โครงการ "การพยากรณ์ความเสี่ยงบนพื้นฐานของการอินทิเกรตแบบเศษส่วน"

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รายงานวิจัยฉบับสมบูรณ์

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย

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สารบัญ

บทคัดย่อ

งานวิจัยนี้ทำการศึกษาวิธีการปรับแต่ง อนุกรมเวลาต่อเนื่อง GARCH(1,1) ให้เป็นอนุกรมเวลาที่มีหน่วยความจำยาวมากขึ้น ด้วยวิธีการเปลี่ยนการเคลื่อนที่บราวเนี่ยนมาตราฐานให้เป็นการเคลื่อนที่บราวเนี่ยนเศษส่วนซึ่งที่มีความจำยาว และเรียกอนุกรมเวลาใหม่นี้ว่า FIGARCH(1,1) แต่ด้วยเหตุที่อนุกรมเวลาที่มีความจำยาวมีธรรมชาติของความเป็น อาร์บิทาจ ปนอยู่ จึงไม่เหมาะที่จะนำมาใช้ในการกำหนดราคาของออพชัน เราจึงได้เสนอแนวทางแก้ไขด้วยการสร้างตัวแบบประมาณค่าของอนุกรมเวลาที่มีหน่วยความจำยาวและไม่มีอาร์บิทาจ ต่อจากนั้นกิจกรรมต่างๆในทางคณิตศาสตร์การเงิน ดังเช่น การกำหนดราคาของออพชัน การประมาณค่าความผันผวน การคำนวณค่าพารามิเตอร์ของตัวแบบอัตราดอกเบี้ย จะทำบนตัวแบบประมาณค่านี้ ได้มีการพิจารณาตัวแบบหน่วยความจำยาวที่มีการกระโดดด้วย และได้มีการพัฒนาโปรแกรมประยุกต์เพื่อการคำนวณหาค่าของพารามิเตอร์และราคาพันธบัตรจากสูตรต่างๆที่ได้พัฒนาไว้ข้างต้น

คำสำคัญ กระบานการมีหน่วยความจำยาว กระบวนการเศษส่วน ตัวแบบGARCH(1,1) อาร์บิทาจ กระบวนการเศษส่วนที่มีการกระโดด

Abstract

The purpose of this research project is to modify the continuous time GARCH (1, 1) model in such a way that it can explain a long memory effect. To do this, we change the standard Brownian motion into a fractional Brownian motion and call it FIGARCH (1, 1) model. Since the long memory process has an arbitrage. We then develop an approximate model of FIGARCH (1, 1) which do not have an arbitrage. After that we shall use this approximate model to find a formula for a European call option, estimate parameters for some interest rate models, and bond pricing. We also discuss the long memory model with jump. An application software for computing parameters of some interest rate models was purposed.

Keywords: Long memory process, fractional process, GARCH(1,1) model, arbitrage, fractional process with jump.

Executive Summary
บทสรุปย่อสำหรับผู้บริหาร

วัตถุประสงค์ของการวิจัย

1. To develop a method for adding long memory effects into continuous time GARCH(1,1) models. This new model is called FIGARCH (1,1).
2. To construct an approximate model for FIGARCH (1,1) and find a European call option by using the approximate model
3. To investigate FIGARCH model with jump.
4. To find a formula for European call option when the asset prices follows a stochastic volatility Levy model.
5. To develop an application software for calculating parameters of the CIR model, Vasicek interest rate models, and bond pricing.

สิ่งที่ได้ดำเนินการไปแล้ว

1. We constructed an approximate model for continuous time FIGARCH (1,1) model. We proved the convergence of the approximate model to the FIGARCH (1,1) model.
2. By using FIGARCH (1,1) model, we developed a formula for pricing a European call option. We showed by simulation that pricing in FIGARCH (1,1) model can reduce error significantly when compare with the original GARCH model.
3. We investigated FIGARCH (1,1) model with jump.
4. We investigated the option pricing when the underlying asset follows a stochastic Levy model with stochastic interest rate.
5. We develop an application software for calculating parameters of the CIR model, Vasicek interest rate models, and bond pricing.

สิ่งที่ได้พบ

We found that pricing of the contingent claims under FIGARCH model is more realistic than the original GARCH model in the sense that error of the approximation can be reduced significantly and this is consistent with the nature of the return series which is not Markovian.

END OF EXECUTIVE SUMMARY

เนื้อหางานวิจัย

1. ตามวัตถุประสงค์ข้อที่หนึ่ง

By a Fractional Integrated GARCH (1,1) model of continuous time (FIGARCH), we shall mean a process of the form

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dB_t^H \tag{1}$$

where $0 \le t \le T,\ \theta, \xi$ are weight parameters and B_t^H is a fractional Brownian motion. B_t^H is defined by

$$B_t^H = \int_0^t (t-s)^{-\alpha} dB_t ,$$

where $\alpha = (1/2) - H$ and B_t is the standard Brownian motion. For each $\varepsilon > 0,$ an approximate model of FIGARCH is a process of the form

$$dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon)dt + \xi v_t^\varepsilon dB_t^\varepsilon , \tag{2}$$

where B_t^ε is the approximate process of is B_t^H and is defined by

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{-\alpha} dB_s$$

One can show that B_t^ε converges to B_t^H in $L_2(\Omega)$ as $\varepsilon \to 0$, uniformly with respect to $t \in [0,T].$ We have proved the following two main results.

Theorem 1 For any $\varepsilon > 0,$ a solution of the approximate model (2) is given by

$$v_t^\varepsilon = \exp\left[\xi B_t^\varepsilon - \left(\theta + \frac{1}{2}\xi^2 \varepsilon^{2\alpha}\right)t\right]\left[v_0^\varepsilon + \omega \int_0^t e^{\left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)s - \xi B_s^\varepsilon} ds\right]$$

where $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$ and $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s$.

Theorem 2 The solution (3) converges to the solution of (1) in $L^2(\Omega)$ and uniformly with respect to $t \in [0,T]$ as $\varepsilon \to 0$.

Moreover, we do numerical simulation to confirm that the volatility that come from the approximate model give a better approximation to the volatility of SCB stock price than the volatility that come from the continuous time GARCH model when ε is small enough. [The proof can be found in Ref. 1]

2. ตามวัตถุประสงค์ข้อที่สองและข้อที่สาม

In the second paper, we are interested in the option pricing when the stock pricing follows jump diffusion model and their stochastic volatility follows a fractional stochastic volatility model, i.e. our dynamic system is of the form:

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}\,dB_t\right) + S_{t-}YdN_t \qquad (3)$$
$$dv_t = (\omega - \theta v_t)dt + \xi v_t dB_t^H .$$

We can not price a European call option by using the dynamic (3) directly since it contains a fractional process and hence an arbitrage exists. To solve the problem, we consider an approximate model of the form

$$dS_t^\varepsilon = S_t^\varepsilon\left(\mu dt + \sqrt{v_t^\varepsilon}\,dB_t\right) + S_{t-}^\varepsilon YdN_t \qquad (4)$$
$$dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon)dt + \xi v_t^\varepsilon dB_t^\varepsilon .$$

Once again, one can show that the solution processes S_t^ε and v_t^ε of equation (4) converge to the solutions S_t^ε and v_t^ε of equation (3) in $L^2(\Omega)$ and uniformly with respect to $t \in [0,T]$ as $\varepsilon \to 0$. We derived a formula for European call option on the approximate model (4) and we get the following main result.

Theorem 3 For each $\varepsilon > 0$, the value of a European call option written on the model (4) is

$$\hat{C}(S_t^\varepsilon, v_t^\varepsilon, t, K, T) = S_t^\varepsilon \hat{P}_1(S_t^\varepsilon, v_t^\varepsilon, t, K, T) - Ke^{-r(T-t)}\hat{P}_2(S_t^\varepsilon, v_t^\varepsilon, t, K, T)$$

where P_1 is the risk neutral probability that $S_T > K$ and P_2 is the risk neutral in the money probability. [The proof can be found in Ref. 2]

3. ตามวัตถุประสงค์ข้อที่สี่

In the third paper, we are interested in the option pricing again but the jump process is not the compound Poisson process as appeared in the second paper. Here, we assume that the jump process is a pure Levy process and the interest rate is not constant but it is stochastic and satisfies the Hull-White process.

Our models are of the form:

$$S_t = S_0 \exp(r_t + (\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t) + J_{T_t}) \qquad (5)$$
$$dr_t = (\alpha(t) - \beta r_t)dt + \sigma_r dB_t^r \qquad (6)$$

where B_t^r is a standard Brownian motion with respect to the process r_t, $\sigma_r > 0$ the volatility coefficient of the interest rate process (6), and J_{T_t} the pure Levy process.

The process T_t is defined by

$$T_t = \int_0^t v_s ds ,$$

where v_t follows the CIR process

$$dv_t = \gamma(1 - v_t)dt + \sigma_v \sqrt{v_t}\,dB_t^v . \qquad (7)$$

When working with the T-forward measure Q^T , we can find a formula of European call option for which the asset price, the interest rate, and the volatility satisfy the dynamic (5),(6), and (7) respectively. Hence, we have our main results.

Theorem 4 The value of a European call option of SDE (5)-(7) is

$$C(t,S_t,r_t,\nu_t;T,K)=S_t\tilde{P}_1(t,X_t,r_t,\nu_t;T,\kappa)-KP^*(t,T)\tilde{P}_2(t,X_t,r_t,\nu_t;T,\kappa).$$

[The proof can be found in Ref. 3]

4. ตามวัตถุประสงค์ข้อ 5

The fourth paper involved in writing a manual for an application software for computing parameters for CIR model, Vasicek interest rate model, and also finding the bond price. One CD which contains the software is attached to this report. [The detail can be found in Ref. 4].

Up to this point, we have finished the work according to the objective that was written in the research project. However, we would like to submit another 3 supplementary research articles which were undertaken under this research project.

Supplementary research articles

5. The fifth paper studied an insurance model where the surplus process can be controlled by two activities, one is reinsurance for which the reinsurance company has an opportunity to default and other is an investment in a financial market. We prove the existence of an optimal plan and derive a formula for the value function which is the minimum of total discounted cost function in the framework of discrete-time surplus process. The main theorem is as follow:

Theorem 5 Let $x\in S$ be an initial state. Then there exists $u^*=(b^*,\delta^*)\in U$ such that

$$G(u^*)=\min_{(b,\delta)\in U}E[e^{-\beta(c(b)Z-\{h(b,Y)K+Y(1-K)+\langle\delta,R\rangle}]<\infty$$

and, moreover, $u^*-\textit{stationary}$ is an optimal plan. Here $b^*\in[\underline{b},\overline{b}]$ represents the retention level of reinsurance and $\delta^*=(\delta_1,\delta_2,...,\delta_m)$ is the portfolio vector. The cost function $G(u):=G(b,\delta)$ is a function of retention level and the portfolio vector. So this theorem say that, with some assumptions as given in the paper, the optimal control (or minimum plan) is stationary, i.e. we should select the same retention level and the same investment on every period of time. The proof of this theorem can be found in . [The proof can be found in Ref. 5]

6. The sixth paper studied an insurance model under the condition that the claims can be control by reinsurance and an insurance company requires a sufficient initial capital to ensure a ruin probability will not exceed a given quantity α . We prove the main theorem is about the existence of the minimum initial capital which was stated as follow.

Theorem 6 Let $N \in \{1,2,3,...\}$, $\pi \in P(N,[\underline{b},\overline{b}])$ and let $\alpha \in (0,1)$. Then there exists $x^* \geq 0$ such that

$$x^* = \min_{x \geq 0} \{ x : \Phi_N(x,\pi) \leq \alpha \},$$

Where is $P(N,[\underline{b},\overline{b}])$ the set of all plans and $\Phi_N(x,\pi)$ is the ruin probability at time n.

[The proof can be found in Ref. 6]

7. The seven studied forecasting the volatility of gold price using Markov Regime Switching GARCH models. These models allow volatility to have different dynamics according to unobserved regime variables. The main purpose is to find out whether MRS-GARCH models are an improvement on the GARCH type models in terms of modeling and forecasting gold price volatility. The MRS-GARCH is the best performance model for gold price volatility in some loss function. Moreover, we forecast the closing prices of gold price to trade future contract. MRS-GARCH got the most cumulative return same as GIR model. [The details are in Ref. 7].

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3. S. Pinkam, and **P. Sattayatham**, Option pricing for a stochastic volatility Levy model with stochastic interest rate. To be published in the Journal of Korean Statistical Society, 2012.
4. S. Pinkam and **P. Sattayatham**, Manual and One CD which contains as application software for computing parameters for CIR model, Vasicek interest rate model, and also finding the bond price.
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6. **P. Sattayatham,** K. Chuarkham, and W. Klongdee, Run probability based initial capital of the discrete time surplus process in insurance under reinsurance as a control parameter, Submitted to Scandinavian Actuarial Journal 2011.
7. N. Sopipan, **P. Sattayatham,** and B. Premanode, Forecasting volatility of gold price using Markov regime switching and trading strategy, Accepted in the Journal of mathematical finance 2011.

ผลที่ได้จากโครงการ

We wrote 6 articles and developed an application software for computing parameters of CIR model, Vasicek interest rate model, and also finding the bond price.

The details are as follows:

1. **Articles according to the objective of this research project**

   1.1. T. Plienpanich, **P. Sattayatham**, and T. Thao, Fractional integrated GARCH diffusion limit model, Journal of Korean Statistical Society, 38: 231-238, 2009

   1.2. **P. Sattayatham**, A. Intarasit, An approximate formula of European option for fractional stochastic volatility jump-diffusion model, Journal of Mathematics and Statistics, 7(3): 230-238, 2011.

   1.3. S. Pinkam, and **P. Sattayatham**, Option pricing for a stochastic volatility Levy model with stochastic interest rate. Submitted to Journal of Korean Statistical Society, 2011.

   1.4. S. Pinkam and **P. Sattayatham**, Manual and One CD which contains as application software for computing parameters for CIR model, Vasicek interest rate model, and also finding the bond price.

2. **Subplementary articles**

   2.1. W. Klongdee, **P. Sattayatham**, and K. Sangaroon, A value function of discrete-time surplus process in insurance under investment and reinsurance credit risk, Far East Journal of Theoretical Statistics, 32(2): 183-198, 2010.

   2.2. **P. Sattayatham**, K. Chuarkham, W. Klongdee, Ruin probability based initial capital of the discrete time surplus process in insurance under reinsurance as a control parameter, Submitted to Scandinavian Actuarial Journal. 2011.

   2.3. N. Sopipan, **P. Sattayatham**, B. Premanode, Forecasting volatility of Gold price using Markov regime switching and trading strategy, Accepted in Journal of Mathematical Finance, 2011.

ภาคผนวก

(Appendix)

ประกอบด้วยเอกสารแนบ 7 ฉบับดังนี้

1. **Articles according to the objective of this research project**

1.1. T. Plienpanich, **P. Sattayatham**, and T. Thao, Fractional integrated GARCH diffusion limit model, Journal of Korean Statistical Society, 38: 231-238, 2009

1.2. **P. Sattayatham**, A. Intarasit, An approximate formula of European option for fractional stochastic volatility jump-diffusion model, Journal of Mathematics and Statistics, 7(3): 230-238, 2011.

1.3. S. Pinkam, and **P. Sattayatham**, Option pricing for a stochastic volatility Levy model with stochastic interest rate. Submitted to Journal of Korean Statistical Society, 2011.

1.4. S. Pinkam and **P. Sattayatham**, Manual and One CD which contains as application software for computing parameters for CIR model, Vasicek interest rate model, and also finding the bond price.

2. **Subplementary articles**

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1.2. **P. Sattayatham**, K. Chuarkham, W. Klongdee, Ruin probability based initial capital of the discrete time surplus process in insurance under reinsurance as a control parameter, Submitted to Scandinavian Actuarial Journal. 2011.

1.3. N. Sopipan, **P. Sattayatham**, B. Premanode, Forecasting volatility of Gold price using Markov regime switching and trading strategy, Accepted in Journal of Mathematical Finance, 2011.


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Fractional integrated GARCH diffusion limit models☆

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ABSTRACT

In this paper, we introduce an approximate approach to the fractional integrated GARCH(1,1) model of continuous time perturbed by fractional noise. Based on the L^2-approximation of this noise by semimartingales, we proved a convergence theorem concerning an approximate solution. A simulation example shows a significant reduction of error in a fractional stock price model as compared to the classical stock price model.

1. Introduction

Risk in a financial market is measured by using volatility. So predictability of volatility has important implications for risk management. If volatility increases, so will Value At Risk (VAR). Investors may want to adjust their portfolio to reduce their exposure to those assets whose volatility is predicted to increase. One method that is widely employed for volatility estimation is to use GARCH models. A discrete time GARCH(1,1) model is a model of the form

$$\nu_{k+1} = \omega_0 + \beta\nu_{k+1} + \alpha\nu_k U_k^2, \qquad X_k = \sigma_k U_k \tag{1}$$

where $\sigma_k = \sqrt{\nu_k}$, and α, β are weight parameters, ω_0 is a parameter related to the long-term variance, and U_K is a sequence of independent normal random variables with zero mean and variance of 1.

It is well known that GARCH models are not designed for long range-dependence (LRD). But there are some empirical studies showing that log-return series (X_t) of foreign exchange rates, stock indices and share prices exhibit the LRD effect (see, for example, Mikosch and Starica (2003, page 445)). In 1990, Nelson (1990) showed that as the time interval decreases and become infinitesimal, Eq. (1) can be changed to

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}W_t \tag{2}$$

where $v_t = \sigma_t^2$ is the stock-return variance, ω, θ and ξ are weight parameters and W_t is a standard Brownian motion process. To be more accurate, there is a weak convergence of the discrete GARCH process to the continuous diffusion limit. The purpose of this paper is to introduce LRD effect into GARCH models of continuous time (i.e., into Eq. (2)). The importance of this process in finance is that it can be used to forecast volatility and risk of some financial instruments.

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Recall that a fractional Brownian motion process W_t^H, with Hurst index H, is a centered Gaussian process such that its covariance function $R(t,s) = EW_t^H W_s^H$ is given by

$$R(s,t) = \frac{1}{2}(|t|^\gamma + |s|^\gamma - |t-s|^\gamma)$$

where $\gamma = 2H$ and $0 < H < 1$. If $H = \frac{1}{2}$, then W_t^H is the usual standard Brownian motion process. For $H \neq \frac{1}{2}$, W_t^H is neither a semimartingale nor a Markov process so we cannot apply the standard stochastic calculus for this process. It is a process of long range dependence in the following sense: If $\rho_n = E[W_1^H(W_{n+1}^H) - W_n^H]$, then the series $\sum_{n=0}^{\infty}\rho_n$ is either divergent or convergent with very late rate. It is known that a fractional Brownian motion W_t^H can be decomposed as follows:

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}\left[Z_t + \int_0^t (t-s)^\alpha \mathrm{d}W_s\right],$$

where Γ is the gamma function, $Z_t = \int_{-\infty}^0 [(t-s)^\alpha - (-s)^\alpha]\mathrm{d}W_s$, $\alpha = H - \frac{1}{2}$, and W_t is a standard Brownian motion.

We suppose from now on $0 < \alpha < \frac{1}{2}$ so that $\frac{1}{2} < H < 1$. Then Z_t has absolutely continuous trajectories and it is the term $B_t^H := \int_0^t (t-s)^\alpha \mathrm{d}W_s$ that exhibits long range dependence. We will use B_t^H instead of W_t^H in fractional stochastic calculus.

In Thao (2006) constructed an approximate process B_t^ε of B_t^H as follows:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} \mathrm{d}W_s$$

where $\frac{1}{2} < H < 1$, and W_t is a standard Brownian motion. He also proved that $B_t^\varepsilon \to B_t^H$ in $L^2(\Omega)$ as $\varepsilon \to 0$ (uniformly in t) and B_t^ε is a semimartingale. These results give us a convenient way to study fractional Brownian motions since we can use the standard Ito integrals and then it is easy to do numerical approximation.

By a fractional integrated GARCH model of continuous time (FIGARCH), we shall mean a process of the form

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}B_t^H \tag{3}$$

where $0 \le t \le T$, ω, θ and ξ are weight parameters, and B_t^H is a fractional Brownian motion. For each $\epsilon > 0$, an approximate model of the FIGARCH model is a process of the form

$$\mathrm{d}v_t^\varepsilon = (\omega - \theta v_t^\varepsilon)\mathrm{d}t + \xi v_t^\varepsilon \mathrm{d}B_t^\varepsilon \tag{4}$$

where B_t^ε is the approximate process of B_t^H. We shall show that its solution converges to the solution of the FIGARCH model (3).

Moreover, geometric Brownian motion for the asset price was used to simulate the SCB stock prices where the volatility of this model was predicted from an approximate fractional variance process of GARCH(1,1) model in continuous time and classical GARCH(1,1) model in continuous time. And both of them were compared with the empirical historical stock prices of SCB.

2. Solutions of the approximate models

In this section, we are interested in finding a solution of the approximate model (4) together with initial condition $v_{t(t=0)}^\varepsilon = v_0$.

Let $\varepsilon > 0$. Recall that an approximated process B_t^ε is defined by

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \mathrm{d}W_s,$$

where $\alpha = H - \frac{1}{2}$, $0 < H < 1$, and W_t is a Brownian motion process. By an application of the stochastic Fubini Theorem, one gets

$$\begin{aligned}
\int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1}\mathrm{d}W_u \mathrm{d}s &= \int_0^t \int_u^t (s-u+\varepsilon)^{\alpha-1}\mathrm{d}s\mathrm{d}W_u \\
&= \frac{1}{\alpha}\int_0^t ((t-u+\varepsilon)^\alpha - \varepsilon^\alpha)\mathrm{d}W_u \\
&= \frac{1}{\alpha}\left[\int_0^t (t-u+\varepsilon)^\alpha \mathrm{d}W_u - \varepsilon^\alpha \int_0^t \mathrm{d}W_u\right] \\
&= \frac{1}{\alpha}[B_t^\varepsilon - \varepsilon^\alpha W_t].
\end{aligned}$$

Consequently

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon \mathrm{d}s + \varepsilon^\alpha W_t$$

where

$$\varphi_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha-1} \mathrm{d}W_s.$$

Thus we have

$$\mathrm{d}B_t^{\varepsilon} = \alpha\varphi_t^{\varepsilon}\mathrm{d}t + \varepsilon^{\alpha}\mathrm{d}W_t. \tag{5}$$

Substituting $\mathrm{d}B_t^{\varepsilon}$ from (5) into (4), then Eq. (4) can be rewritten into the following form

$$\begin{aligned}\mathrm{d}v_t^{\varepsilon} &= (\omega - \theta v_t^{\varepsilon})\mathrm{d}t + \xi v_t^{\varepsilon}(\alpha\varphi_t^{\varepsilon}\mathrm{d}t + \varepsilon^{\alpha}\mathrm{d}W_t),\\ &= (\omega - \theta v_t^{\varepsilon} + \xi\alpha\varphi_t^{\varepsilon}v_t^{\varepsilon})\mathrm{d}t + \xi v_t^{\varepsilon}\varepsilon^{\alpha}\mathrm{d}W_t.\end{aligned} \tag{6}$$

Theorem 1. *For any $\varepsilon > 0$, a solution of the approximate model (4) is given by*

$$v_t^{\varepsilon} = \exp\left(\xi B_t^{\varepsilon} - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t\right)\left(v_0^{\varepsilon} + \omega\int_0^t \mathrm{e}^{\left(\theta+\frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)s - \xi B_s^{\varepsilon}}\mathrm{d}s\right), \tag{7}$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha}\mathrm{d}W_s$.

Proof. To find a solution of (6), we look for a solution of the form

$$v_t^{\varepsilon} = U_t V_t$$

where

$$\mathrm{d}U_t = (-\theta + \xi\alpha\varphi_t^{\varepsilon})U_t\mathrm{d}t + \xi\varepsilon^{\alpha}U_t\mathrm{d}W_t$$

and

$$\mathrm{d}V_t = a_t\mathrm{d}t + b_t\mathrm{d}W_t.$$

Firstly, we shall find a solution of $\mathrm{d}U_t = (-\theta + \xi\alpha\varphi_t^{\varepsilon})U_t\mathrm{d}t + \xi\varepsilon^{\alpha}U_t\mathrm{d}W_t$. By an application of the Ito formula to the function $f(u) = \ln u$ for $u = U_t$, one gets

$$\begin{aligned}\mathrm{d}(\ln U_t) &= \frac{1}{U_t}\mathrm{d}U_t - \frac{1}{2U_t^2}(\mathrm{d}U)^2\\ &= \frac{1}{U_t}\left((-\theta + \xi\alpha\varphi_t^{\varepsilon})U_t\mathrm{d}t + \xi\varepsilon^{\alpha}U_t\mathrm{d}W_t\right) - \frac{1}{2U_t^2}(\xi^2\varepsilon^{2\alpha}U_t^2\mathrm{d}t)\\ &= \left(-\theta + \xi\alpha\varphi_t^{\varepsilon} - \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)\mathrm{d}t + \xi\varepsilon^{\alpha}\mathrm{d}W_t\end{aligned}$$

or, equivalently,

$$\ln U_t - \ln U_0 = \xi\alpha\int_0^t \varphi_s^{\varepsilon}\mathrm{d}s - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t + \xi\varepsilon^{\alpha}W_t.$$

That is

$$U_t = U_0\exp\left(\xi\alpha\int_0^t \varphi_s^{\varepsilon}\mathrm{d}s - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t + \xi\varepsilon^{\alpha}W_t\right). \tag{8}$$

Set $U_0 = 1$ and $V_0 = v_0^{\varepsilon}$. Taking the differential of the product, we get

$$\begin{aligned}\mathrm{d}(U_tV_t) &= U_t\mathrm{d}V_t + V_t\mathrm{d}U_t + \mathrm{d}U_t\mathrm{d}V_t\\ &= U_t(a_t\mathrm{d}t + b_t\mathrm{d}W_t) + V_t\left((-\theta + \xi\alpha\varphi_t^{\varepsilon})U_t\mathrm{d}t + \xi\varepsilon^{\alpha}U_t\mathrm{d}W_t\right) + \xi\varepsilon^{\alpha}U_tb_t\mathrm{d}t\\ &= \left(U_ta_t + V_t(-\theta + \xi\alpha\varphi_t^{\varepsilon})U_t + \xi\varepsilon^{\alpha}U_tb_t\right)\mathrm{d}t + (U_tb_t + V_t\xi\varepsilon^{\alpha}U_t)\,\mathrm{d}W_t.\end{aligned}$$

Since $v_t^{\varepsilon} = U_tV_t$ then

$$\mathrm{d}v_t^{\varepsilon} = \left(U_ta_t + (-\theta + \xi\alpha\varphi_t^{\varepsilon})v_t^{\varepsilon} + \xi\varepsilon^{\alpha}U_tb_t\right)\mathrm{d}t + \left(U_tb_t + \xi\varepsilon^{\alpha}v_t^{\varepsilon}\right)\mathrm{d}W_t. \tag{9}$$

Comparing the coefficients of Eq. (9) with Eq. (6), we see that the desired coefficients a_t and b_t turn out to satisfy the following equations

$$U_tb_t = 0$$

and

$$U_t a_t + \xi\varepsilon^{\alpha} U_t b_t = \omega.$$

Then $b_t = 0$ and $a_t = \frac{\omega}{U_t}$. Hence

$$V_t := V_0 + \int_0^t a_t \mathrm{d}t + \int_0^t b_t \mathrm{d}W_t = v_0^{\varepsilon} + \int_0^t \frac{\omega}{U_s} \mathrm{d}s.$$

Moreover, v_t^{ε} is found to be

$$v_t^{\varepsilon} := U_t V_t = U_t \left(v_0^{\varepsilon} + \int_0^t \frac{\omega}{U_s} \mathrm{d}s \right).$$

Hence, with $U_0 = 1$ and using U_t as in (8), the solution of v_t^{ε} is given by

$$v_t^{\varepsilon} = \exp\left(\xi\alpha \int_0^t \varphi_s^{\varepsilon} \mathrm{d}s - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right) t + \xi\varepsilon^{\alpha} W_t \right) \left(v_0^{\varepsilon} + \omega \int_0^t \mathrm{e}^{\left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)s - \xi\varepsilon^{\alpha} W_s - \xi\alpha \int_0^s \varphi_u^{\varepsilon} \mathrm{d}u} \mathrm{d}s \right).$$

This proves Theorem 1. ∎

3. Convergence of the solutions of an approximate model

To prove the convergence of v_t^{ε}, firstly, let us consider the process v_t which satisfies Eq. (2). Let $X_t = \ln v_t$. It follows from the Ito formula that

$$\mathrm{d}X_t = \left(\omega \mathrm{e}^{-X_t} - \frac{\xi^2}{2} - \theta \right) \mathrm{d}t + \xi \mathrm{d}W_t. \tag{10}$$

The fractional model of the process X_t is a process which is of the form

$$\mathrm{d}X_t = \left(\omega \mathrm{e}^{-X_t} - \frac{\xi^2}{2} - \theta \right) \mathrm{d}t + \xi \mathrm{d}B_t^H, \tag{11}$$

where B_t^H is a fractional Brownian motion. Then an approximated model of (11) is of the form

$$\mathrm{d}X_t^{\varepsilon} = \left(\omega \mathrm{e}^{-X_t^{\varepsilon}} - \frac{\xi^2}{2} - \theta \right) \mathrm{d}t + \xi \mathrm{d}B_t^{\varepsilon} \tag{12}$$

where B_t^{ε} has already been defined in Section 1.

Theorem 2. *The solution of (12) converges to the solution of (11) in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$.*

Proof. We note that Eqs. (11) and (12) give

$$X_t - X_t^{\varepsilon} = \omega \int_0^t \left(\mathrm{e}^{-X_s} - \mathrm{e}^{-X_s^{\varepsilon}} \right) \mathrm{d}s + \xi \left(B_t^H - B_t^{\varepsilon} \right). \tag{13}$$

Let $\|\cdot\|$ denote the norm in $L^2(\Omega)$. It follows from (13) that

$$\begin{aligned} \left\| X_t - X_t^{\varepsilon} \right\| &= \left\| \omega \int_0^t \left(\mathrm{e}^{-X_s} - \mathrm{e}^{-X_s^{\varepsilon}} \right) \mathrm{d}s + \xi \left(B_t^H - B_t^{\varepsilon} \right) \right\| \\ &\leqslant |\omega| \int_0^t \left\| \mathrm{e}^{-X_s} - \mathrm{e}^{-X_s^{\varepsilon}} \right\| \mathrm{d}s + |\xi| \left\| B_t^H - B_t^{\varepsilon} \right\|. \end{aligned}$$

Since e^{-x} is differentiable and bounded on every compact interval, then

$$\left\| X_t - X_t^{\varepsilon} \right\| \leqslant |\omega| \int_0^t K_1 \left\| X_s - X_s^{\varepsilon} \right\| \mathrm{d}s + |\xi| \left\| B_t^H - B_t^{\varepsilon} \right\| \tag{14}$$

for some constants $K_1 > 0$. Referring to a result Thao (2006, page 127), one gets

$$\left\| B_t^H - B_t^{\varepsilon} \right\| \le C(\alpha)\varepsilon^{\frac{1}{2}+\alpha}, \tag{15}$$

where $0 < \alpha < \frac{1}{2}$ for $\frac{1}{2} < H < 1$ and $-\frac{1}{2} < \alpha < 0$ for $0 < H < \frac{1}{2}$, and $C(\alpha)$ is a positive constant depending only on α.

It follows from (14) and (15) that

$$\left\|X_t - X_t^\varepsilon\right\| \leqslant |\omega| K_1 \int_0^t \left\|X_s - X_s^\varepsilon\right\| \mathrm{d}s + |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha}. \tag{16}$$

Applying Gronwall's lemma to (16), then

$$\left\|X_t - X_t^\varepsilon\right\| \leqslant |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 t}.$$

Therefore

$$\sup_{0\le t\le T} \left\|X_t - X_t^\varepsilon\right\| \leqslant |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 T} \to 0$$

as $\varepsilon \to 0$. So $X_t^\varepsilon \to X_t$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and uniformly with respect to t. ■

Theorem 3. *If $X_t^\varepsilon \to X_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$, then $v_t^\varepsilon \to v_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$.*

Proof. It follows from $X_t = \ln v_t$, so $v_t = \mathrm{e}^{X_t}$ that

$$\left\|v_t - v_t^\varepsilon\right\| = \left\|\mathrm{e}^{X_t} - \mathrm{e}^{X_t^\varepsilon}\right\|.$$

Since e^x is differentiable and bounded in some closed interval, then

$$\left\|v_t - v_t^\varepsilon\right\| \le K_2 \left\|X_t - X_t^\varepsilon\right\|$$

for some positive constant K_2. From (15), we obtain

$$\left\|v_t - v_t^\varepsilon\right\| \leqslant K_2 |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 t}.$$

Therefore

$$\sup_{0\le t\le T} \left\|v_t - v_t^\varepsilon\right\| \leqslant K_2 |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 T} \to 0$$

as $\varepsilon \to 0$. The proof is now complete. ■

4. Applications

In this section, volatilities of the stock of Siam Commerical Bank (SCB) are computed by using FIGARCH(1,1) model and classical GARCH(1,1) model of continuous time. Then SCB stock prices are simulated by using these volatilities. After that both simulated SCB stock prices are compared with the empirical historical prices of SCB.

4.1. SCB simulated stock prices

A model for the dynamic of an asset price that will be considered here is of the form

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t,$$

where μ is known as the drift rate or rate of return of the price S_t and W_t is a Brownian motion. The stochastic volatility σ_t (which measures the standard deviation of the return $\frac{\mathrm{d}S_t}{S_t}$) is defined by $\sigma_t := \sqrt{v_t}$ where v_t is the FIGARCH model of continuous time as in Eq. (3). For comparative purposes, we shall compute the percentage error (PE) between two sets of data by the following formula

$$PE = \frac{1}{K} \sum_{i=1}^{K} \frac{|X_i - Y_i|}{X_i} \times 100,$$

where K is sample size, $X = (X_i, i \ge 1)$ is the market prices and $Y = (Y_i, i \ge 1)$ is the model prices. We use $K = 245$ when we sample data for 12 months.

For simulation purposes, we consider an approximate model

$$\mathrm{d}S_t^\varepsilon = \mu S_t^\varepsilon \mathrm{d}t + \sigma_t^\varepsilon S_t^\varepsilon \mathrm{d}W_t, \tag{17}$$

with $\epsilon > 0$ and $\sigma_t^\varepsilon = \sqrt{v_t^\varepsilon}$. The fractional variance process v_t^ε will be simulated by using Eq. (7), i.e.,

$$v_t^\varepsilon = \exp\left(\xi B_t^\varepsilon - \left(\theta + \frac{1}{2}\xi^2 \varepsilon^{2\alpha}\right) t\right) \left(v_0^\varepsilon + \omega \int_0^t \mathrm{e}^{\left(\theta + \frac{1}{2}\xi^2 \varepsilon^{2\alpha}\right)s - \xi B_s^\varepsilon} \mathrm{d}s\right). \tag{18}$$

Table 1
Discrete parameters ω_h, β_h and α_h of each dataset.

Months	Dataset (DD/MM/YY)	ω_h	β_h	α_h
1	1/12/2006–28/12/2006	0.0033	0	0.1689
3	2/10/2006–28/12/2006	0.0012	0	0.1154
6	3/7/2006–28/12/2006	0.00077659	0	0.0887
9	3/4/2006–28/12/2006	0.00071251	0	0.0726
12	3/1/2006–28/12/2006	0.00062672	0	0.0692

Table 2
Parameters ω, θ and ξ obtained from each dataset.

Months	Dataset (DD/MM/YY)	ω	θ	ξ
1	1/12/2006–28/12/2006	0.0033	0.8311	0.2389
3	2/10/2006–28/12/2006	0.0012	0.8846	0.1632
6	3/7/2006–28/12/2006	0.00077659	0.9113	0.1254
9	3/4/2006–28/12/2006	0.00071251	0.9274	0.1027
12	3/1/2006–28/12/2006	0.00062672	0.9308	0.0979

Table 3
Average *PE* for each set of parameters.

Parameters	ω	θ	ξ	Average of *PE* (%)
1	0.0033	0.8311	0.2389	40.5099
2	0.0012	0.8846	0.1632	24.0161
3	0.00077659	0.9113	0.1254	19.3196
4	0.00071251	0.9274	0.1027	18.3600
5	**0.00062672**	**0.9308**	**0.0979**	**17.3366**

The actual stock prices of Siam Commercial Bank (SCB) were obtained from http://www.tiscoetrade.com. Using the dataset from January 3, 2006 to December 28, 2007. We divide these data into two disjoint sets. The first one, from January 3, 2006 to December 28, 2006, will be used to estimate parameters ω, θ, and ξ for Eq. (18). The second set (January 3, 2007–December 28, 2007) will be used for comparison with the simulated prices.

We begin by estimating parameters ω, θ and ξ. To do this, we firstly enter the following 5 datasets, i.e., 1 month (December 1, 2006–December 28, 2006), 3 months (October 2, 2006–December 28, 2006), 6 months (July 3, 2006–December 28, 2006) and 12 months (January 3, 2006–December 28, 2006) into Matlab 6.5 (GARCH Toolbox) with COMPAQ Presario B1908TU to obtain discrete parameters of GARCH(1,1) model (ω_h, β_h and α_h). Those discrete parameters from each datasets are shown in Table 1.

Secondly, we utilize the formulas between discrete parameters and continuous parameters which have been given by Nelson (1990) to estimate the parameters ω, θ and ξ. The formulas are as follows:

$$\omega = h^{-1}\omega_h,$$
$$\theta = h^{-1}(1 - \beta_h - \alpha_h),$$
$$\xi = \sqrt{2h^{-1}}\alpha_h,$$

where h is the time lag between two consecutive data. Here we use $h = 1$. Thus the estimated parameters ω, θ and ξ for each dataset (1, 3, 6, 9 and 12 months) are given in Table 2.

From the information in Table 2, we look for those parameters which can give us the mimum average of *PE*. In order to solve this problem, we simulated v_t^{ε} (see, Eq. (18)) by using the parameters ω, θ and ξ from each dataset (1, 3, 6, 9 and 12 months). Here, we choose $\varepsilon = 0.0001$, $\alpha = 0.15$, $\mu = 0.0017819$ and $v_0^{\varepsilon} = 0$. Then, by using $\sigma_t^{\varepsilon} = \sqrt{v_t^{\varepsilon}}$, the SCB stock prices from January 3, 2007 to December 28, 2007 were forecast by the pricing model S_t^{ϵ} (see, Eq. (17)). Next, we compute *PE* by using the information from the simulation and the empirical data of SCB closing prices (January 3, 2007–December 28, 2007). For each set of parameters, we calculated the average of *PE* for 5000 paths. The results are shown in Table 3.

