



รายงานวิจัยฉบับสมบูรณ์

โครงการ

ทฤษฎีบทจุดตรึงและการประยุกต์ไปยังปัญหาสมดุล ปัญหาค่าต่ำสุด และ
ปัญหาอสมการแปรผัน

Fixed Point Theory and Applications to Equilibrium Problems, Minimization
and Variational Inequality Problems

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สิงหาคม 2556

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยเชียงใหม่
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. และมหาวิทยาลัยเชียงใหม่ไม่จำเป็นต้องเห็นด้วยเสมอไป)

สารบัญ

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บทคัดย่อ

ในโครงการวิจัยนี้ เราได้ศึกษาและค้นคว้าหาองค์ความรู้ใหม่เกี่ยวกับทฤษฎีบทจุดตรึงของการส่งแบบไม่เชิงเส้นต่าง ๆ ทั้งในปริภูมิฮิลแบร์ตและปริภูมิบานาค ตลอดจนการสร้างระเบียบวิธีใหม่ ๆ เพื่อประมาณค่าจุดตรึงร่วมของการส่งแบบต่าง ๆ ที่สนใจศึกษา เรายังได้สร้างระเบียบวิธีใหม่ ๆ อีกหลายวิธีสำหรับการหาสมาชิกร่วมระหว่างเซตของผลเฉลยของปัญหาสมมูล เซตของจุดตรึงร่วมของการส่งแบบต่าง ๆ และเซตของผลเฉลยคำตอบของสมการแปรผัน เราได้ทฤษฎีบทการลู่เข้าแบบเข้มและแบบอ่อนของระเบียบวิธีดังกล่าวข้างต้นที่ลู่เข้าสู่ผลเฉลยร่วมกันของสามปัญหาดังกล่าวข้างต้นภายใต้เงื่อนไขควบคุมบางอย่าง เราสามารถนำเอาองค์ความรู้ดังกล่าวไปสร้างโปรแกรมในการประมาณค่าผลเฉลยของปัญหาดังกล่าวได้ด้วย ผลงานที่ได้ทั้งหมดนับเป็นทฤษฎีใหม่ที่มีประโยชน์ในการประยุกต์สำหรับแก้ปัญหาการหาคำตอบของสมการ อสมการ ปัญหาสมมูล ปัญหาอสมการแปรผัน

Abstract

In this research, we study and investigate fixed point theorems of various nonlinear mappings in both Hilbert spaces and Banach spaces and introduce various new iterative schemes for finding common fixed points of those nonlinear mappings. We also introduce and study several new approximation methods for finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of those nonlinear mappings and the set of solutions of the variational inequality. Several weak and strong convergence theorems of the proposed iterative algorithms to a common element of those three sets are given under some suitable control conditions. Moreover, we apply our main results to the equilibrium problems, minimization problems and variational inequality problems in both Hilbert and Banach spaces. Finally, by using our new approximation methods, we create new software for solving several problems in science and applied science. The results obtained in this work improve and extend many results of the previous works.

Keywords : Fixed point problems, Equilibrium problems, Variational inequalities, Minimization problems, Nonlinear mappings, Hilbert spaces and Banach spaces, Existence and approximation problems

Executive Summary

Fixed Point Theory and Applications to Equilibrium Problems, Minimization and Variational Inequality Problems

ทฤษฎีบทจุดตรึงและการประยุกต์ไปยังปัญหาสมมูล ปัญหาค่าต่ำสุด และ
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1. Importance of the research problem, Principles, Theory, Rationale

and/or Hypotheses

Various problems in science, applied science, physics, optimization, and economics can be formulated in forms of equations, inequalities, system of equations and system of inequalities. Those equations, inequalities, system of equations and system of inequalities are concerned with nonlinear operators equations or inequalities. There are mainly two important questions arising for solving those problems. The first one is **the existence of solutions of those problems**, and the second one is that **how can we find or approximate those solutions of such problems**. Having in view that many of the most important nonlinear problems of science and applied science reduce to solving a given equation or inequality which in turn may be reduced to finding the fixed points of a certain operator and fixed point theory plays an important role in solving solutions of those problems. The metrical fixed point theory has developed significantly in the second part of the 20th century. There are big group of mathematicians who are interested in studying and investigating those two problems. They can discover many new theorems which are very useful for solving many problems in science and applied science, optimization and economics.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \rightarrow X$, in order to obtain results on;

1. the existence (and uniqueness) of fixed points;
2. the structure of the fixed point sets;
3. the approximation of fixed points.

The ambient spaces X involved in fixed point theorems cover a variety of spaces: metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator T are generally metrical or compactness type conditions.

A plethora of metrical fixed point theorems have been obtained, more or less important from a theoretical point of view, which establish usually the existence, or the existence and uniqueness of fixed points for a certain nonlinear mappings. Among these fixed point theorems, only a small number are important from a practical point of view, that, they offer a constructive method for finding the fixed points.

However, from a practical point of view it is important not only to know the fixed point exists (and, possible, is unique), but also to be able to construct that fixed points. As the constructive methods used in metrical fixed point theory are prevailingly iterative procedures, that is, *approximate* methods, it is also of crucial importance to have a priori or/and a posteriori error estimates (or rate of convergence) for such method. For example, the Banach fixed point theorem concerns certain mappings (contractions) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point and it also given a constructive procedure for obtaining better and better approximations to the fixed point (the solution of the practical problem). By definition, this is a method such that we choose an arbitrary x_0 in a given set and calculate recursively a sequence $x_0, x_1, x_2, \dots$ from a relation of the form

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, 3, \dots$$

That is, we choose an arbitrary x_0 and determine successively $x_1 = Tx_0, x_2 = Tx_1, \dots$

It is also known as the *Picard iteration* starting at x_0 .

Iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach fixed point theorem (or more difficult fixed point theorems).

Many researchers are interested in obtaining (additional) condition on T and X as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed

point of T . Moreover, if the Picard iteration converges to a fixed point of T , they will be interested in evaluating the error estimate (or alternatively, the rate of convergence) of the method, that is, in obtaining a stopping criterion for the sequence of successive approximations. However, the Picard iteration may not converge even in the weak topology.

Construction of fixed point iteration processes of nonlinear mappings are an important subject in the theory of nonlinear mappings, and find application in a number of applied areas. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings, relatively nonexpansive mappings, hemi-relatively nonexpansive mappings, generalized nonexpansive mappings and maximal monotone operators in various space have been studied by many mathematicians. Let $(X, \|\cdot\|)$ be a real normed space and $C \subset X$ be a closed and convex. Three classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $T : C \rightarrow C$ and studied by many researchers.

Halpern' iteration

The first one is introduced by Halpern [17] which is defined as follows: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geq 0, \dots\dots\dots(2)$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Mann' iteration

The second iteration process is now known as Mann's iteration process [30] and is defined as follows: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \dots\dots\dots(1)$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Ishikawa' iteration

The third iteration process is referred to as Ishikawa's iteration [20] which is defined recursively by;

$x_0 \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(\beta_n x_n + (1 - \beta_n)Tx_n), \quad n \geq 0, \dots\dots\dots(3)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

In general not much has been known regarding the convergence of the iteration processes (1)-(3) unless the underlying space X has elegant properties which be briefly mention here.

Process (3) is indeed more general than process (2). But research has been concentrated on the latter due probably to the reasons that the formulation of process (2) is simpler then that of (3) and that a convergence theorem for process (2) may possibly lead to a convergence theorem for process (3) provided the sequence $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of process (3) has its own right. As a matter of fact, process (2) may fail to converge while process (3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space (see [20]). Both processes (2) and (3) have only weak convergence, in general. For example, Reich [52] proved that if X is a uniformly convex Banach space with a Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the Mann's iteration converges weakly to a fixed point of T . However, we note that Mann's iteration have only weak convergence even in a Hilbert space.

Normal Hybrid Method (or CQ method)

Attempts to modify the Mann's iteration method (1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [34] proposed the following modification of the Mann's iteration method (1) by using the hybrid method in mathematical programming, for a single nonexpansive mapping T in a Hilbert space:

$$\begin{cases} x_1 = x \in C, \\ u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|u_n - z\| \leq \|x_n - z\|\}, \dots\dots\dots(4) \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x. \end{cases}$$

The iteration process (1) has been proved to be strong convergent in both Hilbert space [17, 28] and uniformly smooth Banach spaces [53, 58] unless the sequence $\{\alpha_n\}$ satisfies conditions:

- (i) $\alpha_n \rightarrow 0$;

$$(ii) \sum_{n=0}^{\infty} \alpha_n = \infty ;$$

$$(iii) \text{ either } \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \text{ or } \lim_{n \rightarrow \infty} (\alpha_n / \alpha_{n+1}) = 1.$$

Due to the restriction of condition (ii), process (1) is widely believed to have slow convergence though the rate of convergence has not be determined. Moreover, Halpern [17] prove that conditions (i) and (ii) are indeed necessary in the sense that if process (1) is strongly convergent for all closed convex subsets C of a Hilbert space H and all nonexpansive mappings T on C , then the sequence $\{\alpha_n\}$ must satisfy conditions (i) and (ii). (However, it is unknown whether these two conditions are also sufficient). Martinez and Xu [32] develop the normal hybrid method for process (1) and proved the strong convergence of this method under condition (i) only. Moreover they extend Nakajo and Takahashi' s iteration process (4) to the Ishikawa iteration process. In 2005, Matsushita and Takahashi [33] extend the results of Nakajo and Takahashi [34] to a Banach space for a relative nonexpansive mapping.

Shrinking Projection Method

Kubota, Takahashi and Takeuchi [26] introduced another hybrid method called the *shrinking projection method* for nonexpansive mapping T in a Hilbert space as follows:

$$\begin{cases} x_1 = x \in C, \\ u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} : \|u_n - z\| \leq \|x_n - z\|\}, \dots\dots\dots(5) \\ x_{n+1} = P_{C_n} x. \end{cases}$$

and they proposed the following modification the iteration method (4) and (5) for a countable family of nonexpansive mappings $\{T_n\}$ satisfies NST-condition (see [35]) in a Hilbert space H .

Monotone Hybrid Method

Qin and Su [50] modify the iteration method (4), so call *the monotone hybrid method* for nonexpansive mapping T in a Hilbert space as follows:

$$\begin{cases} x_1 = x \in C, & C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|u_n - z\| \leq \|x_n - z\|\}, \dots\dots\dots(6) \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x. \end{cases}$$

By using this method, they proved strong convergence theorem under a control condition on the sequence $\{\alpha_n\}$ but the technique they used in this paper is different from Nakajo and Takahashi [34]. More precisely, they can show that the sequence $\{x_n\}$ generated by (6) is a Cauchy sequence, without the use of demiclosedness principle, Opial's condition and the Kadec-Klee property. Moreover, they extend the results to a Banach space for a relative nonexpansive mapping by using same method (see [51]).

There are many new nonlinear mappings arising from various problems in science and applied science, optimization and economics. So the fixed point problems concerning those mappings are interesting for investigating. So we aim to investigate the existence problem, structure of fixed points set, and approximation method for finding fixed point of those new mappings. The mappings that we are interested in studying and have more applications are nonexpansive mappings, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, quasi-asymptotically nonexpansive mappings, pseudocontractive mappings, quasi-pseudocontractive mappings, relatively nonexpansive mappings, hemi-relatively nonexpansive mappings, generalized nonexpansive mappings, maximal monotone operators, and other generalizations of these mappings.

There are many problems in economics, finance, transportation, network and structural analysis, elasticity, and optimization which can be solved by using equilibrium theory. The equilibrium problems cover many important problems such as minimization problems, variational inequality problems. In 1997, Combettes and Hirstoaga first studied and constructed an iterative method for finding a solution of the equilibrium problem in a Hilbert space. They proved strong convergence theorem of their iterative method under some sufficient conditions. Their main results can be applied for solving variational inequality problem and they showed that there is a close relation between the fixed

point problem and the equilibrium problem. So, many techniques in fixed point theory can be applied for the equilibrium problems. Many problems in science and applied science, physics, economics and others areas are more complicated. The solutions of those problems concern intersection of fixed point problems, variational inequality and equilibrium problems. So, the problem of finding a common solutions of those problems are very useful for applications and widely studied by many mathematicians later.

Motivated by many previous works concerning the equilibrium problems, fixed point problems and other important problems in science, applied science and economics, we are interested to study and construct new methods for solving those problems.

2. Research Objectives

1. To investigate various properties of important classes of nonlinear mappings that related to fixed point problem and give some sufficient condition those mappings in order to obtain the new existence theorems for those nonlinear mappings in both Hilbert and Banach spaces.
2. To construct new iterative methods for approximation fixed points of various kinds of nonlinear mappings and finding a common fixed point of a finite and infinite family of those nonlinear mappings studied in (1), and give some sufficient conditions for proving weak and strong convergence theorems of our iterative methods for those mappings.
3. To construct new iterative methods for finding a common element of the set of equilibrium problems, variational inequality problems and fixed point problems, and give some control conditions to prove weak and strong convergence theorems of those new methods.
4. To apply our main results obtained from (2) and (3) for solving the equilibrium problem, variational inequality problems, and fixed point problems in the

problem of science and applied science and economics. Moreover, software for solving those problems will be constructed.

5. To build and develop new researchers in the area of fixed point theory and applications for developing human resource and high standard academic researches.
6. To create research network both inside and outside of the country for making cooperation in doing research in this area.

3. Research Methodology

- (1) Collect all research papers, books and all articles in various journals related to various important kinds of nonlinear mappings and their properties related to fixed point property
- (2) Collect and study various knowledge concerning fixed point theory and applications by attending seminars, meeting, and conferences related to fixed point theory and applications and discuss research problems with research network and team research.
- (3) Study various iterative methods for the fixed point problems, equilibrium problems and fixed point problems of various nonlinear mappings in both Hilbert spaces and Banach spaces.
- (4) Using basic knowledge and various methods from 9.1 – 9.3 to study and construct new methods for obtaining the research objectives.
- (5) Prove weak and strong convergence theorems of our iterative schemes defined in 9.4 by finding some suitable conditions on the control sequences.
- (6) Apply our main results obtained by 9.5 to the equilibrium problems, variational inequality problems and others problems in science and applied science, and economics.
- (7) Write research papers concerning our research problems and submit to the international journals
- (8) Write research report for every sixth month and report to the TRF.

4. Research Plan and Outputs in each year

The first year

Activities and Process	2010 (the first sixth month)					
	1	2	3	4	5	6
1. Collect research articles related to our research problems from various journals and read and discuss the research problems with the research team.	↔					
2. Study basic properties of those nonlinear mappings mentioned in 10.1 from the references.		↔				
3. Using knowledge from 2. to prove the existence theorems and some useful properties of new mappings mentioned in 10.1 and construct new iterative methods for approximation fixed points or common fixed points of those mappings mentions in 10.1 in Hilbert spaces and Banach spaces				↔		
4. Write papers and submit to international journal and write report to the TRF						↔

Activities and Process	2010 (the last sixth month)					
	7	8	9	10	11	12
1. Collect research articles related to various iterative methods for finding equilibrium problems and discuss research problems and the direction of doing the problems with the team of research	↔					
2. Study basic properties of the mappings related to the equilibrium problems from the references.		↔				
3. Using knowledge from 2. to prove the existence theorems of the equilibrium problems and investigate some useful properties of the mappings related the equilibrium problems.						

Construct new iterative methods for finding common element of the fixed points set or common fixed points of those mappings mentions in 10.1 in Hilbert spaces and the solution of the equilibrium problems.						
4. Write papers and submit to international journal and write report to the TRF						

The second year

Activities and Process	2011 (the first sixth month)					
	1	2	3	4	5	6
1. Collect research articles related to equilibrium, variational inequality problems from various journals and read and discuss the research problems with the research team.						
2. Study iterative scheme for finding a common element of the equilibrium problems and variational inequality problems from the references.						
3. Using knowledge from 2. to construct new iterative methods for finding common element of the fixed points set or common fixed points of those mappings mentions in 10.1 in Banach spaces and the solution of the equilibrium problems.						
4. Write papers and submit to international journal and write report to the TRF						

Activities and Process	2011 (the last sixth month)					
	7	8	9	10	11	12
1. Collect research articles related to various iterative methods for finding equilibrium problems and variational inequality problems and discuss research problems and the direction of doing the problems with the team of research	↔					
2. Study weak and strong convergence theorems of the iterative methods mentioned in 1 from the references.		↔				
3. Using knowledge from 1 and 2 to modify and construct new iterative methods for finding a common element of the common fixed points of finite family of those mappings mentions in 10.1 in Banach spaces and the solution of the equilibrium problems and variational inequality problems.			↔			
4. Write papers and submit to international journal and write report to the TRF						↔

The Third year

Activities and Process	2012 (the first sixth month)					
	7	8	9	10	11	12
1. Collect research articles related to various iterative methods for finding common element of the equilibrium problems, variational inequality problems and fixed point problems and discuss research problems and the direction of doing the problems with the team of research	↔					

2. Study weak and strong convergence theorems of the iterative methods mentioned in 1 from the references.		↔				
3. Using knowledge from 1 and 2 to modify and construct new iterative methods for finding common element of the common fixed points of an infinite family of those mappings mentions in 10.1 in Banach spaces and the solution of the equilibrium problems and variational inequality problems.			↔			
4. Write papers and submit to international journal and write report to the TRF						↔

Activities and Process	2012 (the last sixth month)					
	7	8	9	10	11	12
1. Study computer programming for computing and approximation solutions of various equations or inequalities.	↔					
2. Develop our iterative methods for writing computer program for solving the specific equilibrium, variational inequality and fixed point problems.		↔				
3. Run such computer programs and analyze the algorithm and errors, improve the program and use them for their purposes.					↔	
4. Write papers and submit to international journal and write report to the TRF						↔

7. Research Team:

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Chapter I

Approximation Methods for Fixed Point and Common Fixed Point of Some Nonlinear Mappings

In this chapter, we introduce approximation methods for finding a fixed point and common fixed point for a family of the following nonlinear mappings : nonexpansive mappings, asymptotically quasi-nonexpansive mappings, strict pseudo-contractions, Bregman strongly nonexpansive mappings. We divide this chapter into 9 sections corresponding to the studied mappings.

1.1 A new approximation method for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces

In this section, we consider a new iterative scheme to approximate a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings. We prove several strong and weak convergence results of the proposed iteration in Banach spaces. These results generalize and refine many known results in the current literature.

1. Introduction

In 2008, Khan et al. [1] introduced an iterative process for a finite family of mappings as follows:

Let C be a convex subset of a Banach space X and let $\{T_i : i = 1, 2, \dots, k\}$ be a family of self-mappings of C . Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, 3, \dots$ and $i = 1, 2, \dots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{1}$$

where $y_{0n} = x_n$ for all n . The iterative process (1) is the generalized form of the modified Mann (one-step) iterative process by Schu [2], the modified Ishikawa (two-step) iterative process by Tan and Xu [3], and the three-step iterative process by Xu and Noor [4].

Goebel and Kirk [5], in 1972, introduced the new concept of asymptotically nonexpansive and proved that every asymptotically nonexpansive self-mapping of a nonempty closed bounded and convex subset of a uniformly convex Banach space has a fixed point. In 1978, Bose [6] studied an iterative scheme for fixed points of asymptotically nonexpansive mappings. In 2001, Khan and Takahashi [7] used the modified Ishikawa process to approximate common fixed points of two asymptotically nonexpansive mappings.

Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers [8–10] are interested in studying approximation method for finding common fixed points of nonlinear mapping. Also, approximation methods for finding fixed points for nonexpansive mappings can be seen in [11–18].

In 2003, Sun [19] studied an implicit iterative scheme initiated by Xu and Ori [20] for a finite family of asymptotically quasi-nonexpansive mappings. Shahzad and Udomene [21], in 2006, proved some convergence theorems for the modified Ishikawa iterative process of two asymptotically quasi-nonexpansive mappings to a common fixed point. Nammanee et al. [22] introduced a three-step iteration scheme for asymptotically nonexpansive mappings and proved weak and strong convergence theorems of that iteration scheme under some control conditions. In 2007, Fukhar-ud-din and Khan [23] studied a new three-step iteration scheme for approximating a common fixed point of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Shahzad and Zegeye [24] introduced a new concept of generalized asymptotically nonexpansive mappings and proved some strong convergence theorems for fixed points of finite family of this class. Recently, Khan et al. [1] introduced the iterative sequence (1) for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

Motivated by Khan et al. [1], we introduce a new iterative scheme for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings as follow:

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1 - \alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= (1 - \alpha_{(k-2)n})y_{(k-3)n} + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \quad (2)$$

where $y_{0n} = x_n$ for all n .

The aim of this paper is to obtain some strong and weak convergence results for the iterative process (2) of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

2. Preliminaries

In this section, we review definitions and lemmas used for the rest of the paper as follow:

Let C be a nonempty subset of a real Banach space X and T be a self-mapping of C . The fixed point set of T is denoted by $F(T) = \{x \in C : Tx = x\}$. If $F(T)$ is not empty, then T is called

- (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;
- (ii) *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$;
- (iii) *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ and $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (iv) *asymptotically quasi-nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ and $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, for all $x \in C, p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (v) *generalized quasi-nonexpansive* if there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\|T^n x - p\| \leq \|x - p\| + s_n$, for all $x \in C, p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (vi) *generalized asymptotically quasi-nonexpansive* [24] if there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, \infty)$ with $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C, p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (vii) *uniformly L -Lipschitzian* if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (viii) *$(L - \gamma)$ uniform Lipschitz* if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^\gamma$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (ix) *semi-compact* if for a sequence $\{x_n\}$ in C with $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow p \in C$.

Condition (A''). Let C be a subset of a normed space X . A family of self-mappings $\{T_i : i = 1, 2, \dots, k\}$ of C is said to have Condition (A'') if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T_i x\| \geq f(d(x, F))$ for some $1 \leq i \leq k$ and for all $x \in C$ where $d(x, F) = \inf \{\|x - p\| : p \in F = \bigcap_{i=1}^k F(T_i)\}$.

The Condition (A'') defined above by the authors is the generalization of the Condition (A) [25] when $k = 1$ and Condition (A') [26] for $k = 2$.

The map $T : C \rightarrow X$ is said to be *demiclosed at 0* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and Tx_n converging strongly to 0, we get $Tx = 0$.

A Banach space X is said to have *Opial's property* if for each sequence $\{x_n\}$ converging weakly to $x \in C$ and $x \neq y$, we have the condition

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

Remark 2.1. It is easy to see that,

- (i) a quasi-nonexpansive mapping is generalized quasi-nonexpansive;
- (ii) an asymptotically quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;
- (iii) a generalized quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;
- (iv) a uniformly L -Lipschitzian mapping is $(L - 1)$ uniform Lipschitz.

The next example shows that there is an asymptotically quasi-nonexpansive mapping which is not quasi-nonexpansive.

Example 2.2. Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \quad \text{for all } x = (x_1, x_2, \dots, x_n, \dots) \in X,$$

and a mapping $T : X \rightarrow X$ defined by $Tx = (0, 2x_1, 0, 0, \dots, 0, \dots)$. By letting $Tx = x$ for any $x = (x_1, x_2, \dots, x_n, \dots) \in X$, we have

$$(0, 2x_1, 0, 0, \dots, 0, \dots) = (x_1, x_2, \dots, x_n, \dots),$$

i.e., $F(T) = \{0\}$. Moreover, $T^n x = (0, 0, 0, \dots, 0, \dots)$ for all $n = 2, 3, 4, \dots$

For the sequence $\{r_n\}$, where $r_n = \frac{1}{n}$, and $p \in F(T)$, we have

$$\|Tx - p\| = 2\|x_1\| \leq (1 + r_1)\|x - p\| \quad \text{and} \quad \|T^n x - p\| \leq (1 + r_n)\|x - p\|,$$

for all $n = 2, 3, 4, \dots$. This implies that T is an asymptotically quasi-nonexpansive mapping. However, T is not a quasi-nonexpansive mapping since, for $x^0 = (1, 0, 0, \dots, 0, \dots)$ in X ,

$$\|Tx^0 - p\| = \|(0, 2, 0, 0, \dots, 0, \dots)\| = 2 > 1 = \|x^0 - p\|. \quad \square$$

Lemma 2.3 (Cf. [19, Lemma 2.2]). Let the sequences $\{a_n\}$ and $\{\delta_n\}$ of real numbers satisfy:

$$a_{n+1} \leq (1 + \delta_n)a_n, \quad \text{where } a_n \geq 0, \delta_n \geq 0, \text{ for all } n = 1, 2, 3, \dots$$

and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists;
- (ii) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 (See [2, Lemma 1.3]). Let X be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1$, $n = 1, 2, 3, \dots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, where $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

3. Convergence in Banach spaces

The aim of this section is to establish the strong convergence of the iterative scheme (2) to converge to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space under some appropriate conditions.

Lemma 3.1. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$, are defined by (2). Then for $p \in F$, we get

- (i) $\|x_n - T_i^n x_n\| \leq (2 + r_n)\|x_n - p\|$, for all $i = 1, 2, \dots, k$;
- (ii) $\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n)\|y_{(i-1)n} - p\|$, for all $i = 1, 2, \dots, k$;
- (iii) $\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n)\|y_{(i-1)n} - p\|$, for all $i = 1, 2, \dots, k$;
- (iv) $\|y_{in} - p\| \leq (1 + r_n)^i \|x_n - p\|$, for $i = 1, 2, \dots, k - 1$;
- (v) $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\|$;
- (vi) if $\sum_{n=1}^{\infty} r_n < \infty$, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists,

where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$ and $\delta_n = \binom{k}{1} r_n + \binom{k}{2} r_n^2 + \dots + \binom{k}{k} r_n^k$.

Proof. Let $p \in F$.

(i) For $i = 1, 2, 3, \dots, k$, we have

$$\begin{aligned}\|x_n - T_i^n x_n\| &\leq \|x_n - p\| + \|T_i^n x_n - p\| \\ &\leq \|x_n - p\| + (1 + r_n)\|x_n - p\|, \\ &= (2 + r_n)\|x_n - p\|.\end{aligned}$$

(ii) Similarly to part (i), we have

$$\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n)\|y_{(i-1)n} - p\|, \quad \text{for all } i = 1, 2, \dots, k.$$

(iii) For $i = 1, 2, \dots, k$, we have

$$\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n)\|y_{(i-1)n} - p\|.$$

(iv) By part (i) and $\alpha_{1n} \leq 1$, we obtain

$$\begin{aligned}\|y_{1n} - p\| &= \|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_1^n x_n - p)\| \\ &\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| \\ &\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}(1 + r_n)\|x_n - p\| \\ &\leq (1 + r_n)\|x_n - p\|.\end{aligned}$$

We assume that $\|y_{jn} - p\| \leq (1 + r_n)^j\|x_n - p\|$ holds for some $1 \leq j \leq k - 2$. From part (iii) and $\alpha_{(j+1)n} \leq 1$, we then have

$$\begin{aligned}\|y_{(j+1)n} - p\| &= \|(1 - \alpha_{(j+1)n})(y_{jn} - p) + \alpha_{(j+1)n}(T_{j+1}^n y_{jn} - p)\| \\ &\leq (1 - \alpha_{(j+1)n})\|y_{jn} - p\| + \alpha_{(j+1)n}\|T_{j+1}^n y_{jn} - p\| \\ &\leq (1 - \alpha_{(j+1)n})\|y_{jn} - p\| + \alpha_{(j+1)n}(1 + r_n)\|y_{jn} - p\| \\ &\leq (1 + r_n)\|y_{jn} - p\| \\ &\leq (1 + r_n)(1 + r_n)^j\|x_n - p\| \\ &= (1 + r_n)^{j+1}\|x_n - p\|.\end{aligned}$$

Therefore, by mathematical induction, we obtain

$$\|y_{in} - p\| \leq (1 + r_n)^i\|x_n - p\|, \quad \text{for } i = 1, 2, \dots, k - 1.$$

(v) By part (ii), part (iv), and $\alpha_{kn} \leq 1$, we get

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| \\ &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - p\| \\ &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_n)\|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)\|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)(1 + r_n)^{k-1}\|x_n - p\| \\ &= (1 + r_n)^k\|x_n - p\| \\ &\leq (1 + \delta_n)\|x_n - p\|,\end{aligned}$$

where $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \dots + \binom{k}{k}r_n^k$.

(vi) By (v), we have $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\|$ for all $n \in \mathbb{N}$. From $\sum_{n=1}^{\infty} r_n < \infty$, we also have $\sum_{n=1}^{\infty} r_n^i < \infty$ for $i = 1, 2, 3, \dots, k$. It follows that $\sum_{n=1}^{\infty} \delta_n < \infty$. By Lemma 2.3, we get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. $\square$

Theorem 3.2. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - p_i\| \leq (1 + r_n)\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2). Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

Proof. The necessity is obvious and then we prove only the sufficiency. Let $p \in F$. By Lemma 3.1(vi), $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and hence $\{\|x_n - p\|\}$ is bounded. We let $M = \sup_{n \geq 1} \{\|x_n - p\|\}$. From Lemma 3.1(v), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M\delta_n, \quad n \geq 1.$$

Thus, for positive integers m and n , we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + M\delta_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + M(\delta_{n+m-1} + \delta_{n+m-2}) \\ &\vdots \\ &\leq \|x_n - p\| + M \sum_{i=n}^{n+m-1} \delta_i. \end{aligned} \quad (3)$$

By Lemma 3.1(v), we obtain

$$d(x_{n+1}, F) \leq (1 + \delta_n)d(x_n, F).$$

From the given condition $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ and Lemma 2.3, we get

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (4)$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C . By (4) and $\sum_{n=1}^{\infty} \delta_n < \infty$, we get that for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$d(x_n, F) < \frac{\epsilon}{8} \quad \text{and} \quad \sum_{n=n_0}^{\infty} \delta_n < \frac{\epsilon}{2M}. \quad (5)$$

From the first inequality of (5), there exists $p_0 \in F$ such that

$$\|x_{n_0} - p_0\| < \frac{\epsilon}{4}. \quad (6)$$

For any positive integer m , by (3), (5) and (6), we obtain

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq 2\|x_{n_0} - p_0\| + M \sum_{i=n_0}^{n_0+m-1} \delta_i \\ &< 2\left(\frac{\epsilon}{4}\right) + M\left(\frac{\epsilon}{2M}\right) = \epsilon. \end{aligned} \quad (7)$$

Thus, $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, $\{x_n\} \rightarrow q \in X$. Actually, $q \in C$ because $\{x_n\} \subset C$ and C is a closed subset of X . Next we show that $q \in F$. Since $F(T_i)$ is a closed subset in C for all $i = 1, 2, \dots, k$, so is $F = \bigcap_{i=1}^k F(T_i)$. From the continuity of $d(x, F)$ with $d(x_n, F) \rightarrow 0$ and $x_n \rightarrow q$ as $n \rightarrow \infty$, we get $d(q, F) = 0$ and then $q \in F$. Therefore, the proof is complete. $\square$

Since any quasi-nonexpansive mapping is asymptotically quasi-nonexpansive, the next corollary is obtained immediately from Theorem 3.2.

Corollary 3.3. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - p_i\| \leq \|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2). Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

4. Convergence in uniformly convex Banach spaces

In this section, we prove some strong and weak convergence results for the iterative process (2) on uniformly convex Banach spaces without using the condition $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ appearing in Section 3. Instead, we consider $(L - \gamma)$ uniform Lipschitz mappings, Condition (A''), semi-compact mappings, Opial property and demiclosed mappings at 0.

Theorem 4.1. Let C be a nonempty closed convex subset of an uniformly convex real Banach space X . Let $\{T_i : i = 1, 2, \dots, k\}$ be a family of uniformly $(L - \gamma_i)$ Lipschitzian and asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{1/\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $\{T_i : i = 1, 2, \dots, k\}$ satisfies Condition (A'') and $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

Proof. Let $p \in F$. By Lemma 3.1(vi), we get that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Then there is a real number $c \geq 0$ such that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (8)$$

By Lemma 3.1(iv), we have

$$\|y_{in} - p\| \leq (1 + r_n)^i \|x_n - p\|, \quad \text{for } i = 1, 2, \dots, k-1.$$

By taking $\limsup$ on both sides of the above inequality, we get

$$\limsup_{n \rightarrow \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (9)$$

Therefore, by Lemma 3.1(iii) and (8), we obtain

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k. \quad (10)$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c$, we have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = c.$$

Using (8), (10) and Lemma 2.4, we conclude that

$$\lim_{n \rightarrow \infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0.$$

We assume that

$$\lim_{n \rightarrow \infty} \|y_{(j-1)n} - T_j^n y_{(j-1)n}\| = 0, \quad \text{for some } 2 \leq j \leq k. \quad (11)$$

By Lemma 3.1(iii), we have

$$\|x_{n+1} - p\| \leq (1 + r_n)^{k-i} \|y_{in} - p\|, \quad \text{for all } i = 1, 2, \dots, k-1.$$

This together with (11) and $r_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$c \leq \liminf_{n \rightarrow \infty} \|y_{(j-1)n} - p\|. \quad (12)$$

By Lemma 3.1(iv), (2) and (12), we get

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{(j-1)n})(y_{(j-2)n} - p) + \alpha_{(j-1)n}(T_{j-1}^n y_{(j-2)n} - p)\| = \lim_{n \rightarrow \infty} \|y_{(j-1)n} - p\| = c.$$

Using (9), (10) and Lemma 2.4, we conclude that

$$\lim_{n \rightarrow \infty} \|y_{(j-2)n} - T_{j-1}^n y_{(j-2)n}\| = 0.$$

Therefore, by mathematical induction, we obtain

$$\lim_{n \rightarrow \infty} \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (13)$$

From (2), we have

$$\|y_{in} - y_{(i-1)n}\| = \alpha_{in} \|T_i^n y_{(i-1)n} - y_{(j-1)n}\|, \quad \text{for } i = 1, 2, \dots, k-1.$$

By (13), we obtain that

$$\|y_{in} - y_{(i-1)n}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for } i = 1, 2, \dots, k-1. \quad (14)$$

From

$$\|x_n - y_{in}\| \leq \|x_n - y_{1n}\| + \|y_{1n} - y_{2n}\| + \dots + \|y_{(i-1)n} - y_{in}\|,$$

for $i = 1, 2, \dots, k-1$. It follows by (14) that

$$\|x_n - y_{in}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for } i = 1, 2, \dots, k-1. \quad (15)$$

From (13), when $i = 1$ we get $\lim_{n \rightarrow \infty} \|x_n - T_1^n x_n\| = 0$. For $2 \leq i \leq k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - T_i^n x_n\| \\ &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + L \|y_{(i-1)n} - x_n\|^{y_i}. \end{aligned}$$

From (13) and (15), we conclude that

$$\lim_{n \rightarrow \infty} y_{in} = \lim_{n \rightarrow \infty} \|x_n - T_i^n x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k \quad (16)$$

where $\gamma_{in} = \|x_n - T_i^n x_n\|$. From (2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - x_n\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - x_n\| \\ &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - x_n\| + \alpha_{kn} (\|T_k^n y_{(k-1)n} - y_{(k-1)n}\| + \|y_{(k-1)n} - x_n\|) \\ &= \|y_{(k-1)n} - x_n\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - y_{(k-1)n}\|. \end{aligned}$$

From (13) and (15),

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (17)$$

For $i = 1, 2, \dots, k$, we have

$$\begin{aligned} \|x_{n+1} - T_i x_{n+1}\| &\leq \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\leq \gamma_{i(n+1)} + L \|x_{n+1} - T_i^n x_{n+1}\|^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L (\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|)^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L (\|x_{n+1} - x_n\| + \gamma_{in} + L \|x_n - x_{n+1}\|^{\gamma_i})^{\gamma_i}. \end{aligned}$$

Using (16) and (17), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_i x_{n+1}\| = 0, \quad \text{for } i = 1, 2, \dots, k.$$

Therefore, by using Condition (A''), there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - T_j x_n\| = 0,$$

for some $1 \leq j \leq k$. That is

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

By Theorem 3.2, we conclude that $\{x_n\}$ converges strongly to a point $p \in F$. $\square$

Lemma 4.2. Let C be a nonempty closed convex subset of a uniformly convex real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - T_i^n y\| \leq L \|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in}) \|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then,

- (i) $\lim_{n \rightarrow \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0$, for all $i = 1, 2, \dots, k$;
- (ii) $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, for all $i = 1, 2, \dots, k$.

Proof. (i) Let $p \in F$. By Lemma 3.1(vi), we obtain that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and we then suppose that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (18)$$

By (18) and Lemma 3.1(iv), we have

$$\limsup_{n \rightarrow \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (19)$$

By (2), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - p\| \\ &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn} (1 + r_n) \|y_{(k-1)n} - p\| \\ &\leq (1 + r_n) \|y_{(k-1)n} - p\| \\ &= (1 + r_n) \|(1 - \alpha_{(k-1)n}) (y_{(k-2)n} - p) + \alpha_{(k-1)n} (T_{k-1}^n y_{(k-2)n} - p)\| \\ &\leq (1 + r_n) ((1 - \alpha_{(k-1)n}) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} (1 + r_n) \|y_{(k-2)n} - p\|) \\ &\leq (1 + r_n)^2 \|y_{(k-2)n} - p\| \\ &\vdots \\ &\leq (1 + r_n)^{k-i} \|y_{in} - p\|, \end{aligned}$$

for some $i = 1, 2, \dots, k-1$. It follows that

$$c \leq \liminf_{n \rightarrow \infty} \|y_{in} - p\|, \quad \text{for } i = 1, 2, \dots, k-1. \quad (20)$$

From (19) and (20), we obtain

$$\lim_{n \rightarrow \infty} \|y_{in} - p\| = c, \quad (21)$$

and then

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{in})(y_{(i-1)n} - p) + \alpha_{in}(T_i^n y_{(i-1)n} - p)\| = c, \quad (22)$$

for $i = 1, 2, \dots, k-1$.

By Lemma 3.1(iii) and (21), we get

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (23)$$

From (19), (22), (23) and Lemma 2.4, we obtain

$$\lim_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k-1. \quad (24)$$

Now we want to show that (24) is also true for $i = k$. By Lemma 3.1(ii) and (iv), we have

$$\begin{aligned} \|T_k^n y_{(k-1)n} - p\| &\leq (1 + r_n) \|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)(1 + r_n)^{k-1} \|x_n - p\| \\ &= (1 + r_n)^k \|x_n - p\|. \end{aligned}$$

This implies by (18) that

$$\limsup_{n \rightarrow \infty} \|T_k^n y_{(k-1)n} - p\| \leq c. \quad (25)$$

We also have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c.$$

Hence, by (19), (25) and Lemma 2.4, we obtain

$$\lim_{n \rightarrow \infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0. \quad (26)$$

Then, (24) and (26) give us

$$\lim_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (27)$$

From

$$\|x_n - T_i^n y_{(i-1)n}\| \leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\|,$$

it implies by (15) and (27) that

$$\lim_{n \rightarrow \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0, \quad (28)$$

for some $i = 1, 2, 3, \dots, k$.

(ii) From part (i), for $i = 1$, we have

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - x_n\| = 0. \quad (29)$$

For $i = 2, 3, 4, \dots, k$, we get

$$\begin{aligned} \|T_i^n x_n - x_n\| &\leq \|T_i^n x_n - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - x_n\| \\ &\leq L \|x_n - y_{(i-1)n}\|^{y^i} + \|T_i^n y_{(i-1)n} - x_n\|. \end{aligned}$$

By part (i) and (15), we conclude that

$$\lim_{n \rightarrow \infty} \|T_i^n x_n - x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (30)$$

For $1 \leq i \leq k$, we obtain

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + L \|x_{n+1} - x_n\|^{y^i} + L \|T_i^n x_n - x_n\|^{y^i}. \end{aligned}$$

From (16) and (30), we then have

$$\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad \square$$

Theorem 4.3. Under the hypotheses of Lemma 4.2, assume that T_j^m is semi-compact for some positive integers m and $1 \leq j \leq k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, k\}$.

Proof. Suppose that T_j^m is semi-compact for some positive integers $m \geq 1$ and $1 \leq j \leq k$. We have

$$\begin{aligned} \|T_j^m x_n - x_n\| &\leq \|T_j^m x_n - T_j^{m-1} x_n\| + \|T_j^{m-1} x_n - T_j^{m-2} x_n\| + \dots + \|T_j^2 x_n - T_j x_n\| + \|T_j x_n - x_n\| \\ &\leq (m-1)L\|T_j x_n - x_n\|^{\gamma_j} + \|T_j x_n - x_n\|. \end{aligned}$$

Then, by Lemma 4.2(ii), we get $\|T_j^m x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded and T_j^m is semi-compact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightarrow q \in C$ as $l \rightarrow \infty$.

By continuity of T_i and Lemma 4.2(ii), we obtain

$$\|q - T_j q\| = \lim_{l \rightarrow \infty} \|x_{n_l} - T_j x_{n_l}\| = 0, \quad \text{for all } j = 1, 2, \dots, k.$$

Therefore, $q \in F$ and then Theorem 3.2 implies that $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1, 2, \dots, k\}$. $\square$

We note that in practical Theorem 4.3 is very useful in the case that one of T_i , $i = 1, 2, 3, \dots, k$, is semi-compact.

Theorem 4.4. Let C be a nonempty closed convex subset of an uniformly convex real Banach space X satisfying the Opial property, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. If $I - T_i$, $i = 1, 2, \dots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.

Proof. Let $p \in F$. By Lemma 3.1(vi), we get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Then we follow the proof of Theorem 3.2 by Khan et al. [1] until we can conclude that $\{x_n\}$ converges weakly to a common fixed point $p \in F$. $\square$

5. Concluding remarks

The following remarks are obtained directly from the results in Sections 3 and 4.

Remark 5.1. It is hard to show that Theorems 4.1, 4.3 and 4.4 can be extended to a finite family of generalized asymptotically quasi-nonexpansive mappings.

Remark 5.2. It is clear that Theorems 3.2, 4.1, 4.3 and 4.4 can be used for any quasi-nonexpansive mapping.

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1.2 Convergence theorems from monotone hybrid methods for an infinitely countable family of Lipschitz asymptotically quasi-nonexpansive mappings

In this section, we prove a weak convergence theorem for the modified Mann iteration process for a uniformly Lipschitzian and asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space. We also introduce two new kinds of monotone hybrid methods and obtain strong convergence theorems for an infinitely countable family of uniformly Lipschitzian and asymptotically quasi-nonexpansive mappings in a Hilbert space. The results of this section improve on and extend corresponding ones announced by many authors.

1. Introduction

Let C be a nonempty closed and convex subset of a real Banach space X and let $T : C \rightarrow C$. We denote as $F(T)$ the set of fixed points of T , that is, $F(T) = \{x \in C : x = Tx\}$. Then T is said to be:

- (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
- (ii) *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$;
- (iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n \rightarrow \infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$;

- (iv) *asymptotically quasi-nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n \rightarrow \infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\|,$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

- (v) *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$.

From the above definitions, it is clear that:

- (i) a nonexpansive mapping is an asymptotically quasi-nonexpansive mapping;
- (ii) a quasi-nonexpansive mapping is an asymptotically quasi-nonexpansive mapping;
- (iii) an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping.

However, the converse of each of above statements may be not true.

The map $T : C \rightarrow C$ is said to be *demi-closed* at 0 if for each sequence $\{x_n\}$ in C converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

A Banach space X is said to satisfy *Opial's property* (see [1]) if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x , the following condition holds for all $x \neq y$:

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

In 1953, Mann [2] introduced the following iterative procedure for approximating a fixed point of a nonexpansive mapping T in a Hilbert space H :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \quad (1.1)$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see [3,4].

In 2003, Nakajo and Takahashi [5] proposed the following modification of the Mann iteration for a single nonexpansive mapping T in a Hilbert space H . They proved the following theorem:

Theorem 1.1. *Let C be a closed and convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1)$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C using the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \quad (1.2)$$

Then $\{x_n\}$ defined by (1.2) converges strongly to $P_{F(T)}x_0$.

Recently, Kim and Xu [6] extended the result of Nakajo and Takahashi [5] from a single nonexpansive mapping to a single asymptotically nonexpansive mapping in a Hilbert space. They proved the following theorem:

Theorem 1.2. *Let C be a nonempty bounded, closed and convex subset of a Hilbert space H and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \rightarrow 1$ as $n \rightarrow \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C using the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (1.3)$$

where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\text{diam}C)^2 \rightarrow 0$, as $n \rightarrow \infty$. Then $\{x_n\}$ defined by (1.3) converges strongly to $P_{F(T)}x_0$.

Since 2003, the strong convergence problems of the CQ method for fixed point iteration processes in a Hilbert space or a Banach space have been studied by many authors; see [7,8,6,9–11,5,12–16].

Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of uniformly L_i -Lipschitzian and asymptotically quasi-nonexpansive mappings and let $F := \bigcap_{i=1}^{\infty} F(T_i)$. In this paper, motivated by Kim and Xu [6], Nakajo and Takahashi [5], we introduce two kinds of new algorithms for finding a common fixed point of a countable family of uniformly Lipschitzian and asymptotically quasi-nonexpansive mappings which are modifications of the normal Mann iterative scheme. Our iterative schemes are defined as follows:

Algorithm 1.3. For an initial point $x_0 \in C$, compute the sequence $\{x_n\}$ by means of the iterative process

$$\begin{cases} y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n})T_i^n x_n, \\ C_{i,n} = \{z \in C : \|y_{i,n} - z\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1 - \alpha_{i,n})\|T_i^n x_n - x_n\|^2 + \theta_{i,n}\}, \\ C_n = \bigcap_{i=1}^{\infty} C_{i,n}, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle z - x_n, x_0 - x_n \rangle \leq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_{i,n} = (1 - \alpha_{i,n})(k_{i,n}^2 - 1)\nabla_n^2$, $\nabla_n = \sup_{n \in \mathbb{N}} \{\|x_n - z\| : z \in F\} < \infty$.

Algorithm 1.4. For an initial point $x_0 \in H$, compute the sequence $\{x_n\}$ by means of the iterative process

$$\begin{cases} C_{i,1} = C, & C_1 = \bigcap_{i=1}^{\infty} C_{i,1}, & x_1 = P_{C_1}x_0, \\ y_{i,n} = \alpha_{i,n}x_n + (1 - \alpha_{i,n})T_i^n x_n, \\ C_{i,n+1} = \{z \in C_{i,n} : \|y_{i,n} - z\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1 - \alpha_{i,n})\|T_i^n x_n - x_n\|^2 + \theta_{i,n}\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, & n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0, & n \geq 0, \end{cases}$$

where $\theta_{i,n} = (1 - \alpha_{i,n})(k_{i,n}^2 - 1)\nabla_n^2$, $\nabla_n = \sup_{z \in F} \{\|x_n - z\| : z \in F\} < \infty$.

2. Preliminaries

In this section, we collect and give some useful lemmas which will be used in our main results.

Lemma 2.1 ([17]). Let the sequences $\{a_n\}$ and $\{u_n\}$ of real numbers satisfy:

$$a_{n+1} \leq (1 + u_n)a_n, \quad \forall n \geq 1,$$

where $a_n \geq 0$, $u_n \geq 0$ and $\sum_{n=1}^{\infty} u_n < \infty$. Then:

- (i) $\lim_{n \rightarrow \infty} a_n$ exists;
- (ii) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2 ([18]). Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \omega_p(\lambda)g(\|x - y\|),$$

for all $x, y \in B_r(0) = \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0, 1]$ where $\omega_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 2.3. Let C be a nonempty subset of a Banach space X and $T : C \rightarrow C$ be uniformly L -Lipschitzian. Let $\{x_n\}$ be a sequence in C such that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$. Then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Proof. Observe that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_n - x_{n+1}\| + L\|T^n x_n - x_n\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$, it follows that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. $\square$

The following lemmas give some characterizations and a useful property of the metric projection P_C in a Hilbert space.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H . For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C . We know that P_C is a nonexpansive mapping of H onto C .

Lemma 2.4 ([9]). Let C be a closed and convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C . Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the following holds:

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.5 ([5]). Let C be a nonempty closed and convex subset of a real Hilbert space H and $P_C : H \rightarrow C$ be the metric projection from H onto C . Then the following inequality holds:

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in C.$$

Lemma 2.6 ([9]). Let H be a real Hilbert space. Then the following equations hold:

- (i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$, $\forall x, y \in H$;
- (ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, $\forall t \in [0, 1]$ and $x, y \in H$.

Lemma 2.7 ([19]). Let C be a nonempty closed and convex subset of a real Hilbert space H . Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex and closed.

Lemma 2.8. Let C be a closed and convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian and asymptotically quasi-nonexpansive mapping. Then $F(T)$ is a closed and convex subset of C .

Proof. Since T is continuous, $F(T)$ is closed. Next, we show that $F(T)$ is convex. Let $p = tp_1 + (1 - t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0, 1)$. By Lemma 2.6, we have

$$\begin{aligned} \|p - T^n p\|^2 &= \|t(T^n p - p_1) + (1 - t)(T^n p - p_2)\|^2 \\ &= t\|T^n p - p_1\|^2 + (1 - t)\|T^n p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &\leq tk_n^2\|p - p_1\|^2 + (1 - t)k_n^2\|p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &= t(1 - t)^2k_n^2\|p_1 - p_2\|^2 + (1 - t)t^2k_n^2\|p_1 - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &= ((1 - t)t(k_n^2 - 1))\|p_1 - p_2\|^2. \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \|p - T^n p\| = 0$. Since T is continuous, we have $\lim_{n \rightarrow \infty} \|Tp - T^{n+1}p\| = 0$, so $p = Tp$. Hence $p \in F(T)$. $\square$

3. Main results

First, we prove a weak convergence theorem for a uniformly Lipschitzian and asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space.

Theorem 3.1. Let X be a uniformly convex Banach space which satisfies Opial's property. Let C be a nonempty, closed and convex subset of X , and $T : C \rightarrow C$ a uniformly L -Lipschitzian and asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $\{k_n\} \subset [1, \infty)$ for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} k_n = 1$. Assume that $I - T$ is demi-closed at 0, where I is the identity mapping of C , $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C generated by the modified Mann iteration process:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Proof. Let $p \in F(T)$; we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|T^n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) k_n \|x_n - p\| \\ &\leq (1 + (k_n - 1)(1 - \alpha_n)) \|x_n - p\|. \end{aligned} \tag{3.1}$$

Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, then by Lemma 2.1 and (3.1), we have that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded. It follows that

$$\|T^n x_n - p\| \leq k_n \|x_n - p\|.$$

Hence $\{T^n x_n - p\}$ is bounded. Put $r = \sup_{n \in \mathbb{N}} \{\|T^n x_n - p\|, \|x_n - p\|\}$. By Lemma 2.2, there is a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(T^n x_n - p)\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|T^n x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|T^n x_n - x_n\|) \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)k_n^2 \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|T^n x_n - x_n\|). \end{aligned}$$

It follows that

$$\alpha_n(1 - \alpha_n)g(\|T^n x_n - x_n\|) \leq (k_n^2 - \alpha_n(k_n^2 - 1))\|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

We get that $\lim_{n \rightarrow \infty} g(\|T^n x_n - x_n\|) = 0$. Since g is continuous strictly increasing with $g(0) = 0$, we can conclude that $\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0$. Observe that $\|x_{n+1} - x_n\| \leq \alpha_n \|T^n x_n - x_n\| \rightarrow 0$. Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0$ and T is uniformly L -Lipschitzian, we obtain from Lemma 2.3 that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $q_1 \in C$. From $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $I - T$ being demi-closed at 0, we obtain that $Tq_1 = q_1$. That is $q_1 \in F$. Next, we show that $\{x_n\}$ converges weakly to q_1 .

Take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ converging weakly to some $q_2 \in C$. Again, as above, we can conclude that $q_2 \in F$. Finally, we show that $q_1 = q_2$. Assume $q_1 \neq q_2$. Then by the Opial's property of X , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q_1\| &= \lim_{k \rightarrow \infty} \|x_{m_k} - q_1\| < \lim_{k \rightarrow \infty} \|x_{m_k} - q_2\| = \lim_{n \rightarrow \infty} \|x_n - q_2\| \\ &= \lim_{k \rightarrow \infty} \|x_{m_k} - q_2\| < \lim_{k \rightarrow \infty} \|x_{m_k} - q_1\| = \lim_{n \rightarrow \infty} \|x_n - q_1\|, \end{aligned}$$

which is a contradiction. Therefore $q_1 = q_2$. This shows that $\{x_n\}$ converges weakly to $q_1 \in F(T)$. $\square$

Next, we prove strong convergence theorems for a countable family of uniformly Lipschitzian and asymptotically quasi-nonexpansive mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of uniformly L_i -Lipschitzian and asymptotically quasi-nonexpansive mappings of C into itself with $\{k_{i,n}\} \subset [1, \infty)$ such that $\lim_{n \rightarrow \infty} k_{i,n} = 1$ for all $i \in \mathbb{N}$. Assume that $F := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and the sequence $\{\alpha_{i,n}\} \subset [0, 1]$ is such that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.3 converges strongly to $P_F x_0$.*

Proof. We split the proof into seven steps.

Step 1. Show that $P_F x_0$ is well defined for every $x_0 \in C$.

By Lemma 2.8, we obtain that $F(T_i)$ is a closed convex subset of C for every $i \in \mathbb{N}$. Hence, $F := \bigcap_{i=1}^\infty F(T_i)$ is a nonempty closed convex subset of C ; consequently, $P_F x_0$ is well defined for every $x_0 \in C$.

Step 2. Show that both C_n and Q_n are closed and convex for all $n \geq 0$.

From the definition of C_n and Q_n , it is obvious that Q_n is closed and convex for each $n \in \mathbb{N}$. It follows from Lemma 2.7 that C_n is closed and convex for each $n \in \mathbb{N}$, so $P_{C_n \cap Q_n} x_0$ is well defined.

Step 3. Show that $F \subset C_n \cap Q_n$, for all $n \geq 0$.

First, we prove that $F \subset C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. Since T_i is an asymptotically quasi-nonexpansive mapping for all $i \in \mathbb{N}$, we obtain for any $p \in F$ that

$$\begin{aligned} \|y_{i,n} - p\|^2 &= \|\alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n - p\|^2 \\ &= \alpha_{i,n} \|x_n - p\|^2 + (1 - \alpha_{i,n}) \|T_i^n x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\ &\leq \alpha_{i,n} \|x_n - p\|^2 + (1 - \alpha_{i,n}) k_{i,n}^2 \|x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + (1 - \alpha_{i,n}) (k_{i,n}^2 - 1) \|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + \theta_{i,n}. \end{aligned} \quad (3.2)$$

We have $p \in C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. This proves that $F \subset C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. Hence, $F \subset C_n$ for all $n \geq 0$.

As shown in Marino and Xu [9], by induction, we can show that $F \subset Q_n$ for all $n \geq 0$. Hence $F \subset C_n \cap Q_n$, for all $n \geq 0$.

Step 4. Show that $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

From $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0. \quad (3.3)$$

On the other hand, as $F \subset Q_n$, we obtain

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad \forall n \geq 0, \forall z \in F. \quad (3.4)$$

Combining (3.3) and (3.4), the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

Step 5. Show that $x_n \rightarrow q$, where $q \in C$.

For $m > n$, by the definition of Q_n , we see that $Q_m \subset Q_n$. Noting that $x_m = P_{Q_m} x_0$ and $x_n = P_{Q_n} x_0$, by Lemma 2.5, we conclude that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

From Step 4, we obtain that $x_m - x_n \rightarrow 0$ as $m, n \rightarrow \infty$, which means that $\{x_n\}$ is Cauchy. Since H is complete and C is closed, we can assume that $x_n \rightarrow q \in C$ as $n \rightarrow \infty$. In particular, we have that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Step 6. Show that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$.

Let $i, n \in \mathbb{N}$. Since $x_{n+1} \in C_n$, it follows that

$$\|y_{i,n} - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + \theta_{i,n}. \quad (3.5)$$

Moreover, since $y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n$ and by Lemma 2.6, we have

$$\begin{aligned} \|y_{i,n} - x_{n+1}\|^2 &= \|\alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n - x_{n+1}\|^2 \\ &= \alpha_{i,n} \|x_n - x_{n+1}\|^2 + (1 - \alpha_{i,n}) \|T_i^n x_n - x_{n+1}\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2. \end{aligned} \quad (3.6)$$

Substitute (3.6) into (3.5) to get

$$(1 - \alpha_{i,n}) \|T_i^n x_n - x_{n+1}\|^2 \leq (1 - \alpha_{i,n}) \|x_n - x_{n+1}\|^2 + \theta_{i,n}. \quad (3.7)$$

From (3.7) and our assumptions, we have

$$\lim_{n \rightarrow \infty} \|T_i^n x_n - x_{n+1}\| = 0.$$

It follows that

$$\|T_i^n x_n - x_n\| \leq \|T_i^n x_n + x_{n+1}\| + \|x_{n+1} - x_n\| \rightarrow 0$$

for all $i \in \mathbb{N}$. From Lemma 2.3, we get that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$.

Step 7. Show that $q = P_F x_0$.

Since $x_n \rightarrow q$ and $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$, we have $q = T_i q$, $\forall i \in \mathbb{N}$. Hence $q \in F$. By Lemma 2.4, we obtain

$$\langle z - x_n, x_0 - x_n \rangle \leq 0$$

for all $z \in Q_n$. This leads to

$$\langle z - q, x_0 - q \rangle \leq 0$$

for all $z \in F$. By Lemma 2.4, we conclude that $q = P_F x_0$. This completes the proof. $\square$

Theorem 3.3. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of uniformly L_i -Lipschitzian and asymptotically quasi-nonexpansive mappings of C into itself with $\{k_{i,n}\} \subset [1, \infty)$ such that $\lim_{n \rightarrow \infty} k_{i,n} = 1$ for all $i \in \mathbb{N}$. Assume that $F := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and the sequence $\{\alpha_{i,n}\} \subset [0, 1]$ is such that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.4 converges strongly to $P_F x_0$.

Proof. We divide our proof into four steps.

Step 1. Show that C_n is closed and convex and $F \subset C_n$ for all $n \geq 1$.

It suffices to show that $\forall i \in \mathbb{N}$, $C_{i,n}$ is closed and convex and $F \subset C_{i,n}$ for every $n \geq 1$. We prove this by induction on n . In fact, for $n = 1$, $C_{i,1} = C$ is closed and convex. Assume that $C_{i,n}$ is closed and convex for some $n \geq 1$; from the definition $C_{i,n+1}$, we obtain that $C_{i,n+1}$ is also closed and convex for some $n \geq 1$, and hence $C_{i,n}$ is closed and convex for all $n \geq 1$. For $n = 1$, $F \subset C_{i,1} = C$. Assume that $F \subset C_{i,n}$ for some $n \geq 1$; it follows from the induction assumption, (3.2) and the definition of $C_{i,n+1}$ that we have that $F \subset C_{i,n+1}$; hence $F \subset C_{i,n}$.

Step 2. Show that $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

From $x_n = P_{C_n} x_0$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \geq 0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0. \quad (3.8)$$

On the other hand, as $F \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad \forall n \geq 0, \forall z \in F. \quad (3.9)$$

Combining (3.8) and (3.9), the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

Step 3. Show that $x_n \rightarrow q$, where $q \in C$.

For $m > n$, by the definition of C_n , we see that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By Lemma 2.5, we obtain that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence $x_n \rightarrow q \in C$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. By using the same proof as in Step 6 of Theorem 3.2, we can show that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$.

Step 4. Show that $q = P_F x_0$.

Since $x_n \rightarrow q$ and $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$, we have $q = T_i q$ $\forall i \in \mathbb{N}$. Hence $q \in F \subset C_n$ and since $x_n = P_{C_n} x_0$, by Lemma 2.4, we have

$$\langle z - x_n, x_0 - x_n \rangle \leq 0,$$

for all $z \in C_n$, and hence,

$$\langle z - q, x_0 - q \rangle \leq 0,$$

for all $z \in F$. This shows that $q = P_F x_0$, which completes the proof. $\square$

Remark 3.4. Theorems 3.2 and 3.3 extend and improve the main result of [6,5].

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1.3 Strong convergence of composite iterative schemes for a countable family of nonexpansive mappings in Banach spaces

In this section, we propose a new modified viscosity approximation method for approximating common fixed points for a countable family of nonexpansive mappings in a Banach space. We prove strong convergence theorems for a countable family nonexpansive mappings in a reflexive Banach space with uniformly Gateaux differentiable norm under some control conditions. These results improve and extend the results of Jong Soo Jung [J.S. Jung, Convergence on composite iterative schemes for nonexpansive mappings in Banach spaces, Fixed Point Theory and Appl. 2008 (2008) 14 pp., Article ID 167535]. Further, we apply our result to the problem of finding a zero of an accretive operator and extend the results of Kim and Xu [T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 5160], Ceng, et al. [L.-C. Ceng, A.R. Khan, Q.H. Ansari, J.-C. Yao, Strong convergence of composite iterative schemes for zeros of m -accretive operators in Banach space, Nonlinear Anal. 70 (2009) 1830-1840] and Chen and Zhu [R. Chen, Z. Zhu, Viscosity approximation methods for accretive operator in Banach space, Nonlinear Anal. 69 (2008) 1356-1363].

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E . Recall that a mapping $T : C \rightarrow C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C \mid x = Tx\}$, and a self-mapping $f : C \rightarrow C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C , that is $\Pi_C = \{f : C \rightarrow C \mid f \text{ is a contraction with a constant } \alpha\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C , and, for any fixed element $x_0 \in C$, Picard's iteration $x_{n+1} = f^n(x_0)$ converges strongly to a unique fixed point of f . However, a simple example shows that Picard's iteration cannot be used in the case of nonexpansive mappings. One method used for nonexpansive mappings is to employ a Halpern-type iterative scheme which produces a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \end{aligned} \tag{1.1}$$

where $u \in C$ is arbitrary (but fixed) and $\{\alpha_n\} \subset [0, 1]$.

In 1967, Halpern [1] was the first to study the convergence of the iterative scheme (1.1) in a framework of Hilbert space. Lions [2] and Wittmann [3] improved the result of Halpern. Reich [4] and Shioji and Takahashi [5] extended the result of

Wittmann [3] to the case of Banach space. They proved the strong convergence of the sequence $\{x_n\}$ defined by (1.1) under certain control conditions.

In 2000, Moudafi [6] introduced a viscosity approximation method for a nonexpansive mapping as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \end{aligned} \quad (1.2)$$

where $f \in \Pi_C$ and $\{\alpha_n\} \subset [0, 1]$. He proved that if E is a real Hilbert space and $\{\alpha_n\}$ satisfies certain control conditions then the sequence $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T which is the unique solution to the variational inequality $\langle (I - f)\bar{x}, x - \bar{x} \rangle \geq 0$ for all $x \in F(T)$. In 2004, Xu [7] extended the results of Moudafi [6] for the iterative scheme (1.2) to a Banach space setting.

In 2005, Kim and Xu [8] studied the sequence $\{x_n\}$ generated as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n u + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \quad (1.3)$$

where $u \in C$ is arbitrary (but fixed) and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. They proved the strong convergence of iterative scheme (1.3) in the framework of a uniformly smooth Banach space. In 2008, Yao et al. [9] proposed a new modified iteration by combining the Mann iteration and the viscosity approximation method introduced by Moudafi [6]. This iterative scheme is defined as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \quad (1.4)$$

where $f \in \Pi_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. They proved that under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ the iterative scheme (1.4) strongly converges to a fixed point of T .

Recently, Jung [10] introduced a new composite iterative scheme for a nonexpansive mapping T as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n Ty_n, \quad n \geq 0 \end{aligned} \quad (1.5)$$

where $f \in \Pi_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. He proved the strong convergence of the sequence $\{x_n\}$ defined by (1.5) under suitable conditions of the control parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a uniformly Gateaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

On the other hand, Aoyama et al. [11] introduced a Halpern-type iterative scheme for finding a common fixed point of a countable family of nonexpansive mappings as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)T_n x_n, \quad n \geq 0 \end{aligned} \quad (1.6)$$

where $u \in C$ is arbitrary (but fixed), $\{\alpha_n\} \subset [0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved that the sequence $\{x_n\}$ defined by (1.6) converges strongly to a common fixed point of $\{T_n\}$.

Recently, Takahashi [12] studied the iterative scheme (1.2) for finite countable families of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n, \quad n \geq 0 \end{aligned} \quad (1.7)$$

where $f \in \Pi_C$, $\{\alpha_n\} \subset [0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. He proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of $\{T_n\}$.

Recently, Plubtieng and Wangkeeree [13] modified the iterative scheme (1.4) for a countable families of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)T_n x_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \quad (1.8)$$

where $f \in \Pi_C$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved, under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that the sequence $\{x_n\}$ defined by (1.8) converges strongly to a common fixed point of $\{T_n\}$.

Inspired and motivated by these facts, a new iterative scheme for countable family nonexpansive mappings is introduced and studied in this paper. This iterative scheme can be viewed as an extension of the iterative scheme (1.5). We introduce the following iterative scheme:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0 \end{aligned} \quad (1.9)$$

where C is a nonempty closed subset of a Banach space, $f \in \Pi_C$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. If $T_n = T$ for all $n \geq 0$, then (1.9) reduces to (1.5) and if $T_n = T$ for all $n \geq 0$ and $\beta_n = 0$, then (1.9) reduces to (1.2); finally, if $\beta_n = 0$, then (1.9) reduces to (1.7). The main propose of this paper is to prove the strong convergence of the sequence $\{x_n\}$ defined by (1.9) to a common fixed point of $\{T_n\}$ under certain suitable conditions of the control parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and the asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a uniformly Gateaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Further, we apply our result to the problem of finding a zero of an accretive operator.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E . The value of $f \in E^*$ at $x \in E$ is denoted by $\langle x, f \rangle$. Recall that the duality map J from E into 2^{E^*} is defined by $J(x) = \{f \in E^* | \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$ for every $x \in E$. Let $S = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gateaux differentiable if, for each $x, y \in S$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists. In this case, the Banach space E is said to be smooth. The norm of E is said to be uniformly Gateaux differentiable if, for each $y \in S$, the limit (2.1) is attained uniformly for $x \in S$. We know that, if E is smooth, then the duality mapping J is single valued. The norm of E is said to be uniformly Frechet differentiable if the limit (2.1) is attained uniformly for $x, y \in S$, and in this case E said to be uniformly smooth. It is known that E is uniformly smooth if and only if the duality mapping J is single valued and norm-to-norm uniformly continuous on each bounded subset of E . It also known that, if E has a uniformly Gateaux differentiable norm, J is uniformly norm to weak continuous on each bounded subsets of E . We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and write $x_n \rightarrow x$ to indicate that $\{x_n\}$ converges strongly to x .

Recall that, if C is nonempty closed convex subset of E , then C is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D . We need the following lemmas for the proof of our main results.

Lemma 2.1 ([7]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, $n \geq 0$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$, (ii) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2. Let J be the duality map of a Banach space E . Suppose that X is smooth. Then, for all $x, y \in X$, there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle.$$

Lemma 2.3 ([11]). Let C be a nonempty closed convex subset of E . Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C . Moreover, let T be a mapping of C into itself defined by $Tz = \lim_n T_n z$ for all $z \in C$. Then $\lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

Lemma 2.4 ([14]). Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C a nonempty closed convex subset of E , $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then the unique fixed point $x_t \in C$ of the contraction $x \mapsto tf(x) + (1 - t)Tx$ converges strongly to a fixed point of T as $t \rightarrow 0^+$. If we define $Q : \Pi_C \rightarrow F(T)$ by $Q(f) = \lim_{t \rightarrow 0^+} x_t$, then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, \quad p \in F(T).$$

Finally, a sequence $\{x_n\}$ in E is said to be asymptotically regular if $\lim_n \|x_{n+1} - x_n\| = 0$.

3. Main results

In this section, we prove several strong convergence theorems of the composite iterative scheme (1.9) to a common fixed point of a countable family of nonexpansive mappings.

Theorem 3.1. Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E , and $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ which satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0. \end{aligned}$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C . Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, take an arbitrary fixed $p \in \bigcap_{n=1}^{\infty} F(T_n)$. By the definition of $\{y_n\}$ and $\{x_n\}$, we have

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1 - \alpha_n) T_n x_n - p\| \\ &= \|\alpha_n (f(x_n) - f(p)) + \alpha_n (f(p) - p) + (1 - \alpha_n) (T_n x_n - p)\| \\ &\leq \alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &= (1 - (1 - \alpha) \alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned} \quad (3.1)$$

We obtain that $\|y_n - p\| \leq \max\{\|x_n - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\}$.

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \beta_n) y_n + \beta_n T_n y_n - p\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|T_n y_n - p\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|y_n - p\| \\ &= \|y_n - p\| \leq \max\left\{\|x_n - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\right\}. \end{aligned}$$

By induction on n , we obtain that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\}$ for all $n \geq 0$ and all $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence, the sequence $\{x_n\}$ is bounded and so $\{y_n\}$, $\{T_n x_n\}$ and $\{f(x_n)\}$ are bounded sequences. Moreover, it follows from condition (i) that

$$\|y_n - T_n x_n\| = \alpha_n \|f(x_n) - T_n x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

Observing that

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|(1 - \beta_n) y_n + \beta_n T_n y_n - y_n\| \\ &= \beta_n \|T_n y_n - y_n\| \\ &\leq \beta_n (\|T_n y_n - T_n x_n\| + \|T_n x_n - y_n\|) \\ &\leq b (\|y_n - x_n\| + \|T_n x_n - y_n\|) \\ &\leq b (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|T_n x_n - y_n\|), \end{aligned}$$

we obtain

$$\|x_{n+1} - y_n\| \leq \frac{b}{1-b} (\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|). \quad (3.3)$$

So, by (3.2) and the asymptotic regularity of $\{x_n\}$, we have $\|x_{n+1} - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.4)$$

Next, we show that $\|T x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

By the asymptotic regularity of $\{x_n\}$ and Lemma 2.3, we have

$$\lim_n \|x_{n+1} - x_n\| = 0 = \lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in \{x_k\}\}. \quad (3.5)$$

By (3.3), we have

$$\begin{aligned}\|T_n x_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \frac{b}{1-b}(\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|) + \|y_n - T_n x_n\| \\ &\leq \frac{1}{1-b}(\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|).\end{aligned}$$

It follows by (3.2) that

$$\|T_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

Therefore, by (3.5) and (3.6), we have

$$\begin{aligned}\|T x_n - x_n\| &\leq \|T x_n - T_n x_n\| + \|T_n x_n - x_n\| \\ &\leq \sup\{\|T z - T_n z\| : z \in \{x_k\}\} + \|T_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Now, observe that

$$\begin{aligned}\|y_n - T_n y_n\| &\leq \|y_n - x_n\| + \|x_n - T_n x_n\| + \|T_n x_n - T_n y_n\| \\ &\leq \|y_n - x_n\| + \|x_n - T_n x_n\| + \|x_n - y_n\| \\ &= 2\|x_n - y_n\| + \|x_n - T_n x_n\|.\end{aligned} \quad (3.7)$$

It follows by (3.4) and (3.6) that $\|y_n - T_n y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we have

$$\begin{aligned}\|T y_n - y_n\| &\leq \|T y_n - T_n y_n\| + \|T_n y_n - y_n\| \\ &\leq \sup\{\|T z - T_n z\| : z \in \{y_k\}\} + \|T_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned} \quad (3.8)$$

Next, we claim that $\limsup_{n \rightarrow \infty} \langle f(q) - q, J(y_n - q) \rangle \leq 0$, where $q = Q(f)$ is defined as in Lemma 2.4. To prove this, let a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ be such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(y_n - q) \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, J(y_{n_k} - q) \rangle \quad (3.9)$$

and $y_{n_k} \rightharpoonup p$ for some $p \in C$. By the demiclosed principle and the fact that $\|T y_{n_k} - y_{n_k}\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain $p \in F(T)$. By Lemma 2.4 we have $x_t \rightarrow q = Q(f) \in F(T)$, where x_t is the fixed point of the contraction $x \mapsto t f(x) + (1-t)Tx$. By Lemma 2.2, we obtain

$$\begin{aligned}\|x_t - y_{n_k}\|^2 &= \|(1-t)(T x_t - y_{n_k}) + t(f(x_t) - y_{n_k})\|^2 \\ &\leq (1-t)^2 \|T x_t - y_{n_k}\|^2 + 2t \langle f(x_t) - y_{n_k}, J(x_t - y_{n_k}) \rangle \\ &\leq (1-t)^2 (\|T x_t - T y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|)^2 + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + 2t \langle x_t - y_{n_k}, J(x_t - y_{n_k}) \rangle \\ &\leq (1-t)^2 (\|x_t - y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|)^2 + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + 2t \|x_t - y_{n_k}\|^2 \\ &\leq (1-t)^2 \|x_t - y_{n_k}\|^2 + \|T y_{n_k} - y_{n_k}\| (2\|x_t - y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|) \\ &\quad + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + 2t \|x_t - y_{n_k}\|^2 \\ &= (1+t^2) \|x_t - y_{n_k}\|^2 + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + a_k(t)\end{aligned} \quad (3.10)$$

where $a_k(t) = \|T y_{n_k} - y_{n_k}\| (2\|x_t - y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|)$. It follows by (3.8) that $a_k(t) \rightarrow 0$ as $k \rightarrow \infty$. By (3.10), we obtain

$$\langle f(x_t) - x_t, J(y_{n_k} - x_t) \rangle \leq \frac{t}{2} \|x_t - y_{n_k}\|^2 + \frac{1}{2t} a_k(t). \quad (3.11)$$

Letting $k \rightarrow \infty$ in (3.11), we obtain

$$\limsup_{k \rightarrow \infty} \langle f(x_t) - x_t, J(x_{n_k} - x_t) \rangle \leq \frac{t}{2} M, \quad (3.12)$$

where $M > 0$ is a constant such that $M \geq \|x_t - y_{n_k}\|^2$ for all $t \in (0, 1)$ and $k \geq 0$. Taking the limsup as $t \rightarrow 0$ in (3.12), we immediately get

$$\limsup_{t \rightarrow 0} \limsup_{k \rightarrow \infty} \langle f(x_t) - x_t, J(y_{n_k} - x_t) \rangle \leq 0.$$

Since E is a reflexive Banach space with uniformly Gateaux differentiable norm, then the duality map is norm-to-weak uniformly continuous on bounded subsets of C . By the same argument as in the proof of [10, Theorem 3.2], we obtain

$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(y_n - q) \rangle \leq 0$. Finally, we claim that $x_n \rightarrow q$ as $n \rightarrow \infty$. Indeed, using Lemma 2.2 and (3.1) again, we obtain that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 = \|\alpha_n(f(x_n) - q) + (1 - \alpha_n)(T_n x_n - q)\|^2 \\ &\leq (1 - \alpha_n)^2 \|T_n x_n - q\|^2 + 2\alpha_n \langle f(x_n) - q, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q), J(y_n - q) \rangle + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \|x_n - q\| \|y_n - q\| + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &\quad + 2\alpha_n \|x_n - q\| ((1 - (1 - \alpha)\alpha_n) \|x_n - q\| + \alpha_n \|f(q) - q\|) \\ &= (1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n(1 - (1 - \alpha)\alpha_n)) \|x_n - q\|^2 \\ &\quad + 2\alpha_n^2 \|x_n - q\| \|f(q) - q\| + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &= (1 - \gamma_n) \|x_n - q\|^2 + \delta_n, \end{aligned}$$

where $\gamma_n = \alpha_n(2 - \alpha_n - 2\alpha(1 - (1 - \alpha)\alpha_n))$ and $\delta_n = \alpha_n(2\alpha\alpha_n \|x_n - q\| \|f(q) - q\| + 2\langle f(q) - q, J(y_n - q) \rangle)$. It is easily seen that $\lim_{n \rightarrow \infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$. By Lemma 2.1, we conclude that $x_n \rightarrow q$. $\square$

Theorem 3.2. Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E . Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ which satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0. \end{aligned}$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C . Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. Since E is a uniformly smooth Banach space, E is reflexive and the norm is uniformly Gateaux differentiable norm and its every nonempty weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Thus, the conclusion of the theorem follows from Theorem 3.1 immediately. $\square$

Theorem 3.3. Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E , and $\{T_n\}$ a sequence nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C . Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0. \end{aligned}$$

If $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ which satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$,
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ which satisfy conditions (i)–(iii). We shall show that $\{x_n\}$ is asymptotic regular. By the definition of $\{y_n\}$ and $\{x_n\}$, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|(\alpha_n f(x_n) + (1 - \alpha_n) T_n x_n) - (\alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1}) T_{n-1} x_{n-1})\| \\ &\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + (1 - \alpha_n) \|T_n x_n - T_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - T_{n-1} x_{n-1}\| \\ &\leq \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n) \|T_n x_n - T_{n-1} x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - T_{n-1} x_{n-1}\| \\ &= (1 - (1 - \alpha)\alpha_n) \|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - T_{n-1} x_{n-1}\|, \end{aligned}$$

and

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|((1 - \beta_n)y_n + \beta_n T_n y_n) - ((1 - \beta_{n-1})y_{n-1} + \beta_{n-1} T_{n-1} y_{n-1})\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|T_n y_n - T_{n-1} y_{n-1}\| + |\beta_n - \beta_{n-1}|\|T_{n-1} y_{n-1} - y_{n-1}\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|T_n y_n - T_n y_{n-1}\| + \beta_n\|T_n y_{n-1} - T_{n-1} y_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}|\|T_{n-1} y_{n-1} - y_{n-1}\| \\
 &\leq \|y_n - y_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\| + |\beta_n - \beta_{n-1}|\|T_{n-1} y_{n-1} - y_{n-1}\| \\
 &\leq (1 - (1 - \alpha)\alpha_n)\|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\
 &\quad + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\|
 \end{aligned}$$

where $M = \max\{\sup_n \|f(x_{n-1}) - T_{n-1} x_{n-1}\|, \sup_n \|T_{n-1} y_{n-1} - y_{n-1}\|\}$.

Put

$$\mu_n = M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\|, \quad n \geq 2.$$

We have

$$\begin{aligned}
 \sum_{n=2}^{\infty} \mu_n &\leq M \sum_{n=2}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) + \sum_{n=2}^{\infty} \sup\{\|T_n z - T_{n-1} z\| : z \in \{x_k\}\} \\
 &\quad + \sum_{n=2}^{\infty} \sup\{\|T_n z - T_{n-1} z\| : z \in \{y_k\}\} < \infty.
 \end{aligned}$$

It follows by Lemma 2.1 that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$; hence $\{x_n\}$ is asymptotic regular. By Theorem 3.1, we obtain that $\{x_n\}$ strongly converges to a common fixed point of $\{T_n\}$. $\square$

If $T_n = T$ for all $n \in \mathbb{N}$, then the following result is obtained directly from Theorem 3.1, and we get the result of Jung [10].

Corollary 3.4 ([10], Theorem 3.2). *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E , and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ which satisfy the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
 x_0 &= x \in C, \\
 y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 0 \\
 x_{n+1} &= (1 - \beta_n) y_n + \beta_n T y_n, \quad n \geq 0.
 \end{aligned}$$

If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to a fixed point of T .

4. Applications

In this section, we apply Theorem 3.1 for finding a zero of an accretive operator. Let E be a real Banach space. Recall that an operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ in E is called accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ ($i = 1, 2$), there exists a $j \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j \rangle \geq 0$, where $J : E \rightarrow 2^{E^*}$ is the duality mapping. An accretive operator A is m -accretive if $R(I + rA) = E$ for each $r \geq 0$. Throughout this section we always assume that A is m -accretive and has a zero. The set of zeros of A is denoted by $A^{-1}0$, that is $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$. For each $r \geq 0$, we denote by J_r the resolvent of A , that is $J_r = (I + rA)^{-1}$. Note that, if A is m -accretive, then $J_r : E \rightarrow E$ is a nonexpansive mapping and $F(J_r) = A^{-1}0$ for all $r \geq 0$. It is well known that, for each $r, s \geq 0$ and $x \in R(I + rA) \cap R(I + sA)$, it holds that

$$\|J_r x - J_s x\| \leq \frac{|r - s|}{r} \|x - J_r x\|. \quad (4.1)$$

From the resolvent identity, we have the following lemma:

Lemma 4.1 ([11]). *Let E be a Banach space and C a nonempty closed convex subset of E . Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$. Suppose that $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\inf\{r_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$. Then*

- (i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| : z \in B\} < \infty$ for any bounded subset B of C .
- (ii) $\lim_{n \rightarrow \infty} J_{r_n} z = J_r z$ for all $z \in C$ and $F(J_r) = \bigcap_{n=1}^{\infty} F(J_{r_n})$, where $r_n \rightarrow r$ as $n \rightarrow \infty$.

From Lemma 4.1 and Theorem 3.3, we obtain the following results.

Theorem 4.2. Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E , and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Given sequences $\{\alpha_n\}$ in $(0, 1)$, $\{\beta_n\}$ in $[0, 1]$ and $\{r_n\}$ in $(0, \infty)$, suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\beta_n \in [0, b)$ for some $b \in (0, 1)$,
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n J_{r_n} y_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

If $f(x) = u$ for all $x \in C$ in Theorem 4.2, we obtain the result of Ceng et al. [15].

Theorem 4.3 ([15], Theorem 2.1). Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Given sequences $\{\alpha_n\}$ in $(0, 1)$, $\{\beta_n\}$ in $[0, 1]$ and $\{r_n\}$ in $(0, \infty)$, suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\beta_n \in [0, b)$ for some $b \in (0, 1)$,
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n J_{r_n} y_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

If $\beta_n = 0$ in Theorem 4.2, we obtain the result of Chen and Zhu [16].

Theorem 4.4 ([16], Theorem 3.2). Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ and $\{r_n\}$ in $(0, \infty)$ which satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $f \in \Pi_C$ and x_0 be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

If $\beta_n = 0$ and $f(x) = u$ for all $x \in C$ in Theorem 4.2, we obtain the result of Kim and Xu [8].

Theorem 4.5 ([8], Theorem 2). Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ and $\{r_n\}$ in $(0, \infty)$ which satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

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1.4 Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces

In this section, we investigate strong convergence for Bregman strongly nonexpansive mappings by Halpern's iteration in the framework of reflexive Banach spaces. Using the obtained result, convergence for a family of Bregman strongly nonexpansive mappings is also discussed. As applications, we apply our main result to problems of finding zeros of maximal monotone operators and equilibrium problems in reflexive Banach spaces.

1. Introduction

Let E be a real reflexive Banach space and C a nonempty, closed and convex subset of E . Let $T : C \rightarrow C$ be a nonlinear mapping. The fixed points set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. A mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In recent years, several types of iterative schemes have been constructed and proposed in order to get strong convergence results for nonexpansive mappings in various settings. One classical and effective iteration process is defined as follows: $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (1.1)$$

where $\{\alpha_n\} \subset (0, 1)$. Such a method was introduced, in 1967, by Halpern [1] and is often called *Halpern's iteration*. In fact, he proved, in a real Hilbert space, strong convergence of $\{x_n\}$ to a fixed point of a nonexpansive mapping T , where $\alpha_n = n^{-a}$, $a \in (0, 1)$.

Now, because of a simple construction, Halpern's iteration is widely used to approximate a solution of fixed points for nonexpansive mappings and other classes of nonlinear mappings by mathematicians in different styles.

In 1977, Lions [2] obtained a strong convergence provided the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1: \lim_{n \rightarrow \infty} \alpha_n = 0; C2: \sum_{n=1}^{\infty} \alpha_n = \infty; C3: \lim_{n \rightarrow \infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}^2} = 0.$$

Reich [3] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lions' conditions imposed on the real sequence $\{\alpha_n\}$ exclude the canonical choice $\alpha_n = 1/n$ for all $n \in \mathbb{N}$.

Subsequently, Wittmann [4] overcome the problem mentioned above by proving strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the conditions C1, C2 and C4:

$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

In 1997, Shioji–Takahashi [5] extended Wittmann's result from Hilbert spaces to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings.

In 2002, Xu [6] introduced another control condition C5: $\frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}} \rightarrow 0$ instead of the conditions C3 or C4 and proved strong convergence of the sequence $\{x_n\}$.

In 2005, Cho–Kang–Zhou [7] pointed out that the control conditions C4 and C5 are not comparable, in general. They gave some examples which satisfy the conditions C1, C2, C3, C4 and C5, and also presented the control condition C6:

$$|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$. This includes the conditions C3, C4 and C5 as special cases.

One question arises in literature naturally: Is it possible to get strong convergence of (1.1) when the sequence $\{\alpha_n\}$ satisfies only the conditions C1 and C2?

Recently, Chidume–Chidume [8] and Suzuki [9] independently gave an affirmative answer to the above question. To be more precise, they introduced the following Halpern-type iteration: $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n), \quad \forall n \geq 1, \quad (1.2)$$

and obtained strong convergence results for the sequence $\{x_n\}$ generated by (1.2) when only the conditions C1 and C2 are imposed on the sequence $\{\alpha_n\}$.

Very recently, Saejung [10] focused in studying Halpern's iteration for an important subclass of nonexpansive mappings which is the so-called *strongly nonexpansive* [11], i.e., a mapping $T : C \rightarrow C$ satisfying T is nonexpansive and

$$x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0.$$

He proved that the sequence generated by Halpern's iteration converges strongly to a fixed point of T if the sequence $\{\alpha_n\} \subset (0, 1)$ just satisfies the conditions C1 and C2. This shows that a class of strongly nonexpansive mappings works for Halpern's iteration with the conditions C1 and C2.

The purpose of this work is to consider strong convergence results for Bregman strongly nonexpansive mappings in reflexive Banach spaces by Halpern's iteration with only the conditions C1 and C2. We note that there are many examples which are Bregman strongly nonexpansive such as the Bregman projection, the resolvents of maximal monotone operators, the resolvents of equilibrium problems, the resolvents of variational inequality problems and others (see, for example, [12–15]). Furthermore, using the main result, some convergence theorems for a family of Bregman strongly nonexpansive are also established. Finally, we give some applications concerning the problems of finding zeros of maximal monotone operators and equilibrium problems.

2. Preliminaries and lemmas

In this section, we begin by recalling some preliminaries and lemmas which will be used in the proof.

Let E be a real reflexive Banach space with the norm $\|\cdot\|$ and E^* the dual space of E . Throughout this paper, $f : E \rightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\text{dom } f$ the domain of f , that is, the set $\{x \in E : f(x) < +\infty\}$.

Let $x \in \text{int dom } f$. The *subdifferential* of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y) \quad \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \rightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup \{ \langle x^*, x \rangle - f(x) : x \in E \}.$$

We know that the Young–Fenchel inequality holds:

$$\langle x^*, x \rangle \leq f(x) + f^*(x^*) \quad \forall x \in E, x^* \in E^*.$$

Furthermore, equality holds if $x^* \in \partial f(x)$ (see also [16, Theorem 23.5]).

The set $\text{lev}_{\leq}^f(r) = \{x \in E : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a *sublevel* of f .

A function f on E is *coercive* [17] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty.$$

A function f on E is said to be *strongly coercive* [18] if

$$\lim_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \text{int dom } f$ and $y \in E$, the *right-hand derivative* of f at x in the direction y is defined by

$$f^o(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x if $\lim_{t \rightarrow 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y . In this case, $f^o(x, y)$ coincides with $\nabla f(x)$, the value of the *gradient* ∇f of f at x . The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \text{int dom } f$. The function f is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int dom } f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $\text{int dom } f$ (see also [19,20]).

Definition 2.1 ([21, Definition 5.2]). The function f is said to be:

- (i) *essentially smooth*, if ∂f is both locally bounded and single-valued on its domain.
- (ii) *essentially strictly convex*, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\text{dom } \partial f$.
- (iii) *Legendre*, if it is both essentially smooth and essentially strictly convex.

Remark 2.2. Let E be a reflexive Banach space. Then we have

- (a) f is essentially smooth if and only if f^* is essentially strictly convex (see [21, Theorem 5.4]).
- (b) $(\partial f)^{-1} = \partial f^*$ (see [20]).
- (c) f is Legendre if and only if f^* is Legendre. (see [21, Corollary 5.5]).
- (d) If f is Legendre, then ∇f is a bijection satisfying

$\nabla f = (\nabla f^*)^{-1}$, $\text{ran } \nabla f = \text{dom } \nabla f^* = \text{int dom } f^*$ and $\text{ran } \nabla f^* = \text{dom } \nabla f = \text{int dom } f$ (see [21, Theorem 5.10] and [14]).

Examples of Legendre functions were given in [22,21]. One important and interesting Legendre function is $\frac{1}{p} \|\cdot\|^p$ ($1 < p < \infty$) when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E , i.e., $\nabla f = J_p$ ($1 < p < \infty$). In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \rightarrow (-\infty, +\infty]$ is Legendre.

The following crucial lemma was proved by Reich–Sabach [23] (see also [18, Proposition 3.6.3]).

Lemma 2.3 ([23, Proposition 2.1]). If $f : E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E , then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .

Let $f : E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom } f \times \text{int dom } f \rightarrow [0, +\infty)$ defined as follows:

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance* with respect to f [24].

Recall that the *Bregman projection* [25] of $x \in \text{int dom } f$ onto the nonempty closed and convex set $C \subset \text{dom } f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}.$$

Let $f : E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The *modulus of total convexity* of f at $x \in \text{int dom } f$ is the function $v_f(x, \cdot) : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$v_f(x, t) := \inf \{D_f(y, x) : y \in \text{dom } f, \|y - x\| = t\}.$$

The function f is called *totally convex* at x if $v_f(x, t) > 0$ whenever $t > 0$. The function f is called *totally convex* if it is totally convex at any point $x \in \text{int dom } f$ and is said to be *totally convex on bounded sets* if $v_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$v_f(B, t) := \inf \{v_f(x, t) : x \in B \cap \text{dom } f\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [26, Theorem 2.10]).

Recall that the function f is said to be *sequentially consistent* [26] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded,

$$\lim_{n \rightarrow \infty} D_f(y_n, x_n) = 0 \implies \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

The following crucial lemma was proved by Butnariu–Iusem [27].

Lemma 2.4 ([27, Lemma 2.1.2]). *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Let C be a convex subset of $\text{int dom } f$ and let T be a self-mapping of C . A point $p \in C$ is called an *asymptotic fixed point* of T [28,29] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T .

Definition 2.5. A mapping T with a nonempty fixed point set is said to be:

(i) *Bregman strongly nonexpansive* [11,14] with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in \widehat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{n \rightarrow \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n \rightarrow \infty} D_f(x_n, Tx_n) = 0.$$

(ii) *Bregman firmly nonexpansive* [12,30,15] if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

The existence and approximation of Bregman firmly nonexpansive mappings was studied in [15]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E , then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [15]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [26] that $z = P_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \quad \forall y \in C.$$

We also know the following:

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall x \in E, y \in C.$$

Let $f : E \rightarrow \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [31] and [24], we make use of the function $V_f : E \times E^* \rightarrow [0, +\infty)$ associated with f , which is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x \rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*) \quad (2.1)$$

for all $x \in E$ and $x^*, y^* \in E^*$ (see also [32, Lemmas 3.2 and 3.3]). In addition, if $f : E \rightarrow (-\infty, +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \rightarrow (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function (see [33]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i), \quad (2.2)$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

The properties of the Bregman projection and the relative projection operators were studied in [26,34].

The following lemmas give us some nice properties of real sequences.

Lemma 2.6 ([35]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

- (a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
- (b) $\limsup_{n \rightarrow \infty} \frac{b_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < +\infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7 ([36]). Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$\gamma_{m_k} \leq \gamma_{m_k+1} \quad \text{and} \quad \gamma'_k \leq \gamma_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. Main results

In this section, we now prove our main result in this paper. To this end, we first prove the following proposition.

Proposition 3.1. Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $T : C \rightarrow C$ be a mapping such that $\widehat{F}(T) = F(T)$ is nonempty, closed and convex. Suppose that $u \in C$ and $\{x_n\}$ is a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. Then

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle \leq 0,$$

where $p = P_{F(T)}^f(u)$ and $P_{F(T)}^f$ is the Bregman projection of C onto $F(T)$.

Proof. Let $u \in C$ and put $p = P_{F(T)}^f(u)$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$ and

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle = \langle \nabla f(u) - \nabla f(p), v - p \rangle.$$

On the other hand, since $\|x_{n_k} - Tx_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, we have $v \in \widehat{F}(T) = F(T)$. It follows from definition of the Bregman projection that

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle = \langle \nabla f(u) - \nabla f(p), v - p \rangle \leq 0.$$

This completes the proof. $\square$

Theorem 3.2. Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let T be a Bregman strongly nonexpansive mapping on E such that $F(T) = \widehat{F}(T) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}^f(u)$, where $P_{F(T)}^f$ is the Bregman projection of E onto $F(T)$.

Proof. We note, by Reich and Sabach [15], that $F(T)$ is closed and convex. Let $p = P_{F(T)}^f(u) \in F(T) = \widehat{F}(T)$. Using (2.2), we have

$$\begin{aligned} D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n) \\ &\leq \max \{D_f(p, u), D_f(p, x_n)\}. \end{aligned}$$

By induction, the sequence $\{D_f(p, x_n)\}$ is bounded.

We next show that the sequence $\{x_n\}$ is also bounded. We follow the proof line as in [15]. Since $\{D_f(p, x_n)\}$ is bounded, there exists $M > 0$ such that

$$f(p) - \langle \nabla f(x_n), p \rangle + f^*(\nabla f(x_n)) = V_f(p, \nabla f(x_n)) = D_f(p, x_n) \leq M.$$

Hence $\{\nabla f(x_n)\}$ is contained in the sublevel set $lev_{\leq}^{\psi}(M - f(p))$, where $\psi = f^* - \langle \cdot, p \rangle$. Since f is lower semicontinuous, f^* is weak* lower semicontinuous. Hence the function ψ is coercive by Moreau–Rockafellar Theorem (see [37, Theorem 7A])

and [38]). This shows that $\{\nabla f(x_n)\}$ is bounded. Since f is strongly coercive, f^* is bounded on bounded sets (see [18, Lemma 3.6.1] and [21, Theorem 3.3]). Hence ∇f^* is also bounded on bounded subsets of E^* (see [27, Proposition 1.1.11]). Since f is a Legendre function, it follows that $x_n = \nabla f^*(\nabla f(x_n))$ is bounded for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded. So are $\{Tx_n\}$ and $\{\nabla f(Tx_n)\}$. Indeed, since f is bounded on bounded subsets of E , ∇f is also bounded on bounded subsets of E (see [27, Proposition 1.1.11]). Therefore $\{\nabla f(Tx_n)\}$ is bounded.

We next show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \rightarrow \infty} (D_f(p, x_{n_k+1}) - D_f(p, x_{n_k})) = 0,$$

then

$$\lim_{k \rightarrow \infty} (D_f(p, Tx_{n_k}) - D_f(p, x_{n_k})) = 0.$$

Since $\{\nabla f(Tx_{n_k})\}$ is bounded and $\alpha_{n_k} \rightarrow 0$, we obtain

$$\lim_{k \rightarrow \infty} \|\nabla f(x_{n_k+1}) - \nabla f(Tx_{n_k})\| = \lim_{k \rightarrow \infty} \alpha_{n_k} \|\nabla f(u) - \nabla f(Tx_{n_k})\| = 0. \quad (3.1)$$

Since f is strongly coercive and uniformly convex on bounded subsets of E , f^* is uniformly Fréchet differentiable on bounded subsets of E^* (see [18, Proposition 3.6.2]). Moreover, f^* is bounded on bounded sets (see [18, Lemma 3.6.1] and [21, Theorem 3.3]). Since f is Legendre, applying Lemma 2.3, we have

$$\lim_{k \rightarrow \infty} \|x_{n_k+1} - Tx_{n_k}\| = \lim_{k \rightarrow \infty} \|\nabla f^*(\nabla f(x_{n_k+1})) - \nabla f^*(\nabla f(Tx_{n_k}))\| = 0. \quad (3.2)$$

On the other hand, if f is uniformly Fréchet differentiable on bounded subsets of E , then f is uniformly continuous on bounded subsets of E (see [39, Theorem 1.8]). It follows that

$$\lim_{k \rightarrow \infty} |f(x_{n_k+1}) - f(Tx_{n_k})| = 0. \quad (3.3)$$

We now consider the following equality.

$$\begin{aligned} D_f(p, Tx_{n_k}) - D_f(p, x_{n_k}) &= f(p) - f(Tx_{n_k}) - \langle \nabla f(Tx_{n_k}), p - Tx_{n_k} \rangle - D_f(p, x_{n_k}) \\ &= f(p) - f(x_{n_k+1}) + f(x_{n_k+1}) - f(Tx_{n_k}) - \langle \nabla f(x_{n_k+1}), p - x_{n_k+1} \rangle \\ &\quad + \langle \nabla f(x_{n_k+1}), p - x_{n_k+1} \rangle - \langle \nabla f(Tx_{n_k}), p - Tx_{n_k} \rangle - D_f(p, x_{n_k}) \\ &= D_f(p, x_{n_k+1}) + (f(x_{n_k+1}) - f(Tx_{n_k})) + \langle \nabla f(x_{n_k+1}), p - x_{n_k+1} \rangle \\ &\quad - \langle \nabla f(Tx_{n_k}), p - Tx_{n_k} \rangle - D_f(p, x_{n_k}) \\ &= (D_f(p, x_{n_k+1}) - D_f(p, x_{n_k})) + (f(x_{n_k+1}) - f(Tx_{n_k})) \\ &\quad + \langle \nabla f(x_{n_k+1}) - \nabla f(Tx_{n_k}), p - x_{n_k+1} \rangle - \langle \nabla f(Tx_{n_k}), x_{n_k+1} - Tx_{n_k} \rangle. \end{aligned}$$

It follows from (3.1) to (3.3) that

$$\lim_{k \rightarrow \infty} (D_f(p, Tx_{n_k}) - D_f(p, x_{n_k})) = 0.$$

We next consider the following two cases.

