty};a_n=\infty$

(iv) $|\alpha_1^{n+1,j}-\alpha_1^{n,j}|\to 0$, *and* $|\alpha_3^{n+1,j}-\alpha_3^{n,j}|\to 0$ *as* $n\to\infty$, *for all* $j\in\{1,2,3,\dots,N\}$.

Then $\{x_n\}$ converges strongly to $z\in\bigcap_{i=1}^N F(T_i)\bigcap EP(F)$, where $z=P_{\bigcap_{i=1}^N F(T_i)\bigcap EP(F)}u$.

Proof. Put $A\equiv 0$ in Theorem 3.1. Then, from Theorem 3.1, we can get the desired conclusion. $\square$

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space and let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let A be an α-inverse strongly monotone mapping of C into H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i)\bigcap EP\neq\emptyset$. For $j=1,2,\dots,N$, let $\{\alpha_1^{n,j}\}_{j=1}^N\in[0,1]$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1}\subset[\eta_1,\theta_1]$ with $0<\eta_1\le\theta_1<1$, $\{\alpha_1^{n,N}\}\subset[\eta_N,1]$ with $0<\eta_N\le 1$, $\forall n\in\mathbb{N}$. Let W_n be the W-mappings generated by $T_1,T_2,\dots,T_N$ and $\alpha_1^{n,1},\alpha_1^{n,2},\dots,\alpha_1^{n,N}$. Let $u\in C$ and $x_1\in C$ and let $\{z_n\}\subset C$ and $\{x_n\}\subset C$ be sequences generated by*

$$\begin{cases}F(z_n,y)+\langle Ax_n,y-z_n\rangle+\dfrac{1}{\lambda_n}\langle y-z_n,z_n-x_n\rangle\ge 0, & \forall y\in C,\\ x_{n+1}=\beta_nx_n+(1-\beta_n)W_n(a_nu+(1-a_n)z_n), & \forall n\in\mathbb{N},\end{cases}\tag{4.2}$$

where $\{a_n\}\in[0,1]$, $\{\beta_n\}\subset[0,1]$ and $\{\lambda_n\}\subset[0,2\alpha]$ satisfy the following conditions:

(i) $0<a\le\lambda_n\le b<2\alpha,\ 0<c\le\beta_n\le d<1$;

(ii) $\lim_{n\to\infty}\dfrac{\lambda_n}{\lambda_{n+1}}=1$;

(iii) $\lim_{n\to\infty} a_n = 0,\ \sum_{n=1}^{\infty} a_n = \infty;$

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0,$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \dots, N\}$.

Then $\{x_n\}$ *converges strongly to* $z \in \bigcap_{i=1}^{N} F(T_i) \bigcap EP$, *where* $z = P_{\bigcap_{i=1}^{N} F(T_i) \bigcap EP}u$.

Proof. Put $\alpha_2^{n,j} = 0$ for all $j \in \{1, 2, 3, \dots, N\}$, and all $n \in \mathbb{N}$ in Theorem 3.1. Then, by Theorem 3.1 the conclusion follows. □

Corollary 4.3 *([7], Theorem 3.1). Let C be a closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let A be an α-inverse strongly monotone mapping of C into H and let T be nonexpansive mappings of C into itself with $F(T) \bigcap EP \neq \emptyset$. Let $u, x_1 \in C$ and let $\{z_n\}, \{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T_1(a_n u + (1 - a_n)z_n), & \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $\{a_n\} \in [0, 1]$, $\{\beta_n\} \subset [0, 1]$ *and* $\{\lambda_n\} \subset [0, 2\alpha]$ *satisfy the following conditions:*

(i) $0 < a \leq \lambda_n \leq b < 2\alpha,\ 0 < c \leq \beta_n \leq d < 1;$

(ii) $\lim_{n\to\infty} \dfrac{\lambda_n}{\lambda_{n+1}} = 1;$

(iii) $\lim_{n\to\infty} a_n = 0,\ \sum_{n=1}^{\infty} a_n = \infty.$

Then $\{x_n\}$ *converges strongly to* $z \in \bigcap_{i=1}^{N} F(T_i) \bigcap EP$, *where* $z = P_{\bigcap_{i=1}^{N} F(T_i) \bigcap EP}u$.

Proof. Put $N = 1$ and $T_1 = T$ and $\alpha_2^{n,1}, \alpha_3^{n,1} = 0\ \forall n \in \mathbb{N}$ in Theorem 3.1. Then $S_n = T$. Hence, we obtain the desired result from Theorem 3.1. □

Remark. In Theorem 3.1, by taking $N = 1$ and $\alpha_2^{n,1}, \alpha_3^{n,1} = 0$ for all $n \in \mathbb{N}$, one can easily see that Theorems 4.1, 4.2, 4.3 of Takahashi and Takahashi [7] are special cases of Theorem 3.1.

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Research Article

A New Iterative Method for Common Fixed Points of a Finite Family of Nonexpansive Mappings

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Let X be a real uniformly convex Banach space and C a closed convex nonempty subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C. For a given $x_1 \in C$, let $\{x_n\}$ and $\{x_n^{(i)}\}$, $i = 1,2,\ldots,r$, be sequences defined $x_n^{(0)} = x_n$, $x_n^{(1)} = a_{n1}^{(1)}T_1x_n^{(0)} + (1 - a_{n1}^{(1)})x_n^{(0)}$, $x_n^{(2)} = a_{n2}^{(2)}T_2x_n^{(1)} + a_{n1}^{(2)}T_1x_n + (1 - a_{n2}^{(2)} - a_{n1}^{(2)})x_n$, $\ldots$, $x_{n+1} = x_n^{(r)} = a_{nr}^{(r)}T_rx_n^{(r-1)} + a_{n(r-1)}^{(r)}T_{r-1}x_n^{(r-2)} + \cdots + a_{n1}^{(r)}T_1x_n + (1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)})x_n$, $n \geq 1$, where $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\ldots,j$. In this paper, weak and strong convergence theorems of the sequence $\{x_n\}$ to a common fixed point of a finite family of nonexpansive mappings T_i $(i = 1,2,\ldots,r)$ are established under some certain control conditions.

1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, $y \in C$. Let $T_i : C \to C$, $i = 1,2,\ldots,r$, be nonexpansive mappings. Let $\mathrm{Fix}(T_i)$ denote the fixed points set of T_i, that is, $\mathrm{Fix}(T_i) := \{x \in C : T_ix = x\}$, and let $F := \bigcap_{i=1}^r \mathrm{Fix}(T_i)$.

For a given $x_1 \in C$, and a fixed $r \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(0)}\}, \{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(r)}\}$ by

$$x_n^{(0)} = x_n,$$

$$x_n^{(1)} = a_{n1}^{(1)}T_1x_n^{(0)} + \left(1 - a_{n1}^{(1)}\right)x_n^{(0)}$$

$$x_n^{(2)} = a_{n2}^{(2)} T_2 x_n^{(1)} + a_{n1}^{(2)} T_1 x_n + \left(1 - a_{n2}^{(2)} - a_{n1}^{(2)}\right) x_n,$$

$$\vdots$$

$$\begin{aligned} x_{n+1} = x_n^{(r)} &= a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n \\ &+ \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) x_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\ldots,j$. If $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\ldots,r-1\}$ and $i = 1,2,\ldots,j$, then (1.1) reduces to the iterative scheme

$$x_{n+1} = S_n x_n, \quad n \geq 1, \tag{1.2}$$

where $S_n := a_{nr}^{(r)} T_r + a_{n(r-1)}^{(r)} T_{r-1} + \cdots + a_{n1}^{(r)} T_1 + (1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}) I$, $a_{ni}^{(r)} \in [0,1]$ for all $i = 1,2,\ldots,r$ and $n \in \mathbb{N}$.

If $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\ldots,r-1\}$, $i = 1,2,\ldots,j$ and $a_{ni}^{(r)} := \alpha_i$, for all $n \in \mathbb{N}$ for all $i = 1,2,\ldots,r$, then (1.1) reduces to the iterative scheme defined by Liu et al. [1]

$$x_{n+1} = S x_n, \quad n \geq 1, \tag{1.3}$$

where $S := \alpha_r T_r + \alpha_{r-1} T_{r-1} + \cdots + \alpha_1 T_1 + (1 - \alpha_r - \alpha_{r-1} - \cdots - \alpha_1) I$, $\alpha_i \geq 0$ for all $i = 2,3,\ldots,r$ and $1 - \alpha_r - \alpha_{r-1} - \cdots - \alpha_1 > 0$. They showed that $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of T_i, $i = 1,2,\ldots,r$, in Banach spaces, provided that T_i, $i = 1,2,\ldots,r$ satisfy condition A. The result improves the corresponding results of Kirk [2], Maiti and Saha [3] and Sentor and Dotson [4].

If $r = 2$ and $a_{n1}^{(2)} := 0$ for all $n \in \mathbb{N}$, then (1.1) reduces to a generalization of Mann and Ishikawa iteration given by Das and Debata [5] and Takahashi and Tamura [6]. This scheme deals with two mappings:

$$\begin{aligned} x_n^{(1)} &= a_{n1}^{(1)} T_1 x_n + \left(1 - a_{n1}^{(1)}\right) x_n, \\ x_{n+1} = x_n^{(2)} &= a_{n2}^{(2)} T_2 x_n^{(1)} + \left(1 - a_{n2}^{(2)}\right) x_n, \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $\{a_{n1}^{(1)}\}, \{a_{n2}^{(2)}\}$ are appropriate sequences in $[0,1]$.

The purpose of this paper is to establish strong convergence theorems in a uniformly convex Banach space of the iterative sequence $\{x_n\}$ defined by (1.1) to a common fixed point of T_i $(i = 1,2,\ldots,r)$ under some appropriate control conditions in the case that one of T_i $(i = 1,2,\ldots,r)$ is completely continuous or semicompact or $\{T_i\}_{i=1}^r$ satisfies condition (B). Moreover, weak convergence theorem of the iterative scheme (1.1) to a common fixed point of T_i $(i = 1,2,\ldots,r)$ is also established in a uniformly convex Banach spaces having the Opial's condition.

2. Preliminaries

In this section, we recall the well-known results and give a useful lemma that will be used in the next section.

Recall that a Banach space X is said to satisfy *Opial's condition* [7] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$. A finite family of mappings $T_i : C \to C$ $(i = 1,2,\ldots,r)$ with $F := \bigcap_{i=1}^{r}\mathrm{Fix}(T_i) \neq \emptyset$ is said to satisfy *condition (B)* [8] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\max_{1\le i\le r}\{\|x - T_i x\|\} \ge f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf\{\|x - p\| : p \in F\}$.

Lemma 2.1 (see [9, Theorem 2]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|), \tag{2.1}$$

for all x, y in $B_r = \{x \in X : \|x\| \le r\}$, $\lambda \in [0,1]$, where

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda). \tag{2.2}$$

Lemma 2.2 (see [10, Lemma 1.6]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in \mathrm{Fix}(T)$.*

Lemma 2.3 (see [11, Lemma 2.7]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let u, $v \in X$ be such that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.4. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. Then for each $n \in \mathbb{N}$, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|^2 \le \sum_{i=1}^{n}\alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|), \tag{2.3}$$

for all $x_i \in B_r$ and all $\alpha_i \in [0,1]$ $(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

Proof. Clearly (2.3) holds for $n = 1,2$, by Lemma 2.1. Next, suppose that (2.3) is true when $n = k-1$. Let $x_i \in B_r$ and $\alpha_i \in [0,1]$, $i = 1,2,\ldots,k$ with $\sum_{i=1}^{k}\alpha_i = 1$. Then $\alpha_{k-1}/(1 - \sum_{i=1}^{k-2}\alpha_i)x_{k-1} + \alpha_k/(1 - \sum_{i=1}^{k-2}\alpha_i)x_k \in B_r$. By Lemma 2.1, we obtain that

$$\left\|\frac{\alpha_{k-1}}{1 - \sum_{i=1}^{k-2}\alpha_i}x_{k-1} + \frac{\alpha_k}{1 - \sum_{i=1}^{k-2}\alpha_i}x_k\right\|^2 \le \frac{\alpha_{k-1}}{1 - \sum_{i=1}^{k-2}\alpha_i}\|x_{k-1}\|^2 + \frac{\alpha_k}{1 - \sum_{i=1}^{k-2}\alpha_i}\|x_k\|^2. \tag{2.4}$$

By the inductive hypothesis, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\left\| \sum_{i=1}^{k-1} \beta_i y_i \right\|^2 \leq \sum_{i=1}^{k-1} \beta_i \|y_i\|^2 - \beta_1 \beta_2 g(\|y_1 - y_2\|) \tag{2.5}$$

for all $y_i \in B_r$ and all $\beta_i \in [0,1]$, $i = 1,2,\ldots,k-1$ with $\sum_{i=1}^{k-1} \beta_i = 1$. It follows that

$$\begin{aligned} \left\| \sum_{i=1}^{k} \alpha_i x_i \right\|^2 &= \left\| \sum_{i=1}^{k-2} \alpha_i x_i + \left(1 - \sum_{i=1}^{k-2} \alpha_i\right) \left(\frac{\alpha_{k-1} x_{k-1}}{1 - \sum_{i=1}^{k-2} \alpha_i} + \frac{\alpha_k x_k}{1 - \sum_{i=1}^{k-2} \alpha_i} \right) \right\|^2 \\ &\leq \sum_{i=1}^{k-2} \alpha_i \|x_i\|^2 + \left(1 - \sum_{i=1}^{k-2} \alpha_i\right) \left\| \frac{\alpha_{k-1} x_{k-1}}{1 - \sum_{i=1}^{k-2} \alpha_i} + \frac{\alpha_k x_k}{1 - \sum_{i=1}^{k-2} \alpha_i} \right\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) \\ &\leq \sum_{i=1}^{k-2} \alpha_i \|x_i\|^2 + \left(1 - \sum_{i=1}^{k-2} \alpha_i\right) \left(\frac{\alpha_{k-1} \|x_{k-1}\|^2}{1 - \sum_{i=1}^{k-2} \alpha_i} + \frac{\alpha_k \|x_k\|^2}{1 - \sum_{i=1}^{k-2} \alpha_i} \right) - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) \\ &= \sum_{i=1}^{k} \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|). \end{aligned} \tag{2.6}$$

Hence, we have the lemma. □

3. Main Results

In this section, we prove weak and strong convergence theorems of the iterative scheme (1.1) for a finite family of nonexpansive mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

The next lemma is crucial for proving the main theorems.

Lemma 3.1. *Let X be a Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C. Let $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\ldots,j$. For a given $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.1). If $F \neq \emptyset$, then $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. For each $n \geq 1$, we note that

$$\begin{aligned} \left\| x_n^{(1)} - p \right\| &= \left\| a_{n1}^{(1)} T_1 x_n + \left(1 - a_{n1}^{(1)}\right) x_n - p \right\| \\ &\leq a_{n1}^{(1)} \|T_1 x_n - p\| + \left(1 - a_{n1}^{(1)}\right) \|x_n - p\| \\ &\leq a_{n1}^{(1)} \|x_n - p\| + \left(1 - a_{n1}^{(1)}\right) \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$\begin{aligned} \left\|x_n^{(2)} - p\right\| &= \left\|a_{n2}^{(2)} T_2 x_n^{(1)} + a_{n1}^{(2)} T_1 x_n + \left(1 - a_{n2}^{(2)} - a_{n1}^{(2)}\right) x_n - p\right\| \\ &\le a_{n2}^{(2)} \left\|T_2 x_n^{(1)} - p\right\| + a_{n1}^{(2)} \|T_1 x_n - p\| + \left(1 - a_{n2}^{(2)} - a_{n1}^{(2)}\right) \|x_n - p\| \\ &\le a_{n2}^{(2)} \left\|x_n^{(1)} - p\right\| + a_{n1}^{(2)} \|x_n - p\| + \left(1 - a_{n2}^{(2)} - a_{n1}^{(2)}\right) \|x_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned} \left\|x_n^{(3)} - p\right\| &= \left\|a_{n3}^{(3)} T_3 x_n^{(2)} + a_{n2}^{(3)} T_2 x_n^{(1)} + a_{n1}^{(3)} T_1 x_n + \left(1 - a_{n3}^{(3)} - a_{n2}^{(3)} - a_{n1}^{(3)}\right) x_n - p\right\| \\ &\le a_{n3}^{(3)} \left\|T_3 x_n^{(2)} - p\right\| + a_{n2}^{(3)} \left\|T_2 x_n^{(1)} - p\right\| + a_{n1}^{(3)} \|T_1 x_n - p\| \\ &\quad + \left(1 - a_{n3}^{(3)} - a_{n2}^{(3)} - a_{n1}^{(3)}\right) \|x_n - p\| \\ &\le a_{n3}^{(3)} \left\|x_n^{(2)} - p\right\| + a_{n2}^{(3)} \left\|x_n^{(1)} - p\right\| + a_{n1}^{(3)} \|x_n - p\| \\ &\quad + \left(1 - a_{n3}^{(3)} - a_{n2}^{(3)} - a_{n1}^{(3)}\right) \|x_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{3.3}$$

By continuing the above argument, we obtain that

$$\left\|x_n^{(i)} - p\right\| \le \|x_n - p\| \quad \forall i = 1, 2, \ldots, r. \tag{3.4}$$

In particular, we get $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $n \in \mathbb{N}$, which implies that $\lim_{n\to\infty}\|x_n - p\|$ exists. □

Lemma 3.2. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\ldots,j$ such that $\sum_{i=1}^{j} a_{ni}^{(j)}$ are in $[0,1]$ for all $j \in \{1,2,\ldots,r\}$ and $n \in \mathbb{N}$. For a given $x_1 \in C$, let $\{x_n\}$ be defined by (1.1). If $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \le \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then*

(i) $\lim_{n\to\infty}\|T_i x_n^{(i-1)} - x_n\| = 0$ *for all* $i = 1,2,\ldots,r$,

(ii) $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ *for all* $i = 1,2,\ldots,r$,

(iii) $\lim_{n\to\infty}\|x_n^{(i)} - x_n\| = 0$ *for all* $i = 1,2,\ldots,r$.

Proof. (i) Let $p \in F$, by Lemma 3.1, $\sup_n\|x_n - p\| < \infty$. Choose a number $s > 0$ such that $\sup_n\|x_n - p\| < s$, it follows by (3.4) that $\{x_n^{(i)} - p\}, \{T_i x_n^{(i-1)} - p\} \subseteq B_s$, for all $i \in \{1,2,\ldots,r\}$. □

By Lemma 2.4, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \leq \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|), \tag{3.5}$$

for all $x_i \in B_s$, $\alpha_i \in [0,1]$ $(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n} \alpha_i = 1$. By (3.4) and (3.5), we have for $i = 1,2,\ldots,r$,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \Big\| a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n \\
&\quad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) x_n - p \Big\|^2 \\
&\leq a_{nr}^{(r)} \left\| T_r x_n^{(r-1)} - p \right\|^2 + a_{n(r-1)}^{(r)} \left\| T_{r-1} x_n^{(r-2)} - p \right\|^2 + \cdots \\
&\quad + a_{n1}^{(r)} \|T_1 x_n - p\|^2 + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\quad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \\
&\leq a_{nr}^{(r)} \left\| x_n^{(r-1)} - p \right\|^2 + a_{n(r-1)}^{(r)} \left\| x_n^{(r-2)} - p \right\|^2 + \cdots + a_{n1}^{(r)} \|x_n - p\|^2 \\
&\quad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\quad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \\
&\leq a_{nr}^{(r)} \|x_n - p\|^2 + a_{n(r-1)}^{(r)} \|x_n - p\|^2 + \cdots + a_{n1}^{(r)} \|x_n - p\|^2 \\
&\quad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\quad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \\
&= \|x_n - p\|^2 - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right).
\end{aligned} \tag{3.6}$$

Therefore

$$a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{3.7}$$

for all $i = 1,2,\ldots,r$. Since $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty} (a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, it implies by Lemma 3.1 that $\lim_{n\to\infty} g(\|T_i x_n^{(i-1)} - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T_i x_n^{(i-1)} - x_n\| = 0$ for all $i = 1,2,\ldots,r$.

(ii) For $i \in \{1,2,\ldots,r\}$, we have

$$\begin{aligned}\|T_i x_n - x_n\| &\le \left\|T_i x_n - T_i x_n^{(i-1)}\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\| \\ &\le \left\|x_n - x_n^{(i-1)}\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\| \\ &\le \sum_{j=1}^{i-1} a_{nj}^{(i-1)} \left\|T_j x_n^{(j-1)} - x_n\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\|.\end{aligned} \tag{3.8}$$

It follows from (i) that

$$\|T_i x_n - x_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.9}$$

(iii) For $i \in \{1,2,\ldots,r\}$, it follows from (i) that

$$\left\|x_n^{(i)} - x_n\right\| \le \sum_{j=1}^{i} a_{nj}^{(i)} \left\|T_j x_n^{(j-1)} - x_n\right\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.10}$$

Theorem 3.3. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let the sequence $\{a_{ni}^{(j)}\}_{n=1}^{\infty}$ be as in Lemma 3.2. For a given $x_1 \in C$, let sequences $\{x_n\}$ and $\{x_n^{(i)}\}$ $(i = 0,1,\ldots,r)$ be defined by (1.1). If one of $\{T_i\}_{i=1}^r$ is completely continuous then $\{x_n\}$ and $\{x_n^{(j)}\}$ converge strongly to a common fixed point of $\{T_i\}_{i=1}^r$ for all $j = 1,2,\ldots,r$.*

Proof. Suppose that T_{i_0} is completely continuous where $i_0 \in \{1,2,\ldots,r\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_{i_0} x_{n_k}\}$ converges. □

Let $\lim_{k\to\infty} T_{i_0} x_{n_k} = q$ for some $q \in C$. By Lemma 3.2 (ii), $\lim_{n\to\infty}\|T_{i_0} x_n - x_n\| = 0$. It follows that $\lim_{k\to\infty} x_{n_k} = q$. Again by Lemma 3.2(ii), we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1,2,\ldots,r$. It implies that $\lim_{k\to\infty} T_i x_{n_k} = q$. By continuity of T_i, we get $T_i q = q$, $i = 1,2,\ldots,r$. So $q \in F$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - q\|$ exists, it follows that $\lim_{n\to\infty}\|x_n - q\| = 0$. By Lemma 3.2(iii), we have $\lim_{n\to\infty}\|x_n^{(j)} - x_n\| = 0$ for each $j \in \{1,2,\ldots,r\}$. It follows that $\lim_{n\to\infty} x_n^{(j)} = q$ for all $j = 1,2,\ldots,r$.

Theorem 3.4. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let the sequence $\{a_{ni}^{(j)}\}_{n=1}^{\infty}$ be as in Lemma 3.2. For a given $x_1 \in C$, let sequences $\{x_n\}$ and $\{x_n^{(i)}\}$ $(i = 0,1,\ldots,r)$ be defined by (1.1). If the family $\{T_i\}_{i=1}^r$ satisfies condition (B) then $\{x_n\}$ and $\{x_n^{(j)}\}$ converge strongly to a common fixed point of $\{T_i\}_{i=1}^r$ for all $j = 1,2,\ldots,r$.*

Proof. Let $p \in F$. Then by Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists and $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $n \ge 1$. This implies that $d(x_{n+1}, F) \le d(x_n, F)$ for all $n \ge 1$, therefore, we get $\lim_{n\to\infty} d(x_n, F)$ exists. By Lemma 3.2(ii), we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for each $i = 1,2,\ldots,r$. It follows, by the condition (B) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing and $f(0) = 0$, therefore, we get $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since

$\lim_{n\to\infty} d(x_n, F) = 0$, given any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \epsilon/2$ for all $n \geq n_0$. In particular, $d(x_{n_0}, F) < \epsilon/2$. Then there exists $q \in F$ such that $\|x_{n_0} - q\| < \epsilon/2$. For all $n \geq n_0$ and $m \geq 1$, it follows by Lemma 3.1 that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\| \leq \|x_{n_0} - q\| + \|x_{n_0} - q\| < \epsilon. \tag{3.11}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C, hence it must converge to a point of C. Let $\lim_{n\to\infty} x_n = p^*$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and F is closed, we obtain $p^* \in F$. By Lemma 3.2(iii), $\lim_{n\to\infty}\|x_n^{(j)} - x_n\| = 0$ for each $j \in \{1,2,\ldots,r\}$. It follows that $\lim_{n\to\infty} x_n^{(j)} = p^*$ for all $j = 1,2,\ldots,r$. □

In Theorem 3.4, if $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\ldots,r-1\}$ and $i = 1,2,\ldots,j$, we obtain the following result.

Corollary 3.5. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(r)} \in [0,1]$ for all $i = 1,2,\ldots,r$ and $n \in \mathbb{N}$ such that $\sum_{i=1}^r a_{ni}^{(r)}$ are in $[0,1]$ for all $n \in \mathbb{N}$. For a given $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.2). If the family $\{T_i\}_{i=1}^r$ satisfies condition (B) and $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Remark 3.6. In Corollary 3.5, if $a_{ni}^{(r)} = a_i$, for all $n \in \mathbb{N}$ and for all $i = 1,2,\ldots,r$, the iterative scheme (1.2) reduces to the iterative scheme (1.3) defined by Liu et al. [1] and we obtain strong convergence of the sequence $\{x_n\}$ defined by Liu et al. when $\{T_i\}_{i=1}^r$ satisfies condition (B) which is different from the condition (A) defined by Liu et al. and we note that the result of Senter and Dotson [4] is a special case of Theorem 3.4 when $r = 1$.

In the next result, we prove weak convergence for the iterative scheme (1.1) for a finite family of nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.7. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined by (1.1). If the sequence $\{a_{ni}^{(j)}\}_{n=1}^\infty$ is as in Lemma 3.2, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Proof. By Lemma 3.2(ii), $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1,2,\ldots,r$. Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality we may assume that $x_n \to u$ weakly as $n \to \infty$ for some $u \in C$. By Lemma 2.2, we have $u \in F$. Suppose that there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ that converge weakly to u and v, respectively. From Lemma 2.2, we have u, $v \in F$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 2.3 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$. □

For $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\ldots,r-1\}$ and $i = 1,2,\ldots,j$ in Theorem 3.7, we obtain the following result.

Corollary 3.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(r)} \in [0,1]$ for all $i = 1,2,\ldots,r$ and $n \in \mathbb{N}$ such that $\sum_{i=1}^r a_{ni}^{(r)}$ are in $[0,1]$ for all $n \in \mathbb{N}$. For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined by (1.2). If $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Remark 3.9. In Corollary 3.8, if $a_{ni}^{(r)} = \alpha_i$, for all $n \in \mathbb{N}$ and for all $i = 1,2,\ldots,r$, then we obtain weak convergence of the sequence $\{x_n\}$ defined by Liu et al. [1].

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RESEARCH OUTPUTS

1. There are 11 papers accepted for publication in international journals.
2. There are 2 papers submitted for publication in international journals
3. There are 12 Ph.D students doing research under this project.
4. Four new researchers are built from this project.

Here are the list of 11 papers published in international journals.

1. A. Kananthai and K. Nonlaopon, On the generalized nonlinear ultra-hyperbolic heat equation related to the spectrum, Computational and Applied Mathematics, Volume 28 N. 2, pp. 1-10, 2009.

2. W. Satsanit and A. Kananthai, On the ultra-hyperbolic wave operator, International Journal of Pure and Applied Mathematics, Volume 52 N. 1, pp. 117-126, 2009.

3. C. Bunpog and A. Kananthai, On the Green Function of the Operator Related to the Bessel Helmholtz Operator and the Bessel Klein-Gordon Operator, Journal of Applied Functional Analysis, Volume 4 pp 10-19, 2009.

4. W. Satsanit and A. Kananthai, Diamond operator related to Bihamonic equation, Far East Journal of Applied Mathematics.

5. W. Satsanit and A. Kananthai, The operator and its spectrum related to heat equation, International Journal of Pure and Applied Mathematics.

6. S. Thianwan and S. Suantai, Convergence Criteria of a New Three-step Iteration with Errors for Nonexpansive- Nonself- Mappings, Computers and Mathematics with Applications 52 (2006) 1107 – 1118.

7. K. Nammanee and S. Suantai, The Modified Noor Iterations with Errors for Non-Lipschitzian Mappings in Banach Sapces, Applied Mathematics and Computation 187 (2007),669 – 679.

8. N. petrot and S. Suantai, The Criteria of Stric Monotonicity and Rotundinty points in generalized Calderon-Lozanovski Spaces, Nonlinear Analysis ,2009

9. A. Kangtunyakarn and S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis : Theory and Methods

10. Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis : Hybrid Method , 2009.

11. S. Imnang and S. Suantai, A new iterative method for common fixed points of a finite family of nonexpansive mappings, International Journal of Mathematical and Mathematical Sciences, Vol. 2009, Article ID 391839, 9 pages doi : 10.1155/2009/391839.

Here are 2 papers submitted for publication in international journals

1. Amnuay Kananthai, On the Diamond-Wave Operator, submitted to Journal of Applied Mathematics and Computation.

2. Amnuay Kananthai, On the Nonlinear heat equation related to the operator, submitted to Nonlinear Analisis and Application.

Here are 12 Ph.D students doing research under this project.

1. Mr. Sornsak Thianwan, Naresuan University

2. Mr. Kamonrat Nammanee, Naresuan University

3. Mr. Chakkrid Klin-Eam , Naresuan University

4. Mr. Siwicha Imnang, Taksin University

5. Mr. Wanchak Satsanit

6. Mr. Chalermpon Bunpog, Chiang Mai University

7. Mrs. Watcharaporn Cholamjiak

8. Miss. Urailuk Singthong

9. นายพรศักดิ์ ยตะโคตร

10. นาย เอกชัย สุนทรศิลสังวร

11. นายสมบูรณ์ นิยม

12. นายธวัชชัย ปัญญาติ๊บ

Here are four new researchers who are built from this project.

1. Assoc. Dr. Utith Inprasit, Ubon Rajathanee University

2. Dr. Hathaikarn Wattanataweekul, Ubon Rajathanee University

3. Assoc. Dr. Chantana Hattakosol, Prince Sonkla University

4. Assist. Chamnian Nantadilok, Rachapat Lampang University

APPENDIX

Reprints of papers published in international journals

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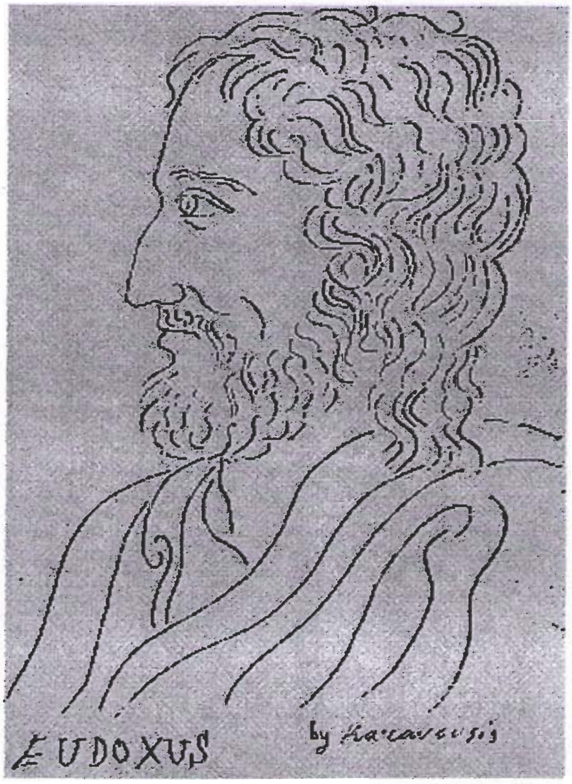

JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

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On the Green Function of the $(\diamondsuit_B + m^4)^k$ Operator Related to the Bessel-Helmholtz Operator and the Bessel Klein-Gordon Operator

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Abstract

In this paper, we study the Green function of the operator $(\diamondsuit_B + m^4)^k$ which is iterated k-times and is defined by

$$(\diamondsuit_B + m^4)^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2 + m^4\right]^k, \qquad (0.1)$$

where m is a positive real number and $p+q=n$ is the dimension of $\mathbb{R}_n^+$ and k is a nonnegative integer and $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0$. At first we study the Green function of the operator $(\diamondsuit_B + m^4)^k$, we have that such a Green function related to the elementary solutions of the Bessel-Helmholtz operator $(\triangle_B + m^2)^k$ iterated $k-$times and the Bessel Klein-Gordon operator $(\square_B + m^2)^k$ iterated $k-$times. We also apply such a Green function to solve the solution of the equation $(\diamondsuit_B + m^4)^k u(x) = f(x)$ where f is a generalized function and $u(x)$ is an unknown function for $x \in \mathbb{R}_n^+$.

Keywords: Green function, Bessel diamond operator, Helmholtz operator, Klein-Gordon operator

1 Introduction

A. Kananthai [1] first introduced the diamond operator $\diamondsuit^k$ iterated $k-$times, defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i{}^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j{}^2}\right)^2\right]^k,$$

the equation $\diamondsuit^k u(x) = f(x)$, see [2], has been already studied and the convolution $u(x) = (-1)^k R_{2k}^H(x) * R_{2k}^e * f(x)$ has been obtained as a solution of such an equation.

Later the equation $(\diamondsuit + m^4)^k u(x) = f(x)$, see [3], has been studied and the convolution $u(x) = \left(W_{2k}^H(u,m) * W_{2k}^e(v,m)\right) * (s^{*k})^{*-1}(x) * f(x)$ has been obtained a solution of such an equation.

Furthermore, Hüseyin Yildirim, Mzeki Sarikaya and Sermin Öztürk [4] first introduced the Bessel diamond operator $\diamondsuit_B^k$ iterated k−times, defined by

$$\diamondsuit_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k \tag{1.1}$$

where $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0$. The operator $\diamondsuit_B^k$ can be expressed by $\diamondsuit_B^k = \triangle_B^k \Box_B^k = \Box_B^k \triangle_B^k$, where

$$\triangle_B^k = \left(\sum_{i=1}^{p} B_{x_i}\right)^k. \tag{1.2}$$

and

$$\Box_B^k = \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k. \tag{1.3}$$

The equation $\diamondsuit_B^k u(x) = \delta(x)$, see([4], p.382), has been already studied and the convolution $u(x) = (-1)^k S_{2k} * R_{2k}$ has been obtained as a solution of such an equation where the function S_{2k} and R_{2k} are defined by (2.1) and (2.2), respectively, with $\alpha = \beta = 2k$. In this work, we study the equation of the form

$$(\diamondsuit_B + m^4)^k G(x) = \delta(x).$$

We obtain the elementary solution $G(x) = (T_{2k}(x) * W_{2k}(x)) * (C^{*k})^{*-1}(x)$, where the symbol $*k$ denotes the convolution of itself k−times and the symbol $*-1$ is an inverse of the convolution algebra, $T_{2k}(x)$ is the elementary solution of the Bessel-Helmholtz operator $(\triangle_B + m^2)^k$ iterated k−times, that is $T_{2k}(x)$ satisfy the equation

$$(\triangle_B + m^2)^k u(x) = \delta(x)$$

and $W_{2k}(x)$ is the elementary solution of the Bessel Klein-Gordon operator $(\Box_B + m^2)^k$ iterated k−times, that is $W_{2k}(x)$ satisfy the equation

$$(\Box_B + m^2)^k u(x) = \delta(x)$$

and $C(x)$ is defined by

$$C(x) = \delta(x) - m^2(T_2(x) + W_2(V)) + 2m^4\left(T_2(x) * W_2(V)\right).$$

Moreover, we apply such a Green function to obtain the solution of the equation

$$(\Diamond_B + m^4)^k u(x) = f(x).$$

where f is a generalized function.

2 Preliminaries

Definition 2.1 Let $x = (x_1, x_2, \ldots, x_n), \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}_n^+$. For any complex number α, we define the function $S_\alpha(x)$ by

$$S_\alpha(x) = \frac{2^{n+2|\nu|-2\alpha}\Gamma(\frac{n+2|\nu|-\alpha}{2})|x|^{\alpha-n-2|\nu|}}{\prod_{i=1}^n 2^{\nu_i-\frac{1}{2}}\Gamma(\nu_i+\frac{1}{2})} \tag{2.1}$$

Definition 2.2 Let $x = (x_1, x_2, \ldots, x_n), \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}_n^+$, and denote by $V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$ the nondegenerated quadratic form. Denote the interior of the forward cone by $\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \ldots, x_n > 0, V > 0\}$. The function $R_\beta(x)$ is defined by

$$R_\beta(x) = \frac{V^{\frac{\beta-n-2|\nu|}{2}}}{K_n(\beta)}, \tag{2.2}$$

where

$$K_n(\beta) = \frac{\pi^{\frac{n+2|\nu|-1}{2}}\Gamma\left(\frac{2+\beta-n-2|\nu|}{2}\right)\Gamma\left(\frac{1-\beta}{2}\right)\Gamma(\beta)}{\Gamma\left(\frac{2+\beta-p-2|\nu|}{2}\right)\Gamma\left(\frac{p-\beta}{2}\right)},$$

where β is a complex number.