It can be seen from Table 3 that the parameters $\omega = 0.00062672$, $\theta = 0.9308$ and $\xi = 0.0979$ give us the minimum value of the average *PE*. We select this set of parameters for forecasting the future stock prices of SCB. In summary, when the SCB stock prices were simulated by Eq. (17) using parameters as mentioned above, the average of *PE* and its variance will be given as follows:

$$\begin{aligned} &\text{average of } PE = 17.3366\% \\ &\text{variance} = 43.0287\%. \end{aligned} \tag{19}$$

Recall that a GARCH(1,1) model of continuous time is of the form

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}W_t, \tag{20}$$

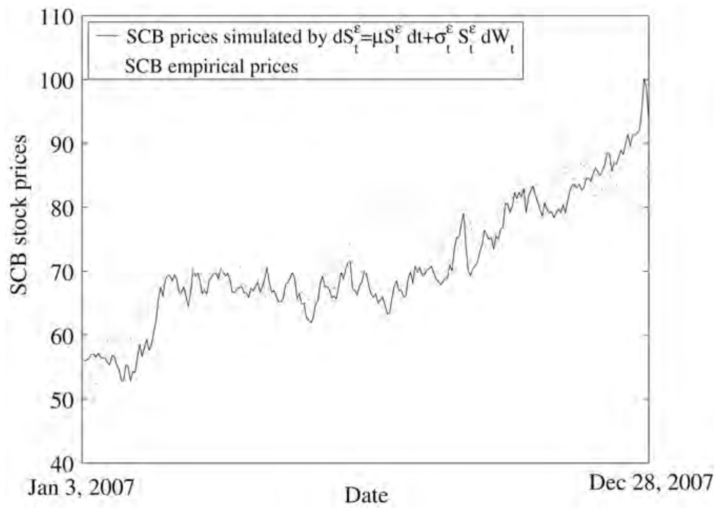

Fig. 1. Price behaviour of SCB, between January 3, 2007 and December 28, 2007, compared with a scenario simulated from fractional pricing model (dashed line := empirical data, solid line := simulated by $dS^{\varepsilon}_t = \mu S^{\varepsilon}_t dt + \sigma^{\varepsilon}_t S^{\varepsilon}_t dW_t$, $PE = 6.0401\%$).

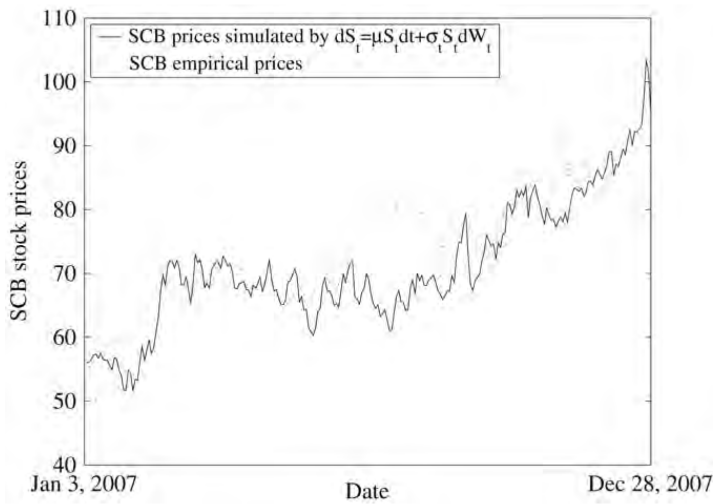

Fig. 2. Price behaviour of SCB, between January 3, 2007 and December 28, 2007, compared with a scenario simulated by pricing model (dashed line := empirical data, solid line := simulated by $dS_t = \mu S_t dt + \sigma_t S_t dW_t$, $PE = 6.9627\%$).

and the pricing model is

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, \tag{21}$$

where $\sigma_t = \sqrt{v_t}$.

We simulated the pricing model (21) by using $\omega = 0.00062672$, $\theta = 0.9308$, $\xi = 0.0979$, $\mu = 0.0017819$, $v_0 = 0$ and $K = 245$. We compute the PE of these simulation prices and the empirical data of SCB closing prices from January 3, 2007 to December 28, 2007. Next we compute the average of PE, by using $N = 5000$, and found that

$$\begin{aligned} &\text{average of } PE = 21.6536\% \\ &\text{variance} = 69.2135\%. \end{aligned} \tag{22}$$

By comparing the average PE and its variance by Eq. (19) and (22), one can see that in the case of SCB, the forecast of the future stock prices by using model (17) (which includes the fractional part) give an average error significantly smaller than using model (21) (which does not includes the fractional part).

For an illustration, Fig. 1 shows the empirical data of SCB as compared to the price simulated by the fractional price model (17). Here we used $\varepsilon = 0.0001$, $\alpha = 0.15$, $\theta = 0.9308$, $\omega = 0.00062672$, $\xi = 0.0979$ and $v^{\varepsilon}_0 = 0$. The percentage error $PE = 6.0401\%$.

Fig. 2, shows the empirical data of SCB as compared to the price simulated by the price model (21). Here we used $\theta = 0.9308$, $\omega = 0.00062672$, $\xi = 0.0979$, $\mu = 0.0017819$, $v_0 = 0$ and $\sigma_t = \sqrt{v_t}$. The percentage error $PE = 6.9627\%$.

By comparing PE and variance of Figs. 1 and 2, one can see that in the case of SCB the sample path from the fractional pricing model gives a better fit with the data than the ordinary pricing model, since the percentage error is smaller.

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An Approximate Formula of European Option for Fractional Stochastic Volatility Jump-Diffusion Model

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Abstract: Problem statement: We presented option pricing when the stock prices follows a jump-diffusion model and their stochastic volatility follows a fractional stochastic volatility model. This proposed model exhibits the a memory of a stochastic volatility model that is not expressed in the classical stochastic volatility model. **Approach:** We introduce an approximated method to fractional stochastic volatility model perturbed by the fractional Brownian motion. A relationship between stochastic differential equations and partial differential equations for a bivariate model is presented. **Results:** By using an approximate method, we provide the approximate solution of the fractional stochastic volatility model. And European options are priced by using the risk-neutral valuation. **Conclusion/Recommendations:** The formula of European option is calculated by using the technique base on the characteristic function of an underlying asset which can be expressed in an explicit formula. A numerical integration technique to simulation fractional stochastic volatility are presented in this study.

Key words: Fractional Brownian motion, approximate method, fractional stochastic volatility, jump diffusion model, option pricing model

INTRODUCTION

Let (Ω, F, P) be a probability space with filtration $\mathbb{F} = (F_t)_{0\le t\le T}$. All processes that we shall consider in this section will be defined in this space. For $t \in [0, T]$ and $T < \infty$ a geometric Brownian motion (gBm) model with jumps and with fractional stochastic volatility is a model of the form:

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}\, dW_t\right) + S_{t-} Y dN_t \qquad (1)$$

where $\mu \in \Re, S = (S_t)_{t\in[0,T]}$ is a process representing a price of the underlying risky assets, $W = (W_t)_{t\in[0,T]}$ is the standard Brownian motion, $N = (N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ and $S_{t-}Y_t$ represents the amplitude of the jump which occurs at time t. We assume that the processes W and N are independent. The volatility process $v_t := \sigma_t^2$ in (1) is modeled by:

$$dv_t = \left(\omega - \theta v_t\right)dt + \xi v_t dB_t \qquad (2)$$

where, $\omega > 0$ is the mean long-term volatility, $\theta \in \Re$ is the rate at which the volatility reverts toward its long-term mean, $\xi>0$ is the volatility of the volatility process and $(B_t)_{t\in[0,T]}$ is a fractional Brownian motion.

Assume that the processes (S_t) and (v_t) are F_t-measurable.

The notation S_{t-} means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula.

The fractional version of Eq. 1 is given by:

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}\, dB_t\right) + S_{t-} Y_t\, dN_t \qquad (3)$$

The process S_t in (3) can be approximated in $L_2(\Omega)$ by a semimartingale S_t^ε in the sense that $\left\|S_t^\varepsilon - S_t\right\|_{L_2(\Omega)} \to 0$ as $\varepsilon \to 0$, where S_t^ε satisfies the following equation (Intarasit and Sattayatham, 2010 for more details):

$$dS_t^\varepsilon = S_t^\varepsilon\left(\mu dt + \sqrt{v_t^\varepsilon}\, dB_t^\varepsilon\right) + S_{t-}^\varepsilon Y_t\, dN_t$$

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The purpose of this study is to consider the problem of option pricing for the gBm model with jumps (1) and with fractional stochastic volatility (2). Since driving process B_t of v_t in Eq. 2 is not a semimartingale, thus we cannot apply Itô calculus directly. We shall thus work in another direction by introducing an approximate model of SDE (1) and (2) then using it to price European call option. The advantage of these approximate model is there no more arbitrage. In order to find such a formula, we shall work in the space of a risk-neutral probability measure. Indeed, there are some authors who have investigated this problem before but not in the fractional case, for example (Heston, 1993). In fact, there are many author studies a volatility and fractional volatility process. For example (Magnus and Fosu, 2006) use GARCH to model and forecast volatility returns on the Ghana stock exchange and (Shamiri and Isa, 2009) study modeling and forecasting of volatility of the Malaysian stock markets. An empirical study of fractional volatility are presented in (Cheong, 2008) for example.

Recall that the fractional Brownian motion with Hurst coefficient is a Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with zero mean and the covariance function is given by:

$$R(t,s) = E\left[B_t^H B_s^H\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

If H = 1/2, then R(t, s) = min(t, s) and B_t^H is the usual standard Brownian motion. In the case 1/2 < H < 1 the fractional Brownian motion exhibits statistical long-range dependency in the sense that $\rho_n := E\left[B_1^H\left(B_{n+1}^H - B_n^H\right)\right] > 0$ for all n = 1, 2, 3, … and $\sum_{n=1}^{\infty}\rho_n = \infty$. Hence, in financial modeling, one usually assumes that $H \in (1/2, 1)$. Put α= 1/2 –H. It is known that a fractional Brownian motion B_t^H can be decomposed as follows:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)}\left\{Z_t + \int_0^t (t-s)^{-\alpha} dW_s\right\}$$

where, Γ is the gamma function:

$$Z_t = \int_{-\infty}^{0} [(t-s)^{-\alpha} - (s)^{-\alpha}] dW_s$$

We suppose from now on that 0 < α < 1/2. The process Z_t has absolutely continuous trajectories, so it suffices to consider only the term:

$$B_t = \int_0^t (t-s)^{-\alpha} dW_s \tag{4}$$

that has a long-range dependence.

Note that B_t can be approximated by:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{-\alpha} dw_s \tag{5}$$

in the sense that B_t^ε converges to B_t in $L_2(\Omega)$ as $\varepsilon \to 0$, uniform with respect to $t \in [0,T]$ (Thao, 2006).

Since $(B_t^\varepsilon)_{t\in[0,T]}$ is a continuous semimartingale then Itô calculus can be applied to the following Stochastic Differential Equation (SDE):

$$dS_t^\varepsilon = S_t^\varepsilon\left(\mu dt + \sigma dB_t^\varepsilon\right), 0 \leq t \leq T$$

Let S_t^ε be the solution of the above equation. Because of the convergence of B_t^ε to B_t in $L_2(\Omega)$ when $\varepsilon \to 0$, we shall define the solution of a fractional stochastic differential equation of the form:

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right), 0 \leq t \leq T$$

to be a process S_t^* defined on the probability space (Ω, F, P) such that the process S_t^ε converges to S_t^* in $L_2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to $t \in [0,T]$. This definition will be applied to the other similar fractional stochastic differential equations which will appear later.

A risk-neutral model for a gBm model combining jumps with stochastic volatility is introduced next. Its solution will also be discussed. Firstly, let us rewrite the model (1) into an integral form as follows:

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sqrt{v_s} S_s dW_s + \int_0^t S_{s-} Y_s dN_s \tag{6}$$

Note that the last term on the right hand side of Eq. 6 is defined by:

$$\int_0^t S_{s-} Y_s dN_s := \sum_{n=1}^{N_t} \Delta S_n$$

Where:

$$\Delta S_n := ST_n - ST_{n-} = S_{n-} Y_n$$

The assumption $Y_n > 0$ always leads to positive values of the stock prices. The process $(Y_n)_{n\in N}$ is assumed to be independently identically distributed (i.i.d.) with density ϕ_Y (y) and $(T_n)_{n\in N}$ is a sequence of jump time.

In order to solve Eq. 6 with an initial condition $S_{t(t=0)} = S_0$ we assume that $E\left[\int_0^T v_s S_s^2 ds\right] < \infty$. Then, by an application of Itô's formula for the jump process (Cont and Tankov, 2009, Theorem 8.14) on Eq. 6 with $f(S_t, t) = \log(S_t)$ we get:

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s}\, dW_s + \int_0^t \log(1+Y_s) dN_s\right)$$

It is assumed that a risk-neutral probability measure M exists; the asset price S_t, under this risk-neutral measure, follows a jump-diffusion process, with zero-mean, risk-free rate r:

$$dS_t = S_t\left(r - \lambda E_M[Y_t]\right)dt + \sqrt{v_t}\, dW_t\,) + S_{t-} Y_t\, dN_t \qquad (7)$$

and the stochastic variance v_t satisfies the following fractional SDE:

$$dv_t = \left(\omega - \theta v_t\right)dt + \xi v_t dB_t \qquad (8)$$

with an initial condition $v_{t(t=0)} = v_0 \in L_2(\Omega)$.

It is only necessary to know that the risk-neutral measure exists (Cont and Tankov, 2009). Hence, all processes to be discussed after this will be the processes under the risk-neutral probability measure M.

Using an initial condition $S_{t(t=0)} = S_0 \in L_2(\Omega)$, the solution of Eq. 7 is given by:

$$S_t = S_0 \exp\begin{pmatrix} \int_0^t (r - \lambda E_M[Y_s])ds - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s}\, dW_s \\ + \int_0^t \log(1+Y_s) dN_s \end{pmatrix}. \quad (9)$$

Under approximate method, for each $\varepsilon > 0$, consider an approximate model of Eq. 7 and 8 respectively:

$$dS_t^\varepsilon = S_t^\varepsilon\left((r - \lambda E_M[Y_t])dt + \sqrt{v_t^\varepsilon} W_t\right) + S_{t-}^\varepsilon Y_t dN_t \qquad (10)$$

$$dv_t^\varepsilon = \left(\omega - \theta v_t^\varepsilon\right)dt + \xi v_t^\varepsilon dB_t^\varepsilon \qquad (11)$$

By using the same initial condition as in Eq. 10, we have:

$$S_t^\varepsilon = S_0 \exp\begin{pmatrix} \int_0^t (r - \lambda E_M[Y_s])ds - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \\ \int_0^t \sqrt{v_s^\varepsilon}\, dW_s + \int_0^t \log(1+Y_s\, dN_s) \end{pmatrix} \qquad (12)$$

and one can prove that S_t^ε converges to S_t of Eq. 9 in $L_2(\Omega)$ as $\varepsilon \to 0$ and uniformly on $t \in [0, T]$. Moreover, one can show that the solution v_t^ε of Eq. 11 converges in $L_2(\Omega)$ to the process:

$$v_t = \left(v_0 + \omega\int_0^t \exp(\gamma s - \xi B_s) ds\right)\exp(\xi B_t - \gamma t)$$

for some real constant γ. Hence, by definition, v_t is the solution of Eq. 8 (Intarasit and Sattayatham, 2010, Lemma 2).

MATERIALS AND METHODS

The relationship between the stochastic deferential equation and the partial differential equation for bivarate model is presented.

Consider the process $\vec{X}_t = (X_t^1, X_t^2)$ where X_t^1 and X_t^2 are processes in $\Re$ and satisfy the following equations:

$$\begin{aligned} dX_t^1 &= f_1(t)dt + g_1(t)dW_t + X_{t-}^1 Y_t\, dN_t \\ dX_t^2 &= f_2(t)dt + g_2(t)d\overline{W}_t \end{aligned} \qquad (13)$$

where, f_1, g_1, f_2 and g_2 are all continuous functions from [0, T] into $\Re$.

Since every compound Poisson process can be represented as an integral form of Poisson random measure (Cont and Tankov, 2009) then the last term on the right hand side of Eq. 13 can be written as follows:

$$\begin{aligned} \int_0^t X_{s-}^1 Y_s\, dN_s &= \sum_{n=1}^{N_t} X_{n-}^1 Y_n = \sum_{n=1}^{N_t} [X_{T_n}^1 - X_{T_n -}^1] \\ &= \int_0^t \int_{\Re} X_{s-}^1 z J_z(ds\, dz) \end{aligned}$$

where, Y_n are i.i.d. random variables with density ϕ_Y (y) and J_Z is a Poisson random measure of the process $Z_t = \sum_{n=1}^{N_t} Y_n$ with intensity measure $\lambda\phi_Y(d_z)dt$.

Let $U(\vec{x})$ be a bounded real function on $\Re^2$ and twice continuously differentiable in $\vec{x} = (x_1, x_2) \in \Re^2$ and:

$$u(\vec{x}, t) = E\left[u\left(\vec{X}_T\right)\vec{X}_t = \vec{x}\right] \tag{14}$$

By the two dimensional Dynkin's formula (Hanson, 2007, Theorem 7.7), u is a solution of the Partial Integro-Differential Equation (PIDE):

$$0 = \frac{\partial v(\vec{x}, t)}{\partial t} + Av(\vec{x}, t) + \lambda \int_{\Re} \left[v(\vec{x} + \vec{y}, t) - v(\vec{x}, t)\right]\phi_Y(y)dy$$

subject to the final condition $u(\vec{x}, T) = U(\vec{x})$ and $\vec{y} = (y, 0)$. The notation A is defined by:

$$Au\left(\vec{x}, t\right) = f_1(t)\frac{\partial u(\vec{x}, t)}{\partial x_1} + f_2(t)\frac{\partial u(\vec{x}, t)}{\partial x_2} + \frac{1}{2}g_1^2(t)\frac{\partial^2 u(\vec{x}, t)}{\partial x_1^2}$$
$$+\rho g_1(t)g_2(t)\frac{\partial^2 u(\vec{x}, t)}{\partial x_1 \partial x_2} + \frac{1}{2}g_2^2(t)\frac{\partial^2 u(\vec{x}, t)}{\partial x_2^2}$$

and the correlation ρ defined by $\rho = \mathrm{Corr}[dW_t, dW_t]$.

Next, we present the classical method to pricing of European call option. The European call option formula in terms of characteristic function is given in the next section.

Let C denote the price at time t of a European style call option on the current price of the underlying asset S_t with strike price K and expiration time T. The terminal payoff of a European call option on the underling stock S_t with strike price K is max (S_T- K; 0). This means that the holder will exercise his right only if S_T > K and then his gain is S_T -K. Otherwise, if S_T> K, then the holder will buy the underlying asset from the market and the value of the option is zero. Assuming the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff:

$$\begin{aligned} C\left(S_t, v_t, t; K, T\right) &= e^{-r(T-t)}E_M\left[\max(S_T - K, 0)\middle|S_t, v_t\right] \\ &= e^{-r(T-t)}\left(\int_K^{\infty}(S_T - K)P_M(S_T \mid S_t, v_t)dS_T\right) \\ &= S_t\left(\frac{1}{E_M\left[S_T\middle|S_t, t\right]}\int_K^{\infty} S_T P_M(S_T \mid S_t, v_t)dS_T\right) \\ &\quad -Ke^{-r(T-t)}\int_K^{\infty} P_M(S_T \mid S_t, v_t)dS_T \\ &= S_t P_1(S_t, v_t, t; K < T) - Ke^{-r(T-t)}P_2(S_t, v_t, t; K < T) \end{aligned} \tag{15}$$

where, E_M is the expectation with respect to the risk-neutral probability measure, $P_M(S_T \mid S_t, v_t)$ is the corresponding conditional density given (S_t, v_t) and:

$$P_1(S_t, v_t, t; K, T) = \left(\int_K^{\infty} S_T P_M(S_T \middle| S_t, v_t)dS_T\right) / E_M[S_T \mid S_t, v_t]$$

Note that P_1 is the risk-neutral probability that S_T > K (since the integrand is nonnegative and the integral over [0, ∞) is one) and finally, that:

$$P_2(S_t, v_t, t; K, T) = \int_K^{\infty} P_M(S_t, v_t)dS_T = \mathrm{Prob}(S_T > K \mid S_t, v_t)$$

is the risk-neutral in-the-money probability. Moreover, $E_M[S_T \mid S_t, v_t] = e^{r(T-t)}S_t \text{ for } t \geq 0$.

Note that we do not have a formulation for these probabilities thus we will calculate some approximations of P_1 and P_2. Indeed, these probabilities are related to characteristic functions which have formulation as will be seen in Lemma 2.

RESULTS

In order to calculate the price of a European call option with strike price K and maturity T of the model (7) for which its fractional stochastic volatility satisfies Eq. 8, we consider the approximate model (10) and (11). Firstly, we consider logarithm of S_t^{ε} namely L_t^{ε}, i.e. $L_t^{\varepsilon} = \log(S_t^{\varepsilon})$ where S_t^{ε} satisfies Eq. 12 (the solution of Eq. 10) and its inverse $S_t^{\varepsilon} = \exp(L_t^{\varepsilon})$. Denote k = log (K) the logarithm of the strike price. Secondly, we now refer to SDE (11), since this approximate model is driven by a semimartingale B_t^{ε} and hence there is no opportunity of arbitrage (for more details (Thao, 2006)). This is the advantage of our approximate approach and we will use this model for pricing the European call option instead of SDE (8).

Note that we can write:

$$dB_t^{\varepsilon} = \alpha\varphi_t^{\varepsilon}dt + \varepsilon^{\alpha}dW_t \tag{16}$$

where $\varphi_t^{\varepsilon} = \int_0^t (t - u + \varepsilon)^{1-\alpha}dW_u$, $\alpha = 1/2 - H$ and 0< α < 1/2 ((Thao, 2006), Lemma 2.1).

Substituting (16) into Eq. 11, we obtain:

$$dv_t^{\varepsilon} = (\omega + (\alpha\xi\varphi_t^{\varepsilon} - \theta)v_t^{\varepsilon})dt + \xi\varepsilon^{\alpha}v_t^{\varepsilon}dW_t \tag{17}$$

Consider the SDE (10) and (17). Define a function U on $\Re^2$ as follows:

$$U(x_1, x_2) = e^{-r(T-t)} \max(\exp(x_1 - \kappa), 0).$$

By virtue of Eq. 14:

$$\begin{aligned} u(\vec{x}, t) &= E_M\left[U\left(\vec{X}_T\right)\middle|\vec{X}_t = \vec{x}\right] \\ &= e^{-r(T-t)} E_M\left[\max(\exp((L_t^\varepsilon) - \kappa), 0)\middle| L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right] \\ &:= C(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) \end{aligned}$$

satisfies the following PIDE:

$$\begin{aligned} 0 &= \frac{\partial C}{\partial t} + f_1 \frac{\partial C}{\partial \ell^\varepsilon} + f_2 \frac{\partial C}{\partial v^\varepsilon} + \frac{1}{2} g_1^2 \frac{\partial^2 C}{(\partial \ell^\varepsilon)^2} \\ &+ \rho g_1 g_2 \frac{\partial^2 C}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} g_2^2 \frac{\partial^2 C}{\partial (v^\varepsilon)^2} - rC \\ &+ \lambda \int_\Re \left[C(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - C(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right] \phi_Y(y) dy. \end{aligned} \tag{18}$$

In the current state variable, the last line of Eq. 15 becomes:

$$\begin{aligned} C(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) &= e^{\ell^\varepsilon} P_1(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) \\ &- e^{\kappa - r(T-t)} P_2(\ell^\varepsilon, v^\varepsilon, t; \kappa, T). \end{aligned} \tag{19}$$

The following lemma shows the relationship between P_1 and P_2 in the option value of the Eq. 19.

Lemma 1: The probability P_1 in the option value of the Eq. 19 satisfies the following PIDE:

$$\begin{aligned} 0 &= \frac{\partial P_1}{\partial t} + A[P_1](\ell^\varepsilon, v^\varepsilon, t; \kappa, T) + v^\varepsilon \frac{\partial P_1}{\partial \ell^\varepsilon} + \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial P_1}{\partial v^\varepsilon} \\ &+ (r - \lambda E_M(Y_t)) P_1 \\ &+ \lambda \int_\Re \left[(e^y - 1) P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T)\right] \phi_Y(y) dy \\ &:= \frac{\partial p_1}{\partial t} + A_1[P_1](\ell^\varepsilon, v^\varepsilon, t; \kappa, T) \end{aligned} \tag{20}$$

subject to the boundary condition at expiration time t = T:

$$P_1(\ell^\varepsilon, v^\varepsilon, T; \kappa, T) = 1_{\ell^\varepsilon > \kappa}. \tag{21}$$

And the probability P_2 in the option value of the Eq. 19 satisfies the following PIDE:

$$\begin{aligned} 0 &= \frac{\partial P_2}{\partial t}(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) + rP_2 \\ &:= \frac{\partial P_2}{\partial t} + A_2\left[P_2\right](\ell^\varepsilon, v^\varepsilon, t; \kappa, T) \end{aligned} \tag{22}$$

subject to the boundary condition at expiration time t = T:

$$P_2(\ell^\varepsilon, v^\varepsilon, T; \kappa, T) = 1_{\ell^\varepsilon > \kappa} \tag{23}$$

Where:

$$\begin{aligned} A\left[f\right](\ell^\varepsilon, v^\varepsilon, t; \kappa, T) &:= (r - \lambda E[Y_t] - \frac{1}{2} v^\varepsilon) \frac{\partial f}{\partial \ell^\varepsilon} \\ &+ \left(\omega + (\alpha \xi \varphi^\varepsilon - \theta) v^\varepsilon\right) \frac{\partial f}{\partial v^\varepsilon} + \frac{1}{2} v^\varepsilon \frac{\partial^2 f}{\partial (\ell^\varepsilon)^2} \\ &+ \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial^2 f}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} \xi^2 \varepsilon^{2\alpha} (v^\varepsilon)^2 \frac{\partial^2 f}{\partial (v^\varepsilon)^2} \\ &- rf + \lambda \int_\Re \begin{bmatrix} f(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) \\ -f(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) \end{bmatrix} \phi_Y(y) dy \end{aligned} \tag{24}$$

Note that $1_{\ell^\varepsilon > \kappa} = 1$ if $\ell^\varepsilon > \kappa$ and otherwise $1_{\ell^\varepsilon > \kappa} = 0$.

Proof: Calculating the partial derivatives of function $C(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$ in Eq. 19 and substituting it's in Eq. 18 then separating it by assumed independent terms P_1 and P_2. This gives two PIDEs for the risk-neutralized probability $P_j(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$, j = 1, 2. For j=1 we have:

$$\begin{aligned} 0 &= \frac{\partial P_1}{\partial t} + \left(r - \lambda E_M(Y_t) - \frac{1}{2} v^\varepsilon\right)\left(\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1\right) \\ &+ \left(\omega + (\alpha \xi \varphi_t^\varepsilon - \theta) v^\varepsilon\right) \frac{\partial P_1}{\partial v^\varepsilon} + \frac{1}{2} v^\varepsilon \left(\frac{\partial^2 P_1}{(\ell^\varepsilon)^2} + 2 \frac{\partial P_1}{\partial \ell^\varepsilon} + P_1\right) \\ &+ \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \left(\frac{\partial^2 P_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{\partial P_1}{\partial v^\varepsilon}\right) + \frac{1}{2} \xi^2 \varepsilon^{2\alpha} (v^\varepsilon)^2 \frac{\partial^2 P_1}{\partial (v^\varepsilon)^2} - rP_1 \\ &+ \lambda \int_\Re \begin{bmatrix} (e^y - 1) P_1(\ell^\varepsilon + y, v^\varepsilon, t; T) \\ +(P_1(\ell^\varepsilon + y, v^\varepsilon, t; T) - P_1(\ell^\varepsilon + y, v^\varepsilon, t; T)) \end{bmatrix} \phi Y(y) dy \end{aligned} \tag{25}$$

subject to the boundary condition at the expiration time t = T according to Eq. 21. By using the notation in Eq. 24 to PIDE (25) we get Eq. 20:

For $P_2(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$, we have:

$$\begin{aligned} 0 &= \frac{\partial P_2}{\partial t} + rP_2 + \left(r - \lambda E_M(Y_t) - \frac{1}{2} v^\varepsilon\right)\left(\frac{\partial P_2}{\partial \ell^\varepsilon}\right) \\ &+ (\omega(\alpha \xi \varphi_t^\varepsilon - \theta) v^\varepsilon) \frac{\partial P_2}{\partial v^\varepsilon} + \frac{1}{2} v^\varepsilon \frac{\partial^2 P_2}{\partial (\ell^\varepsilon)^2} \\ &+ \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial^2 P_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} \xi^2 \varepsilon^{2\alpha} (v^\varepsilon)^2 \frac{\partial^2 P_1}{\partial (v^\varepsilon)^2} \\ &- rP_2 + \lambda \int_\Re \begin{bmatrix} P_2(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) \\ -P_2(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) \end{bmatrix} \phi_Y(y) dy \end{aligned} \tag{26}$$

subject to the boundary condition at expiration time t = T according to Eq. 23. Again, by using the notation (24) to PIDE (26) we get Eq. 22. The proof is now completed.

Next, an approximate formula of European call option is calculated. For j = 1, 2 the characteristic functions for $P_j(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$ with respect to the variable k are defined by:

$$f_j(\ell^\varepsilon, v^\varepsilon, t: x, T) := -\int_{-\infty}^{\infty} e^{ix\kappa} dP_j(\ell^\varepsilon, v^\varepsilon, t: \kappa, T)$$

with a minus sign to account for the negativity of the measure dP_j. Note that f_j also satisfies similar PIDEs:

$$\frac{\partial f_j}{\partial t} + A_j\left[f_j\right](\ell^\varepsilon, v^\varepsilon, t; \kappa, T) = 0 \quad (27)$$

with the respective boundary conditions:

$$f_j(\ell^\varepsilon, v^\varepsilon, T; x, T) = -\int_{-\infty}^{\infty} e^{jxk} dP_j(\ell^\varepsilon, v^\varepsilon, T; \kappa, T)$$

$$= -\int_{-\infty}^{\infty} e^{ix\kappa}(-\delta(\ell^\varepsilon - \kappa)d\kappa) = e^{ix\ell^\varepsilon}$$

Since:

$$dP_j(\ell^\varepsilon, v^\varepsilon, T: \kappa, T) = d1_{\ell^\varepsilon > \kappa} = dH(\ell^\varepsilon - \kappa) = -\delta(\ell^\varepsilon - \kappa)d\kappa$$

Note that the probabilities P_j, j = 1, 2 are the conditional probabilities that the option expires in-the-money that is:

$$P_j = M\{L_T^\varepsilon \geq \log K \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v\}$$

where again $L_t^\varepsilon = \log S_t^\varepsilon$ and $(S_t^\varepsilon, v_t^\varepsilon)$ evolves according to Eq. 10 and 11 respectively.

Using a Fourier transform method one gets:

$$dP_j(\ell^\varepsilon, v^\varepsilon, t: \kappa, T) = \frac{1}{2} + \frac{1}{\pi}\int_{0^+}^{+\infty} \mathrm{Re}\left[\frac{e^{-ix\kappa} f_j(\ell^\varepsilon, v^\varepsilon, t: x, T)}{ix}\right]dx \quad (28)$$

where, j = 1, 2 and the characteristic function $f_j(\ell^\varepsilon, v^\varepsilon, t; x, T)$ also satisfy the PIDEs in lemma 1, namely Eq. 20 and 22 and Re[.] denoting the real component of a complex number. The practice to solving of this kind of equations is to guess the general form of the solution. The following lemma shows how to calculate the probabilities P_1 and P_2 as they appeared in Lemma 1.

Lemma 2: The probabilities P_1 and P_2 can be calculated by Eq. 28 where the explicit expressions of the characteristic functions is given as follows. (i) The characteristic function f_1 is given by:

$$f_j(\ell^\varepsilon, v^\varepsilon, t; x, t+\tau) = \exp(g_1(\tau) + v^\varepsilon h_1(\tau) + jx\ell^\varepsilon)$$

where, $\tau = T - t$:

$$g_1(\tau) = \left[r - \lambda E_M(Y_t)jx - \lambda E_M(Y_t)\right]\tau$$
$$+\tau\lambda\int_{\Re}(e^{(ix+1)y} - 1)\phi_Y(y)dy$$
$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1 - \frac{(\Delta_1 + \eta_1) + (1 - e^{\Delta_1\tau})}{2\Delta_1}\right) + (\Delta_1 + \eta_1)\tau\right],$$
$$h_1(\tau) = \frac{(\eta_1^2 - \Delta_1^2)(e^{\Delta_1\tau} - 1)}{\xi^2\varepsilon^{2\alpha\varepsilon}v^\varepsilon(\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau})},$$
$$\eta_1 = \rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}(1 + ix) + (\alpha\xi\ell_t^\varepsilon - \theta)$$

and $\Delta_1 = \sqrt{\eta_1^2 - \xi^2\varepsilon^{2\alpha}ix(ix+1)}$

(ii) The characteristic function f_2 is given by:

$$f_2\left(\ell^\varepsilon, v^\varepsilon, t; x, t+\tau\right) = \exp(g_2(\tau) + v^\varepsilon h_2(\tau) + ix\ell^\varepsilon + r\tau)$$

Where:

$$g_2(\tau) = [r - \lambda E_M[Y_t]iy - r]\tau + \tau\lambda\int_{\Re}(e^{ixy} - 1)\phi_Y(y)dy$$
$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1 - \frac{(\Delta_2 + \eta_2) + (1 - e^{\Delta_2\tau})}{2\Delta_2}\right) + (\Delta_2 + \eta_2)\tau\right],$$
$$h_2(\tau) = \frac{(\eta_2^2 - \Delta_2^2)(e^{\Delta_2\tau} - 1)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon(\eta_2 + \Delta_2 - (\eta_2 - \Delta_2)e^{\Delta_2\tau})},$$
$$\eta_2 = \rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}ix + (\alpha\xi\varphi_t^\varepsilon - \theta)$$

and $\Delta_2 = \sqrt{\eta_2^2 + \xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix-1)}$

Proof: Proof of (i). To solve for the characteristic explicitly, letting $\tau = T - t$ be the time-to-go. Following (Heston, 1993), we conjecture that the function f_1 is given by:

$$f_1(\ell^\varepsilon, v^\varepsilon, t; x, t+\tau) = \exp\left((g_1)(\tau) + v^\varepsilon h_1(\tau) + ix\ell^\varepsilon\right) \quad (29)$$

and the boundary condition $g_1(0) = 0 = h_1(0)$. This conjecture exploits the linearity of the coefficient in PIDE (27).

Note that the characteristic functions of f_1 always exists. In order to substitute (29) into (27), firstly, we calculate the partial derivative of f_1 and substitute it's into Eq. 27. After canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned}0 &= g_1'(\tau) - v^\varepsilon h_1'(\tau) + (r - \lambda E_M[Y_t] + \frac{1}{2}v^\varepsilon)ix \\ &+(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v^\varepsilon) + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2} h_1(\tau) \\ &-\frac{1}{2}v^\varepsilon x^2 + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2} ixh_1(\tau) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^\varepsilon)^2 h_1^2(\tau) \\ &-\lambda E_M[Y_t] + \lambda\int_{\Re}(e^{(ix+1)y} - 1)\phi_Y(y)dy\end{aligned}$$

By separating the order v^ε and ordering the remaining terms, we can reduce it to two Ordinary Differential Equations (ODEs):

$$\begin{aligned}h_1'(\tau) &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^\varepsilon h_1^2(\tau) + (\rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}(1+ix) \\ &+(\alpha\xi\varphi_t^\varepsilon - \theta))h_1(\tau) + \frac{1}{2}ix - \frac{1}{2}x^2\end{aligned} \tag{30}$$

$$\begin{aligned}g_1'(\tau) &= \omega h_1(\tau) + (r - \lambda E_M[Y_t])ix - \lambda E_M[Y_t] \\ &+\lambda\int_{\Re}(e^{(ix+1)y} - 1)\,\phi_Y(y)dy\end{aligned} \tag{31}$$

Let $\eta_1 = \rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}(1+ix) + (\alpha\xi\varphi_t^\varepsilon - \theta)$ and substitute it to Eq. 30.

We get:

$$\begin{aligned}h_1'(\tau) &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(h_1^2(\tau) + \frac{2\eta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}h_1(\tau) + \frac{1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}ix(ix+1)\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}\left(h_1(\tau) + \frac{2\eta_1 + \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right) \\ &\times\left(h_1(\tau) + \frac{2\eta_1 - \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(h_1(\tau) + \frac{\eta_1 + \Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right)\left(h_1(\tau)\frac{\eta_1 - \Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right)\end{aligned}$$

Where:

$$\Delta_1 = \sqrt{\eta_1^2 - \xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix+1)}$$

By method of variable separation, we have:

$$\frac{2dh_1(\tau)}{\left(h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right)\left(h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right)} = \xi^2\varepsilon^{2\alpha}v^\varepsilon d\tau$$

Using partial fractions, we get:

$$\frac{1}{\Delta_1}\left(\frac{1}{h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}} - \frac{1}{h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}}\right)dh_1(\tau) = d\tau$$

Integrating both sides, we obtain:

$$\log\left(\frac{h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^{2\varepsilon}\alpha v^\varepsilon}}{h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^{2\varepsilon}\alpha v^\varepsilon}}\right) = \Delta_1\tau + C$$

Using boundary condition $h_1(\tau = 0) = 0$ we get:

$$C = \log\left(\frac{\eta_1 - \Delta_1}{\eta_1 + \Delta_1}\right)$$

Solving for h_1, we obtain:

$$h_1(\tau) = \frac{(\eta_1^2 - \Delta_1^2)(e^{\Delta_1\tau} - 1)}{\xi^2\varepsilon^{2\alpha}c^\varepsilon(\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau})}$$

In order to solve $g_1(\tau)$ explicitly, we substitute h_1 into Eq. 31 and integrate with respect to T on both sides.

Then we get:

$$\begin{aligned}g_1(\tau) &= \left[(r - \lambda E_M(Y_t))ix - \lambda E(Y_t)\right]\tau \\ &+\tau\lambda\int_{\Re}(e^{(ix+1)y} - 1)\phi_Y(y)dy \\ &-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1 - \frac{(\Delta_1 + \eta_1) + (1 - e^{\Delta_1\tau})}{}\right) + (\Delta_1 + \eta_1)\tau\right]\end{aligned}$$

Proof of (ii). The details of the proof are similar to case (i). Hence, we have:

$$f_2(\ell^\varepsilon, v^\varepsilon, t; y, t+\tau) = \exp(g_2(\tau) + v^\varepsilon h_2(\tau) + iy\ell^\varepsilon + r\tau)$$

where, $g_2(\tau), h_2(\tau), \eta_2$ and Δ_2 are as given in the Lemma.