Case 1. $D_f(p, x_{n+1}) \leq D_f(p, x_n)$ for all sufficiently large n . Hence the sequence $\{D_f(p, x_n)\}$ is bounded and nonincreasing. So we have $\lim_{n \rightarrow \infty} D_f(p, x_n)$ exists. This shows that $\lim_{n \rightarrow \infty} (D_f(p, x_{n+1}) - D_f(p, x_n)) = 0$ and hence

$$\lim_{n \rightarrow \infty} (D_f(p, Tx_n) - D_f(p, x_n)) = 0.$$

Since T is Bregman strongly nonexpansive,

$$\lim_{n \rightarrow \infty} D_f(x_n, Tx_n) = 0.$$

Since f is totally convex on bounded subsets of E , by Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

Proposition 3.1 yields that

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle \leq 0.$$

Finally, we show that $x_n \rightarrow p$. Using (2.1), we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n)) \\ &\leq V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n) - \alpha_n (\nabla f(u) - \nabla f(p))) \\ &\quad + \langle \alpha_n (\nabla f(u) - \nabla f(p)), x_{n+1} - p \rangle \\ &= V_f(p, \alpha_n \nabla f(p) + (1 - \alpha_n) \nabla f(Tx_n)) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &\leq \alpha_n V_f(p, \nabla f(p)) + (1 - \alpha_n) V_f(p, \nabla f(Tx_n)) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &= (1 - \alpha_n) D_f(p, Tx_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &\leq (1 - \alpha_n) D_f(p, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle. \end{aligned}$$

By Lemma 2.6, we can conclude that $\lim_{n \rightarrow \infty} D_f(p, x_n) = 0$. Therefore, by Lemma 2.4, $x_n \rightarrow p$ since f is totally convex on bounded subsets of E .

Case 2. There exists a subsequence $\{D_f(p, x_{n_j})\}$ of $\{D_f(p, x_n)\}$ such that $D_f(p, x_{n_j}) < D_f(p, x_{n_j+1})$ for all $j \in \mathbb{N}$. By Lemma 2.7, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1}) \quad \text{and} \quad D_f(p, x_k) \leq D_f(p, x_{m_k+1}).$$

So we have

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} (D_f(p, x_{m_k+1}) - D_f(p, x_{m_k})) \\ &\leq \limsup_{n \rightarrow \infty} (D_f(p, x_{n+1}) - D_f(p, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, Tx_n) - D_f(p, x_n)) \\ &= \limsup_{n \rightarrow \infty} (\alpha_n (D_f(p, u) - D_f(p, Tx_n)) + (D_f(p, Tx_n) - D_f(p, x_n))) \\ &\leq \limsup_{n \rightarrow \infty} \alpha_n (D_f(p, u) - D_f(p, Tx_n)) = 0. \end{aligned}$$

This implies that

$$\lim_{k \rightarrow \infty} (D_f(p, x_{m_k+1}) - D_f(p, x_{m_k})) = 0. \quad (3.4)$$

Following the proof line in Case 1, we can verify

$$\limsup_{k \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_{m_k} - p \rangle \leq 0$$

and

$$D_f(p, x_{m_k+1}) \leq (1 - \alpha_{m_k}) D_f(p, x_{m_k}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p \rangle.$$

This implies

$$\begin{aligned} \alpha_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p \rangle \\ &\leq \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p \rangle. \end{aligned}$$

Hence $\lim_{k \rightarrow \infty} D_f(p, x_{m_k}) = 0$. Using this and (3.4) together, we conclude that

$$\limsup_{k \rightarrow \infty} D_f(p, x_k) \leq \lim_{k \rightarrow \infty} D_f(p, x_{m_k+1}) = 0.$$

This completes the proof. $\square$

As a direct consequence of Theorem 3.2, we obtain the convergence result concerning strongly relatively nonexpansive mappings [40,29] in a uniformly smooth and uniformly convex Banach space.

Corollary 3.3. Let E be a uniformly smooth and uniformly convex Banach space. Let T be a strongly relatively nonexpansive mapping on E such that $F(T) = \widehat{F}(T) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n J(u) + (1 - \alpha_n) J(Tx_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}(u)$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

4. Convergence theorems for a family of mappings

In this section, using [Theorem 3.2](#), we present two strong convergence theorems for a family of Bregman strongly nonexpansive mappings.

Firstly, we consider the following condition: Let C be a subset of a real Banach space E , $f : E \rightarrow \mathbb{R}$ a convex and Gâteaux differentiable function and $\{T_n\}_{n=1}^\infty$ a sequence of mappings of C such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^\infty$ is said to satisfy the AKTT-condition [\[41\]](#) if, for any bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} < \infty.$$

We follow [\[41,42\]](#) and give details of proof here.

Proposition 4.1. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a Legendre and Fréchet differentiable function. Let $\{T_n\}_{n=1}^\infty$ be a sequence of mappings from C into E such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^\infty$ satisfies the AKTT-condition. Then there exists the mapping $T : B \rightarrow E$ such that*

$$Tx = \lim_{n \rightarrow \infty} T_n x, \quad \forall x \in B \quad (4.1)$$

and $\lim_{n \rightarrow \infty} \sup_{z \in B} \|\nabla f(Tz) - \nabla f(T_n z)\| = 0$.

Proof. To complete this, we show that $\{\nabla f(T_n x)\}$ is a Cauchy sequence in E^* for all $x \in B$. Let $\epsilon > 0$ be given. By the AKTT-condition, there exists $\ell_0 \in \mathbb{N}$ such that

$$\sum_{n=\ell_0}^{\infty} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} < \epsilon.$$

In particular, if $k > \ell \geq \ell_0$, then

$$\begin{aligned} \|\nabla f(T_k x) - \nabla f(T_\ell x)\| &\leq \sum_{n=\ell}^{k-1} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} \\ &\leq \sum_{n=\ell_0}^{\infty} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} < \epsilon. \end{aligned}$$

This shows that $\{\nabla f(T_n x)\}$ is a Cauchy sequence in E^* and $\lim_{n \rightarrow \infty} \nabla f(T_n x)$ exists for all $x \in B$. It is easily seen that the convergence is uniform on B . Since E is reflexive and f is Legendre, ∇f is bijective and $\nabla f^* = (\nabla f)^{-1}$. So we can define the mapping $T : B \rightarrow E$ as follows:

$$Tx = \nabla f^* \left(\lim_{n \rightarrow \infty} \nabla f(T_n x) \right) \quad \forall x \in B.$$

Since f is Fréchet differentiable, ∇f is norm-to-norm continuous. Hence

$$Tx = \nabla f^* \nabla f \left(\lim_{n \rightarrow \infty} T_n x \right) = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in B.$$

This completes that proof. $\square$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^\infty$ satisfies the AKTT-condition and T is defined by (4.1) with $\bigcap_{n=1}^\infty F(T_n) = F(T)$.

Theorem 4.2. *Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $\{T_n\}_{n=1}^\infty$ be a sequence of Bregman strongly nonexpansive mappings on E such that $F(T_n) = \widehat{F}(T_n)$ for all $n \in \mathbb{N}$ and $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^* (\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(T_n x_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ converges strongly to $P_F^f(u)$, where P_F^f is the Bregman projection of E onto F .

Proof. As in [Theorem 3.2](#), we can prove that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. Since f is uniformly Fréchet differentiable and bounded on bounded subsets of E , it follows that $\lim_{n \rightarrow \infty} \|\nabla f(x_n) - \nabla f(T_n x_n)\| = 0$ by [Lemma 2.3](#).

To this end, it suffices to show that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. By Proposition 4.1, we see that

$$\begin{aligned} \|\nabla f(x_n) - \nabla f(Tx_n)\| &\leq \|\nabla f(x_n) - \nabla f(T_n x_n)\| + \|\nabla f(T_n x_n) - \nabla f(Tx_n)\| \\ &\leq \|\nabla f(x_n) - \nabla f(T_n x_n)\| + \sup_{z \in \{x_n\}} \|\nabla f(T_n z) - \nabla f(Tz)\| \\ &\rightarrow 0. \end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. This completes the proof. $\square$

Next, employing the idea in [29], we consider the mapping $T : E \rightarrow E$ defined by $T = T_N T_{N-1} \cdots T_1$, where T_i ($i = 1, 2, \dots, N$) are Bregman strongly nonexpansive mappings on E .

Theorem 4.3. Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let T_i ($i = 1, 2, \dots, N$) be Bregman strongly nonexpansive mappings on E such that $F(T_i) = \widehat{F}(T_i)$ ($i = 1, 2, \dots, N$) and $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(T_N T_{N-1} \cdots T_1 x_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_F^f(u)$, where P_F^f is the Bregman projection of E onto F .

Proof. Put $T = T_N T_{N-1} \cdots T_1$. We see that $\emptyset \neq F(T) \subset \widehat{F}(T)$ since $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. By Lemma 1 in [29], we know that $\widehat{F}(T) \subset \bigcap_{i=1}^N \widehat{F}(T_i) = \bigcap_{i=1}^N F(T_i)$. It follows that

$$F(T) \subset \widehat{F}(T) \subset \bigcap_{i=1}^N \widehat{F}(T_i) = \bigcap_{i=1}^N F(T_i) \subset F(T).$$

Therefore $F(T) = \widehat{F}(T)$. On the other hand, by Lemma 2 in [29], we also know that T is Bregman strongly nonexpansive. Using Theorem 3.2, we deduce that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$. $\square$

5. Applications

In this section, we give some applications of Theorem 3.2 in the framework of reflexive Banach spaces.

5.1. Zeros of maximal monotone operators

Let $A : E \rightarrow 2^{E^*}$ be a set-valued mapping. The domain of A is denoted by $\text{dom } A = \{x \in E : Ax \neq \emptyset\}$ and also the graph of A is denote by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A is said to be *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for each $(x, x^*), (y, y^*) \in G(A)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operators on E . It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$ is closed and convex. Now, we apply Theorem 3.2 to the problem of finding $x \in E$ such that $0^* \in Ax$ which strongly relates to the convex minimization problems in optimization, economics and applied sciences.

The resolvent of A , denoted by $\text{Res}_A^f : E \rightarrow 2^E$, is defined as follows [12]:

$$\text{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(\text{Res}_A^f) = A^{-1}(0^*)$, and Res_A^f is single-valued and Bregman firmly nonexpansive (see [12]). If f is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E , then $\widehat{F}(\text{Res}_A^f) = F(\text{Res}_A^f)$ (see [15]). The Yosida approximation $A_\lambda : E \rightarrow E$, $\lambda > 0$, is also defined by

$$A_\lambda(x) = \frac{1}{\lambda} \left(\nabla f(x) - \nabla f(\text{Res}_{\lambda A}^f(x)) \right)$$

for all $x \in E$. From Proposition 2.7 in [43], we know that $(\text{Res}_{\lambda A}^f(x), A_\lambda(x)) \in G(A)$, and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$. Using these facts, we obtain the following result by replacing $T = \text{Res}_{\lambda A}^f$, $\lambda > 0$ in Theorem 3.2.

Theorem 5.1. Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $A : E \rightarrow 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(\text{Res}_{\lambda A}^f x_n)), \quad \forall n \geq 1,$$

where $\lambda > 0$ and $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{A^{-1}(0^*)}^f(u)$, where $P_{A^{-1}(0^*)}^f$ is the Bregman projection of E onto $A^{-1}(0^*)$.

5.2. Equilibrium problems

Let C be a nonempty, closed and convex subset of E and $G : C \times C \rightarrow \mathbb{R}$ a bifunction. Finally, we apply Theorem 3.2 to the problem of finding $x \in C$ such that $G(x, y) \geq 0$ for all $y \in C$. Such a problem is called *equilibrium problem* and the solutions set is denoted by $EP(G)$. Numerous problems in economics, physics and applied sciences can be reduced to find solutions of equilibrium problems.

In order to solve the equilibrium problem, let us assume that a bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions [44]:

- (A1) $G(x, x) = 0$ for all $x \in C$;
- (A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;
- (A4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semi-continuous.

The resolvent of a bifunction G [45] is the operator $Res_G^f : E \rightarrow 2^C$ defined by

$$Res_G^f(x) = \{z \in C : G(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \forall y \in C\}.$$

From Lemma 1 in [14], if $f : E \rightarrow (-\infty, +\infty]$ is a strongly coercive and Gâteaux differentiable function, and G satisfies conditions (A1)–(A4), then $\text{dom}(Res_G^f) = E$. We also know the following lemma which give us some characterizations of the resolvent Res_G^f .

Lemma 5.2 ([14, Lemma 2]). *Let $f : E \rightarrow (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of E . If the bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfies conditions (A1)–(A4), then*

- (i) Res_G^f is single-valued;
- (ii) Res_G^f is a Bregman firmly nonexpansive operator;
- (iii) $F(Res_G^f) = EP(G)$;
- (iv) $EP(G)$ is a closed and convex subset of C ;
- (v) for all $x \in E$ and for all $p \in F(Res_G^f)$, we have

$$D_f(p, Res_G^f(x)) + D_f(Res_G^f(x), x) \leq D_f(p, x).$$

In addition, by Reich and Sabach [15], if f is uniformly Fréchet differentiable and bounded on bounded subsets of E , then we have from Lemma 5.2 that $F(Res_G^f) = \widehat{F}(Res_G^f) = EP(G)$ is closed and convex. Also, by replacing $T = Res_G^f$ in Theorem 3.2, we obtain the following result.

Theorem 5.3. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $G : C \times C \rightarrow \mathbb{R}$ be a bifunction which satisfies the conditions (A1)–(A4) such that $EP(G) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Res_G^f x_n) \right), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{EP(G)}^f(u)$, where $P_{EP(G)}^f$ is the Bregman projection of E onto $EP(G)$.

Remark 5.4. Theorem 3.2 can be applied to convex feasibility problems, variational inequality problems and the problems of finding zeros of Bregman inverse strongly monotone operators in reflexive Banach spaces (see Sections 4, 7, 8 in [14]).

Remark 5.5. Theorem 4.3 can be applied to a system of variational inequality problems and a system of equilibrium problems.

Remark 5.6. Our results are new even if the convex function f is chosen so that $f(\cdot) = \frac{1}{p} \|\cdot\|^p$ ($1 < p < \infty$) in uniformly smooth and uniformly convex Banach spaces.

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1.5 Strong convergence for a countable family of strict pseudocontractions in q -uniformly smooth Banach spaces

In this section, we introduce a new iterative algorithm for finding a common fixed point of a countable family of strict pseudocontractions in q -uniformly smooth and uniformly convex Banach spaces. We then prove that the sequence generated by the proposed algorithm converges strongly to a common fixed point of an infinite family of strict pseudocontractions. Our results mainly improve and extend the results announced by Yao et al. [Y. Yao, Y.-C. Liou, G. Marino, Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2009 (2009) 7 pages. doi:10.1155/2009/279058. Art. ID 279058].

1. Introduction

Let E and E^* be a real Banach space and the space dual to E , respectively. Let J_q ($q > 1$) denote the generalized duality mapping from E into 2^{E^*} given by $J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \forall x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^* . It is well known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \neq 0$. If E is smooth then J_q is single-valued, which is denoted by j_q . The duality mapping J from a smooth Banach space E into E^* is said to be weakly sequentially continuous if x_n weak convergent to x implies Jx_n weak* convergent to Jx . We denote the fixed points set of a nonlinear mapping $T : C \rightarrow E$ by $F(T) = \{x \in C : Tx = x\}$ and write $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called:

- (i) λ -strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q - \lambda \|(I - T)x - (I - T)y\|^q,$$

or equivalently

$$\langle (I - T)x - (I - T)y, j_q(x - y) \rangle \geq \lambda \|(I - T)x - (I - T)y\|^q;$$

- (ii) L -Lipschitzian if for all $x, y \in D(T)$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping. By the definition, we know that every λ -strictly pseudocontractive mapping is $(\frac{1+\lambda}{\lambda})$ -Lipschitzian (see [2]).

Remark 1.2. Let C be a nonempty subset of a real Hilbert space and $T : C \rightarrow C$ be a mapping. Then T is said to be κ -strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2. \quad (1.1)$$

It is well known that (1.1) is equivalent to the following:

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2.$$

It is worth mentioning that the class of strict pseudocontractions includes properly the class of nonexpansive mappings. In 1953, Mann [3] introduced an iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \quad (1.2)$$

where $\alpha_n \in [0, 1]$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [4]).

In 1967, Browder and Petryshyn [1] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using the Mann iterative algorithm (1.2) with a constant sequence $\alpha_n = \alpha$ for all $n \geq 0$. Recently, Marino and Xu [5] and Zhou [6] extended the results of Browder and Petryshyn [1] to Mann's iteration process (1.2). Zhou [7] also investigated the weak convergence in a 2-uniformly smooth Banach space. In a much more general setting, Osilike and Udomene [8], Zhang and Su [9], Zhang and Guo [10] and Zhou [11] investigated the weak convergence in a q -uniformly smooth Banach space. Since 1967, the construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors (see, e.g., [12–15]).

Due to the weak convergence of the Mann iterative algorithm, in 1967, Halpern [16] introduced a modified Mann's iteration (1.2) which is the so called *Halpern iterative scheme* as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \quad (1.3)$$

where $\alpha_n \in [0, 1]$ and $u \in C$. He proved, in a real Hilbert space, the convergence of $\{x_n\}$ to a fixed point of T where $\alpha_n := n^{-a}$, $a \in (0, 1)$.

In 1977, Lions [17] obtained a strong convergence provided the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0; \quad C2 : \sum_{n=0}^{\infty} \alpha_n = \infty; \quad C3 : \lim_{n \rightarrow \infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$$

Reich [18] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lions' conditions imposed on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$.

In 1992, Wittmann [19] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0; \quad C2 : \sum_{n=0}^{\infty} \alpha_n = \infty; \quad C3 : \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Later, Shioji and Takahashi [20] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of the Halpern iterative scheme has been widely used to approximate the fixed points for nonexpansive mappings (see, e.g., [21–26] and the references cited therein).

In 2003, Nakajo and Takahashi [27] introduced another modified Mann iterative algorithm for a nonexpansive mapping T in a real Hilbert space as follows: $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 1, \end{cases} \quad (1.4)$$

where $\{\alpha_n\} \subset [0, 1]$ and P_K is a metric projection from H into a nonempty, closed and convex subset K . Such an algorithm is called the *CQ method*. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T under suitable conditions.

Recently, Takahashi et al. [28] introduced a new modification of the Mann iterative scheme called the *shrinking projection method* for a nonexpansive mapping T in a Hilbert space as follows: for $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad n \geq 1, \end{cases} \quad (1.5)$$

where $\{\alpha_n\} \subset [0, 1]$. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T under suitable conditions.

The concepts of the CQ method and the shrinking projection method have also been extensively investigated in order to get strong convergence results by many authors (see, e.g., [29–33]). A natural question arises:

Question. Besides the concepts mentioned before, could one construct a new modification of the Mann iterative algorithm in order to get strong convergence?

In 2009, Yao et al. [34] introduced a new modified Mann iterative algorithm which is different from those in the literature for a nonexpansive mapping in a real Hilbert space. To be more precise, they proved the following theorem.

Theorem 1.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. For $x_0 \in C$ given arbitrarily, let the sequence $\{x_n\}$, $n \geq 0$, be generated iteratively by*

$$\begin{aligned} y_n &= P_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T y_n. \end{aligned} \quad (1.6)$$

Suppose that the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;

then the sequence $\{x_n\}$ generated by (1.6) strongly converges to a fixed point of T .

Let C be a nonempty, closed and convex subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty} : C \rightarrow C$ a countable family of strict pseudocontractions.

Motivated and inspired by Marino and Xu [5] and Yao et al. [34], we consider the following modified Mann-type iteration: $x_1 \in C$ and

$$\begin{aligned} y_n &= Q_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n[(1 - \gamma_n)y_n + \gamma_n T_n y_n], \quad n \geq 1, \end{aligned} \quad (1.7)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ and Q_C is a sunny, nonexpansive retraction of E onto C .

In this paper, first we prove path convergence for a nonexpansive mapping in uniformly convex and q -uniformly smooth Banach spaces. Then we prove a strong convergence of the modified Mann-type iteration process (1.7) for a countable family of strict pseudocontractions. Using the result obtained, the strong convergence is also established for an infinite strict pseudocontractions. Our results extend the main result of Yao et al. [34] in several aspects.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The *modulus of convexity* of E is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\},$$

for all $\epsilon \in [0, 2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case E is called *smooth*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. The norm of E is called *uniformly Fréchet differentiable* if the limit is attained uniformly for $x, y \in S(E)$. It is well known that (uniform) *Fréchet differentiability* of the norm of E implies (uniform) *Gâteaux differentiability* of the norm of E .

Let $\rho_E : [0, \infty) \rightarrow [0, \infty)$ be the *modulus of smoothness* of E defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t \right\}.$$

A Banach space E is said to be *uniformly smooth* if $\frac{\rho_E(t)}{t} \rightarrow 0$ as $t \rightarrow 0$. Suppose that $q > 1$; then E is said to be *q -uniformly smooth* if there exists $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q -uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable, and hence the norm of E is Fréchet differentiable. For more details, we refer the reader to [35,36].

In 1972, Gossez and Lami [37] gave some geometric properties related to the fixed point theory for nonexpansive mappings. They proved that a space with a weakly continuous duality map satisfies Opial's condition [38]. Conversely, if a space satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous duality mapping at zero.

Let E be a real Banach space, C a nonempty, closed and convex subset of E , and K a nonempty subset of C . Let $Q : C \rightarrow K$. Then Q is said to be:

(1) *sunny* if for each $x \in C$ and $t \in [0, 1]$, we have

$$Q(tx + (1-t)Qx) = Qx;$$

(2) *a retraction of C onto K* if

$$Qx = x, \quad \forall x \in K;$$

(3) *a sunny nonexpansive retraction* if Q is sunny, nonexpansive and a retraction onto K (see also Bruck [39], Goebel and Reich [40] and Reich [24]).

In the sequel, we need the following crucial lemmas.

Lemma 2.1 ([41]). *Let E be a real q -uniformly smooth Banach space. Then the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + C_q \|y\|^q,$$

for all $x, y \in E$ and for some $C_q > 0$.

Lemma 2.2 ([42]). *Suppose that $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b .

Lemma 2.3 ([43]). *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space E such that*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n, \quad n \geq 1,$$

where $\{\beta_n\}$ satisfies the following condition: $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. If $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Lemma 2.4 ([44,45]). *Let E be a real uniformly convex Banach space and let C be a nonempty, closed and convex subset of E and $S : C \rightarrow C$ a nonexpansive mapping such that $F(S) \neq \emptyset$; then $I - S$ is demiclosed at zero.*

Lemma 2.5 ([26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$(a) \sum_{n=1}^{\infty} \gamma_n = \infty; \quad (b) \limsup_{n \rightarrow \infty} \delta_n \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty.$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

To deal with a family of mappings, the following conditions are introduced. Let C be a subset of a real Banach space E and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [21] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Lemma 2.6 ([21]). *Let C be a nonempty and closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to a point in C . Moreover, let the mapping T be defined by*

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in C.$$

Then for each bounded subset B of C ,

$$\limsup_{n \rightarrow \infty} \{\|Tz - T_n z\| : z \in B\} = 0.$$

The following results can be found in [46,47].

Lemma 2.7 ([46,47]). Let C be a closed and convex subset of a smooth Banach space E . Suppose that $\{T_n\}_{n=1}^\infty$ is a family of λ -strictly pseudocontractive mappings from C into E with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\mu_n\}_{n=1}^\infty$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^\infty \mu_n = 1$. Then the following conclusions hold:

- (1) $G := \sum_{n=1}^\infty \mu_n T_n : C \rightarrow E$ is a λ -strictly pseudocontractive mapping;
- (2) $F(G) = \bigcap_{n=1}^\infty F(T_n)$.

Lemma 2.8 ([47]). Let C be a closed and convex subset of a smooth Banach space E . Suppose that $\{S_k\}_{k=1}^\infty$ is a countable family of λ -strictly pseudocontractive mappings of C into itself with $\bigcap_{k=1}^\infty F(S_k) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \rightarrow C$ by

$$T_n x = \sum_{k=1}^n \mu_n^k S_k x, \quad x \in C,$$

where $\{\mu_n^k\}$ is a family of nonnegative numbers satisfying:

- (i) $\sum_{k=1}^n \mu_n^k = 1$ for all $n \in \mathbb{N}$; (ii) $\mu^k := \lim_{n \rightarrow \infty} \mu_n^k > 0$ for all $k \in \mathbb{N}$;
- (iii) $\sum_{n=1}^\infty \sum_{k=1}^n |\mu_{n+1}^k - \mu_n^k| < \infty$.

Then:

- (1) Each T_n is a λ -strictly pseudocontractive mapping.
- (2) $\{T_n\}$ satisfies the AKTT-condition.
- (3) If $T : C \rightarrow C$ is defined by

$$Tx = \sum_{k=1}^\infty \mu^k S_k x, \quad x \in C,$$

then $Tx = \lim_{n \rightarrow \infty} T_n x$ and $F(T) = \bigcap_{n=1}^\infty F(T_n) = \bigcap_{k=1}^\infty F(S_k)$.

In the sequel, we will write that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.6 with $F(T) = \bigcap_{n=1}^\infty F(T_n)$.

3. The path convergence theorem

Let C be a nonempty, closed and convex subset of a real Banach space E . Let $S : C \rightarrow C$ be a nonexpansive mapping. For each $t \in (0, 1)$, we define the mapping $S_t : C \rightarrow C$ by

$$S_t x = SQ_C[(1-t)x], \quad \forall x \in C. \quad (3.1)$$

It is easy to see that S_t is a contraction. By the Banach contraction principle, there exists a unique fixed point x_t of S_t in C , that is,

$$x_t = SQ_C[(1-t)x_t]. \quad (3.2)$$

Theorem 3.1. Let E be a real q -uniformly smooth and uniformly convex Banach space which admits the weakly sequentially continuous duality mapping j_q and C be a nonempty, closed and convex subset of E . Let $S : C \rightarrow C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. For each $t \in (0, 1)$, let $\{x_t\}$ be generated by (3.2), then as $t \rightarrow 0$, $\{x_t\}$ converges strongly to a fixed point of S .

Proof. First, we prove that $\{x_t\}$ is bounded. Let $x^* \in F(S)$. From (3.2) we have

$$\begin{aligned} \|x_t - x^*\| &= \|SQ_C[(1-t)x_t] - SQ_C x^*\| \\ &\leq (1-t)\|x_t - x^*\| + t\|x^*\|. \end{aligned}$$

This implies that $\|x_t - x^*\| \leq \|x^*\|$ and hence $\{x_t\}$ is bounded.

Next, we prove that $\|x_t - Sx_t\| \rightarrow 0$ as $t \rightarrow 0$. From (3.2) we also have

$$\begin{aligned} \|x_t - Sx_t\| &= \|SQ_C[(1-t)x_t] - Sx_t\| \\ &\leq t\|x_t\| \rightarrow 0, \end{aligned} \quad (3.3)$$

as $t \rightarrow 0$.

Next, we prove that $\{x_t\}$ is relatively norm compact as $t \rightarrow 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Put $x_n := x_{t_n}$. From (3.3) we get that

$$\|x_n - Sx_n\| \rightarrow 0, \quad (3.4)$$

as $n \rightarrow \infty$. Using Lemma 2.1, we obtain for each $u \in F(S)$ that

$$\begin{aligned}\|x_t - u\|^q &= \|SQ_C[(1-t)x_t] - SQ_C u\|^q \\ &\leq \|(x_t - u) - tx_t\|^q \\ &\leq \|x_t - u\|^q - qt \langle x_t, j_q(x_t - u) \rangle + C_q t^q \|x_t\|^q \\ &= \|x_t - u\|^q - qt \langle x_t - u, j_q(x_t - u) \rangle - qt \langle u, j_q(x_t - u) \rangle + C_q t^q \|x_t\|^q \\ &= \|x_t - u\|^q - qt \|x_t - u\|^q - qt \langle u, j_q(x_t - u) \rangle + C_q t^q \|x_t\|^q.\end{aligned}$$

This shows that

$$\|x_t - u\|^q \leq \langle u, j_q(u - x_t) \rangle + \frac{C_q t^{q-1}}{q} \|x_t\|^q.$$

In particular, we obtain that

$$\|x_n - u\|^q \leq \langle u, j_q(u - x_n) \rangle + \frac{C_q t_n^{q-1}}{q} \|x_n\|^q.$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n \rightarrow z^* \in C$. By Lemma 2.4 and (3.4), we know that $z^* \in F(S)$. So we have

$$\|x_n - z^*\|^q \leq \langle z^*, j_q(z^* - x_n) \rangle + \frac{C_q t_n^{q-1}}{q} \|x_n\|^q.$$

Since j_q is weakly sequential continuous, we obtain $x_n \rightarrow z^*$ as $n \rightarrow \infty$. Next, we prove that $x_t \rightarrow z^*$ as $t \rightarrow 0$. Assume that $x_{t_m} \rightarrow x^*$, where $t_m \rightarrow 0$. Put $x_m := x_{t_m}$. Using the same argument as above, we can show that $x^* \in F(S)$ and

$$\|x_m - z^*\|^q \leq \langle z^*, j_q(z^* - x_m) \rangle + \frac{C_q t_m^{q-1}}{q} \|x_m\|^q.$$

Therefore,

$$\|x^* - z^*\|^q \leq \langle z^*, j_q(z^* - x^*) \rangle. \quad (3.5)$$

Interchanging x^* and z^* , we have

$$\|z^* - x^*\|^q \leq \langle x^*, j_q(x^* - z^*) \rangle. \quad (3.6)$$

Combining (3.5) and (3.6) we obtain

$$\begin{aligned}2\|x^* - z^*\|^q &\leq \langle z^*, j_q(z^* - x^*) \rangle + \langle x^*, j_q(x^* - z^*) \rangle \\ &= \|x^* - z^*\|^q.\end{aligned}$$

This implies that $x^* = z^*$ and hence the proof is complete. $\square$

4. The main results

Theorem 4.1. Let E be a real q -uniformly smooth and uniformly convex Banach space which admits the weakly sequentially continuous duality mapping j_q and C be a nonempty, closed and convex subset of E . Let $\{T_n\}_{n=1}^\infty : C \rightarrow C$ be a family of λ -strict pseudocontractions, $0 < \lambda < 1$, such that $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$ which satisfy the conditions:

- (C1) $\sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C3) $\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty$;
- (C4) $0 < a \leq \gamma_n \leq \mu$, $\mu = \min \left\{ 1, \left(\frac{q\lambda}{C_q} \right)^{\frac{1}{q-1}} \right\}$.

Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a common fixed point of $\{T_n\}_{n=1}^\infty$.

Proof. First, we prove that $\{x_n\}$ is bounded. Put $z_n = (1 - \gamma_n)y_n + \gamma_n T_n y_n$ for all $n \geq 1$. For each $u \in F$, it follows from Lemma 2.1 that

$$\begin{aligned} \|z_n - u\|^q &= \|(y_n - u) + \gamma_n(T_n y_n - y_n)\|^q \\ &\leq \|y_n - u\|^q + q\gamma_n \langle T_n y_n - y_n, j_q(y_n - u) \rangle + C_q \gamma_n^q \|y_n - T_n y_n\|^q \\ &\leq \|y_n - u\|^q - q\lambda \gamma_n \|y_n - T_n y_n\|^q + C_q \gamma_n^q \|y_n - T_n y_n\|^q \\ &= \|y_n - u\|^q - \gamma_n(q\lambda - C_q \gamma_n^{q-1}) \|y_n - T_n y_n\|^q. \end{aligned}$$

This implies by (C4) that

$$\|z_n - u\| \leq \|y_n - u\|.$$

Hence

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1 - \beta_n)(x_n - u) + \beta_n(z_n - u)\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n\|z_n - u\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n\|y_n - u\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n((1 - \alpha_n)\|x_n - u\| + \alpha_n\|u\|) \\ &= (1 - \alpha_n \beta_n)\|x_n - u\| + \alpha_n \beta_n\|u\| \\ &\leq \max\{\|x_n - u\|, \|u\|\}. \end{aligned}$$

By induction, we get that $\{x_n\}$ is bounded, and so is $\{y_n\}$. First, we observe that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|(1 - \alpha_{n+1})x_{n+1} - (1 - \alpha_n)x_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\|. \end{aligned} \quad (4.1)$$

For each $n \geq 1$, define $S_n x = (1 - \gamma_n)x + \gamma_n T_n x$ for all $x \in C$. By (C4), it is easy to verify that S_n is nonexpansive and $F(T_n) = F(S_n)$ for all $n \geq 1$. Moreover, $\{S_n\}$ satisfies the AKTT-condition. In fact, for any bounded subset B of C ,

$$\begin{aligned} \sup_{z \in B} \|S_{n+1}z - S_n z\| &= \sup_{z \in B} \|((1 - \gamma_{n+1})z + \gamma_{n+1}T_{n+1}z) - ((1 - \gamma_n)z + \gamma_n T_n z)\| \\ &\leq |\gamma_{n+1} - \gamma_n| \sup_{z \in B} \|z\| + \gamma_{n+1} \sup_{z \in B} \|T_{n+1}z - T_n z\| + |\gamma_{n+1} - \gamma_n| \sup_{z \in B} \|T_n z\| \\ &\leq |\gamma_{n+1} - \gamma_n| \left(\sup_{z \in B} \|z\| + \sup_{z \in B} \|T_n z\| \right) + \sup_{z \in B} \|T_{n+1}z - T_n z\|. \end{aligned}$$

From condition (C3) and $\{T_n\}$ satisfying the AKTT-condition, we get that

$$\sum_{n=1}^{\infty} \sup_{z \in B} \|S_{n+1}z - S_n z\| < \infty. \quad (4.2)$$

Next, we show that $\|x_{n+1} - x_n\| \rightarrow 0$ and $\|y_n - T_n y_n\| \rightarrow 0$ as $n \rightarrow \infty$. We observe that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|S_{n+1}y_{n+1} - S_n y_n\| \\ &\leq \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_n y_n\| \\ &\leq \|y_{n+1} - y_n\| + \sup_{z \in \{y_n\}} \|S_{n+1}z - S_n z\|. \end{aligned} \quad (4.3)$$

It follows from (4.1) and (4.3) that

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{z \in \{y_n\}} \|S_{n+1}z - S_n z\|.$$

From (C1) and (4.2) we obtain

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.3 and (C2), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (4.4)$$

Hence

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \beta_n \|z_n - x_n\| = 0. \quad (4.5)$$

From (1.7) and (C1) we see that

$$\begin{aligned}\|y_n - x_n\| &= \|Q_C[(1 - \alpha_n)x_n] - Q_Cx_n\| \\ &\leq \alpha_n \|x_n\| \rightarrow 0,\end{aligned}\quad (4.6)$$

as $n \rightarrow \infty$. So from (4.4) and (4.6) we have $\|z_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Observe that

$$\|T_n y_n - y_n\| = \frac{1}{\gamma_n} \|z_n - y_n\|,$$

which implies by condition (C4) that

$$\lim_{n \rightarrow \infty} \|T_n y_n - y_n\| = 0. \quad (4.7)$$

Since $\{S_n\}$ satisfies the AKTT-condition, we can define $S : C \rightarrow C$ by $Sx = \lim_{n \rightarrow \infty} S_n x$ for all $x \in C$. It is easy to see that S is nonexpansive. Since $\{\gamma_n\}$ is bounded, there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} \rightarrow \gamma$ as $j \rightarrow \infty$. It follows that

$$Sx = \lim_{j \rightarrow \infty} S_{n_j} x = \lim_{j \rightarrow \infty} [(1 - \gamma_{n_j})x + \gamma_{n_j} T_{n_j} x] = (1 - \gamma)x + \gamma T x, \quad \forall x \in C.$$

This shows that $F(S) = F(T) = F$. We observe that

$$\begin{aligned}\|y_n - S y_n\| &\leq \|y_n - S_n y_n\| + \|S_n y_n - S y_n\| \\ &\leq \|y_n - z_n\| + \sup_{z \in \{y_n\}} \|S_n z - S z\| \rightarrow 0,\end{aligned}$$

as $n \rightarrow \infty$. On the other hand, we see that

$$\begin{aligned}\|x_n - S x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_n y_n\| + \|S_n y_n - S_n x_n\| + \|S_n x_n - S x_n\| \\ &\leq \|x_n - x_{n+1}\| + (1 - \beta_n) \|x_n - z_n\| + \|y_n - x_n\| + \sup_{z \in \{x_n\}} \|S_n z - S z\|.\end{aligned}$$

From (4.4)–(4.6), we have

$$\lim_{n \rightarrow \infty} \|S x_n - x_n\| = 0. \quad (4.8)$$

Let $\{x_t\}$ be defined by (3.2). By Theorem 3.1, we have $x_t \rightarrow x^* \in F(S)$ as $t \rightarrow 0$. Next, we prove that

$$\limsup_{n \rightarrow \infty} \langle x^*, j_q(x^* - x_n) \rangle \leq 0.$$

From Lemma 2.1 and (3.2), we obtain that

$$\begin{aligned}\|x_t - x_n\|^q &= \|x_t - S x_n + S x_n - x_n\|^q \\ &\leq \|x_t - S x_n\|^q + q \langle S x_n - x_n, j_q(x_t - S x_n) \rangle + C_q \|S x_n - x_n\|^q \\ &\leq \|x_t - S x_n\|^q + q \|S x_n - x_n\| \|x_t - S x_n\|^{q-1} + C_q \|S x_n - x_n\|^q \\ &\leq \|S Q_C(1 - t)x_t - S Q_C x_n\|^q + M \|S x_n - x_n\| \\ &\leq \|x_t - x_n - t x_t\|^q + M \|S x_n - x_n\| \\ &\leq \|x_t - x_n\|^q - q t \langle x_t, j_q(x_t - x_n) \rangle + C_q t^q \|x_t\|^q + M \|S x_n - x_n\|\end{aligned}$$

for some $M > 0$. It follows that

$$\langle x_t, j_q(x_t - x_n) \rangle \leq \frac{C_q t^{q-1}}{q} \|x_t\|^q + \frac{M}{q t} \|S x_n - x_n\|.$$

From (4.8) we obtain that

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle x_t, j_q(x_t - x_n) \rangle \leq 0. \quad (4.9)$$

On the other hand, we observe that

$$\begin{aligned}\langle x^*, j_q(x^* - x_n) \rangle &= \langle x^*, j_q(x^* - x_n) - j_q(x_t - x_n) \rangle + \langle x^* - x_t, j_q(x_t - x_n) \rangle + \langle x_t, j_q(x_t - x_n) \rangle \\ &\leq \langle x^*, j_q(x^* - x_n) - j_q(x_t - x_n) \rangle + \|x^* - x_t\| \|x_t - x_n\|^{q-1} + \langle x_t, j_q(x_t - x_n) \rangle.\end{aligned}\quad (4.10)$$

Since E is q -uniformly smooth, j_q is uniformly continuous on bounded sets of E . So the limits $\limsup_{n \rightarrow \infty}$ and $\limsup_{t \rightarrow 0}$ are interchangeable. Hence, from (4.9) and (4.10), we have

$$\limsup_{n \rightarrow \infty} \langle x^*, j_q(x^* - x_n) \rangle \leq 0. \quad (4.11)$$

From (1.7), we have

$$\begin{aligned} \|y_n - x^*\|^q &\leq \|(1 - \alpha_n)(x_n - x^*) - \alpha_n x^*\|^q \\ &\leq (1 - \alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - q \alpha_n \langle x^*, j_q((1 - \alpha_n)(x_n - x^*)) \rangle \\ &= (1 - \alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - q \alpha_n (1 - \alpha_n)^{q-1} \langle x^*, j_q(x_n - x^*) \rangle \\ &\leq (1 - \alpha_n) \|x_n - x^*\|^q + \alpha_n (C_q \alpha_n^{q-1} \|x^*\|^q - q (1 - \alpha_n)^{q-1} \langle x^*, j_q(x_n - x^*) \rangle) \\ &= (1 - \alpha_n) \|x_n - x^*\|^q + \alpha_n \delta_n, \end{aligned} \quad (4.12)$$

where $\delta_n = C_q \alpha_n^{q-1} \|x^*\|^q + q(1 - \alpha_n)^{q-1} \langle x^*, j_q(x^* - x_n) \rangle$. From (C1) and (4.11), it is easy to see that $\limsup_{n \rightarrow \infty} \delta_n \leq 0$.

Finally, we prove that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. By Lemma 2.2 and (1.7), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^q &= (1 - \beta_n) \langle x_n - x^*, j_q(x_{n+1} - x^*) \rangle + \beta_n \langle z_n - x^*, j_q(x_{n+1} - x^*) \rangle \\ &\leq (1 - \beta_n) \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \beta_n \|z_n - x^*\| \|x_{n+1} - x^*\|^{q-1} \\ &\leq (1 - \beta_n) \left(\frac{1}{q} \|x_n - x^*\|^q + \frac{q-1}{q} \|x_{n+1} - x^*\|^q \right) + \beta_n \left(\frac{1}{q} \|y_n - x^*\|^q + \frac{q-1}{q} \|x_{n+1} - x^*\|^q \right) \\ &= \frac{(1 - \beta_n)}{q} \|x_n - x^*\|^q + \frac{q-1}{q} \|x_{n+1} - x^*\|^q + \frac{\beta_n}{q} \|y_n - x^*\|^q. \end{aligned} \quad (4.13)$$

It follows from (4.12) and (4.13) that

$$\begin{aligned} \|x_{n+1} - x^*\|^q &\leq (1 - \beta_n) \|x_n - x^*\|^q + \beta_n \|y_n - x^*\|^q \\ &\leq (1 - \beta_n) \|x_n - x^*\|^q + \beta_n ((1 - \alpha_n) \|x_n - x^*\|^q + \alpha_n \delta_n) \\ &= (1 - \alpha_n \beta_n) \|x_n - x^*\|^q + \alpha_n \beta_n \delta_n. \end{aligned} \quad (4.14)$$

Applying Lemma 2.5 to (4.14), we conclude that $x_n \rightarrow x^* \in F$ as $n \rightarrow \infty$. This completes the proof. $\square$

Remark 4.2. Theorem 4.1 extends the main result of Yao et al. in the following senses:

- (i) from real Hilbert spaces to q -uniformly smooth and uniformly convex Banach spaces;
- (ii) from a single nonexpansive mapping to an infinite family of strict pseudocontractions.

As a direct consequence of Lemmas 2.7 and 2.8 and Theorem 4.1, we obtain the following result.

Theorem 4.3. Let E be a real q -uniformly smooth and uniformly convex Banach space which admits the weakly sequentially continuous duality mapping j_q and C be a nonempty, closed and convex subset of E . Let $\{S_k\}_{k=1}^\infty$ be a sequence of λ_k -strict pseudocontractions of C into itself such that $\bigcap_{k=1}^\infty F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define the sequence $\{x_n\}$ by $x_1 \in C$,

$$\begin{aligned} y_n &= Q_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n \left((1 - \gamma_n)y_n + \gamma_n \sum_{k=1}^n \mu_n^k S_k y_n \right), \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$ which satisfy (C1)–(C4) of Theorem 4.1 and $\{\mu_n^k\}$ is a real sequence which satisfies (i)–(iii) of Lemma 2.8. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{S_k\}_{k=1}^\infty$.

Remark 4.4. By Gossez and Lami [37], we know that if E satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then E has a weakly continuous duality mapping at zero. Thus, Theorems 4.1 and 4.3 hold in uniformly convex and q -uniformly smooth Banach spaces which satisfy Opial's condition and also hold in real Hilbert spaces.

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1.6 Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions

In this section, we introduce a new iterative scheme for finding a common fixed point of a finite family of strict pseudo-contractions. The results obtained in this paper extend and improve the corresponding results of Marino, Colao, Qin and Kang [G. Marino, V. Colao, X. Qin, S. M. Kang, Strong convergence of the modified Mann iterative method for strict pseudo-contractions, Comput. Math. Appl. 57 (2009) 455-465].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2.$$

Recall that the mapping T is said to be κ -strict pseudo-contractive if there exist $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in D(T). \quad (1.1)$$

Note that the class of κ -strict pseudo-contractions strictly includes the class of nonexpansive mappings. If $\kappa = 1$, T is said to be a *pseudo-contractive mapping*. T is *strong pseudo-contractive* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. In a real Hilbert space H (1.1) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in D(T). \quad (1.2)$$

The class of κ -strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings and class of strong pseudo-contractive mappings is independent of the class of κ -strict pseudo-contractions.

In 1953, Mann [2] introduced an iteration procedure for approximating fixed points of a nonexpansive self-mapping T of a nonempty closed convex subset K of a Hilbert space as follows: $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1, \quad (1.3)$$

where $\{\alpha_n \in [0, 1]\}$. Later in 1967, Halpern [3] introduced the iterative scheme for a nonexpansive self-mapping T of a nonempty closed convex subset K of a Hilbert space,

$$x_{n+1} = (1 - \alpha_n)u + \alpha_n Tx_n, \quad n \geq 1, \quad (1.4)$$

where $u, x_1 \in C$ are fixed and $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$. He also pointed out that the conditions $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of $\{x_n\}$ to a fixed point of T .

Many authors try to modify the iteration (1.3) process to have a strong convergence: see, e.g., [4–10].

In 2006, Marino and Xu [11] introduced a new iterative scheme by the viscosity approximation method which was first introduced by Moudafi [12],

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (1.5)$$

where S is a nonexpansive mapping of H into itself and the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality,

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(S). \quad (1.6)$$

In 2008, Yao, Chen and Yao [14] introduced iterative scheme as follows;

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Sx_n, \\ x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n)y_n, \end{cases} \quad n \geq 0, \quad (1.7)$$

where S is a nonexpansive mapping on a closed convex subset of Banach space and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. They proved under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of S .

Recently, Marino, Colao, Qin and Kang [15] introduced an iterative algorithm as follows;

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \end{cases} \quad n \geq 0, \quad (1.8)$$

where T is κ -strict pseudo-contractive on H . Under suitable conditions of $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(T)$ which solves the following variational inequality $\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T)$.

In 1999, Atsushiba and Takahashi [16] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I, \\ U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_N - 1U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I, \end{aligned} \quad (1.9)$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the W -mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [17] proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \dots, N$.

In this paper, motivated by Takahashi and Shimoji [17], we introduce a new mapping for finding a common fixed point of a finite family of strict pseudo-contractive mappings. Let X be a real Hilbert space and C a nonempty closed convex subset of X and let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contractive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \dots, N$,

let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define the mapping $S_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I \\ S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \end{aligned} \quad (1.10)$$

We shall show that $F(S_n) = \bigcap_{i=1}^N F(T_i)$ under some conditions and motivated by Marino, Colao, Qin and Kang [15], we introduce an iterative scheme as follows:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \quad (1.11)$$

where $u \in K$ is a given point, f is an α -contraction and A is a linear bounded strongly positive operator. We prove under some control conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{a_n\}$ that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \bigcap_{i=1}^N F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i). \quad (1.12)$$

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main results in the next section.

Lemma 2.1 (See [4]). Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \rightarrow C$ be a self-mapping of C . If S is a κ -strict pseudo-contraction mapping, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1 + \kappa}{1 - \kappa} \|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.2 (See [13]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1 - \alpha_n) s_n + \delta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.3. In a real Hilbert space H , there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad x, y \in H.$$

Lemma 2.4 (See [11]). Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.

Lemma 2.5. Let H be a real Hilbert space. There hold the following identities

- (i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$ and $\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$
- (ii) $\|\sum_{i=0}^m \alpha_i x_i\|^2 = \sum_{i=0}^m \alpha_i \|x_i\|^2 - \sum_{i=0}^m \alpha_i \alpha_j \|x_i - x_j\|^2$

for $\sum_{i=0}^m \alpha_i = 1$, $\alpha_i \in [0, 1]$, $\forall i \in \{0, 1, 2, \dots, m\}$.

Lemma 2.6 (See [11]). Let H be a real Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Let f be an α -contraction. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : H \rightarrow H$ be a nonexpansive mapping. For $0 < t \leq \|A\|^{-1}$, let x_t be the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point $\bar{x}$ of T which solves the variational inequality

$$\langle \gamma f(\bar{x}) - A\bar{x}, z - \bar{x} \rangle \leq 0, \quad \forall z \in F(T). \quad (2.1)$$

Lemma 2.7 (See [18]). Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that

$$a_{n+1} = (1 - \alpha_n)a_n + b_n + c_n, \quad \text{for all } n \geq 0. \quad (2.2)$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

- (a) if $b_n \leq \alpha_n C$ where $C \geq 0$, then $\{a_n\}$ is a bounded sequence.
(b) if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 2.1. Let C be a nonempty convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contractions of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \rightarrow C$ as follows:

$$\begin{aligned} U_0 &= I \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\ &\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned} \quad (2.3)$$

This mapping is called S -mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.8. Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ -strict pseudo-contractive mapping of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, 3, \dots, N$, where $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (\kappa, 1), \alpha_3^N \in [\kappa, 1), \alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

Proof. It is clear that $\bigcap_{i=1}^N F(T_i) \subseteq F(S)$. Next we show that $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. To show this, let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. Then we have

$$\begin{aligned} \|Sx_0 - x^*\|^2 &= \|\alpha_1^N T_N U_{N-1} x_0 + \alpha_2^N U_{N-1} x_0 + \alpha_3^N x_0 - x^*\|^2 \\ &= \|\alpha_1^N (T_N U_{N-1} x_0 - x^*) + \alpha_2^N (U_{N-1} x_0 - x^*) + \alpha_3^N (x_0 - x^*)\|^2 \\ &= \alpha_1^N \|T_N U_{N-1} x_0 - x^*\|^2 + \alpha_2^N \|U_{N-1} x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2 - \alpha_1^N \alpha_2^N \|T_N U_{N-1} x_0 - U_{N-1} x_0\|^2 \\ &\quad - \alpha_1^N \alpha_3^N \|T_N U_{N-1} x_0 - x_0\|^2 - \alpha_2^N \alpha_3^N \|U_{N-1} x_0 - x_0\|^2 \\ &\leq \alpha_1^N (\|U_{N-1} x_0 - x^*\|^2 + \kappa \|(I - T_N) U_{N-1} x_0 - (I - T_N) x^*\|^2) \\ &\quad + \alpha_2^N \|U_{N-1} x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2 - \alpha_1^N \alpha_2^N \|T_N U_{N-1} x_0 - U_{N-1} x_0\|^2 \\ &= (\alpha_1^N + \alpha_2^N) \|U_{N-1} x_0 - x^*\|^2 + \alpha_1^N (\kappa - \alpha_2^N) \|(I - T_N) U_{N-1} x_0\|^2 + \alpha_3^N \|x_0 - x^*\|^2 \\ &\leq (1 - \alpha_3^N) \|U_{N-1} x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) \|\alpha_1^{N-1} T_{N-1} U_{N-2} x_0 + \alpha_2^{N-1} U_{N-2} x_0 + \alpha_3^{N-1} x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) \|\alpha_1^{N-1} (T_{N-1} U_{N-2} x_0 - x^*) + \alpha_2^{N-1} (U_{N-2} x_0 - x^*) \\ &\quad + \alpha_3^{N-1} (x_0 - x^*)\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) (\alpha_1^{N-1} \|T_{N-1} U_{N-2} x_0 - x^*\|^2 + \alpha_2^{N-1} \|U_{N-2} x_0 - x^*\|^2 \\ &\quad + \alpha_3^{N-1} \|x_0 - x^*\|^2 - \alpha_1^{N-1} \alpha_2^{N-1} \|T_{N-1} U_{N-2} x_0 - U_{N-2} x_0\|^2 \\ &\quad - \alpha_1^{N-1} \alpha_3^{N-1} \|T_{N-1} U_{N-2} x_0 - x_0\|^2 - \alpha_2^{N-1} \alpha_3^{N-1} \|U_{N-2} x_0 - x_0\|^2) + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &\leq (1 - \alpha_3^N) (\alpha_1^{N-1} (\|U_{N-2} x_0 - x^*\|^2 + \kappa \|(I - T_{N-1}) U_{N-2} x_0 - (I - T_{N-1}) x^*\|^2) \\ &\quad + \alpha_2^{N-1} \|U_{N-2} x_0 - x^*\|^2 + \alpha_3^{N-1} \|x_0 - x^*\|^2 - \alpha_1^{N-1} \alpha_2^{N-1} \|T_{N-1} U_{N-2} x_0 - U_{N-2} x_0\|^2) \\ &\quad + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) ((\alpha_1^{N-1} + \alpha_2^{N-1}) \|U_{N-2} x_0 - x^*\|^2 + \alpha_1^{N-1} (\kappa - \alpha_2^{N-1}) \|(I - T_{N-1}) U_{N-2} x_0\|^2 \end{aligned} \quad (2.4)$$

$$\begin{aligned}
& + \alpha_3^{N-1} \|x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\
& \leq (1 - \alpha_3^N)(1 - \alpha_3^{N-1}) \|U_{N-2}x_0 - x^*\|^2 + \alpha_3^{N-1}(1 - \alpha_3^N) \|x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\
& = \prod_{j=N-1}^N (1 - \alpha_3^j) \|U_{N-2}x_0 - x^*\|^2 + \left(1 - \prod_{j=N-1}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& \vdots \\
& \leq \prod_{j=3}^N (1 - \alpha_3^j) \|U_2x_0 - x^*\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \tag{2.5}
\end{aligned}$$

$$\leq \prod_{j=2}^N (1 - \alpha_3^j) \|U_1x_0 - x^*\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \tag{2.6}$$

$$\begin{aligned}
& = \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 T_1 U_0 x_0 + \alpha_2^1 U_0 x_0 + \alpha_3^1 x_0 - x^*\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 (T_1 x_0 - x^*) + (1 - \alpha_1^1)(x_0 - x^*)\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|T_1 x_0 - x^*\|^2 + (1 - \alpha_1^1) \|x_0 - x^*\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|T_1 x_0 - x_0\|^2) \\
& \quad + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& \leq \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 (\|x_0 - x^*\|^2 + \kappa \|(I - T_1)x_0 - (I - T_1)x^*\|^2) \\
& \quad + (1 - \alpha_1^1) \|x_0 - x^*\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|T_1 x_0 - x_0\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|x_0 - x^*\|^2 + \kappa \alpha_1^1 \|(I - T_1)x_0\|^2 + (1 - \alpha_1^1) \|x_0 - x^*\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|T_1 x_0 - x_0\|^2) \\
& \quad + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) (\|x_0 - x^*\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x_0\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2. \tag{2.7}
\end{aligned}$$

For each $k = 1, 2, \dots, N - 1$, we obtain

$$\alpha_1^k ((1 - \alpha_1^k) - \kappa) \geq \alpha_1^k ((1 - \alpha_1^k) - \alpha_2^k) = \alpha_1^k (1 - (\alpha_1^k + \alpha_2^k)) = \alpha_1^k \alpha_3^k > 0. \tag{2.8}$$

This implies by (2.7) that

$$\prod_{j=2}^N (1 - \alpha_3^j) \|x_0 - x^*\|^2 \leq \prod_{j=2}^N (1 - \alpha_3^j) (\|x_0 - x^*\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x_0\|^2).$$

Hence, we obtain

$$\|x_0 - x^*\|^2 \leq \|x_0 - x^*\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x_0\|^2,$$

which implies

$$\alpha_1^1 ((1 - \alpha_1^1) - \kappa) \|x_0 - T_1 x_0\|^2 \leq 0.$$

It follows from (2.8) that

$$\|x_0 - T_1 x_0\|^2 = 0.$$

Hence $T_1x_0 = x_0$, that is $x_0 \in F(T_1)$. It implies that

$$U_1x_0 = \alpha_1^1 T_1 U_0 x_0 + \alpha_2^1 U_0 x_0 + \alpha_3^1 x_0 = x_0.$$

By (2.5), we have

$$\begin{aligned} \|x_0 - x^*\|^2 &= \|Sx_0 - x^*\|^2 \leq \prod_{j=3}^N (1 - \alpha_3^j) \|U_2x_0 - x^*\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) \|\alpha_1^2 (T_2 U_1 x_0 - x^*) + \alpha_2^2 (U_1 x_0 - x^*) + \alpha_3^2 (x_0 - x^*)\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) \|\alpha_1^2 (T_2 x_0 - x^*) + (1 - \alpha_1^2) (x_0 - x^*)\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) (\alpha_1^2 \|T_2 x_0 - x^*\|^2 + (1 - \alpha_1^2) \|x_0 - x^*\|^2 - \alpha_1^2 (1 - \alpha_1^2) \|T_2 x_0 - x_0\|^2) \\ &\quad + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &\leq \prod_{j=3}^N (1 - \alpha_3^j) (\alpha_1^2 (\|x_0 - x^*\|^2 + \kappa \|(I - T_2)x_0 - (I - T_2)x^*\|^2) \\ &\quad + (1 - \alpha_1^2) \|x_0 - x^*\|^2 - \alpha_1^2 (1 - \alpha_1^2) \|T_2 x_0 - x_0\|^2) + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) (\alpha_1^2 \|x_0 - x^*\|^2 + \kappa \alpha_1^2 \|(I - T_2)x_0\|^2 \\ &\quad + (1 - \alpha_1^2) \|x_0 - x^*\|^2 - \alpha_1^2 (1 - \alpha_1^2) \|T_2 x_0 - x_0\|^2) + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) (\|x_0 - x^*\|^2 + \alpha_1^2 (\kappa - (1 - \alpha_1^2)) \|(I - T_2)x_0\|^2) + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2, \end{aligned}$$

it follows that

$$\|x_0 - x^*\|^2 \leq \|x_0 - x^*\|^2 + \alpha_1^2 (\kappa - (1 - \alpha_1^2)) \|(I - T_2)x_0\|^2$$

which implies

$$\alpha_1^2 ((1 - \alpha_1^2) - \kappa) \|x_0 - T_2 x_0\|^2 \leq 0.$$

It follows by (2.8) that

$$\|x_0 - T_2 x_0\|^2 = 0.$$

Hence, we have $T_2 x_0 = x_0$, that is $x_0 \in F(T_2)$.

This implies that $U_2 x_0 = \alpha_1^2 T_2 U_1 x_0 + \alpha_2^2 U_1 x_0 + \alpha_3^2 x_0 = x_0$.

By continuing in this way, we can show that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1, \dots, N - 1$.

Finally, we shall show that $x_0 \in F(T_N)$.

Since

$$\begin{aligned} 0 &= Sx_0 - x_0 = \alpha_1^N T_N U_{N-1} x_0 + \alpha_2^N U_{N-1} x_0 + \alpha_3^N x_0 - x_0 \\ &= \alpha_1^N (T_N x_0 - x_0), \end{aligned}$$

and $\alpha_1^N \in (\kappa, 1]$, we obtain that $T_N x_0 = x_0$, that is $x_0 \in F(T_N)$. Hence $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$.

Next we show that S is nonexpansive. To show this let $x, y \in C$. Then

$$\begin{aligned} \|Sx - Sy\|^2 &= \|\alpha_1^N T_N U_{N-1} x + \alpha_2^N U_{N-1} x + \alpha_3^N x - \alpha_1^N T_N U_{N-1} y - \alpha_2^N U_{N-1} y - \alpha_3^N y\|^2 \\ &= \|\alpha_1^N (T_N U_{N-1} x - T_N U_{N-1} y) + \alpha_2^N (U_{N-1} x - U_{N-1} y) + \alpha_3^N (x - y)\|^2 \end{aligned}$$

$$\begin{aligned}
&= \alpha_1^N \|T_N U_{N-1}x - T_N U_{N-1}y\|^2 + \alpha_2^N \|U_{N-1}x - U_{N-1}y\|^2 + \alpha_3^N \|x - y\|^2 \\
&\quad - \alpha_1^N \alpha_2^N \|U_{N-1}x - U_{N-1}y - T_N U_{N-1}x + T_N U_{N-1}y\|^2 \\
&\quad - \alpha_1^N \alpha_3^N \|T_N U_{N-1}x - T_N U_{N-1}y - (x - y)\|^2 - \alpha_2^N \alpha_3^N \|U_{N-1}x - U_{N-1}y - (x - y)\|^2 \\
&\leq \alpha_1^N (\|U_{N-1}x - U_{N-1}y\|^2 + \kappa \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2) + \alpha_2^N \|U_{N-1}x - U_{N-1}y\|^2 \\
&\quad + \alpha_3^N \|x - y\|^2 - \alpha_1^N \alpha_2^N \|U_{N-1}x - U_{N-1}y - T_N U_{N-1}x + T_N U_{N-1}y\|^2 \\
&= (\alpha_1^N + \alpha_2^N) \|U_{N-1}x - U_{N-1}y\|^2 + \alpha_1^N (\kappa - \alpha_2^N) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 + \alpha_3^N \|x - y\|^2 \\
&\leq (1 - \alpha_3^N) \|U_{N-1}x - U_{N-1}y\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) \|\alpha_1^{N-1} T_{N-1} U_{N-2}x + \alpha_2^{N-1} U_{N-2}x + \alpha_3^{N-1} x - \alpha_1^{N-1} T_{N-1} U_{N-2}y \\
&\quad - \alpha_2^{N-1} U_{N-2}y - \alpha_3^{N-1} y\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) \|\alpha_1^{N-1} (T_{N-1} U_{N-2}x - T_{N-1} U_{N-2}y) + \alpha_2^{N-1} (U_{N-2}x - U_{N-2}y) \\
&\quad + \alpha_3^{N-1} (x - y)\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) (\alpha_1^{N-1} \|T_{N-1} U_{N-2}x - T_{N-1} U_{N-2}y\|^2 + \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y\|^2 \\
&\quad + \alpha_3^{N-1} \|x - y\|^2 - \alpha_1^{N-1} \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y - T_{N-1} U_{N-2}x \\
&\quad + T_{N-1} U_{N-2}y\|^2 - \alpha_1^{N-1} \alpha_3^{N-1} \|T_{N-1} U_{N-2}x - T_{N-1} U_{N-2}y - (x - y)\|^2 \\
&\quad - \alpha_2^{N-1} \alpha_3^{N-1} \|U_{N-2}x - U_{N-2}y - (x - y)\|^2) + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&\leq (1 - \alpha_3^N) (\alpha_1^{N-1} (\|U_{N-2}x - U_{N-2}y\|^2 + \kappa \|(I - T_{N-1})U_{N-2}x \\
&\quad - (I - T_{N-1})U_{N-2}y\|^2) + \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y\|^2 + \alpha_3^{N-1} \|x - y\|^2 \\
&\quad - \alpha_1^{N-1} \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y - T_{N-1} U_{N-2}x + T_{N-1} U_{N-2}y\|^2) + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) ((\alpha_1^{N-1} + \alpha_2^{N-1}) \|U_{N-2}x - U_{N-2}y\|^2 + \alpha_1^{N-1} (\kappa - \alpha_2^{N-1}) \|(I - T_{N-1})U_{N-2}x \\
&\quad - (I - T_{N-1})U_{N-2}y\|^2 + \alpha_3^{N-1} \|x - y\|^2) + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&\leq (1 - \alpha_3^N) (1 - \alpha_3^{N-1}) \|U_{N-2}x - U_{N-2}y\|^2 + \alpha_3^{N-1} (1 - \alpha_3^N) \|x - y\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= \prod_{j=N-1}^N (1 - \alpha_3^j) \|U_{N-2}x - U_{N-2}y\|^2 + \left(1 - \prod_{j=N-1}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&\leq \\
&\vdots \\
&\leq \prod_{j=2}^N (1 - \alpha_3^j) \|U_1x - U_1y\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 (T_1 U_0x - T_1 U_0y) + \alpha_2^1 (U_0x - U_0y) + \alpha_3^1 (x - y)\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 (T_1x - T_1y) + (1 - \alpha_1^1) (x - y)\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|T_1x - T_1y\|^2 + (1 - \alpha_1^1) \|x - y\|^2 \\
&\quad - \alpha_1^1 (1 - \alpha_1^1) \|x - y - Tx + Ty\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&\leq \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 (\|x - y\|^2 + \kappa \|(I - T_1)x - (I - T_1)y\|^2) \\
&\quad + (1 - \alpha_1^1) \|x - y\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|x - y - Tx + Ty\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2
\end{aligned}$$

$$\begin{aligned}
&= \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|x - y\|^2 + \kappa \alpha_1^1 \|(I - T_1)x - (I - T_1)y\|^2 \\
&\quad + (1 - \alpha_1^1) \|x - y\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|(I - T_1)x - (I - T_1)y\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) (\|x - y\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x - (I - T_1)y\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&\leq \prod_{j=2}^N (1 - \alpha_3^j) \|x - y\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \|x - y\|^2. \quad \square
\end{aligned}$$

3. Main results

In this section, we prove a strong convergence theorem of the iterative scheme (1.11) to a common fixed point of $\bigcap_{i=1}^N F(T_i)$.

Theorem 3.1. Let H be a Hilbert space, let f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contraction of H into itself for some $\kappa_i \in [0, 1)$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1, 2, \dots, N$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \end{cases} \quad n \geq 1, \quad (3.1)$$

where $\{\beta_n\}, \{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 0$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$, for all $j \in \{1, 2, 3, \dots, N\}$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in \bigcap_{i=1}^N F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

Proof. Let $p \in \bigcap_{i=1}^N F(T_i)$. Since $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, we may assume that $\alpha_n < \|A\|^{-1}$ for all $n \in \mathbb{N}$. By Lemma 2.4 we can conclude that $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$. By Lemma 2.8, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|\alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n - p\| \\
&= \|\alpha_n (\gamma (a_n u + (1 - a_n) f(x_n)) - Ap) + (I - \alpha_n A) (y_n - p)\| \\
&\leq \alpha_n \|\gamma (a_n u + (1 - a_n) f(x_n)) - Ap\| + \|I - \alpha_n A\| \|y_n - p\| \\
&\leq \alpha_n \gamma \|(a_n u + (1 - a_n) f(x_n)) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \bar{\gamma}) \|\beta_n x_n \\
&\quad + (1 - \beta_n) S_n x_n - p\| \\
&\leq \alpha_n \gamma (a_n \|u - f(p)\| + (1 - a_n) \|f(x_n) - f(p)\|) + \alpha_n \|\gamma f(p) - Ap\| \\
&\quad + (1 - \alpha_n \bar{\gamma}) (\beta_n \|x_n - p\| + (1 - \beta_n) \|S_n x_n - p\|) \\
&\leq \alpha_n \gamma (a_n \|u - f(p)\| + (1 - a_n) \alpha \|x_n - p\|) + \alpha_n \|\gamma f(p) - Ap\| \\
&\quad + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\
&= \alpha_n \gamma a_n \|u - f(p)\| + (1 - \alpha_n \bar{\gamma} + (1 - a_n) \alpha \gamma) \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| \\
&\leq (1 - \alpha_n (\bar{\gamma} - \gamma \alpha)) \|x_n - p\| + \alpha_n \gamma a_n \|u - f(p)\| + \alpha_n \|\gamma f(p) - Ap\|.
\end{aligned}$$

It follows by Lemma 2.7 that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$. From condition (i) and (1.11), we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = \lim_{n \rightarrow \infty} \alpha_n \|\gamma(a_n u + (1 - a_n)f(x_n)) - Ay_n\| = 0. \quad (3.2)$$

Next, we show that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.3)$$

By definition of $\{x_n\}$ and Lemma 2.4, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n \gamma(a_n u + (1 - a_n)f(x_n)) + (I - \alpha_n A)y_n - \alpha_{n-1} \gamma(a_{n-1} u \\ &\quad + (1 - a_{n-1})f(x_{n-1})) - (I - \alpha_{n-1} A)y_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})\gamma a_n u + \alpha_{n-1} \gamma u(a_n - a_{n-1}) + \alpha_n \gamma(1 - a_n)(f(x_n) - f(x_{n-1})) \\ &\quad + \alpha_n \gamma(1 - a_n)f(x_{n-1}) + (I - \alpha_n A)y_n - \alpha_{n-1} \gamma(1 - a_{n-1})f(x_{n-1}) - (I - \alpha_{n-1} A)y_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})\gamma a_n u + \alpha_{n-1} \gamma u(a_n - a_{n-1}) + \alpha_n \gamma(1 - a_n)(f(x_n) - f(x_{n-1})) \\ &\quad + \alpha_n \gamma f(x_{n-1}) - \alpha_{n-1} \gamma f(x_{n-1})a_n + (I - \alpha_n A)(y_n - y_{n-1}) + y_{n-1} - \alpha_n A y_{n-1} \\ &\quad - \alpha_{n-1} \gamma f(x_{n-1}) + \alpha_{n-1} \gamma f(x_{n-1})a_{n-1} - y_{n-1} + \alpha_{n-1} A y_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})\gamma a_n u + \alpha_{n-1} \gamma u(a_n - a_{n-1}) + \alpha_n \gamma(1 - a_n)(f(x_n) - f(x_{n-1})) \\ &\quad + (\alpha_n - \alpha_{n-1})(\gamma f(x_{n-1}) - A y_{n-1}) + (I - \alpha_n A)(y_n - y_{n-1}) \\ &\quad + (\alpha_{n-1}(a_{n-1} - a_n) + (\alpha_{n-1} - \alpha_n)a_n)\gamma f(x_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|\gamma a_n u\| + \alpha_{n-1} \|\gamma u\| |a_n - a_{n-1}| + \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(x_{n-1}) - A y_{n-1}\| + \|I - \alpha_n A\| \|y_n - y_{n-1}\| \\ &\quad + \alpha_{n-1} |a_{n-1} - a_n| \|\gamma f(x_{n-1})\| + |\alpha_{n-1} - \alpha_n| a_n \|\gamma f(x_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|\gamma u\| + \gamma \|u\| |a_n - a_{n-1}| + \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(x_{n-1}) - A y_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| \\ &\quad + |a_{n-1} - a_n| \|\gamma f(x_{n-1})\| + |\alpha_{n-1} - \alpha_n| \|\gamma f(x_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| K + K |a_n - a_{n-1}| + \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| K + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |a_{n-1} - a_n| K + |\alpha_{n-1} - \alpha_n| K \\ &= \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K, \end{aligned} \quad (3.4)$$

where $K = \max\{\gamma \|u\|, \gamma \sup_{n \in \mathbb{N}} \|f(x_n)\|, \sup_{n \in \mathbb{N}} \|\gamma f(x_n) - A y_n\|\}$. By definition of $\{y_n\}$ and nonexpansiveness of S_n , we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\beta_n x_n + (1 - \beta_n) S_n x_n - \beta_{n-1} x_{n-1} - (1 - \beta_{n-1}) S_{n-1} x_{n-1}\| \\ &= \|\beta_n (x_n - x_{n-1}) + (1 - \beta_n) S_n x_n + (\beta_n - \beta_{n-1}) x_{n-1} - (1 - \beta_{n-1}) S_{n-1} x_{n-1}\| \\ &= \|\beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} + (1 - \beta_n) (S_n x_n - S_{n-1} x_{n-1}) \\ &\quad + (S_n x_{n-1} - S_{n-1} x_{n-1}) + (\beta_{n-1} - \beta_n) S_{n-1} x_{n-1}\| \\ &\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) (\|x_n - x_{n-1}\| \\ &\quad + \|S_n x_{n-1} - S_{n-1} x_{n-1}\|) + |\beta_{n-1} - \beta_n| \|S_{n-1} x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + 2M |\beta_n - \beta_{n-1}| + (1 - \beta_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\|, \end{aligned} \quad (3.5)$$

where $M = \max\{\sup_{n \in \mathbb{N}} \|x_n\|, \sup_{n \in \mathbb{N}} \|S_n x_n\|\}$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|U_{n+1,1} x_n - U_{n,1} x_n\| &= \|\alpha_1^{n+1,1} T_1 x_n + (1 - \alpha_1^{n+1,1}) x_n - \alpha_1^{n,1} T_1 x_n - (1 - \alpha_1^{n,1}) x_n\| \\ &= \|\alpha_1^{n+1,1} T_1 x_n - \alpha_1^{n+1,1} x_n - \alpha_1^{n,1} T_1 x_n + \alpha_1^{n,1} x_n\| \\ &= \|(\alpha_1^{n+1,1} - \alpha_1^{n,1}) T_1 x_n - (\alpha_1^{n+1,1} - \alpha_1^{n,1}) x_n\| \\ &= |\alpha_1^{n+1,1} - \alpha_1^{n,1}| \|T_1 x_n - x_n\| \end{aligned} \quad (3.6)$$

and for $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned} \|U_{n+1,k} x_n - U_{n,k} x_n\| &= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n + \alpha_3^{n+1,k} x_n \\ &\quad - \alpha_1^{n,k} T_k U_{n,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n - \alpha_3^{n,k} x_n\| \\ &= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} x_n + \alpha_3^{n+1,k} x_n - \alpha_1^{n,k} T_k U_{n,k-1} x_n - \alpha_3^{n,k} x_n \\ &\quad + \alpha_2^{n+1,k} U_{n+1,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \end{aligned}$$

$$\begin{aligned}
&= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} x_n - \alpha_1^{n+1,k} T_k U_{n,k-1} x_n + \alpha_1^{n+1,k} T_k U_{n,k-1} x_n \\
&\quad - \alpha_1^{n,k} T_k U_{n,k-1} x_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \\
&= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) T_k U_{n,k-1} x_n \\
&\quad + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \\
&= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_k U_{n,k-1} x_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n \\
&\quad - \alpha_2^{n+1,k} U_{n,k-1} x_n + \alpha_2^{n+1,k} U_{n,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \\
&= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_k U_{n,k-1} x_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} (U_{n+1,k-1} x_n \\
&\quad - U_{n,k-1} x_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k}) U_{n,k-1} x_n\| \\
&\leq \alpha_1^{n+1,k} \|T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&\quad + \alpha_2^{n+1,k} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}| \|U_{n,k-1} x_n\| \\
&= \alpha_1^{n+1,k} \|T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + \alpha_2^{n+1,k} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |1 - \alpha_1^{n+1,k} \\
&\quad - \alpha_3^{n+1,k} - 1 + \alpha_1^{n,k} + \alpha_3^{n,k}| \|U_{n,k-1} x_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&\leq \alpha_1^{n+1,k} \frac{1+\kappa}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + \alpha_2^{n+1,k} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + (|\alpha_1^{n,k} \\
&\quad - \alpha_1^{n+1,k}| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|) \|U_{n,k-1} x_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&\leq \frac{1+\kappa}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + \frac{1-\kappa}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + (|\alpha_1^{n,k} - \alpha_1^{n+1,k}| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|) \|U_{n,k-1} x_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&= \frac{2}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| (\|T_k U_{n,k-1} x_n\| \\
&\quad + \|U_{n,k-1} x_n\|) + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| (\|U_{n,k-1} x_n\| + \|x_n\|). \tag{3.7}
\end{aligned}$$

By (3.6) and (3.7), we have

$$\begin{aligned}
\|S_{n+1} x_n - S_n x_n\| &= \|U_{n+1,N} x_n - U_{n,N} x_n\| \\
&\leq \frac{2}{1-\kappa} \|U_{n+1,N-1} x_n - U_{n,N-1} x_n\| + |\alpha_1^{n+1,N} - \alpha_1^{n,N}| (\|T_N U_{n,N-1} x_n\| \\
&\quad + \|U_{n,N-1} x_n\|) + |\alpha_3^{n+1,N} - \alpha_3^{n,N}| (\|U_{n,N-1} x_n\| + \|x_n\|) \\
&\leq \frac{2}{1-\kappa} \left(\frac{2}{1-\kappa} \|U_{n+1,N-2} x_n - U_{n,N-2} x_n\| \right. \\
&\quad + |\alpha_1^{n+1,N-1} - \alpha_1^{n,N-1}| (\|T_{N-1} U_{n,N-2} x_n\| + \|U_{n,N-2} x_n\|) \\
&\quad \left. + |\alpha_3^{n+1,N-1} - \alpha_3^{n,N-1}| (\|U_{n,N-2} x_n\| + \|x_n\|) \right) \\
&\quad + |\alpha_1^{n+1,N} - \alpha_1^{n,N}| (\|T_N U_{n,N-1} x_n\| + \|U_{n,N-1} x_n\|) + |\alpha_3^{n+1,N} \\
&\quad - \alpha_3^{n,N}| (\|U_{n,N-1} x_n\| + \|x_n\|) \\
&= \left(\frac{2}{1-\kappa} \right)^2 \|U_{n+1,N-2} x_n - U_{n,N-2} x_n\| + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa} \right)^{N-j} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| (\|T_j U_{n,j-1} x_n\|
\end{aligned}$$

$$\begin{aligned}
& + \|U_{n,j-1}x_n\|) + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa}\right)^{N-j} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| (\|U_{n,j-1}x_n\| + \|x_n\|) \\
& \leq \\
& \vdots \\
& \leq \left(\frac{2}{1-\kappa}\right)^{N-1} \|U_{n+1,1}x_n - U_{n,1}x_n\| + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| (\|T_j U_{n,j-1}x_n\| \\
& + \|U_{n,j-1}x_n\|) + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| (\|U_{n,j-1}x_n\| + \|x_n\|) \\
& = \left(\frac{2}{1-\kappa}\right)^{N-1} |\alpha_1^{n+1,1} - \alpha_1^{n,1}| \|T_1 x_n - x_n\| + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} \\
& |\alpha_1^{n+1,j} - \alpha_1^{n,j}| (\|T_j U_{n,j-1}x_n\| + \|U_{n,j-1}x_n\|) + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} \\
& |\alpha_3^{n+1,j} - \alpha_3^{n,j}| (\|U_{n,j-1}x_n\| + \|x_n\|).
\end{aligned}$$

This implies by assumption (ii) that

$$\sum_{n=1}^{\infty} \|S_{n+1}x_n - S_n x_n\| < \infty.$$

From (3.4) and (3.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| & \leq \alpha_n \gamma (1 - a_n) \alpha \|x_n - x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K \\
& \leq \alpha_n \gamma \alpha \|x_n - x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) (\|x_n - x_{n-1}\| + 2M|\beta_n - \beta_{n-1}| \\
& + (1 - \beta_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\|) + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K \\
& \leq (1 - \alpha_n (\bar{\gamma} - \gamma \alpha)) \|x_n - x_{n-1}\| + 2M|\beta_n - \beta_{n-1}| \\
& + (1 - \beta_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\| + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K.
\end{aligned}$$

It follows from Lemma 2.2 that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. By definition of $\{y_n\}$, we obtain

$$\|x_n - y_n\| = (1 - \beta_n) \|S_n x_n - x_n\|.$$

This implies by (3.2) and (3.3)

$$(1 - \beta_n) \|S_n x_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By condition (iii), we obtain

$$\|S_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

Let $q = \lim_{t \rightarrow 0} x_t$, where x_t is being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)S_n x$ for $t \in (0, \|A\|^{-1})$. By Lemma 2.8, S_n is nonexpansive and $F(S_n) = \bigcap_{i=1}^N F(T_i)$. From Lemma 2.6 we can conclude that $q \in \bigcap_{i=1}^N F(T_i)$ and

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

By definition of $\{x_t\}$, we can write

$$x_t = t\gamma f(x_t) + (I - tA)S_n x_t.$$

Then, we have

$$\|x_t - x_n\| = \|(I - tA)(S_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|.$$

By Lemma 2.3, we have

$$\begin{aligned}
\|x_t - x_n\|^2 & = \|(I - tA)(S_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\
& \leq (1 - \bar{\gamma}t)^2 \|S_n x_t - x_n\|^2 + 2t \langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\
& \leq (1 - \bar{\gamma}t)^2 (\|S_n x_t - S_n x_n\| + \|S_n x_n - x_n\|)^2 + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\
& \leq (1 - 2\bar{\gamma}t + \bar{\gamma}^2 t^2) \|x_t - x_n\|^2 + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\| \\
& + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle.
\end{aligned} \quad (3.9)$$

Since A is a linear strongly positive operator with coefficient $\bar{\gamma}$, it follows by (3.9) that

$$\begin{aligned} 2t \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\leq (\bar{\gamma}^2 t^2 - 2\bar{\gamma}t) \|x_t - x_n\|^2 + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \\ &\quad \|S_n x_n - x_n\| + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &= \bar{\gamma}(\bar{\gamma}t^2 - 2t) \|x_t - x_n\|^2 + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \\ &\quad \|S_n x_n - x_n\| + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &\leq (\bar{\gamma}t^2 - 2t) \langle Ax_t - Ax_n, x_t - x_n \rangle + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| \\ &\quad + \|S_n x_n - x_n\|) \|S_n x_n - x_n\| + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &= \bar{\gamma}t^2 \langle Ax_t - Ax_n, x_t - x_n \rangle + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\|, \end{aligned}$$

which implies

$$\begin{aligned} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\leq \frac{\bar{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{(1 - \bar{\gamma}t)^2}{2t} (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\| \\ &\leq \frac{\bar{\gamma}t}{2} M' + \frac{(1 - \bar{\gamma}t)^2}{2t} (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\|, \end{aligned}$$

where $M' > 0$ is an appropriate constant such that $M' \geq \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0, \|A\|^{-1})$ and for all $n \in \mathbb{N}$. It implies by (3.8) that

$$\limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0. \quad (3.10)$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0. \quad (3.11)$$

Since

$$\begin{aligned} \langle \gamma f(q) - Aq, x_n - q \rangle &= \langle \gamma f(q) - Aq, x_n - q \rangle - \langle \gamma f(q) - Aq, x_n - x_t \rangle + \langle \gamma f(q) \\ &\quad - Aq, x_n - x_t \rangle - \langle \gamma f(q) - Ax_t, x_n - x_t \rangle + \langle \gamma f(q) \\ &\quad - Ax_t, x_n - x_t \rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\ &= \langle \gamma f(q) - Aq, x_t - q \rangle + \langle Ax_t - Aq, x_n - x_t \rangle \\ &\quad + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\ &\leq \|\gamma f(q) - Aq\| \|x_t - q\| + \|A\| \|x_t - q\| \|x_n - x_t\| \\ &\quad + \gamma \alpha \|q - x_t\| \|x_n - x_t\| + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle, \end{aligned} \quad (3.12)$$

it follows from (3.10) and (3.12) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &= \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\ &\leq \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} (\|\gamma f(q) - Aq\| \|x_t - q\|) + \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} (\|A\| \|x_t - q\| \|x_n - x_t\|) \\ &\quad + \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} (\gamma \alpha \|q - x_t\| \|x_n - x_t\|) + \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\ &\leq 0. \end{aligned}$$

Finally, we will show that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \bigcap_{i=1}^N F(T_i)$. Since

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n \gamma(a_n u + (1 - a_n)f(x_n)) + (I - \alpha_n A)y_n - \alpha_n Aq - (I - \alpha_n A)q\|^2 \\ &= \|\alpha_n (\gamma(a_n u + (1 - a_n)f(x_n)) - Aq) + (I - \alpha_n A)(y_n - q)\|^2 \\ &\leq \|(I - \alpha_n A)(y_n - q)\|^2 + 2\alpha_n \langle \gamma(a_n u + (1 - a_n)f(x_n)) - Aq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|y_n - q\|^2 + 2\alpha_n \langle \gamma(a_n u + (1 - a_n)f(x_n)) - \gamma f(q) + \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= (1 - \alpha_n \bar{\gamma})^2 \|y_n - q\|^2 + 2\alpha_n \langle \gamma(a_n u + (1 - a_n)f(x_n)) - f(q) \rangle + \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma \langle a_n(u - f(q)) + (1 - a_n)(f(x_n) \\ &\quad - f(q)), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma \langle a_n(u - f(q)), x_{n+1} - q \rangle \\ &\quad + \langle (1 - a_n)(f(x_n) - f(q)), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma a_n \langle u - f(q), x_{n+1} - q \rangle \end{aligned}$$

$$\begin{aligned}
& + 2\alpha_n \gamma (1 - a_n) \langle f(x_n) - f(q), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
\leq & (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma a_n \|u - f(q)\| \|x_{n+1} - q\| \\
& + 2\alpha_n \gamma \alpha \|x_n - q\| \|x_{n+1} - q\| + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
\leq & (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma a_n \|u - f(q)\| \|x_{n+1} - q\| \\
& + \alpha_n \gamma \alpha (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle,
\end{aligned}$$

we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 & \leq \frac{(1 - 2\alpha_n \bar{\gamma} + \alpha_n^2 \bar{\gamma}^2)}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 + \frac{\alpha_n \gamma \alpha}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n \gamma a_n}{(1 - \alpha_n \gamma \alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n \gamma \alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \frac{(1 - \alpha_n \gamma \alpha + \alpha_n \gamma \alpha - 2\alpha_n \bar{\gamma} + \alpha_n^2 \bar{\gamma}^2)}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 + \frac{\alpha_n \gamma \alpha}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n \gamma a_n}{(1 - \alpha_n \gamma \alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n \gamma \alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \frac{((1 - \alpha_n \gamma \alpha) + \alpha_n \gamma \alpha - 2\alpha_n \bar{\gamma} + \alpha_n^2 \bar{\gamma}^2 + \alpha_n \gamma \alpha)}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n \gamma a_n}{(1 - \alpha_n \gamma \alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n \gamma \alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \frac{((1 - \alpha_n \gamma \alpha) - 2\alpha_n (\bar{\gamma} - \gamma \alpha) + \alpha_n^2 \bar{\gamma}^2)}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n \gamma a_n}{(1 - \alpha_n \gamma \alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n \gamma \alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \left(1 - \frac{2\alpha_n (\bar{\gamma} - \gamma \alpha)}{(1 - \alpha_n \gamma \alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n^2 \bar{\gamma}^2}{(1 - \alpha_n \gamma \alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n \gamma a_n}{(1 - \alpha_n \gamma \alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n \gamma \alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \left(1 - \frac{2\alpha_n (\bar{\gamma} - \gamma \alpha)}{(1 - \alpha_n \gamma \alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n}{(1 - \alpha_n \gamma \alpha)} (\alpha_n \bar{\gamma}^2 \|x_n - q\|^2 \\
& + 2\gamma a_n \|u - f(q)\| \|x_{n+1} - q\| + 2\langle \gamma f(q) - Aq, x_{n+1} - q \rangle) \\
& = \left(1 - \frac{2\alpha_n (\bar{\gamma} - \gamma \alpha)}{(1 - \alpha_n \gamma \alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n}{(1 - \alpha_n \gamma \alpha)} \frac{2(\bar{\gamma} - \gamma \alpha)}{2(\bar{\gamma} - \gamma \alpha)} (\alpha_n \bar{\gamma}^2 \|x_n - q\|^2 \\
& + 2\gamma a_n \|u - f(q)\| \|x_{n+1} - q\| + 2\langle \gamma f(q) - Aq, x_{n+1} - q \rangle) \\
& = \left(1 - \frac{2\alpha_n (\bar{\gamma} - \gamma \alpha)}{(1 - \alpha_n \gamma \alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n 2(\bar{\gamma} - \gamma \alpha)}{(1 - \alpha_n \gamma \alpha)} \left(\frac{\alpha_n \bar{\gamma}^2}{2(\bar{\gamma} - \gamma \alpha)} \|x_n - q\|^2\right. \\
& \left. + \frac{\gamma a_n}{(\bar{\gamma} - \gamma \alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{1}{(\bar{\gamma} - \gamma \alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle\right).
\end{aligned}$$

This implies by Lemma 2.7 that the sequence $\{x_n\}$ converges strongly to $q \in \bigcap_{i=1}^N$. By (3.2), we have

$$\|y_n - q\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - q\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof. $\square$

4. Application

Using our main result, we obtain the following strong convergence theorems in a Hilbert space.

Theorem 4.1. Let H be a Hilbert space, let f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strictly pseudo-contractions of H into itself for some $\kappa_i \in [0, 1)$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let W_n be the W -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{n,1}, \alpha_1^{n,2}, \dots, \alpha_1^{n,N}$, where $\alpha_1^{n,j} \in [0, 1]$, and $\kappa < a \leq \alpha_1^{n,j} \leq b < 1$ for all

$j = 1, 2, \dots, N-1, \kappa < c \leq \alpha_1^{n,N} \leq 1$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \quad (4.1)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 0$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, for all $j \in \{1, 2, 3, \dots, N\}$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in \bigcap_{i=1}^N F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

Proof. Put $\alpha_2^{n,j} = 0$ for all $j \in \{1, 2, 3, \dots, N\}$, and all $n \in \mathbb{N}$ in Theorem 3.1. Then, by Theorem 3.1 the conclusion follows. $\square$

Theorem 4.2. Let H be a Hilbert space, let f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let T be a κ -strict pseudo-contraction of H into itself for some $\kappa \in [0, 1)$ and $F(T) \neq \emptyset$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \quad (4.2)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 0$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in F(T)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

Proof. Put $N = 1$ and $T_1 = T$ and $\alpha_2^{n,1}, \alpha_3^{n,1} = 0 \forall n \in \mathbb{N}$ in Theorem 3.1. Then $S_n = T$ and we obtain the desired result from Theorem 3.1. $\square$

Corollary 4.3 ([15], Theorem 2.1). Let H be a Hilbert space, f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let T be a κ -strict pseudo-contraction of H into itself for some $\kappa \in [0, 1)$ and $F(T) \neq \emptyset$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \quad (4.3)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in F(T)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

Proof. Put $N = 1$ and $T_1 = T$ and $a_n, \alpha_2^{n,1}, \alpha_3^{n,1} = 0 \forall n \in \mathbb{N}$ in Theorem 3.1. Then $S_n = T$ and the desired result is directly obtained by Theorem 3.1. $\square$

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1.7 Composite iterative schemes for maximal monotone operators in reflexive Banach spaces

In this section, we introduce composite iterative schemes for finding a zero point of a finite family of maximal monotone operators in a reflexive Banach space. Then, we prove strong convergence theorems by using a shrinking projection method. Moreover, we also apply our results to a system of convex minimization problems in reflexive Banach spaces.

Introduction

Let E be a real Banach space and C a nonempty subset of E . Let E^* be the dual space of E . We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x \rangle$. Let $A : E \rightarrow 2^{E^*}$ be a set-valued mapping. We denote $\text{dom } A$ by *domain* of A , that is, $\text{dom } A = \{x \in E : Ax \neq \emptyset\}$ and also denote $G(A)$ by the *graph* of A , that is, $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A set-valued mapping A is said to be *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $(x, x^*) \in G(A)$; $(y, y^*) \in G(A)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operator on E . It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$ is closed and convex.

The problem of finding zero points for maximal monotone operators plays an important role in optimizations. This is because it can be reduced to a convex minimization problem and a variational inequality problem. Many authors have studied the convergence of such problems in several settings, (see [1-6]). Initiated by Martinet [7], in a Hilbert space, Rockafellar [8] introduced the following iterative schemes:

$$\begin{cases} x_1 = x \in E, \\ x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \geq 1, \end{cases} \quad (1.1)$$

where $\{\lambda_n\} \subset (0, \infty)$ and J_λ is the resolvent of A defined by $J_\lambda = (I + \lambda A)^{-1}$ for all $\lambda > 0$, and A is a maximal monotone operator on E . Such an algorithm is called the *proximal point algorithm*. He proved that the sequence $\{x_n\}$ generated by (1.1) converges weakly to an element in $A^{-1}(0)$ provided $\liminf_{n \rightarrow \infty} \lambda_n > 0$. Later, Kamimura and Takahashi [9] introduced the following iteration in a Hilbert space:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 1, \quad (1.2)$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$. The weak convergence theorems are also established in a Hilbert space under suitable conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2005, Kohsaka and Takahashi [10] studied the above iteration process in a more general setting, reflexive Banach spaces. In fact, those authors proposed the following algorithm:

$$x_{n+1} = \nabla f^* (\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(J_{\lambda_n} x_n)), \quad \forall n \geq 1, \quad (1.3)$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, \infty)$, $f: E \rightarrow \mathbb{R}$ is a Bregman function and $J_\lambda = (\nabla f + \lambda A)^{-1} \nabla f$ for all $\lambda > 0$. They also proved a weak convergence theorem of the proposed algorithm.

Very recently, in 2010, Reich and Sabach [11] proposed an algorithm for finding a zero point of maximal monotone operators $A_i: E \rightarrow 2^{E^*}$ ($i = 1, 2, \dots, N$) in a general reflexive Banach space E as follows:

$$\begin{cases} x_0 \in E, \\ \gamma_n^i = \text{Res}_{\lambda_n^i A_i}^f(x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, \gamma_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}^f(x_0), \quad \forall n \geq 0, \end{cases} \quad (1.4)$$

where $\{\lambda_n^i\}_{i=1}^N \subset (0, \infty)$, $\{e_n^i\}_{i=1}^N$ is an error sequence in E with $e_n^i \rightarrow 0$ and P_K^f the Bregman projection with respect to f from E onto a closed and convex subset K of E . Those authors showed that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common element in $\bigcap_{i=1}^N A_i^{-1}(0^*)$ under some mild conditions.

Motivated by the previous ones, we first introduce a composite iterative scheme which is different from (1.4) for finding a zero point of maximal monotone operators $A_i: E \rightarrow 2^{E^*}$ ($i = 1, 2, \dots, N$) in reflexive Banach spaces. Using the shrinking projection technique, introduced by Takahashi et al. [12], we then prove that a sequence generated by the proposed algorithm converges strongly to an element in $\bigcap_{i=1}^N A_i^{-1}(0^*)$ under some appropriate control conditions. Finally, we also apply our result to a system of convex minimization problems.

Preliminaries and lemmas

Let E be a real reflexive Banach space with a norm $\|\cdot\|$ and E^* be the dual space of E . Throughout this article, $f: E \rightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous, and convex function, and the Fenchel conjugate of f is the function $f^*: E^* \rightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup \{ \langle x^*, x \rangle - f(x) : x \in E \}.$$

We denote by $\text{dom } f$ the domain of f , that is, the set $\{x \in E : f(x) < +\infty\}$. For any $x \in \text{int dom } f$ and $y \in E$, the *right-hand derivative* of f at x in the direction y is defined by

$$f^o(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x if $\lim_{t \rightarrow 0^+} \frac{f(x+ty) - f(x)}{t}$ exists for any y . In this case, $f'(x, y)$ coincides with $\nabla f(x)$, the value of the *gradient* ∇f of f at x . The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \text{int dom } f$. The function f is said to be *Fréchet differentiable at x* if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let E be a reflexive Banach space. The Legendre function is defined from a general Banach space E into $(-\infty, +\infty]$ (see [13]). According to [13], the function f is *Legendre* if and only if it satisfies the following conditions:

(L1) The interior of the domain of f (denoted by $\text{int dom } f$) is nonempty, f^* is Gâteaux differentiable on $\text{int dom } f$, and $\text{dom } \nabla f = \text{int dom } f$;

(L2) The interior of the domain f^* (denoted by $\text{int dom } f^*$) is nonempty, f^* is Gâteaux differentiable on $\text{int dom } f^*$, and $\text{dom } \nabla f^* = \text{int dom } f^*$.

Since E is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [14]). This fact, when combined with the conditions (L1) and (L2), implies the following equalities [15]:

$$\begin{aligned}\nabla f &= (\nabla f^*)^{-1}, \\ \text{ran } \nabla f &= \text{dom } \nabla f^* = \text{int dom } f^*, \\ \text{ran } \nabla f^* &= \text{dom } \nabla f = \text{int dom } f.\end{aligned}$$

Also, the conditions (L1) and (L2), in conjunction with [13], imply that the functions f and f^* are strictly convex on the interior of their respective domains. Several interesting examples of the Legendre functions are presented in [13,16]. Especially, the functions $\frac{1}{s} \|\cdot\|^s$ with $s \in (1, \infty)$ are Legendre, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space. Throughout this article, we assume that the convex function $f: E \rightarrow (-\infty, +\infty]$ is Legendre.

Lemma 2.1. [17] *If $f: E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E , then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .*

Let $f: E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f: \text{dom } f \times \text{int dom } f \rightarrow [0, +\infty)$ is defined as follows:

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance* with respect to f [18].

Recall that the *Bregman projection* [19] of $x \in \text{int dom } f$ onto the nonempty, closed, and convex set $C \subset \text{dom } f$ is necessarily the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}.$$

Let $f: E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function f is said to be *totally convex* at $x \in \text{int dom } f$ if its modulus of total convexity at x , that is, the function $v_f: \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty]$ defined by

$$v_f(x, t) := \inf \{D_f(y, x) : y \in \text{dom } f, \|y - x\| = t\}$$

is positive, whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in \text{int dom } f$. In addition, the function f is said to be

totally convex on bounded sets if $v_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f: \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty]$ defined by

$$v_f(B, t) := \inf \{v_f(x, t) : x \in B \cap \text{dom } f\}.$$

Let C be a nonempty, closed, and convex subset of E . Let $f: E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [20] that $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0$ for all $y \in C$. We also have

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall x \in E, y \in C. \quad (2.1)$$

Recall that the function f is said to be *sequentially consistent* [20] if, for any two sequences, $\{x_n\}$ and $\{y_n\}$, in E such that the first is bounded:

$$\lim_{n \rightarrow \infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

The following lemmas were proved by Reich and Sabach [11].