Definition 2.3 Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, For any complex number α, we define the function

$$T_\alpha(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma\left(\frac{\eta}{2}+r\right)}{r!\Gamma\left(\frac{\eta}{2}\right)}(m^2)^r(-1)^{\frac{\alpha}{2}+r}S_{\alpha+2r}(x), \tag{2.3}$$

where η is a complex number and $S_{\alpha+2r}(x)$ is defined in definition 2.1.

Definition 2.4 Let $x = (x_1, x_2, \ldots, x_n)$, For any complex number β, we define the function

$$W_\beta(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma\left(\frac{\eta}{2}+r\right)}{r!\Gamma\left(\frac{\eta}{2}\right)}(m^2)^r R_{\beta+2r}(x), \tag{2.4}$$

where η is a complex number and $R_{\beta+2r}(x)$ is defined in definition 2.2.

Lemma 2.1 *Given the equation* $\triangle_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\triangle_B^k$ *is defined by (1.2). Then*

$$u(x) = (-1)^k S_{2k}(x)$$

where $S_{2k}(x)$ *is defined by (2.1), with* $\alpha = 2k$.

Proof. See ([4], p.379). □

Lemma 2.2 *Given the equation* $\square_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\square_B^k$ *is defined by (1.3). Then*

$$u(x) = R_{2k}(x)$$

where $R_{2k}(x)$ *is defined by (2.2), with* $\beta = 2k$

Proof. See ([4], p.379). □

Lemma 2.3 (The elementary solution of the Bessel-Helmholtz operator). *Given the equation* $(\triangle_B + m^2)^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\triangle_B$ *is defined by (1.2) with* $k = 1$. *Then*

$$u(x) = T_{2k}(x)$$

where $T_{2k}(x)$ *is defined by (2.3), with* $\alpha = 2k$.

Proof. At first, the following formula is valid ([5], p.3),

$$\Gamma\left(\frac{\eta}{2} + r\right) = \frac{\eta}{2}\left(\frac{\eta}{2} + 1\right) \cdots \left(\frac{\eta}{2} + r - 1\right)\Gamma\left(\frac{\eta}{2}\right).$$

Equivalently,

$$\begin{aligned}(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2} + r\right) &= \frac{(-1)^r \frac{\eta}{2}\left(\frac{\eta}{2} + 1\right) \cdots \left(\frac{\eta}{2} + r - 1\right)\Gamma\left(\frac{\eta}{2}\right)}{r!} \\ &= \frac{\left(-\frac{\eta}{2}\right)\left(-\frac{\eta}{2} - 1\right) \cdots \left[-\left(\frac{\eta}{2} + r - 1\right)\right]}{r!}\Gamma\left(\frac{\eta}{2}\right).\end{aligned}$$

We have,

$$(-1)^r \frac{1}{r!}\Gamma\left(\frac{\eta}{2} + r\right) = \binom{-\frac{\eta}{2}}{r}\Gamma\left(\frac{\eta}{2}\right).$$

Then, we obtain the function $T_\alpha(x)$ is defined by Definition 2.3 become

$$T_\alpha(x) = \sum_{r=0}^{\infty} \binom{-\frac{\eta}{2}}{r} (m^2)^r (-1)^{\frac{\alpha}{2} + r} S_{\alpha + 2r}(x). \tag{2.5}$$

Putting $\alpha = \eta = 2k$ in (2.5), we have

$$T_{2k}(x) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} S_{2k+2r}(x).$$

Since the operator $\triangle_B$ is linearly continuous and has $1-1$ mapping, then it has inverse, by Lemma 2.1 we obtain

$$T_{2k}(x) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \delta(x) * \triangle_B^{-k-r}$$
$$= (\triangle_B + m^2)^{-k}\delta(x), \tag{2.6}$$

where $(\triangle_B + m^2)^{-k}$ is the inverse operator of the operator $(\triangle_B + m^2)^k$. By applying the operator $(\triangle_B + m^2)^k$ to both sides of (2.6), we obtain

$$(\triangle_B + m^2)^k T_{2k}(x) = (\triangle_B + m^2)^k.(\triangle_B + m^2)^{-k}\delta(x).$$

Thus

$$(\triangle_B + m^2)^k T_{2k}(x) = \delta(x).$$

□

Lemma 2.4 (The elementary solution of the Bessel Klein-Gordon operator). *Given the equation $(\Box_B + m^2)^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\Box_B$ is defined by (1.3) with $k = 1$. Then*

$$u(x) = W_{2k}(x)$$

where $W_{2k}(x)$ is defined by (2.4), with $\alpha = 2k$.

Proof. The proof of lemma 2.4 is similar to the proof of Lemma 2.3. □

Lemma 2.5 *Let $T_{2k}(x)$ and $W_{2k}(x)$ be defined by (2.3) and (2.4) respectively,where $\alpha = \beta = 2k$. Then the convolution $T_{2k}(x) * W_{2k}(x)$ exist and it is lie in S', where S' is a space of tempered distribution.*

Proof. From (2.3) and (2.4) with $\alpha = \beta = 2k$, we have

$$T_{2k}(x) * W_{2k}(x) = \left(\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r (-1)^{k+r} S_{2k+2r}(x) \right)$$
$$* \left(\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r R_{2k+2r}(x) \right)$$
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(k+s)}{s!\Gamma(k)} (m^2)^s \cdot \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r \cdot$$
$$(-1)^{k+r} S_{2k+2r}(x) * R_{2k+2r}(x).$$

Hüseyin Yildirim, Mzeki Sarikaya and Sermin Öztürk ([4],p.380) has shown that $S_{2k+2r}(x) * R_{2k+2r}(x)$ exists and is a tempered distribution. It follows that $T_{2k}(x) * W_{2k}(x)$ exists and also is a tempered distribution. □

Lemma 2.6 *Let $T_2(x)$ and $W_2(x)$ be defined by (2.3) and (2.4) respectively, where $\alpha = \beta = 2$. Then*

$$\left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right](T_2(x) * W_2(x)) = C(x), \tag{2.7}$$

where $C(x) = \delta(x) - m^2(T_2(x) + W_2(x)) + 2m^4 (T_2(x) * W_2(x))$

Proof. We have

$$\begin{aligned}\left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right](T_2(x) * W_2(x)) = \\ \left[(\triangle_B + m^2)(\Box_B + m^2)(T_2(x) * W_2(x)) - m^2(\triangle_B + \Box_B)(T_2(x) * W_2(x))\right] = \\ \left[(\triangle_B + m^2)T_2(x) * (\Box_B + m^2)W_2(x) - m^2(\triangle_B T_2(x) * W_2(x) + T_2(x) * \Box_B W_2(x))\right].\end{aligned} \tag{2.8}$$

From Lemma 2.3 and Lemma 2.4, for $k = 1$ we have

$$(\triangle_B + m^2)T_2(x) = \delta(x) \quad \text{and} \quad (\Box_B + m^2)W_2(x) = \delta(x),$$

respectively. Moreover,

$$\triangle_B T_2(x) = \delta(x) - m^2 T_2(x)$$

and

$$\Box_B W_2(x) = \delta(x) - m^2 W_2(x),$$

thus(2.8) become

$$\begin{aligned}\left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right](T_2(x) * W_2(x)) = \\ \delta(x) * \delta(x) - m^2\left[(\delta(x) - m^2T_2(x)) * W_2(x) + T_2(x) * (\delta(x) - m^2W_2(x))\right] = \\ \delta(x) - m^2\left[W_2(x) - m^2T_2(x) * W_2(x) + T_2(x) - m^2T_2(x) * W_2(x)\right] = \\ \delta(x) - m^2(T_2(x) + W_2(x)) - 2m^4(T_2(x) * W_2(x)) = C(x).\end{aligned}$$

□

Lemma 2.7 *Let $S_\alpha(x)$ be the function, defined by (2.1). Then*

$$S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x),$$

where α and β are a positive even numbers.

Proof. See([4],p.380) □

Lemma 2.8 *Let $R_\beta(x)$ be the function, defined by (2.2). Then*

$$R_\beta(x) * R_\alpha(x) = R_{\beta+\alpha}(x),$$

where α and β are a positive even numbers.

Proof. Since $R_\beta(x)$ and $R_\alpha(x)$ are tempered distributions (see [4], p.380). Let $\text{Supp}R_\beta(x) = K \subset \overline{\Gamma}_+$, where K is a compact set and $\overline{\Gamma}_+$ is a closure of Γ_+ appears in Definition 2.2, then $R_\beta(x) * R_\alpha(x)$ exists and is well defined. To show that $R_\beta(x) * R_\alpha(x) = R_{\beta+\alpha}(x)$, by Lemma 2.2 $\Box_B^k u(x) = \delta(x)$ Then $u(x) = R_{2k}(x)$.
Now, $\Box_B^k u(x) = \Box_B^r \Box_B^{k-r} u(x) = \delta(x)$ for $r < k$, then by Lemma 2.2 $\Box_B^{k-r} u(x) = R_{2r}(x)$. Convolving both sides by $R_{2(k-r)}(x)$ we obtain

$$R_{2(k-r)}(x) * \Box_B^{k-r} u(x) = R_{2(k-r)}(x) * R_{2r}(x)$$

or,

$$\Box_B^{k-r} R_{2(k-r)}(x) * u(x) = R_{2(k-r)}(x) * R_{2r}(x)$$

by Lemma 2.2 again, we have

$$\delta(x) * u(x) = R_{2(k-r)}(x) * R_{2r}(x).$$

It follow that

$$u(x) = R_{2(k-r)}(x) * R_{2r}(x).$$

Since $u(x) = R_{2k}(x)$, thus

$$R_{2(k-r)}(x) * R_{2r}(x) = R_{2k}(x).$$

Let $\beta = 2(k-r)$ and $\alpha = 2r$, actually β and α are positive even numbers. It follows that $R_\beta(x) * R_\alpha(x) = R_{\beta+\alpha}(x)$ as required. □

3 Main Results

Theorem 3.1 *Given the equation*

$$(\Diamond_B + m^4)^k G(x) = \delta(x) \tag{3.1}$$

where $(\Diamond_B + m^4)^k$ is the operator iterated k-times defined by (0.1), δ is the Dirac-delta distribution, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain $G(x) = T_{2k}(x) * W_{2k}(x) * \left(C^{*k}(x)\right)^{*-1}$ is a Green function for the operator $(\Diamond_B + m^4)^k$ iterated k-time where $\Diamond_B$ is defined by (1.1) with $k = 1$, m is a nonnegative real number and

$$C(x) = \delta(x) - m^2(T_2(x) + W_2(x)) + 2m^4\,(T_2(x) * W_2(x)) \tag{3.2}$$

$C^{*k}(x)$ denote the convolution of C it self k-time, $\left(C^{*k}(x)\right)^{*-1}$ denote the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $C(x)$ is a tempered distribution.

Proof. Sine $(\Diamond_B + m^4)^k = ((\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B))^k$.

$$\left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right] \cdot \left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right]^{k-1} G(x) = \delta(x) \quad (3.3)$$

From Lemma 2.5 we have $T_2(x) * W_2(x)$ exists and is a tempered distribution. Convolving both sides of the above equation by $T_2(x) * W_2(x)$,we obtain

$$\left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right] (T_2(x) * W_2(x)) * \left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right]^{k-1} G(x) = (T_2(x) * W_2(x)) * \delta(x)$$

by Lemma 2.6, we have

$$C(x) * \left[(\triangle_B + m^2)(\Box_B + m^2) - m^2(\triangle_B + \Box_B)\right]^{k-1} G(x) = (T_2(x) * W_2(x)) * \delta(x).$$

Keeping on convolving both sides of the above equation by $T_2(x) * W_2(x)$ up to $k-1$ times, we have

$$C^{*k}(x) * G(x) = (T_2(x) * W_2(x))^{*k},$$

where $*k$ denotes the convolution of itself $k-$times.
By Lemma 2.7, Lemma 2.8 and definitions of $T_\alpha(x)$ and $W_\beta(x)$, we have

$$(T_2(x) * W_2(x))^{*k} = T_{2k}(x) * W_{2k}(x),$$

then

$$C^{*k}(x) * G(x) = T_{2k}(x) * W_{2k}(x).$$

Now, consider the function $C^{*k}(x)$, since $\delta(x)$, $T_2(x)$, $W_2(x)$ and $T_2(x) * W_2(x)$ are lies in $\mathcal{S}'$ where $\mathcal{S}'$ is a space of tempered distribution, then $C(x) \in \mathcal{S}'$, moreover by ([6], p.152) we obtain $C^{*k}(x) \in \mathcal{S}'$. Since $T_{2k}(x) * W_{2k}(x) \in \mathcal{S}'$, choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution. Thus $T_{2k}(x) * W_{2k}(x) \in \mathcal{D}'_{\mathcal{R}}$, it follow that $T_{2k}(x) * W_{2k}(x)$ is an element of convolution algebra, thus by ([7], p.150-151), we have that the equation (2.8) has a unique solution

$$G(x) = T_{2k}(x) * W_{2k}(x) * \left(C^{*k}(x)\right)^{*-1}$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of C^{*k} in the convolution algebra, $G(x)$ is called the Green function of the operator $(\Diamond_B + m^4)^k$. Since $T_{2k}(x) * W_{2k}(x)$ and $\left(C^{*k}(x)\right)^{*-1}$ are lies in $\mathcal{S}'$, then by ([6], p.152) again, we have $T_{2k}(x) * W_{2k}(x) * \left(C^{*k}(x)\right)^{*-1} \in \mathcal{S}'$. Hence $G(x)$ is a tempered distribution. $\Box$

Theorem 3.2 *Given the equation*

$$(\Diamond_B + m^4)^k u(x) = f(x) \tag{3.4}$$

where f is a given generalized function and $u(x)$ is an unknown function, we obtain

$$u(x) = G(x) * f(x)$$

is a unique solution of the equation (3.4) where $G(x)$ is a Green function for $(\Diamond_B+m^4)^k$.

Proof. Convolving both sides of (3.4) by $G(x)$ where $G(x)$ is a Green function for $(\Diamond_B + m^4)^k$ in theorem 3.1, we obtain

$$G(x) * (\Diamond_B + m^4)^k u(x) = G(x) * f(x)$$

or,

$$(\Diamond_B + m^4)^k G(x) * u(x) = G(x) * f(x)$$

applying the Theorem 3.1,we have

$$\delta(x) * u(x) = G(x) * f(x).$$

Therefor,

$$u(x) = G(x) * f(x).$$

Sine $G(x)$ is unique. Hence $u(x)$ is a unique solution of the equation (3.4). □

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DIAMOND OPERATOR RELATED TO BIHARMONIC EQUATIONS

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Abstract

In this paper, we study the generalized wave equation of the form

$$\frac{\partial^2}{\partial t^2} u(x, t) - c^2(\Diamond)^k u(x, t) = 0$$

with the initial conditions

$$u(x, 0) = f(x), \quad \frac{\partial}{dt} u(x, 0) = g(0),$$

where $u(x, t) \in \mathbb{R}^n \times [0, \infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Diamond^k$ is the Diamond operator iterated k-times defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k,$$

$\Diamond$ can be written as the product of the operators in the form $\Diamond = \Delta\Box = \Box\Delta$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}$

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$-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. $p+q=n$, c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable functions. We obtain $u(x, t)$ as a solution for such equation. Moreover, by ε-approximation we also obtain the asymptotic solution $u(x, t) = O(\varepsilon^{-n/2k})$. In particularly, if we put $n = 1$, $k = 2$ and $p = 0$, the $u(x, t)$ reduces to the solution of the biharmonic wave equation

$$\frac{\partial^2}{\partial t^2}u(x, t) + c^2(\Delta)^4 u(x, t) = 0.$$

1. Introduction

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x, t) = c^2\frac{\partial^2}{\partial x^2}u(x, t), \tag{1.1}$$

we obtain $u(x, t) = f(x+ct) + g(x-ct)$ as a solution of the equation where f and g are continuous.

Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x, t) + c^2\Delta\, u(x, t) = 0, \tag{1.2}$$

with the initial condition

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x, 0) = g(x),$$

where f and g are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\hat{u}(\xi, t) = \hat{f}(\xi)\cos(2\pi|\xi|)t + \hat{g}(\xi)\frac{\sin(2\pi|\xi|)t}{2\pi|\xi|},$$

where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$, $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$ (see [1, p. 177]). By using the inverse Fourier transform, we obtain $u(x, t)$ in the convolution form, that is,

$$u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x), \tag{I.3}$$

where Φ_t is an inverse Fourier transform of $\hat{\Phi}_t(\xi) = \frac{\sin(2\pi|\xi|)t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\hat{\Psi}_t(\xi) = \cos(2\pi|\xi|)t = \frac{\partial}{\partial t}\hat{\Phi}(\xi)$.

In 1996, Kananthai [2] introduced the *Diamond operator* $\Diamond$ defined by

$$\Diamond = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2, \quad p + q = n$$

or $\Diamond$ can be written as the product of the operators in the form $\Diamond = \Delta\Box = \Box\Delta$ where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. The Fourier transform of the Diamond operator has also been studied and the elementary solution of such operator, see [3]. Next, G. Sritantana, A. Kananthai study the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2(-\Delta)^k u(x, t) = 0$$

see [7, pp. 23-29], where

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^k.$$

Next, W. Satsanit, A. Kananthai study the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2(\Box)^k u(x, t) = 0$$

see [6], where

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

we obtain the solution related to the beam equation.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\Diamond)^k u(x, t) = 0 \tag{1.4}$$

with $u(x, 0) = f(x)$ and $\partial/\partial t\, u(x, 0) = g(x)$, where c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable. The equation (1.4) is motivated by the heat equation of the form

$$\frac{\partial}{\partial t} u(x, t) = -c^2 (\Diamond)^k u(x, t)$$

(see [4, 1-4]). We obtain

$$u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{1.5}$$

as a solution of (1.4), where Φ_t is an inverse Fourier transform of $\hat{\Phi}_t(\xi) = \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k}$ and Ψ_t is an inverse Fourier transform of $\hat{\Psi}_t(\xi) = \cos c(\sqrt{s^4 - r^4})^k t = \frac{\partial}{\partial t} \hat{\Phi}_t(\xi)$, where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$. Moreover, if we put $k = 2$ and $p = 0$ in (1.4), then (1.5) reduces to the solution of the n-dimensional biharmonic wave equation and also if $k = 1$, $n = 1$ and $p = 0$ in (1.4), then (1.5) reduces to the solution of beam equation.

We also study the asymptotic form of $u(x, t)$ in (1.5) by using ε-approximation and obtain $u(x, t) = O(\varepsilon^{-n/2k})$.

2. Preliminaries

We shall need the following definitions

Definition 2.1. Let $f \in L_1(\mathbb{R}^n)$ the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} f(x) dx, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{f}(x) dx. \tag{2.2}$$

Lemma 2.1. *Given the function*

$$f(x) = \exp\left[-\sqrt{-\left(\sum_{i=1}^{p} x_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^2} \right],$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. *Then*

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\Omega_p \Omega_q}{8} \cdot \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)},$$

where Γ *denotes the Gamma function. That is,* $\int_{\mathbb{R}^n} f(x) dx$ *is bounded.*

Proof. First note that

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{-\left(\sum_{i=1}^{p} x_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^2} \right] dx.$$

Now, we transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \quad x_2 = r\omega_2, \ldots, \quad x_p = r\omega_p,$$

$$dx_1 = rd\omega_1, \quad dx_2 = rd\omega_2, \ldots, \quad dx_p = rd\omega_p$$

and

$$x_{p+1} = s\omega_{p+1}, \quad x_{p+2} = s\omega_{p+2}, \ldots, \quad x_{p+q} = s\omega_{p+q},$$

$$dx_{p+1} = sd\omega_{p+1}, \quad dx_{p+2} = sd\omega_{p+2}, \ldots, \quad dx_{p+q} = sd\omega_{p+q},$$

where $\omega_1^2 + \omega_2^2 + \cdots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \omega_{p+q}^2 = 1$.

Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q,$$

where $dx = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \int_{\mathbb{R}^n} \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q.$$

By a direct computation, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p\Omega_q \int_0^\infty \int_0^s \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drds,$$

where $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s \exp[-\sqrt{s^4 - r^4}]r^{p-1}s^{q-1}drds.$$

Put $r^2 = s^2\sin\theta$, $2rdr = s^2\cos\theta d\theta$ and $0 \leq \theta \leq \dfrac{\pi}{2}$, to have

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s e^{-\sqrt{s^4 - s^4\sin^2\theta}} s^{p-2}(\sin\theta)^{\frac{p-2}{2}} s^{q+1}\cos\theta d\theta ds$$

$$= \frac{\Omega_p\Omega_q}{2} \int_0^\infty \int_0^s e^{-s^2\cos\theta} s^{p+q-1}(\sin\theta)^{\frac{p-2}{2}}\cos\theta d\theta ds.$$

Put $y = s^2\cos\theta$, $ds = \dfrac{dy}{2s\cos\theta}$, to have

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p\Omega_q}{4} \int_0^{\pi/2} \int_0^\infty e^{-y}\left(\frac{y}{\cos\theta}\right)^{\frac{n-2}{2}}(\sin\theta)^{\frac{p-2}{2}}\cos\theta d\theta \frac{dy}{\cos\theta}$$

$$= \frac{\Omega_p\Omega_q}{4} \int_0^{\pi/2} \int_0^\infty e^{-y} y^{\frac{n-2}{2}}(\cos\theta)^{\frac{2-n}{2}}(\sin\theta)^{\frac{p-2}{2}} dyd\theta$$

$$= \frac{\Omega_p \Omega_q}{4} \Gamma\left(\frac{n}{2}\right) \int_0^{\pi/2} (\cos\theta)^{\frac{2-n}{2}} (\sin\theta)^{\frac{p-2}{2}} d\theta$$

$$= \frac{\Omega_p \Omega_q}{8} \Gamma\left(\frac{n}{2}\right) \beta\left(\frac{p}{4}, \frac{4-n}{4}\right).$$

Therefore,

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\Omega_p \Omega_q}{8} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{p}{4}\right) \Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.$$

Thus it follows that $\int_{\mathbb{R}^n} f(x) dx$ is bounded.

3. Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (\Diamond)^k u(x, t) = 0 \tag{3.1}$$

with initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = g(x), \tag{3.2}$$

where $u(x, t) \in \mathbb{R}^n \times [0, \infty)$, $\Diamond^k$ *is the Diamond operator iterated k-times, c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable for* $x \in \mathbb{R}^n$. *Then* (3.1) *has a unique solution*

$$u(x, t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{3.3}$$

and satisfies the condition (3.2) *where* Φ_t *is the inverse Fourier transform of*

$$\hat{\Phi}_t(\xi) = \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k}$$

and Ψ_t *is the inverse Fourier transform of*

$$\hat{\Psi}_t(\xi) = \cos c(\sqrt{s^4 - r^4})^k t = \frac{\partial}{\partial t} \hat{\Phi}(\xi),$$

with $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ *and* $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Proof. By applying the Fourier transform defined by (2.1) to (3.1), we obtain

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi, t) + c^2\left(-\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k \hat{u}(\xi, t) = 0.$$

Let $s > r$. Thus

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi, t) + c^2(s^4 - r^4)^k \hat{u}(\xi, t) = 0,$$

$$\hat{u}(\xi, t) = A(\xi)\cos c(\sqrt{s^4 - r^4})^k t + B(\xi)\sin c(\sqrt{s^4 - r^4})^k t.$$

By (3.2), $\hat{u}(\xi, 0) = A(\xi) = \hat{f}(\xi)$,

$$\frac{\partial \hat{u}(\xi, t)}{\partial t} = -c(\sqrt{s^4 - r^4})^k A(\xi)\sin c(\sqrt{s^4 - r^4})^k t$$

$$+ c(\sqrt{s^4 - r^4})^k B(\xi)\cos c(\sqrt{s^4 - r^4})^k t.$$

$$\frac{\partial \hat{u}(\xi, 0)}{\partial t} = 0 + c(\sqrt{s^4 - r^4})^k B(\xi) = \hat{g}(\xi),$$

$$B(\xi) = \frac{\hat{g}(\xi)}{c(\sqrt{s^4 - r^4})^k},$$

$$\hat{u}(\xi, t) = \hat{f}(\xi)\cos c(\sqrt{s^4 - r^4})^k t + \frac{\hat{g}(\xi)}{c(\sqrt{s^4 - r^4})^k}\sin c(\sqrt{s^4 - r^4})^k t. \qquad (3.4)$$

By applying the inverse Fourier transform (3.4), we obtain the solution $u(x, t)$ in the convolution form of (3.1). Now, we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$. Consider the Fourier transforms

$$\widehat{\Phi_t}(x) = \frac{\sin c(\sqrt{s^4 - r^4})^k t}{c(\sqrt{s^4 - r^4})^k} \quad \text{and} \quad \Psi_t(x) = \cos c(\sqrt{s^4 - r^4})^k t.$$

These are all tempered distributions not lying in the space $L_1(\mathbb{R}^n)$ of integrable functions. So we cannot compute the inverse Fourier transforms $\Phi_t(x)$ and $\Psi_t(x)$

directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of ε-approximation.

Define

$$\widehat{\phi_t^{\varepsilon}}(\xi) = e^{-\varepsilon c(\sqrt{s^4-r^4})^k}\,\widehat{\phi_t}(\xi) = e^{-\varepsilon c(\sqrt{s^4-r^4})^k}\,\frac{\sin c(\sqrt{s^4-r^4})^k t}{c(\sqrt{s^4-r^4})^k} \quad \text{for } \varepsilon > 0. \tag{3.5}$$

We see that $\phi_t^{\varepsilon}(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi_t^{\varepsilon}}(x) \to \widehat{\phi_t}(x)$ uniformly as $\varepsilon \to 0$. So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi_t^{\varepsilon}(x)$. Now

$$\Phi_t^{\varepsilon}(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,\, x)}\widehat{\Phi_t^{\varepsilon}}(\xi)\,d\xi$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{i(\xi,x)}e^{-\varepsilon c(\sqrt{s^4-r^4})^k}\,\frac{\sin c(\sqrt{s^4-r^4})^k t}{c(\sqrt{s^4-r^4})^k}\,d\xi,$$

$$|\,\Phi_t^{\varepsilon}(x)\,| \le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \frac{e^{-\varepsilon c(\sqrt{s^4-r^4})^k}}{c(\sqrt{s^4-r^4})^k}\,d\xi. \tag{3.6}$$

By changing to bipolar coordinates and putting

$$\xi_1 = rw_1, \quad \xi_2 = rw_2, \ldots, \quad \xi_p = rw_p,$$

and

$$\xi_{p+1} = sw_{p+1}, \quad \xi_{p+2} = sw_{p+2}, \ldots, \quad \xi_p = sw_{p+q}, \quad p + q = n,$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$, we obtain

$$|\,\Phi_t^{\varepsilon}(x)\,| \le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \frac{e^{-\varepsilon c(\sqrt{s^4-r^4})^k}}{c(\sqrt{s^4-r^4})^k}\,r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}drds\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit spheres in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, with $\Omega_p = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{(2\pi)^{q/2}}{\Gamma(q/2)}$. Now,

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\int_0^{\infty}\int_0^{s}\frac{e^{-\varepsilon c(\sqrt{s^4-r^4})^k}}{c(\sqrt{s^4-r^4})^k}r^{p-1}s^{q-1}drds.$$

Putting $r^2 = s^2\sin\theta$, $2rdr = s^2\cos\theta d\theta$ and $0 \le \theta \le \frac{\pi}{2}$, we get

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}}\int_0^{\infty}\int_0^{\pi/2}\frac{e^{-\varepsilon c(\sqrt{s^4-s^4\sin^2\theta})^k}}{c(\sqrt{s^4-s^4\sin^2\theta})^k}(\sin\theta)^{\frac{p-2}{2}}s^{p+q-1}\cos\theta d\theta ds$$

$$= \frac{\Omega_p\Omega_q}{2c(2\pi)^{n/2}}\int_0^{\infty}\int_0^{\pi/2}\frac{e^{-\varepsilon c(s^2\cos\theta)^k}}{c(s^2\cos\theta)^k}s^{p+q-1}(\sin\theta)^{\frac{p-2}{2}}\cos\theta d\theta ds.$$

Putting $y = \varepsilon c(s^2\cos\theta)^k = \varepsilon cs^{2k}\cos^k\theta$, $s^{2k} = \frac{y}{c\varepsilon\cos^k\theta}$, $ds = \frac{sdy}{2ky}$, it follows that

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{4c(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^{\infty}\frac{e^{-y}s^{n-1}}{y/(\varepsilon c)}(\sin\theta)^{\frac{p-2}{2}}\cos\theta\frac{s}{ky}dyd\theta$$

$$= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^{\infty}\frac{e^{-y}\varepsilon}{ky^2}\left(\frac{y}{c\varepsilon\cos^k\theta}\right)^{n/2k}(\sin\theta)^{p-2/2}\cos\theta dyd\theta$$

$$= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^{\infty}\frac{e^{-y}y^{n/2k-2}\varepsilon}{c^{n/2k}k\varepsilon^{n/2k-1}}(\sin\theta)^{\frac{p-2}{2}}(\cos\theta)^{\frac{2-n}{2}}dyd\theta$$

$$= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\frac{\Gamma\left(\frac{n}{2k}-1\right)}{k\varepsilon^{\frac{n}{2k}-1}c^{n/2k}}\int_0^{\pi/2}(\sin\theta)^{\frac{p-2}{2}}(\cos\theta)^{\frac{2-n}{2}}d\theta$$

$$= \frac{\Omega_p\Omega_q}{8c^{n/2k}(2\pi)^{n/2}k\varepsilon^{n/2k-1}}\Gamma\left(\frac{n}{2k}-1\right)\beta\left(\frac{p}{4},\frac{4-n}{4}\right),$$

and

$$|\Phi_t^{\varepsilon}(x)| \le \frac{\Omega_p\Omega_q}{8c^{n/2k}(2\pi)^{n/2}k\varepsilon^{n/2k-1}}\frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.$$

Similarly, we define $\widehat{\Psi_t^{\varepsilon}}(\xi) = e^{-\varepsilon c(\sqrt{s^4-r^4})^k} \cos c(\sqrt{s^4 - r^4})^k\, t$ and

$$\Psi_t^{\varepsilon}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,\, x)} \widehat{\Psi_t^{\varepsilon}}(\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,\, x)} e^{-\varepsilon c(\sqrt{s^4-r^4})^k} \cos c(\sqrt{s^4 - r^4})^k\, t d\xi,$$

$$|\Psi_t^{\varepsilon}(x)| \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\varepsilon c(\sqrt{s^4-r^4})^k} d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} \int_0^{s} e^{-\varepsilon c(\sqrt{s^4-r^4})^k} r^{p-1} s^{q-1} dr ds.$$

Putting $r^2 = s^2 \sin\theta$, $2r dr = s^2 \cos\theta d\theta$ and $0 \le \theta \le \frac{\pi}{2}$, we obtain

$$|\Psi_t^{\varepsilon}(x)| \le \frac{\Omega_p \Omega_q}{2(2\pi)^{n/2}} \int_0^{\infty} \int_0^{\pi/2} e^{-\varepsilon c(s^2 \cos\theta)^k} (\sin\theta)^{\frac{p-2}{2}} s^{p+q-1} \cos\theta d\theta ds$$

$$= \frac{\Omega_p \Omega_q}{2(2\pi)^{n/2}} \int_0^{\infty} \int_0^{\pi/2} e^{-\varepsilon c(s^2 \cos\theta)^k} s^{p+q-1} (\sin\theta)^{p-2/2} \cos\theta d\theta ds.$$

Next, putting $y = \varepsilon c(s^2 \cos\theta)^k$, $ds = s\frac{dy}{2ky}$, we have

$$|\Psi_t^{\varepsilon}(x)| \le \frac{\Omega_p \Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-y}}{y} \left(\frac{y}{c\varepsilon \cos^k\theta}\right)^{n/2k} (\sin\theta)^{\frac{p-2}{2}} \cos\theta dy d\theta$$

$$= \frac{\Omega_p \Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-y} y^{n/2k-1}}{c^{n/2k} \varepsilon^{n/2k}} (\sin\theta)^{\frac{p-2}{2}} (\cos\theta)^{\frac{2-n}{2}} dy d\theta$$

$$= \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2} k c^{n/2k} \varepsilon^{n/2k}} \Gamma\left(\frac{n}{2k}\right) \int_0^{\pi/2} (\sin\theta)^{\frac{p-2}{2}} (\cos\theta)^{\frac{2-n}{2}} d\theta,$$

$$|\Psi_t^{\varepsilon}(x)| \le \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k c^{n/2k} \varepsilon^{n/2k}} \cdot \frac{\Gamma\left(\frac{n}{2k}\right) \Gamma\left(\frac{p}{4}\right) \Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.$$

Set

$$u^{\varepsilon}(x, t) = f(x) * \Psi_t^{\varepsilon}(x) + g(x) * \Phi_t^{\varepsilon}(x) \tag{3.7}$$

which is an ε-approximation of $u(x, t)$ in (3.7). For $\varepsilon \to 0$, $u^{\varepsilon}(x, t) \to u(x, t)$ uniformly. Now

$$u^{\varepsilon}(x, t) = \int_{\mathbb{R}^n} f(r)\Psi_t^{\varepsilon}(x - r)dr + \int_{\mathbb{R}^n} g(r)\Phi_t^{\varepsilon}(x - r)dr.$$

Thus

$$|u^{\varepsilon}(x, t)| \leq |\Psi_t^{\varepsilon}(x - r)| \int_{\mathbb{R}^n} |f(r)|dr + |\Phi_t^{\varepsilon}(x - r)| \int_{\mathbb{R}^n} |g(r)|dr$$

$$\leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}\varepsilon^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M$$

$$+ \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}\varepsilon^{n/2k-1}} \frac{\Gamma\left(\frac{n}{2k} - 1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{2-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N,$$

$$\varepsilon^{n/2k}|u^{\varepsilon}(x, t)| \leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M$$

$$+ \frac{\Omega_p\Omega_q\varepsilon}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k} - 1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N,$$

where $M = \int_{\mathbb{R}^n} |f(r)|dr$ and $N = \int_{\mathbb{R}^n} |g(r)|dr$. Since f and g are absolutely integrable,

$$\lim_{\varepsilon \to 0} \varepsilon^{n/2k}|u^{\varepsilon}(x, t)| \leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} = K.$$

It follows that $u(x, t) = O(\varepsilon^{-n/2k})$ for $n \neq k$ as $\varepsilon \to 0$.

In particular, if we put $k = 2$, $n = 1$ and $p = 0$, then (3.1) reduces to the solution of the beam equation, see [5, p. 47],

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = g(x),$$

where f and g are continuous and absolutely integrable for $x \in \mathbb{R}^n$.

Thus we obtain $u(x, t) = O(\varepsilon^{-1/4})$ which is a solution of such a biharmonic wave equation.

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On the generalized nonlinear ultra-hyperbolic heat equation related to the spectrum

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Abstract. In this paper, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Box^k u(x,t) = f(x,t,u(x,t))$$

where $\Box^k$ is *the ultra-hyperbolic operator* iterated k-times, defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, k is a positive integer and c is a positive constant.

On the suitable conditions for f, u and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$. Moreover, if we put $k = 1$ and $q = 0$ we obtain the solution of nonlinear equation related to the heat equation.

Mathematical subject classification: author, please, provide the AMS classif.

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1 Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Delta u(x,t) \tag{1.1}$$

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with the initial condition

$$u(x,0) = f(x)$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, and f is a continuous function, we obtain the solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|x-y|^2}{4c^2 t}\right] f(y)dy \tag{1.2}$$

as the solution of (1.1).

Now, (1.2) can be written as $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left[-\frac{|x|^2}{4c^2 t}\right]. \tag{1.3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$ and $t > 0$, see [1, p. 208–209].

Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta distribution. We also have extended (1.1) to be the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t) \tag{1.4}$$

where $\Box$ is *the ultra-hyperbolic operator*, defined by

$$\Box = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}\right).$$

We obtain *the ultra-hyperbolic heat kernel*

$$E(x,t) = \frac{(i)^q}{(4c^2\pi t)^{n/2}} \exp\left[\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right]$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and $i = \sqrt{-1}$. For finding the kernel $E(x,t)$ see [4].

In this paper, we extend (1.4) to be the general of the nonlinear form

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Box^k u(x,t) = f(x,t,u(x,t)) \tag{1.5}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and with the following conditions on u and f as follows,

(1) $u(x,t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$-derivatives.