We can thus evaluate the characteristic functions in explicit form. However, we are interested in the risk-neutral probabilities P_j. These can be inverted from the characteristic functions by performing the following integration:

$$\hat{P}_j(S_t^\varepsilon, v_t^\varepsilon; K, T) = P_j(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{+\infty} \mathrm{Re}\left[\frac{e^{-ix\kappa} f_j(\ell^\varepsilon, v_t^\varepsilon t; x, T)}{ix}\right] dx$$

for j = 1, 2, where $\ell^\varepsilon = \log(S_t^\varepsilon)$, $v^\varepsilon = \log(v_t^\varepsilon)$, and $\kappa = \log(K)$.

To verify the above equation, firstly we note that:

$$E_M\left[e^{ix(\log(S_t^\varepsilon) - \log(K))} \middle| \log(S_t^\varepsilon) = L_t^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$
$$= E_M\left[e^{ix(\ell^\varepsilon - K)} \middle| L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$

The computation of the right of above equation are:

$$\int_{-\infty}^{+\infty} e^{-ix(\ell^\varepsilon - \kappa)} dP_j(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) = e^{-ix\kappa}\int_{-\infty}^{+\infty} e^{-ix\ell^\varepsilon} dP_j(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$$
$$= e^{-ix\kappa}\int_{-\infty}^{+\infty} e^{ix\ell^\varepsilon}(-\delta(\ell^\varepsilon - \kappa)d\kappa) = e^{-ix\kappa} f_j(\ell^\varepsilon, \varepsilon, t; x, T)$$

Then:

$$\frac{1}{2} + \frac{1}{\pi}\int_{0+}^{+\infty} \mathrm{Re}\left[\frac{e^{-i\infty x} f_j(\ell^\varepsilon, v_t^\varepsilon t; x, T)}{ix}\right] dx$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{+\infty} \mathrm{Re}\left[\frac{E_M[e^{ix(\log(S_t^\varepsilon) - \log(\kappa))} | \log(S_t^\varepsilon) = L_t^\varepsilon, v_t^\varepsilon = v^\varepsilon]}{ix} dx\right]$$
$$= E_M\left[\frac{1}{2} + \frac{1}{\pi}\int_{0+}^{+\infty} \mathrm{Re}\left[\frac{e^{ix(\ell^\varepsilon - \kappa)}}{ix}\right] dx \middle| L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$
$$= E_M\left[\frac{1}{2} + \frac{1}{\pi}\int_{0+}^{+\infty} \frac{\sin x(\ell^\varepsilon - \kappa)}{x} dx \middle| L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$
$$= E_M\left[\frac{1}{2} + \mathrm{sgn}(\kappa^\varepsilon - \kappa)\frac{1}{\pi}\int_{0+}^{+\infty} \frac{\sin(x)}{x} dx \middle| L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$
$$= E_M\left[\frac{1}{2} + \mathrm{sgn}(\ell^\varepsilon - \kappa) \middle| L_t^\varepsilon = v^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$
$$= E_M\left[1_{\ell^\varepsilon \geq \kappa} \middle| L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right]$$

where we have used the Dirichlet formula $\int_{-\infty}^{+\infty} \frac{\sin(x)}{x} dx = 1$ and the sgn function is defined as sgn $\mathrm{sgn}(x) = 1$ if $x > 0$, 0 if $x = 0$ and and -1 if $x < 0$.

In summary, we have just proved the following main theorem.

Theorem 3: For each ε > 0; the value of a European call option written on the model (10) and (11) is:

$$\hat{C}(S_t^\varepsilon, v_t^\varepsilon, t, K, T) = S_t^\varepsilon \hat{P}_1(S_t^\varepsilon, v_t^\varepsilon, t, K, T)$$
$$-Ke^{-r(T-t)}\hat{P}_2(S_t^\varepsilon, v_t^\varepsilon, t, K, T)$$

where, P_1 and P_2 are as given in Lemma 2.

DISCUSSION

A simple and efficient numerical scheme for determining the approximate process S_t^ε and v_t^ε is presented.

In order to compute the value of $\hat{C}(S_t^\varepsilon, v_t^\varepsilon, t; K, T)$ according to the formula as given in Theorem 3, we firstly choose a real number $\varepsilon > 0$, the solution that we get is the value of a European call option of the approximation model (10) with (11) and this value can be used as an approximating value of a call option of the fractional model (7) including model (8) as ε approaches zero. As the Monte-Carlo based technique, it will generate discrete sample values S_i^ε and v_i^ε of the stock and its variance respectively, by discretizing the associated SDEs (10) and (11). A natural choice for this purpose is the Euler scheme:

$$\begin{aligned}\hat{S}_{i+1}^\varepsilon &= S_{i+1}^\varepsilon((r - \lambda E_M[\hat{Y}_i])dt + \sqrt{\hat{v}_i^\varepsilon}\Delta W_t) + S_{i-}^\varepsilon Y_i \Delta N_i \\ \hat{v}_{i+1}^\varepsilon &= (\omega - \theta\hat{v}_i^\varepsilon)h + \xi\hat{v}_i^\varepsilon \Delta B_i^\varepsilon\end{aligned} \tag{32}$$

Where:

ΔW_t = Standard normal random variable with variance h, which is defined as the time mesh-size

ΔN_i = A Poisson process with intensity λh

These processes, W and N are assumed independent. However, (Glasserman, 2004) suggests that the second-order scheme has a better convergence (less bias) for option pricing applications but this scheme quite complex. For the simulation of Brownian motion there are numerous procedures see (Glasserman, 2004). For a sample path of fractional Brownian motion in Eq. 10, we can be simulated, for fixed t > 0, as:

$$B_t \simeq \sum_{k=1}^{N}(t - \frac{k}{N}t)^\alpha [W_{(k+1)\frac{t}{N}} - W_{k\frac{t}{N}}]$$
$$= \sum_{k=1}^{N}(t - \frac{k}{N}t)^\alpha \sqrt{\frac{t}{N}}[W_{(k+1)} - W_k]$$
$$= \sqrt{\frac{t}{N}}\sum_{k=1}^{N}(t - \frac{k}{N}t)^\alpha g_k$$

where, $g_k \sim N(0,1)$ and $0 < \alpha < 1/2$.

There are two basic estimation of the volatility process of Eq. 32 in the cast the volatility process is constant. The first method considers the function of density of transition from solution of Eq. 32. The second method proposes the estimate of the parameters of the model via the observation. Khaled and Samia (2010) for more details). In our case, the volatility of Eq. 32 is the stochastic process. There are many articles provided the estimation procedure for example see (Fiorentini *et al*., 2002).

CONCLUSION

An alternative fractional stochastic volatility model with jump is proposed in this study which the stock prices follows a geometric Brownian motion combining a compound Poisson processes and a stochastic volatility perturbed by a fractional Brownian motion. This proposed model exhibits a long memory of a stochastic volatility model that is not expressed in the classical stochastic volatility model. By using a fundamental result of the L^2-approximation of a fractional Brownian motion, we provide an approximate solution of bivariate diffusion model. A relationship between stochastic differential equations and partial differential equations for a bivariate model is presented. The risk-neutral method for valuation of options are reviewed. By using the technique base on the characteristic function of an underlying assets, an approximate formula of a European options is derived in an explicit formula. Finally a numerical integration technique to simulation the fractional stochastic volatility are present.

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Dear the Editor in Chief (Journal of the Korean Statistical Society)

Referring to your email letter on April 22, 2012 about the accepting the paper (with reference above) to publish in Korean statistical Society journal. I would like to thank you very much for your kind consideration. I did not change the title of this paper since I agree with your comment.

Moreover, I and the second author have carefully read and checked this paper again line by line. We have already corrected a little bit typing mistakes which we found while preparing the final version.

Now I have finished the process of preparing the draft of a final version and I just uploaded this final draft of the paper to you. I am glad to see the published article soon.

Yours Sincerely,

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Option Pricing for a Stochastic Volatility Lévy model with Stochastic Interest Rates

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Abstract. An alternative option pricing model under a forward measure is proposed, in which asset prices follow a stochastic volatility Lévy model with stochastic interest rate. The stochastic interest rate is driven by the Hull White process. By using an approximate method, we find a formulation for the European option in term of the characteristic function of the tail probabilities.

Keywords: Time-change Lévy process, Stochastic interest rate, Option pricing.

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1. Introduction

Let (Ω, F, P) be a probability space. A stochastic process L_t is a Lévy process if it has independent and stationary increments and has a stochastically continuous sample path, i.e. for any $\varepsilon > 0$, $\lim_{h \downarrow 0} P(|L_{t+h} - L_t| > \varepsilon) \to 0$. The simplest possible Lévy processes are the standard Brownian motion W_t, Poisson process N_t, and compound Poisson process $\sum_{i=1}^{N_t} Y_i$ where Y_i are i.i.d. random variables. Of course, we can build a new Lévy process from known ones by using the technique of linear transformation. For example, the jump diffusion process $\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where μ, σ are constants, is a Lévy process which comes from a linear transformation of two independent Lévy processes, i.e. a Brownian motion with drift and a compound Poisson process.

Assume that a risk-neutral probability measure Q exists and all processes in section 1 and 2 will be considered under this risk-neutral measure.

In the Black - Scholes model, the price of a risky asset S_t under a risk-neutral measure Q and with non dividend payment follows

$$S_t = S_0 \exp(\tilde{L}_t) = S_0 \exp\left(rt + \left(\sigma W_t - \tfrac{1}{2}\sigma^2 t\right)\right), \tag{1.1}$$

where $r \in \Re$ is a risk-free interest rate, $\sigma \in \Re$ is a volatility coefficient of the stock price.

Instead of modeling the log returns $\tilde{L}_t = rt + \left(\sigma W_t - \tfrac{1}{2}\sigma^2 t\right)$ with a normal distribution, we now replace it with a more sophisticated process L_t which is a Lévy process of the form

$$L_t = rt + \left(\sigma W_t - \tfrac{1}{2}\sigma^2 t\right) + J_t, \tag{1.2}$$

where J_t denotes a pure Lévy jump component, (i.e. a Lévy process with no Brownian motion part). We assume that the processes W_t and J_t are independent.

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To incorporate the volatility effect to the model Eq. (1.2), we follow the technique of Carr and Wu (2004) by subordinating a standard Brownian motion component $\sigma W_t - \frac{1}{2}\sigma^2 t$ and a pure jump Lévy process J_t by the time integral of a mean reverting Cox Ingersoll Ross (CIR) process

$$T_t = \int_0^t v_s ds ,$$

where v_t follows the CIR process

$$dv_t = \gamma(1 - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v . \tag{1.3}$$

Here W_t^v is a standard Brownian motion which corresponds to the process v_t. The constant $\gamma \in \mathbb{R}$ is the rate at which the process v_t reverts toward its long term mean and $\sigma_v > 0$ is the volatility coefficient of the process v_t.

Hence, the model (1.2) has been changed to

$$L_t = rt + \left(\sigma W_{T_t} - \tfrac{1}{2}\sigma^2 T_t\right) + J_{T_t} \tag{1.4}$$

and this new process is called a stochastic volatility Lévy process. One can interpret T_t as the stochastic clock process with activity rate process v_t. By replacing $\tilde{L}_t$ in (1.1) with L_t, we obtain a model of an underlying asset under the risk-neutral measure Q with stochastic volatility as follows:

$$S_t = S_0 \exp(L_t) = S_0 \exp(rt + \left(\sigma W_{T_t} - \tfrac{1}{2}\sigma^2 T_t\right) + J_{T_t}). \tag{1.5}$$

In this paper, we shall consider the problem of finding a formula for European call options based on the underlying asset model (1.5) for which the constant interest rates r is replaced by the stochastic interest rates r_t, i.e. the model under our consideration is given by

$$S_t = S_0 \exp\left(r_t t + \left(\sigma W_{T_t} - \tfrac{1}{2}\sigma^2 T_t\right) + J_{T_t}\right). \tag{1.6}$$

Here, we assume that r_t follows the Hull-White process

$$dr_t = (\alpha(t) - \beta r_t)dt + \sigma_r dW_t^r , \tag{1.7}$$

W_t^r is a standard Brownian motion with respect to the process r_t, and $dW_t^r dW_t = 0$. The constant $\beta \in \mathbb{R}$ is the rate at which the interest rate reverts toward its long term mean, $\sigma_r > 0$ is the volatility coefficient of the interest rate process (1.7), $\alpha(t)$ is a deterministic function, and is well defined in a time interval $[0, T]$. We also assume that the interest rate process r_t and the activity rate process v_t are independent.

The problem of option pricing under stochastic interest rates has been investigated for along time. Kim (2001) constructed the option pricing formula based on Black-Scholes model under several stochastic interest rate processes, i.e., Vasicek, CIR, Ho-Lee type. He found that by incorporating stochastic interest rates into the Black-Scholes model, for a short maturity option, does not contribute to improvement in the performance of the original Black-Scholes' pricing formula. Brigo and Mercurio (2006, page 883) mention that the stochastic feature of interest rates has a stronger impact on the option price when pricing for a long maturity option. Carr and Wu (2004) continue this study by giving the option pricing formula based on a time-changed Levy process model. But they still use constant interest rates in the model.

In this paper, we give an analysis on the option pricing model based on a time-changed Levy process with stochastic interest rates.

The rest of the paper is organized as follows. The dynamics under the forward measure is described in section 2. The option pricing formula is given in section 3. Finally, the close form solution for a European call option in terms of the characteristic function is given in section 4.

2. The dynamics under the Forward Measure

We begin by giving a brief review of the definition of a correlated Brownian motion and some of its properties (see Brummelhuis (2008) page 70). Recalling that a *standard Brownian motion in* R^n is a stochastic process $(\vec{Z}_t)_{t\geq 0}$ whose value at time t is simply a vector of n independent Brownian motions at t:

$$\vec{Z}_t = (Z_{1,t},...Z_{n,t}).$$

We use Z instead of W, since we would like to reserve the latter for the more general case of correlated Brownian motion, which will be defined as follows:

Let $\rho = (\rho_{ij})_{1\leq i,j\leq n}$ be a (constant) positive symmetric matrix satisfying $\rho_{ii} = 1$ and $-1 \leq \rho_{ij} \leq 1$. By Cholesky's decomposition theorem, one can find an upper triangular $n\times n$ matrix $\mathrm{H} = (h_{ij})$ such that $\rho = \mathrm{H}\mathrm{H}^t$, where H^t is the transpose of the matrix H. Let $\vec{Z}_t = (Z_{1,t},...Z_{n,t})$ be a standard Brownian motion as introduced above, we define a new vector-valued process $\vec{W}_t = (W_{1,t},...,W_{n,t})$ by $\vec{W}_t = \mathrm{H}\vec{Z}_t$ or, in term of components,

$$W_{i,t} = \sum_{j=1}^{n} h_{ij} Z_{j,t},\ i = 1,...,n.$$

The process $(\vec{W}_t)_{t\geq 0}$ is called a *correlated Brownian motion* with a (constant) correlation matrix ρ. Each component-process $(W_{i,t})_{t\geq 0}$ is itself a standard Brownian motion. Note that if $\rho = Id$ (the identity matrix) then $\vec{W}_t$ is a standard Brownian motion. For example, if we let a symmetric matrix

$$\rho = \begin{bmatrix} 1 & \rho_v & 0 \\ \rho_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.1}$$

Then ρ has a *Cholesky decomposition* of the form $\rho = HH^T$ where H is an upper triangular matrix of the form

$$H = \begin{bmatrix} \sqrt{1-\rho_v^2} & \rho_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\vec{Z}_t = (Z_t, Z_t^r, Z_t^v)$ be three independent Brownian motions then $\vec{W}_t = (W_t, W_t^r, W_t^v)$ defined by $\vec{W}_t = \mathrm{H}\vec{Z}_t$, or in terms of components,

$$W_t = (\sqrt{1-\rho_v^2})Z_t + \rho_v Z_t^v, \quad W_t^v = Z_t^v, \quad W_t^r = Z_t^r, \tag{2.2}$$

is a correlated Brownian motion with correlation matrix ρ as given in Eq. (2.1).

Now let us turn to our problem. Note that, by Ito's lemma, the model (1.6) has the dynamic given by

$$\begin{aligned} dS_t &= S_{t-}r_t dt + \sigma S_{t-} dW_{T_t} + S_{t-} dJ_{T_t}^*, \\ dr_t &= (\alpha(t) - \beta r_t)dt + \sigma_r dW_t^r, \\ dv_t &= \gamma(1-v_t)dt + \sigma_v \sqrt{v_t}\, dW_t^v, \end{aligned} \tag{2.3}$$

where $dJ_{T_t}^* = dJ_{T_t} + (e^{\Delta J_{T_t}} - 1 - \Delta J_{T_t})$, $dW_t dW_t^r = dW_t^r dW_t^v = 0$, and $dW_t dW_t^v = \rho_v dt$.

We can re-write the system (2.3) in terms of three independent Brownian motions (Z_t, Z_t^v, Z_t^r) as follows:

$$dS_t = S_{t-}r_t dt + \sigma S_{t-}\left(\rho_v dZ_{T_t}^v + \sqrt{1-\rho_v^2}dZ_{T_t}\right) + S_{t-}dJ_{T_t}^*, \tag{2.4}$$

$$dr_t = (\alpha(t) - \beta r_t)dt + \sigma_r dZ_t^r, \tag{2.5}$$

$$dv_t = \gamma(1-v_t)dt + \sigma_v\sqrt{v_t}dZ_t^v. \tag{2.6}$$

This decomposition makes it easier to perform a measure transformation. In fact, for any fixed maturity *T*, let us denote by Q^T the T-forward measure, i.e. the probability measure that is defined by the Radon-Nikodym derivative,

$$\frac{dQ^T}{dQ} = \frac{\exp\left(-\int_0^T r_u du\right)}{P(0,T)}. \tag{2.7}$$

Here, $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T and is defined as

$$P(t,T) = E_Q\left[e^{-\int_t^T r_s ds} \mid F_t\right]. \tag{2.8}$$

We denote $f(0,t)$ to be the market instantaneous forward rate at time 0 for the maturity time $t \geq 0$ and it is defined by

$$f(0,t) := -\frac{\partial}{\partial t}\ln P(0,t), \quad 0 \leq t \leq T. \tag{2.9}$$

Poulsen (2005) gave a relation between the coefficients of Eq. (2.5) and the forward rate $f(0,t)$ as follows:

$$\alpha(t) = \frac{\partial f(0,t)}{\partial t} + \beta f(0,t) + \frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta t}\right). \tag{2.10}$$

Lemma 1 *The process r_t which satisfies the dynamic in (2.5) can be written in the form*

$$r_t = x_t + \varphi(t), \quad 0 \leq t \leq T, \tag{2.11}$$

where the process x_t satisfies

$$dx_t = -\beta x_t dt + \sigma_r dZ_t^r,\ x_0 = 0. \tag{2.12}$$

Moreover, the function φ is deterministic, well defined in the time interval [0,T], and satisfies

$$\varphi(t) = f(0,t) + \frac{\sigma_r^2}{2\beta^2}\left(1 - e^{-\beta t}\right)^2. \tag{2.13}$$

In particular, $\varphi(0) = r_0$.

Proof To find a solution of SDE (2.5), we let $g(t,r) = e^{\beta t}r$. By using Ito's Lemma, we have

$$dg = de^{\beta t}r_t = \alpha(t)e^{\beta t}dt + e^{\beta t}\sigma_r dZ_t^r. \tag{2.14}$$

Integrating on both sides of the above equation from 0 to t, we obtain

$$r_t = r_0 e^{-\beta t} + \int_0^t \alpha(u)e^{\beta(u-t)}du + \int_0^t e^{\beta(u-t)}\sigma_r dZ_u^r. \tag{2.15}$$

Substituting the value of $\alpha(t)$ from Eq. (2.10) into (2.15), we have

$$r_t = r_0 e^{-\beta t} + \int_0^t \left(\frac{\partial f(0,u)}{\partial u} + \beta f(0,u) + \frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta u}\right)\right)e^{-\beta(t-u)}du + \sigma_r\int_0^t e^{-\beta(t-u)}dZ_u^r. \tag{2.16}$$

Applying integration by parts formula to Eq. (2.16) and after simplifying, we obtain

$$r_t = r_0 e^{-\beta t} + f(0,t) - f(0,0)e^{-\beta t} + \frac{\sigma_r^2}{2\beta^2}\left(e^{-\beta t} - 1\right)^2 + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r. \tag{2.17}$$

By using the definition of φ_t from Eq. (2.13), we can write Eq.(2.17) into a compact form as follows:

$$r_t = r_0 e^{-\beta t} - f(0,0)e^{-\beta t} + \varphi(t) + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r = \varphi(t) + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r, \tag{2.18}$$

because of $f(0,0) = r_0$, see Andrew(2004) page 89.
Note that the solution of Eq. (2.12) is

$$x_t = x_0 e^{-\beta t} + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r = \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r. \tag{2.19}$$

Hence, $r_t = \varphi(t) + x_t,\ 0 \le t \le T$. The proof is now complete.

Now we are ready to calculate the Radon-Nikodym derivative as appears in Eq. (2.7). By virtue of Lemma 1, $r_t = \varphi_t + x_t$. Substituting r_t and $P(0,T) = \exp\left(-\int_0^T f(0,u)du\right)$ into Eq. (2.7), one gets

$$\frac{dQ^T}{dQ} = \exp\left(-\int_0^T x_u du - \frac{\sigma_r^2}{2\beta^2}\int_0^T \left(1 - e^{-\beta(T-u)}\right)^2 du\right). \tag{2.20}$$

Stochastic integration by parts implies

$$\int_0^T x_u du = Tx_T - \int_0^T u dx_u = \int_0^T (T-u) dx_u . \tag{2.21}$$

By substituting the expression for dx_u from Eq.(2.12), we have

$$\int_0^T (T-u) dx_u = -\beta \int_0^T (T-u) x_u du + \sigma_r \int_0^T (T-u) dZ_u^r . \tag{2.22}$$

Moreover, by substituting the expression for x_u from Eq. (2.19) into the right hand side of Eq. (2.22), one gets

$$-\beta \int_0^T (T-u) x_u du = -\beta\sigma_r \int_0^T \left((T-u) \int_0^u e^{-\beta(u-s)} dZ_u^r \right) du. \tag{2.23}$$

Using integration by parts, we have

$$-\beta\sigma_r \int_0^T \left((T-u) \int_0^u e^{-\beta(u-s)} dZ_u^r \right) du = -\frac{\sigma_r}{\beta}\left[\int_0^T \left(e^{-\beta(T-u)} - 1\right) dZ_u^r\right] - \sigma_r \int_0^T (T-u) dZ_u^r. \tag{2.24}$$

Substituting Eq. (2.24) into (2.22) and Eq. (2.21) becomes

$$\int_0^T x_u du = -\frac{\sigma_r}{\beta}\left[\int_0^T \left(e^{-\beta(T-u)} - 1\right) dZ_u^r\right]. \tag{2.25}$$

Substituting Eq. (2.25) into (2.20), we obtain

$$\frac{dQ^T}{dQ} = \exp\left(-\frac{\sigma_r}{\beta}\int_0^T \left(1 - e^{-\beta(T-u)}\right) dZ_u^r - \frac{\sigma_r^2}{2\beta^2}\int_0^T \left(1 - e^{-\beta(T-u)}\right)^2 du\right). \tag{2.26}$$

Hence, by Girsanov's theorem, the three processes Z_t^{rT}, Z_t^{vT} and Z_t^T defined by

$$dZ_t^{rT} = dZ_t^r + \frac{\sigma_r}{\beta}\left(1 - e^{-\beta(T-t)}\right) dt, \quad dZ_t^{vT} = dZ_t^v , \quad dZ_t^T = dZ_t, \tag{2.27}$$

are three independent Brownian motions under the T-forward measure Q^T. Therefore, the dynamics of r_t, v_t and S_t under Q^T are given by

$$dS_t = S_{t-} r_t dt + \sigma S_{t-}\left(\rho_v dZ_{T_t}^{vT} + \sqrt{1-\rho_v^2}\, dZ_{T_t}^{T}\right) + S_{t-} dJ_{T_t}^{*}, \tag{2.28}$$

$$dr_t = \left(\alpha(t) - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)dt + \sigma_r dZ_t^{rT}, \tag{2.29}$$

$$dv_t = \gamma(1-v_t)dt + \sigma_v \sqrt{v_t}\, dZ_t^{vT}. \tag{2.30}$$

3. The Pricing of a European Call Option on the Given Asset

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $\left(\Omega, F, F_t, Q^T\right)$, and F_t is usually taken to be the price history up to time t. All processes in this section will be defined in this space. We denote C the price at time t of a European call option on the current price of an underlying asset S_t with strike price K and expiration time T.

The terminal payoff of a European option on the underlying stock S_t with strike price K is

$$\max(S_T - K, 0). \tag{3.1}$$

This means the holder will exercise his right only $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \le K$ then the holder will buy the underlying asset from the market and the value of the option is zero.

We would like to find a formula for pricing a European call option with strike price K and maturity T based on the model (2.28) - (2.30). Consider a continuous-time economy where interest rates are stochastic and the price of the European call option at time t under the T-forward measure Q^T is

$$\begin{aligned} C(t, S_t, r_t, v_t; T, K) &= P^*(t,T) E_{Q^T}\left(\max\left(S_T - K, 0\right) \mid S_t, r_t, v_t\right) \\ &= P^*(t,T)\int_0^\infty \max\left(S_T - K, 0\right) p_{Q^T}(S_T \mid S_t, r_t, v_t) dS_T. \end{aligned}$$

Here E_{Q^T} is the expectation with respect to the T-forward probability measure, p_{Q^T} is the corresponding conditional density given $\left(S_t, r_t, v_t\right)$, and P^* is a zero coupon bond which is defined by

$$P^*(t,T) := E_{Q^T}\left[\exp\left(-\int_t^T r_s ds\right) \mid F_t\right]. \tag{3.2}$$

With a change in variable $X_t = \ln S_t$,

$$\begin{aligned} C(t, S_t, r_t, v_t; T, K) &= P^*(t,T)\int_{-\infty}^{\infty} \max\left(e^{X_T} - K, 0\right) p_{Q^T}\left(X_T \mid X_t, r_t, v_t,\right) dX_T \\ &= e^{X_t} \mathrm{P}_1\left(t, X_t, r_t, v_t; T, K\right) - K P^*(t,T) \mathrm{P}_2\left(t, X_t, r_t, v_t; T, K\right) \\ &= e^{X_t} \Pr\left(X_T > \ln K \mid X_t, r_t, v_t\right) - K P^*(t,T) \Pr\left(X_T > \ln K \mid X_t, r_t, v_t\right), \end{aligned} \tag{3.3}$$

where those probabilities in Eq. (3.3) are calculated under the probability measure Q^T.

The European call option for log asset price $X_t = \ln S_t$, will be denoted by

$$\hat{C}(t, X_t, r_t, v_t; T, \kappa) = e^{X_t} \tilde{\mathrm{P}}_1\left(t, X_t, r_t, v_t; T, \kappa\right) - e^{\kappa} P^*(t,T) \tilde{\mathrm{P}}_2\left(t, X_t, r_t, v_t; T, \kappa\right), \tag{3.4}$$

where $\kappa = \ln K$ and $\tilde{P}_j(t, X_t, r_t, v_t; T, \kappa) := P_j(t, X_t, r_t, v_t; T, K), \quad j = 1, 2.$

Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 4.

Next, consider a continuous-time economy where interest rates are stochastic and satisfy Eq. (2.29). Since the SDE in Eq. (2.29) satisfies all the necessary conditions of Theorem 32, see Protter (2005) page 238, then the solution has Markov property. As a consequence, the zero coupon bond price at time *t* under the forward measure Q^T in Eq. (3.2) satisfies

$$P^*(t,T) = E_{Q^T}\left[\exp\left(-\int_t^T r_s ds\right) \mid r_t\right]. \tag{3.5}$$

Note that $P^*(t,T)$ depends on r_t then it becomes a function $F(t,r_t)$ of r_t. This means that the calculation of $P^*(t,T)$ can now be formulated as a search for the function $F(t,r_t)$.

Lemma 2 *The price of a zero coupon bond can be derived by computing the expectation (3.5). We obtain*

$$P^*(t,T) = \exp\left(a(t,T) + b(t,T)r_t\right) \tag{3.6}$$

where $b(t,T) = \dfrac{1}{\beta}\left(e^{-\beta(T-t)} - 1\right),\ a(t,T) = -f(0,t)b(t,T) + \ln\left(\dfrac{P^*(0,T)}{P^*(0,t)}\right) - \dfrac{3\sigma_r^2}{4\beta}\left[b(t,T)^2\left(1 - e^{-2\beta t}\right)\right].$

Proof Under the T-forward measure Q^T, the interest rate is given by Eq. (2.29). The specification of the interest rate means that the model (2.29) belong to the affine class of interest rate models. Thus the bond price at time *t* with maturity *T* is of the form Eq. (3.6) where $a(t,T)$ and $b(t,T)$ are functions to be determined under the condition $a(T,T) = 0$ and $b(T,T) = 0$. We will now find explicit formulas for the functions $a(t,T)$ and $b(t,T)$ in Eq. (3.6).
The zero coupon bond price PDE satisfies (the proof is similar to Privault (2008) Prop. 4.1)

$$\frac{\partial F(t,r_t)}{\partial t} + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right)\frac{\partial F(t,r_t)}{\partial r_t} + \frac{1}{2}\frac{\partial^2 F(t,r_t)}{\partial r_t^2}\sigma_r^2 - r_t F(t,r_t) = 0\,. \tag{3.7}$$

Note that $F(t,r_t) = P^*(t,T)$. We substitute the value $F(t,r_t)$ from (3.6) into the above equation and after canceling some common factors, we have

$$\left(\frac{\partial a(t,T)}{\partial t} + r_t\frac{\partial b(t,T)}{\partial t}\right) + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right)b(t,T) + \frac{1}{2}b^2(t,T)\sigma_r^2 - r_t = 0.$$

We can reduce it to two ordinary differential equations

$$\frac{\partial a(t,T)}{\partial t} + \frac{\sigma_r^2}{2}b^2(t,T) + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)b(t,T) = 0, \tag{3.8}$$

$$\frac{\partial b(t,T)}{\partial t} - \beta b(t,T) - 1 = 0, \tag{3.9}$$

with boundary conditions $a(T,T) = 0, \quad b(T,T) = 0.$
Firstly, we note that the solution of Eq. (3.9) which satisfies the boundary conditions $b(T,T) = 0$ is

$$b(t,T) = \frac{1}{\beta}\left(e^{-\beta(T-t)} - 1\right). \tag{3.10}$$

Secondly, we try to solve Eq. (3.8). Note that

$$\int_t^T \frac{\partial a(u,T)}{\partial u}du = \left[a(u,T)\right]_{u=t}^{u=T} = a(T,T) - a(t,T) = -a(t,T)\,. \tag{3.11}$$

Thus

$$a(t,T) = \left(\frac{3\sigma_r^2}{2}\right)\int_t^T \left(b(u,T)\right)^2 du + \int_t^T \alpha(u)b(u,T)du. \tag{3.12}$$

It follows from Eq. (2.9) and (3.6) that the forward rate at time 0 with the maturity T can be written as

$$f(0,T) = -\frac{\partial}{\partial T}\ln P^*(0,T) = -\frac{\partial a(0,T)}{\partial T} - r_0\frac{\partial b(0,T)}{\partial T}. \tag{3.13}$$

Differentiate $a(0,T)$ with respect to T and using $a(T,T)=0,\ b(T,T)=0$, we obtain from Eq. (3.12) that

$$\frac{\partial a(0,T)}{\partial T} = 3\sigma_r^2\int_0^T b(u,T)\frac{\partial b(u,T)}{\partial T}du + \int_0^T \alpha(u)\frac{\partial b(u,T)}{\partial T}du.$$

Substituting the value of $b(u,T)$ from Eq. (3.10) into the above equation and after some calculations, we get

$$\frac{\partial a(0,T)}{\partial T} = \frac{\left(3\sigma_r^2\right)}{2\beta^2}\left(\mathrm{e}^{-\beta T}-1\right)^2 - \int_0^T \mathrm{e}^{-\beta(T-u)}\alpha(u)du.$$

Now substitute the value of $\frac{\partial a(0,T)}{\partial T}$ and the value of $\frac{\partial b(0,T)}{\partial T}$ into Eq. (3.13), we have

$$f(0,T) = -\frac{3\sigma_r^2}{2\beta^2}\left(\mathrm{e}^{-\beta T}-1\right)^2 + \int_0^T \mathrm{e}^{-\beta(T-u)}\alpha(u)du + r_0\left(\mathrm{e}^{-\beta T}\right). \tag{3.14}$$

To isolate $\alpha(T)$, we differentiate $f(0,T)$ with respect to T and get

$$\frac{\partial f(0,T)}{\partial T} = \frac{3\sigma_r^2}{\beta}\left(\mathrm{e}^{-2\beta T}-\mathrm{e}^{-\beta T}\right) - r_0\beta\,\mathrm{e}^{-\beta T} + \left(\alpha(T) - \beta\left(\int_0^T \mathrm{e}^{-\beta(T-u)}\alpha(u)du\right)\right).$$

Using Eq. (3.14) to rewrite the above equation and after simplifying , we get

$$\alpha(T) = \frac{\partial f(0,T)}{\partial T} + \beta f(0,T) - \frac{3\sigma_r^2}{2\beta}\left(\mathrm{e}^{-2\beta T}-1\right). \tag{3.15}$$

Next, we shall find a formula for $a(t,T)$ in Eq. (3.12). Note that

$$\frac{3\sigma_r^2}{2}\int_t^T b(u,T)^2 du = \frac{3\sigma_r^2}{2\beta}\left(-\frac{1}{2}b(t,T)^2 + \frac{1}{\beta}\left(b(t,T)+T-t\right)\right),$$

and

$$\int_t^T \alpha(u)b(u,T)du = \int_t^T\left(\frac{\partial f(0,T)}{\partial T} + \beta f(0,T) - \frac{3\sigma_r^2}{2\beta}\left(\mathrm{e}^{-2\beta T}-1\right)\right)b(u,T)du$$

$$= -f(0,t)b(t,T) - \int_t^T f(0,u)du - \frac{3\sigma_r^2}{2\beta^2}(T-t) + \frac{3\sigma_r^2}{4\beta^3}\left[\mathrm{e}^{-2\beta T} - 2\,\mathrm{e}^{-\beta(T+t)} - 2\,\mathrm{e}^{-\beta(T-t)} + \mathrm{e}^{-2\beta t} + 2\right].$$

Therefore

$$a(t,T) = -f(0,t)b(t,T) - \int_t^T f(0,u)du - \frac{3\sigma_r^2}{4\beta}b^2(t,T)(1-e^{-2\beta t}).$$

By definition, $P^*(0,T) = e^{-\int_0^T f(0,u)du}$. Thus $-\int_t^T f(0,u)du = \ln\left(\frac{P^*(0,T)}{P^*(0,t)}\right)$.

Finally, we have

$$a(t,T) = -f(0,t)b(t,T) + \ln\left(\frac{P^*(0,T)}{P^*(0,t)}\right) - \frac{3\sigma_r^2}{4\beta}b^2(t,T)(1-e^{-2\beta t}).$$

The proof is now complete.

The following lemma shows the relationship between $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of Eq. (3.4).

Lemma 3. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ in the option values of Eq. (3.4) satisfy the PIDEs:*

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_1}{\partial t} + A[P_1] + \frac{1}{2}\sigma^2 v \frac{\partial \tilde{P}_1}{\partial x} + \rho_v \sigma v \sigma_v \frac{\partial \tilde{P}_1}{\partial v} \\ &+ v\int_{-\infty}^{\infty} \left[(e^y - 1)\left(\tilde{P}_1(t, x+y, r, v; T, \kappa) - \tilde{P}_1(x, t, r, v; T, \kappa) \right) \right] k(y) dy \end{aligned} \tag{3.16}$$

and subject to the boundary condition at expiration t= T , $\tilde{P}_1(T, x, r, v; T, \kappa) = 1_{x>\kappa}.$ (3.17)

Moreover, $\tilde{P}_2$ *satisfies the equation*

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_2}{\partial t} + A[\tilde{P}_2] - \frac{\sigma^2 v}{2}\frac{\partial \tilde{P}_2}{\partial x} + \sigma_r^2 b(t,T)\frac{\partial \tilde{P}_2}{\partial r} + \left(\frac{\partial a(t,T)}{\partial t} + r\frac{\partial b(t,T)}{\partial t} \right)\tilde{P}_2 \\ &+ \left(\frac{3\sigma_r^2}{2} b^2(t,T) - r + \left(\alpha(t) - \beta r \right) b(t,T) \right)\tilde{P}_2 \end{aligned} \tag{3.18}$$

and subject to the boundary condition at expiration t= T, $\tilde{P}_2(T, x, r, v; T, \kappa) = 1_{x>\kappa}.$ (3.19)

Here, for i=1,2,

$$\begin{aligned} A[\tilde{P}_i] &= r\frac{\partial \tilde{P}_i}{\partial x} + \left(\alpha(t) - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) \right)\frac{\partial \tilde{P}_i}{\partial r} + \gamma(1-v)\frac{\partial \tilde{P}_i}{\partial v} + \frac{\sigma_v^2 v}{2}\frac{\partial^2 \tilde{P}_i}{\partial v^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_i}{\partial x^2} \\ &+ \frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_i}{\partial r^2} + \left(\rho_v \sigma v \sigma_v \right)\frac{\partial \tilde{P}_i}{\partial v \partial x} + v\int_{-\infty}^{\infty} \left(\tilde{P}_i(t, x+y, r, v; T, \kappa) - \tilde{P}_i(x, t, r, v; T, \kappa) - \left(\frac{\partial \tilde{P}_i}{\partial x} \right)(e^y - 1) \right) k(y) dy. \end{aligned} \tag{3.20}$$

Note that $1_{x>\kappa} = 1$ *if* $x > \kappa$ *and zero otherwise. We assume that the jump kernel k(y) exists.*

Proof. See Appendix.

4. The Closed-Form Solution for European call options

For j =1,2, the characteristic function for $\tilde{P}_j(t, x, r, v; T, \kappa)$, with respect to the variable κ, are defined by

$$f_j(t, x, r, v; T, u) := -\int_{-\infty}^{\infty} e^{iu\kappa} d\tilde{P}_j(t, x, r, v; T, \kappa), \tag{4.1}$$

with a minus sign to account for the negativity of the measure $d\tilde{P}_j$. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t} + A_j\left[f_j \right](t, x, r, v; T, \kappa) = 0, \tag{4.2}$$

with the respective boundary conditions

$$f_j(T, x, r, v; T, u) = -\int_{-\infty}^{\infty} e^{iu\kappa} d\tilde{P}_j(t, x, r, v; T, \kappa) = -\int_{-\infty}^{\infty} e^{iu\kappa}(-\delta(\kappa - x)) d\kappa = e^{iux}.$$

The following lemma shows how to calculate the characteristic functions for $\tilde{P}_1$ and $\tilde{P}_2$ as they appeared in Lemma 3.