Lemma 2.2. [11] *Let $f: E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}_{n=1}^\infty$ is also bounded.*

We know that the resolvent of A , denoted by $\text{Res}_A^f: E \rightarrow 2^E$, is defined as follows [21]:

$$\text{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(\text{Res}_A^f) = A^{-1}(0^*)$, and Res_A^f is single-valued (see [21]). If f is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded, subsets of E , then $\hat{F}(\text{Res}_A^f) = F(\text{Res}_A^f)$ (see [22]). The *Yosida approximation* $A_\lambda: E \rightarrow E$, $\lambda > 0$, is also defined by

$$A_\lambda(x) = \frac{1}{\lambda} \left(\nabla f(x) - \nabla f(\text{Res}_{\lambda A}^f(x)) \right)$$

for all $x \in E$. From Proposition 2.7 in [11], we know that $(\text{Res}_{\lambda A}^f(x), A_\lambda(x)) \in G(A)$ and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$.

Lemma 2.3. [11] *Let $A: E \rightarrow 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Then,*

$$D_f(p, \text{Res}_{\lambda A}^f(x)) + D_f(\text{Res}_{\lambda A}^f(x), x) \leq D_f(p, x)$$

for all $\lambda > 0$, $p \in A^{-1}(0^*)$ and $x \in E$.

Strong convergence theorems

Now, in this section, we prove our main results of this article.

Theorem 3.1. *Let E be a real reflexive Banach space and $f: E \rightarrow \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $A_i: E \rightarrow 2^{E^*}$ ($i = 1, 2, \dots, N$) be maximal monotone operators such that $F := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset E$ be such that $\lim_{n \rightarrow \infty} e_n = 0$. Define a sequence $\{x_n\}_{n=1}^\infty$ in E as follows:*

$$\begin{cases} x_1 \in E, \\ C_1 = E, \\ \gamma_n = \text{Res}_{\lambda_n^{N A_N}}^f \circ \text{Res}_{\lambda_n^{N-1 A_{N-1}}}^f \circ \cdots \circ \text{Res}_{\lambda_n^{1 A_1}}^f (x_n + e_n), \\ C_{n+1} = \{z \in C_n : D_f(z, \gamma_n) \leq D_f(z, x_n + e_n)\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \quad \forall n \geq 1. \end{cases} \quad (3.1)$$

If $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$ for each $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ converges strongly to a point $P_F^f(x_1)$

Proof. We divide our proof into six steps as follows:

Step 1. $F \subset C_n$ for all $n \geq 1$.

Since $A_i^{-1}(0^*)$ is closed and convex for each $i = 1, 2, \dots, N$, we get that $F := \bigcap_{i=1}^N A_i^{-1}(0^*)$ is a nonempty, closed and convex subset of E . It is easy to see that C_n is closed and convex for all $n \geq 1$. Indeed, for each $z \in C_n$, it follows that $D_f(z, \gamma_n) \leq D_f(z, x_n + e_n)$ is equivalent to

$$\langle \nabla f(x_n + e_n) - \nabla f(\gamma_n), z \rangle \leq f(\gamma_n) - f(x_n + e_n) + \langle \nabla f(x_n + e_n), x_n + e_n \rangle - \langle \nabla f(\gamma_n), \gamma_n \rangle.$$

This shows that C_n is closed and convex for all $n \geq 1$. It is obvious that $F \subset C_1 = E$. Now, suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. For any $p \in F$, by Lemma 2.3, we have

$$\begin{aligned} D_f(p, \gamma_k) &= D_f\left(p, \text{Res}_{\lambda_k^{N A_N}}^f \circ \text{Res}_{\lambda_k^{N-1 A_{N-1}}}^f \circ \cdots \circ \text{Res}_{\lambda_k^{1 A_1}}^f (x_k + e_k)\right) \\ &\leq D_f\left(p, \text{Res}_{\lambda_k^{N-1 A_{N-1}}}^f \circ \text{Res}_{\lambda_k^{N-2 A_{N-2}}}^f \circ \cdots \circ \text{Res}_{\lambda_k^{1 A_1}}^f (x_k + e_k)\right) \\ &\quad \dots \\ &\leq D_f\left(p, \text{Res}_{\lambda_k^{2 A_2}}^f \circ \text{Res}_{\lambda_k^{1 A_1}}^f (x_k + e_k)\right) \\ &\leq D_f\left(p, \text{Res}_{\lambda_k^{1 A_1}}^f (x_k + e_k)\right) \\ &\leq D_f(p, x_k + e_k). \end{aligned} \quad (3.2)$$

This implies that $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$.

Step 2. $\lim_{n \rightarrow \infty} D_f(x_n, x_0)$ exists.

From $x_n = P_{C_n}^f(x_1)$ and $x_{n+1} = P_{C_{n+1}}^f(x_1) \in C_{n+1} \subset C_n$ we have

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1), \quad \forall n \geq 1. \quad (3.3)$$

By (2.1), for any $p \in F \subset C_m$ we have

$$D_f(x_n, x_1) = D_f(P_{C_n}^f(x_1), x_1) \leq D_f(p, x_1) - D_f(p, x_n) \leq D_f(p, x_1). \quad (3.4)$$

Combining (3.3) and (3.4), we know that $\lim_{n \rightarrow \infty} D_f(x_n, x_1)$ exists.

Step 3. $\lim_{n \rightarrow \infty} \|\nabla f(\gamma_n) - \nabla f(x_n + e_n)\| = 0$

Since $x_m = P_{C_m}^f(x_1) \in C_m \subset C_n$ for $m > n \geq 1$, by (2.1), it follows that

$$\begin{aligned} D_f(x_m, x_n) &= D_f\left(x_m, P_{C_n}^f(x_1)\right) \leq D_f\left(x_m, x_1\right) - D_f\left(P_{C_n}^f(x_1), x_1\right) \\ &= D_f(x_m, x_1) - D_f(x_n, x_1). \end{aligned}$$

Letting $m, n \rightarrow \infty$, we have $D_f(x_m, x_n) \rightarrow 0$. Since f is totally convex on bounded subsets of E , f is sequentially consistent by Butnariu and Resmerita [20]. It follows that $\|x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. By the

completeness of the space E , we can assume that $x_n \rightarrow q \in E$ as $n \rightarrow \infty$. In particular, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Since $e_n \rightarrow 0$, we also obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - (x_n + e_n)\| = 0. \quad (3.5)$$

Since $x_{n+1} = P_{C_{n+1}}^f(x_1) \in C_{n+1}$

$$\begin{aligned} D_f(x_{n+1}, \gamma_n) &\leq D_f(x_{n+1}, x_n + e_n) \\ &= f(x_{n+1}) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), x_{n+1} - (x_n + e_n) \rangle. \end{aligned}$$

We know from [23] that, if f is bounded on bounded subsets of E , then ∇f is also bounded on bounded subsets of E . Moreover, if f is uniformly Fréchet differentiable on bounded subsets of E , then f is uniformly continuous on bounded subsets of E (see [24]). Using (3.5), we have

$$\lim_{n \rightarrow \infty} D_f(x_{n+1}, \gamma_n) = 0.$$

Also, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - \gamma_n\| = 0$$

and hence,

$$\lim_{n \rightarrow \infty} \|\gamma_n - x_n\| = 0$$

and, since $e_n \rightarrow 0$,

$$\lim_{n \rightarrow \infty} \|\gamma_n - (x_n + e_n)\| = 0. \quad (3.6)$$

Since f is uniformly Fréchet differentiable on bounded subsets of E , ∇f is norm-to-norm uniformly continuous on bounded subsets of E by Lemma 2.1. Hence, we have

$$\lim_{n \rightarrow \infty} \|\nabla f(\gamma_n) - \nabla f(x_n + e_n)\| = 0. \quad (3.7)$$

Step 4. $\lim_{n \rightarrow \infty} \|\nabla f(\Theta_n^i(x_n + e_n)) - \nabla f(\Theta_n^{i-1}(x_n + e_n))\| = 0 \quad \forall i = 1, 2, \dots, N.$

Denote $\Theta_n^i = \text{Res}_{\lambda_n^i A_i}^f \circ \text{Res}_{\lambda_n^{i-1} A_{i-1}}^f \circ \dots \circ \text{Res}_{\lambda_n^1 A_1}^f$ for each $i \in \{1, 2, \dots, N\}$ and $\Theta_n^0 = I$ for each $n \geq 1$. We note that $\gamma_n = \Theta_n^N(x_n + e_n)$ for each $n \geq 1$. For any $p \in F$, by (3.2), it follows that

$$\begin{aligned} D_f(p, \Theta_n^{N-1}(x_n + e_n)) &\leq D_f(p, \Theta_n^{N-2}(x_n + e_n)) \\ &\leq D_f(p, \Theta_n^{N-3}(x_n + e_n)) \\ &\dots \\ &\leq D_f(p, (x_n + e_n)). \end{aligned} \quad (3.8)$$

Since $p \in A_N^{-1}(0^*)$, by Lemma 2.3 and (3.8), it follows that

$$\begin{aligned} & D_f(\gamma_n, \Theta_n^{N-1}(x_n + e_n)) \\ & \leq D_f(p, \Theta_n^{N-1}(x_n + e_n)) - D_f(p, \gamma_n) \\ & \leq D_f(p, (x_n + e_n)) - D_f(p, \gamma_n) \\ & = f(\gamma_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), p - (x_n + e_n) \rangle + \langle \nabla f(\gamma_n), p - \gamma_n \rangle \\ & = f(\gamma_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), p - \gamma_n \rangle \\ & \quad + \langle \nabla f(x_n + e_n), (x_n + e_n) - \gamma_n \rangle + \langle \nabla f(\gamma_n), p - \gamma_n \rangle \\ & = f(\gamma_n) - f(x_n + e_n) + \langle \nabla f(\gamma_n) - \nabla f(x_n + e_n), p - \gamma_n \rangle \\ & \quad + \langle \nabla f(x_n + e_n), (x_n + e_n) - \gamma_n \rangle. \end{aligned}$$

From (3.6) and (3.7), we get that $\lim_{n \rightarrow \infty} D_f(\gamma_n, \Theta_n^{N-1}(x_n + e_n)) = 0$. Since f is sequentially consistent,

$$\lim_{n \rightarrow \infty} \|\gamma_n - \Theta_n^{N-1}(x_n + e_n)\| = 0. \quad (3.9)$$

Thus, from (3.6) and (3.9), it follows that

$$\lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^{N-1}(x_n + e_n)\| = 0 \quad (3.10)$$

and hence,

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-1}(x_n + e_n))\| = 0. \quad (3.11)$$

Again, since $p \in A_N^{-1}(0^*)$, by Lemma 2.3 and (3.8), we know that

$$\begin{aligned} & D_f(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)) \\ & \leq D_f(p, \Theta_n^{N-2}(x_n + e_n)) - D_f(p, \Theta_n^{N-1}(x_n + e_n)) \\ & \leq D_f(p, (x_n + e_n)) - D_f(p, \Theta_n^{N-1}(x_n + e_n)). \end{aligned}$$

From (3.10) and (3.11), we have

$$\lim_{n \rightarrow \infty} D_f(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)) = 0.$$

Since f is sequentially consistent, it follows that

$$\lim_{n \rightarrow \infty} \|\Theta_n^{N-1}(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n)\| = 0. \quad (3.12)$$

From (3.10) and (3.12), we have

$$\lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n)\| = 0,$$

and hence,

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-2}(x_n + e_n))\| = 0.$$

In a similar way, we can show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\Theta_n^{N-2}(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n)\| &= \dots = \lim_{n \rightarrow \infty} \|\Theta_n^1(x_n + e_n) - (x_n + e_n)\| = 0, \\ \lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n)\| &= \dots = \lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^1(x_n + e_n)\| = 0 \text{ and} \end{aligned}$$

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-3}(x_n + e_n))\| \\
 &= \lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-4}(x_n + e_n))\| \\
 &\quad \dots \\
 &= \lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^1(x_n + e_n))\| \\
 &= 0.
 \end{aligned}$$

Hence, we can conclude that

$$\lim_{n \rightarrow \infty} \|\nabla f(\Theta_n^i(x_n + e_n)) - \nabla f(\Theta_n^{i-1}(x_n + e_n))\| = 0 \quad (3.13)$$

for each $i = 1, 2, \dots, N$.

Step 5. $q \in \bigcap_{i=1}^N A_i^{-1}(0^*)$.

For each $i = 1, 2, \dots, N$, we note that $\Theta_n^i(x_n + e_n) = \text{Res}_{\lambda_n^i A_i}^f \Theta_n^{i-1}(x_n + e_n)$ and so

$$\|A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)\| = \frac{1}{\lambda_n^i} \|\nabla f(\Theta_n^{i-1}(x_n + e_n)) - \nabla f(\Theta_n^i(x_n + e_n))\|.$$

From (3.13) and $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$, we have

$$\lim_{n \rightarrow \infty} \|A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)\| = 0. \quad (3.14)$$

We note that $(\Theta_n^i(x_n + e_n), A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)) \in G(A_i)$ for each $i = 1, 2, \dots, N$. If $(w, w^*) \in G(A_i)$ for each $i = 1, 2, \dots, N$, then it follows from the monotonicity of A_i that

$$\langle w^* - A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n), w - \Theta_n^i(x_n + e_n) \rangle \geq 0.$$

Since $x_n \rightarrow q$ and $e_n \rightarrow 0$, $x_n + e_n \rightarrow q$. Therefore, $\Theta_n^i(x_n + e_n) \rightarrow q$ for each $i = 1, 2, \dots, N$. Thus, from (3.14), we have

$$\langle w^*, w - q \rangle \geq 0.$$

By the maximality of A_i , we have $q \in A_i^{-1}(0^*)$ for each $i = 1, 2, \dots, N$. Hence, $q \in F := \bigcap_{i=1}^N A_i^{-1}(0^*)$.

Step 6. $q = P_F^f(x_1)$.

From $x_n = P_{C_n}^f(x_1)$, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), x_n - z \rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\langle \nabla f(x_1) - \nabla f(x_n), x_n - z \rangle \geq 0, \quad \forall z \in F. \quad (3.15)$$

Letting $n \rightarrow \infty$ in (3.15), we obtain

$$\langle \nabla f(x_1) - \nabla f(q), q - z \rangle \geq 0, \quad \forall z \in F.$$

Hence, we have $q = P_F^f(x_1)$. This completes the proof.

As a direct consequence of Theorem 3.1, we also obtain the following result concerning a system of convex minimization problems in reflexive Banach spaces:

Theorem 3.2. Let E be a real reflexive Banach space and $f: E \rightarrow \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E . Let $g_i: E \rightarrow (-\infty, \infty]$ ($i = 1, 2, \dots, N$) be proper lower semi-continuous convex functions such that $F := \bigcap_{i=1}^N (\partial g_i^{-1})(0) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset E$ be a sequence in E such that $\lim_{n \rightarrow \infty} e_n = 0$. Define a sequence $\{x_n\}_{n=1}^\infty$ in E as follows:

$$\begin{cases} x_1 \in E, \\ C_1 = E, \\ z_n^1 = \arg \min_{y \in E} \left\{ g_1(y) + \frac{1}{\lambda_n^1} D_f(y, x_n + e_n) \right\}, \\ \dots \\ z_n^{N-1} = \arg \min_{y \in E} \left\{ g_{N-1}(y) + \frac{1}{\lambda_n^{N-1}} D_f(y, z_n^{N-2}) \right\}, \\ \gamma_n = \arg \min_{y \in E} \left\{ g_N(y) + \frac{1}{\lambda_n^N} D_f(y, z_n^{N-1}) \right\}, \\ C_{n+1} = \{z \in C_n : D_f(z, \gamma_n) \leq D_f(z, x_n + e_n)\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \quad \forall n \geq 1. \end{cases} \quad (3.16)$$

If $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$ for each $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ converges strongly to a point $P_F^f(x_1)$.

Proof. By Rockafellar's theorem [25,26], ∂g_i are maximal monotone operators for each $i = 1, 2, \dots, N$. Let $\lambda^i > 0$ for each $i = 1, 2, \dots, N$. Then $z^i = \text{Res}_{\lambda^i \partial g_i}^f(x)$ if and only if

$$\begin{aligned} 0 &\in \partial g_i(z^i) + \frac{1}{\lambda^i} (\nabla f(z^i) - \nabla f(x)) \\ &= \partial \left(g_i + \frac{1}{\lambda^i} (f - \nabla f(x)) \right) (z^i), \end{aligned}$$

which is equivalent to

$$\begin{aligned} z^i &= \arg \min_{y \in E} \left\{ g_i(y) + \frac{1}{\lambda^i} (f(y) - \langle y, \nabla f(x) \rangle) \right\} \\ &= \arg \min_{y \in E} \left\{ g_i(y) + \frac{1}{\lambda^i} D_f(y, x) \right\}. \end{aligned}$$

Using Theorem 3.1, we can complete the proof.

Remark 3.3. By means of the composite iterative scheme together with the shrinking projection method, we can construct the proximal point algorithms for finding a common element in the set $\bigcap_{i=1}^N A_i^{-1}(0^*)$. Moreover, our algorithm is different from that of Reich and Sabach [11] which is based on a finite intersection of sets.

Remark 3.4. Theorems 3.1 and 3.2 also hold in a uniformly convex and uniformly smooth Banach space with the generalized duality mapping.

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Authors' contributions

PC designed of the study, performed the nonlinear and convex analysis and also wrote the article. YJC participated in the design of the study, carried out the materials and helped to check the manuscript. SS conceived of the study, participated in its design and also helped to draft the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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1.8 Common fixed points of an infinite family of nonexpansive mappings in uniformly convex metric spaces

In this section, we generalize the notion of K -mapping due to Kangtunyakarn and Suantai (2009) [1] for an infinite family of nonexpansive mappings on a convex metric space and study some of its useful properties. We also establish strong convergence of a Mann type iterative sequence to a common fixed point of an infinite family of nonexpansive mappings on a uniformly convex metric space.

1. Introduction

Let C be a nonempty closed convex subset of a metric space (X, d) , and let $T : C \rightarrow C$ be a mapping. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. A mapping T is called *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$.

Kangtunyakarn and Suantai [1] introduced the K -mapping in a Banach space as follows. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_i\}_{i=1}^N$ be a real sequence in $[0, 1]$. Let $K : C \rightarrow C$ be a mapping defined by

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3)U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}, \end{aligned} \tag{1.1}$$

where I is the identity mapping of C . Such a mapping K is called the K -mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$. They proved that if X is a strictly convex Banach space, then $F(K) = \bigcap_{i=1}^N F(T_i)$ where $0 < \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$.

Takahashi [2] introduced a notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \rightarrow X$ is said to be a *convex structure* on X if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$$

for all $z \in X$. A metric space (X, d) together with a convex structure W is called a *convex metric space* which will be denoted by (X, d, W) . A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. Clearly, a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold.

Motivated by the above results, we consider the K -mapping in a convex metric space as follows: For $\{\lambda_n\}$ a sequence in $[0, 1]$ and $x \in X$, mappings $U_1, U_2, \dots, U_n$ and K_n are defined by

$$\begin{aligned} U_1x &= W(T_1x, x, \lambda_1), \\ U_2x &= W(T_2U_1x, U_1x, \lambda_2), \\ U_3x &= W(T_3U_2x, U_2x, \lambda_3), \\ &\vdots \\ U_{n-1}x &= W(T_{n-1}U_{n-2}x, U_{n-2}x, \lambda_{n-1}), \\ K_nx &= U_nx = W(T_nU_{n-1}x, U_{n-1}x, \lambda_n). \end{aligned}$$

Such a mapping K_n is called the K -mapping generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$.

In this paper, we consider the K -mapping generated by an infinite family of nonexpansive mappings in convex metric spaces. We give some useful properties of the common fixed point set of the family of nonexpansive mappings and then prove a Mann type convergence theorem in a uniformly convex metric space.

2. Preliminaries

A convex metric space (X, d, W) has the following properties, see more in [2,3]. For each $x, y \in X$ and $\lambda \in [0, 1]$,

- (i) $W(x, x, \lambda) = x$, $W(x, y, 0) = y$ and $W(x, y, 1) = x$;
- (ii) $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$ and $d(y, W(x, y, \lambda)) = \lambda d(x, y)$;
- (iii) $d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)$.

In 1996, Shimizu and Takahashi [4] introduced the concept of uniform convexity in convex metric spaces and studied some properties of these spaces. A convex metric space (X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, 1]$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$ imply that $d(z, W(x, y, \frac{1}{2})) \leq (1 - \delta)r$. Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

We say that a convex metric space (X, d, W) has the *property (H)* if

$$d(W(x, y, \lambda), W(x, z, \lambda)) \leq (1 - \lambda)d(y, z)$$

for all $x, y, z \in X$ and $\lambda \in [0, 1]$. It is obvious that the property (H) implies continuity of a convex structure W . By the corresponding results of Aoyama et al. [3], Shimizu [5] and Leuştean [6], we conclude that a uniformly convex metric space with the property (H) satisfies the following inequality:

$$d(W(x, y, \lambda), W(z, w, \lambda)) \leq \lambda d(x, z) + (1 - \lambda)d(y, w)$$

for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$.

The following lemma is important to prove our main results.

Lemma 2.1 ([5, Lemma 1]). *Let (X, d, W) be a uniformly convex metric space with continuous convex structure. Then for arbitrary positive number ε , there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$d(z, W(x, y, \lambda)) \leq (1 - 2 \min\{\lambda, 1 - \lambda\}\eta)r$$

for all $r > 0$ and $x, y, z \in X$, $d(z, x) \leq r$, $d(z, y) \leq r$, $d(x, y) \geq r\varepsilon$ and $\lambda \in [0, 1]$.

3. Main results

Lemma 3.1. *Let C be a nonempty closed convex subset of a complete convex metric space (X, d, W) . Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Then for every $x \in C$, $\lim_{n \rightarrow \infty} K_nx$ exists.*

Proof. Let $z \in \bigcap_{i=1}^{\infty} F(T_i)$ and $x \in C$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(K_nx, z) &= d(W(T_nU_{n-1}x, U_{n-1}x, \lambda_n), z) \\ &\leq \lambda_n d(T_nU_{n-1}x, z) + (1 - \lambda_n)d(U_{n-1}x, z) \\ &\leq d(U_{n-1}x, z) \\ &\vdots \\ &\leq d(U_1x, z) \\ &\leq \lambda_1 d(T_1x, z) + (1 - \lambda_1)d(x, z) \\ &\leq d(x, z), \end{aligned}$$

which implies that $\{K_n x\}$ is bounded, so is $\{T_{n+1} K_n x\}$. It follows by $K_n x = U_n x$ that

$$K_{n+1} x = W(T_{n+1} U_n x, U_n x, \lambda_{n+1}) = W(T_{n+1} K_n x, K_n x, \lambda_{n+1}).$$

Thus

$$d(K_{n+1} x, K_n x) = \lambda_{n+1} d(T_{n+1} K_n x, K_n x). \quad (3.1)$$

For $m > n$, it follows by (3.1) that

$$\begin{aligned} d(K_m x, K_n x) &\leq d(K_m x, K_{m-1} x) + \cdots + d(K_{n+1} x, K_n x) \\ &= \lambda_m d(T_m K_{m-1} x, K_{m-1} x) + \cdots + \lambda_{n+1} d(T_{n+1} K_n x, K_n x) \\ &\leq M \sum_{k=n+1}^m \lambda_k, \end{aligned}$$

where $M = \sup \{d(T_{n+1} K_n x, K_n x) : n \in \mathbb{N}\}$. This implies that $\{K_n x\}$ is a Cauchy sequence. Hence $\lim_{n \rightarrow \infty} K_n x$ exists. $\square$

From the above, we can define a mapping $K : C \rightarrow C$ as follows

$$Kx = \lim_{n \rightarrow \infty} K_n x, \quad x \in C.$$

Such a mapping K is called the K -mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$ and we show in the next theorem that $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.

Theorem 3.2. Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K be the K -mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$. Then, K is a nonexpansive mapping and $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. First, we show that K is a nonexpansive mapping. For $x, y \in C$, we have

$$\begin{aligned} d(K_n x, K_n y) &= d(W(T_n U_{n-1} x, U_{n-1} x, \lambda_n), W(T_n U_{n-1} y, U_{n-1} y, \lambda_n)) \\ &\leq \lambda_n d(T_n U_{n-1} x, T_n U_{n-1} y) + (1 - \lambda_n) d(U_{n-1} x, U_{n-1} y) \\ &\leq d(U_{n-1} x, U_{n-1} y) \\ &\vdots \\ &\leq d(U_1 x, U_1 y) \\ &\leq \lambda_1 d(T_1 x, T_1 y) + (1 - \lambda_1) d(x, y) \\ &\leq d(x, y). \end{aligned}$$

This implies that K_n is a nonexpansive mapping and we have

$$d(Kx, Ky) = \lim_{n \rightarrow \infty} d(K_n x, K_n y) \leq d(x, y).$$

Thus K is also a nonexpansive mapping. Next, we show that $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$. Let $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Then, it is obvious that $U_i p = p$ for all $i \in \mathbb{N}$, so $p \in F(K_n)$ for all $n \in \mathbb{N}$, which implies that $p \in F(K)$. Hence $\bigcap_{i=1}^{\infty} F(T_i) \subseteq F(K)$. We now show that $F(K) \subseteq \bigcap_{i=1}^{\infty} F(T_i)$. Let $k \in \mathbb{N}$ be fixed with $n > k$. For each $x \in F(K)$ and $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we observe that

$$\begin{aligned} d(p, K_n x) &= d(p, W(T_n U_{n-1} x, U_{n-1} x, \lambda_n)) \\ &\leq \lambda_n d(p, T_n U_{n-1} x) + (1 - \lambda_n) d(p, U_{n-1} x) \\ &\leq d(p, U_{n-1} x) \\ &\vdots \\ &\leq d(p, U_{k+1} x) \\ &= d(p, W(T_{k+1} U_k x, U_k x, \lambda_{k+1})) \\ &\leq \lambda_{k+1} d(p, T_{k+1} U_k x) + (1 - \lambda_{k+1}) d(p, U_k x) \\ &\leq d(p, U_k x) \\ &\vdots \\ &\leq d(p, U_1 x) \\ &= d(p, W(T_1 x, x, \lambda_1)) \\ &\leq \lambda_1 d(p, T_1 x) + (1 - \lambda_1) d(p, x) \\ &\leq d(p, x). \end{aligned} \quad (3.2)$$

This implies that $d(p, x) = \lim_{n \rightarrow \infty} d(p, K_n x) \leq d(p, W(T_1 x, x, \lambda_1)) \leq d(p, x)$. Thus, we have

$$d(p, W(T_1 x, x, \lambda_1)) = d(p, x) \quad \text{and} \quad d(p, T_1 x) = d(p, x).$$

Suppose that $x \neq T_1 x$. Then $d(x, T_1 x) > 0$. It follows from Lemma 2.1 that

$$d(p, W(T_1 x, x, \lambda_1)) < d(p, x) \quad \text{for } \lambda_1 \in (0, 1),$$

which is a contradiction. Hence $x = T_1 x$, that is $x \in F(T_1)$. Also we have $U_1 x = x$. By the inequality (3.2), we get $d(p, K_n x) \leq d(p, W(T_2 U_1 x, U_1 x, \lambda_2)) = d(p, W(T_2 x, x, \lambda_2))$. This implies that $d(p, x) = \lim_{n \rightarrow \infty} d(p, K_n x) \leq d(p, W(T_2 x, x, \lambda_2)) \leq d(p, x)$. Thus, we have

$$d(p, W(T_2 x, x, \lambda_2)) = d(p, x) \quad \text{and} \quad d(p, T_2 x) = d(p, x).$$

Suppose that $x \neq T_2 x$. Then $d(x, T_2 x) > 0$. Again, by Lemma 2.1, we have

$$d(p, W(T_2 x, x, \lambda_2)) < d(p, x) \quad \text{for } \lambda_2 \in (0, 1),$$

which is a contradiction. Hence $x = T_2 x$, that is $x \in F(T_2)$. Also we have $U_2 x = x$. By using the same argument, we conclude that $T_i x = x$ and $U_i x = x$ for $i = 1, 2, \dots, k$. By the inequality (3.2), we get $d(p, K_n x) \leq d(p, W(T_{k+1} U_k x, U_k x, \lambda_{k+1})) = d(p, W(T_{k+1} x, x, \lambda_{k+1}))$. It follows that

$$\begin{aligned} d(p, x) &= \lim_{n \rightarrow \infty} d(p, K_n x) \leq d(p, W(T_{k+1} x, x, \lambda_{k+1})) \\ &\leq \lambda_{k+1} d(p, T_{k+1} x) + (1 - \lambda_{k+1}) d(p, x) \\ &\leq d(p, x). \end{aligned}$$

Thus, we have

$$d(p, W(T_{k+1} x, x, \lambda_{k+1})) = d(p, x) \quad \text{and} \quad d(p, T_{k+1} x) = d(p, x).$$

Suppose that $x \neq T_{k+1} x$. Then $d(x, T_{k+1} x) > 0$. By Lemma 2.1, we have

$$d(p, W(T_{k+1} x, x, \lambda_{k+1})) < d(p, x) \quad \text{for } \lambda_{k+1} \in (0, 1).$$

This is a contradiction. Hence $x = T_{k+1} x$, that is $x \in F(T_{k+1})$. By induction, we have $x \in F(T_n)$ for all $n \in \mathbb{N}$. Hence $x \in \bigcap_{i=1}^{\infty} F(T_i)$. Therefore, $F(K) \subseteq \bigcap_{i=1}^{\infty} F(T_i)$. $\square$

Next, we prove a convergence theorem for the K -mapping in uniformly convex metric spaces.

Theorem 3.3. Let C be a nonempty compact convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K -mappings generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$. Assume that $x_1 \in C$ and the sequence $\{x_n\}$ generated by

$$x_{n+1} = W(x_n, K_n x_n, \alpha_n),$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges to a common fixed point of $\{T_i\}$.

Proof. Suppose that $\inf_n d(x_n, K_n x_n) = r_1 > 0$. Let $z \in \bigcap_{i=1}^{\infty} F(T_i)$. Since K_n is a nonexpansive mapping, therefore we have

$$\begin{aligned} d(z, x_{n+1}) &= d(z, W(x_n, K_n x_n, \alpha_n)) \\ &\leq \alpha_n d(z, x_n) + (1 - \alpha_n) d(z, K_n x_n) \\ &\leq d(z, x_n). \end{aligned}$$

Hence $\{d(z, x_n)\}$ is nonincreasing, and also $\lim_{n \rightarrow \infty} d(z, x_n)$ exists. Thus, we may assume that $\lim_{n \rightarrow \infty} d(z, x_n) = r_2 > 0$. Since $d(z, K_n x_n) \leq d(z, x_n)$ and

$$d(x_n, K_n x_n) \geq d(z, x_n) \frac{r_1}{d(z, x_n)} \geq d(z, x_n) \frac{r_1}{d(z, x_1)} > 0,$$

for all $n \geq 1$, we have by Lemma 2.1 that there exists $\eta \left(\frac{r_1}{d(z, x_1)} \right) > 0$ such that

$$\begin{aligned} d(z, x_{n+1}) &= d(z, W(x_n, K_n x_n, \alpha_n)) \\ &\leq d(z, x_n) (1 - 2 \min \{\alpha_n, 1 - \alpha_n\} \eta) \\ &\leq d(z, x_n) (1 - 2\alpha_n (1 - \alpha_n) \eta). \end{aligned}$$

Hence, we have

$$2\eta\alpha_n(1 - \alpha_n)r_2 \leq 2\eta\alpha_n(1 - \alpha_n)d(z, x_n) \leq d(z, x_n) - d(z, x_{n-1})$$

for all $n \geq 1$. Therefore, we get

$$2r_2\eta \sum_{n=1}^m \alpha_n(1 - \alpha_n) \leq d(z, x_1) - d(z, x_m)$$

for all $m \geq 1$. By letting $m \rightarrow \infty$ in the above inequality, we have $\infty \leq d(z, x_1) - r_2 < \infty$, which is a contradiction. Hence, $\inf_n d(x_n, K_n x_n) = 0$. Thus there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i \rightarrow \infty} d(x_{n_i}, K_{n_i} x_{n_i}) = 0$. Since C is compact, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ and $p \in C$ such that $\lim_{j \rightarrow \infty} x_{n_{ij}} = p$. Let K be the K -mappings generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$. Thus, we have

$$\begin{aligned} d(p, Kp) &\leq d(p, x_{n_{ij}}) + d(x_{n_{ij}}, K_{n_{ij}} x_{n_{ij}}) + d(K_{n_{ij}} x_{n_{ij}}, K_{n_{ij}} p) + d(K_{n_{ij}} p, Kp) \\ &\leq 2d(p, x_{n_{ij}}) + d(x_{n_{ij}}, K_{n_{ij}} x_{n_{ij}}) + d(K_{n_{ij}} p, Kp) \rightarrow 0. \end{aligned}$$

Hence $p \in F(K)$. By $\lim_{n \rightarrow \infty} d(p, x_n)$ exists, we have $\{x_n\}$ converges to $p \in F(K) = \bigcap_{i=1}^{\infty} F(T_i)$. $\square$

Remark 3.4.

- (i) Uniformly convex hyperbolic spaces and CAT(0) spaces are examples of uniformly convex metric spaces having the property (H); see more details in [6–8]. Therefore, Lemma 3.1, Theorems 3.2 and 3.3 are also true for these two classes of spaces.
- (ii) The reader interested in convergence of Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a complete uniformly convex metric space is referred to Khan et al. [9, Theorem 2.2].

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1.9 On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval

In this section, we propose a new iteration, called the SP-iteration, for approximating a fixed point of continuous functions on an arbitrary interval. Then, a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval is given. We also compare the convergence speed of Mann, Ishikawa, Noor and SP-iterations. It is proved that the SP-iteration is equivalent to and converges faster than the others. Our results extend and improve the corresponding results of Borwein and Borwein [D. Borwein, J. Borwein, Fixed point iterations for real functions, J. Math. Anal. Appl. 157 (1991) 112–126], Qing and Qihou [Y. Qing, L. Qihou, The necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval, J. Math. Anal. Appl. 323 (2006) 1383–1386], Rhoades [B.E. Rhoades, Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56 (1976) 741–750], and many others. Moreover, we also present numerical examples for the SP-iteration to compare with the Mann, Ishikawa and Noor iterations.

1. Introduction

Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. A point $p \in E$ is a *fixed point* of f if $f(p) = p$. We denote by $F(f)$ the set of fixed points of f . It is known that if E is also bounded, then $F(f)$ is nonempty. The *Mann iteration* (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \quad (1.1)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$, and will be denoted by $M(u_1, \alpha_n, f)$. The *Ishikawa iteration* (see [2]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1 - \beta_n)s_n + \beta_n f(s_n) \\ s_{n+1} = (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{cases} \quad (1.2)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will be denoted by $I(s_1, \alpha_n, \beta_n, f)$. The *Noor iteration* (see [3]) is defined by $w_1 \in E$ and

$$\begin{cases} r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n) \\ q_n = (1 - \beta_n)w_n + \beta_n f(r_n) \\ w_{n+1} = (1 - \alpha_n)w_n + \alpha_n f(q_n) \end{cases} \quad (1.3)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$, and will be denoted by $N(w_1, \alpha_n, \beta_n, \gamma_n, f)$. Clearly the Mann and Ishikawa iterations are special cases of the Noor iteration.

In 1976, Rhoades [4] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and nondecreasing functions on unit closed interval. Further, Borwein and Borwein [5] proved the convergence of the Mann iteration of continuous functions on a bounded closed interval. Recently, Qing and Qihou [6] extended their results to an arbitrary interval and to the Ishikawa iteration and gave some control conditions for the convergence of Ishikawa iteration on an arbitrary interval.

It was shown in [7] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [8] showed that the Mann iteration converges faster than the Ishikawa iteration for this class of operators. Two years later, Qing and Rhoades [9] provided an example to show that the claim in [8] is false.

Motivated by the above results, we propose a new iteration as follows:

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n) \\ y_n = (1 - \beta_n)z_n + \beta_n f(z_n) \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n f(y_n) \end{cases} \quad (1.4)$$

for all $n \geq 1$, where $x_1 \in E$, $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$, and it will be denoted by $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and is called the SP-iteration.

In this paper, we give a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval. We also prove that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges faster than the others for the class of continuous and nondecreasing functions. Moreover, we present numerical examples for the SP-iteration to compare with the Mann, Ishikawa and Noor iterations.

2. Convergence theorems

In this section, we prove the convergence theorems of the SP-iteration for continuous functions on an arbitrary interval.

Theorem 2.1. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $x_1 \in E$, let the SP-iteration $\{x_n\}_{n=1}^\infty$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$, (iii) $\sum_{n=1}^\infty \beta_n < \infty$ and (iv) $\sum_{n=1}^\infty \gamma_n < \infty$.

Then $\{x_n\}_{n=1}^\infty$ is bounded if and only if $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. It is obvious that if $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f , then it is bounded. Now, assume that $\{x_n\}_{n=1}^\infty$ is bounded. We shall show that $\{x_n\}_{n=1}^\infty$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n \rightarrow \infty} x_n$, $b = \limsup_{n \rightarrow \infty} x_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Because $f(x)$ is a continuous function, there exists $\delta, 0 < \delta < b - a$, such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \quad (2.1)$$

Since $\{x_n\}_{n=1}^\infty$ is bounded, $\{x_n\}_{n=1}^\infty$ belongs to a bounded closed interval. By continuity of f , we have that $\{f(x_n)\}_{n=1}^\infty$ belongs to another bounded closed interval, so $\{f(x_n)\}_{n=1}^\infty$ is bounded, and since $z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n)$, so $\{z_n\}_{n=1}^\infty$ is bounded, and thus $\{f(z_n)\}_{n=1}^\infty$ is bounded. Similarly, since $y_n = (1 - \beta_n)z_n + \beta_n f(z_n)$, we have $\{y_n\}_{n=1}^\infty$ and $\{f(y_n)\}_{n=1}^\infty$ are bounded. Using (1.4), we have $x_{n+1} - y_n = \alpha_n(f(y_n) - y_n)$, $y_n - z_n = \beta_n(f(z_n) - z_n)$ and $z_n - x_n = \gamma_n(f(x_n) - x_n)$. By (i), (iii) and (iv), we get $|x_{n+1} - y_n| \rightarrow 0$, $|y_n - z_n| \rightarrow 0$ and $|z_n - x_n| \rightarrow 0$. Since $|x_{n+1} - x_n| \leq |x_{n+1} - y_n| + |y_n - z_n| + |z_n - x_n|$, we have $|x_{n+1} - x_n| \rightarrow 0$. Thus there exists N such that

$$|x_{n+1} - x_n| < \frac{\delta}{3}, \quad |z_n - x_n| < \frac{\delta}{3}, \quad |y_n - z_n| < \frac{\delta}{3} \quad (2.2)$$

for all $n > N$. Since $b = \limsup_{n \rightarrow \infty} x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $x_k > m$. For x_k , there exist only two cases:

(A1) $x_k \geq m + \frac{\delta}{3}$, then by (2.2), we have $x_{k+1} - x_k > -\frac{\delta}{3}$, then $x_{k+1} > x_k - \frac{\delta}{3} \geq m$, so $x_{k+1} > m$.

(A2) $m < x_k < m + \frac{\delta}{3}$, then by (2.2), we have $m - \frac{\delta}{3} < z_k < m + \frac{2\delta}{3}$ and $m - \frac{2\delta}{3} < y_k < m + \delta$. So we have $|x_k - m| < \frac{\delta}{3} < \delta$, $|z_k - m| < \frac{2\delta}{3} < \delta$ and $|y_k - m| < \delta$. Using (2.1), we get

$$f(x_k) - x_k > 0, \quad f(z_k) - z_k > 0, \quad f(y_k) - y_k > 0. \quad (2.3)$$

By (1.4), we have

$$\begin{aligned} x_{k+1} &= y_k + \alpha_k(f(y_k) - y_k) \\ &= z_k + \beta_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\ &= x_k + \gamma_k(f(x_k) - x_k) + \beta_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k). \end{aligned} \quad (2.4)$$

By (2.3), we have $x_{k+1} \geq x_k$. Thus $x_{k+1} > m$.

By (A1) and (A2), we can conclude that $x_{k+1} > m$. By using the above argument, we obtain $x_{k+2} > m$, $x_{k+3} > m$, $x_{k+4} > m$, $\dots$. Thus we get $x_n > m$, for all $n > k = n_{k_1}$. So $a = \liminf_{n \rightarrow \infty} x_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For $\{x_n\}_{n=1}^\infty$, there exist only two cases:

(B1) There exists x_M such that $a < x_M < b$.

Then $f(x_M) = x_M$. Thus

$$z_M = (1 - \gamma_M)x_M + \gamma_M f(x_M) = x_M,$$

$$y_M = (1 - \beta_M)z_M + \beta_M f(z_M) = (1 - \beta_M)x_M + \beta_M f(x_M) = x_M,$$

$$x_{M+1} = (1 - \alpha_M)y_M + \alpha_M f(y_M) = (1 - \alpha_M)x_M + \alpha_M f(x_M) = x_M.$$

Analogously, we have $x_M = x_{M+1} = x_{M+2} = x_{M+3} = \dots$, so $x_n \rightarrow x_M$. It follows that $x_M = a$ and $x_n \rightarrow a$, which is a contradiction with the assumption.

(B2) For all n , $x_n \leq a$ or $x_n \geq b$.

Because $b - a > 0$ and $|x_{n+1} - x_n| \rightarrow 0$, so there exists N_0 such that $|x_{n+1} - x_n| < \frac{b-a}{3}$, for all $n > N_0$. It implies that either $x_n \leq a$ for all $n > N_0$ or $x_n \geq b$ for all $n > N_0$. If $x_n \leq a$ for $n > N_0$, then $b = \limsup_{n \rightarrow \infty} x_n \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > N_0$, so we have $a = \liminf_{n \rightarrow \infty} x_n \geq b$, which is a contradiction with $a < b$.

Hence, we have $\{x_n\}_{n=1}^\infty$ is convergent.

Next, we prove that $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f . Let $x_n \rightarrow p$ and suppose $f(p) \neq p$. By continuity of f , we have $\{f(x_n)\}_{n=1}^\infty$ is bounded. From $z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n)$ and $\gamma_n \rightarrow 0$, we obtain $z_n \rightarrow p$. Similarly, by $y_n = (1 - \beta_n)z_n + \beta_n f(z_n)$ and $\beta_n \rightarrow 0$, it follows that $y_n \rightarrow p$. Let $h_k = f(y_k) - y_k$, $q_k = f(z_k) - z_k$ and $s_k = f(x_k) - x_k$. By continuity of f , we have $\lim_{k \rightarrow \infty} h_k = \lim_{k \rightarrow \infty} (f(y_k) - y_k) = f(p) - p \neq 0$, $\lim_{k \rightarrow \infty} q_k = \lim_{k \rightarrow \infty} (f(z_k) - z_k) = f(p) - p \neq 0$ and $\lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} (f(x_k) - x_k) = f(p) - p \neq 0$. Put $w = f(p) - p$. Then $w \neq 0$. By (2.4), we have

$$\sum_{k=1}^{n-1} (x_{k+1} - x_k) = \sum_{k=1}^{n-1} (\alpha_k h_k + \beta_k q_k + \gamma_k s_k).$$

It follows that

$$x_n = x_1 + \sum_{k=1}^{n-1} (\alpha_k h_k + \beta_k q_k + \gamma_k s_k). \quad (2.5)$$

By (ii)–(iv) and $h_k \rightarrow w \neq 0$, we have that $\sum_{k=1}^\infty \alpha_k h_k$ is divergent, $\sum_{k=1}^\infty \beta_k q_k$ and $\sum_{k=1}^\infty \gamma_k s_k$ are convergent. This implies by (2.5) that $\{x_n\}_{n=1}^\infty$ is divergent, which is a contradiction with $x_n \rightarrow p$. Thus $f(p) = p$, that is $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f . $\square$

The following result give a convergence theorem for the Noor iteration, for continuous functions on a closed interval of a real line under some control conditions which are weaker than those in Theorem 2.1.

Theorem 2.2. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $w_1 \in E$, let the Noor iteration $\{w_n\}_{n=1}^\infty$ be defined by (1.3), where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=1}^\infty \alpha_n = \infty$, (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (iii) $\lim_{n \rightarrow \infty} \beta_n = 0$ and (iv) $\lim_{n \rightarrow \infty} \gamma_n = 0$.

Then $\{w_n\}_{n=1}^\infty$ is bounded if and only if $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. It is clear that if $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f , so it is bounded. Now, assume that $\{w_n\}_{n=1}^\infty$ is bounded. We shall show that $\{w_n\}_{n=1}^\infty$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n \rightarrow \infty} w_n$, $b = \limsup_{n \rightarrow \infty} w_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Because $f(x)$ is a continuous function, there exists δ , $0 < \delta < b - a$, such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \quad (2.6)$$

Since $\{w_n\}_{n=1}^\infty$ is bounded, $\{w_n\}_{n=1}^\infty$ belongs to a bounded closed interval. By continuity of f , we have that $\{f(w_n)\}_{n=1}^\infty$ belongs to another bounded closed interval, so $\{f(w_n)\}_{n=1}^\infty$ is bounded, and since $r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n)$, so $\{r_n\}_{n=1}^\infty$ is bounded, and thus $\{f(r_n)\}_{n=1}^\infty$ is bounded. Similarly, since $q_n = (1 - \beta_n)r_n + \beta_n f(r_n)$, we have $\{q_n\}_{n=1}^\infty$ and $\{f(q_n)\}_{n=1}^\infty$ are bounded. Using (1.3), we have $w_{n+1} - w_n = \alpha_n(f(q_n) - w_n)$, $q_n - w_n = \beta_n(f(r_n) - w_n)$ and $r_n - w_n = \gamma_n(f(w_n) - w_n)$. By (i), (iii) and (iv), we get $|w_{n+1} - w_n| \rightarrow 0$, $|q_n - w_n| \rightarrow 0$ and $|r_n - w_n| \rightarrow 0$. Thus there exists N such that

$$|w_{n+1} - w_n| < \frac{\delta}{3}, \quad |q_n - w_n| < \frac{\delta}{3}, \quad |r_n - w_n| < \frac{\delta}{3} \quad (2.7)$$

for all $n > N$. Since $b = \limsup_{n \rightarrow \infty} w_n > m$, there exists $k_1 > N$ such that $w_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $w_k > m$. For w_k , there exist only two cases:

(A1) $w_k \geq m + \frac{\delta}{3}$, then by (2.7), we have $w_{k+1} - w_k > -\frac{\delta}{3}$, then $w_{k+1} > w_k - \frac{\delta}{3} \geq m$, so $w_{k+1} > m$.

(A2) $m < w_k < m + \frac{\delta}{3}$, then by (2.7), we have $m - \frac{\delta}{3} < q_k < m + \frac{2\delta}{3}$ and $m - \frac{\delta}{3} < r_k < m + \frac{2\delta}{3}$. So we have $|w_k - m| < \frac{\delta}{3} < \delta$, $|q_k - m| < \frac{2\delta}{3} < \delta$ and $|r_k - m| < \frac{2\delta}{3} < \delta$. Using (2.6), we get

$$f(w_k) - w_k > 0, \quad f(q_k) - q_k > 0, \quad f(r_k) - r_k > 0. \quad (2.8)$$

By (1.3), we have

$$\begin{aligned} w_{k+1} &= w_k + \alpha_k(f(q_k) - w_k) \\ &= w_k + \alpha_k(f(q_k) - q_k) + \alpha_k\beta_k(f(r_k) - w_k) \\ &= w_k + \alpha_k(f(q_k) - q_k) + \alpha_k\beta_k(f(r_k) - r_k) + \alpha_k\beta_k\gamma_k(f(w_k) - w_k). \end{aligned}$$

By (2.8), we have $w_{k+1} \geq w_k$. Thus $w_{k+1} > m$.

By (A1) and (A2), we can conclude that $w_{k+1} > m$. By using the above argument, we obtain $w_{k+2} > m$, $w_{k+3} > m$, $w_{k+4} > m$, $\dots$. Thus we get $w_n > m$, for all $n > k = n_{k_1}$. So $a = \liminf_{n \rightarrow \infty} w_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For $\{w_n\}_{n=1}^\infty$, there exist only two cases:

(B1) There exists w_M such that $a < w_M < b$.

Then $f(w_M) = w_M$. Thus

$$\begin{aligned} r_M &= (1 - \gamma_M)w_M + \gamma_M f(w_M) = w_M, \\ q_M &= (1 - \beta_M)w_M + \beta_M f(r_M) = (1 - \beta_M)w_M + \beta_M f(w_M) = w_M, \\ w_{M+1} &= (1 - \alpha_M)w_M + \alpha_M f(q_M) = (1 - \alpha_M)w_M + \alpha_M f(w_M) = w_M. \end{aligned}$$

Analogously, we have $w_M = w_{M+1} = w_{M+2} = w_{M+3} = \dots$, so $w_n \rightarrow w_M$. It follows that $w_M = a$ and $w_n \rightarrow a$, which is a contradiction with the assumption.

(B2) For all n , $w_n \leq a$ or $w_n \geq b$.

Because $b - a > 0$ and $|w_{n+1} - w_n| \rightarrow 0$, so there exists N_0 such that $|w_{n+1} - w_n| < \frac{b-a}{3}$, for all $n > N_0$. It implies that either $w_n \leq a$ for all $n > N_0$ or $w_n \geq b$ for all $n > N_0$. If $w_n \leq a$ for $n > N_0$, then $b = \limsup_{n \rightarrow \infty} w_n \leq a$, which is a contradiction with $a < b$. If $w_n \geq b$ for $n > N_0$, so we have $a = \liminf_{n \rightarrow \infty} w_n \geq b$, which is a contradiction with $a < b$.

Hence, we have $\{w_n\}_{n=1}^\infty$ is convergent.

Next, we prove that $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f . Let $w_n \rightarrow p$ and suppose $f(p) \neq p$. By continuity of f , we have $\{f(w_n)\}_{n=1}^\infty$ is bounded. Since $r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n)$ and $\gamma_n \rightarrow 0$, it follows that $r_n \rightarrow p$. From $q_n = (1 - \beta_n)r_n + \beta_n f(r_n)$ and $\beta_n \rightarrow 0$, we obtain $q_n \rightarrow p$. Let $h_k = f(q_k) - w_k$. By continuity of f , we have $\lim_{k \rightarrow \infty} h_k = \lim_{k \rightarrow \infty} (f(q_k) - w_k) = f(p) - p \neq 0$. Let $d = f(p) - p$. Then $d \neq 0$. Using (1.3), we get $w_{k+1} = w_k + \alpha_k(f(q_k) - w_k)$. It follows that

$$w_n = w_1 + \sum_{k=1}^{n-1} \alpha_k h_k. \quad (2.9)$$

By $h_k \rightarrow d \neq 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$, it implies by (2.9) that $\{w_n\}_{n=1}^\infty$ is divergent, which is a contradiction with $w_n \rightarrow p$. Thus $f(p) = p$, that is $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f . $\square$

The following three corollaries are obtained directly by Theorem 2.1.

Corollary 2.3. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $x_1 \in E$, let $\{x_n\}_{n=1}^\infty$ be the sequence defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n f(x_n) \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n f(y_n), \end{cases} \quad (2.10)$$

where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$ and (iii) $\sum_{n=1}^\infty \beta_n < \infty$.

Then $\{x_n\}_{n=1}^\infty$ is bounded if and only if $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. By putting $\gamma_n = 0$ for all $n \geq 1$ in Theorem 2.1, then we obtain the required result directly from Theorem 2.1. $\square$

Corollary 2.4 ([6]). Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $u_1 \in E$, let the Mann iteration $\{u_n\}_{n=1}^\infty$ be defined by (1.1), where $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (ii) $\sum_{n=1}^\infty \alpha_n = \infty$.

Then $\{u_n\}_{n=1}^\infty$ is bounded if and only if $\{u_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. By putting $\gamma_n = 0$ and $\beta_n = 0$ for all $n \geq 1$ in Theorem 2.1, we obtain the desired result. $\square$

Corollary 2.5. Let $f : [a, b] \rightarrow [a, b]$ be a continuous function. For $x_1 \in [a, b]$, let the SP-iteration $\{x_n\}_{n=1}^\infty$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$, (iii) $\sum_{n=1}^\infty \beta_n < \infty$ and (iv) $\sum_{n=1}^\infty \gamma_n < \infty$.

Then $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f .

The following result is obtained directly from Theorem 2.2.

Corollary 2.6 ([6]). Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $s_1 \in E$, let the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ be defined by (1.2), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\sum_{n=1}^\infty \alpha_n = \infty$, (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (iii) $\lim_{n \rightarrow \infty} \beta_n = 0$.

Then $\{s_n\}_{n=1}^\infty$ is bounded if and only if $\{s_n\}_{n=1}^\infty$ converges to a fixed point of f .

3. Rate of Convergence

In this section, we study the rate of convergence of the SP-iteration for continuous and nondecreasing functions; we also compare the rate of convergence of the SP-iteration with the Mann, Ishikawa and Noor iterations. We show that the SP-iteration converges faster than the others. For analysis of the rate of convergence, we use the concept introduced by Rhoades [4] as follows.

Definition 3.1. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two iterations which converge to the fixed point p of f . Then $\{x_n\}_{n=1}^\infty$ is said to converge faster than $\{y_n\}_{n=1}^\infty$ if

$$|x_n - p| \leq |y_n - p| \quad \text{for all } n \geq 1.$$

The following lemmas are useful and crucial for our main results.

Lemma 3.2. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1]$. Let $\{u_n\}_{n=1}^\infty, \{s_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty, \{x_n\}_{n=1}^\infty$ be defined by (1.1)–(1.4), respectively. Then the following hold:

- (i) If $f(u_1) < u_1$, then $f(u_n) < u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^\infty$ is nonincreasing.
- (ii) If $f(u_1) > u_1$, then $f(u_n) > u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^\infty$ is nondecreasing.
- (iii) If $f(s_1) < s_1$, then $f(s_n) < s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^\infty$ is nonincreasing.
- (iv) If $f(s_1) > s_1$, then $f(s_n) > s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^\infty$ is nondecreasing.
- (v) If $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^\infty$ is nonincreasing.
- (vi) If $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^\infty$ is nondecreasing.
- (vii) If $f(x_1) < x_1$, then $f(x_n) < x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^\infty$ is nonincreasing.
- (viii) If $f(x_1) > x_1$, then $f(x_n) > x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^\infty$ is nondecreasing.

Proof. (i) Let $f(u_1) < u_1$. Then $f(u_1) < u_2 \leq u_1$. Since f is nondecreasing, we have $f(u_2) \leq f(u_1)$. Thus $f(u_2) < u_2$. Assume that $f(u_k) < u_k$. Then $f(u_k) < u_{k+1} \leq u_k$. Since $f(u_{k+1}) \leq f(u_k)$, we have $f(u_{k+1}) < u_{k+1}$. By mathematical induction, we obtain $f(u_n) < u_n$ for all $n \geq 1$. It follows that $u_{n+1} \leq u_n$ for all $n \geq 1$, that is $\{u_n\}_{n=1}^\infty$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(s_1) < s_1$. Then $f(s_1) < t_1 \leq s_1$. Since f is nondecreasing, we have $f(t_1) \leq f(s_1) < t_1 \leq s_1$. This implies $f(t_1) < s_2 \leq s_1$. Thus $f(s_2) \leq f(s_1) < t_1 \leq s_1$. If $f(t_1) < s_2 \leq t_1$, then $f(s_2) \leq f(t_1) < s_2$. Otherwise, if $t_1 < s_2 \leq s_1$, then $f(s_2) \leq f(s_1) < t_1 < s_2$. Hence, we have $f(s_2) < s_2$. By continuing in this way, we can show that $f(s_n) < s_n$ for all $n \geq 1$. This implies $t_n \leq s_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(t_n) \leq f(s_n) < s_n$ for all $n \geq 1$. It follows that $s_{n+1} \leq s_n$ for all $n \geq 1$, that is $\{s_n\}_{n=1}^\infty$ is nonincreasing.

(iv) By using the same argument as in (iii), we obtain the desired result.

(v) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \leq w_1$. Since f is nondecreasing, we have $f(r_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(r_1) < q_1 \leq w_1$. Thus $f(q_1) \leq f(w_1) < r_1 \leq w_1$. For q_1 , we consider the following two cases:

Case 1: $f(r_1) < q_1 \leq r_1$. Then $f(q_1) \leq f(r_1) < q_1 \leq r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$, if $q_1 < w_2 \leq r_1$, then $f(w_2) \leq f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < w_2$. Thus, we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \leq w_1$. Then $f(q_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 < q_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$ and if $q_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Cases 1 and 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \geq 1$. This implies $r_n \leq w_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(r_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Thus $q_n \leq w_n$ for all $n \geq 1$, then $f(q_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Hence, we have $w_{n+1} \leq w_n$ for all $n \geq 1$, that is $\{w_n\}_{n=1}^\infty$ is nonincreasing.

(vi) By using the same argument as in (v), we obtain the desired result.

(vii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1$. By (1.4), we have $f(z_1) < y_1 \leq z_1$. This implies $f(y_1) \leq f(z_1) < y_1$. Since $f(y_1) < x_2 \leq y_1$, we have $f(x_2) \leq f(y_1) < x_2$. Assume that $f(x_k) < x_k$. Then $f(x_k) < z_k \leq x_k$. Since f is nondecreasing, we have $f(z_k) \leq f(x_k) < z_k$. By (1.4), we have $f(z_k) < y_k \leq z_k$. This implies $f(y_k) \leq f(z_k) < y_k$. Since $f(y_k) < x_{k+1} \leq y_k$, we have $f(x_{k+1}) \leq f(y_k) < x_{k+1}$. By induction, we can conclude that $f(x_n) < x_n$ for all $n \geq 1$. Thus together with (1.4), we have $z_n \leq x_n$ for all $n \geq 1$. It follows that $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. This implies that $y_n \leq x_n$ for all $n \geq 1$. Hence, we have $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. It follows that $x_{n+1} = (1 - \alpha_n)y_n + \alpha_nf(y_n) \leq x_n$ for all $n \geq 1$. Thus $\{x_n\}_{n=1}^\infty$ is nonincreasing.

(viii) By using the same argument as in (vii), we obtain the desired result. $\square$

Lemma 3.3. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function. Let $\{x_n\}_{n=1}^\infty$ be the sequence defined by (1.1)–(1.3) or (1.4), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1)$. Then the following are satisfied:

- (i) If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.
- (ii) If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.

Proof. We shall prove only the case that $\{x_n\}_{n=1}^\infty$ is defined by (1.4) because other cases can be proved similarly.

- (i) Suppose that $p \in F(f)$ and $x_1 > p$. Since f is nondecreasing, we have $f(x_1) \geq f(p) = p$. By (1.4), we have $z_1 \geq p$. Thus $f(z_1) \geq p$. This implies by (1.4) that $y_1 \geq p$. Thus $f(y_1) \geq p$, it follows that $x_2 \geq p$. Assume that $x_k \geq p$. Thus $f(x_k) \geq f(p) = p$. By (1.4), we have $z_k \geq p$. Thus $f(z_k) \geq p$. This implies that $y_k \geq p$. Thus $f(y_k) \geq p$, it follows that $x_{k+1} \geq p$. By induction, we can conclude that $x_n \geq p$ for all $n \geq 1$.
- (ii) Suppose that $p \in F(f)$ and $x_1 < p$. By using the same argument as in (i), we can show that $x_n \leq p$ for all $n \geq 1$. $\square$

Lemma 3.4. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^\infty, \{s_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty, \{x_n\}_{n=1}^\infty$ be the sequences defined by (1.1)–(1.4), respectively. Then the following are satisfied:

- (i) If $f(u_1) < u_1$, then $x_n \leq w_n \leq s_n \leq u_n$ for all $n \geq 1$.
- (ii) If $f(u_1) > u_1$, then $x_n \geq w_n \geq s_n \geq u_n$ for all $n \geq 1$.

Proof. (i) Let $f(u_1) < u_1$. Since $u_1 = s_1 = w_1 = x_1$, we get $f(s_1) < s_1, f(w_1) < w_1$ and $f(x_1) < x_1$. First, we show that $x_n \leq w_n$ for all $n \geq 1$. By (1.4), we have $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1$. This implies $f(z_1) < y_1 \leq z_1 \leq x_1$. Using (1.3) and (1.4), we have $z_1 - r_1 = (1 - \gamma_1)(x_1 - w_1) + \gamma_1(f(x_1) - f(w_1)) = 0$, that is $z_1 = r_1$, and we get $y_1 - q_1 = (1 - \beta_1)(z_1 - w_1) + \beta_1(f(z_1) - f(r_1)) \leq 0$. Since f is nondecreasing, we have $f(y_1) \leq f(q_1)$. This implies

$$\begin{aligned} x_2 - w_2 &= (1 - \alpha_1)(y_1 - w_1) + \alpha_1(f(y_1) - f(q_1)) \\ &= (1 - \alpha_1)(y_1 - x_1) + \alpha_1(f(y_1) - f(q_1)) \\ &\leq 0, \end{aligned}$$

that is $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. By Lemma 3.2(v) and (vii), we have $f(w_k) < w_k$ and $f(x_k) < x_k$. This implies $f(x_k) < z_k \leq x_k \leq w_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus $y_k - w_k = (z_k - w_k) + \beta_k(f(z_k) - z_k) \leq 0$ and $z_k - r_k = (1 - \gamma_k)(x_k - w_k) + \gamma_k(f(x_k) - f(w_k)) \leq 0$. That is $y_k \leq w_k$ and $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have $y_k - q_k = (1 - \beta_k)(z_k - w_k) + \beta_k(f(z_k) - f(r_k)) \leq 0$, so $y_k \leq q_k$, which implies $f(y_k) \leq f(q_k)$. It follows that

$$\begin{aligned} x_{k+1} - w_{k+1} &= (1 - \alpha_k)(y_k - w_k) + \alpha_k(f(y_k) - f(q_k)) \\ &\leq 0, \end{aligned}$$

that is $x_{k+1} \leq w_{k+1}$. By mathematical induction, we obtain $x_n \leq w_n$ for all $n \geq 1$.

Next, we show that $w_n \leq s_n$ for all $n \geq 1$. Using (1.2) and (1.3), we have $r_1 - s_1 = (w_1 - s_1) + \gamma_1(f(w_1) - w_1) \leq 0$, that is $r_1 \leq s_1$. Since $f(r_1) \leq f(s_1)$, we obtain $q_1 - t_1 = (1 - \beta_1)(w_1 - s_1) + \beta_1(f(r_1) - f(s_1)) \leq 0$, so $q_1 \leq t_1$, which implies $f(q_1) \leq f(t_1)$. It follows that

$$\begin{aligned} w_2 - s_2 &= (1 - \alpha_1)(w_1 - s_1) + \alpha_1(f(q_1) - f(t_1)) \\ &\leq 0, \end{aligned}$$

that is $w_2 \leq s_2$. Assume that $w_k \leq s_k$. Thus $f(w_k) \leq f(s_k)$. By Lemma 3.2(iii) and (v), we have $f(s_k) < s_k$ and $f(w_k) < w_k$. This implies $r_k - s_k = (w_k - s_k) + \gamma_k(f(w_k) - w_k) \leq 0$, thus $r_k \leq s_k$. Since $f(r_k) \leq f(s_k)$, we have $q_k - t_k = (1 - \beta_k)(w_k - s_k) + \beta_k(f(r_k) - f(s_k)) \leq 0$, so $q_k \leq t_k$, which implies $f(q_k) \leq f(t_k)$. It follows that

$$\begin{aligned} w_{k+1} - s_{k+1} &= (1 - \alpha_k)(w_k - s_k) + \alpha_k(f(q_k) - f(t_k)) \\ &\leq 0, \end{aligned}$$

that is $w_{k+1} \leq s_{k+1}$. By mathematical induction, we obtain $w_n \leq s_n$ for all $n \geq 1$.

Finally, we show that $s_n \leq u_n$ for all $n \geq 1$. Using (1.1) and (1.2), we have $t_1 - u_1 = (s_1 - u_1) + \beta_1(f(s_1) - s_1) \leq 0$, thus $t_1 \leq u_1$. Since f is nondecreasing, we have $f(t_1) \leq f(u_1)$. This implies

$$\begin{aligned} s_2 - u_2 &= (1 - \alpha_1)(s_1 - u_1) + \alpha_1(f(t_1) - f(u_1)) \\ &\leq 0, \end{aligned}$$

that is $s_2 \leq u_2$. Assume that $s_k \leq u_k$. Thus $f(s_k) \leq f(u_k)$. By Lemma 3.2(i) and (iii), we have $f(u_k) < u_k$ and $f(s_k) < s_k$. This implies $t_k - u_k = (s_k - u_k) + \beta_k(f(s_k) - s_k) \leq 0$, so $t_k \leq u_k$, which implies $f(t_k) \leq f(u_k)$. It follows that

$$\begin{aligned} s_{k+1} - u_{k+1} &= (1 - \alpha_k)(s_k - u_k) + \alpha_k(f(t_k) - f(u_k)) \\ &\leq 0, \end{aligned}$$

that is $s_{k+1} \leq u_{k+1}$. By mathematical induction, we obtain $s_n \leq u_n$ for all $n \geq 1$.

(ii) By using Lemma 3.2(ii, iv, vi, viii) and the same argument as in (i), we can show that $x_n \geq w_n \geq s_n \geq u_n$ for all $n \geq 1$. $\square$

Proposition 3.5. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. If $f(x_1) > x_1$, then the sequence $\{x_n\}_{n=1}^\infty$ defined by one of the following iteration methods: $M(x_1, \alpha_n, f)$, $I(x_1, \alpha_n, \beta_n, f)$, $N(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ does not converge to a fixed point of f .

Proof. By Lemma 3.2(ii, iv, vi, viii), we have that $\{x_n\}_{n=1}^\infty$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^\infty$ does not converge to a fixed point of f .

Proposition 3.6. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. If $f(x_1) < x_1$, then the sequence $\{x_n\}_{n=1}^\infty$ defined by one of the following iteration method: $M(x_1, \alpha_n, f)$, $I(x_1, \alpha_n, \beta_n, f)$, $N(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ does not converge to a fixed point of f .

Proof. By Lemma 3.2(i, iii, v, vii), we have that $\{x_n\}_{n=1}^\infty$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^\infty$ does not converge to a fixed point of f . $\square$

Theorem 3.7. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^\infty, \{s_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty, \{x_n\}_{n=1}^\infty$ be the sequences defined by (1.1)–(1.4), respectively. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. Then the following are satisfied:

- (i) The Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to $p \in F(f)$ if and only if the Mann iteration $\{u_n\}_{n=1}^\infty$ converges to p . Moreover, the Ishikawa iteration converges faster than the Mann iteration.
- (ii) The Noor iteration $\{w_n\}_{n=1}^\infty$ converges to $p \in F(f)$ if and only if the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to p . Moreover, the Noor iteration converges faster than the Ishikawa iteration.
- (iii) The SP-iteration $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$ if and only if the Noor iteration $\{w_n\}_{n=1}^\infty$ converges to p . Moreover, the SP-iteration converges faster than the Noor iteration.

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$.

(i) $(\Rightarrow)$ If the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then set $\beta_n = 0$ for all $n \geq 1$ in (1.2), we can get the convergence of the Mann iteration.

$(\Leftarrow)$ Suppose that the Mann iteration $\{u_n\}_{n=1}^\infty$ converges to $p \in F(f)$. We divide our proof into the following three cases: Case 1: $u_1 = s_1 > U$, Case 2: $u_1 = s_1 < L$, Case 3: $L \leq u_1 = s_1 \leq U$.

Case 1: $u_1 = s_1 > U$. By Proposition 3.5, we get $f(u_1) < u_1$ and $f(s_1) < s_1$. This implies by Lemma 3.4(i) that $s_n \leq u_n$ for all $n \geq 1$. We note that $U < s_1$ and by using (1.2) and mathematical induction, we can show that $U \leq s_n$ for all $n \geq 1$. Then, we have $0 \leq s_n - p \leq u_n - p$, so

$$|s_n - p| \leq |u_n - p| \quad \text{for all } n \geq 1. \quad (3.1)$$

It follows that $s_n \rightarrow p$. That is the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to the same fixed point p . Moreover, by (3.1), we see that the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$.

Case 2: $u_1 = s_1 < L$. By Proposition 3.6, we get $f(u_1) > u_1$ and $f(s_1) > s_1$. This implies by Lemma 3.4(ii) that $s_n \geq u_n$ for all $n \geq 1$. We note that $s_1 < L$ and by using (1.2) and mathematical induction, we can show that $s_n \leq L$ for all $n \geq 1$. Then, we have $|s_n - p| \leq |u_n - p|$ for all $n \geq 1$. It follows that $s_n \rightarrow p$ and the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$.

Case 3: $L \leq u_1 = s_1 \leq U$. Suppose that $f(u_1) \neq u_1$. If $f(u_1) < u_1$, we have by Lemma 3.2(i) that $\{u_n\}_{n=1}^\infty$ is nonincreasing with limit p . By Lemmas 3.3(i) and 3.4(i), we have $p \leq s_n \leq u_n$ for all $n \geq 1$. It follows that $|s_n - p| \leq |u_n - p|$ for all $n \geq 1$. Hence, we have that $s_n \rightarrow p$ and the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$. If $f(u_1) > u_1$, we have by Lemma 3.2(ii) that $\{u_n\}_{n=1}^\infty$ is nondecreasing with limit p . By Lemmas 3.3(ii) and 3.4(ii), we have $p \geq s_n \geq u_n$ for all $n \geq 1$. It follows that $|s_n - p| \leq |u_n - p|$ for all $n \geq 1$. Hence, we have that $s_n \rightarrow p$ and the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$.

(ii) $(\Rightarrow)$ If the Noor iteration $\{w_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then set $\gamma_n = 0$ for all $n \geq 1$ in (1.3), we can get the convergence of the Ishikawa iteration.

$(\Leftarrow)$ Suppose that the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to $p \in F(f)$. By using Lemmas 3.2(iii, iv), 3.3 and 3.4, Propositions 3.5, 3.6 and the same proof as in (i), we obtain the desired result.

(iii) $(\Rightarrow)$ If the SP-iteration $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then set $\beta_n = 0$ and $\gamma_n = 0$ for all $n \geq 1$ in (1.4), we can get the convergence of the Mann iteration. Then, the result is obtained directly by (i) and (ii).

$(\Leftarrow)$ Suppose that the Noor iteration $\{w_n\}_{n=1}^\infty$ converges to $p \in F(f)$. By using Lemma 3.2(v, vi), Lemmas 3.3 and 3.4, Propositions 3.5 and 3.6 and the same proof as in (i), we obtain the desired result. $\square$

The speed of convergence for the Mann, Ishikawa, Noor and SP-iterations also depends on the choice of $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ in the interval $[0, 1)$. We present such result only for the SP-iteration. The others are very similar.

Theorem 3.8. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$, $\{\alpha_n^*\}_{n=1}^\infty$, $\{\beta_n^*\}_{n=1}^\infty$, $\{\gamma_n^*\}_{n=1}^\infty$ be sequences in $[0, 1)$ such that $\alpha_n \leq \alpha_n^*$, $\beta_n \leq \beta_n^*$ and $\gamma_n \leq \gamma_n^*$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ be defined by $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$, respectively. If $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then $\{x_n^*\}_{n=1}^\infty$ converges to p . Moreover, $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$, provided that $x_1^* = x_1 \in E$.

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$. Suppose that $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$. We divide our proof into the following three cases:

Case 1: $x_1 = x_1^* > U$. By Proposition 3.5, we have $f(x_1) < x_1$. This implies by Lemma 3.2(vii) that $f(x_n) < x_n$ for all $n \geq 1$. It follows by (1.4), we can show that $f(z_n) < z_n$ and $f(y_n) < y_n$ for all $n \geq 1$. Using (1.4), we have

$$\begin{aligned} z_1^* - z_1 &= (x_1^* - x_1) + \gamma_1^*(f(x_1^*) - x_1^*) + \gamma_1(x_1 - f(x_1)) \\ &= (\gamma_1^* - \gamma_1)(f(x_1) - x_1) \\ &\leq 0, \end{aligned}$$

that is $z_1^* \leq z_1$. Since f is nondecreasing, we have $f(z_1^*) \leq f(z_1)$. By $z_1 - f(z_1) > 0$, it follows that

$$\begin{aligned} y_1^* - y_1 &= (z_1^* - z_1) + \beta_1^*(f(z_1^*) - z_1^*) + \beta_1(z_1 - f(z_1)) \\ &\leq (z_1^* - z_1) + \beta_1^*(f(z_1^*) - z_1^*) + \beta_1(z_1 - f(z_1)) \\ &= (1 - \beta_1^*)(z_1^* - z_1) + \beta_1^*(f(z_1^*) - f(z_1)) \\ &\leq 0, \end{aligned}$$

that is $y_1^* \leq y_1$. Since $f(y_1^*) \leq f(y_1)$ and $y_1 - f(y_1) > 0$, we have

$$\begin{aligned} x_2^* - x_2 &= (y_1^* - y_1) + \alpha_1^*(f(y_1^*) - y_1^*) + \alpha_1(y_1 - f(y_1)) \\ &\leq (y_1^* - y_1) + \alpha_1^*(f(y_1^*) - y_1^*) + \alpha_1(y_1 - f(y_1)) \\ &= (1 - \alpha_1^*)(y_1^* - y_1) + \alpha_1^*(f(y_1^*) - f(y_1)) \\ &\leq 0, \end{aligned}$$

that is $x_2^* \leq x_2$. Assume that $x_k^* \leq x_k$. Since $f(x_k^*) \leq f(x_k) < x_k$, we have $z_k^* - z_k \leq (1 - \gamma_k^*)(x_k^* - x_k) + \gamma_k^*(f(x_k^*) - f(x_k)) \leq 0$, that is $z_k^* \leq z_k$. Since $f(z_k^*) \leq f(z_k) < z_k$, we have $y_k^* - y_k \leq (1 - \beta_k^*)(z_k^* - z_k) + \beta_k^*(f(z_k^*) - f(z_k)) \leq 0$, so $y_k^* \leq y_k$, then $f(y_k^*) \leq f(y_k) < y_k$. It follows that

$$\begin{aligned} x_{k+1}^* - x_{k+1} &\leq (1 - \alpha_k^*)(y_k^* - y_k) + \alpha_k^*(f(y_k^*) - f(y_k)) \\ &\leq 0, \end{aligned}$$

that is $x_{k+1}^* \leq x_{k+1}$. By mathematical induction, we obtain $x_n^* \leq x_n$ for all $n \geq 1$. We note that $U < x_1^*$ and by using (1.4) and mathematical induction, we can show that $U \leq x_n^*$ for all $n \geq 1$. Hence, we have $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. It follows that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$.

Case 2: $x_1 = x_1^* < L$. By Proposition 3.6, we get $f(x_1) > x_1$. In the same way as Case 1, we can show that $x_n^* \geq x_n$ for all $n \geq 1$. We note that $x_1^* < L$ and by using (1.4) and mathematical induction, we can show that $x_n^* \leq L$ for all $n \geq 1$. This implies $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. It follows that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$.

Case 3: $L \leq x_1 = x_1^* \leq U$. Suppose that $f(x_1) \neq x_1$. If $f(x_1) < x_1$, we have by Lemma 3.2(vii) that $\{x_n\}_{n=1}^\infty$ is nonincreasing with limit p . By Lemma 3.3(i), we have $p \leq x_n^*$ for all $n \geq 1$. By using the same argument as in Case 1, we can show that $x_n^* \leq x_n$ for all $n \geq 1$, so $p \leq x_n^* \leq x_n$ for all $n \geq 1$. It follows that $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. Hence, we have that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$. If $f(x_1) > x_1$, we have by Lemma 3.2(viii) that $\{x_n\}_{n=1}^\infty$ nondecreasing with limit p . By Lemma 3.3(ii), we have $p \geq x_n^*$ for all $n \geq 1$. By using the same argument as in Case 1, we can show that $x_n^* \geq x_n$ for all $n \geq 1$, so $p \geq x_n^* \geq x_n$ for all $n \geq 1$. It follows that $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. Hence, we obtain that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$.

Table 1

Comparison of rate of convergence of the Mann, Ishikawa, Noor and SP-iterations for the given function in Example 3.10.

n	Mann	Ishikawa	Noor	SP-iteration		
	u_n	s_n	w_n	x_n	$ f(x_n) - x_n $	$ \frac{x_n - p}{x_{n-1} - p} $
10	1.13516510	1.10003535	1.09340527	1.02246989	1.7925E–02	6.7448E–01
⋮	⋮	⋮	⋮	⋮	⋮	⋮
33	1.00006355	1.00004644	1.00004331	1.00000919	7.3542E–06	7.3219E–01
34	1.00004667	1.00003410	1.00003181	1.00000674	5.3932E–06	7.3335E–01
35	1.00003433	1.00002508	1.00002339	1.00000495	3.9611E–06	7.3447E–01

Table 2

Comparison of rate of convergence of the SP-iteration for the given function in Example 3.11.

n	$\alpha_n = \frac{1}{n^{0.2}+1}$	$\alpha_n^* = 0.7$				
	x_n	$ f(x_n) - x_n $	$ \frac{x_n - x_{n-1}}{x_n} $	x_n^*	$ f(x_n^*) - x_n^* $	$ \frac{x_n^* - x_{n-1}^*}{x_n^*} $
10	3.01562105	1.3019E–02	2.6712E–03	3.00031983	2.6652E–04	1.5452E–04
⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	3.00073037	6.0865E–04	1.0716E–04	3.00000027	2.2357E–07	1.2644E–07
19	3.00050902	4.2418E–04	7.3772E–05	3.00000011	9.2676E–08	5.2357E–08
20	3.00035599	2.9666E–04	5.1004E–05	3.00000005	3.8437E–08	2.1695E–08

Remark 3.9. From Theorem 3.8, the control sequences $\{\alpha_n^*\}_{n=1}^\infty$, $\{\beta_n^*\}_{n=1}^\infty$ and $\{\gamma_n^*\}_{n=1}^\infty$ in $[0, 1)$ can be chosen such that $SP(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$ converges to a fixed point of f but those sequences may not satisfy the control condition in Theorem 2.1.

Next, we will present two numerical examples. The first example shows that our iteration converges faster than the Mann, Ishikawa and Noor iterations for continuous and nondecreasing functions on an arbitrary interval and the second example compares the speed of convergence of the SP-iteration with respect to various control conditions. Throughout this section, we use $\alpha_n = \frac{1}{n^{0.2}+1}$, $\beta_n = \frac{1}{n^2+1}$ and $\gamma_n = \frac{1}{n^2+1}$.

Example 3.10. Let $f : [0, 8] \rightarrow [0, 8]$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is a continuous and nondecreasing function. The comparison of the convergences of the Mann, Ishikawa, Noor and SP-iterations to the exact fixed point $p = 1$ are given in Table 1, with the initial point $u_1 = s_1 = w_1 = x_1 = 4$.

From Table 1, we see that the SP-iteration converges faster than the Mann, Ishikawa and Noor iterations. The value $|\frac{x_n - p}{x_{n-1} - p}|$ is called the *order of convergence*. If $|\frac{x_n - p}{x_{n-1} - p}| \rightarrow K$ with $0 < K < 1$, the sequence $\{x_n\}_{n=1}^\infty$ is said to be *linear convergent*. From Table 1, the sequence $\{x_n\}_{n=1}^\infty$ is seen to be linear convergent.

Example 3.11. Let $f : [-6, \infty) \rightarrow [-6, \infty)$ be defined by $f(x) = \sqrt{x+6}$. Then f is a continuous and nondecreasing function. The comparison of the convergence for SP-iterations with new control conditions to the exact fixed point $p = 3$ is given in Table 2, with initial point $x_1 = x_1^* = 9$.

From Table 2, we see that $SP(x_1^*, \alpha_n^*, \beta_n, \gamma_n, f)$ converges faster than $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$. It clear that $\{\alpha_n^*\}_{n=1}^\infty$ does not satisfy the control condition in Theorem 2.1.

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Chapter II

Equilibrium Problems, General System of Variational Inequality Problems and Fixed Point Problems

In this chapter, we introduce several iterative methods for finding a common element of the set of solutions of a general system of variational inequalities, the set of solutions of a mixed equilibrium problem and the set of fixed points of some nonlinear mappings in both Hilbert and Banach spaces. We divide this chapter into 5 sections corresponding to the studied problems and mappings.

2.1 A Hybrid Iterative Scheme for Mixed Equilibrium Problems, General System of Variational Inequality Problems, and Fixed Point Problems in Hilbert Spaces

In this section, we introduce a new iterative scheme for finding a common element of the set of solutions of a general system of variational inequalities, the set of solutions of a mixed equilibrium problem, and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Using the demiclosedness principle for nonexpansive mappings, we prove that the iterative sequence converges strongly to a common element of the above three sets under some control conditions, and we also give some examples for mappings which satisfy conditions of the main result.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H . A mapping $T : C \rightarrow C$ is said to be *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Halpern [1] studied the following iteration formula for approximating a fixed point of T . For an arbitrary $v \in C$, let the sequence $\{x_n\}$ be defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n v + (1 - \alpha_n)Tx_n, \quad n \geq 1, \quad (1.1)$$

where $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$ that satisfies the following conditions: $C1 : \lim_{n \rightarrow \infty} \alpha_n = 0$ and $C2 : \sum_{n=1}^{\infty} \alpha_n = \infty$.

Actually, Halpern studied the special case of (1.1) in which $\alpha_n = n^{-\sigma}$, $\sigma \in (0, 1)$, and $v = 0$ and proved that $\{x_n\}$ converges strongly to a fixed point of T . Under a different restriction on the parameter $\{\alpha_n\}$, Lions [2] improved the result of Halpern, still in Hilbert spaces. He proved strong convergence of $\{x_n\}$ to a fixed point of T , where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0, \quad C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \quad C3 : \lim_{n \rightarrow \infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}^2} = 0. \quad (1.2)$$

Reich [3] proved that the result of Halpern remains true when H is uniformly smooth. It was observed that both Halpern's and Lion's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$. This was overcome by Wittmann [4] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0, \quad C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \quad C3 : \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \quad (1.3)$$

Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (1.4)$$

The variational inequality, nonconvex variational inequality, and mixed variational inequality have been widely studied in the literature; see, for example, Ceng et al. [5–14], Chang et al. [15], Noor [16–18], Peng and Yao [19–21], Plubtieng and Punpaeng [22], Yao et al. [23], Zeng and Yao [24], Zhao and He [25], and the references therein.

For solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k -Lipschitz continuous, and $VI(C, A)$ is nonempty, Korpelevich [26] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda Ax_n), \\ x_{n+1} &= P_C(x_n - \lambda Ay_n), \end{aligned} \quad (1.5)$$

for every $n = 0, 1, 2, \dots$, where $\lambda \in (0, 1/k)$ and P_C is the projection of $\mathbb{R}^n$ onto C . He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in VI(C, A)$. In 2007, Y. Yao and J. C. Yao [27] introduced a new iterative scheme for finding the common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for α -inverse-strongly monotone mappings in a real Hilbert space.

Let C be a nonempty closed convex subset of a real Hilbert space H .

Let $A, B : C \rightarrow H$ be two mappings. In 2008, Ceng et al. [12] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.6)$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then the problem (1.6) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.7)$$

which was defined by Verma [28] and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then the problem (1.7) reduces to the classical variational inequality $VI(C, A)$. Ceng et al. [12] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of the problem (1.6) for the α - and β -inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $x_1 = v \in C$, and $\{x_n\}, \{y_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \mu B x_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{aligned} \quad (1.8)$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{a_n\}, \{b_n\} \subset [0, 1]$. Then, they proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the problem (1.6) under some control conditions.

Let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and F a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [13] considered the following mixed equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (1.9)$$

The set of solutions of (1.9) is denoted by $MEP(F, \varphi)$. It is easy to see that x is a solution of the problem (1.9) implying that $x \in \text{dom } \varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the mixed equilibrium problem (1.9) becomes the following equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (1.10)$$

If $F = 0$, then the mixed equilibrium problem (1.9) reduces to the convex minimization problem

$$\text{find } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (1.11)$$

The set of solutions of (1.10) is denoted by $EP(F)$.

If $\varphi = 0$ and $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$, where A is a mapping from C into H , then (1.9) reduces to the classical variational inequality and $EP(F) = VI(C, A)$.

The equilibrium problems and variational inequality problems can be applied for the problems in science and technology, economics, optimizations, and control theory. The problems of finding a solution of the systems of variational inequalities which is also a solution of the mixed equilibrium problems and the problem of finding a solution of the mixed equilibrium problems under the constraints that it is also a solution of the system of variational inequalities are very useful and important for studying problems in science and applied science.

Inspired and motivated by these facts, we introduce a new iteration process for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a general system of variational inequalities in a real Hilbert space. Start with an arbitrary $v \in C$, and let $x_1 \in C$, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be the sequences generated by

$$\begin{aligned} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= P_C(u_n - \mu B u_n), \\ x_{n+1} &= a_n v + (1 - a_n) T P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{aligned} \quad (1.12)$$

where $\lambda > 0$ and $\mu > 0$ are two constants, $\{r_n\} \subset (0, \infty)$ and $\{a_n\} \subset [0, 1]$. Using the demiclosedness principle for nonexpansive mappings, we will show that the sequence $\{x_n\}$ converges strongly to a common element of the above three sets under some control conditions.

2. Preliminaries

In this section, we recall the well-known results and give some useful lemmas that will be used in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H .

A mapping $A : C \rightarrow H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \quad (2.1)$$

A is called α -strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C. \quad (2.2)$$

This implies that

$$\|Ax - Ay\| \geq \alpha \|x - y\|, \quad (2.3)$$

that is, A is α -expansive and, when $\alpha = 1$, it is *expansive*.

We can see easily that the following implications in monotonicity, strong monotonicity, and expansiveness hold:

$$\begin{aligned} & \text{strong monotonicity} \Rightarrow \text{monotonicity} \\ & \Downarrow \\ & \text{expansiveness.} \end{aligned} \tag{2.4}$$

The mapping A is called *L-Lipschitz continuous* (or *Lipschitzian*) if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

A is called *α -inverse-strongly monotone* (or *α -cocoercive*) if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.6}$$

It is obvious that every α -inverse-strongly monotone mapping A is monotone and Lipschitz continuous. Also, if A is α -strongly and L -Lipschitz continuous, then A is (α/L^2) -inverse-strongly monotone, but inverse-strongly monotone need not be strongly monotone.

A mapping A is called *relaxed c -cocoercive*, if there exists a constant $c > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-c) \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.7}$$

A mapping A is called *relaxed (c, d) -cocoercive*, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-c) \|Ax - Ay\|^2 + d \|x - y\|^2, \quad \forall x, y \in C. \tag{2.8}$$

For $c = 0$, A is d -strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. As a result, we have the following implication: d -strong monotonicity $\Rightarrow$ relaxed (c, d) -cocoercivity.

It is known that if the operator is Lipschitz continuous, then the relaxed cocoercivity is strongly monotone, but the strongly monotone does not imply the cocoercivity as shown in the following example.

Example 2.1. Let $H = \mathbb{R}$, $C = [1, \infty)$, and $A : C \rightarrow H$ be defined by $Ax = x^2$, $x \in C$. For $x, y \in C$, we have

$$\begin{aligned} \langle Ax - Ay, x - y \rangle &= (x^2 - y^2)(x - y) \\ &= (x + y)|x - y|^2 \\ &\geq 2|x - y|^2. \end{aligned} \tag{2.9}$$

Hence, A is 2-strongly monotone. If A is μ -cocoercive for some $\mu > 0$, then $\langle Ax - Ay, x - y \rangle = (x+y)|x-y|^2 \geq \mu|x^2 - y^2|^2$. This implies $x+y \leq 1/\mu$ for all $x, y \in [1, \infty)$ which is a contradiction. Hence, A is not μ -cocoercive for any $\mu > 0$.

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \quad (2.10)$$

P_C is called the *metric projection* of H onto C . It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.11)$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.12)$$

Recall that $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} \langle x - P_C x, y - P_C x \rangle &\leq 0, \\ \|x - y\|^2 &\geq \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{aligned} \quad (2.13)$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [29] for more details.

For solving the mixed equilibrium problem, let us assume the following assumptions for the bifunction F, φ and the set C :

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex;
- (A5) for each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
- (B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z); \quad (2.14)$$

- (B2) C is a bounded set.

In the sequel we will need to use the following lemma.

Lemma 2.2 (see [30]). *Let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \forall y \in C \right\} \quad (2.15)$$

for all $x \in H$. Then the following conclusions hold:

- (1) for each $x \in H$, $T_r(x) \neq \emptyset$;
- (2) T_r is single valued;
- (3) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \quad (2.16)$$

- (4) $F(T_r) = \text{MEP}(F, \varphi)$;
- (5) $\text{MEP}(F, \varphi)$ is closed and convex.

We also need the following lemmas.

Lemma 2.3 (see [31]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + t_n c_n, \quad \forall n \geq 1, \quad (2.17)$$

where $\{t_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions

$$(i) \sum_{n=1}^{\infty} t_n = \infty, \quad (ii) \sum_{n=1}^{\infty} b_n < \infty, \quad (iii) \limsup_{n \rightarrow \infty} c_n \leq 0. \quad (2.18)$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 (see [32]). *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \|x - y\|^2 - \alpha \gamma \|x - z\|^2 - \beta \gamma \|y - z\|^2. \quad (2.19)$$

Lemma 2.5 (see [29] (demiclosedness principle)). *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$) and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \rightarrow y$), it follows that $(I - T)x = y$. Here I is the identity operator of H .*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \quad (2.20)$$

In 2009, Kangtunyakarn and Suantai [33] introduced a new mapping called the S -mapping as follows.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \dots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. Very recently, in 2009, Kangtunyakarn and Suantai [33] introduced the new mapping $S_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I, \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I, \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I, \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I, \\ S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \end{aligned} \quad (2.21)$$

The mapping S_n is called the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. Nonexpansivity of each T_i ensures the nonexpansivity of S_n . They also showed the following useful fact.

Lemma 2.7 (see [33]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \dots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$, $\alpha_1^N \in (0, 1]$ and $\alpha_2^j, \alpha_3^j \in [0, 1]$ for all $j = 1, 2, \dots, N$. Let S be the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.8 (see [12]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.6) if and only if x^* is a fixed point of the mapping $G : C \rightarrow C$ defined by*

$$G(x) = P_C [P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C, \quad (2.22)$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $\text{GVI}(C, A, B)$.

Next, we prove a lemma which is very useful for our consideration.

Lemma 2.9. Let $G : C \rightarrow C$ be defined by

$$G(x) = P_C [P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C, \quad (2.23)$$

where $\lambda, \mu > 0$ and $A, B : C \rightarrow H$ are two mappings. If $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings, then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned} \|G(x) - G(y)\| &= \|P_C [P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)] \\ &\quad - P_C [P_C(y - \mu By) - \lambda A P_C(y - \mu By)]\|^2 \\ &\leq \|P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx) - (P_C(y - \mu By) - \lambda A P_C(y - \mu By))\| \\ &= \|(I - \lambda A)P_C(I - \mu B)x - (I - \lambda A)P_C(I - \mu B)y\| \\ &\leq \|x - y\|. \end{aligned} \quad (2.24)$$

This shows that G is a nonexpansive mapping. $\square$

3. Main Results

In this section, we prove strong convergence theorems of the iterative scheme (1.12) to a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a general system of variational inequality in a real Hilbert space.

Theorem 3.1. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper lower semi-continuous and convex function. Let $\lambda, \mu > 0$, and let $A, B : C \rightarrow H$ be such that $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be defined by (1.12), $\{r_n\} \subset (0, \infty)$ and $\{a_n\} \subset [0, 1]$ such that

$$(C1) \lim_{n \rightarrow \infty} a_n = 0, \lim_{n \rightarrow \infty} (a_{n+1} / a_n) = 1, \text{ and } \sum_{n=1}^{\infty} a_n = \infty,$$

$$(C2) \liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

If $\lim_{n \rightarrow \infty} \|Ay_n - Ay^*\| = 0$ and $\lim_{n \rightarrow \infty} \|Bu_n - Bx^*\| = 0$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$, then $\{x_n\}$ converges strongly to $\bar{x} = P_{\Omega}v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ a sequence of mappings defined as in Lemma 2.2. It follows from Lemma 2.8 that

$$x^* = P_C [P_C(x^* - \mu Bx^*) - \lambda A P_C(x^* - \mu Bx^*)]. \quad (3.1)$$

Put $y^* = P_C(x^* - \mu Bx^*)$ and $t_n = P_C(y_n - \lambda Ay_n)$, then $x^* = P_C(y^* - \lambda Ay^*)$ and

$$x_{n+1} = a_nv + (1 - a_n)Tt_n. \quad (3.2)$$

Since $I - \lambda A$, $I - \mu B$, T_{r_n} , and P_C are nonexpansive mappings, we have

$$\begin{aligned} \|t_n - x^*\|^2 &= \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ &\leq \|y_n - y^*\|^2 \\ &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\ &\leq \|x_n - x^*\|^2, \end{aligned} \quad (3.3)$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_nv + (1 - a_n)Tt_n - x^*\| \\ &\leq a_n\|v - x^*\| + (1 - a_n)\|t_n - x^*\| \\ &\leq a_n\|v - x^*\| + (1 - a_n)\|x_n - x^*\| \\ &\leq \max\{\|v - x^*\|, \|x_1 - x^*\|\}. \end{aligned} \quad (3.4)$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bu_n\}$, and $\{Tt_n\}$ are also bounded. Also, observe that

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|P_C(y_{n+1} - \lambda Ay_{n+1}) - P_C(y_n - \lambda Ay_n)\| \\ &\leq \|y_{n+1} - y_n\| \\ &= \|P_C(u_{n+1} - \mu Bu_{n+1}) - P_C(u_n - \mu Bu_n)\| \\ &\leq \|u_{n+1} - u_n\|. \end{aligned} \quad (3.5)$$

On the other hand, from $u_n = T_{r_n}x_n \in \text{dom } \varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \text{dom } \varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (3.6)$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \quad (3.7)$$

Putting $y = u_{n+1}$ in (3.6) and $y = u_n$ in (3.7), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle &\geq 0, \\ F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle &\geq 0. \end{aligned} \quad (3.8)$$

From the monotonicity of F , we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \quad (3.9)$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \quad (3.10)$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right), \end{aligned} \quad (3.11)$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \end{aligned} \quad (3.12)$$

It follows from (3.5) and (3.12) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \quad (3.13)$$

Next, we show that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (3.13), we have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &= \|a_{n+1}v + (1 - a_{n+1})Tt_{n+1} - a_nv - (1 - a_n)Tt_n\| \\ &= \|(a_{n+1} - a_n)v + (a_n - a_{n+1})Tt_{n+1} + (1 - a_n)(Tt_{n+1} - Tt_n)\| \\ &\leq |a_{n+1} - a_n|(\|v\| + \|Tt_{n+1}\|) + (1 - a_n)\|t_{n+1} - t_n\| \end{aligned}$$

$$\begin{aligned}
&\leq |a_{n+1} - a_n|(\|v\| + \|Tt_{n+1}\|) + (1 - a_n)\|x_{n+1} - x_n\| \\
&\quad + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\leq (1 - a_n)\|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n| M + 2M|a_{n+1} - a_n|,
\end{aligned} \tag{3.14}$$

where $M = \max\{\|v\|, \sup_n \|Tt_{n+1}\|, \sup_n \|u_{n+1} - x_{n+1}\|\}$. By the assumptions on $\{a_n\}$ and $\{r_n\}$ and Lemma 2.3, we conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

From (C1), (C2), (3.5), (3.12), and (3.15), we also have $\|u_{n+1} - u_n\| \rightarrow 0$, $\|t_{n+1} - t_n\| \rightarrow 0$, and $\|y_{n+1} - y_n\| \rightarrow 0$, as $n \rightarrow \infty$.

Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n)(Tt_n - x_n), \tag{3.16}$$

we have that

$$\|Tt_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

Next, we prove that $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. From Lemma 2.2(3), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \leq \langle T_{r_n} x_n - T_{r_n} x^*, x_n - x^* \rangle \\
&= \langle u_n - x^*, x_n - x^* \rangle = \frac{1}{2} \left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \right).
\end{aligned} \tag{3.18}$$

Hence,

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2. \tag{3.19}$$

From Lemma 2.4, (3.3), and (3.19), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|u_n - x^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|x_n - x^*\|^2 - \|x_n - u_n\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n) \|x_n - u_n\|^2.
\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned} (1 - a_n)\|x_n - u_n\|^2 &\leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq a_n\|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|. \end{aligned} \quad (3.21)$$

From conditions (C1) and (3.15), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.22)$$

By (3.17) and (3.22), we have

$$\|Tt_n - u_n\| \leq \|Tt_n - x_n\| + \|x_n - u_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.23)$$

Next, we prove that $\|Tt_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (2.11) and nonexpansiveness of $I - \mu B$, we get

$$\begin{aligned} \|y_n - y^*\|^2 &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\leq \langle (u_n - \mu Bu_n) - (x^* - \mu Bx^*), y_n - y^* \rangle \\ &= \frac{1}{2} \left[\|(u_n - \mu Bu_n) - (x^* - \mu Bx^*)\|^2 + \|y_n - y^*\|^2 \right. \\ &\quad \left. - \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*) - (y_n - y^*)\|^2 \right] \\ &\leq \frac{1}{2} \left[\|u_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \right. \\ &\quad \left. + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \right]. \end{aligned} \quad (3.24)$$

By (3.3), we obtain

$$\begin{aligned} \|y_n - y^*\|^2 &\leq \|u_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \\ &\quad + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \\ &\quad + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2. \end{aligned} \quad (3.25)$$

Hence,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|y_n - y^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \\
&\quad \times \left[\|x_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \right. \\
&\quad \left. + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n) \|(u_n - x^*) - (y_n - y^*)\|^2 \\
&\quad + (1 - a_n) 2\mu \|(u_n - x^*) - (y_n - y^*)\| \|Bu_n - Bx^*\|,
\end{aligned} \tag{3.26}$$

which implies that

$$\begin{aligned}
(1 - a_n) \|(u_n - x^*) - (y_n - y^*)\|^2 &\leq a_n \|v - x^*\|^2 \\
&\quad + (1 - a_n) 2\mu \|(u_n - x^*) - (y_n - y^*)\| \|Bu_n - Bx^*\| \\
&\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.27}$$

By this together with (C1), (3.15), and $\lim_{n \rightarrow \infty} \|Bu_n - Bx^*\| = 0$, we obtain

$$\|(u_n - x^*) - (y_n - y^*)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.28}$$

From Lemma 2.6 and (2.12), it follows that

$$\begin{aligned}
&\|(y_n - t_n) + (x^* - y^*)\|^2 \\
&= \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)] + \lambda(Ay_n - Ay^*)\|^2 \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)]\|^2 \\
&\quad + 2\lambda \langle Ay_n - Ay^*, (y_n - t_n) + (x^* - y^*) \rangle \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\
&\quad + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|TP_C(y_n - \lambda Ay_n) - TP_C(y^* - \lambda Ay^*)\|^2 \\
&\quad + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - (Tt_n - x^*)\| \\
&\quad \times [\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|Tt_n - x^*\|] \\
&\quad + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\|
\end{aligned}$$

$$\begin{aligned}
 &= \|u_n - Tt_n + x^* - y^* - (u_n - y_n) - \lambda(Ay_n - Ay^*)\| \\
 &\quad \times [\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|Tt_n - x^*\|] \\
 &\quad + 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\|.
 \end{aligned} \tag{3.29}$$

By this together with (3.23), (3.28), and $\lim_{n \rightarrow \infty} \|Ay_n - Ay^*\| = 0$, we obtain $\|(y_n - t_n) + (x^* - y^*)\| \rightarrow 0$ as $n \rightarrow \infty$. This together with (3.17), (3.22), and (3.28), we obtain that

$$\begin{aligned}
 \|Tt_n - t_n\| &\leq \|Tt_n - x_n\| + \|x_n - u_n\| + \|(u_n - y_n) - (x^* - y^*)\| \\
 &\quad + \|(y_n - t_n) + (x^* - y^*)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.30}$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle \leq 0, \tag{3.31}$$

where $\bar{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{Tt_n\}$ are two bounded sequences in C , we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n \rightarrow \infty} \langle v - \bar{x}, Tt_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle v - \bar{x}, Tt_{n_i} - \bar{x} \rangle. \tag{3.32}$$

Since $\lim_{n \rightarrow \infty} \|Tt_n - t_n\| = 0$, we obtain that $Tt_{n_i} \rightharpoonup z$ as $i \rightarrow \infty$.

Next, we show that $z \in \Omega$.

(a) We first show $z \in F(T)$.

Since $t_{n_i} \rightharpoonup z$ and $\|Tt_n - t_n\| \rightarrow 0$, we obtain by Lemma 2.5 that $z \in F(T)$.

(b) Now, we show that $z \in \text{GVI}(C, A, B)$.

From (3.30) and (3.17), we have

$$\|t_n - x_n\| \leq \|Tt_n - t_n\| + \|Tt_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.33}$$

Furthermore, by Lemma 2.9, we have that $G : C \rightarrow C$ is nonexpansive. Then, we have

$$\begin{aligned}
 \|t_n - G(t_n)\| &= \|P_C(y_n - \lambda Ay_n) - G(t_n)\| \\
 &= \|P_C[P(u_n - \mu Bu_n) - \lambda AP(u_n - \mu Bu_n)] - G(t_n)\| \\
 &= \|G(u_n) - G(t_n)\| \leq \|u_n - t_n\| \\
 &\leq \|u_n - x_n\| + \|x_n - t_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.34}$$

Again by Lemma 2.5, we have $z \in \text{GVI}(C, A, B)$.

(c) We show that $z \in \text{MEP}(F, \varphi)$. Since $t_{n_i} \rightarrow z$ and $\|x_n - t_n\| \rightarrow 0$, we obtain that $x_{n_i} \rightarrow z$. From $\|u_n - x_n\| \rightarrow 0$, we also obtain that $u_{n_i} \rightarrow z$. By using the same argument as that in the proof of [30, Theorem 3.1, page 1825], we can show that $z \in \text{MEP}(F, \varphi)$. Therefore, there holds $z \in \Omega$.

On the other hand, it follows from (2.13), (3.17), and $Tt_{n_i} \rightarrow z$ as $i \rightarrow \infty$ that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle &= \limsup_{n \rightarrow \infty} \langle v - \bar{x}, Tt_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle v - \bar{x}, Tt_{n_i} - \bar{x} \rangle \\ &= \langle v - \bar{x}, z - \bar{x} \rangle \leq 0. \end{aligned} \quad (3.35)$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= \langle a_n v + (1 - a_n)Tt_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - a_n) \langle Tt_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2}(1 - a_n) \left(\|t_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2 \right) \\ &\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2}(1 - a_n) \left(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2 \right), \end{aligned} \quad (3.36)$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - a_n) \|x_n - \bar{x}\|^2 + 2a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle. \quad (3.37)$$

By this together with (C1) and (3.35), we have by Lemma 2.3 that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. $\square$

The following examples provide mappings A and B which satisfy those conditions in Theorem 3.1.

Example 3.2. Let $A, B : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. If $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$, then we have

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,
- (2) $\|Ay_n - Ay^*\| \rightarrow 0$ and $\|Bu_n - Bx^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 + \lambda^2 \|Ax - Ay\|^2 - 2\lambda \langle x - y, Ax - Ay \rangle \\ &\leq \|x - y\|^2 + \lambda^2 \|Ax - Ay\|^2 - 2\lambda\alpha \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned} \quad (3.38)$$

hence, $I - \lambda A$ is nonexpansive. Similarly, we can show that $I - \mu B$ is nonexpansive.

(2) Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.3), we have

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
 &= a_n \|v - x^*\|^2 + (1 - a_n) \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \right] \\
 &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2, \\
 \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|y_n - y^*\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|u_n - x^*\|^2 + \mu(\mu - 2\beta) \|B u_n - B x^*\|^2 \right] \\
 &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \mu(\mu - 2\beta) \|B u_n - B x^*\|^2.
 \end{aligned} \tag{3.39}$$

Therefore, we have

$$\begin{aligned}
 -(1 - a_n) \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 &\leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|, \\
 -(1 - a_n) \mu(\mu - 2\beta) \|B u_n - B x^*\|^2 &\leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
 \end{aligned} \tag{3.40}$$

From (3.15) and (C1), we obtain

$$\|A y_n - A y^*\| \longrightarrow 0, \quad \|B u_n - B x^*\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.41}$$

□

Example 3.3. Let $A : C \rightarrow H$ be an L_A -Lipschitzian and relaxed (c, d) -cocoercive mapping and $B : C \rightarrow H$ an L_B -Lipschitzian and relaxed (c', d') -cocoercive mapping. If $0 < \lambda < (2(d - cL_A^2))/L_A^2$ and $0 < \mu < (2(d' - c'L_B^2))/L_B^2$, then we have

(1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,

(2) $\|A y_n - A y^*\| \rightarrow 0$ and $\|B u_n - B x^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu B x^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned}
\|(I - \mu B)x - (I - \mu B)y\|^2 &= \|(x - y) - \mu(Bx - By)\|^2 \\
&= \|x - y\|^2 + \mu^2\|Bx - By\|^2 - 2\mu\langle x - y, Bx - By \rangle \\
&\leq \|x - y\|^2 + \mu^2\|Bx - By\|^2 \\
&\quad - 2\mu\left[-c'\|Bx - By\|^2 + d'\|x - y\|^2\right] \\
&= \|x - y\|^2 + \mu^2\|Bx - By\|^2 \\
&\quad + 2\mu c'\|Bx - By\|^2 - 2\mu d'\|x - y\|^2 \\
&\leq \|x - y\|^2 + \mu^2 L_B^2 \|x - y\|^2 \\
&\quad + 2\mu c' L_B^2 \|x - y\|^2 - 2\mu d' \|x - y\|^2 \\
&= \left(1 + 2\mu c' L_B^2 - 2\mu d' + \mu^2 L_B^2\right) \|x - y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned} \tag{3.42}$$

hence, $I - \mu B$ is nonexpansive. Similarly, we can show that $I - \lambda A$ is nonexpansive.

(2) Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.3), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|y_n - y^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \\
&\quad \times \left[\|u_n - x^*\|^2 - 2\mu \langle u_n - x^*, B u_n - B x^* \rangle + \mu^2 \|B u_n - B x^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \\
&\quad \times \left[\|u_n - x^*\|^2 + 2\mu c' \|B u_n - B x^*\|^2 - 2\mu d' \|u_n - x^*\|^2 + \mu^2 \|B u_n - B x^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|x_n - x^*\|^2 + \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|B u_n - B x^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|B u_n - B x^*\|^2,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
&= a_n \|v - x^*\|^2 + (1 - a_n) \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|x_n - x^*\|^2 + \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|A y_n - A y^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|A y_n - A y^*\|^2.
\end{aligned} \tag{3.43}$$

Therefore, we have

$$\begin{aligned}
& - (1 - a_n) \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|B u_n - B x^*\|^2 \\
& \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|, \\
& - (1 - a_n) \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|A y_n - A y^*\|^2 \\
& \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.44}$$

From (3.15) and (C1), we obtain

$$\|A y_n - A y^*\| \rightarrow 0, \quad \|B u_n - B x^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.45}$$

□

By using the same proof as in Examples 3.2 and 3.3, we can obtain the following example.

Example 3.4. Let A be an α -inverse-strongly monotone mapping of C into H and B be a L -Lipschitzian and relaxed (c, d) -cocoercive mapping of C into H . If $\lambda \in (0, 2\alpha)$ and $0 < \mu < (2(d - cL^2))/L^2$, then we have

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,
- (2) $\|A y_n - A y^*\| \rightarrow 0$ and $\|B u_n - B x^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu B x^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Let $\mathcal{A}$ be the class of all α -inverse-strongly monotone mappings from C into H , $\mathcal{B}$ the class of all β -inverse-strongly monotone mappings from C into H , $\mathcal{C}$ the class of all L -Lipschitzian and relaxed (c, d) -cocoercive mappings from C into H , and $\mathcal{D}$ the class of all L_1 -Lipschitzian and relaxed (c', d') -cocoercive mappings from C into H .

Theorem 3.5. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \bigcap_{i=1}^N F(T_i) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \dots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$, $\alpha_1^N \in (0, 1]$ and $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let S be the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ the sequences generated by

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$y_n = P_C(u_n - \mu B u_n), \quad (3.46)$$

$$x_{n+1} = a_n v + (1 - a_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1.$$

If one of the following conditions is satisfied:

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, and $\mu \in (0, 2\beta)$,
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, and $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, and $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$ and $\{a_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Proof. By Lemma 2.7, we obtain that S is nonexpansive and $F(S) = \bigcap_{i=1}^N F(T_i)$. Hence, the result is obtained directly from Theorem 3.1 and Examples 3.2–3.4. $\square$

From Theorem 3.1 and Examples 3.2–3.4, we obtain the following result.

Corollary 3.6. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . If one of the following conditions is satisfied:

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, and $\mu \in (0, 2\beta)$,
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, and $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, and $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$, $\{a_n\}$ are as in Theorem 3.1, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Let $\varphi = 0$ in Theorem 3.1. From Theorem 3.1 and Examples 3.2–3.4, we obtain the following result.

Corollary 3.7. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5). Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \text{GVI}(C, A, B) \cap \text{EP}(F) \neq \emptyset$. Let $x_1, v \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= P_C(u_n - \mu B u_n), \\ x_{n+1} &= a_n v + (1 - a_n) T P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{aligned} \tag{3.47}$$

If one of the following conditions is satisfied:

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, and $\mu \in (0, 2\beta)$,
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, and $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, and $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}, \{a_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\bar{x} = P_{\Omega}v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Remark 3.8. In Theorem 3.5, if $F \equiv 0$, then the sequence $\{x_n\}$ generated by (3.46) converges strongly to a solution of the minimization problem which is also a solution of a system of variational inequalities.

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2.2 Strong Convergence to Solutions of Generalized Mixed Equilibrium Problems with Applications

In this section, we introduce a Halpern-type iteration for a generalized mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. Strong convergence theorems are also established in this paper. As applications, we apply our main result to mixed equilibrium, generalized equilibrium, and mixed variational inequality problems in Banach spaces. Finally, examples and numerical results are also given.

1. Introduction

Let E be a real Banach space, C a nonempty, closed, and convex subset of E , and E^* the dual space of E . Let $T : C \rightarrow C$ be a nonlinear mapping. The fixed points set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

One classical way often used to approximate a fixed point of a nonlinear self-mapping T on C was firstly introduced by Halpern [1] which is defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (1.1)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. He proved, in a real Hilbert space, a strong convergence theorem for a nonexpansive mapping T when $\alpha_n = n^{-a}$ for any $a \in (0, 1)$.

Subsequently, motivated by Halpern [1], many mathematicians devoted time to study algorithm (1.1) in different styles. Several strong convergence results for nonlinear mappings were also continuously established in some certain Banach spaces (see also [2–9]).

Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction, $A : C \rightarrow E^*$ a mapping, and $\varphi : C \rightarrow \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.2)$$

The solutions set of (1.2) is denoted by $\text{GMEP}(f, A, \varphi)$ (see Peng and Yao [10]).

If $A \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following mixed equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.3)$$

The solutions set of (1.3) is denoted by $\text{MEP}(f, \varphi)$ (see Ceng and Yao [11]).

If $f \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following mixed variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.4)$$

The solutions set of (1.4) is denoted by $\text{VI}(C, A, \varphi)$ (see Noor [12]).

If $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following generalized equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \quad (1.5)$$

The solutions set of (1.5) is denoted by $\text{GEP}(f, A)$ (see Moudafi [13]).

If $\varphi \equiv 0$, then the mixed equilibrium problem (1.3) reduces to the following equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (1.6)$$

The solutions set of (1.6) is denoted by $\text{EP}(f)$ (see Combettes and Hirstoaga [14]).

If $f \equiv 0$, then the mixed equilibrium problem (1.3) reduces to the following convex minimization problem: finding $\hat{x} \in C$ such that

$$\varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.7)$$

The solutions set of (1.7) is denoted by $\text{CMP}(\varphi)$.

If $\varphi \equiv 0$, then the mixed variational inequality problem (1.4) reduces to the following variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \quad (1.8)$$

The solutions set of (1.8) is denoted by $VI(C, A)$ (see Stampacchia [7]).

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others. For more details on these topics, see, for instance, [14–34].

For solving the generalized mixed equilibrium problem, let us assume the following [25]:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The purpose of this paper is to investigate strong convergence of Halpern-type iteration for a generalized mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. As applications, our main result can be deduced to mixed equilibrium, generalized equilibrium, mixed variational inequality problems, and so on. Examples and numerical results are also given in the last section.

2. Preliminaries and Lemmas

In this section, we need the following preliminaries and lemmas which will be used in our main theorem.

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *strictly convex* if, for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1. \quad (2.1)$$

It is also said to be *uniformly convex* if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| < 1 - \delta. \quad (2.2)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0, 2] \rightarrow [0, 1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}. \quad (2.3)$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.4)$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in U$. The *normalized duality mapping* $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (2.5)$$

for all $x \in E$. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E (see [35]).

Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.6)$$

Remark 2.1. We know the following: for any $x, y, z \in E$,

- (1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$;
- (2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$;
- (3) $\phi(x, y) = \|x - y\|^2$ in a real Hilbert space.

Lemma 2.2 (see [36]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, and smooth Banach space and let C be a nonempty closed and convex subset of E . The *generalized projection mapping*, introduced by Alber [37], is a mapping $\Pi_C : E \rightarrow C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}. \quad (2.7)$$

In fact, we have the following result.

Lemma 2.3 (see [37]). *Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

Lemma 2.4 (see [36, 37]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E , $x \in E$, and $z \in C$. Then $z = \Pi_C x$ if and only if*

$$\langle Jx - Jz, y - z \rangle \leq 0, \quad \forall y \in C. \quad (2.8)$$

Lemma 2.5 (see [36, 37]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \quad (2.9)$$

Lemma 2.6 (see [38]). *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty, closed, and convex subset of E . Then Π_C is uniformly norm-to-norm continuous on every bounded set.*

We make use of the following mapping V studied in Alber [37]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \quad (2.10)$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.7 (see [39]). *Let E be a reflexive, strictly convex, smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*) \quad (2.11)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.8 (see [25]). *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq 0, \quad \forall y \in C. \quad (2.12)$$

Following [25, 40], we know the following lemma.

Lemma 2.9 (see [41]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex, and reflexive Banach space E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. For all $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq \varphi(z), \quad \forall y \in C. \quad (2.13)$$

Define the mapping $T_r : E \rightarrow 2^C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq \varphi(z), \quad \forall y \in C \right\}. \quad (2.14)$$

Then, the followings hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive-type mapping [42], that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \quad (2.15)$$

- (3) $F(T_r) = \text{GMEP}(f, A, \varphi)$;
- (4) $\text{GMEP}(f, A, \varphi)$ is closed and convex.

Remark 2.10. It is known that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x) \quad (2.16)$$

for all $x, y \in \text{dom } T$ (see [42]).

The following lemmas give us some nice properties of real sequences.

Lemma 2.11 (see [43]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n, \quad \forall n \geq 1, \quad (2.17)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

- (a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
- (b) $\limsup_{n \rightarrow \infty} b_n / \alpha_n \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < +\infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.12 (see [44]). Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \geq 1$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \geq 1$:

$$\gamma_{m_k} \leq \gamma_{m_k+1}, \quad \gamma_k \leq \gamma_{m_k+1}. \quad (2.18)$$

In fact, m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. Main Results

In this section, we prove our main theorem in this paper. To this end, we need the following proposition.

Proposition 3.1. Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and uniformly smooth Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \rightarrow E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function

from C to $\mathbb{R}$ such that $\text{GMEP}(f, A, \varphi) \neq \emptyset$. Let $\{r_n\} \subset (0, \infty)$ be such that $\liminf_{n \rightarrow \infty} r_n > 0$. For each $n \geq 1$, let T_{r_n} be defined as in Lemma 2.9. Suppose that $x \in C$ and $\{x_n\}$ is a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|x_n - T_{r_n} x_n\| = 0$. Then

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle \leq 0, \quad (3.1)$$

where $p = \Pi_{\text{GMEP}(f, A, \varphi)} x$ and $\Pi_{\text{GMEP}(f, A, \varphi)}$ is the generalized projection of C onto $\text{GMEP}(f, A, \varphi)$.

Proof. Let $x \in C$ and put $p = \Pi_{\text{GMEP}(f, A, \varphi)} x$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$ and

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle = \langle Jx - Jp, v - p \rangle. \quad (3.2)$$

Put $y_n = T_{r_n} x_n$. Since $\lim_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| = 0$, we have $y_{n_k} \rightharpoonup v$. On the other hand, since E is uniformly smooth, J is uniformly norm-to-norm continuous on bounded subsets of E . So we have

$$\lim_{k \rightarrow \infty} \|Jx_{n_k} - Jy_{n_k}\| = 0. \quad (3.3)$$

Since $\liminf_{k \rightarrow \infty} r_{n_k} > 0$,

$$\lim_{k \rightarrow \infty} \frac{\|Jx_{n_k} - Jy_{n_k}\|}{r_{n_k}} = 0. \quad (3.4)$$

By the definition of $T_{r_{n_k}}$, for any $y \in C$, we see that

$$f(y_{n_k}, y) + \langle Ay_{n_k}, y - y_{n_k} \rangle + \varphi(y) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y - y_{n_k} \rangle \geq \varphi(y_{n_k}). \quad (3.5)$$

By (A2), for each $y \in C$, we obtain

$$\begin{aligned} f(y, y_{n_k}) + \varphi(y_{n_k}) &\leq -f(y_{n_k}, y) + \varphi(y_{n_k}) \\ &\leq \langle Ay_{n_k}, y - y_{n_k} \rangle + \varphi(y) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y - y_{n_k} \rangle. \end{aligned} \quad (3.6)$$

For any $t \in (0, 1)$ and $y \in C$, we define $y_t = ty + (1 - t)v$. Then $y_t \in C$. It follows by the monotonicity of A that

$$\begin{aligned} f(y_t, y_{n_k}) + \varphi(y_{n_k}) &\leq \langle Ay_{n_k} - Ay_t, y_t - y_{n_k} \rangle + \langle Ay_t, y_t - y_{n_k} \rangle \\ &\quad + \varphi(y_t) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y_t - y_{n_k} \rangle \\ &\leq \langle Ay_t, y_t - y_{n_k} \rangle + \varphi(y_t) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y_t - y_{n_k} \rangle. \end{aligned} \quad (3.7)$$

By (A4), (3.4), and the weakly lower semicontinuity of φ , letting $k \rightarrow \infty$, we obtain

$$f(y_t, v) + \varphi(v) \leq \langle Ay_t, y_t - v \rangle + \varphi(y_t). \quad (3.8)$$

By (A1), (A4), and the convexity of φ , we have

$$\begin{aligned} 0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, v) + t\varphi(y) + (1-t)\varphi(v) - \varphi(y_t) \\ &= t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)(f(y_t, v) + \varphi(v) - \varphi(y_t)) \\ &\leq t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)\langle Ay_t, y_t - v \rangle \\ &= t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)t\langle Ay_t, y - v \rangle. \end{aligned} \quad (3.9)$$

It follows that

$$f(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle Ay_t, y - v \rangle \geq 0. \quad (3.10)$$

By (A3), the weakly lower semicontinuity of φ , and the continuity of A , letting $t \rightarrow 0$, we obtain

$$f(v, y) + \varphi(y) - \varphi(v) + \langle Av, y - v \rangle \geq 0, \quad \forall y \in C. \quad (3.11)$$

This shows that $v \in \text{GMEP}(f, A, \varphi)$. By Lemma 2.4, we have

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle = \langle Jx - Jp, v - p \rangle \leq 0. \quad (3.12)$$

This completes the proof. $\square$

Theorem 3.2. Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \rightarrow E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\text{GMEP}(f, A, \varphi) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$\begin{aligned} f(y_n, y) + \langle Ay_n, y - y_n \rangle + \varphi(y) + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.13)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{GMEP}(f,A,\varphi)}x$, where $\Pi_{\text{GMEP}(f,A,\varphi)}$ is the generalized projection of C onto $\text{GMEP}(f, A, \varphi)$.

Proof. From Lemma 2.9(4), we know that $\text{GMEP}(f, A, \varphi)$ is closed and convex. Let $p = \Pi_{\text{GMEP}(f,A,\varphi)}x$. Put $y_n = T_{r_n}x_n$ and $z_n = J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n)$ for all $n \in \mathbb{N}$. So, by Lemma 2.5, we have

$$\begin{aligned}\phi(p, x_{n+1}) &\leq \phi(p, z_n) \\ &\leq \alpha_n \phi(p, x) + (1 - \alpha_n) \phi(p, y_n) \\ &\leq \alpha_n \phi(p, x) + (1 - \alpha_n) \phi(p, x_n).\end{aligned}\tag{3.14}$$

By induction, we can show that $\phi(p, x_n) \leq \phi(p, x)$ for each $n \in \mathbb{N}$. Hence $\{\phi(p, x_n)\}$ is bounded and thus $\{x_n\}$ is also bounded.

We next show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \rightarrow \infty} (\phi(p, x_{n_{k+1}}) - \phi(p, x_{n_k})) = 0,\tag{3.15}$$

then

$$\lim_{k \rightarrow \infty} (\phi(p, y_{n_k}) - \phi(p, x_{n_k})) = 0.\tag{3.16}$$

Since $\alpha_{n_k} \rightarrow 0$,

$$\lim_{k \rightarrow \infty} \|Jz_{n_k} - Jy_{n_k}\| = \lim_{k \rightarrow \infty} \alpha_{n_k} \|Jx - Jy_{n_k}\| = 0.\tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E , so is J^{-1} . It follows that

$$\lim_{k \rightarrow \infty} \|z_{n_k} - y_{n_k}\| = 0.\tag{3.18}$$

Since E is uniformly smooth and uniformly convex, by Lemma 2.6, Π_C is uniformly norm-to-norm continuous on bounded sets. So we obtain

$$\lim_{k \rightarrow \infty} \|x_{n_{k+1}} - y_{n_k}\| = \lim_{k \rightarrow \infty} \|\Pi_C z_{n_k} - \Pi_C y_{n_k}\| = 0,\tag{3.19}$$

and hence

$$\lim_{k \rightarrow \infty} \|Jx_{n_{k+1}} - Jy_{n_k}\| = 0.\tag{3.20}$$

Furthermore, $\lim_{k \rightarrow \infty} \phi(x_{n_k+1}, y_{n_k}) = 0$. Indeed, by the definition of ϕ , we observe that

$$\begin{aligned} \phi(x_{n_k+1}, y_{n_k}) &= \|x_{n_k+1}\|^2 - 2\langle x_{n_k+1}, Jy_{n_k} \rangle + \|y_{n_k}\|^2 \\ &= \langle x_{n_k+1}, Jx_{n_k+1} - Jy_{n_k} \rangle + \langle y_{n_k} - x_{n_k+1}, Jy_{n_k} \rangle. \end{aligned} \quad (3.21)$$

It follows from (3.19) and (3.20) that $\lim_{k \rightarrow \infty} \phi(x_{n_k+1}, y_{n_k}) = 0$. On the other hand, from Remark 2.1(2), we have

$$\begin{aligned} \phi(p, y_{n_k}) - \phi(p, x_{n_k}) &= (\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})) + (\phi(p, y_{n_k}) - \phi(p, x_{n_k+1})) \\ &= (\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})) \\ &\quad + \phi(x_{n_k+1}, y_{n_k}) + 2\langle p - x_{n_k+1}, Jx_{n_k+1} - Jy_{n_k} \rangle. \end{aligned} \quad (3.22)$$

It follows from (3.20) and (3.21) that $\lim_{k \rightarrow \infty} (\phi(p, y_{n_k}) - \phi(p, x_{n_k})) = 0$.

We next consider the following two cases.

Case 1. $\phi(p, x_{n+1}) \leq \phi(p, x_n)$ for all sufficiently large n . Hence the sequence $\{\phi(p, x_n)\}$ is bounded and nonincreasing. So $\lim_{n \rightarrow \infty} \phi(p, x_n)$ exists. This shows that $\lim_{n \rightarrow \infty} (\phi(p, x_{n+1}) - \phi(p, x_n)) = 0$ and hence

$$\lim_{n \rightarrow \infty} (\phi(p, y_n) - \phi(p, x_n)) = 0. \quad (3.23)$$

Since T_{r_n} is of firmly nonexpansive type, by Remark 2.10, we have

$$\phi(y_n, p) + \phi(p, y_n) + \phi(y_n, x_n) + \phi(T_{r_n}p, p) \leq \phi(y_n, p) + \phi(p, x_n), \quad (3.24)$$

which implies

$$\phi(p, y_n) + \phi(y_n, x_n) \leq \phi(p, x_n). \quad (3.25)$$

Hence

$$\phi(y_n, x_n) \leq \phi(p, x_n) - \phi(p, y_n) \longrightarrow 0 \quad (3.26)$$

as $n \rightarrow \infty$. By Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.27)$$

Proposition 3.1 yields that

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle \leq 0. \quad (3.28)$$

It also follows that

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, y_n - p \rangle \leq 0. \quad (3.29)$$

Finally, we show that $x_n \rightarrow p$. Using Lemma 2.7, we see that

$$\begin{aligned} \phi(p, x_{n+1}) &\leq \phi(p, z_n) \\ &= V(p, \alpha_n Jx + (1 - \alpha_n) Jy_n) \\ &\leq V(p, \alpha_n Jx + (1 - \alpha_n) Jy_n - \alpha_n (Jx - Jp)) + \langle \alpha_n (Jx - Jp), z_n - p \rangle \\ &= V(p, \alpha_n Jp + (1 - \alpha_n) Jy_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &\leq \alpha_n V(p, Jp) + (1 - \alpha_n) V(p, Jy_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &= (1 - \alpha_n) \phi(p, y_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &\leq (1 - \alpha_n) \phi(p, x_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &= (1 - \alpha_n) \phi(p, x_n) + \alpha_n (\langle Jx - Jp, z_n - y_n \rangle + \langle Jx - Jp, y_n - p \rangle). \end{aligned} \quad (3.30)$$

Set $a_n = \phi(p, x_n)$ and $b_n = \alpha_n (\langle Jx - Jp, z_n - y_n \rangle + \langle Jx - Jp, y_n - p \rangle)$. We see that $\limsup_{n \rightarrow \infty} b_n / \alpha_n \leq 0$. By Lemma 2.11, since $\sum_{n=1}^{\infty} \alpha_n = +\infty$, we conclude that $\lim_{n \rightarrow \infty} \phi(p, x_n) = 0$. Hence $x_n \rightarrow p$ as $n \rightarrow \infty$.

Case 2. There exists a subsequence $\{\phi(p, x_{n_j})\}$ of $\{\phi(p, x_n)\}$ such that $\phi(p, x_{n_j}) < \phi(p, x_{n_j+1})$ for all $j \in \mathbb{N}$. By Lemma 2.12, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1}), \quad \phi(p, x_k) \leq \phi(p, x_{m_k+1}). \quad (3.31)$$

So we have

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} (\phi(p, x_{m_k+1}) - \phi(p, x_{m_k})) \\ &\leq \limsup_{n \rightarrow \infty} (\phi(p, x_{n+1}) - \phi(p, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\phi(p, z_n) - \phi(p, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n \phi(p, x) + (1 - \alpha_n) \phi(p, y_n) - \phi(p, x_n)) \\ &= \limsup_{n \rightarrow \infty} (\alpha_n (\phi(p, x) - \phi(p, y_n)) + (\phi(p, y_n) - \phi(p, x_n))) \\ &\leq \limsup_{n \rightarrow \infty} \alpha_n (\phi(p, x) - \phi(p, y_n)) = 0. \end{aligned} \quad (3.32)$$

This shows that

$$\lim_{k \rightarrow \infty} (\phi(p, x_{m_k+1}) - \phi(p, x_{m_k})) = 0. \quad (3.33)$$

Following the proof line in Case 1, we can show that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle Jx - Jp, y_{m_k} - p \rangle &\leq 0, \\ \phi(p, x_{m_k+1}) &\leq (1 - \alpha_{m_k})\phi(p, x_{m_k}) + \alpha_{m_k}(\langle Jx - Jp, z_{m_k} - y_{m_k} \rangle + \langle Jx - Jp, y_{m_k} - p \rangle). \end{aligned} \quad (3.34)$$

This implies

$$\begin{aligned} \alpha_{m_k}\phi(p, x_{m_k}) &\leq \phi(p, x_{m_k}) - \phi(p, x_{m_k+1}) \\ &\quad + \alpha_{m_k}(\langle Jx - Jp, z_{m_k} - y_{m_k} \rangle + \langle Jx - Jp, y_{m_k} - p \rangle) \\ &\leq \alpha_{m_k}(\langle Jx - Jp, z_{m_k} - y_{m_k} \rangle + \langle Jx - Jp, y_{m_k} - p \rangle). \end{aligned} \quad (3.35)$$

Hence $\lim_{k \rightarrow \infty} \phi(p, x_{m_k}) = 0$. Using this and (3.33) together, we conclude that

$$\limsup_{k \rightarrow \infty} \phi(p, x_k) \leq \lim_{k \rightarrow \infty} \phi(p, x_{m_k+1}) = 0. \quad (3.36)$$

This completes the proof. □

As a direct consequence of Theorem 3.2, we obtain the following results.

Corollary 3.3. *Let C be nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\text{MEP}(f, \varphi) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and*

$$\begin{aligned} f(y_n, y) + \varphi(y) + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.37)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{MEP}(f, \varphi)} x$, where $\Pi_{\text{MEP}(f, \varphi)}$ is the generalized projection of C onto $\text{MEP}(f, \varphi)$.

Corollary 3.4. Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and $A : C \rightarrow E^*$ a continuous and monotone mapping such that $\text{GEP}(f, A) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$\begin{aligned} f(y_n, y) + \langle Ay_n, y - y_n \rangle + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.38)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{GEP}(f, A)} x$, where $\Pi_{\text{GEP}(f, A)}$ is the generalized projection of C onto $\text{GEP}(f, A)$.

Corollary 3.5. Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\text{VI}(C, A, \varphi) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$\begin{aligned} \langle Ay_n, y - y_n \rangle + \varphi(y) + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.39)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{VI}(C, A, \varphi)} x$, where $\Pi_{\text{VI}(C, A, \varphi)}$ is the generalized projection of C onto $\text{VI}(C, A, \varphi)$.

4. Examples and Numerical Results

In this section, we give examples and numerical results for our main theorem.

Example 4.1. Let $E = \mathbb{R}$ and $C = [-1, 1]$. Let $f(x, y) = -9x^2 + xy + 8y^2$, $\varphi(x) = 3x^2$, and $Ax = 2x$. Find $\hat{x} \in [-1, 1]$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in [-1, 1]. \quad (4.1)$$

Solution. It is easy to check that f , φ , and A satisfy all conditions in Theorem 3.2. For each $r > 0$ and $x \in [-1, 1]$, Lemma 2.9 ensures that there exists $z \in [-1, 1]$ such that, for any $y \in [-1, 1]$,

$$\begin{aligned} f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r} \langle z - x, y - z \rangle &\geq \varphi(z) \\ \iff -9z^2 + yz + 8y^2 + 2z(y - z) + 3y^2 + \frac{1}{r}(z - x)(y - z) &\geq 3z^2 \\ \iff 11ry^2 + (3rz + z - x)y - (14rz^2 + z^2 - xz) &\geq 0. \end{aligned} \quad (4.2)$$

Put $G(y) = 11ry^2 + (3rz + z - x)y - (14rz^2 + z^2 - xz)$. Then G is a quadratic function of y with coefficient $a = 11r$, $b = (3rz + z - x)$, and $c = -(14rz^2 + z^2 - xz)$. We next compute the discriminant Δ of G as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= [(3r + 1)z - x]^2 + 44r(14rz^2 + z^2 - xz) \\ &= x^2 - 2(3r + 1)xz + (3r + 1)^2z^2 + 616r^2z^2 + 44rz^2 - 44rxz \\ &= x^2 - 50rxz - 2xz + 625r^2z^2 + 50rz^2 + z^2 \\ &= x^2 - 2(25rz + z)x + (625r^2z^2 + 50rz^2 + z^2) \\ &= [x - (25rz + z)]^2. \end{aligned} \quad (4.3)$$

We know that $G(y) \geq 0$ for all $y \in [-1, 1]$ if it has at most one solution in $[-1, 1]$. So $\Delta \leq 0$ and hence $x = 25rz + z$. Now we have $z = T_r x = x / (25r + 1)$.

Let $\{x_n\}_{n=1}^\infty$ be the sequence generated by $x_1 = x \in [-1, 1]$ and

$$\begin{aligned} f(y_n, y) + \langle Ay_n, y - y_n \rangle + \varphi(y) + \frac{1}{r_n} \langle y_n - x_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in [-1, 1], \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n)y_n, \quad \forall n \geq 1, \end{aligned} \quad (4.4)$$

and, equivalently,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_{r_n} x_n, \quad \forall n \geq 1. \quad (4.5)$$

We next give two numerical results for algorithm (4.5).

Algorithm 4.2. Let $\alpha_n = 1/80n$ and $r_n = n/(n + 1)$. Choose $x_1 = x = 1$. Then algorithm (4.5) becomes

$$x_{n+1} = \frac{1}{80n} + \left(1 - \frac{1}{80n}\right) \left(\frac{n+1}{26n+1}\right) x_n, \quad \forall n \geq 1. \quad (4.6)$$

Table 1

n	x_n
1	1.0000
2	0.0856
3	0.0111
4	0.0047
5	0.0033
$\vdots$	$\vdots$
261	0.0001
262	0.0000

Table 2

n	x_n
1	-1.0000
2	-0.0481
3	-0.0074
4	-0.0038
5	-0.0027
$\vdots$	$\vdots$
217	-0.0001
218	0.0000

Numerical Result I

See Table 1.

Algorithm 4.3. Let $\alpha_n = 1/100n$ and $r_n = (n+1)/2n$. Choose $x_1 = x = -1$. Then algorithm (4.5) becomes

$$x_{n+1} = -\frac{1}{100n} + \left(1 - \frac{1}{100n}\right) \left(\frac{2n}{27n+25}\right) x_n, \quad \forall n \geq 1. \quad (4.7)$$

Numerical Result II

See Table 2.

5. Conclusion

Tables 1 and 2 show that the sequence $\{x_n\}$ converges to 0 which solves the generalized mixed equilibrium problem. On the other hand, using Lemma 2.9(3), we can check that $\text{GMEP}(f, A, \varphi) = F(T_r) = \{0\}$.

Remark 5.1. In the view of computation, our algorithm is simple in order to get strong convergence for generalized mixed equilibrium problems.

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2.3 CONVERGENCE THEOREMS OF HYBRID METHODS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF AN INFINITE FAMILY OF LIPSCHITZIAN QUASI-NONEXPANSIVE MAPPINGS IN HILBERT SPACES

In this section, we use a hybrid iterative method to find a common element of the set of fixed points of an infinite family of lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in real Hilbert spaces. We also show that our main strong convergence theorem for finding that common element can be deduced for nonexpansive mappings and applied for strict pseudo-contraction mappings.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H . Let $B : C \rightarrow H$ be a nonlinear mapping, $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function and $f : C \times C \rightarrow \mathbb{R}$ be a bifunction. Peng and Yao [10] considered the following *generalized mixed equilibrium problem*:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \quad (1.1)$$

In this paper, we denote the set of solutions of (1.1) by $GMEP(f, \varphi, B)$. It is obvious that if u is a solution of (1.1), it implies that $u \in \text{dom } \varphi = \{u \in C : \varphi(u) < +\infty\}$.

If $B = 0$ in (1.1), we obtain the following *mixed equilibrium problem* [3]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) \geq \varphi(u), \quad \forall y \in C. \quad (1.2)$$

We denote the set of solutions of (1.2) by $MEP(f, \varphi)$.

If $\varphi = 0$ in (1.1), we obtain the following *generalized equilibrium problem* [16]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \quad (1.3)$$

We denote the set of solutions of (1.3) by $GEP(f, B)$.

If $\varphi = 0$ and $B = 0$ in (1.1), we obtain the following *equilibrium problem* [2]:

$$\text{Finding } u \in C \text{ such that } f(u, y) \geq 0, \quad \forall y \in C. \quad (1.4)$$

We denote the set of solutions of (1.4) by $EP(f)$.

If $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *generalized variational inequality problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \quad (1.5)$$

We denote the set of solutions of (1.5) by $GVI(C, \varphi, B)$.

If $\varphi = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *variational inequality problem* (see also [1, 5]):

$$\text{Finding } u \in C \text{ such that } \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \quad (1.6)$$

We denote the set of solutions of (1.6) by $VI(C, B)$.

If $B = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *minimization problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) \geq \varphi(u), \quad \forall y \in C. \quad (1.7)$$

We denote the set of solutions of (1.7) by $MP(C, \varphi)$.

In 1994, Blum and Oettli showed that the formulation of (1.4) covered monotone inclusive problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems and Nash equilibria in noncooperative games. Several problems in physics, optimization and economics can be reduced to find solutions of (1.4). The existence of equilibrium problems has been discovered by many authors (see, for example, [1, 6, 8, 15] and the references therein). Also, some solution methods have been studied by some authors (see, for example, [6, 15, 13] and the references therein).

In 2003, Takahashi and Toyoda [16] introduced the method for finding an element of $F(S) \cap VI(C, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \rightarrow H$ is a nonexpansive mapping and $A : C \rightarrow H$ is an inverse-strongly monotone mapping. Their iteration is the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), n \geq 0,$$

where $x_0 \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and P_C is the metric projection from H onto C . They proved that, if $F(S) \cap VI(C, A) \neq \emptyset$, $\{x_n\}$ converges weakly to a point $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap VI(C, A)} x_n$.

Later, Takahashi and Takahashi [15] studied the contraction method for finding $F(S) \cap EP(f)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \rightarrow H$

is a nonexpansive mapping, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions. Their algorithm is the following:

$$\begin{aligned} x_1 &\in H, \\ f(y_n, u) + \frac{1}{r_n} \langle u - y_n, y_n - x_n \rangle &\geq 0, \quad \forall u \in C, \\ x_{n+1} &= \alpha_n f_1(x_n) + (1 - \alpha_n) S y_n \quad \forall n \geq 1, \end{aligned}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and some appropriate conditions. They proved that, if $F(S) \cap EP(f) \neq \emptyset$, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in F(S) \cap EP(f)$, where $z = P_{F(S) \cap EP(f)} f(z)$.

Recently, Cho et al. [4] introduced a hybrid projection method for finding $F := F(S) \cap VI(C, B) \cap GEP(f, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \rightarrow C$ is a k -strict pseudo-contraction with a fixed point, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions, $A : C \rightarrow H$ is an α -inverse-strongly monotone mapping and $B : C \rightarrow H$ is an β -inverse-strongly monotone mapping. Their iterative scheme is the following:

$$\begin{aligned} x_1 &\in C, \\ C_1 &= C, \\ f(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ z_n &= P_C(u_n - \lambda_n B u_n), \\ y_n &= \alpha_n x_n + (1 - \alpha_n) S_k z_n, \\ C_{n+1} &= \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{aligned}$$

where $S_k x = kx + (1 - k)Sx$ for all $x \in C$, $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0, 2\alpha)$ and some appropriate conditions. They proved that, if $F \neq \emptyset$, $\{x_n\}$ converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

In this paper, motivated by the above result, we prove a strong convergent theorem of a hybrid projection iterative method defined by (3.1) for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces. Our main result can be deduced for nonexpansive mappings applied for strict pseudo-contraction mappings. It is clear that our result generalizes the work by [4].

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H . Let $B : C \rightarrow H$. $x_n \rightarrow x$ means $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ implies $\{x_n\}$ converges weakly to x . We denote the set of fixed points of T by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.

Recall the following definitions:

- (1) A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(2) A mapping $T : C \rightarrow C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \in F(P).$$

(3) A mapping $T : C \rightarrow C$ is said to be *Lipschitzian* if there is a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

(4) A mapping $T : C \rightarrow C$ is said to be *strictly pseudo-contractive* with the coefficient $k \in [0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

(5) B is said to be *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(6) B is said to be α -*strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(7) B is said to be α -*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

(8) A set-valued mapping $T : H \rightarrow 2^H$ is said to be *monotone* if, for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$.

(9) A monotone mapping $T : H \rightarrow 2^H$ is said to be *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

In the other words, a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $B : C \rightarrow H$ be a monotone mapping and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \quad \forall u \in C\}$. Define a mapping T on C by

$$Tv = \begin{cases} Bv + N_C v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C. \end{cases} \quad (2.1)$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Bv, u - v \rangle \geq 0$ for all $u \in C$ (see [14]).

Let B be a β -inverse-strongly monotone mapping of C into H . It is easy to show that B is $\frac{1}{\beta}$ -Lipschitz. For $\lambda \in (0, 2\beta]$, it is known that $I - \lambda B$ is a nonexpansive mapping of C into H .

Let C be a nonempty closed convex subset of H . Therefore, for any $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H on C . We know that for $x \in H$ and $z \in C, z = P_C x$ is equivalent to $\langle x - z, z - y \rangle \geq 0$ for all $y \in C$. It is also known that a Hilbert space H satisfies the *Opial condition*, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightarrow x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ and $x \neq y$.

Let A be a monotone mapping from C into H . In the context of the variational inequality problem, it is known that

$$u \in \text{VI}(C, A) \Rightarrow u = P_C(u - \lambda Au), \quad \text{for all } \lambda > 0,$$

and

$$u = P_C(u - \lambda Au), \quad \text{for some } \lambda > 0 \Rightarrow u \in \text{VI}(C, A).$$

Let C be a nonempty closed subset of a Hilbert space H . Let $\{T_n\}$ and Γ be two families of nonlinear mappings of C into itself such that $F(\Gamma) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(\Gamma) = \bigcap_{T \in \Gamma} F(T)$. $\{T_n\}$ is said to satisfy the *NST-condition* ([9]) with Γ if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \implies \lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \text{ for all } T \in \Gamma.$$

In the case $\Gamma \in \{T\}$, i.e., Γ consists of one mapping T , $\{T_n\}$ is said to satisfy the NST-condition with T .

For solving the generalized mixed equilibrium problem, we may assume the following conditions for the bifunction f , the function φ and the set C :

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous;
- (A4) for each $x \in C$, $y \mapsto f(x, y)$ is weakly upper semicontinuous;
- (B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \text{dom}(\varphi)$ such that for any $z \in C \setminus D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z);$$

- (B2) C is a bounded set.

The following lemmas are useful for proving some convergence results in next two sections.

Lemma 2.1. ([10, 11, 12]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \quad \forall y \in C\}$$

for all $x \in H$. Then the following conclusions hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle;$$

- (3) $F(T_r) = \text{MEP}(F, \varphi)$;
- (4) $\text{MEP}(F, \varphi)$ is closed and convex.

Lemma 2.2. ([7]). *Let $A : C \rightarrow H$ be α -inverse-strongly monotone. If $r \in (0, 2\alpha)$, then we have $I - rA$ is nonexpansive.*

Lemma 2.3. ([17]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ be a k -strict pseudo-contraction. Define a mapping $S : C \rightarrow C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

We would like to mention the following remark since our result is very interesting. It shows that a monotone mapping maps all points in a generalized mixed equilibrium problem to the same point.

Remark 2.4. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A2) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. Let A be a monotone mapping of C into H . Then $Au = Av$ for all $u, v \in GMEP(f, \varphi, A)$.

Proof. Let $u, v \in GMEP(f, \varphi, A)$. We then get

$$f(u, y) + \varphi(y) + \langle Au, y - u \rangle \geq \varphi(u), \quad \forall y \in C \quad (2.2)$$

and

$$f(v, y) + \varphi(y) + \langle Av, y - v \rangle \geq \varphi(v), \quad \forall y \in C. \quad (2.3)$$

By letting $y = v$ in (2.2) and $y = u$ in (2.3), we get

$$f(u, v) + \varphi(v) + \langle Au, v - u \rangle \geq \varphi(u) \quad (2.4)$$

and

$$f(v, u) + \varphi(u) + \langle Av, u - v \rangle \geq \varphi(v). \quad (2.5)$$

By (2.4), (2.5) and the condition (A2), we have

$$\langle Av - Au, u - v \rangle \geq f(u, v) + f(v, u) + \langle Au, v - u \rangle + \langle Av, u - v \rangle \geq 0. \quad (2.6)$$

From A is a α -inverse-strongly monotone mapping,

$$0 \leq \alpha \|Au - Av\|^2 \leq \langle Au - Av, u - v \rangle \leq 0.$$

That is $Au = Av$. □

By letting $f = 0$ and $\varphi = 0$ in Lemma 2.4, we obtain the following remark.

Remark 2.5. Let C be a closed convex subset of a real Hilbert space H and A be a monotone mapping of C into H . Then $Au = Av$ for all $u, v \in VI(C, A)$.

3. MAIN RESULT

In this section, we show a strong convergent theorem of hybrid methods for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces under some appropriate conditions.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of Lipschitzian quasi-nonexpansive mappings of C into itself such that $\lim_{n \rightarrow \infty} \|S_n x - S_n y\| \leq L_n \|x - y\|$ for all $x, y \in C$, $\sup_n L_n = L$, $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \quad (3.1)$$

where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Proof. We divide our proof into 5 steps.

Step 1: We show that $F \subset C_n$ and C_n is closed and convex for all $n \geq 1$.

From the assumption, $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \geq 1$. Next, we show that C_{m+1} is closed and convex. For any $w \in C_m$, we see that

$$\|y_m - w\| \leq \|x_m - w\|$$

is equivalent to

$$\|x_m\|^2 - \|y_m\|^2 - 2\langle w, x_m - y_m \rangle \geq 0.$$

Therefore, C_{m+1} is closed and convex.

Since A is α -inverse-strongly monotone and B is β -inverse-strongly monotone, by Lemma 2.2, we get that $I - r_n A$ and $I - \lambda_n B$ are nonexpansive.

By nonexpansiveness of T_{r_n} and $I - r_n A$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(p - r_n A p)\|^2 \\ &\leq \|(x_n - r_n A x_n) - (p - r_n A p)\|^2 \\ &= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\ &= \|x_n - p\|^2 - 2r_n \langle x_n - p, Ax_n - Ap \rangle + r_n^2 \|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n \alpha \|Ax_n - Ap\|^2 + r_n^2 \|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 + r_n(r_n - 2\alpha) \|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

We are now ready to show that $F \subset C_n$ for each $n \geq 1$. From the assumption, we have that $F \subset CC_1$. Suppose $F \subset C_m$ for some $m \geq 1$. For any $w \in F \subset C_m$, by nonexpansiveness of $I - \lambda_m B$, we have

$$\begin{aligned} \|y_m - w\| &= \|\alpha_m x_m + (1 - \alpha_m) S_m z_m - w\| \\ &\leq \alpha_m \|x_m - w\| + (1 - \alpha_m) \|z_m - w\| \\ &= \alpha_m \|x_m - w\| + (1 - \alpha_m) \|P_C(I - \lambda_m B)u_m - P_C(I - \lambda_m B)w\| \\ &\leq \alpha_m \|x_m - w\| + (1 - \alpha_m) \|u_m - w\| \\ &\leq \alpha_m \|x_m - w\| + (1 - \alpha_m) \|x_m - w\| \\ &= \|x_m - w\|. \end{aligned}$$

That is $w \in C_{m+1}$. By mathematical induction, we conclude that $F \subset C_n$ for each $n \geq 1$.

Step 2: We show that $\{x_n\}$ is bounded.

Since $x_n = P_{C_n} x_1$ and $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we get

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\| \|x_1 - x_{n+1}\|. \end{aligned} \tag{3.3}$$

Thus

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|. \quad (3.4)$$

Since $x_n = P_{C_n}x_1$, for any $w \in F \subset C_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - w\|. \quad (3.5)$$

In particular, we obtain

$$\|x_1 - x_n\| \leq \|x_1 - P_Fx_1\|. \quad (3.6)$$

By (3.4) and (3.6), we get that $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists. It implies that $\{x_n\}$ is bounded.

Step 3: We show that $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$ for all $S \in \mathcal{S}$.

By using (3.3), we obtain that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2 \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0. \quad (3.7)$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$$

and then

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|.$$

By (3.7), we get

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.8)$$

On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n)S_n z_n\| = (1 - \alpha_n)\|x_n - S_n z_n\|.$$

It follows from (3.8) and the assumption $0 \leq \alpha_n \leq a < 1$ that

$$\lim_{n \rightarrow \infty} \|x_n - S_n z_n\| = 0. \quad (3.9)$$

For any $w \in F$, we have

$$\begin{aligned} \|y_n - w\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)S_n z_n - w\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|S_n z_n - w\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|z_n - w\|^2. \end{aligned} \quad (3.10)$$

From (3.2), we obtain

$$\begin{aligned} \|y_n - w\|^2 &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|P_C(I - \lambda_n B)u_n - w\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|(I - \lambda_n B)u_n - (I - \lambda_n B)w\|^2 \\ &= \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|u_n - w\|^2 + \lambda_n^2 \|Bu_n - Bw\|^2 \\ &\quad - 2\lambda_n \langle u_n - w, Bu_n - Bw \rangle) \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|u_n - w\|^2 + \lambda_n(\lambda_n - 2\beta) \|Bu_n - Bw\|^2) \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|x_n - w\|^2 + \lambda_n(\lambda_n - 2\beta) \|Bu_n - Bw\|^2) \end{aligned}$$

$$\leq \|x_n - w\|^2 + (1 - \alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2.$$

We then have

$$\begin{aligned} (1 - a)b(2\beta - c)\|Bu_n - Bw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|). \end{aligned}$$

By (3.8), we obtain that

$$\lim_{n \rightarrow \infty} \|Bu_n - Bw\| = 0. \quad (3.11)$$

On the other hand, since P_C is firmly nonexpansive and $I - \lambda_n B$ is nonexpansive, we have

$$\begin{aligned} \|z_n - w\|^2 &= \|P_C(I - \lambda_n B)u_n - P_C(I - \lambda_n B)w\|^2 \\ &\leq \langle (I - \lambda_n B)u_n - (I - \lambda_n B)w, z_n - w \rangle \\ &= \frac{1}{2} \{ \|(I - \lambda_n B)u_n - (I - \lambda_n B)w\|^2 + \|z_n - w\|^2 \\ &\quad - \|(I - \lambda_n B)u_n - (I - \lambda_n B)w - (z_n - w)\|^2 \} \\ &\leq \frac{1}{2} \{ \|u_n - w\|^2 + \|z_n - w\|^2 \\ &\quad - \|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2 \} \\ &= \frac{1}{2} \{ \|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2\lambda_n \langle u_n - z_n, Bu_n - Bw \rangle - \lambda_n^2 \|Bu_n - Bw\|^2 \}. \end{aligned} \quad (3.12)$$

From (3.2), it implies that

$$\begin{aligned} \|z_n - w\|^2 &\leq \|u_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle u_n - z_n, Bu_n - Bw \rangle \\ &\quad - \lambda_n^2 \|Bu_n - Bw\|^2 \\ &\leq \|x_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Bu_n - Bw\|. \end{aligned} \quad (3.13)$$

By (3.10) and (3.13), we get

$$\begin{aligned} (1 - \alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\quad + 2(1 - \alpha_n)\lambda_n \|u_n - z_n\| \|Bu_n - Bw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad + 2(1 - \alpha_n)\lambda_n \|u_n - z_n\| \|Bu_n - Bw\|. \end{aligned}$$

By (3.8), (3.11) and the assumption $0 \leq \alpha_n \leq a < 1$, we get

$$\lim_{n \rightarrow \infty} \|u_n - z_n\| = 0. \quad (3.14)$$

Also, by (3.10) and (3.12), we obtain that

$$\begin{aligned} \|y_n - w\|^2 &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|u_n - w\|^2 \\ &= \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(w - r_n Aw)\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|(x_n - r_n Ax_n) - (w - r_n Aw)\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|x_n - w\|^2 - 2r_n \langle x_n - w, Ax_n - Aw \rangle \\ &\quad + r_n^2 \|Ax_n - Aw\|^2) \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|x_n - w\|^2 \\ &\quad + r_n(r_n - 2\alpha) \|Ax_n - Aw\|^2) \\ &\leq \|x_n - w\|^2 + (1 - \alpha_n) r_n(r_n - 2\alpha) \|Ax_n - Aw\|^2. \end{aligned} \quad (3.15)$$

From the assumptions $0 \leq \alpha_n \leq a < 1$ and $0 < d \leq r_n \leq e < 2\alpha$, we have

$$\begin{aligned} (1-a)d(2\alpha-e)\|Ax_n - Aw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|). \end{aligned}$$

By (3.8), we obtain

$$\lim_{n \rightarrow \infty} \|Ax_n - Aw\| = 0. \quad (3.16)$$

On the other hand, by using Lemma 2.1, we have T_{r_n} is firmly nonexpansive. Then we get

$$\begin{aligned} \|u_n - w\|^2 &= \|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)w\|^2 \\ &\leq \langle (I - r_n A)x_n - (I - r_n A)w, u_n - w \rangle \\ &= \frac{1}{2}(\|(I - r_n A)x_n - (I - r_n A)w\|^2 + \|u_n - w\|^2 \\ &\quad - \|(I - r_n A)x_n - (I - r_n A)w - (u_n - w)\|^2) \\ &\leq \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2) \\ &= \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n \langle x_n - u_n, Ax_n - Aw \rangle - r_n^2 \|Ax_n - Aw\|^2) \end{aligned}$$

and so

$$\begin{aligned} \|u_n - w\|^2 &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n \langle x_n - u_n, Ax_n - Aw \rangle \\ &\quad - r_n^2 \|Ax_n - Aw\|^2 \\ &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Aw\|. \end{aligned} \quad (3.17)$$

By (3.15) and (3.17), we get

$$\begin{aligned} \|y_n - w\|^2 &\leq \|x_n - w\|^2 - (1 - \alpha_n) \|x_n - u_n\|^2 \\ &\quad + 2(1 - \alpha_n)r_n \|x_n - u_n\| \|Ax_n - Aw\|, \end{aligned}$$

which implies that

$$\begin{aligned} (1 - \alpha_n) \|x_n - u_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Aw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad + 2r_n \|x_n - u_n\| \|Ax_n - Aw\|. \end{aligned}$$

From the assumptions $0 \leq \alpha_n \leq a < 1$, $0 < d \leq r_n \leq e < 2\alpha$, (3.8) and (3.16), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.18)$$

On the other hand, we have

$$\begin{aligned} \|x_n - S_n x_n\| &\leq \|S_n x_n - S_n z_n\| + \|S_n z_n - x_n\| \\ &\leq L \|x_n - z_n\| + \|S_n z_n - x_n\| \\ &\leq L \|x_n - u_n\| + L \|u_n - z_n\| + \|S_n z_n - x_n\|. \end{aligned}$$

Using (3.9), (3.14) and (3.18), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \quad (3.19)$$

From the assumption $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$, we have

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0, \quad \forall S \in \mathcal{S}. \quad (3.20)$$

Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ .

Step 4: We show that $\xi \in F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A)$.

First, we show that $\xi \in F(\mathcal{S})$. Suppose that $\xi \neq S\xi$ for some $S \in \mathcal{S}$. From Opial's condition and (3.20), we obtain

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|x_{n_i} - \xi\| &< \liminf_{i \rightarrow \infty} \|x_{n_i} - S\xi\| \\ &= \liminf_{i \rightarrow \infty} \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - S\xi\| \\ &\leq \liminf_{i \rightarrow \infty} \|x_{n_i} - \xi\|, \end{aligned}$$

which give us a contradiction. Hence, $\xi \in F(\mathcal{S})$.

Now, we prove that $\xi \in VI(C, B)$. Let T be the maximal monotone mapping defined by (2.1):

$$Tx = \begin{cases} Bx + N_c x & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, we get $y - Bx \in N_C x$. By $z_n \in C$ and the definition of N_C , we have

$$\langle x - z_n, y - Bx \rangle \geq 0. \quad (3.21)$$

On the other hand, since $z_n = P_C(I - \lambda_n B)u_n$, we obtain

$$\langle x - z_n, z_n - (I - \lambda_n B)u_n \rangle \geq 0$$

and then

$$\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Bu_n \rangle \geq 0. \quad (3.22)$$

Since $u_n - z_n \rightarrow 0$ as $n \rightarrow \infty$, we have that

$$Bu_n - Bz_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.23)$$

From (3.21), (3.22) and the β -inverse monotonicity of B , we obtain

$$\begin{aligned} \langle x - z_{n_i}, y \rangle &\geq \langle x - z_{n_i}, Bx \rangle \\ &\geq \langle x - z_{n_i}, Bx \rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i} \rangle \\ &= \langle x - z_{n_i}, Bx - Bz_{n_i} \rangle + \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i} \rangle \\ &\quad - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i} \rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &= \langle x, Bz_{n_i} - Bu_{n_i} \rangle - \langle z_{n_i}, Bz_{n_i} - Bu_{n_i} \rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle x, Bz_{n_i} - Bu_{n_i} \rangle - \|z_{n_i}\| \|Bz_{n_i} - Bu_{n_i}\| \\ &\quad - \frac{1}{\lambda_{n_i}} \|x - z_{n_i}\| \|z_{n_i} - u_{n_i}\|. \end{aligned} \quad (3.24)$$

By (3.14) and (3.18), we get

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

Since $x_{n_i} \rightharpoonup \xi$, we obtain $z_{n_i} \rightharpoonup \xi$. From (3.14), (3.23) and (3.24), we obtain

$$\langle x - \xi, y \rangle = \lim_{n_i \rightarrow \infty} \langle x - z_{n_i}, y \rangle \geq 0.$$

Since T is maximal monotone, we obtain that $0 \in T\xi$. It follows that $\xi \in VI(C, B)$.

Next, we show that $\xi \in GMEP(f, \varphi, A)$. For any $y \in C$,

$$f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n).$$

From the condition (A2), we get that

$$\varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq f(y, u_n) + \varphi(u_n).$$

Replacing n by n_i , we obtain

$$\varphi(y) + \langle Ax_{n_i}, y - u_{n_i} \rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq f(y, u_{n_i}) + \varphi(u_{n_i}). \quad (3.25)$$

For any t with $0 < t \leq 1$ and $y \in C$, put $\rho_t = ty + (1 - t)\xi$. Since $y \in C$ and $\xi \in C$, we obtain $\rho_t \in C$. It follows from (3.25) and the monotonicity of A that

$$\begin{aligned} \langle \rho_t - u_{n_i}, A\rho_t \rangle &\geq \langle \rho_t - u_{n_i}, A\rho_t \rangle - \langle Ax_{n_i}, \rho_t - u_{n_i} \rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \\ &\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\ &= \langle \rho_t - u_{n_i}, A\rho_t - Au_{n_i} \rangle + \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle \\ &\quad - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\ &\geq \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \\ &\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t). \end{aligned} \quad (3.26)$$

Since A is Lipschitzian, by (3.18), we get $Au_{n_i} - Ax_{n_i} \rightarrow 0$ as $i \rightarrow \infty$. From (A3) and (3.26), we arrive at

$$\langle \rho_t - \xi, A\rho_t \rangle \geq f(\rho_t, \xi) + \varphi(\xi) - \varphi(\rho_t). \quad (3.27)$$

From (A1), (A3), (3.27) and convexity of φ , we have that

$$\begin{aligned} 0 &= f(\rho_t, \rho_t) \leq tf(\rho_t, y) + (1 - t)f(\rho_t, \xi) \\ &\leq tf(\rho_t, y) + (1 - t)(\langle \rho_t - \xi, A\rho_t \rangle + \varphi(\rho_t) - \varphi(\xi)) \\ &\leq tf(\rho_t, y) + (1 - t)t(\langle y - \xi, A\rho_t \rangle + \varphi(y) - \varphi(\xi)), \end{aligned}$$

which implies that

$$f(\rho_t, y) + (1 - t)(\langle y - \xi, A\rho_t \rangle + \varphi(y) - \varphi(\xi)) \geq 0.$$

Letting $t \rightarrow 0$, by (A4), we arrive at

$$f(\xi, y) + \langle y - \xi, A\xi \rangle + \varphi(y) - \varphi(\xi) \geq 0.$$

This shows that $\xi \in GMEP(f, \varphi, A)$.

Step 5: We show that $x_n \rightarrow P_F x_1$.

Let $\bar{x} = P_F x_1$. Since $\bar{x} = P_F x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_1$, we get

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \bar{x}\|.$$

On the other hand, we have

$$\|x_1 - \bar{x}\| \leq \|x_1 - \xi\|$$

$$\begin{aligned}
 &\leq \liminf_{i \rightarrow \infty} \|x_1 - x_{n_i}\| \\
 &\leq \limsup_{i \rightarrow \infty} \|x_1 - x_{n_i}\| \\
 &\leq \|x_1 - \bar{x}\|.
 \end{aligned}$$

Therefore, we get

$$\|x_1 - \xi\| = \lim_{i \rightarrow \infty} \|x_1 - x_{n_i}\| = \|x_1 - \bar{x}\|.$$

This implies that $\bar{x} = \xi$. Since H has the Kadec-Klee property and $x_1 - x_{n_i} \rightarrow x_1 - \bar{x}$, it follows that $x_{n_i} \rightarrow \bar{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. The proof is now complete. $\square$

4. DEDUCED THEOREMS AND APPLICATIONS

Theorem 3.1 can be reduced to many different results. By putting $S_n = S$ for all $n \geq 1$ in Theorem 3.1, we obtain the following theorem:

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $S : C \rightarrow C$ be a L -Lipschitzian quasi-nonexpansive mapping such that $F = F(S) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases}
 x_1 \in C, \\
 C_1 = C, \\
 f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \forall y \in C, \\
 z_n = P_C(u_n - \lambda_n B u_n), \\
 y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\
 C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\
 x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1,
 \end{cases} \quad (4.1)$$

where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

When $\{S_n\}$ and $\mathcal{S}$ are families of nonexpansive mappings, we get the following theorem:

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Now we show how to apply Theorem 4.2 for families of strict pseudo-contraction mappings.

Theorem 4.3. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $\{R_n\}$ and $\mathcal{R}$ be families of k -strict pseudo-contraction mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(R_n) = F(\mathcal{R})$ and $F = F(\mathcal{R}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S_n : C \rightarrow C$ by $S_n x = kx + (1-k)R_n x$ for all $x \in C$. Suppose that $\{R_n\}$ satisfies the NST-condition with $\mathcal{R}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Proof. By Lemma 2.3, we obtain that S_n is nonexpansive for all positive integer n . We also get that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S} = \{kI + (1-k)T : T \in \mathcal{R}\}$. The proof is now complete because of the direct result of Theorem 4.2. $\square$

By putting $S_n = S$ for all $n \geq 1$ in Theorem 4.3, we obtain the following corollary.

Corollary 4.4. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $R : C \rightarrow C$ be a k -strict pseudo-contraction such that $F = F(R) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S : C \rightarrow C$ by $Sx = kx + (1-k)Rx$ for all $x \in C$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm (4.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Remark 4.5. By letting $\varphi = 0$ in Corollary 4.4, we obtain Theorem 2.1 of [4].

Remark 4.6. Since Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 4.4 are for finding a common element of the set of fixed points, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem, we can reduce each theorem or corollary by letting $B = 0$, $A = 0$, $\varphi = 0$ or $f(x, y) = 0$.

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2.4 Strong Convergence Theorem for a New General System of Variational Inequalities in Banach Spaces

In this section, we introduce a new system of general variational inequalities in Banach spaces. The equivalence between this system of variational inequalities and fixed point problems concerning the nonexpansive mapping is established. By using this equivalent formulation, we introduce an iterative scheme for finding a solution of the system of variational inequalities in Banach spaces.

1. Introduction

Let X be a real Banach space, and X^* be its dual space. Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X . X is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$ there exists a constant $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (1.1)$$

The norm on X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (1.2)$$

exists for each $x, y \in U$ and in this case X is said to have a *uniformly Frechet differentiable norm* if the limit (1.2) is attained uniformly for $x, y \in U$ and in this case X is said to be *uniformly smooth*. We define a function $\rho : [0, \infty) \rightarrow [0, \infty)$, called the *modulus of smoothness* of X , as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2(\|x+y\| + \|x-y\|)} - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}. \quad (1.3)$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \rightarrow 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For $q > 1$, the generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in X. \quad (1.4)$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and usually, we write $J_2 = J$. If X is a Hilbert space, then $J = I$. Further, we have the following properties of the generalized duality mapping J_q :

- (1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$,
- (2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0, \infty)$,
- (3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is known that if X is smooth, then J is single-valued, which is denoted by j . Recall that the duality mapping j is said to be *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \rightarrow x$ weakly, we have $j(x_n) \rightarrow j(x)$ weakly-*. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth. For the details, see the work of Gossez and Lami Dozo in [1].

Let C be a nonempty closed convex subset of a smooth Banach space X . Recall that a mapping $A : C \rightarrow X$ is said to be *accretive* if

$$\langle Ax - Ay, j(x - y) \rangle \geq 0 \quad (1.5)$$

for all $x, y \in C$. A mapping $A : C \rightarrow X$ is said to be *α -strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2 \quad (1.6)$$

for all $x, y \in C$. A mapping $A : C \rightarrow X$ is said to be *α -inverse strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|Ax - Ay\|^2 \quad (1.7)$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Let D be a nonempty subset of C . A mapping $Q : C \rightarrow D$ is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx, \quad (1.8)$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \rightarrow D$ is called a *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a *sunny nonexpansive retraction* from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D . It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C .

Conveying an idea of the *classical variational inequality*, denoted by $VI(C, A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \quad (1.9)$$

where $X = H$ is a Hilbert space and A is a mapping from C into H . The variational inequality has been widely studied in the literature; see, for example, the work of Chang et al. in [2], Zhao and He [3], Plubtieng and Punpaeng [4], Yao et al. [5] and the references therein.

Let $A, B : C \rightarrow H$ be two mappings. In 2008, Ceng et al. [6] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.10)$$

which is called a *general system of variational inequalities*, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.10) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.11)$$

which is defined by Verma [7] and is called the *new system of variational inequalities*. Further, if we add up the requirement that $x^* = y^*$, then problem (1.11) reduces to the classical variational inequality $VI(C, A)$.

In 2006, Aoyama et al. [8] first considered the following generalized variational inequality problem in Banach spaces. Let $A : C \rightarrow X$ be an accretive operator. Find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \quad (1.12)$$

The problem (1.12) is very interesting as it is connected with the fixed point problem for nonlinear mapping and the problem of finding a zero point of an accretive operator in Banach spaces, see [9–11] and the references therein.

Aoyama et al. [8] introduced the following iterative algorithm in Banach spaces:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= Q_C(x_n - \lambda_n A)x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n)y_n, \quad n \geq 0, \end{aligned} \quad (1.13)$$

where Q_C is a sunny nonexpansive retraction from X onto C . Then they proved a weak convergence theorem which is generalized simultaneously theorems of Browder and Petryshyn [12] and Gol'shtein and Tret'yakov [13]. In 2008, Hao [14] obtained a strong convergence theorem by using the following iterative algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= b_n x_n + (1 - b_n)Q_C(I - \lambda_n A x_n), \\ x_{n+1} &= a_n u + (1 - a_n)y_n, \quad n \geq 0, \end{aligned} \quad (1.14)$$

where $\{a_n\}$, $\{b_n\}$ are two sequences in $(0, 1)$ and $u \in C$.

Very recently, in 2009, Yao et al. [5] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A, B : C \rightarrow X$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle Ay^* + x^* - y^*, j(x - x^*) \rangle &\geq 0, \quad \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*) \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.15)$$

which is called *the system of general variational inequalities in a real Banach space*. They proved a strong convergence theorem by using the following iterative algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= Q_C(x_n - Bx_n), \\ x_{n+1} &= a_n u + b_n x_n + c_n Q_C(y_n - Ay_n), \quad n \geq 0, \end{aligned} \quad (1.16)$$

where $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three sequences in $(0, 1)$ and $u \in C$.

In this paper, motivated and inspired by the idea of Yao et al. [5] and Cheng et al. [6]. First, we introduce the following system of variational inequalities in Banach spaces.

Let C be a nonempty closed convex subset of a real Banach space X . Let $A_i : C \rightarrow X$ for all $i = 1, 2, 3$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned} \langle \lambda_1 A_1 y^* + x^* - y^*, j(x - x^*) \rangle &\geq 0, \quad \forall x \in C, \\ \langle \lambda_2 A_2 z^* + y^* - z^*, j(x - y^*) \rangle &\geq 0, \quad \forall x \in C, \\ \langle \lambda_3 A_3 x^* + z^* - x^*, j(x - z^*) \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.17)$$

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which is called a *new general system of variational inequalities in Banach spaces*, where $\lambda_i > 0$ for all $i = 1, 2, 3$. In particular, if $A_3 = 0$, $z^* = x^*$, and $\lambda_i = 1$ for $i = 1, 2, 3$, then problem (1.17) reduces to problem (1.15). Further, if $A_3 = 0$, $z^* = x^*$, then problem (1.17) reduces to the problem (1.10) in a real Hilbert space. Second, we introduce iteration process for finding a solution of a new general system of variational inequalities in a real Banach space. Starting with arbitrary points $v, x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{aligned} z_n &= Q_C(x_n - \lambda_3 A_3 x_n), \\ y_n &= Q_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) Q_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1, \end{aligned} \tag{1.18}$$

where $\lambda_i > 0$ for all $i = 1, 2, 3$ and $\{a_n\}$, $\{b_n\}$ are two sequences in $(0, 1)$. Using the demiclosedness principle for nonexpansive mapping, we will show that the sequence $\{x_n\}$ converges strongly to a solution of a new general system of variational inequalities in Banach spaces under some control conditions.

2. Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Lemma 2.1 (see [15]). *Let X be a q -uniformly smooth Banach space with $1 \leq q \leq 2$. Then*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + 2 \|Ky\|^q \tag{2.1}$$

for all $x, y \in X$, where K is the q -uniformly smooth constant of X .

The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.2 (see [16, 17]). *Let C be a closed convex subset of a smooth Banach space X . Let D be a nonempty subset of C and $Q : C \rightarrow D$ be a retraction. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu) \rangle \leq 0, \tag{2.2}$$

for all $u \in C$ and $y \in D$.

The first result regarding the existence of sunny nonexpansive retractions on the fixed point set of a nonexpansive mapping is due to Bruck [18].

Remark 2.3. If X is strictly convex and uniformly smooth and if $T : C \rightarrow C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then there exists a sunny nonexpansive retraction of C onto $F(T)$.

Lemma 2.4 (see [19]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 1, \quad (2.3)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.5 (see [20]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.6 (see [21]). Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X , and $T : C \rightarrow C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if $x_n \rightarrow x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly, then $x \in F(T)$.

3. Main Results

In this section, we establish the equivalence between the new general system of variational inequalities (1.17) and some fixed point problem involving a nonexpansive mapping. Using the demiclosedness principle for nonexpansive mapping, we prove that the iterative scheme (1.18) converges strongly to a solution of a new general system of variational inequalities (1.17) in a Banach space under some control conditions. In order to prove our main result, the following lemmas are needed.

The next lemmas are crucial for proving the main theorem.

Lemma 3.1. Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let the mapping $A : C \rightarrow X$ be α -inverse strongly accretive. Then, we have

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2, \quad (3.1)$$

where K is the 2-uniformly smooth constant of X . In particular, if $\alpha \geq \lambda K^2$, then $I - \lambda A$ is a nonexpansive mapping.

Proof. Indeed, for all $x, y \in C$, from Lemma 2.1, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda \langle (Ax - Ay), j(x - y) \rangle \\ &\quad + 2K^2\lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2. \end{aligned} \quad (3.2)$$

It is clear that, if $\alpha \geq \lambda K^2$, then $I - \lambda A$ is a nonexpansive mapping. □

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Lemma 3.2. Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let Q_C be the sunny nonexpansive retraction from X onto C . Let $A_i : C \rightarrow X$ be an α_i -inverse strongly accretive mapping for $i = 1, 2, 3$. Let $G : C \rightarrow C$ be a mapping defined by

$$G(x) = Q_C[Q_C(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x)) - \lambda_1 A_1 Q_C(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x))], \quad \forall x \in C. \quad (3.3)$$

If $\alpha_i \geq \lambda_i K^2$ for all $i = 1, 2, 3$, then $G : C \rightarrow C$ is nonexpansive.

Proof. For all $x, y \in C$, we have

$$\begin{aligned} \|G(x) - G(y)\| &= \|Q_C[Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x)) \\ &\quad - Q_C[Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y))] \| \\ &\leq \|Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\quad - [Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y))] \| \\ &= \|(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)x \\ &\quad - (I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)y\|. \end{aligned} \quad (3.4)$$

From Lemma 3.1, we have $(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)$ is nonexpansive which implies by (3.4) that G is nonexpansive. $\square$

Lemma 3.3. Let C be a nonempty closed convex subset of a real smooth Banach space X . Let Q_C be the sunny nonexpansive retraction from X onto C . Let $A_i : C \rightarrow X$ be three nonlinear mappings. For given $(x^*, y^*, z^*) \in C \times C \times C$, (x^*, y^*, z^*) is a solution of problem (1.17) if and only if $x^* \in F(G)$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$, where G is the mapping defined as in Lemma 3.2.

Proof. Note that we can rewrite (1.17) as

$$\begin{aligned} \langle x^* - (y^* - \lambda_1 A_1 y^*), j(t - x^*) \rangle &\geq 0, \quad \forall t \in C, \\ \langle y^* - (z^* - \lambda_2 A_2 z^*), j(t - y^*) \rangle &\geq 0, \quad \forall t \in C, \\ \langle z^* - (x^* - \lambda_3 A_3 x^*), j(t - z^*) \rangle &\geq 0, \quad \forall t \in C. \end{aligned} \quad (3.5)$$

From Lemma 2.2, we can deduce that (3.5) is equivalent to

$$\begin{aligned} x^* &= Q_C(y^* - \lambda_1 A_1 y^*), \\ y^* &= Q_C(z^* - \lambda_2 A_2 z^*), \\ z^* &= Q_C(x^* - \lambda_3 A_3 x^*). \end{aligned} \quad (3.6)$$

It is easy to see that (3.6) is equivalent to $x^* = Gx^*$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. $\square$

From now on we denote by Ω^* the set of all fixed points of the mapping G . Now we prove the strong convergence theorem of algorithm (1.18) for solving problem (1.17).

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C . Let the mappings $A_i : C \rightarrow X$ be α_i -inverse strongly accretive with $\alpha_i \geq \lambda_i K^2$, for all $i = 1, 2, 3$ and $\Omega^* \neq \emptyset$. For given $x_1, v \in C$, let the sequence $\{x_n\}$ be generated iteratively by (1.18). Suppose the sequences $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0, 1)$ such that*

$$\begin{aligned} \text{(C1)} \quad & \lim_{n \rightarrow \infty} a_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty; \\ \text{(C2)} \quad & 0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $Q'v$ where Q' is the sunny nonexpansive retraction of C onto Ω^* .

Proof. Let $x^* \in \Omega^*$ and $t_n = Q_C(y_n - \lambda_1 A_1 y_n)$, it follows from Lemma 3.3 that

$$\begin{aligned} x^* &= Q_C[Q_C(Q_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 Q_C(x^* - \lambda_3 A_3 x^*)) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 Q_C(x^* - \lambda_3 A_3 x^*))]. \end{aligned} \quad (3.7)$$

Put $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. Then $x^* = Q_C(y^* - \lambda_1 A_1 y^*)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) t_n. \quad (3.8)$$

From Lemma 3.1, we have $I - \lambda_i A_i$ ($i = 1, 2, 3$) is nonexpansive. Therefore

$$\begin{aligned} \|t_n - x^*\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C(y^* - \lambda_1 A_1 y^*)\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(z_n - \lambda_2 A_2 z_n) - Q_C(z^* - \lambda_2 A_2 z^*)\| \\ &\leq \|z_n - z^*\| \\ &= \|Q_C(x_n - \lambda_3 A_3 x_n) - Q_C(x^* - \lambda_3 A_3 x^*)\| \\ &\leq \|x_n - x^*\|. \end{aligned} \quad (3.9)$$

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It follows that

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|a_n v + b_n x_n + (1 - a_n - b_n)t_n - x^*\| \\
 &\leq a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|t_n - x^*\| \\
 &\leq a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|x_n - x^*\| \\
 &= a_n \|v - x^*\| + (1 - a_n) \|x_n - x^*\|.
 \end{aligned} \tag{3.10}$$

By induction, we have

$$\|x_{n+1} - x^*\| \leq \max\{\|v - x^*\|, \|x_1 - x^*\|\}. \tag{3.11}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{A_1 y_n\}$, $\{A_2 z_n\}$, and $\{A_3 x_n\}$ are also bounded. By nonexpansiveness of Q_C and $I - \lambda_i A_i$ ($i = 1, 2, 3$), we have

$$\begin{aligned}
 \|t_{n+1} - t_n\| &= \|Q_C(y_{n+1} - \lambda_1 A_1 y_{n+1}) - Q_C(y_n - \lambda_1 A_1 y_n)\| \\
 &\leq \|y_{n+1} - y_n\| \\
 &= \|Q_C(z_{n+1} - \lambda_2 A_2 z_{n+1}) - Q_C(z_n - \lambda_2 A_2 z_n)\| \\
 &\leq \|z_{n+1} - z_n\| \\
 &= \|Q_C(x_{n+1} - \lambda_3 A_3 x_{n+1}) - Q_C(x_n - \lambda_3 A_3 x_n)\| \\
 &\leq \|x_{n+1} - x_n\|.
 \end{aligned} \tag{3.12}$$

Let $w_n = (x_{n+1} - b_n x_n) / (1 - b_n)$, $n \in \mathbb{N}$. Then $x_{n+1} = b_n x_n + (1 - b_n)w_n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
 w_{n+1} - w_n &= \frac{x_{n+2} - b_{n+1}x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\
 &= \frac{a_{n+1}v + (1 - a_{n+1} - b_{n+1})t_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n)t_n}{1 - b_n} \\
 &= \frac{a_{n+1}}{1 - b_{n+1}}(v - t_{n+1}) + \frac{a_n}{1 - b_n}(t_n - v) + t_{n+1} - t_n.
 \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have

$$\begin{aligned}
 \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| &\leq \frac{a_{n+1}}{1 - b_{n+1}} \|v - t_{n+1}\| + \frac{a_n}{1 - b_n} \|t_n - v\| \\
 &\quad + \|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \\
 &\leq \frac{a_{n+1}}{1 - b_{n+1}} \|v - t_{n+1}\| + \frac{a_n}{1 - b_n} \|t_n - v\|.
 \end{aligned} \tag{3.14}$$

This together with (C1) and (C2), we obtain that

$$\limsup_{n \rightarrow \infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq 0. \quad (3.15)$$

Hence, by Lemma 2.5, we get $\|x_n - w_n\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - b_n) \|w_n - x_n\| = 0. \quad (3.16)$$

Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n - b_n)(t_n - x_n), \quad (3.17)$$

therefore

$$\|t_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.18)$$

Furthermore, by Lemma 3.2, we have $G : C \rightarrow C$ is nonexpansive. Thus, we have

$$\begin{aligned} \|t_n - G(t_n)\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - G(t_n)\| \\ &= \|Q_C[Q_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 Q_C(z_n - \lambda_2 A_2 z_n)] - G(t_n)\| \\ &= \|Q_C[Q_C(Q_C(x_n - \lambda_3 A_3 x_n) - \lambda_2 A_2 Q_C(x_n - \lambda_3 A_3 x_n)) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(x_n - \lambda_3 A_3 x_n) - \lambda_2 A_2 Q_C(x_n - \lambda_3 A_3 x_n))] - G(t_n)\| \\ &= \|G(x_n) - G(t_n)\| \leq \|x_n - t_n\|, \end{aligned} \quad (3.19)$$

which implies $\|t_n - G(t_n)\| \rightarrow 0$ as $n \rightarrow \infty$.

Since

$$\begin{aligned} \|x_n - G(x_n)\| &\leq \|x_n - t_n\| + \|t_n - G(t_n)\| + \|G(t_n) - G(x_n)\| \\ &\leq \|x_n - t_n\| + \|t_n - G(t_n)\| + \|t_n - x_n\|, \end{aligned} \quad (3.20)$$

therefore

$$\lim_{n \rightarrow \infty} \|x_n - G(x_n)\| = 0. \quad (3.21)$$

Let Q' be the sunny nonexpansive retraction of C onto Ω^* . Now we show that

$$\limsup_{n \rightarrow \infty} \langle v - Q'v, j(x_n - Q'v) \rangle \leq 0. \quad (3.22)$$

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To prove (3.22), since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\bar{x}$ and

$$\limsup_{n \rightarrow \infty} \langle v - Q'v, j(x_n - Q'v) \rangle = \lim_{i \rightarrow \infty} \langle v - Q'v, j(x_{n_i} - Q'v) \rangle. \quad (3.23)$$

From Lemma 2.6 and (3.21), we obtain $\bar{x} \in \Omega^*$. Now, from Lemma 2.2, (3.23), and the weakly sequential continuity of the duality mapping j , we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle v - Q'v, j(x_n - Q'v) \rangle &= \lim_{i \rightarrow \infty} \langle v - Q'v, j(x_{n_i} - Q'v) \rangle \\ &= \langle v - Q'v, j(\bar{x} - Q'v) \rangle \leq 0. \end{aligned} \quad (3.24)$$

From (3.9), we have

$$\begin{aligned} \|x_{n+1} - Q'v\|^2 &= \langle a_nv + b_nx_n + (1 - a_n - b_n)t_n - Q'v, j(x_{n+1} - Q'v) \rangle \\ &= a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + b_n \langle x_n - Q'v, j(x_{n+1} - Q'v) \rangle \\ &\quad + (1 - a_n - b_n) \langle t_n - Q'v, j(x_{n+1} - Q'v) \rangle \\ &\leq a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + b_n (\|x_n - Q'v\| \|j(x_{n+1} - Q'v)\|) \\ &\quad + (1 - a_n - b_n) (\|t_n - Q'v\| \|j(x_{n+1} - Q'v)\|) \\ &= a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + b_n (\|x_n - Q'v\| \|x_{n+1} - Q'v\|) \\ &\quad + (1 - a_n - b_n) (\|t_n - Q'v\| \|x_{n+1} - Q'v\|) \\ &\leq a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + \frac{1}{2} b_n (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &\quad + \frac{1}{2} (1 - a_n - b_n) (\|t_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &\leq a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + \frac{1}{2} b_n (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &\quad + \frac{1}{2} (1 - a_n - b_n) (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &= a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + \frac{1}{2} (1 - a_n) (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2), \end{aligned} \quad (3.25)$$

which implies that

$$\|x_{n+1} - Q'v\|^2 \leq (1 - a_n) \|x_n - Q'v\|^2 + 2a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle. \quad (3.26)$$

It follows from Lemma 2.4, (3.24), and (3.26) that $\{x_n\}$ converges strongly to $Q'v$. This completes the proof. $\square$

Letting $A_3 = 0$ and $\lambda_i = 1$ for $i = 1, 2, 3$ in Theorem 3.4, we obtain the following result.

Corollary 3.5 (see [5, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C . Let the mappings $A_i : C \rightarrow X$ be α_i -inverse strongly accretive with $\alpha_i \geq K^2$, for all $i = 1, 2$ and $\Omega^* \neq \emptyset$. For given $x_1, v \in C$, and let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= Q_C(x_n - A_2 x_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) Q_C(y_n - A_1 y_n), \quad n \geq 1, \end{aligned} \quad (3.27)$$

where $\{a_n\}, \{b_n\}$ are two sequences in $(0, 1)$ such that

- (C1) $\lim_{n \rightarrow \infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;
 (C2) $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $Q'v$ where Q' is the sunny nonexpansive retraction of C onto Ω^* .

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2.5 An iterative method for equilibrium problems and a finite family of relatively nonexpansive mappings in a Banach space

In this section, we introduce a new iterative method for finding a common element of the set of fixed points of a finite family of relatively nonexpansive mappings and the set of solutions of an equilibrium problem in uniformly convex and uniformly smooth Banach spaces. Then we prove a strong convergence theorem by using the generalized projection.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E . Let C be a nonempty, closed and convex subset of E . Let f be a bifunction from $C \times C$ to the set of real numbers $\mathbb{R}$. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $EP(f)$. Let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T .

Numerous problems in physics, optimization and economics can be reduced to find a solution of the equilibrium problem (1.1) under the constraint on the common fixed points set of a family of mappings $\{T_i\}_{i=1}^m$. Such problems are equivalent to the problems of finding a common element of the solutions set of the equilibrium problem and the fixed points set of a family of mappings.

In this paper, we deal with a class of relatively nonexpansive mappings introduced by Matsushita and Takahashi [10]. There are a lot of important examples of relatively nonexpansive mappings such as the generalized projections, resolvents of a maximal monotone operators, and others; see [9,10].

In 2005, Matsushita and Takahashi [9] introduced an iterative process to approximate a fixed point of a single relatively nonexpansive mapping. They proved strong convergence theorems by using the generalized projection in a uniformly convex and uniformly smooth Banach space. Two years later, Plubtieng and Ungchittrakool in [14] generalized the previous iterative method with two relatively nonexpansive mappings.

In 2007, Tada and Takahashi [18] introduced strong convergence theorems for finding common elements of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Later, Takahashi and Zembayashi in [22] modified the iterative method to find a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a Banach space. Recently, the generalized equilibrium problem was also considered by Takahashi and Takahashi [21].

Motivated and inspired by the previous authors, we introduce a new iterative scheme and prove a strong convergence theorem of such a scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a finite family of relatively nonexpansive mappings in a Banach space. Our main result extends the results in [9,11,14,18,22] in several aspects.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E . The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \rightarrow \infty} \left\|\frac{x_n + y_n}{2}\right\| = 1$. Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *smooth* provided that $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in S$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in S$. Moreover, E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$. The norm of E is said to be *Fréchet differentiable*, if for each $x \in S$, the limit is attained uniformly for $y \in S$. It is well-known that if E is smooth, then the duality mapping J is single valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subsets of E , i.e., for a bounded set C of E and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow \|Jx - Jy\| < \varepsilon$$

for all $x, y \in C$; see [5,15–17,19] for more details. It is also known that, if E is uniformly convex, then E has the *Kadec–Klee property*, that is, $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ imply $x_n \rightarrow x$ for $x \in E$ where $x_n \rightharpoonup x$ and $x_n \rightarrow x$ denote the weak convergence and strong convergence of the sequence $\{x_n\}$ to x , respectively. Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$; see [19,20] for more details. Then, it is obvious by the definition that $\phi(x, y) \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$. Moreover, $\phi(x, y) = \|x - y\|^2$ for all x, y in a Hilbert space H . Let C be a closed and convex subset of E and let T be a mapping from C into itself. A mapping T is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and *relatively nonexpansive* if T satisfies the following conditions:

- (1) $F(T)$ is nonempty;
- (2) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$ and $x \in C$;
- (3) $\hat{F}(T) = F(T)$ where $\hat{F}(T)$ is the set of all asymptotic fixed points of T ; see [9,10,22] for more details.

The following lemmas are so useful in proving our main theorem.

Lemma 2.1 (Kamimura and Takahashi [7]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty, closed and convex subset of E . Suppose that E is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : E \rightarrow C$ defined by $\Pi_C x = x_0$ is called the *generalized projection* [1,2,7]. The following are well-known results. For example, see [1,2,7].

Lemma 2.2. [Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E , let $x \in E$ and let $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle y - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.3. [Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall x \in C \text{ and } y \in E.$$

Lemma 2.4. [Kamimura and Takahashi [7]] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.5. [Cho et al. [4]] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.6. [Nammanee et al. [12]] Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\|\alpha x + \beta y + \gamma z + \lambda v\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 + \lambda \|v\|^2 - \alpha \beta g(\|x - y\|)$$

for all $x, y, z, v \in B_r(0)$ and $\alpha, \beta, \gamma, \lambda \in [0, 1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

Next, we prove our lemmas.

Lemma 2.7. Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|),$$

where $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$.

Proof. We first observe that $\frac{\sum_{i=k+1}^m \alpha_i x_i}{1 - \sum_{i=1}^k \alpha_i} \in B_r(0)$ where $x_{k+1}, x_{k+2}, \dots, x_m \in B_r(0)$. Indeed,

$$\left\| \frac{\sum_{i=k+1}^m \alpha_i x_i}{1 - \sum_{i=1}^k \alpha_i} \right\| \leq \frac{\sum_{i=k+1}^m \alpha_i \|x_i\|}{1 - \sum_{i=1}^k \alpha_i} \leq \frac{\sum_{i=k+1}^m \alpha_i r}{1 - \sum_{i=1}^k \alpha_i} = r.$$

Therefore, $\frac{\sum_{i=k+1}^m \alpha_i x_i}{1 - \sum_{i=1}^k \alpha_i} \in B_r(0)$. We set $k = 3$ and then use Lemma 2.6. Thus, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned} \left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 &= \left\| \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \left(1 - \sum_{i=1}^3 \alpha_i\right) \left[\frac{\sum_{i=4}^m \alpha_i x_i}{1 - \sum_{i=1}^3 \alpha_i} \right] \right\|^2 \\ &\leq \sum_{i=1}^3 \alpha_i \|x_i\|^2 + \left(1 - \sum_{i=1}^3 \alpha_i\right) \left\| \frac{\sum_{i=4}^m \alpha_i x_i}{1 - \sum_{i=1}^3 \alpha_i} \right\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) \\ &\leq \sum_{i=1}^3 \alpha_i \|x_i\|^2 + \left(1 - \sum_{i=1}^3 \alpha_i\right) \left[\frac{\sum_{i=4}^m \alpha_i \|x_i\|^2}{1 - \sum_{i=1}^3 \alpha_i} \right] - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|), \end{aligned}$$

where $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$. $\square$

By interchanging the roles of vectors x_i in Lemma 2.7 and summing the inequalities together, we obtain the following lemma.

Lemma 2.8. Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1} \left[\sum_{i=1}^m \alpha_i g(\|x_j - x_i\|) \right]$$

for some $j \in \{1, 2, \dots, m\}$, $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$.

Proof. Now, we fix $\alpha_1 x_1$. Then, by Lemma 2.7, we also obtain

$$\begin{aligned} \|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_m x_m\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) \\ \|\alpha_1 x_1 + \alpha_3 x_3 + \alpha_4 x_4 + \dots + \alpha_2 x_2\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_3 g(\|x_1 - x_3\|) \\ \|\alpha_1 x_1 + \alpha_4 x_4 + \alpha_5 x_5 + \dots + \alpha_3 x_3\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_4 g(\|x_1 - x_4\|) \\ &\vdots \\ \|\alpha_1 x_1 + \alpha_m x_m + \alpha_2 x_2 + \dots + \alpha_{m-1} x_{m-1}\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_m g(\|x_1 - x_m\|). \end{aligned}$$

By combining the above inequalities, we have

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \frac{\alpha_1}{m-1} \left[\sum_{i=2}^m \alpha_i g(\|x_1 - x_i\|) \right].$$

In the same manner, we also have

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1} \left[\sum_{i=1}^m \alpha_i g(\|x_j - x_i\|) \right]$$

for some $j \in \{1, 2, \dots, m\}$. For $m = 4$, we obtain Lemma 4 of [13]. $\square$

Lemma 2.9. [Plubtieng and Ungchittrakool [14]] *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed convex subset of E . Then, for points $w, x, y, z \in E$ and a real number $a \in \mathbb{R}$, the set $K := \{v \in C : \phi(v, y) \leq \phi(v, x) + \langle v, Jz - Jw \rangle + a\}$ is closed and convex.*

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions; see [3,6,22]:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Lemma 2.10. [Takahashi and Zembayashi [22]] *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in E$. Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is a firmly nonexpansive-type mapping; see [8], i.e., for all $x, y \in E$,
 $\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, J x - J y \rangle$;
- (3) $F(T_r) = EP(f)$;
- (4) $EP(f)$ is closed and convex.

Lemma 2.11. [Takahashi and Zembayashi [22]] *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.12. [Matsushita and Takahashi [10]] *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3. Main result

In this section, we prove a strong convergence theorem.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty, closed and convex subset of E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Let $T_1, T_2, \dots, T_m$ be relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i) \cap EP(f) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1} \left(\sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right), \quad T_1 = I, \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, J x - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \dots, \end{cases}$$

where $\{\lambda_n^{(i)}\} \subset [0, 1]$, $\sum_{i=1}^m \lambda_n^{(i)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\liminf_{n \rightarrow \infty} \lambda_n^{(i)} \lambda_n^{(j)} > 0$ for some $j \in \{1, 2, \dots, m\}$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F .

Proof. First, we will show that the sequence $\{x_n\}$ is well-defined. By the definition of W_n , it is obvious that W_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.9, we also know that H_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $H_n \cap W_n$ is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.10 (4) and Lemma 2.12, we know that $F := \bigcap_{i=1}^m F(T_i) \cap EP(f)$ is also closed and convex.