(2) f satisfies the Lipchitz condition, that is

$$|f(x,t,u) - f(x,t,w)| \leq A|u-w|$$

where A is constant and $0 < A < 1$.

(3)
$$\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dx\, dt < \infty$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0,\infty)$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Under such conditions of f, u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$ and $E(x,t)$ is an elementary solution defined by (2.5).

2 Preliminaries

Definition 2.1. *Let $f(x) \in \mathbb{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by*

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1\, dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{2.2}$$

Definition 2.2. *The spectrum of the kernel $E(x,t)$ defined by (2.5) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.*

Definition 2.3. *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and we write*

$$u = \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2, \quad p + q = n.$$

Denote by

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by Definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E(\xi,t)} = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{2.3}$$

Lemma 2.1. *Let L be the operator defined by*

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2 \Box^k \tag{2.4}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated k-times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p + q = n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$, k is a positive integer and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi \tag{2.5}$$

as a elementary solution of (2.4) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $\mathrm{L}E(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x,t) - c^2 \Box^k E(x,t) = \delta(x)\delta(t).$$

Take the Fourier transform defined by (2.1) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k\right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k\right]$$

which has been already defined by (2.3). Thus

$$E(x,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi = \frac{1}{(2\pi)^{n/2}}\int_{\Omega} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi$$

where Ω is the spectrum of $E(x,t)$. Thus from (2.3)

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(\xi,x)\right]d\xi \quad \text{for } t > 0.$$

□

Definition 2.4. *Let us extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting*

$$E(x,t) = \begin{cases} \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(\xi,x)\right]d\xi & \text{for } t > 0, \\ 0 & \text{for } t \le 0, \end{cases}$$

3 Main Results

Theorem 3.1. *The kernel $E(x,t)$ defined by (2.5) have the following properties:*

(1) *$E(x,t) \in C^\infty$-the space infinitely differentiable.*

(2) $\left(\frac{\partial}{\partial t} - c^2\Box^k\right) E(x,t) = 0$ *for* $t > 0$.

(3)

$$|E(x,t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}, \quad \textit{for } t > 0,$$

where $M(t)$ is a function of t in the spectrum Ω and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t > 0$.

(4) $\lim_{t\to 0} E(x,t) = \delta$.

Proof.

(1) From (2.5), since

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \frac{\partial^n}{\partial x^n} \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi.$$

Thus $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} - c^2\Box^k\right) E(x,t) = 0.$$

(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi.$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p \quad \text{and}$$
$$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q}$$

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ON THE ULTRA-HYPERBOLIC WAVE OPERATOR

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Abstract: In this paper, we study the generalized wave equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\Box)^k u(x,t) = 0$$

with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x),$$

where $u(x,t) \in \mathbb{R}^n \times [0,\infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Box^k$ is the ultra-hyperbolic operator iterated $k-$times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p+q=n$, c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable functions. We obtain $u(x,t)$ as a solution for such equation. Moreover, by ϵ-approximation we also obtain the asymptotic solution $u(x,t) = O(\epsilon^{-n/k})$. In particularly, if we put $n=1$, $k=2$ and $q=0$, the $u(x,t)$ reduces to the solution of the beam equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\frac{\partial^4}{\partial x^4}u(x,t) = 0.$$

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1. Introduction

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \tag{1}$$

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation, where f and g are continuous. Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\Delta u(x,t) = 0, \tag{2}$$

with the initial condition

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)\,,$$

where f and g are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos\left(2\pi|\xi|\right)t + \widehat{g}(\xi)\frac{\sin\left(2\pi|\xi|\right)t}{2\pi|\xi|}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ (see [2, p. 177]). By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{3}$$

where Φ_t is an inverse Fourier transform of $\widehat{\Phi}_t(\xi) = \frac{\sin(2\pi|\xi|)t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi) = \cos\left(2\pi|\xi|\right)t = \frac{\partial}{\partial t}\widehat{\Phi}(\xi)$.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\left(\Box\right)^k u(x,t) = 0 \tag{4}$$

with $u(x,0) = f(x)$ and $\frac{\partial}{\partial t}u(x,0) = g(x)$, where c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable. The equation (4) is motivated by the heat equation of the form

$$\frac{\partial}{\partial t}u(x,t) = -c^2\left(\Box\right)^k u(x,t)$$

(see [3], more general: [1]-[4]). We obtain

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{5}$$

as a solution of (4) where Φ_t is an inverse Fourier transform of

$$\widehat{\Phi}_t(\xi) = \frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$$

and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi) = \cos c\left(\sqrt{s^2 - r^2}\right)^k t = \frac{\partial}{\partial t}\widehat{\Phi}_t(\xi)$ where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$. Moreover, if we put $k = 1$ and $q = 0$ in (4) then (5) reduces to the solution of the $n-$dimensional wave equation and also if $k = 2, n = 1$ and $q = 0$ in (4) then (5) reduces to the solution of beam equation.

We also study the asymptotic form of $u(x,t)$ in (5) by using ϵ approximation and obtain $u(x,t) = O(\epsilon^{-n/k})$.

2. Preliminaries

We shall need the following definitions.

Definition 1. Let $f \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx\,, \tag{6}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(x)dx. \tag{7}$$

Lemma 2. *Given the function*

$$f(x) = \exp\left[-\sqrt{-\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2}\right],$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. *Then*

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p \Omega_q}{2} \cdot \frac{\Gamma(n)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-q}{2})},$$

where Γ *denotes the Gamma function. That is* $\int_{\mathbb{R}^n} f(x)dx$ *is bounded.*

Proof.

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{-\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2}\right] dx\,.$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \quad x_2 = r\omega_2, \ldots, \quad x_p = r\omega_p\,,$$

$$dx_1 = rd\omega_1, \quad dx_2 = rd\omega_2, \ldots, \quad dx_p = rd\omega_p \,,$$

and

$$x_{p+1} = s\omega_{p+1}, \quad x_{p+2} = s\omega_{p+2}, \ldots, \quad x_{p+q} = s\omega_{p+q} \,,$$

$$dx_{p+1} = sd\omega_{p+1}, \quad dx_{p+2} = sd\omega_{p+2}, \ldots, \quad dx_{p+q} = sd\omega_{p+q} \,,$$

where $\omega_1^2 + \omega_2^2 + \ldots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \ldots + \omega_{p+q}^2 = 1$. Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q \,,$$

where $dx = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively,

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q \,.$$

By computing directly, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p \Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1} s^{q-1} dr ds \,,$$

where $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \Omega_p \Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1} s^{q-1} dr ds.$$

Put $r = s\sin\theta$, $dr = s\cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$,

$$\begin{aligned}
|\int_{\mathbb{R}^n} f(x)dx| &\leq \Omega_p \Omega_q \int_0^\infty \int_0^s e^{-\sqrt{s^2 - s^2\sin^2\theta}} (s\sin\theta)^{p-1} s^{q-1} s\cos\theta d\theta ds \\
&= \Omega_p \Omega_q \int_0^\infty \int_0^s e^{-s\cos\theta} s^{p+q-1} (\sin\theta)^{p-1} \cos\theta d\theta ds.
\end{aligned}$$

Put $y = s\cos\theta$, $ds = \frac{dy}{\cos\theta}$,

$$\begin{aligned}
|\int_{\mathbb{R}^n} f(x)dx| &\leq \Omega_p \Omega_q \int_0^{\pi/2} \int_0^\infty e^{-y} (\frac{y}{\cos\theta})^{n-1} (\sin\theta)^{p-1} \cos\theta d\theta \frac{dy}{\cos\theta} \\
&= \Omega_p \Omega_q \int_0^{\pi/2} \int_0^\infty e^{-y} y^{n-1} (\cos\theta)^{1-n} (\sin\theta)^{p-1} dy d\theta \\
&= \Omega_p \Omega_q \Gamma(n) \int_0^{\pi/2} (\cos\theta)^{1-n} (\sin\theta)^{p-1} d\theta \\
&= \frac{\Omega_p \Omega_q}{2} \Gamma(n) \beta\left(\frac{p}{2}, \frac{2-n}{2}\right), \\
|\int_{\mathbb{R}^n} f(x)dx| &\leq \frac{\Omega_p \Omega_q}{2} \frac{\Gamma(n)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-n}{2})}.
\end{aligned}$$

That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded. □

3. Main Results

Theorem 3. *Given the equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2 (\Box)^k u(x,t) = 0 \tag{8}$$

with initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)\,, \tag{9}$$

where $u(x,t) \in \mathbb{R}^n \times [0,\infty)$, $\Box^k$ *is the ultra-hyperbolic operator iterated* $k-$*times,* c *is a positive constant,* k *is a nonnegative integer,* f *and* g *are continuous functions and absolutely integrable for* $x \in \mathbb{R}^n$*. Then (8) has a unique solution*

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{10}$$

and satisfy the condition (9), where Φ_t *is an inverse Fourier transform of*

$$\widehat{\Phi}_t(\xi) = \frac{\sin c\left(\sqrt{s^2 - r^2}\right)^k t}{c\left(\sqrt{s^2 - r^2}\right)^k}$$

and Ψ_t *is an inverse Fourier transform of*

$$\widehat{\Psi}_t(\xi) = \cos c\left(\sqrt{s^2 - r^2}\right)^k t = \frac{\partial}{\partial t}\widehat{\Phi}(\xi)\,,$$

where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ *and* $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Proof. By applying the Fourier transform defined by (6) to (8) and obtain

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(-\xi_1^2 - \xi_2^2 - \cdots - \xi_p^2 + \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right)^k \widehat{u}(\xi,t) = 0\,.$$

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(-\sum_{i=1}^{p}\xi_i^2 + \sum_{j=p+1}^{p+q}\xi_j^2\right)^k \widehat{u}(\xi,t) = 0$$

and let $s > r$. Thus we have

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(s^2 - r^2\right)^k \widehat{u}(\xi,t) = 0$$

$$\widehat{u}(\xi,t) = A(\xi)\cos c\left(\sqrt{s^2 - r^2}\right)^k t + B(\xi)\sin c\left(\sqrt{s^2 - r^2}\right)^k t.$$

By (9), $\widehat{u}(\xi,0) = A(\xi) = \widehat{f}(\xi)$

$$\begin{aligned}\frac{\partial\widehat{u}(\xi,t)}{\partial t} &= -c\left(\sqrt{s^2 - r^2}\right)^k A(\xi)\sin c\left(\sqrt{s^2 - r^2}\right)^k t \\ &\quad + c\left(\sqrt{s^2 - r^2}\right)^k B(\xi)\cos c\left(\sqrt{s^2 - r^2}\right)^k t,\end{aligned}$$

$$\frac{\partial \widehat{u}(\xi,0)}{\partial t} = 0 + c\left(\sqrt{s^2-r^2}\right)^k B(\xi) = \widehat{g}(\xi)\,,$$

$$B(\xi) = \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^2-r^2}\right)^k}\,,$$

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos c\left(\sqrt{s^2-r^2}\right)^k t + \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^2-r^2}\right)^k}\sin c\left(\sqrt{s^2-r^2}\right)^k t\,. \quad (11)$$

By applying the inverse Fourier transform (11), we obtain the solution $u(x,t)$ in the convolution form of (8). Now we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$.

Let us consider the Fourier transform

$$\widehat{\Phi_t}(x) = \frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} \quad \text{and} \quad \Psi_t(x) = \cos c\left(\sqrt{s^2-r^2}\right)^k t.$$

They are all tempered distributions but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform $\Phi_t(x)$ and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of $\epsilon-$approximation.

Let us define

$$\widehat{\phi_t^\epsilon}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\widehat{\phi_t}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$$

$$\text{for} \quad \epsilon > 0. \quad (12)$$

We see that $\phi_t^\epsilon(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi_t^\epsilon}(x) \to \widehat{\phi_t}(x)$ uniformly as $\epsilon \to 0$. So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi_t^\epsilon(x)$. Now

$$\begin{aligned}
\Phi_t^\epsilon(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\Phi_t^\epsilon}(\xi)d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi \\
|\Phi_t^\epsilon(x)| &\le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi\,. \qquad (13)
\end{aligned}$$

By changing to bipolar coordinates. Now, put

$$\xi_1 = rw_1, \xi_2 = rw_2, \dots, \xi_p = rw_p$$

and $\qquad \xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \dots, \xi_p = sw_{p+q},\ p+q = n,$

where $w_1^2 + w_2^2 + \dots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \dots + w_{p+q}^2 = 1$.

$$|\Phi_t^\epsilon(x)| \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, where $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$.

$$|\Phi_t^\epsilon(x)| \le \frac{\Omega_p \Omega_q}{(2\pi)^{n/2}} \int_0^\infty \int_0^s \frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1} s^{q-1} dr ds\,.$$

Put $r = s\sin\theta, dr = s\cos\theta d\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-\epsilon c\left(\sqrt{s^2-s^2\sin^2\theta}\right)^k}}{c\left(\sqrt{s^2-s^2\sin^2\theta}\right)^k} (s\sin\theta)^{p-1} s^{q-1} s\cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{c(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-\epsilon c(s\cos\theta)^k}}{(s\cos\theta)^k} (s)^{p-1} s^{q-1} s (\sin\theta)^{p-1} \cos\theta d\theta ds.
\end{aligned}$$

Put $y = \epsilon c\,(s\cos\theta)^k = \epsilon c s^k \cos^k\theta$, $s^k = \frac{y}{c\epsilon\cos^k\theta}$, $ds = \frac{dy}{cks^{k-1}\epsilon\cos^k\theta} = \frac{sdy}{ky}$, thus

$$\begin{aligned}
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{c(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y} s^{n-1}}{y/(\epsilon c)} (\sin\theta)^{p-1} \cos\theta \frac{s}{ky} dy d\theta \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}\epsilon}{ky^2} \left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/k} (\sin\theta)^{p-1} \cos\theta dy d\theta \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y} y^{n/k-2}}{c^{n/k} k \epsilon^{n/k-1}} (\sin\theta)^{p-1} (\cos\theta)^{1-n} dy d\theta \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \frac{\Gamma\left(\frac{n}{k}-1\right)}{k\epsilon^{\frac{n}{k}-1} c^{n/k}} \int_0^{\pi/2} (\sin\theta)^{p-1} (\cos\theta)^{1-n} d\theta \\
&= \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^{n/2} k\epsilon^{n/k-1}} \Gamma\left(\frac{n}{k}-1\right) \beta\left(\frac{p}{2}, \frac{2-n}{2}\right), \\
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^{n/2} k\epsilon^{n/k-1}} \frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}.
\end{aligned}$$

Similarly, we defined $\widehat{\Psi_t^\epsilon}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\cos c\left(\sqrt{s^2-r^2}\right)^k t$ and

$$\begin{aligned}
\Psi_t^\epsilon(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\Psi_t^\epsilon}(\xi)d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\cos c\left(\sqrt{s^2-r^2}\right)^k t d\xi\,, \\
|\Psi_t^\epsilon(x)| &\leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k} d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_0^\infty\int_0^s e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drds,
\end{aligned}$$

Put $r = s\sin\theta$, $dr = s\cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned}
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2} e^{-\epsilon c(s\cos\theta)^k}(s\sin\theta)^{p-1}s^{q-1}s\cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2} e^{-\epsilon c(s\cos\theta)^k}s^{p+q-1}(\sin\theta)^{p-1}\cos\theta d\theta ds\,.
\end{aligned}$$

Put $y = \epsilon c(s\cos\theta)^k$, $ds = s\frac{dy}{ky}$,

$$\begin{aligned}
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{k(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}}{y}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/k}(\sin\theta)^{p-1}\cos\theta dy d\theta \\
&= \frac{\Omega_p\Omega_q}{k(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}y^{n/k-1}}{c^{n/k}\epsilon^{n/k}}(\sin\theta)^{p-1}(\cos\theta)^{1-n}dy d\theta \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k}}\Gamma\left(\frac{n}{k}\right)\int_0^{\pi/2}(\sin\theta)^{p-1}(\cos\theta)^{1-n}d\theta\,, \\
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}.
\end{aligned}$$

Set

$$u^\epsilon(x,t) = f(x) * \Psi_t^\epsilon(x) + g(x) * \Phi_t^\epsilon(x) \qquad (14)$$

which is $\epsilon-$approximation of $u(x,t)$ in (14) for $\epsilon \to 0$, $u^\epsilon(x,t) \to u(x,t)$ uniformly. Now

$$u^\epsilon(x,t) = \int_{\mathbb{R}^n} f(r)\Psi_t^\epsilon(x-r)dr + \int_{\mathbb{R}^n} g(r)\Phi_t^\epsilon(x-r)dr\,.$$

Thus

$$|u^\epsilon(x,t)| \leq |\Psi_t^\epsilon(x-r)|\int_{\mathbb{R}^n}|f(r)|dr + |\Phi_t^\epsilon(x-r)|\int_{\mathbb{R}^n}|g(r)|dr$$

$$\leq \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}M$$
$$+\frac{\Omega_p\Omega_q}{(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k-1}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}N,$$

$$\epsilon^{n/k}|u^\epsilon(x,t)| \leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}M$$
$$+\frac{\Omega_p\Omega_q\epsilon}{2(2\pi)^{n/2}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}N,$$

where $M = \int_{\mathbb{R}^n}|f(r)|dr$ and $N = \int_{\mathbb{R}^n}|g(r)|dr$, since f and g are absolutely integrable.

$$\lim_{\epsilon\to 0}\epsilon^{n/k}|u^\epsilon(x,t)| \leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)} = K.$$

It follows that $u(x,t) = O\left(\epsilon^{-n/k}\right)$ for $n \neq k$ as $\epsilon \to 0$.

In particular, if we put $k = 2, n = 1$ and $q = 0$ then (8) reduces to the solution of the beam equation, see [1, p. 47]

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\frac{\partial^4}{\partial x^4}u(x,t) = 0,$$

with the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x),$$

where f and g are continuous and absolutely integrable for $x \in \mathbb{R}^n$. Thus we obtain $u(x,t) = O(\epsilon^{-1/2})$ which is a solution of such beam equation.

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Convergence Criteria of a New Three-Step Iteration with Errors for Nonexpansive Nonself-Mappings

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Abstract—A new three-step iteration with errors for nonexpansive nonself-mappings in Banach spaces is introduced and studied. Weak and strong convergence theorems of such iterations are established. The results obtained in this paper extend and improve the several recent results in this area.

Keywords—Nonexpansive nonself-mappings, Completely continuous, Uniformly convex, Opial's condition, Condition (A).

1. INTRODUCTION

Let X be a normed space, C be a nonempty convex subset of X, $P : X \to C$ be the nonexpansive retraction of X onto C, and $T : C \to X$ be a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P(a_n T x_n + (1 - a_n - \gamma_n) x_n + \gamma_n u_n), \\ y_n &= P(b_n T z_n + c_n T x_n + (1 - b_n - c_n - \mu_n) x_n + \mu_n v_n), \qquad (1.1) \\ x_{n+1} &= P(\alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n - \lambda_n) x_n + \lambda_n w_n), \qquad n \geq 1, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are appropriate sequences in [0, 1] and $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are bounded sequences in C.

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If $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the iteration scheme defined by Shahzad [1]

$$\begin{aligned} y_n &= P(b_nTx_n + (1-b_n)x_n), \\ x_{n+1} &= P(\alpha_nTy_n + (1-\alpha_n)x_n), \qquad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

If $T : C \to C$, then the iterative scheme (1.1) reduces to the three-step iterations with errors

$$\begin{aligned} z_n &= a_nTx_n + (1-a_n-\gamma_n)x_n + \gamma_nu_n, \\ y_n &= b_nTz_n + c_nTx_n + (1-b_n-c_n-\mu_n)x_n + \mu_nv_n, \\ x_{n+1} &= \alpha_nTy_n + \beta_nTz_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_nw_n, \qquad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are bounded sequences in C.

If $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then the iterative scheme (1.3) reduces to the Ishikawa iterative scheme

$$\begin{aligned} y_n &= b_nTx_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nTy_n + (1-\alpha_n)x_n, \qquad n \geq 1, \end{aligned} \tag{1.4}$$

where $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

Fixed-point iteration processes for approximating the fixed point of nonexpansive mapping in Banach spaces have been studied by various authors, using the Mann iteration process (see [2]) or the Ishikawa iteration process (see [3–6]). In 2000, Noor [7] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. In 1998, Jung and Kim [8] proved the existence of a fixed point for nonexpansive nonself-mapping in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. In [5], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive self-mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space. In [9], Zhou *et al.* gave criteria for weak convergence theorems of the Ishikawa iterative scheme (1.4) for nonexpansive self-mapping in a uniformly convex Banach space which satisfies Opial's condition, and for strong convergence theorems for nonexpansive self-mapping in a uniformly convex Banach space which satisfies Condition (A). In 2004, Cho, Zhou and Guo [10] defined and studied a new three-step iteration with errors for asymptotically nonexpansive mappings in a uniformly convex Banach space. Suantai [11] defined a new three-step iteration which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in a uniformly convex Banach space. Recently, Shahzad [1] extended Tan and Xu results [5, Theorem 1, p. 305] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. Inspired and motivated by research going on in this area, we define and study a new three-step iteration with errors for nonexpansive nonself-mapping. This scheme can be viewed as an extension for the two-step iterative schemes of Shahzad [1].

The purpose of this paper is to establish weak and strong convergence results of the iterative scheme (1.1) for nonexpansive nonself-mappings in a uniformly convex Banach space. Our results extend and improve the corresponding ones announced by Shahzad [1], Tan and Xu [5], and others.

Now, we recall the well-known concepts and results.

Recall that a Banach space X is said to satisfy *Opial's condition* [12] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

LEMMA 1.1. *(See [5, Lemma 1].) Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \qquad \forall n = 1, 2, \dots .$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ *exists.*
(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

LEMMA 1.2. *(See [13, Lemma 1.4].) Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x-y\|),$$

for all $x, y, z, w \in B_r$, and all $\alpha, \beta, \mu, \lambda \in [0,1]$ with $\alpha + \beta + \mu + \lambda = 1$.

LEMMA 1.3. *(See [14].) Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

LEMMA 1.4. *(See [11, Lemma 2.7].) Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2. MAIN RESULTS

Weak and strong convergence theorems of the new three-step iterative scheme (1.1) for nonexpansive nonself-mapping in a uniformly convex Banach space are given in this section. The following lemma is needed.

LEMMA 2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \ne \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$, and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and let $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences defined as in (1.1).*

(i) *If q is a fixed point of T, then $\lim_{n\to\infty}\|x_n - q\|$ exists.*
(ii) *If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty}\|Ty_n - x_n\| = 0$.*
(iii) *If either $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ or $0 < \liminf_{n\to\infty} a_n$ and $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then $\lim_{n\to\infty}\|Tz_n - x_n\| = 0$.*
(iv) *If the following conditions:*
   (1) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and*
   (2) *either* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$

   are satisfied, then $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

PROOF. Letting $q \in F(T)$, by boundedness of the sequence $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$, we can put

$$M = \max\left\{\sup_{n\geq 1}\|u_n - q\|, \sup_{n\geq 1}\|v_n - q\|, \sup_{n\geq 1}\|w_n - q\|\right\}.$$

(i) For each $n \geq 1$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P(\alpha_n T y_n + \beta_n T z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n) - P(q)\| \\
&= \|\alpha_n T y_n + \beta_n T z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n - q\| \\
&\leq \alpha_n\|Ty_n - q\| + \beta_n\|Tz_n - q\| \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + \lambda_n\|w_n - q\| \\
&\leq \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + M\lambda_n,
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\|z_n - q\| &= \|P(a_n T x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n) - P(q)\| \\
&\leq a_n\|Tx_n - q\| + (1-a_n-\gamma_n)\|x_n - q\| + \gamma_n\|u_n - q\| \\
&\leq a_n\|x_n - q\| + (1-a_n-\gamma_n)\|x_n - q\| + M\gamma_n \\
&\leq \|x_n - q\| + M\gamma_n,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\|y_n - q\| &= \|P(b_n T z_n + c_n T x_n + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n) - P(q)\| \\
&\leq b_n\|Tz_n - q\| + c_n\|Tx_n - q\| \\
&\quad + (1-b_n-c_n-\mu_n)\|x_n - q\| + \mu_n\|v_n - q\| \\
&\leq b_n\|z_n - q\| + c_n\|x_n - q\| + (1-b_n-c_n-\mu_n)\|x_n - q\| + M\mu_n \\
&\leq b_n\|z_n - q\| + (1-b_n)\|x_n - q\| + M\mu_n.
\end{aligned}$$

From (2.2) we get

$$\begin{aligned}
\|y_n - q\| &\leq b_n(\|x_n - q\| + M\gamma_n) + (1-b_n)\|x_n - q\| + M\mu_n \\
&= \|x_n - q\| + \epsilon_{(1)}^n,
\end{aligned} \tag{2.3}$$

where $\epsilon_{(1)}^n = Mb_n\gamma_n + M\mu_n$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$, we have $\sum_{n=1}^{\infty}\epsilon_{(1)}^n < \infty$.

From (2.1)–(2.3) we get

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \alpha_n\left(\|x_n - q\| + \epsilon_{(1)}^n\right) + \beta_n(\|x_n - q\| + M\gamma_n) \\
&\qquad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + M\lambda_n \\
&= \alpha_n\|x_n - q\| + \alpha_n\epsilon_{(1)}^n + \beta_n\|x_n - q\| + M\beta_n\gamma_n \\
&\qquad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - q\| + M\lambda_n \\
&\leq \|x_n - q\| + \epsilon_{(2)}^n,
\end{aligned} \tag{2.4}$$

where $\epsilon_{(2)}^n = \alpha_n\epsilon_{(1)}^n + M\beta_n\gamma_n + M\lambda_n$. Since $\sum_{n=1}^{\infty}\epsilon_{(2)}^n < \infty$ we obtained from (2.4) and Lemma 1.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(ii) By (i) we have that $\lim_{n\to\infty}\|x_n - q\|$ exists for any $q \in F(T)$. It follows from (2.2) and (2.3) that $\{x_n - q\}$, $\{Tx_n - q\}$, $\{z_n - q\}$, $\{Tz_n - q\}$, $\{y_n - q\}$, and $\{Ty_n - q\}$ are bounded sequences. This allows us to put

$$K = \max\left\{M, \sup_{n\geq 1}\|x_n - q\|, \sup_{n\geq 1}\|Tx_n - q\|, \sup_{n\geq 1}\|z_n - q\|, \sup_{n\geq 1}\|Tz_n - q\|, \sup_{n\geq 1}\|y_n - q\|, \sup_{n\geq 1}\|Ty_n - q\|\right\}.$$

Since $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, it follows from (2.2) and (2.3) that

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 + \epsilon^n_{(3)}, \tag{2.5}$$

$$\|y_n - q\|^2 \leq \|x_n - q\|^2 + \epsilon^n_{(4)}, \tag{2.6}$$

where $\epsilon^n_{(3)} = M^2\gamma_n^2 + 2MK\gamma_n$, and $\epsilon^n_{(4)} = (\epsilon^n_{(1)})^2 + 2K\epsilon^n_{(1)}$. Since $\sum_{n=1}^{\infty}\epsilon^n_{(3)} < \infty$ and $\sum_{n=1}^{\infty}\epsilon^n_{(4)} < \infty$, by Lemma 1.2, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that

$$\|\lambda x + \beta y + \gamma z + \mu w\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \mu\|w\|^2 - \lambda\beta g(\|x - y\|), \tag{2.7}$$

for all $x, y, z, w \in B_K$ and all $\lambda, \beta, \gamma, \mu \in [0,1]$ with $\lambda + \beta + \gamma = 1$. By (2.5)–(2.7), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P(\alpha_n Ty_n + \beta_n Tz_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\|^2 \\
&\leq \|\alpha_n(Ty_n - q) + \beta_n(Tz_n - q) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - q) + \lambda_n(w_n - q)\|^2 \\
&\leq \alpha_n\|Ty_n - q\|^2 + \beta_n\|Tz_n - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + \lambda_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g\|Ty_n - x_n\| \\
&\leq \alpha_n\|y_n - q\|^2 + \beta_n\|z_n - q\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 \\
&\quad + K^2\lambda_n - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g\|Ty_n - x_n\| \qquad (2.8) \\
&\leq \alpha_n\left(\|x_n - q\|^2 + \epsilon^n_{(4)}\right) + \beta_n\left(\|x_n - q\|^2 + \epsilon^n_{(3)}\right) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g\|Ty_n - x_n\| \\
&= \alpha_n\|x_n - q\|^2 + \alpha_n\epsilon^n_{(4)} + \beta_n\|x_n - q\|^2 + \beta_n\epsilon^n_{(3)} \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g\|Ty_n - x_n\| \\
&\leq \|x_n - q\|^2 + \epsilon^n_{(5)} - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g\|Ty_n - x_n\|,
\end{aligned}$$

where $\epsilon^n_{(5)} = \alpha_n\epsilon^n_{(4)} + \beta_n\epsilon^n_{(3)} + K^2\lambda_n$. It is worth noting here that $\sum_{n=1}^{\infty}\epsilon^n_{(5)} < \infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(4)} < \infty$, $\sum_{n=1}^{\infty}\epsilon^n_{(3)} < \infty$, and $\sum_{n=1}^{\infty}\lambda_n < \infty$. Since $0 < \liminf_{n\to\infty}\alpha_n \leq$

$\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, there exists $n_0\in\mathbb{N}$ and $\delta_1,\delta_2\in(0,1)$ such that $0<\delta_1<\alpha_n$ and $\alpha_n+\beta_n+\lambda_n<\delta_2<1$ for all $n\geq n_0$. Hence, by (2.8), we have

$$\begin{aligned}\delta_1(1-\delta_2)\sum_{n=n_0}^{m} g\|Ty_n-x_n\| &< \sum_{n=n_0}^{m}\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2\right)+\sum_{n=n_0}^{m}\epsilon_{(5)}^n\\ &=\|x_{n_0}-q\|^2+\sum_{n=n_0}^{m}\epsilon_{(5)}^n.\end{aligned}\tag{2.9}$$

Since $\sum_{n=n_0}^{\infty}\epsilon_{(5)}^n<\infty$, by letting $m\to\infty$ in (2.9) we get $\sum_{n=n_0}^{\infty}g\|Ty_n-x_n\|<\infty$, and therefore $\lim_{n\to\infty}g\|Ty_n-x_n\|=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Ty_n-x_n\|=0$.

(iii) First, we assume that $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$. By (2.7), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \alpha_n\|y_n-q\|^2+\beta_n\|z_n-q\|^2+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n\\ &\quad-\beta_n(1-\alpha_n-\beta_n-\lambda_n)g\|Tz_n-x_n\|\\ &\leq \alpha_n\left(\|x_n-q\|^2+\epsilon_{(4)}^n\right)+\beta_n\left(\|x_n-q\|^2+\epsilon_{(3)}^n\right)\\ &\quad+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n\\ &\quad-\beta_n(1-\alpha_n-\beta_n-\lambda_n)g\|Tz_n-x_n\|\\ &=\alpha_n\|x_n-q\|^2+\alpha_n\epsilon_{(4)}^n+\beta_n\|x_n-q\|^2+\beta_n\epsilon_{(3)}^n\\ &\quad+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n\\ &\quad-\beta_n(1-\alpha_n-\beta_n-\lambda_n)g\|Tz_n-x_n\|\\ &\leq\|x_n-q\|^2+\epsilon_{(5)}^n-\beta_n(1-\alpha_n-\beta_n-\lambda_n)g\|Tz_n-x_n\|,\end{aligned}\tag{2.10}$$

where $\epsilon_{(5)}^n=\alpha_n\epsilon_{(4)}^n+\beta_n\epsilon_{(3)}^n+K^2\lambda_n$. Since $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, there exist $n_0\in\mathbb{N}$ and $\delta_1,\delta_2\in(0,1)$ such that $0<\delta_1<\beta_n$ and $\alpha_n+\beta_n+\lambda_n<\delta_2<1$ for all $n\geq n_0$. Hence, by (2.10), we have $\epsilon_{(5)}^n=\alpha_n\epsilon_{(4)}^n+\beta_n\epsilon_{(3)}^n+K^2\lambda_n$.

$$\begin{aligned}\delta_1(1-\delta_2)\sum_{n=n_0}^{m} g\|Tz_n-x_n\| &< \sum_{n=n_0}^{m}\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2\right)+\sum_{n=n_0}^{m}\epsilon_{(5)}^n\\ &=\|x_{n_0}-q\|^2+\sum_{n=n_0}^{m}\epsilon_{(5)}^n.\end{aligned}\tag{2.11}$$

Since $\sum_{n=n_0}^{\infty}\epsilon_{(5)}^n<\infty$, by letting $m\to\infty$ in (2.11) we get $\sum_{n=n_0}^{\infty}g\|Tz_n-x_n\|<\infty$, and therefore $\lim_{n\to\infty}g\|Tz_n-x_n\|=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tz_n-x_n\|=0$.

Next, we assume that $0<\liminf_{n\to\infty}\alpha_n$ and $\liminf_{n\to\infty}b_n\leq\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$. By (2.5) and (2.7), we have

$$\begin{aligned}\|y_n-q\|^2 &= \|P(b_nTz_n+c_nTx_n+(1-b_n-c_n-\mu_n)x_n+\mu_nv_n)-P(q)\|^2\\ &\leq\|b_n(Tz_n-q)+c_n(Tx_n-q)+(1-b_n-c_n-\mu_n)(x_n-q)+\mu_n(v_n-q)\|^2\\ &\leq b_n\|Tz_n-q\|^2+c_n\|Tx_n-q\|^2\end{aligned}\tag{2.12}$$

$$
\begin{aligned}
&+ (1-b_n-c_n-\mu_n)\|x_n-q\|^2+\mu_n\|v_n-q\|^2\\
&- b_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\|\\
&\le b_n\|z_n-q\|^2+c_n\|x_n-q\|^2+(1-b_n-c_n-\mu_n)\|x_n-q\|^2+\mu_nK^2\\
&- b_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\| \qquad (2.12)\text{(cont.)}\\
&\le b_n\left(\|x_n-q\|^2+\epsilon^n_{(3)}\right)+c_n\|x_n-q\|^2+(1-b_n-c_n-\mu_n)\|x_n-q\|^2+\mu_nK^2\\
&\le b_n\left(\|x_n-q\|^2+\epsilon^n_{(3)}\right)+c_n\|x_n-q\|^2+(1-b_n-c_n-\mu_n)\|x_n-q\|^2+\mu_nK^2\\
&- b_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\|\\
&\le \|x_n-q\|^2+\epsilon^n_{(6)}-b_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\|,
\end{aligned}
$$

where $\epsilon^n_{(6)}=b_n\epsilon^n_{(3)}+\mu_nK^2$.

By (2.5), (2.7), and (2.12), we also have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P(\alpha_nTy_n+\beta_nTz_n+(1-\alpha_n-\beta_n-\lambda_n)x_n+\lambda_nw_n)-P(q)\|^2\\
&\le \|\alpha_n(Ty_n-q)+\beta_n(Tz_n-q)\\
&\quad +(1-\alpha_n-\beta_n-\lambda_n)(x_n-q)+\lambda_n(w_n-q)\|^2\\
&\le \alpha_n\|y_n-q\|^2+\beta_n\|z_n-q\|^2+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n\\
&= \alpha_n\left(\|x_n-q\|^2+\epsilon^n_{(6)}-b_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\|\right) \qquad (2.13)\\
&\quad +\beta_n\left(\|x_n-q\|^2+\epsilon^n_{(3)}\right)+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n\\
&= \alpha_n\|x_n-q\|^2+\alpha_n\epsilon^n_{(6)}-\alpha_nb_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\|\\
&\quad +\beta_n\|x_n-q\|^2+\beta_n\epsilon^n_{(3)}+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n\\
&\le \|x_n-q\|^2+\epsilon^n_{(7)}-\alpha_nb_n(1-b_n-c_n-\mu_n)g\|Tz_n-x_n\|,
\end{aligned}
$$

where $\epsilon^n_{(7)}=\alpha_n\epsilon^n_{(6)}+\beta_n\epsilon^n_{(3)}+K^2\lambda_n$.

It is worth noting here that $\sum_{n=1}^{\infty}\epsilon^n_{(7)}<\infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(6)}<\infty$, $\sum_{n=1}^{\infty}\epsilon^n_{(3)}<\infty$, and $\sum_{n=1}^{\infty}\lambda_n<\infty$.