Lemma 4 *The functions* $\tilde{P}_1$ *and* $\tilde{P}_2$ *can be calculated by the inverse Fourier transformations of the characteristic function, i.e.*

$$\tilde{P}_j(t,x,r,v;T,\kappa)=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{\infty}\mathrm{Re}\left[\frac{e^{iu\kappa}f_j(t,x,r,v;T,u)}{iu}\right]du,$$

for $j=1,2,$ *with* $Re[.]$ *denoting the real component of a complex number.*
By letting $\tau=T-t$, *the characteristic function* f_j *is given by*

$$f_j(t,x,r,v;t+\tau,u)=\exp\left(iux+B_j(\tau)+rC_j(\tau)+vE_j(\tau)-(j-1)\ln P^*(t,t+\tau)\right),$$

where $\tilde{b}_{1j}=b_{2j}+\nabla_j$, $\tilde{b}_{2j}=b_{2j}-\nabla_j$, $\nabla_j=\sqrt{b_{2j}{}^2-4b_{0j}b_1}$, $b_{22}=\rho_v\sigma\sigma_v iu-\gamma,$

$$b_1=\frac{\sigma_v^2}{2},\quad b_{21}=\rho_v\sigma\sigma_v\left(1+iu\right)-\gamma,\quad \alpha(t)=\frac{\partial f(0,t)}{\partial t}+\beta f(0,t)-\frac{3\sigma_r^2}{2\beta}\left(\mathrm{e}^{-2\beta t}-1\right),$$

$$b_{01}=\frac{\sigma^2}{2}\left(iu-u^2\right)+\int_{-\infty}^{\infty}\left[e^{iux+y}-iu(e^y-1)\right]k(y)dy,\ C_1(\tau)=\frac{iu}{\beta}(1-e^{-\beta\tau}),C_2(\tau)=\frac{iu-1}{\beta}\left(1-e^{\beta\tau}\right),$$

$$b_{02}=-\frac{\sigma^2}{2}(iu+u^2)+\int_{-\infty}^{\infty}\left(e^{iux}-iu(e^y-1)\right)k(y)dy,\ E_j(\tau)=\frac{\left(e^{\tau\nabla_j}-1\right)\tilde{b}_{1j}\tilde{b}_{2j}}{2b_1\left(\tilde{b}_{1j}-e^{\tau\nabla_j}\tilde{b}_{2j}\right)},$$

$$B_1(\tau)=\int_{T-\tau}^{T}\alpha(t)C_1(T-t)dt+\frac{\sigma_r^2}{2\beta^3}\left(\frac{u^2}{2}+\frac{iu}{\beta}\right)\left(\left(e^{\beta\tau}-2\right)^2+2\beta\tau-1\right)+\frac{\gamma^2\left(\tilde{b}_2-\tilde{b}_1\right)}{2b_1}\ln\left(\frac{\tilde{b}_{21}-\tilde{b}_{11}}{e^{\tau\nabla}\tilde{b}_{21}-\tilde{b}_{11}}\right)+\frac{\gamma^2\tilde{b}_{21}\nabla_1\tau}{2b_1},$$

$$B_2(\tau)=\int_{T-\tau}^{T}\alpha(t)C_2(T-t)dt+\frac{\sigma_r^2\left(iu-1\right)}{2\beta^3}\left(\frac{1}{\beta}-\frac{(iu-1)}{2}\right)\left(\left(e^{\beta\tau}-2\right)^2+2\beta\tau-1\right)+\frac{\gamma^2\tilde{b}_{22}\nabla_2\tau}{2b_1}$$
$$+\frac{\gamma^2\left(\tilde{b}_{22}-\tilde{b}_{12}\right)}{2b_1}\ln\left(\frac{\tilde{b}_{22}-\tilde{b}_{12}}{e^{\tau\nabla}\tilde{b}_{22}-\tilde{b}_{12}}\right).$$

Proof. To solve the characteristic function explicitly, letting $\tau=T-t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1(t,x,r,v;t+\tau,u)=\exp\left(iux+B_1(\tau)+rC_1(\tau)+vE_1(\tau)\right), \tag{4.3}$$

and the boundary condition $B_1(0)=C_1(0)=E_1(0)=0$. This conjecture exploits the linearity of the coefficient in PIDEs (4.2). Note that the characteristic function of f_1 always exists. In order to substitute Eq. (4.3) into (4.2), firstly, we compute

$$\frac{\partial f_1}{\partial t}=-\left(B_1'(\tau)+rC_1'(\tau)+vE_1'(\tau)\right)f_1,\quad \frac{\partial f_1}{\partial x}=iuf_1,\quad \frac{\partial f_1}{\partial r}=C_1(\tau)f_1,\quad \frac{\partial f_1}{\partial v}=E_1(\tau)f_1,$$
$$\frac{\partial^2 f_1}{\partial x^2}=-u^2f_1,\quad \frac{\partial^2 f_1}{\partial v^2}=E_1^2(\tau)f_1,\quad \frac{\partial^2 f_1}{\partial r^2}=C_1{}^2(\tau)f_1,\quad \frac{\partial^2 f_1}{\partial v\partial x}=iuE_1(\tau)f_1,$$
$$f_1(t,x+y,r,v;t+\tau,u)-f_1(t,x,r,v;t+\tau,u)=e^{iux}f_1(t,x,r,v;t+\tau,u).$$

Substituting all the above terms into Eq. (4.2), after cancelling the common factor of f_1, we get a simplified form as follows:

$$0=v\left(-E_1'(\tau)+\frac{\sigma_v^2}{2}E_1^2(\tau)+\left(\rho_v\sigma\sigma_v\left(1+iu\right)-\gamma\right)E_1(\tau)+\frac{\sigma^2\left(iu-u^2\right)}{2}+\int_{-\infty}^{\infty}\left[e^{iux+y}-iu(e^y-1)\right]k(y)dy\right)$$
$$+\left(-B_1'(\tau)+\left(\alpha(t)-\frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)C_1(\tau)+\gamma E_1(\tau)+\frac{\sigma_r^2}{2}C_1{}^2(\tau)\right)+r\left(-C_1'(\tau)+iu-\beta C_1(\tau)\right).$$

By separating the order r, v and ordering the remaining terms, we can reduce it to three ordinary differential equations (ODEs) as follows:

$$C_1'(\tau) = -\beta C_1(\tau) + iu, \tag{4.4}$$

$$E_1'(\tau) = \frac{\sigma_v^2}{2} E_1^2(\tau) + \left[\rho_v \sigma\sigma_v \left(1+iu\right) - \gamma\right] E_1(\tau) + \frac{\sigma^2(iu-u^2)}{2} + \int_{-\infty}^{\infty} \left(e^{iux+y} - iu(e^y - 1)\right) k(y)dy, \tag{4.5}$$

$$B_1'(\tau) = \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right) C_1(\tau) + \frac{\sigma_r^2}{2} C_1^{\,2}(\tau) + \gamma E_1(\tau). \tag{4.6}$$

It is clear from Eq. (4.4) and $C(0)=0$ that $C_1(\tau) = \frac{iu}{\beta}(1-e^{-\beta\tau})$. (4.7)

Let $b_0 = \frac{\sigma^2\left(iu-u^2\right)}{2} + \int_{-\infty}^{\infty}\left[e^{iux+y} - iu(e^y-1)\right]k(y)dy,\ b_1 = \frac{\sigma_v^2}{2}$ and $b_2 = \left(\rho_v\sigma\sigma_v\left(1+iu\right) - \gamma\right).$

Substitute these constants into Eq. (4.5), one gets

$$E_1'(\tau) = b_1\left(E_1^2(\tau) + \frac{b_2}{b_1}E_1(\tau) + \frac{b_0}{b_1}\right) = b_1\left[\left(E_1(\tau) - \tfrac{-b_2+\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}\right)\left(E_1(\tau) - \tfrac{-b_2-\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}\right)\right].$$

By method of variable separation, we have

$$\frac{dE_1(\tau)}{\left(E_1(\tau) - \tfrac{-b_2+\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}\right)\left(E_1(\tau) - \tfrac{-b_2-\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}\right)} = b_1 d\tau.$$

Using partial fraction on the left hand side, one obtains

$$\left(\frac{1}{\left(E_1(\tau) - \tfrac{-b_2+\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}\right)} - \frac{1}{\left(E_1(\tau) - \tfrac{-b_2-\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}\right)}\right) dE_1(\tau) = \sqrt{b_2^{\,2} - 4b_0b_1}\, d\tau.$$

Integrating both sides, we obtain

$$\ln\left(\frac{E_1(\tau) - \tfrac{-b_2+\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}}{E_1(\tau) - \tfrac{-b_2-\sqrt{b_2^{\,2}-4b_0b_1}}{2b_1}}\right) = \tau\sqrt{b_2^{\,2} - 4b_0b_1} + E_0.$$

Applying boundary condition $E_1(\tau=0)=0,$ we get $E_0 = \ln\left(\frac{-b_2+\sqrt{b_2^{\,2}-4b_0b_1}}{-b_2-\sqrt{b_2^{\,2}-4b_0b_1}}\right).$

Solving for E_1, we have $E_1(\tau) = \dfrac{\left(e^{\tau\sqrt{b_2^{\,2}-4b_0b_1}} - 1\right)\tilde{b}_1\tilde{b}_2}{2b_1\left(\tilde{b}_1 - e^{\tau\sqrt{b_2^{\,2}-4b_0b_1}}\tilde{b}_2\right)},$

where $\tilde{b}_1 = b_2 + \sqrt{b_2^{\,2} - 4b_0b_1}$, and $\tilde{b}_2 = b_2 - \sqrt{b_2^{\,2} - 4b_0b_1}$.

In order to solve $B_1(\tau)$, we substitute $C_1(\tau)$ and $E_1(\tau)$ into Eq. (4.6) to get

$$B_1(\tau) = \int_{T-\tau}^{T} \alpha(t) C_1(T-t)dt + \frac{\sigma^2{}_r}{2\beta^3}\left(\frac{u^2}{2} + \frac{iu}{\beta}\right)\left(\left(e^{\beta\tau} - 2\right)^2 + 2\beta\tau - 1\right) + \frac{\gamma^2\left(\tilde{b}_2 - \tilde{b}_1\right)}{2b_1}\ln\left(\frac{\tilde{b}_2 - \tilde{b}_1}{e^{\tau\nabla}\tilde{b}_2 - \tilde{b}_1}\right) + \frac{\gamma^2\tilde{b}_2\nabla\tau}{2b_1},$$

where $\nabla = \sqrt{b_2^{\,2} - 4b_0b_1}$ and $\alpha(t)$ is defined in Eq. (3.15).

The details of the proof for the characteristic function f_2 are similar to f_1. Hence, we have

$$f_2(t,x,r,v;t+\tau,u) = \exp\left[iux + B_2(\tau) + rC_2(\tau) + vE_2(\tau) - \ln P^*(t,t+\tau)\right],$$

where $B_2(\tau), C_2(\tau),$ and $E_2(\tau)$ are as given in the Lemma.

Up to this point, we obtained the characteristic functions in close form. However, we are interested in the probability $\tilde{P}_j$. These can be inverted from the characteristic functions by performing the following integration

$$\tilde{P}_j(t,x,r,v;T,\kappa) = \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{\infty} \mathrm{Re}\left(\frac{e^{iu\kappa} f_j(t,x,v.r;T,u)}{iu}\right) du,\ j = 1,2, \tag{4.14}$$

where $X_t = \ln S_t$ and $\kappa = \ln K$ (see Sattayatham and Intarasit (2011)).
The proof is now complete.

In summary, we have just proved the following main theorem.

Theorem 5 *The value of a European call option of SDE* (2.28) *is*

$$C(t,S_t,r_t,v_t;T,K) = S_t\tilde{P}_1(t,X_t,r_t,v_t;T,\kappa) - KP^*(t,T)\tilde{P}_2(t,X_t,r_t,v_t;T,\kappa)$$

where $\tilde{P}_1$ *and* $\tilde{P}_2$ *are given in Lemma 4 and* $P^*(t,T)$ *is given in Lemma 2.*

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Appendix: Proof of Lemma 3

By Ito's lemma, $\hat{C}(t,x,r,v)$ follows the partial integro - differential equation (PIDE)

$$r\hat{C}=\frac{\partial\hat{C}}{\partial t}+\left(r-\frac{1}{2}\sigma^2 v\right)\frac{\partial\hat{C}}{\partial x}+\left(\alpha(t)-\beta r-\frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)\frac{\partial\hat{C}}{\partial r}+\gamma(1-v)\frac{\partial C}{\partial v}+\frac{\sigma_r^2}{2}\frac{\partial^2\hat{C}}{\partial r^2}+\frac{\sigma_v^2 v}{2}\frac{\partial^2\hat{C}}{\partial v^2}$$
$$+\frac{\sigma^2 v}{2}\frac{\partial^2\hat{C}}{\partial x^2}+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial^2\hat{C}}{\partial x\partial v}+v\int_{-\infty}^{\infty}\left(\hat{C}(t,x+y,r,v)-\hat{C}(t,x,r,v)-\frac{\partial\hat{C}}{\partial x}(e^y-1)\right)k(y)dy. \tag{A1}$$

We plan to substitute Eq. (3.4) into (A1). Firstly, we compute

$$\frac{\partial\hat{C}}{\partial t}=e^x\frac{\partial\tilde{P}_1}{\partial t}-e^{\kappa}P^*(t,T)\left[\frac{\partial\tilde{P}_2}{\partial t}+\tilde{P}_2\frac{\partial}{\partial t}\big(a(t,T)+b(t,T)r\big)\right],\ \frac{\partial\hat{C}}{\partial x}=e^x\left(\frac{\partial\tilde{P}_1}{\partial x}+\tilde{P}_1\right)-e^{\kappa}P^*(t,T)\frac{\partial\tilde{P}_2}{\partial x},$$

$$\frac{\partial\hat{C}}{\partial v}=e^x\frac{\partial\tilde{P}_1}{\partial v}-e^{\kappa}P^*(t,T)\frac{\partial\tilde{P}_2}{\partial v},\ \frac{\partial\hat{C}}{\partial r}=e^x\frac{\partial\tilde{P}_1}{\partial r}-e^{\kappa}P^*(t,T)\left(\frac{\partial\tilde{P}_2}{\partial r}+\tilde{P}_2 b(t,T)\right),$$

$$\frac{\partial^2\hat{C}}{\partial x^2}=e^x\left(\frac{\partial^2\tilde{P}_1}{\partial x^2}+2\frac{\partial\tilde{P}_1}{\partial x}+\tilde{P}_1\right)-e^{\kappa}P^*(t,T)\frac{\partial^2\tilde{P}_2}{\partial x^2},\ \frac{\partial^2\hat{C}}{\partial v^2}=e^x\frac{\partial^2\tilde{P}_1}{\partial v^2}-e^{\kappa}P^*(t,T)\frac{\partial^2\tilde{P}_2}{\partial v^2},$$

$$\frac{\partial^2\hat{C}}{\partial r^2}=e^x\frac{\partial^2\tilde{P}_1}{\partial r^2}-e^{\kappa}P^*(t,T)\left(\frac{\partial^2\tilde{P}_2}{\partial r^2}+2b(t,T)\frac{\partial\tilde{P}_2}{\partial r}+\tilde{P}_2 b^2(t,T)\right),\frac{\partial^2\hat{C}}{\partial v\partial x}=e^x\left(\frac{\partial\tilde{P}_1}{\partial v\partial x}+\frac{\partial\tilde{P}_1}{\partial v}\right)-e^{\kappa}P^*(t,T)\frac{\partial\tilde{P}_2}{\partial v\partial x},$$

$$\begin{aligned}&\hat{C}(t,x+y,r,v,;T,\kappa)-\hat{C}(t,x,r,v,;T,\kappa)\\&\qquad=e^x\left[(e^y-1)\tilde{P}_1(t,x+y,r,v;T,\kappa)+\left(\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]\\&\qquad\quad-e^{\kappa}P^*(t,T)\left[\tilde{P}_2(t,x+y,r,v;T,\kappa)-\tilde{P}_2(t,x,r,v;T,\kappa)\right].\end{aligned}$$

Substitute all terms above into Eq. (A1) and separate it by assumed independent terms of $\tilde{P}_1$ and $\tilde{P}_2$. This gives two PIDEs for the forward probability for $\tilde{P}_i(t,x,r,v;T,\kappa),\ j=1,2$:

$$\begin{aligned}0=&\frac{\partial\tilde{P}_1}{\partial t}+\left(r+\frac{1}{2}\sigma^2 v\right)\frac{\partial\tilde{P}_1}{\partial x}+\left[\gamma(1-v)+\left(\rho_v\sigma v\sigma_v\right)\right]\frac{\partial\tilde{P}_1}{\partial v}+\frac{\sigma_r^2}{2}\frac{\partial^2\tilde{P}_1}{\partial r^2}\\&+\left(\alpha(t)-\beta r-\frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)\frac{\partial\tilde{P}_1}{\partial r}+\frac{\sigma_v^2 v}{2}\frac{\partial^2\tilde{P}_1}{\partial v^2}+\frac{\sigma^2 v}{2}\frac{\partial^2\tilde{P}_1}{\partial x^2}+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial\tilde{P}_1}{\partial v\partial x}\\&+v\int_{-\infty}^{\infty}\left[\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)-\left(\frac{\partial\tilde{P}_1}{\partial x}\right)(e^y-1)\right]k(y)dy\\&+v\int_{-\infty}^{\infty}\left[(e^y-1)\left(\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]k(y)dy,\end{aligned}\tag{A2}$$

and subject to the boundary condition at the expiration time $t = T$ according to Eq. (3.17).
By using the notation in Eq. (3.20), Eq. (A2) becomes

$$\begin{aligned}0=&\frac{\partial\tilde{P}_1}{\partial t}+A[P_1]+\left(\frac{1}{2}\sigma^2 v\right)\tilde{P}_1+\sigma^2 v\frac{\partial\tilde{P}_1}{\partial x}+\rho_v\sigma v\sigma_v\frac{\partial\tilde{P}_1}{\partial v}\\&+v\int_{-\infty}^{\infty}\left[\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)-\left(\frac{\partial\tilde{P}_1}{\partial x}\right)(e^y-1)\right]k(y)dy:=\frac{\partial\tilde{P}_1}{\partial t}+A_1[\tilde{P}_1].\end{aligned}$$

For $\tilde{P}_2(t,x,r,v;T,\kappa)$:

$$\begin{aligned}0=&\frac{\partial\tilde{P}_2}{\partial t}+\left(r-\frac{1}{2}\sigma^2 v\right)\frac{\partial\tilde{P}_2}{\partial x}+\left(\alpha(t)-\beta r-\frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)+\sigma_r^2 b(t,T)\right)\frac{\partial\tilde{P}_2}{\partial r}+\gamma(1-v)\frac{\partial\tilde{P}_2}{\partial v}+\frac{\sigma_v^2 v}{2}\frac{\partial^2\tilde{P}_2}{\partial v^2}\\&+\frac{\sigma_r^2}{2}\frac{\partial^2\tilde{P}_2}{\partial r^2}+\frac{\sigma^2 v}{2}\frac{\partial^2\tilde{P}_2}{\partial x^2}+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial\tilde{P}_2}{\partial v\partial x}+\left(\frac{3\sigma_r^2}{2}b^2(t,T)-r+\big(\alpha(t)-\beta r\big)b(t,T)\right)\tilde{P}_2\\&+\left(\frac{\partial a(t,T)}{\partial t}+r\frac{\partial b(t,T)}{\partial t}\right)\tilde{P}_2+v\int_{-\infty}^{\infty}\left(\tilde{P}_2(t,x+y,r,v;T,\kappa)-\tilde{P}_2(t,x,r,v;T,\kappa)-\frac{\partial\tilde{P}_2}{\partial x}(e^y-1)\right)k(y)dy,\end{aligned}\tag{A3}$$

and subject to the boundary condition at expiration time $t = T$ according to Eq. (3.19). Again, by using the notation (3.20), Eq. (A3) becomes

$$\begin{aligned}0 &= \frac{\partial \tilde{P}_2}{\partial t} + A[\tilde{P}_2] - \frac{\sigma^2 v}{2}\frac{\partial \tilde{P}_2}{\partial x} + \sigma_r^2 b(t,T)\frac{\partial \tilde{P}_2}{\partial r} + \left(\frac{\partial a(t,T)}{\partial t} + r\frac{\partial b(t,T)}{\partial t}\right)\tilde{P}_2 \\ &\quad + \left(\frac{3\sigma_r^2}{2} b^2(t,T) - r + \big(\alpha(t) - \beta r\big) b(t,T)\right)\tilde{P}_2 := \frac{\partial \tilde{P}_2}{\partial t} + A_2[\tilde{P}_2].\end{aligned}$$

The proof is now completed.

การใช้งานโปรแกรมคำนวณค่าพารามิเตอร์ของโมเดลอัตราดอกเบี้ย และราคาพันธบัตรที่ไม่ระบุดอกเบี้ย (Zero Coupon Bond Price)

S. Pinkham and P.Sattayatham

โปรแกรมที่ใช้งาน : MATLAB 7.2

ไฟล์ที่ใช้ดำเนินงาน : runestimation.m, R2measure.m, GMMweigthsNW.m, GMMobjective.m, GMMestimation.m, MomentJacobian.m

(รายละเอียด MATLAB code ใน Appendix A)

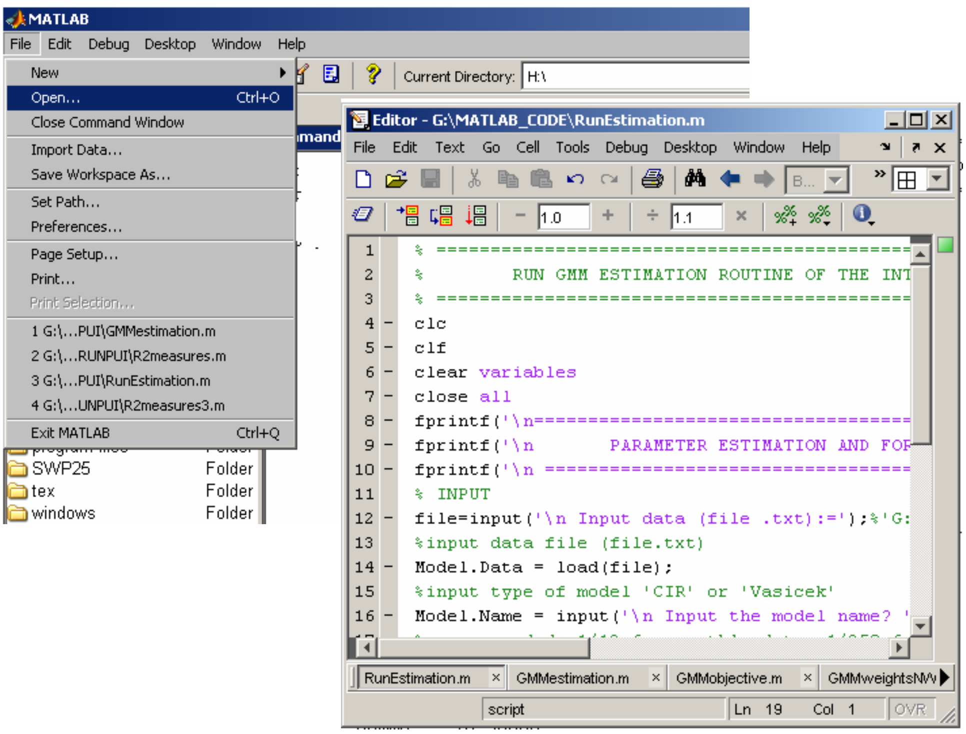
ไฟล์ข้อมูลที่ใช้ดำเนินงาน : file.txt

ผลการดำเนินการโปรแกรม :

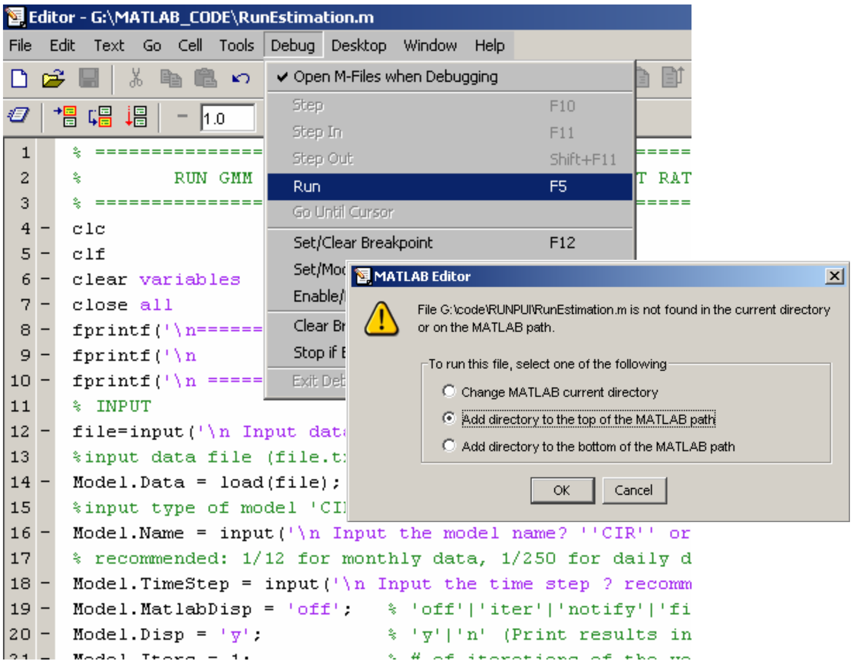
- คำนวณค่าพารามิเตอร์ของโมเดลอัตราดอกเบี้ย ชนิด CIR และ Vasicek และทดสอบสมมติฐาน ของค่าพารามิเตอร์
- จำลองข้อมูลโดยใช้ค่าพารามิเตอร์ที่คำนวณได้
- แสดงกราฟของข้อมูลที่ได้จากการจำลองเปรียบเทียบกับข้อมูลจริง และคำนวณค่าเฉลี่ยความคลาดเคลื่อนกำลังสอง(MSE)
- คำนวณค่าราคาพันธบัตรที่ไม่ระบุดอกเบี้ย (Zero coupon bond) โดยใช้สูตรการคำนวณ (Exact formula) และคำนวณโดยใช้เทคนิคของ Monte Carlo

การใช้งานโปรแกรม :

1. เปิดโปรแกรม MATLAB 7.2 และเปิดไฟล์ runestimation.m



2. ที่ แถบเมนู ให้เลือก Debug และ เลือก Run หรือ กดปุ่ม F5 ที่ คีย์บอร์ด เพื่อดำเนินงานโปรแกรมคำนวณ และ เลือก add directory to the top of the MATLAB path



3. ผลการดำเนินการจะแสดงบนหน้าต่างคำสั่ง (command window) ดังนี้

==

PARAMETER ESTIMATION AND FORCASTING INTERST RATE MODEL

==

Input data (file .txt):= ………………………

(ระบุไฟล์ข้อมูลที่ต้องการคำนวณ ตัวอย่างเช่น 'G:\MATLAB_CODE\DTB33.txt')

Input the model name? := 'CIR' or 'Vasicek':= ……………………

(ระบุรูปแบบของโมเดล 'CIR' หรือ 'Vasicek')

Input the time step ? := ……………………….

recommended:'1/12' for monthly or '1/250'for daily :=1/250 :=

(ระบุ time step โดยแนะนำว่าถ้า ข้อมูลเป็นรายวันให้ ระบุ 1/12 หรือข้อมูลรายเดือนให้ ระบุ 1/250)

4. ผลการคำนวณจะแสดงดังนี้

Parameters etimates :

First run without weighting matrix

alpha = +0.52044 beta = -4.68300

sigma2 = +0.38250 gamma = +0.50000

Parameters etimates, t-statistic in parentheses Second run with weighting matrix, Iteration #1

alpha = +0.44570 (+2.36)

beta = -3.71553 (-2.16)

sigma2 = +0.30599 (+6.64)

gamma = +0.50000

Chi2 statistic = +9.0847

p-value = +0.0026

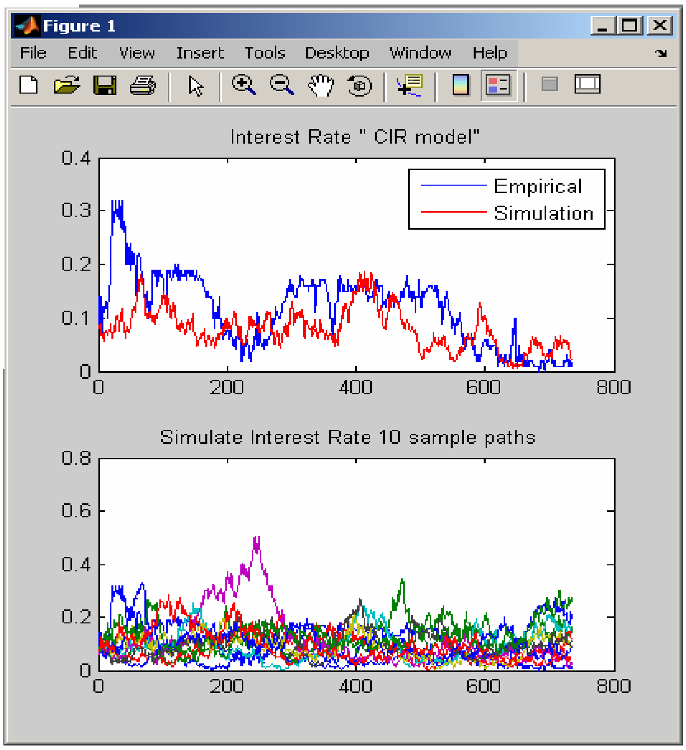
Objective function = 1.234e-002

Mean of data = 1.160734e-001

Variance of data = 4.457622e-003

Number of data = 736

MSE = +0.01244



ผลการวิเคราะห์จะได้ค่าพารามิเตอร์ ค่าสถิติทดสอบ และ ค่าเฉลี่ยความคลาดเคลื่อนกำลังสอง รวมถึงแสดงกราฟของข้อมูลจริงเปรียบเทียบกับการจำลองข้อมูลโดยอาศัยพารามิเตอร์ที่ได้จากการคำนวณ (สามารถดูรายละเอียดของวิธีการประมาณค่าพารามิเตอร์โดยใช้ GMM ที่ Appendix C)

5. คำนวณค่าราคาพันธบัตรที่ไม่ระบุดอกเบี้ย (Zero coupon bond) โดยจะต้องระบุ ค่าอัตราดอกเบี้ยเริ่มต้น (r_0) และ อายุของพันธบัตร (maturity time , T) ยกตัวอย่างเช่น $r_0 = 0.08\%,\ T = 1$ ผลการคำนวณจะแสดงบนหน้าต่างคำสั่ง (command window) ดังนี้

ZERO COUPON BOND PRICE

Input initial interest rate (r0) := 0.08

Input matuarity time (T) := 1

Exact CIR Price = 0.896508

Monte Carlo Price = 0.902940

ผลลัพธ์ที่ได้จะแสดงค่า Exact Price ซึ่งเป็นค่าของ zero coupon bond ที่ได้จากสูตร และ ค่า Monte Carlo Price ซึ่งเป็นค่าที่ได้จาก การคำนวณโดยใช้เทคนิคของ Monte Carlo