Next, we will show that $F \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. Now, let $u \in F$ and put $u_n = T_{r_n} y_n$ for each $n \in \mathbb{N} \cup \{0\}$. We note by [22] that T_{r_n} are relatively nonexpansive. Then, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, T_{r_n} y_n) \leq \phi(u, y_n) = \phi \left(u, J^{-1} \left(\sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right) \right) = \|u\|^2 - 2 \left\langle u, \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \\ &= \|u\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle u, J T_i x_n \rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \leq \|u\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle u, J T_i x_n \rangle + \sum_{i=1}^m \lambda_n^{(i)} \|T_i x_n\|^2 \\ &= \sum_{i=1}^m \lambda_n^{(i)} \phi(u, T_i x_n) \leq \sum_{i=1}^m \lambda_n^{(i)} \phi(u, x_n) = \phi(u, x_n). \end{aligned}$$

Thus $u \in H_n$ and $F \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$. By utilizing Lemmas 2.2 and 2.3 and using the same argument presented in the proof of Theorem 3.1 in [22], we obtain that $F \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$ and hence the sequence $\{x_n\}$ is well-defined. Moreover, $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$. By Lemma 2.1, we also have $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x \in H_n$,

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \rightarrow 0$$

as $n \rightarrow \infty$. Then, we get that $\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (3.1)$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E , we obtain

$$\lim_{n \rightarrow \infty} \|J u_n - J x_n\| = 0. \quad (3.2)$$

Next, we prove that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, m$. Since $\{x_n\}$ is bounded, so are $\{T_i x_n\}$ for all $i = 1, 2, \dots, m$. Note that $T_1 = I$. Now, let $r = \sup_{n \geq 0} \{\|x_n\|, \|T_2 x_n\|, \dots, \|T_m x_n\|\}$ and let $j \in \{1, 2, \dots, m\}$. Note that E^* is uniformly convex. By Lemma 2.8, we have for each $p \in F$ that there exists a continuous, strictly increasing, and convex function $g^*: [0, \infty) \rightarrow [0, \infty)$ such that $g^*(0) = 0$ and

$$\begin{aligned} \phi(p, u_n) &= \phi(p, T_{r_n} y_n) \leq \phi(p, y_n) = \phi \left(p, J^{-1} \left(\sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right) \right) = \|p\|^2 - 2 \left\langle p, \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \\ &= \|p\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle p, J T_i x_n \rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \leq \|p\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle p, J T_i x_n \rangle \\ &\quad + \sum_{i=1}^m \lambda_n^{(i)} \|T_i x_n\|^2 - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^*(\|J T_j x_n - J T_i x_n\|) \right] = \sum_{i=1}^m \lambda_n^{(i)} \phi(p, T_i x_n) \\ &\quad - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^*(\|J T_j x_n - J T_i x_n\|) \right] \leq \sum_{i=1}^m \lambda_n^{(i)} \phi(p, x_n) - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^*(\|J T_j x_n - J T_i x_n\|) \right] \\ &= \phi(p, x_n) - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^*(\|J T_j x_n - J T_i x_n\|) \right]. \end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^*(\|JT_j x_n - JT_i x_n\|) \right] &\leq \phi(p, x_n) - \phi(p, u_n) = \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n \rangle \\ &\leq \|x_n\|^2 - \|u_n\|^2 + 2|\langle p, Jx_n - Ju_n \rangle| \leq \|x_n\| - \|u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Since $\liminf_{n \rightarrow \infty} \lambda_n^{(i)} \lambda_n^{(j)} > 0$ for all $i = 1, 2, \dots, m$, we obtain that

$$\lim_{n \rightarrow \infty} g^*(\|JT_j x_n - JT_i x_n\|) = 0$$

for all $i = 1, 2, \dots, m$. Since g^* is a continuous function with $g^*(0) = 0$, we have

$$\lim_{n \rightarrow \infty} \|JT_j x_n - JT_i x_n\| = 0$$

for all $i = 1, 2, \dots, m$. Since J is uniformly norm-to-norm continuous on bounded sets, so is J^{-1} . Thus

$$\lim_{n \rightarrow \infty} \|T_j x_n - T_i x_n\| = \lim_{n \rightarrow \infty} \|J^{-1}(JT_j x_n) - J^{-1}(JT_i x_n)\| = 0.$$

Therefore,

$$\|x_n - T_i x_n\| \leq \|x_n - T_j x_n\| + \|T_j x_n - T_i x_n\| \rightarrow 0$$

as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in \bigcap_{i=1}^m \hat{F}(T_i) = \bigcap_{i=1}^m F(T_i)$.

Next, we prove that $\hat{x} \in EP(f)$. Let $u \in F$. Note that $u_n = T_{r_n} y_n$ and $\phi(u, y_n) \leq \phi(u, x_n)$. Hence, by Lemma 2.11, we obtain that

$$\phi(u_n, y_n) = \phi(T_{r_n} y_n, y_n) \leq \phi(u, y_n) - \phi(u, T_{r_n} y_n) \leq \phi(u, x_n) - \phi(u, T_{r_n} y_n) = \phi(u, x_n) - \phi(u, u_n).$$

Let $r = \sup_{n \geq 0} \{\|y_n\|, \|u_n\|\}$. Then, by Lemma 2.4, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and

$$g(\|u_n - y_n\|) \leq \phi(u_n, y_n) \leq \phi(u, x_n) - \phi(u, u_n) \leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.$$

It follows from (3.1) and (3.2) that

$$g(\|u_n - y_n\|) \rightarrow 0,$$

as $n \rightarrow \infty$. Therefore, from the property of g we get that

$$\|u_n - y_n\| \rightarrow 0,$$

as $n \rightarrow \infty$. By the uniform continuity of J , we also obtain

$$\|Ju_n - Jy_n\| \rightarrow 0,$$

as $n \rightarrow \infty$. Since $r_n \geq a > 0$,

$$\frac{\|Ju_n - Jy_n\|}{r_n} \rightarrow 0,$$

as $n \rightarrow \infty$. Since $x_{n_k} \rightharpoonup \hat{x}$, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ and $\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0$, there exist subsequences $\{y_{n_k}\}$ of $\{y_n\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that $y_{n_k} \rightarrow \hat{x}$ and $u_{n_k} \rightarrow \hat{x}$. By using the same argument as the proof of Theorem 3.1 in [22], we also have $\hat{x} \in EP(f)$. Therefore $\hat{x} \in F := \bigcap_{i=1}^m F(T_i) \cap EP(f)$. Let $w = \Pi_F x$. Then $w \in F \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x$, we have

$$\phi(x_{n+1}, x) = \min_{y \in H_n \cap W_n} \phi(y, x) \leq \phi(w, x).$$

Since the norm is weakly lower semicontinuous, we have

$$\phi(\hat{x}, x) = \|\hat{x}\|^2 - 2\langle \hat{x}, Jx \rangle + \|x\|^2 \leq \liminf_{k \rightarrow \infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx \rangle + \|x\|^2) = \liminf_{k \rightarrow \infty} \phi(x_{n_k}, x) \leq \limsup_{k \rightarrow \infty} \phi(x_{n_k}, x) \leq \phi(w, x).$$

From the definition of Π_F , we have $\hat{x} = w$. Thus $\lim_{k \rightarrow \infty} \phi(x_{n_k}, x) = \phi(w, x)$. By the definition of ϕ , we also have $\lim_{k \rightarrow \infty} \|x_{n_k}\| = \|w\|$. Since E is uniformly convex, E has the Kadec–Klee property. We obtain that $x_{n_k} \rightarrow w \in \Pi_F x$. Therefore, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x$. This completes the proof. $\square$

Remark 3.2

- (i) Theorem 3.1 extends the results of [9,22] from a single relatively nonexpansive mapping to a finite family of relatively nonexpansive mappings.

- (ii) Theorem 3.1 extends the results of [14] from two relatively nonexpansive mappings to a finite family of relatively nonexpansive mappings.
- (iii) Theorem 3.1 extends the results of [11,18] from real Hilbert spaces to uniformly smooth and uniformly convex Banach spaces.

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Chapter III

Fixed Point Theorems for Multi-valued Mappings and Equilibrium Problems in Banach Spaces

In this chapter, we first introduce an iterative scheme for equilibrium problem and fixed point problem in Banach spaces and we define a new type of multi-valued mappings and prove the existence of its fixed point in the setting of Hilbert spaces.

3.1 An iterative method for equilibrium problems and a finite family of relatively nonexpansive mappings in a Banach space

In this section, we introduce a new iterative method for finding a common element of the set of fixed points of a finite family of relatively nonexpansive mappings and the set of solutions of an equilibrium problem in uniformly convex and uniformly smooth Banach spaces. Then we prove a strong convergence theorem by using the generalized projection.

1. Introduction and preliminaries

In this work, let D be a nonempty convex subset of a Banach spaces X . The metric projection operator (the nearest point projection) P_D defined on X is a mapping from X to 2^D such that

$$P_D x = \{y \in D : \|x - y\| = d(x, D)\}, \quad \forall x \in X.$$

If $P_D x \neq \emptyset$ for every $x \in X$, then D is called a *proximal set*. If $P_D x$ is a singleton, for every $x \in X$, then D is said to be a *Chebyshev set*. Let $CB(D)$, $K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximal bounded subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for $A, B \in CB(D)$. A single-valued mapping $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued mapping $T : D \rightarrow CB(D)$ is called (i) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$ and (ii) *hemicompact* if for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \rightarrow CB(D)$ is *hemicompact*. An element $p \in D$ is called a *fixed point* of $T : D \rightarrow D$ (respectively, $T : D \rightarrow CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$.

A family $\{T_i\}_{i=1}^N : D \rightarrow CB(D)$ is said to satisfy the condition (A) if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, T_j x) \geq f \left(d \left(x, \bigcap_{i=1}^N F(T_i) \right) \right)$$

for some $j \in \{1, 2, \dots, N\}$ and for all $x \in D$.

In 2001, Xu and Ori [1] introduced the following implicit iteration process for solving a fixed point problem of a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \dots, N\}$) in Hilbert spaces. For $x_0 \in D$ and $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \quad (1.1)$$

where $T_n = T_{n \bmod N}$ (here the $n \bmod N$ function takes values in I). Xu and Ori proved the weak convergence of this process (1.1) to a common fixed point of a finite family $\{T_i : i \in I\}$.

In 2002, Zhou and Chang [2] studied the theorems of weak and strong convergence of the implicit iteration process to a common fixed point of a finite family of asymptotically nonexpansive mappings in a uniformly convex Banach space with $\alpha_n \in (b, c)$ for some constants $b, c \in (0, 1)$.

In 2005, Chidume and Shahzad [3] generalized Zhou's result, still in the uniformly convex Banach space, with $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$.

However, the studying of nonexpansive multi-valued mappings is harder than that of single-valued nonexpansive mappings in both Hilbert spaces and Banach spaces; see also [4,5].

Let $T : D \rightarrow P(D)$ and $P_T x = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Here P_T is called *the best approximation operator*.

It is remarked that Hussain and Khan [6], in 2003, employed the best approximation operator P_T to study fixed points of a $*$ -nonexpansive multi-valued mapping T and the strong convergence of its iterates to a fixed point of T defined on a closed and convex subset of a Hilbert space.

The following example shows that P_T is quasi-nonexpansive but T is not nonexpansive.

Example 1.1 ([7]). Consider $D = [0, 1]$ with the usual norm. Define $T : D \rightarrow CB(D)$ by

$$Tx = \begin{cases} [0, x], & x \in \left[0, \frac{1}{2}\right], \\ \left\{\frac{1}{2}\right\}, & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

We do indeed have that P_T is nonexpansive.

Case 1: if $x, y \in [0, \frac{1}{2}]$ then $H(P_T x, P_T y) = \|x - y\|$.

Case 2: if $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$ then $H(P_T x, P_T y) = \|x - \frac{1}{2}\| \leq \|x - y\|$.

Case 3: if $x, y \in (\frac{1}{2}, 1]$ then $H(P_T x, P_T y) = 0$.

It would be interesting to study the convergence of the multi-valued mapping T by using the best approximation operator P_T .

Let X be a real Banach space and let J be the normalized duality mapping from X to 2^{X^*} defined by $J(x) = \{f \in X^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}$, $\forall x \in X$, where X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair. It is well known that if X is a smooth Banach space, then J is single-valued. A Banach space X is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. For all $\epsilon \in [0, 2]$, the *modulus of convexity* of X is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x + y\| \leq 1, \|x - y\| \geq \epsilon \right\}.$$

Lemma 1.2 ([8]). Let X be a uniformly convex Banach space with the modulus of uniform convexity δ_X . Then $\delta_X : [0, 2] \rightarrow [0, 1]$ is continuous increasing, $\delta_X(0) = 0$, and $\delta_X(t) > 0$ for $t > 0$, while

$$\|cu + (1 - c)v\| \leq 1 - 2 \min\{c, 1 - c\} \delta_X(\|u - v\|)$$

where $0 \leq c \leq 1$ and $\|u\|, \|v\| \leq 1$.

Lemma 1.3 ([9]). Let $x, y \in X$. Then the following conditions are equivalent:

- (i) $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \geq 0$; (ii) there exists $f \in J(x)$ with $\langle y, f \rangle \geq 0$.

2. Strong convergence theorems

In this section, we first introduce an implicit iteration process for finding a common fixed point of a finite family of multi-valued mappings. Then the strong convergence theorems are also established. Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. For $x_0 \in D$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$x_n \in \alpha_n x_{n-1} + (1 - \alpha_n) P_{T_n} x_n, \quad n \geq 1, \quad (2.1)$$

where $T_n = T_{n \bmod N}$ (here the $n \bmod N$ function takes values in $\{1, 2, \dots, N\}$) and P_{T_n} is nonexpansive.

The following well-known result was proved in Nadler [10].

Theorem 2.1 ([10]). *Let D be a nonempty closed subset of a Banach space X and $T : D \rightarrow CB(D)$ be a multi-valued contraction. Then T has a fixed point.*

Let D be a nonempty closed convex subset of a real Banach space X and $T : D \rightarrow CB(D)$ be a nonexpansive multi-valued mapping. For every $u \in D$ and $t \in (0, 1)$, the mapping $G_t : D \rightarrow CB(D)$ defined by $G_t = tu + (1 - t)T$ is a contraction mapping; by Theorem 2.1, there exists $x_t \in D$ (non-unique, in general) such that $x_t \in tu + (1 - t)Tx_t$. This implies that the implicit iteration process (2.1) introduced above can be employed for the approximation of common fixed points of a finite family of multi-valued mappings.

Next, we prove our main results.

Lemma 2.2. *Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. For arbitrary $x_0 \in D$, the sequence $\{x_n\}_{n=1}^\infty$ is generated by (2.1). Assume that the best approximation operator P_{T_i} is nonexpansive for each $i \in \{1, 2, \dots, N\}$. Then:*

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \bigcap_{i=1}^N F(T_i)$,
- (ii) $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for each $i \in \{1, 2, \dots, N\}$.

Proof. (i) By the property of the best approximation operator P_{T_i} and our assumption, we have

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + (1 - \alpha_n) z_n - p\| \quad \text{for some } z_n \in P_{T_n} x_n \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|z_n - p\| \\ &= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) d(z_n, P_{T_n} p) \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) H(P_{T_n} x_n, P_{T_n} p) \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|x_n - p\| \end{aligned}$$

for all $p \in \bigcap_{i=1}^N F(T_i)$. This implies that $\|x_n - p\| \leq \|x_{n-1} - p\|$ and $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \bigcap_{i=1}^N F(T_i)$.

(ii) For each $p \in \bigcap_{i=1}^N F(T_i)$, we have

$$\begin{aligned} \langle z_n - p, j(x_n - p) \rangle &\leq \|z_n - p\| \|x_n - p\| \\ &= d(z_n, P_{T_n} p) \|x_n - p\| \\ &\leq H(P_{T_n} x_n, P_{T_n} p) \|x_n - p\| \\ &\leq \|x_n - p\|^2. \end{aligned}$$

This implies that $\langle x_n - z_n, j(x_n - p) \rangle \geq 0$. From Lemma 1.3, we have

$$\begin{aligned} \|x_n - p\| &\leq \left\| x_n - p + \frac{1 - \alpha_n}{2\alpha_n} (x_n - z_n) \right\| \\ &= \left\| x_n - p + \frac{1 - \alpha_n}{2} (x_{n-1} - z_n) \right\| \\ &= \left\| \alpha_n x_{n-1} + (1 - \alpha_n) z_n - p + \frac{1 - \alpha_n}{2} (x_{n-1} - z_n) \right\| \\ &= \left\| \frac{x_{n-1} + x_n}{2} - p \right\| \\ &\leq \|x_{n-1} - p\| \left[1 - \delta_X \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - p\|} \right) \right]. \end{aligned}$$

By the properties of δ_X , hence $\|x_n - x_{n-1}\| \rightarrow 0$ as $n \rightarrow \infty$. We observe that

$$\|x_{n-1} - z_n\| = \frac{1}{1 - \alpha_n} \|x_n - x_{n-1}\|.$$

Using the assumption that $\limsup_{n \rightarrow \infty} \alpha_n < 1$, we have $\|x_{n-1} - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Hence $\|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. It follows that $d(x_n, P_{T_n} x_n) \leq \|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $\forall i \in \{1, 2, \dots, N\}$,

$$\begin{aligned} d(x_n, P_{T_{n+i}} x_n) &\leq \|x_n - x_{n+i}\| + d(x_{n+i}, P_{T_{n+i}} x_{n+i}) + H(P_{T_{n+i}} x_{n+i}, P_{T_{n+i}} x_n) \\ &\leq 2\|x_n - x_{n+i}\| + d(x_{n+i}, P_{T_{n+i}} x_{n+i}) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, $\forall \ell \in \{1, 2, \dots, N\}$, there exists some $i \in \{1, 2, \dots, N\}$ such that $n + i = \ell \pmod N$. It follows that

$$d(x_n, P_{T_\ell} x_n) = d(x_n, P_{T_{n+i}} x_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Hence

$$d(x_n, T_\ell x_n) \leq d(x_n, P_{T_\ell} x_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This completes the proof. $\square$

Theorem 2.3. Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that the best approximation operator P_{T_i} is nonexpansive for each $i \in \{1, 2, \dots, N\}$. If $\{T_i\}_{i=1}^N$ satisfies the condition (A), then the sequence $\{x_n\}_{n=1}^\infty$ generated by (2.1) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

Proof. Since $\{T_i\}_{i=1}^N$ satisfies the condition (A), we have $d(x_n, \bigcap_{i=1}^N F(T_i)) \rightarrow 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for some $\{p_k\} \subset \bigcap_{i=1}^N F(T_i)$ and all k . So we obtain

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

Next, we shall show that $\{p_k\}$ is a Cauchy sequence in D . Notice that

$$\|p_{k+1} - p_k\| \leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This implies that $\{p_k\}$ is a Cauchy sequence in D and thus converges to $q \in D$. Since

$$d(p_k, T_i q) \leq d(p_k, P_{T_i} q) \leq H(P_{T_i} q, P_{T_i} p_k) \leq \|q - p_k\|$$

for all $i = 1, 2, \dots, N$ and $p_k \rightarrow q$ as $k \rightarrow \infty$, it follows that $d(q, T_i q) = 0$ for all $i = 1, 2, \dots, N$ and thus $q \in \bigcap_{i=1}^N F(T_i)$. Hence $\{x_{n_k}\}$ converges strongly to q . Since $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists, it follows that $\{x_n\}_{n=1}^\infty$ converges strongly to q . This completes the proof. $\square$

Theorem 2.4. Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that the best approximation operator P_{T_i} is nonexpansive for each $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}_{n=1}^\infty$ generated by (2.1) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$.

Proof. Since the necessity is obvious, it suffices to show the sufficiency. Suppose that $\liminf_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$; then by Lemma 2.2(i), we have that $\lim_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i))$ exists. This implies that $\lim_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$. By using the same argument as in the proof of Theorem 2.3, we obtain that $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of a finite family $\{T_i\}_{i=1}^N$. $\square$

Theorem 2.5. Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that the best approximation operator P_{T_i} is nonexpansive and hemicompact for each $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}_{n=1}^\infty$ generated by (2.1) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

Proof. Since $\lim_{n \rightarrow \infty} d(x_n, P_{T_i} x_n) = 0$ and P_{T_i} is hemicompact for each $i \in \{1, 2, \dots, N\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in D$. It follows that

$$\begin{aligned} d(p, P_{T_i} p) &\leq \|p - x_{n_k}\| + d(x_{n_k}, P_{T_i} x_{n_k}) + H(P_{T_i} x_{n_k}, P_{T_i} p) \\ &\leq 2\|p - x_{n_k}\| + d(x_{n_k}, P_{T_i} x_{n_k}) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Hence $p \in \bigcap_{i=1}^N F(T_i)$. By Lemma 2.2(i), we obtain $x_n \rightarrow p$ as $n \rightarrow \infty$. This completes the proof. $\square$

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3.2 Fixed Point Theorems for Nonspreading-type Multi-valued Mappings in Hilbert Spaces

In this section, we define a new class of nonlinear multi-valued mappings called *k*-nonspreading in a Hilbert space. We then prove fixed point theorems for 1/2-nonspreading multi-valued mappings. Furthermore, we obtain a necessary and sufficient condition for the existence of such mappings in a Hilbert space. Finally, convergence theorems are also discussed.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and D be a nonempty convex subset of H . Let $CB(D)$ and $K(D)$ be the families of nonempty closed bounded subsets and nonempty compact subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for $A, B \in CB(D)$. A single-valued mapping $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued mapping $T : D \rightarrow CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a *fixed point* of $T : D \rightarrow D$ (resp. $T : D \rightarrow CB(D)$) if $p = Tp$ (resp. $p \in Tp$). The fixed points set of T is denoted by $F(T)$. If $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$, then T is called *quasi-nonexpansive*.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H . We denote the weak convergence of $\{x_n\}_{n=1}^{\infty}$ to a point $x \in H$ by $x_n \rightharpoonup x$, and the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to a point $x \in H$ by $x_n \rightarrow x$.

A multi-valued mapping $T : D \rightarrow CB(D)$ is said to be *demiclosed* at $y \in D$ if $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \rightharpoonup x$ and $y_n \in Tx_n$ such that $y_n \rightarrow y$ imply $y \in Tx$.

Fixed point theorems of nonlinear mappings have been intensively studied and considered by many authors (see, for examples, [1, 4, 10, 11, 14, 27]).

In 2008, Kohsaka-Takahashi [16, 17] introduced a new class of nonlinear mappings as follows: let H be a Hilbert space and C a nonempty, closed and convex subset of H . Then, a mapping $T : C \rightarrow C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$.

Later, Takahashi [25] defined a class of nonlinear mappings called *hybrid*, i.e.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$.

Recently, Aoyama-Iemoto-Kohsaka-Takahashi [2] introduced a new class of nonlinear mappings containing the class of nonexpansive mappings, nonspreading mappings and hybrid mappings as follows:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda \langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$ and $\lambda \in \mathbb{R}$. Such a mapping is called λ -*hybrid*. They proved fixed point theorems of such mappings in Hilbert spaces.

The fixed point theory of multi-valued mappings is much more complicated and interesting than the corresponding theory of single-valued mappings. However, some classical fixed point theorems for single-valued mappings have already been extended to multi-valued mappings.

In 1968, Markin [20] firstly established the nonexpansive multi-valued convergence results in Hilbert spaces. Banach's Contraction Principle was extended to a multi-valued contraction in 1969 by Nadler [21]. In 1974, one breakthrough was achieved by Lim using Edelstein's method of asymptotic centers [18]. In 1999, Kirk-Massa [15] obtained another important result for the existence fixed point theorem of nonexpansive multi-valued mappings. In 1999, Sahu [24] obtained the strong convergence theorems of the nonexpansive type and nonself multi-valued mappings in uniformly convex Banach spaces. Recently, Xu [30] extended the result of Kirk-Massa [15] to a nonexpansive multi-valued nonself mapping and obtained the fixed point theorem in 2001. The recent fixed point results for nonexpansive multi-valued mappings can be found in [3, 5, 6, 7, 8, 9, 13, 23, 29] and references cited therein.

2 Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In a real Hilbert space H , we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$ (see [26]) and

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$$

for all $x, y, z, w \in H$ (see [12]).

Let D be a closed and convex subset of H . For every point $x \in H$, there exists a unique nearest point in D , denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall y \in D.$$

P_D is called the *metric projection* of H onto D . We know that P_D is a nonexpansive mapping of H onto D .

The following lemmas will be used for the proof of our main results in the sequel.

Lemma 2.1. *Let H be a real Hilbert space. Then the following well known results hold:*

$$(i) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

(ii) *If $\{x_n\}_{n=1}^\infty$ is a sequence in H which converges weakly to $z \in H$ then*

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Lemma 2.2. [19] *Let D be a closed and convex subset of a real Hilbert space H and let P_D be the metric projection from H onto D . Given $x \in H$ and $z \in D$. Then $z = P_D x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D.$$

Lemma 2.3. [28] *Let D be a nonempty closed convex subset of a Hilbert space H , let $\{x_n\}$ be a bounded sequence in H and let μ be a Banach limit. If $g : D \rightarrow \mathbb{R}$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in D,$$

then there exists a unique $z_0 \in D$ such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

3 Main results

Let H be a real Hilbert space, and D a nonempty and convex subset of H , we say that a multi-valued mapping $T : D \rightarrow CB(D)$ is *k-nonspreading* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \leq k(d(Tx, y)^2 + d(x, Ty)^2) \quad (3.1)$$

for all $x, y \in D$.

It is easy to see that if T is $\frac{1}{2}$ -nonspreading, then T is nonspreading in the case of single-valued mappings (see [16, 17]). Moreover, if T is $\frac{1}{2}$ -nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in D$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\leq d(Tx, p)^2 + d(x, Tp)^2 \\ &\leq H(Tx, Tp)^2 + \|x - p\|^2. \end{aligned}$$

It follows that

$$H(Tx, Tp) \leq \|x - p\|. \quad (3.2)$$

We now give an example of a $\frac{1}{2}$ -nonspreading multi-valued mapping which is not a nonexpansive multi-valued mapping.

Example 3.1. Consider $D = [-3, 3]$ with the usual norm. Define $T : D \rightarrow CB(D)$ by

$$Tx = \begin{cases} \{0\} & , x \in [-2, 2], \\ \left[-\frac{|x|}{|x|+1}, \frac{|x|}{|x|+1} \right] & , x \notin [-2, 2]. \end{cases}$$

To see that T is $\frac{1}{2}$ -nonspreading, we observe that

Case 1: if $x, y \in [-2, 2]$, then $H(Tx, Ty) = 0$.

Case 2: if $x \in [-2, 2]$ and $y \notin [-2, 2]$, then $Tx = \{0\}$ and $Ty = \left[-\frac{|y|}{|y|+1}, \frac{|y|}{|y|+1} \right]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|y|}{|y|+1}\right)^2 < 2 < y^2 \leq d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: if $x, y \notin [-2, 2]$, then $Tx = \left[-\frac{|x|}{|x|+1}, \frac{|x|}{|x|+1} \right]$ and $Ty = \left[-\frac{|y|}{|y|+1}, \frac{|y|}{|y|+1} \right]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\right)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

T is not nonexpansive since for $x = 2$ and $y = \frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = \left[-\frac{5}{7}, \frac{5}{7} \right]$. This implies that $H(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = \left| 2 - \frac{5}{2} \right| = \|x - y\|$.

Let D be a nonempty set. We define $T(D) = \cup_{x \in D} Tx$ and $(ST)x = S(Tx)$ for all $x \in D$. We next present some crucial properties of $\frac{1}{2}$ -nonspreading multi-valued mappings.

Lemma 3.2. Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow CB(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. We shall show that $x \in F(T)$. Observe

$$\begin{aligned} d(x, Tx) &\leq \|x - x_n\| + d(x_n, Tx) \\ &\leq \|x - x_n\| + H(Tx_n, Tx) \\ &\leq 2\|x - x_n\|. \end{aligned}$$

It follows that $d(x, Tx) = 0$ and hence $x \in F(T)$. □

Lemma 3.3. *Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow CB(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping with $F(T) \neq \emptyset$. If $Tp = \{p\}$ for all $p \in F(T)$, then $F(T)$ is convex.*

Proof. Let $p = tp_1 + (1-t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0, 1)$. We shall show that $p \in F(T)$. Let $z \in Tp$. From (3.2) we have

$$\begin{aligned}
 \|p - z\|^2 &= \|t(z - p_1) + (1-t)(z - p_2)\|^2 \\
 &= t\|z - p_1\|^2 + (1-t)\|z - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\
 &= td(z, Tp_1)^2 + (1-t)d(z, Tp_2)^2 - t(1-t)\|p_1 - p_2\|^2 \\
 &\leq tH(Tp, Tp_1)^2 + (1-t)H(Tp, Tp_2)^2 - t(1-t)\|p_1 - p_2\|^2 \\
 &\leq t\|p - p_1\|^2 + (1-t)\|p - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\
 &= t(1-t)^2\|p_1 - p_2\|^2 + (1-t)t^2\|p_1 - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\
 &= 0,
 \end{aligned}$$

which yields $p = z$. Therefore $p \in F(T)$. □

Lemma 3.4. *Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. If $x, y \in D$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \leq H(Tx, Ty)^2 \leq \|x - y\|^2 + 2\langle x - a, y - b \rangle.$$

Proof. Let $x, y \in D$ and $a \in Tx$. From Nadler Theorem (see [21]), there exists $b \in Ty$ such that

$$\|a - b\|^2 \leq H(Tx, Ty)^2.$$

It follows that

$$\begin{aligned}
 2H(Tx, Ty)^2 &\leq d(Tx, y)^2 + d(x, Ty)^2 \\
 &\leq \|a - y\|^2 + \|x - b\|^2 \\
 &\leq \|a - x\|^2 + 2\langle a - x, x - y \rangle + \|x - y\|^2 + \|x - a\|^2 + 2\langle x - a, a - b \rangle + \|a - b\|^2 \\
 &= 2\|a - x\|^2 + \|x - y\|^2 + \|a - b\|^2 + 2\langle a - x, x - a - (y - b) \rangle \\
 &\leq 2\|a - x\|^2 + \|x - y\|^2 + H(Tx, Ty)^2 + 2\langle a - x, x - a - (y - b) \rangle.
 \end{aligned}$$

This shows that

$$H(Tx, Ty)^2 \leq \|x - y\|^2 + 2\langle x - a, y - b \rangle.$$

□

Lemma 3.5. *Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. Then $(I - T)$ is demiclosed at 0.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence in D which converges weakly to p and $y_n \in Tx_n$ such that $\{x_n - y_n\}_{n=1}^\infty$ converges strongly to 0. We shall show that $p \in F(T)$. By Lemma 3.4, there exists $z_n \in Tp$ such that $\|y_n - z_n\|^2 \leq \|x_n - p\|^2 + 2\langle x_n - y_n, p - z_n \rangle$. Since Tp is compact and $z_n \in Tp$,

there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \rightarrow z \in Tp$. Since $\{x_n\}_{n=1}^\infty$ converges weakly to p , it is bounded. For each $x \in H$, define $f : H \rightarrow [0, \infty)$ by

$$f(x) := \limsup_{i \rightarrow \infty} \|x_{n_i} - x\|^2.$$

Then from Lemma 2.1 (ii), we obtain

$$f(x) = \limsup_{i \rightarrow \infty} \|x_{n_i} - p\|^2 + \|p - x\|^2, \quad \forall x \in H.$$

Thus $f(x) = f(p) + \|p - x\|^2, \forall x \in H$. It also follows that

$$f(z) = f(p) + \|p - z\|^2. \quad (3.3)$$

We observe that

$$f(z) = \limsup_{i \rightarrow \infty} \|x_{n_i} - z\|^2 = \limsup_{i \rightarrow \infty} \|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \leq \limsup_{i \rightarrow \infty} \|y_{n_i} - z\|^2$$

This implies that

$$\begin{aligned} f(z) &\leq \limsup_{i \rightarrow \infty} \|y_{n_i} - z\|^2 \\ &= \limsup_{i \rightarrow \infty} [\|y_{n_i} - z_{n_i} + z_{n_i} - z\|]^2 \\ &\leq \limsup_{i \rightarrow \infty} [\|x_{n_i} - p\|^2 + 2\langle x_{n_i} - y_{n_i}, p - z_{n_i} \rangle] \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \quad (3.4)$$

Hence it follows from (3.3) and (3.4) that $\|p - z\| = 0$. This completes the proof. $\square$

Theorem 3.6. *Let D be a nonempty, closed and convex subset of a real Hilbert space H , let T be a multi-valued mapping of D into $CB(D)$. Suppose that there exists an element $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded and*

$$\mu_n \|z_n - z\|^2 \leq \mu_n \|z_n - y\|^2, \quad \forall y \in D, \forall z \in Ty$$

for some Banach limit μ . Then, T has a fixed point in D .

Proof. Using a Banach limit μ on ℓ^∞ , we can define $g : D \rightarrow \mathbb{R}$ as follows:

$$g(z) := \mu_n \|z_n - z\|^2, \quad \forall z \in D.$$

From Lemma 2.3, there exists a unique $z_0 \in D$ such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

So, for each $a \in Tz_0$, we have

$$g(a) = \mu_n \|z_n - a\|^2 \leq \mu_n \|z_n - z_0\|^2 = g(z_0).$$

Since $a \in D$ and $z_0 \in D$ is unique element such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Hence $z_0 = a \in Tz_0$. This completes the proof. $\square$

Theorem 3.7. *Let D be a nonempty, closed and convex subset of a real Hilbert space H , let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. Then, the following are equivalent:*

- (i) *There exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded.*
- (ii) *$F(T)$ is nonempty.*

Proof. It is obvious that (ii) implies (i). We shall show that (i) implies (ii). Assume that there exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded. Let $x, y \in D$ and $z_{n+1} \in Tz_n$. From Lemma 3.4, there exists $b \in Ty$ such that

$$\begin{aligned} \|z_{n+1} - b\|^2 &\leq \|z_n - y\|^2 + 2\langle z_n - z_{n+1}, y - b \rangle \\ &\leq \|z_n - y\|^2 + (\|z_n - b\|^2 + \|z_{n+1} - y\|^2 - \|z_n - y\|^2 - \|z_{n+1} - b\|^2), \end{aligned}$$

which implies

$$2\|z_{n+1} - b\|^2 - \|z_n - b\|^2 \leq \|z_{n+1} - y\|^2.$$

Let μ be a Banach limit on ℓ^∞ . For any $n \in \mathbb{N}$, we have

$$2\mu_n\|z_{n+1} - b\|^2 - \mu_n\|z_n - b\|^2 \leq \mu_n\|z_{n+1} - y\|^2.$$

This implies that

$$\mu_n\|z_n - b\|^2 \leq \mu_n\|z_n - y\|^2.$$

From Theorem 3.6, T has a fixed point in D . □

We next prove the following weak convergence theorem for a $\frac{1}{2}$ -nonspreading mapping in a Hilbert space.

Theorem 3.8. *Let D be a nonempty, closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. Define the sequence $\{x_n\}$ in D as follows: $x_1 = x \in D$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad z_n \in Tx_n$$

for all $n \in \mathbb{N}$, where $0 \leq \alpha_n < 1$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$.

If $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$, then $\{x_n\}$ converges weakly to $q \in F(T)$.

Proof. Since a $\frac{1}{2}$ -nonexpansive multi-valued mapping T with a fixed point is quasi-nonexpansive,

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|z_n - p\| \\ &= \alpha_n \|x_n - p\| + (1 - \alpha_n) d(z_n, Tp) \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) H(Tx_n, Tp) \\ &\leq \|x_n - p\| \end{aligned}$$

for all $p \in F(T)$. Therefore $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded. For all $p \in F(T)$, we observe that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|\alpha_n(x_n - p) + (1 - \alpha_n)(z_n - p)\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\| \\ &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) d(z_n, Tp)^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\| \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) H(Tx_n, Tp)^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\| \\ &\leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\|. \end{aligned}$$

This implies that

$$\alpha_n(1 - \alpha_n) \|x_n - z_n\| \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to q . From Lemma 3.5, we obtain $q \in F(T)$. To complete our conclusion, it is sufficient to show that for another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $w \in F(T)$, $q = w$. Suppose that $q \neq w$. We have from Opial's theorem [22] that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - q\| < \lim_{i \rightarrow \infty} \|x_{n_i} - w\| = \lim_{n \rightarrow \infty} \|x_n - w\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - w\| < \lim_{j \rightarrow \infty} \|x_{n_j} - q\| = \lim_{n \rightarrow \infty} \|x_n - q\|. \end{aligned}$$

This is a contradiction. So, $\{x_n\}$ converges weakly to $q \in F(T)$. □

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Research Article

A New Approximation Method for Common Fixed Points of a Finite Family of Generalized Asymptotically Quasinonexpansive Mappings in Banach Spaces

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We introduce a new iterative scheme to approximate a common fixed point for a finite family of generalized asymptotically quasinonexpansive mappings. Several strong and weak convergence theorems of the proposed iteration in Banach spaces are established. The main results obtained in this paper generalize and refine many known results in the current literature.

1. Introduction

Let C be a convex subset of a Banach space X , and let $\{T_i : i = 1, 2, \dots, k\}$ be a family of self-mappings of C . Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, 3, \dots$ and $i = 1, 2, \dots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all n . The iterative process (1.1) for a finite family of mappings introduced by Khan et al. [1], and the iterative process is the generalized form of the modified Mann (one-step) iterative process by Schu [2], the modified Ishikawa (two-step) iterative process by Tan and Xu [3], and the three-step iterative process by Xu and Noor [4].

Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers [1, 5–19] are interested in studying approximation method for finding common fixed points of nonlinear mapping. Also, approximation methods for finding fixed points for nonexpansive mappings can be seen in [12–16, 20, 21].

In 2003, Sun [17] studied an implicit iterative scheme initiated by Xu and Ori [22] for a finite family of asymptotically quasicontractive mappings. Shahzad and Udomene [18], in 2006, proved some convergence theorems for the modified Ishikawa iterative process of two asymptotically quasicontractive mappings to a common fixed point. Nammanee et al. [23] introduced a three-step iteration scheme for asymptotically nonexpansive mappings and proved weak and strong convergence theorems of that iteration scheme under some control conditions. In 2007, Fukhar-ud-din and Khan [24] studied a new three-step iteration scheme for approximating a common fixed point of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Shahzad and Zegeye [19] introduced a new concept of generalized asymptotically nonexpansive mappings and proved some strong convergence theorems for fixed points of finite family of this class. Recently, Khan et al. [1] introduced the iterative sequence (1.1) for a finite family of asymptotically quasicontractive mappings in Banach spaces.

Motivated by Khan et al. [1], we introduce a new iterative scheme for finding a common fixed point of a finite family of generalized asymptotically quasicontractive mappings as follows:

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1 - \alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= (1 - \alpha_{(k-2)n})y_{(k-3)n} + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{1.2}$$

where $y_{0n} = x_n$ for all n .

The aim of this paper is to obtain strong and weak convergence results for the iterative process (1.2) of a finite family of generalized asymptotically quasicontractive mappings in Banach spaces.

2. Preliminaries

In this section, we give some definitions and lemmas used in the main results.

Let C be a nonempty subset of a real Banach space X , and let T be a self-mapping of C . The fixed point set of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Then let T is called

- (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;
- (ii) *quasinonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$;
- (iii) *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ and $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (iv) *asymptotically quasinonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ and $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (v) *generalized quasinonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\|T^n x - p\| \leq \|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (vi) *generalized asymptotically quasinonexpansive* [19] if $F(T) \neq \emptyset$ and there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, \infty)$ with $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (vii) *uniformly L-Lipschitzian* if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (viii) *(L- γ) uniform Lipschitz* if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^\gamma$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (ix) *semicompact* if for a sequence $\{x_n\}$ in C with $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow p \in C$.

From the definition of these mappings, it can be seen that

- (i) a *quasinonexpansive mapping* is *generalized quasinonexpansive*;
- (ii) an *asymptotically quasinonexpansive mapping* is *generalized asymptotically quasinonexpansive*;
- (iii) a *generalized quasinonexpansive mapping* is *generalized asymptotically quasinonexpansive*;
- (iv) a *uniformly L-Lipschitzian mapping* is *(L - 1) uniform Lipschitz*.

The map $T : C \rightarrow X$ is said to be *demiclosed at 0* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and $T(x_n)$ converging strongly to 0, we get $Tx = 0$.

A Banach space X is said to have *Opial's property* if for each sequence $\{x_n\}$ converging weakly to $x \in C$ and $x \neq y$, we have the condition

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|. \quad (2.1)$$

Condition (A''). Let C be a subset of a normed space X . A family of self-mappings $\{T_i : i = 1, 2, \dots, k\}$ of C is said to have *Condition (A'')* if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T_i x\| \geq f(d(x, F))$ for some $1 \leq i \leq k$ and for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F = \bigcap_{i=1}^k F(T_i)\}$.

The condition (A'') defined above by the authors is the generalization of the condition (A) [25] when $k = 1$ and condition (A') [26] for $k = 2$.

The following lemmas are needed for proving our main results.

Lemma 2.1 (cf. [17, Lemma 2.2]). *Let the sequences $\{a_n\}$, $\{\delta_n\}$ and $\{c_n\}$ of real numbers satisfy:*

$$a_{n+1} \leq (1 + \delta_n)a_n + c_n, \quad \text{where } a_n \geq 0, \delta_n \geq 0, c_n \geq 0 \quad \forall n = 1, 2, 3, \dots \quad (2.2)$$

and $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$.

(i) $\lim_{n \rightarrow \infty} a_n$ exists;

(ii) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2 (see [2, Lemma 1.3]). *Let X be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1$, $n = 1, 2, 3, \dots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, where $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3. Convergence in Banach Spaces

The aim of this section is to establish the strong convergence of the iterative scheme (1.2) to converge to a common fixed point of a finite family of asymptotically quasicontractive mappings in a Banach space under some appropriate conditions.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a finite family of generalized asymptotically quasicontractive self-mappings of C , that is, $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}$, for all $x \in C$ with the sequence $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$ is defined by (1.2). Let $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$ and $s_n = \max_{1 \leq i \leq k} \{s_{in}\}$. Then for $p \in F$, we get the following:*

- (i) $\|x_n - T_i^n x_n\| \leq (2 + r_n)\|x_n - p\| + s_n$, for all $i = 1, 2, \dots, k$;
- (ii) $\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n)\|y_{(i-1)n} - p\| + s_n$, for all $i = 1, 2, \dots, k$;
- (iii) $\|y_{kn} - p\| \leq (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^k (1 + r_n)^{k-i}$;
- (iv) $\|x_{n+1} - p\| \leq (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^k (1 + r_n)^{(k-i)}$;
- (v) $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n$, for all $n \in \mathbb{N}$,
where $c_n = s_n \sum_{i=1}^k (1 + r_n)^{(k-i)}$ and $\delta_n = \binom{k}{1} r_n + \binom{k}{2} r_n^2 + \dots + \binom{k}{k} r_n^k$.
- (vi) If $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Proof. Let $p \in F$.

(i) For $i = 1, 2, 3, \dots, k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \|x_n - p\| + \|T_i^n x_n - p\| \\ &\leq \|x_n - p\| + (1 + r_n)\|x_n - p\| + s_{in}, \\ &\leq (2 + r_n)\|x_n - p\| + s_n. \end{aligned} \quad (3.1)$$

(ii) Similarly to part (i), we have

$$\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n) \|y_{(i-1)n} - p\| + s_n, \quad \forall i = 1, 2, \dots, k. \quad (3.2)$$

(iii) By part (i) and $0 \leq \alpha_{in} \leq 1$, we have

$$\begin{aligned} \|y_{1n} - p\| &= \|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_1^n x_n - p)\| \\ &\leq (1 - \alpha_{1n}) \|x_n - p\| + \alpha_{1n} \|T_1^n x_n - p\| \\ &\leq (1 - \alpha_{1n}) \|x_n - p\| + \alpha_{1n}(1 + r_{1n}) \|x_n - p\| + \alpha_{1n} s_{1n} \\ &\leq (1 + r_n) \|x_n - p\| + s_n, \\ \|y_{2n} - p\| &= \|(1 - \alpha_{2n})(y_{1n} - p) + \alpha_{2n}(T_2^n y_{1n} - p)\| \\ &\leq (1 - \alpha_{2n}) \|y_{1n} - p\| + \alpha_{2n} \|T_2^n y_{1n} - p\| \\ &\leq (1 - \alpha_{2n}) \|y_{1n} - p\| + \alpha_{2n}(1 + r_{2n}) \|y_{1n} - p\| + \alpha_{2n} s_{2n} \\ &\leq (1 + r_n) \|y_{1n} - p\| + s_n \\ &\leq (1 + r_n)(1 + r_n) \|x_n - p\| + s_n(1 + r_n) + s_n \\ &\leq (1 + r_n)^2 \|x_n - p\| + s_n((1 + r_n) + 1), \\ \|y_{3n} - p\| &= \|(1 - \alpha_{3n})(y_{2n} - p) + \alpha_{3n}(T_3^n y_{2n} - p)\| \\ &\leq (1 - \alpha_{3n}) \|y_{2n} - p\| + \alpha_{3n} \|T_3^n y_{2n} - p\| \\ &\leq (1 - \alpha_{3n}) \|y_{2n} - p\| + \alpha_{3n}(1 + r_{3n}) \|y_{2n} - p\| + \alpha_{3n} s_{3n} \\ &\leq (1 + r_n) \|y_{2n} - p\| + s_n \\ &\leq (1 + r_n)^3 \|x_n - p\| + s_n((1 + r_n)^2 + (1 + r_n) + 1), \\ &\vdots \\ \|y_{kn} - p\| &\leq (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^k (1 + r_n)^{(k-i)}. \end{aligned} \quad (3.3)$$

(iv) By part (ii) and part (iii), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| \\ &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - p\| \\ &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_n) \|y_{(k-1)n} - p\| + \alpha_{kn} s_{kn} \\ &\leq (1 + r_n) \|y_{(k-1)n} - p\| + s_n \\ &\leq (1 + r_n)(1 + r_n)^{k-1} \|x_n - p\| + (1 + r_n) s_n \sum_{i=1}^{k-1} (1 + r_n)^{(k-1-i)} + s_n \\ &= (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^k (1 + r_n)^{(k-i)}. \end{aligned} \quad (3.4)$$

- (v) Put $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$ and $c_n = s_n \sum_{i=1}^k (1+r_n)^{(k-i)}$. Then (v) is directly obtained by (iv).
- (vi) By (v), we have $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n$ for all $n \in N$, where $c_n = s_n \sum_{i=1}^k (1+r_n)^{(k-i)}$ and $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$. From $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. By Lemma 2.1, we get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. $\square$

Theorem 3.2. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a finite family of generalized asymptotically quasinonexpansive self-mappings of C , that is, $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$ is closed, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (1.2). Assume that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$ and $s_n = \max_{1 \leq i \leq k} \{s_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

Proof. The necessity is obvious and then we prove only the sufficiency. Let $p \in F$. Since $1 + t \leq e^t$ for $t \geq 0$, we obtain $(1 + t)^k \leq e^{kt}$, for $k = 1, 2, \dots$. Thus by Lemma 3.1(iv) and (v), for positive integers m and n , we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq (1 + r_{(n+m-1)})^k \|x_{(n+m-1)} - p\| + c_{(n+m-1)} \\ &\leq \exp\{kr_{(n+m-1)}\} \|x_{(n+m-1)} - p\| + c_{(n+m-1)} \\ &\leq \cdots \leq \exp\left\{k \sum_{i=n}^{n+m-1} r_i\right\} \|x_n - p\| + \sum_{i=n}^{n+m-1} c_i \\ &\leq \exp\left\{k \sum_{i=1}^{\infty} r_i\right\} \|x_n - p\| + \sum_{i=n}^{\infty} c_i \\ &= M \|x_n - p\| + \sum_{i=n}^{\infty} c_i, \end{aligned} \quad (3.5)$$

where $M = \exp\{k \sum_{i=1}^{\infty} r_i\}$.

By Lemma 3.1(v), we have

$$\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n, \quad \forall p \in F, \quad (3.6)$$

where $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \cdots + \binom{k}{k}r_n^k$ and $c_n = s_n \sum_{i=1}^k (1+r_n)^{(k-i)}$.

It follows that

$$d(x_{n+1}, F) \leq (1 + \delta_n)d(x_n, F). \quad (3.7)$$

From the given condition $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ and Lemma 2.1, we get

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (3.8)$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C . By (3.8) and $\sum_{n=1}^{\infty} c_n < \infty$, we get that for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$d(x_n, F) < \frac{\epsilon}{3M}, \quad \sum_{n=n_0}^{\infty} c_n < \frac{\epsilon}{3}. \quad (3.9)$$

From $d(x_n, F) < \epsilon/3M$, there exists $p_0 \in F$ such that

$$\|x_{n_0} - p_0\| < \frac{\epsilon}{3M}. \quad (3.10)$$

For any positive integer m , by (3.5), (3.9), and (3.10), we have

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq M\|x_{n_0} - p_0\| + \sum_{i=n_0}^{\infty} c_i + \|x_{n_0} - p_0\| \\ &< M\frac{\epsilon}{3M} + \frac{\epsilon}{3} + M\frac{\epsilon}{3M} = \epsilon. \end{aligned} \quad (3.11)$$

Thus, $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, $x_n \rightarrow q \in X$. Actually, $q \in C$ because $\{x_n\} \subset C$ and C is a closed subset of X . Next we show that $q \in F$. Since $F = \bigcap_{i=1}^k F(T_i)$ is closed, by the continuity of $d(x, F)$ with $d(x_n, F) \rightarrow 0$ and $x_n \rightarrow q$ as $n \rightarrow \infty$, we get $d(q, F) = 0$ and then $q \in F$. Therefore, the proof is complete. $\square$

Since any asymptotically quasinonexpansive mapping is generalized asymptotically quasinonexpansive, the next corollary is obtained immediately from Theorem 3.2.

Corollary 3.3 (see [5, Theorem 3.2]). *Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a finite family of asymptotically quasinonexpansive self-mappings of C , that is, $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (1.2). Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.*

4. Convergence in Uniformly Convex Banach Spaces

In this section, strong and weak convergence results for the iterative process (1.2) on uniformly convex Banach spaces are proved without using the condition $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ appearing in Section 3.

Theorem 4.1. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X . Let $\{T_i : i = 1, 2, \dots, k\}$ be a finite family of uniformly $(L - \gamma_i)$ Lipschitzian and generalized asymptotically quasinonexpansive self-mappings of C , that is, $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{Y_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $\{T_i : i = 1, 2, \dots, k\}$ satisfies condition (A'') and $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (1.2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$,*

$\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$ and $s_n = \max_{1 \leq i \leq k} \{s_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

Proof. Let $p \in F$. By Lemma 3.1(vi), we get that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Then there is a real number $c \geq 0$ such that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (4.1)$$

By Lemma 3.1(iii), we have

$$\|y_{kn} - p\| \leq (1 + r_n)^k \|x_n - p\| + s_n \sum_{i=1}^k (1 + r_n)^{(k-i)}, \quad \forall p \in F. \quad (4.2)$$

By taking $\limsup$ on both sides of the above inequality, we get

$$\limsup_{n \rightarrow \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.3)$$

Since $\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n) \|y_{(i-1)n} - p\| + s_n$ and (4.3), we obtain

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k. \quad (4.4)$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c$, we have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = c. \quad (4.5)$$

Using (4.1), (4.4), and Lemma 2.2, we conclude that

$$\lim_{n \rightarrow \infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0. \quad (4.6)$$

We assume that

$$\lim_{n \rightarrow \infty} \|y_{(j-1)n} - T_j^n y_{(j-1)n}\| = 0, \quad \text{for some } 2 \leq j \leq k. \quad (4.7)$$

It follows from (4.5) and (4.7) that

$$c \leq \liminf_{n \rightarrow \infty} \|y_{(j-1)n} - p\|, \quad \text{for } 2 \leq j \leq k. \quad (4.8)$$

By Lemma 3.1(iv), (1.2), and (4.8), we get

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{(j-1)n})(y_{(j-2)n} - p) + \alpha_{(j-1)n}(T_{j-1}^n y_{(j-2)n} - p)\| = \lim_{n \rightarrow \infty} \|y_{(j-1)n} - p\| = c. \quad (4.9)$$

Using (4.3), (4.4) and Lemma 2.2, we conclude that

$$\lim_{n \rightarrow \infty} \|y_{(j-2)n} - T_{j-1}^n y_{(j-2)n}\| = 0. \quad (4.10)$$

Therefore, by mathematical induction, we obtain

$$\lim_{n \rightarrow \infty} \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (4.11)$$

From (1.2), we have

$$\|y_{in} - y_{(i-1)n}\| = \alpha_{in} \|T_i^n y_{(i-1)n} - y_{(j-1)n}\|, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.12)$$

By (4.11), we obtain that

$$\|y_{in} - y_{(i-1)n}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.13)$$

From

$$\|x_n - y_{in}\| \leq \|x_n - y_{1n}\| + \|y_{1n} - y_{2n}\| + \dots + \|y_{(i-1)n} - y_{in}\|, \quad (4.14)$$

for $i = 1, 2, \dots, k-1$, it follows by (4.13) that

$$\|x_n - y_{in}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.15)$$

From (4.11), when $i = 1$, we get $\lim_{n \rightarrow \infty} \|x_n - T_1^n x_n\| = 0$. For $2 \leq i \leq k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - T_i^n x_n\| \\ &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + L \|y_{(i-1)n} - x_n\|^{\gamma_i}. \end{aligned} \quad (4.16)$$

From (4.11) and (4.15), we conclude that

$$\lim_{n \rightarrow \infty} \gamma_{in} = 0, \quad \text{for } i = 1, 2, \dots, k, \quad (4.17)$$

where $\gamma_{in} = \|x_n - T_i^n x_n\|$. From (1.2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - x_n\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - x_n\| \\ &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - x_n\| + \alpha_{kn} (\|T_k^n y_{(k-1)n} - y_{(k-1)n}\| + \|y_{(k-1)n} - x_n\|) \\ &= \|y_{(k-1)n} - x_n\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - y_{(k-1)n}\|. \end{aligned} \quad (4.18)$$

From (4.11) and (4.15),

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (4.19)$$

For $i = 1, 2, \dots, k$, we have

$$\begin{aligned} \|x_{n+1} - T_i x_{n+1}\| &\leq \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\leq \gamma_{i(n+1)} + L \|x_{n+1} - T_i^n x_{n+1}\|^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L (\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|)^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L (\|x_{n+1} - x_n\| + \gamma_{in} + L \|x_n - x_{n+1}\|^{\gamma_i})^{\gamma_i}. \end{aligned} \quad (4.20)$$

Using (4.17) and (4.19), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_i x_{n+1}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (4.21)$$

Therefore, by using condition (A''), there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - T_j x_n\| = 0, \quad (4.22)$$

for some $1 \leq j \leq k$, that is

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (4.23)$$

By Theorem 3.2, we conclude that $\{x_n\}$ converges strongly to a point $p \in F$. $\square$

Lemma 4.2. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and generalized asymptotically quasinonexpansive self-mappings of C , that is, $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\| + s_{in}$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (1.2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$ and $s_n = \max_{1 \leq i \leq k} \{s_{in}\}$. Then,*

- (i) $\lim_{n \rightarrow \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0$, for all $i = 1, 2, \dots, k$;
- (ii) $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, for all $i = 1, 2, \dots, k$.

Proof. (i) Let $p \in F$. By Lemma 3.1(vi), we obtain that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and we then suppose that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (4.24)$$

By (4.24) and Lemma 3.1(iii), we have

$$\limsup_{n \rightarrow \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.25)$$

By (1.2), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn} \|T_k^n y_{(k-1)n} - p\| \\ &\leq (1 - \alpha_{kn}) \|y_{(k-1)n} - p\| + \alpha_{kn} (1 + r_n) \|y_{(k-1)n} - p\| + \alpha_{kn} s_{kn} \\ &\leq (1 + r_n) \|y_{(k-1)n} - p\| + s_n \\ &= (1 + r_n) \|(1 - \alpha_{(k-1)n})(y_{(k-2)n} - p) + \alpha_{(k-1)n}(T_{k-1}^n y_{(k-2)n} - p)\| + s_n \\ &\leq (1 + r_n) [(1 - \alpha_{(k-1)n}) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} \|T_{k-1}^n y_{(k-2)n} - p\|] + s_n \\ &\leq (1 + r_n) [(1 - \alpha_{(k-1)n}) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} (1 + r_n) \|y_{(k-2)n} - p\| + \alpha_{(k-1)n} s_n] + s_n \\ &\leq (1 + r_n)^2 \|y_{(k-2)n} - p\| + s_n (1 + r_n) + s_n \\ &\vdots \\ &\leq (1 + r_n)^{k-i} \|y_{in} - p\| + s_n \sum_{i=1}^{k-1} (1 + r_n)^{k-i}, \end{aligned} \quad (4.26)$$

for $i = 1, 2, \dots, k-1$. It follows that

$$c \leq \liminf_{n \rightarrow \infty} \|y_{in} - p\|, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.27)$$

From (4.25) and (4.27), we obtain

$$\lim_{n \rightarrow \infty} \|y_{in} - p\| = c, \quad \text{for } i = 1, 2, \dots, k-1, \quad (4.28)$$

and then

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{in})(y_{(i-1)n} - p) + \alpha_{in}(T_i^n y_{(i-1)n} - p)\| = c, \quad (4.29)$$

for $i = 1, 2, \dots, k-1$.

Since $\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n) \|y_{(i-1)n} - p\| + s_n$, for $i = 1, 2, \dots, k-1$, we have

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.30)$$

From (4.25), (4.29), (4.30) and Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k-1. \quad (4.31)$$

Now we want to show that (4.31) is also true for $i = k$.

Since $\|T_k^n y_{(k-1)n} - p\| \leq (1 + r_n)\|y_{(k-1)n} - p\| + s_n$, $\lim_{n \rightarrow \infty} r_n = 0$ and $\lim_{n \rightarrow \infty} s_n = 0$, it follows by (4.28),

$$\limsup_{n \rightarrow \infty} \|T_k^n y_{(k-1)n} - p\| \leq c. \quad (4.32)$$

We also have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c. \quad (4.33)$$

Hence, by (4.25), (4.32), and Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0. \quad (4.34)$$

Then, (4.31) and (4.34) give us

$$\lim_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (4.35)$$

From

$$\|x_n - T_i^n y_{(i-1)n}\| \leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\|, \quad (4.36)$$

it implies by (4.15) and (4.35) that

$$\lim_{n \rightarrow \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0, \quad (4.37)$$

for $i = 1, 2, 3, \dots, k$.

(ii) From part (i), for $i = 1$, we have

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - x_n\| = 0. \quad (4.38)$$

For $i = 2, 3, 4, \dots, k$, we get

$$\begin{aligned} \|T_i^n x_n - x_n\| &\leq \|T_i^n x_n - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - x_n\| \\ &\leq L\|x_n - y_{(i-1)n}\|^{r^i} + \|T_i^n y_{(i-1)n} - x_n\|. \end{aligned} \quad (4.39)$$

By part (i) and (4.15),

$$\lim_{n \rightarrow \infty} \|T_i^n x_n - x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (4.40)$$

For $1 \leq i \leq k$, we obtain

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\quad + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\quad + L\|x_{n+1} - x_n\|^{y_i} + L\|T_i^n x_n - x_n\|^{y_i}. \end{aligned} \quad (4.41)$$

From (4.19) and (4.40), we then have

$$\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (4.42)$$

□

Theorem 4.3. *Under the hypotheses of Lemma 4.2, assume that T_j^m is semicompact for some positive integers m and $1 \leq j \leq k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, k\}$.*

Proof. Suppose that T_j^m is semicompact for some positive integers $m \geq 1$ and $1 \leq j \leq k$. We have

$$\begin{aligned} \|T_j^m x_n - x_n\| &\leq \|T_j^m x_n - T_j^{m-1} x_n\| + \|T_j^{m-1} x_n - T_j^{m-2} x_n\| \\ &\quad + \dots + \|T_j^2 x_n - T_j x_n\| + \|T_j x_n - x_n\| \\ &\leq (m-1)L\|T_j x_n - x_n\|^{y_j} + \|T_j x_n - x_n\|. \end{aligned} \quad (4.43)$$

Then, by Lemma 4.2(ii), we get $\|T_j^m x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded and T_j^m is semicompact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightarrow q \in C$ as $l \rightarrow \infty$.

By continuity of T_i and Lemma 4.2(ii), we obtain

$$\|q - T_j q\| = \lim_{l \rightarrow \infty} \|x_{n_l} - T_j x_{n_l}\| = 0, \quad \forall j = 1, 2, \dots, k. \quad (4.44)$$

Therefore, $q \in F$ and then Theorem 3.2 implies that $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1, 2, \dots, k\}$. □

We note that in practical Theorem 4.3 is very useful in the case that one of T_i , $i = 1, 2, 3, \dots, k$, is semicompact.

Theorem 4.4. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X satisfying the Opial property, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and*

generalized asymptotically quasicontractive self-mappings of C , that is, $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{r_i}$ and $\|T_i^n x - p_i\| \leq (1+r_{in})\|x - p_i\| + s_{in}$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (1.2) with $\{\alpha_n\}_{n=1}^\infty \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^\infty r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. If $I - T_i$, $i = 1, 2, \dots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.

Proof. Let $p \in F$. By Lemma 3.1(v), we get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Then we follow the proof of Theorem 3.2 by Khan et al. [1] until we can conclude that $\{x_n\}$ converges weakly to a common fixed point $p \in F$. $\square$

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Strong convergence of a new two-step iterative scheme for two quasi-nonexpansive multi-valued maps in Banach spaces

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ABSTRACT: In this paper, a new two-step iterative scheme is introduced for two quasi-nonexpansive multi-valued maps in Banach spaces. Strong convergence theorem of the purposed iterative scheme is established for quasi-nonexpansive multi-valued maps in Banach spaces. The result obtained in this paper improve and extend the corresponding one announced by Shahzad and Zegeye [N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, *Nonlinear Analysis* 71 (2009) 838-844.].

KEYWORDS: Quasi-nonexpansive multi-valued map; Nonexpansive multi-valued map; Common fixed point; Strong convergence; Banach space.

1. Introduction

Let D be a nonempty convex subset of a Banach spaces E . The set D is called *proximal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D)$, $K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for $A, B \in CB(D)$. A single-valued map $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued map $T : D \rightarrow CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \rightarrow D$ (respectively, $T : D \rightarrow CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. The mapping $T : D \rightarrow CB(D)$ is called *quasi-nonexpansive*[13] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive, see [12].

The mapping $T : D \rightarrow CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \rightarrow CB(D)$ is *hemicompact*.

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A mapping $T : D \rightarrow CB(D)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T)))$$

for all $x \in D$.

A family $\{T_i : D \rightarrow CB(D), i = 1, 2, \dots, N\}$ is said to satisfy *Condition (II)* if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, T_i x) \geq f(d(x, \bigcap_{i=1}^N F(T_i)))$$

for all $i = 1, 2, \dots, N$ and $x \in D$.

In 1953, Mann [6] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H :

$$(1) \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

However, we note that Mann's iteration process (1) has only weak convergence, in general; for instance, see [1, 3, 9].

In 2005, Sastry and Babu [10] proved that the Mann and Ishikawa iteration schemes for multi-valued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p . More precisely, they proved the following result for nonexpansive multi-valued map with compact domain.

In 2007, Panyanak [8] extended the above result of Sastry and Babu [10] to uniformly convex Banach spaces but the domain of T remains compact.

Later, Song and Wang [14] noted that there was a gap in the proofs of Theorem 3.1 (see [8]) and Theorem 5 (see [12]). They further solved/revised the gap and also gave the affirmative answer to Panyanak [8] question using the following Ishikawa iteration scheme. In the main results, domain of T is still compact, which is a strong condition (see [14], Theorem 1) and T satisfies condition(I) (see [14], Theorem 1).

In 2009, Shahzad and Zegeye [10] extended and improved the results of Panyanak [8], Sastry and Babu [12] and Song and Wang [14] to quasi-nonexpansive multi-valued maps. They also relaxed compactness of the domain of T . The results provided an affirmative answer to Panyanak [8] question in a more general setting. They introduced a new iteration as follows: Let D be a nonempty convex subset of a Banach space E and $\alpha_n, \alpha'_n \in [0, 1]$. Let $T : D \rightarrow P(D)$ and $P_T x = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$(2) \quad \begin{aligned} y_n &= \alpha'_n z'_n + (1 - \alpha'_n) x_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n z_n + (1 - \alpha_n) x_n, \quad n \geq 0, \end{aligned}$$

where $z'_n \in P_T x_n$ and $z_n \in P_T y_n$.

Since 2003, the iterative schemes with errors for a single-valued map in Banach spaces have been studied by many authors, see [2, 4, 5, 7].

Question: How can we modify Mann and Ishikawa iterative schemes with errors to obtain convergence theorems for finding a common fixed point of two multi-valued nonexpansive maps?

Motivated by Shahzad and Zegeye [12], we propose a new two-step iterative scheme for two multi-valued quasi-nonexpansive maps in Banach spaces and prove strong convergence theorems of the purposed iteration.

2. Main Results

We use the following iteration scheme:

Let D be a nonempty convex subset of a Banach space E , $\alpha_n, \beta_n, \alpha'_n, \beta'_n \in [0, 1]$ and $\{u_n\}, \{v_n\}$ are bounded sequences in D .

Let T_1, T_2 be two quasi-nonexpansive multi-valued maps from D into $P(D)$ and $P_{T_i}x = \{y \in T_i x : \|x - y\| = d(x, T_i x)\}$, $i = 1, 2$. Let $\{x_n\}$ be the sequence defined by $x_0 \in D$,

$$(3) \quad \begin{aligned} y_n &= \alpha'_n z'_n + \beta'_n x_n + (1 - \alpha'_n - \beta'_n)u_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n z_n + \beta_n x_n + (1 - \alpha_n - \beta_n)v_n, \quad n \geq 0, \end{aligned}$$

where $z'_n \in P_{T_1}x_n$ and $z_n \in P_{T_2}y_n$.

We shall make use of the following results.

Lemma 2.1. [15] Let $\{s_n\}, \{t_n\}$ be two nonnegative sequences satisfying

$$s_{n+1} \leq s_n + t_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n \rightarrow \infty} s_n$ exists.

Lemma 2.2. [11] Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n . Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Theorem 2.3. Let E be a uniformly convex Banach space, D a nonempty, closed and convex subset of E , and T_1, T_2 be two multi-valued maps from D into $P(D)$ with $F(T_1) \cap F(T_2) \neq \emptyset$ such that P_{T_1}, P_{T_2} are nonexpansive. Assume that

- (i) $\{T_1, T_2\}$ satisfies condition (II);
- (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$ and $\sum_{n=1}^{\infty} (1 - \alpha'_n - \beta'_n) < \infty$;
- (iii) $0 < \ell \leq \alpha_n, \alpha'_n \leq k < 1$.

Then the sequence $\{x_n\}$ generated by (3) converges strongly to some elements in $F(T_1) \cap F(T_2)$.

Proof. We split the proof into three steps.

Step 1. Show that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2)$.

Let $p \in F(T_1) \cap F(T_2)$. Then $P_{T_1}p = \{p\}$ and $P_{T_2}p = \{p\}$. Since u_n, v_n are bounded, therefore exists $M > 0$ such that $\max\{\sup_{n \in \mathbb{N}} \|u_n - p\|, \sup_{n \in \mathbb{N}} \|v_n - p\|\} \leq M$. Then

$$(4) \quad \begin{aligned} \|y_n - p\| &\leq \alpha'_n \|z'_n - p\| + \beta'_n \|x_n - p\| + (1 - \alpha'_n - \beta'_n) \|u_n - p\| \\ &\leq \alpha'_n d(z'_n, P_{T_1}p) + \beta'_n \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M \\ &\leq \alpha'_n H(P_{T_1}x_n, P_{T_1}p) + \beta'_n \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M \\ &\leq (\alpha'_n + \beta'_n) \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M \\ &\leq \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M. \end{aligned}$$

It follows that

$$(5) \quad \begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|z_n - p\| + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n) \|v_n - p\| \\ &= \alpha_n d(z_n, P_{T_2}p) + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)M \\ &\leq \alpha_n H(P_{T_2}y_n, P_{T_2}p) + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)M \\ &\leq \alpha_n \|y_n - p\| + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)M \\ &\leq \alpha_n (\|x_n - p\| + (1 - \alpha'_n - \beta'_n)M) + \beta_n \|x_n - p\| \\ &\quad + (1 - \alpha_n - \beta_n)M \\ &= (\alpha_n + \beta_n) \|x_n - p\| + (\alpha_n (1 - \alpha'_n - \beta'_n) + (1 - \alpha_n - \beta_n))M \\ &\leq \|x_n - p\| + (\alpha_n (1 - \alpha'_n - \beta'_n) + (1 - \alpha_n - \beta_n))M \\ &= \|x_n - p\| + \varepsilon_n, \end{aligned}$$

where $\varepsilon_n = (\alpha_n (1 - \alpha'_n - \beta'_n) + (1 - \alpha_n - \beta_n))M$. By (ii), we have $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Thus by Lemma 2.1, we have $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2)$.

Step 2. Show that $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|z'_n - x_n\|$.

Let $p \in F(T_1) \cap F(T_2)$. By Step 1, By Step 1, there is a real number $c > 0$ such that $\lim_{n \rightarrow \infty} \|x_n - p\| = c$. Let $S = \max\{\sup_{n \in \mathbb{N}} \|v_n - y_n\|, \sup_{n \in \mathbb{N}} \|u_n - x_n\|\}$. From 4, we get

$$(6) \quad \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c.$$

Next, we consider

$$\begin{aligned}\|z_n - p + (1 - \alpha_n - \beta_n)(v_n - x_n)\| &\leq \|z_n - p\| + (1 - \alpha_n - \beta_n)\|v_n - x_n\| \\ &\leq d(z_n, P_{T_2}p) + (1 - \alpha_n - \beta_n)S \\ &\leq H(P_{T_2}y_n, P_{T_2}p) + (1 - \alpha_n - \beta_n)S \\ &\leq \|y_n - p\| + (1 - \alpha_n - \beta_n)S\end{aligned}$$

It follows that

$$\limsup_{n \rightarrow \infty} \|z_n - p + (1 - \alpha_n - \beta_n)(v_n - x_n)\| \leq c.$$

Also

$$\begin{aligned}\|x_n - p + (1 - \alpha_n - \beta_n)(v_n - x_n)\| &\leq \|x_n - p\| + (1 - \alpha_n - \beta_n)\|v_n - x_n\| \\ &\leq \|x_n - p\| + (1 - \alpha_n - \beta_n)S\end{aligned}$$

which implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + (1 - \alpha_n - \beta_n)(v_n - x_n)\| \leq c.$$

Since

$$\begin{aligned}\lim_{n \rightarrow \infty} \| &\alpha_n(z_n - p + (1 - \alpha_n - \beta_n)(v_n - x_n)) \\ &+ (1 - \alpha_n)(x_n - p + (1 - \alpha_n - \beta_n)(v_n - x_n)) \| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c,\end{aligned}$$

by Lemma 2.2, we obtain that

$$(7) \quad \lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

By the nonexpansiveness of P_{T_2} , we have

$$\begin{aligned}\|x_n - p\| &\leq \|x_n - z_n\| + \|z_n - p\| \\ &= \|x_n - z_n\| + d(z_n, P_{T_2}p) \\ &\leq \|x_n - z_n\| + H(P_{T_2}y_n, P_{T_2}p) \\ &\leq \|x_n - z_n\| + \|y_n - p\|\end{aligned}$$

which implies

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c.$$

Hence $\lim_{n \rightarrow \infty} \|y_n - p\| = c$. Since

$$\begin{aligned}y_n - p &= \alpha'_n(z'_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)) \\ &\quad + (1 - \alpha'_n)(x_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)),\end{aligned}$$

we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \| &\alpha'_n(z'_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)) \\ &+ (1 - \alpha'_n)(x_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)) \| = c.\end{aligned}$$

Moreover, we get

$$\begin{aligned}\|z'_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)\| &\leq \|z'_n - p\| + (1 - \alpha'_n - \beta'_n)\|u_n - x_n\| \\ &\leq d(z'_n, P_{T_1}p) + (1 - \alpha'_n - \beta'_n)S \\ &\leq H(P_{T_1}x_n, P_{T_1}p) + (1 - \alpha'_n - \beta'_n)S \\ &\leq \|x_n - p\| + (1 - \alpha'_n - \beta'_n)S.\end{aligned}$$

This yields that

$$\limsup_{n \rightarrow \infty} \|z'_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)\| \leq c.$$

Also

$$\begin{aligned}\|x_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)\| &\leq \|x_n - p\| + (1 - \alpha'_n - \beta'_n)\|u_n - x_n\| \\ &\leq \|x_n - p\| + (1 - \alpha'_n - \beta'_n)S.\end{aligned}$$

This implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + (1 - \alpha'_n - \beta'_n)(u_n - x_n)\| \leq c.$$

Again by Lemma 2.2, we have

$$(8) \quad \lim_{n \rightarrow \infty} \|z'_n - x_n\| = 0.$$

Step 3. Show that $\{x_n\}$ converges strongly to q for some $q \in F(T_1) \cap F(T_2)$

From Step 2, we know that $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0 = \lim_{n \rightarrow \infty} \|z'_n - x_n\|$. Also $d(x_n, T_1 x_n) \leq d(x_n, P_{T_1} x_n) \leq \|z'_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since $\{x_n\}, \{u_n\}$ are bounded, so is $\{u_n - z'_n\}$. Now, let $K = \sup_{n \in \mathbb{N}} \|u_n - z'_n\|$. By assumption and (8), we get

$$\begin{aligned}\|y_n - z'_n\| &\leq \|\alpha'_n z'_n + \beta'_n x_n + (1 - \alpha'_n - \beta'_n)u_n - z'_n\| \\ &\leq \beta'_n \|x_n - z'_n\| + (1 - \alpha'_n - \beta'_n)\|u_n - z'_n\| \\ &\leq \beta'_n \|x_n - z'_n\| + (1 - \alpha'_n - \beta'_n)K \\ (9) \quad &\rightarrow 0\end{aligned}$$

as $n \rightarrow \infty$. It follows from (8) and (9) that

$$(10) \quad \begin{aligned}\|y_n - x_n\| &\leq \|y_n - z'_n\| + \|z'_n - x_n\| \\ &\rightarrow 0\end{aligned}$$

as $n \rightarrow \infty$. It follows from (7) and (10) that

$$\begin{aligned}d(x_n, T_2 x_n) &\leq d(x_n, P_{T_2} x_n) \\ &\leq d(x_n, P_{T_2} y_n) + H(P_{T_2} y_n, P_{T_2} x_n) \\ &\leq \|x_n - z_n\| + \|y_n - x_n\| \\ &\rightarrow 0.\end{aligned}$$

Since that $\{T_1, T_2\}$ satisfies the condition (II), we have $d(x_n, F(T_1) \cap F(T_2)) \rightarrow 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(T_1) \cap F(T_2)$ such that

$$(11) \quad \|x_{n_k} - p_k\| < \frac{1}{2^k}$$

for all k . From (5), we obtain

$$\begin{aligned}\|x_{n_{k+1}} - p\| &\leq \|x_{n_{k+1}-1} - p\| + \varepsilon_{n_{k+1}-1} \\ &\leq \|x_{n_{k+1}-2} - p\| + \varepsilon_{n_{k+1}-2} + \varepsilon_{n_{k+1}-1} \\ &\vdots \\ &\leq \|x_{n_k} - p\| + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}\end{aligned}$$

for all $p \in F(T_1) \cap F(T_2)$. This implies that

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i} < \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}.$$

Next, we shall show that $\{p_k\}$ is Cauchy sequence in D . Notice that

$$\begin{aligned}\|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i} \\ &< \frac{1}{2^{k-1}} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}.\end{aligned}$$

This implies that $\{p_k\}$ is Cauchy sequence in D and thus converges to $q \in D$. Since

$$d(p_k, T_i q) \leq d(p_k, P_{T_i} q) \leq H(P_{T_i} q, P_{T_i} p_k) \leq \|q - p_k\|$$

for all $i = 1, 2$ and $p_k \rightarrow q$ as $n \rightarrow \infty$, it follows that $d(q, T_i q) = 0$ for all $i = 1, 2$ and thus $q \in F(T_1) \cap F(T_2)$. It implies by (11) that $\{x_{n_k}\}$ converges strongly to q . Since $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q . This completes the proof. $\square$

For $T_1 = T_2 = T$ and $\alpha_n + \beta_n = 1 = \alpha'_n + \beta'_n$ in Theorem 2.3, we obtain the following result.

Theorem 2.4. (See [12], Theorem 2.7) *Let E be a uniformly convex Banach space, D a nonempty, closed and convex subset of E , and $T : D \rightarrow P(D)$ a multi-valued map with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T satisfies condition (I) and $\alpha_n, \alpha'_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T .*

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Research Article

Convergence Analysis for a System of Equilibrium Problems and a Countable Family of Relatively Quasi-Nonexpansive Mappings in Banach Spaces

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We introduce a new hybrid iterative scheme for finding a common element in the solutions set of a system of equilibrium problems and the common fixed points set of an infinitely countable family of relatively quasi-nonexpansive mappings in the framework of Banach spaces. We prove the strong convergence theorem by the shrinking projection method. In addition, the results obtained in this paper can be applied to a system of variational inequality problems and to a system of convex minimization problems in a Banach space.

1. Introduction

Let E be a real Banach space, and let E^* be the dual of E . Let C be a closed and convex subset of E . Let $\{f_j\}_{j \in \Lambda}$ be bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and Λ is an arbitrary index set. The system of equilibrium problems is to find $\hat{x} \in C$ such that

$$f_j(\hat{x}, y) \geq 0, \quad \forall y \in C, j \in \Lambda. \quad (1.1)$$

If Λ is a singleton, then problem (1.1) reduces to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (1.2)$$

The set of solutions of the equilibrium problem (1.2) is denoted by $EP(f)$.

Combettes and Hirstoaga [1] introduced an iterative scheme for finding a common element in the solutions set of problem (1.1) in a Hilbert space and obtained a weak convergence theorem.

In 2004, Matsushita and Takahashi [2] introduced the following algorithm for a relatively nonexpansive mapping T in a Banach space E : for any initial point $x_0 \in C$, define the sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \quad n \geq 0, \quad (1.3)$$

where J is the duality mapping on E , Π_C is the generalized projection from E onto C , and $\{\alpha_n\}$ is a sequence in $[0, 1]$. They proved that the sequence $\{x_n\}$ converges weakly to fixed point of T under some suitable conditions on $\{\alpha_n\}$.

In 2008, Takahashi and Zembayashi [3] introduced the following iterative scheme which is called the shrinking projection method for a relatively nonexpansive mapping T and an equilibrium problem in a Banach space E :

$$\begin{aligned} x_0 &= x \in C, \quad C_0 = C, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \quad \forall y \in C, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 0. \end{aligned} \quad (1.4)$$

They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T) \cap \text{EP}(f)} x_0$ under some appropriate conditions.

2. Preliminaries and Lemmas

Let E be a real Banach space, and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *strictly convex* if, for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x + y}{2} \right\| < 1. \quad (2.1)$$

It is also said to be *uniformly convex* if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x + y}{2} \right\| < 1 - \delta. \quad (2.2)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. The function $\delta : [0, 2] \rightarrow [0, 1]$ which is called the *modulus of convexity* of E is defined as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon \right\}. \quad (2.3)$$

The space E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.4)$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in U$. The *duality mapping* $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (2.5)$$

for all $x \in E$. If E is a Hilbert space, then $J = I$, where I is the identity operator. It is also known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subset of E (see [4] for more details).

Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad (2.6)$$

for all $x, y \in E$. In a Hilbert space H , we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

Let C be a closed and convex subset of E , and let T be a mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T is said to be *relatively nonexpansive* [6–8] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [6, 7]. T is said to be *relatively quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. It is obvious that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings. The class of relatively quasi-nonexpansive mappings was studied by many authors (see, for example, [9–12]). Recall that T is closed if

$$x_n \longrightarrow x, \quad Tx_n \longrightarrow y \text{ imply } Tx = y. \quad (2.7)$$

The aim of this paper is to introduce a new hybrid projection algorithm for finding a common element in the solutions set of a system of equilibrium problems and the common fixed points set of an infinitely countable family of closed and relatively quasi-nonexpansive mappings in the frameworks of Banach spaces.

We will need the following lemmas.

Lemma 2.1 (Kamimura and Takahashi [8]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Let E be a reflexive, strictly convex, and smooth Banach space, and let C be a nonempty, closed, and convex subset of E . The *generalized projection mapping*, introduced by Alber [13], is a mapping $\Pi_C : E \rightarrow C$ that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$; that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}. \quad (2.8)$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional ϕ and strict monotonicity of the duality mapping J (see, for instance, [4, 8, 13–15]). In a Hilbert space, Π_C is coincident with the metric projection.

Lemma 2.2 (Alber [13], Kamimura and Takahashi [8]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E , let $x \in E$, and let $z \in C$. Then $z = \Pi_C x$ if and only if*

$$\langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C. \quad (2.9)$$

Lemma 2.3 (Alber [13], Kamimura and Takahashi [8]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall x \in C, y \in E. \quad (2.10)$$

Lemma 2.4 (Qin et al. [16]). *Let E be a uniformly convex, smooth Banach space, and let C be a closed and convex subset of E . Let T be a closed and relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone; that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Lemma 2.5 (Blum and Oettli [17]). *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.11)$$

Lemma 2.6 (Takahashi and Zembayashi [18]). *Let C be a closed and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). For all $r > 0$ and $x \in E$, define the mapping $T_r^f : E \rightarrow C$ as follows:*

$$T_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \quad (2.12)$$

Then, the following statements hold:

- (1) T_r^f is single valued;
- (2) T_r^f is of firmly nonexpansive type [19]; that is, for all $x, y \in E$,

$$\langle T_r^f x - T_r^f y, JT_r^f x - JT_r^f y \rangle \leq \langle T_r^f x - T_r^f y, Jx - Jy \rangle; \quad (2.13)$$

- (3) $F(T_r^f) = EP(f)$;
- (4) $EP(f)$ is closed and convex.

Lemma 2.7 (Takahashi and Zembayashi [18]). *Let C be a closed and convex subset of a smooth, strictly, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r^f)$,*

$$\phi(q, T_r^f x) + \phi(T_r^f x, x) \leq \phi(q, x). \quad (2.14)$$

3. Strong Convergence Theorems

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E . Let $\{f_j\}_{j=1}^M$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself. Assume that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M EP(f_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_{n,i} &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} &= T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \quad (3.1)$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \dots, M$ are sequences which satisfy the following conditions:

- (B1) $\limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (B2) $\liminf_{n \rightarrow \infty} r_{j,n} > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. We divide our proof into six steps as follows.

Step 1. $F \subset C_n$ for all $n \geq 1$.

From Lemma 2.4 we know that $F(T_i)$ is closed, and convex for all $i \geq 1$. From Lemma 2.6(4), we also know that $\text{EP}(f_j)$ is closed and convex for each $j = 1, 2, \dots, M$. Hence $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^M \text{EP}(f_j))$ is a nonempty, closed and convex subset of C . Clearly $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For each $z \in C_k$ and $i \geq 1$, we see that $\phi(z, u_{k,i}) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k \rangle - 2\langle z, Ju_{k,i} \rangle \leq \|x_k\|^2 - \|u_{k,i}\|^2. \quad (3.2)$$

By the construction of the set C_{k+1} , we see that

$$\begin{aligned} C_{k+1} &= \left\{ z \in C_k : \sup_{i \geq 1} \phi(z, u_{k,i}) \leq \phi(z, x_k) \right\} \\ &= \bigcap_{i=1}^{\infty} \{ z \in C_k : \phi(z, u_{k,i}) \leq \phi(z, x_k) \}. \end{aligned} \quad (3.3)$$

Hence C_{k+1} is also closed and convex.

It is obvious that $F \subset C_1 = C$. Now, suppose that $F \subset C_k$ for some $k \in \mathbb{N}$, and let $p \in F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^M \text{EP}(f_j))$. Then

$$\begin{aligned} \phi(p, u_{k,i}) &= \phi\left(p, T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{k,i}\right) \\ &\leq \phi\left(p, T_{r_{M-1,n}}^{f_{M-1}} T_{r_{M-2,n}}^{f_{M-2}} \cdots T_{r_{1,n}}^{f_1} y_{k,i}\right) \\ &\vdots \\ &\leq \phi\left(p, T_{r_{1,n}}^{f_1} y_{k,i}\right) \\ &\leq \phi(p, y_{k,i}) \\ &= \phi\left(p, J^{-1}(\alpha_k Jx_k + (1 - \alpha_k)JT_i x_k)\right) \\ &= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1 - \alpha_k)JT_i x_k \rangle \\ &\quad + \|\alpha_k Jx_k + (1 - \alpha_k)JT_i x_k\|^2 \\ &\leq \|p\|^2 - 2\alpha_k \langle p, Jx_k \rangle - 2(1 - \alpha_k) \langle p, JT_i x_k \rangle \\ &\quad + \alpha_k \|x_k\|^2 + (1 - \alpha_k) \|T_i x_k\|^2 \\ &= \alpha_k \phi(p, x_k) + (1 - \alpha_k) \phi(p, T_i x_k) \\ &\leq \phi(p, x_k). \end{aligned} \quad (3.4)$$

Hence $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$.

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Step 2. $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists.

From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad n \geq 1. \quad (3.5)$$

From Lemma 2.3 we get that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0). \quad (3.6)$$

Combining (3.5) and (3.6), we get that $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists.

Step 3. $\{x_n\}$ is a Cauchy sequence in C .

Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for $m > n$, we obtain from Lemma 2.3 that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \quad (3.7)$$

We see that $\phi(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$, which implies with Lemma 2.1 that $\|x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore $\{x_n\}$ is a Cauchy sequence. By the completeness of the space E and the closedness of the set C , we can assume that $x_n \rightarrow q \in C$ as $n \rightarrow \infty$. Moreover, we get that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0. \quad (3.8)$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have for all $i \geq 1$ that

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) \rightarrow 0. \quad (3.9)$$

Applying Lemma 2.1 to (3.8) and (3.9), we derive

$$\lim_{n \rightarrow \infty} \|u_{n,i} - x_n\| = 0, \quad \forall i \geq 1. \quad (3.10)$$

This shows that $u_{n,i} \rightarrow q$ as $n \rightarrow \infty$ for all $i \geq 1$. Since J is uniformly norm-to-norm continuous on bounded subsets of E , we obtain that

$$\lim_{n \rightarrow \infty} \|Ju_{n,i} - Jx_n\| = 0, \quad \forall i \geq 1. \quad (3.11)$$

Step 4. $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Denote $\Theta_n^j = T_{r_{j,n}}^{f_j} T_{r_{j-1,n}}^{f_{j-1}} \cdots T_{r_{1,n}}^{f_1}$ for any $j \in \{1, 2, \dots, M\}$ and $\Theta_n^0 = I$ for all $n \geq 1$. We note that $u_{n,i} = \Theta_n^M y_{n,i}$ for all $i \geq 1$. From (3.4) we observe that

$$\phi(p, \Theta_n^{M-1} y_{n,i}) \leq \phi(p, \Theta_n^{M-2} y_{n,i}) \leq \cdots \leq \phi(p, y_{n,i}) \leq \phi(p, x_n), \quad \forall i \geq 1. \quad (3.12)$$

Since $p \in \text{EP}(f_M) = F(T_{r_{M,n}}^{f_M})$ for all $n \geq 1$, it follows from (3.12) and Lemma 2.7 that

$$\begin{aligned}\phi(u_{n,i}, \Theta_n^{M-1} y_{n,i}) &\leq \phi(p, \Theta_n^{M-1} y_{n,i}) - \phi(p, u_{n,i}) \\ &\leq \phi(p, x_n) - \phi(p, u_{n,i}).\end{aligned}\quad (3.13)$$

From (3.10) and (3.11), we get that $\lim_{n \rightarrow \infty} \phi(u_{n,i}, \Theta_n^{M-1} y_{n,i}) = 0$ for all $i \geq 1$. From Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \|u_{n,i} - \Theta_n^{M-1} y_{n,i}\| = 0, \quad \forall i \geq 1. \quad (3.14)$$

From (3.10) and (3.14), we have

$$\lim_{n \rightarrow \infty} \|x_n - \Theta_n^{M-1} y_{n,i}\| = 0, \quad \forall i \geq 1, \quad (3.15)$$

and hence,

$$\lim_{n \rightarrow \infty} \|Jx_n - J\Theta_n^{M-1} y_{n,i}\| = 0, \quad \forall i \geq 1. \quad (3.16)$$

Again, since $p \in \text{EP}(f_{M-1}) = F(T_{r_{M-1,n}}^{f_{M-1}})$ for all $n \geq 1$, it follows from (3.12) and Lemma 2.7 that

$$\begin{aligned}\phi(\Theta_n^{M-1} y_{n,i}, \Theta_n^{M-2} y_{n,i}) &\leq \phi(p, \Theta_n^{M-2} y_{n,i}) - \phi(p, \Theta_n^{M-1} y_{n,i}) \\ &\leq \phi(p, x_n) - \phi(p, \Theta_n^{M-1} y_{n,i}).\end{aligned}\quad (3.17)$$

From (3.15) and (3.16), we also have

$$\lim_{n \rightarrow \infty} \|\Theta_n^{M-1} y_{n,i} - \Theta_n^{M-2} y_{n,i}\| = 0, \quad \forall i \geq 1. \quad (3.18)$$

Hence, from (3.15) and (3.18), we get

$$\lim_{n \rightarrow \infty} \|x_n - \Theta_n^{M-2} y_{n,i}\| = 0, \quad \forall i \geq 1, \quad (3.19)$$

$$\lim_{n \rightarrow \infty} \|Jx_n - J\Theta_n^{M-2} y_{n,i}\| = 0, \quad \forall i \geq 1. \quad (3.20)$$

In a similar way, we can verify that

$$\lim_{n \rightarrow \infty} \|\Theta_n^{M-2} y_{n,i} - \Theta_n^{M-3} y_{n,i}\| = \cdots = \lim_{n \rightarrow \infty} \|\Theta_n^1 y_{n,i} - y_{n,i}\| = 0 \quad (3.21)$$

for all $i \geq 1$,

$$\lim_{n \rightarrow \infty} \|x_n - \Theta_n^{M-3} y_{n,i}\| = \cdots = \lim_{n \rightarrow \infty} \|x_n - y_{n,i}\| = 0 \quad (3.22)$$

for all $i \geq 1$,

$$\lim_{n \rightarrow \infty} \|Jx_n - J\Theta_n^{M-3} y_{n,i}\| = \cdots = \lim_{n \rightarrow \infty} \|Jx_n - Jy_{n,i}\| = 0 \quad (3.23)$$

for all $i \geq 1$. Hence, we can conclude that

$$\lim_{n \rightarrow \infty} \|\Theta_n^j y_{n,i} - \Theta_n^{j-1} y_{n,i}\| = 0 \quad (3.24)$$

for each $j = 1, 2, \dots, M$ and $i \geq 1$. Observe that

$$\begin{aligned} \|Jy_{n,i} - Jx_n\| &= \|\alpha_n Jx_n + (1 - \alpha_n) JT_i x_n - Jx_n\| \\ &= (1 - \alpha_n) \|JT_i x_n - Jx_n\|, \end{aligned} \quad (3.25)$$

then we obtain from (B1) and (3.23) that

$$\lim_{n \rightarrow \infty} \|JT_i x_n - Jx_n\| = 0, \quad \forall i \geq 1. \quad (3.26)$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets, we get that

$$\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0, \quad \forall i \geq 1. \quad (3.27)$$

Since T_i is closed for all $i \geq 1$ and $x_n \rightarrow q$, we conclude that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Step 5. $q \in \bigcap_{j=1}^M \text{EP}(f_j)$.

From (3.24) and (B2), we have that $\|J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i}\|/r_{j,n} \rightarrow 0$ as $n \rightarrow \infty$. Then, for each $j = 1, 2, \dots, M$, we obtain that

$$f_j(\Theta_n^j y_{n,i}, y) + \frac{1}{r_{j,n}} \langle y - \Theta_n^j y_{n,i}, J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i} \rangle \geq 0, \quad \forall y \in C. \quad (3.28)$$

From (A2) we have that

$$\begin{aligned} \|y - \Theta_n^j y_{n,i}\| \frac{\|J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i}\|}{r_{j,n}} &\geq \frac{1}{r_{j,n}} \langle y - \Theta_n^j y_{n,i}, J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i} \rangle \\ &\geq -f_j(\Theta_n^j y_{n,i}, y) \geq f_j(y, \Theta_n^j y_{n,i}), \quad \forall y \in C. \end{aligned} \quad (3.29)$$

From (A4) and the fact that $\Theta_n^j y_{n,i} \rightarrow q$ for $i \geq 1$, we get $f_j(y, q) \leq 0$ for all $y \in C$. For each $0 < t < 1$ and $y \in C$, denote $y_t = ty + (1-t)q$. Then $y_t \in C$, which implies that $f_j(y_t, q) \leq 0$. From (A1) and (A4), we obtain that $0 = f_j(y_t, y_t) \leq tf_j(y_t, y) + (1-t)f_j(y_t, q) \leq tf_j(y_t, y)$. Thus, $f_j(y_t, y) \geq 0$. From (A3), we have $f_j(q, y) \geq 0$ for all $y \in C$ and $j = 1, 2, \dots, M$. Hence $q \in \bigcap_{j=1}^M \text{EP}(f_j)$.

Step 6. $q = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0 \quad \forall z \in C_n. \quad (3.30)$$

Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - p \rangle \geq 0 \quad \forall p \in F. \quad (3.31)$$

Letting $n \rightarrow \infty$ in (3.31), we obtain that

$$\langle Jx_0 - Jq, q - p \rangle \geq 0 \quad \forall p \in F. \quad (3.32)$$

From Lemma 2.2 we conclude that $q = \Pi_F x_0$. This completes the proof. $\square$

Remark 3.2. Theorem 3.1 improves and extends Theorem 3.1 of Takahashi and Zembayashi in [3] in the following senses:

- (i) from the case of an equilibrium problem to a finite family of equilibrium problems;
- (ii) from a single relatively nonexpansive mapping to an infinitely countable family of relatively quasi-nonexpansive mappings;
- (iii) if $M = 1$ and $T_i = T$ for all $i \geq 1$, then our restriction on $\{\alpha_n\}$ is weaker than that of Theorem 3.1 of [3].

Remark 3.3. The iteration (3.1) is a modification of (1.4) in the following ways.

- (i) We use the composition of mappings $\{T_{r_{j,n}}^{f_j}\}_{j=1}^M$ in the second step.
- (ii) We construct the set C_{n+1} by using the concept of supremum concerning an infinitely countable family of closed and relatively quasi-nonexpansive mappings $\{T_i\}_{i=1}^\infty$. If $M = 1$ and $T_i = T$ for all $i \geq 1$, then the iteration (3.1) reduces to that of (1.4).

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.1, then we have the following corollary.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E . Let $\{f_j\}_{j=1}^M$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies*

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conditions (A1)–(A4), and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself. Assume that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M EP(f_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} y_{n,i} &= T_i x_n, \\ u_{n,i} &= T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \tag{3.33}$$

If $\liminf_{n \rightarrow \infty} r_{j,n} > 0$ for each $j = 1, 2, \dots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

4. Applications

In this section, we give several applications of Theorem 3.1 in the framework of Banach spaces and Hilbert spaces.

Let $A : C \rightarrow E^*$ be a nonlinear mapping. The classical variational inequality problem is to find that $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0 \quad \forall y \in C. \tag{4.1}$$

The solutions set of (4.1) is denoted by $VI(C, A)$. For each $r > 0$ and $x \in E$, define the mapping $T_r^A : E \rightarrow C$ as follows:

$$T_r^A(x) = \left\{ z \in C : \langle Az, y - z \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \tag{4.2}$$

Theorem 4.1. Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E . Let $\{A_j\}_{j=1}^M$ be continuous and monotone operators from C to E^* , and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M VI(C, A_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} y_{n,i} &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} &= T_{r_{M,n}}^{A_M} T_{r_{M-1,n}}^{A_{M-1}} \cdots T_{r_{1,n}}^{A_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \tag{4.3}$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \dots, M$ are sequences which satisfy conditions (B1) and (B2) of Theorem 3.1.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Define $f_j(x, y) = \langle A_j x, y - x \rangle$ for all $x, y \in C$ and $j = 1, 2, \dots, M$. First, we see that $F(T_{r_j}^{f_j}) = \text{EP}(f_j) = \text{VI}(C, A_j) = F(T_{r_j}^{A_j})$ for each $j = 1, 2, \dots, M$.

Next, we show that $\{f_j\}_{j=1}^M$ satisfy conditions (A1)–(A4).

(A1) Consider $f_j(x, x) = \langle A_j x, x - x \rangle = 0$ for all $x \in C$ and $j = 1, 2, \dots, M$.

(A2) For each $x, y \in C$ and $j = 1, 2, \dots, M$, we observe that

$$\begin{aligned} f_j(x, y) + f_j(y, x) &= \langle A_j x, y - x \rangle + \langle A_j y, x - y \rangle \\ &= \langle A_j x - A_j y, y - x \rangle. \end{aligned} \quad (4.4)$$

By the monotonicity of A_j , we obtain that f_j is monotone. Thus $\{f_j\}_{j=1}^M$ satisfy condition (A2).

(A3) For each $x, y, z \in C$ and $j = 1, 2, \dots, M$, we have by the continuity of A_j that

$$\begin{aligned} \limsup_{t \downarrow 0} f_j(tz + (1-t)x, y) &= \limsup_{t \downarrow 0} \langle A_j(tz + (1-t)x), y - (tz + (1-t)x) \rangle \\ &= \langle A_j x, y - x \rangle \\ &= f_j(x, y). \end{aligned} \quad (4.5)$$

This shows that $\{f_j\}_{j=1}^M$ satisfy condition (A3).