By our assumption $0<\liminf_{n\to\infty}\alpha_n$ and $0<\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$, there exist $n_0\in\mathbb{N}$ and $\delta_1,\delta_2\in(0,1)$ such that $0<\delta_1<\alpha_n$, $0<\delta_1<b_n$, and $b_n+c_n+\mu_n<\delta_2<1$ for all $n\ge n_0$. Hence, by (2.13), we have

$$
\begin{aligned}
\delta_1^2(1-\delta_2)\sum_{n=n_0}^{m}g\|Tz_n-x_n\| &< \sum_{n=n_0}^{m}\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2\right)+\sum_{n=n_0}^{m}\epsilon^n_{(7)} \qquad (2.14)\\
&= \|x_{n_0}-q\|^2+\sum_{n=n_0}^{m}\epsilon^n_{(7)}.
\end{aligned}
$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(7)}<\infty$, by letting $m\to\infty$ in (2.14) we get $\sum_{n=n_0}^{\infty}g\|Tz_n-x_n\|<\infty$, and therefore $\lim_{n\to\infty}g\|Tz_n-x_n\|=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tz_n-x_n\|=0$.

(iv) Suppose that Conditions (1) and (2) are satisfied. Then by (ii) and (iii), we have

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Tz_n - x_n\| = 0. \tag{2.15}$$

From $z_n = P(a_nTx_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n)$ and $y_n = P(b_nTz_n + c_nTx_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n)$, we have $\|z_n - x_n\| \le a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\|$ and $\|y_n - x_n\| \le b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\|$.
It follows that

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Tz_n\| + \|Tz_n - x_n\| \\ &\le \|x_n - z_n\| + \|Tz_n - x_n\| \\ &\le a_n\|Tx_n - x_n\| + \gamma_n\|u_n - z_n\| + \|Tz_n - x_n\|,\end{aligned}$$

which implies

$$(1 - a_n)\|Tx_n - x_n\| \le \gamma_n\|u_n - z_n\| + \|Tz_n - x_n\|.$$

If $\limsup_{n\to\infty} a_n < 1$, this together with (2.15) and $\lim_{n\to\infty}\gamma_n = 0$ imply that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

If $\limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, there exist a positive integer N_0 and $\eta \in (0,1)$ such that

$$c_n \le b_n + c_n + \mu_n < \eta, \qquad \forall n \ge N_0.$$

Then for $n \ge N_0$, we have

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \\ &\le \|x_n - y_n\| + \|Ty_n - x_n\| \\ &\le b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| \\ &\quad + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\| \\ &\le b_n\|Tz_n - x_n\| + \eta\|Tx_n - x_n\| \\ &\quad + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\|.\end{aligned}$$

Hence,

$$(1 - \eta)\|Tx_n - x_n\| \le b_n\|Tz_n - x_n\| + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\|.$$

This together with (2.15) and the fact that $\mu_n \to 0$ as $n \to \infty$ imply

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

∎

THEOREM 2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \ne \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + \gamma_n \in [0,1]$, $b_n + c_n + \mu_n \in [0,1]$, and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \ge 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$. If*

(i) $0 < \min\{\liminf_{n\to\infty}\alpha_n, \liminf_{n\to\infty}\beta_n\} \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and* $\limsup_{n\to\infty} a_n < 1$, *or*

(ii) $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by the iterative scheme (1.1) converge strongly to a fixed point of T.*

PROOF. It follows from Lemma 2.1(i) that $\{x_n\}$ is bounded. Again by Lemma 2.1, we have

$$\begin{aligned} &\lim_{n\to\infty} \|Ty_n - x_n\| = 0, \\ &\lim_{n\to\infty} \|Tz_n - x_n\| = 0, \\ &\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \end{aligned} \tag{2.16}$$

Since T is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Hence, by $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, it follows that $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = q$. By continuity of T and (2.16) we have that $Tq = q$, so q is a fixed point of T. By Lemma 2.1(i), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$, so $\lim_{n\to\infty} \|x_n - q\| = 0$. By (2.16), we have

$$\begin{aligned} \|y_n - x_n\| &= \|P(b_nTz_n + c_nTx_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n) - P(x_n)\| \\ &\le b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\| \\ &\to 0, \qquad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| &= \|P\left(a_nTx_n + (1 - a_n - \gamma_n)\, x_n + \gamma_n u_n\right) - P(x_n)\| \\ &\le a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\| \\ &\to 0, \qquad \text{as } n \to \infty. \end{aligned}$$

It follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. ∎

If T is a self-mapping, then the iterative scheme (1.1) reduces to that of (1.3) and the following result is directly obtained by Theorem 2.2.

THEOREM 2.3. *Let X be a uniformly convex Banach space, and C a nonempty closed convex subset of X. Let T be a completely continuous nonexpansive self-mapping of C with $F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \ge 1$. If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and* $\limsup_{n\to\infty} a_n < 1$, *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by iterations (1.3) converge strongly to a fixed point of T.

When $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.2, the following result is obtained.

THEOREM 2.4. *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$z_n = P(a_n T x_n + (1 - a_n)x_n),$$
$$y_n = P(b_n T z_n + (1 - b_n)x_n), \qquad n \geq 1,$$
$$x_{n+1} = P(\alpha_n T y_n + (1 - \alpha_n)x_n).$$

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to a fixed point of T.

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.2, we obtain the following result.

THEOREM 2.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{b_n\}$, $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$y_n = P(b_n T x_n + (1 - b_n)x_n),$$
$$x_{n+1} = P(\alpha_n T y_n + (1 - \alpha_n)x_n), \qquad n \geq 1.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy Condition (A) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in C$,

$$\|x - Tx\| \geq f(d(x, F(T))).$$

The following result gives a strong convergence theorem for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying Condition (A).

THEOREM 2.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + \gamma_n \in [0,1]$, $b_n + c_n + \mu_n \in [0,1]$, and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose that T satisfies Condition (A). If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and* $\limsup_{n\to\infty} a_n < 1$, *or*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequences $\{x_n\}$ defined by the iterative scheme (1.1) converge strongly to some fixed point of T.

PROOF. Let $q \in F(T)$. Then, as in Lemma 2.1, $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - q\|$ exists, and

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \epsilon_{(2)}^n,$$

where $\sum_{n=1}^{\infty} \epsilon_{(2)}^n < \infty$ for all $n \geq 1$. This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \epsilon_{(2)}^n$ and so, by Lemma 1.1, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 2.1, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T satisfies Condition (A), we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \epsilon_{(2)}^{n} < \infty$, given any $\epsilon < 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \epsilon/4$ and $\sum_{k=n_0}^{n} \epsilon_{(2)}^{k}\epsilon/2$ for all $n \geq n_0$. So we can find $y^* \in F(T)$ such that $\|x_{n_0} - y^*\| < \epsilon/4$. For $n \geq n_0$ and $m \geq 1$, we have

$$
\begin{aligned}
\|x_{n+m} - x_n\| &= \|x_{n+m} - y^*\| + \|x_n - y^*\| \\
&\leq \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| + \sum_{k=n_0}^{n} \epsilon_{(2)}^{k} \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. Then $d(u, F(T)) = 0$. It follows that $u \in F(T)$. This completes the proof. ∎

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, the iterative scheme (1.1) reduces to that of (1.2) and the following result is directly obtained by Theorem 2.6.

THEOREM 2.7. *(See [1, Theorem 3.6].) Let X be a uniformly convex Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{b_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Suppose that T satisfies Condition (A). Then the sequences $\{x_n\}$ defined by the iterative scheme (1.2) converge strongly to some fixed point of T.*

In the next result, we prove weak convergence of the iterative scheme (1.1) for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying Opial's condition.

THEOREM 2.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + \gamma_n$, $b_n + c_n + \mu_n$, and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, and $\limsup_{n\to\infty} a_n < 1$, or

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

then the sequence $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by the iterative scheme (1.1) converges weakly to a fixed point of T.

PROOF. It follows from Lemma 2.1 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty} \|Tz_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.3, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.3, $u, v \in F(T)$. By Lemma 2.1(i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point u of T. Since $\|y_n - x_n\| \leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\| \to 0$ (as $n \to \infty$) and $\|z_n - x_n\| \leq a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0$ (as $n \to \infty$) and $x_n \to u$ weakly as $n \to \infty$, it follows that $y_n \to u$ and $z_n \to u$ weakly as $n \to \infty$. ∎

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The modified Noor iterations with errors for non-Lipschitzian mappings in Banach spaces

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Abstract

In this paper, weak and strong convergence theorems are established for the modified Noor iterations with errors for asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space. Mann-type and Ishikawa-type iterations are included by the modified Noor iterations with errors. The results obtained in this paper extend and improve the recent ones announced by Schu [J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991) 407–413; J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153–159], Xu and Noor [B.L. Xu, M.A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 (2002) 444–453], Cho et al. [Y.J. Cho, H.Y. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (2004) 707–717], Suantai [S. Suantai, Weak and strong convergence criteria of Noor Iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 311 (2005) 506–517], Nammanee et al. [K. Nammanee, M.A. Noor, S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 314 (2006) 320–334], and many others.

Keywords: Asymptotically nonexpansive mapping in the intermediate sense; Completely continuous; Modified Noor iteration; Opial's condition; Uniformly convex Banach space

1. Introduction

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [7] in 1992. In 2000 Noor [8,9] have introduced the three-step iterations and studied the approximate solutions of variational inclusion and variational inequalities in Hilbert spaces. Glowinski and Le Tallec [10] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [10] that the three-step iterative schemes give better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [11] studied the

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convergence analysis of the three-step schemes of Glowinski and Le Tallec [10] and applied these schemes to obtain new spitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also prove that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step schemes play an important and significant part in solving various problems, which arise in pure and applied science.

The concept of asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [12]. This concept is a generalization of asymptotically nonexpansiveness. Let C be a subset of real normed linear space X, and let T be a self-mapping on C. The fixed point set of T, $F(T)$, is defined by $F(T) = \{x \in C : Tx = x\}$. T is said to be *nonexpansive* provided $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \geqslant 1$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|$$

for all $x, y \in C$ and each $n \geqslant 1$.

T is called *asymptotically nonexpansive in the intermediate sense* [12] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leqslant 0.$$

It is known [13] that if X is a uniformly convex Banach space and T is asymptotically nonexpansive in the intermediate sense, then $F(T) \neq \emptyset$.

The modified Noor iterations with errors is defined as follows.

Let X be a normed space, C be a nonempty subset of X, and $T : C \to C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n - \gamma_n) x_n + \gamma_n u_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n) x_n + \mu_n v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n - \lambda_n) x_n + \lambda_n w_n, \quad n \geqslant 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ are bounded sequences in C.

The iterative schemes (1.1) are called the *modified Noor iterations with errors*. Noor iterations include the Mann–Ishikawa iterations as spacial cases. If $\gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the modified Noor iterations defined by Suantai [5]

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \quad n \geqslant 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

We note that the usual Ishikawa and Mann iterations are special cases of (1.1) and if $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the Noor iterations defined by Xu and Noor [3]

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geqslant 1, \end{aligned} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the usual Ishikawa iterative schemes

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geqslant 1, \end{aligned} \tag{1.4}$$

where $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad n \geqslant 1. \tag{1.5}$$

where $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$. See [1,2] for more details about Mann iterative scheme.

The purpose of this paper is to establish several strong convergence theorems for the modified Noor iterations with errors (1.1) for completely continuous asymptotically nonexpansive mappings in the intermediate sense, and weak convergence theorems for asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space with Opial's condition.

Recall that a Banach space X is said to satisfy *Opial's condition* [14] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 [15, Lemma 1]. *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1+\delta_n)a_n + b_n, \quad \forall n = 1, 2, \ldots$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ *exists.*
(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 1.2 [4, Lemma 1.6]. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 1.3 [5, Lemma 2.7]. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 1.4 [4, Lemma 1.4]. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leqslant r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|)$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 1.5 [6, Lemma 1.4]. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leqslant r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leqslant \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|)$$

for all $x, y, z, w \in B_r$, and all $\alpha, \beta, \mu, \lambda \in [0, 1]$ with $\alpha + \beta + \mu + \lambda = 1$.

2. Main results

In this section, we prove strong convergence theorems for the modified Noor iterations with errors (1.1) for asymptotically nonexpansive mapping in the intermediate sense in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

The next lemma is crucial for proving the main theorems.

Lemma 2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty bounded closed and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0, \quad \forall n \geqslant 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0, 1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geqslant 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$, *and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in* (1.1).

(i) *If $p \in F(T)$ then $\lim_{n\to\infty}\|x_n - p\|$ exists.*
(ii) *If $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty}\|T^n y_n - x_n\| = 0$.*
(iii) *If $0 < \liminf_{n\to\infty}\alpha_n$ and $0 < \liminf_{n\to\infty}b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then $\lim_{n\to\infty}\|T^n z_n - x_n\| = 0$.*
(iv) *If $0 < \liminf_{n\to\infty}b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ and $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty}\|T^n x_n - x_n\| = 0$.*

Proof. (i) By [13] $F(T) \neq \emptyset$. Let $p \in F(T)$. Since $\{G_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C, we put

$$M = \sup_{n\geqslant 1} G_n \vee \sup_{n\geqslant 1}\|u_n - p\| \vee \sup_{n\geqslant 1}\|v_n - p\| \vee \sup_{n\geqslant 1}\|w_n - p\|.$$

For each $n \geqslant 1$, we note that

$$\begin{aligned}
\|z_n - p\| &= \|a_n T^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n - p\| \\
&\leqslant (1 - a_n - \gamma_n)\|x_n - p\| + a_n\|T^n x_n - p\| + \gamma_n\|u_n - p\| \\
&\leqslant a_n\|x_n - p\| + a_n G_n + (1 - a_n - \gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\
&\leqslant \|x_n - p\| + G_n + M\gamma_n, \qquad (2.1)\\
\|y_n - p\| &= \|b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n - p\| \\
&\leqslant (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n\|T^n z_n - p\| \\
&\quad + c_n\|T^n x_n - p\| + \mu_n\|v_n - p\| \\
&\leqslant (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n[\|z_n - p\| + G_n] \\
&\quad + c_n[\|x_n - p\| + G_n] + \mu_n\|v_n - p\| \\
&\leqslant (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n[(\|x_n - p\| + G_n + M\gamma_n) + G_n] \\
&\quad + c_n[\|x_n - p\| + G_n] + M\mu_n \\
&\leqslant \|x_n - p\| + 3G_n + M\gamma_n + M\mu_n. \qquad (2.2)\\
\|x_{n+1} - p\| &= \|\alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n - p\| \\
&\leqslant (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n\|T^n y_n - p\| \\
&\quad + \beta_n\|T^n z_n - p\| + \lambda_n\|w_n - p\| \\
&\leqslant (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n[\|y_n - p\| + G_n] \\
&\quad + \beta_n[\|z_n - p\| + G_n] + \lambda_n\|w_n - p\| \\
&\leqslant (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| \\
&\quad + \alpha_n[\|(\|x_n - p\| + 3G_n + M\gamma_n + M\mu_n) + G_n] \\
&\quad + \beta_n[\|(\|x_n - p\| + G_n + M\gamma_n) + G_n] + M\lambda_n \\
&\leqslant \|x_n - p\| + 6G_n + M\gamma_n + M\mu_n + M\lambda_n. \qquad (2.3)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}G_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, and $\sum_{n=1}^{\infty}\lambda_n < \infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_n - p\|$ exists.

(ii) By [13], T has a fixed point $p \in C$. Choose a number $r > 0$ such that $C \subseteq B_r$ and $C - C \subseteq B_r$. By Lemma 1.4, there is a continuous, strictly increasing, and convex function $g_1 : [0,\infty) \to [0,\infty)$, $g_1(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|) \qquad (2.4)$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

It follows from (2.4) that

$$\begin{aligned}
\|z_n - p\|^2 &= \|a_nT^nx_n + (1 - a_n - \gamma_n)x_n + \gamma_nu_n - p\|^2 \\
&= \|a_n(T^nx_n - p) + (1 - a_n - \gamma_n)(x_n - p) + \gamma_n(u_n - p)\|^2 \\
&\leqslant a_n\|T^nx_n - p\|^2 + (1 - a_n - \gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 - a_n(1 - a_n - \gamma_n)g_1(\|T^nx_n - x_n\|) \\
&\leqslant a_n[\|x_n - p\| + G_n]^2 + (1 - a_n - \gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 - a_n(1 - a_n - \gamma_n)g_1(\|T^nx_n - x_n\|) \\
&= a_n[\|x_n - p\|^2 + 2G_n\|x_n - p\| + G_n^2] + (1 - a_n - \gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1(\|T^nx_n - x_n\|) \\
&\leqslant \|x_n - p\|^2 + 2G_n\|x_n - p\| + G_n^2 + M^2\gamma_n - a_n(1 - a_n - \gamma_n)g_1(\|T^nx_n - x_n\|).
\end{aligned} \tag{2.5}$$

By Lemma 1.5, there exists a continuous strictly increasing convex function $g_2:[0,\infty) \to [0,\infty)$, $g_2(0) = 0$ such that

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leqslant \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|) \tag{2.6}$$

for all $x,y,z,w \in B_r$, and all $\alpha,\beta,\mu,\lambda \in [0,1]$ with $\alpha + \beta + \mu + \lambda = 1$. It follows from (2.6) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|b_nT^nz_n + c_nT^nx_n + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n - p\|^2 \\
&= \|b_n(T^nz_n - p) + (1 - b_n - c_n - \mu_n)(x_n - p) + c_n(T^nx_n - p) + \mu_n(v_n - p)\|^2 \\
&\leqslant (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n\|T^nz_n - p\|^2 + c_n\|T^nx_n - p\|^2 \\
&\quad + \mu_n\|v_n - p\|^2 - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^nz_n - x_n\|) \\
&\leqslant (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n[\|z_n - p\| + G_n]^2 + c_n[\|x_n - p\| + G_n]^2 + \mu_n\|v_n - p\|^2 \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^nz_n - x_n\|) \\
&= (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n[\|z_n - p\|^2 + 2G_n\|z_n - p\| + G_n^2] \\
&\quad + c_n[\|x_n - p\|^2 + 2G_n\|x_n - p\| + G_n^2] + \mu_n\|v_n - p\|^2 \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^nz_n - x_n\|) \\
&\leqslant (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n[(\|x_n - p\|^2 + 2G_n\|x_n - p\| + G_n^2 + M^2\gamma_n) \\
&\quad + 2G_n(\|x_n - p\| + G_n + M\gamma_n) + G_n^2] + c_n[\|x_n - p\|^2 + 2G_n\|x_n - p\| + G_n^2] + M^2\mu_n \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^nz_n - x_n\|) \\
&\leqslant \|x_n - p\|^2 + 6G_n\|x_n - p\| + 5G_n^2 + M^2(\gamma_n + \mu_n) + 2MG_n \\
&\quad - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^nz_n - x_n\|),
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_nT^nx_n + \beta_nT^nz_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_nw_n - p\|^2 \\
&= \|\alpha_n(T^nx_n - p) + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - p) + \beta_n(T^nz_n - p) + \lambda_n(w_n - p)\|^2 \\
&\leqslant (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 + \alpha_n\|T^ny_n - p\|^2 + \beta_n\|T^nz_n - p\|^2 \\
&\quad + \lambda_n\|w_n - p\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g_2(\|T^ny_n - x_n\|) \\
&\leqslant (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 \\
&\quad + \alpha_n[\|y_n - p\| + G_n]^2 + \beta_n[\|z_n - p\| + G_n]^2 + \lambda_n\|w_n - p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g_2(\|T^ny_n - x_n\|) \\
&= (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\|^2 + \alpha_n[\|y_n - p\|^2 + 2G_n\|y_n - p\| + G_n^2] \\
&\quad + \beta_n[\|z_n - p\|^2 + 2G_n\|z_n - p\| + G_n^2] + \lambda_n\|w_n - p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g_2(\|T^ny_n - x_n\|)
\end{aligned}$$

$$\begin{aligned}
&\leqslant (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\|^2\\
&\quad+\alpha_n\{(\|x_n-p\|^2+6G_n\|x_n-p\|+5G_n^2+M^2(\gamma_n+\mu_n)+2MG_n)\\
&\quad+2G_n(\|x_n-p\|+3G_n+M\gamma_n+M\mu_n)+G_n^2\}\\
&\quad+\beta_n[(\|x_n-p\|^2+2G_n\|x_n-p\|+G_n^2+M^2\gamma_n)\\
&\quad+2G_n(\|x_n-p\|+G_n+M\gamma_n)+G_n^2]+M^2\lambda_n\\
&\quad-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^ny_n-x_n\|)\\
&\leqslant \|x_n-p\|^2+12G_n\|x_n-p\|+16G_n^2+M^2(2\gamma_n+\mu_n)+8MG_n\\
&\quad-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)g_2(\|T^ny_n-x_n\|),
\end{aligned}\tag{2.8}$$

which imply that

$$\alpha_n(1-\alpha_n-\beta_n-\lambda)g_2(\|T^ny_n-x_n\|)\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2+12LG_n+16G_n^2+M^2(2\gamma_n+\mu_n)+8MG_n,\tag{2.9}$$

and

$$\alpha_nb_n(1-b_n-c_n-\mu_n)g_2(\|T^nz_n-x_n\|)\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2+12LG_n+16G_n^2+M^2(2\gamma_n+\mu_n)+8MG_n,\tag{2.10}$$

where $L=\sup\{\|x_n-p\|:n\geqslant 1\}$.

If $0<\liminf_{n\to\infty}\alpha_n\leqslant\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, then there exist a positive integer n_0 and $\eta,\eta'\in(0,1)$ such that

$$0<\eta<\alpha_n \text{ and } \alpha_n+\beta_n+\lambda_n<\eta'<1 \quad \text{for all } n\geqslant n_0.$$

This implies by (2.9) that

$$\begin{aligned}
\eta(1-\eta')g_2(\|T^nz_n-x_n\|)&\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2+12LG_n+16G_n^2+M^2(2\gamma_n+\mu_n)+8MG_n\\
&\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2+12LG_n+16MG_n+M^2(2\gamma_n+\mu_n)+8MG_n\\
&\leqslant\|x_n-p\|^2-\|x_{n+1}-p\|^2+12KG_n+5KG_n+M^2(2\gamma_n+\mu_n)+8KG_n\\
&=\|x_n-p\|^2-\|x_{n+1}-p\|^2+17KG_n+M^2(2\gamma_n+\mu_n),
\end{aligned}\tag{2.11}$$

where $K=\max\{M,L\}$, for all $n\geqslant n_0$. It follows from (2.11) that for $m\geqslant n_0$

$$\begin{aligned}
\sum_{n=n_0}^{m}g_2(\|T^nz_n-x_n\|)&\leqslant\frac{1}{\eta(1-\eta')}\left(\sum_{n=n_0}^{m}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)+\sum_{n=n_0}^{m}(17KG_n+M^2(2\gamma_n+\mu_n))\right)\\
&\leqslant\frac{1}{\eta(1-\eta')}\left(\|x_{n_0}-p\|^2+17K\sum_{n=n_0}^{m}G_n+M^2\sum_{n=n_0}^{m}(2\gamma_n+\mu_n)\right).
\end{aligned}\tag{2.12}$$

Since $\sum_{n=1}^{\infty}G_n<\infty$. Let $m\to\infty$ in inequality (2.12) we get that $\sum_{n=n_0}^{\infty}g_2(\|T^nz_n-x_n\|)<\infty$, and therefore $\lim_{n\to\infty}g_2(\|T^nz_n-x_n\|)=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|T^nz_n-x_n\|=0$.

(iii) If $0<\liminf_{n\to\infty}\alpha_n$ and $0<\liminf_{n\to\infty}b_n\leqslant\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$, then by the using a similar method together with inequality (2.10), it can be shown that

$$\lim_{n\to\infty}\|T^ny_n-x_n\|=0.$$

(iv) If $0<\liminf_{n\to\infty}b_n\leqslant\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$ and $0<\liminf_{n\to\infty}\alpha_n\leqslant\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, then by (ii) and (iii) we have

$$\lim_{n\to\infty}\|T^nz_n-x_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|T^ny_n-x_n\|=0.\tag{2.13}$$

From $y_n = (1 - b_n - c_n - \mu_n)x_n + b_nT^nz_n + c_nT^nx_n + \mu_nv_n$, we have

$$\begin{aligned}\|y_n - x_n\| &= \|(1 - b_n - c_n - \mu_n)x_n + b_nT^nz_n + c_nT^nx_n + \mu_nv_n - x_n\| \\ &= \|b_n(T^nz_n - x_n) + c_nT^n(x_n - x_n) + \mu_n(v_n - x_n)\| \\ &\leqslant b_n\|T^nz_n - x_n\| + c_n\|T^nx_n - x_n\| + \mu_n\|x_n - v_n\|. \end{aligned} \quad (2.14)$$

Thus

$$\begin{aligned}\|T^nx_n - x_n\| &= \|T^nx_n - T^ny_n + T^ny_n - x_n\| \leqslant \|T^nx_n - T^ny_n\| + \|T^ny_n - x_n\| \\ &\leqslant \|x_n - y_n\| + G_n + \|T^ny_n - x_n\| \\ &\leqslant b_n\|T^nz_n - x_n\| + c_n\|T^nx_n - x_n\| + \mu_n\|x_n - v_n\| + G_n + \|T^ny_n - x_n\|,\end{aligned} \quad (2.15)$$

and so

$$(1 - c_n)\|T^nx_n - x_n\| \leqslant b_n\|T^nz_n - x_n\| + \mu_n\|x_n - v_n\| + G_n + \|T^ny_n - x_n\|. \quad (2.16)$$

Since $\limsup_{n\to\infty}c_n < 1$, it follows from (2.13) and $\sum_{n=1}^{\infty}G_n < \infty$ that

$$\lim_{n\to\infty}\|T^nx_n - x_n\| = 0. \qquad \square$$

Theorem 2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty bounded closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C}(\|T^nx - T^ny\| - \|x - y\|) \vee 0 \quad \forall n \geqslant 1,$$

so that $\sum_{n=1}^{\infty}G_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geqslant 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1) and

(i) $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and*
(ii) $0 < \liminf_{n\to\infty}b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. By Lemma 2.1, we have

$$\lim_{n\to\infty}\|T^ny_n - x_n\| = 0,$$
$$\lim_{n\to\infty}\|T^nz_n - x_n\| = 0,$$
$$\lim_{n\to\infty}\|T^nx_n - x_n\| = 0.$$

It follows from (2.14) that $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

From $x_{n+1} = (1 - \alpha_n - \beta_n - \lambda_n)x_n + \alpha_nT^ny_n + \beta_nT^nz_n + \lambda_nw_n$, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1 - \alpha_n - \beta_n - \lambda_n)x_n + \alpha_nT^ny_n + \beta_nT^nz_n + \lambda_nw_n - x_n\| \\ &\leqslant \alpha_n\|T^ny_n - x_n\| + \beta_n\|T^nz_n - x_n\| + \lambda_n\|w_n - x_n\| \to 0.\end{aligned}$$

And

$$\begin{aligned}\|x_{n+1} - T^nx_{n+1}\| &\leqslant \|x_{n+1} - x_n\| + \|T^nx_{n+1} - T^nx_n\| + \|T^nx_n - x_n\| \\ &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - x_n\| + G_n + \|T^nx_n - x_n\| \to 0.\end{aligned}$$

Since

$$\|x_{n+1} - Tx_{n+1}\| \leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\|$$

and by uniform continuity of T and $\lim_{n\to\infty}\|T^nx_n - x_n\| = 0$, it follows that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty}x_{n_k} = p$. By continuity of T and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, we have that $Tp = p$, so p is a fixed point of T. By Lemma 2.1 (i), $\lim_{n\to\infty}\|x_n - p\|$ exists. But $\lim_{k\to\infty}\|x_{n_k} - p\| = 0$. Thus $\lim_{n\to\infty}\|x_n - p\| = 0$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, and

$$\|z_n - x_n\| = \|a_nT^nx_n + (1 - a_n - \gamma_n)x_n + \gamma_nu_n - x_n\| \leqslant \|T^nx_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0 \quad \text{as } n \to \infty,$$

it follows that $\lim_{n\to\infty}y_n = p$ and $\lim_{n\to\infty}z_n = p$. □

From Theorem 2.2, we have the following results.

Corollary 2.3 [6, Theorem 2.3]. *Let X be a uniformly convex Banach space, and C a nonempty bounded, closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geqslant 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty}b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations with errors (1.1). *Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Corollary 2.4 [5, Theorem 2.3]. *Let X be a uniformly convex Banach space, and C a nonempty bounded, closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geqslant 1$, and*

(i) $0 < \liminf_{n\to\infty}b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by the three-step iterative scheme (1.2). *Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

For $c_n = \beta_n \equiv 0$ in Theorem 2.2, we obtain the following result.

Corollary 2.5 [3, Theorem 2.1]. *Let X be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty}b_n \leqslant \limsup_{n\to\infty}b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}\alpha_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= a_nT^nx_n + (1 - a_n)x_n, \\ y_n &= b_nT^nz_n + (1 - b_n)x_n, \quad n \geqslant 1, \\ x_{n+1} &= \alpha_nT^ny_n + (1 - \alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

When $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.2, we can obtain Ishikawa-type convergence result.

Corollary 2.6. *Let X be a uniformly convex Banach space, and let C be a bounded, closed and convex subs… Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant$ … $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}$, $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$y_n = b_n T^n z_n + (1 - b_n)x_n,$$
$$x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

In the next result, we prove weak convergence of the modified Noor iterations with errors for as… cally nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.7. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and le… nonempty bounded, closed and convex subset of X. Let T be an asymptotically nonexpansive in the inte… sense. Put*

$$G_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0, \quad \forall n \geqslant 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1…

(i) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and*
(ii) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. It follows from Theorem 2.2 that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since X is uniformly convex an… bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.… $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respecti… Lemma 1.2, $u, v \in F(T)$. By Lemma 2.1 (i), $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows fr… 1.3 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. □

From Theorem 2.7, we have the following results.

Corollary 2.8 [6, Theorem 2.8]. *Let X be a uniformly convex Banach space which satisfies Opial's co… C a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive se… with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ be … real numbers in $[0,1]$ with $a_n + \gamma_n, b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geqslant 1$, and $\sum$… $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}$ be the sequence defined by modified Noor iterations with errors (1.1). Then $\{x_n\}$ converg… a fixed point of T.

Corollary 2.9 [5, Theorem 2.3]. *Let X be a uniformly convex Banach space which satisfies Opial's co… C a nonempty bounded, closed and convex subset of X. Let T be an asymptotically nonexpansive s… with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be sequences o… in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geqslant 1$, and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}$ be the sequence defined by three-step iterative scheme (1.2). Then $\{x_n\}$ converges weakly to a fixed point of T.

When $c_n = \beta_n \equiv 0$ in Theorem 2.7, we obtain the following result.

Corollary 2.10. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty bounded, closed and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be sequences of real numbers in $[0, 1]$ and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \quad n \geqslant 1, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

When $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.7, we obtain Ishikawa-type weak convergence theorem as follows:

Corollary 2.11. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty bounded, closed and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}$, $\{\alpha_n\}$ be sequences of real numbers in $[0, 1]$ such that*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1. \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

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The criteria of strict monotonicity and rotundity points in generalized Calderón–Lozanovskiĭ spaces☆

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Abstract

In this paper, some basic properties of the general modular space are proven. Criteria for strictly monotone points, extreme points and SU-points in generalized Calderón–Lozanovskiĭ spaces are obtained. Consequently, the sufficient and necessary conditions for the rotundity properties of such spaces are given.

MSC: 46A45; 46B20; 46B30; 46C05; 46E30

Keywords: Musielak–Orlicz function; Generalized Calderón–Lozanovskiĭ spaces; Point of lower(upper) monotonicity; Extreme point; SU-point; Rotundity

1. Introduction

Throughout the paper $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the sets of reals, nonnegative reals and natural numbers, respectively. For a real vector space X, a function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\rho(0) = 0$ and $x = 0$ whenever $\rho(\lambda x) = 0$ for any $\lambda > 0$;
(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If we replace (iii) by

(iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$,

then the modular ρ is called convex modular. Moreover, for arbitrary $x \in X$ we define

$$\xi(x) := \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) < \infty\right\}.$$

We put $\inf \emptyset = \infty$ by the definition.

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For any modular ρ on X, the space

$$X_\rho = \left\{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\right\},$$

is called the *modular space*. If ρ is a convex modular, the functional

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\right\},$$

is a norm on X_ρ, which is called the *Luxemburg norm* (see [35]). A modular ρ is called right-continuous (left-continuous) [continuous] if $\lim_{\lambda\to 1^+} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$ ($\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$) [it is both right- and left-continuous].

Remark 1.1. If ρ is a convex modular and $\rho(\lambda_o x) < \infty$ for some $x \in X_\rho$ and $\lambda_o > 0$, then ρ is right-continuous at λx for any $\lambda \in [0, \lambda_o)$ and left-continuous at λx for any $\lambda \in (0, \lambda_o]$. Indeed, this follows from the fact that the function $f(t) = \rho(tx)$ is convex on $\mathbb{R}^+$ and has finite values on the interval $[0, \lambda_o]$ so it is a continuous function on $[0, \lambda_o]$.

A triple (T, Σ, μ) stands for a nonatomic, positive, complete and σ-finite measure space, while $L^0 = L^0(\mu)$ denotes the space of all (equivalence classes of) σ-measurable functions $x : T \to \mathbb{R}$. In what follows we will identify measurable functions which differ only on a set of measure zero. For $x, y \in L^0$, we write $x \le y$ if $x(t) \le y(t)$ for μ-a.e. $t \in T$ and the notion $x < y$ is used for $x \le y$ and $x \ne y$. Moreover, for any $x \in L^0$, we denote by $|x|$ the absolute value of x, i.e. $|x|(t) = |x(t)|$ for μ-a.e. $t \in T$.

By E we denote a *Köthe space* over the measure space (T, Σ, μ), i.e. $E \subset L^0$ which satisfies the following conditions:

(i) if $x \in E$, $y \in L^0$ and $|y| \le |x|$ for μ-a.e. then $y \in E$ and $\|y\|_E \le \|x\|_E$,
(ii) there exists a function x in E which is strictly positive on the whole T.

A function $\varphi : T \times \mathbb{R} \to [0, \infty)$ is said to be a *Musielak–Orlicz function* if $\varphi(t, \cdot)$ is a nonzero function, it vanishes at zero, it is convex and even for μ-a.e. $t \in T$ and $\varphi(\cdot, u)$ as well as $\varphi^{-1}(\cdot, u)$ are Σ-measurable functions for any $u \in R^+$, where $\varphi^{-1}(t, \cdot)$ is the generalized inverse function of $\varphi(t, \cdot)$ defined on $[0, \infty)$ by

$$\varphi^{-1}(t, u) = \inf\{v \ge 0 : \varphi(t, v) > u\}$$

for each $t \in T$ (see [35]). For Musielak–Orlicz function φ we define a measurable function with respect to $t \in T$ by

$$a(t) = \sup\{u \ge 0 : \varphi(t, u) = 0\},$$

see [6, page 175].

Remark 1.2. Let $\varphi : T \times \mathbb{R} \to [0, \infty)$ be a Musielak–Orlicz function. Then

(i) $\varphi^{-1}(t, \cdot)$ vanishes only at zero;
(ii) $\varphi(t, \varphi^{-1}(t, u)) = u$ for all $u \in [0, \infty)$ and

$$\varphi^{-1}(t, \varphi(t, u)) = \begin{cases} 0, & \text{if } u \in [0, a(t)], \\ u, & \text{if } u \in (a(t), \infty); \end{cases}$$

for μ-a.e. $t \in T$.

Given any Musielak–Orlicz function φ, we define on L^0 a convex modular ϱ_φ by

$$\varrho_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise;} \end{cases}$$

and *the generalized Calderón–Lozanovskiĭ space* is defined by

$$E_\varphi = \{x \in L^0 : \varphi \circ \lambda x \in E \text{ for some } \lambda > 0\}.$$

Then $E_\varphi = (E_\varphi, \|\cdot\|_\varphi)$ becomes a normed space, where $\|\cdot\|_\varphi$ denotes for the Luxemburg norm induced by ϱ_φ (see [4,9]).

As for the investigations of generalized Calderón–Lozanovskiĭ space we refer to [8–10,27].

In the case when φ is an Orlicz function, i.e. there is a set $A \in \Sigma$ with $\mu(A) = 0$ such that $\varphi(t_1, \cdot) = \varphi(t_2, \cdot)$ for all $t_1, t_2 \in T \setminus A$, these Calderón–Lozanovskiĭ spaces were investigated in [3,4,30] and the investigations were continued in the papers [5,11,15,17,20,26,28,29,32–34,36,37].

We say a Musielak–Orlicz function φ satisfies the condition Δ_2^E if there exist a set $A \in \Sigma$ with $\mu(A) = 0$, a constant $K > 0$ and a nonnegative function $h \in E$ such that the inequality

$$\varphi(t, 2u) \leq K\varphi(t, u) + h(t)$$

holds for all $t \in T \setminus A$ and $u \in \mathbb{R}$ (see [35] when $E = L^1$ and [9] in general).