ตัวอย่างการวิเคราะห์ ดูรายละเอียดใน Appendix B

Appendix A : MATLAB Code

```
% ======================================================================
%         RUN GMM ESTIMATION ROUTINE OF THE INTEREST RATE MODEL
% ======================================================================
Clc
clf
clear variables
close all
fprintf('\n=========================================================' );
fprintf('\n    RAMETER ESTIMATION AND FORCASTING NTERST RATE MODEL ' );
fprintf('\n ========================================================' );
%-----------------------------------------------------------------------
%                          INPUT The Details
%-----------------------------------------------------------------------
file=input('\n Input data (file .txt)');%'E:/code/test3.txt'
Model.Data = load(file);
Model.Name = input('\n Input the model name?''CIR''or''Vasicek'':);
Model.TimeStep = input('\n Input the time step ?
%recommended:''1/12''for monthly or ''1/250''for daily :' );
Model.MatlabDisp = 'off';  % 'off'|'iter'|'notify'|'final'(default:off)
Model.Disp = 'y';          % 'y'|'n' (Print results in Matlab's command
                              window, draws graphs)
Model.Iters = 1;            % # of iterations of the weighting matrix
                              (traditionally = 1)
Model.q = 12;               % # of lags in the spectral density matrix
                               estimation,
                            % Model.q = 0 reduces the spectral density
                              matrix to the sample covariance matrix

%-----------------------------------------------------------------------
%                              ESTIMATION
%-----------------------------------------------------------------------
Results = GMMestimation(Model);
%-----------------------------------------------------------------------
%               PLOT GRAPH AND CALCULATE MEAN SQUARE ERROR
%-----------------------------------------------------------------------
R2 = R2measures(Model, Results);


function J = GMMobjective(Params, Model, W)

% =====================================================================
% Objective function for Interest rate models
%
% INPUT: Params, vector, vector of estimated parameters
%        Model, structure
%        W, matrix, weighting matrix
% OUTPUT: d, Jacobian matrix
=======================================================================

Data = Model.Data;
DataF = Data(2:end);
DataL = Data(1:end-1);
Nobs = length(DataL);
Nobs = Nobs-1;
TimeStep = Model.TimeStep;
a = Params(1);
b = Params(2);
```

```
%-----------------------------------------------------------------------
% Calculate the sample moments
%-----------------------------------------------------------------------
switch Model.Name
    case 'CIR'
        sigma = Params(3);
        g1 = sum(DataF - a - b*DataL);
        g2 = sum((DataF - a - b*DataL).^2 - sigma^2*DataL.*TimeStep);
        g3 = sum((DataF - a -b*DataL).*DataL);
        g4 = sum(((DataF - a - b*DataL).^2 -
sigma^2*DataL.*TimeStep).*DataL);
        g1 = g1/Nobs; g2 = g2/Nobs; g3 = g3/Nobs; g4 = g4/Nobs;
    case 'Vasicek'
        sigma = Params(3);
        g1 = sum(DataF - a - b*DataL);
        g2 = sum((DataF - a - b*DataL).^2 - sigma^2*TimeStep);
        g3 = sum((DataF - a -b*DataL).*DataL);
        g4 = sum(((DataF - a - b*DataL).^2 - sigma^2*TimeStep).*DataL);
        g1 = g1/Nobs; g2 = g2/Nobs; g3 = g3/Nobs; g4 = g4/Nobs;
end
g = [g1 g2 g3 g4];
J = g*W*g';
end

function W = GMMweightsNW(Params, Model)

% ======================================================================
% Optimal weighting matrix for CKLS nested models
%
% INPUT: Params, vector, vector of estimated parameters
%        Model, structure, see RunAssignment2
% OUTPUT: W, matrix, optimal weighting matrix
% ======================================================================

Data = Model.Data;
TimeStep = Model.TimeStep;
q = Model.q;
a = Params(1);
b = Params(2);
DataF = Data(2:end);
DataL = Data(1:end-1);
Gamma = zeros(4,4,q+1);
%-----------------------------------------------------------------------
% Construct sample moment functions
%-----------------------------------------------------------------------
switch Model.Name
    case 'CIR'
        sigma = Params(3);
        g1t = DataF - a - b*DataL;
        g2t = (DataF - a - b*DataL).^2 - sigma^2*DataL.*TimeStep;
        g3t = (DataF - a -b*DataL).*DataL;
        g4t = ((DataF - a - b*DataL).^2 - sigma^2*DataL.*TimeStep).*DataL;
    case 'Vasicek'
        sigma = Params(3);
        g1t = DataF - a - b*DataL;
        g2t = (DataF - a - b*DataL).^2 - sigma^2*TimeStep;
        g3t = (DataF - a -b*DataL).*DataL;
        g4t = ((DataF - a - b*DataL).^2 - sigma^2*TimeStep).*DataL;
end
```

```
gt = [g1t g2t g3t g4t];
Nobs = length(g1t);
%-----------------------------------------------------------------------
% Calculate the Newey-West estimate of the spectral density matrix with q
lags
%-----------------------------------------------------------------------
gt = gt - repmat(mean(gt), Nobs, 1);
for v = 0 : q
    gtF = gt(1+v:end, :);
    gtL = gt(1:end-v, :);
    Gamma(:,:,v+1) = (gtF'*gtL)./Nobs;
end
S = Gamma(:,:,1);
for v = 1 : q
    Snext = (1-v/(q+1))*(Gamma(:,:,v+1) + Gamma(:,:,v+1)');
    S = S + Snext;
end
W = inv(S);
End
%-----------------------------------------------------------------------

function d = MomentsJacobian(Params, Model)

% ======================================================================
%   Jacobian matrix for testing parameters significance (t-test)
%
% INPUT: Params, vector, vector of estimated parameters
%        Model, structure, see RunAssignment2
% OUTPUT: d, Jacobian matrix
% ======================================================================

TimeStep = Model.TimeStep;
Data = Model.Data;
DataF = Data(2:end);
DataL = Data(1:end-1);
Nobs = length(DataL);
%-----------------------------------------------------------------------
switch Model.Name
    case 'CIR'
        a = Params(1);
        b = Params(2);
        g1a = -Nobs;
        g2a = -2*sum(DataF - a - b*DataL);
        g3a = -sum(DataL);
        g4a = -2*sum((DataF - a - b*DataL).*DataL);
        g1b = -sum(DataL);
        g2b = -2*sum((DataF - a - b*DataL).*DataL);
        g3b = -sum(DataL.^2);
        g4b = -2*sum((DataF - a - b*DataL).*DataL.^2);
        g1s = 0;
        g2s = -sum(TimeStep*DataL);
        g3s = 0;
        g4s = -sum(TimeStep*DataL.*DataL);
        d = [g1a g1b g1s;...
             g2a g2b g2s;...
             g3a g3b g3s;...
             g4a g4b g4s];
        d = d./Nobs;
    case 'Vasicek'
        a = Params(1);
```

```
        b = Params(2);
        g1a = -Nobs;
        g2a = -2*sum(DataF - a - b*DataL);
        g3a = -sum(DataL);
        g4a = -2*sum((DataF - a - b*DataL).*DataL);
        g1b = -sum(DataL);
        g2b = -2*sum((DataF - a - b*DataL).*DataL);
        g3b = -sum(DataL.^2);
        g4b = -2*sum((DataF - a - b*DataL).*DataL.^2);
        g1s = 0;
        g2s = -TimeStep*Nobs;
        g3s = 0;
        g4s = -sum(TimeStep*DataL);
        d = [g1a g1b g1s;...
             g2a g2b g2s;...
             g3a g3b g3s;...
             g4a g4b g4s];
        d = d./Nobs;
end end
%----------------------------------------------------------------------
function Results = GMMestimation(Model)

% ====================================================================
% GMM estimation routine for CKLS nested models
% % INPUT: Model,structure, see RunAssignment2
% OUTPUT: Results, structure
%         Results.Params, estimated parameters
%         Results.Fval, objective function value
%         Results.Exitflag, Matlab's optimization result
%         Results.Tstat, t-statistics for individual parameters
%         Results.Chi2statisitcs, Chi2 test of model specificaion
% USES: GMMobjective, GMMweightsNW, MomentsJacobian
% ================================================================

TimeStep = Model.TimeStep;

% Initial Parameters for optimization
% Must be set manually. But the fmnisearch optimization algorithm seems to
be quite robust
switch Model.Name
        case {'CIR', 'Vasicek'}
        alpha = 0.01;
        beta = -0.01;
        sigma = 0.01;
        a = alpha*TimeStep;
        b = beta*TimeStep + 1;
        InitialParams = [a b sigma];
end

% ======   First run, with identity weighting matrix =================
W = eye(4);
options = OPTIMSET('LargeScale', 'off', 'MaxIter', 2500, 'MaxFunEvals',
3500, 'Display', Model.MatlabDisp, 'TolFun', 1e-40, 'TolX', 1e-40);
[Params, Fval, Exitflag] =  fminsearch(@(Params) GMMobjective(Params,
Model, W), InitialParams, options);
switch Model.Name
    case 'CIR'
        Ralpha = Params(1)/TimeStep;
        Rbeta  = (Params(2)-1)/TimeStep;
```

```
        Rsigma2 = Params(3)^2;
        Rgamma = 0.5;
    case {'Vasicek'}
        Ralpha = Params(1)/TimeStep;
        Rbeta  = (Params(2)-1)/TimeStep;
        Rsigma2 = Params(3)^2;
        Rgamma = 0;
end
if strcmp(Model.Disp, 'y')
    fprintf('\n Parameters etimates\n');
    fprintf(' First run without weighting matrix');
    fprintf('\n alpha  = %+3.5f\n beta   = %+3.5f\n sigma2 = %+3.5f\n gamma
= %+3.5f\n ------------------------------------------------------------
\n',...
    Ralpha, Rbeta, Rsigma2, Rgamma);
end

% ========= Second run, with optimal weighting matrix W ==========
if Model.Iters > 0
    for i = 1 : Model.Iters
        InitialParams = Params;
        W = GMMweightsNW(Params, Model);
        options = OPTIMSET('LargeScale', 'off', 'MaxIter', 2500,
'MaxFunEvals', 3500, 'Display', Model.MatlabDisp, 'TolFun', 1e-8, 'TolX',
1e-8);
        [Params, Fval, Exitflag] =  fminsearch(@(Params)
GMMobjective(Params, Model, W), InitialParams, options);
        switch Model.Name
         case 'CIR'
            Ralpha = Params(1)/TimeStep;
            Rbeta  = (Params(2)-1)/TimeStep;
            Rsigma2 = Params(3)^2;
            Rgamma = 0.5;
         case {'Vasicek'}
            Ralpha = Params(1)/TimeStep;
            Rbeta  = (Params(2)-1)/TimeStep;
            Rsigma2 = Params(3)^2;
            Rgamma = 0;
        end
        switch Model.Name
        case {'CIR', 'Vasicek','TF'}
%Chi2 statistics of the overidentified model. Are the empirical moments
%sufficiently close to 0?
            Chi2statistic = Fval*length(Model.Data);
            Chi2pvalue = 1-chi2cdf(Chi2statistic,1);
            Results.Chi2statisitcs = Chi2statistic;
            Results.Chi2pvalue = Chi2pvalue;
        end
        % t-statistic
        Nobs = length(Model.Data)-1;
        d = MomentsJacobian(Params, Model);
        VarParams = diag(inv(d'*W*d))./Nobs;
        Params(2) = Params(2)-1;
        Params(3) = Params(3)^2;
        Tstat = Params'./sqrt(VarParams);
        if strcmp(Model.Disp, 'y')
      fprintf('\n Parameters etimates, t-statistic in parentheses\n');
      fprintf(' Second run with weighting matrix, Iteration #%d\n', i);
            switch Model.Name
            case {'CIR', 'Vasicek'}
```

```
                fprintf('\n alpha  = %+3.5f (%+3.2f) \n beta   = %+3.5f
(%+3.2f)\n sigma2 = %+3.5f (%+3.2f) \n gamma  = % +3.5f\n',...
                Ralpha, Tstat(1), Rbeta, Tstat(2), Rsigma2, Tstat(3),
Rgamma);
                fprintf(' Chi2 statistic = %+2.4f\n', Chi2statistic);
                fprintf(' p-value        = %+2.4f\n', Chi2pvalue);
            end
            fprintf(' Objective function = %2.3e\n', Fval);
            fprintf('-------------------------------------------- \n');
        end
    end
Results.Tstat = Tstat;
Results.VarParams = VarParams;
end
Results.Params = [Ralpha Rbeta Rsigma2 Rgamma];
Results.Fval = Fval;
Results.Exitflag = Exitflag;
End
%-------------------------------------------------------------------
function R2 = R2measures(Model, Results)

% ==================================================================
% R2 measures
%
% INPUT: Model, structure
%        Results, structure, see GMMestimation
% OUTPUT: R2, structure
%         R2.Rsquare1, forecast power for interest rate changes,
%         R2.Rsquare2, forecast power for squared interest rate changes
%         Graph of empirical data vs forcasting data
%         MSE
%
% ==================================================================

Data = Model.Data;
TimeStep = Model.TimeStep;
alpha = Results.Params(1);
beta = Results.Params(2);
sigma2 = Results.Params(3);
gamma = Results.Params(4);

CondEdata = alpha*TimeStep + (1+beta*TimeStep).*Data;
CondEddata = CondEdata - Data;
Realddata = diff(Data);

CondVARdata = sigma2*Data.^(2*gamma)*TimeStep;
RealVARdata = Realddata.^2;
%===================================================================
% "forecast power" - calcluate R2 of the OLS of conditional values
%  implied by the selected model on observed values
%===================================================================

y = Realddata;
x = CondEddata(1:end-1);
b1 = x\y;
Rsquare1 = sum((b1.*x-mean(y)).^2)/sum((y-mean(y)).^2);

y = RealVARdata;
x = CondVARdata(1:end-1);
```

```
b1 = x\y;
Rsquare2 = sum((b1.*x-mean(y)).^2)/sum((y-mean(y)).^2);

R2.Rsquare1 = Rsquare1;
R2.Rsquare2 = Rsquare2;
%======================================================================
%                  Forcasting Interest rate model
%======================================================================
fdata(1)=Data(1);
fdata2(1)=Data(1);
fdata3(1)=Data(1);
NN=length(Data);
for j=1:NN
    ex2(j)=exp(-1*beta);
end
dt=1/NN;
for i=2:length(Data)
    fdata(i)=alpha*TimeStep + (1+beta*TimeStep)*fdata(i-1)+...
             sqrt(sigma2*fdata(i-1)^(2*gamma)*TimeStep)*randn(1);
    fdata2(i)=alpha*TimeStep + (1+beta*TimeStep)*fdata2(i-1)-...
            sigma2^2*(1-ex2(i))*TimeStep/beta+...
            sqrt(sigma2*TimeStep)*randn(1);
    fdata3(i)=alpha*dt+ (1+beta*dt)*fdata(i-1)+...
             sqrt(sigma2*fdata(i-1)^(2*gamma)*dt)*randn(1);
end

%======================================================================
%                Calculate Mean Square Error
%======================================================================
 SumErrsqrt=0;
 SumErrsqrt2=0;
 SumErrsqrt3=0;
 for i=1:length(Data)
    Err(i)=Data(i)-fdata(i);
    Err2(i)=Data(i)-fdata2(i);
    Err3(i)=Data(i)-fdata3(i);
    Errsqrt(i)=Err(i)^2;
    Errsqrt2(i)=Err2(i)^2;
    Errsqrt3(i)=Err3(i)^2;
    SumErrsqrt=Errsqrt(i)+SumErrsqrt;
    SumErrsqrt2=Errsqrt2(i)+SumErrsqrt2;
    SumErrsqrt3=Errsqrt3(i)+SumErrsqrt3;
 end
 MSE= SumErrsqrt/length(Data);
 MSE2= SumErrsqrt2/length(Data);
 MSE3= SumErrsqrt3/length(Data);
 MeanIR=mean(Data);
%======================================================================
%                           Display Graph
%======================================================================
if strcmp(Model.Disp, 'y')
        switch Model.Name
         case 'CIR'
              ns=10;
              for j=1:ns
                fdata4(1,j)=Data(1);
                for i=2:length(Data)
                    fdata4(i,j)=alpha*TimeStep + ...
                    1+beta*TimeStep)*fdata4(i-1,j)+ ...
            sqrt(sigma2*fdata4(i-1,j)^(2*gamma)*TimeStep)*randn(1);
```

```
                end
              subplot(2,1,2)
              plot(1:length(Data), Data, 'b', 1:length(fdata4), fdata4)
              hold on
              title('Simulate Interest Rate 10 sample paths  ')
              end
          subplot(2,1,1)
          plot(1:length(Data), Data, 'b', 1:length(fdata), fdata, '-r')
              title('Interest Rate " CIR model"')
              legend('Empirical', 'Simulation')
             fprintf('\n Mean of data = %d ',MeanIR);
            fprintf('\n Variance of data = %d',var(Data));
            fprintf('\n Number of data  = %d ',length(Data));
            fprintf('\n MSE  = %+3.5f \n',MSE);

fprintf('\n=========================================================' );
fprintf('\n                   ZERO COUPON BOND PRICE     \n        ' );
fprintf ('\n =====================================================' );
             % Calculate bond price by using formula
             r0=input('\n Input initial interest rate (r0) := ');
             T=input('\n Input matuarity time (T) := ');
             h=sqrt(beta^2+2*sigma2^2);
             b1=exp(h*T)-1;
             b2=2*h+((h-beta)*(exp(h*T)-1));
             btT=2*b1/b2;
             atT=(2*h*exp((h-beta)*T/2)/b2)^(2*alpha/sigma2^2);
             Zerob=atT*exp(-1*btT*r0);
             fprintf('\n Exact CIR Price  = %f \n',Zerob );
             %------------------------------------------------------
             % Calculate bond price by using  Monte Carlo Simulation
             TS=T/length(fdata);
             sfdata=exp(-1*sum(fdata*TS));
             MC=mean(sfdata);
             fprintf('\n Monte Carlo Price  = %f \n',MC );
%-----------------------------------------------------------------

         case 'Vasicek'
             ns=10;
              for j=1:ns
                fdata4(1,j)=Data(1);
                for i=2:length(Data)
                    fdata4(i,j)=alpha*TimeStep +
(1+beta*TimeStep)...*fdata4(i-1,j)+...
                     sqrt(sigma2*fdata4(i-
1,j)^(2*gamma)*TimeStep)*randn(1);
                 end
              subplot(2,1,2)
              plot(1:length(Data), Data, 'b', 1:length(fdata4), fdata4)
              hold on
              title('Simulate Interest Rate 10 sample paths  ')
              end
              subplot(2,1,1)
          plot(1:length(Data), Data, 'b', 1:length(fdata), fdata, '-r')
          plot(1:length(Data), Data, 'b', 1:length(fdata), fdata, 'r')
             title('Interest Rate "Vasicek model"')
             legend('Empirical', 'ForecastVS')
             fprintf('\n Mean of data = %d ',MeanIR);
             fprintf('\n Variance of data = %d',var(Data));
             fprintf('\n Number of data  = %d ',length(Data));
```

```
            fprintf('\n MSE of Vasicek model  = %+3.5f ',MSE);

fprintf('\n=========================================================' );
fprintf('\n                     ZERO COUPON BOND PRICE             ' );
fprintf('\n=========================================================' );
            % Calculate bond price by using formula
            r0=input('\n Input initial interest rate (r0) := ');
            T=input('\n Input matuarity time (T) := ');
            btT=-1/beta*(1-exp(beta*T));
            atT=(-1*alpha/beta-sigma2^2/(2*beta^2))*(btT-
T)+(sigma2*beta)^2/(4*beta);
            Zerob=exp(atT-btT*r0);
            fprintf('\n Exact Vasicek Price  = %f \n',Zerob );
            %--------------------------------------------------------
            % Calculate bond price by using  Monte Carlo Simulation
            TS=T/length(fdata);
            sfdata=exp(-1*sum(fdata*TS));
            MC=mean(sfdata);
            fprintf('\n Monte Carlo Price  = %f \n',MC );
        end

end end
```

Appendix B : The empirical Example

The Treasury bill yield data used are daily T-bill form Board of Governors of the Federal Reserve System. The data are daily cover period form 4 January 2009 to 8 December 2011. The data set is saved in **DBT33.txt.**

The estimation and pricing routine is execute by running the runestimattion.m with MATLAB program version 7.2 (see MATLAB code in appendix A) . The results are displayed in the MATLAB's command window. The results are follow :

Model	**Parameter**			χ^2 **test**	**P-value**
	α	β	σ^2		
CIR	0.44570	-3.71553	0.30599	9.0847	0.0026
Vasieck	0.57989	-5.45376	0.04105	2.1127	0.1461

Table 1   Parameter estimator and Statistic test

Model	Number of data	Mean	Variance	MSE	Zero coupon bond price with $r_0 = 0.08$ and $T = 1$	
					Exact	Monte Carlo
CIR	736	0.1160734	0.004457622	0.00478	0.896508	0.910618
Vasieck	736	0.1160734	0.004457622	0.00844	0.901410	0.922795

Table 2 The statistics description and Bond price

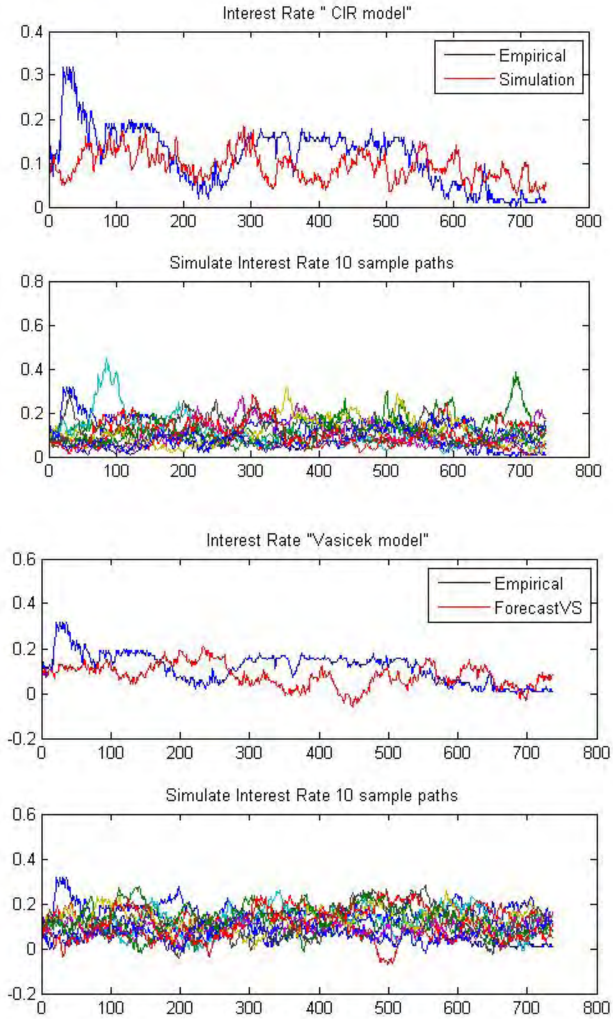


Figure 1 Empirical data VS Forecast data

Appendix C : Parameter Estimation and Application in Finance

1. The interest rate models

In this work , we focus on two specifications of the dynamics of the short-term interest rate as following

$$\text{CIR model : } dr_t = (\alpha + \beta r_t)dt + \sigma\sqrt{r_t}dW_t \tag{1}$$

$$\text{Varicek model : } dr_t = (\alpha + \beta r_t)dt + \sigma\sqrt{r_t}dW_t \tag{2}$$

where r_t is the interest rate, W_t is the Brownion motion, α, β and σ are parameter. The model (1) and (2) are define the parameter vector $\theta = \left(\alpha, \beta, \sigma^2\right)$. We consider to estimate parameters by using the Generalize Method of Moments (GMM) technique which can be used for financial models and application the model in financial problem with MATLAB program.

2. Generalized method of moments

To estimate parameter of (1) and (2) using GMM , we have to discretized the SDE by applied Euler discretization scheme are follow

$$\text{CIR model : } r_{t+1} - r_t = (\alpha + \beta r_t)\Delta t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \equiv \sigma\sqrt{r_t \Delta t}N(0,1) \tag{3}$$

$$\text{Vasicek model : } r_{t+1} - r_t = (\alpha + \beta r_t)\Delta t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \equiv \sigma\sqrt{\Delta t}N(0,1) \tag{4}$$

where $N(0,1)$ is normal variable with zero mean and unit variance, and Δt is time step. Form (3) and (4) we can derive a set of four moment function

$$\text{CIR model : } \quad f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^2 - \sigma^2 r_t \\ \varepsilon_{t+1} r_t \\ \left(\varepsilon_{t+1}^2 - \sigma^2 r_t\right) r_t \end{bmatrix}, \qquad \text{Vasicek model : } f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^2 - \sigma^2 \\ \varepsilon_{t+1} r_t \\ \left(\varepsilon_{t+1}^2 - \sigma^2\right) r_t \end{bmatrix}. \tag{5}$$

The moment function are constructed so that $E\left[f_t(\theta)\right] = 0$ and the sample moment are defines as

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} f_t(\theta)$$

where T is a number of observations. Then the sample moments of (5) as follows

$$\text{CIR model : } \quad g_T(\theta) = \frac{1}{T}\begin{bmatrix} \sum_{t=1}^{T} \varepsilon_{t+1} \\ \sum_{t=1}^{T} \left(\varepsilon_{t+1}^2 - \sigma^2 r_t\right) \\ \sum_{t=1}^{T} \varepsilon_{t+1} r_t \\ \sum_{t=1}^{T} \left(\varepsilon_{t+1}^2 - \sigma^2 r_t\right) r_t \end{bmatrix}, \text{ Vasicek model : } g_T(\theta) = \begin{bmatrix} \sum_{t=1}^{T} \varepsilon_{t+1} \\ \sum_{t=1}^{T} \varepsilon_{t+1}^2 - \sigma^2 \\ \sum_{t=1}^{T} \varepsilon_{t+1} r_t \\ \sum_{t=1}^{T} \left(\varepsilon_{t+1}^2 - \sigma^2\right) r_t \end{bmatrix}. \tag{6}$$

The GMM objective function is defined as

$$J_T(\theta) = g_T^{'}(\theta) W_T(\theta) g_T(\theta) \tag{7}$$

where $W_T(\theta)$ is positive definite weight matrix , or , equivalently, by solving the system of equations

$$D_0^{'}(\theta)' W_T g_T(\theta) = 0, \tag{8}$$

where $D_0^{'}(\theta)$ is the Jacobian matrix of $g_T(\theta)$ wish respect to θ. We find the parameter by solving

$$\hat{\theta} = \arg\min_{\theta} J_T \ . \tag{9}$$

For the unrestricted model, the parameter are just identified and $J_T(\theta)$ attains zero for all choices of $W_T(\theta)$. For the interest rate models, the GMM estimates of the over-identified parameter sub-vector of θ do depend on the choice of $W_T(\theta)$. Hanson [5] shows the optimal choice of the weighting matrix in term minimizing the asymptotic covariance matrix of the estimator is

$$W_T(\theta) = \left(\sum_{k=-\infty}^{\infty} E\left[f_t(\theta) f_{t-k}(\theta)' \right] \right)^{-1} . \tag{10}$$

In our consideration, GMM is a two stage estimator. We usually proceed in the following way: First, we minimize (7) using identity weighting matrix $W_T = I$. This means that we consider all moments equally important. We plug estimated parameter vector into (10) to get W_T . And secondly , we minimize (7) again, but time using the W_T form the previous step.

When the number of moment condition is greater than the dimension of parameter vector, the model is said to be over identified. Over identification allows us to check whether the model's moment conditions match the data well or not. We can check whether sample moment is sufficiently close to zero to suggest that the model fits the data well. The test statistics is asymptotically χ^2 distributed ,

$$T \times g_T^{'}(\hat{\theta}) W_T(\hat{\theta}) g_T(\hat{\theta}) \xrightarrow{d} \chi^2_{m-p} \tag{11}$$

where m is a number of moment condition and p is a number of parameters. If the test statistic reject, then the underlying model that generated the system of moment condition is declared invalid.

3. **Pricing of Zero Coupon Bond**

A zero coupon bond is a contract priced $P(t,T)$ at time $t < T$ to deliver $P(T,T) = 1$ at time T . The computation of the arbitrage price $P(t,T)$ of a zero coupon bond base on as underlying short term interest rate process r_t is a basic and important issue in interest rate modeling. We may distinguish three different situations:

- The short rate is a deterministic constants $r > 0$. In this case $P(t,T)$ should satisfy $e^{-r(T-t)} P(t,T) = P(T,T) = 1,$ which leads to

$$P(t,T) = e^{-r(T-t)}, \quad 0 \le t \le T \ . \tag{12}$$

- The short rate is a deterministic function $(r_t)_{t\in R+}$. In this case, an argument similar to the above shows that

$$P(t,T)=\exp(-\int_t^T r_s ds), \quad 0\le t\le T. \tag{13}$$

- The short rate is a deterministic process $(r_t)_{t\in R+}$. In this case, formula (13) no longer makes sense because the price $P(t,T)$ being set at time t, can depend only on information known up to time t. This is contradiction with (13) in which $P(t,T)$ depends on the future values of r_s for $s\in[t,T]$. Then $P(t,T)$ should be

$$P(t,T)=E\left[\exp(-\int_t^T r_s ds)\mid F_t\right], \quad 0\le t\le T. \tag{14}$$

Theorem 1 (Zero coupon bond in the Vasicek model). *In the Vasicek model , the price of a zero coupon bond with maturity T at time* $t\in[0,T]$ *is given by*

$$P(t,T)=A(t,T)\exp(-r_t B(t,T)),$$

where $B(t,T)=-\frac{1}{\beta}\left(1-\exp\left(\beta(T-t)\right)\right)$

and $A(t,T)=\exp\left[\left(-\frac{\alpha}{\beta}-\frac{\sigma^2}{2\beta^2}\right)\left(B(t,T)-T+t\right)+\frac{\sigma^2}{4\beta}B^2(t,T)\right].$

Proof. See Brigo [2] pp.58-59.

Theorem 2 (Zero coupon bond in the CIR model). *In the CIR model , the price of a zero coupon bond with maturity T at time* $t\in[0,T]$ *is given by*

$$P(t,T)=A(t,T)\exp(-r_t B(t,T)),$$

where $B(t,T)=\dfrac{2(e^{h(T-t)}-1)}{2h+(h-\beta)(e^{h(T-t)}-1)}$,

and $A(t,T)=\left(\dfrac{2he^{(h-\beta)(T-t)/2}}{2h+(h-\beta)(e^{h(T-t)}-1)}\right)^{2\alpha/\sigma^2}$ *with* $h=\sqrt{\beta^2+2\sigma^2}$.

Proof. See Bohner [1].

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A VALUE FUNCTION OF DISCRETE-TIME SURPLUS PROCESS IN INSURANCE UNDER INVESTMENT AND REINSURANCE CREDIT RISK

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Abstract

This paper has studied an insurance model where the surplus process can be controlled by two activities, one is reinsurance for which the reinsurance company has an opportunity to default and other is an investment in a financial market. We prove the existence of an optimal plan and derive a formula for the value function which is the minimum of total discounted cost function in the framework of discrete-time surplus process.

1. Introduction

In recent years, risk models have attracted much attention in the insurance business, in connection with the possible insolvency and the capital reserves of the insurance company. The main interest from the point of view of an insurance

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company is claim arrival and claim size, which affect the capital of the company. Claims happen at the times T_i, satisfying $0 = T_0 \leq T_1 \leq T_2 \leq \cdots$. We call them *claim arrivals* or, simply, *arrivals*. The nth claim arriving at time causes the claim size Y_n. The *inter-arrival times* or, simply, *inter-arrival*, $Z_n := T_n - T_{n-1}$ is the length of time between $(n-1)$th claim and nth claim. By period n, we shall mean the random interval $[T_{n-1}, T_n)$, $n \geq 1$.

Now let the constant c_0 represent the premium rate for one unit time; the random variable $c_0 \sum_{i=1}^{n+1} Z_i = c_0 T_{n+1}$ describes the inflow of capital into the business by time T_{n+1}, and $\sum_{i=1}^{n+1} Y_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_{n+1}]$. Therefore, the quantity

$$X_{n+1} = x + c_0 T_{n+1} - \sum_{i=1}^{n+1} Y_i \tag{1.1}$$

is the insurer's balance (or surplus) at time T_{n+1} with the constant $X_0 = x > 0$ as the initial capital. In summary, the discrete-time surplus process will be defined as follows:

$$X_0 = x, \quad X_{n+1} = X_n + c_0 Z_{n+1} - Y_{n+1}, \quad n = 0, 1, 2, \dots. \tag{1.2}$$

Reinsurance and investment are a normally activity of insurance company because reinsurance can reduce the risk (ruin probability) arising from claims, and the investment can make more profit for the company. Thus there are many papers which studied their effect in the insurance business. For example, the effect of reinsurance on ruin probability was studied by Dickson and Waters [3], minimizing ruin probability in a continuous-time model considered by Browne [2], Hipp and Plum [4], Hipp and Vogt [5], Højgaard and Taksar [6, 7], Schmidli [9], and exponential utility and minimizing ruin probability in a discrete-time model considered by Schäl [8].

In this paper, we shall prove the existence of an optimal plan (i.e., the strategy or policy of choosing reinsurance and investment for minimizing a value function) and derive a formula of the value function under the condition that a reinsurer has opportunity to default and an investment in risky assets. Let $\{X_n, n \geq 1\}$ be the surplus process which can be controlled by choosing the retention level b of reinsurance for one period, and for each level b, the insurer has to pay the premium rate to the reinsurer which is deducted from c_0, as a result of which the insurer's income rate will be represented by the function $c(b)$. The level $\overline{b}$ stands for the control action without reinsurance, so that $c_0 = c(\overline{b})$ and the level $\underline{b}$ is the smallest retention level which can be chosen. Of course, we obtain the *net income rate* $c(b)$, where $0 \leq c(b) \leq c_0$, for all $b \in [\underline{b}, \overline{b}]$ and $c(b)$ is increasing. By the *expected value principle*, $c(b)$ can be calculated as follows:

$$c(b) = c_0 - (1 + \theta_0) \cdot \frac{E[Y - h(b, Y)]}{E[Z]}, \tag{1.3}$$

where θ_0 is the safety loading of the reinsurer and the function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$ which called *reinsurance recovery* is paid by the reinsurer.

Next, we shall recall the *reinsurance credit risk* (RCR) which is the risk of the reinsurance counterparty failing to pay reinsurance recoveries in full to the ceding company (insurer) in a timely manner, i.e., unwillingness to pay, or even not paying them at all. Therefore, we assume that for each retention level $b \in [\underline{b}, \overline{b}]$, the reinsurer has an opportunity to default, i.e., the insurer has to pay

$$\begin{cases} y, & \text{if reinsurer default with probability } P(K = 0) = p, \\ h(b, y), & \text{if reinsurer dose not default with probability } P(K = 1) = 1 - p, \end{cases}$$

where K is a random variable with value in $\{0, 1\}$ and $p \in [0, 1)$ is constant. The random variable K is said to be *binary recovery*. Let $\{T_n, n \geq 0\}$ be a sequence of arrival and let K_n be a binary recovery at time T_n.

In addition, the insurer can invest the surplus (capital) in a financial market with m risky assets, called *stocks*, described by the price process

$$\{S_n = (S_n^1, S_n^2, ..., S_n^m), n \geq 1\},$$

where $S_n^k > 0$ is the price of one share of stock k at the time T_{n-1}. We now define the return process $\{R_n = (R_n^1, R_n^2, ..., R_n^m), n \geq 1\}$ by $R_n^k = (S_n^k - S_{n-1}^k)/S_{n-1}^k, 1 \leq k \leq m$. For each $n \geq 1$, a portfolio vector $\delta_{n-1} = (\delta_{n-1}^1, \delta_{n-1}^2, ..., \delta_{n-1}^m) \in \mathbb{R}^m$ specifies the time T_{n-1} and the component δ_{n-1}^{k} represents the amount invested in stock k during period n. This means that the insurance company holds $\delta_{n-1}^{k}/S_{n-1}^{k}$ shares of stock k during period n, so that the value of these shares at the time T_n is $\delta_{n-1}^{k} \cdot S_n^k / S_{n-1}^{k}$.

In this situation, we will allow for a negative value for δ_{n-1}^{k}, that is, we admit the short selling of stocks. Letting X_n be a surplus and (b_n, δ_n) be a control action at the time T_n, therefore, we can adapt the surplus process (1.2) as follows:

$$\begin{aligned} X_{n+1} &= X_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1})K_{n+1} - Y_{n+1}(1-K_{n+1}) - \sum_{k=1}^{m} \delta_n^k + \sum_{k=1}^{m} \frac{\delta_n^k}{S_n^k} S_{n+1}^k \\ &= X_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1})K_{n+1} - Y_{n+1}(1-K_{n+1}) + \sum_{k=1}^{m} \delta_n^k \frac{(S_{n+1}^k - S_n^k)}{S_n^k} \\ &= X_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1})K_{n+1} - Y_{n+1}(1-K_{n+1}) + \sum_{k=1}^{m} \delta_n^k R_{n+1}^k \\ &= X_n + c(b_n)Z_{n+1} - \{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1-K_{n+1})\} + \langle \delta_n, R_{n+1} \rangle, \end{aligned}$$

where $X_0 = x$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$. It is convenient to set

$$X_0 = x, \quad X_{n+1} = X_n + L(b_n, \delta_n, K_{n+1}, R_{n+1}, Y_{n+1}, Z_{n+1}), \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where

$$L(b, \delta, k, r, y, z) = c(b) \cdot z - \{h(b, y)k + y(1-k)\} + \langle \delta, r \rangle. \tag{1.5}$$

If we let $f(x, b, \delta, k, r, y, z) = x + L(b, \delta, r, y, z)$, then f is the *system function* as mentioned in Bersekas and Shreve [1]. We see that the process $\{X_n, n \geq 0\}$ is driven by the control action $u_n = (b_n, \delta_n)$ and the sequence of random vector $\{W_n, n \geq 1\}$, where $W_n = (K_n, R_n, Y_n, Z_n)$ (the disturbance for period n) is the source of the randomness of the model. It is natural to assume that the process W_n is iid, i.e., we make the following assumption:

Assumption 1. Independence assumption (IA)

$W_n = (K_n, R_n, Y_n, Z_n)$, $n \geq 1$ are independent and identically distributed random variables (iid). In addition, it is assumed that (K_n, Y_n, Z_n) and R_n are independent for all $n \geq 1$.

As a consequence, $\{R_n, n \geq 1\}$ as well as $\{K_n, n \geq 1\}$, $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ are also iid. We set $K = K_1$, $R = R_1$, $Y = Y_1$, $Z = Z_1$ and $W = W_1$ for a typical binary recovery, typical return, typical period length, typical claim, and typical disturbance, respectively.

2. Dynamic Programming with Finite Horizon

In this section, we assume that all the processes are defined in a probability space $(\Omega, \mathcal{F}, P)$. Let $\{X_n, n \geq 0\}$ be a surplus process (as in Section 1) with value in a state space $(S, \mathcal{S})$ which is a measurable space. Suppose that $\{X_n, n \geq 0\}$ is driven by a sequence of iid random variables $\{W_n, n \geq 1\}$ with values in a measurable space $(E, \mathcal{E})$. Therefore, $(E, \mathcal{E})$ is called *disturbance space*. The surplus process can be controlled at the beginning of every period in a measurable space $(U, \mathcal{U})$ which is called the *control action space*. In addition, the model is specified by the following quantities:

- $\alpha \in [0, 1]$ is the *discount factor*;
- $g : S \times U \to (-\infty, \infty]$ is the *one-period cost function*, which is measurable and bounded from below;
- $N \in \{1, 2, 3, \ldots\}$ is a *time horizon* (number of periods) and
- $\hat{V}_N : S \to (-\infty, \infty]$ is the *terminal cost function* for time horizon N, which is measurable and bounded from below.

Definition 2.1. A *plan* for the time horizon N over action space U is a (finite) sequence $\pi = (u_i)_{i=0}^{N-1}$ of control action $u_i \in U$, for all $i \in \{0, 1, 2, ..., N-1\}$. A set of all plans for the time horizon N over action space U is denoted $\mathcal{P}(N, U)$. A plan $\pi \in \mathcal{P}(N, U)$ is said to be *u-stationary*, if $\pi = \underbrace{(u, u, ..., u)}_{N \text{ terms}}$ for some $u \in U$.

For each initial state $x \in S$ and plan $\pi = (u_i)_{i=0}^{N-1}$, the surplus process (1.4) can be written by

$$X_{n+1} = X_n + L(u_n, W_{n+1}) = x + \sum_{k=0}^{n} L(u_k, W_{k+1}), \quad n = 0, 1, 2, ..., N-1 \quad (2.1)$$

and $X_0 = x$.

For the state $X_n = x_n$, the cost at the time T_n will be $g(x_n, u_n)$ and the next state

$$x_{n+1} = x_n + c(b_n)z_n - (h(b_n, y_{n+1})k_{n+1} + y_{n+1}(1 - k_{n+1})) + \langle \delta_n, r_{n+1} \rangle \quad (2.2)$$

which will result in a cost $g(x_{n+1}, u_{n+1})$ at the time T_{n+1}. Thus, the present value of the costs at the time T_{n+1} will be $\alpha \cdot g(x_{n+1}, u_{n+1})$, i.e., $g(x_{n+1}, u_{n+1})$ is discounted by α.

Definition 2.2. Let N be the time horizon. Then the *total discounted cost function* and the *valued function for the time horizon N* are defined by

$$\Phi^{(N)}(x, \pi) = E\left[\sum_{i=0}^{N-1} \alpha^i g(X_i, u_i) + \alpha^N \hat{V}_N(X_N) | X_0 = x\right], \quad \text{where} \quad \pi = (u_i)_{i=0}^{N-1}, \quad (2.3)$$

and

$$V^{(N)}(x) = \inf_{\pi \in \mathcal{P}(N, U)} \Phi^{(N)}(x, \pi), \quad \text{respectively.} \quad (2.4)$$

A plan $\pi \in \mathcal{P}(N, U)$ is said to be *optimal*, if $V^{(N)}(x) = \Phi^{(N)}(x, \pi)$. If π is u-stationary, then we write $\Phi^{(N)}(x, \pi) = \Phi^{(N)}(x, u)$.

3. Main Results

In this section, we study the insurance model introduced in Section 1 under the assumption that the insurer can borrow an unlimited amount of money. Let the state space $S = \mathbb{R}$ and the control space $U = [\underline{b}, \overline{b}] \times \mathbb{R}^m$. Thus, for each state $x \in S$, we can choose any control actions $u = (b, \delta) \in [\underline{b}, \overline{b}] \times \mathbb{R}^m$, where b is the retention level of reinsurance and $\delta = (\delta^1, \delta^2, \ldots, \delta^m)$ is the portfolio vector.

We will study the cost structure which is given by the idea that the insurance company is not insolvency (ruined) but only penalized if the size of the surplus is negative or small. The penalty cost of being in state x is of the form $\text{const} \times e^{-\beta x}$ for some $\beta > 0$. Therefore, we define the cost functions as

$$g(x, u) = \gamma \cdot e^{-\beta x}, \quad \hat{V}_N(x) = \nu_0 \cdot e^{-\beta x}, \quad \text{for some} \quad \gamma, \nu_0 \geq 0, \quad (3.1)$$

when $x \in S$, $u \in U$. Thus, we obtain the total discounted cost function of model (1.4) as

$$\Phi^{(N)}(x, \pi) = E\left[\sum_{i=0}^{N-1} \alpha^i \gamma \cdot e^{-\beta x_i} + \alpha^N \nu_0 \cdot e^{-\beta X_N} \mid X_0 = x\right], \tag{3.2}$$

where $\pi \in \mathcal{P}(N, U)$.

In this paper, we will use the method of dynamic programming to prove the main theorem. In order to do this, we define $\Phi_n^{(N)}(x, \pi)$ and $V_n^{(N)}(x)$ as follows:

$$\Phi_n^{(N)}(x, \pi) = E\left[\sum_{i=n}^{N-1} \alpha^{i-n} \gamma \cdot e^{-\beta X_i} + \alpha^{N-n} \nu_0 \cdot e^{-\beta X_N} \mid X_n = x\right], \quad n = 0, 1, 2, \ldots, N-1 \tag{3.3}$$

$$\Phi_N^{(N)}(x, \pi) = \nu_0 \cdot e^{-\beta x}, \quad \text{where} \quad \pi = (u_i)_{i=0}^{N-1}, \tag{3.4}$$

and

$$V_n^{(N)}(x) = \inf_{\pi \in \mathcal{P}(N, U)} \Phi_n^{(N)}(x, \pi), \quad n = 0, 1, 2, \ldots, N-1, \tag{3.5}$$

$$V_N^{(N)}(x) = \Phi_N^{(N)}(x, \pi). \tag{3.6}$$

It is obvious to see that $\Phi^{(N)}(x) = \Phi_0^{(N)}(x)$ and $V^{(N)}(x) = V_0^{(N)}(x)$. For each $\pi = (u_0, u_1, u_2, \ldots, u_{N-1}) \in \mathcal{P}(N, U)$, we can see from equation (3.3) that

$$\Phi_n(x, \pi) = \Phi_n(x, (u_0, u_1, u_2, \ldots, u_{n-1}, u_n, \ldots, u_{N-1}))$$

does not depend on the control actions $u_0, u_1, \ldots, u_{n-1}$. Therefore, (3.5) becomes

$$V_n^{(N)}(x) = \inf_{u_n, u_{n+1}, \ldots, u_{N-1} \in U} \Phi_n^{(N)}(x, (u_0, u_1, u_2, \ldots, u_{n-1}, u_n, \ldots, u_{N-1})). \tag{3.7}$$

Next, we define a function $G : U \to [0, \infty]$ by

$$G(u) := E[e^{-\beta L(u, W_1)}], \tag{3.8}$$

for all $u \in U$, where W_1 is given in Assumption 1 (IA). Thus, by Assumption 1 (IA), we have

$$E[e^{-\beta L(u, W_n)}] = E[e^{-\beta L(u, W_1)}], \tag{3.9}$$

for all $u \in U$ and $n \in \{1, 2, 3, \ldots\}$.