(A4) Let $u, v \in C$ and $s \in (0, 1)$. Then, for each $x \in C$ and $j = 1, 2, \dots, M$, we have

$$\begin{aligned} f_j(x, su + (1-s)v) &= \langle A_j x, su + (1-s)v - x \rangle \\ &= s \langle A_j x, u - x \rangle + (1-s) \langle A_j x, v - x \rangle \\ &= s f_j(x, u) + (1-s) f_j(x, v). \end{aligned} \quad (4.6)$$

Thus f_j is convex in the second variable. Let $u_n \in C$ and $\lim_{n \rightarrow \infty} u_n = u$. Then

$$\begin{aligned} f_j(x, u) &= \langle A_j x, u - x \rangle \\ &= \lim_{n \rightarrow \infty} \langle A_j x, u_n - x \rangle \\ &= \lim_{n \rightarrow \infty} f_j(x, u_n). \end{aligned} \quad (4.7)$$

This shows that f_j is lower semicontinuous in the second variable. Hence $\{f_j\}_{j=1}^M$ satisfy condition (A4). From Theorem 3.1 we obtain the desired result. $\square$

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.1, then we have the following corollary.

Corollary 4.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E . Let $\{A_j\}_{j=1}^M$ be continuous and monotone operators from C to E^* , and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M VI(C, A_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_{n,i} &= T_i x_n, \\ u_{n,i} &= T_{r_{M,n}}^{A_M} T_{r_{M-1,n}}^{A_{M-1}} \cdots T_{r_{1,n}}^{A_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \tag{4.8}$$

If $\liminf_{n \rightarrow \infty} r_{j,n} > 0$ for each $j = 1, 2, \dots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function. The convex minimization problem is to find that $\hat{x} \in C$ such that

$$\varphi(\hat{x}) \leq \varphi(y) \quad \forall y \in C. \tag{4.9}$$

The solutions set of (4.9) is denoted by $CMP(\varphi)$. For each $r > 0$ and $x \in E$, define the mapping $T_r^\varphi : E \rightarrow C$ as follows:

$$T_r^\varphi(x) = \left\{ z \in C : \varphi(y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq \varphi(z), \forall y \in C \right\}. \tag{4.10}$$

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E . Let $\{\varphi_j\}_{j=1}^M$ be lower semicontinuous and convex functions from C to $\mathbb{R}$, and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M CMP(\varphi_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_{n,i} &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} &= T_{r_{M,n}}^{\varphi_M} T_{r_{M-1,n}}^{\varphi_{M-1}} \cdots T_{r_{1,n}}^{\varphi_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \tag{4.11}$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \dots, M$ are sequences which satisfy conditions (B1) and (B2) of Theorem 3.1.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Define $f_j(x, y) = \varphi_j(y) - \varphi_j(x)$ for all $x, y \in C$ and $j = 1, 2, \dots, M$. Then $F(T_{r_j}^{f_j}) = \text{EP}(f_j) = \text{CMP}(\varphi_j) = F(T_{r_j}^{\varphi_j})$ for each $j = 1, 2, \dots, M$, and therefore $\{f_j\}_{j=1}^M$ satisfy conditions (A1) and (A2).

Next, we show that $\{f_j\}_{j=1}^M$ satisfy conditions (A3) and (A4). For each $x, y, z \in C$, we have by the lower semicontinuity of φ_j that

$$\begin{aligned} \lim_{t \downarrow 0} \sup f_j(tz + (1-t)x, y) &= \lim_{t \downarrow 0} \sup (\varphi_j(y) - \varphi_j(tz + (1-t)x)) \\ &\leq \varphi_j(y) - \varphi_j(x) \\ &= f_j(x, y). \end{aligned} \quad (4.12)$$

This implies that $\{f_j\}_{j=1}^M$ satisfy condition (A3).

Let $u, v \in C$ and $s \in (0, 1)$. For each $x \in C$, we have by the convexity of φ_j that

$$\begin{aligned} f_j(x, su + (1-s)v) &= \varphi_j(su + (1-s)v) - \varphi_j(x) \\ &\leq s\varphi_j(u) + (1-s)\varphi_j(v) - \varphi_j(x) \\ &= s(\varphi_j(u) - \varphi_j(x)) + (1-s)(\varphi_j(v) - \varphi_j(x)) \\ &= sf_j(x, u) + (1-s)f_j(x, v). \end{aligned} \quad (4.13)$$

On the other hand, let $u_n \in C$ and $\lim_{n \rightarrow \infty} u_n = u$. By the lower semicontinuity of φ_j we have

$$\begin{aligned} f_j(x, u) &= \varphi_j(u) - \varphi_j(x) \\ &\leq \liminf_{n \rightarrow \infty} (\varphi_j(u_n) - \varphi_j(x)) \\ &= \liminf_{n \rightarrow \infty} f_j(x, u_n). \end{aligned} \quad (4.14)$$

Thus $\{f_j\}_{j=1}^M$ satisfy condition (A4). From Theorem 3.1 we also obtain the desired result. $\square$

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.3, then we have the following corollary.

Corollary 4.4. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E . Let $\{\varphi_j\}_{j=1}^M$ be lower semicontinuous and convex*

functions from C to $\mathbb{R}$, and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M \text{CMP}(\varphi_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} y_{n,i} &= T_i x_n, \\ u_{n,i} &= T_{r_{M,n}}^{\varphi_M} T_{r_{M-1,n}}^{\varphi_{M-1}} \cdots T_{r_{1,n}}^{\varphi_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \tag{4.15}$$

If $\liminf_{n \rightarrow \infty} r_{j,n} > 0$ for each $j = 1, 2, \dots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

As a direct consequence of Theorem 3.1, we obtain the following application in a Hilbert space.

Theorem 4.5. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $\{f_j\}_{j=1}^M$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^\infty F(T_i)) \cap (\bigcap_{j=1}^M \text{EP}(f_j)) \neq \emptyset$. For any initial point $x_0 \in H$ with $x_1 = P_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} y_{n,i} &= \alpha_n x_n + (1 - \alpha_n) T_i x_n, \\ u_{n,i} &= T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \|z - u_{n,i}\| \leq \|z - x_n\| \right\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 1, \end{aligned} \tag{4.16}$$

where P is the metric projection. Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \dots, M$ are sequences which satisfy conditions (B1) and (B2) of Theorem 3.1.

Then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. Taking $E = H$ in Theorem 3.1, the result is obtained immediately. $\square$

Remark 4.6. Theorem 4.5 improves and extends the main results of [20–22] in the following senses:

- (i) from the case of an equilibrium problem to a finite family of equilibrium problems;
- (ii) from the class of nonexpansive mappings to the class of an infinitely countable family of quasi-nonexpansive mappings.

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Strong convergence theorems for a general system of variational inequality problems, mixed equilibrium problems and fixed points problems with applications

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ABSTRACT

In this paper, we introduce a new iterative scheme for finding a common element of the set of solutions of a general system of variational inequalities, the set of solutions of a mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Using the demi-closedness principle for nonexpansive mapping, we prove that the iterative sequence converges strongly to a common element of the above three sets under some control conditions. Our main result extends a recent result of Ceng, Wang and Yao [L.C. Ceng, C.Y. Wang and J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res. 67 (2008) 375–390].

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C be a nonempty closed convex subset of H . A mapping $A : C \rightarrow H$ is called α -inverse-strongly monotone, if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that every α -inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping A is called relaxed (c, d) -cocoercive, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \quad \forall x, y \in C.$$

Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C.$$

The variational inequality has been widely studied in the literature; see, e.g., [1–4] and the references therein.

For solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k -Lipschitz-continuous and $VI(C, A)$ is nonempty, Korpelevich [5] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

for every $n = 0, 1, 2, \dots$, where $\lambda \in (0, 1/k)$.

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In 2007, Yao and Yao [6] introduced the following iterative scheme: Let C be a closed convex subset of a real Hilbert space H . Let A be an α -inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = v \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$ and $\{a_n\}, \{b_n\} \subset [0, 1]$. They proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for α -inverse-strongly monotone mapping under some control conditions.

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $A, B : C \rightarrow H$ be two mappings. In 2008, Ceng et al. [7] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \quad (1.1)$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \quad (1.2)$$

which is defined by Verma [8], and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.2) reduces to the classical variational inequality $VI(C, A)$. Ceng et al. [7] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of problem (1.1) for the α and β -inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $x_1 = v \in C$ and let $\{x_n\}$ and $\{y_n\}$ be given by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{a_n\}, \{b_n\} \subset [0, 1]$. Then, they proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of problem (1.1) under some control conditions.

Let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [9] considered the following mixed equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (1.3)$$

The set of solutions of (1.3) is denoted by $MEP(F, \varphi)$. It is easy to see that x is a solution of problem (1.3) and implies that $x \in \text{dom } \varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the mixed equilibrium problem (1.3) becomes the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (1.4)$$

The set of solutions of (1.4) is denoted by $EP(F)$.

If $\varphi = 0$ and $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$, where A is a mapping from C into H , then (1.3) reduces to the classical variational inequality.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. In 1999, Atsushiba and Takahashi [10] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1} T_1 + (1 - \lambda_{n,1}) I, \\ U_{n,2} &= \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) I, \\ U_{n,3} &= \lambda_{n,3} T_3 U_{n,2} + (1 - \lambda_{n,3}) I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\ W_n &= U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I, \end{aligned}$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the W -mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [11] proved that if X is a strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1$, $i = 1, 2, \dots, N$.

For each $n \in \mathbb{N}$, and $j = 1, 2, \dots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. Very recently, in 2009, Kangtunyakarn and Suantai [12] introduced the new mapping $S_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I, \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I, \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I, \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I, \\ S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \end{aligned}$$

The mapping S_n is called the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. Nonexpansivity of each T_i ensures the nonexpansivity of S_n .

In this paper, we introduce a new iterative scheme for finding a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of a general system of variational inequalities in a real Hilbert space. Using the demi-closedness principle for nonexpansive mapping, we prove the strong convergence of the proposed iterative method under suitable conditions.

2. Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H . For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C . It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.1)$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.2)$$

Recall that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} \langle x - P_C x, y - P_C x \rangle &\leq 0, \\ \|x - y\|^2 &\geq \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{aligned} \quad (2.3)$$

for all $x \in H$ and $y \in C$; see [13] for more details.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F , φ and the set C :

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
- (A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;
- (A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
- (B1) For each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z).$$

- (B2) C is a bounded set.

In the sequel we shall need to use the following lemmas.

Lemma 2.1 ([14]). Let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows.

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \quad \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:

- (1) For each $x \in H$, $T_r(x) \neq \emptyset$;
- (2) T_r is single-valued;
- (3) T_r is firmly nonexpansive, i.e. for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (4) $F(T_r) = \text{MEP}(F, \varphi)$;
- (5) $\text{MEP}(F, \varphi)$ is closed and convex.

Lemma 2.2 ([15]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.3 ([16]). Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2.$$

Lemma 2.4 ([17]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.5 ([13]). Demi-closedness principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H . If T has a fixed point, then $I - T$ is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \rightarrow y$), it follows that $(I - T)x = y$. Here I is the identity operator of H .

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. In a real Hilbert space H , there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 2.7 ([12]). Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \dots, N$, where $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N - 1$, $\alpha_1^N \in (0, 1]$ and $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let S be the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.

Lemma 2.8 ([12]). Let C be a nonempty closed convex subset of a Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and for all $n \in \mathbb{N}$ and all $j \in \{1, 2, \dots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Suppose $\alpha_i^{n,j} \rightarrow \alpha_i^j$ as $n \rightarrow \infty$ for all $i \in \{1, 3\}$ and all $j = 1, 2, 3, \dots, N$. Let S and S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ and $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. Then $\lim_{n \rightarrow \infty} \|S_n x - Sx\| = 0$ for every $x \in C$.

Lemma 2.9 ([7]). For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.1) if and only if x^* is a fixed of the mapping $G : C \rightarrow C$ defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $\text{GVI}(C, A, B)$.

Next, we prove a lemma which is very useful for our consideration.

Lemma 2.10. Let C be a nonempty closed and convex subset of a real Hilbert space H and let $\lambda, \mu > 0$ and $A, B : C \rightarrow H$ be two mappings. Let $G : C \rightarrow C$ be defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C.$$

If $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings, then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned}\|G(x) - G(y)\| &= \|P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)] - P_C[P_C(y - \mu By) - \lambda AP_C(y - \mu By)]\|^2 \\ &\leq \|P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx) - (P_C(y - \mu By) - \lambda AP_C(y - \mu By))\| \\ &= \|(I - \lambda A)P_C(I - \mu B)x - (I - \lambda A)P_C(I - \mu B)y\| \\ &\leq \|x - y\|.\end{aligned}$$

This shows that G is a nonexpansive mapping. $\square$

3. Main results

In this section, we introduce a new iterative scheme for finding the common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of a general system of variational inequalities in a real Hilbert space. Using the demi-closedness principle for nonexpansive mapping, we show that the iterative sequence converges strongly to a common element of the three sets.

Theorem 3.1. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $\lambda, \mu > 0$ and let $A, B : C \rightarrow H$ be two mappings. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C . Let S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $j = 1, 2, \dots, N$. Assume that either (B1) or (B2) holds and $\Omega := \bigcap_{i=1}^N F(T_i) \cap GVI(C, A, B) \cap MEP(F, \varphi) \neq \emptyset$. Let $x_1 = v \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n P_C(y_n - \lambda A y_n), & n \geq 1, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $\{a_n\}, \{b_n\}$ are two sequences in $[0, 1]$.

Suppose that the following conditions are satisfied :

- (C1) $\lim_{n \rightarrow \infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;
- (C2) $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$;
- (C3) $\liminf_{n \rightarrow \infty} r_n > 0$ and $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$;
- (C4) $\lim_{n \rightarrow \infty} |\alpha_1^{n+1,i} - \alpha_1^{n,i}| = 0$ for all $i \in \{1, 2, \dots, N\}$,
 $\lim_{n \rightarrow \infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| = 0$ for all $j \in \{2, 3, \dots, N\}$ and
- (C5) $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_2]$ with $0 \leq \theta_2 < 1$.

If $I - \lambda A$ and $I - \mu B$ are nonexpansive and $\lim_{n \rightarrow \infty} \|A y_n - A y^*\| = \lim_{n \rightarrow \infty} \|B u_n - B x^*\| = 0$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu B x^*)$, then $\{x_n\}$ converges strongly to $\bar{x} = P_{\Omega} v$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.1), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. It follows from Lemma 2.9 that

$$x^* = P_C[P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)].$$

Put $y^* = P_C(x^* - \mu B x^*)$ and $t_n = P_C(y_n - \lambda A y_n)$, then $x^* = P_C(y^* - \lambda A y^*)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n.$$

Since $I - \lambda A, I - \mu B, T_{r_n}$ and P_C are nonexpansive mappings, we have

$$\begin{aligned}\|t_n - x^*\|^2 &= \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\ &\leq \|y_n - y^*\|^2 \\ &= \|P_C(u_n - \mu B u_n) - P_C(x^* - \mu B x^*)\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \\ &\leq \|x_n - x^*\|^2,\end{aligned}\tag{3.1}$$

which, implies that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n - x^*\| \\ &\leq a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|t_n - x^*\| \\ &\leq a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|x_n - x^*\| \\ &\leq \max\{\|v - x^*\|, \|x_1 - x^*\|\}.\end{aligned}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bu_n\}$ and $\{S_n t_n\}$ are also bounded. Also, observe that

$$\begin{aligned}\|t_{n+1} - t_n\| &= \|P_C(y_{n+1} - \lambda A y_{n+1}) - P_C(y_n - \lambda A y_n)\| \\ &\leq \|y_{n+1} - y_n\| \\ &= \|P_C(u_{n+1} - \mu B u_{n+1}) - P_C(u_n - \mu B u_n)\| \\ &\leq \|u_{n+1} - u_n\|.\end{aligned}\tag{3.2}$$

On the other hand, from $u_n = T_{r_n} x_n \in \text{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}} x_{n+1} \in \text{dom}\varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,\tag{3.3}$$

and

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C.\tag{3.4}$$

Putting $y = u_{n+1}$ in (3.3) and $y = u_n$ in (3.4), we have

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0,$$

and

$$F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

From the monotonicity of F , we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0,$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\},\end{aligned}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|.\tag{3.5}$$

It follows from (3.2) and (3.5) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|.\tag{3.6}$$

Let $x_{n+1} = b_n x_n + (1 - b_n) z_n$. Then, we obtain

$$\begin{aligned}z_{n+1} - z_n &= \frac{x_{n+2} - b_{n+1} x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\ &= \frac{a_{n+1} v + (1 - a_{n+1} - b_{n+1}) S_{n+1} t_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n) S_n t_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}} (v - S_{n+1} t_{n+1}) + \frac{a_n}{1 - b_n} (S_n t_n - v) + S_{n+1} t_{n+1} - S_n t_n.\end{aligned}\tag{3.7}$$

Next, we estimate $\|S_{n+1}t_{n+1} - S_n t_n\|$.

For each $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}
 \|U_{n+1,k}t_n - U_{n,k}t_n\| &= \|\alpha_1^{n+1,k}T_k U_{n+1,k-1}t_n + \alpha_2^{n+1,k}U_{n+1,k-1}t_n + \alpha_3^{n+1,k}t_n - \alpha_1^{n,k}T_k U_{n,k-1}t_n - \alpha_2^{n,k}U_{n,k-1}t_n - \alpha_3^{n,k}t_n\| \\
 &= \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}t_n - T_k U_{n,k-1}t_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_k U_{n,k-1}t_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k})t_n \\
 &\quad + \alpha_2^{n+1,k}(U_{n+1,k-1}t_n - U_{n,k-1}t_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}t_n\| \\
 &\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}t_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| \\
 &\quad + \alpha_2^{n+1,k}\|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}t_n\| \\
 &= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}t_n\| \\
 &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}t_n\| \\
 &\leq \|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}t_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| \\
 &\quad + |(\alpha_1^{n,k} - \alpha_1^{n+1,k}) + (\alpha_3^{n,k} - \alpha_3^{n+1,k})|\|U_{n,k-1}t_n\| \\
 &\leq \|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}t_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| \\
 &\quad + |\alpha_1^{n,k} - \alpha_1^{n+1,k}|\|U_{n,k-1}t_n\| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|\|U_{n,k-1}t_n\| \\
 &= \|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_k U_{n,k-1}t_n\| + \|U_{n,k-1}t_n\|) \\
 &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|(\|t_n\| + \|U_{n,k-1}t_n\|). \tag{3.8}
 \end{aligned}$$

It follows from (3.8) that

$$\begin{aligned}
 \|S_{n+1}t_n - S_n t_n\| &= \|U_{n+1,N}t_n - U_{n,N}t_n\| \\
 &\leq \|U_{n+1,1}t_n - U_{n,1}t_n\| + \sum_{j=2}^N |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_j U_{n,j-1}t_n\| + \|U_{n,j-1}t_n\|) \\
 &\quad + \sum_{j=2}^N |\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|t_n\| + \|U_{n,j-1}t_n\|) \\
 &= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1 t_n - t_n\| + \sum_{j=2}^N |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_j U_{n,j-1}t_n\| + \|U_{n,j-1}t_n\|) \\
 &\quad + \sum_{j=2}^N |\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|t_n\| + \|U_{n,j-1}t_n\|).
 \end{aligned}$$

This together with the condition (C4), we obtain

$$\lim_{n \rightarrow \infty} \|S_{n+1}t_n - S_n t_n\| = 0. \tag{3.9}$$

It follows from (3.6) that

$$\begin{aligned}
 \|S_{n+1}t_{n+1} - S_n t_n\| &\leq \|t_{n+1} - t_n\| + \|S_{n+1}t_n - S_n t_n\| \\
 &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| + \|S_{n+1}t_n - S_n t_n\|. \tag{3.10}
 \end{aligned}$$

By (3.7) and (3.10), we have

$$\begin{aligned}
 \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{a_{n+1}}{1 - b_{n+1}}\|v - S_{n+1}t_{n+1}\| + \frac{a_n}{1 - b_n}\|S_n t_n - v\| + \|S_{n+1}t_{n+1} - S_n t_n\| - \|x_{n+1} - x_n\| \\
 &\leq \frac{a_{n+1}}{1 - b_{n+1}}\|v - S_{n+1}t_{n+1}\| + \frac{a_n}{1 - b_n}\|S_n t_n - v\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
 &\quad + \|S_{n+1}t_n - S_n t_n\|.
 \end{aligned}$$

This together with (C1)–(C3) and (3.9), we obtain

$$\limsup_{n \rightarrow \infty} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0.$$

Hence, by Lemma 2.4, we get $\|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - b_n)\|z_n - x_n\| = 0. \tag{3.11}$$

From (C3), (3.2) and (3.5), we also have $\|u_{n+1} - u_n\| \rightarrow 0$, $\|t_{n+1} - t_n\| \rightarrow 0$ and $\|y_{n+1} - y_n\| \rightarrow 0$, as $n \rightarrow \infty$. Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n - b_n)(S_n t_n - x_n),$$

we have

$$\|S_n t_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.12)$$

Next, we prove that $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. From Lemma 2.1(3), we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \leq \langle T_{r_n} x_n - T_{r_n} x^*, x_n - x^* \rangle \\ &= \langle u_n - x^*, x_n - x^* \rangle = \frac{1}{2} \{ \|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \}. \end{aligned}$$

Hence

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2. \quad (3.13)$$

From Lemma 2.3, (3.1) and (3.13), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|t_n - x^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|u_n - x^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) [\|x_n - x^*\|^2 - \|x_n - u_n\|^2] \\ &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n - b_n) \|x_n - u_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - a_n - b_n) \|x_n - u_n\|^2 &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|. \end{aligned}$$

From the conditions (C1), (C2) and (3.11), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.14)$$

Since

$$\|S_n t_n - u_n\| \leq \|S_n t_n - x_n\| + \|x_n - u_n\|,$$

it follows from (3.12) and (3.14) that

$$\lim_{n \rightarrow \infty} \|S_n t_n - u_n\| = 0. \quad (3.15)$$

Next, we prove that $\|S_n t_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (2.1) and the nonexpansiveness of $I - \mu B$, we get

$$\begin{aligned} \|y_n - y^*\|^2 &= \|P_C(u_n - \mu B u_n) - P_C(x^* - \mu B x^*)\|^2 \\ &\leq \langle (u_n - \mu B u_n) - (x^* - \mu B x^*), y_n - y^* \rangle \\ &= \frac{1}{2} [\|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 + \|y_n - y^*\|^2 - \|(u_n - \mu B u_n) - (x^* - \mu B x^*) - (y_n - y^*)\|^2] \\ &\leq \frac{1}{2} [\|u_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2] \\ &\quad + 2\mu \langle (u_n - x^*) - (y_n - y^*), B u_n - B x^* \rangle - \mu^2 \|B u_n - B x^*\|^2. \end{aligned}$$

By (3.1), we obtain

$$\begin{aligned} \|y_n - y^*\|^2 &\leq \|u_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 + 2\mu \langle (u_n - x^*) - (y_n - y^*), B u_n - B x^* \rangle - \mu^2 \|B u_n - B x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 + 2\mu \langle (u_n - x^*) - (y_n - y^*), B u_n - B x^* \rangle - \mu^2 \|B u_n - B x^*\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|y_n - y^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) [\|x_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2] \\ &\quad + 2\mu \langle (u_n - x^*) - (y_n - y^*), B u_n - B x^* \rangle - \mu^2 \|B u_n - B x^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n - b_n) \|(u_n - x^*) - (y_n - y^*)\|^2 \\ &\quad + (1 - a_n - b_n) 2\mu \|(u_n - x^*) - (y_n - y^*)\| \|B u_n - B x^*\|, \end{aligned}$$

which implies

$$(1 - a_n - b_n)\|(u_n - x^*) - (y_n - y^*)\|^2 \leq a_n\|v - x^*\|^2 + (1 - a_n - b_n)2\mu\|(u_n - x^*) - (y_n - y^*)\| \|Bu_n - Bx^*\| \\ + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|.$$

This together with (C1), (3.11) and $\lim_{n \rightarrow \infty} \|Bu_n - Bx^*\| = 0$, we obtain

$$\|(u_n - x^*) - (y_n - y^*)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.16)$$

From Lemma 2.6 and (2.2), it follows that

$$\|(y_n - t_n) + (x^* - y^*)\|^2 = \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)] + \lambda(Ay_n - Ay^*)\|^2 \\ \leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)]\|^2 \\ + 2\lambda \langle Ay_n - Ay^*, (y_n - t_n) + (x^* - y^*) \rangle \\ \leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\ \leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|S_n P_C(y_n - \lambda Ay_n) - S_n P_C(y^* - \lambda Ay^*)\|^2 \\ + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\ \leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - (S_n t_n - x^*)\| [\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| \\ + \|S_n t_n - x^*\|] + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\ = \|u_n - S_n t_n + x^* - y^* - (u_n - y_n) - \lambda(Ay_n - Ay^*)\| [\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| \\ + \|S_n t_n - x^*\|] + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\|.$$

This together with (3.15), (3.16) and $\lim_{n \rightarrow \infty} \|Ay_n - Ay^*\| = 0$, we obtain $\|(y_n - t_n) + (x^* - y^*)\| \rightarrow 0$ as $n \rightarrow \infty$. This together with (3.12), (3.14) and (3.16), we obtain

$$\|S_n t_n - t_n\| \leq \|S_n t_n - x_n\| + \|x_n - u_n\| + \|(u_n - y_n) - (x^* - y^*)\| \\ + \|(y_n - t_n) + (x^* - y^*)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.17)$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle \leq 0,$$

where $\bar{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{S_n t_n\}$ are two bounded sequences in C , we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n \rightarrow \infty} \langle v - \bar{x}, S_n t_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle v - \bar{x}, S_{n_i} t_{n_i} - \bar{x} \rangle.$$

Since $\lim_{n \rightarrow \infty} \|S_n t_n - t_n\| = 0$, we obtain that $S_{n_i} t_{n_i} \rightharpoonup z$ as $i \rightarrow \infty$.

Next, we show that $z \in \Omega$.

(a) We first show that $z \in \bigcap_{i=1}^N F(T_i)$.

By (C5), we can assume that $\alpha_1^{n,j} \rightarrow \alpha_1^j \in (0, 1)$ as $n \rightarrow \infty$ for all $j \in \{1, 2, \dots, N-1\}$, $\alpha_1^{n,N} \rightarrow \alpha_1^N \in (0, 1]$ as $n \rightarrow \infty$ and $\alpha_2^{n,j} \rightarrow \alpha_2^j \in [0, 1)$ and $\alpha_3^{n,j} \rightarrow \alpha_3^j \in [0, 1)$ as $n \rightarrow \infty$ for $j = 1, 2, \dots, N$. Let S be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, for $j = 1, 2, \dots, N$. From Lemma 2.8, we have $\|S_n t_n - S t_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\|S t_n - t_n\| \leq \|S t_n - S_n t_n\| + \|S_n t_n - t_n\|,$$

it follows by (3.17) that $\|S t_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Since $t_{n_i} \rightharpoonup z$ and $\|S t_n - t_n\| \rightarrow 0$, we obtain by Lemmas 2.5 and 2.7 that $z \in F(S) = \bigcap_{i=1}^N F(T_i)$.

(b) Now, we show that $z \in GVI(C, A, B)$.

Since

$$\|t_n - x_n\| \leq \|S_n t_n - t_n\| + \|S_n t_n - x_n\|,$$

it follows from (3.17) and (3.12) that $\|t_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, by Lemma 2.10, we have that $G : C \rightarrow C$ is nonexpansive. Then, we have

$$\begin{aligned}\|t_n - G(t_n)\| &= \|P_C(y_n - \lambda Ay_n) - G(t_n)\| \\ &= \|P_C[P(u_n - \mu Bu_n) - \lambda AP(u_n - \mu Bu_n)] - G(t_n)\| \\ &= \|G(u_n) - G(t_n)\| \leq \|u_n - t_n\| \\ &\leq \|u_n - x_n\| + \|x_n - t_n\|,\end{aligned}$$

which implies $\|t_n - G(t_n)\| \rightarrow 0$ as $n \rightarrow \infty$. Again by Lemma 2.5, we have $z \in GVI(C, A, B)$.

(c) We show that $z \in MEP(F, \varphi)$. Since $t_{n_i} \rightarrow z$ and $\|x_n - t_n\| \rightarrow 0$, we obtain that $x_{n_i} \rightarrow z$. From $\|u_n - x_n\| \rightarrow 0$, we also obtain that $u_{n_i} \rightarrow z$. By using the same argument as that in the proof of [14, Theorem 3.1, pp. 1825], we can show that $z \in MEP(F, \varphi)$. Therefore there holds $z \in \Omega$.

On the other hand, it follows from (2.3) and $S_{n_i} t_{n_i} \rightarrow z$ as $i \rightarrow \infty$ that

$$\begin{aligned}\limsup_{n \rightarrow \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle &= \limsup_{n \rightarrow \infty} \langle v - \bar{x}, S_n t_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle v - \bar{x}, S_{n_i} t_{n_i} - \bar{x} \rangle \\ &= \langle v - \bar{x}, z - \bar{x} \rangle \leq 0.\end{aligned}\tag{3.18}$$

Hence, we have

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \langle a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + b_n \langle x_n - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - a_n - b_n) \langle S_n t_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} b_n (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \frac{1}{2} (1 - a_n - b_n) (\|t_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\ &\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} b_n (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \frac{1}{2} (1 - a_n - b_n) (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\ &= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} (1 - a_n) (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2),\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - a_n) \|x_n - \bar{x}\|^2 + 2a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle.$$

It follows from Lemma 2.2 and (3.18) that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. $\square$

The following examples show that there are mappings A and B which satisfy those conditions in Theorem 3.1.

Example 1. Let $A, B : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. If $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$, then we have

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive and
- (2) $\|Ay_n - Ay^*\| \rightarrow 0$ and $\|Bu_n - Bx^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$ where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 + \lambda^2 \|Ax - Ay\|^2 - 2\lambda \langle x - y, Ax - Ay \rangle \\ &\leq \|x - y\|^2 + \lambda^2 \|Ax - Ay\|^2 - 2\lambda \alpha \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2,\end{aligned}$$

hence $I - \lambda A$ is nonexpansive. Similarly, we can show that $I - \mu B$ is nonexpansive.

- (2) Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|t_n - x^*\|^2 \\ &= a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) [\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha) \|Ay_n - Ay^*\|^2] \\ &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n - b_n) \lambda(\lambda - 2\alpha) \|Ay_n - Ay^*\|^2,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|t_n - x^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|y_n - y^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*)\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) [\|u_n - x^*\|^2 + \mu(\mu - 2\beta) \|Bu_n - Bx^*\|^2], \\ &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n - b_n) \mu(\mu - 2\beta) \|Bu_n - Bx^*\|^2.\end{aligned}$$

Therefore, we have

$$-(1 - a_n - b_n) \lambda(\lambda - 2\alpha) \|Ay_n - Ay^*\|^2 \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|,$$

and

$$-(1 - a_n - b_n) \mu(\mu - 2\beta) \|Bu_n - Bx^*\|^2 \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.$$

From (3.11) and (C1), we obtain

$$\|Ay_n - Ay^*\| \rightarrow 0 \quad \text{and} \quad \|Bu_n - Bx^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \square$$

Example 2. Let $A : C \rightarrow H$ be a L_A -Lipschitzian and relaxed (c, d) -cocoercive mapping and $B : C \rightarrow H$ be a L_B -Lipschitzian and relaxed (c', d') -cocoercive mapping. If $0 < \lambda < \frac{2(d - cL_A^2)}{L_A^2}$ and $0 < \mu < \frac{2(d' - c'L_B^2)}{L_B^2}$, then we have

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive and
- (2) $\|Ay_n - Ay^*\| \rightarrow 0$ and $\|Bu_n - Bx^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$ where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned}\|(I - \mu B)x - (I - \mu B)y\|^2 &= \|(x - y) - \mu(Bx - By)\|^2 \\ &= \|x - y\|^2 + \mu^2 \|Bx - By\|^2 - 2\mu \langle x - y, Bx - By \rangle \\ &\leq \|x - y\|^2 + \mu^2 \|Bx - By\|^2 - 2\mu [-c' \|Bx - By\|^2 + d' \|x - y\|^2] \\ &= \|x - y\|^2 + \mu^2 \|Bx - By\|^2 + 2\mu c' \|Bx - By\|^2 - 2\mu d' \|x - y\|^2 \\ &\leq \|x - y\|^2 + \mu^2 L_B^2 \|x - y\|^2 + 2\mu c' L_B^2 \|x - y\|^2 - 2\mu d' \|x - y\|^2 \\ &= (1 + 2\mu c' L_B^2 - 2\mu d' + \mu^2 L_B^2) \|x - y\|^2 \\ &\leq \|x - y\|^2,\end{aligned}$$

hence $I - \mu B$ is nonexpansive. Similarly, we can show that $I - \lambda A$ is nonexpansive.

- (2) Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|t_n - x^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|y_n - y^*\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*)\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) [\|u_n - x^*\|^2 - 2\mu \langle u_n - x^*, Bu_n - Bx^* \rangle \\ &\quad + \mu^2 \|Bu_n - Bx^*\|^2] \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n) [\|u_n - x^*\|^2 + 2\mu c' \|Bu_n - Bx^*\|^2 - 2\mu d' \|u_n - x^*\|^2 + \mu^2 \|Bu_n - Bx^*\|^2] \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \left[\|x_n - x^*\|^2 + \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|Bu_n - Bx^*\|^2 \right] \\ &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n - b_n) \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|Bu_n - Bx^*\|^2,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|t_n - x^*\|^2 \\ &= a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2\end{aligned}$$

$$\begin{aligned} &\leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n) \left[\|x_n - x^*\|^2 + \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|Ay_n - Ay^*\|^2 \right] \\ &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n - b_n) \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|Ay_n - Ay^*\|^2. \end{aligned}$$

Therefore, we have

$$-(1 - a_n - b_n) \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|Bu_n - Bx^*\|^2 \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|,$$

and

$$-(1 - a_n - b_n) \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|Ay_n - Ay^*\|^2 \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.$$

From (3.11) and (C1), we obtain

$$\|Ay_n - Ay^*\| \rightarrow 0 \quad \text{and} \quad \|Bu_n - Bx^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \square$$

By using the same proof as in Examples 1 and 2, we can obtain the following example.

Example 3. Let A be an α -inverse-strongly monotone mapping of C into H and B be a L -Lipschitzian and relaxed (c, d) -cocoercive mapping of C into H . If $\lambda \in (0, 2\alpha)$ and $0 < \mu < \frac{2(d-cl^2)}{l^2}$, then we have that

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive and
- (2) $\|Ay_n - Ay^*\| \rightarrow 0$ and $\|Bu_n - Bx^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$ where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Let $\mathcal{A}$ be the class of all α -inverse-strongly monotone mappings from C into H , $\mathcal{B}$ the class of all β -inverse-strongly monotone mappings from C into H , $\mathcal{C}$ the class of all L -Lipschitzian and relaxed (c, d) -cocoercive mappings from C into H and $\mathcal{D}$ the class of all L_1 -Lipschitzian and relaxed (c', d') -cocoercive mappings from C into H .

From Theorem 3.1 and Examples 1–3, we obtain the following results.

Corollary 3.2. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A and B be two mappings satisfying one of the following conditions :

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$;
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < \frac{2(d-cl^2)}{l^2}$ and $0 < \mu < \frac{2(d'-c'l_1^2)}{l_1^2}$;
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$ and $0 < \lambda < \frac{2(d-cl^2)}{l^2}$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C . For all $j \in \{1, 2, \dots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be as in Theorem 3.1. Let S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. Assume that either (B1) or (B2) holds and $\Omega := \bigcap_{i=1}^N F(T_i) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $x_1 = v \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n P_C(y_n - \lambda A y_n), & n \geq 1 \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $\{a_n\}, \{b_n\} \subset [0, 1]$.

If the conditions (C1)–(C5) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.1), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Let $\alpha_2^{n,j} = 0$ for all $j \in \{1, 2, \dots, N\}$ and all $n \in \mathbb{N}$. Then $K_n = W_n$ for all $n \in \mathbb{N}$. By Theorem 3.1 and Examples 1–3, we obtain the following result.

Corollary 3.3. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A and B be two mappings satisfying one of the following conditions :

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$;
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < \frac{2(d-cl^2)}{l^2}$ and $0 < \mu < \frac{2(d'-c'l_1^2)}{l_1^2}$;
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$ and $0 < \lambda < \frac{2(d-cl^2)}{l^2}$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C . For all $n \in \mathbb{N}$, let $\{\alpha_1^{n,j}\}_{j=1}^N \subset (0, b]$ with $0 < b < 1$. Let W_n be the W -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{n,1}, \alpha_1^{n,2}, \dots, \alpha_1^{n,N}$. Assume that either (B1) or (B2) holds and $\Omega := \bigcap_{i=1}^N F(T_i) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $x_1 = v \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) W_n P_C(y_n - \lambda A y_n), & n \geq 1, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $\{a_n\}, \{b_n\} \subset [0, 1]$.

If the conditions (C1)–(C4) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.1), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Let $N = 1, T_1 = S, \alpha_2^{n,1} = \alpha_3^{n,1} = 0, \varphi = 0, F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ all $n \in \mathbb{N}$ in Theorem 3.1. Then $u_n = P_C x_n = x_n$ and $W_n = S$. By Corollary 3.2, we obtain the following result.

Corollary 3.4 ([7, Theorem 3.1]). Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $A, B : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. Let S be a nonexpansive self-mapping of C such that $\Omega = F(S) \cap \text{GVI}(C, A, B) \neq \emptyset$. Let $x_1 = v \in C$ and $\{x_n\}$ and $\{y_n\}$ be the sequences generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda A y_n), & n \geq 1. \end{cases}$$

If $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ and the sequences $\{a_n\}, \{b_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.1), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

4. Applications

Using Corollary 3.2, we obtain the following three theorems in a real Hilbert space.

We recall that a mapping $T : C \rightarrow C$ is called strictly pseudocontractive if there exists some k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \rightarrow C$ is a strictly pseudocontractive mapping with k . Then we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we have

$$\|(I - T)x - (I - T)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \|Ax - Ay\|^2.$$

Hence we have

$$\langle x - y, Ax - Ay \rangle \geq \frac{1 - k}{2} \|Ax - Ay\|^2.$$

Then, A is $\frac{1-k}{2}$ -inverse-strongly monotone.

Theorem 4.1. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $T, V : C \rightarrow C$ be strictly pseudocontractive with constants k and l , respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C . Let S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)}$ ($j = 1, 2, \dots, N$) are as in Theorem 3.1. Assume that either (B1) or (B2) holds and $\Omega := \bigcap_{i=1}^N F(T_i) \cap \text{GVI}(C, I - T, I - V) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $x_1 = v \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = (1 - \mu)u_n + \mu V u_n, \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n((1 - \lambda)y_n + \lambda T y_n), & n \geq 1, \end{cases}$$

where $\lambda \in (0, 1 - k), \mu \in (0, 1 - l), \{r_n\} \subset (0, \infty)$ and $\{a_n\}$ and $\{b_n\} \subset [0, 1]$.

If the conditions (C1)–(C5) are satisfied, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.1), where $\bar{y} = (1 - \mu)\bar{x} + \mu V \bar{x}$.

Proof. Put $A = I - T$ and $B = I - V$. Then A is $\frac{1-k}{2}$ -inverse-strongly monotone and B is $\frac{1-l}{2}$ -inverse-strongly monotone, respectively. We have

$$P_C(u_n - \mu Bu_n) = (1 - \mu)u_n + \mu Vu_n$$

and

$$P_C(y_n - \lambda Ay_n) = (1 - \lambda)y_n + \lambda Ty_n.$$

Therefore, the conclusion follows immediately from Corollary 3.2. $\square$

Theorem 4.2. Let H be a real Hilbert space. Let F be a function from $H \times H$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : H \rightarrow H$ be an α -inverse-strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of H . Let S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\{\alpha_j^{(n)}\}$ ($j = 1, \dots, N$) are as in Theorem 3.1. Assume that either (B1) or (B2) holds and $\Omega = \bigcap_{i=1}^N F(T_i) \cap A^{-1}0 \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $x_1 = v \in H$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in H, \\ y_n = u_n - \lambda Au_n, \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n(y_n - \lambda Ay_n), & n \geq 1, \end{cases}$$

where $\lambda \in (0, 1 - k)$, $\mu \in (0, 1 - l)$, $\{r_n\} \subset (0, \infty)$ and $\{a_n\}$ and $\{b_n\} \subset [0, 1]$.

If the conditions (C1)–(C5) are satisfied, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$.

Proof. Put $\lambda = \mu$, $C = H$, $B = A$ and $P_H = I$ in Corollary 3.2. By using the same argument as that in the proof of [7, Theorem 4.1, pp. 388–389], we can show that $A^{-1}0 = \text{GVI}(C, A, B) = \text{VI}(A, H)$. Thus, by Corollary 3.2, we obtain the desired result. $\square$

Recall that the resolvent of the maximal monotone mapping $B : H \rightarrow 2^H$ is defined by $J_r^B = (I + rB)^{-1}$ for all $r > 0$, it is known that $F(J_r^B) = B^{-1}0$ and J_r^B is nonexpansive.

Theorem 4.3. Let H be a real Hilbert space. Let F be a function from $H \times H$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : H \rightarrow H$ be an α -inverse-strongly monotone mapping and let $B_1, B_2, \dots, B_N : H \rightarrow 2^H$ be maximal monotone mappings. For all $i \in \{1, 2, \dots, N\}$ and $r_i > 0$, let $J_{r_i}^{B_i}$ be the resolvent of B_i . For all $j \in \{1, 2, \dots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be as in Theorem 3.1. Let S_n be the S -mappings generated by $J_{r_1}^{B_1}, J_{r_2}^{B_2}, \dots, J_{r_N}^{B_N}$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. Assume that either (B1) or (B2) holds and $\Omega := \bigcap_{i=1}^N B_i^{-1}0 \cap A^{-1}0 \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $x_1 = v \in H$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in H, \\ y_n = u_n - \lambda Au_n, \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n(y_n - \lambda Ay_n), & n \geq 1, \end{cases}$$

where $\lambda \in (0, 2\alpha)$. If the conditions (C1)–(C5) are satisfied, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$.

Proof. For all $i = 1, 2, \dots, N$ and $r_i > 0$, we have $F(J_{r_i}^{B_i}) = B_i^{-1}0$. Putting $P_H = I$ and $T_i = J_{r_i}^{B_i}$ for all $i = 1, 2, \dots, N$, by Theorem 4.2, we obtain the desired result. $\square$

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A MODIFIED HALPERN-TYPE ITERATION PROCESS FOR AN EQUILIBRIUM PROBLEM AND A FAMILY OF RELATIVELY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, based on a generalized projection, we introduce a new modified Halpern-type iteration algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of a common fixed point of an infinitely countable family of relatively quasi-nonexpansive mappings in the framework of Banach spaces. We establish the strong convergence theorem and obtain some applications. Our main results improve and extend the corresponding results announced by many authors.

1. INTRODUCTION

Let H be a real Hilbert space and C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a mapping. Recall that T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T .

In 1967, Halpern [10] proposed a classical iterative process to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H . The following algorithm is known as Halpern's iteration process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (1.1)$$

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where $u \in C$ is a fixed point and $\{\alpha_n\}$ is a sequence in $[0,1]$. Halpern [10] proved that the following conditions:

$$(i) : \lim_{n \rightarrow \infty} \alpha_n = 0, \quad (ii) : \sum_{n=1}^{\infty} \alpha_n = \infty$$

are indeed necessary for the strong convergence of the algorithm (1.1) for all closed convex subsets C of a Hilbert space H and for all nonexpansive mappings T on C ; see also [29].

However, due to the restriction of condition (ii), the convergence of $\{x_n\}$ is believed to be slow. So to improve the rate of convergence of algorithm (1.1), one has to perform some additional step of iteration.

Question 1. Can we construct algorithms for modifying the iterative process (1.1) to have strong convergence under the condition (i) only ?

Recently, Martinez-Yanes and Xu [13] has adapted Nakajo and Takahashi's [16] idea to modify the process (1.1) for a nonexpansive mapping T in a Hilbert space H :

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle) \right\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \quad n \geq 1, \end{cases} \quad (1.2)$$

where P_K is the metric projection from H onto a closed convex subset K of H . They proved that the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to $P_{F(T)}x_0$ provided the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Question 2. Can we extend the iterative process (1.2) from Hilbert spaces to Banach spaces ?

Very recently, Qin et al. [19], introduced the following modification of the process (1.2) for a closed relatively quasi-nonexpansive mapping T in a Banach space E :

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \quad (1.3)$$

where J is the duality mapping on E and Π_K is the generalized projection from E onto K . They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$ provided $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Question 3. Can we extend the above algorithm to finding a common element of the set of solutions of an equilibrium problem and the set of a common fixed

point of an infinitely countable family of relatively quasi-nonexpansive mappings in the framework of Banach spaces ?

Let f be a bifunction from $C \times C$ to the set of real numbers $\mathbb{R}$. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (1.4)$$

The set of solutions of (1.4) is denoted by $EP(f)$.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The problem of finding a common element of the set of fixed points of nonexpansive, relatively nonexpansive or relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; for instance, see [7, 9, 14, 15, 17, 22, 23, 25, 27, 28, 30] and the references cited therein.

Motivated and inspired by Martinez-Yanes and Xu [13], and Qin et al. [19], we construct a new hybrid projection algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of a common fixed point of a family of closed relatively quasi-nonexpansive mappings in the framework of Banach spaces.

2. PRELIMINARIES AND LEMMAS

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1.$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0, 2] \rightarrow [0, 1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E ; see [24] for more details.

Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. In a Hilbert space H , we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

Let C be a closed convex subset of E , and let T be mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of T [21] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\hat{F}(T)$. A mapping T is said to be *relatively nonexpansive* [3, 4, 14] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [3, 4]. T is said to be *ϕ -nonexpansive*, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be *relatively quasi-nonexpansive* or *quasi- ϕ -nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. It is obvious that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [3, 4, 6, 14]. Recall that T is closed if

$$x_n \rightarrow x, Tx_n \rightarrow y \quad \text{imply} \quad Tx = y.$$

We give some examples which are closed relatively quasi-nonexpansive; see [18].

Example 2.1. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0 \neq \emptyset$. Then, $J_r = (J + rA)^{-1}J$ is a closed relatively quasi-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.2. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E . Then, Π_C is a closed relatively quasi-nonexpansive mapping with $F(\Pi_C) = C$.

Lemma 2.3 (Kamimura and Takahashi [11]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed and convex subset of E . The *generalized projection mapping*,

introduced by Alber [1], is a mapping $\Pi_C : E \rightarrow C$, that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}.$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional ϕ and strict monotonicity of the duality mapping J ; for instance, see [1, 2, 8, 11, 24]. In a Hilbert space, Π_C is coincident with the metric projection.

Lemma 2.4 (Alber [1] and Kamimura and Takahashi [11]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let $x \in E$ and let $z \in C$. Then $z = \Pi_C x$ if and only if*

$$\langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.5 (Alber [1] and Kamimura and Takahashi [11]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall x \in C \text{ and } y \in E.$$

Lemma 2.6 (Qin et al. [18]). *Let E be a uniformly convex, smooth Banach space, let C be a closed convex subset of E , let T be a closed and relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C .*

Lemma 2.7 (Blum and Oettli [5]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8 (Takahashi and Zembayashi [26]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is a firmly nonexpansive-type mapping [12], i.e., for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

- (3) $F(T_r) = EP(f)$;
- (4) $EP(f)$ is closed and convex.

Lemma 2.9 (Takahashi and Zembayashi [26]). *Let C be a closed convex subset of a smooth, strictly, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4), let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

3. MAIN RESULTS

In this section, we prove the strong convergence theorem.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed relatively quasi-nonexpansive mappings from C into itself. Assume that $F := \bigcap_{i=1}^\infty F(T_i) \cap EP(f) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} \in C \text{ such that } f(u_{n,i}, y) + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where J is the duality mapping on E . Assume that $\{\alpha_n\}$ and $\{r_n\}$ are two sequences satisfying the restrictions:

(B1) $\{\alpha_n\} \subset [0, 1]$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$;

(B2) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. We split our proof into seven steps.

Step 1. Show that C_n is closed and convex for all $n \geq 1$.

By Lemma 2.6, we know that $F(T_i)$ is closed and convex for all $i = 1, 2, \dots$. By Lemma 2.8 (4), we also know that $EP(f)$ is closed and convex. Hence $F := \bigcap_{i=1}^\infty F(T_i) \cap EP(f)$ is a nonempty, closed and convex subset of C ; consequently, $\Pi_F x_1$ is well-defined. Clearly, $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for $k \in \mathbb{N}$. For each $z \in C_k$ and $i = 1, 2, \dots$, we observe that $\phi(z, u_{k,i}) \leq \alpha_k \phi(z, x_1) + (1 - \alpha_k) \phi(z, x_k)$ is equivalent to

$$2\alpha_k \langle z, Jx_1 \rangle + 2(1 - \alpha_k) \langle z, Jx_k \rangle - 2 \langle z, Ju_{k,i} \rangle \leq \alpha_k \|x_1\|^2 + (1 - \alpha_k) \|x_k\|^2 - \|u_{k,i}\|^2.$$

By the construction of the set C_{k+1} , we see that

$$\begin{aligned} C_{k+1} &= \left\{ z \in C_k : \sup_{i \geq 1} \phi(z, u_{k,i}) \leq \alpha_k \phi(z, x_1) + (1 - \alpha_k) \phi(z, x_k) \right\} \\ &= \bigcap_{i=1}^\infty \left\{ z \in C_k : \phi(z, u_{k,i}) \leq \alpha_k \phi(z, x_1) + (1 - \alpha_k) \phi(z, x_k) \right\}. \end{aligned}$$

Hence, C_{k+1} is closed and convex. By induction, we get that C_n is closed and convex for all $n \geq 1$.

Step 2. Show that $F \subset C_n$ for all $n \geq 1$.

$F \subset C_1 = C$ is obvious. Suppose that $F \subset C_k$ for $k \in \mathbb{N}$. We note that $u_{k,i} = T_{r_k} y_{k,i}$ for $i = 1, 2, \dots$. Then, for each $u \in F$, we have

$$\begin{aligned}
 \phi(u, u_{k,i}) &= \phi(u, T_{r_k} y_{k,i}) \leq \phi(u, y_{k,i}) \\
 &= \phi\left(u, J^{-1}(\alpha_k Jx_1 + (1 - \alpha_k)JT_i x_k)\right) \\
 &= \|u\|^2 - 2\langle u, \alpha_k Jx_1 + (1 - \alpha_k)JT_i x_k \rangle \\
 &\quad + \|\alpha_k Jx_1 + (1 - \alpha_k)JT_i x_k\|^2 \\
 &\leq \|u\|^2 - 2\alpha_k \langle u, Jx_1 \rangle - 2(1 - \alpha_k) \langle u, JT_i x_k \rangle \\
 &\quad + \alpha_k \|x_1\|^2 + (1 - \alpha_k) \|T_i x_k\|^2 \\
 &\leq \alpha_k \phi(u, x_1) + (1 - \alpha_k) \phi(u, x_k).
 \end{aligned} \tag{3.1}$$

Hence, $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$. Hence $\Pi_{C_{n+1}} x_1$ is well-defined.

Step 3. Show that $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$ exists.

From $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1) \quad \forall n \geq 1. \tag{3.2}$$

From Lemma 2.5, we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \tag{3.3}$$

Combining (3.2) and (3.3), we get that $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$ exists.

Step 4. Show that $\{x_n\}$ is a Cauchy sequence in C .

Since $x_m = \Pi_{C_m} x_1 \in C_m \subset C_n$ for $m > n$, by Lemma 2.5, we have

$$\begin{aligned}
 \phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) &\leq \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\
 &= \phi(x_m, x_1) - \phi(x_n, x_1).
 \end{aligned}$$

Taking $m, n \rightarrow \infty$, we obtain that $\phi(x_m, x_n) \rightarrow 0$. From Lemma 2.3, we have $\|x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of E and the closedness of C , we can assume that $x_n \rightarrow q \in C$ as $n \rightarrow \infty$. Further, we obtain

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.4}$$

Note that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_{n,i}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) \rightarrow 0, \tag{3.5}$$

as $n \rightarrow \infty$. Applying Lemma 2.3 to (3.4) and (3.5), we get

$$\lim_{n \rightarrow \infty} \|u_{n,i} - x_n\| = 0 \quad \forall i = 1, 2, \dots \tag{3.6}$$

This shows that $u_{n,i} \rightarrow q$ as $n \rightarrow \infty$ and $i=1,2,\dots$. Since J is uniformly norm-to-norm continuous on bounded subsets of E , we also obtain

$$\lim_{n \rightarrow \infty} \|Ju_{n,i} - Jx_n\| = 0 \quad \forall i = 1, 2, \dots \quad (3.7)$$

Step 5. Show that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

From (3.1), we observe that

$$\phi(u, y_{n,i}) \leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n). \quad (3.8)$$

Note that $u_{n,i} = T_{r_n} y_{n,i}$. Hence, it follows from (3.8) and Lemma 2.9 that

$$\begin{aligned} \phi(u_{n,i}, y_{n,i}) &= \phi(T_{r_n} y_{n,i}, y_{n,i}) \\ &\leq \phi(u, y_{n,i}) - \phi(u, T_{r_n} y_{n,i}) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) - \phi(u, u_{n,i}) \\ &= \alpha_n \left(\phi(u, x_1) - \phi(u, x_n) \right) + \left(\phi(u, x_n) - \phi(u, u_{n,i}) \right). \end{aligned}$$

From (3.6) and (B1), we get $\lim_{n \rightarrow \infty} \phi(u_{n,i}, y_{n,i}) = 0$ for all $i = 1, 2, \dots$. By Lemma 2.3, we also obtain

$$\lim_{n \rightarrow \infty} \|u_{n,i} - y_{n,i}\| = 0 \quad \forall i = 1, 2, \dots \quad (3.9)$$

Again, from (3.6) and (3.9), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_{n,i}\| = 0 \quad \forall i = 1, 2, \dots$$

and hence,

$$\lim_{n \rightarrow \infty} \|Jx_n - Jy_{n,i}\| = 0 \quad \forall i = 1, 2, \dots \quad (3.10)$$

Observing

$$\begin{aligned} \|Jy_{n,i} - Jx_n\| &= \|\alpha_n Jx_1 + (1 - \alpha_n) JT_i x_n - Jx_n\| \\ &= \|- \alpha_n (JT_i x_n - Jx_1) + (JT_i x_n - Jx_n)\| \\ &\geq -\alpha_n \|JT_i x_n - Jx_1\| + \|JT_i x_n - Jx_n\|, \end{aligned}$$

we obtain, by (B1) and (3.10), that

$$\|JT_i x_n - Jx_n\| \leq \|Jy_{n,i} - Jx_n\| + \alpha_n \|JT_i x_n - Jx_1\| \rightarrow 0,$$

as $n \rightarrow \infty$ and $i = 1, 2, \dots$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0 \quad \forall i = 1, 2, \dots$$

Since T_i is closed for $i = 1, 2, \dots$ and $x_n \rightarrow q$, we can conclude that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Step 6. Show that $q \in EP(f)$.

From (3.9) and $r_n \geq a > 0$, we have $\frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_n} \rightarrow 0$. From $u_{n,i} = T_{r_n} y_{n,i}$ for $i = 1, 2, \dots$, we get

$$f(u_{n,i}, y) + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \quad \forall y \in C.$$

By (A2), we have

$$\begin{aligned} \|y - u_{n,i}\| \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_n} &\geq \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \\ &\geq -f(u_{n,i}, y) \geq f(y, u_{n,i}), \quad \forall y \in C. \end{aligned}$$

From (A4) and $u_{n,i} \rightarrow q$ for $i = 1, 2, \dots$, we get $f(y, q) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$, Define $y_t = ty + (1-t)q$. Then $y_t \in C$, which implies that $f(y_t, q) \leq 0$. From (A1), we obtain that $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, q) \leq tf(y_t, y)$. Thus, $f(y_t, y) \geq 0$. From (A3), we have $f(q, y) \geq 0$ for all $y \in C$. Hence $q \in EP(f)$.

Step 7. Show that $q = \Pi_F x_1$.

From $x_n = \Pi_{C_n} x_1$, we have

$$\langle Jx_1 - Jx_n, x_n - z \rangle \geq 0 \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\langle Jx_1 - Jx_n, x_n - u \rangle \geq 0 \quad \forall u \in F. \quad (3.11)$$

By taking limit in (3.11), we obtain that

$$\langle Jx_1 - Jq, q - u \rangle \geq 0 \quad \forall u \in F.$$

By Lemma 2.4, we can conclude that $q = \Pi_F x_1$. This completes the proof. $\square$

Remark 3.2. If we take $f \equiv 0$ and $T_i = T$ for all $i = 1, 2, \dots$, then Theorem 3.1 reduces to Theorem 3.1 of Qin et al. [19].

Remark 3.3. If we take $f \equiv 0$ and $T_i = T$ for all $i = 1, 2, \dots$, then Theorem 3.1 improves on Theorem 3.2 of Qin and Su [20] from the class of relatively nonexpansive mappings to the class of relatively quasi-nonexpansive mappings; that is, we relax the strong restriction: $\hat{F}(T) = F(T)$. Moreover, this algorithm is also simpler to compute than the one given in [20].

4. APPLICATIONS

In this section, we give some applications of Theorem 3.1 in the framework of Banach spaces.

Let $A : C \rightarrow E^*$ be a nonlinear mapping. The classical variational inequality problem is to find $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0 \quad \forall y \in C. \quad (4.1)$$

The set of solutions of (4.1) is denoted by $VI(C, A)$.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E . Let A be a continuous monotone mapping from C to E^* , and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed relatively quasi-nonexpansive mappings from C into itself such that*

$F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} \in C \text{ such that } \langle Au_{n,i}, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where J is the duality mapping on E . Assume that $\{\alpha_n\}$ and $\{r_n\}$ are sequences satisfying (B1) and (B2) of Theorem 3.1.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. Define $f(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then, by Theorem 3.1, we obtain the desired result. $\square$

Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function. The convex minimization problem is to find $\hat{x} \in C$ such that

$$\varphi(\hat{x}) \leq \varphi(y) \quad \forall y \in C. \quad (4.2)$$

The set of solutions of (4.2) is denoted by $CMP(\varphi)$.

Theorem 4.2. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E . Let φ be a proper lower semi-continuous and convex function from C to $\mathbb{R}$, and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed relatively quasi-nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap CMP(\varphi) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} \in C \text{ such that } \varphi(y) + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq \varphi(u_{n,i}), \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where J is the duality mapping on E . Assume that $\{\alpha_n\}$ and $\{r_n\}$ are sequences satisfying (B1) and (B2) of Theorem 3.1.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. Define $f(x, y) = \varphi(y) - \varphi(x)$ for all $x, y \in C$. Then, by Theorem 3.1, we obtain the desired result. $\square$

As a direct consequence of Theorem 3.1, we also obtain the following application in a Hilbert space.

Theorem 4.3. Let C be a nonempty and closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed quasi-nonexpansive mappings

from C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(f) \neq \emptyset$. For an initial point $x_0 \in H$ with $x_1 = P_{C_1}x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} u_{n,i} \in C \text{ such that } f(u_{n,i}, y) + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - y_{n,i} \rangle \geq 0, \quad \forall y \in C, \\ y_{n,i} = \alpha_n x_1 + (1 - \alpha_n) T_i x_n, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \|z - u_{n,i}\|^2 \leq \alpha_n \|z - x_1\|^2 + (1 - \alpha_n) \|z - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where P is a metric projection. Assume that $\{\alpha_n\}$ and $\{r_n\}$ are sequences satisfying (B1) and (B2) of Theorem 3.1.

Then, the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. By taking $E = H$ in Theorem 3.1, we obtain the desired result. $\square$

Remark 4.4. If we take $f \equiv 0$ and $T_i = T$ for all $i = 1, 2, \dots$ in Theorem 4.3, then Theorem 4.3 improves and extends Theorem 3.1 of Martinez-Yanes and Xu [13].

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Research Article

Weak Convergence Theorems for a Countable Family of Strict Pseudocontractions in Banach Spaces

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We investigate the convergence of Mann-type iterative scheme for a countable family of strict pseudocontractions in a uniformly convex Banach space with the Fréchet differentiable norm. Our results improve and extend the results obtained by Marino-Xu, Zhou, Osilike-Udomene, Zhang-Guo and the corresponding results. We also point out that the condition given by Chidume-Shahzad (2010) is not satisfied in a real Hilbert space. We show that their results still are true under a new condition.

1. Introduction

Let E and E^* be a real Banach space and the dual space of E , respectively. Let K be a nonempty subset of E . Let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$, for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^* . If E is smooth or E^* is strictly convex, then J is single-valued.

Throughout this paper, we denote the single valued duality mapping by j and denote the set of fixed points of a nonlinear mapping $T : K \rightarrow E$ by

$$F(T) = \{x \in K : Tx = x\}. \quad (1.1)$$

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called

(i) *pseudocontractive* [1] if, for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2, \quad (1.2)$$

- (ii) λ -strictly pseudocontractive [2] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(I - T)x - (I - T)y\|^2, \quad (1.3)$$

or equivalently

$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq \lambda \|(I - T)x - (I - T)y\|^2, \quad (1.4)$$

- (iii) L -Lipschitzian if, for all $x, y \in D(T)$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|. \quad (1.5)$$

Remark 1.2. It is obvious by the definition that

- (1) every strictly pseudocontractive mapping is pseudocontractive,
- (2) every λ -strictly pseudocontractive mapping is $((1 + \lambda)/\lambda)$ -Lipschitzian; see [3].

Remark 1.3. Let K be a nonempty subset of a real Hilbert space and $T : K \rightarrow K$ a mapping. Then T is said to be κ -strictly pseudocontractive [2] if, for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2. \quad (1.6)$$

It is well known that (1.6) is equivalent to the following:

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2. \quad (1.7)$$

It is worth mentioning that the class of strict pseudocontractions includes properly the class of nonexpansive mappings. Moreover, we know from [4] that the class of pseudocontractions also includes properly the class of strict pseudocontractions. A mapping $A : E \rightarrow E$ is called *accretive* if, for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y) \rangle \geq 0$. It is also known that A is accretive if and only if $T := I - A$ is pseudocontractive. Hence, a solution of the equation $Au = 0$ is a solution of the fixed point of $T := I - A$. Note that if $T := I - A$, then A is λ -strictly accretive if and only if T is λ -strictly pseudocontractive.

In 1953, Mann [5] introduced the iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \quad (1.8)$$

where $\alpha_n \in [0, 1]$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.8) converges

weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [6]). However, if T is a Lipschitzian pseudocontractive mapping, then Mann iteration defined by (1.8) may fail to converge in a Hilbert space; see [4].

In 1967, Browder-Petryshyn [2] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using Mann's iteration (1.8) with a constant sequence $\alpha_n = \alpha$ for all n . Respectively, Marino-Xu [7] and Zhou [8] extended the results of Browder-Petryshyn [2] to Mann's iteration process (1.8). To be more precise, they proved the following theorem.

Theorem 1.4 (see [7]). *Let K be a closed convex subset of a real Hilbert space H . Let $T : K \rightarrow K$ be a κ -strict pseudocontraction for some $0 \leq \kappa < 1$, and assume that T admits a fixed point in K . Let a sequence $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm (1.8). Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty} (\alpha_n - \kappa)(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T .*

Meanwhile, Marino, and Xu raised the open question: whether Theorem 1.4 can be extended to Banach spaces which are uniformly convex and have a Fréchet differentiable norm. Later, Zhou [9] and Zhang-Su [10], respectively, extended the result above to 2-uniformly smooth and q -uniformly smooth Banach spaces which are uniformly convex or satisfy Opial's condition.

In 2001, Osilike-Udomene [11] proved the convergence theorems of the Mann [5] and Ishikawa [12] iteration methods in the framework of q -uniformly smooth and uniformly convex Banach spaces. They also obtained that a sequence $\{x_n\}$ defined by (1.8) converges weakly to a fixed point of T under suitable control conditions. However, the sequence $\{\alpha_n\} \subset [0, 1]$ excluded the canonical choice $\alpha_n = 1/n$, $n \geq 1$. This was a motivation for Zhang-Guo [13] to improve the results in the same space. Observe that the results of Osilike-Udomene [11] and Zhang-Guo [13] hold under the assumption that

$$C_q < \frac{q^\lambda}{b^{q-1}}, \quad (1.9)$$

for some $b \in (0, 1)$ and C_q is a constant depending on the geometry of the space.

Lemma 1.5 (see [14–16]). *Let E be a uniformly smooth real Banach space. Then there exists a nondecreasing continuous function $\beta : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \rightarrow 0^+} \beta(t) = 0$ and $\beta(ct) \leq c\beta(t)$ for $c \geq 1$ such that, for all $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + \max\{\|x\|, 1\} \|y\| \beta(\|y\|). \quad (1.10)$$

Recently, Chidume-Shahzad [17] extended the results of Osilike-Udomene [11] and Zhang-Guo [13] by using Reich's inequality (1.10) to the much more general real Banach spaces which are uniformly smooth and uniformly convex. Under the assumption that

$$\beta(t) \leq \frac{\lambda t}{\max\{2r, 1\}}, \quad (1.11)$$

for some $r > 0$, they proved the following theorem.

Theorem 1.6 (see [17]). *Let E be a uniformly smooth real Banach space which is also uniformly convex and K a nonempty closed convex subset of E . Let $T : K \rightarrow K$ be a λ -strict pseudocontraction for some $0 \leq \lambda < 1$ with $x^* \in F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \quad (1.12)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:

- (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then, $\{x_n\}$ converges weakly to a fixed point of T .

However, we would like to point out that the results of Chidume-Shahzad [17] do not hold in real Hilbert spaces. Indeed, we know from Chidume [14] that

$$\beta(t) = \sup \left\{ \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x) \rangle : \|x\| \leq 1, \|y\| \leq 1 \right\}. \quad (1.13)$$

If E is a real Hilbert space, then we have

$$\begin{aligned} \beta(t) &= \sup \left\{ \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, x \rangle : \|x\| \leq 1, \|y\| \leq 1 \right\} \\ &= \sup \left\{ \frac{\|x\|^2 + 2t\langle x, y \rangle + t^2\|y\|^2 - \|x\|^2}{t} - 2\langle y, x \rangle : \|x\| \leq 1, \|y\| \leq 1 \right\} \\ &= \sup \left\{ t\|y\|^2 : \|y\| \leq 1 \right\} = t. \end{aligned} \quad (1.14)$$

On the other hand, by assumption (1.11), we see that

$$\beta(t) \leq \frac{\lambda t}{\max\{2r, 1\}} < t, \quad (1.15)$$

which is a contradiction.

It is known that one can extend his result from a single strict pseudocontraction to a finite family of strict pseudocontractions by replacing the convex combination of these mappings in the iteration under suitable conditions. The construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors; see also [18–22] and the references therein.

Our motivation in this paper is the following:

- (1) to modify the normal Mann iteration process for finding common fixed points of an infinitely countable family of strict pseudocontractions,

- (2) to improve and extend the results of Chidume-Shahzad [17] from a real uniformly smooth and uniformly convex Banach space to a real uniformly convex Banach space which has the Fréchet differentiable norm.

Motivated and inspired by Marino-Xu [7], Osilike-Udomene [11], Zhou [8], Zhang-Guo [13], and Chidume-Shahzad [17], we consider the following Mann-type iteration: $x_1 \in K$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \quad (1.16)$$

where α_n is a real sequence in $[0, 1]$ and $\{T_n\}_{n=1}^\infty$ is a countable family of strict pseudocontractions on a closed and convex subset K of a real Banach space E .

In this paper, we prove the weak convergence of a Mann-type iteration process (1.16) in a uniformly convex Banach space which has the Fréchet differentiable norm for a countable family of strict pseudocontractions under some appropriate conditions. The results obtained in this paper improve and extend the results of Chidume-Shahzad [17], Marino-Xu [7], Osilike-Udomene [11], Zhou [8], and Zhang-Guo [13] in some aspects.

We will use the following notation:

- (i) $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.
 (ii) $\omega_\omega(x_n) = \{x : x_{n_i} \rightharpoonup x\}$ denotes the weak ω -limit set of $\{x_n\}$.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that, for $x, y \in E$ with $\|x\|, \|y\| \leq 1$, and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}, \quad (2.1)$$

for all $\epsilon \in [0, 2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.2)$$

exists for each $x, y \in S(E)$. In this case E is called *smooth*. The norm of E is said to be *Fréchet differentiable* or E is *Fréchet smooth* if, for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. In other words, there exists a function $\varepsilon_x(s)$ with $\varepsilon_x(s) \rightarrow 0$ as $s \rightarrow 0$ such that

$$|\|x + ty\| - \|x\| + t\langle y, j(x) \rangle| \leq |t|\varepsilon_x(|t|) \quad (2.3)$$

for all $y \in S(E)$. In this case the norm is *Gâteaux differentiable* and

$$\lim_{t \rightarrow 0} \sup_{y \in S(E)} \left| \frac{1/2 \|x + ty\|^2 - 1/2 \|x\|^2}{t} - \langle y, j(x) \rangle \right| = 0 \quad (2.4)$$

for all $x \in E$. On the other hand,

$$\frac{1}{2} \|x\|^2 + \langle h, j(x) \rangle \leq \frac{1}{2} \|x + h\|^2 \leq \frac{1}{2} \|x\|^2 + \langle h, j(x) \rangle + b(\|h\|) \quad (2.5)$$

for all $x, h \in E$, where b is a function defined on $\mathbb{R}^+$ such that $\lim_{t \rightarrow 0^+} (b(t)/t) = 0$. The norm of E is called *uniformly Fréchet differentiable* if the limit is attained uniformly for $x, y \in S(E)$.

Let $\rho_E : [0, \infty) \rightarrow [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t \right\}. \quad (2.6)$$

A Banach space E is said to be *uniformly smooth* if $\rho_E(t)/t \rightarrow 0$ as $t \rightarrow 0$. Let $q > 1$, then E is said to be *q -uniformly smooth* if there exists $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q -uniformly smooth, then E is uniformly smooth. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable, and hence the norm of E is Fréchet differentiable, and it is also known that if E is Fréchet smooth, then E is smooth. Moreover, every uniformly smooth Banach space is reflexive. For more details, we refer the reader to [14, 23]. A Banach space E is said to satisfy *Opial's condition* [24] if $x \in E$ and $x_n \rightharpoonup x$; then

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in E, x \neq y. \quad (2.7)$$

In the sequel, we will need the following lemmas.

Lemma 2.1 (see [23]). *Let E be a Banach space and $J : E \rightarrow 2^{E^*}$ the duality mapping. Then one has the following:*

- (i) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, j(x) \rangle$ for all $x, y \in E$, where $j(x) \in J(x)$;
- (ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$ for all $x, y \in E$, where $j(x + y) \in J(x + y)$.

Lemma 2.2 (see [25]). *Let E be a real uniformly convex Banach space, K a nonempty, closed, and convex subset of E , and $T : K \rightarrow K$ a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero, that is, for all sequence $\{x_n\} \subset K$ with $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \rightarrow 0$ it follows that $p = Tp$.*

Lemma 2.3 (see [25]). *Let E be a real reflexive Banach space which satisfies Opial's condition, K a nonempty, closed and convex subset of E and $T : K \rightarrow K$ a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4 (see [26]). *Let E be a real uniformly convex Banach space with a Fréchet differentiable norm. Let K be a closed and convex subset of E and $\{S_n\}_{n=1}^\infty$ a family of L_n -Lipschitzian self-mappings*

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on K such that $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. For arbitrary $x_1 \in K$, define $x_{n+1} = S_n x_n$ for all $n \geq 1$. Then for every $p, q \in F$, $\lim_{n \rightarrow \infty} \langle x_n, j(p - q) \rangle$ exists, in particular, for all $u, v \in \omega_{\omega}(x_n)$ and $p, q \in F$, $\langle u - v, j(p - q) \rangle = 0$.

Lemma 2.5 (see [17, 27]). Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$, be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 0. \quad (2.8)$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n \rightarrow \infty} a_n = 0$.

To deal with a family of mappings, the following conditions are introduced. Let K be a subset of a real Banach space E , and let $\{T_n\}$ be a family of mappings of K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the AKTT-condition [28] if for each bounded subset B of K ,

$$\sum_{n=1}^{\infty} \sup \{ \|T_{n+1}z - T_n z\| : z \in B \} < \infty. \quad (2.9)$$

Lemma 2.6 (see [28]). Let K be a nonempty and closed subset of a Banach space E , and let $\{T_n\}$ be a family of mappings of K into itself which satisfies the AKTT-condition, then the mapping $T : K \rightarrow K$ defined by

$$Tx = \lim_{n \rightarrow \infty} T_n x, \quad \forall x \in K \quad (2.10)$$

satisfies

$$\limsup_{n \rightarrow \infty} \{ \|Tz - T_n z\| : z \in B \} = 0 \quad (2.11)$$

for each bounded subset B of K .

So we have the following results proved by Boonchari-Saejung [29, 30].

Lemma 2.7 (see [29, 30]). Let K be a closed and convex subset of a smooth Banach space E . Suppose that $\{T_n\}_{n=1}^{\infty}$ is a family of λ -strictly pseudocontractive mappings from K into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \beta_n = 1$. Then the following conclusions hold:

- (1) $G := \sum_{n=1}^{\infty} \beta_n T_n : K \rightarrow E$ is a λ -strictly pseudocontractive mapping;
- (2) $F(G) = \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.8 (see [30]). Let K be a closed and convex subset of a smooth Banach space E . Suppose that $\{S_k\}_{k=1}^{\infty}$ is a countable family of λ -strictly pseudocontractive mappings of K into itself with $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : K \rightarrow K$ by

$$T_n x = \sum_{k=1}^n \beta_n^k S_k x, \quad x \in K, \quad (2.12)$$

where $\{\beta_n^k\}$ is a family of nonnegative numbers satisfying

- (i) $\sum_{k=1}^n \beta_n^k = 1$ for all $n \in \mathbb{N}$;
- (ii) $\beta^k := \lim_{n \rightarrow \infty} \beta_n^k > 0$ for all $k \in \mathbb{N}$;
- (iii) $\sum_{n=1}^{\infty} \sum_{k=1}^n |\beta_{n+1}^k - \beta_n^k| < \infty$.

Then

- (1) each T_n is a λ -strictly pseudocontractive mapping;
- (2) $\{T_n\}$ satisfies AKTT-condition;
- (3) If $T : K \rightarrow K$ is defined by

$$Tx = \sum_{k=1}^{\infty} \beta^k S_k x, \quad x \in K, \quad (2.13)$$

then $Tx = \lim_{n \rightarrow \infty} T_n x$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

For convenience, we will write that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.6 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

3. Main Results

Lemma 3.1. Let E be a real Banach space, and let K be a nonempty, closed, and convex subset of E . Let $\{T_n\}_{n=1}^{\infty} : K \rightarrow K$ be a family of λ -strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \quad (3.1)$$

where $\{\alpha_n\} \subset [0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. If $\{T_n\}$ satisfies the AKTT-condition, then

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$;
- (ii) $\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.

Proof. Let $p \in F$, and put $L = (\lambda + 1)/\lambda$. First, we observe that

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T_n x_n - p\| \leq (1 + L)\|x_n - p\|, \\ \|x_{n+1} - x_n\| &= \alpha_n\|T_n x_n - x_n\| \leq \alpha_n(1 + L)\|x_n - p\|. \end{aligned} \quad (3.2)$$

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Since T_n is a λ -strict pseudocontraction, there exists $j(x_{n+1} - p) \in J(x_{n+1} - p)$. By Lemma 2.1 we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|(x_n - p) + \alpha_n(T_n x_n - x_n)\|^2 \\
 &\leq \|x_n - p\|^2 + 2\alpha_n \langle T_n x_n - x_n, j(x_{n+1} - p) \rangle \\
 &= \|x_n - p\|^2 + 2\alpha_n \langle T_n x_n - T_n x_{n+1}, j(x_{n+1} - p) \rangle \\
 &\quad + 2\alpha_n \langle T_n x_{n+1} - x_{n+1}, j(x_{n+1} - p) \rangle + 2\alpha_n \langle x_{n+1} - x_n, j(x_{n+1} - p) \rangle \\
 &\leq \|x_n - p\|^2 + 2\alpha_n L \|x_n - x_{n+1}\| \|x_{n+1} - p\| \\
 &\quad - 2\alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2 + 2\alpha_n \|x_n - x_{n+1}\| \|x_{n+1} - p\| \\
 &\leq \|x_n - p\|^2 + 2\alpha_n^2 L(1+L)^2 \|x_n - p\|^2 \\
 &\quad - 2\alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2 + 2\alpha_n^2 (1+L)^2 \|x_n - p\|^2 \\
 &= \|x_n - p\|^2 + 2\alpha_n^2 (1+L)^3 \|x_n - p\|^2 - 2\alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2.
 \end{aligned} \tag{3.3}$$

This implies that

$$\|x_{n+1} - p\|^2 \leq \left(1 + 2\alpha_n^2 (1+L)^3\right) \|x_n - p\|^2. \tag{3.4}$$

Hence, by $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, we have from Lemma 2.5 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists; consequently, $\{x_n\}$ is bounded. Moreover, by (3.3), we also have

$$\sum_{n=1}^{\infty} \alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2 \leq \sum_{n=1}^{\infty} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \right) + 2(1+L)^3 M_1^2 \sum_{n=1}^{\infty} \alpha_n^2 < \infty, \tag{3.5}$$

where $M_1 = \sup_{n \geq 1} \{\|x_n - p\|\}$. It follows that $\liminf_{n \rightarrow \infty} \|T_n x_{n+1} - x_{n+1}\| = 0$. Since $\{x_n\}$ is bounded,

$$\begin{aligned}
 \|x_{n+1} - T_{n+1} x_{n+1}\| &\leq \|x_{n+1} - T_n x_{n+1}\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
 &\leq \|x_{n+1} - T_n x_{n+1}\| + \sup_{z \in \{x_n\}} \|T_n z - T_{n+1} z\|.
 \end{aligned} \tag{3.6}$$

Since $\{T_n\}$ satisfies the AKTT-condition, it follows that $\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. This completes the proof of (i) and (ii). $\square$

Lemma 3.2. *Let E be a real Banach space with the Fréchet differentiable norm. For $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by*

$$\beta^*(t) = \sup_{y \in S(E)} \left| \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x) \rangle \right|. \tag{3.7}$$

Then, $\lim_{t \rightarrow 0^+} \beta^*(t) = 0$, and

$$\|x + h\|^2 \leq \|x\|^2 + 2\langle h, j(x) \rangle + \|h\|\beta^*(\|h\|) \quad (3.8)$$

for all $h \in E \setminus \{0\}$.

Proof. Let $x \in E$. Since E has the Fréchet differentiable norm, it follows that

$$\lim_{t \rightarrow 0} \sup_{y \in S(E)} \left| \frac{1/2\|x + ty\|^2 - 1/2\|x\|^2}{t} - \langle y, j(x) \rangle \right| = 0. \quad (3.9)$$

Then $\lim_{t \rightarrow 0^+} \beta^*(t) = 0$, and hence

$$\left| \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x) \rangle \right| \leq \beta^*(t), \quad \forall y \in S(E) \quad (3.10)$$

which implies that

$$\|x + ty\|^2 \leq \|x\|^2 + 2t\langle y, j(x) \rangle + t\beta^*(t), \quad \forall y \in S(E). \quad (3.11)$$

Suppose that $h \neq 0$. Put $y = h/\|h\|$ and $t = \|h\|$. By (3.11), we have

$$\|x + h\|^2 \leq \|x\|^2 + 2\langle h, j(x) \rangle + \|h\|\beta^*(\|h\|). \quad (3.12)$$

This completes the proof. $\square$

Remark 3.3. In a real Hilbert space, we see that $\beta^*(t) = t$ for $t > 0$.

In our more general setting, throughout this paper we will assume that

$$\beta^*(t) \leq 2t, \quad (3.13)$$

where β^* is a function appearing in (3.8).

So we obtain the following results.

Lemma 3.4. *Let E be a real Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E . Let $\{T_n\}_{n=1}^\infty : K \rightarrow K$ be a family of λ -strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \quad (3.14)$$

where $\{\alpha_n\} \subset [0, 1]$ satisfying $\sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty \alpha_n^2 < \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = \lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$.

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Proof. Let $p \in F$, and put $M_2 = \sup_{n \geq 1} \{\|x_n - T_n x_n\|\} > 0$. Then by (3.8) and (3.13) we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|(x_n - p) + \alpha_n(T_n x_n - x_n)\|^2 \\
 &\leq \|x_n - p\|^2 + 2\alpha_n \langle T_n x_n - x_n, j(x_n - p) \rangle \\
 &\quad + \alpha_n \|T_n x_n - x_n\| \beta^*(\alpha_n \|T_n x_n - x_n\|) \\
 &\leq \|x_n - p\|^2 - 2\alpha_n \lambda \|x_n - T_n x_n\|^2 + 2\alpha_n^2 \|x_n - T_n x_n\|^2 \\
 &\leq \|x_n - p\|^2 - 2\alpha_n \lambda \|x_n - T_n x_n\|^2 + 2\alpha_n^2 M_2^2.
 \end{aligned} \tag{3.15}$$

It follows that

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - T_n x_n\|^2 < \infty. \tag{3.16}$$

Observe that

$$\begin{aligned}
 \|x_n - T_{n+1} x_{n+1}\|^2 &= \|(x_n - T_n x_n) + (T_n x_n - T_{n+1} x_{n+1})\|^2 \\
 &\leq \|x_n - T_n x_n\|^2 + 2 \langle T_n x_n - T_{n+1} x_{n+1}, j(x_n - T_{n+1} x_{n+1}) \rangle \\
 &= \|x_n - T_n x_n\|^2 + 2 \langle T_n x_n - T_n x_{n+1}, j(x_n - T_{n+1} x_{n+1}) \rangle \\
 &\quad + 2 \langle T_n x_{n+1} - T_{n+1} x_{n+1}, j(x_n - T_{n+1} x_{n+1}) \rangle \\
 &\leq \|x_n - T_n x_n\|^2 + 2L \|x_n - x_{n+1}\| \|x_n - T_{n+1} x_{n+1}\| \\
 &\quad + 2 \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \|x_n - T_{n+1} x_{n+1}\| \\
 &\leq \|x_n - T_n x_n\|^2 + 2L \|x_n - x_{n+1}\| \|x_n - T_n x_n\| \\
 &\quad + 2L \|x_n - x_{n+1}\| \|T_n x_n - T_n x_{n+1}\| \\
 &\quad + 2L \|x_n - x_{n+1}\| \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
 &\quad + 2 \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \|x_n - x_{n+1}\| \\
 &\quad + 2 \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \|x_{n+1} - T_{n+1} x_{n+1}\| \\
 &\leq \|x_n - T_n x_n\|^2 + \left(2L\alpha_n + 2L^2\alpha_n^2\right) \|x_n - T_n x_n\|^2 \\
 &\quad + (2LM_2\alpha_n + 2M_2\alpha_n + 2M_2) \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
 &\leq \|x_n - T_n x_n\|^2 + 2L(1 + L)\alpha_n \|x_n - T_n x_n\|^2 \\
 &\quad + 2M_2(L + 2) \|T_n x_{n+1} - T_{n+1} x_{n+1}\|.
 \end{aligned} \tag{3.17}$$

By (3.17), we have

$$\begin{aligned}
\|x_{n+1} - T_{n+1}x_{n+1}\|^2 &\leq (1 - \alpha_n)\|x_n - T_{n+1}x_{n+1}\|^2 + \alpha_n\|T_nx_n - T_{n+1}x_{n+1}\|^2 \\
&\leq \|x_n - T_{n+1}x_{n+1}\|^2 \\
&\quad + \alpha_n(\|T_nx_n - T_nx_{n+1}\| + \|T_nx_{n+1} - T_{n+1}x_{n+1}\|)^2 \\
&= \|x_n - T_{n+1}x_{n+1}\|^2 + \alpha_n\|T_nx_n - T_nx_{n+1}\|^2 \\
&\quad + 2\alpha_n\|T_nx_n - T_nx_{n+1}\|\|T_nx_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + \alpha_n\|T_nx_{n+1} - T_{n+1}x_{n+1}\|^2 \\
&\leq \|x_n - T_{n+1}x_{n+1}\|^2 + \alpha_n^2L^2\|x_n - T_nx_n\|^2 \\
&\quad + 2\alpha_n^2L\|x_n - T_nx_n\|\|T_nx_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + \alpha_n\|T_nx_{n+1} - T_{n+1}x_{n+1}\|^2 \\
&\leq \|x_n - T_{n+1}x_{n+1}\|^2 + \alpha_n^2L^2M_2^2 \\
&\quad + 2LM_2\|T_nx_{n+1} - T_{n+1}x_{n+1}\| + \|T_nx_{n+1} - T_{n+1}x_{n+1}\|^2 \quad (3.18) \\
&\leq \|x_n - T_nx_n\|^2 + 2L(1 + L)\alpha_n\|x_n - T_nx_n\|^2 \\
&\quad + 2M_2(L + 2)\|T_nx_{n+1} - T_{n+1}x_{n+1}\| + \alpha_n^2L^2M_2^2 \\
&\quad + 2LM_2\|T_nx_{n+1} - T_{n+1}x_{n+1}\| + \|T_nx_{n+1} - T_{n+1}x_{n+1}\|^2 \\
&\leq \|x_n - T_nx_n\|^2 + 2L(1 + L)\alpha_n\|x_n - T_nx_n\|^2 \\
&\quad + \alpha_n^2L^2M_2^2 + 2M_2(2L + 2)\|T_nx_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + \|T_nx_{n+1} - T_{n+1}x_{n+1}\|^2 \\
&\leq \|x_n - T_nx_n\|^2 + 2L(1 + L)\alpha_n\|x_n - T_nx_n\|^2 \\
&\quad + \alpha_n^2L^2M_2^2 + 2M_2(2L + 2) \sup_{z \in \{x_n\}} \|T_nz - T_{n+1}z\| \\
&\quad + \sup_{z \in \{x_n\}} \|T_nz - T_{n+1}z\|^2.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n\|x_n - T_nx_n\|^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, and $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in \{x_n\}\} < \infty$, it follows from Lemma 2.5 that $\lim_{n \rightarrow \infty} \|x_n - T_nx_n\|$ exists. Hence, by Lemma 3.1(ii), we can conclude that $\lim_{n \rightarrow \infty} \|x_n - T_nx_n\| = 0$. Since

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\| \\
&\leq \|x_n - T_nx_n\| + \sup_{z \in \{x_n\}} \|T_nz - Tz\|, \quad (3.19)
\end{aligned}$$

it follows from Lemma 2.6 that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. This completes the proof. $\square$

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Now, we prove our main result.

Theorem 3.5. *Let E be a real uniformly convex Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E . Let $\{T_n\}_{n=1}^{\infty} : K \rightarrow K$ be a family of λ -strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \quad (3.20)$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}$.

Proof. Let $p \in F$, and define $S_n : K \rightarrow K$ by

$$S_n x = (1 - \alpha_n)x + \alpha_n T_n x, \quad x \in K. \quad (3.21)$$

Then $\bigcap_{n=1}^{\infty} F(S_n) = F = F(T)$. By (3.8), we have for bounded $x, y \in K$ that

$$\begin{aligned} \|S_n x - S_n y\|^2 &= \|x - y - \alpha_n[x - y - (T_n x - T_n y)]\|^2 \\ &\leq \|x - y\|^2 - 2\alpha_n \langle (I - T_n)x - (I - T_n)y, j(x - y) \rangle \\ &\quad + \alpha_n \|x - y - (T_n x - T_n y)\| \beta^*(\alpha_n \|x - y - (T_n x - T_n y)\|) \\ &\leq \|x - y\|^2 - 2\alpha_n \lambda \|x - y - (T_n x - T_n y)\|^2 \\ &\quad + 2\alpha_n^2 \|x - y - (T_n x - T_n y)\|^2 \\ &= \|x - y\|^2 - 2\alpha_n (\lambda - \alpha_n) \|x - y - (T_n x - T_n y)\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \quad (3.22)$$

This implies that S_n is nonexpansive. By Lemma 3.1(i), we know that $\{x_n\}$ is bounded. By Lemma 3.4, we also know that $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$. Applying Lemma 2.2, we also have $\omega_\omega(x_n) \subset F(T)$.

Finally, we will show that $\omega_\omega(x_n)$ is a singleton. Suppose that $x^*, y^* \in \omega_\omega(x_n) \subset F(T)$. Hence $x^*, y^* \in \bigcap_{n=1}^{\infty} F(S_n)$. By Lemma 2.4, $\lim_{n \rightarrow \infty} \langle x_n, j(x^* - y^*) \rangle$ exists. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. Then

$$\|x^* - y^*\|^2 = \langle x^* - y^*, j(x^* - y^*) \rangle = \lim_{k \rightarrow \infty} \langle x_{n_k} - x_{m_k}, j(x^* - y^*) \rangle = 0. \quad (3.23)$$

Hence $x^* = y^*$; consequently, $x_n \rightharpoonup x^* \in \bigcap_{n=1}^{\infty} F(S_n) = F$ as $n \rightarrow \infty$. This completes the proof. $\square$

As a direct consequence of Theorem 3.5, Lemmas 2.7 and 2.8 we also obtain the following results.

Theorem 3.6. Let E be a real uniformly convex Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E . Let $\{S_k\}_{k=1}^\infty$ be a sequence of λ_k -strict pseudocontractions of K into itself such that $\bigcap_{k=1}^\infty F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define a sequence $\{x_n\}$ by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \sum_{k=1}^n \beta_n^k S_k x_n, \quad n \geq 1, \quad (3.24)$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty \alpha_n^2 < \infty$ and $\{\beta_n^k\}$ satisfies conditions (i)–(iii) of Lemma 2.8. Then, $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^\infty$.

Remark 3.7. (i) Theorems 3.5 and 3.6 extend and improve Theorems 3.3 and 3.4 of Chidume-Shahzad [17] in the following senses:

- (i) from real uniformly smooth and uniformly convex Banach spaces to real uniformly convex Banach spaces with Fréchet differentiable norms;
- (ii) from finite strict pseudocontractions to infinite strict pseudocontractions.

Using Opial's condition, we also obtain the following results in a real reflexive Banach space.

Theorem 3.8. Let E be a real Fréchet smooth and reflexive Banach space which satisfies Opial's condition, and let K be a nonempty, closed, and convex subset of E . Let $\{T_n\}_{n=1}^\infty$ be a family of λ -strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \quad (3.25)$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty \alpha_n^2 < \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}$.

Proof. Let $p \in F$. By Lemma 3.1(i), we know that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Since E has the Fréchet differentiable norm, by Lemma 3.4, we know that $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$. It follows from Lemma 2.3 that $\omega_\omega(x_n) \subset F(T) = F$. Finally, we show that $\omega_\omega(x_n)$ is a singleton. Let $x^*, y^* \in \omega_\omega(x_n)$, and let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be subsequences of $\{x_n\}$ chosen so that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. If $x^* \neq y^*$, then Opial's condition of E implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - x^*\| &= \lim_{k \rightarrow \infty} \|x_{n_k} - x^*\| < \lim_{k \rightarrow \infty} \|x_{n_k} - y^*\| = \lim_{k \rightarrow \infty} \|x_{m_k} - y^*\| \\ &< \lim_{k \rightarrow \infty} \|x_{m_k} - x^*\| = \lim_{n \rightarrow \infty} \|x_n - x^*\|. \end{aligned} \quad (3.26)$$

This is a contradiction, and thus the proof is complete. $\square$

Theorem 3.9. Let E be a real Fréchet smooth and reflexive Banach space which satisfies Opial's condition, and let K be a nonempty, closed, and convex subset of E . Let $\{S_k\}_{k=1}^\infty$ be a sequence of

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λ_k -strict pseudocontractions of K into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define a sequence $\{x_n\}$ by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \sum_{k=1}^n \beta_n^k S_k x_n, \quad n \geq 1, \quad (3.27)$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\{\beta_n^k\}$ satisfies conditions (i)–(iii) of Lemma 2.8. Then, $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.

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Research Article

A New General Iterative Method for a Finite Family of Nonexpansive Mappings in Hilbert Spaces

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We introduce a new general iterative method by using the K -mapping for finding a common fixed point of a finite family of nonexpansive mappings in the framework of Hilbert spaces. A strong convergence theorem of the purposed iterative method is established under some certain control conditions. Our results improve and extend the results announced by many others.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H . A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T provided that $Tx = x$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Recall that a self-mapping $f : C \rightarrow C$ is a contraction on C , if there exists a constant $\alpha \in (0, 1)$ such that $\|fx - fy\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \quad (1.1)$$

In 1953, Mann [1] introduced a well-known classical iteration to approximate a fixed point of a nonexpansive mapping. This iteration is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n), \quad n \geq 0, \quad (1.2)$$

where the initial guess x_0 is taken in C arbitrarily, and the sequence $\{\alpha_n\}_{n=0}^\infty$ is in the interval $[0, 1]$. But Mann's iteration process has only weak convergence, even in a Hilbert space setting. In general for example, Reich [2] showed that if E is a uniformly convex Banach space and has a Frechet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\sum_{n=1}^\infty \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by process (1.2) converges weakly to a point in $F(T)$. Therefore, many authors try to modify Mann's iteration process to have strong convergence.

In 2005, Kim and Xu [3] introduced the following iteration process:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n. \end{aligned} \quad (1.3)$$

They proved in a uniformly smooth Banach space that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T under some appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

In 2008, Yao et al. [4] alsomodified Mann's iterative scheme 1.2 to get a strong convergence theorem.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings with $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. There are many authors introduced iterative method for finding an element of F which is an optimal point for the minimization problem. For $n > N$, T_n is understood as $T_{(n \bmod N)}$ with the mod function taking values in $\{1, 2, \dots, N\}$. Let u be a fixed element of H .

In 2003, Xu [5] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1 - \epsilon_n A) T_{n+1} x_n + \epsilon_{n+1} u \quad (1.4)$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle, \quad (1.5)$$

under suitable hypotheses on ϵ_n and under the additional hypothesis

$$F = F(T_1 T_2 \cdots T_N) = F(T_N T_1 \cdots T_{N-1}) = \cdots = F(T_2 T_3 \cdots T_N T_1). \quad (1.6)$$

In 1999, Atsushiba and Takahashi [6] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \gamma_{n,1} T_1 + (1 - \gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1 - \gamma_{n,2}) I, \\ U_{n,3} &= \gamma_{n,3} T_3 U_{n,2} + (1 - \gamma_{n,3}) I, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1} T_N - 1 U_{n,N-2} + (1 - \gamma_{n,N-1}) I, \\ W_n &= U_{n,N} = \gamma_{n,N} T_N U_{n,N-1} + (1 - \gamma_{n,N}) I, \end{aligned} \quad (1.7)$$

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where $\{\gamma_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the W -mapping generated by $T_1, T_2, \dots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \dots, \gamma_{n,N}$.

In 2000, Takahashi and Shimoji [7] proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \dots, N$.

In 2007, Shang et al. [8] introduced a composite iteration scheme as follows:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \end{aligned} \quad (1.8)$$

where $f \in \Pi_C$ is a contraction, and A is a linear bounded operator.

Note that the iterative scheme (1.8) is not well-defined, because $x_n (n \geq 1)$ may not lie in C , so $W_n x_n$ is not defined. However, if $C = H$, the iterative scheme (1.8) is well-defined and Theorem 2.1 [8] is obtained. In the case $C \neq H$, we have to modify the iterative scheme (1.8) in order to make it well-defined.

In 2009, Kangtunyakarn and Suantai [9] introduced a new mapping, called K -mapping, for finding a common fixed point of a finite family of nonexpansive mappings. For a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and sequence $\{\gamma_{n,i}\}_i^N$ in $[0, 1]$, the mapping $K_n : C \rightarrow C$ is defined as follows:

$$\begin{aligned} U_{n,1} &= \gamma_{n,1} T_1 + (1 - \gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1 - \gamma_{n,2}) U_{n,1}, \\ U_{n,3} &= \gamma_{n,3} T_3 U_{n,2} + (1 - \gamma_{n,3}) U_{n,2}, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \gamma_{n,N-1}) U_{n,N-2}, \\ K_n &= U_{n,N} = \gamma_{n,N} T_N U_{n,N-1} + (1 - \gamma_{n,N}) U_{n,N-1}. \end{aligned} \quad (1.9)$$

The mapping K_n is called the K -mapping generated by $T_1, \dots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \dots, \gamma_{n,N}$.

In this paper, motivated by Kim and Xu [3], Marino and Xu [10], Xu [5], Yao et al. [4], and Shang et al. [8], we introduce a composite iterative scheme as follows:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1 - \beta_n) K_n x_n, \\ x_{n+1} &= P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n), \end{aligned} \quad (1.10)$$

where $f \in \Pi_C$ is a contraction, and A is a bounded linear operator. We prove, under certain appropriate conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of the finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, which solves a variational inequality problem.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. For all $x, y \in H$, there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad x, y \in H. \quad (1.11)$$

Lemma 1.2 (see [11]). Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X , and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose that

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n \quad (1.12)$$

for all integer $n \geq 0$, and

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (1.13)$$

Then $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Lemma 1.3 (see [5]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, $n \geq 0$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
- (ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 1.4 (see [10]). Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.