Lemma 1.3 (*[9, Lemma 5]*). *The property that $\|x\|_\varphi = 1$ if and only if $\varrho_\varphi(x) = 1$ holds true for any $x \in E_\varphi$ if and only if $\varphi \in \Delta_2^E$.*

Lemma 1.4 (*[19, Lemma 1]*). *For any Musielak–Orlicz function φ the inequality*

$$\varphi(t, u + v) \geq \varphi(t, u) + \varphi(t, a(t) + v)$$

holds for μ-a.e. $t \in T$ and any $u \geq a(t)$, $v \geq 0$.

Lemma 1.5 (*[9, Corollary 7]*). *If $\varphi \in \Delta_2^E$ then $\mu(\{t \in T : a(t) > 0\}) = 0$.*

By $S(E)$, $B(E)$ and $E^+(= \{x \in E : x \geq 0\})$ we denote the unit sphere, the closed unit ball and the positive cone of the Köthe space E. For any $x \in E$, define $\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$.

A point $x \in E^+$ is called a point of *upper monotonicity* (*UM*-point for short) if for every $y \in E^+ \setminus \{0\}$ we have $\|x\|_E < \|x + y\|_E$. A point $x \in E^+ \setminus \{0\}$ is called a point of *lower monotonicity* (*LM*-point for short) if for every $y \in E^+ \setminus \{0\}$, such that $y < x$, we have $\|x - y\|_E < \|x\|_E$. If every point of $S(E^+)$ is a *UM*-point (or an *LM*-point), then we say that the space E is *strictly monotone*. It is easy to see that $x \in E^+ \setminus \{0\}$ in any Köthe space E is a *UM*-point (*LM*-point) if and only if $x/\|x\|$ is a *UM*-point (*LM*-point). Therefore, it is enough to formulate the criteria of monotonicity for points in $S(E^+)$ only.

A point $x \in S(E)$ is said to be an *extreme point* of $B(E)$ ($x \in \operatorname{ext} B(E)$ for short) if for any $y, z \in B(E)$ such that $2x = y + z$ we have $y = z$. If any point of $S(E)$ is an extreme point of $B(E)$, we say that the space E is rotund ($E \in (R)$).

A point $x \in S(E)$ is called a *strong U-point* (*SU*-point for short) of $B(E)$ if for any $y \in S(E)$ with $\|x + y\|_E = 2$, we have $x = y$. It is obvious that a Banach space E is rotund if and only if any $x \in S(E)$ is an *SU*-point, but the notions of an extreme point and an *SU*-point are different (see [7]).

It is well known that rotundity properties of Banach spaces have applications in various branches of mathematics, such as, Fixed point Theory, Approximation Theory, Ergodic Theory, and many others. Moreover, if the focus of the study is Banach lattices, then there are strong relationships between rotundity properties and monotonicity properties (see [2,13,14,16,18,21,24,25]). Specially, in [17,20] the local rotundity and local monotonicity structures of a certain Banach lattice, namely Calderón–Lozanovskiĭ spaces, were studied. The results of our paper will be a generalization of two such excellent papers [17,20] by considering Orlicz function with parameter called Musielak–Orlicz function instead of Orlicz function. Of course, some ideas from those papers are also applied in our paper. However, because of the different properties among functions, in many parts of the proofs of our results new methods and techniques are developed.

Let us note that if E has the Fatou property, i.e. for any $x \in L^0$ and $(x_n)_{n=1}^\infty$ in E such that $0 \leq x_n \nearrow x$ μ-a.e. and $\sup_n \|x_n\|_E < \infty$ we have that $x \in E$ and $\|x\|_E = \lim_{n\to\infty} \|x_n\|_E$ (see [1,23,31]), then E_φ also has this property, and moreover, the modular ϱ_φ is left-continuous (see [9, Theorem 12]). Consequently, E_φ is a Banach space. So, in the whole paper we will assume that E is a Köthe space with the Fatou property. Moreover, we will denote $(\varphi \circ x)(t) = \varphi(t, x(t))$ for each $t \in T$.

The paper is organized as follows. In Section 2 we give some basic auxiliary results of general modular space and E_φ. Section 3 is devoted to the strictly monotone points of E_φ. We study rotundity points of E_φ in Section 4. Finally, in Section 5 we give a characterization of rotundity structure in E_φ.

2. Auxiliary lemmas

We start by proving some facts in any modular space.

Lemma 2.1. *Let X_ρ be a modular space generated by a convex modular ρ and $x, y \in B(X_\rho)$. If $\xi(x) < 1$ then $\xi\left(\frac{x+y}{2}\right) < 1$.*

Proof. Since $\xi(x) < 1$, we take a real number $a \in (\xi(x), 1)$ and put $\varepsilon = \frac{1-a}{1+a}$. Then $\varepsilon > 0$ and $\frac{(1+\varepsilon)a}{2} + \frac{1+\varepsilon}{2} = 1$. Thus,

$$\rho\left((1+\varepsilon)\left(\frac{x+y}{2}\right)\right) = \rho\left(\frac{1+\varepsilon}{2}\cdot x + \frac{1+\varepsilon}{2}\cdot y\right)$$
$$= \rho\left(\frac{(1+\varepsilon)a}{2}\cdot\frac{x}{a} + \frac{1+\varepsilon}{2}\cdot y\right)$$
$$\le \frac{(1+\varepsilon)a}{2}\rho\left(\frac{x}{a}\right) + \frac{1+\varepsilon}{2}\rho(y) < \infty,$$

which implies that $\xi\left(\frac{x+y}{2}\right) < 1$. This completes the proof. □

Lemma 2.2. *Let X_ρ be the modular space generated by a convex modular ρ and $x \in B\left(X_\rho\right)$ be such that $\xi(x) < 1$. If y is any element in $B\left(X_\rho\right)$ satisfying $\left\|\frac{x+y}{2}\right\|_\rho = 1$, then $\rho\left(\frac{x+y}{2}\right) = 1$.*

Proof. By $\xi(x) < 1$ and Lemma 2.1, we have $\xi\left(\frac{x+y}{2}\right) < 1$. Put $I = \left[0, \frac{1}{\xi\left(\frac{x+y}{2}\right)}\right)$ and define a function $f : I \to \mathbb{R}$ by $f(t) = \rho\left(t\frac{x+y}{2}\right)$. Then f is a convex function and has finite values on I, which imply that f is a continuous function on I. Assuming that $\rho\left(\frac{x+y}{2}\right) < 1$, there exists a $\lambda > 1$ such that $\rho\left(\lambda\frac{x+y}{2}\right) < 1$ whence $\left\|\frac{x+y}{2}\right\|_\rho \le \frac{1}{\lambda} < 1$, a contradiction. □

We close this section by giving a basic result on the generalized Calderón–Lozanovskiĭ space as follows:

Lemma 2.3. *For any $x \in E_\varphi$ and any measurable partition $\{T_i\}_{i=1}^n$ of T we have,*

$$\xi(x) = \max_{1\le i\le n}\{\xi(x\chi_{T_i})\}.$$

Proof. Put $\alpha = \max_{1\le i\le n}\{\xi(x\chi_{T_i})\}$, then it is obvious that $\alpha \le \xi(x)$. We now show that the converse inequality holds. If not, then a real number $\beta \in (\alpha, \xi(x))$ can be found and consequently,

$$\varrho_\varphi\left(\frac{x}{\beta}\right) = \left\|\varphi\circ\left(\frac{x}{\beta}\right)\right\|_E = \left\|\sum_{i=1}^n \varphi\circ\left(\frac{x}{\beta}\chi_{T_i}\right)\right\|_E \le \sum_{i=1}^n \left\|\varphi\circ\left(\frac{x}{\beta}\chi_{T_i}\right)\right\|_E = \sum_{i=1}^n \varrho_\varphi\left(\frac{x}{\beta}\chi_{T_i}\right) < \infty,$$

which contradicts the definition of the number $\xi(x)$. □

3. Points of monotonicity in E_φ

In this section, we give some criteria for upper and lower monotonicity points in E_φ.

Theorem 3.1. *A point $x \in S(E_\varphi^+)$ is upper monotone if and only if*

(i) $\varrho_\varphi(x) = 1$;
(ii) $\mu(\{t \in T : x(t) < a(t)\}) = 0$;
(iii) *$\varphi \circ x$ is an upper monotone point of E.*

Proof. *Necessity.* If condition (i) does not hold, then $\varrho_\varphi(x) =: r < 1$. Let D be a subset of A such that $\mu(D) > 0$ and $\chi_D \in E$. Let u be a nonnegative measurable function defined by

$$u(t) = \varphi^{-1}\left(t, \frac{1-r}{\|\chi_D\|_E}\right)\chi_D(t).$$

Then $\varphi \circ u = \frac{1-r}{\|\chi_D\|_E}\chi_D$ which implies $\varphi \circ u \in E$, and moreover,

$$\|\varphi \circ u\|_E = \left\| \frac{(1-r)}{\|\chi_D\|_E}\chi_D \right\|_E = 1 - r.$$

Since $u > 0$, there exist a real number $\lambda > 0$ and a measurable function $y > 0$ with supp $y = D$ satisfying

$$\varphi(t, x(t) + y(t)) \le \varphi(t, x(t)) + \varphi(t, u(t)), \quad y(t) \le \lambda$$

for μ-a.e. $t \in T$. On the other hand, an ascending sequence $(T_n)_{n=1}^{\infty}$ such that $\bigcup_n T_n = T$ and $\sup_{t\in T_n} \varphi(t, u) < \infty$ for each $n \in \mathbb{N}$ and $u \in \mathbb{R}^+$ can be found (see [22]), which allows us to obtain a nonnegative real number d_λ such that

$$d_\lambda = \sup\{\varphi(t, \lambda) : t \in D\}.$$

Consequently, $\varphi \circ y \le d_\lambda \chi_D$ which implies that $y \in E_\varphi$. Moreover,

$$\begin{aligned}\varrho_\varphi(x + y) &= \|\varphi \circ x\chi_{T\setminus D} + \varphi \circ (x + y)\chi_D\|_E \le \|\varphi \circ x\chi_{T\setminus D} + \varphi \circ x\chi_D + \varphi \circ u\|_E \\ &= \|\varphi \circ x + \varphi \circ u\|_E \le \|\varphi \circ x\|_E + \|\varphi \circ u\|_E = r + (1 - r) = 1.\end{aligned}$$

Hence, $1 = \|x\|_\varphi \le \|x + y\|_\varphi \le 1$ and therefore, x is not an upper monotone point.

Suppose that (ii) is not satisfied. Then the set $A = \{t \in T : x(t) < a(t)\}$ has a positive measure. Let us define $y(t) = (a - x)(t)\chi_A(t)$ for all $t \in T$. We see that $y \in E_\varphi^+ \setminus \{0\}$ and

$$\begin{aligned}\varrho_\varphi(x + y) = \|\varphi \circ (x + y)\|_E &= \|\varphi \circ x\chi_{T\setminus A} + \varphi \circ (x + y)\chi_A\|_E \\ &= \|\varphi \circ x\chi_{T\setminus A} + \varphi \circ a\chi_A\|_E \\ &= \|\varphi \circ x\chi_{T\setminus A}\|_E \le \varrho_\varphi(x) \le 1.\end{aligned}$$

Hence, $\|x + y\|_\varphi \le 1$. But, since $y \in E_\varphi^+ \setminus \{0\}$ the fact that $\|x + y\|_\varphi \ge \|x\|_\varphi = 1$ is always true, we obtain $\|x + y\|_\varphi = 1$. This means that x is not an upper monotone point.

It remains to show the necessity of condition (iii). Let us assume that $x \in S(E_\varphi^+)$ is an upper monotone point. Since the necessity of (i) has been proved, we may assume that $\varphi \circ x \in S(E)$ and suppose that condition (iii) is not satisfied, i.e. there exists $y \in E^+ \setminus \{0\}$ such that $\|\varphi \circ x + y\|_E = 1$. Let us define $z \in E_\varphi^+ \setminus \{0\}$ by $z(t) = \varphi^{-1}(t, y(t))$ for all $t \in T$. Hence there exists a nonnegative measurable function h such that supp $h \subset$ supp z and

$$\varphi(t, x(t) + h(t)) \le \varphi(t, x(t)) + \varphi(t, z(t)), \quad h(t) \le \lambda$$

for all $t \in T$. Thus $h \in E_\varphi$ and

$$\varrho_\varphi(x + h) = \|\varphi \circ (x + h)\|_E \le \|\varphi \circ x + \varphi \circ z\|_E = \|\varphi \circ x + y\|_E = 1,$$

which implies that $\|x + h\|_\varphi = 1$. This contradicts the upper monotonicity of x and the proof is completed.

Sufficiency. Let $x \in S(E_\varphi^+)$ and assume that conditions (i)–(iii) are satisfied. Let $y \in E^+ \setminus \{0\}$ be given. In view of Lemma 1.4, condition (ii) gives

$$\varphi(t, x(t) + y(t)) \ge \varphi(t, x(t)) + \varphi(t, a(t) + y(t))$$

for μ-a.e. $t \in T$. Since $\mu(\{t \in T : \varphi(t, a(t) + y(t)) > 0\}) > 0$ and $\varphi \circ x$ is an upper monotone point in E, we have

$$\varrho_\varphi(x + y) = \|\varphi \circ (x + y)\|_E \ge \|\varphi \circ x + \varphi \circ (a + y)\|_E > \|\varphi \circ x\|_E = \varrho_\varphi(x) = 1,$$

that is, $\|x + y\|_\varphi > 1$. This completes the proof. □

Theorem 3.2. *A point $x \in S(E_\varphi^+)$ is a lower monotone point if and only if*

(i) $\xi(x) < 1$;

(ii) $\mu(\{t \in \operatorname{supp} x : x(t) \leq a(t)\}) = 0$;
(iii) *$\varphi \circ x$ is a lower monotone point of E.*

Proof. *Necessity.* Let $x \in S(E^+)$ be a lower monotone point. Suppose that condition (i) is not satisfied, i.e. $\xi(x) = 1$. Take $A, B \in \Sigma$, both of positive measure, such that $A \cap B = \emptyset$ and $A \cup B = \operatorname{supp} x$. Thus by Lemma 2.3 we obtain $\xi(x\chi_A) = 1$ or $\xi(x\chi_B) = 1$. Without loss of generality we may assume that $\xi(x\chi_A) = 1$, and it would be $\xi(x - x\chi_B) = \xi(x\chi_A) = 1$. This implies $\|x - x\chi_B\|_\varphi \geq 1$, a contradiction.

If condition (ii) does not hold, then the set $A = \{t \in \operatorname{supp} x : x(t) \leq a(t)\}$ has positive measure. By (i), the necessity of which has been already proved, we have $\xi(x) < 1$, and consequently $\varrho_\varphi(x) = 1$ by Lemma 2.2. Define $y(t) = x(t)\chi_A(t)$, then we have $0 < y < x$, and

$$\varrho_\varphi(x - y) = \|\varphi \circ x\chi_{T \setminus A}\|_E = \|\varphi \circ x\|_E = \varrho_\varphi(x) = 1.$$

This implies that $\|x - y\|_\varphi = 1$, a contradiction.

Now we will show that condition (iii) holds. By (i), we have $\varphi \circ x \in S(E)$. Let us take $y \in E$ such that $0 < y < \varphi \circ x$ and choose a measurable function z such that $0 < z < x$ with $\varphi \circ x - y \leq \varphi \circ (x - z)$. Since x is a lower monotone point, we have

$$\|\varphi \circ x - y\|_E \leq \|\varphi \circ (x - z)\|_E = \varrho_\varphi(x - z) \leq \|x - z\|_\varphi < 1.$$

This shows that $\varphi \circ x$ is then a lower monotone point of E.

Sufficiency. Let $x \in S(E_\varphi^+)$, $y \in E^+ \setminus \{0\}$ be such that $y < x$ and conditions (i)–(iii) are satisfied. Obviously, supp $y \subset \operatorname{supp} x$ which together with condition (ii) imply that for $z = \varphi \circ x - \varphi \circ (x - y)$ we have $z > 0$. Moreover, by condition (i), we have $\varrho_\varphi(x) = 1$. Since $\varphi \circ x$ is a lower monotone point of E and $z \leq \varphi \circ x$, so

$$\varrho_\varphi(x - y) = \|\varphi \circ (x - y)\|_E = \|\varphi \circ x - z\|_E < \|\varphi \circ x\|_E = \varrho_\varphi(x) = 1. \tag{3.1}$$

Using Eq. (3.1) together with $\xi(x - y) < 1$ (by condition (i)) and the continuity of ϱ_φ, in light of Lemma 2.2, we have $\|x - y\|_\varphi < 1$. This completes the proof. □

4. Points of rotundity in E_φ

We will study the points of rotundity, such as extreme point and SU-point in this Section. We begin with the following definition:

A point $x \in S(E^+)$ is said to be an extreme point of $B(E^+)$ ($x \in \operatorname{ext} B(E^+)$ for short) if for any $x, y \in S(E^+)$ such that $x = (y + z)/2$, we have $y = z = x$.

Lemma 4.1 (*[17, Lemma 4]*). *In any Köthe space E, $x \in S(E)$ is an extreme point of $B(E)$ if and only if $|x|$ is a UM-point of E and $|x| \in \operatorname{ext} B(E^+)$.*

Theorem 4.2. *A point $x \in S(E_\varphi)$ is an extreme point of $B(E_\varphi)$ if and only if*

(i) $\varrho_\varphi(x) = 1$;
(ii) $\mu(\{t \in T : |x(t)| < a(t)\}) = 0$;
(iii) *$\varphi \circ |x|$ is a UM-point;*
(iv) *if $u, v \in S(E)$ satisfy $\frac{u+v}{2} = \varphi \circ |x|$ then either*

$$u = v \quad \text{or} \quad \varphi \circ \left(\frac{y + z}{2}\right) < \frac{1}{2}(\varphi \circ y + \varphi \circ z),$$

where $y(t) = \varphi^{-1}(t, |u(t)|)$, $z(t) = \varphi^{-1}(t, |v(t)|)$ for all $t \in T$.

Proof. *Sufficiency.* Assume that conditions (i)–(iv) are satisfied. Let $x \in S(E_\varphi)$ and $y, z \in B(E_\varphi)$ be such that $2x = y + z$. We shall show that $y = z$. First, we will show that

$$\varphi \circ |x|(t) = \varphi \circ \frac{|y + z|}{2}(t) = \varphi \circ \left[\frac{|y| + |z|}{2}\right](t) = \frac{1}{2}\left[\varphi \circ |y|(t) + \varphi \circ |z|(t)\right] \tag{4.1}$$

for μ-a.e. $t \in T$. Note that, we always have

$$\varphi \circ |x|(t) = \varphi \circ \frac{|y+z|}{2}(t) \le \varphi \circ \left[\frac{|y|+|z|}{2}\right](t) \le \frac{1}{2}[\varphi \circ |y|(t) + \varphi \circ |z|(t)]$$

for μ-a.e. $t \in T$. Let $A = \{t \in T : \varphi \circ |x|(t) < \frac{1}{2}[\varphi \circ |y|(t) + \varphi \circ |z|(t)]\}$. If $\mu(A) > 0$ then by conditions (i) and (iii) we have

$$1 = \varrho_\varphi(x) = \|\varphi \circ |x|\|_E < \left\|\frac{1}{2}\varphi \circ |y| + \frac{1}{2}\varphi \circ |z|\right\|_E \le \frac{1}{2}(\|\varphi \circ |y|\|_E + \|\varphi \circ |z|\|_E) \le 1,$$

which is a contradiction. Consequently, Eq. (4.1) holds.

Let $C_\varphi = \{t \in T : \varphi(t, \cdot) \text{ is a convex and even function}\}$. It is clear that $\mu(T \setminus C_\varphi) = 0$. Next for each $t \in T$ we define $\hat{y}(t) = \varphi^{-1}(t, \varphi(t, |y(t)|))$ and $\hat{z}(t) = \varphi^{-1}(t, (\varphi(t, |z(t)|)))$. Using condition (ii) together with Eq. (4.1), in light of Remark 1.2(ii), we have $\hat{y}(t) = |y(t)|$ and $\hat{z}(t) = |z(t)|$ for μ-a.e. $t \in C_\varphi$. Consequently, by Eq. (4.1) and condition (iv) we conclude that $\varphi \circ |y|(t) = \varphi \circ |z|(t)$ for μ-a.e. $t \in C_\varphi$. We claim that $|y| = |z|$. Put $B = \{t \in C_\varphi : |y|(t) \ne |z|(t)\}$ and suppose that $\mu(B) > 0$. Thus, since $\varphi(t, \cdot)$ is an injective function on the set $[a(t), \infty)$ for all $t \in C_\varphi$ we should have

$$|y(t)| \vee |z(t)| \le a(t) \quad \text{and} \quad |y(t)| \wedge |z(t)| < a(t) \tag{4.2}$$

for all $t \in B \subset C_\varphi$. So

$$\varphi \circ |x|(t) = \frac{1}{2}[\varphi \circ |y|(t) + \varphi \circ |z|(t)] = 0$$

for all $t \in B$. Combining this equation with Eq. (4.2) and the assumption that $2x = y + z$ we obtain $|x(t)| < |a(t)|$ for all $t \in B$, which contradicts condition (ii). Hence, we have the claim. Finally, by condition (ii) and the fact that $\varphi(t, \cdot)$ is an injective function on $[a(t), \infty)$ for all $t \in C_\varphi$, in view of Eq. (4.1), we obtain that $|y(t) + z(t)| = |y(t)| + |z(t)|$ for μ-a.e. $t \in T$. This together with $|y(t)| = |z(t)|$ for μ-a.e. $t \in T$ implies that $y = z$.

Necessity. Let $x \in S(E_\varphi)$ be an extreme point of $B(E_\varphi)$. By, Lemma 4.1 we obtain that $|x|$ is a UM-point in E_φ. Thus by Theorem 3.1 we have $x(t) \ge a(t)$ for μ-a.e. $t \in T$, $\varrho_\varphi(x) = 1$ and $\varphi \circ x$ is an upper monotone point of E. Therefore, it remains only to prove that if $x \in \text{ext } B(E_\varphi)$ then condition (iv) holds. If not, there are $u, v \in S(E)$ such that

$$u(t) \ne v(t) \quad \text{and} \quad \varphi \circ \left[\frac{y+z}{2}\right](t) = \frac{1}{2}[\varphi \circ y(t) + \varphi \circ z(t)] = \frac{u(t)+v(t)}{2} = \varphi \circ |x|(t),$$

for μ-a.e. $t \in T$, where $y(t), z(t)$ are defined in condition (iv). Clearly, $y, z \in S(E_\varphi)$ with $y \ne z$. Consequently, $|x| \notin \text{ext } B(E_\varphi^+)$. Finally, Lemma 4.1 yields that $x \notin \text{ext } B(E_\varphi)$. □

Recall that a point $x \in S(E^+)$ is called a *strong U-point*(an SU-point for short) of $B(E^+)$ if for any $y \in S(E^+)$ with $\|x + y\|_E = 2$, we have $x = y$.

Remark 4.3 (*[17, page 387]*). If a point $x \in S(E^+)$ is an SU-point of $B(E^+)$, then x is a LM-point of E and x is an UM-point of E.

Lemma 4.4 (*[17, Lemma 7]*). *A point $x \in S(E)$ is an SU-point of $B(E)$ if and only if $|x|$ is an SU-point of $B(E^+)$.*

Theorem 4.5. *Let E be a strictly monotone Köthe space and $x \in S(E_\varphi)$. Then x is an SU-point of $B(E_\varphi)$ if and only if*

(i) $\xi(x) < 1$;

(ii) $\mu(\{t \in \text{supp}\, x : |x|(t) \le a(t)\}) = 0$;

(iii) *if $u \in S(E^+)$ satisfies $\|u + \varphi \circ |x|\|_E = 2$ then either*

$$u = \varphi \circ |x| \quad \text{or} \quad \varphi \circ \left(\frac{|x|+y}{2}\right) < \frac{1}{2}(\varphi \circ |x| + \varphi \circ y),$$

where $y(t) = \varphi^{-1}(t, u(t))$ for all $t \in T$.

Proof. *Necessity*. Assume that x is an SU-point of $B(E_\varphi)$. Applying Lemma 4.4, Remark 4.3 and Theorem 3.2 we see that the remainder is condition (iii). Suppose the converse, that is, there are $u \in S(E^+)$ such that $\|u + \varphi \circ |x|\|_E = 2$, $u \neq \varphi \circ |x|$ and $\varphi \circ \left(\frac{|x|+y}{2}\right) = \frac{1}{2}[\varphi \circ |x| + \varphi \circ y]$, where $y(t)$ is defined as in condition (iii). Then,

$$\varrho_\varphi(y) = \|\varphi \circ y\|_E = \|u\|_E = 1,$$

and consequently,

$$\begin{aligned} 2 = \|u + \varphi \circ |x|\|_E &= \|\varphi \circ y + \varphi \circ |x|\|_E \\ &\leq \|\varphi \circ y\|_E + \|\varphi \circ |x|\|_E \\ &\leq \varrho_\varphi(y) + \varrho_\varphi(x) \leq 2. \end{aligned}$$

This implies that

$$\begin{aligned} \varrho_\varphi\left(\frac{|x|+y}{2}\right) &= \left\|\varphi \circ \left(\frac{x+y}{2}\right)\right\|_E \\ &= \frac{1}{2}\left[\|\varphi \circ |x| + \varphi \circ y\|_E\right] \\ &= \frac{1}{2}\left[\|\varphi \circ |x|\|_E + \|\varphi \circ y\|_E\right] \\ &= \frac{1}{2}[\varrho_\varphi(|x|) + \varrho_\varphi(y)] = 1, \end{aligned}$$

so $\left\|\frac{|x|+y}{2}\right\|_\varphi = 1$. Since $u \neq \varphi \circ |x|$, we have $|x| \neq y$, which implies that $|x|$ is not an SU-point of $B(E_\varphi^+)$. Thus Lemma 4.4 finishes the proof of the necessity.

Sufficiency. Let $y \in S(E_\varphi)$ be such that

$$\left\|\frac{x+y}{2}\right\|_\varphi = 1. \tag{4.3}$$

We shall show that $x = y$. Combining Eq. (4.3) with condition (i), and applying Lemma 2.2, we get $\varrho_\varphi\left(\frac{x+y}{2}\right) = 1$. This gives

$$\begin{aligned} 1 = \varrho_\varphi\left(\frac{x+y}{2}\right) &= \left\|\varphi \circ \left(\frac{x+y}{2}\right)\right\|_E \\ &\leq \frac{1}{2}\|\varphi \circ x + \varphi \circ y\|_E \\ &\leq \frac{1}{2}\left[\varrho_\varphi(x) + \varrho_\varphi(y)\right] \\ &\leq 1, \end{aligned} \tag{4.4}$$

whence

$$\|\varphi \circ x + \varphi \circ y\|_E = 2. \tag{4.5}$$

Using this equation together with the strict monotonicity of E, the fact $\varrho_\varphi\left(\frac{x+y}{2}\right) = 1$ and the convexity of $\varphi(t, \cdot)$ on $\mathbb{R}$ for all $t \in C_\varphi$, where C_φ defined as in Theorem 4.2 it is easy to see that

$$\varphi \circ \left(\frac{|x|+|y|}{2}\right)(t) = \frac{\varphi \circ |x|(t) + \varphi \circ |y|(t)}{2} \tag{4.6}$$

for μ-a.e. $t \in C_\varphi$. Put $u(t) = \varphi \circ |y|(t)$ for all $t \in T$. Then $u \in E^+$ and $\|u\|_E = \|\varphi \circ y\|_E = \varrho_\varphi(y) = 1$ by Eq. (4.4). Moreover, by virtue of condition (iii), Eqs. (4.5) and (4.6) imply that $\varphi \circ |x|(t) = \varphi \circ |y|(t)$ for μ-a.e. $t \in C_\varphi$. Since $\mu(\{t \in \operatorname{supp} x : |x|(t) \leq a(t)\}) = 0$ and $\varphi(t, \cdot)$ is an injective function on the interval $[a(t), \infty)$ for μ-a.e. $t \in C_\varphi$ we get $|x|(t) = |y|(t)$ for μ-a.e. $t \in T$. Then $|x + y| \leq |x| + |y| = 2|x|$. If $|x + y| < |x| + |y| = 2|x|$

then $\|(x+y)/2\|_\varphi < 1$ (since $|x|$ is an *LM*-point of E_φ by Theorem 3.2). This contradicts Eq. (4.3) and proves that $|x-y| = |x| + |y|$. Combining this equality with $|x| = |y|$, we get $x = y$. □

5. Rotundity of E_φ

In this final section we present a result concerning the rotundity structure of E_φ.

Theorem 5.1. *Let E be a Köthe space and φ be a Musielak–Orlicz function. Then $E_\varphi \in (R)$ if and only if*

(i) *$E \in (SM)$;*
(ii) *$\varphi \in \Delta_2^E$;*
(iii) *if $u, v \in S(E^+)$ with $u \neq v$ then either*

$$\left\|\frac{u+v}{2}\right\|_E < 1 \quad or \quad \varphi \circ \left(\frac{x+y}{2}\right) < \frac{1}{2}(\varphi \circ x + \varphi \circ y),$$

where $x(t) = \varphi^{-1}(t, u(t))$ and $y(t) = \varphi^{-1}(t, v(t))$ for all $t \in T$.

Proof. *Necessity.* Suppose on the contrary that $E_\varphi \in (R)$ and $E \notin (SM)$. Then an element $u \in S(E^+)$ which is not a *UM*-point can be found. Put $x(t) = \varphi^{-1}(t, u(t))$. Then $\varrho_\varphi(x) = \|\varphi \circ x\|_E = \|u\|_E = 1$, so $x \in S(E_\varphi)$ and hence $x \in \text{ext } B(E_\varphi)$. However, $\varphi \circ x$ is not a *UM*-point in E, thus Theorem 4.2 yields a contradiction.

Suppose that $E_\varphi \in (R)$ and $\varphi \notin \Delta_2^E$. By Lemma 1.3, there exists $x \in S(E_\varphi)$ with $\varrho_\varphi(x) < 1$. By $E_\varphi \in (R)$, $x \in \text{ext } B(E_\varphi)$ and Theorem 4.2 yields a contradiction.

Suppose that condition (iii) is not satisfied. Then there are $u, v \in S(E^+)$ with $u \neq v$ such that $\|u+v\|_E = 2$ and $\varphi \circ \left(\frac{x+y}{2}\right) = \frac{1}{2}(\varphi \circ x + \varphi \circ y) = \frac{u+v}{2}$, where $x(t), y(t)$ are defined in condition (iii). Putting $z = \frac{x+y}{2}$, we have $\varrho_\varphi(z) = 1$, thus $z \in \text{ext } B(E_\varphi)$. Since $x \in \text{ext } B(E_\varphi)$, Theorem 4.2 yields a contradiction.

Sufficiency. Let $x \in S(E_\varphi)$ be arbitrary. We shall show that $x \in \text{ext } B(E_\varphi)$, by proving that conditions (i)–(iv) in Theorem 4.2 are satisfied. First, by $\varphi \in \Delta_2^E$ we have $\varrho_\varphi(x) = 1$ and $|x(t)| \geq a(t)$ for μ-a.e. $t \in T$ by Lemmas 1.3 and 1.5, respectively. Next, $\varphi \circ |x|$ is a *UM*-point in E, because $E \in (SM)$. Finally, we will show that condition (iv) of Theorem 4.2 holds. Let $u, v \in S(E)$ be such that $\frac{u+v}{2} = \varphi \circ |x|$. By condition (iii) in our assumptions, we get $\varphi \circ \left(\frac{y+z}{2}\right) < \frac{1}{2}(\varphi \circ y + \varphi \circ z)$, where $\varphi \circ y = u$ and $\varphi \circ z = v$, which means that condition (iv) from Theorem 4.2 is satisfied. Hence, our theorem is proved. □

Note that, if $E = L^1$ then $E_\varphi = \{x \in L^0 : \int_T \varphi(t, \lambda x(t)) \mathrm{d}\mu < \infty \text{ for some } \lambda > 0\} =: L^\varphi$, which is called the Musielak–Orlicz space. Therefore, a direct consequence of Theorem 5.1, we have the following result.

Corollary 5.2. *Let φ be a Musielak–Orlicz function and L^φ be the Musielak–Orlicz space generated by φ. Then $L^\varphi \in (R)$ if and only if*

(i) *$\varphi \in \Delta_2^{L_1}$;*
(ii) *if $u, v \in S(L_1^+)$ with $u \neq v$ then*

$$\varphi \circ \left(\frac{x+y}{2}\right) < \frac{1}{2}(\varphi \circ x + \varphi \circ y),$$

where $x(t) = \varphi^{-1}(t, u(t))$ and $y(t) = \varphi^{-1}(t, v(t))$ for all $t \in T$.

Proof. Since $L^1 \in (SM)$ and for any $u, v \in S(L_1^+)$ we must have $\|\frac{u+v}{2}\|_{L_1} = 1$, thus, the conclusion of Corollary 5.2 follows exactly from Theorem 5.1. This completes the proof. □

Remark 5.3. Rotundity properties of Musielak–Orlicz space, L^φ, equipped with the Luxemburg norm were given by Hudzik [12], in terms of the strict convexity of Musielak–Orlicz function φ. Since condition (ii) in Corollary 5.2 means that $\varphi(t, \cdot)$ is a strictly convex Musielak–Orlicz function for μ-a.e. $t \in T$, therefore, Corollary 5.2 gives a result from [12].

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Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings

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ABSTRACT

In this paper, we introduce a new mapping and a Hybrid iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a Hilbert space. Then, we prove the strong convergence of the proposed iterative algorithm to a common fixed point of a finite family of nonexpansive mappings which is a solution of the generalized equilibrium problem. The results obtained in this paper extend the recent ones of Takahashi and Takahashi [S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025–1033].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H and $A : C \to H$ be a nonlinear mapping and let P_C be the projection of H onto the convex subset C. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}. \tag{1.1}$$

Many problems in physics, optimization, and economics require some elements of $EP(F)$, see [2–7]. Several iterative methods have been proposed to solve the equilibrium problem, see for instance [3,5–7]. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$.

For a bifunction $F : C \times C \to \mathbb{R}$ and a nonlinear mapping $A : C \to H$, we consider the following equilibrium problem:

$$\text{Find } z \in C \text{ such that } F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \ \forall y \in C\}.$$

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In the case of $A \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is also denoted by $VI(C, A)$. Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, economics reduce to finding a solution of (1.3) see, for instance, [2,4].

A mapping A of C into H is called *α-inverse strongly* monotone, see [8], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

For $r > 0$, let $T_r : H \to C$ be defined by

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}. \tag{1.4}$$

Combettes and Hirstoaga [9] showed that under some suitable conditions of F, T_r is single-valued and firmly nonexpansive and satisfies $F(T_r) = EP(F)$.

In 2007, Takahashi and Takahashi [6] introduced a hybrid viscosity approximation method in the framework of a real Hilbert space H. They defined the iterative sequences $\{x_n\}$ and $\{u_n\}$ as follows:

$$\begin{cases} x_1 \in H, \text{ arbitrarily;} \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $f : H \longrightarrow H$ is a contraction mapping with a constant $\alpha \in (0, 1)$ and $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$. They proved, under some suitable conditions on the sequence $\{\alpha_n\}$, $\{r_n\}$ and bifunction F, that $\{x_n\}$ and $\{u_n\}$ strongly converge to $z \in F(T) \bigcap EP(F)$, where $z = P_{F(T)\bigcap EP(F)} f(z)$.

Recently, in 2008, Takahashi and Takahashi [7] introduced a hybrid iterative method for finding a common element of EP and $F(T)$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} u, x_1 \in C, \text{ arbitrarily;} \\ F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T(a_n u + (1 - a_n)z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where A be an α-inverse strongly monotone mapping of C into H with positive real number α, and $\{a_n\} \in [0, 1]$, $\{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset [0, 2\alpha]$, and proved strong convergence of the scheme (1.6) to $z \in \bigcap_{i=1}^N F(T_i) \bigcap EP$, where $z = P_{\bigcap \ldots}$ in the framework of a Hilbert space, under some suitable conditions on $\{a_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and bifunction F.

In 1999, Atsushiba and Takahashi [10] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I, \\ U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I, \\ &\ \ \vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_N - 1U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I, \end{aligned} \tag{1.7}$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the *W-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [11] proved that if X is a strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1$, $i = 1, 2, \ldots, N$.