Remark 3.1. By Assumption 1 (IA), for each $\pi = (u_i)_{i=0}^{N-1} \in \mathcal{P}(N, U)$, (3.3) becomes

$$\Phi_n^{(N)}(x, \pi) = \gamma e^{-\beta x} + \alpha G(u_n)\Phi_{n+1}^{(N)}(x, \pi), \tag{3.10}$$

for all $n = 0, 1, 2, \ldots, N-1$.

Proof of Remark 3.1. Let $\pi = (u_i)_{i=0}^{N-1} \in \mathcal{P}(N, U)$. In the case of $n = N-1$, we have

$$\begin{aligned}\Phi_{N-1}^{(N)}(x, \pi) &= E[\gamma e^{-\beta x} + \alpha \nu_0 \cdot e^{-\beta(x + L(u_{N-1}, W_N))}] \\ &= \gamma e^{-\beta x} + \alpha E[e^{-\beta L(u_{N-1}, W_N)}]\nu_0 \cdot e^{-\beta x} \\ &= \gamma e^{-\beta x} + \alpha G(u_{N-1})\Phi_N^{(N)}(x, \pi). \end{aligned} \tag{3.11}$$

In the case of $0 \le n < N-1$. Consider

$$\begin{aligned}&\Phi_n^{(N)}(x, \pi) \\ &= E\left[\sum_{i=n}^{N-1} \alpha^{i-n}\gamma e^{-\beta X_i} + \alpha^{N-n}\nu_0 \cdot e^{-\beta X_N} \mid X_n = x\right] \\ &= E\left[\gamma e^{-\beta x} + \alpha\gamma e^{-\beta(x + L(u_n, W_{n+1}))} + \sum_{i=n+2}^{N-1} \alpha^{i-n}\gamma e^{-\beta\left(x + \sum_{j=n}^{i-1} L(u_j, W_{j+1})\right)}\right. \\ &\qquad \left. + \nu_0\alpha^{N-n} e^{-\beta\left(x + \sum_{j=n}^{N-1} L(u_j, W_{j+1})\right)}\right] \\ &= \gamma e^{-\beta x} + \alpha E\left[e^{-\beta L(u_n, W_{n+1})}\left\{\gamma e^{-\beta x} + \sum_{i=n+2}^{N-1} \gamma\alpha^{i-(n+1)}\gamma e^{-\beta\left(x + \sum_{j=n+1}^{i-1} L(u_j, W_{j+1})\right)}\right.\right. \\ &\qquad \left.\left. + \nu_0\alpha^{N-(n+1)} e^{-\beta\left(x + \sum_{j=n+1}^{N-1} L(u_j, W_{j+1})\right)}\right\}\right]. \end{aligned} \tag{3.12}$$

Since the $\{W_{n+1}, n \geq 0\}$ is an independent sequence, $\{L(u_n, W_{n+1}), n \geq 0\}$ is also an independent sequence. Thus, we obtain

$$\Phi_n^{(N)}(x, \pi)$$

$$= \gamma e^{-\beta x} + \alpha E[e^{-\beta L(u_n, W_{n+1})}]E\left[\gamma e^{-\beta x} + \sum_{i=n+2}^{N-1} \gamma \alpha^{i-(n+1)} e^{-\beta\left(x + \sum_{j=n+1}^{i-1} L(u_j, W_{j+1})\right)} + \nu_0 \alpha^{N-(n+1)} e^{-\beta\left(x + \sum_{j=n+1}^{N-1} L(u_j, W_{j+1})\right)}\right]$$

$$= \gamma e^{-\beta x} + \alpha E[e^{-\beta L(u_n, W_{n+1})}]E\left[\sum_{i=n+1}^{N-1} \alpha^{i-(n+1)} \gamma e^{-\beta X_i} + \alpha^{N-n} \nu_0 e^{-\beta X_N} \mid X_{n+1} = x\right]$$

$$= \gamma e^{-\beta x} + \alpha G(u_n) \Phi_{n+1}^{(N)}(x, \pi). \tag{3.13}$$

This proves Remark 3.1. $\square$

Remark 3.1 leads to the following lemma:

Lemma 3.2. *Under Assumption* 1, *let* $x \in S$ *be an initial state and* $u \in U$ *be a control action. If* $G(u) < \infty$, *then*

$$\Phi_n^{(N)}(x, u) = \begin{cases} (\gamma - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-n}) \cdot \dfrac{e^{-\beta x}}{1 - \alpha G(u)}, & \alpha G(u) \neq 1, \\[2ex] (\gamma(N - n) + \nu_0) \cdot e^{-\beta x}, & \alpha G(u) = 1, \end{cases}$$

for all $n = 0, 1, 2, ..., N$.

Proof. In the case of $\alpha G(u) \neq 1$, we will prove this lemma by using mathematical induction. Obviously, the case $n = N$ holds. Now assume that

$$\Phi_{n+1}^{(N)}(x, u) = (\gamma - [\gamma - \nu_0 \cdot (1 - \alpha G(u))](\alpha G(u))^{N-(n+1)}) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}, \tag{3.14}$$

where $n + 1 < N$. By virtue of Remark 3.1, we get

$$\Phi_n^{(N)}(x, u)$$

$$= \gamma e^{-\beta x} + \alpha G(u)\Phi_{n+1}^{(N)}(x, u)$$

$$= \gamma e^{-\beta x} + \alpha G(u)(\gamma - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-(n+1)}) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}$$

$$= (\gamma(1 - \alpha G(u)) + (\gamma\alpha G(u) - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-n})) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}$$

$$= (\gamma - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-n}) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}. \tag{3.15}$$

Obviously, the case $\alpha G(u) = 1$ holds. This proves Lemma 3.2. □

Lemma 3.3. *Under Assumption* 1, *let* $x \in S$ *be an initial state. If there exists* $u^* \in U$ *such that* $G(u^*) = \min_{u \in U} E[e^{-\beta L(u, W)}] < \infty$, *then*

$$V_n^{(N)}(x) = \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{n+1}^{(N)}(x) \tag{3.16}$$

and the u^**-stationary is an optimal plan, i.e.,* $V^{(N)}(x) = \Phi^{(N)}(x, u^*)$.

Proof. Let $n \in \{0, 1, 2, ..., N-1\}$. Then, by equation (3.7) and Remark 3.1, we have

$$V_n^{(N)}(x) = \inf_{u_n, u_{n+1}, ..., u_{N-1} \in U} \Phi_n^{(N)}(x, (u_0, u_1, u_2, ..., u_{N-1})) \tag{3.17}$$

$$= \gamma e^{-\beta x} + \alpha \inf_{u_n, u_{n+1}, ..., u_{N-1} \in U} \{G(u_n)\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, ..., u_{N-1}))\}. \tag{3.18}$$

For each $(u_i)_{i=0}^{N-1} \in \mathcal{P}(N, U)$, we have $\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, ..., u_{N-1})) \geq 0$ and $G(u_n) \geq 0$, for all $n \in \{0, 1, 2, ..., N-1\}$, and $\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, ..., u_{N-1}))$ does not depend on the control actions $u_0, u_1, ..., u_n$. Therefore, (3.18) becomes

$$V_n^{(N)}(x) = \gamma e^{-\beta x} + \alpha \inf_{u_n \in U} G(u_n) \cdot \inf_{u_{n+1}, ..., u_{N-1} \in U} \Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, ..., u_{N-1})), \tag{3.19}$$

$$= \gamma e^{-\beta x} + \alpha G(u^*) \cdot \inf_{\pi \in \mathcal{P}(N, U)} \Phi_{n+1}^{(N)}(x, \pi) \tag{3.20}$$

$$= \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{n+1}^{(N)}(x). \tag{3.21}$$

From Remark 3.1 and (3.21), since $V_N^{(N)}(x) = \Phi_N^{(N)}(x, u^*)$, we conclude that

$$V_n^{(N)}(x) = \Phi_n^{(N)}(x, u^*), \tag{3.22}$$

for all $n \in \{0, 1, 2, 3, \ldots, N-1\}$. Therefore, $V^{(N)}(x) = \Phi^{(N)}(x, u^*)$. This means that u^*-stationary is an optimal plan. □

From Lemma 3.3, we need the condition for the existence of $\min_{u \in U} G(u)$ which can be shown by using the extreme value theorem. First, we need the property that $u \to G(u)$ is continuous, so we make the following assumption:

Assumption 2. Continuity assumption (CA)

The functions $c(b)$ and $h(b, y)$ are continuous in b (for each y) and

$$E[e^{\beta \cdot Y}] < \infty, \quad E[e^{\varepsilon \cdot \| R \|}] < \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{3.23}$$

Since (K, Z, Y) and R are assumed to be independent as in Assumption 1 (IA) and since $0 \le h(b, Y)K + Y(1-K) \le YK + Y(1-K) = Y$,

$$\begin{aligned} E[e^{-\beta(c(b)Z - \{h(b,Y)K + Y(1-K)\} + \langle \delta, R \rangle)}] &\le E[e^{\beta\{h(b,Y)K + Y(1-K)\} - \beta\langle \delta, R \rangle}] \\ &\le E[e^{\beta\{h(b,Y)K + Y(1-K)\}}] E[e^{-\beta\langle \delta, R \rangle}] \\ &\le E[e^{\beta Y}] \cdot E[e^{|\beta| \cdot \| \delta \| \cdot \| R \|}] < \infty. \end{aligned}$$

We now conclude that $G(u)$ is continuous by using the dominated convergence theorem. Moreover, $b \mapsto E[e^{-\beta(c(b)Z - \{h(b,Y)K + Y(1-K)\})}]$ and $\delta \mapsto [e^{-\beta\langle \delta, R \rangle}]$ are also continuous.

Recall that, we have already set $K = K_1$, $R = R_1$, $Z = Z_1$ and $Y = Y_1$ but sometimes, in Assumption 3, we write K_1, R_1, Z_1 and Y_1 instead of K, R, Z and Y, respectively, to emphasize the balance's surplus at time T_1.

Assumption 3. No-arbitrage assumption (NA)

For any portfolio vector $\delta \in \mathbb{R}^m$, $P(\langle \delta, R \rangle \ge 0) = 1$, implies $P(\langle \delta, R \rangle = 0) = 1$.

In the investment, the investor will look for the *arbitrage opportunity*, i.e., they want to hold the portfolio $\delta_0 \in \mathbb{R}^m$ such that $P(\langle\delta_0, R_1\rangle \geq 0) = 1$, which implies that for the initial surplus $X_0 = x$, we have

$$\begin{aligned} X_1 &= x + c(b_0)Z_1 - \{h(b_0, Y_1)K_1 + Y_1(1 - K_1)\} + \langle\delta_0, R_1\rangle \\ &\geq x + c(b_0)Z_1 - \{h(b_0, Y_1)K_1 + Y_1(1 - K_1)\} \quad \text{a.s.} \end{aligned} \tag{3.24}$$

which means that the portfolio $\delta_0 \in \mathbb{R}^m$ has no risk. Of course, the investor would like to use this opportunity because the quantity $P(\langle\delta_0, R_1\rangle > 0)$ may be positive which indicates an arbitrage opportunity. Note that Assumption 3 (NA) is equivalent to

"for any portfolio $\delta \in \mathbb{R}^m$, $0 < P(\langle\delta, R\rangle < 0) < 1$ or $\langle\delta, R\rangle = 0$ a.s." (NA*)

By using (NA*), we have

$$\mathbb{R}^m = \mathfrak{I} \cup \mathfrak{I}^* \quad \text{and} \quad \mathfrak{I} \cup \mathfrak{I}^* \neq \varnothing,$$

where $\mathfrak{I} = \{\delta \in \mathbb{R}^m : \langle\delta, R\rangle = 0 \text{ a.s.}\}$ and $\mathfrak{I}^* = \{\delta \in \mathbb{R}^m : 0 < P(\langle\delta, R\rangle < 0) < 1\}$. It is easy to see that $\mathfrak{I}$ is a linear subspace of $\mathbb{R}^m$. Thus, there exists a linear subspace $\mathfrak{I}^\perp$ of $\mathbb{R}^m$ such that $\mathbb{R}^m = \mathfrak{I} \oplus \mathfrak{I}^\perp$ and $\mathfrak{I} \cap \mathfrak{I}^\perp = \{0\}$ ($\mathbb{R}^m$ is the direct sum of $\mathfrak{I}$ and $\mathfrak{I}^\perp$) which implies $\mathfrak{I}^\perp \setminus \{0\} \subset \mathfrak{I}^*$.

Lemma 3.4. *Under Assumptions* 1-3, *let* $\delta \in \mathbb{R}^m$ *be given. If* $\delta \in \mathfrak{I}^\perp \setminus \{0\}$, *then there exists an* $\varepsilon > 0$ *such that* $E[-\langle\delta, R\rangle 1_{(\langle\delta, R\rangle < 0)}] \geq \varepsilon \cdot P(\langle\delta, R\rangle \leq -\varepsilon) > 0$.

Proof. Let $\delta \in \mathfrak{I}^\perp \setminus \{0\}$. Then, by (NA*), we have $P(\langle\delta, R\rangle < 0) := q$ for some $q > 0$. Let $A_n := \{\delta \in \Omega : \langle\delta, R\rangle \leq -1/n\}$ and $A_\infty := \{\delta \in \Omega : \langle\delta, R\rangle < 0\}$. Obviously, $A_n \subset A_{n+1} \subset A_\infty$, for all $n = 1, 2, 3, \ldots$ and $\bigcup_{n=1}^{\infty} A_n = A_\infty$. Thus, $\{P(A_n)\}_{n=1}^{\infty}$ is an increasing sequence and then

$$\lim_{l \to \infty} P(A_l) = \lim_{l \to \infty} P\left(\bigcup_{n=1}^{l} A_n\right) = P(A_\infty) = q.$$

So that there exists $l_0 \in \mathbb{N}$ such that $P(A_{l_0}) > q/2$, i.e., $P(\langle \delta, R \rangle \leq -1/l_0) > q/2$. By Markov's inequality, we have

$$\begin{aligned} l_0 E[-\langle \delta, R\rangle 1_{(\langle \delta, R\rangle < 0)}] &\geq P(-\langle \delta, R\rangle 1_{(\langle \delta, R\rangle < 0)} \geq 1/l_0) \\ &= P(\langle \delta, R\rangle 1_{(\langle \delta, R\rangle < 0)} \leq -1/l_0) \\ &= P(\langle \delta, R\rangle \leq -1/l_0) \\ &> q_0/2 > 0. \end{aligned}$$

Choose $\varepsilon = 1/l_0$. The lemma follows. □

Theorem 3.1. *Under Assumptions* 1-3, *let* $x \in S$ *be an initial state. Then there exists* $u^* = (b^*, \delta^*) \in U$ *such that*

$$G(u^*) = \min_{(b, \delta) \in U} E[e^{-\beta(c(b)Z - \{h(b, Y)K + Y(1-K)\} + \langle \delta, R\rangle)}] < \infty$$

and

$$V^{(N)}(x) = \begin{cases} (\gamma - [\gamma - \nu_0 \cdot (1 - \alpha G(u^*))](\alpha G(u^*))^N) \cdot \dfrac{e^{-\beta x}}{1 - \alpha G(u^*)}, & \alpha G(u^*) \neq 1, \\ (\gamma(N-n) + \nu_0) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

Moreover, u^**-stationary is an optimal plan.*

Proof. By Assumption 1 (IA), we have

$$\inf_{u \in U} G(u) = \inf_{b \in [\underline{b}, \overline{b}]} E[e^{-\beta(c(b)Z - \{h(b, Y)K + Y(1-K)\})}] \inf_{\delta \in \mathbb{R}^m} E[e^{-\beta\langle \delta, R\rangle}].$$

Since $[\underline{b}, \overline{b}]$ is compact and $b \mapsto E[e^{-\beta(c(b)Z - \{h(b, Y)K + Y(1-K)\})}]$ is continuous, by using extreme value theorem, there exists $b^* \in [\underline{b}, \overline{b}]$ such that

$$E[e^{-\beta(c(b^*)Z - \{h(b^*, Y)K + Y(1-K)\})}] = \min_{b \in [\underline{b}, \overline{b}]} E[e^{-\beta(c(b)Z - \{h(b, Y)K + Y(1-K)\})}]. \quad (3.25)$$

Next, we will find the minimizer of $E[e^{-\beta\langle \delta, R\rangle}]$ over $\mathbb{R}^m$. We consider the following cases:

Case 1. If $\Im = \mathbb{R}^m$, then by (NA*), we can see that $E[e^{-\beta\langle\delta, R\rangle}] = 1$, for all $\delta \in \mathbb{R}^m$.

Case 2. If $\Im \subset \mathbb{R}^m$, then $\Im^{\perp} \neq \{0\}$. By using Lemma 3.4, we can show that for each $\delta \in \Im^{\perp}\backslash\{0\}$, there exists $\varepsilon > 0$ such that

$$E[-\langle\delta, R\rangle 1_{(\langle\delta, R\rangle<0)}] \geq \varepsilon \cdot P(\langle\delta, R\rangle \leq -\varepsilon) > 0.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} E[e^{-\beta\langle n\cdot\delta, R\rangle}] &= \lim_{n\to\infty} E[e^{-\beta\langle n\cdot\delta, R\rangle}1_{(\langle\delta, R\rangle<0)} + e^{-\beta\langle n\cdot\delta, R\rangle}1_{(\langle\delta, R\rangle\geq 0)}] \\
&\geq \lim_{n\to\infty} E[e^{-\beta\langle n\cdot\delta, R\rangle} 1_{(\langle\delta, R\rangle<0)}] \\
&\geq \lim_{n\to\infty} e^{\beta n E[-\langle\delta, R\rangle 1_{(\langle\delta, R\rangle<0)}]} \\
&\geq \lim_{n\to\infty} e^{\beta n\varepsilon P(\langle\delta, R\rangle<-\varepsilon)} \\
&= \infty. \qquad (3.26)
\end{aligned}$$

Next, for each $\kappa > 0$, we define $F_\kappa := \{\delta \in \Im^{\perp} : \|\delta\| = 1, E[e^{-\beta\langle\kappa\cdot\delta, R\rangle}] \leq 2\}$. Let κ_1 and κ_2 be two real numbers such that $\kappa_2 > \kappa_1 > 0$. If $F_{\kappa_2} \neq \varnothing$, then

$$\begin{aligned}
E[e^{-\beta\langle\kappa_1\cdot\delta, R\rangle}] &= E\left[e^{-\beta\frac{\kappa_1}{\kappa_2}\cdot\kappa_2\langle\delta, R\rangle+\frac{\kappa_2-\kappa_1}{\kappa_2}\cdot 0}\right] \\
&\leq \frac{\kappa_1}{\kappa_2}E[e^{-\beta\kappa_2\langle\delta, R\rangle}] + \frac{\kappa_2-\kappa_1}{\kappa_2} \\
&\leq \frac{2\kappa_1}{\kappa_2} + \frac{\kappa_2-\kappa_1}{\kappa_2} = \frac{\kappa_2+\kappa_1}{\kappa_2} < 2, \quad \text{for all} \quad \delta \in F_{\kappa_2}.
\end{aligned}$$

This means that $F_{\kappa_1} \supset F_{\kappa_2}$, for all $\kappa_2 > \kappa_1 > 0$. Since F_κ is a compact for all $\kappa > 0$, and by inequality (3.26), we have $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$. Thus, by the nested sequence property, there exists an $n_0 \in \mathbb{N}$ such that $F_n = \varnothing$, for all $n \geq n_0$. This implies that $F_\kappa = \varnothing$, for all $\kappa \geq n_0$ and this is equivalent to $\partial B_\kappa := \{\delta \in \Im^{\perp} : \|\delta\| = \kappa, E[e^{-\beta\langle\delta, R\rangle}] \leq 2\} = \varnothing$, for all $\kappa \geq n_0$. Therefore, we have

$$\begin{aligned}\inf_{\delta\in\mathbb{R}^m} E[e^{-\beta\langle\delta, R\rangle}] &= \inf_{\delta\in\mathbb{R}^m} E[e^{-\beta(\langle\rho(\delta), R\rangle+\langle\delta-\rho(\delta), R\rangle)}] \\ &= \inf_{\delta\in\mathbb{R}^m} E[e^{-\beta\langle\rho(\delta), R\rangle}] \\ &= \inf_{\delta\in\mathfrak{I}^\perp} E[e^{-\beta\langle\delta, R\rangle}] \\ &= \inf_{\delta\in\mathfrak{I}^\perp, \|\delta\|\le n_0} E[e^{-\beta\langle\delta, R\rangle}], \end{aligned} \tag{3.27}$$

where $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is an orthogonal projection on $\mathfrak{I}^\perp$. Since $\{\delta \in \mathfrak{I}^\perp : \|\delta\| \le n_0\}$ is compact and $\delta \mapsto E[e^{-\beta\langle\delta, R\rangle}]$ is continuous, there exists $\delta^* \in \{\delta \in \mathfrak{I}^\perp : \|\delta\| \le n_0\}$ such that

$$E[e^{-\beta\langle\delta^*, R\rangle}] = \inf_{\delta\in\mathfrak{I}^\perp, \|\delta\|\le n_0} E[e^{-\beta\langle\delta, R\rangle}]. \tag{3.28}$$

Therefore, $u^* = (b^*, \delta^*)$ is a minimizer of $G(u)$. By Lemma 3.4, we see that u^*-stationary is an optimal plan. Also, from Lemma 3.2, we obtain

$$V^{(N)}(x) = \begin{cases} (\gamma - [\gamma - \nu_0 \cdot (1 - \alpha G(u^*))](\alpha G(u^*))^N) \cdot \dfrac{e^{-\beta x}}{1 - \alpha G(u^*)}, & \alpha G(u^*) \ne 1, \\ (\gamma(N - n) + \nu_0) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

This completes the proof.

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Ruin Probability-Based Initial Capital of the Discrete-Time Surplus Process in Insurance under Reinsurance as a Control Parameter[1]

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Abstract

This paper studied the surplus process model as a premium income minus and the claims are iid random variables. The insurer is allowed to buy reinsurance with retention level b for the period of time between two claims. Whereas the general approach is to consider the ruin probability as a function of initial capital, the authors suggest to study the initial capital via ruin probability. The objective is to find the minimum initial capital for a given boundary for the ruin probability.

Keywords : *Insurance*, *Reinsurance*, *Capital reserve*, *Ruin probability.*
2011 Mathematics Subject Classification : *Primary* 91B30; *Secondary* 93E20.

1. Introduction

In recent years, risk models have been attracted much attention in an insurance business, in connection with any possible insolvency and the capital reserves of an insurance company. The main interest from the point of view of an insurance company is claim arrival and claim size, which affect the capital of the company.

In this paper, we assume that all processes are defined in a probability space $(\Omega, \Im, P)$. Claims happen at the times T_i, satisfying $0 = T_0 \leq T_1 \leq T_2 \leq \cdots$. We call them *arrivals*. The n^{th} claim arriving at time T_n causes the *claim size* Y_n. The *interarrival*, $Z_n := T_n - T_{n-1}$ is the length of time between the $(n-1)^{th}$ claim and the n^{th} claim. By a *period* n, we shall mean the random interval $[T_{n-1}, T_n), n \geq 1$.

Now let a constant c_0 represent the premium rate for one unit time; the random variable $c_0 \sum_{i=1}^{n} Z_i = c_0 T_n$ describes the inflow of capital into the business in $[0, T_n]$, and $\sum_{i=1}^{n} Y_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_n]$. Therefore, the quantity

$$X_0 = x,\ X_n = x + c_0 \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} Y_i,\ n = 1, 2, 3, \ldots \tag{1}$$

is the discrete-time surplus process at time T_n with the constant $x \geq 0$ as initial capital.

The general approach for studying ruin probability in the discrete-time surplus process is the so-called *Gerber − Shiu discounted penalty function*; as found in, Pavlovao and Willmot [11], Dickson [4] and Li [8][9]. These articles study the ruin probability as a function of the initial capital x.

In this paper, we study the initial capital for the discrete-time surplus process via the ruin probability. The objective is to find the minimum initial capital for a given boundary for the ruin probability.

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2. Model Descriptions

Let $\{X_n, n \geq 0\}$ be the surplus process which can be controlled by choosing a retention level $b \in [\underline{b}, \overline{b}], 0 \leq \underline{b} \leq b \leq \overline{b} \leq \infty$, of a reinsurance for one period. Next, for each level b, an insurer pays a premium rate to a reinsurer which is deducted from c_0. As a result, the insurer's income rate will be represented by a function $c(b)$. The level $\overline{b}$ stands for the control action without reinsurance, so that $c_0 = c(\overline{b})$ and the level $\underline{b}$ is the smallest retention level which can be chosen. As a consequence, we obtain the *net income rate* $c(b)$ where $0 \leq c(b) \leq c_0$ for all $b \in [\underline{b}, \overline{b}]$ and $c(b)$ is non-decreasing. The premium rate for one unit time c_0 and the net income rate $c(b)$ are assumed to be satisfied the following:

$$c_0 > \frac{E[Y]}{E[Z]} \quad \text{and} \quad c(b) > \frac{E[h(b,Y)]}{E[Z]} \tag{2}$$

where Y is a claim size and Z is an interarrival.

Moreover, by the *expected value principle*, c_0 and $c(b)$ can be calculated as follows:

$$c_0 = (1+\theta_0)\frac{E[Y]}{E[Z]} \quad \text{and} \quad c(b) = c_0 - (1+\theta_1)\frac{E[Y - h(b,Y)]}{E[Z]} \tag{3}$$

where $0 < \theta_0 < 1$ and $0 < \theta_1 < 1$ are the *safety loadings* of the insurer and the reinsurer respectively. The measurable function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$ which is called *reinsurance recovery* paid by the reinsurer. In the case of an *excess of loss reinsurance*, we have

$$h(b,y) = \min\{b, y\} \quad \text{with retention level} \quad 0 \leq \underline{b} \leq b \leq \overline{b} = \infty.$$

In the case of a *proportional reinsurance*, we have

$$h(b,y) = by \quad \text{with retention level} \quad 0 \leq \underline{b} \leq b \leq \overline{b} = 1.$$

For each $n \in \{1, 2, 3, \ldots\}$, let b_{n-1} be a retention level (control action) at the time T_{n-1} and let $Z_n = 1$. Therefore, we can modify the surplus process (1) to be the following:

$$X_n = x + \sum_{i=1}^{n} c(b_{i-1}) - \sum_{i=1}^{n} h(b_{i-1}, Y_i) \tag{4}$$

where $X_0 = x$.

We see that the process $\{X_n, n \geq 0\}$ is driven by the sequence of retention level (control actions) $\{b_{n-1}, n \geq 1\}$ and the sequence of claims $\{Y_n, n \geq 1\}$. So, we make the following assumption:

Assumption 1. Independence Assumption (IA)
The sequence of claims $\{Y_n, n \geq 1\}$ is independent and identically distributed (iid) random variables.

From Assumption IA, it follows that $\{h(b_{n-1}, Y_n), n \geq 1\}$ is an independent sequence.

Definition 1. *Let $N \in \{1, 2, 3, \ldots\}$ be a time horizon (number of periods). A plan for the time N is a (finite) sequence $\pi = \{b_{n-1}\}_{n=1}^{N}$ of $b_{n-1} \in [\underline{b}, \overline{b}]$ for $n = 1, 2, 3, \ldots, N$. A set of all plans for the time horizon N over a control space $[\underline{b}, \overline{b}]$ is denoted by $\mathcal{P}(N, [\underline{b}, \overline{b}])$. A plan $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ is said to be stationary, if $b_0 = b_1 = \cdots = b_{N-1}$.*

3. Main Results

In this section, we consider a finite-time ruin probability of the discrete-time surplus process as in equation (4) where the sequence of claims $\{Y_n, n \geq 1\}$ satisfy Assumption IA. Let F_{Y_1} be the distribution function of Y_1, i.e.,

$$F_{Y_1}(y) = P(Y_1 \leq y).$$

Let $N \in \{1, 2, 3, \ldots\}$ be a time horizon and $x \geq 0$ be an initial capital. The *survival probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ is defined by

$$\varphi_n(x, \pi) := P(X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, \ldots, X_n \geq 0 | X_0 = x) \tag{5}$$

where $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$. Moreover, the *ruin probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ is defined by

$$\Phi_n(x, \pi) = 1 - \varphi_n(x, \pi). \tag{6}$$

Definition 2. *Let $\{X_n, n \geq 0\}$ be the surplus process as in equation (4), driven by the sequence of control actions $\{b_{n-1}, n \geq 1\}$ and the sequence of claims $\{Y_n, n \geq 1\}$. Let $\{c(b_{n-1})\}_{n\geq 1}$ be a sequence of net income rates and $x \geq 0$ be an initial capital. For each time horizon $N \in \{1, 2, 3, \ldots\}$, let $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and $\alpha \in (0, 1)$. If $\Phi_N(x, \pi) \leq \alpha$, then x is called an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. Particularly, if*

$$x^* = \min_{x \geq 0} \{x : \Phi_N(x, \pi) \leq \alpha\}$$

exists, x^* is called the minimum initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$ and is written as

$$x^* := \mathbf{MIC}(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1}).$$

3.1 Ruin and Survival Probability

We defined a *total claim process* by

$$S_n := h(b_0, Y_1) + h(b_1, Y_2) + \cdots + h(b_{n-1}, Y_n)$$

for all $n \in \{1, 2, 3, \ldots\}$. The survival probability at the time horizon N as mentioned in equation (5) can be expressed as follows:

$$\begin{aligned} \varphi_N(x, \pi) &= P\left(S_1 \leq x + c(b_0), S_2 \leq x + \sum_{n=1}^{2} c(b_{n-1}), \ldots, S_N \leq x + \sum_{n=1}^{N} c(b_{n-1})\right) \\ &= P\left(\bigcap_{n=1}^{N} \left\{S_n \leq x + \sum_{k=1}^{n} c(b_{k-1})\right\}\right). \end{aligned} \tag{7}$$

From equation (7), we have

$$\varphi_N(x, \pi) = E\left[\prod_{n=1}^{N} 1_{(-\infty, 0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - x\right)\right],$$

where

$$1_A(x) = \begin{cases} 1 & , \ x \in A \\ 0 & , \ \text{else} \ , \end{cases}$$

for all $A \subseteq R$. For each $a \in R$ and $x \geq 0$, we obtain

$$1_{(-\infty,0]}(a - x) = \begin{cases} 1 & , \ x \geq a \\ 0 & , \ x < a. \end{cases}$$

Then $1_{(-\infty,0]}(a-x)$ is non-decreasing in x and right continuous on $(0,\infty]$. This implies that $\prod_{n=1}^{N} 1_{(-\infty,0]}(a_n - x)$ is also non-decreasing in x and right continuous on $(0,\infty]$ where $a_n \in R, n = 1,2,3,\ldots,N$. For each plan $\pi = \{b_0, b_1, b_2, \ldots, b_{N-1}\}$, by the Dominated Convergence Theorem, we get

$$\begin{aligned}
\lim_{u\to x^+} \varphi_N(u,\pi) &= \lim_{u\to x^+} E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - u\right)\right] \\
&= E\left[\lim_{u\to x^+} \prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - u\right)\right] \\
&= E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - x\right)\right] \\
&= \varphi_N(x,\pi).
\end{aligned}$$

Therefore, $\varphi_N(x,\pi)$ is non-decreasing in x and right continuous on $(0,\infty)$. This implies that $\Phi_N(x,\pi) = 1 - \varphi_N(x,\pi)$ is non-increasing in x and also right continuous on $(-\infty,\infty)$.

Theorem 1. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$, and let $x \geq 0$ be given. Then*

$$\lim_{x\to\infty} \varphi_N(x,\pi) = 1 \ \textit{and} \ \lim_{x\to\infty} \Phi_N(x,\pi) = 0.$$

Proof : Firstly, we will show the following relation

$$\bigcap_{n=1}^{N} \{\omega : h(b_{n-1},Y_n)(\omega) \leq x + c(b_{n-1})\} \subseteq \bigcap_{n=1}^{N} \left\{\omega : S_n(\omega) \leq Nx + \sum_{k=1}^{n} c(b_{k-1})\right\}. \quad (8)$$

Let $\omega_0 \in \bigcap_{n=1}^{N} \{\omega : h(b_{n-1},Y_n)(\omega) \leq x + c(b_{n-1})\}$ be given. For each $n \in \{1,2,3,\ldots,N\}$, we have $h(b_{n-1},Y_n)(\omega_0) \leq x + c(b_{n-1})$. Thus, $S_n(\omega_0) = \sum_{k=1}^{n} h(b_{k-1},Y_k)(\omega_0) \leq nx + \sum_{k=1}^{n} c(b_{k-1}) \leq Nx + \sum_{k=1}^{n} c(b_{k-1})$. That is $\omega_0 \in \left\{\omega : S_n(\omega) \leq Nx + \sum_{k=1}^{n} c(b_{k-1})\right\}$. Therefore (8) follows. By Assumption IA, the process $\{h(b_{n-1},Y_n), n \geq 1\}$ is an independent sequence, then we have

$$P\left(\bigcap_{n=1}^{N} \{h(b_{n-1},Y_n) \leq x + c(b_{n-1})\}\right) = \prod_{n=1}^{N} P\left(h(b_{n-1},Y_n) \leq x + c(b_{n-1})\right). \quad (9)$$

Note that $Y_n \geq h(b_{n-1}, Y_n)$ for all $n \in \{1, 2, 3, \ldots, N\}$, then

$$\{\omega : Y_n(\omega) \leq x + c(b_{n-1})\} \subseteq \{\omega : h(b_{n-1}, Y_n)(\omega) \leq x + c(b_{n-1})\}.$$

From equation (9), we get

$$\begin{aligned} P\left(\bigcap_{n=1}^{N} \{h(b_{n-1}, Y_n) \leq x + c(b_{n-1})\}\right) &\geq \prod_{n=1}^{N} P\left(Y_n \leq x + c(b_{n-1})\right) \\ &= \prod_{n=1}^{N} F_{Y_n}\left(x + c(b_{n-1})\right). \end{aligned} \tag{10}$$

Moreover, it follows from equation (7) that

$$\varphi_N(Nx, \pi) = P\left(\bigcap_{n=1}^{N} \left\{S_n \leq Nx + \sum_{k=1}^{n} c(b_{k-1})\right\}\right). \tag{11}$$

Thus

$$\begin{aligned} \prod_{n=1}^{N} F_{Y_n}\left(x + c(b_{n-1})\right) &\leq P\left(\bigcap_{n=1}^{N} \{h(b_{n-1}, Y_n) \leq x + c(b_{n-1})\}\right) \\ &\leq P\left(\bigcap_{n=1}^{N} \left\{S_n \leq Nx + \sum_{k=1}^{n} c(b_{k-1})\right\}\right) \qquad \text{(By (8))} \\ &= \varphi_N(Nx, \pi) \leq 1. \qquad \text{(By equation (11))} \end{aligned}$$

Since $F_{Y_n}(x + c(b_{n-1})) \to 1$ as $x \to \infty$ for $n = 1, 2, 3, \ldots, N$, then $\prod_{n=1}^{N} F_{Y_n}\left(x + c(b_{n-1})\right) \to 1$ as $x \to \infty$. Hence $\varphi_N(x, \pi) \to 1$ and $\Phi_N(x, \pi) = 1 - \varphi_N(x, \pi) \to 0$ for $x \to \infty$. The proof is now complete.

Corollary 2. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$, $\alpha \in (0, 1)$, and let $x \geq 0$ be given. Then there exists $\tilde{x} \geq 0$ such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.*

Proof : We consider by cases:

Case 1. $0 \leq \Phi_N(0, \pi) \leq \alpha$. Since $\Phi_N(x, \pi)$ is non-increasing in x, then $\Phi_N(x, \pi) \leq \Phi_N(0, \pi) \leq \alpha$ for all $x > 0$. In this case choose $\tilde{x} = 0$

Case 2. $\Phi_N(0, \pi) > \alpha$. By Theorem 1, we have $\Phi_N(x, \pi) \to 0$ as $x \to \infty$. Thus there exists $\tilde{x} > 0$ such that $\Phi_N(\tilde{x}, \pi) \leq \alpha$. Since $\Phi_N(x, \pi)$ is non-increasing in x, then $\Phi_N(x, \pi) \leq \Phi_N(\tilde{x}, \pi) \leq \alpha$ for all $x \geq \tilde{x}$.

3.2 Bounds of the Ruin Probability

In this part, we shall describe the upper bound of the ruin probability with negative exponential. In order to prove the following lemma, we shall use an equivalent definition of the ruin probability which will be given as follows:

$$\Phi_n(x, \pi) = P\left(\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}(h(b_{i-1}, Y_i) - c(b_{i-1}))\right) > x\right), \ n = 1, 2, 3, \ldots.$$

Lemma 3. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ be stationary, $\alpha \in (0,1)$, and let $x \geq 0$ be given. Then the ruin probability at the time N satisfies the following equation*

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + \int\limits_{\{y: 0 \leq h(b_0,y) \leq x + c(b_0)\}} \Phi_{N-1}(x + c(b_0) - h(b_0,y), \pi) dF_{Y_1}(y) \qquad (12)$$

where $\Phi_0(x,\pi) = 0$.