Lemma 1.5 (see [10]). Let H be a Hilbert space. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma} / \alpha$. Let $T : C \rightarrow C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto \gamma f(x) + (1 - tA)Tx$. Then x_t converges strongly as $t \rightarrow 0$ to a fixed point $\bar{x}$ of T , which solves the variational inequality

$$\langle (A - \gamma f)\bar{x}, z - \bar{x} \rangle \geq 0, \quad z \in F(T). \quad (1.14)$$

Lemma 1.6 (see [1]). Demiclosedness principle. Assume that T is nonexpansive self-mapping of closed convex subset C of a Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y , it follows that $(I - T)x = y$. Here, I is identity mapping of H .

Lemma 1.7 (see [9]). Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \dots, N - 1$ and $0 < \lambda_N \leq 1$. Let K be the K -mapping of C into itself generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.

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By using the same argument as in [9, Lemma 2.10], we obtain the following lemma.

Lemma 1.8. *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0, 1]$ such that $\lambda_{n,i} \rightarrow \lambda_i$, as $n \rightarrow \infty$, ($i = 1, 2, \dots, N$). Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K -mappings generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$, and $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$, respectively. Then, for every bounded sequence $x_n \in C$, one has $\lim_{n \rightarrow \infty} \|K_n x_n - K x_n\| = 0$.*

Let H be real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, C a nonempty closed convex subset of H . Recall that the metric (nearest point) projection P_C from a real Hilbert space H to a closed convex subset C of H is defined as follows. Given that $x \in H$, $P_C x$ is the only point in C with the property $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. Below Lemma 1.9 can be found in any standard functional analysis book.

Lemma 1.9. *Let C be a closed convex subset of a real Hilbert space H . Given that $x \in H$ and $y \in C$ then*

- (i) $y = P_C x$ if and only if the inequality $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$,
- (ii) P_C is nonexpansive,
- (iii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for all $x, y \in H$,
- (iv) $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $x \in H$ and $y \in C$.

2. Main Result

In this section, we prove strong convergence of the sequences $\{x_n\}$ defined by the iteration scheme (1.10).

Theorem 2.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H . Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, and let $f \in \prod_C$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let K_n be defined by (1.9). Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \rightarrow 0$;
- (C2) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, for all $i = 1, 2, \dots, N$ and $\{\gamma_{n,i}\}_{i=1}^N \subset [a, b]$, where $0 < a \leq b < 1$;
- (C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by (1.10), then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad p \in F. \quad (2.1)$$

Proof. First, we observe that $\{x_n\}_{n=0}^\infty$ is bounded. Indeed, take a point $u \in F$, and notice that

$$\|y_n - u\| \leq \beta_n \|x_n - u\| + (1 - \beta_n) \|K_n x_n - u\| \leq \|x_n - u\|. \quad (2.2)$$

Since $\alpha_n \rightarrow 0$, we may assume that $\alpha_n \leq \|A^{-1}\|$ for all n . By Lemma 1.4, we have $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$ for all n .

It follows that

$$\begin{aligned} \|x_{n+1} - u\| &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(u)\| \\ &\leq \|\alpha_n(\gamma f(x_n) - Au) + (I - \alpha_n A)(y_n - u)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Au\| + (1 - \alpha_n \bar{\gamma}) \|y_n - u\| \\ &\leq \alpha_n \|\gamma f(x_n) - \gamma f(u)\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma}) \|y_n - u\| \\ &\leq \alpha \gamma \alpha_n \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma}) \|x_n - u\| \\ &= (1 - (\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| \\ &= (1 - (\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - u\| + (\bar{\gamma} - \gamma \alpha) \alpha_n \frac{\|\gamma f(u) - Au\|}{\bar{\gamma} - \gamma \alpha} \\ &\leq \max \left\{ \|x_n - u\|, \frac{\|\gamma f(u) - Au\|}{\bar{\gamma} - \gamma \alpha} \right\}. \end{aligned} \quad (2.3)$$

By simple inductions, we have

$$\|x_n - u\| \leq \max \left\{ \|x_0 - u\|, \frac{\|\gamma f(u) - Au\|}{\bar{\gamma} - \gamma \alpha} \right\}, \quad n \geq 0. \quad (2.4)$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{f(x_n)\}$. Since K_n is nonexpansive and $y_n = \beta_n x_n + (1 - \beta_n) K_n x_n$, we also have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|(\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) K_{n+1} x_{n+1}) - (\beta_n x_n + (1 - \beta_n) K_n x_n)\| \\ &= \|\beta_{n+1} x_{n+1} - \beta_n x_n + \beta_{n+1} x_n - \beta_n x_n + (1 - \beta_{n+1})(K_{n+1} x_{n+1} - K_{n+1} x_n) \\ &\quad + (1 - \beta_{n+1})(K_{n+1} x_n - K_n x_n) + (1 - \beta_{n+1}) K_n x_n - (1 - \beta_n) K_n x_n\| \\ &\leq \beta_{n+1} \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_n\| + (1 - \beta_{n+1}) \|K_{n+1} x_{n+1} - K_{n+1} x_n\| \\ &\quad + (1 - \beta_{n+1}) \|K_{n+1} x_n - K_n x_n\| + |\beta_n - \beta_{n+1}| \|K_n x_n\| \\ &\leq \beta_{n+1} \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_n\| + (1 - \beta_{n+1}) \|x_{n+1} - x_n\| \\ &\quad + (1 - \beta_{n+1}) \|K_{n+1} x_n - K_n x_n\| + |\beta_n - \beta_{n+1}| \|K_n x_n\| \\ &= \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_n\| + (1 - \beta_{n+1}) \|K_{n+1} x_n - K_n x_n\| + |\beta_n - \beta_{n+1}| \|K_n x_n\|. \end{aligned} \quad (2.5)$$

By using the inequalities (2.6) and (2.11) of [9, Lemma 2.11], we can conclude that

$$\|K_n x_{n-1} - K_{n-1} x_{n-1}\| \leq M \sum_{j=1}^N |\gamma_{n,j} - \gamma_{n-1,j}|, \quad (2.6)$$

where $M = \sup\{\sum_{j=2}^N (\|T_j U_{n,j-1} x_n\| + \|U_{n,j-1} x_n\|) + \|T_1 x_n\| + \|x_n\|\}$.

By (2.5) and (2.6), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n)) - (P_C(\alpha_{n-1} \gamma f(x_{n-1}) + (I - \alpha_{n-1} A)y_{n-1}))\| \\ &\leq \|(I - \alpha_n A)(y_n - y_{n-1}) - (\alpha_n - \alpha_{n-1})A y_{n-1} \\ &\quad + \gamma \alpha_n (f(x_n) - f(x_{n-1})) + \gamma (\alpha_n - \alpha_{n-1}) f(x_{n-1})\| \\ &\leq (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|A y_{n-1}\| \\ &\quad + \gamma \alpha_n \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\ &\leq (1 - \alpha_n \bar{\gamma}) [\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\ &\quad + |1 - \beta_n| \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\beta_{n-1} - \beta_n| \|K_{n-1} x_{n-1}\|] \\ &\quad + |\alpha_n - \alpha_{n-1}| \|A y_{n-1}\| + \gamma \alpha_n \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\ &\leq (1 - \alpha_n \bar{\gamma}) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\ &\quad + |1 - \beta_n| \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\beta_{n-1} - \beta_n| \|K_{n-1} x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|A y_{n-1}\| + \gamma \alpha_n \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\ &= (1 - (\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - x_{n-1}\| + L |\beta_{n-1} - \beta_n| + M' |\alpha_n - \alpha_{n-1}| \\ &\quad + |1 - \beta_n| M \sum_{j=1}^N |\gamma_{n,j} - \gamma_{n-1,j}|, \end{aligned} \quad (2.7)$$

where $L = \sup\{\|x_{n-1}\| + \|K_{n-1} x_{n-1}\| : n \in \mathbb{N}\}$, $M' = \max\{\|A y_{n-1}\| + \gamma \|f(x_{n-1})\|\}$. Since $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |\gamma_{n,j} - \gamma_{n-1,j}| < \infty$, for all $j = 1, 2, \dots, N$, by Lemma 1.3, we obtain $\|x_{n+1} - x_n\| \rightarrow 0$. It follows that

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(y_n)\| \\ &\leq \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - y_n\| \\ &= \alpha_n \|\gamma f(x_n) + A y_n\|. \end{aligned} \quad (2.8)$$

Since $\alpha_n \rightarrow 0$ and $\{f(x_n)\}, \{Ay_n\}$ are bounded, we have $\|x_{n+1} - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, \quad (2.9)$$

it implies that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

On the other hand, we have

$$\|K_n x_n - x_n\| \leq \|x_n - y_n\| + \|y_n - K_n x_n\| = \|x_n - y_n\| + \beta_n \|x_n - K_n x_n\|, \quad (2.10)$$

which implies that $(1 - \beta_n)\|K_n x_n - x_n\| \leq \|x_n - y_n\|$.

From condition (C3) and $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\|K_n x_n - x_n\| \rightarrow 0. \quad (2.11)$$

By (C4), we have $\lim_{n \rightarrow \infty} \gamma_{n,i} = \gamma_i \in [a, b]$ for all $i = 1, 2, \dots, N$. Let K be the K -mapping generated by $T_1, \dots, T_N$ and $\gamma_1, \dots, \gamma_N$. Next, we show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0, \quad (2.12)$$

where $q = \lim_{t \rightarrow 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Kx$. Thus, x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Kx_t$. By Lemma 1.5 and Lemma 1.7, we have $q \in F$ and $\langle \gamma f(q) - Aq, p - q \rangle \geq 0$ for all $p \in F$. It follows by (2.11) and Lemma 1.8 that $\|Kx_n - x_n\| \rightarrow 0$. Thus, we have $\|x_t - x_n\| = \|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$. It follows from Lemma 1.1 that for $0 < t < \|A\|^{-1}$,

$$\begin{aligned} \|x_t - x_n\|^2 &= \|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\ &\leq (1 - \bar{\gamma}t)^2 \|Kx_t - x_n\|^2 + 2t \langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\ &\leq (1 - \bar{\gamma}t)^2 (\|Kx_t - Kx_n\|^2 + 2\|Kx_t - Kx_n\| \|Kx_n - x_n\| + \|Kx_n - x_n\|^2) \\ &\quad + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &\leq (1 - 2\bar{\gamma}t + (\bar{\gamma}t)^2) \|x_t - x_n\|^2 + f_n(t) + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle \\ &\quad + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle, \end{aligned} \quad (2.13)$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - Kx_n\|)\|x_n - Kx_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.14)$$

It follows that

$$\begin{aligned}
 \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\leq \left(\frac{-2\bar{\gamma}t + (\bar{\gamma}t)^2}{2t} \right) \|x_t - x_n\|^2 + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
 &\leq \left(\frac{-2 + \bar{\gamma}t}{2} \right) \bar{\gamma} \|x_t - x_n\|^2 + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
 &\leq \left(-1 + \frac{\bar{\gamma}t}{2} \right) \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
 &\leq \frac{\bar{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t).
 \end{aligned} \tag{2.15}$$

Letting $n \rightarrow \infty$ in (2.15) and (2.14), we get

$$\limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{t}{2} M_0, \tag{2.16}$$

where $M_0 > 0$ is a constant such that $M_0 \geq \bar{\gamma} \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0, 1)$ and $n \geq 1$. Taking $t \rightarrow 0$ in (2.16), we have

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0. \tag{2.17}$$

On the other hand, one has

$$\begin{aligned}
 \langle \gamma f(q) - Aq, x_n - q \rangle &= \langle \gamma f(q) - Aq, x_n - q \rangle - \langle \gamma f(q) - Aq, x_n - x_t \rangle \\
 &\quad + \langle \gamma f(q) - Aq, x_n - x_t \rangle - \langle \gamma f(q) - Ax_t, x_n - x_t \rangle \\
 &\quad + \langle \gamma f(q) - Ax_t, x_n - x_t \rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
 &\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle. \\
 &= \langle \gamma f(q) - Aq, x_t - q \rangle + \langle Ax_t - Aq, x_n - x_t \rangle \\
 &\quad + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
 &\leq \|\gamma f(q) - Aq\| \|x_t - q\| + (\|A\| \|x_t - q\| + \gamma \alpha \|x_t - q\|) \|x_n - x_t\| \\
 &\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
 &= \|\gamma f(q) - Aq\| \|x_t - q\| + (\|A\| + \gamma \alpha) \|x_t - q\| \|x_n - x_t\| \\
 &\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
 \end{aligned} \tag{2.18}$$

It follows that

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &\leq \|\gamma f(q) - Aq\| \|x_t - q\| + (\|A\| + \gamma \alpha) \|x_t - q\| \limsup_{n \rightarrow \infty} \|x_n - x_t\| \\
 &\quad + \limsup_{n \rightarrow \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
 \end{aligned} \tag{2.19}$$

Therefore, from (2.17) and $\lim_{t \rightarrow 0} \|x_t - q\| = 0$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &\leq \limsup_{t \rightarrow 0} \left(\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \right) \\ &\leq \limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \leq 0. \end{aligned} \quad (2.20)$$

Hence (2.12) holds. Finally, we prove that $x_n \rightarrow q$. By using (2.2) and together with the Schwarz inequality, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(q)\|^2 \\ &\leq \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(y_n - q)\|^2 \\ &= \|(I - \alpha_n A)(y_n - q)\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\ &\quad + 2\alpha_n \langle (I - \alpha_n A)(y_n - q), \gamma f(x_n) - Aq \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|y_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\ &\quad + 2\alpha_n \langle y_n - q, \gamma f(x_n) - Aq \rangle - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\ &\quad + 2\alpha_n \langle y_n - q, \gamma f(x_n) - \gamma f(q) \rangle + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\ &\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\ &\quad + 2\alpha_n \|y_n - q\| \|\gamma f(x_n) - \gamma f(q)\| + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\ &\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\ &\quad + 2\gamma \alpha_n \|y_n - q\| \|x_n - q\| + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\ &\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\ &\quad + 2\gamma \alpha_n \|x_n - q\|^2 + 2\alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle \\ &\quad - 2\alpha_n^2 \langle A(y_n - q), \gamma f(x_n) - Aq \rangle \\ &\leq \left((1 - \alpha_n \bar{\gamma})^2 + 2\gamma \alpha_n \right) \|x_n - q\|^2 + 2\alpha_n \langle y_n - q, \gamma f(x_n) - Aq \rangle \\ &\quad + \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 + 2\alpha_n^2 \|A(y_n - q)\| \|\gamma f(x_n) - Aq\| \\ &= (1 - 2(\bar{\gamma} - \gamma \alpha) \alpha_n) \|x_n - q\|^2 \\ &\quad + \alpha_n \left(2 \langle y_n - q, \gamma f(q) - Aq \rangle \right. \\ &\quad \left. + \alpha_n \left(\|\gamma f(x_n) - Aq\|^2 + 2 \|A(y_n - q)\| \|\gamma f(x_n) - Aq\| + \bar{\gamma}^2 \|x_n - q\|^2 \right) \right). \end{aligned} \quad (2.21)$$

Fixed Point Theory and Applications

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{y_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \geq \|\gamma f(x_n) - Aq\|^2 + 2\|A(y_n - q)\| \|\gamma f(x_n) - Aq\| + \bar{\gamma}^2 \|x_n - q\|^2 \quad (2.22)$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq (1 - 2(\bar{\gamma} - \gamma\alpha_n))\|x_n - q\|^2 + \alpha_n\beta_n, \quad (2.23)$$

where $\beta_n = 2\langle y_n - q, \gamma f(q) - Aq \rangle + \eta\alpha_n$. By $\limsup_{n \rightarrow \infty} \langle (\gamma f - A)q, y_n - q \rangle \leq 0$, we get $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. By applying Lemma 1.3 to (2.23), we can conclude that $x_n \rightarrow q$. This completes the proof. $\square$

If $A = I$ and $\gamma = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let H be a Hilbert space, C a closed convex nonempty subset of H , and let $f \in \Pi_C$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let K_n be defined by (1.9). Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \rightarrow 0$;
- (C2) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, for all $i = 1, 2, \dots, N$ and $\{\gamma_{n,i}\}_{i=1}^N \subset [a, b]$, where $0 < a \leq b < 1$;
- (C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) K_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{aligned} \quad (2.24)$$

then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle (f - I)q, p - q \rangle \leq 0, \quad p \in F. \quad (2.25)$$

If $N = 1$, $A = I$, $\gamma = 1$, and $f \equiv u \in C$ is a constant in Theorem 2.1, we get the results of Kim and Xu [3].

Corollary 2.3. *Let H be a Hilbert space, C a closed convex nonempty subset of H , and let $f \in \Pi_C$. Let T be a nonexpansive mapping of C into itself. $F(T) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \rightarrow 0$;
- (C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

$$(C3) \ 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1;$$

$$(C4) \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(C5) \ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

If $\{x_n\}_{n=1}^{\infty}$ is the composite process defined by

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} &= \alpha_n u + (I - \alpha_n) y_n, \end{aligned} \quad (2.26)$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle u - q, p - q \rangle \leq 0, \quad p \in F. \quad (2.27)$$

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Research Article

Convergence Analysis for a System of Generalized Equilibrium Problems and a Countable Family of Strict Pseudocontractions

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We introduce a new iterative algorithm for a system of generalized equilibrium problems and a countable family of strict pseudocontractions in Hilbert spaces. We then prove that the sequence generated by the proposed algorithm converges strongly to a common element in the solutions set of a system of generalized equilibrium problems and the common fixed points set of an infinitely countable family of strict pseudocontractions.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H . Let $\{f_k\}_{k \in \Lambda} : C \times C \rightarrow \mathbb{R}$ be a family of bifunctions, and let $\{A_k\}_{k \in \Lambda} : C \rightarrow H$ be a family of nonlinear mappings, where Λ is an arbitrary index set. The system of generalized equilibrium problems is to find $\hat{x} \in C$ such that

$$f_k(\hat{x}, y) + \langle A_k \hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C, k \in \Lambda. \quad (1.1)$$

If Λ is a singleton, then (1.1) reduces to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A \hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \quad (1.2)$$

The solutions set of (1.2) is denoted by $\text{GEP}(f, A)$. If $f \equiv 0$, then the solutions set of (1.2) is denoted by $\text{VI}(C, A)$, and if $A \equiv 0$, then the solutions set of (1.2) is denoted by $\text{EP}(f)$.

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and the Nash equilibrium problem in noncooperative games; see also [1, 2]. Some methods have been constructed to solve the system of equilibrium problems (see, e.g., [3–7]). Recall that a mapping $A : C \rightarrow H$ is said to be

(1) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C, \quad (1.3)$$

(2) *α -inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (1.4)$$

It is easy to see that if A is α -inverse-strongly monotone, then A is monotone and $1/\alpha$ -Lipschitz.

For solving the equilibrium problem, let us assume that f satisfies the following conditions:

- (A1) $f(x, x) = 0$ for all $x \in C$,
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$,
- (A3) for each $x, y, z \in C$, $\lim_{t \rightarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$,
- (A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Throughout this paper, we denote the fixed points set of a nonlinear mapping $T : C \rightarrow C$ by $F(T) = \{x \in C : Tx = x\}$. Recall that T is said to be a κ -strict pseudocontraction if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2. \quad (1.5)$$

It is well known that (1.5) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2. \quad (1.6)$$

It is worth mentioning that the class of strict pseudocontractions includes properly the class of nonexpansive mappings. It is also known that every κ -strict pseudocontraction is $((1 + \kappa)/(1 - \kappa))$ -Lipschitz; see [8].

In 1953, Mann [9] introduced the iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Sx_n, \quad n \geq 0, \quad (1.7)$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$. If S is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined

by (1.7) converges weakly to a fixed point of S (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [10]).

In 1967, Browder and Petryshyn [11] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using Mann iterative algorithm (1.7) with a constant sequence $\alpha_n = \alpha$ for all $n \geq 0$. Recently, Marino and Xu [8] and Zhou [12] extended the results of Browder and Petryshyn [11] to Mann's iteration process (1.7). Since 1967, the construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors (see, e.g., [13–22]).

Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a nonexpansive mapping, $f : C \times C \rightarrow \mathbb{R}$ a bifunction, and let $A : C \rightarrow H$ be an inverse-strongly monotone mapping.

In 2008, Moudafi [23] introduced an iterative method for approximating a common element of the fixed points set of a nonexpansive mapping S and the solutions set of a generalized equilibrium problem $\text{GEP}(f, A)$ as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$\begin{aligned} f(y_n, y) + \langle Ax_n, y - y_n \rangle + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) S y_n, \quad n \geq 1, \end{aligned} \quad (1.8)$$

where $\{\alpha_n\}_{n=0}^\infty \subset (0, 1)$ and $\{r_n\}_{n=0}^\infty \subset (0, \infty)$. He proved that the sequence $\{x_n\}$ generated by (1.8) converges weakly to an element in $\text{GEP}(f, A) \cap F(S)$ under suitable conditions.

Due to the weak convergence, recently, S. Takahashi and W. Takahashi [24] introduced another modification iterative method of (1.8) for finding a common element of the fixed points set of a nonexpansive mapping and the solutions set of a generalized equilibrium problem in the framework of a real Hilbert space. To be more precise, they proved the following theorem.

Theorem 1.1 (see [24]). *Let C be a closed convex subset of a real Hilbert space H , and let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let A be an α -inverse-strongly monotone mapping of C into H , and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \text{GEP}(f, A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$, and let $\{y_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{aligned} f(y_n, y) + \langle Ax_n, y - y_n \rangle + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) y_n], \quad n \geq 1, \end{aligned} \quad (1.9)$$

where $\{\alpha_n\}_{n=1}^\infty \subset [0, 1]$, $\{\beta_n\}_{n=1}^\infty \subset [0, 1]$ and $\{r_n\}_{n=1}^\infty \subset [0, 2\alpha]$ satisfy

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$,
- (ii) $0 < c \leq \beta_n \leq d < 1$,
- (iii) $0 < a \leq r_n \leq b < 2\alpha$,
- (iv) $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$.

Then, $\{x_n\}$ converges strongly to $z = P_{F(S) \cap \text{GEP}(f, A)} u$.

Recently, Yao et al. [25] introduced a new modified Mann iterative algorithm which is different from those in the literature for a nonexpansive mapping in a real Hilbert space. To be more precise, they proved the following theorem.

Theorem 1.2 (see [25]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^\infty$, and let $\{\beta_n\}_{n=0}^\infty$ be two real sequences in $(0, 1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$, $n \geq 0$, be generated iteratively by*

$$\begin{aligned} y_n &= P_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n S y_n. \end{aligned} \quad (1.10)$$

Suppose that the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$,
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$,

then, the sequence $\{x_n\}$ generated by (1.10) strongly converges to a fixed point of S .

We know the following crucial lemmas concerning the equilibrium problem in Hilbert spaces.

Lemma 1.3 (see [1]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \quad (1.11)$$

Lemma 1.4 (see [26]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $x \in H$ and $r > 0$, define the mapping $T_r^f : H \rightarrow 2^C$ as follows:*

$$T_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}. \quad (1.12)$$

Then, the following statements hold:

- (1) T_r^f is single-valued,
- (2) T_r^f is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r^f x - T_r^f y\|^2 \leq \langle T_r^f x - T_r^f y, x - y \rangle, \quad (1.13)$$

- (3) $F(T_r^f) = \text{EP}(f)$,
- (4) $\text{EP}(f)$ is closed and convex.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $r_k > 0$ for each $k \in \{1, 2, \dots, M\}$. Let $\{f_k\}_{k=1}^M : C \times C \rightarrow \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \rightarrow H$ be a family of α_k -inverse-strongly monotone mappings, and let $\{T_n\}_{n=1}^\infty : C \rightarrow C$ be a countable family of κ -strict pseudocontractions. For each $k \in \{1, 2, \dots, M\}$, denote the mapping $T_{r_k}^{f_k, A_k} : C \rightarrow C$ by $T_{r_k}^{f_k, A_k} := T_{r_k}^{f_k}(I - r_k A_k)$, where $T_{r_k}^{f_k} : H \rightarrow C$ is the mapping defined as in Lemma 1.4.

Motivated and inspired by Marino and Xu [8], Moudafi [23], S. Takahashi and W. Takahashi [24], and Yao et al. [25], we consider the following iteration: $x_1 \in C$ and

$$\begin{aligned} y_n &= P_C[(1 - \alpha_n)x_n], \\ u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \dots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n)[\gamma u_n + (1 - \gamma)T_n u_n], \quad n \geq 1, \end{aligned} \quad (1.14)$$

where $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, $\{\beta_n\}_{n=1}^\infty \subset (0, 1)$ and $\gamma \in (0, 1)$.

In this paper, we first prove a path convergence result for a nonexpansive mapping and a system of generalized equilibrium problems. Then, we prove a strong convergence theorem of the iteration process (1.14) for a system of generalized equilibrium problems and a countable family of strict pseudocontractions in a real Hilbert space. Our results extend the main results obtained by Yao et al. [25] in several aspects.

2. Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space H . For each $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. P_C is called the metric projection of H onto C . It is also known that for $x \in H$ and $z \in C$, $z = P_C x$ is equivalent to $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$. Furthermore,

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad (2.1)$$

for all $x \in H, y \in C$. In a real Hilbert space, we also know that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad (2.2)$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

In the sequel, we need the following lemmas.

Lemma 2.1 (see [27, 28]). *Let E be a real uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E , and let $S : C \rightarrow C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$, then $I - S$ is demiclosed at zero.*

Lemma 2.2 (see [29]). *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space E such that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, \quad n \geq 1, \quad (2.3)$$

where $\{\beta_n\}_{n=1}^\infty$ satisfies conditions: $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. If $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2.3 (see [30]). Assume that $\{a_n\}_{n=1}^\infty$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 1, \quad (2.4)$$

where $\{\gamma_n\}_{n=1}^\infty$ is a sequence in $(0, 1)$ and $\{\delta_n\}_{n=1}^\infty$ is a sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^\infty \gamma_n = \infty$; (b) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^\infty |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 (see [31]). Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let the mapping $A : C \rightarrow H$ be α -inverse-strongly monotone, and let $r > 0$ be a constant. Then, we have

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \quad (2.5)$$

for all $x, y \in C$. In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

To deal with a family of mappings, the following conditions are introduced: let C be a subset of a real Hilbert space H , and let $\{T_n\}_{n=1}^\infty$ be a family of mappings of C such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Then, $\{T_n\}$ is said to satisfy the AKTT-condition [32] if for each bounded subset B of C ,

$$\sum_{n=1}^\infty \sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty. \quad (2.6)$$

Lemma 2.5 (see [32]). Let C be a nonempty and closed subset of a Hilbert space H , and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_nx\}$ converges strongly to a point in C . Moreover, let the mapping T be defined by

$$Tx = \lim_{n \rightarrow \infty} T_nx, \quad \forall x \in C. \quad (2.7)$$

Then, for each bounded subset B of C ,

$$\limsup_{n \rightarrow \infty} \{\|Tz - T_nz\| : z \in B\} = 0. \quad (2.8)$$

The following results can be found in [33, 34].

Lemma 2.6 (see [33, 34]). Let C be a closed, and convex subset of a Hilbert space H . Suppose that $\{T_n\}_{n=1}^\infty$ is a family of κ -strictly pseudocontractive mappings from C into H with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\mu_n\}_{n=1}^\infty$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^\infty \mu_n = 1$. Then, the following conclusions hold:

- (1) $G := \sum_{n=1}^\infty \mu_n T_n : C \rightarrow H$ is a κ -strictly pseudocontractive mapping,
- (2) $F(G) = \bigcap_{n=1}^\infty F(T_n)$.

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Lemma 2.7 (see [34]). *Let C be a closed and convex subset of a Hilbert space H . Suppose that $\{S_i\}_{i=1}^\infty$ is a countable family of κ -strictly pseudocontractive mappings of C into itself with $\bigcap_{i=1}^\infty F(S_i) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \rightarrow C$ by*

$$T_n x = \sum_{i=1}^n \mu_n^i S_i x, \quad x \in C, \quad (2.9)$$

where $\{\mu_n^i\}$ is a family of nonnegative numbers satisfying

- (i) $\sum_{i=1}^n \mu_n^i = 1$ for all $n \in \mathbb{N}$,
- (ii) $\mu^i := \lim_{n \rightarrow \infty} \mu_n^i > 0$ for all $i \in \mathbb{N}$,
- (iii) $\sum_{n=1}^\infty \sum_{i=1}^n |\mu_{n+1}^i - \mu_n^i| < \infty$.

Then,

- (1) Each T_n is a κ -strictly pseudocontractive mapping.
- (2) $\{T_n\}$ satisfies AKTT-condition.
- (3) If $T : C \rightarrow C$ is defined by

$$Tx = \sum_{i=1}^\infty \mu^i S_i x, \quad x \in C, \quad (2.10)$$

then $Tx = \lim_{n \rightarrow \infty} T_n x$ and $F(T) = \bigcap_{n=1}^\infty F(T_n) = \bigcap_{i=1}^\infty F(S_i)$.

In the sequel, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.5 with $F(T) = \bigcap_{n=1}^\infty F(T_n)$.

3. Path Convergence Results

Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a nonexpansive mapping. Let $\{f_k\}_{k=1}^M : C \times C \rightarrow \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \rightarrow H$ be a family of α_k -inverse-strongly monotone mappings, and let $r_k \in (0, 2\alpha_k)$. For each $k \in \{1, 2, \dots, M\}$, we denote the mapping $T_{r_k}^{f_k, A_k} : C \rightarrow C$ by

$$T_{r_k}^{f_k, A_k} := T_{r_k}^{f_k} (I - r_k A_k), \quad (3.1)$$

where $T_{r_k}^{f_k}$ is the mapping defined as in Lemma 1.4. For each $t \in (0, 1)$, we define the mapping $S_t : C \rightarrow C$ as follows:

$$S_t x = S T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \dots T_{r_1}^{f_1, A_1} P_C [(1-t)x], \quad \forall x \in C. \quad (3.2)$$

By Lemmas 1.4(2) and 2.4, we know that $T_{r_k}^{f_k}$ and $I - r_k A_k$ are nonexpansive for each $k \in \{1, 2, \dots, M\}$. So, the mapping $T_{r_k}^{f_k, A_k}$ is also nonexpansive for each $k \in \{1, 2, \dots, M\}$.

Moreover, we can check easily that S_t is a contraction. Then, the Banach contraction principle ensures that there exists a unique fixed point x_t of S_t in C , that is,

$$x_t = ST_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \dots T_{r_1}^{f_1, A_1} P_C[(1-t)x_t], \quad t \in (0, 1). \quad (3.3)$$

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a nonexpansive mapping. Let $\{f_k\}_{k=1}^M : C \times C \rightarrow \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \rightarrow H$ be a family of α_k -inverse-strongly monotone mappings, and let $r_k \in (0, 2\alpha_k)$. For each $k \in \{1, 2, \dots, M\}$, let the mapping $T_{r_k}^{f_k, A_k}$ be defined by (3.1). Assume that $F := (\bigcap_{k=1}^M \text{GEP}(f_k, A_k)) \cap (\bigcap_{n=1}^\infty F(T_n)) \neq \emptyset$. For each $t \in (0, 1)$, let the net $\{x_t\}$ be generated by (3.3). Then, as $t \rightarrow 0$, the net $\{x_t\}$ converges strongly to an element in F .*

Proof. First, we show that $\{x_t\}$ is bounded. For each $t \in (0, 1)$, let $y_t = P_C[(1-t)x_t]$ and $u_t = T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \dots T_{r_1}^{f_1, A_1} y_t$. From (3.3), we have for each $p \in F$ that

$$\|x_t - p\| = \|Su_t - Sp\| \leq \|u_t - p\| \leq \|y_t - p\| \leq (1-t)\|x_t - p\| + t\|p\|. \quad (3.4)$$

It follows that

$$\|x_t - p\| \leq \|p\|. \quad (3.5)$$

Hence, $\{x_t\}$ is bounded and so are $\{y_t\}$ and $\{u_t\}$. Observe that

$$\|y_t - x_t\| \leq t\|x_t\| \rightarrow 0, \quad (3.6)$$

as $t \rightarrow 0$ since $\{x_t\}$ is bounded.

Next, we show that $\|u_t - x_t\| \rightarrow 0$ as $t \rightarrow 0$. Denote $\Theta^k = T_{r_k}^{f_k, A_k} T_{r_{k-1}}^{f_{k-1}, A_{k-1}} \dots T_{r_1}^{f_1, A_1}$ for any $k \in \{1, 2, \dots, M\}$ and $\Theta^0 = I$. We note that $u_t = \Theta^M y_t$ for each $t \in (0, 1)$. From Lemma 2.4, we have for each $k \in \{1, 2, \dots, M\}$ and $p \in F$ that

$$\begin{aligned} \|\Theta^k y_t - p\|^2 &= \|T_{r_k}^{f_k, A_k} \Theta^{k-1} y_t - T_{r_k}^{f_k, A_k} \Theta^{k-1} p\|^2 \\ &= \|T_{r_k}^{f_k} (\Theta^{k-1} y_t - r_k A_k \Theta^{k-1} y_t) - T_{r_k}^{f_k} (\Theta^{k-1} p - r_k A_k \Theta^{k-1} p)\|^2 \\ &\leq \|(\Theta^{k-1} y_t - r_k A_k \Theta^{k-1} y_t) - (\Theta^{k-1} p - r_k A_k \Theta^{k-1} p)\|^2 \\ &\leq \|\Theta^{k-1} y_t - p\|^2 + r_k(r_k - 2\alpha_k) \|A_k \Theta^{k-1} y_t - A_k p\|^2. \end{aligned} \quad (3.7)$$

It follows that

$$\begin{aligned}
 \|u_t - p\|^2 &= \|\Theta^M y_t - p\|^2 \\
 &\leq \|y_t - p\|^2 + \sum_{i=1}^M r_i(r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_t - A_i p\|^2 \\
 &= \|P_C[(1-t)x_t] - p\|^2 + \sum_{i=1}^M r_i(r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_t - A_i p\|^2 \\
 &\leq (\|x_t - p\| + t\|x_t\|)^2 + \sum_{i=1}^M r_i(r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_t - A_i p\|^2 \\
 &\leq \|x_t - p\|^2 + tM_1 + \sum_{i=1}^M r_i(r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_t - A_i p\|^2,
 \end{aligned} \tag{3.8}$$

where $M_1 = \sup_{0 < t < 1} \{2\|x_t - p\|\|x_t\| + t\|x_t\|^2\}$. So, we have

$$\begin{aligned}
 \|x_t - p\|^2 &\leq \|u_t - p\|^2 \\
 &\leq \|x_t - p\|^2 + tM_1 + \sum_{i=1}^M r_i(r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_t - A_i p\|^2,
 \end{aligned} \tag{3.9}$$

which implies that

$$\lim_{t \rightarrow 0} \|A_k \Theta^{k-1} y_t - A_k p\| = 0, \tag{3.10}$$

for each $k \in \{1, 2, \dots, M\}$. Since $T_{r_k}^{f_k}$ is firmly nonexpansive for each $k \in \{1, 2, \dots, M\}$, we have for each $p \in F$ and $k \in \{1, 2, \dots, M\}$ that

$$\begin{aligned}
 \|\Theta^k y_t - p\|^2 &= \|T_{r_k}^{f_k, A_k} \Theta^{k-1} y_t - T_{r_k}^{f_k, A_k} \Theta^{k-1} p\|^2 \\
 &= \|T_{r_k}^{f_k}(\Theta^{k-1} y_t - r_k A_k \Theta^{k-1} y_t) - T_{r_k}^{f_k}(\Theta^{k-1} p - r_k A_k \Theta^{k-1} p)\|^2 \\
 &\leq \langle \Theta^{k-1} y_t - r_k A_k \Theta^{k-1} y_t - (p - r_k A_k p), \Theta^k y_t - p \rangle \\
 &= \frac{1}{2} \left(\|\Theta^{k-1} y_t - r_k A_k \Theta^{k-1} y_t - (p - r_k A_k p)\|^2 + \|\Theta^k y_t - p\|^2 \right. \\
 &\quad \left. - \|\Theta^{k-1} y_t - r_k A_k \Theta^{k-1} y_t - (p - r_k A_k p) - (\Theta^k y_t - p)\|^2 \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left(\left\| \Theta^{k-1} y_t - p \right\|^2 + \left\| \Theta^k y_t - p \right\|^2 - \left\| \Theta^{k-1} y_t - \Theta^k y_t - r_k (A_k \Theta^{k-1} y_t - A_k p) \right\|^2 \right) \\
&\leq \frac{1}{2} \left(\left\| \Theta^{k-1} y_t - p \right\|^2 + \left\| \Theta^k y_t - p \right\|^2 - \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\|^2 \right. \\
&\quad \left. + 2r_k \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\| \left\| A_k \Theta^{k-1} y_t - A_k p \right\| \right).
\end{aligned} \tag{3.11}$$

This implies that

$$\begin{aligned}
\left\| \Theta^k y_t - p \right\|^2 &\leq \left\| \Theta^{k-1} y_t - p \right\|^2 - \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\|^2 \\
&\quad + 2r_k \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\| \left\| A_k \Theta^{k-1} y_t - A_k p \right\| \\
&\leq \left\| \Theta^{k-1} y_t - p \right\|^2 - \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\|^2 + M_2 \left\| A_k \Theta^{k-1} y_t - A_k p \right\|,
\end{aligned} \tag{3.12}$$

where $M_2 = \max_{1 \leq k \leq M} \sup_{0 < t < 1} \{2r_k \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\|\}$. This shows that

$$\begin{aligned}
\left\| u_t - p \right\|^2 &= \left\| \Theta^M y_t - p \right\|^2 \\
&\leq \left\| y_t - p \right\|^2 - \sum_{i=1}^M \left\| \Theta^{i-1} y_t - \Theta^i y_t \right\|^2 + M_2 \sum_{i=1}^M \left\| A_i \Theta^{i-1} y_t - A_i p \right\| \\
&\leq \left\| x_t - p \right\|^2 + tM_1 - \sum_{i=1}^M \left\| \Theta^{i-1} y_t - \Theta^i y_t \right\|^2 + M_2 \sum_{i=1}^M \left\| A_i \Theta^{i-1} y_t - A_i p \right\|.
\end{aligned} \tag{3.13}$$

Hence,

$$\begin{aligned}
\left\| x_t - p \right\|^2 &\leq \left\| u_t - p \right\|^2 \\
&\leq \left\| x_t - p \right\|^2 + tM_1 - \sum_{i=1}^M \left\| \Theta^{i-1} y_t - \Theta^i y_t \right\|^2 + M_2 \sum_{i=1}^M \left\| A_i \Theta^{i-1} y_t - A_i p \right\|.
\end{aligned} \tag{3.14}$$

From (3.10), we obtain

$$\sum_{i=1}^M \left\| \Theta^{i-1} y_t - \Theta^i y_t \right\| \longrightarrow 0, \tag{3.15}$$

as $t \rightarrow 0$. So, we can conclude that

$$\lim_{t \rightarrow 0} \left\| \Theta^{k-1} y_t - \Theta^k y_t \right\| = 0, \tag{3.16}$$

for each $k \in \{1, 2, \dots, M\}$. Observing

$$\begin{aligned} \|u_n - y_t\| &= \|\Theta^M y_t - y_t\| \\ &\leq \|\Theta^M y_t - \Theta^{M-1} y_t\| + \|\Theta^{M-1} y_t - \Theta^{M-2} y_t\| + \dots + \|\Theta^1 y_t - y_t\|, \end{aligned} \quad (3.17)$$

it follows by (3.16) that

$$\lim_{t \rightarrow 0} \|u_t - y_t\| = 0. \quad (3.18)$$

From (3.6) and (3.18), we have

$$\lim_{t \rightarrow 0} \|u_t - x_t\| = 0. \quad (3.19)$$

Hence,

$$\|x_t - Sx_t\| = \|Su_t - Sx_t\| \leq \|u_t - x_t\| \rightarrow 0, \quad (3.20)$$

as $t \rightarrow 0$.

Next, we show that $\{x_t\}$ is relatively norm compact as $t \rightarrow 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Put $x_n := x_{t_n}$. From (3.20), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \quad (3.21)$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to $x^* \in C$. Applying Lemma 2.1 to (3.21), we can conclude that $x^* \in F(S)$.

Next, we show that $x^* \in \bigcap_{k=1}^M \text{GEP}(f_k, A_k)$. Note that $\Theta^k y_n = T_{r_k}^{f_k, A_k} \Theta^{k-1} y_n = T_{r_k}^{f_k}(\Theta^{k-1} y_n - r_k A_k \Theta^{k-1} y_n)$ for each $k \in \{1, 2, \dots, M\}$. Hence, for each $y \in C$ and $k \in \{1, 2, \dots, M\}$, we obtain

$$f_k(\Theta^k y_n, y) + \frac{1}{r_k} \langle y - \Theta^k y_n, \Theta^k y_n - (\Theta^{k-1} y_n - r_k A_k \Theta^{k-1} y_n) \rangle \geq 0. \quad (3.22)$$

From (A2), we have

$$\frac{1}{r_k} \langle y - \Theta^k y_n, \Theta^k y_n - (\Theta^{k-1} y_n - r_k A_k \Theta^{k-1} y_n) \rangle \geq f_k(y, \Theta^k y_n), \quad \forall y \in C. \quad (3.23)$$

Therefore,

$$\left\langle y - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}}{r_k} + A_k \Theta^{k-1} y_{n_j} \right\rangle \geq f_k(y, \Theta^k y_{n_j}), \quad \forall y \in C. \quad (3.24)$$

For each $t \in (0, 1)$ and $y \in C$, put $z_t = ty + (1 - t)x^*$. Then, we have $z_t \in C$. From (3.24), we get that

$$\begin{aligned}
 \langle z_t - \Theta^k y_{n_j}, A_k z_t \rangle &\geq \langle z_t - \Theta^k y_{n_j}, A_k z_t \rangle \\
 &\quad - \left\langle z_t - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}}{r_k} + A_k \Theta^{k-1} y_{n_j} \right\rangle + f_k(z_t, \Theta^k y_{n_j}) \\
 &= \langle z_t - \Theta^k y_{n_j}, A_k z_t - A_k \Theta^k y_{n_j} \rangle + \langle z_t - \Theta^k y_{n_j}, A_k \Theta^k y_{n_j} - A_k \Theta^{k-1} y_{n_j} \rangle \\
 &\quad - \left\langle z_t - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}}{r_k} \right\rangle + f_k(z_t, \Theta^k y_{n_j}).
 \end{aligned} \tag{3.25}$$

We note that $\|A_k \Theta^k y_{n_j} - A_k \Theta^{k-1} y_{n_j}\| \leq (1/\alpha_k) \|\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}\| \rightarrow 0$, $\Theta^k y_{n_j} \rightarrow x^*$ as $j \rightarrow \infty$, and $\{A_k\}_{k=1}^M$ is a family of monotone mappings. It follows from (3.25) that

$$\langle z_t - x^*, A_k z_t \rangle \geq f_k(z_t, x^*). \tag{3.26}$$

So, by (A1), (A4) and (3.26), we have for each $y \in C$ and $k \in \{1, 2, \dots, M\}$ that

$$\begin{aligned}
 0 &= f_k(z_t, z_t) \leq t f_k(z_t, y) + (1 - t) f_k(z_t, x^*) \\
 &\leq t f_k(z_t, y) + (1 - t) \langle z_t - x^*, A_k z_t \rangle \\
 &= t f_k(z_t, y) + t(1 - t) \langle y - x^*, A_k z_t \rangle.
 \end{aligned} \tag{3.27}$$

This implies that

$$f_k(z_t, y) + (1 - t) \langle y - x^*, A_k z_t \rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

Letting $t \rightarrow 0$ in (3.28), it follows from (A3) that

$$f_k(x^*, y) + \langle y - x^*, A_k x^* \rangle \geq 0, \quad \forall y \in C. \tag{3.29}$$

Hence $x^* \in \bigcap_{k=1}^M \text{GEP}(f_k, A_k)$; consequently, $x^* \in F$. Further, we see that

$$\begin{aligned}
 \|x_t - x^*\|^2 &= \|Su_t - x^*\|^2 \\
 &\leq \|u_t - x^*\|^2 \\
 &\leq \|y_t - x^*\|^2 \\
 &\leq \|x_t - x^* - tx_t\|^2 \\
 &= \|x_t - x^*\|^2 - 2t\langle x_t, x_t - x^* \rangle + t^2\|x_t\|^2 \\
 &= \|x_t - x^*\|^2 - 2t\langle x_t - x^*, x_t - x^* \rangle - 2t\langle x^*, x_t - x^* \rangle + t^2\|x_t\|^2.
 \end{aligned} \tag{3.32}$$

So, we have

$$\|x_t - x^*\|^2 \leq \langle x^*, x^* - x_t \rangle + \frac{t}{2}\|x_t\|^2. \tag{3.31}$$

In particular,

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - x_n \rangle + \frac{t_n}{2}\|x_n\|^2. \tag{3.32}$$

Since $x_n \rightharpoonup x^*$, we have $x_n \rightarrow x^*$ as $n \rightarrow \infty$. By using the same argument as in the proof of Theorem 3.1 of [25], we can show that $x_t \rightarrow x^* \in F$ as $t \rightarrow 0$. This completes the proof. $\square$

4. Strong Convergence Results

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $\{f_k\}_{k=1}^M : C \times C \rightarrow \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \rightarrow H$ be a family of α_k -inverse-strongly monotone mappings and let $\{T_n\}_{n=1}^\infty : C \rightarrow C$ be a countable family of κ -strict pseudocontractions for some $0 < \kappa < 1$ such that $F := (\bigcap_{k=1}^M \text{GEP}(f_k, A_k)) \cap (\bigcap_{n=1}^\infty F(T_n)) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, $\{\beta_n\}_{n=1}^\infty \subset (0, 1)$, $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \dots, M\}$ satisfy the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$,
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}$ generated by (1.14) converges strongly to an element in F .

Proof. For each $n \in \mathbb{N}$, define $S_n : C \rightarrow C$ by $S_n x = \gamma x + (1 - \gamma)T_n x$, $x \in C$. Then, $F(S_n) = F(T_n) = F(T)$, since $\gamma \in (0, 1)$. Moreover, we know that $\{S_n\}$ satisfies the AKTT-condition, since $\{T_n\}$ satisfies the AKTT-condition. By Lemma 2.5, we can define the mapping $S : C \rightarrow C$ by $Sx = \lim_{n \rightarrow \infty} S_n x$ for $x \in C$. Hence, $Sx = \gamma x + (1 - \gamma)Tx$, $x \in C$, since $T_n x \rightarrow Tx$ for

$x \in C$. Further, we know that S_n is nonexpansive for each $n \in \mathbb{N}$. Indeed, for each $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
 \|S_n x - S_n y\|^2 &= \|\gamma x + (1 - \gamma)T_n x - \gamma y - (1 - \gamma)T_n y\|^2 \\
 &= \|\gamma(x - y) + (1 - \gamma)(T_n x - T_n y)\|^2 \\
 &= \gamma\|x - y\|^2 + (1 - \gamma)\|T_n x - T_n y\|^2 - \gamma(1 - \gamma)\|(I - T_n)x - (I - T_n)y\|^2 \\
 &\leq \gamma\|x - y\|^2 + (1 - \gamma)\|x - y\|^2 + (1 - \gamma)\kappa\|(I - T_n)x - (I - T_n)y\|^2 \\
 &\quad - \gamma(1 - \gamma)\|(I - T_n)x - (I - T_n)y\|^2 \\
 &= \|x - y\|^2 + (1 - \gamma)(\kappa - \gamma)\|(I - T_n)x - (I - T_n)y\|^2 \\
 &\leq \|x - y\|^2.
 \end{aligned} \tag{4.1}$$

Hence, S_n is nonexpansive for each $n \in \mathbb{N}$ and so is S .

Next, we show that $\{x_n\}$ is bounded. Denote $\Theta^k = T_{r_k}^{f_k, A_k} T_{r_{k-1}}^{f_{k-1}, A_{k-1}} \dots T_{r_1}^{f_1, A_1}$ for any $k \in \{1, 2, \dots, M\}$ and $\Theta^0 = I$. We note that $u_n = \Theta^M y_n$. From (1.14), we have for each $p \in F$ that

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|\beta_n x_n + (1 - \beta_n)S_n u_n\| \\
 &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|S_n u_n - p\| \\
 &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|u_n - p\| \\
 &= \beta_n \|x_n - p\| + (1 - \beta_n)\|\Theta^M y_n - p\| \\
 &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|y_n - p\| \\
 &\leq \beta_n \|x_n - p\| + (1 - \beta_n)[(1 - \alpha_n)\|x_n - p\| + \alpha_n\|p\|] \\
 &= (1 - \alpha_n(1 - \beta_n))\|x_n - p\| + \alpha_n(1 - \beta_n)\|p\| \\
 &\leq \max\{\|x_n - p\|, \|p\|\}.
 \end{aligned} \tag{4.2}$$

Hence, by induction, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$.

Next, we show that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{4.3}$$

Since $u_n = \Theta^M y_n$ and $u_{n+1} = \Theta^M y_{n+1}$,

$$\begin{aligned}
 \|u_{n+1} - u_n\| &= \|\Theta^M y_{n+1} - \Theta^M y_n\| \\
 &\leq \|y_{n+1} - y_n\|.
 \end{aligned} \tag{4.4}$$

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Set $z_n = S_n u_n$, $n \in \mathbb{N}$. So, we have from (1.14) and (4.4) that

$$\begin{aligned}
 \|z_{n+1} - z_n\| &= \|S_{n+1}u_{n+1} - S_n u_n\| \\
 &\leq \|S_{n+1}u_{n+1} - S_{n+1}u_n\| + \|S_{n+1}u_n - S_n u_n\| \\
 &\leq \|u_{n+1} - u_n\| + \|S_{n+1}u_n - S_n u_n\| \\
 &\leq \|y_{n+1} - y_n\| + \|S_{n+1}u_n - S_n u_n\| \\
 &\leq \|(1 - \alpha_{n+1})x_{n+1} - (1 - \alpha_n)x_n\| + \sup_{z \in \{u_n\}} \|S_{n+1}z - S_n z\| \\
 &\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{z \in \{u_n\}} \|S_{n+1}z - S_n z\|.
 \end{aligned} \tag{4.5}$$

Since $\{S_n\}$ satisfies the AKTT-condition and $\lim_{n \rightarrow \infty} \alpha_n = 0$, it follows that

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{4.6}$$

So, by Lemma 2.2 and (ii), we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{4.7}$$

Hence,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|z_n - x_n\| = 0. \tag{4.8}$$

Observe that

$$\|y_n - x_n\| = \|P_C[(1 - \alpha_n)x_n] - P_C x_n\| \leq \alpha_n \|x_n\| \longrightarrow 0, \tag{4.9}$$

as $n \rightarrow \infty$. Similar to the proof of Theorem 3.1, we obtain for each $p \in F$ that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \alpha_n M'_1 + \sum_{i=1}^M r_i(r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_n - A_i p\|^2, \tag{4.10}$$

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \alpha_n M'_1 - \sum_{i=1}^M \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 + M'_2 \sum_{i=1}^M \|A_i \Theta^{i-1} y_n - A_i p\|^2, \tag{4.11}$$

for some $M'_1 > 0$ and $M'_2 > 0$. Then, from (4.10), we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|S_n u_n - p\|^2 \\
 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|u_n - p\|^2 \\
 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \\
 &\quad \times \left(\|x_n - p\|^2 + \alpha_n M'_1 + \sum_{i=1}^M r_i (r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_n - A_i p\|^2 \right) \\
 &\leq \|x_n - p\|^2 + \alpha_n M'_1 + (1 - \beta_n) \sum_{i=1}^M r_i (r_i - 2\alpha_i) \|A_i \Theta^{i-1} y_n - A_i p\|^2,
 \end{aligned} \tag{4.12}$$

which implies that

$$(1 - \beta_n) \sum_{i=1}^M r_i (2\alpha_i - r_i) \|A_i \Theta^{i-1} y_n - A_i p\|^2 \leq \|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \alpha_n M'_1. \tag{4.13}$$

So, from (4.8), (i), (ii) and $0 < r_k < 2\alpha_k$ for each $k = 1, 2, \dots, M$, we have

$$\lim_{n \rightarrow \infty} \|A_k \Theta^{k-1} y_n - A_k p\| = 0, \tag{4.14}$$

for each $k \in \{1, 2, \dots, M\}$. Similarly, from (4.11), we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|S_n u_n - p\|^2 \\
 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|u_n - p\|^2 \\
 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \\
 &\quad \times \left(\|x_n - p\|^2 + \alpha_n M'_1 - \sum_{i=1}^M \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 + M'_2 \sum_{i=1}^M \|A_i \Theta^{i-1} y_n - A_i p\| \right) \\
 &\leq \|x_n - p\|^2 + \alpha_n M'_1 - (1 - \beta_n) \sum_{i=1}^M \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 + M'_2 \sum_{i=1}^M \|A_i \Theta^{i-1} y_n - A_i p\|.
 \end{aligned} \tag{4.15}$$

This implies that

$$\begin{aligned}
 (1 - \beta_n) \sum_{i=1}^M \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 \\
 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M'_1 + M'_2 \sum_{i=1}^M \|A_i \Theta^{i-1} y_n - A_i p\|.
 \end{aligned} \tag{4.16}$$

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From (i), (ii), (4.8), and (4.14), it follows that

$$\lim_{n \rightarrow \infty} \|\Theta^{k-1}y_n - \Theta^k y_n\| = 0, \quad (4.17)$$

for each $k \in \{1, 2, \dots, M\}$.

Next, we show that

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \quad (4.18)$$

Observing

$$\begin{aligned} \|u_n - y_n\| &= \|\Theta^M y_n - y_n\| \\ &\leq \|\Theta^M y_n - \Theta^{M-1} y_n\| + \|\Theta^{M-1} y_n - \Theta^{M-2} y_n\| + \dots + \|\Theta^1 y_n - y_n\|, \end{aligned} \quad (4.19)$$

it follows, by (4.17), that

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (4.20)$$

From (4.9) and (4.20), we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (4.21)$$

We see that

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - S_n u_n\| + \|S_n u_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\ &\leq \|x_n - S_n u_n\| + \|u_n - x_n\| + \sup_{z \in \{x_n\}} \|S_n z - Sz\|. \end{aligned} \quad (4.22)$$

So, by (4.7), (4.21), and Lemma 2.5, we have

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \quad (4.23)$$

Let the net $\{x_t\}$ be defined by (3.3). By Theorem 3.1, we have $x_t \rightarrow x^* \in F$ as $t \rightarrow 0$. Moreover, by proving in the same manner as in Theorem 3.2 of [25], we can show that

$$\limsup_{n \rightarrow \infty} \langle x^*, x^* - x_n \rangle \leq 0. \quad (4.24)$$

Finally, we show that $x_n \rightarrow x^* \in F$ as $n \rightarrow \infty$. From (1.14), we have

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|S_n u_n - x^*\|^2 \\
 &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|u_n - x^*\|^2 \\
 &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|y_n - x^*\|^2 \\
 &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|(1 - \alpha_n)(x_n - x^*) - \alpha_n x^*\|^2 \\
 &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \\
 &\quad \times \left((1 - \alpha_n) \|x_n - x^*\|^2 - 2\alpha_n(1 - \alpha_n) \langle x^*, x_n - x^* \rangle + \alpha_n^2 \|x^*\|^2 \right) \\
 &= (1 - \alpha_n(1 - \beta_n)) \|x_n - x^*\|^2 \\
 &\quad + \alpha_n(1 - \beta_n) \left(2(1 - \alpha_n) \langle x^*, x^* - x_n \rangle + \alpha_n \|x^*\|^2 \right).
 \end{aligned} \tag{4.25}$$

By (i) and (4.24), it follows from Lemma 2.3 that $x_n \rightarrow x^* \in F$. This completes the proof. $\square$

As a direct consequence of Lemmas 2.6 and 2.7 and Theorem 4.1, we obtain the following result.

Theorem 4.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $\{f_k\}_{k=1}^M : C \times C \rightarrow \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \rightarrow H$ be a family of α_k -inverse-strongly monotone mappings, and let $\{S_i\}_{i=1}^\infty$ be a sequence of κ_i -strict pseudocontractions of C into itself such that $F := (\bigcap_{k=1}^M \text{GEP}(f_k, A_k)) \cap (\bigcap_{i=1}^\infty F(S_i)) \neq \emptyset$ and $\sup\{\kappa_i : i \in \mathbb{N}\} = \kappa > 0$. Assume that $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \dots, M\}$. Define the sequence $\{x_n\}$ by $x_1 \in C$ and*

$$\begin{aligned}
 y_n &= P_C[(1 - \alpha_n)x_n], \\
 u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \dots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\
 x_{n+1} &= \beta_n x_n + (1 - \beta_n) \left[\gamma u_n + (1 - \gamma) \sum_{i=1}^n \mu_n^i S_i u_n \right], \quad n \geq 1,
 \end{aligned} \tag{4.26}$$

where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0, 1)$ which satisfy (i)-(ii) of Theorem 4.1 and $\{\mu_n^i\}$ is a real sequence which satisfies (i)-(iii) of Lemma 2.7. Then, $\{x_n\}$ converges strongly to an element in F .

Remark 4.3. Theorems 4.1 and 4.2 extend the main results in [25] from a nonexpansive mapping to an infinite family of strict pseudocontractions and a system of generalized equilibrium problems.

Remark 4.4. If we take $A_k \equiv 0$ and $f_k \equiv 0$ for each $k = 1, 2, \dots, M$, then Theorems 3.1, 4.1, and 4.2 can be applied to a system of equilibrium problems and to a system of variational inequality problems, respectively.

Remark 4.5. Let $S_1, S_2, \dots$ be an infinite family of nonexpansive mappings of C into itself, and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i < 1$ for all $i \in \mathbb{N}$. Moreover, let W_n and W be the W -mappings [35] generated by $S_1, S_2, \dots, S_n$ and $\xi_1, \xi_2, \dots, \xi_n$ and $S_1, S_2, \dots$ and $\xi_1, \xi_2, \dots$. Then, we know from [7, 35] that $(\{W_n\}, W)$ satisfies the AKTT-condition. Therefore, in Theorem 4.1, the mapping T_n can be also replaced by W_n .

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An iterative method for equilibrium problems and a finite family of relatively nonexpansive mappings in a Banach space

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ABSTRACT

In this paper, we introduce a new iterative method for finding a common element of the set of fixed points of a finite family of relatively nonexpansive mappings and the set of solutions of an equilibrium problem in uniformly convex and uniformly smooth Banach spaces. Then we prove a strong convergence theorem by using the generalized projection.

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1. Introduction

Let E be a real Banach space and let E^* be the dual space of E . Let C be a nonempty, closed and convex subset of E . Let f be a bifunction from $C \times C$ to the set of real numbers $\mathbb{R}$. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $EP(f)$. Let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T .

Numerous problems in physics, optimization and economics can be reduced to find a solution of the equilibrium problem (1.1) under the constraint on the common fixed points set of a family of mappings $\{T_i\}_{i=1}^m$. Such problems are equivalent to the problems of finding a common element of the solutions set of the equilibrium problem and the fixed points set of a family of mappings.

In this paper, we deal with a class of relatively nonexpansive mappings introduced by Matsushita and Takahashi [10]. There are a lot of important examples of relatively nonexpansive mappings such as the generalized projections, resolvents of a maximal monotone operators, and others; see [9,10].

In 2005, Matsushita and Takahashi [9] introduced an iterative process to approximate a fixed point of a single relatively nonexpansive mapping. They proved strong convergence theorems by using the generalized projection in a uniformly convex and uniformly smooth Banach space. Two years later, Plubtieng and Ungchittrakool in [14] generalized the previous iterative method with two relatively nonexpansive mappings.

In 2007, Tada and Takahashi [18] introduced strong convergence theorems for finding common elements of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Later, Takahashi and Zembayashi in [22] modified the iterative method to find a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a Banach space. Recently, the generalized equilibrium problem was also considered by Takahashi and Takahashi [21].

Motivated and inspired by the previous authors, we introduce a new iterative scheme and prove a strong convergence theorem of such a scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a finite family of relatively nonexpansive mappings in a Banach space. Our main result extends the results in [9,11,14,18,22] in several aspects.

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2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E . The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \rightarrow \infty} \|\frac{x_n + y_n}{2}\| = 1$. Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *smooth* provided that $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in S$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in S$. Moreover, E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$. The norm of E is said to be *Fréchet differentiable*, if for each $x \in S$, the limit is attained uniformly for $y \in S$. It is well-known that if E is smooth, then the duality mapping J is single valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subsets of E , i.e., for a bounded set C of E and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow \|Jx - Jy\| < \varepsilon$$

for all $x, y \in C$; see [5,15–17,19] for more details. It is also known that, if E is uniformly convex, then E has the *Kadec–Klee property*, that is, $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ imply $x_n \rightarrow x$ for $x \in E$ where $x_n \rightharpoonup x$ and $x_n \rightarrow x$ denote the weak convergence and strong convergence of the sequence $\{x_n\}$ to x , respectively. Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$; see [19,20] for more details. Then, it is obvious by the definition that $\phi(x, y) \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$. Moreover, $\phi(x, y) = \|x - y\|^2$ for all x, y in a Hilbert space H . Let C be a closed and convex subset of E and let T be a mapping from C into itself. A mapping T is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and *relatively nonexpansive* if T satisfies the following conditions:

- (1) $F(T)$ is nonempty;
- (2) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$ and $x \in C$;
- (3) $\hat{F}(T) = F(T)$ where $\hat{F}(T)$ is the set of all asymptotic fixed points of T ; see [9,10,22] for more details.

The following lemmas are so useful in proving our main theorem.

Lemma 2.1 (Kamimura and Takahashi [7]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty, closed and convex subset of E . Suppose that E is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : E \rightarrow C$ defined by $\Pi_C x = x_0$ is called the *generalized projection* [1,2,7]. The following are well-known results. For example, see [1,2,7].

Lemma 2.2. [Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E , let $x \in E$ and let $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle y - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.3. [Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall x \in C \text{ and } y \in E.$$

Lemma 2.4. [Kamimura and Takahashi [7]] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.5. [Cho et al. [4]] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.6. [Nammanee et al. [12]] Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\|\alpha x + \beta y + \gamma z + \lambda v\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 + \lambda \|v\|^2 - \alpha \beta g(\|x - y\|)$$

for all $x, y, z, v \in B_r(0)$ and $\alpha, \beta, \gamma, \lambda \in [0, 1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

Next, we prove our lemmas.

Lemma 2.7. Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|),$$

where $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$.

Proof. We first observe that $\frac{\sum_{i=k+1}^m \alpha_i x_i}{1 - \sum_{i=1}^k \alpha_i} \in B_r(0)$ where $x_{k+1}, x_{k+2}, \dots, x_m \in B_r(0)$. Indeed,

$$\left\| \frac{\sum_{i=k+1}^m \alpha_i x_i}{1 - \sum_{i=1}^k \alpha_i} \right\| \leq \frac{\sum_{i=k+1}^m \alpha_i \|x_i\|}{1 - \sum_{i=1}^k \alpha_i} \leq \frac{\sum_{i=k+1}^m \alpha_i r}{1 - \sum_{i=1}^k \alpha_i} = r.$$

Therefore, $\frac{\sum_{i=k+1}^m \alpha_i x_i}{1 - \sum_{i=1}^k \alpha_i} \in B_r(0)$. We set $k = 3$ and then use Lemma 2.6. Thus, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned} \left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 &= \left\| \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \left(1 - \sum_{i=1}^3 \alpha_i\right) \left[\frac{\sum_{i=4}^m \alpha_i x_i}{1 - \sum_{i=1}^3 \alpha_i} \right] \right\|^2 \\ &\leq \sum_{i=1}^3 \alpha_i \|x_i\|^2 + \left(1 - \sum_{i=1}^3 \alpha_i\right) \left\| \frac{\sum_{i=4}^m \alpha_i x_i}{1 - \sum_{i=1}^3 \alpha_i} \right\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) \\ &\leq \sum_{i=1}^3 \alpha_i \|x_i\|^2 + \left(1 - \sum_{i=1}^3 \alpha_i\right) \left[\frac{\sum_{i=4}^m \alpha_i \|x_i\|^2}{1 - \sum_{i=1}^3 \alpha_i} \right] - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|), \end{aligned}$$

where $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$. $\square$

By interchanging the roles of vectors x_i in Lemma 2.7 and summing the inequalities together, we obtain the following lemma.

Lemma 2.8. Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1} \left[\sum_{i=1}^m \alpha_i g(\|x_j - x_i\|) \right]$$

for some $j \in \{1, 2, \dots, m\}$, $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \dots, m$ with $\sum_{i=1}^m \alpha_i = 1$.

Proof. Now, we fix $\alpha_1 x_1$. Then, by Lemma 2.7, we also obtain

$$\begin{aligned} \|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_m x_m\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|) \\ \|\alpha_1 x_1 + \alpha_3 x_3 + \alpha_4 x_4 + \dots + \alpha_2 x_2\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_3 g(\|x_1 - x_3\|) \\ \|\alpha_1 x_1 + \alpha_4 x_4 + \alpha_5 x_5 + \dots + \alpha_3 x_3\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_4 g(\|x_1 - x_4\|) \\ &\vdots \\ \|\alpha_1 x_1 + \alpha_m x_m + \alpha_2 x_2 + \dots + \alpha_{m-1} x_{m-1}\|^2 &\leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \alpha_1 \alpha_m g(\|x_1 - x_m\|). \end{aligned}$$

By combining the above inequalities, we have

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \frac{\alpha_1}{m-1} \left[\sum_{i=2}^m \alpha_i g(\|x_1 - x_i\|) \right].$$

In the same manner, we also have

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 \leq \sum_{i=1}^m \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1} \left[\sum_{i=1}^m \alpha_i g(\|x_j - x_i\|) \right]$$

for some $j \in \{1, 2, \dots, m\}$. For $m = 4$, we obtain Lemma 4 of [13]. $\square$

Lemma 2.9. [Plubtieng and Ungchittrakool [14]] *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed convex subset of E . Then, for points $w, x, y, z \in E$ and a real number $a \in \mathbb{R}$, the set $K := \{v \in C : \phi(v, y) \leq \phi(v, x) + \langle v, Jz - Jw \rangle + a\}$ is closed and convex.*

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions; see [3,6,22]:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Lemma 2.10. [Takahashi and Zembayashi [22]] *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in E$. Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is a firmly nonexpansive-type mapping; see [8], i.e., for all $x, y \in E$,
 $\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, J x - J y \rangle$;
- (3) $F(T_r) = EP(f)$;
- (4) $EP(f)$ is closed and convex.

Lemma 2.11. [Takahashi and Zembayashi [22]] *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.12. [Matsushita and Takahashi [10]] *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3. Main result

In this section, we prove a strong convergence theorem.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty, closed and convex subset of E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Let $T_1, T_2, \dots, T_m$ be relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i) \cap EP(f) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1} \left(\sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right), \quad T_1 = I, \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, J x - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \dots, \end{cases}$$

where $\{\lambda_n^{(i)}\} \subset [0, 1]$, $\sum_{i=1}^m \lambda_n^{(i)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\liminf_{n \rightarrow \infty} \lambda_n^{(i)} \lambda_n^{(j)} > 0$ for some $j \in \{1, 2, \dots, m\}$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F .

Proof. First, we will show that the sequence $\{x_n\}$ is well-defined. By the definition of W_n , it is obvious that W_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.9, we also know that H_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $H_n \cap W_n$ is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.10 (4) and Lemma 2.12, we know that $F := \bigcap_{i=1}^m F(T_i) \cap EP(f)$ is also closed and convex.

Next, we will show that $F \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. Now, let $u \in F$ and put $u_n = T_{r_n} y_n$ for each $n \in \mathbb{N} \cup \{0\}$. We note by [22] that T_{r_n} are relatively nonexpansive. Then, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, T_{r_n} y_n) \leq \phi(u, y_n) = \phi \left(u, J^{-1} \left(\sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right) \right) = \|u\|^2 - 2 \left\langle u, \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \\ &= \|u\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle u, J T_i x_n \rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \leq \|u\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle u, J T_i x_n \rangle + \sum_{i=1}^m \lambda_n^{(i)} \|T_i x_n\|^2 \\ &= \sum_{i=1}^m \lambda_n^{(i)} \phi(u, T_i x_n) \leq \sum_{i=1}^m \lambda_n^{(i)} \phi(u, x_n) = \phi(u, x_n). \end{aligned}$$

Thus $u \in H_n$ and $F \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$. By utilizing Lemmas 2.2 and 2.3 and using the same argument presented in the proof of Theorem 3.1 in [22], we obtain that $F \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$ and hence the sequence $\{x_n\}$ is well-defined. Moreover, $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$. By Lemma 2.1, we also have $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x \in H_n$,

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \rightarrow 0$$

as $n \rightarrow \infty$. Then, we get that $\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (3.1)$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E , we obtain

$$\lim_{n \rightarrow \infty} \|J u_n - J x_n\| = 0. \quad (3.2)$$

Next, we prove that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, m$. Since $\{x_n\}$ is bounded, so are $\{T_i x_n\}$ for all $i = 1, 2, \dots, m$. Note that $T_1 = I$. Now, let $r = \sup_{n \geq 0} \{\|x_n\|, \|T_2 x_n\|, \dots, \|T_m x_n\|\}$ and let $j \in \{1, 2, \dots, m\}$. Note that E^* is uniformly convex. By Lemma 2.8, we have for each $p \in F$ that there exists a continuous, strictly increasing, and convex function $g^*: [0, \infty) \rightarrow [0, \infty)$ such that $g^*(0) = 0$ and

$$\begin{aligned} \phi(p, u_n) &= \phi(p, T_{r_n} y_n) \leq \phi(p, y_n) = \phi \left(p, J^{-1} \left(\sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right) \right) = \|p\|^2 - 2 \left\langle p, \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \\ &= \|p\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle p, J T_i x_n \rangle + \left\| \sum_{i=1}^m \lambda_n^{(i)} J T_i x_n \right\|^2 \leq \|p\|^2 - 2 \sum_{i=1}^m \lambda_n^{(i)} \langle p, J T_i x_n \rangle \\ &\quad + \sum_{i=1}^m \lambda_n^{(i)} \|T_i x_n\|^2 - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^* (\|J T_j x_n - J T_i x_n\|) \right] = \sum_{i=1}^m \lambda_n^{(i)} \phi(p, T_i x_n) \\ &\quad - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^* (\|J T_j x_n - J T_i x_n\|) \right] \leq \sum_{i=1}^m \lambda_n^{(i)} \phi(p, x_n) - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^* (\|J T_j x_n - J T_i x_n\|) \right] \\ &= \phi(p, x_n) - \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^* (\|J T_j x_n - J T_i x_n\|) \right]. \end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{\lambda_n^{(j)}}{m-1} \left[\sum_{i=1}^m \lambda_n^{(i)} g^*(\|JT_j x_n - JT_i x_n\|) \right] &\leq \phi(p, x_n) - \phi(p, u_n) = \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n \rangle \\ &\leq \|x_n\|^2 - \|u_n\|^2 + 2|\langle p, Jx_n - Ju_n \rangle| \leq \|x_n\| - \|u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Since $\liminf_{n \rightarrow \infty} \lambda_n^{(i)} \lambda_n^{(j)} > 0$ for all $i = 1, 2, \dots, m$, we obtain that

$$\lim_{n \rightarrow \infty} g^*(\|JT_j x_n - JT_i x_n\|) = 0$$

for all $i = 1, 2, \dots, m$. Since g^* is a continuous function with $g^*(0) = 0$, we have

$$\lim_{n \rightarrow \infty} \|JT_j x_n - JT_i x_n\| = 0$$

for all $i = 1, 2, \dots, m$. Since J is uniformly norm-to-norm continuous on bounded sets, so is J^{-1} . Thus

$$\lim_{n \rightarrow \infty} \|T_j x_n - T_i x_n\| = \lim_{n \rightarrow \infty} \|J^{-1}(JT_j x_n) - J^{-1}(JT_i x_n)\| = 0.$$

Therefore,

$$\|x_n - T_i x_n\| \leq \|x_n - T_j x_n\| + \|T_j x_n - T_i x_n\| \rightarrow 0$$

as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in \bigcap_{i=1}^m \hat{F}(T_i) = \bigcap_{i=1}^m F(T_i)$.

Next, we prove that $\hat{x} \in EP(f)$. Let $u \in F$. Note that $u_n = T_{r_n} y_n$ and $\phi(u, y_n) \leq \phi(u, x_n)$. Hence, by Lemma 2.11, we obtain that

$$\phi(u_n, y_n) = \phi(T_{r_n} y_n, y_n) \leq \phi(u, y_n) - \phi(u, T_{r_n} y_n) \leq \phi(u, x_n) - \phi(u, T_{r_n} y_n) = \phi(u, x_n) - \phi(u, u_n).$$

Let $r = \sup_{n \geq 0} \{\|y_n\|, \|u_n\|\}$. Then, by Lemma 2.4, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and

$$g(\|u_n - y_n\|) \leq \phi(u_n, y_n) \leq \phi(u, x_n) - \phi(u, u_n) \leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.$$

It follows from (3.1) and (3.2) that

$$g(\|u_n - y_n\|) \rightarrow 0,$$

as $n \rightarrow \infty$. Therefore, from the property of g we get that

$$\|u_n - y_n\| \rightarrow 0,$$

as $n \rightarrow \infty$. By the uniform continuity of J , we also obtain

$$\|Ju_n - Jy_n\| \rightarrow 0,$$

as $n \rightarrow \infty$. Since $r_n \geq a > 0$,

$$\frac{\|Ju_n - Jy_n\|}{r_n} \rightarrow 0,$$

as $n \rightarrow \infty$. Since $x_{n_k} \rightharpoonup \hat{x}$, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ and $\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0$, there exist subsequences $\{y_{n_k}\}$ of $\{y_n\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that $y_{n_k} \rightarrow \hat{x}$ and $u_{n_k} \rightarrow \hat{x}$. By using the same argument as the proof of Theorem 3.1 in [22], we also have $\hat{x} \in EP(f)$. Therefore $\hat{x} \in F := \bigcap_{i=1}^m F(T_i) \cap EP(f)$. Let $w = \Pi_F x$. Then $w \in F \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x$, we have

$$\phi(x_{n+1}, x) = \min_{y \in H_n \cap W_n} \phi(y, x) \leq \phi(w, x).$$

Since the norm is weakly lower semicontinuous, we have

$$\phi(\hat{x}, x) = \|\hat{x}\|^2 - 2\langle \hat{x}, Jx \rangle + \|x\|^2 \leq \liminf_{k \rightarrow \infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx \rangle + \|x\|^2) = \liminf_{k \rightarrow \infty} \phi(x_{n_k}, x) \leq \limsup_{k \rightarrow \infty} \phi(x_{n_k}, x) \leq \phi(w, x).$$

From the definition of Π_F , we have $\hat{x} = w$. Thus $\lim_{k \rightarrow \infty} \phi(x_{n_k}, x) = \phi(w, x)$. By the definition of ϕ , we also have $\lim_{k \rightarrow \infty} \|x_{n_k}\| = \|w\|$. Since E is uniformly convex, E has the Kadec–Klee property. We obtain that $x_{n_k} \rightarrow w \in \Pi_F x$. Therefore, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x$. This completes the proof. $\square$

Remark 3.2

- (i) Theorem 3.1 extends the results of [9,22] from a single relatively nonexpansive mapping to a finite family of relatively nonexpansive mappings.

- (ii) Theorem 3.1 extends the results of [14] from two relatively nonexpansive mappings to a finite family of relatively nonexpansive mappings.
- (iii) Theorem 3.1 extends the results of [11,18] from real Hilbert spaces to uniformly smooth and uniformly convex Banach spaces.

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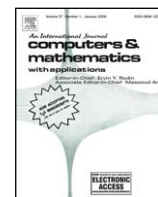
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A new approximation method for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces

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ABSTRACT

In this paper, we consider a new iterative scheme to approximate a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings. We prove several strong and weak convergence results of the proposed iteration in Banach spaces. These results generalize and refine many known results in the current literature.

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1. Introduction

In 2008, Khan et al. [1] introduced an iterative process for a finite family of mappings as follows:

Let C be a convex subset of a Banach space X and let $\{T_i : i = 1, 2, \dots, k\}$ be a family of self-mappings of C . Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, 3, \dots$ and $i = 1, 2, \dots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \quad (1)$$

where $y_{0n} = x_n$ for all n . The iterative process (1) is the generalized form of the modified Mann (one-step) iterative process by Schu [2], the modified Ishikawa (two-step) iterative process by Tan and Xu [3], and the three-step iterative process by Xu and Noor [4].

Goebel and Kirk [5], in 1972, introduced the new concept of asymptotically nonexpansive and proved that every asymptotically nonexpansive self-mapping of a nonempty closed bounded and convex subset of a uniformly convex Banach space has a fixed point. In 1978, Bose [6] studied an iterative scheme for fixed points of asymptotically nonexpansive mappings. In 2001, Khan and Takahashi [7] used the modified Ishikawa process to approximate common fixed points of two asymptotically nonexpansive mappings.

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Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers [8–10] are interested in studying approximation method for finding common fixed points of nonlinear mapping. Also, approximation methods for finding fixed points for nonexpansive mappings can be seen in [11–18].

In 2003, Sun [19] studied an implicit iterative scheme initiated by Xu and Ori [20] for a finite family of asymptotically quasi-nonexpansive mappings. Shahzad and Udomene [21], in 2006, proved some convergence theorems for the modified Ishikawa iterative process of two asymptotically quasi-nonexpansive mappings to a common fixed point. Nammanee et al. [22] introduced a three-step iteration scheme for asymptotically nonexpansive mappings and proved weak and strong convergence theorems of that iteration scheme under some control conditions. In 2007, Fukhar-ud-din and Khan [23] studied a new three-step iteration scheme for approximating a common fixed point of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Shahzad and Zegeye [24] introduced a new concept of generalized asymptotically nonexpansive mappings and proved some strong convergence theorems for fixed points of finite family of this class. Recently, Khan et al. [1] introduced the iterative sequence (1) for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

Motivated by Khan et al. [1], we introduce a new iterative scheme for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings as follow:

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1 - \alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= (1 - \alpha_{(k-2)n})y_{(k-3)n} + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \quad (2)$$

where $y_{0n} = x_n$ for all n .

The aim of this paper is to obtain some strong and weak convergence results for the iterative process (2) of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

2. Preliminaries

In this section, we review definitions and lemmas used for the rest of the paper as follow:

Let C be a nonempty subset of a real Banach space X and T be a self-mapping of C . The fixed point set of T is denoted by $F(T) = \{x \in C : Tx = x\}$. If $F(T)$ is not empty, then T is called

- (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;
- (ii) *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$;
- (iii) *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ and $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (iv) *asymptotically quasi-nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ and $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, for all $x \in C, p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (v) *generalized quasi-nonexpansive* if there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\|T^n x - p\| \leq \|x - p\| + s_n$, for all $x \in C, p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (vi) *generalized asymptotically quasi-nonexpansive* [24] if there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, \infty)$ with $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C, p \in F(T)$ and $n = 1, 2, 3, \dots$;
- (vii) *uniformly L -Lipschitzian* if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (viii) *$(L - \gamma)$ uniform Lipschitz* if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^\gamma$, for all $x, y \in C$ and $n = 1, 2, 3, \dots$;
- (ix) *semi-compact* if for a sequence $\{x_n\}$ in C with $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow p \in C$.

Condition (A''). Let C be a subset of a normed space X . A family of self-mappings $\{T_i : i = 1, 2, \dots, k\}$ of C is said to have Condition (A'') if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T_i x\| \geq f(d(x, F))$ for some $1 \leq i \leq k$ and for all $x \in C$ where $d(x, F) = \inf \{\|x - p\| : p \in F = \bigcap_{i=1}^k F(T_i)\}$.

The Condition (A'') defined above by the authors is the generalization of the Condition (A) [25] when $k = 1$ and Condition (A') [26] for $k = 2$.

The map $T : C \rightarrow X$ is said to be *demiclosed at 0* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and Tx_n converging strongly to 0, we get $Tx = 0$.

A Banach space X is said to have *Opial's property* if for each sequence $\{x_n\}$ converging weakly to $x \in C$ and $x \neq y$, we have the condition

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

Remark 2.1. It is easy to see that,

- (i) a quasi-nonexpansive mapping is generalized quasi-nonexpansive;
- (ii) an asymptotically quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;
- (iii) a generalized quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;
- (iv) a uniformly L -Lipschitzian mapping is $(L - 1)$ uniform Lipschitz.

The next example shows that there is an asymptotically quasi-nonexpansive mapping which is not quasi-nonexpansive.

Example 2.2. Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \quad \text{for all } x = (x_1, x_2, \dots, x_n, \dots) \in X,$$

and a mapping $T : X \rightarrow X$ defined by $Tx = (0, 2x_1, 0, 0, \dots, 0, \dots)$. By letting $Tx = x$ for any $x = (x_1, x_2, \dots, x_n, \dots) \in X$, we have

$$(0, 2x_1, 0, 0, \dots, 0, \dots) = (x_1, x_2, \dots, x_n, \dots),$$

i.e., $F(T) = \{0\}$. Moreover, $T^n x = (0, 0, 0, \dots, 0, \dots)$ for all $n = 2, 3, 4, \dots$

For the sequence $\{r_n\}$, where $r_n = \frac{1}{n}$, and $p \in F(T)$, we have

$$\|Tx - p\| = 2\|x_1\| \leq (1 + r_1)\|x - p\| \quad \text{and} \quad \|T^n x - p\| \leq (1 + r_n)\|x - p\|,$$

for all $n = 2, 3, 4, \dots$. This implies that T is an asymptotically quasi-nonexpansive mapping. However, T is not a quasi-nonexpansive mapping since, for $x^0 = (1, 0, 0, \dots, 0, \dots)$ in X ,

$$\|Tx^0 - p\| = \|(0, 2, 0, 0, \dots, 0, \dots)\| = 2 > 1 = \|x^0 - p\|. \quad \square$$

Lemma 2.3 (Cf. [19, Lemma 2.2]). Let the sequences $\{a_n\}$ and $\{\delta_n\}$ of real numbers satisfy:

$$a_{n+1} \leq (1 + \delta_n)a_n, \quad \text{where } a_n \geq 0, \delta_n \geq 0, \text{ for all } n = 1, 2, 3, \dots$$

and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists;
- (ii) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 (See [2, Lemma 1.3]). Let X be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1$, $n = 1, 2, 3, \dots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$, where $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

3. Convergence in Banach spaces

The aim of this section is to establish the strong convergence of the iterative scheme (2) to converge to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space under some appropriate conditions.

Lemma 3.1. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$, are defined by (2). Then for $p \in F$, we get

- (i) $\|x_n - T_i^n x_n\| \leq (2 + r_n)\|x_n - p\|$, for all $i = 1, 2, \dots, k$;
- (ii) $\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n)\|y_{(i-1)n} - p\|$, for all $i = 1, 2, \dots, k$;
- (iii) $\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n)\|y_{(i-1)n} - p\|$, for all $i = 1, 2, \dots, k$;
- (iv) $\|y_{in} - p\| \leq (1 + r_n)^i \|x_n - p\|$, for $i = 1, 2, \dots, k - 1$;
- (v) $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\|$;
- (vi) if $\sum_{n=1}^{\infty} r_n < \infty$, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists,

where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$ and $\delta_n = \binom{k}{1} r_n + \binom{k}{2} r_n^2 + \dots + \binom{k}{k} r_n^k$.

Proof. Let $p \in F$.

(i) For $i = 1, 2, 3, \dots, k$, we have

$$\begin{aligned}\|x_n - T_i^n x_n\| &\leq \|x_n - p\| + \|T_i^n x_n - p\| \\ &\leq \|x_n - p\| + (1 + r_n)\|x_n - p\|, \\ &= (2 + r_n)\|x_n - p\|.\end{aligned}$$

(ii) Similarly to part (i), we have

$$\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \leq (2 + r_n)\|y_{(i-1)n} - p\|, \quad \text{for all } i = 1, 2, \dots, k.$$

(iii) For $i = 1, 2, \dots, k$, we have

$$\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n)\|y_{(i-1)n} - p\|.$$

(iv) By part (i) and $\alpha_{1n} \leq 1$, we obtain

$$\begin{aligned}\|y_{1n} - p\| &= \|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_1^n x_n - p)\| \\ &\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| \\ &\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}(1 + r_n)\|x_n - p\| \\ &\leq (1 + r_n)\|x_n - p\|.\end{aligned}$$

We assume that $\|y_{jn} - p\| \leq (1 + r_n)^j \|x_n - p\|$ holds for some $1 \leq j \leq k - 2$. From part (iii) and $\alpha_{(j+1)n} \leq 1$, we then have

$$\begin{aligned}\|y_{(j+1)n} - p\| &= \|(1 - \alpha_{(j+1)n})(y_{jn} - p) + \alpha_{(j+1)n}(T_{j+1}^n y_{jn} - p)\| \\ &\leq (1 - \alpha_{(j+1)n})\|y_{jn} - p\| + \alpha_{(j+1)n}\|T_{j+1}^n y_{jn} - p\| \\ &\leq (1 - \alpha_{(j+1)n})\|y_{jn} - p\| + \alpha_{(j+1)n}(1 + r_n)\|y_{jn} - p\| \\ &\leq (1 + r_n)\|y_{jn} - p\| \\ &\leq (1 + r_n)(1 + r_n)^j \|x_n - p\| \\ &= (1 + r_n)^{j+1} \|x_n - p\|.\end{aligned}$$

Therefore, by mathematical induction, we obtain

$$\|y_{in} - p\| \leq (1 + r_n)^i \|x_n - p\|, \quad \text{for } i = 1, 2, \dots, k - 1.$$

(v) By part (ii), part (iv), and $\alpha_{kn} \leq 1$, we get

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| \\ &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - p\| \\ &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_n)\|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)\|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)(1 + r_n)^{k-1} \|x_n - p\| \\ &= (1 + r_n)^k \|x_n - p\| \\ &\leq (1 + \delta_n)\|x_n - p\|,\end{aligned}$$

where $\delta_n = \binom{k}{1} r_n + \binom{k}{2} r_n^2 + \dots + \binom{k}{k} r_n^k$.

(vi) By (v), we have $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\|$ for all $n \in \mathbb{N}$. From $\sum_{n=1}^{\infty} r_n < \infty$, we also have $\sum_{n=1}^{\infty} r_n^i < \infty$ for $i = 1, 2, 3, \dots, k$. It follows that $\sum_{n=1}^{\infty} \delta_n < \infty$. By Lemma 2.3, we get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. $\square$

Theorem 3.2. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - p_i\| \leq (1 + r_n)\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2). Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

Proof. The necessity is obvious and then we prove only the sufficiency. Let $p \in F$. By Lemma 3.1(vi), $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and hence $\{\|x_n - p\|\}$ is bounded. We let $M = \sup_{n \geq 1} \{\|x_n - p\|\}$. From Lemma 3.1(v), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M\delta_n, \quad n \geq 1.$$

Thus, for positive integers m and n , we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + M\delta_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + M(\delta_{n+m-1} + \delta_{n+m-2}) \\ &\vdots \\ &\leq \|x_n - p\| + M \sum_{i=n}^{n+m-1} \delta_i. \end{aligned} \quad (3)$$

By Lemma 3.1(v), we obtain

$$d(x_{n+1}, F) \leq (1 + \delta_n)d(x_n, F).$$

From the given condition $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ and Lemma 2.3, we get

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (4)$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C . By (4) and $\sum_{n=1}^{\infty} \delta_n < \infty$, we get that for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$d(x_n, F) < \frac{\epsilon}{8} \quad \text{and} \quad \sum_{n=n_0}^{\infty} \delta_n < \frac{\epsilon}{2M}. \quad (5)$$

From the first inequality of (5), there exists $p_0 \in F$ such that

$$\|x_{n_0} - p_0\| < \frac{\epsilon}{4}. \quad (6)$$

For any positive integer m , by (3), (5) and (6), we obtain

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq 2\|x_{n_0} - p_0\| + M \sum_{i=n_0}^{n_0+m-1} \delta_i \\ &< 2\left(\frac{\epsilon}{4}\right) + M\left(\frac{\epsilon}{2M}\right) = \epsilon. \end{aligned} \quad (7)$$

Thus, $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, $\{x_n\} \rightarrow q \in X$. Actually, $q \in C$ because $\{x_n\} \subset C$ and C is a closed subset of X . Next we show that $q \in F$. Since $F(T_i)$ is a closed subset in C for all $i = 1, 2, \dots, k$, so is $F = \bigcap_{i=1}^k F(T_i)$. From the continuity of $d(x, F)$ with $d(x_n, F) \rightarrow 0$ and $x_n \rightarrow q$ as $n \rightarrow \infty$, we get $d(q, F) = 0$ and then $q \in F$. Therefore, the proof is complete. $\square$

Since any quasi-nonexpansive mapping is asymptotically quasi-nonexpansive, the next corollary is obtained immediately from Theorem 3.2.

Corollary 3.3. Let C be a nonempty closed convex subset of a real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - p_i\| \leq \|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2). Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

4. Convergence in uniformly convex Banach spaces

In this section, we prove some strong and weak convergence results for the iterative process (2) on uniformly convex Banach spaces without using the condition $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ appearing in Section 3. Instead, we consider $(L - \gamma)$ uniform Lipschitz mappings, Condition (A''), semi-compact mappings, Opial property and demiclosed mappings at 0.

Theorem 4.1. Let C be a nonempty closed convex subset of an uniformly convex real Banach space X . Let $\{T_i : i = 1, 2, \dots, k\}$ be a family of uniformly $(L - \gamma_i)$ Lipschitzian and asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{1/\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_n)\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $\{T_i : i = 1, 2, \dots, k\}$ satisfies Condition (A'') and $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (2) with $\{\alpha_n\}_{n=1}^{\infty} \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

Proof. Let $p \in F$. By Lemma 3.1(vi), we get that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Then there is a real number $c \geq 0$ such that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (8)$$

By Lemma 3.1(iv), we have

$$\|y_{in} - p\| \leq (1 + r_n)^i \|x_n - p\|, \quad \text{for } i = 1, 2, \dots, k-1.$$

By taking $\limsup$ on both sides of the above inequality, we get

$$\limsup_{n \rightarrow \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (9)$$

Therefore, by Lemma 3.1(iii) and (8), we obtain

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k. \quad (10)$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c$, we have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = c.$$

Using (8), (10) and Lemma 2.4, we conclude that

$$\lim_{n \rightarrow \infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0.$$

We assume that

$$\lim_{n \rightarrow \infty} \|y_{(j-1)n} - T_j^n y_{(j-1)n}\| = 0, \quad \text{for some } 2 \leq j \leq k. \quad (11)$$

By Lemma 3.1(iii), we have

$$\|x_{n+1} - p\| \leq (1 + r_n)^{k-i} \|y_{in} - p\|, \quad \text{for all } i = 1, 2, \dots, k-1.$$

This together with (11) and $r_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$c \leq \liminf_{n \rightarrow \infty} \|y_{(j-1)n} - p\|. \quad (12)$$

By Lemma 3.1(iv), (2) and (12), we get

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{(j-1)n})(y_{(j-2)n} - p) + \alpha_{(j-1)n}(T_{j-1}^n y_{(j-2)n} - p)\| = \lim_{n \rightarrow \infty} \|y_{(j-1)n} - p\| = c.$$

Using (9), (10) and Lemma 2.4, we conclude that

$$\lim_{n \rightarrow \infty} \|y_{(j-2)n} - T_{j-1}^n y_{(j-2)n}\| = 0.$$

Therefore, by mathematical induction, we obtain

$$\lim_{n \rightarrow \infty} \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (13)$$

From (2), we have

$$\|y_{in} - y_{(i-1)n}\| = \alpha_{in} \|T_i^n y_{(i-1)n} - y_{(j-1)n}\|, \quad \text{for } i = 1, 2, \dots, k-1.$$

By (13), we obtain that

$$\|y_{in} - y_{(i-1)n}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for } i = 1, 2, \dots, k-1. \quad (14)$$

From

$$\|x_n - y_{in}\| \leq \|x_n - y_{1n}\| + \|y_{1n} - y_{2n}\| + \dots + \|y_{(i-1)n} - y_{in}\|,$$

for $i = 1, 2, \dots, k-1$. It follows by (14) that

$$\|x_n - y_{in}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for } i = 1, 2, \dots, k-1. \quad (15)$$

From (13), when $i = 1$ we get $\lim_{n \rightarrow \infty} \|x_n - T_1^n x_n\| = 0$. For $2 \leq i \leq k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - T_i^n x_n\| \\ &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + L \|y_{(i-1)n} - x_n\|^{y_i}. \end{aligned}$$

From (13) and (15), we conclude that

$$\lim_{n \rightarrow \infty} y_{in} = \lim_{n \rightarrow \infty} \|x_n - T_i^n x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k \quad (16)$$

where $\gamma_{in} = \|x_n - T_i^n x_n\|$. From (2), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - x_n\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - x_n\| \\ &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - x_n\| + \alpha_{kn}(\|T_k^n y_{(k-1)n} - y_{(k-1)n}\| + \|y_{(k-1)n} - x_n\|) \\ &= \|y_{(k-1)n} - x_n\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - y_{(k-1)n}\|.\end{aligned}$$

From (13) and (15),

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (17)$$

For $i = 1, 2, \dots, k$, we have

$$\begin{aligned}\|x_{n+1} - T_i x_{n+1}\| &\leq \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\leq \gamma_{i(n+1)} + L\|x_{n+1} - T_i^n x_{n+1}\|^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L(\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|)^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L(\|x_{n+1} - x_n\| + \gamma_{in} + L\|x_n - x_{n+1}\|^{\gamma_i})^{\gamma_i}.\end{aligned}$$

Using (16) and (17), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_i x_{n+1}\| = 0, \quad \text{for } i = 1, 2, \dots, k.$$

Therefore, by using Condition (A''), there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - T_j x_n\| = 0,$$

for some $1 \leq j \leq k$. That is

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

By Theorem 3.2, we conclude that $\{x_n\}$ converges strongly to a point $p \in F$. $\square$

Lemma 4.2. Let C be a nonempty closed convex subset of a uniformly convex real Banach space X , and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then,

- (i) $\lim_{n \rightarrow \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0$, for all $i = 1, 2, \dots, k$;
- (ii) $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, for all $i = 1, 2, \dots, k$.

Proof. (i) Let $p \in F$. By Lemma 3.1(vi), we obtain that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and we then suppose that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (18)$$

By (18) and Lemma 3.1(iv), we have

$$\limsup_{n \rightarrow \infty} \|y_{in} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (19)$$

By (2), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - p\| \\ &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_n)\|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)\|y_{(k-1)n} - p\| \\ &= (1 + r_n)\|(1 - \alpha_{(k-1)n})(y_{(k-2)n} - p) + \alpha_{(k-1)n}(T_{k-1}^n y_{(k-2)n} - p)\| \\ &\leq (1 + r_n)((1 - \alpha_{(k-1)n})\|y_{(k-2)n} - p\| + \alpha_{(k-1)n}(1 + r_n)\|y_{(k-2)n} - p\|) \\ &\leq (1 + r_n)^2\|y_{(k-2)n} - p\| \\ &\vdots \\ &\leq (1 + r_n)^{k-i}\|y_{in} - p\|,\end{aligned}$$

for some $i = 1, 2, \dots, k-1$. It follows that

$$c \leq \liminf_{n \rightarrow \infty} \|y_{in} - p\|, \quad \text{for } i = 1, 2, \dots, k-1. \quad (20)$$

From (19) and (20), we obtain

$$\lim_{n \rightarrow \infty} \|y_{in} - p\| = c, \quad (21)$$

and then

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{in})(y_{(i-1)n} - p) + \alpha_{in}(T_i^n y_{(i-1)n} - p)\| = c, \quad (22)$$

for $i = 1, 2, \dots, k-1$.

By Lemma 3.1(iii) and (21), we get

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - p\| \leq c, \quad \text{for } i = 1, 2, \dots, k-1. \quad (23)$$

From (19), (22), (23) and Lemma 2.4, we obtain

$$\lim_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k-1. \quad (24)$$

Now we want to show that (24) is also true for $i = k$. By Lemma 3.1(ii) and (iv), we have

$$\begin{aligned} \|T_k^n y_{(k-1)n} - p\| &\leq (1 + r_n) \|y_{(k-1)n} - p\| \\ &\leq (1 + r_n)(1 + r_n)^{k-1} \|x_n - p\| \\ &= (1 + r_n)^k \|x_n - p\|. \end{aligned}$$

This implies by (18) that

$$\limsup_{n \rightarrow \infty} \|T_k^n y_{(k-1)n} - p\| \leq c. \quad (25)$$

We also have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c.$$

Hence, by (19), (25) and Lemma 2.4, we obtain

$$\lim_{n \rightarrow \infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0. \quad (26)$$

Then, (24) and (26) give us

$$\lim_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (27)$$

From

$$\|x_n - T_i^n y_{(i-1)n}\| \leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\|,$$

it implies by (15) and (27) that

$$\lim_{n \rightarrow \infty} \|x_n - T_i^n y_{(i-1)n}\| = 0, \quad (28)$$

for some $i = 1, 2, 3, \dots, k$.