Let X be a real Hilbert space and C a nonempty closed convex subset of X and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define mapping $S_n : C \to C$ as follows:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \alpha_1^{n,1}T_1U_{n,0} + \alpha_2^{n,1}U_{n,0} + \alpha_3^{n,1}I \\ U_{n,2} &= \alpha_1^{n,2}T_2U_{n,1} + \alpha_2^{n,2}U_{n,1} + \alpha_3^{n,2}I \\ U_{n,3} &= \alpha_1^{n,3}T_3U_{n,2} + \alpha_2^{n,3}U_{n,2} + \alpha_3^{n,3}I \\ &\ \ \vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1}T_{N-1}U_{n,N-2} + \alpha_2^{n,N-1}U_{n,N-2} + \alpha_3^{n,N-1}I \\ S_n &= U_{n,N} = \alpha_1^{n,N}T_NU_{n,N-1} + \alpha_2^{n,N}U_{n,N-1} + \alpha_3^{n,N}I. \end{aligned}$$

The mapping S_n is called the *S-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$.
For given $u \in C$ and $x_1 \in C$, let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n(a_n u + (1 - a_n)z_n). & \forall n \in \mathbb{N}. \end{cases} \tag{1.8}$$

In this paper, we show that if X is strictly convex, then $F(S_n) = \bigcap_{i=1}^N F(T_i)$ if $\alpha_1^{n,j} \in (0, 1)$ for all $j = 1, 2, \ldots, N - 1$, $\alpha_1^{n,N} \in (0, 1]$ and $\alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1)$ for all $j = 1, 2, \ldots, N$, and we prove that under some suitable conditions, the sequence $\{x_n\}$ converges strongly to a point $z = P_{\bigcap_{i=1}^N F(T_i) \bigcap EP} u$.

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.
Let C be the closed convex subset of a real Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1 (See [12]). *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality $\langle x - y, y - z\rangle \geq 0$ $\forall z \in C$.*

Lemma 2.2 (See [11]). *In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$$

for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Lemma 2.3 (See [13]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying $s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n$, $\forall n \geq 0$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions*

$$(1)\ \{\alpha_n\} \subset [0, 1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad (2)\ \limsup_{n\to\infty} \beta_n \leq 0.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 (See [14]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ for all integer $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x, x) = 0$ $\forall x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;
(A3) $\forall x, y, z \in C$,

$$\lim_{t\to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) $\forall x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.
The following lemma appears implicitly in [2].

Lemma 2.5 (See [2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \tag{2.1}$$

for all $y \in C$.

Lemma 2.6 (See [9]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\} \tag{2.2}$$

for all $z \in H$. Then, the following hold:

(1) T_r *is single-valued;*
(2) T_r *is firmly nonexpansive i.e.*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$;
(4) $EP(F)$ *is closed and convex.*

Definition 2.7. Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\
&\;\;\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\
S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned} \tag{2.3}$$

This mapping is called *S-mapping* generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Next, we prove a lemma which is very useful for our consideration.

Lemma 2.8. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j), j = 1, 2, 3, \ldots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0, 1]$, $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \ldots, N$. Let S be the mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Proof. It is clear that $\bigcap_{i=1}^N F(T_i) \subseteq F(S)$. Next, we show that $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. To show this, let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. Then we have

$$\begin{aligned}
\|x_0 - x^*\| &= \|Sx_0 - x^*\| = \|\alpha_1^N (T_N U_{N-1} x_0 - x^*) + \alpha_2^N (U_{N-1} x_0 - x^*) + \alpha_3^N (x_0 - x^*)\| \\
&\leq \alpha_1^N \|T_N U_{N-1} x_0 - x^*\| + \alpha_2^N \|U_{N-1} x_0 - x^*\| + \alpha_3^N \|x_0 - x^*\| \\
&\leq (1 - \alpha_3^N)\|U_{N-1} x_0 - x^*\| + (1 - (1 - \alpha_3^N))\|x_0 - x^*\| \\
&= (1 - \alpha_3^N)\|\alpha_1^{N-1}(T_{N-1} U_{N-2} x_0 - x^*) + \alpha_2^{N-1}(U_{N-2} x_0 - x^*) + \alpha_3^{N-1}(x_0 - x^*)\| \\
&\quad + (1 - (1 - \alpha_3^N))\|x_0 - x^*\| && (2.4) \\
&\leq (1 - \alpha_3^N)(\alpha_1^{N-1}\|T_{N-1} U_{N-2} x_0 - x^*\| + \alpha_2^{N-1}\|U_{N-2} x_0 - x^*\| + \alpha_3^{N-1}\|x_0 - x^*\|) \\
&\quad + (1 - (1 - \alpha_3^N))\|x_0 - x^*\| \\
&\leq \prod_{j=N-1}^{N} (1 - \alpha_3^j)\|U_{N-2} x_0 - x^*\| + \left(1 - \prod_{j=N-1}^{N} (1 - \alpha_3^j)\right)\|x_0 - x^*\| && (2.5) \\
&= \prod_{j=N-1}^{N} (1 - \alpha_3^j)\|\alpha_1^{N-2}(T_{N-2} U_{N-3} x_0 - x^*) + \alpha_2^{N-2}(U_{N-3} x_0 - x^*) + \alpha_3^{N-2}(x_0 - x^*)\| \\
&\quad + \left(1 - \prod_{j=N-1}^{N} (1 - \alpha_3^j)\right)\|x_0 - x^*\| && (2.6) \\
&\leq \prod_{j=N-1}^{N} (1 - \alpha_3^j)(\alpha_1^{N-2}\|T_{N-2} U_{N-3} x_0 - x^*\| + \alpha_2^{N-2}\|U_{N-3} x_0 - x^*\| + \alpha_3^{N-2}\|x_0 - x^*\|) \\
&\quad + \left(1 - \prod_{j=N-1}^{N} (1 - \alpha_3^j)\right)\|x_0 - x^*\|
\end{aligned}$$

$$\le \prod_{j=N-2}^{N}(1-\alpha_3^j)\|U_{N-3}x_0 - x^*\| + \left(1 - \prod_{j=N-2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|$$
$$\le$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\le \prod_{j=3}^{N}(1-\alpha_3^j)\|\alpha_1^2(T_2U_1x_0 - x^*) + \alpha_2^2(U_1x_0 - x^*) + \alpha_3^2(x_0 - x^*)\|$$
$$+ \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\| \tag{2.7}$$
$$\le \prod_{j=3}^{N}(1-\alpha_3^j)(\alpha_1^2\|T_2U_1x_0 - x^*\| + \alpha_2^2\|U_1x_0 - x^*\| + \alpha_3^2\|x_0 - x^*\|)$$
$$+ \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|$$
$$\le \prod_{j=2}^{N}(1-\alpha_3^j)\|U_1x_0 - x^*\| + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\| \tag{2.8}$$
$$= \prod_{j=2}^{N}(1-\alpha_3^j)\|\alpha_1^1(T_1x_0 - x^*) + (1-\alpha_1^1)(x_0 - x^*)\| + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\| \tag{2.9}$$
$$\le \prod_{j=2}^{N}(1-\alpha_3^j)(\alpha_1^1\|T_1x_0 - x^*\| + (1-\alpha_1^1)\|x_0 - x^*\|)$$
$$+ \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\| \tag{2.10}$$
$$\le \prod_{j=2}^{N}(1-\alpha_3^j)\|x_0 - x^*\| + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|$$
$$= \|x_0 - x^*\|.$$

This implies by (2.9) that

$$\|x_0 - x^*\| = \prod_{j=2}^{N}(1-\alpha_3^j)\|\alpha_1^1(T_1x_0 - x^*) + (1-\alpha_1^1)(x_0 - x^*)\| + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|,$$

hence

$$\|x_0 - x^*\| = \|\alpha_1^1(T_1x_0 - x^*) + (1-\alpha_1^1)(x_0 - x^*)\|. \tag{2.11}$$

By (2.10), we obtain

$$\|x_0 - x^*\| = \prod_{j=2}^{N}(1-\alpha_3^j)[\alpha_1^1\|T_1x_0 - x^*\| + (1-\alpha_1^1)\|x_0 - x^*\|] + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|,$$

which implies

$$\|x_0 - x^*\| = \alpha_1^1\|T_1x_0 - x^*\| + (1-\alpha_1^1)\|x_0 - x^*\|.$$

It follows that

$$\|x_0 - x^*\| = \|T_1x_0 - x^*\|. \tag{2.12}$$

From (2.11) and (2.12), we have by Lemma 2.2 that $T_1x_0 = x_0$, that is $x_0 \in F(T_1)$.

It implies that

$$U_1x_0 = \lambda_1T_1x_0 + (1-\lambda_1)x_0 = x_0.$$

By (2.7), we have

$$\|x_0 - x^*\| = \prod_{j=3}^{N}(1-\alpha_3^j)\|\alpha_1^2(T_2U_1x_0 - x^*) + \alpha_2^2(U_1x_0 - x^*) + \alpha_3^2(x_0 - x^*)\| + \left[1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right]\|x_0 - x^*\|.$$

It follows that

$$\begin{aligned}\|x_0 - x^*\| &= \|\alpha_1^2(T_2U_1x_0 - x^*) + \alpha_2^2(U_1x_0 - x^*) + \alpha_3^2(x_0 - x^*)\| \\ &= \|\alpha_1^2(T_2x_0 - x^*) + (1-\alpha_1^2)(x_0 - x^*)\|.\end{aligned} \tag{2.13}$$

By (2.8), we have

$$\|x_0 - x^*\| = \prod_{j=3}^{N}(1-\alpha_3^j)(\alpha_1^2\|T_2U_1x_0 - x^*\| + \alpha_2^2\|U_1x_0 - x^*\| + \alpha_3^2\|x_0 - x^*\|) + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|,$$

which implies

$$\begin{aligned}\|x_0 - x^*\| &= \alpha_1^2\|T_2U_1x_0 - x^*\| + \alpha_2^2\|U_1x_0 - x^*\| + \alpha_3^2\|x_0 - x^*\| \\ &= \alpha_1^2\|T_2x_0 - x^*\| + (1-\alpha_1^2)\|x_0 - x^*\|.\end{aligned}$$

Hence, we obtain

$$\|x_0 - x^*\| = \|T_2x_0 - x^*\|. \tag{2.14}$$

From (2.13) and (2.14), we have by Lemma 2.2 that $T_2x_0 = x_0$, that is $x_0 \in F(T_2)$.

This implies that $U_2x_0 = \alpha_1^2T_2U_1x_0 + \alpha_2^2U_1x_0 + \alpha_3^2x_0 = x_0$.

By continuing in this way, we can show that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1, 2, \dots, N-1$.

Finally, we shall show that $x_0 \in F(T_N)$.

Since

$$\begin{aligned}0 = Sx_0 - x_0 &= \alpha_1^N T_N U_{N-1}x_0 + \alpha_2^N U_{N-1}x_0 + \alpha_3^N x_0 - x_0 \\ &= \alpha_1^N(T_Nx_0 - x_0),\end{aligned}$$

and $\alpha_1^N \in (0, 1]$, we obtain $T_Nx_0 = x_0$ so that $x_0 \in F(T_N)$. Hence $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. □

Lemma 2.9. *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and for each $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Suppose $\alpha_i^{n,j} \to \alpha_i^j$ as $n \to \infty$ for $i = 1, 3$ and $j = 1, 2, 3, \dots, N$. Let S and S_n be the S-mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ and $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. Then $\lim_{n\to\infty}\|S_nx - Sx\| = 0$ for every $x \in C$.*

Proof. Let $x \in C$, U_k and $U_{n,k}$ be generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ and $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. For each $n \in \mathbb{N}$ and for $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}\|U_{n,1}x - U_1x\| &= \|\alpha_1^{n,1}T_1x + (1-\alpha_1^{n,1})x - \alpha_1^1T_1x - (1-\alpha_1^1)x\| \\ &= |\alpha_1^{n,1} - \alpha_1^1|\,\|T_1x - x\|,\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\|U_{n,k}x - U_kx\| &= \|\alpha_1^{n,k}T_kU_{n,k-1}x + \alpha_2^{n,k}U_{n,k-1}x + \alpha_3^{n,k}x - \alpha_1^kT_kU_{k-1}x - \alpha_2^kU_{k-1}x - \alpha_3^kx\| \\ &= \|\alpha_1^{n,k}(T_kU_{n,k-1}x - T_kU_{k-1}x) + (\alpha_1^{n,k} - \alpha_1^k)T_kU_{k-1}x \\ &\quad + (\alpha_3^{n,k} - \alpha_3^k)x + \alpha_2^{n,k}(U_{n,k-1}x - U_{k-1}x) + (\alpha_2^{n,k} - \alpha_2^k)U_{k-1}x\| \\ &\le \alpha_1^{n,k}\|T_kU_{n,k-1}x - T_kU_{k-1}x\| + |\alpha_1^{n,k} - \alpha_1^k|\,\|T_kU_{k-1}x\| \\ &\quad + |\alpha_3^{n,k} - \alpha_3^k|\,\|x\| + \alpha_2^{n,k}\|U_{n,k-1}x - U_{k-1}x\| + |\alpha_2^{n,k} - \alpha_2^k|\,\|U_{k-1}x\| \\ &\le \alpha_1^{n,k}\|U_{n,k-1}x - U_{k-1}x\| + |\alpha_1^{n,k} - \alpha_1^k|\,\|T_kU_{k-1}x\| \\ &\quad + \alpha_2^{n,k}\|U_{n,k-1}x - U_{k-1}x\| + (|\alpha_1^k - \alpha_1^{n,k}| + |\alpha_3^{n,k} - \alpha_3^k|)\|U_{k-1}x\| + |\alpha_3^{n,k} - \alpha_3^k|\,\|x\| \\ &\le \|U_{n,k-1}x - U_{k-1}x\| + |\alpha_1^{n,k} - \alpha_1^k|(\|T_kU_{k-1}x\| + \|U_{k-1}x\|) \\ &\quad + |\alpha_3^{n,k} - \alpha_3^k|(\|U_{k-1}x\| + \|x\|).\end{aligned} \tag{2.16}$$

By (2.15) and (2.16), we have

$$\|S_nx - Sx\| = \|U_{n,N}x - U_Nx\|$$
$$\leq |\alpha_1^{n,1} - \alpha_1^1|\|T_1x - x\| + \sum_{j=2}^{N} |\alpha_1^{n,j} - \alpha_1^j|(\|T_jU_{j-1}x\| + \|U_{N-j}x\|) + \sum_{j=2}^{N} |\alpha_3^{n,j} - \alpha_3^j|(\|U_{j-1}x\| + \|x\|).$$

This together with our assumption, we can conclude that

$$\lim_{n\longrightarrow\infty} \|S_nx - Sx\| = 0. \quad \square$$

3. Main result

In this section, we prove a strong convergence theorem of the iterative scheme (3.1) to a common element of EP and $\bigcap_{i=1}^N F(T_i)$ under some control conditions.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let A be an α-inverse strongly monotone mapping of C into H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \bigcap EP \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N$, $\{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_nx_n + (1 - \beta_n)S_n(a_nu + (1 - a_n)z_n), & \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{a_n\} \in [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions:

(i) $0 < a \leq \lambda_n \leq b < 2\alpha$, $0 < c \leq \beta_n \leq d < 1$;

(ii) $\lim_{n\to\infty} \dfrac{\lambda_n}{\lambda_{n+1}} = 1$;

(iii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$;

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$, *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as $n \to \infty$, for all $j \in \{1, 2, 3, \ldots, N\}$.*

Then $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^N F(T_i) \bigcap EP$, where $z = P_{\bigcap_{i=1}^N F(T_i)\bigcap EP}u$.

Proof. First, we show that $(I - \lambda_nA)$ is nonexpansive. Let $x, y \in C$. Since A is α-strongly monotone and $\lambda_n < 2\alpha$ $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - \lambda_nA)x - (I - \lambda_nA)y\|^2 &= \|x - y - \lambda_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha\lambda_n\|Ax - Ay\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{3.2}$$

Thus $(I - \lambda_nA)$ is nonexpansive.

Since

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

we obtain

$$F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (I - \lambda_nA)x_n\rangle \geq 0, \quad \forall y \in C.$$

By Lemma 2.6, we have $z_n = T_{\lambda_n}(x_n - \lambda_nAx_n)$ $\forall n \in \mathbb{N}$.

Let $z \in \bigcap_{i=1}^N F(T_i) \bigcap EP$. Then $F(z, y) + \langle y - z, Az\rangle \geq 0$, $\forall y \in C$.

So $F(z,y) + \frac{1}{\lambda_n}\langle y - z, z - z + \lambda_n Az\rangle \geq 0, \quad \forall y \in C$.

Again by Lemma 2.6, we have $z = T_{\lambda_n}(z - \lambda_n Az)$. Since $I - \lambda_n A$ and T_{λ_n} are nonexpansive, we have

$$\begin{aligned} \|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \\ &\leq \|x_n - z\|^2, \end{aligned} \tag{3.3}$$

hence $\|z_n - z\| \leq \|x_n - z\|$.

Putting $y_n = a_n u + (1 - a_n)z_n$. Then we have

$$\begin{aligned} \|y_n - z\| &= \|a_n(u - z) + (1 - a_n)(z_n - z)\| \\ &\leq a_n\|u - z\| + (1 - a_n)\|x_n - z\|. \end{aligned} \tag{3.4}$$

This implies that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1 - \beta_n)(S_n y_n - z)\| \\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)\|y_n - z\| \\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)(a_n\|u - z\| + (1 - a_n)\|x_n - z\|). \end{aligned} \tag{3.5}$$

Putting $K = \max\{\|x_1 - z\|, \|u - z\|\}$. By (3.5), we can show by induction that $\|x_n - z\| \leq K, \ \forall n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded. Hence $\{Ax_n\}$, $\{y_n\}$, $\{S_n y_n\}$, $\{z_n\}$ are bounded.

Next we will show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Putting $u_n = x_n - \lambda_n Ax_n$. Then, we have $z_{n+1} = T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) = T_{\lambda_{n+1}}u_{n+1}$, $z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n) = T_{\lambda_n}u_n$. So we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|a_{n+1}u + (1 - a_{n+1})z_{n+1} - a_n u - (1 - a_n)z_n\| \\ &= \|a_{n+1}u + (1 - a_{n+1})T_{\lambda_{n+1}}u_{n+1} - a_n u - (1 - a_n)T_{\lambda_n}u_n\| \\ &= \|(a_{n+1} - a_n)u + (1 - a_{n+1})(T_{\lambda_{n+1}}u_{n+1} - T_{\lambda_{n+1}}u_n + T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n + T_{\lambda_n}u_n) - (1 - a_n)T_{\lambda_n}u_n\| \\ &= \|(a_{n+1} - a_n)u + (1 - a_{n+1})(T_{\lambda_{n+1}}u_{n+1} - T_{\lambda_{n+1}}u_n) \\ &\quad + (1 - a_{n+1})(T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n) + (1 - a_{n+1})T_{\lambda_n}u_n - (1 - a_n)T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + (1 - a_{n+1})\|u_{n+1} - u_n\| \\ &\quad + (1 - a_{n+1})\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\|. \end{aligned} \tag{3.7}$$

Since $I - \lambda_{n+1}A$ is nonexpansive, we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|x_{n+1} - \lambda_{n+1}Ax_{n+1} - x_n + \lambda_n Ax_n\| \\ &= \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n + (\lambda_n - \lambda_{n+1})Ax_n\| \\ &\leq \|x_{n+1} - x_n + |\lambda_n - \lambda_{n+1}|Ax_n\|. \end{aligned} \tag{3.8}$$

By Lemma 2.6, we have

$$F(T_{\lambda_n}u_n, y) + \frac{1}{\lambda_n}\langle y - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0, \quad \forall y \in C$$

and

$$F(T_{\lambda_{n+1}}u_n, y) + \frac{1}{\lambda_{n+1}}\langle y - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle \geq 0, \quad \forall y \in C.$$

In particular, we have

$$F(T_{\lambda_n}u_n, T_{\lambda_{n+1}}u_n) + \frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0, \tag{3.9}$$

and

$$F(T_{\lambda_{n+1}}u_n, T_{\lambda_n}u_n) + \frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle \geq 0. \tag{3.10}$$

Summing up (3.9) and (3.10) and using (A2), we obtain

$$\frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n\rangle + \frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n\rangle \geq 0, \quad \forall y \in C.$$

It then follows that

$$\left\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, \frac{T_{\lambda_{n+1}}u_n - u_n}{\lambda_{n+1}} - \frac{T_{\lambda_n}u_n - u_n}{\lambda_n}\right\rangle \geq 0.$$

This implies

$$\begin{aligned} 0 &\leq \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n - \frac{\lambda_n}{\lambda_{n+1}}(T_{\lambda_{n+1}}u_n - u_n)\right\rangle \\ &= \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(T_{\lambda_{n+1}}u_n - u_n)\right\rangle. \end{aligned}$$

It follows that

$$\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| (\|T_{\lambda_{n+1}}u_n\| + \|u_n\|).$$

Hence, we obtain

$$\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L, \tag{3.11}$$

where $L = \sup\{\|u_n\| + \|T_{\lambda_{n+1}}u_n\| : n \in \mathbb{N}\}$.

By (3.7), (3.8) and (3.11), we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq |a_{n+1} - a_n|\|u\| + (1 - a_{n+1})\|u_{n+1} - u_n\| + (1 - a_{n+1})\|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + (1 - a_{n+1})(\|x_{n+1} - x_n + |\lambda_n - \lambda_{n+1}|Ax_n\|) \\ &\quad + (1 - a_{n+1})\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + \left\|x_{n+1} - x_n + \lambda_{n+1}\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| Ax_n\right\| \\ &\quad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| \\ &\leq |a_{n+1} - a_n|\|u\| + \left\|x_{n+1} - x_n + b\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| Ax_n\right\| \\ &\quad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\|. \end{aligned} \tag{3.12}$$

We can rewrite x_{n+1} by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n, \tag{3.13}$$

where $y_n = a_n u + (1 - a_n)z_n$.

Next, we show that

$$\lim_{n\to\infty} \|S_n y_n - x_n\| = 0. \tag{3.14}$$

For $k \in \{2, 3, \ldots, N\}$, we have

$$\begin{aligned} \|U_{n+1,k}y_n - U_{n,k}y_n\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}y_n + \alpha_2^{n+1,k}U_{n+1,k-1}y_n + \alpha_3^{n+1,k}y_n \\ &\quad - \alpha_1^{n,k}T_kU_{n,k-1}y_n - \alpha_2^{n,k}U_{n,k-1}y_n - \alpha_3^{n,k}y_n\| \\ &= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}y_n - T_kU_{n,k-1}y_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}y_n \\ &\quad + (\alpha_3^{n+1,k} - \alpha_3^{n,k})y_n + \alpha_2^{n+1,k}(U_{n+1,k-1}y_n - U_{n,k-1}y_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}y_n\| \\ &\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\| \\ &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|y_n\| + \alpha_2^{n+1,k}\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}y_n\| \\ &= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\| \\ &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|y_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}y_n\| \\ &\leq \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\| \end{aligned}$$

$$+|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|y_n\| + |(\alpha_1^{n,k} - \alpha_1^{n+1,k}) + (\alpha_3^{n,k} - \alpha_3^{n+1,k})|\|U_{n,k-1}y_n\|$$
$$\le \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}y_n\|$$
$$+|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|y_n\| + |\alpha_1^{n,k} - \alpha_1^{n+1,k}|\|U_{n,k-1}y_n\| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|\|U_{n,k-1}y_n\|$$
$$= \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_kU_{n,k-1}y_n\| + \|U_{n,k-1}y_n\|)$$
$$+|\alpha_3^{n+1,k} - \alpha_3^{n,k}|(\|y_n\| + \|U_{n,k-1}y_n\|). \quad (3.15)$$

By (3.15), we obtain that for each $n \in \mathbb{N}$,

$$\|S_{n+1}y_n - S_ny_n\| = \|U_{n+1,N}y_n - U_{n,N}y_n\|$$
$$\le \|U_{n+1,1}y_n - U_{n,1}y_n\| + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}y_n\| + \|U_{n,j-1}y_n\|)$$
$$+\sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|y_n\| + \|U_{n,j-1}y_n\|)$$
$$= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1y_n - y_n\| + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}y_n\| + \|U_{n,j-1}y_n\|)$$
$$+\sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|y_n\| + \|U_{n,j-1}y_n\|).$$

This together with condition (iv), we obtain

$$\lim_{n\to\infty} \|S_{n+1}y_n - S_ny_n\| = 0. \quad (3.16)$$

By (3.12), we have

$$\|S_{n+1}y_{n+1} - S_ny_n\| \le \|y_{n+1} - y_n\| + \|S_{n+1}y_n - S_ny_n\|$$
$$\le |a_{n+1} - a_n|\|u\| + \|x_{n+1} - x_n\| + b\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|Ax_n\|$$
$$+\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + |a_{n+1} - a_n|\|T_{\lambda_n}u_n\| + \|S_{n+1}y_n - S_ny_n\|. \quad (3.17)$$

This together with (3.16) and conditions (ii) and (iii), we obtain

$$\limsup_{n\to\infty}(\|S_{n+1}y_{n+1} - S_ny_n\| - \|x_{n+1} - x_n\|) \le 0. \quad (3.18)$$

It follows from (3.13) and (3.17) and Lemma 2.4, $\lim_{n\to\infty} \|S_ny_n - x_n\| = 0$.

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|S_ny_n - x_n\| = 0. \quad (3.19)$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \quad (3.20)$$

By monotonicity of A and nonexpansiveness of T_{λ_n}, we have

$$\|x_{n+1} - z\|^2 = \|\beta_n(x_n - z) + (1 - \beta_n)(S_ny_n - z)\|^2$$
$$\le \beta_n\|x_n - z\|^2 + (1 - \beta_n)\|y_n - z\|^2$$
$$= \beta_n\|x_n - z\|^2 + (1 - \beta_n)\|a_n(u - z) + (1 - a_n)(z_n - z)\|^2$$
$$\le \beta_n\|x_n - z\|^2 + (1 - \beta_n)(a_n\|u - z\|^2 + (1 - a_n)\|z_n - z\|^2)$$
$$= \beta_n\|x_n - z\|^2 + (1 - \beta_n)(a_n\|u - z\|^2 + (1 - a_n)\|T_{\lambda_n}(x_n - \lambda_nAx_n) - T_{\lambda_n}(z - \lambda_nAz)\|^2)$$
$$\le \beta_n\|x_n - z\|^2 + (1 - \beta_n)(a_n\|u - z\|^2 + (1 - a_n)\|(x_n - \lambda_nAx_n) - (z - \lambda_nAz)\|^2)$$
$$= \beta_n\|x_n - z\|^2 + (1 - \beta_n)(a_n\|u - z\|^2 + (1 - a_n)\|(x_n - z) - \lambda_n(Ax_n - Az)\|^2)$$
$$= \beta_n\|x_n - z\|^2 + (1 - \beta_n)(a_n\|u - z\|^2 + (1 - a_n)(\|x_n - z\|^2$$
$$- 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + \lambda_n^2\|Ax_n - Az\|^2)) \quad (3.21)$$

$$\begin{aligned}&\le \beta_n\|x_n - z\|^2 + (1-\beta_n)(a_n\|u - z\|^2 + (1-a_n)(\|x_n - z\|^2 \\ &\quad - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2)) \\ &= \beta_n\|x_n - z\|^2 + (1-\beta_n)(a_n\|u - z\|^2 + (1-a_n)(\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2)) \\ &\le \|x_n - z\|^2 + (1-\beta_n)a_n\|u - z\|^2 + (1-a_n)(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2. \end{aligned} \tag{3.22}$$

By (3.22), we have

$$(1-a_n)(1-\beta_n)\lambda_n(2\alpha - \lambda_n)\|Ax_n - Az\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\beta_n)a_n\|u - z\|^2. \tag{3.23}$$

Since $0 < a \le \lambda_n \le b < 2\alpha$ and $0 < c \le \beta_n \le d < 1$, we have

$$\begin{aligned}(1-a_n)(1-d)a(2\alpha - \lambda_n)\|Ax_n - Az\|^2 &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1-\beta_n)a_n\|u - z\|^2 \\ &\le \|x_{n+1} - x_n\|(\|x_n - z\| + \|x_{n+1} - z\|) + (1-\beta_n)a_n\|u - z\|^2. \end{aligned} \tag{3.24}$$

This implies, by (3.19) and condition (*iii*), that

$$\lim_{n\to\infty} \|Ax_n - Az\| = 0. \tag{3.25}$$

Since T_{λ_n} is a firmly nonexpansive, we have

$$\begin{aligned}\|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \\ &\le \langle (x_n - \lambda_n Ax_n) - (z - \lambda_n Az), z_n - z\rangle \\ &= \frac{1}{2}(\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 + \|z_n - z\|^2 - \|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az) - (z_n - z)\|^2) \\ &\le \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Az)\|^2) \\ &= \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Az\rangle - \lambda_N^2\|Ax_n - Az\|^2). \end{aligned} \tag{3.26}$$

It follows that

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|. \tag{3.27}$$

By (3.21) and (3.27), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)[a_n\|u - z\|^2 + (1-a_n)\|z_n - z\|^2] \\ &\le \beta_n\|x_n - z\|^2 + a_n\|u - z\|^2 + (1-\beta_n)\|z_n - z\|^2 \\ &\le \beta_n\|x_n - z\|^2 + a_n\|u - z\|^2 + (1-\beta_n)(\|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|) \\ &\le \|x_n - z\|^2 + a_n\|u - z\|^2 - (1-\beta_n)\|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|. \end{aligned} \tag{3.28}$$

This implies

$$(1-\beta_n)\|x_n - z_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + a_n\|u - z\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|.$$

Hence

$$(1-d)\|x_n - z_n\|^2 \le \|x_{n+1} - x_n\|(\|x_n - z\| + \|x_{n+1} - z\|) + a_n\|u - z\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|.$$

By (3.19) and (3.25), we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.29}$$

Since $y_n = a_n u + (1-a_n)z_n$, we have $\|y_n - z_n\| = a_n\|u - z_n\|$.
This implies $\lim_{n\to\infty} \|y_n - z_n\| = 0$.
By (3.14) and (3.29), we have

$$\|S_n y_n - y_n\| \le \|S_n y_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.30}$$

Next, putting $z_0 = P_{\bigcap_{i=1}^N F(T_i)\bigcap EP}u$, we shall show that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0. \tag{3.31}$$

To show this inequality, take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle = \limsup_{k\to\infty}\langle u - z_0, y_{n_k} - z_0\rangle. \tag{3.32}$$

Without loss of generality, we may assume that $y_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. We first show $\omega \in EP$. We have $z_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Since $z_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$, we obtain

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), we have $\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq F(y, z_n)$. Then

$$\langle Ax_{n_k}, y - z_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle y - z_{n_k}, z_{n_i} - x_{n_k}\rangle \geq F(y, z_{n_k}), \quad \forall y \in C. \tag{3.33}$$

Put $z_t = ty + (1-t)\omega$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.33) we have

$$\begin{aligned}\langle z_t - z_{n_k}, Az_t\rangle &\geq \langle z_t - z_{n_k}, Az_t\rangle - \langle z_t - z_{n_k}, Ax_{n_i}\rangle - \left\langle z_t - z_{n_k}, \frac{z_{n_i} - x_{n_k}}{\lambda_{n_k}}\right\rangle + F(z_t, z_{n_k})\\ &= \langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle - \left\langle z_t - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\lambda_{n_k}}\right\rangle + F(z_t, z_{n_k}).\end{aligned}$$

Since $\|z_{n_k} - x_{n_k}\| \to 0$, we have $\|Az_{n_k} - Ax_{n_k}\| \to 0$. Further, from the monotonicity of A, we have $\langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle \geq 0$. So, from (A4) we have

$$\langle z_t - \omega, Az_t\rangle \geq F(z_t, \omega) \quad \text{as } k \to \infty. \tag{3.34}$$

From (A1), (A4) and (3.34), we also have

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, \omega)\\ &\leq tF(z_t, y) + (1-t)\langle z_t - \omega, Az_t\rangle\\ &= tF(z_t, y) + (1-t)t\langle y - \omega, Az_t\rangle,\end{aligned}$$

hence

$$0 \leq F(z_t, y) + (1-t)\langle y - \omega, Az_t\rangle.$$

Letting $t \to 0$, we have

$$0 \leq F(\omega, y) + \langle y - \omega, A\omega\rangle \quad \forall y \in C. \tag{3.35}$$

Therefore $\omega \in EP$.

Next, we show that $\omega \in \bigcap_{i=1}^N F(T_i)$. We can assume that

$$\alpha_1^{n_k,j} \to \alpha_1^j \in (0, 1) \quad \text{and} \quad \alpha_1^{n_k,N} \to \alpha_1^N \in (0, 1] \quad \text{as } k \to \infty \text{ for } j = 1, 2, \ldots, N-1 \tag{3.36}$$

and

$$\alpha_3^{n_k,j} \to \alpha_3^j \in [0, 1) \quad \text{as } k \to \infty \text{ for } j = 1, 2, \ldots, N. \tag{3.37}$$

Let S be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ where $\beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, for $j = 1, 2, \ldots, N$. By Lemma 2.9, we have

$$\lim_{k\to\infty} \|S_{n_k}x - Sx\| = 0 \tag{3.38}$$

for all $x \in C$.

By Lemma 2.8, we have $\bigcap_{i=1}^N F(T_i) = F(S)$. Assume that $S\omega \neq \omega$. By using the Opial property and (3.30) and (3.38), we have

$$\begin{aligned}\liminf_{k\to\infty} \|y_{n_k} - \omega\| &< \liminf_{k\to\infty} \|y_{n_k} - S\omega\|\\ &\leq \liminf_{k\to\infty}(\|y_{n_k} - S_{n_k}y_{n_k}\| + \|S_{n_k}y_{n_k} - S_{n_k}\omega\| + \|S_{n_k}\omega - S\omega\|)\\ &\leq \liminf_{k\to\infty} \|y_{n_k} - \omega\|,\end{aligned}$$

which is a contradiction. Thus $S\omega = \omega$, so $\omega \in F(S) = \bigcap_{i=1}^N F(T_i)$.

Hence $\omega \in \bigcap_{i=1}^N F(T_i) \bigcap EP$.

Since $y_{n_k} \rightharpoonup \omega$ and $\omega \in \bigcap_{i=1}^N F(T_i) \bigcap EP$, we have

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle = \limsup_{k\to\infty}\langle u - z_0, y_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.39}$$

By using (3.3), we have

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &= \|\beta_n(x_n-z_0)+(1-\beta_n)(S_ny_n-z_0)\|^2\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|y_n-z_0\|^2\\ &= \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|a_nu+(1-a_n)z_n-z_0\|^2\\ &= \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|a_n(u-z_0)+(1-a_n)(z_n-z_0)\|^2\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)((1-a_n)^2\|z_n-z_0\|^2+2a_n\langle u-z_0,y_n-z_0\rangle)\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)(1-a_n)\|z_n-z_0\|^2+2(1-\beta_n)a_n\langle u-z_0,y_n-z_0\rangle\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)(1-a_n)\|x_n-z_0\|^2+2(1-\beta_n)a_n\langle u-z_0,y_n-z_0\rangle\\ &= (1-(1-\beta_n)a_n)\|x_n-z_0\|^2+2(1-\beta_n)a_n\langle u-z_0,y_n-z_0\rangle.\end{aligned}$$

Since $\sum_{i=1}^{\infty}(1-\beta_n)a_n=\infty$ and $\limsup_{n\to\infty}2\langle u-z_0,y_n-z_0\rangle\le 0$, we can conclude from Lemma 2.3 that

$$\lim_{n\to\infty}\|x_n-z_0\|=0. \qquad \square$$

4. Applications

Using our main theorem (Theorem 3.1), we obtain the following strong convergence theorems in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space and let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i)\bigcap EP(F)\ne\emptyset$. For $j=1,2,\dots,N$, let $\alpha_j^{(n)}=(\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j})$ be such that $\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j}\in[0,1]$, $\alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j}=1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1}\subset[\eta_1,\theta_1]$ with $0<\eta_1\le\theta_1<1$, $\{\alpha_1^{n,N}\}\subset[\eta_N,1]$ with $0<\eta_N\le 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N$, $\{\alpha_3^{n,j}\}_{j=1}^N\subset[0,\theta_3]$ with $0\le\theta_3<1$. Let S_n be the S-mappings generated by $T_1,T_2,\dots,T_N$ and $\alpha_1^{(n)},\alpha_2^{(n)},\dots,\alpha_N^{(n)}$. Let $u\in C$ and $x_1\in C$ and let $\{z_n\}\subset C$ and $\{x_n\}\subset C$ be sequences generated by*

$$\begin{cases}F(z_n,y)+\dfrac{1}{\lambda_n}\langle y-z_n,z_n-x_n\rangle\ge 0, & \forall y\in C,\\ x_{n+1}=\beta_nx_n+(1-\beta_n)S_n(a_nu+(1-a_n)z_n), & \forall n\in\mathbb{N},\end{cases}\tag{4.1}$$

where $\{a_n\}\in[0,1]$, $\{\beta_n\}\subset[0,1]$ and $\{\lambda_n\}\subset[0,2\alpha]$ satisfy the following conditions:

(i) $0<a\le\lambda_n\le b<2\alpha,\ 0<c\le\beta_n\le d<1$;

(ii) $\lim_{n\to\infty}\dfrac{\lambda_n}{\lambda_{n+1}}=1$;

(iii) $\lim_{n\to\infty}a_n=0,\ \sum_{n=1}^{\infty};a_n=\infty$

(iv) $|\alpha_1^{n+1,j}-\alpha_1^{n,j}|\to 0$, *and* $|\alpha_3^{n+1,j}-\alpha_3^{n,j}|\to 0$ *as* $n\to\infty$, *for all* $j\in\{1,2,3,\dots,N\}$.