Proof : We will prove equation (12) by induction. We start with $N = 1$. Since $\Phi_0(x,\pi) = 0$ for all $x \geq 0$, then

$$\int\limits_{\{y: 0 \leq h(b_0,y) \leq x + c(b_0)\}} \Phi_0(x + c(b_0) - h(b_0,y), \pi) dF_{Y_1}(y) = 0.$$

This proves equation (12) for $N = 1$. Now assume that equation (12) holds for $1 < n \leq N-1$. Then

$$\Phi_N(x,\pi) = P\left(\max_{1 \leq n \leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right)$$

$$= P\left(\left\{\max_{1 \leq n \leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right\}\bigcap \Omega\right)$$

$$= P\left(\left\{\max_{1 \leq n \leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right\}\bigcap \{\{h(b_0,Y_1) - c(b_0) > x\}\bigcup\{h(b_0,Y_1) - c(b_0) \leq x\}\}\right)$$

$$= P\left(\max_{1 \leq n \leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) > x\right)$$

$$+ P\left(\max_{1 \leq n \leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) \leq x\right).$$

Since π is stationary and $\{Y_n\}_{n \geq 1}$ is an iid sequence, then

$$\left\{\omega \in \Omega : \max_{1 \leq n \leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)(\omega) - c(b_{i-1}))\right) > x, h(b_0,Y_1)(\omega) - c(b_0) > x\right\}$$
$$= \{\omega \in \Omega : h(b_0,Y_1)(\omega) - c(b_0) > x\}.$$

This result implies

$$\Phi_N(x,\pi) = P(h(b_0,Y_1) - c(b_0) > x)$$

$$+ P\left(\max_{2 \leq n \leq N}\left(h(b_0,Y_1) - c(b_0) + \sum_{i=2}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) \leq x\right)$$

$$= \Phi_1(x,\pi)$$

$$+ P\left(h(b_0,Y_1) - c(b_0) + \max_{2 \leq n \leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) \leq x\right)$$

$$= \Phi_1(x,\pi)$$

$$+ E\left[1_{h(b_0,Y_1) - c(b_0) \leq x,\ h(b_0,Y_1) - c(b_0) + \max_{2 \leq n \leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x}\right]$$

$$= \Phi_1(x,\pi)$$
$$+ E\left[1_{h(b_0,Y_1)-c(b_0)\le x}\cdot 1_{h(b_0,Y_1)-c(b_0)+\max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}\right]$$
$$= \Phi_1(x,\pi)$$
$$+ E\left[E\left[1_{h(b_0,Y_1)-c(b_0)\le x}\cdot 1_{h(b_0,Y_1)-c(b_0)+\max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}|\sigma(Y_1)\right]\right]$$
$$= \Phi_1(x,\pi)$$
$$+ E\left[1_{h(b_0,Y_1)\le x+c(b_0)}\cdot E\left[1_{\max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)+(h(b_0,Y_1)-x-c(b_0))>0}|\sigma(Y_1)\right]\right]$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\le x+c(b_0)}\cdot E\left[1_{(0,\infty)}(Z+W)|\sigma(Y_1)\right]\right] \tag{13}$$

where $Z = \max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)$ and $W = h(b_0,Y_1) - x - c(b_0)$. Since $\{h(b_{n-1},Y_n)\}_{n\ge 1}$ is an independent sequence, then Z and W are independent. It follows from [5, exercise 9, page 341] that

$$\begin{aligned} E\left[1_{(0,\infty)}(Z+W)|\sigma(Y_1)\right] &= \int\limits_{\omega\in\ \Omega} 1_{(0,\infty)}(Z(\omega)+W|\sigma(Y_1))dP_Z(\omega) \\ &= \int\limits_{R} 1_{(0,\infty)}(z+W)dF_Z(z). \end{aligned}$$

This implies that

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\le x+c(b_0)}\cdot\left(\int\limits_R 1_{(0,\infty)}(z+W)dF_Z(z)\right)\right]$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\le x+c(b_0)}\cdot\left(\int\limits_R 1_{(0,\infty)}(z+h(b_0,Y_1)-x-c(b_0))dF_Z(z)\right)\right]$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega: h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}}\left(\int\limits_R 1_{(0,\infty)}(z+h(b_0,Y_1)(\omega)-x-c(b_0))dF_Z(z)\right)dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega: h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} E\left[1_{Z>x+c(b_0)-h(b_0,Y_1)(\omega)}\right]dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega: h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} P\left(Z>x+c(b_0)-h(b_0,Y_1)(\omega)\right)dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{y\in R: 0\le h(b_0,y)\le x+c(b_0)\}} P\left(Z>x+c(b_0)-h(b_0,y)\right)dF_{Y_1}(y)$$

$$= \Phi_1(x,\pi) + \int_{\{y:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_{N-1}(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y).$$

This proves equation (12).

Remark 4. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\bar{b}])$ be stationary, $\alpha \in (0,1)$. Assume that $\{Y_n, n \ge 1\}$ is an iid sequence of exponential distribution with intensity $\lambda > 0$, i.e., Y_1 has the probability density function*

$$f(y) = \lambda e^{-\lambda y}.$$

By Lemma 3, the ruin probability can be written in a recursive form as follows:

Case 1: For an excess of loss reinsurance, we get

$$\Phi_0(x,\pi) = 0 \text{ and}$$

$$\Phi_n(x,\pi) \quad = \quad \Phi_{n-1}(x,\pi) + \frac{[\lambda(x+nc(b_0))]^{n-1}}{(n-1)!} e^{-\lambda[x+nc(b_0)]} \frac{x+c(b_0)}{x+nc(b_0)} \tag{14}$$

for $b_0 \ge x + c(b_0)$ and $n = 1,2,3,\ldots,N$.

Case 2: For a proportional reinsurance, we get

$$\Phi_0(x,\pi) = 0 \text{ and}$$

$$\Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{1}{(n-1)!}\left[\frac{\lambda}{b_0}(x+nc(b_0))\right]^{n-1} e^{-\frac{\lambda}{b_0}(x+nc(b_0))} \frac{x+c(b_0)}{x+nc(b_0)} \tag{15}$$

for all $n = 1,2,3,\ldots,N$. Further, for $b_0 = \bar{b}_0 = 1$, we also obtained the recursive form as follows :

$$\Phi_0(x,\pi) = 0 \text{ and } \Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{1}{(n-1)!}\left[\lambda(x+nc_0)\right]^{n-1} e^{-\lambda(x+nc_0)} \frac{x+c_0}{x+nc_0}$$

for all $n = 1,2,3,\ldots,N$.

Definition 3. *(Sub-adjustment coefficient). Let $s > 0$ and Y be a non-negative random variable. If there exists $d_0 > 0$ such that*

$$E\left[e^{d_0 Y}\right] \le e^{d_0 s}, \tag{16}$$

then d_0 is called a sub-adjustment coefficient of (s,Y). Specifically, if (16) is an equality then d_0 is called an adjustment coefficient of (s,Y).

Theorem 5. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\bar{b}])$ be stationary, and let $c(b_0) > 0$ be a net income rate. If $d_0 > 0$ is a sub-adjustment coefficient of $(c(b_0), h(b_0,Y_1))$, then*

$$\Phi_n(x,\pi) \le e^{-d_0 x}, \tag{17}$$

for all $x \ge 0$ and all $n = 1,2,3,\ldots,N$.

Proof : Let $x \geq 0$ and $d_0 > 0$ be a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, i.e.,

$$E\left[e^{d_0 h(b_0,Y_1)}\right] \leq e^{d_0 c(b_0)}.$$

We shall prove this theorem by induction. We start with $n = 1$,

$$\begin{aligned}
\Phi_1(x,\pi) &= P(h(b_0,Y_1) > x + c(b_0)) \\
&= P(d_0 h(b_0,Y_1) > d_0(x + c(b_0))) \\
&= P(e^{d_0 h(b_0,Y_1)} > e^{d_0(x+c(b_0))}) \\
&\leq \frac{E\left[e^{d_0 h(b_0,Y_1)}\right]}{e^{d_0(x+c(b_0))}} \qquad \text{(By Markov's inequality)} \\
&\leq \frac{e^{d_0 c(b_0)}}{e^{d_0(x+c(b_0))}} = e^{-d_0 x}.
\end{aligned}$$

Let $k \leq N-1$. Assume that inequality (17) holds for $1 < n \leq k$. Next, we shall show that inequality (17) holds for $n = k+1$. By Lemma 3 and inductive assumption, we get

$$\begin{aligned}
&\Phi_{k+1}(x,\pi) \\
&= \Phi_1(x,\pi) + \int\limits_{\{y:0\leq h(b_0,y)\leq x+c(b_0)\}} \Phi_k(x + c(b_0) - h(b_0,y),\pi) dF_{Y_1}(y) \\
&\leq \Phi_1(x,\pi) + \int\limits_{\{y:0\leq h(b_0,y)\leq x+c(b_0)\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y). \qquad (18)
\end{aligned}$$

Next, we shall calculate the first term of right-hand side of inequality (18).

$$\begin{aligned}
&\Phi_1(x,\pi) \\
&= P(h(b_0,Y_1) > x + c(b_0)) \\
&= P\left(e^{d_0 h(b_0,Y_1)} 1_{(x+c(b_0),\infty)}(h(b_0,Y_1)) > e^{d_0(x+c(b_0))}\right) \\
&\leq \frac{E\left[e^{d_0 h(b_0,Y_1)} 1_{(x+c(b_0),\infty)}(h(b_0,Y_1))\right]}{e^{d_0(x+c(b_0))}} \qquad \text{(By Markov's inequality)} \\
&= \frac{\int\limits_R e^{d_0 h(b_0,y)} 1_{(x+c(b_0),\infty)}(h(b_0,y)) dF_{Y_1}(y)}{e^{d_0(x+c(b_0))}} \\
&= \frac{\int\limits_{\{y:x+c(b_0)<h(b_0,y)<\infty\}} e^{d_0 h(b_0,y)} dF_{Y_1}(y)}{e^{d_0(x+c(b_0))}} \\
&= \int\limits_{\{y:x+c(b_0)<h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y).
\end{aligned}$$

Thus inequality (18) can be modified to be the following

$\Phi_{k+1}(x,\pi)$

$$\leq \int\limits_{\{y:x+c(b_0)<h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$+ \int\limits_{\{y:0\leq h(b_0,y)\leq x+c(b_0)\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$= \int\limits_{\{y:0\leq h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$= \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} \int\limits_{\{y:0\leq h(b_0,y)<\infty\}} e^{d_0 h(b_0,y)} dF_{Y_1}(y)$$

$$= \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} E\left[e^{d_0 h(b_0,Y_1)}\right]$$

$$\leq \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} e^{d_0 c(b_0)} = e^{-d_0 x}.$$

This proves equation (17) for $n = k+1$ and concludes the proof.

Corollary 6. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ be stationary, $\alpha \in (0,1)$, and let $c(b_0) > 0$ be a net income rate. Assume that $d_0 > 0$ is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, then there exists an acceptable initial capital $x (x \geq 0)$ corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$ such that*

$$0 \leq x \leq -\frac{ln\alpha}{d_0} \; or \;\; \alpha \leq e^{-d_0 x}.$$

Proof : Let $d_0 > 0$ be a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$. By Theorem 5, we have

$$\Phi_N(u, \pi) \leq e^{-d_0 u},$$

for all $u \geq 0$. Let $\alpha \in (0,1)$. By Corollary 2, there exists $v \geq 0$ which is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$. By Definition 2, we have

$$\Phi_N(v, \pi) \leq \alpha.$$

Since $\Phi_N(v, \pi)$ is non-increasing in v for each π, then there exists $0 \leq x \leq v$ such that $\alpha = \Phi_N(x, \pi) \leq e^{-d_0 x}$. Hence x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$. The proof is now complete.

Note:It's known that a large initial capital results in a small ruin probability. However, an insurance company usually does not posses unlimited initial capital, but only a small initial capital, that must be sufficient for a predetermined solvency (not ruin) condition for the firm is preferable. If an acceptable ruin probability is fixed, the firm can find an interval of acceptable initial capital by virtue of Corollary 6.

Example 1. *(Exponential claims under the proportional reinsurance). We assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$, and $\{X_n\}_{n\geq 0}$ is a sequence of surplus which satisfies the model (4). Let $N \in \{1,2,3,\ldots\}$, and $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ be* *stationary. Suppose that $h(b_0, y)$ is the proportional reinsurance with retention level b_0, and $c(b_0) > 0$ is a net income rate which is calculated by the expected value principle, i.e.,*

$$c(b_0) = c_0 - (1 + \theta_1)E[Y_1 - h(b_0, Y_1)] = \theta_0 - \theta_1 + b_0(1 + \theta_1). \tag{19}$$

Assume that $\alpha = 0.05$, $\theta_0 = \theta_1 = 0.1$, and $b_0 = 0.6$. Then there exists an adjustment coefficient $d_0 = 0.2935569060$ of $(c(b_0), b_0Y_1)$ such that

$$0 \leq x \leq \frac{-ln 0.05}{0.2935569060} = 10.20494566$$

which is an interval of acceptable initial capital with corresponding to $(1, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq1}, \{b_0Y_n\}_{n\geq1})$

Let

$$f(d) := E\left[e^{db_0Y_1}\right] - e^{dc(b_0)}.$$

Note that

$$E\left[e^{db_0Y_1}\right] = \int_0^\infty e^{db_0y} f_{Y_1}(y)dy = \int_0^\infty e^{db_0y}e^{-y}dy = \frac{1}{1 - db_0} \quad \text{and} \quad e^{dc(b_0)} = e^{db_0(1+\theta_1)}. \tag{20}$$

By Definition 3, d_0 is an adjustment coefficient of $(c(b_0), b_0Y_1)$ if $f(d_0) = 0$. Hence $E\left[e^{d_0b_0Y_1}\right] = e^{d_0c(b_0)}$. By substitute b_0 and θ_1 into equation (20), we get

$$\frac{1}{1 - 0.6d_0} = e^{0.66d_0}.$$

Solving for d_0, we get $d_0 = 0.2935569060$. By Corollary 6, we get

$$0 \leq x \leq \frac{-ln 0.05}{0.2935569060} = 10.20494566$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = 0.66\}_{n\geq1}, \{0.6Y_n\}_{n\geq1})$. This means that $\Phi_N(x, \pi) \leq 0.05$ for all $0 \leq x \leq 10.20494566$.

Example 2. *(Exponential claims under the excess of loss reinsurance). We assume that $\{Y_n\}_{n\geq1}$ and $\{X_n\}_{n\geq0}$ are the sequences given in example 1. Let $N \in \{1, 2, 3, \ldots\}$, and $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ be stationary. Suppose that $h(b_0, y)$ is the excess of loss reinsurance with retention level b_0. By expected value principle, the net income rate $c(b_0)$ satisfies the following equation*

$$c(b_0) = c_0 - (1 + \theta_1)E[Y_1 - h(b_0, Y_1)] = \theta_0 - \theta_1 + (1 + \theta_1)[1 - e^{-b_0}]. \tag{21}$$

Assume that $\alpha = 0.05$, $\theta_0 = \theta_1 = 0.1$ and $b_0 = 100$. Then there exists a sub-adjustment coefficient $d_0 = 0.17$ of $(c(b_0), h(b_0, Y_1))$ such that

$$0 \leq x \leq -\frac{ln 0.05}{0.17} = 17.6220$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq1}, \{h(b_0, Y_n)\}_{n\geq1})$

Let

$$f(d) := E\left[e^{dh(b_0,Y_1)}\right] - e^{dc(b_0)}.$$

Note that

$$E\left[e^{dh(b_0,Y_1)}\right] = \int_0^\infty e^{dh(b_0,y)}e^{-y}dy = \int_0^{b_0} e^{dy}e^{-y}dy + \int_{b_0}^\infty e^{b_0 d}e^{-y}dy = \frac{de^{b_0(d-1)}-1}{d-1},$$

$$\text{and} \quad e^{dc(b_0)} = e^{d(1+\theta_1)\left[1-e^{-b_0}\right]}. \tag{22}$$

By Definition 3, d_0 is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$ if $f(d_0) \leq 0$. Hence $E\left[e^{d_0 h(b_0,Y_1)}\right] \leq e^{d_0 c(b_0)}$. By substitute b_0, θ_0 and θ_1 into equation (22), we get

$$\frac{d_0 e^{100(d_0-1)}-1}{d_0-1} \leq e^{1.1d_0\left[1-e^{-100}\right]}.$$

Solving for d_0, we get $d_0 = 0.17$. By Corollary 6, we get

$$0 \leq x \leq -\frac{ln 0.05}{0.17} = 17.6220$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = 1.1\}_{n\geq 1}, \{h(100, Y_n)\}_{n\geq 1})$. This means that $\Phi_N(x,\pi) \leq 0.05$ for all $0 \leq x \leq 17.6220$.

3.3 Existence of Minimal Capital

Let $\alpha \in (0,1)$. As a result of Corollary 4.6 that $\{x \geq 0 : \Phi_N(x,\pi) \leq \alpha\}$ is a non-empty set. Since the set $\{x \geq 0 : \Phi_N(x,\pi) \leq \alpha\}$ is an infinite set, then there are many acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. In this section, we will prove the existence of a minimum initial capital that correspond to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.

Lemma 7. *Let a, b and α be real numbers such that $a \leq b$. If f is non-increasing and right continuous on $[a,b]$ and $\alpha \in [f(b), f(a)]$, then there exists $d \in [a,b]$ such that*

$$d = \min\{x \in [a,b] : f(x) \leq \alpha\}.$$

Proof : Let

$$S := \{x \in [a,b] : f(x) \leq \alpha\}.$$

Since $\alpha \in [f(b), f(a)]$, i.e., $f(b) \leq \alpha \leq f(a)$, then we have $b \in S$. Hence S is a non empty set. Since S is a subset of the closed and bounded interval $[a,b]$, then there exists $d \in [a,b]$ such that $d = \inf S$. Next, we consider the following cases:

Case 1. $d = b$. We know that $b \in S$, thus $b = \min S$.

Case 2. $a \leq d < b$. Since $d = \inf S$, then there exists $d_n \in S$ such that

$$d \leq d_n < d + 1/n$$

for all $n \in \{1, 2, 3, \ldots\}$. Since f is non-increasing and $d_n \in S$, then

$$f(d_n) \leq \alpha.$$

Since f is right continuous at d, we have

$$f(d) = \lim_{n\to\infty} f(d_n) \le \alpha.$$

Therefore, $d \in S$, i.e., $d = \min S$. This completes the proof.

Theorem 8. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and let $\alpha \in (0, 1)$. Then there exists $x^* \ge 0$ such that*

$$x^* = \min_{x\ge 0}\{x : \Phi_N(x, \pi) \le \alpha\}.$$

Proof : Let $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$. We consider by case.

Case 1: For $\Phi_N(0, \pi) \le \alpha$. We get $\min\limits_{x\ge 0}\{x : \Phi_N(x, \pi) \le \alpha\} = 0$.

Case 2: For $\Phi_N(0, \pi) > \alpha$. By Corollary 2, there exists $\tilde{x} > 0$ such that $\Phi_N(\tilde{x}, \pi) \le \alpha$. Hence $\alpha \in [\Phi_N(\tilde{x}, \pi), \Phi_N(0, \pi)]$. Since $\Phi_N(x, \pi)$ is non-increasing in x and right continuous on $[0, \infty)$. Then ,by Lemma 7, there exists $x^* \in [0, \tilde{x}]$ such that

$$x^* = \min_{x\in[0,\tilde{x}]}\{x : \Phi_N(x, \pi) \le \alpha\} = \min_{x\in[0,\infty)}\{x : \Phi_N(x, \pi) \le \alpha\}.$$

From case 1 and 2, we have $x^* = \min\limits_{x\ge 0}\{x : \Phi_N(x, \pi) \le \alpha\}$.

Next, we will approximate the minimal initial capital x^* by the bisection method.

Theorem 9. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and let $\alpha \in (0, 1)$. Assume that $v_0, x_0 \ge 0$ such that $v_0 < x_0$. Let $\{v_m\}_{m\ge 1}$ and $\{x_m\}_{m\ge 1}$ be two real sequences defined by*

$$\begin{cases} v_m = v_{m-1} \quad \text{and} \quad x_m = \dfrac{x_{m-1} + v_{m-1}}{2}, & \text{if } \Phi_N\left(\dfrac{x_{m-1} + v_{m-1}}{2}, \pi\right) \le \alpha \\[2ex] v_m = \dfrac{v_{m-1} + x_{m-1}}{2} \quad \text{and} \quad x_m = x_{m-1}, & \text{if } \Phi_N\left(\dfrac{x_{m-1} + v_{m-1}}{2}, \pi\right) > \alpha \end{cases}$$

for all $m = 1, 2, 3, \ldots$. If $\Phi_N(x_0, \pi) \le \alpha < \Phi_N(v_0, \pi)$, then

$$\lim_{m\to\infty} x_m = \min_{x\ge 0}\{x : \Phi_N(x, \pi) \le \alpha\} = x^*.$$

Proof : Obviously, $\{x_m\}_{m\ge 1}$ is non-increasing and $\{v_m\}_{m\ge 1}$ is non-decreasing. Moreover, $v_m \le x_m$ for all $m \in \{1, 2, 3, \ldots\}$. Thus, $\{x_m\}_{m\ge 1}$ and $\{v_m\}_{m\ge 1}$ are convergent. Since

$$0 \le x_m - v_m = \frac{x_0 - v_0}{2^m} \to 0 \text{ as } m \to \infty,$$

then there exists $x^* \in [v_0, x_0]$ such that

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} v_m := x^*.$$

Since $\Phi_N(x, \pi)$ is right continuous in x for each π and $\Phi_N(x_m, \pi) \le \alpha$ for all m, then

$$\Phi_N(x^*, \pi) = \lim_{m\to\infty} \Phi_N(x_m, \pi) \le \alpha.$$

Since $\Phi_N(x, \pi)$ is non-increasing in x for each π and $\Phi_N(v_m, \pi) > \alpha$ for all m, then $\Phi_N(x, \pi) > \alpha$ for $x < x^*$. Therefore

$$x^* = \min_{x\ge 0}\{x : \Phi_N(x, \pi) \le \alpha\}. \tag{23}$$

This completes the proof.

4. Numerical Results

In this section, we provide numerical illustration of main results. We approximate the minimal initial capital of the discrete-time surplus process (4) by using Theorem 9 according to the following cases:

(a). **Proportional Reinsurance**.

We assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$ and $h(b_0, y)$ is the proportional reinsurance with retention level b_0. Let $N \in \{1, 2, 3, \ldots\}$ be the time horizon and $\pi = \{b_{n-1} = 0.6\}_{n=1}^{N}$ be stationary. We choose model parameters as follows: $\theta_0 = \theta_1 = 0.10$ which give $c(b_0) = 0.66$ and $\theta_0 = \theta_1 = 0.25$ which give $c(b_0) = 0.75$. Moreover, we choose $\alpha = 0.05, \alpha = 0.1$ and $\alpha = 0.2$. As a result, we get the table of the minimum initial capital as below :

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
N	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$
10	3.3909 : 2.7854	2.5919 : 2.0384	1.7358 : 1.2562
20	4.4983 : 3.3728	3.4846 : 2.4796	2.3918 : 1.5524
30	5.2438 : 3.6605	4.0747 : 2.6854	2.8148 : 1.6829
40	5.8067 : 3.8215	4.5137 : 2.7963	3.1233 : 1.7504
50	6.2558 : 3.9175	4.8593 : 2.8605	3.3619 : 1.7884
100	7.6364 : 4.0664	5.8902 : 2.9559	4.0471 : 1.8426
200	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
300	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
400	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
500	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
$1,000$	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
$5,000$	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
$10,000$	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497

Table 1: Minimum initial capital in the case proportional reinsurance.

Table 1 shows an approximation of $\min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$ with $m = 25$, $v_0 = 0, x_0 = 20$ as mentioned in Theorem 9 and $\Phi_N(x, \pi)$ is computed by using the recursive form as mentioned in equation (15). The numerical results in Table 1 show a minimum initial capital $x = 3.3909$ for $\alpha = 0.05$, $N = 10$ and $\theta_0 = \theta_1 = 0.1$ etc.

(b). **Excess of Loss Reinsurance**.

Again we assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$ and $h(b_0, y)$ is the excess of loss reinsurance with retention level $b_0 = 100$. Let $N \in \{1, 2, 3, \ldots\}$ be the time horizon and $\pi = \{b_{n-1} = 100\}_{n=1}^{N}$ be stationary. We choose model parameters as follows: $\theta_0 = \theta_1 = 0.10$ which give $c(b_0) = 1.1$ and $\theta_0 = \theta_1 = 0.25$ which give $c(b_0) = 1.25$. Moreover, we choose $\alpha = 0.05, \alpha = 0.1$ and $\alpha = 0.2$. As a result, we get the table of the minimum initial capital as below :

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
N	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$
10	5.6515 : 4.6424	4.3198 : 3.3973	2.8930 : 2.0936
20	7.4972 : 5.6213	5.8076 : 4.1327	3.9863 : 2.5874
30	8.7396 : 6.1009	6.7911 : 4.4756	4.6913 : 2.8048
40	9.6779 : 6.3692	7.5229 : 4.6605	5.2054 : 2.9174
50	10.4264 : 6.5291	8.0989 : 4.7675	5.6031 : 2.9806
100	12.7273 : 6.7773	9.8169 : 4.9265	6.7452 : 3.0709
200	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
300	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
400	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
500	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
$1,000$	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
$5,000$	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
$10,000$	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828

Table 2: Minimum initial capital in the case excess of loss reinsurance.

Table 2 shows an approximation of $\min_{x\geq 0}\{x : \Phi_N(x,\pi) \leq \alpha\}$ with $m = 25$, $v_0 = 0, x_0 = 20$ as mentioned in Theorem 9 and $\Phi_N(x,\pi)$ is computed by using the recursive form as mentioned in equation (14). The numerical results in Table 2 show a minimum initial capital $x = 5.6515$ for $\alpha = 0.05$, $N = 10$ and $\theta_0 = \theta_1 = 0.1$ etc.

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European Option Pricing for a Stochastic Volatility Lévy Model with Stochastic Interest Rates

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Abstract

We present a European option pricing when the underlying asset price dynamics is governed by a linear combination of the time-change Lévy process and a stochastic interest rate which follows the Vasicek process. We obtain an explicit formula for the European call option in term of the characteristic function of the tail probabilities.

Keywords: Time-Change Lévy Process, Stochastic Interest Rate, Vasicek Process, Forward Measure, Option Pricing

1. Introduction

Let (Ω, F, P) be a probability space. A stochastic process L_t is a Lévy process if it has independent and stationary increments and has a stochastically continuous sample path, *i.e*. for any $\varepsilon > 0$, $\lim_{h \downarrow 0} P(|L_{t+h} - L_t| > \varepsilon) \to 0$. The simplest possible Lévy processes are the standard Brownian motion W_t, Poisson process N_t, and compound Poisson process $\sum_{i=1}^{N_t} Y_i$ where N_t is Poisson process with intensity λt and Y_i are i.i.d. random variables. Of course, we can build a new Lévy process from known ones by using the technique of linear transformation. For example, the jump diffusion process $\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where μ, σ are constants, is a Lévy process which comes from a linear transformation of two independent Lévy processes, *i.e*. a Brownian motion with drift and a compound Poisson process.

Assume that a risk-neutral probability measure Q exists and all processes in section 1 will be considered under this risk-neutral measure.

In the Black-Scholes model, the price of a risky asset S_t under a risk-neutral measure Q and with non dividend payment follows

$$S_t = S_0 \exp\left(\tilde{L}_t\right) = S_0 \exp\left(rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)\right) \quad (1.1)$$

where $r \in \mathbb{R}$ is a risk-free interest rates, $\sigma \in \mathbb{R}$ is a volatility coefficient of the stock price. Instead of modeling the log returns

$$\tilde{L}_t = rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$$

with a normal distribution. We now replace it with a more sophisticated process L_t which is a Lévy process of the form

$$L_t = rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) + \left(J_t - \zeta t\right), \quad (1.2)$$

where J_t and ζ_t denotes a pure Lévy jump component, (*i.e*. a Lévy process with no Brownian motion part) and its convexity adjustment. We assume that the processes W_t and J_t are independent. To incorporate the volatileity effect to the model (1.2), we follow the technique of Carr and Wu [1] by subordinating a part of a standard Brownian motion $\sigma W_t - \frac{1}{2}\sigma^2 t$ and a part of jump Lévy process $J_t - \zeta t$ by the time integral of a mean reverting Cox Ingersoll Ross (CIR) process

$$T_t = \int_0^t v_s \mathrm{d}s,$$

where v_t follows the CIR process

$$\mathrm{d}v_t = \gamma(1 - v_t)\mathrm{d}t + \sigma_v \sqrt{v_t}\mathrm{d}W_t^v \quad (1.3)$$

Here W_t^v is a standard Brownian motion which corresponds to the process v_t. The constant $\gamma \in \mathbb{R}$ is the rate at which the process v_t reverts toward its long term mean and $\sigma_v > 0$ is the volatility coefficient of the process v_t.

Hence, the model (1.2) has been changed to

$$L_t = rt + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + \left(J_{T_t} - \zeta T_t\right), \qquad (1.4)$$

and this new process is called a stochastic volatility Levy process. One can interpret T_t as the stochastic clock process with activity rate process v_t. By replacing $\tilde{L}_t$ in (1.1) with L_t, we obtain a model of an underlying asset under the risk-neutral measure Q with stochastic volatility as follows:

$$S_t = S_0 \exp\left(rt + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + \left(J_{T_t} - \zeta T_t\right)\right) (1.5)$$

In this paper, we shall consider the problem of finding a formula for European call options based on the underlying asset model (1.5) for which the constant interest rates r is replaced by the stochastic interest rates r_t, and J_t is compound Poisson process, *i.e*. the model under our consideration is given by

$$S_t = S_0 \exp\left(r_t t + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + \left(J_{T_t} - \zeta T_t\right)\right) \quad (1.6)$$

Here, we assume that r_t follows the Vasicek process

$$\mathrm{d}r_t = (\alpha - \beta r_t)\mathrm{d}t + \sigma_r \mathrm{d}W_t^r, \qquad (1.7)$$

W_t^r is a standard Brownian motion with respect to the process r_t and $\mathrm{d}W_t^r \mathrm{d}W_t^v = \mathrm{d}W_t^r \mathrm{d}W_t = 0$. The constant $\beta > 0$ is the rate at which the interest rate reverts toward its long term mean, $\sigma_r > 0$ is the volatility coefficient of the interest rate process (1.7), The constant $\alpha > 0$ is a speed reversion.

2. Literature Reviews

Many financial engineering studies have been undertaken to modify and improve the Black-Scholes model. For example, The jump diffusion models of Merton [2], the stochastic Volatility jump diffusion model of Bates [3] and Yan and Hanson [4]. Furthermore, the time change Lévy models proposed by Carr and Wu [1].

The problem of option pricing under stochastic interest rates has been investigated for along time. Kim [5] constructed the option pricing formula based on Black-Scholes model under several stochastic interest rate processes, *i.e*., Vasicek, CIR, Ho-Lee type. He found that by incurporating stochastic interest rates into the Black-Scholes model, for a short maturity option, does not contribute to improvement in the performance of the original Black-Scholes' pricing formula. Brigo and Mercurio [6] mention that the stochastic feature of interest rates has a stronger impact on the option price when pricing for a long maturity option. Carr and Wu [1] continue this study by giving the option pricing formula based on a time-changed Lévy process model. But they still use constant interest rates in the model.

In this paper, we give an analysis on the option pricing model based on a time-changed Lévy process with stochastic interest rates.

The rest of the paper is organized as follows. The dynamics under the forward measure is described in Section 3. The option pricing formula is given in Section 4. Finally, the close form solution for a European call option in terms of the characteristic function is given in Section 5.

3. The Ddynamics under the Forward Measure

We begin by giving a brief review of the definition of a correlated Brownian motion and some of its properties (for more details one see Brummelhuis [7]). Recalling that a standard Brownian motion in R^n is a stochastic process $(\boldsymbol{Z}_t)_{t\geq 0}$ whose value at time t is simply a vector of n independent Brownian motions at t,

$$\boldsymbol{Z}_t = \left(Z_{1,t}, \cdots, Z_{n,t}\right).$$

We use Z instead of W since we would like to reserve the latter for the more general case of correlated Brownian motion, which will be defined as follows:

Let $\rho = (\rho_{ij})_{1\leq i,j\leq n}$ be a (constant) positive symmetric matrix satisfying $\rho_{ii} = 1$ and $-1 \leq \rho_{ij} \leq 1$ By Cholesky's decomposition theorem, one can find an upper triangul $n \times n$ matrix $H = (h_{ij})$ such that $\rho = HH^t$, where H^t is the transpose of the matrix H. Let $\boldsymbol{Z}_t = \left(Z_{1,t}, \cdots, Z_{n,t}\right)$ be a standard Brownian motion as introduced above, we define a new vector-valued process $\boldsymbol{W}_t = \left(W_{1,t}, \cdots, W_{n,t}\right)$ by $\boldsymbol{W}_t = H\boldsymbol{Z}_t$ or in term of components,

$$W_{i,t} = \sum_{j=1}^{n} h_{ij} Z_{j,t}, \ i = 1, \cdots, n$$

The process $(\boldsymbol{W}_t)_{t\geq 0}$ is called a correlated Brownian motion with a (constant) correlation matrix ρ. Each component process $(W_{i,t})_{t\geq 0}$ is itself a standard Brownian motion. Note that if $\rho = Id$ (the identity matrix) then $\boldsymbol{W}_t$ is a standard Brownian motion. For example, if we let a symmetric matrix

$$\rho = \begin{bmatrix} 1 & \rho_v & 0 \\ \rho_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.1)$$

Then ρ has a *Cholesky decomposition* of the form $\rho = HH^T$ where H is an upper triangular matrix of the form

$$H = \begin{bmatrix} \sqrt{1-\rho_v^2} & \rho_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\boldsymbol{Z}_t = \left(Z_t, Z_t^r, Z_t^v\right)$ be three independent Brownian motions then $\boldsymbol{W}_t = \left(W_t, W_t^r, W_t^v\right)$ defined by $\boldsymbol{W}_t = \mathrm{H}\boldsymbol{Z}_t$, or in terms of components,

$$W_t = \left(\sqrt{1-\rho_v^2}\right) Z_t + \rho_v Z_t^v, W_t^v = Z_t^v, W_t^r = Z_t^r \quad (3.2)$$

Now let us turn to our problem. Note that, by Ito's lemma, the model (1.6) has the dynamic given by

$$\begin{aligned} \mathrm{d}S_t &= S_t\left(\left(r_t - \lambda_m v_t\right)\mathrm{d}t + \sigma \mathrm{d}W_{T_t}\right) + S_{t-}\left(\mathrm{e}^{Y_t} - 1\right)\mathrm{d}N_{T_t}, \\ \mathrm{d}r_t &= \left(\alpha - \beta r_t\right)\mathrm{d}t + \sigma_r \mathrm{d}W_t^r, \\ \mathrm{d}v_t &= \gamma\left(1 - v_t\right)\mathrm{d}t + \sigma_v \sqrt{v_t}\mathrm{d}W_t^v, \end{aligned} \quad (3.3)$$

where $\lambda_m = \lambda E\left(\mathrm{e}^{Y_t} - 1\right)$, $\mathrm{d}W_t \mathrm{d}W_t^r = \mathrm{d}W_t \mathrm{d}W_t^r = 0$ and $\mathrm{d}W_t \mathrm{d}W_t^v = \rho_v \mathrm{d}t$.

We can re-write the dynamic (3.3) in terms of three independent Brownian motions $\left(Z_t, Z_t^v, Z_t^r\right)$ follows (3.2), we get

$$\begin{aligned} \mathrm{d}S_t &= S_t\left(\left(r_t - \lambda_m v_t\right)\mathrm{d}t + \sigma\sqrt{v_t}\left(\rho_v dZ_t^v + \sqrt{1-\rho_v^2}\mathrm{d}Z_t\right)\right) \\ &\quad + S_{t-}\left(\mathrm{e}^{Y_t} - 1\right)\mathrm{d}N_{T_t}, \end{aligned} \quad (3.4)$$

$$\mathrm{d}r_t = \left(\alpha - \beta r_t\right)\mathrm{d}t + \sigma_r \mathrm{d}Z_t^r, \quad (3.5)$$

$$\mathrm{d}v_t = \gamma\left(1 - v_t\right)\mathrm{d}t + \sigma_v\sqrt{v_t}\mathrm{d}Z_t^v, \quad (3.6)$$

This decomposition makes it easier to perform a measure transformation. In fact, for any fixed maturity T, let us denote by Q^T the T-forward measure, *i.e*. the probability measure that is defined by the Radon-Nikodym derivative,

$$\frac{\mathrm{d}Q^T}{\mathrm{d}Q} = \frac{\exp\left(-\int_0^T r_u \mathrm{d}u\right)}{P(0,T)}. \quad (3.7)$$

Here, $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T and is defined as

$$P(t,T) = E_Q\left[\mathrm{e}^{-\int_t^T r_s ds} \middle| F_t\right]. \quad (3.8)$$

Next, Consider a continuous-time economy where interest rates are stochastic and satisfy (3.5). Since the SDE (3.5) satisfies all the necessary conditions of Theorem 32, see Protter [8], then the solution of (3.5) has the Markov property. As a consequence, the zero coupon bond price at time t under the measure Q in (3.8) satisfies

$$P(t,T) = E_Q\left[\exp\left(-\int_t^T r_s \mathrm{d}s\right) \middle\| r_t\right] \quad (3.9)$$

Note that $P(t,T)$ depends on r_t only instead of depending on all information available in F_t up to time t. As such, it becomes a function $F(t,r_t)$ of r_t,

$$P(t,T) = F(t,r_t),$$

meaning that the pricing problem can now be formulated as a search for the function $F(t,r_t)$.

Lemma 1 *The price of a zero coupon bond can be derived by computing the expectation* (3.9). *We obtain*

$$P(t,T) = \exp\left(a(t,T) + b(t,T)r_t\right) \quad (3.10)$$

where $$b(t,r_t) = \frac{1}{\beta}\left(\mathrm{e}^{-\beta(T-t)} - 1\right),$$

$$\begin{aligned} a(t,T) &= \left(\frac{\alpha}{\beta^2} - \frac{3\sigma_r^2}{4\beta^3}\right) - \frac{\sigma_r^2}{4\beta^3}\mathrm{e}^{-2\beta(T-t)} \\ &\quad + \left(\frac{\sigma_r^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)\mathrm{e}^{-\beta(T-t)} + \left(\frac{\sigma_r^2}{2\beta^2} - \frac{\alpha}{\beta^2}\right)(T-t) \end{aligned}.$$

Proof. See Privault [9] (pp. 38-39).

Lemma 2 *The process* r_t *following the dynamics in* (3.5) *can be written in the form*

$$r_t = x_t + w(t), \textit{ for each } t \quad (3.11)$$

where the process x_t *satisfies*

$$\mathrm{d}x_t = -\beta x_t \mathrm{d}t + \sigma_r \mathrm{d}Z_t^r, \; x_0 = 0. \quad (3.12)$$

Moreover, the function $w(t)$ *is deterministic and well defined in the time interval* $[0,T]$ *which satisfied*

$$w(t) = r_0\mathrm{e}^{-\beta t} + \frac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right) \quad (3.13)$$

In particular, $w(0) = r_0$.

Proof. To solve the solution of SDE (3.5),

Let $g(t,r) = e^{\beta t}r$ and using Ito's Lemma

$$\mathrm{d}g = \frac{\partial g}{\partial t}\mathrm{d}t + \frac{\partial g}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\left(\mathrm{d}r\right)^2,$$

Then,

$$\begin{aligned} \mathrm{d}e^{\beta t}r_t &= \beta e^{\beta t}r_t \mathrm{d}t + e^{\beta t}\left(\left(\alpha - \beta r_t\right)\mathrm{d}t + \sigma_r \mathrm{d}Z_t^r\right) \\ &= \alpha e^{\beta t}\mathrm{d}t + e^{\beta t}\sigma_r \mathrm{d}Z_t^r, \end{aligned} \quad (3.14)$$

Integrated on both side the above equation from 0 to t where $0 < t \le T$ and simplified, one get

$$r_t = r_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma_r\int_0^t e^{-\beta(t-u)}\mathrm{d}Z_u^r$$

By using the definition of $w(t)$ form (3.13),

$$r_t = w(t) + \sigma_r \int_0^t e^{-\beta(t-u)} \mathrm{d}Z_u^r \tag{3.15}$$

where $w(t) = r_0 \mathrm{e}^{-\beta t} + \frac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right)$.