Then $\{x_n\}$ converges strongly to $z\in\bigcap_{i=1}^N F(T_i)\bigcap EP(F)$, where $z=P_{\bigcap_{i=1}^N F(T_i)\bigcap EP(F)}u$.

Proof. Put $A\equiv 0$ in Theorem 3.1. Then, from Theorem 3.1, we can get the desired conclusion. $\square$

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space and let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let A be an α-inverse strongly monotone mapping of C into H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i)\bigcap EP\ne\emptyset$. For $j=1,2,\dots,N$, let $\{\alpha_1^{n,j}\}_{j=1}^N\in[0,1]$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1}\subset[\eta_1,\theta_1]$ with $0<\eta_1\le\theta_1<1$, $\{\alpha_1^{n,N}\}\subset[\eta_N,1]$ with $0<\eta_N\le 1$, $\forall n\in\mathbb{N}$. Let W_n be the W-mappings generated by $T_1,T_2,\dots,T_N$ and $\alpha_1^{n,1},\alpha_1^{n,2},\dots,\alpha_1^{n,N}$. Let $u\in C$ and $x_1\in C$ and let $\{z_n\}\subset C$ and $\{x_n\}\subset C$ be sequences generated by*

$$\begin{cases}F(z_n,y)+\langle Ax_n,y-z_n\rangle+\dfrac{1}{\lambda_n}\langle y-z_n,z_n-x_n\rangle\ge 0, & \forall y\in C,\\ x_{n+1}=\beta_nx_n+(1-\beta_n)W_n(a_nu+(1-a_n)z_n), & \forall n\in\mathbb{N},\end{cases}\tag{4.2}$$

where $\{a_n\}\in[0,1]$, $\{\beta_n\}\subset[0,1]$ and $\{\lambda_n\}\subset[0,2\alpha]$ satisfy the following conditions:

(i) $0<a\le\lambda_n\le b<2\alpha,\ 0<c\le\beta_n\le d<1$;

(ii) $\lim_{n\to\infty}\dfrac{\lambda_n}{\lambda_{n+1}}=1$;

(iii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$;

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$, *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \dots, N\}$.

Then $\{x_n\}$ *converges strongly to* $z \in \bigcap_{i=1}^{N} F(T_i) \bigcap EP$, *where* $z = P_{\bigcap_{i=1}^{N} F(T_i) \bigcap EP}u$.

Proof. Put $\alpha_2^{n,j} = 0$ for all $j \in \{1, 2, 3, \dots, N\}$, and all $n \in \mathbb{N}$ in Theorem 3.1. Then, by Theorem 3.1 the conclusion follows. □

Corollary 4.3 (*[7], Theorem 3.1*). *Let C be a closed convex subset of a real Hilbert space and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions (A1)–(A4). Let A be an* α*-inverse strongly monotone mapping of C into H and let T be nonexpansive mappings of C into itself with* $F(T) \bigcap EP \neq \emptyset$. *Let* $u, x_1 \in C$ *and let* $\{z_n\}, \{x_n\} \subset C$ *be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T_1(a_n u + (1-a_n)z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $\{a_n\} \in [0, 1]$, $\{\beta_n\} \subset [0, 1]$ *and* $\{\lambda_n\} \subset [0, 2\alpha]$ *satisfy the following conditions:*

(i) $0 < a \leq \lambda_n \leq b < 2\alpha$, $0 < c \leq \beta_n \leq d < 1$;

(ii) $\lim_{n\to\infty} \dfrac{\lambda_n}{\lambda_{n+1}} = 1$;

(iii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $z \in \bigcap_{i=1}^{N} F(T_i) \bigcap EP$, *where* $z = P_{\bigcap_{i=1}^{N} F(T_i) \bigcap EP}u$.

Proof. Put $N = 1$ and $T_1 = T$ and $\alpha_2^{n,1}, \alpha_3^{n,1} = 0$ $\forall n \in \mathbb{N}$ in Theorem 3.1. Then $S_n = T$. Hence, we obtain the desired result from Theorem 3.1. □

Remark. In Theorem 3.1, by taking $N = 1$ and $\alpha_2^{n,1}, \alpha_3^{n,1} = 0$ for all $n \in \mathbb{N}$, one can easily see that Theorems 4.1, 4.2, 4.3 of Takahashi and Takahashi [7] are special cases of Theorem 3.1.

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A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings

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ABSTRACT

In this paper, we introduce and study a new mapping generated by a finite family of nonexpansive mappings and finite real numbers and introduce a general iterative method concerning the new mappings for finding a common element of the set of solutions of an equilibrium problem and of the set of common fixed points of a finite family of nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem of the proposed iterative method for a finite family of nonexpansive mappings to the unique solution of variational inequality which is the optimality condition for a minimization problem. Our main result can be applied to obtain strong convergence of the general iterative methods which are modifications of those in [G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318 (1) (2006) 43–52; S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 336 (1) (2007) 455–469; S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (1) (2007) 506–515] to a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping T of H into itself is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2.$$

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings with $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Many authors (see [2–7]) introduced iterative methods for finding an element of F which is an optimal point for the minimization problem. For $n > N$, T_n is understood as $T_{(n \bmod N)}$ with the mod function taking values in $\{1, 2, \dots, N\}$. Let u be a fixed element of H. In 2003, Xu [8] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1 - \epsilon_n A)T_{n+1}x_n + \epsilon_n u$$

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converges strongly to the solution of the quadratic minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle$$

under suitable hypotheses on $\{\epsilon_n\}$ and under the additional hypothesis,

$$F = F(T_1T_2 \dots T_N) = F(T_NT_1 \dots T_{N-1}) = \dots = F(T_2T_3 \dots T_NT_1).$$

In 2000, Moudafi [9] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H and $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1-\sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geq 0, \tag{1.1}$$

where $\{\sigma_n\}$ is a sequence in (0, 1). He proved that under the certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.1) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

In 2006, Marino and Xu [10] introduced the following general iterative method:

$$x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n\gamma f(x_n), \quad n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in (0, 1) satisfying the following conditions:

(C1) $\alpha_n \to 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) either $\sum_{n=0}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

They proved the following theorem:

Theorem 1.1. *Let $\{x_n\}$ be generated by algorithm (1.3) with the sequence $\{\alpha_n\}$ of parameters satisfying conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to x^* where x^* is the unique solution of the following variation inequality:*

$$\langle (A-\gamma f)x^*, x^*-z\rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, we have $P_{F(T)}(I-A+\gamma f)x^* = x^*$.

Let $G : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for G is to determine its equilibrium points, i.e. the set

$$EP(G) = \{x \in G : G(x,y) \geq 0, \forall y \in C\}. \tag{1.4}$$

Many problems in physics, optimization, and economics are seeking some elements of $EP(G)$, see [11,12]. Several iterative methods have been proposed to solve the equilibrium problem, see, for instance, [4,12–15]. In 2005, Combettes and Hirstoaga [12] introduced some iterative schemes of finding the best approximation to the initial data when $EP(G)$ is nonempty and proved the strong convergence theorem.

Also in [12] Combettes and Hirstoaga, following [11] define $S_r : H \to C$ by

$$S_r(x) = \left\{z \in C : G(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0\, \forall y \in C\right\}. \tag{1.5}$$

They prove that under suitable hypotheses G, S_r is single-valued and firmly nonexpansive with $F(S_r) = EP(G)$.

In 2007, Takahashi and Takahashi [15] proved the following theorem:

Theorem 1.2. *Let C be a nonempty closed convex subset of H. Let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying*

(A1) $G(x,x) = 0\, \forall x \in C$;
(A2) *G is monotone, i.e.* $G(x,y) + G(y,x) \leq 0\, \forall x, y \in C$;
(A3) $\forall x, y, z \in C$,

$$\lim_{t\to 0^+} G(tz + (1-t)x, y) \leq G(x,y).$$

(A4) $\forall x \in C$, $y \mapsto G(x,y)$ *is convex and lower semicontinuous;*

and let S be a nonexpansive mapping of C into H such that $F(S)\bigcap EP(G) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$G(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n$$

for all $n \in N$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,1)$ satisfy (C1)–(C3) and $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1}-r_n| < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S)\bigcap EP(G)$, where $z = P_{F(S)\bigcap EP(G)}f(z)$.

In 2007, Plubtieng and Punpaeng [13] introduced a general iterative method for finding a common element of $EP(G)$ and $F(S)$. They proved the following theorem.

Theorem 1.3. *Let H be a real Hilbert space, let G be a bifunction from $H \times H \to \mathbb{R}$ satisfying (A1)–(A4) and let S be a nonexpansive mapping on H such that $F(S) \bigcap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive bounded linear operator on H with coefficients $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$*

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 & \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Su_n, & \forall n \in \mathbb{N}, \end{cases}$$

where $u_n = S_{r_n}x_n$, $\{r_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset [0, 1]$ satisfy (C1)–(C3) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \bigcap EP(F)$ which solves the variational inequality:

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(S) \bigcap EP(G).$$

Equivalently, we have $P_{F(S)\bigcap EP(G)}(I - A + \gamma f)z = z$.

Question 1. Are the conditions (C1) and (C2) in Theorems 1.2 and 1.3 sufficient for strong convergence of the sequence $\{x_n\}$?

In 1999, Atsushiba and Takahashi [16] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I, \\ U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_N - 1U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I, \end{aligned} \tag{1.6}$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the W-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. In 2000 Takahashi and Shimoji [14] proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \dots, N$.

Very recently, Colao, Marino and Xu [17], introduced a new general iterative method for finding a common element of the set of solutions of equilibrium problem and the set of common fixed points of finite family of nonexpansive mappings in a Hilbert space. They proved that under some sufficient suitable conditions, the sequences $\{u_n\}$ and $\{x_n\}$ generated by $x_1 \in H$ and

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + [(1 - \beta)I - \epsilon_n A]W_nu_n \end{cases} \tag{1.7}$$

converge strongly to a point $x^* \in F$ which is an equilibrium point for G and is the unique solution of the variational inequality,

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0 \quad \forall x \in F \bigcap EP(G). \tag{1.8}$$

Motivated by Atsushiba and Takahashi [16], Plubtieng and Punpaeng [13], Colao, Marino and Xu [17], we introduce a new mapping and apply it to the iteration scheme (1.7) to obtain strong convergence to a common element of $EP(G)$ and F.

Let X be a real Banach space and C a nonempty closed convex subset of X. For a finite family of nonexpansive mappings $T_1, T_2, \dots, T_N$ and sequence $\{\lambda_{n,i}\}_i^N$ in $[0,1]$, we define the mapping $K_n : C \to C$ as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})U_{n,1}, \\ U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})U_{n,2}, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_N - 1U_{n,N-2} + (1 - \lambda_{n,N-1})U_{n,N-2}, \\ K_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})U_{n,N-1}. \end{aligned} \tag{1.9}$$

For $x_1 \in H$, let $\{u_n\}$ and $\{x_n\}$ be the sequence defined by

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 & \forall y \in C, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1 - \beta)I - \epsilon_n A)K_nu_n. \end{cases} \tag{1.10}$$

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In this paper, we prove that if X is strictly convex, then $F(K_n) = \bigcap_{i=1}^{N} F(T_i)$ where $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$, and under the conditions (C1) and (C2) and some other suitable conditions, the sequences $\{x_n\}$ and $\{u_n\}$ strongly converge to a point $x^* = P_{F\bigcap EP(G)}(I - (A - \gamma f))x^*$, where $P_{F\bigcap EP(G)} : H \to F \bigcap EP(G)$ is the metric projection of H onto $F \bigcap EP(G)$.

2. Preliminaries

In this section, we give some useful lemmas that will be used for the main result in the next section.

Let C be closed convex subset of a Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1 (See [18]). *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2 (See [8]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

$$(1) \quad \{\alpha_n\} \subset [0, 1], \qquad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2) \quad \limsup_{n\to\infty} \beta_n \leq 0 \quad or \quad \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (See [19]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$$

for all integer $n \geq 0$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.4 (See [10]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.5 (See [12]). *Let C be a nonempty closed convex subset of a Hilbert space H and $G : C \times C \to \mathbb{R}$ satisfy*

(A1) $G(x, x) = 0 \; \forall x \in C$;
(A2) G is monotone, i.e. $G(x, y) + G(y, x) \leq 0 \; \forall x, y \in C$;
(A3) $\forall x, y, z \in C$,

$$\lim_{t\to 0^+} G(tz + (1 - t)x, y) \leq G(x, y);$$

(A4) $\forall x \in C, y \mapsto G(x, y)$ is convex and lower semicontinuous.

For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be

$$S_r(x) = \left\{z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}.$$

Then S_r is well defined and the following hold:

(1) S_r is single-valued;
(2) S_r is firmly nonexpansive, i.e.

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle \quad \forall x, y \in H;$$

(3) $F(S_r) = EP(G)$;
(4) $EP(G)$ is closed and convex.

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Lemma 2.6 (See [18]). *Demiclosedness principle. Assume that T is a nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y it follows that $(I-T)x = y$. Here, I is the identity mapping of H.*

Lemma 2.7. *Let H be a real Hilbert space. Then, for all $x, y \in H$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle.$$

Lemma 2.8 (See [20]). *In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1-\lambda)y\|$$

for all $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$.

Definition 2.1. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, \dots, N$. We define a mapping $K : C \to C$ as follows:

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1-\lambda_1)I, \\ U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}. \end{aligned} \tag{2.1}$$

Such a mapping K is called the *K-mapping* generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$.

Lemma 2.9. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \dots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.*

Proof. It easy to see that $\bigcap_{i=1}^N F(T_i) \subset F(K)$. Let $x_0 \in F(K)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. By the definition of K, we have

$$\begin{aligned} \|x_0 - x^*\| = \|Kx_0 - x^*\| &= \|\lambda_N(T_N U_{N-1}x_0 - x^*) + (1-\lambda_N)(U_{N-1}x_0 - x^*)\| \\ &\leq \lambda_N \|T_N U_{N-1}x_0 - x^*\| + (1-\lambda_N)\|U_{N-1}x_0 - x^*\| \\ &\leq \lambda_N \|U_{N-1}x_0 - x^*\| + (1-\lambda_N)\|U_{N-1}x_0 - x^*\| \\ &= \|U_{N-1}x_0 - x^*\| \\ &= \|\lambda_{N-1}(T_{N-1}U_{N-2}x_0 - x^*) + (1-\lambda_{N-1})(U_{N-2}x_0 - x^*)\| \\ &\leq \lambda_{N-1}\|T_{N-1}U_{N-2}x_0 - x^*\| + (1-\lambda_{N-1})\|U_{N-2}x_0 - x^*\| \\ &\leq \lambda_{N-1}\|U_{N-2}x_0 - x^*\| + (1-\lambda_{N-1})\|U_{N-2}x_0 - x^*\| \\ &= \|U_{N-2}x_0 - x^*\| \\ &\;\;\vdots \\ &\leq \|U_1 x_0 - x^*\| \\ &= \|\lambda_1(T_1x_0 - x^*) + (1-\lambda_1)(x_0 - x^*)\| \\ &\leq \lambda_1\|T_1x_0 - x^*\| + (1-\lambda_1)\|x_0 - x^*\| \\ &\leq \lambda_1\|x_0 - x^*\| + (1-\lambda_1)\|x_0 - x^*\| \\ &= \|x_0 - x^*\|. \end{aligned} \tag{2.2}$$

This implies that $\|x_0 - x^*\| = \|\lambda_1(T_1x_0 - x^*) + (1-\lambda_1)(x_0 - x^*)\|$ and $\|x_0 - x^*\| = \|T_1x_0 - x^*\|$.
By Lemma 2.8, we have $T_1x_0 = x_0$, that is $x_0 \in F(T_1)$.
It follows that $U_1x_0 = x_0$.
By (2.2), we have

$$\begin{aligned} \|x_0 - x^*\| = \|U_2x_0 - x^*\| &= \|\lambda_2(T_2U_1x_0 - x^*) + (1-\lambda_2)(U_1x_0 - x^*)\| \\ &= \|\lambda_2(T_2x_0 - x^*) + (1-\lambda_2)(x_0 - x^*)\|. \end{aligned}$$

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Again by (2.2) together with $U_1x_0 = x_0$, we have

$$\begin{aligned}\|x_0 - x^*\| &= \lambda_2\|T_2U_1x_0 - x^*\| + (1-\lambda_2)\|U_1x_0 - x^*\| \\ &= \lambda_2\|T_2x_0 - x^*\| + (1-\lambda_2)\|x_0 - x^*\|,\end{aligned}$$

which implies $\|x_0 - x^*\| = \|T_2x_0 - x^*\|$.

By Lemma 2.8, we have $T_2x_0 = x_0$.

It follows that $U_2x_0 = x_0$.

By using the same argument, we can conclude that $T_ix_0 = x_0$ and $U_ix_o = x_o$ for $i = 1, 2, \dots, N-1$.

This implies that $0 = x_0 - x_0 = \lambda_N(T_Nx_0 - x_0)$.

It follows that $x_0 \in F(T_N)$. Therefore $x_0 \in \bigcap_{i=1}^N F(T_i)$. □

Lemma 2.10. *Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0, 1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1, 2, \dots, N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$, and $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ respectively. Then, for every $x \in C$, we have*

$$\lim_{n\to\infty} \|K_nx - Kx\| = 0.$$

Proof. Let $x \in C$ and U_k and $U_{n,k}$ be generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$, and $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ respectively. Note that

$$\begin{aligned}\|U_{n,1}x - U_1x\| &= \|(\lambda_{n,1} - \lambda_1)T_1x - (\lambda_{n,1} - \lambda_1)x\| \\ &\le |\lambda_{n,1} - \lambda_1|\|T_1x - x\|.\end{aligned}$$

For $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}\|U_{n,k}x - U_kx\| &= \|\lambda_{n,k}T_kU_{n,k-1}x + (1-\lambda_{n,k})U_{n,k-1}x - \lambda_kT_kU_{k-1}x - (1-\lambda_k)U_{k-1}x\| \\ &= \|\lambda_{n,k}T_kU_{n,k-1}x + \lambda_{n,k}T_kU_{k-1}x - \lambda_{n,k}T_kU_{k-1}x + \lambda_{n,k}U_{k-1}x - \lambda_{n,k}U_{k-1}x \\ &\quad + (1-\lambda_{n,k})U_{n,k-1}x - \lambda_kT_kU_{k-1}x - (1-\lambda_k)U_{k-1}x\| \\ &= \|\lambda_{n,k}(T_kU_{n,k-1}x - T_kU_{k-1}x) + (\lambda_{n,k} - \lambda_k)T_kU_{k-1}x - (1-\lambda_{n,k})U_{k-1}x \\ &\quad + (\lambda_k - \lambda_{n,k})U_{k-1}x + (1-\lambda_{n,k})U_{n,k-1}x\| \\ &\le \lambda_{n,k}\|T_kU_{n,k-1}x - T_kU_{k-1}x\| + |\lambda_{n,k} - \lambda_k|\|T_kU_{k-1}x\| \\ &\quad + (1-\lambda_{n,k})\|U_{n,k-1}x - U_{k-1}x\| + |\lambda_k - \lambda_{n,k}|\|U_{k-1}x\| \\ &\le \lambda_{n,k}\|U_{n,k-1}x - U_{k-1}x\| + (1-\lambda_{n,k})\|U_{n,k-1}x - U_{k-1}x\| + |\lambda_{n,k} - \lambda_k|(\|T_kU_{k-1}x\| + \|U_{k-1}x\|) \\ &= \|U_{n,k-1}x - U_{k-1}x\| + |\lambda_{n,k} - \lambda_k|(\|T_kU_{k-1}x\| + \|U_{k-1}x\|).\end{aligned}$$

It follows that

$$\begin{aligned}\|K_nx - Kx\| &= \|U_{n,N}x - U_Nx\| \le \|U_{n,N-1}x - U_{N-1}x\| + |\lambda_{n,N} - \lambda_N|(\|T_NU_{N-1}x\| + \|U_{N-1}x\|) \\ &\le \|U_{n,N-2}x - U_{N-2}x\| + |\lambda_{n,N-1} - \lambda_{N-1}|(\|T_{N-1}U_{N-2}x\| + \|U_{N-2}x\|) \\ &\quad + |\lambda_{n,N} - \lambda_N|(\|T_NU_{N-1}x\| + \|U_{N-1}x\|) \\ &= \|U_{n,N-2}x - U_{N-2}x\| + \sum_{j=N-1}^{N} |\lambda_{n,j} - \lambda_j|(\|T_jU_{j-1}x\| + \|U_{j-1}x\|) \\ &\;\;\vdots \\ &\le \|U_{n,1}x - U_1x\| + \sum_{j=2}^{N} |\lambda_{n,j} - \lambda_j|(\|T_jU_{j-1}x\| + \|U_{j-1}x\|) \\ &\le |\lambda_{n,1} - \lambda_1|\|T_1x - x\| + \sum_{j=2}^{N} |\lambda_{n,j} - \lambda_j|(\|T_jU_{j-1}x\| + \|U_{j-1}x\|).\end{aligned}$$

Since $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1, 2, \dots, N)$ it follows that $\lim_{n\to\infty} \|K_nx - Kx\| = 0$. □

Lemma 2.11. *Let H be a Hilbert space, C a closed convex nonempty subset of H, $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4). *For every $n \in \mathbb{N}$, let*

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K_n be a K-mapping generated by $T_1, \ldots, T_N$ and $\lambda_{n,1}, \ldots, \lambda_{n,N}$ with $\{\lambda_{n,i}\}_{i=1}^N \subset [a, b]$ where $0 < a \le b < 1$. For a sequence $\{r_n\}$ in $(0, \infty)$, let $S_{r_n} : H \to C$ be defined by

$$S_{r_n}(x) = \left\{z \in C : G(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \ge 0, \forall y \in C\right\}.$$

If $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$ and $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0\ \forall i \in \{1, 2, 3, \ldots, N\}$, then

(1) $\lim_{n\to\infty} \|K_{n+1}S_{r_{n+1}}w_n - K_{n+1}S_{r_n}w_n\| = 0,$

(2) $\lim_{n\to\infty} \|K_{n+1}w_n - K_nw_n\| = 0$

for every bounded sequence $\{w_n\}$ in H.

Proof. By using the nonexpansivity of K_{n+1} and the proof of Step 2 in Theorem 3.1 of [17], it can be shown that (1) is satisfied. Next, we show (2). For $j \in \{2, \ldots, N-2\}$, we have

$$\begin{aligned}\|U_{n+1,N-j}w_n - U_{n,N-j}w_n\| &= \|\lambda_{n+1,N-j}T_{N-j}U_{n+1,N-j-1}w_n + (1-\lambda_{n+1,N-j})U_{n+1,N-j-1}w_n \\ &\quad - \lambda_{n,N-j}T_{N-j}U_{n,N-j-1}w_n - (1-\lambda_{n,N-j})U_{n,N-j-1}w_n\| \\ &= \|\lambda_{n+1,N-j}T_{N-j}U_{n+1,N-j-1}w_n - \lambda_{n+1,N-j}T_{N-j}U_{n,N-j-1}w_n \\ &\quad + \lambda_{n+1,N-j}T_{N-j}U_{n,N-j-1}w_n - \lambda_{n+1,N-j}U_{n,N-j-1}w_n \\ &\quad + \lambda_{n+1,N-j}U_{n,N-j-1}w_n + (1-\lambda_{n+1,N-j})U_{n+1,N-j-1}w_n \\ &\quad - \lambda_{n,N-j}T_{N-j}U_{n,N-j-1}w_n - (1-\lambda_{n,N-j})U_{n,N-j-1}w_n\| \\ &\le \lambda_{n+1,N-j}\|T_{N-j}U_{n+1,N-j-1}w_n - T_{N-j}U_{n,N-j-1}w_n\| \\ &\quad + (1-\lambda_{n+1,N-j})\|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| \\ &\quad + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|T_{N-j}U_{n,N-j-1}w_n\| + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|U_{n,N-j-1}w_n\| \\ &\le \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + M|\lambda_{n+1,N-j} - \lambda_{n,N-j}| \end{aligned} \tag{2.3}$$

where $M = \sup\{\sum_{j=2}^N(\|T_jU_{n,j-1}w_n\| + \|U_{n,j-1}w_n\|) + \|T_1w_n\| + \|w_n\|\} < \infty$.
By (2.3), we have

$$\begin{aligned}\|K_{n+1}w_n - K_nw_n\| &= \|U_{n+1,N}w_n - U_{n,N}w_n\| \\ &\le \|U_{n+1,N-1}w_n - U_{n,N-1}w_n\| + M|\lambda_{n+1,N} - \lambda_{n,N}| \\ &\le \|U_{n+1,N-2}w_n - U_{n,N-2}w_n\| + M|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + M|\lambda_{n+1,N} - \lambda_{n,N}| \\ &\ \ \vdots \\ &\le M\sum_{j=2}^N|\lambda_{n+1,j} - \lambda_{n,j}| + \|U_{n+1,1}w_n - U_{n,1}w_n\|,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\|U_{n+1,1}w_n - U_{n,1}w_n\| &= \|\lambda_{n+1,1}T_1w_n + (1-\lambda_{n+1,1})w_n - \lambda_{n,1}T_1w_n - (1-\lambda_{n,1})w_n\| \\ &\le |\lambda_{n+1,1} - \lambda_{n,1}|\|T_1w_n\| + |\lambda_{n+1,1} - \lambda_{n,1}|\|w_n\| \\ &\le |\lambda_{n+1,1} - \lambda_{n,1}|M.\end{aligned} \tag{2.5}$$

By (2.4), (2.5) and the condition $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$, we can conclude that

$$\|K_{n+1}w_n - K_nw_n\| \le M\sum_{j=1}^N|\lambda_{n+1,j} - \lambda_{n,j}| \to 0 \quad \text{as } n \to \infty.$$

Hence (2) is satisfied. □

3. Main result

In this section, we prove the strong convergence of the sequences $\{u_n\}$ and $\{x_n\}$ defined by the iteration scheme (1.10).

Theorem 3.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H, $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, $G : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)–(A4) with $F \bigcap EP(G) \neq \emptyset$, A a strongly positive bounded linear operator on H with coefficient $\bar{\gamma}$ and f an α-contraction on H for some $0 < \alpha < 1$. Moreover, let $\{\epsilon_n\}$*

be a sequence in $(0, 1)$, $\{\lambda_{n,i}\}_{i=1}^N$ *sequences in* $[a, b]$ *with* $0 < a \le b < 1$, $\{r_n\}$ *a sequence in* $(0, \infty)$ *and let* γ *and* β *be two real numbers such that* $0 < \beta < 1$ *and* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Assume that*

(i) *the sequence* $\{r_n\}$ *satisfies*

$$\textbf{(D1)}\ \liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad \text{(D2)}\ \lim_{n\to\infty}\frac{r_n}{r_{n+1}} = 1,$$

(ii) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^N$ *satisfies*

$$\textbf{(E1)}\ \lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0, \quad \forall i = \{1, 2, 3, \ldots, N\},$$

(iii) *the sequence* $\{\epsilon_n\}$ *satisfies*

$$\textbf{(C1)}\ \lim_{n\to\infty}\epsilon_n = 0, \qquad \textbf{(C2)}\ \sum_{n=1}^{\infty}\epsilon_n = \infty.$$

For every $n \in N$, *let* K_n *be a* K*-mapping generated by* $T_1, \ldots, T_N$ *and* $\lambda_{n,1}, \ldots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)K_n u_n, \end{cases} \tag{3.1}$$

where $f : H \to H$ *is an* α*-contraction. Then both* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in F = \bigcap_{i=1}^N F(T_i)$ *where* x^* *is an equilibrium point for* G *and is the unique solution of the variational inequality* (1.8), *i.e.*,

$$x^* = P_{F\bigcap EP(G)}(I - (A - \gamma f))x^*.$$

Proof. By Lemma 2.5, it follows that for every $n \in N$, there exists a nonexpansive mapping $S_{r_n} : H \to H$ such that $u_n = S_{r_n}x_n$ and $EP(G) = F(S_{r_n})$. Whenever needed, we shall write scheme (3.1) as

$$x_{n+1} = \epsilon_n\gamma f(x_n) + \beta x_n + [(1-\beta)I - \epsilon_n A]K_n S_{r_n}x_n.$$

Moreover, we shall assume that $\epsilon_n \le (1-\beta)\|A\|^{-1}$ and $1 - \epsilon_n(\overline{\gamma} - \alpha\gamma) > 0$.

Observe that, if $\|u\| = 1$, then

$$\langle((1-\beta)I - \epsilon_n A)u, u\rangle = (1-\beta) - \epsilon_n\langle Au, u\rangle \ge (1 - \beta - \epsilon_n\|A\|) \ge 0.$$

By Lemma 2.4, we have

$$\|(1-\beta)I - \epsilon_n A\| \le 1 - \beta - \epsilon_n\overline{\gamma}.$$

We shall divide our proof into 7 steps.

Step 1. We shall show that the sequence $\{x_n\}$ is bounded.

Let $v \in EP(G)\bigcap F$. Then

$$\begin{aligned} \|x_{n+1} - v\| &= \|\epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)K_n u_n - v\| \\ &= \|((1-\beta)I - \epsilon_n A)(K_n u_n - v) + \beta(x_n - v) + \epsilon_n(\gamma f(x_n) - Av)\| \\ &= \|((1-\beta)I - \epsilon_n A)(K_n u_n - v) + \beta(x_n - v) + \epsilon_n(\gamma f(x_n) - \gamma f(v)) + \epsilon_n(\gamma f(v) - Av)\| \\ &\le \|(1-\beta)I - \epsilon_n A\|\|K_n S_{r_n}x_n - K_n S_{r_n}v\| + \beta\|x_n - v\| + \epsilon_n\gamma\alpha\|x_n - v\| + \epsilon_n\|\gamma f(v) - Av\| \\ &\le (1 - \beta - \epsilon_n\overline{\gamma})\|x_n - v\| + \beta\|x_n - v\| + \epsilon_n\gamma\alpha\|x_n - v\| + \epsilon_n\|\gamma f(v) - Av\| \\ &= (1 - \epsilon_n(\overline{\gamma} - \gamma\alpha))\|x_n - v\| + \epsilon_n\|\gamma f(v) - Av\| \\ &\quad + (1 - \epsilon_n(\overline{\gamma} - \gamma\alpha))\|x_n - v\| + \frac{\epsilon_n(\overline{\gamma} - \gamma\alpha)}{\overline{\gamma} - \gamma\alpha}\|\gamma f(v) - Av\| \\ &\le \max\left\{\|x_n - v\|, \frac{\|\gamma f(v) - Av\|}{\overline{\gamma} - \gamma\alpha}\right\}. \end{aligned}$$

By induction we can prove that $\{x_n\}$ is bounded and also $\{Ax_n\}$ and $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Define sequence $\{z_n\}$ by $z_n = \frac{1}{1-\beta}(x_{n+1} - \beta x_n)$.

Then $x_{n+1} = \beta x_n + (1-\beta)z_n$.

Since $\{x_n\}$ is bounded, we have, for some big enough constant $M > 0$,

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \frac{1}{1-\beta}\|x_{n+2} - \beta x_{n+1} - (x_{n+1} - \beta x_n)\| \\
&= \frac{1}{1-\beta}\|\epsilon_{n+1}\gamma f(x_{n+1}) + ((1-\beta)I - \epsilon_{n+1}A)K_{n+1}u_{n+1} - (\epsilon_n\gamma f(x_n) + ((1-\beta)I - \epsilon_n A)K_n u_n)\| \\
&= \frac{1}{1-\beta}\|\gamma(\epsilon_{n+1}f(x_{n+1}) - \epsilon_n f(x_n)) + ((1-\beta)I - \epsilon_{n+1}A)K_{n+1}u_{n+1} - ((1-\beta)I - \epsilon_n A)K_n u_n\| \\
&= \frac{1}{1-\beta}\|\gamma(\epsilon_{n+1}f(x_{n+1}) - \epsilon_n f(x_n)) + (1-\beta)(K_{n+1}u_{n+1} - K_n u_n) - (\epsilon_{n+1}AK_{n+1}u_{n+1} - \epsilon_n AK_n u_n)\| \\
&= \left\|\frac{\gamma}{1-\beta}(\epsilon_{n+1}f(x_{n+1}) - \epsilon_n f(x_n)) + (K_{n+1}u_{n+1} - K_n u_n) - \frac{1}{1-\beta}(\epsilon_{n+1}AK_{n+1}u_{n+1} - \epsilon_n AK_n u_n)\right\| \\
&\le \frac{\gamma}{1-\beta}(\epsilon_{n+1}\|f(x_{n+1})\| + \epsilon_n\|f(x_n)\|) + \|K_{n+1}u_{n+1} - K_n u_n\| + \frac{1}{1-\beta}(\epsilon_{n+1}\|AK_{n+1}u_{n+1}\| + \epsilon_n\|AK_n u_n\|) \\
&\le \|K_{n+1}S_{r_{n+1}}x_{n+1} - K_n S_{r_n}x_n\| + M(\epsilon_n + \epsilon_{n+1}) \\
&\le \|K_{n+1}S_{r_{n+1}}x_{n+1} - K_{n+1}S_{r_{n+1}}x_n\| + \|K_{n+1}S_{r_{n+1}}x_n - K_n S_{r_n}x_n\| + M(\epsilon_n + \epsilon_{n+1}) \\
&\le \|x_{n+1} - x_n\| + \|K_{n+1}S_{r_{n+1}}x_n - K_{n+1}S_{r_n}x_n\| + \|K_{n+1}S_{r_n}x_n - K_n S_{r_n}x_n\| + M(\epsilon_n + \epsilon_{n+1}).
\end{aligned}$$

By condition on $\{\epsilon_n\}$ and by Lemma 2.11, we can conclude that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

By Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

This implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = (1-\beta)\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

Step 3. We will show that $\lim_{n\to\infty}\|x_n - K_n u_n\| = 0$ where $u_n = S_{r_n}x_n$.

Since

$$\begin{aligned}
\|x_n - K_n u_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - K_n u_n\| \\
&= \|x_n - x_{n+1}\| + \|\epsilon_n\gamma f(x_n) + \beta x_n + (1-\beta)K_n u_n - \epsilon_n AK_n u_n - K_n u_n\| \\
&\le \|x_n - x_{n+1}\| + \epsilon_n\|\gamma f(x_n) - AK_n u_n\| + \beta\|x_n - K_n u_n\|,
\end{aligned}$$

we have

$$\|x_n - K_n u_n\| \le \frac{1}{(1-\beta)}(\|x_n + x_{n+1}\| + \epsilon_n\|\gamma f(x_n) - AK_n u_n\|).$$

By (C1) and Step 2, we obtain $\lim_{n\to\infty}\|x_n - K_n u_n\| = 0$.

Step 4. We shall show that $\lim_{n\to\infty}\|x_n - S_{r_n}x_n\| = 0$.

Let $v \in F\bigcap EP(G)$. Since S_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|v - S_{r_n}x_n\|^2 &= \|S_{r_n}v - S_{r_n}x_n\|^2 \\
&\le \langle S_{r_n}v - S_{r_n}x_n, v - x_n\rangle \\
&= \frac{1}{2}(\|S_{r_n}x_n - v\|^2 + \|x_n - v\|^2 - \|S_{r_n}x_n - x_n\|^2).
\end{aligned}$$

Hence

$$\|S_{r_n}x_n - v\|^2 \le \|x_n - v\|^2 - \|S_{r_n}x_n - x_n\|^2. \tag{3.2}$$

Set $y_n = \gamma f(x_n) - AK_n u_n$ and $\lambda > 0$ be a constant such that

$$\lambda > \sup_k\{\|y_k\|, \|x_k - v\|\}. \tag{3.3}$$

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By (3.2) and (3.3), we have

$$\begin{aligned}\|x_{n+1}-v\|^2 &= \|\epsilon_n\gamma f(x_n)+\beta x_n+((1-\beta)I-\epsilon_nA)K_nu_n-v\|^2\\ &= \|[(1-\beta)I-\epsilon_nA](K_nu_n-v)+\beta(x_n-v)+\epsilon_n(\gamma f(x_n)-Av)\|^2\\ &= \|(1-\beta)(K_nu_n-v)-\epsilon_nA(K_nu_n-v)+\beta(x_n-v)+\epsilon_n(\gamma f(x_n)-Av)\|^2\\ &= \|(1-\beta)(K_nu_n-v)+\beta(x_n-v)+\epsilon_n(\gamma f(x_n)-A(K_nu_n))\|^2\\ &\le \|(1-\beta)(K_nu_n-v)+\beta(x_n-v)\|^2+2\epsilon_n\langle y_n,x_{n+1}-v\rangle\\ &\le \|(1-\beta)(K_nS_{r_n}x_n-v)+\beta(x_n-v)\|^2+2\epsilon_n\lambda^2\\ &\le (1-\beta)\|K_nS_{r_n}x_n-v\|^2+\beta\|x_n-v\|^2+2\epsilon_n\lambda^2\\ &\le (1-\beta)\|S_{r_n}x_n-v\|^2+\beta\|x_n-v\|^2+2\epsilon_n\lambda^2\\ &\le \|x_n-v\|^2-(1-\beta)\|S_{r_n}x_n-x_n\|^2+2\epsilon_n\lambda^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|S_{r_n}x_n-x_n\|^2 &\le \frac{1}{1-\beta}(\|x_n-v\|^2-\|x_{n+1}-v\|^2+2\epsilon_n\lambda^2)\\ &= \frac{1}{1-\beta}((\|x_n-v\|-\|x_{n+1}-v\|)(\|x_n-v\|+\|x_{n+1}-v\|)+2\epsilon_n\lambda^2)\\ &\le \frac{1}{1-\beta}(\|x_{n+1}-x_n\|(\|x_n-v\|+\|x_{n+1}-v\|)+2\epsilon_n\lambda^2).\end{aligned}$$

By $\|x_{n+1}-x_n\|\to 0$ and $\epsilon_n\to 0$, as $n\to\infty$, we obtain that

$$\lim_{n\to\infty}\|x_n-S_{r_n}x_n\|=0.$$

Step 5. Let $\omega(x_n)$ be the set of all weak ω-limits of $\{x_n\}$. We shall show that $\omega(x_n)\subset F\bigcap EP(G)$. It is a consequence of Step 4 and [12, Lemma 2.13] that $\omega(x_n)\subset EP(G)$.

So, it remains to prove that $z\in F$. To see this, we observe that we may assume that

$$\lambda_{n_m,k}\to\lambda_k\in(0,1)\quad \text{as } m\to\infty\ (k=1,2,\ldots,N).$$

Let K be the K-mapping generated by $T_1,T_2,\ldots,T_N$ and $\lambda_1,\ldots,\lambda_N$, then by Lemma 2.10, we have, for every $x\in C$,

$$K_{n_m}x\to Kx\quad \text{as } m\to\infty. \tag{3.4}$$

We will show that $z\in F=\bigcap_{i=1}^N F(T_i)$. Assume that there exists $j\in\{1,2,\ldots,N\}$ such that $z\neq T_jz$. By Lemma 2.9, we have $z\neq Wz$. Since $z\in EP(G)=F(S_{r_n})$, by Step 3, (3.4) and Opial's property of Hilbert space, we have

$$\begin{aligned}\liminf_{m\to\infty}\|x_{n_m}-z\| &< \liminf_{m\to\infty}\|x_{n_m}-Kz\|\\ &\le \liminf_{m\to\infty}(\|x_{n_m}-K_{n_m}S_{r_{n_m}}x_{n_m}\|+\|K_{n_m}S_{r_{n_m}}x_{n_m}-K_{n_m}S_{r_{n_m}}z\|+\|K_{n_m}S_{r_{n_m}}z-Kz\|)\\ &\le \liminf_{m\to\infty}\|x_{n_m}-z\|.\end{aligned}$$

This is a contradiction, then $z\in F=\bigcap_{i=1}^N F(T_i)$.

Step 6. Let x^* be the unique solution of the variational inequality,

$$\langle(A-\gamma f)x^*,x-x^*\rangle\ge 0,\quad \forall x\in F\bigcap EP(G). \tag{3.5}$$

We shall show that $\limsup_{n\to\infty}\langle(\gamma f-A)x^*,x_n-x^*\rangle\le 0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle(\gamma f-A)x^*,x_{n_k}-x^*\rangle=\limsup_{n\to\infty}\langle(\gamma f-A)x^*,x_n-x^*\rangle. \tag{3.6}$$

Without loss of generality, we may assume that $\{x_{n_k}\}$ weakly converges to some z in H. By Step 5, $z\in F\bigcap EP(G)$. Thus combining (3.5) and (3.6), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f-A)x^*,x_n-x^*\rangle &= \lim_{k\to\infty}\langle(\gamma f-A)x^*,x_{n_k}-x^*\rangle\\ &= \langle(\gamma f-A)x^*,z-x^*\rangle\le 0\end{aligned} \tag{3.7}$$

as required.

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Step 7. Finally, we will show that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F\bigcap EP(G)$. Let x^* be the unique fixed point of the mapping $P_{F\bigcap EP(G)}(I-(A-\gamma f))$, i.e the unique solution of the variational inequality (1.8). By Lemmas 2.4 and 2.7, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\epsilon_n\gamma f(x_n)+\beta x_n+((1-\beta)I-\epsilon_nA)K_nu_n-x^*\|^2\\
&= \|((1-\beta)I-\epsilon_nA)(K_nu_n-x^*)+\beta(x_n-x^*)+\epsilon_n(\gamma f(x_n)-Ax^*)\|^2\\
&\le \|((1-\beta)I-\epsilon_nA)(K_nu_n-x^*)+\beta(x_n-x^*)\|^2+2\epsilon_n\langle\gamma f(x_n)-Ax^*,x_{n+1}-x^*\rangle\\
&= \left\|\frac{(1-\beta)((1-\beta)I-\epsilon_nA)(K_nu_n-x^*)}{(1-\beta)}+\beta(x_n-x^*)\right\|^2\\
&\quad+2\epsilon_n\gamma\langle f(x_n)-f(x^*),x_{n+1}-x^*\rangle+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&\le (1-\beta)\left\|\frac{((1-\beta)I-\epsilon_nA)(K_nu_n-x^*)}{(1-\beta)}\right\|^2+\beta\|x_n-x^*\|^2\\
&\quad+2\epsilon_n\gamma\alpha\|x_n-x^*\|\|x_{n+1}-x^*\|+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&\le \frac{\|((1-\beta)I-\epsilon_nA)(K_nu_n-x^*)\|^2}{(1-\beta)}+\beta\|x_n-x^*\|^2\\
&\quad+2\epsilon_n\gamma\alpha\|x_n-x^*\|\|x_{n+1}-x^*\|+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&\le \frac{\|(1-\beta)I-\epsilon_nA\|^2}{(1-\beta)}\|K_nu_n-x^*\|^2+\beta\|x_n-x^*\|^2\\
&\quad+\epsilon_n\gamma\alpha[\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2]+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&\le \frac{(1-\beta-\epsilon_n\overline{\gamma})^2}{(1-\beta)}\|x_n-x^*\|^2+\beta\|x_n-x^*\|^2+\epsilon_n\gamma\alpha(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\
&\quad+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&= \left(\frac{(1-\beta-\epsilon_n\overline{\gamma})^2}{(1-\beta)}+\beta+\epsilon_n\gamma\alpha\right)\|x_n-x^*\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2\\
&\quad+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&= \left(\frac{(1-\beta)^2-2(1-\beta)\epsilon_n\overline{\gamma}+\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}+\beta+\epsilon_n\gamma\alpha\right)\|x_n-x^*\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2\\
&\quad+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&= \left((1-\beta)-2\epsilon_n\overline{\gamma}+\frac{\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}+\beta+\epsilon_n\gamma\alpha\right)\|x_n-x^*\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2\\
&\quad+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\\
&= \left(1-(2\overline{\gamma}-\alpha\gamma)\epsilon_n+\frac{\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}\right)\|x_n-x^*\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2+2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{1}{1-\epsilon_n\gamma\alpha}\left(1-(2\overline{\gamma}-\alpha\gamma)\epsilon_n+\frac{\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}\right)\|x_n-x^*\|^2+\frac{1}{1-\epsilon_n\gamma\alpha}(2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle)\\
&= \frac{1}{1-\epsilon_n\gamma\alpha}((1-(2\overline{\gamma}-\alpha\gamma)\epsilon_n))\|x_n-x^*\|^2\\
&\quad+\frac{1}{1-\epsilon_n\gamma\alpha}\left(2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle+\frac{\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}\|x_n-x^*\|^2\right)\\
&= \frac{1}{1-\epsilon_n\gamma\alpha}(1-2\epsilon_n\overline{\gamma}+\alpha\gamma\epsilon_n)\|x_n-x^*\|^2\\
&\quad+\frac{1}{1-\epsilon_n\gamma\alpha}\left(2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle+\frac{\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}\|x_n-x^*\|^2\right)\\
&= \frac{1}{1-\epsilon_n\gamma\alpha}(1-2\epsilon_n\overline{\gamma}+2\alpha\gamma\epsilon_n-\alpha\gamma\epsilon_n)\|x_n-x^*\|^2\\
&\quad+\frac{1}{1-\epsilon_n\gamma\alpha}\left(2\epsilon_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle+\frac{\epsilon_n^2\overline{\gamma}^2}{(1-\beta)}\|x_n-x^*\|^2\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{1-\epsilon_n\gamma\alpha}(1-\alpha\gamma\epsilon_n - 2\epsilon_n(\overline{\gamma}-\alpha\gamma))\|x_n - x^*\|^2 \\
&\quad + \frac{\epsilon_n}{1-\epsilon_n\gamma\alpha}\left(2\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle + \frac{\epsilon_n\overline{\gamma}^2}{(1-\beta)}\|x_n - x^*\|^2\right) \\
&= \left(1 - \frac{2\epsilon_n(\overline{\gamma}-\alpha\gamma)}{1-\epsilon_n\gamma\alpha}\right)\|x_n - x^*\|^2 + \frac{\epsilon_n}{1-\epsilon_n\gamma\alpha}\left(2\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle + \frac{\epsilon_n\overline{\gamma}^2}{(1-\beta)}\|x_n - x^*\|^2\right) \\
&= \left(1 - \frac{2\epsilon_n(\overline{\gamma}-\alpha\gamma)}{1-\epsilon_n\gamma\alpha}\right)\|x_n - x^*\|^2 + \frac{2(\overline{\gamma}-\alpha\gamma)}{2(\overline{\gamma}-\alpha\gamma)}\frac{\epsilon_n}{1-\epsilon_n\gamma\alpha} \\
&\quad \times \left(2\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle + \frac{\epsilon_n\overline{\gamma}^2}{(1-\beta)}\|x_n - x^*\|^2\right) \\
&= \left(1 - \frac{2\epsilon_n(\overline{\gamma}-\alpha\gamma)}{1-\epsilon_n\gamma\alpha}\right)\|x_n - x^*\|^2 + \frac{2\epsilon_n(\overline{\gamma}-\alpha\gamma)}{1-\epsilon_n\gamma\alpha} \\
&\quad \times \left(\frac{\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle}{(\overline{\gamma}-\alpha\gamma)} + \frac{\epsilon_n\overline{\gamma}^2}{2(1-\beta)(\overline{\gamma}-\alpha\gamma)}\|x_n - x^*\|^2\right).
\end{aligned} \tag{3.8}
$$

We can rewrite (3.8) as

$$\|x_{n+1} - x^*\|^2 \le (1-\xi_n)\|x_n - x^*\|^2 + \xi_n\delta_n$$

where $\xi_n = \frac{2\epsilon_n(\overline{\gamma}-\alpha\gamma)}{1-\epsilon_n\gamma\alpha}$ and $\delta_n = (\frac{\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle}{(\overline{\gamma}-\alpha\gamma)} + \frac{\epsilon_n\overline{\gamma}^2}{2(1-\beta)(\overline{\gamma}-\alpha\gamma)}\|x_n - x^*\|^2)$.

By our hypotheses it is easily verified that $\sum_{n=1}^{\infty}\xi_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$.

Therefore, by Lemma 2.2, we can conclude that $\|x_n - x^*\| \to 0$.

Since $\|u_n - x^*\| = \|S_{r_n}x_n - x^*\| \le \|x_n - x^*\|$, it follows that $u_n \to x^*$ in norm. This completes the proof. □

Remark. (1) If we take $N = 1$, $T_1 = S$ and $G(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$, then the iterative scheme (3.1) reduces to the following scheme:

$$x_1 \in H, \quad x_{n+1} = \epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)Sx_n, \tag{3.9}$$

which is a modification of the iterative scheme (1.3) and by Theorem 3.1 we observe that the conditions (C1) and (C2) are sufficient for strong convergence of the sequence $\{x_n\}$ generated by (3.9) to a fixed point of S.

(2) If we take $N = 1$, $T_1 = S$ and $A = I$, then the iterative scheme (3.1) reduces to the following scheme:

$$
\begin{cases}
x_1 \in C, \\
G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\
x_{n+1} = \epsilon_n f(x_n) + \beta x_n + (1 - \beta - \epsilon_n)Su_n,
\end{cases} \tag{3.10}
$$

which is a modification of the scheme in Theorem 1.2 defined by Takahashi and Takahashi [15], and by Theorem 3.1, we obtain strong convergence of the sequence $\{x_n\}$ generated by (3.10) under the sufficient conditions of Theorem 1.2 but without the condition (C3).

(3) If we take $N = 1$ and $T_1 = S$ in Theorem 3.1, the iterative scheme (3.1) reduces to the following scheme:

$$
\begin{cases}
x_1 \in H, G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in H, \\
x_{n+1} = \epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)Su_n,
\end{cases} \tag{3.11}
$$

which is a modification of the scheme in Theorem 1.3, and by Theorem 3.1, we obtain strong convergence of the sequence $\{x_n\}$ generated by (3.11) under some sufficient conditions without the condition (C3).

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Research Article

A New Iterative Method for Common Fixed Points of a Finite Family of Nonexpansive Mappings

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Recommended by Jie Xiao

Let X be a real uniformly convex Banach space and C a closed convex nonempty subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C. For a given $x_1 \in C$, let $\{x_n\}$ and $\{x_n^{(i)}\}$, $i = 1,2,\ldots,r$, be sequences defined $x_n^{(0)} = x_n$, $x_n^{(1)} = a_{n1}^{(1)}T_1x_n^{(0)} + (1 - a_{n1}^{(1)})x_n^{(0)}$, $x_n^{(2)} = a_{n2}^{(2)}T_2x_n^{(1)} + a_{n1}^{(2)}T_1x_n + (1 - a_{n2}^{(2)} - a_{n1}^{(2)})x_n$, $\ldots$, $x_{n+1} = x_n^{(r)} = a_{nr}^{(r)}T_rx_n^{(r-1)} + a_{n(r-1)}^{(r)}T_{r-1}x_n^{(r-2)} + \cdots + a_{n1}^{(r)}T_1x_n + (1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)})x_n$, $n \geq 1$, where $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\ldots,j$. In this paper, weak and strong convergence theorems of the sequence $\{x_n\}$ to a common fixed point of a finite family of nonexpansive mappings T_i $(i = 1,2,\ldots,r)$ are established under some certain control conditions.

1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let $T_i : C \to C$, $i = 1,2,\ldots,r$, be nonexpansive mappings. Let $\text{Fix}(T_i)$ denote the fixed points set of T_i, that is, $\text{Fix}(T_i) := \{x \in C : T_ix = x\}$, and let $F := \bigcap_{i=1}^r \text{Fix}(T_i)$.

For a given $x_1 \in C$, and a fixed $r \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(0)}\}, \{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(r)}\}$ by

$$x_n^{(0)} = x_n,$$

$$x_n^{(1)} = a_{n1}^{(1)}T_1x_n^{(0)} + \left(1 - a_{n1}^{(1)}\right)x_n^{(0)}$$

$$x_n^{(2)} = a_{n2}^{(2)} T_2 x_n^{(1)} + a_{n1}^{(2)} T_1 x_n + \left(1 - a_{n2}^{(2)} - a_{n1}^{(2)}\right) x_n,$$

$$\vdots$$

$$x_{n+1} = x_n^{(r)} = a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n$$

$$+ \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) x_n, \quad n \geq 1, \tag{1.1}$$

where $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\ldots,j$. If $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\ldots,r-1\}$ and $i = 1,2,\ldots,j$, then (1.1) reduces to the iterative scheme

$$x_{n+1} = S_n x_n, \quad n \geq 1, \tag{1.2}$$

where $S_n := a_{nr}^{(r)} T_r + a_{n(r-1)}^{(r)} T_{r-1} + \cdots + a_{n1}^{(r)} T_1 + (1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}) I$, $a_{ni}^{(r)} \in [0,1]$ for all $i = 1,2,\ldots,r$ and $n \in \mathbb{N}$.

If $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\ldots,r-1\}$, $i = 1,2,\ldots,j$ and $a_{ni}^{(r)} := \alpha_i$, for all $n \in \mathbb{N}$ for all $i = 1,2,\ldots,r$, then (1.1) reduces to the iterative scheme defined by Liu et al. [1]

$$x_{n+1} = S x_n, \quad n \geq 1, \tag{1.3}$$

where $S := \alpha_r T_r + \alpha_{r-1} T_{r-1} + \cdots + \alpha_1 T_1 + (1 - \alpha_r - \alpha_{r-1} - \cdots - \alpha_1) I$, $\alpha_i \geq 0$ for all $i = 2,3,\ldots,r$ and $1 - \alpha_r - \alpha_{r-1} - \cdots - \alpha_1 > 0$. They showed that $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of T_i, $i = 1,2,\ldots,r$, in Banach spaces, provided that T_i, $i = 1,2,\ldots,r$ satisfy condition A. The result improves the corresponding results of Kirk [2], Maiti and Saha [3] and Sentor and Dotson [4].

If $r = 2$ and $a_{n1}^{(2)} := 0$ for all $n \in \mathbb{N}$, then (1.1) reduces to a generalization of Mann and Ishikawa iteration given by Das and Debata [5] and Takahashi and Tamura [6]. This scheme dealts with two mappings:

$$x_n^{(1)} = a_{n1}^{(1)} T_1 x_n + \left(1 - a_{n1}^{(1)}\right) x_n,$$
$$x_{n+1} = x_n^{(2)} = a_{n2}^{(2)} T_2 x_n^{(1)} + \left(1 - a_{n2}^{(2)}\right) x_n, \quad n \geq 1, \tag{1.4}$$

where $\{a_{n1}^{(1)}\}$, $\{a_{n2}^{(2)}\}$ are appropriate sequences in $[0,1]$.

The purpose of this paper is to establish strong convergence theorems in a uniformly convex Banach space of the iterative sequence $\{x_n\}$ defined by (1.1) to a common fixed point of T_i $(i = 1,2,\ldots,r)$ under some appropriate control conditions in the case that one of T_i $(i = 1,2,\ldots,r)$ is completely continuous or semicompact or $\{T_i\}_{i=1}^r$ satisfies condition (B). Moreover, weak convergence theorem of the iterative scheme (1.1) to a common fixed point of T_i $(i = 1,2,\ldots,r)$ is also established in a uniformly convex Banach spaces having the Opial's condition.

2. Preliminaries

In this section, we recall the well-known results and give a useful lemma that will be used in the next section.

Recall that a Banach space X is said to satisfy *Opial's condition* [7] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$. A finite family of mappings $T_i : C \to C$ $(i = 1,2,\ldots,r)$ with $F := \bigcap_{i=1}^{r}\mathrm{Fix}(T_i) \neq \emptyset$ is said to satisfy *condition (B)* [8] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\max_{1\le i\le r}\{\|x - T_i x\|\} \ge f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf\{\|x - p\| : p \in F\}$.

Lemma 2.1 (see [9, Theorem 2]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|), \tag{2.1}$$

for all x, y in $B_r = \{x \in X : \|x\| \le r\}$, $\lambda \in [0,1]$, where

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda). \tag{2.2}$$

Lemma 2.2 (see [10, Lemma 1.6]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in Fix(T)$.*

Lemma 2.3 (see [11, Lemma 2.7]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let u, $v \in X$ be such that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.4. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. Then for each $n \in \mathbb{N}$, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|^2 \le \sum_{i=1}^{n}\alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|), \tag{2.3}$$

for all $x_i \in B_r$ and all $\alpha_i \in [0,1]$ $(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

Proof. Clearly (2.3) holds for $n = 1,2$, by Lemma 2.1. Next, suppose that (2.3) is true when $n = k-1$. Let $x_i \in B_r$ and $\alpha_i \in [0,1]$, $i = 1,2,\ldots,k$ with $\sum_{i=1}^{k}\alpha_i = 1$. Then $\alpha_{k-1}/(1 - \sum_{i=1}^{k-2}\alpha_i)x_{k-1} + \alpha_k/(1 - \sum_{i=1}^{k-2}\alpha_i)x_k \in B_r$. By Lemma 2.1, we obtain that

$$\left\|\frac{\alpha_{k-1}}{1-\sum_{i=1}^{k-2}\alpha_i}x_{k-1} + \frac{\alpha_k}{1-\sum_{i=1}^{k-2}\alpha_i}x_k\right\|^2 \le \frac{\alpha_{k-1}}{1-\sum_{i=1}^{k-2}\alpha_i}\|x_{k-1}\|^2 + \frac{\alpha_k}{1-\sum_{i=1}^{k-2}\alpha_i}\|x_k\|^2. \tag{2.4}$$

By the inductive hypothesis, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\left\|\sum_{i=1}^{k-1}\beta_i y_i\right\|^2 \le \sum_{i=1}^{k-1}\beta_i\|y_i\|^2 - \beta_1\beta_2 g(\|y_1 - y_2\|) \tag{2.5}$$

for all $y_i \in B_r$ and all $\beta_i \in [0,1]$, $i = 1,2,\dots,k-1$ with $\sum_{i=1}^{k-1}\beta_i = 1$. It follows that

$$\begin{aligned}
\left\|\sum_{i=1}^{k}\alpha_i x_i\right\|^2 &= \left\|\sum_{i=1}^{k-2}\alpha_i x_i + \left(1-\sum_{i=1}^{k-2}\alpha_i\right)\left(\frac{\alpha_{k-1}x_{k-1}}{1-\sum_{i=1}^{k-2}\alpha_i} + \frac{\alpha_k x_k}{1-\sum_{i=1}^{k-2}\alpha_i}\right)\right\|^2 \\
&\le \sum_{i=1}^{k-2}\alpha_i\|x_i\|^2 + \left(1-\sum_{i=1}^{k-2}\alpha_i\right)\left\|\frac{\alpha_{k-1}x_{k-1}}{1-\sum_{i=1}^{k-2}\alpha_i} + \frac{\alpha_k x_k}{1-\sum_{i=1}^{k-2}\alpha_i}\right\|^2 - \alpha_1\alpha_2 g(\|x_1-x_2\|) \\
&\le \sum_{i=1}^{k-2}\alpha_i\|x_i\|^2 + \left(1-\sum_{i=1}^{k-2}\alpha_i\right)\left(\frac{\alpha_{k-1}\|x_{k-1}\|^2}{1-\sum_{i=1}^{k-2}\alpha_i} + \frac{\alpha_k\|x_k\|^2}{1-\sum_{i=1}^{k-2}\alpha_i}\right) - \alpha_1\alpha_2 g(\|x_1-x_2\|) \\
&= \sum_{i=1}^{k}\alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1-x_2\|).
\end{aligned} \tag{2.6}$$

Hence, we have the lemma. □

3. Main Results

In this section, we prove weak and strong convergence theorems of the iterative scheme (1.1) for a finite family of nonexpansive mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

The next lemma is crucial for proving the main theorems.

Lemma 3.1. *Let X be a Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^{r}$ be a finite family of nonexpansive self-mappings of C. Let $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\dots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\dots,j$. For a given $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.1). If $F \neq \emptyset$, then $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. For each $n \ge 1$, we note that

$$\begin{aligned}
\left\|x_n^{(1)} - p\right\| &= \left\|a_{n1}^{(1)}T_1x_n + \left(1-a_{n1}^{(1)}\right)x_n - p\right\| \\
&\le a_{n1}^{(1)}\|T_1x_n - p\| + \left(1-a_{n1}^{(1)}\right)\|x_n - p\| \\
&\le a_{n1}^{(1)}\|x_n - p\| + \left(1-a_{n1}^{(1)}\right)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$\begin{aligned}\left\|x_n^{(2)}-p\right\| &= \left\|a_{n2}^{(2)}T_2x_n^{(1)}+a_{n1}^{(2)}T_1x_n+\left(1-a_{n2}^{(2)}-a_{n1}^{(2)}\right)x_n-p\right\|\\ &\le a_{n2}^{(2)}\left\|T_2x_n^{(1)}-p\right\|+a_{n1}^{(2)}\|T_1x_n-p\|+\left(1-a_{n2}^{(2)}-a_{n1}^{(2)}\right)\|x_n-p\|\\ &\le a_{n2}^{(2)}\left\|x_n^{(1)}-p\right\|+a_{n1}^{(2)}\|x_n-p\|+\left(1-a_{n2}^{(2)}-a_{n1}^{(2)}\right)\|x_n-p\|\\ &\le \|x_n-p\|.\end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}\left\|x_n^{(3)}-p\right\| &= \left\|a_{n3}^{(3)}T_3x_n^{(2)}+a_{n2}^{(3)}T_2x_n^{(1)}+a_{n1}^{(3)}T_1x_n+\left(1-a_{n3}^{(3)}-a_{n2}^{(3)}-a_{n1}^{(3)}\right)x_n-p\right\|\\ &\le a_{n3}^{(3)}\left\|T_3x_n^{(2)}-p\right\|+a_{n2}^{(3)}\left\|T_2x_n^{(1)}-p\right\|+a_{n1}^{(3)}\|T_1x_n-p\|\\ &\quad+\left(1-a_{n3}^{(3)}-a_{n2}^{(3)}-a_{n1}^{(3)}\right)\|x_n-p\|\\ &\le a_{n3}^{(3)}\left\|x_n^{(2)}-p\right\|+a_{n2}^{(3)}\left\|x_n^{(1)}-p\right\|+a_{n1}^{(3)}\|x_n-p\|\\ &\quad+\left(1-a_{n3}^{(3)}-a_{n2}^{(3)}-a_{n1}^{(3)}\right)\|x_n-p\|\\ &\le \|x_n-p\|.\end{aligned} \tag{3.3}$$

By continuing the above argument, we obtain that

$$\left\|x_n^{(i)}-p\right\| \le \|x_n-p\| \quad \forall i=1,2,\ldots,r. \tag{3.4}$$

In particular, we get $\|x_{n+1}-p\| \le \|x_n-p\|$ for all $n \in \mathbb{N}$, which implies that $\lim_{n\to\infty}\|x_n-p\|$ exists. □

Lemma 3.2. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\ldots,r\}$, $n \in \mathbb{N}$ and $i=1,2,\ldots,j$ such that $\sum_{i=1}^j a_{ni}^{(j)}$ are in $[0,1]$ for all $j \in \{1,2,\ldots,r\}$ and $n \in \mathbb{N}$. For a given $x_1 \in C$, let $\{x_n\}$ be defined by (1.1). If $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \le \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then*

(i) $\lim_{n\to\infty}\|T_ix_n^{(i-1)}-x_n\|=0$ *for all* $i=1,2,\ldots,r$,

(ii) $\lim_{n\to\infty}\|T_ix_n-x_n\|=0$ *for all* $i=1,2,\ldots,r$,

(iii) $\lim_{n\to\infty}\|x_n^{(i)}-x_n\|=0$ *for all* $i=1,2,\ldots,r$.

Proof. (i) Let $p \in F$, by Lemma 3.1, $\sup_n\|x_n-p\| < \infty$. Choose a number $s > 0$ such that $\sup_n\|x_n-p\| < s$, it follows by (3.4) that $\{x_n^{(i)}-p\}, \{T_ix_n^{(i-1)}-p\} \subseteq B_s$, for all $i \in \{1,2,\ldots,r\}$. □

By Lemma 2.4, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \le \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|), \tag{3.5}$$

for all $x_i \in B_s$, $\alpha_i \in [0,1]$ $(i = 1,2,\dots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$. By (3.4) and (3.5), we have for $i = 1,2,\dots,r$,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \Big\| a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n \\
&\quad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) x_n - p \Big\|^2 \\
&\le a_{nr}^{(r)} \left\| T_r x_n^{(r-1)} - p \right\|^2 + a_{n(r-1)}^{(r)} \left\| T_{r-1} x_n^{(r-2)} - p \right\|^2 + \cdots \\
&\quad + a_{n1}^{(r)} \|T_1 x_n - p\|^2 + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\quad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\| \right) \\
&\le a_{nr}^{(r)} \left\| x_n^{(r-1)} - p \right\|^2 + a_{n(r-1)}^{(r)} \left\| x_n^{(r-2)} - p \right\|^2 + \cdots + a_{n1}^{(r)} \|x_n - p\|^2 \\
&\quad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\quad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\| \right) \\
&\le a_{nr}^{(r)} \|x_n - p\|^2 + a_{n(r-1)}^{(r)} \|x_n - p\|^2 + \cdots + a_{n1}^{(r)} \|x_n - p\|^2 \\
&\quad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\quad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\| \right) \\
&= \|x_n - p\|^2 - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\| \right).
\end{aligned} \tag{3.6}$$

Therefore

$$a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\| \right) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{3.7}$$

for all $i = 1,2,\dots,r$. Since $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \le \limsup_{n\to\infty} (a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, it implies by Lemma 3.1 that $\lim_{n\to\infty} g(\|T_i x_n^{(i-1)} - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T_i x_n^{(i-1)} - x_n\| = 0$ for all $i = 1,2,\dots,r$.

(ii) For $i \in \{1,2,\dots,r\}$, we have

$$\begin{aligned}\|T_i x_n - x_n\| &\le \left\|T_i x_n - T_i x_n^{(i-1)}\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\| \\ &\le \left\|x_n - x_n^{(i-1)}\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\| \\ &\le \sum_{j=1}^{i-1} a_{nj}^{(i-1)} \left\|T_j x_n^{(j-1)} - x_n\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\|.\end{aligned} \tag{3.8}$$

It follows from (i) that

$$\|T_i x_n - x_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.9}$$

(iii) For $i \in \{1,2,\dots,r\}$, it follows from (i) that

$$\left\|x_n^{(i)} - x_n\right\| \le \sum_{j=1}^{i} a_{nj}^{(i)} \left\|T_j x_n^{(j-1)} - x_n\right\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.10}$$

Theorem 3.3. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let the sequence $\{a_{ni}^{(j)}\}_{n=1}^{\infty}$ be as in Lemma 3.2. For a given $x_1 \in C$, let sequences $\{x_n\}$ and $\{x_n^{(i)}\}$ $(i = 0,1,\dots,r)$ be defined by (1.1). If one of $\{T_i\}_{i=1}^r$ is completely continuous then $\{x_n\}$ and $\{x_n^{(j)}\}$ converge strongly to a common fixed point of $\{T_i\}_{i=1}^r$ for all $j = 1,2,\dots,r$.*

Proof. Suppose that T_{i_0} is completely continuous where $i_0 \in \{1,2,\dots,r\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_{i_0} x_{n_k}\}$ converges. □

Let $\lim_{k\to\infty} T_{i_0} x_{n_k} = q$ for some $q \in C$. By Lemma 3.2 (ii), $\lim_{n\to\infty}\|T_{i_0} x_n - x_n\| = 0$. It follows that $\lim_{k\to\infty} x_{n_k} = q$. Again by Lemma 3.2(ii), we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1,2,\dots,r$. It implies that $\lim_{k\to\infty} T_i x_{n_k} = q$. By continuity of T_i, we get $T_i q = q$, $i = 1,2,\dots,r$. So $q \in F$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - q\|$ exists, it follows that $\lim_{n\to\infty}\|x_n - q\| = 0$. By Lemma 3.2(iii), we have $\lim_{n\to\infty}\|x_n^{(j)} - x_n\| = 0$ for each $j \in \{1,2,\dots,r\}$. It follows that $\lim_{n\to\infty} x_n^{(j)} = q$ for all $j = 1,2,\dots,r$.

Theorem 3.4. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let the sequence $\{a_{ni}^{(j)}\}_{n=1}^{\infty}$ be as in Lemma 3.2. For a given $x_1 \in C$, let sequences $\{x_n\}$ and $\{x_n^{(i)}\}$ $(i = 0,1,\dots,r)$ be defined by (1.1). If the family $\{T_i\}_{i=1}^r$ satisfies condition (B) then $\{x_n\}$ and $\{x_n^{(j)}\}$ converge strongly to a common fixed point of $\{T_i\}_{i=1}^r$ for all $j = 1,2,\dots,r$.*

Proof. Let $p \in F$. Then by Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists and $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $n \ge 1$. This implies that $d(x_{n+1}, F) \le d(x_n, F)$ for all $n \ge 1$, therefore, we get $\lim_{n\to\infty} d(x_n, F)$ exists. By Lemma 3.2(ii), we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for each $i = 1,2,\dots,r$. It follows, by the condition (B) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing and $f(0) = 0$, therefore, we get $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since

$\lim_{n\to\infty} d(x_n, F) = 0$, given any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \epsilon/2$ for all $n \geq n_0$. In particular, $d(x_{n_0}, F) < \epsilon/2$. Then there exists $q \in F$ such that $\|x_{n_0} - q\| < \epsilon/2$. For all $n \geq n_0$ and $m \geq 1$, it follows by Lemma 3.1 that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\| \leq \|x_{n_0} - q\| + \|x_{n_0} - q\| < \epsilon. \tag{3.11}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C, hence it must converge to a point of C. Let $\lim_{n\to\infty} x_n = p^*$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and F is closed, we obtain $p^* \in F$. By Lemma 3.2(iii), $\lim_{n\to\infty} \|x_n^{(j)} - x_n\| = 0$ for each $j \in \{1, 2, \ldots, r\}$. It follows that $\lim_{n\to\infty} x_n^{(j)} = p^*$ for all $j = 1, 2, \ldots, r$. □

In Theorem 3.4, if $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1, 2, \ldots, r-1\}$ and $i = 1, 2, \ldots, j$, we obtain the following result.

Corollary 3.5. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(r)} \in [0,1]$ for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N}$ such that $\sum_{i=1}^r a_{ni}^{(r)}$ are in $[0,1]$ for all $n \in \mathbb{N}$. For a given $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.2). If the family $\{T_i\}_{i=1}^r$ satisfies condition (B) and $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty} (a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Remark 3.6. In Corollary 3.5, if $a_{ni}^{(r)} = \alpha_i$, for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, r$, the iterative scheme (1.2) reduces to the iterative scheme (1.3) defined by Liu et al. [1] and we obtain strong convergence of the sequence $\{x_n\}$ defined by Liu et al. when $\{T_i\}_{i=1}^r$ satisfies condition (B) which is different from the condition (A) defined by Liu et al. and we note that the result of Senter and Dotson [4] is a special case of Theorem 3.4 when $r = 1$.

In the next result, we prove weak convergence for the iterative scheme (1.1) for a finite family of nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.7. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined by (1.1). If the sequence $\{a_{ni}^{(j)}\}_{n=1}^\infty$ is as in Lemma 3.2, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Proof. By Lemma 3.2(ii), $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, r$. Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality we may assume that $x_n \to u$ weakly as $n \to \infty$ for some $u \in C$. By Lemma 2.2, we have $u \in F$. Suppose that there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ that converge weakly to u and v, respectively. From Lemma 2.2, we have $u, v \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.3 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$. □

For $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1, 2, \ldots, r-1\}$ and $i = 1, 2, \ldots, j$ in Theorem 3.7, we obtain the following result.

Corollary 3.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of X. Let* $\{T_i\}_{i=1}^r$ *be a finite family of nonexpansive self-mappings of C with* $F \neq \emptyset$ *and* $a_{ni}^{(r)} \in [0,1]$ *for all* $i = 1, 2, \ldots, r$ *and* $n \in \mathbb{N}$ *such that* $\sum_{i=1}^r a_{ni}^{(r)}$ *are in* $[0,1]$ *for all* $n \in \mathbb{N}$*. For a given* $x_1 \in C$*, let* $\{x_n\}$ *be the sequence defined by* (1.2)*. If* $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$*, then the sequence* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^r$*.*

Remark 3.9. In Corollary 3.8, if $a_{ni}^{(r)} = \alpha_i$, for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, r$, then we obtain weak convergence of the sequence $\{x_n\}$ defined by Liu et al. [1].

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