Note that the solution of (3.12) is

$$x_t = x_0 \mathrm{e}^{-\beta t} + \sigma_r \int_0^t \mathrm{e}^{-\beta(t-u)} \mathrm{d}Z_u^r = \sigma_r \int_0^t \mathrm{e}^{-\beta(t-u)} \mathrm{d}Z_u^r . \tag{3.16}$$

Hence, $r_t = w(t) + x_t$ for each t. The proof is now complete.

Next we shall calculate the Radon-Nikodym derivative as appear in (3.7). By Lemma 1 and 2, we have $r_t = x_t + w(t)$ and $P(0,T)$. Substituting r_t and $P(0,T)$ into (3.7), we have

$$\begin{aligned}\frac{\mathrm{d}Q^T}{\mathrm{d}Q} &= \frac{\exp\left(-\int_0^T x_u + w(u)\mathrm{d}u\right)}{\exp\left(a(0,T) + b(0,T) r_0\right)} \\ &= \exp\left(-\int_0^T x_u \mathrm{d}u - \frac{\sigma^2}{2\beta^2}\int_0^T \left(1 - e^{-\beta(T-u)}\right)^2 \mathrm{d}u\right).\end{aligned} \tag{3.17}$$

Stochastic integration by parts implies that

$$\int_0^T x_u \mathrm{d}u = T x_T - \int_0^T u \mathrm{d}x_u = \int_0^T (T-u)\mathrm{d}x_u . \tag{3.18}$$

By substituting the expression for dx_u from (3.12),

$$\begin{aligned}&\int_0^T (T-u)\mathrm{d}x_u \\ &= -\beta \int_0^T (T-u) x_u \mathrm{d}u + \sigma_r \int_0^T (T-u)\mathrm{d}Z_u^r\end{aligned} \tag{3.19}$$

Moreover by substituting the expression for x_u from (3.16), the first integral on the right hand side of (3.19) becomes

$$\begin{aligned}&-\beta \int_0^T (T-u) x_u \mathrm{d}u \\ &= -\beta\sigma_r \int_0^T \left((T-u)\int_o^u e^{-\beta(u-s)} \mathrm{d}Z_u^r\right)\mathrm{d}u\end{aligned} \tag{3.20}$$

Using integral by parts, we have (Equation 3.21)
Substituting (3.21) into (3.19), we obtain

$$\int_0^T (T-u)\mathrm{d}x_u = -\frac{\sigma_r}{\beta}\left[\int_0^T \left(e^{-\beta(T-u)} - 1\right)\mathrm{d}Z_u^r\right]$$

Hence, $\int_0^T x_u \mathrm{d}u = -\frac{\sigma_r}{\beta}\left[\int_0^T \left(\mathrm{e}^{-\beta(T-u)} - 1\right)\mathrm{d}Z_u^r\right]$ (3.22)

Substituting (3.22) into (3.17), once get

$$\begin{aligned}&\frac{\mathrm{d}Q^T}{\mathrm{d}Q} \\ &= \exp\left(\frac{\sigma_r}{\beta}\int_0^T \left(1 - \mathrm{e}^{-\beta(T-u)}\right)\mathrm{d}Z_u^r - \frac{\sigma_r^2}{2\beta^2}\int_0^T \left(1 - \mathrm{e}^{-\beta(T-u)}\right)^2 \mathrm{d}u\right)\end{aligned} \tag{3.23}$$

The Girsanov theorem then implies that the three processes Z_t^{rT}, Z_t^{vT} and Z_t^T defined by

$$\begin{aligned}\mathrm{d}Z_t^{rT} &= \mathrm{d}Z_t^r + \frac{\sigma_r}{\beta}\left(1 - \mathrm{e}^{-\beta(T-t)}\right)\mathrm{d}t \\ \mathrm{d}Z_t^{vT} &= \mathrm{d}Z_t^v,\ \mathrm{d}Z_t^T = \mathrm{d}Z_t\end{aligned} \tag{3.24}$$

are three independent Brownian motions under the measure Q^T. Therefore, the dynamics of r_t, v_t and S_t under Q^T are given by

$$\begin{aligned}\mathrm{d}S_t &= S_t\left(\left(r_t - \lambda_m v_t\right)\mathrm{d}t + \rho_v \sigma \sqrt{v_t}\mathrm{d}Z_t^{vT} + \sigma\sqrt{v_t\left(1-\rho_v^2\right)}\mathrm{d}Z_t^T\right) \\ &\quad + S_{t-}\left(e^{Y_t} - 1\right)\mathrm{d}N_{T_t}, \\ \mathrm{d}r_t &= \left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\mathrm{d}t + \sigma_r \mathrm{d}Z_t^{rT}, \\ \mathrm{d}v_t &= \gamma\left(1 - v_t\right)\mathrm{d}t + \sigma_v \sqrt{v_t}\mathrm{d}Z_t^{vT},\end{aligned} \tag{3.25}$$

4. The Pricing of a European Call Option on the Given Asset

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $\left(\Omega, F, F_t, Q^T\right)$, F_t is usually taken to be the price history up to time t. All processes in this section will be defined in this space. We denote C the price at time t of a European u call option on the current price of an underlying asset S_t with strike price K and expiration time T.

$$\begin{aligned}&-\beta\sigma_r \int_0^T \left((T-u)\int_o^u e^{-\beta(u-s)}\mathrm{d}Z_u^r\right)\mathrm{d}u \\ &= -\beta\sigma_r \int_0^T \left(\int_0^u e^{\beta s}\mathrm{d}Z_s^r\right)(T-u)e^{-\beta u}\mathrm{d}u = -\beta\sigma_r \int_0^T \left(\int_0^u e^{\beta s}\mathrm{d}Z_s^r\right)\mathrm{d}\left(\int_0^u (T-v)e^{-\beta v}\mathrm{d}v\right) \\ &= -\beta\sigma_r \left[\left(\int_0^T e^{\beta u}\mathrm{d}Z_u^r\right)\left(\int_0^T (T-v)e^{-\beta v}\mathrm{d}v\right) - \int_0^T \left(\int_0^u (T-v)e^{-\beta v}\mathrm{d}v\right)e^{\beta u}\mathrm{d}Z_u^r\right] \\ &= -\beta\sigma_r \left[\int_0^T e^{\beta u}\left(\int_0^u (T-v)e^{-\beta v}\mathrm{d}v\right)\mathrm{d}Z_u^r\right] = -\frac{\sigma_r}{\beta}\left[\int_0^T \left(e^{-\beta(T-u)} - 1\right)\mathrm{d}Z_u^r\right] - \sigma_r \int_0^T (T-u)\mathrm{d}Z_u^r.\end{aligned} \tag{3.21}$$

The terminal payoff of a European option on the underlying stock S_t with strike price K is

$$\max\left(S_T-K,0\right) \tag{4.1}$$

This means the holder will exercise his right only $S_T>K$ and then his gain is S_T-K. Otherwise, if $S_T\le K$ then the holder will buy the underlying asset from the market and the value of the option is zero.

We would like to find a formula for pricing a European call option with strike price K and maturity T based on the model (3.25). Consider a continuous-time economy where interest rates are stochastic and the price of the European call option at time t under the T-forward measure Q^T is

$$C\left(t,S_t,r_t,v_t;T,K\right)=P\left(t,T\right)E_{Q^T}\left(\max\left(S_T-K,0\right)\middle|S_t,r_t,v_t\right)$$
$$=P(t,T)\int_0^\infty \max\left(S_T-K,0\right)p_{Q^T}\left(S_T\left|S_t,r_t,v_t\right.\right)\mathrm{d}S_T$$

where E_{Q^T} is the expectation with respect to the T-forward probability measure, p_{Q^T} is the corresponding conditional density given $\left(S_t,r_t,v_t\right)$ and P is a zero coupon bond which is defined in Lemma 1.

With a change in variable $X_t=\ln S_t$,

$$\begin{aligned}
&C\left(t,S_t,r_t,v_t;T,K\right)\\
&=P\left(t,T\right)\int_{-\infty}^{\infty}\max\left(\mathrm{e}^{X_T}-K,0\right)p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&=P\left(t,T\right)\int_{\ln K}^{\infty}\left(\mathrm{e}^{X_T}-K\right)1_{X_T\ge\ln K}\,p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&=P\left(t,T\right)\int_{\ln K}^{\infty}\mathrm{e}^{X_T}p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&\quad-KP\left(t,T\right)\int_{\ln K}^{\infty}p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&=\mathrm{e}^{X_t}\left(\frac{1}{E_{Q^T}\left(\mathrm{e}^{X_T}\left|S_t,r_t,v_t\right.\right)}\int_{\ln K}^{\infty}\mathrm{e}^{X_T}p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)vX_T\right)\\
&\quad-KP\left(t,T\right)\int_{\ln K}^{\infty}p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&=\mathrm{e}^{X_t}\left(\int_{\ln K}^{\infty}e^{X_T}\frac{p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)}{E_{Q^T}(\mathrm{e}^{X_T}\left|S_t,r_t,v_t\right.)}\mathrm{d}X_T\right)\\
&\quad-KP\left(t,T\right)\int_{\ln K}^{\infty}p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T
\end{aligned}\tag{4.2}$$

With the first integrand in (4.2) being positive and integrating up to one. The first integrand therefore defines a new probability measure that we denote by q_{Q^T} below

$$\begin{aligned}
&C\left(t,S_t,r_t,v_t;T,K\right)\\
&=\mathrm{e}^{X_t}\int_{\ln K}^{\infty}q_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&\quad-KP\left(t,T\right)\int_{\ln K}^{\infty}p_{Q^T}\left(X_T\left|X_t,r_t,v_t\right.\right)\mathrm{d}X_T\\
&=\mathrm{e}^{X_t}\mathrm{P}_1\left(t,X_t,r_t,v_t;T,K\right)-KP\left(t,T\right)\mathrm{P}_2\left(t,X_t,r_t,v_t;T,K\right)\\
&=\mathrm{e}^{X_t}\,\Pr\left(X_T>\ln K\left|X_t,r_t,v_t\right.\right)\\
&\quad-KP\left(t,T\right)\Pr\left(X_T>\ln K\left|X_t,r_t,v_t\right.\right)
\end{aligned}\tag{4.3}$$

where those probabilities in (4.3) are calculated under the probability measure Q^T.

The European call option for log asset price $X_t=\ln S_t$ will be denoted by

$$\begin{aligned}\hat{C}\left(t,X_t,r_t,v_t;T,\kappa\right)&=\mathrm{e}^{X_t}\tilde{\mathrm{P}}_1\left(t,X_t,r_t,v_t;T,\kappa\right)\\&\quad-\mathrm{e}^{\kappa}P\left(t,T\right)\tilde{\mathrm{P}}_2\left(t,X_t,r_t,v_t;T,\kappa\right)\end{aligned}\tag{4.4}$$

where $\kappa=\ln K$ and

$$\tilde{\mathrm{P}}_j\left(t,X_t,r_t,v_t;T,\kappa\right) := \mathrm{P}_j\left(t,X_t,r_t,v_t;T,K\right),\quad j=1,2.$$

Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 4. The following lemma shows the relationship between $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of (4.4).

Lemma 3 *The functions* $\tilde{P}_1$ *and* $\tilde{P}_2$ *in the option values of* (4.4) *satisfy the PIDEs* (4.5):
and subject to the boundary condition at expiration t= T

$$\tilde{P}_1\left(T,x,r,v;T,\kappa\right)=1_{x>\kappa}. \tag{4.6}$$

Moreover, $\tilde{P}_2$ *satisfies the Equation* (4.7)

$$0=\frac{\partial\tilde{P}_1}{\partial t}+A\left[\tilde{P}_1\right]+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial\tilde{P}_1}{\partial v}+v\int_{-\infty}^{\infty}\left[\left(\mathrm{e}^y-1\right)\left(\tilde{P}_1\left(t,x+y,r,v;T,\kappa\right)-\tilde{P}_1\left(x,t,r,v;T,\kappa\right)\right)\right]k\left(y\right)\mathrm{d}y \tag{4.5}$$

$$\begin{aligned}
0=&\frac{\partial\tilde{P}_2}{\partial t}+A\left[\tilde{P}_2\right]-\sigma^2v\frac{\partial v}{\partial x}+\frac{\sigma^2v}{2}\frac{\partial^2\tilde{P}_2}{\partial x^2}+b\left(t,T\right)\sigma_r^2\frac{\partial\tilde{P}_2}{\partial r}\\
&+\tilde{P}_2\left(\frac{\partial a\left(t,T\right)}{\partial t}+r\left(\frac{\partial b\left(t,T\right)}{\partial t}-1\right)+\frac{\sigma_r^2}{2}b^2\left(t,T\right)\right)+\tilde{P}_2\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-\mathrm{e}^{-\beta(T-t)}\right)\right)b\left(t,T\right)\\
&+v\int_{-\infty}^{\infty}\left(\tilde{P}_2\left(t,x+y,r,v;T,\kappa\right)-\tilde{P}_2\left(t,x,r,v;T,\kappa\right)-\frac{\partial\tilde{P}_2}{\partial x}(\mathrm{e}^y-1)\right)k\left(y\right)\mathrm{d}y
\end{aligned}\tag{4.7}$$

and subject to the boundary condition at expiration t= T

$$\tilde{P}_2(T,x,r,v;T,\kappa)=1_{x>\kappa}. \tag{4.8}$$

where for i=1,2

$$\begin{aligned}A[\tilde{P}_i]=&\left(r+\frac{1}{2}\sigma^2 v\right)\frac{\partial\tilde{P}_{i1}}{\partial x}+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)\frac{\partial\tilde{P}_i}{\partial r}\\&+\gamma(1-v)\frac{\partial\tilde{P}_i}{\partial v}+\frac{\sigma_v^2 v}{2}\frac{\partial^2\tilde{P}_i}{\partial v^2}+\frac{\sigma^2 v}{2}\frac{\partial^2\tilde{P}_i}{\partial x^2}+\frac{\sigma_r^2}{2}\frac{\partial^2\tilde{P}_i}{\partial r^2}+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial\tilde{P}_i}{\partial v\partial x}\\&+v\int_{-\infty}^{\infty}\left[\tilde{P}_i(t,x+y,r,v;T,\kappa)-\tilde{P}_i(x,t,r,v;T,\kappa)-\left(\frac{\partial\tilde{P}_i}{\partial x}\right)\left(e^y-1\right)\right]k(y)\,dy\end{aligned} \tag{4.9}$$

Note that $1_{x>\kappa}=1$ *if* $x>\kappa$ *and otherwise* $1_{x>\kappa}=0$.

Proof. See Appendix A.

5. The Closed-Form Solution for European Call Options

For j =1,2 the characteristic function for $\tilde{P}_j(t,x,r,v;T,\kappa)$, with respect to the variable κ, are defined by

$$f_j(t,x,r,v;T,u):=-\int_{-\infty}^{\infty}e^{iu\kappa}d\tilde{P}_j(t,x,r,v;T,\kappa), \tag{5.1}$$

with a minus sign to account for the negativity of the measure $d\tilde{P}_j$. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t}+A_j\left[f_j\right](t,x,r,v;T,\kappa)=0, \tag{5.2}$$

with the respective boundary conditions

$$\begin{aligned}f_j(T,x,r,v;T,u)&=-\int_{-\infty}^{\infty}e^{iu\kappa}d\tilde{P}_j(t,x,r,v;T,\kappa)\\&=-\int_{-\infty}^{\infty}e^{iu\kappa}\left(-\delta(\kappa-x)\right)d\kappa=e^{iux}.\end{aligned}$$

Since $d\tilde{P}_j(t,x,r,v;T,\kappa)=d1_{x>\kappa}=-\delta(\kappa-x)\,d\kappa$

The following lemma shows how to calculate the characteristic functions for $\tilde{P}_1$ and $\tilde{P}_2$ as they appeared in Lemma 3.

Lemma 4 *The functions* $\tilde{P}_1$ *and* $\tilde{P}_2$ *can be calculated by the inverse Fourier transformations of the characteristic function, i.e.*

$$\tilde{P}_j(t,x,r,v;T,\kappa)=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{\infty}\mathrm{Re}\left[\frac{e^{iu\kappa}f_j(t,x,r,v;T,u)}{iu}\right]du,$$

for $j=1,2,$ *with Re*[.] *denoting the real component of a complex number.*

By letting $\tau=T-t$, *the characteristic function* f_j *is given by*

$$f_j(t,x,r,v;t+\tau,u)=\exp\left(iux+B_j(\tau)+rC_j(\tau)+vE_j(\tau)\right),$$

where

$$\tilde{b}_{j1}=b_{j2}+\nabla_j,\quad \tilde{b}_{j2}=b_{j2}-\nabla_j,\quad b_{j1}=\frac{\sigma_v^2}{2},$$

$$b_{12}=\rho_v\sigma\sigma_v(1+iu)-\gamma,\ b_{22}=\rho_v\sigma\sigma_v iu-\gamma,$$

$$b_{10}=-\left(\frac{1}{2}\sigma^2\left(iu-u^2\right)+\int_{-\infty}^{\infty}\left(e^{iux+y}-iu\left(e^y-1\right)\right)k(y)\,dy\right)$$

$$b_{20}=-\left(\frac{1}{2}\sigma^2\left(u^2+iu\right)-\int_{-\infty}^{\infty}\left(e^{iux}-iu\left(e^y-1\right)\right)k(y)\,dy\right)$$

$$\begin{aligned}B_1(\tau)=&\frac{-\tau\gamma\tilde{b}_{12}}{2b_{11}}+\frac{\gamma\left(\tilde{b}_{11}+\tilde{b}_{12}\right)}{2b_{11}\nabla_1}\ln\left(\frac{\tilde{b}_{11}+e^{\tau\nabla_1}\tilde{b}_{12}}{\tilde{b}_{11}+\tilde{b}_{12}}\right)\\&+\frac{iu\sigma_r^2}{2\beta^3}\left(e^{-\beta\tau}-1\right)^2+\left(2iu\left(\alpha\beta-\sigma_r^2\right)-\sigma_r^2u^2\right)\frac{\tau}{\beta^2}\\&+\frac{\sigma_r^2u^2}{4\beta^3}\left(-4e^{-\beta\tau}+e^{-2\beta\tau}-3\right)\end{aligned}$$

$$C_j(\tau)=\frac{iu}{\beta}\left(1-e^{-\beta\tau}\right),\nabla_j=\sqrt{b_{j2}^2-4b_{j0}b_{j1}}$$

$$E_j(\tau)=\frac{\tilde{b}_{j1}\tilde{b}_{j2}\left(e^{\tau\sqrt{b_{j2}^2-4b_{j0}b_{j1}}}-1\right)}{2b_{j1}\left(\tilde{b}_{j1}+\tilde{b}_{j2}e^{\tau\sqrt{b_{j2}^2-4b_{j0}b_{j1}}}\right)}.$$

$$\begin{aligned}B_2(\tau)=&\frac{-\tau\gamma\tilde{b}_{22}}{2b_{21}}+\frac{\gamma\left(\tilde{b}_{22}+\tilde{b}_{21}\right)}{2b_{21}\nabla_2}\ln\left(\frac{\tilde{b}_{21}+e^{\tau\nabla_2}\tilde{b}_{22}}{\tilde{b}_{21}+\tilde{b}_{22}}\right)\\&+\left(\frac{\sigma_r^2}{\beta^3}\left(u^2+4iu-2\right)-\frac{iu\alpha}{\beta^2}\right)\left(1-e^{-\beta\tau}\right))\\&+\frac{\sigma_r^2}{4\beta^3}\left(u^2+4iu-2\right)\left(e^{-2\beta\tau}-1\right)\\&+\left(\frac{\alpha}{\beta^2}\left(iu\beta-\beta+1\right)-\frac{\sigma_r^2}{2\beta^2}\left(u^2+4iu-2\right)\right)\tau\end{aligned}$$

Proof. See Appendix B.

In summary, we have just proved the following main theorem.

Theorem 5 *The value of a European call option of SDE* (3.25) *is*

$$C(t,S_t,r_t,v_t;T,K)$$
$$= S_t\tilde{P}_1(t,X_t,r_t,v_t;T,\kappa) - KP(t,T)\tilde{P}_2(t,X_t,r_t,v_t;T,\kappa)$$

where $\tilde{P}_1$ *and* $\tilde{P}_2$ *are given in Lemma* 4 *and* $P(t,T)$ *is given in Lemma* 1.

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Appendix A: Proof of Lemma 3

By Ito's lemma, $\hat{C}(t,x,r,v)$ follows the partial integro-differential equation (PIDE)

$$\frac{\partial \hat{C}}{\partial t}+L_t^D\hat{C}+L_t^J\hat{C}=0, \quad (A.1)$$

where

$$L_t^D\hat{C}=\left(r-\frac{1}{2}\sigma^2 v\right)\frac{\partial \hat{C}}{\partial x}+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-\mathrm{e}^{-\beta(T-t)}\right)\right)\frac{\partial \hat{C}}{\partial r}$$
$$+\gamma(1-v)\frac{\partial \hat{C}}{\partial v}+\frac{\sigma_v^2 v}{2}\frac{\partial^2 \hat{C}}{\partial v^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 \hat{C}}{\partial x^2}+\frac{\sigma_r^2}{2}\frac{\partial^2 \hat{C}}{\partial r^2}$$
$$+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial^2 \hat{C}}{\partial x\partial v}-r\hat{C}$$

and

$$L_t^J\hat{C}$$
$$=v\int_{-\infty}^{\infty}\left(\hat{C}(t,x+y,r,v)-\hat{C}(t,x,r,v)-\frac{\partial \hat{C}}{\partial x}\left(\mathrm{e}^y-1\right)\right)k(y)\,\mathrm{d}y$$

where $k(y)$ is the Lévy density.

We plan to substitute (4.4) into (A.1). Firstly, we compute

$$\frac{\partial \hat{C}}{\partial t}=\mathrm{e}^x\frac{\partial \tilde{P}_1}{\partial t}-\mathrm{e}^\kappa P(t,T)\left[\frac{\partial \tilde{P}_2}{\partial t}+\tilde{P}_2\frac{\partial}{\partial t}\left(a(t,T)+b(t,T)r\right)\right],$$

$$\frac{\partial \hat{C}}{\partial x}=\mathrm{e}^x\left(\frac{\partial \tilde{P}_1}{\partial x}+\tilde{P}_1\right)-\mathrm{e}^\kappa P(t,T)\frac{\partial \tilde{P}_2}{\partial x},$$

$$\frac{\partial \hat{C}}{\partial v}=\mathrm{e}^x\frac{\partial \tilde{P}_1}{\partial v}-e^\kappa P(t,T)\frac{\partial \tilde{P}_2}{\partial v},$$

$$\frac{\partial \hat{C}}{\partial r}=\mathrm{e}^x\frac{\partial \tilde{P}_1}{\partial r}-\mathrm{e}^\kappa P(t,T)\left(\frac{\partial \tilde{P}_2}{\partial r}+\tilde{P}_2 b(t,T)\right),$$

$$\frac{\partial^2 \hat{C}}{\partial x^2}=\mathrm{e}^x\left(\frac{\partial^2 \tilde{P}_1}{\partial x^2}+2\frac{\partial \tilde{P}_1}{\partial x}+\tilde{P}_1\right)-\mathrm{e}^\kappa P(t,T)\frac{\partial^2 \tilde{P}_2}{\partial x^2},$$

$$\frac{\partial^2 \hat{C}}{\partial v^2}=\mathrm{e}^x\frac{\partial^2 \tilde{P}_1}{\partial v^2}-\mathrm{e}^\kappa P(t,T)\frac{\partial^2 \tilde{P}_2}{\partial v^2},$$

$$\frac{\partial^2 \hat{C}}{\partial v\partial x}=\mathrm{e}^x\left(\frac{\partial \tilde{P}_1}{\partial v\partial x}+\frac{\partial \tilde{P}_1}{\partial v}\right)-\mathrm{e}^\kappa P(t,T)\frac{\partial \tilde{P}_2}{\partial v\partial x},\ \frac{\partial^2 \hat{C}}{\partial r^2}=\mathrm{e}^x\frac{\partial^2 \tilde{P}_1}{\partial r^2}$$
$$-\mathrm{e}^\kappa P(t,T)\left(\frac{\partial^2 \tilde{P}_2}{\partial r^2}+2b(t,T)\frac{\partial \tilde{P}_2}{\partial r}+\tilde{P}_2 b^2(t,T)\right),.$$

$$\frac{\partial^2 \hat{C}}{\partial x\partial r}=\mathrm{e}^x\left(\frac{\partial^2 \tilde{P}_1}{\partial x\partial r}+\frac{\partial \tilde{P}_1}{\partial r}\right)-\mathrm{e}^\kappa P(t,T)\left(\frac{\partial^2 \tilde{P}_2}{\partial x\partial r}+b(t,T)\frac{\partial \tilde{P}_2}{\partial x}\right),$$

$$\hat{C}(t,x+y,r,v,;T,\kappa)-\hat{C}(t,x,r,v,;T,\kappa)$$
$$=\mathrm{e}^x\left[\left(\mathrm{e}^y-1\right)\tilde{P}_1(t,x+y,r,v;T,\kappa)+\left(\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]$$
$$-\mathrm{e}^\kappa P(t,T)\left[\tilde{P}_2(t,x+y,r,v;T,\kappa)-\tilde{P}_2(t,x,r,v;T,\kappa)\right].$$

Substitute all terms above into (A.1) and separate it by assumed independent terms of $\tilde{P}_1$ and $\tilde{P}_2$. This gives two PIDEs for the T-forward probability for $\tilde{P}_i(t,x,r,v;T,\kappa),\ j=1,2$:

$$\frac{\partial \tilde{P}_1}{\partial t}+\left(r+\frac{1}{2}\sigma^2 v\right)\frac{\partial \tilde{P}_1}{\partial x}+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-\mathrm{e}^{-\beta(T-t)}\right)\right)\frac{\partial \tilde{P}_1}{\partial r}$$
$$+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial \tilde{P}_1}{\partial v\partial x}+\frac{\sigma_v^2 v}{2}\frac{\partial^2 \tilde{P}_1}{\partial v^2}+\frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_1}{\partial x^2}+\frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_1}{\partial r^2}+\left(\gamma(1-v)+\rho_v\sigma v\sigma_v\right)\frac{\partial \tilde{P}_1}{\partial v}$$
$$+v\int_{-\infty}^{\infty}\left[\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)-\left(\frac{\partial \tilde{P}_1}{\partial x}\right)\left(\mathrm{e}^y-1\right)\right]k(y)\,\mathrm{d}y.$$
$$+v\int_{-\infty}^{\infty}\left[\left(\mathrm{e}^y-1\right)\left(\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]k(y)\,\mathrm{d}y.=0 \quad (A.2)$$

and subject to the boundary condition at the expiration time $t=T$ according to (4.6).

By using the notation in (4.9), then (A.2) becomes Equation (A.3)

$$0=\frac{\partial \tilde{P}_1}{\partial t}+A\left[\tilde{P}_1\right]+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial \tilde{P}_1}{\partial v}+v\int_{-\infty}^{\infty}\left[\left(\mathrm{e}^y-1\right)\left(\tilde{P}_1(t,x+y,r,v;T,\kappa)-\tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]k(y)\,\mathrm{d}y.$$
$$:=\frac{\partial \tilde{P}_1}{\partial t}+A_1\left[\tilde{P}_1\right]. \quad (A.3)$$

For $\tilde{P}_2(t,x,r,v;T,\kappa)$:

$$0=\frac{\partial \tilde{P}_2}{\partial t}+\left(r-\frac{1}{2}\sigma^2 v\right)\frac{\partial \tilde{P}_2}{\partial x}+\gamma(1-v)\frac{\partial \tilde{P}_2}{\partial v}+\left(\rho_v\sigma v\sigma_v\right)\frac{\partial \tilde{P}_2}{\partial v\partial x}+\frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_2}{\partial x^2}+\frac{\sigma_v^2 v}{2}\frac{\partial^2 \tilde{P}_2}{\partial v^2}+\frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_2}{\partial r^2}+\left(\frac{\sigma_r^2}{2}b^2(t,T)-r\right)\tilde{P}_2$$
$$+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)+2b(t,T)\frac{\sigma_r^2}{2}\right)\left(\frac{\partial \tilde{P}_2}{\partial r}\right)+\tilde{P}_2\left(\frac{\partial a(t,T)}{\partial t}+r\frac{\partial b(t,T)}{\partial t}+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-\mathrm{e}^{-\beta(T-t)}\right)\right)b(t,T)\right)$$
$$+v\int_{-\infty}^{\infty}\left(\tilde{P}_2(t,x+y,r,v;T,\kappa)-\tilde{P}_2(t,x,r,v;T,\kappa)-\frac{\partial \tilde{P}_2}{\partial x}\left(\mathrm{e}^y-1\right)\right)k(y)\,\mathrm{d}y. \tag{A.4}$$

and subject to the boundary condition at expiration time $t = T$ according to (4.8).

Again, by using the notation (4.9), then (A.4) becomes

$$0=\frac{\partial \tilde{P}_2}{\partial t}+A\left[\tilde{P}_2\right]-\sigma^2 v\frac{\partial \tilde{P}_2}{\partial x}+\frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_2}{\partial x^2}+b(t,T)\sigma_r^2\frac{\partial \tilde{P}_2}{\partial r}+\tilde{P}_2\left(\frac{\partial a(t,T)}{\partial t}+r\left(\frac{\partial b(t,T)}{\partial t}-1\right)+\frac{\sigma_r^2}{2}b^2(t,T)\right)$$
$$+\tilde{P}_2\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}\left(1-\mathrm{e}^{-\beta(T-t)}\right)\right)b(t,T):=\frac{\partial \tilde{P}_2}{\partial t}+A_2\left[\tilde{P}_2\right] \tag{A.5}$$

The proof is now completed.

Appendix B: Proof of Lemma 4

To solve the characteristic function explicitly, letting $\tau = T - t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1(t,x,r,v;t+\tau,u) = \exp\left(\mathrm{iux} + B_1(\tau) + rC_1(\tau) + vE_1(\tau)\right), \tag{B.1}$$

and the boundary condition $B_1(0) = C_1(0) = E_1(0) = 0$. This conjecture exploits the linearity of the coefficient in PIDEs (5.2).

Note that the characteristic function of f_1 always exists. In order to substitute (B.1) into (5.2), firstly, we compute

$$\frac{\partial f_1}{\partial t} = -\left(B_1'(\tau) + rC_1'(\tau) + vE_1'(\tau)\right) f_1,\quad \frac{\partial f_1}{\partial x} = iuf_1,$$

$$\frac{\partial f_1}{\partial r} = C_1(\tau)\, f_1,\quad \frac{\partial f_1}{\partial v} = E_1(\tau)\, f_1,\quad \frac{\partial^2 f_1}{\partial x^2} = -u^2 f_1,$$

$$\frac{\partial^2 f_1}{\partial v^2} = E_1^2(\tau)\, f_1,\ \frac{\partial^2 f_1}{\partial r^2} = C_1^2(\tau)\, f_1,$$

$$\frac{\partial^2 f_1}{\partial x \partial r} = iuC_1(\tau)\, f_1,\quad \frac{\partial^2 f_1}{\partial v \partial x} = iuE_1(\tau)\, f_1,$$

$$e^{iux} f_1(t,x,r,v;t+\tau,u) = f_1(t,x+y,r,v;t+\tau,u) - f_1(t,x,r,v;t+\tau,u)$$

Substituting all the above terms into (5.2), after cancelling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned}0 &= r\left[-C_1'(\tau) + iu - \beta C_1(\tau)\right]\\ &+ v\left[-E_1'(\tau) + \left(\rho_v \sigma\sigma_v(1+iu) - \gamma\right)E_1(\tau) + \frac{\sigma_v^2}{2}E_1^2(\tau) + \frac{\sigma^2}{2}\left(iu - u^2\right) + \int_{-\infty}^{\infty}\left[\mathrm{e}^{iux} - iu\left(\mathrm{e}^y - 1\right) + \left(\mathrm{e}^y - 1\right)\mathrm{e}^{iux}\right]k(y)\,\mathrm{d}y\right]\\ &+ \left[-B_1'(\tau) + \left(\alpha - \frac{\sigma_r^2}{\beta}\left(1 - \mathrm{e}^{-\beta(T-t)}\right)\right)C_1(\tau) + \frac{\sigma_r^2}{2}C_1^2(\tau) + \gamma E_1(\tau)\right]\end{aligned}$$

By separating the order r, v and ordering the remaining terms, we can reduce it to three ordinary differential equations (ODEs) as follows:

$$C_1'(\tau) = -\beta C_1(\tau) + iu, \tag{B.2}$$

$$E_1'(\tau) = \frac{\sigma_v^2}{2}E_1^2(\tau) + \left(\rho_v\sigma\sigma_v(1+iu) - \gamma\right)E_1(\tau) + \frac{\sigma^2}{2}\left(iu - u^2\right) + \int_{-\infty}^{\infty}\left(\mathrm{e}^{iux+y} - iu\left(e^y - 1\right)\right)k(y)\,\mathrm{d}y, \tag{B.3}$$

$$B_1'(\tau) = \left[\alpha - \frac{\sigma_r^2}{\beta}\left(1 - \mathrm{e}^{-\beta(T-t)}\right)\right]C_1(\tau) + \gamma E_1(\tau) + \frac{\sigma_r^2}{2}C_1^2(\tau). \tag{B.4}$$

It is clear from (B.2) and $C(0) = 0$ that

$$C_1(\tau) = \frac{iu}{\beta}\times\left(1 - \mathrm{e}^{-\beta\tau}\right), \tag{B.5}$$

Let

$$b_1 = \frac{\sigma_v^2}{2},$$

$$b_2 = \rho_v\sigma\sigma_v(1+iu) - \gamma,$$

$$b_0 = -\left(\frac{1}{2}\sigma^2\left(iu - u^2\right) + \int_{-\infty}^{\infty}\left(\mathrm{e}^{iux+y} - iu\left(\mathrm{e}^y - 1\right)\right)k(y)\,\mathrm{d}y\right)$$

and substitute all term above into (B.3). we get

$$E_1'(\tau) = b_1\left(E_1(\tau) - \frac{-b_2 + \sqrt{b_2^{\,2} - 4b_0b_1}}{2b_1}\right)\times\left(E_1(\tau) - \frac{-b_2 - \sqrt{b_2^{\,2} - 4b_0b_1}}{2b_1}\right)$$

By method of variable separation, we have

$$\frac{\mathrm{d}E_1(\tau)}{\left(E_1(\tau) - \frac{-b_2 + \sqrt{b_2^{\,2} - 4b_0b_1}}{2b_1}\right)\left(E_1(\tau) - \frac{-b_2 - \sqrt{b_2^{\,2} - 4b_0b_1}}{2b_1}\right)} = b_1\mathrm{d}\tau$$

Using partial fraction on the left hand side, we get

$$\left(\frac{1}{\left(E_1(\tau)-\frac{-b_2+\nabla}{2b_1}\right)}-\frac{1}{\left(E_1(\tau)-\frac{-b_2-\nabla}{2b_1}\right)}\right)\mathrm{d}E_1(\tau)=\nabla\mathrm{d}\tau$$

where $\nabla=\sqrt{b_2{}^2-4b_0b_1}$.

Integrating both sides, we have

$$\ln\left(\frac{E_1(\tau)-\frac{-b_2+\nabla}{2b_1}}{E_1(\tau)-\frac{-b_2-\nabla}{2b_1}}\right)=\tau\nabla+E_0$$

Using boundary condition $E_1(\tau=0)=0$ we get

$$E_0=\ln\left(\frac{-b_2+\nabla}{-b_2-\nabla}\right)$$

Solving for $E_1(\tau)$, we obtain

$$E_1(\tau)=\frac{\left(\mathrm{e}^{\tau\nabla}-1\right)\tilde{b}_1\tilde{b}_2}{2b_1\left(\tilde{b}_1+\mathrm{e}^{\tau\nabla}\tilde{b}_2\right)} \quad \text{(B.6)}$$

where $\tilde{b}_1=b_2+\nabla,\quad \tilde{b}_2=b_2-\nabla$.

In order to solve $B_1(\tau)$ explicitly, we substitute $C_1(\tau)$ and $E_1(\tau)$ in (B.5) and (B.6) into (B.4) .

$$B'_1(\tau)=\left(\frac{iu\alpha}{\beta}-\frac{iu\sigma_r^2}{\beta^2}\right)\left(1-\mathrm{e}^{-\beta\tau}\right)+\frac{iu\sigma_r^2}{\beta^2}\left(\mathrm{e}^{-\beta\tau}-\mathrm{e}^{-2\beta\tau}\right)$$
$$-\frac{\sigma_r^2u^2}{2\beta^2}\left(1-2\mathrm{e}^{-\beta\tau}+\mathrm{e}^{-2\beta\tau}\right)+\frac{\gamma\tilde{b}_1\tilde{b}_2\left(\mathrm{e}^{\tau\nabla}-1\right)}{2b_1\left(\tilde{b}_1+\mathrm{e}^{\tau\nabla}\tilde{b}_2\right)}$$

Integrating with respect to τ and using boundary condition $B_1(\tau=0)=0$, then we get

$$B_1(\tau)=\left(2iu\left(\alpha\beta-\sigma_r^2\right)-\sigma_r^2u^2\right)\frac{\tau}{\beta^2}$$
$$+\frac{iu\sigma_r^2}{2\beta^3}\left(\mathrm{e}^{-\beta\tau}-1\right)^2+\frac{\sigma_r^2u^2}{4\beta^3}\left(-4\mathrm{e}^{-\beta\tau}+\mathrm{e}^{-2\beta\tau}-3\right)$$
$$+\left[\frac{-\tau\gamma\tilde{b}_2}{2b_1}+\frac{\gamma\left(\tilde{b}_1+\tilde{b}_2\right)}{2b_1\nabla}\ln\left(\frac{\tilde{b}_1+\mathrm{e}^{\tau\nabla}\tilde{b}_2}{\tilde{b}_1+\tilde{b}_2}\right)\right]$$

The details of the proof for the characteristic function f_2 are similar to f_1.

Hence, we have

$$f_2(t,x,r,v;T-\tau,u)$$
$$=\exp\left[iux+B_2(\tau)+rC_2(\tau)+vE_2(\tau)\right],$$

where $B_2(\tau)$, $C_2(\tau)$ and $E_2(\tau)$ are as given in this Lemma.

We can thus evaluate the characteristic function in close form. However, we are interested in the probability $\tilde{P}_j$. These can be inverted from the characteristic functions by performing the following integration

$$\tilde{P}_j(t,x,r,v;T,\kappa)$$
$$=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{\infty}\mathrm{Re}\left(\frac{e^{\mathrm{i}u\kappa}f_j(t,x,v.r;T,u)}{iu}\right)\mathrm{d}u$$

for $j=1,2$ where $X_t=\ln S_t$ and $\kappa=\ln K$, see Kendall *et. al*. [10]. The proof is now complete.