(ii) From part (i), for $i = 1$, we have

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - x_n\| = 0. \quad (29)$$

For $i = 2, 3, 4, \dots, k$, we get

$$\begin{aligned} \|T_i^n x_n - x_n\| &\leq \|T_i^n x_n - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - x_n\| \\ &\leq L \|x_n - y_{(i-1)n}\|^{y^i} + \|T_i^n y_{(i-1)n} - x_n\|. \end{aligned}$$

By part (i) and (15), we conclude that

$$\lim_{n \rightarrow \infty} \|T_i^n x_n - x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad (30)$$

For $1 \leq i \leq k$, we obtain

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + L \|x_{n+1} - x_n\|^{y_i} + L \|T_i^n x_n - x_n\|^{y_i}. \end{aligned}$$

From (16) and (30), we then have

$$\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \quad \square$$

Theorem 4.3. Under the hypotheses of Lemma 4.2, assume that T_j^m is semi-compact for some positive integers m and $1 \leq j \leq k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, k\}$.

Proof. Suppose that T_j^m is semi-compact for some positive integers $m \geq 1$ and $1 \leq j \leq k$. We have

$$\begin{aligned} \|T_j^m x_n - x_n\| &\leq \|T_j^m x_n - T_j^{m-1} x_n\| + \|T_j^{m-1} x_n - T_j^{m-2} x_n\| + \dots + \|T_j^2 x_n - T_j x_n\| + \|T_j x_n - x_n\| \\ &\leq (m-1)L\|T_j x_n - x_n\|^{\gamma_j} + \|T_j x_n - x_n\|. \end{aligned}$$

Then, by Lemma 4.2(ii), we get $\|T_j^m x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded and T_j^m is semi-compact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightarrow q \in C$ as $l \rightarrow \infty$.

By continuity of T_i and Lemma 4.2(ii), we obtain

$$\|q - T_j q\| = \lim_{l \rightarrow \infty} \|x_{n_l} - T_j x_{n_l}\| = 0, \quad \text{for all } j = 1, 2, \dots, k.$$

Therefore, $q \in F$ and then Theorem 3.2 implies that $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1, 2, \dots, k\}$. $\square$

We note that in practical Theorem 4.3 is very useful in the case that one of T_i , $i = 1, 2, 3, \dots, k$, is semi-compact.

Theorem 4.4. Let C be a nonempty closed convex subset of an uniformly convex real Banach space X satisfying the Opial property, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (2) with $\{\alpha_{in}\}_{i=1}^n \subset [a, b]$, where $0 < a < b < 1$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. If $I - T_i$, $i = 1, 2, \dots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.

Proof. Let $p \in F$. By Lemma 3.1(vi), we get $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Then we follow the proof of Theorem 3.2 by Khan et al. [1] until we can conclude that $\{x_n\}$ converges weakly to a common fixed point $p \in F$. $\square$

5. Concluding remarks

The following remarks are obtained directly from the results in Sections 3 and 4.

Remark 5.1. It is hard to show that Theorems 4.1, 4.3 and 4.4 can be extended to a finite family of generalized asymptotically quasi-nonexpansive mappings.

Remark 5.2. It is clear that Theorems 3.2, 4.1, 4.3 and 4.4 can be used for any quasi-nonexpansive mapping.

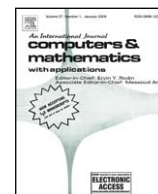
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Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions

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ABSTRACT

The purpose of this paper is to introduce a new iterative scheme for finding a common fixed point of a finite family of strict pseudo-contractions. The results obtained in this paper extend and improve the corresponding results of Marino, Colao, Qin and Kang [G. Marino, V. Colao, X. Qin, S. M. Kang, Strong convergence of the modified Mann iterative method for strict pseudo-contractions, Comput. Math. Appl. 57 (2009) 455–465].

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2.$$

Recall that the mapping T is said to be κ -strict pseudo-contractive if there exist $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in D(T). \quad (1.1)$$

Note that the class of κ -strict pseudo-contractions strictly includes the class of nonexpansive mappings. If $\kappa = 1$, T is said to be a *pseudo-contractive mapping*. T is *strong pseudo-contractive* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. In a real Hilbert space H (1.1) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in D(T). \quad (1.2)$$

T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \forall x, y \in D(T).$$

T is strong pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$

$$\langle Tx - Ty, x - y \rangle \leq (1 - \lambda) \|x - y\|^2 \quad \forall x, y \in D(T).$$

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The class of κ -strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings and class of strong pseudo-contractive mappings is independent of the class of κ -strict pseudo-contractions.

In 1953, Mann [2] introduced an iteration procedure for approximating fixed points of a nonexpansive self-mapping T of a nonempty closed convex subset K of a Hilbert space as follows: $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1, \quad (1.3)$$

where $\{\alpha_n \in [0, 1]\}$. Later in 1967, Halpern [3] introduced the iterative scheme for a nonexpansive self-mapping T of a nonempty closed convex subset K of a Hilbert space,

$$x_{n+1} = (1 - \alpha_n)u + \alpha_n Tx_n, \quad n \geq 1, \quad (1.4)$$

where $u, x_1 \in C$ are fixed and $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$. He also pointed out that the conditions $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of $\{x_n\}$ to a fixed point of T .

Many authors try to modify the iteration (1.3) process to have a strong convergence: see, e.g., [4–10].

In 2006, Marino and Xu [11] introduced a new iterative scheme by the viscosity approximation method which was first introduced by Moudafi [12],

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (1.5)$$

where S is a nonexpansive mapping of H into itself and the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality,

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(S). \quad (1.6)$$

In 2008, Yao, Chen and Yao [14] introduced iterative scheme as follows;

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Sx_n, \\ x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n)y_n, \end{cases} \quad n \geq 0, \quad (1.7)$$

where S is a nonexpansive mapping on a closed convex subset of Banach space and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. They proved under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of S .

Recently, Marino, Colao, Qin and Kang [15] introduced an iterative algorithm as follows;

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \end{cases} \quad n \geq 0, \quad (1.8)$$

where T is κ -strict pseudo-contractive on H . Under suitable conditions of $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(T)$ which solves the following variational inequality $\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T)$.

In 1999, Atsushiba and Takahashi [16] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I, \\ U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_N - 1U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I, \end{aligned} \quad (1.9)$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the W -mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [17] proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \dots, N$.

In this paper, motivated by Takahashi and Shimoji [17], we introduce a new mapping for finding a common fixed point of a finite family of strict pseudo-contractive mappings. Let X be a real Hilbert space and C a nonempty closed convex subset of X and let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contractive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \dots, N$,

let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define the mapping $S_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I \\ S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \end{aligned} \quad (1.10)$$

We shall show that $F(S_n) = \bigcap_{i=1}^N F(T_i)$ under some conditions and motivated by Marino, Colao, Qin and Kang [15], we introduce an iterative scheme as follows:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \quad n \geq 1, \end{cases} \quad (1.11)$$

where $u \in K$ is a given point, f is an α -contraction and A is a linear bounded strongly positive operator. We prove under some control conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{a_n\}$ that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \bigcap_{i=1}^N F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i). \quad (1.12)$$

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main results in the next section.

Lemma 2.1 (See [4]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \rightarrow C$ be a self-mapping of C . If S is a κ -strict pseudo-contraction mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1 + \kappa}{1 - \kappa} \|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.2 (See [13]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n) s_n + \delta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.3. *In a real Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad x, y \in H.$$

Lemma 2.4 (See [11]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.*

Lemma 2.5. *Let H be a real Hilbert space. There hold the following identities*

- (i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$ and $\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$
 - (ii) $\|\sum_{i=0}^m \alpha_i x_i\|^2 = \sum_{i=0}^m \alpha_i \|x_i\|^2 - \sum_{i=0}^m \alpha_i \alpha_j \|x_i - x_j\|^2$
- for $\sum_{i=0}^m \alpha_i = 1$, $\alpha_i \in [0, 1]$, $\forall i \in \{0, 1, 2, \dots, m\}$.

Lemma 2.6 (See [11]). *Let H be a real Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Let f be an α -contraction. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : H \rightarrow H$ be a nonexpansive mapping. For $0 < t \leq \|A\|^{-1}$, let x_t be the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point $\bar{x}$ of T which solves the variational inequality*

$$\langle \gamma f(\bar{x}) - A\bar{x}, z - \bar{x} \rangle \leq 0, \quad \forall z \in F(T). \quad (2.1)$$

Lemma 2.7 (See [18]). Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that

$$a_{n+1} = (1 - \alpha_n)a_n + b_n + c_n, \quad \text{for all } n \geq 0. \quad (2.2)$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

- (a) if $b_n \leq \alpha_n C$ where $C \geq 0$, then $\{a_n\}$ is a bounded sequence.
- (b) if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 2.1. Let C be a nonempty convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contractions of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \rightarrow C$ as follows:

$$\begin{aligned} U_0 &= I \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\ &\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned} \quad (2.3)$$

This mapping is called S -mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.8. Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ -strict pseudo-contractive mapping of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, 3, \dots, N$, where $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (\kappa, 1), \alpha_3^N \in [\kappa, 1), \alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

Proof. It is clear that $\bigcap_{i=1}^N F(T_i) \subseteq F(S)$. Next we show that $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. To show this, let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. Then we have

$$\begin{aligned} \|Sx_0 - x^*\|^2 &= \|\alpha_1^N T_N U_{N-1} x_0 + \alpha_2^N U_{N-1} x_0 + \alpha_3^N x_0 - x^*\|^2 \\ &= \|\alpha_1^N (T_N U_{N-1} x_0 - x^*) + \alpha_2^N (U_{N-1} x_0 - x^*) + \alpha_3^N (x_0 - x^*)\|^2 \\ &= \alpha_1^N \|T_N U_{N-1} x_0 - x^*\|^2 + \alpha_2^N \|U_{N-1} x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2 - \alpha_1^N \alpha_2^N \|T_N U_{N-1} x_0 - U_{N-1} x_0\|^2 \\ &\quad - \alpha_1^N \alpha_3^N \|T_N U_{N-1} x_0 - x_0\|^2 - \alpha_2^N \alpha_3^N \|U_{N-1} x_0 - x_0\|^2 \\ &\leq \alpha_1^N (\|U_{N-1} x_0 - x^*\|^2 + \kappa \|(I - T_N) U_{N-1} x_0 - (I - T_N) x^*\|^2) \\ &\quad + \alpha_2^N \|U_{N-1} x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2 - \alpha_1^N \alpha_2^N \|T_N U_{N-1} x_0 - U_{N-1} x_0\|^2 \\ &= (\alpha_1^N + \alpha_2^N) \|U_{N-1} x_0 - x^*\|^2 + \alpha_1^N (\kappa - \alpha_2^N) \|(I - T_N) U_{N-1} x_0\|^2 + \alpha_3^N \|x_0 - x^*\|^2 \\ &\leq (1 - \alpha_3^N) \|U_{N-1} x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) \|\alpha_1^{N-1} T_{N-1} U_{N-2} x_0 + \alpha_2^{N-1} U_{N-2} x_0 + \alpha_3^{N-1} x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) \|\alpha_1^{N-1} (T_{N-1} U_{N-2} x_0 - x^*) + \alpha_2^{N-1} (U_{N-2} x_0 - x^*) \\ &\quad + \alpha_3^{N-1} (x_0 - x^*)\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) (\alpha_1^{N-1} \|T_{N-1} U_{N-2} x_0 - x^*\|^2 + \alpha_2^{N-1} \|U_{N-2} x_0 - x^*\|^2 \\ &\quad + \alpha_3^{N-1} \|x_0 - x^*\|^2 - \alpha_1^{N-1} \alpha_2^{N-1} \|T_{N-1} U_{N-2} x_0 - U_{N-2} x_0\|^2 \\ &\quad - \alpha_1^{N-1} \alpha_3^{N-1} \|T_{N-1} U_{N-2} x_0 - x_0\|^2 - \alpha_2^{N-1} \alpha_3^{N-1} \|U_{N-2} x_0 - x_0\|^2) + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &\leq (1 - \alpha_3^N) (\alpha_1^{N-1} (\|U_{N-2} x_0 - x^*\|^2 + \kappa \|(I - T_{N-1}) U_{N-2} x_0 - (I - T_{N-1}) x^*\|^2) \\ &\quad + \alpha_2^{N-1} \|U_{N-2} x_0 - x^*\|^2 + \alpha_3^{N-1} \|x_0 - x^*\|^2 - \alpha_1^{N-1} \alpha_2^{N-1} \|T_{N-1} U_{N-2} x_0 - U_{N-2} x_0\|^2) \\ &\quad + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\ &= (1 - \alpha_3^N) ((\alpha_1^{N-1} + \alpha_2^{N-1}) \|U_{N-2} x_0 - x^*\|^2 + \alpha_1^{N-1} (\kappa - \alpha_2^{N-1}) \|(I - T_{N-1}) U_{N-2} x_0\|^2 \end{aligned} \quad (2.4)$$

$$\begin{aligned}
& + \alpha_3^{N-1} \|x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\
& \leq (1 - \alpha_3^N)(1 - \alpha_3^{N-1}) \|U_{N-2}x_0 - x^*\|^2 + \alpha_3^{N-1}(1 - \alpha_3^N) \|x_0 - x^*\|^2 + (1 - (1 - \alpha_3^N)) \|x_0 - x^*\|^2 \\
& = \prod_{j=N-1}^N (1 - \alpha_3^j) \|U_{N-2}x_0 - x^*\|^2 + \left(1 - \prod_{j=N-1}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& \vdots \\
& \leq \prod_{j=3}^N (1 - \alpha_3^j) \|U_2x_0 - x^*\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \tag{2.5}
\end{aligned}$$

$$\leq \prod_{j=2}^N (1 - \alpha_3^j) \|U_1x_0 - x^*\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \tag{2.6}$$

$$\begin{aligned}
& = \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 T_1 U_0 x_0 + \alpha_2^1 U_0 x_0 + \alpha_3^1 x_0 - x^*\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 (T_1 x_0 - x^*) + (1 - \alpha_1^1)(x_0 - x^*)\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|T_1 x_0 - x^*\|^2 + (1 - \alpha_1^1) \|x_0 - x^*\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|T_1 x_0 - x_0\|^2) \\
& \quad + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& \leq \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 (\|x_0 - x^*\|^2 + \kappa \|(I - T_1)x_0 - (I - T_1)x^*\|^2) \\
& \quad + (1 - \alpha_1^1) \|x_0 - x^*\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|T_1 x_0 - x_0\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|x_0 - x^*\|^2 + \kappa \alpha_1^1 \|(I - T_1)x_0\|^2 + (1 - \alpha_1^1) \|x_0 - x^*\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|T_1 x_0 - x_0\|^2) \\
& \quad + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\
& = \prod_{j=2}^N (1 - \alpha_3^j) (\|x_0 - x^*\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x_0\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2. \tag{2.7}
\end{aligned}$$

For each $k = 1, 2, \dots, N - 1$, we obtain

$$\alpha_1^k ((1 - \alpha_1^k) - \kappa) \geq \alpha_1^k ((1 - \alpha_1^k) - \alpha_2^k) = \alpha_1^k (1 - (\alpha_1^k + \alpha_2^k)) = \alpha_1^k \alpha_3^k > 0. \tag{2.8}$$

This implies by (2.7) that

$$\prod_{j=2}^N (1 - \alpha_3^j) \|x_0 - x^*\|^2 \leq \prod_{j=2}^N (1 - \alpha_3^j) (\|x_0 - x^*\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x_0\|^2).$$

Hence, we obtain

$$\|x_0 - x^*\|^2 \leq \|x_0 - x^*\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x_0\|^2,$$

which implies

$$\alpha_1^1 ((1 - \alpha_1^1) - \kappa) \|x_0 - T_1 x_0\|^2 \leq 0.$$

It follows from (2.8) that

$$\|x_0 - T_1 x_0\|^2 = 0.$$

Hence $T_1x_0 = x_0$, that is $x_0 \in F(T_1)$. It implies that

$$U_1x_0 = \alpha_1^1 T_1 U_0 x_0 + \alpha_2^1 U_0 x_0 + \alpha_3^1 x_0 = x_0.$$

By (2.5), we have

$$\begin{aligned} \|x_0 - x^*\|^2 &= \|Sx_0 - x^*\|^2 \leq \prod_{j=3}^N (1 - \alpha_3^j) \|U_2x_0 - x^*\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) \|\alpha_1^2 (T_2 U_1 x_0 - x^*) + \alpha_2^2 (U_1 x_0 - x^*) + \alpha_3^2 (x_0 - x^*)\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) \|\alpha_1^2 (T_2 x_0 - x^*) + (1 - \alpha_1^2) (x_0 - x^*)\|^2 + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) (\alpha_1^2 \|T_2 x_0 - x^*\|^2 + (1 - \alpha_1^2) \|x_0 - x^*\|^2 - \alpha_1^2 (1 - \alpha_1^2) \|T_2 x_0 - x_0\|^2) \\ &\quad + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &\leq \prod_{j=3}^N (1 - \alpha_3^j) (\alpha_1^2 (\|x_0 - x^*\|^2 + \kappa \|(I - T_2)x_0 - (I - T_2)x^*\|^2) \\ &\quad + (1 - \alpha_1^2) \|x_0 - x^*\|^2 - \alpha_1^2 (1 - \alpha_1^2) \|T_2 x_0 - x_0\|^2) + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) (\alpha_1^2 \|x_0 - x^*\|^2 + \kappa \alpha_1^2 \|(I - T_2)x_0\|^2 \\ &\quad + (1 - \alpha_1^2) \|x_0 - x^*\|^2 - \alpha_1^2 (1 - \alpha_1^2) \|T_2 x_0 - x_0\|^2) + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2 \\ &= \prod_{j=3}^N (1 - \alpha_3^j) (\|x_0 - x^*\|^2 + \alpha_1^2 (\kappa - (1 - \alpha_1^2)) \|(I - T_2)x_0\|^2) + \left(1 - \prod_{j=3}^N (1 - \alpha_3^j)\right) \|x_0 - x^*\|^2, \end{aligned}$$

it follows that

$$\|x_0 - x^*\|^2 \leq \|x_0 - x^*\|^2 + \alpha_1^2 (\kappa - (1 - \alpha_1^2)) \|(I - T_2)x_0\|^2$$

which implies

$$\alpha_1^2 ((1 - \alpha_1^2) - \kappa) \|x_0 - T_2 x_0\|^2 \leq 0.$$

It follows by (2.8) that

$$\|x_0 - T_2 x_0\|^2 = 0.$$

Hence, we have $T_2 x_0 = x_0$, that is $x_0 \in F(T_2)$.

This implies that $U_2 x_0 = \alpha_1^2 T_2 U_1 x_0 + \alpha_2^2 U_1 x_0 + \alpha_3^2 x_0 = x_0$.

By continuing in this way, we can show that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1, \dots, N - 1$.

Finally, we shall show that $x_0 \in F(T_N)$.

Since

$$\begin{aligned} 0 &= Sx_0 - x_0 = \alpha_1^N T_N U_{N-1} x_0 + \alpha_2^N U_{N-1} x_0 + \alpha_3^N x_0 - x_0 \\ &= \alpha_1^N (T_N x_0 - x_0), \end{aligned}$$

and $\alpha_1^N \in (\kappa, 1]$, we obtain that $T_N x_0 = x_0$, that is $x_0 \in F(T_N)$. Hence $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$.

Next we show that S is nonexpansive. To show this let $x, y \in C$. Then

$$\begin{aligned} \|Sx - Sy\|^2 &= \|\alpha_1^N T_N U_{N-1} x + \alpha_2^N U_{N-1} x + \alpha_3^N x - \alpha_1^N T_N U_{N-1} y - \alpha_2^N U_{N-1} y - \alpha_3^N y\|^2 \\ &= \|\alpha_1^N (T_N U_{N-1} x - T_N U_{N-1} y) + \alpha_2^N (U_{N-1} x - U_{N-1} y) + \alpha_3^N (x - y)\|^2 \end{aligned}$$

$$\begin{aligned}
&= \alpha_1^N \|T_N U_{N-1}x - T_N U_{N-1}y\|^2 + \alpha_2^N \|U_{N-1}x - U_{N-1}y\|^2 + \alpha_3^N \|x - y\|^2 \\
&\quad - \alpha_1^N \alpha_2^N \|U_{N-1}x - U_{N-1}y - T_N U_{N-1}x + T_N U_{N-1}y\|^2 \\
&\quad - \alpha_1^N \alpha_3^N \|T_N U_{N-1}x - T_N U_{N-1}y - (x - y)\|^2 - \alpha_2^N \alpha_3^N \|U_{N-1}x - U_{N-1}y - (x - y)\|^2 \\
&\leq \alpha_1^N (\|U_{N-1}x - U_{N-1}y\|^2 + \kappa \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2) + \alpha_2^N \|U_{N-1}x - U_{N-1}y\|^2 \\
&\quad + \alpha_3^N \|x - y\|^2 - \alpha_1^N \alpha_2^N \|U_{N-1}x - U_{N-1}y - T_N U_{N-1}x + T_N U_{N-1}y\|^2 \\
&= (\alpha_1^N + \alpha_2^N) \|U_{N-1}x - U_{N-1}y\|^2 + \alpha_1^N (\kappa - \alpha_2^N) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 + \alpha_3^N \|x - y\|^2 \\
&\leq (1 - \alpha_3^N) \|U_{N-1}x - U_{N-1}y\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) \|\alpha_1^{N-1} T_{N-1} U_{N-2}x + \alpha_2^{N-1} U_{N-2}x + \alpha_3^{N-1} x - \alpha_1^{N-1} T_{N-1} U_{N-2}y \\
&\quad - \alpha_2^{N-1} U_{N-2}y - \alpha_3^{N-1} y\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) \|\alpha_1^{N-1} (T_{N-1} U_{N-2}x - T_{N-1} U_{N-2}y) + \alpha_2^{N-1} (U_{N-2}x - U_{N-2}y) \\
&\quad + \alpha_3^{N-1} (x - y)\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) (\alpha_1^{N-1} \|T_{N-1} U_{N-2}x - T_{N-1} U_{N-2}y\|^2 + \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y\|^2 \\
&\quad + \alpha_3^{N-1} \|x - y\|^2 - \alpha_1^{N-1} \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y - T_{N-1} U_{N-2}x \\
&\quad + T_{N-1} U_{N-2}y\|^2 - \alpha_1^{N-1} \alpha_3^{N-1} \|T_{N-1} U_{N-2}x - T_{N-1} U_{N-2}y - (x - y)\|^2 \\
&\quad - \alpha_2^{N-1} \alpha_3^{N-1} \|U_{N-2}x - U_{N-2}y - (x - y)\|^2) + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&\leq (1 - \alpha_3^N) (\alpha_1^{N-1} (\|U_{N-2}x - U_{N-2}y\|^2 + \kappa \|(I - T_{N-1})U_{N-2}x \\
&\quad - (I - T_{N-1})U_{N-2}y\|^2) + \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y\|^2 + \alpha_3^{N-1} \|x - y\|^2 \\
&\quad - \alpha_1^{N-1} \alpha_2^{N-1} \|U_{N-2}x - U_{N-2}y - T_{N-1} U_{N-2}x + T_{N-1} U_{N-2}y\|^2) + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= (1 - \alpha_3^N) ((\alpha_1^{N-1} + \alpha_2^{N-1}) \|U_{N-2}x - U_{N-2}y\|^2 + \alpha_1^{N-1} (\kappa - \alpha_2^{N-1}) \|(I - T_{N-1})U_{N-2}x \\
&\quad - (I - T_{N-1})U_{N-2}y\|^2 + \alpha_3^{N-1} \|x - y\|^2) + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&\leq (1 - \alpha_3^N) (1 - \alpha_3^{N-1}) \|U_{N-2}x - U_{N-2}y\|^2 + \alpha_3^{N-1} (1 - \alpha_3^N) \|x - y\|^2 + (1 - (1 - \alpha_3^N)) \|x - y\|^2 \\
&= \prod_{j=N-1}^N (1 - \alpha_3^j) \|U_{N-2}x - U_{N-2}y\|^2 + \left(1 - \prod_{j=N-1}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&\leq \\
&\vdots \\
&\leq \prod_{j=2}^N (1 - \alpha_3^j) \|U_1x - U_1y\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 (T_1 U_0x - T_1 U_0y) + \alpha_2^1 (U_0x - U_0y) + \alpha_3^1 (x - y)\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) \|\alpha_1^1 (T_1x - T_1y) + (1 - \alpha_1^1) (x - y)\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|T_1x - T_1y\|^2 + (1 - \alpha_1^1) \|x - y\|^2 \\
&\quad - \alpha_1^1 (1 - \alpha_1^1) \|x - y - Tx + Ty\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&\leq \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 (\|x - y\|^2 + \kappa \|(I - T_1)x - (I - T_1)y\|^2) \\
&\quad + (1 - \alpha_1^1) \|x - y\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|x - y - Tx + Ty\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2
\end{aligned}$$

$$\begin{aligned}
&= \prod_{j=2}^N (1 - \alpha_3^j) (\alpha_1^1 \|x - y\|^2 + \kappa \alpha_1^1 \|(I - T_1)x - (I - T_1)y\|^2 \\
&\quad + (1 - \alpha_1^1) \|x - y\|^2 - \alpha_1^1 (1 - \alpha_1^1) \|(I - T_1)x - (I - T_1)y\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \prod_{j=2}^N (1 - \alpha_3^j) (\|x - y\|^2 + \alpha_1^1 (\kappa - (1 - \alpha_1^1)) \|(I - T_1)x - (I - T_1)y\|^2) + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&\leq \prod_{j=2}^N (1 - \alpha_3^j) \|x - y\|^2 + \left(1 - \prod_{j=2}^N (1 - \alpha_3^j)\right) \|x - y\|^2 \\
&= \|x - y\|^2. \quad \square
\end{aligned}$$

3. Main results

In this section, we prove a strong convergence theorem of the iterative scheme (1.11) to a common fixed point of $\bigcap_{i=1}^N F(T_i)$.

Theorem 3.1. Let H be a Hilbert space, let f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contraction of H into itself for some $\kappa_i \in [0, 1)$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1, 2, \dots, N$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \end{cases} \quad n \geq 1, \quad (3.1)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 0$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$, for all $j \in \{1, 2, 3, \dots, N\}$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in \bigcap_{i=1}^N F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

Proof. Let $p \in \bigcap_{i=1}^N F(T_i)$. Since $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, we may assume that $\alpha_n < \|A\|^{-1}$ for all $n \in \mathbb{N}$. By Lemma 2.4 we can conclude that $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$. By Lemma 2.8, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|\alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n - p\| \\
&= \|\alpha_n (\gamma (a_n u + (1 - a_n) f(x_n)) - Ap) + (I - \alpha_n A) (y_n - p)\| \\
&\leq \alpha_n \|\gamma (a_n u + (1 - a_n) f(x_n)) - Ap\| + \|I - \alpha_n A\| \|y_n - p\| \\
&\leq \alpha_n \gamma \|(a_n u + (1 - a_n) f(x_n)) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \bar{\gamma}) \|\beta_n x_n \\
&\quad + (1 - \beta_n) S_n x_n - p\| \\
&\leq \alpha_n \gamma (a_n \|u - f(p)\| + (1 - a_n) \|f(x_n) - f(p)\|) + \alpha_n \|\gamma f(p) - Ap\| \\
&\quad + (1 - \alpha_n \bar{\gamma}) (\beta_n \|x_n - p\| + (1 - \beta_n) \|S_n x_n - p\|) \\
&\leq \alpha_n \gamma (a_n \|u - f(p)\| + (1 - a_n) \alpha \|x_n - p\|) + \alpha_n \|\gamma f(p) - Ap\| \\
&\quad + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\
&= \alpha_n \gamma a_n \|u - f(p)\| + (1 - \alpha_n \bar{\gamma} + (1 - a_n) \alpha_n \gamma \alpha) \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| \\
&\leq (1 - \alpha_n (\bar{\gamma} - \gamma \alpha)) \|x_n - p\| + \alpha_n \gamma a_n \|u - f(p)\| + \alpha_n \|\gamma f(p) - Ap\|.
\end{aligned}$$

It follows by Lemma 2.7 that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$. From condition (i) and (1.11), we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = \lim_{n \rightarrow \infty} \alpha_n \|\gamma(a_n u + (1 - a_n)f(x_n)) - Ay_n\| = 0. \quad (3.2)$$

Next, we show that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.3)$$

By definition of $\{x_n\}$ and Lemma 2.4, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n \gamma(a_n u + (1 - a_n)f(x_n)) + (I - \alpha_n A)y_n - \alpha_{n-1} \gamma(a_{n-1} u \\ &\quad + (1 - a_{n-1})f(x_{n-1})) - (I - \alpha_{n-1} A)y_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})\gamma a_n u + \alpha_{n-1} \gamma u(a_n - a_{n-1}) + \alpha_n \gamma(1 - a_n)(f(x_n) - f(x_{n-1})) \\ &\quad + \alpha_n \gamma(1 - a_n)f(x_{n-1}) + (I - \alpha_n A)y_n - \alpha_{n-1} \gamma(1 - a_{n-1})f(x_{n-1}) - (I - \alpha_{n-1} A)y_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})\gamma a_n u + \alpha_{n-1} \gamma u(a_n - a_{n-1}) + \alpha_n \gamma(1 - a_n)(f(x_n) - f(x_{n-1})) \\ &\quad + \alpha_n \gamma f(x_{n-1}) - \alpha_{n-1} \gamma f(x_{n-1})a_n + (I - \alpha_n A)(y_n - y_{n-1}) + y_{n-1} - \alpha_n A y_{n-1} \\ &\quad - \alpha_{n-1} \gamma f(x_{n-1}) + \alpha_{n-1} \gamma f(x_{n-1})a_{n-1} - y_{n-1} + \alpha_{n-1} A y_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})\gamma a_n u + \alpha_{n-1} \gamma u(a_n - a_{n-1}) + \alpha_n \gamma(1 - a_n)(f(x_n) - f(x_{n-1})) \\ &\quad + (\alpha_n - \alpha_{n-1})(\gamma f(x_{n-1}) - A y_{n-1}) + (I - \alpha_n A)(y_n - y_{n-1}) \\ &\quad + (\alpha_{n-1}(a_{n-1} - a_n) + (\alpha_{n-1} - \alpha_n)a_n)\gamma f(x_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|\gamma a_n u\| + \alpha_{n-1} \|\gamma u\| |a_n - a_{n-1}| + \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(x_{n-1}) - A y_{n-1}\| + \|I - \alpha_n A\| \|y_n - y_{n-1}\| \\ &\quad + \alpha_{n-1} |a_{n-1} - a_n| \|\gamma f(x_{n-1})\| + |\alpha_{n-1} - \alpha_n| a_n \|\gamma f(x_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|\gamma u\| + \gamma \|u\| |a_n - a_{n-1}| + \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|\gamma f(x_{n-1}) - A y_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| \\ &\quad + |a_{n-1} - a_n| \|\gamma f(x_{n-1})\| + |\alpha_{n-1} - \alpha_n| \|\gamma f(x_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| K + K |a_n - a_{n-1}| + \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| K + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |a_{n-1} - a_n| K + |\alpha_{n-1} - \alpha_n| K \\ &= \alpha_n \gamma(1 - a_n) \alpha \|x_n - x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K, \end{aligned} \quad (3.4)$$

where $K = \max\{\gamma \|u\|, \gamma \sup_{n \in \mathbb{N}} \|f(x_n)\|, \sup_{n \in \mathbb{N}} \|\gamma f(x_n) - A y_n\|\}$. By definition of $\{y_n\}$ and nonexpansiveness of S_n , we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\beta_n x_n + (1 - \beta_n) S_n x_n - \beta_{n-1} x_{n-1} - (1 - \beta_{n-1}) S_{n-1} x_{n-1}\| \\ &= \|\beta_n (x_n - x_{n-1}) + (1 - \beta_n) S_n x_n + (\beta_n - \beta_{n-1}) x_{n-1} - (1 - \beta_{n-1}) S_{n-1} x_{n-1}\| \\ &= \|\beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} + (1 - \beta_n) (S_n x_n - S_{n-1} x_{n-1}) \\ &\quad + (S_n x_{n-1} - S_{n-1} x_{n-1}) + (\beta_{n-1} - \beta_n) S_{n-1} x_{n-1}\| \\ &\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) (\|x_n - x_{n-1}\| \\ &\quad + \|S_n x_{n-1} - S_{n-1} x_{n-1}\|) + |\beta_{n-1} - \beta_n| \|S_{n-1} x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + 2M |\beta_n - \beta_{n-1}| + (1 - \beta_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\|, \end{aligned} \quad (3.5)$$

where $M = \max\{\sup_{n \in \mathbb{N}} \|x_n\|, \sup_{n \in \mathbb{N}} \|S_n x_n\|\}$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|U_{n+1,1} x_n - U_{n,1} x_n\| &= \|\alpha_1^{n+1,1} T_1 x_n + (1 - \alpha_1^{n+1,1}) x_n - \alpha_1^{n,1} T_1 x_n - (1 - \alpha_1^{n,1}) x_n\| \\ &= \|\alpha_1^{n+1,1} T_1 x_n - \alpha_1^{n+1,1} x_n - \alpha_1^{n,1} T_1 x_n + \alpha_1^{n,1} x_n\| \\ &= \|(\alpha_1^{n+1,1} - \alpha_1^{n,1}) T_1 x_n - (\alpha_1^{n+1,1} - \alpha_1^{n,1}) x_n\| \\ &= |\alpha_1^{n+1,1} - \alpha_1^{n,1}| \|T_1 x_n - x_n\| \end{aligned} \quad (3.6)$$

and for $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned} \|U_{n+1,k} x_n - U_{n,k} x_n\| &= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n + \alpha_3^{n+1,k} x_n \\ &\quad - \alpha_1^{n,k} T_k U_{n,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n - \alpha_3^{n,k} x_n\| \\ &= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} x_n + \alpha_3^{n+1,k} x_n - \alpha_1^{n,k} T_k U_{n,k-1} x_n - \alpha_3^{n,k} x_n \\ &\quad + \alpha_2^{n+1,k} U_{n+1,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \end{aligned}$$

$$\begin{aligned}
&= \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} x_n - \alpha_1^{n+1,k} T_k U_{n,k-1} x_n + \alpha_1^{n+1,k} T_k U_{n,k-1} x_n \\
&\quad - \alpha_1^{n,k} T_k U_{n,k-1} x_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \\
&= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) T_k U_{n,k-1} x_n \\
&\quad + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \\
&= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_k U_{n,k-1} x_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} U_{n+1,k-1} x_n \\
&\quad - \alpha_2^{n+1,k} U_{n,k-1} x_n + \alpha_2^{n+1,k} U_{n,k-1} x_n - \alpha_2^{n,k} U_{n,k-1} x_n\| \\
&= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
&\quad \times T_k U_{n,k-1} x_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) x_n + \alpha_2^{n+1,k} (U_{n+1,k-1} x_n \\
&\quad - U_{n,k-1} x_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k}) U_{n,k-1} x_n\| \\
&\leq \alpha_1^{n+1,k} \|T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&\quad + \alpha_2^{n+1,k} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}| \|U_{n,k-1} x_n\| \\
&= \alpha_1^{n+1,k} \|T_k U_{n+1,k-1} x_n - T_k U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + \alpha_2^{n+1,k} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |1 - \alpha_1^{n+1,k} \\
&\quad - \alpha_3^{n+1,k} - 1 + \alpha_1^{n,k} + \alpha_3^{n,k}| \|U_{n,k-1} x_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&\leq \alpha_1^{n+1,k} \frac{1+\kappa}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + \alpha_2^{n+1,k} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + (|\alpha_1^{n,k} \\
&\quad - \alpha_1^{n+1,k}| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|) \|U_{n,k-1} x_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&\leq \frac{1+\kappa}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} x_n\| \\
&\quad + \frac{1-\kappa}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + (|\alpha_1^{n,k} - \alpha_1^{n+1,k}| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|) \|U_{n,k-1} x_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|x_n\| \\
&= \frac{2}{1-\kappa} \|U_{n+1,k-1} x_n - U_{n,k-1} x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| (\|T_k U_{n,k-1} x_n\| \\
&\quad + \|U_{n,k-1} x_n\|) + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| (\|U_{n,k-1} x_n\| + \|x_n\|). \tag{3.7}
\end{aligned}$$

By (3.6) and (3.7), we have

$$\begin{aligned}
\|S_{n+1} x_n - S_n x_n\| &= \|U_{n+1,N} x_n - U_{n,N} x_n\| \\
&\leq \frac{2}{1-\kappa} \|U_{n+1,N-1} x_n - U_{n,N-1} x_n\| + |\alpha_1^{n+1,N} - \alpha_1^{n,N}| (\|T_N U_{n,N-1} x_n\| \\
&\quad + \|U_{n,N-1} x_n\|) + |\alpha_3^{n+1,N} - \alpha_3^{n,N}| (\|U_{n,N-1} x_n\| + \|x_n\|) \\
&\leq \frac{2}{1-\kappa} \left(\frac{2}{1-\kappa} \|U_{n+1,N-2} x_n - U_{n,N-2} x_n\| \right. \\
&\quad + |\alpha_1^{n+1,N-1} - \alpha_1^{n,N-1}| (\|T_{N-1} U_{n,N-2} x_n\| + \|U_{n,N-2} x_n\|) \\
&\quad + |\alpha_3^{n+1,N-1} - \alpha_3^{n,N-1}| (\|U_{n,N-2} x_n\| + \|x_n\|) \Big) \\
&\quad + |\alpha_1^{n+1,N} - \alpha_1^{n,N}| (\|T_N U_{n,N-1} x_n\| + \|U_{n,N-1} x_n\|) + |\alpha_3^{n+1,N} \\
&\quad - \alpha_3^{n,N}| (\|U_{n,N-1} x_n\| + \|x_n\|) \\
&= \left(\frac{2}{1-\kappa} \right)^2 \|U_{n+1,N-2} x_n - U_{n,N-2} x_n\| + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa} \right)^{N-j} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| (\|T_j U_{n,j-1} x_n\|
\end{aligned}$$

$$\begin{aligned}
& + \|U_{n,j-1}x_n\|) + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa}\right)^{N-j} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| (\|U_{n,j-1}x_n\| + \|x_n\|) \\
& \leq \\
& \vdots \\
& \leq \left(\frac{2}{1-\kappa}\right)^{N-1} \|U_{n+1,1}x_n - U_{n,1}x_n\| + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| (\|T_j U_{n,j-1}x_n\| \\
& + \|U_{n,j-1}x_n\|) + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| (\|U_{n,j-1}x_n\| + \|x_n\|) \\
& = \left(\frac{2}{1-\kappa}\right)^{N-1} |\alpha_1^{n+1,1} - \alpha_1^{n,1}| \|T_1 x_n - x_n\| + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} \\
& |\alpha_1^{n+1,j} - \alpha_1^{n,j}| (\|T_j U_{n,j-1}x_n\| + \|U_{n,j-1}x_n\|) + \sum_{j=2}^N \left(\frac{2}{1-\kappa}\right)^{N-j} \\
& |\alpha_3^{n+1,j} - \alpha_3^{n,j}| (\|U_{n,j-1}x_n\| + \|x_n\|).
\end{aligned}$$

This implies by assumption (ii) that

$$\sum_{n=1}^{\infty} \|S_{n+1}x_n - S_nx_n\| < \infty.$$

From (3.4) and (3.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| & \leq \alpha_n \gamma (1 - a_n) \alpha \|x_n - x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K \\
& \leq \alpha_n \gamma \alpha \|x_n - x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) (\|x_n - x_{n-1}\| + 2M|\beta_n - \beta_{n-1}| \\
& + (1 - \beta_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\|) + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K \\
& \leq (1 - \alpha_n (\bar{\gamma} - \gamma \alpha)) \|x_n - x_{n-1}\| + 2M|\beta_n - \beta_{n-1}| \\
& + (1 - \beta_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\| + |a_{n-1} - a_n| 2K + |\alpha_{n-1} - \alpha_n| 3K.
\end{aligned}$$

It follows from Lemma 2.2 that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. By definition of $\{y_n\}$, we obtain

$$\|x_n - y_n\| = (1 - \beta_n) \|S_n x_n - x_n\|.$$

This implies by (3.2) and (3.3)

$$(1 - \beta_n) \|S_n x_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By condition (iii), we obtain

$$\|S_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

Let $q = \lim_{t \rightarrow 0} x_t$, where x_t is being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)S_n x$ for $t \in (0, \|A\|^{-1})$. By Lemma 2.8, S_n is nonexpansive and $F(S_n) = \bigcap_{i=1}^N F(T_i)$. From Lemma 2.6 we can conclude that $q \in \bigcap_{i=1}^N F(T_i)$ and

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

By definition of $\{x_t\}$, we can write

$$x_t = t\gamma f(x_t) + (I - tA)S_n x_t.$$

Then, we have

$$\|x_t - x_n\| = \|(I - tA)(S_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|.$$

By Lemma 2.3, we have

$$\begin{aligned}
\|x_t - x_n\|^2 & = \|(I - tA)(S_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\
& \leq (1 - \bar{\gamma}t)^2 \|S_n x_t - x_n\|^2 + 2t \langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\
& \leq (1 - \bar{\gamma}t)^2 (\|S_n x_t - S_n x_n\| + \|S_n x_n - x_n\|)^2 + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\
& \leq (1 - 2\bar{\gamma}t + \bar{\gamma}^2 t^2) \|x_t - x_n\|^2 + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\| \\
& + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle.
\end{aligned} \quad (3.9)$$

Since A is a linear strongly positive operator with coefficient $\bar{\gamma}$, it follows by (3.9) that

$$\begin{aligned} 2t \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\leq (\bar{\gamma}^2 t^2 - 2\bar{\gamma}t) \|x_t - x_n\|^2 + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \\ &\quad \|S_n x_n - x_n\| + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &= \bar{\gamma}(\bar{\gamma}t^2 - 2t) \|x_t - x_n\|^2 + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \\ &\quad \|S_n x_n - x_n\| + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &\leq (\bar{\gamma}t^2 - 2t) \langle Ax_t - Ax_n, x_t - x_n \rangle + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| \\ &\quad + \|S_n x_n - x_n\|) \|S_n x_n - x_n\| + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle \\ &= \bar{\gamma}t^2 \langle Ax_t - Ax_n, x_t - x_n \rangle + (1 - \bar{\gamma}t)^2 (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\|, \end{aligned}$$

which implies

$$\begin{aligned} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\leq \frac{\bar{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{(1 - \bar{\gamma}t)^2}{2t} (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\| \\ &\leq \frac{\bar{\gamma}t}{2} M' + \frac{(1 - \bar{\gamma}t)^2}{2t} (2\|x_t - x_n\| + \|S_n x_n - x_n\|) \|S_n x_n - x_n\|, \end{aligned}$$

where $M' > 0$ is an appropriate constant such that $M' \geq \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0, \|A\|^{-1})$ and for all $n \in \mathbb{N}$. It implies by (3.8) that

$$\limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0. \quad (3.10)$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0. \quad (3.11)$$

Since

$$\begin{aligned} \langle \gamma f(q) - Aq, x_n - q \rangle &= \langle \gamma f(q) - Aq, x_n - q \rangle - \langle \gamma f(q) - Aq, x_n - x_t \rangle + \langle \gamma f(q) \\ &\quad - Aq, x_n - x_t \rangle - \langle \gamma f(q) - Ax_t, x_n - x_t \rangle + \langle \gamma f(q) \\ &\quad - Ax_t, x_n - x_t \rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\ &= \langle \gamma f(q) - Aq, x_t - q \rangle + \langle Ax_t - Aq, x_n - x_t \rangle \\ &\quad + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\ &\leq \|\gamma f(q) - Aq\| \|x_t - q\| + \|A\| \|x_t - q\| \|x_n - x_t\| \\ &\quad + \gamma \alpha \|q - x_t\| \|x_n - x_t\| + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle, \end{aligned} \quad (3.12)$$

it follows from (3.10) and (3.12) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &= \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\ &\leq \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} (\|\gamma f(q) - Aq\| \|x_t - q\|) + \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} (\|A\| \|x_t - q\| \|x_n - x_t\|) \\ &\quad + \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} (\gamma \alpha \|q - x_t\| \|x_n - x_t\|) + \limsup_{t \rightarrow 0^+} \limsup_{n \rightarrow \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\ &\leq 0. \end{aligned}$$

Finally, we will show that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \bigcap_{i=1}^N F(T_i)$. Since

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n \gamma(a_n u + (1 - a_n)f(x_n)) + (I - \alpha_n A)y_n - \alpha_n Aq - (I - \alpha_n A)q\|^2 \\ &= \|\alpha_n (\gamma(a_n u + (1 - a_n)f(x_n)) - Aq) + (I - \alpha_n A)(y_n - q)\|^2 \\ &\leq \|(I - \alpha_n A)(y_n - q)\|^2 + 2\alpha_n \langle \gamma(a_n u + (1 - a_n)f(x_n)) - Aq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|y_n - q\|^2 + 2\alpha_n \langle \gamma(a_n u + (1 - a_n)f(x_n)) - \gamma f(q) + \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= (1 - \alpha_n \bar{\gamma})^2 \|y_n - q\|^2 + 2\alpha_n \langle \gamma(a_n u + (1 - a_n)f(x_n)) - f(q) \rangle + \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma \langle a_n(u - f(q)) + (1 - a_n)(f(x_n) \\ &\quad - f(q)), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma \langle a_n(u - f(q)), x_{n+1} - q \rangle \\ &\quad + \langle (1 - a_n)(f(x_n) - f(q)), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma a_n \langle u - f(q), x_{n+1} - q \rangle \end{aligned}$$

$$\begin{aligned}
& + 2\alpha_n\gamma(1-a_n)\langle f(x_n) - f(q), x_{n+1} - q \rangle + 2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
\leq & (1 - \alpha_n\bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n\gamma a_n \|u - f(q)\| \|x_{n+1} - q\| \\
& + 2\alpha_n\gamma\alpha \|x_n - q\| \|x_{n+1} - q\| + 2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
\leq & (1 - \alpha_n\bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n\gamma a_n \|u - f(q)\| \|x_{n+1} - q\| \\
& + \alpha_n\gamma\alpha (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q \rangle,
\end{aligned}$$

we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 & \leq \frac{(1 - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 + \frac{\alpha_n\gamma\alpha}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n\gamma a_n}{(1 - \alpha_n\gamma\alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \frac{(1 - \alpha_n\gamma\alpha + \alpha_n\gamma\alpha - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2)}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 + \frac{\alpha_n\gamma\alpha}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n\gamma a_n}{(1 - \alpha_n\gamma\alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \frac{((1 - \alpha_n\gamma\alpha) + \alpha_n\gamma\alpha - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2 + \alpha_n\gamma\alpha)}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n\gamma a_n}{(1 - \alpha_n\gamma\alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \frac{((1 - \alpha_n\gamma\alpha) - 2\alpha_n(\bar{\gamma} - \gamma\alpha) + \alpha_n^2\bar{\gamma}^2)}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n\gamma a_n}{(1 - \alpha_n\gamma\alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \left(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\alpha)}{(1 - \alpha_n\gamma\alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n^2\bar{\gamma}^2}{(1 - \alpha_n\gamma\alpha)} \|x_n - q\|^2 \\
& + \frac{2\alpha_n\gamma a_n}{(1 - \alpha_n\gamma\alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
& = \left(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\alpha)}{(1 - \alpha_n\gamma\alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n}{(1 - \alpha_n\gamma\alpha)} (\alpha_n\bar{\gamma}^2 \|x_n - q\|^2 \\
& + 2\gamma a_n \|u - f(q)\| \|x_{n+1} - q\| + 2\langle \gamma f(q) - Aq, x_{n+1} - q \rangle) \\
& = \left(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\alpha)}{(1 - \alpha_n\gamma\alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n}{(1 - \alpha_n\gamma\alpha)} \frac{2(\bar{\gamma} - \gamma\alpha)}{2(\bar{\gamma} - \gamma\alpha)} (\alpha_n\bar{\gamma}^2 \|x_n - q\|^2 \\
& + 2\gamma a_n \|u - f(q)\| \|x_{n+1} - q\| + 2\langle \gamma f(q) - Aq, x_{n+1} - q \rangle) \\
& = \left(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\alpha)}{(1 - \alpha_n\gamma\alpha)}\right) \|x_n - q\|^2 + \frac{\alpha_n 2(\bar{\gamma} - \gamma\alpha)}{(1 - \alpha_n\gamma\alpha)} \left(\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma\alpha)} \|x_n - q\|^2\right. \\
& \left. + \frac{\gamma a_n}{(\bar{\gamma} - \gamma\alpha)} \|u - f(q)\| \|x_{n+1} - q\| + \frac{1}{(\bar{\gamma} - \gamma\alpha)} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle\right).
\end{aligned}$$

This implies by Lemma 2.7 that the sequence $\{x_n\}$ converges strongly to $q \in \bigcap_{i=1}^N$. By (3.2), we have

$$\|y_n - q\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - q\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof. $\square$

4. Application

Using our main result, we obtain the following strong convergence theorems in a Hilbert space.

Theorem 4.1. Let H be a Hilbert space, let f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strictly pseudo-contractions of H into itself for some $\kappa_i \in [0, 1)$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let W_n be the W -mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{n,1}, \alpha_1^{n,2}, \dots, \alpha_1^{n,N}$, where $\alpha_1^{n,j} \in [0, 1]$, and $\kappa < a \leq \alpha_1^{n,j} \leq b < 1$ for all

$j = 1, 2, \dots, N-1, \kappa < c \leq \alpha_1^{n,N} \leq 1$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \end{cases} \quad n \geq 1, \quad (4.1)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 0$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, for all $j \in \{1, 2, 3, \dots, N\}$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in \bigcap_{i=1}^N F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

Proof. Put $\alpha_2^{n,j} = 0$ for all $j \in \{1, 2, 3, \dots, N\}$, and all $n \in \mathbb{N}$ in Theorem 3.1. Then, by Theorem 3.1 the conclusion follows. $\square$

Theorem 4.2. Let H be a Hilbert space, let f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let T be a κ -strict pseudo-contraction of H into itself for some $\kappa \in [0, 1)$ and $F(T) \neq \emptyset$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \end{cases} \quad n \geq 1, \quad (4.2)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} a_n = 0$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in F(T)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

Proof. Put $N = 1$ and $T_1 = T$ and $\alpha_2^{n,1}, \alpha_3^{n,1} = 0 \forall n \in \mathbb{N}$ in Theorem 3.1. Then $S_n = T$ and we obtain the desired result from Theorem 3.1. $\square$

Corollary 4.3 ([15], Theorem 2.1). Let H be a Hilbert space, f be an α -contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let T be a κ -strict pseudo-contraction of H into itself for some $\kappa \in [0, 1)$ and $F(T) \neq \emptyset$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \end{cases} \quad n \geq 1, \quad (4.3)$$

where $\{\beta_n\}$, $\{\alpha_n\}$, and $\{a_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
- (iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ for all $n \geq 1$, for some $\theta \in (0, 1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in F(T)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

Proof. Put $N = 1$ and $T_1 = T$ and $a_n, \alpha_2^{n,1}, \alpha_3^{n,1} = 0 \forall n \in \mathbb{N}$ in Theorem 3.1. Then $S_n = T$ and the desired result is directly obtained by Theorem 3.1. $\square$

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Research Article

Strong Convergence Theorem for a New General System of Variational Inequalities in Banach Spaces

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We introduce a new system of general variational inequalities in Banach spaces. The equivalence between this system of variational inequalities and fixed point problems concerning the nonexpansive mapping is established. By using this equivalent formulation, we introduce an iterative scheme for finding a solution of the system of variational inequalities in Banach spaces. Our main result extends a recent result achieved by Yao, Noor, Noor, Liou, and Yaqoob.

1. Introduction

Let X be a real Banach space, and X^* be its dual space. Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X . X is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$ there exists a constant $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (1.1)$$

The norm on X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (1.2)$$

exists for each $x, y \in U$ and in this case X is said to have a *uniformly Frechet differentiable norm* if the limit (1.2) is attained uniformly for $x, y \in U$ and in this case X is said to be *uniformly smooth*. We define a function $\rho : [0, \infty) \rightarrow [0, \infty)$, called the *modulus of smoothness* of X , as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2(\|x+y\| + \|x-y\|)} - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}. \quad (1.3)$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \rightarrow 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For $q > 1$, the generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in X. \quad (1.4)$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and usually, we write $J_2 = J$. If X is a Hilbert space, then $J = I$. Further, we have the following properties of the generalized duality mapping J_q :

- (1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$,
- (2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0, \infty)$,
- (3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is known that if X is smooth, then J is single-valued, which is denoted by j . Recall that the duality mapping j is said to be *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \rightarrow x$ weakly, we have $j(x_n) \rightarrow j(x)$ weakly-*. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth. For the details, see the work of Gossez and Lami Dozo in [1].

Let C be a nonempty closed convex subset of a smooth Banach space X . Recall that a mapping $A : C \rightarrow X$ is said to be *accretive* if

$$\langle Ax - Ay, j(x - y) \rangle \geq 0 \quad (1.5)$$

for all $x, y \in C$. A mapping $A : C \rightarrow X$ is said to be *α -strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2 \quad (1.6)$$

for all $x, y \in C$. A mapping $A : C \rightarrow X$ is said to be *α -inverse strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|Ax - Ay\|^2 \quad (1.7)$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Let D be a nonempty subset of C . A mapping $Q : C \rightarrow D$ is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx, \quad (1.8)$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \rightarrow D$ is called a *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a *sunny nonexpansive retraction* from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D . It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C .

Conveying an idea of the *classical variational inequality*, denoted by $VI(C, A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \quad (1.9)$$

where $X = H$ is a Hilbert space and A is a mapping from C into H . The variational inequality has been widely studied in the literature; see, for example, the work of Chang et al. in [2], Zhao and He [3], Plubtieng and Punpaeng [4], Yao et al. [5] and the references therein.

Let $A, B : C \rightarrow H$ be two mappings. In 2008, Ceng et al. [6] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.10)$$

which is called a *general system of variational inequalities*, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.10) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.11)$$

which is defined by Verma [7] and is called the *new system of variational inequalities*. Further, if we add up the requirement that $x^* = y^*$, then problem (1.11) reduces to the classical variational inequality $VI(C, A)$.

In 2006, Aoyama et al. [8] first considered the following generalized variational inequality problem in Banach spaces. Let $A : C \rightarrow X$ be an accretive operator. Find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \quad (1.12)$$

The problem (1.12) is very interesting as it is connected with the fixed point problem for nonlinear mapping and the problem of finding a zero point of an accretive operator in Banach spaces, see [9–11] and the references therein.

Aoyama et al. [8] introduced the following iterative algorithm in Banach spaces:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= Q_C(x_n - \lambda_n A)x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n)y_n, \quad n \geq 0, \end{aligned} \quad (1.13)$$

where Q_C is a sunny nonexpansive retraction from X onto C . Then they proved a weak convergence theorem which is generalized simultaneously theorems of Browder and Petryshyn [12] and Gol'shtein and Tret'yakov [13]. In 2008, Hao [14] obtained a strong convergence theorem by using the following iterative algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= b_n x_n + (1 - b_n)Q_C(I - \lambda_n A x_n), \\ x_{n+1} &= a_n u + (1 - a_n)y_n, \quad n \geq 0, \end{aligned} \quad (1.14)$$

where $\{a_n\}, \{b_n\}$ are two sequences in $(0, 1)$ and $u \in C$.

Very recently, in 2009, Yao et al. [5] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A, B : C \rightarrow X$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle Ay^* + x^* - y^*, j(x - x^*) \rangle &\geq 0, \quad \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*) \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.15)$$

which is called *the system of general variational inequalities in a real Banach space*. They proved a strong convergence theorem by using the following iterative algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= Q_C(x_n - Bx_n), \\ x_{n+1} &= a_n u + b_n x_n + c_n Q_C(y_n - Ay_n), \quad n \geq 0, \end{aligned} \quad (1.16)$$

where $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are three sequences in $(0, 1)$ and $u \in C$.

In this paper, motivated and inspired by the idea of Yao et al. [5] and Cheng et al. [6]. First, we introduce the following system of variational inequalities in Banach spaces.

Let C be a nonempty closed convex subset of a real Banach space X . Let $A_i : C \rightarrow X$ for all $i = 1, 2, 3$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned} \langle \lambda_1 A_1 y^* + x^* - y^*, j(x - x^*) \rangle &\geq 0, \quad \forall x \in C, \\ \langle \lambda_2 A_2 z^* + y^* - z^*, j(x - y^*) \rangle &\geq 0, \quad \forall x \in C, \\ \langle \lambda_3 A_3 x^* + z^* - x^*, j(x - z^*) \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.17)$$

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which is called a *new general system of variational inequalities in Banach spaces*, where $\lambda_i > 0$ for all $i = 1, 2, 3$. In particular, if $A_3 = 0$, $z^* = x^*$, and $\lambda_i = 1$ for $i = 1, 2, 3$, then problem (1.17) reduces to problem (1.15). Further, if $A_3 = 0$, $z^* = x^*$, then problem (1.17) reduces to the problem (1.10) in a real Hilbert space. Second, we introduce iteration process for finding a solution of a new general system of variational inequalities in a real Banach space. Starting with arbitrary points $v, x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{aligned} z_n &= Q_C(x_n - \lambda_3 A_3 x_n), \\ y_n &= Q_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) Q_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1, \end{aligned} \tag{1.18}$$

where $\lambda_i > 0$ for all $i = 1, 2, 3$ and $\{a_n\}$, $\{b_n\}$ are two sequences in $(0, 1)$. Using the demiclosedness principle for nonexpansive mapping, we will show that the sequence $\{x_n\}$ converges strongly to a solution of a new general system of variational inequalities in Banach spaces under some control conditions.

2. Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Lemma 2.1 (see [15]). *Let X be a q -uniformly smooth Banach space with $1 \leq q \leq 2$. Then*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + 2 \|Ky\|^q \tag{2.1}$$

for all $x, y \in X$, where K is the q -uniformly smooth constant of X .

The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.2 (see [16, 17]). *Let C be a closed convex subset of a smooth Banach space X . Let D be a nonempty subset of C and $Q : C \rightarrow D$ be a retraction. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu) \rangle \leq 0, \tag{2.2}$$

for all $u \in C$ and $y \in D$.

The first result regarding the existence of sunny nonexpansive retractions on the fixed point set of a nonexpansive mapping is due to Bruck [18].

Remark 2.3. If X is strictly convex and uniformly smooth and if $T : C \rightarrow C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then there exists a sunny nonexpansive retraction of C onto $F(T)$.

Lemma 2.4 (see [19]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 1, \quad (2.3)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.5 (see [20]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.6 (see [21]). Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X , and $T : C \rightarrow C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if $x_n \rightarrow x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly, then $x \in F(T)$.

3. Main Results

In this section, we establish the equivalence between the new general system of variational inequalities (1.17) and some fixed point problem involving a nonexpansive mapping. Using the demiclosedness principle for nonexpansive mapping, we prove that the iterative scheme (1.18) converges strongly to a solution of a new general system of variational inequalities (1.17) in a Banach space under some control conditions. In order to prove our main result, the following lemmas are needed.

The next lemmas are crucial for proving the main theorem.

Lemma 3.1. Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let the mapping $A : C \rightarrow X$ be α -inverse strongly accretive. Then, we have

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2, \quad (3.1)$$

where K is the 2-uniformly smooth constant of X . In particular, if $\alpha \geq \lambda K^2$, then $I - \lambda A$ is a nonexpansive mapping.

Proof. Indeed, for all $x, y \in C$, from Lemma 2.1, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\langle Ax - Ay, j(x - y) \rangle \\ &\quad + 2K^2\lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2. \end{aligned} \quad (3.2)$$

It is clear that, if $\alpha \geq \lambda K^2$, then $I - \lambda A$ is a nonexpansive mapping. □

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Lemma 3.2. Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let Q_C be the sunny nonexpansive retraction from X onto C . Let $A_i : C \rightarrow X$ be an α_i -inverse strongly accretive mapping for $i = 1, 2, 3$. Let $G : C \rightarrow C$ be a mapping defined by

$$G(x) = Q_C[Q_C(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x)) - \lambda_1 A_1 Q_C(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x))], \quad \forall x \in C. \quad (3.3)$$

If $\alpha_i \geq \lambda_i K^2$ for all $i = 1, 2, 3$, then $G : C \rightarrow C$ is nonexpansive.

Proof. For all $x, y \in C$, we have

$$\begin{aligned} \|G(x) - G(y)\| &= \|Q_C[Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x)) \\ &\quad - Q_C[Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y))] \| \\ &\leq \|Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\quad - [Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y))] \| \\ &= \|(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)x \\ &\quad - (I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)y\|. \end{aligned} \quad (3.4)$$

From Lemma 3.1, we have $(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)$ is nonexpansive which implies by (3.4) that G is nonexpansive. $\square$

Lemma 3.3. Let C be a nonempty closed convex subset of a real smooth Banach space X . Let Q_C be the sunny nonexpansive retraction from X onto C . Let $A_i : C \rightarrow X$ be three nonlinear mappings. For given $(x^*, y^*, z^*) \in C \times C \times C$, (x^*, y^*, z^*) is a solution of problem (1.17) if and only if $x^* \in F(G)$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$, where G is the mapping defined as in Lemma 3.2.

Proof. Note that we can rewrite (1.17) as

$$\begin{aligned} \langle x^* - (y^* - \lambda_1 A_1 y^*), j(t - x^*) \rangle &\geq 0, \quad \forall t \in C, \\ \langle y^* - (z^* - \lambda_2 A_2 z^*), j(t - y^*) \rangle &\geq 0, \quad \forall t \in C, \\ \langle z^* - (x^* - \lambda_3 A_3 x^*), j(t - z^*) \rangle &\geq 0, \quad \forall t \in C. \end{aligned} \quad (3.5)$$

From Lemma 2.2, we can deduce that (3.5) is equivalent to

$$\begin{aligned}x^* &= Q_C(y^* - \lambda_1 A_1 y^*), \\y^* &= Q_C(z^* - \lambda_2 A_2 z^*), \\z^* &= Q_C(x^* - \lambda_3 A_3 x^*).\end{aligned}\tag{3.6}$$

It is easy to see that (3.6) is equivalent to $x^* = Gx^*$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. $\square$

From now on we denote by Ω^* the set of all fixed points of the mapping G . Now we prove the strong convergence theorem of algorithm (1.18) for solving problem (1.17).

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C . Let the mappings $A_i : C \rightarrow X$ be α_i -inverse strongly accretive with $\alpha_i \geq \lambda_i K^2$, for all $i = 1, 2, 3$ and $\Omega^* \neq \emptyset$. For given $x_1, v \in C$, let the sequence $\{x_n\}$ be generated iteratively by (1.18). Suppose the sequences $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0, 1)$ such that*

$$\begin{aligned}(\text{C1}) \quad & \lim_{n \rightarrow \infty} a_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty; \\(\text{C2}) \quad & 0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1.\end{aligned}$$

Then $\{x_n\}$ converges strongly to $Q'v$ where Q' is the sunny nonexpansive retraction of C onto Ω^* .

Proof. Let $x^* \in \Omega^*$ and $t_n = Q_C(y_n - \lambda_1 A_1 y_n)$, it follows from Lemma 3.3 that

$$\begin{aligned}x^* &= Q_C[Q_C(Q_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 Q_C(x^* - \lambda_3 A_3 x^*)) \\&\quad - \lambda_1 A_1 Q_C(Q_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 Q_C(x^* - \lambda_3 A_3 x^*))].\end{aligned}\tag{3.7}$$

Put $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. Then $x^* = Q_C(y^* - \lambda_1 A_1 y^*)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) t_n.\tag{3.8}$$

From Lemma 3.1, we have $I - \lambda_i A_i$ ($i = 1, 2, 3$) is nonexpansive. Therefore

$$\begin{aligned}\|t_n - x^*\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C(y^* - \lambda_1 A_1 y^*)\| \\&\leq \|y_n - y^*\| \\&= \|Q_C(z_n - \lambda_2 A_2 z_n) - Q_C(z^* - \lambda_2 A_2 z^*)\| \\&\leq \|z_n - z^*\| \\&= \|Q_C(x_n - \lambda_3 A_3 x_n) - Q_C(x^* - \lambda_3 A_3 x^*)\| \\&\leq \|x_n - x^*\|.\end{aligned}\tag{3.9}$$

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It follows that

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|a_n v + b_n x_n + (1 - a_n - b_n)t_n - x^*\| \\
 &\leq a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|t_n - x^*\| \\
 &\leq a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|x_n - x^*\| \\
 &= a_n \|v - x^*\| + (1 - a_n) \|x_n - x^*\|.
 \end{aligned} \tag{3.10}$$

By induction, we have

$$\|x_{n+1} - x^*\| \leq \max\{\|v - x^*\|, \|x_1 - x^*\|\}. \tag{3.11}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{A_1 y_n\}$, $\{A_2 z_n\}$, and $\{A_3 x_n\}$ are also bounded. By nonexpansiveness of Q_C and $I - \lambda_i A_i$ ($i = 1, 2, 3$), we have

$$\begin{aligned}
 \|t_{n+1} - t_n\| &= \|Q_C(y_{n+1} - \lambda_1 A_1 y_{n+1}) - Q_C(y_n - \lambda_1 A_1 y_n)\| \\
 &\leq \|y_{n+1} - y_n\| \\
 &= \|Q_C(z_{n+1} - \lambda_2 A_2 z_{n+1}) - Q_C(z_n - \lambda_2 A_2 z_n)\| \\
 &\leq \|z_{n+1} - z_n\| \\
 &= \|Q_C(x_{n+1} - \lambda_3 A_3 x_{n+1}) - Q_C(x_n - \lambda_3 A_3 x_n)\| \\
 &\leq \|x_{n+1} - x_n\|.
 \end{aligned} \tag{3.12}$$

Let $w_n = (x_{n+1} - b_n x_n) / (1 - b_n)$, $n \in \mathbb{N}$. Then $x_{n+1} = b_n x_n + (1 - b_n)w_n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
 w_{n+1} - w_n &= \frac{x_{n+2} - b_{n+1}x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\
 &= \frac{a_{n+1}v + (1 - a_{n+1} - b_{n+1})t_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n)t_n}{1 - b_n} \\
 &= \frac{a_{n+1}}{1 - b_{n+1}}(v - t_{n+1}) + \frac{a_n}{1 - b_n}(t_n - v) + t_{n+1} - t_n.
 \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have

$$\begin{aligned}
 \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| &\leq \frac{a_{n+1}}{1 - b_{n+1}} \|v - t_{n+1}\| + \frac{a_n}{1 - b_n} \|t_n - v\| \\
 &\quad + \|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \\
 &\leq \frac{a_{n+1}}{1 - b_{n+1}} \|v - t_{n+1}\| + \frac{a_n}{1 - b_n} \|t_n - v\|.
 \end{aligned} \tag{3.14}$$

This together with (C1) and (C2), we obtain that

$$\limsup_{n \rightarrow \infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq 0. \quad (3.15)$$

Hence, by Lemma 2.5, we get $\|x_n - w_n\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - b_n) \|w_n - x_n\| = 0. \quad (3.16)$$

Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n - b_n)(t_n - x_n), \quad (3.17)$$

therefore

$$\|t_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.18)$$

Furthermore, by Lemma 3.2, we have $G : C \rightarrow C$ is nonexpansive. Thus, we have

$$\begin{aligned} \|t_n - G(t_n)\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - G(t_n)\| \\ &= \|Q_C[Q_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 Q_C(z_n - \lambda_2 A_2 z_n)] - G(t_n)\| \\ &= \|Q_C[Q_C(Q_C(x_n - \lambda_3 A_3 x_n) - \lambda_2 A_2 Q_C(x_n - \lambda_3 A_3 x_n)) \\ &\quad - \lambda_1 A_1 Q_C(Q_C(x_n - \lambda_3 A_3 x_n) - \lambda_2 A_2 Q_C(x_n - \lambda_3 A_3 x_n))] - G(t_n)\| \\ &= \|G(x_n) - G(t_n)\| \leq \|x_n - t_n\|, \end{aligned} \quad (3.19)$$

which implies $\|t_n - G(t_n)\| \rightarrow 0$ as $n \rightarrow \infty$.

Since

$$\begin{aligned} \|x_n - G(x_n)\| &\leq \|x_n - t_n\| + \|t_n - G(t_n)\| + \|G(t_n) - G(x_n)\| \\ &\leq \|x_n - t_n\| + \|t_n - G(t_n)\| + \|t_n - x_n\|, \end{aligned} \quad (3.20)$$

therefore

$$\lim_{n \rightarrow \infty} \|x_n - G(x_n)\| = 0. \quad (3.21)$$

Let Q' be the sunny nonexpansive retraction of C onto Ω^* . Now we show that

$$\limsup_{n \rightarrow \infty} \langle v - Q'v, j(x_n - Q'v) \rangle \leq 0. \quad (3.22)$$

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To prove (3.22), since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\bar{x}$ and

$$\limsup_{n \rightarrow \infty} \langle v - Q'v, j(x_n - Q'v) \rangle = \lim_{i \rightarrow \infty} \langle v - Q'v, j(x_{n_i} - Q'v) \rangle. \quad (3.23)$$

From Lemma 2.6 and (3.21), we obtain $\bar{x} \in \Omega^*$. Now, from Lemma 2.2, (3.23), and the weakly sequential continuity of the duality mapping j , we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle v - Q'v, j(x_n - Q'v) \rangle &= \lim_{i \rightarrow \infty} \langle v - Q'v, j(x_{n_i} - Q'v) \rangle \\ &= \langle v - Q'v, j(\bar{x} - Q'v) \rangle \leq 0. \end{aligned} \quad (3.24)$$

From (3.9), we have

$$\begin{aligned} \|x_{n+1} - Q'v\|^2 &= \langle a_nv + b_nx_n + (1 - a_n - b_n)t_n - Q'v, j(x_{n+1} - Q'v) \rangle \\ &= a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + b_n \langle x_n - Q'v, j(x_{n+1} - Q'v) \rangle \\ &\quad + (1 - a_n - b_n) \langle t_n - Q'v, j(x_{n+1} - Q'v) \rangle \\ &\leq a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + b_n (\|x_n - Q'v\| \|j(x_{n+1} - Q'v)\|) \\ &\quad + (1 - a_n - b_n) (\|t_n - Q'v\| \|j(x_{n+1} - Q'v)\|) \\ &= a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + b_n (\|x_n - Q'v\| \|x_{n+1} - Q'v\|) \\ &\quad + (1 - a_n - b_n) (\|t_n - Q'v\| \|x_{n+1} - Q'v\|) \\ &\leq a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + \frac{1}{2} b_n (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &\quad + \frac{1}{2} (1 - a_n - b_n) (\|t_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &\leq a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + \frac{1}{2} b_n (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &\quad + \frac{1}{2} (1 - a_n - b_n) (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2) \\ &= a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle + \frac{1}{2} (1 - a_n) (\|x_n - Q'v\|^2 + \|x_{n+1} - Q'v\|^2), \end{aligned} \quad (3.25)$$

which implies that

$$\|x_{n+1} - Q'v\|^2 \leq (1 - a_n) \|x_n - Q'v\|^2 + 2a_n \langle v - Q'v, j(x_{n+1} - Q'v) \rangle. \quad (3.26)$$

It follows from Lemma 2.4, (3.24), and (3.26) that $\{x_n\}$ converges strongly to $Q'v$. This completes the proof. $\square$

Letting $A_3 = 0$ and $\lambda_i = 1$ for $i = 1, 2, 3$ in Theorem 3.4, we obtain the following result.

Corollary 3.5 (see [5, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C . Let the mappings $A_i : C \rightarrow X$ be α_i -inverse strongly accretive with $\alpha_i \geq K^2$, for all $i = 1, 2$ and $\Omega^* \neq \emptyset$. For given $x_1, v \in C$, and let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= Q_C(x_n - A_2 x_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) Q_C(y_n - A_1 y_n), \quad n \geq 1, \end{aligned} \quad (3.27)$$

where $\{a_n\}, \{b_n\}$ are two sequences in $(0, 1)$ such that

- (C1) $\lim_{n \rightarrow \infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;
 (C2) $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $Q'v$ where Q' is the sunny nonexpansive retraction of C onto Ω^* .

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On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval

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ABSTRACT

In this paper, we propose a new iteration, called the SP-iteration, for approximating a fixed point of continuous functions on an arbitrary interval. Then, a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval is given. We also compare the convergence speed of Mann, Ishikawa, Noor and SP-iterations. It is proved that the SP-iteration is equivalent to and converges faster than the others. Our results extend and improve the corresponding results of Borwein and Borwein [D. Borwein, J. Borwein, Fixed point iterations for real functions, J. Math. Anal. Appl. 157 (1991) 112–126], Qing and Qihou [Y. Qing, L. Qihou, The necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval, J. Math. Anal. Appl. 323 (2006) 1383–1386], Rhoades [B.E. Rhoades, Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56 (1976) 741–750], and many others. Moreover, we also present numerical examples for the SP-iteration to compare with the Mann, Ishikawa and Noor iterations.

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1. Introduction

Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. A point $p \in E$ is a *fixed point* of f if $f(p) = p$. We denote by $F(f)$ the set of fixed points of f . It is known that if E is also bounded, then $F(f)$ is nonempty. The *Mann iteration* (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \quad (1.1)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$, and will be denoted by $M(u_1, \alpha_n, f)$. The *Ishikawa iteration* (see [2]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1 - \beta_n)s_n + \beta_n f(s_n) \\ s_{n+1} = (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{cases} \quad (1.2)$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will be denoted by $I(s_1, \alpha_n, \beta_n, f)$. The *Noor iteration* (see [3]) is defined by $w_1 \in E$ and

$$\begin{cases} r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n) \\ q_n = (1 - \beta_n)w_n + \beta_n f(r_n) \\ w_{n+1} = (1 - \alpha_n)w_n + \alpha_n f(q_n) \end{cases} \quad (1.3)$$

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for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$, and will be denoted by $N(w_1, \alpha_n, \beta_n, \gamma_n, f)$. Clearly the Mann and Ishikawa iterations are special cases of the Noor iteration.

In 1976, Rhoades [4] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and nondecreasing functions on unit closed interval. Further, Borwein and Borwein [5] proved the convergence of the Mann iteration of continuous functions on a bounded closed interval. Recently, Qing and Qihou [6] extended their results to an arbitrary interval and to the Ishikawa iteration and gave some control conditions for the convergence of Ishikawa iteration on an arbitrary interval.

It was shown in [7] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [8] showed that the Mann iteration converges faster than the Ishikawa iteration for this class of operators. Two years later, Qing and Rhoades [9] provided an example to show that the claim in [8] is false.

Motivated by the above results, we propose a new iteration as follows:

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n) \\ y_n = (1 - \beta_n)z_n + \beta_n f(z_n) \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n f(y_n) \end{cases} \quad (1.4)$$

for all $n \geq 1$, where $x_1 \in E$, $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$, and it will be denoted by $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and is called the SP-iteration.

In this paper, we give a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval. We also prove that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges faster than the others for the class of continuous and nondecreasing functions. Moreover, we present numerical examples for the SP-iteration to compare with the Mann, Ishikawa and Noor iterations.

2. Convergence theorems

In this section, we prove the convergence theorems of the SP-iteration for continuous functions on an arbitrary interval.

Theorem 2.1. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $x_1 \in E$, let the SP-iteration $\{x_n\}_{n=1}^\infty$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$, (iii) $\sum_{n=1}^\infty \beta_n < \infty$ and (iv) $\sum_{n=1}^\infty \gamma_n < \infty$.

Then $\{x_n\}_{n=1}^\infty$ is bounded if and only if $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. It is obvious that if $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f , then it is bounded. Now, assume that $\{x_n\}_{n=1}^\infty$ is bounded. We shall show that $\{x_n\}_{n=1}^\infty$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n \rightarrow \infty} x_n$, $b = \limsup_{n \rightarrow \infty} x_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Because $f(x)$ is a continuous function, there exists $\delta, 0 < \delta < b - a$, such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \quad (2.1)$$

Since $\{x_n\}_{n=1}^\infty$ is bounded, $\{x_n\}_{n=1}^\infty$ belongs to a bounded closed interval. By continuity of f , we have that $\{f(x_n)\}_{n=1}^\infty$ belongs to another bounded closed interval, so $\{f(x_n)\}_{n=1}^\infty$ is bounded, and since $z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n)$, so $\{z_n\}_{n=1}^\infty$ is bounded, and thus $\{f(z_n)\}_{n=1}^\infty$ is bounded. Similarly, since $y_n = (1 - \beta_n)z_n + \beta_n f(z_n)$, we have $\{y_n\}_{n=1}^\infty$ and $\{f(y_n)\}_{n=1}^\infty$ are bounded. Using (1.4), we have $x_{n+1} - y_n = \alpha_n(f(y_n) - y_n)$, $y_n - z_n = \beta_n(f(z_n) - z_n)$ and $z_n - x_n = \gamma_n(f(x_n) - x_n)$. By (i), (iii) and (iv), we get $|x_{n+1} - y_n| \rightarrow 0$, $|y_n - z_n| \rightarrow 0$ and $|z_n - x_n| \rightarrow 0$. Since $|x_{n+1} - x_n| \leq |x_{n+1} - y_n| + |y_n - z_n| + |z_n - x_n|$, we have $|x_{n+1} - x_n| \rightarrow 0$. Thus there exists N such that

$$|x_{n+1} - x_n| < \frac{\delta}{3}, \quad |z_n - x_n| < \frac{\delta}{3}, \quad |y_n - z_n| < \frac{\delta}{3} \quad (2.2)$$

for all $n > N$. Since $b = \limsup_{n \rightarrow \infty} x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $x_k > m$. For x_k , there exist only two cases:

(A1) $x_k \geq m + \frac{\delta}{3}$, then by (2.2), we have $x_{k+1} - x_k > -\frac{\delta}{3}$, then $x_{k+1} > x_k - \frac{\delta}{3} \geq m$, so $x_{k+1} > m$.

(A2) $m < x_k < m + \frac{\delta}{3}$, then by (2.2), we have $m - \frac{\delta}{3} < z_k < m + \frac{2\delta}{3}$ and $m - \frac{2\delta}{3} < y_k < m + \delta$. So we have $|x_k - m| < \frac{\delta}{3} < \delta$, $|z_k - m| < \frac{2\delta}{3} < \delta$ and $|y_k - m| < \delta$. Using (2.1), we get

$$f(x_k) - x_k > 0, \quad f(z_k) - z_k > 0, \quad f(y_k) - y_k > 0. \quad (2.3)$$

By (1.4), we have

$$\begin{aligned} x_{k+1} &= y_k + \alpha_k(f(y_k) - y_k) \\ &= z_k + \beta_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\ &= x_k + \gamma_k(f(x_k) - x_k) + \beta_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k). \end{aligned} \quad (2.4)$$

By (2.3), we have $x_{k+1} \geq x_k$. Thus $x_{k+1} > m$.

By (A1) and (A2), we can conclude that $x_{k+1} > m$. By using the above argument, we obtain $x_{k+2} > m$, $x_{k+3} > m$, $x_{k+4} > m$, $\dots$. Thus we get $x_n > m$, for all $n > k = n_{k_1}$. So $a = \liminf_{n \rightarrow \infty} x_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For $\{x_n\}_{n=1}^\infty$, there exist only two cases:

(B1) There exists x_M such that $a < x_M < b$.

Then $f(x_M) = x_M$. Thus

$$z_M = (1 - \gamma_M)x_M + \gamma_M f(x_M) = x_M,$$

$$y_M = (1 - \beta_M)z_M + \beta_M f(z_M) = (1 - \beta_M)x_M + \beta_M f(x_M) = x_M,$$

$$x_{M+1} = (1 - \alpha_M)y_M + \alpha_M f(y_M) = (1 - \alpha_M)x_M + \alpha_M f(x_M) = x_M.$$

Analogously, we have $x_M = x_{M+1} = x_{M+2} = x_{M+3} = \dots$, so $x_n \rightarrow x_M$. It follows that $x_M = a$ and $x_n \rightarrow a$, which is a contradiction with the assumption.

(B2) For all n , $x_n \leq a$ or $x_n \geq b$.

Because $b - a > 0$ and $|x_{n+1} - x_n| \rightarrow 0$, so there exists N_0 such that $|x_{n+1} - x_n| < \frac{b-a}{3}$, for all $n > N_0$. It implies that either $x_n \leq a$ for all $n > N_0$ or $x_n \geq b$ for all $n > N_0$. If $x_n \leq a$ for $n > N_0$, then $b = \limsup_{n \rightarrow \infty} x_n \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > N_0$, so we have $a = \liminf_{n \rightarrow \infty} x_n \geq b$, which is a contradiction with $a < b$.

Hence, we have $\{x_n\}_{n=1}^\infty$ is convergent.

Next, we prove that $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f . Let $x_n \rightarrow p$ and suppose $f(p) \neq p$. By continuity of f , we have $\{f(x_n)\}_{n=1}^\infty$ is bounded. From $z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n)$ and $\gamma_n \rightarrow 0$, we obtain $z_n \rightarrow p$. Similarly, by $y_n = (1 - \beta_n)z_n + \beta_n f(z_n)$ and $\beta_n \rightarrow 0$, it follows that $y_n \rightarrow p$. Let $h_k = f(y_k) - y_k$, $q_k = f(z_k) - z_k$ and $s_k = f(x_k) - x_k$. By continuity of f , we have $\lim_{k \rightarrow \infty} h_k = \lim_{k \rightarrow \infty} (f(y_k) - y_k) = f(p) - p \neq 0$, $\lim_{k \rightarrow \infty} q_k = \lim_{k \rightarrow \infty} (f(z_k) - z_k) = f(p) - p \neq 0$ and $\lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} (f(x_k) - x_k) = f(p) - p \neq 0$. Put $w = f(p) - p$. Then $w \neq 0$. By (2.4), we have

$$\sum_{k=1}^{n-1} (x_{k+1} - x_k) = \sum_{k=1}^{n-1} (\alpha_k h_k + \beta_k q_k + \gamma_k s_k).$$

It follows that

$$x_n = x_1 + \sum_{k=1}^{n-1} (\alpha_k h_k + \beta_k q_k + \gamma_k s_k). \quad (2.5)$$

By (ii)–(iv) and $h_k \rightarrow w \neq 0$, we have that $\sum_{k=1}^\infty \alpha_k h_k$ is divergent, $\sum_{k=1}^\infty \beta_k q_k$ and $\sum_{k=1}^\infty \gamma_k s_k$ are convergent. This implies by (2.5) that $\{x_n\}_{n=1}^\infty$ is divergent, which is a contradiction with $x_n \rightarrow p$. Thus $f(p) = p$, that is $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f . $\square$

The following result give a convergence theorem for the Noor iteration, for continuous functions on a closed interval of a real line under some control conditions which are weaker than those in Theorem 2.1.

Theorem 2.2. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $w_1 \in E$, let the Noor iteration $\{w_n\}_{n=1}^\infty$ be defined by (1.3), where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=1}^\infty \alpha_n = \infty$, (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (iii) $\lim_{n \rightarrow \infty} \beta_n = 0$ and (iv) $\lim_{n \rightarrow \infty} \gamma_n = 0$.

Then $\{w_n\}_{n=1}^\infty$ is bounded if and only if $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. It is clear that if $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f , so it is bounded. Now, assume that $\{w_n\}_{n=1}^\infty$ is bounded. We shall show that $\{w_n\}_{n=1}^\infty$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n \rightarrow \infty} w_n$, $b = \limsup_{n \rightarrow \infty} w_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Because $f(x)$ is a continuous function, there exists δ , $0 < \delta < b - a$, such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \quad (2.6)$$

Since $\{w_n\}_{n=1}^\infty$ is bounded, $\{w_n\}_{n=1}^\infty$ belongs to a bounded closed interval. By continuity of f , we have that $\{f(w_n)\}_{n=1}^\infty$ belongs to another bounded closed interval, so $\{f(w_n)\}_{n=1}^\infty$ is bounded, and since $r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n)$, so $\{r_n\}_{n=1}^\infty$ is bounded, and thus $\{f(r_n)\}_{n=1}^\infty$ is bounded. Similarly, since $q_n = (1 - \beta_n)r_n + \beta_n f(r_n)$, we have $\{q_n\}_{n=1}^\infty$ and $\{f(q_n)\}_{n=1}^\infty$ are bounded. Using (1.3), we have $w_{n+1} - w_n = \alpha_n(f(q_n) - w_n)$, $q_n - w_n = \beta_n(f(r_n) - w_n)$ and $r_n - w_n = \gamma_n(f(w_n) - w_n)$. By (i), (iii) and (iv), we get $|w_{n+1} - w_n| \rightarrow 0$, $|q_n - w_n| \rightarrow 0$ and $|r_n - w_n| \rightarrow 0$. Thus there exists N such that

$$|w_{n+1} - w_n| < \frac{\delta}{3}, \quad |q_n - w_n| < \frac{\delta}{3}, \quad |r_n - w_n| < \frac{\delta}{3} \quad (2.7)$$

for all $n > N$. Since $b = \limsup_{n \rightarrow \infty} w_n > m$, there exists $k_1 > N$ such that $w_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $w_k > m$. For w_k , there exist only two cases:

(A1) $w_k \geq m + \frac{\delta}{3}$, then by (2.7), we have $w_{k+1} - w_k > -\frac{\delta}{3}$, then $w_{k+1} > w_k - \frac{\delta}{3} \geq m$, so $w_{k+1} > m$.

(A2) $m < w_k < m + \frac{\delta}{3}$, then by (2.7), we have $m - \frac{\delta}{3} < q_k < m + \frac{2\delta}{3}$ and $m - \frac{\delta}{3} < r_k < m + \frac{2\delta}{3}$. So we have $|w_k - m| < \frac{\delta}{3} < \delta$, $|q_k - m| < \frac{2\delta}{3} < \delta$ and $|r_k - m| < \frac{2\delta}{3} < \delta$. Using (2.6), we get

$$f(w_k) - w_k > 0, \quad f(q_k) - q_k > 0, \quad f(r_k) - r_k > 0. \quad (2.8)$$

By (1.3), we have

$$\begin{aligned} w_{k+1} &= w_k + \alpha_k(f(q_k) - w_k) \\ &= w_k + \alpha_k(f(q_k) - q_k) + \alpha_k\beta_k(f(r_k) - w_k) \\ &= w_k + \alpha_k(f(q_k) - q_k) + \alpha_k\beta_k(f(r_k) - r_k) + \alpha_k\beta_k\gamma_k(f(w_k) - w_k). \end{aligned}$$

By (2.8), we have $w_{k+1} \geq w_k$. Thus $w_{k+1} > m$.

By (A1) and (A2), we can conclude that $w_{k+1} > m$. By using the above argument, we obtain $w_{k+2} > m$, $w_{k+3} > m$, $w_{k+4} > m$, ... Thus we get $w_n > m$, for all $n > k = n_{k_1}$. So $a = \liminf_{n \rightarrow \infty} w_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For $\{w_n\}_{n=1}^\infty$, there exist only two cases:

(B1) There exists w_M such that $a < w_M < b$.

Then $f(w_M) = w_M$. Thus

$$\begin{aligned} r_M &= (1 - \gamma_M)w_M + \gamma_M f(w_M) = w_M, \\ q_M &= (1 - \beta_M)w_M + \beta_M f(r_M) = (1 - \beta_M)w_M + \beta_M f(w_M) = w_M, \\ w_{M+1} &= (1 - \alpha_M)w_M + \alpha_M f(q_M) = (1 - \alpha_M)w_M + \alpha_M f(w_M) = w_M. \end{aligned}$$

Analogously, we have $w_M = w_{M+1} = w_{M+2} = w_{M+3} = \dots$, so $w_n \rightarrow w_M$. It follows that $w_M = a$ and $w_n \rightarrow a$, which is a contradiction with the assumption.

(B2) For all n , $w_n \leq a$ or $w_n \geq b$.

Because $b - a > 0$ and $|w_{n+1} - w_n| \rightarrow 0$, so there exists N_0 such that $|w_{n+1} - w_n| < \frac{b-a}{3}$, for all $n > N_0$. It implies that either $w_n \leq a$ for all $n > N_0$ or $w_n \geq b$ for all $n > N_0$. If $w_n \leq a$ for $n > N_0$, then $b = \limsup_{n \rightarrow \infty} w_n \leq a$, which is a contradiction with $a < b$. If $w_n \geq b$ for $n > N_0$, so we have $a = \liminf_{n \rightarrow \infty} w_n \geq b$, which is a contradiction with $a < b$.

Hence, we have $\{w_n\}_{n=1}^\infty$ is convergent.

Next, we prove that $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f . Let $w_n \rightarrow p$ and suppose $f(p) \neq p$. By continuity of f , we have $\{f(w_n)\}_{n=1}^\infty$ is bounded. Since $r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n)$ and $\gamma_n \rightarrow 0$, it follows that $r_n \rightarrow p$. From $q_n = (1 - \beta_n)w_n + \beta_n f(r_n)$ and $\beta_n \rightarrow 0$, we obtain $q_n \rightarrow p$. Let $h_k = f(q_k) - w_k$. By continuity of f , we have $\lim_{k \rightarrow \infty} h_k = \lim_{k \rightarrow \infty} (f(q_k) - w_k) = f(p) - p \neq 0$. Let $d = f(p) - p$. Then $d \neq 0$. Using (1.3), we get $w_{k+1} = w_k + \alpha_k(f(q_k) - w_k)$. It follows that

$$w_n = w_1 + \sum_{k=1}^{n-1} \alpha_k h_k. \quad (2.9)$$

By $h_k \rightarrow d \neq 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$, it implies by (2.9) that $\{w_n\}_{n=1}^\infty$ is divergent, which is a contradiction with $w_n \rightarrow p$. Thus $f(p) = p$, that is $\{w_n\}_{n=1}^\infty$ converges to a fixed point of f . $\square$

The following three corollaries are obtained directly by Theorem 2.1.

Corollary 2.3. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $x_1 \in E$, let $\{x_n\}_{n=1}^\infty$ be the sequence defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n f(x_n) \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n f(y_n), \end{cases} \quad (2.10)$$

where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$ and (iii) $\sum_{n=1}^\infty \beta_n < \infty$.

Then $\{x_n\}_{n=1}^\infty$ is bounded if and only if $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. By putting $\gamma_n = 0$ for all $n \geq 1$ in Theorem 2.1, then we obtain the required result directly from Theorem 2.1. $\square$

Corollary 2.4 ([6]). Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $u_1 \in E$, let the Mann iteration $\{u_n\}_{n=1}^\infty$ be defined by (1.1), where $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (ii) $\sum_{n=1}^\infty \alpha_n = \infty$.

Then $\{u_n\}_{n=1}^\infty$ is bounded if and only if $\{u_n\}_{n=1}^\infty$ converges to a fixed point of f .

Proof. By putting $\gamma_n = 0$ and $\beta_n = 0$ for all $n \geq 1$ in Theorem 2.1, we obtain the desired result. $\square$

Corollary 2.5. Let $f : [a, b] \rightarrow [a, b]$ be a continuous function. For $x_1 \in [a, b]$, let the SP-iteration $\{x_n\}_{n=1}^\infty$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$, (iii) $\sum_{n=1}^\infty \beta_n < \infty$ and (iv) $\sum_{n=1}^\infty \gamma_n < \infty$.

Then $\{x_n\}_{n=1}^\infty$ converges to a fixed point of f .

The following result is obtained directly from Theorem 2.2.

Corollary 2.6 ([6]). Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. For $s_1 \in E$, let the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ be defined by (1.2), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\sum_{n=1}^\infty \alpha_n = \infty$, (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (iii) $\lim_{n \rightarrow \infty} \beta_n = 0$.

Then $\{s_n\}_{n=1}^\infty$ is bounded if and only if $\{s_n\}_{n=1}^\infty$ converges to a fixed point of f .

3. Rate of Convergence

In this section, we study the rate of convergence of the SP-iteration for continuous and nondecreasing functions; we also compare the rate of convergence of the SP-iteration with the Mann, Ishikawa and Noor iterations. We show that the SP-iteration converges faster than the others. For analysis of the rate of convergence, we use the concept introduced by Rhoades [4] as follows.

Definition 3.1. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two iterations which converge to the fixed point p of f . Then $\{x_n\}_{n=1}^\infty$ is said to converge faster than $\{y_n\}_{n=1}^\infty$ if

$$|x_n - p| \leq |y_n - p| \quad \text{for all } n \geq 1.$$

The following lemmas are useful and crucial for our main results.

Lemma 3.2. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1]$. Let $\{u_n\}_{n=1}^\infty, \{s_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty, \{x_n\}_{n=1}^\infty$ be defined by (1.1)–(1.4), respectively. Then the following hold:

- (i) If $f(u_1) < u_1$, then $f(u_n) < u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^\infty$ is nonincreasing.
- (ii) If $f(u_1) > u_1$, then $f(u_n) > u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^\infty$ is nondecreasing.
- (iii) If $f(s_1) < s_1$, then $f(s_n) < s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^\infty$ is nonincreasing.
- (iv) If $f(s_1) > s_1$, then $f(s_n) > s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^\infty$ is nondecreasing.
- (v) If $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^\infty$ is nonincreasing.
- (vi) If $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^\infty$ is nondecreasing.
- (vii) If $f(x_1) < x_1$, then $f(x_n) < x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^\infty$ is nonincreasing.
- (viii) If $f(x_1) > x_1$, then $f(x_n) > x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^\infty$ is nondecreasing.

Proof. (i) Let $f(u_1) < u_1$. Then $f(u_1) < u_2 \leq u_1$. Since f is nondecreasing, we have $f(u_2) \leq f(u_1)$. Thus $f(u_2) < u_2$. Assume that $f(u_k) < u_k$. Then $f(u_k) < u_{k+1} \leq u_k$. Since $f(u_{k+1}) \leq f(u_k)$, we have $f(u_{k+1}) < u_{k+1}$. By mathematical induction, we obtain $f(u_n) < u_n$ for all $n \geq 1$. It follows that $u_{n+1} \leq u_n$ for all $n \geq 1$, that is $\{u_n\}_{n=1}^\infty$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(s_1) < s_1$. Then $f(s_1) < t_1 \leq s_1$. Since f is nondecreasing, we have $f(t_1) \leq f(s_1) < t_1 \leq s_1$. This implies $f(t_1) < s_2 \leq s_1$. Thus $f(s_2) \leq f(s_1) < t_1 \leq s_1$. If $f(t_1) < s_2 \leq t_1$, then $f(s_2) \leq f(t_1) < s_2$. Otherwise, if $t_1 < s_2 \leq s_1$, then $f(s_2) \leq f(s_1) < t_1 < s_2$. Hence, we have $f(s_2) < s_2$. By continuing in this way, we can show that $f(s_n) < s_n$ for all $n \geq 1$. This implies $t_n \leq s_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(t_n) \leq f(s_n) < s_n$ for all $n \geq 1$. It follows that $s_{n+1} \leq s_n$ for all $n \geq 1$, that is $\{s_n\}_{n=1}^\infty$ is nonincreasing.

(iv) By using the same argument as in (iii), we obtain the desired result.

(v) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \leq w_1$. Since f is nondecreasing, we have $f(r_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(r_1) < q_1 \leq w_1$. Thus $f(q_1) \leq f(w_1) < r_1 \leq w_1$. For q_1 , we consider the following two cases:

Case 1: $f(r_1) < q_1 \leq r_1$. Then $f(q_1) \leq f(r_1) < q_1 \leq r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$, if $q_1 < w_2 \leq r_1$, then $f(w_2) \leq f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < w_2$. Thus, we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \leq w_1$. Then $f(q_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 < q_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$ and if $q_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Cases 1 and 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \geq 1$. This implies $r_n \leq w_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(r_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Thus $q_n \leq w_n$ for all $n \geq 1$, then $f(q_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Hence, we have $w_{n+1} \leq w_n$ for all $n \geq 1$, that is $\{w_n\}_{n=1}^\infty$ is nonincreasing.

(vi) By using the same argument as in (v), we obtain the desired result.

- (vii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1$. By (1.4), we have $f(z_1) < y_1 \leq z_1$. This implies $f(y_1) \leq f(z_1) < y_1$. Since $f(y_1) < x_2 \leq y_1$, we have $f(x_2) \leq f(y_1) < x_2$. Assume that $f(x_k) < x_k$. Then $f(x_k) < z_k \leq x_k$. Since f is nondecreasing, we have $f(z_k) \leq f(x_k) < z_k$. By (1.4), we have $f(z_k) < y_k \leq z_k$. This implies $f(y_k) \leq f(z_k) < y_k$. Since $f(y_k) < x_{k+1} \leq y_k$, we have $f(x_{k+1}) \leq f(y_k) < x_{k+1}$. By induction, we can conclude that $f(x_n) < x_n$ for all $n \geq 1$. Thus together with (1.4), we have $z_n \leq x_n$ for all $n \geq 1$. It follows that $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. This implies that $y_n \leq x_n$ for all $n \geq 1$. Hence, we have $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. It follows that $x_{n+1} = (1 - \alpha_n)y_n + \alpha_nf(y_n) \leq x_n$ for all $n \geq 1$. Thus $\{x_n\}_{n=1}^\infty$ is nonincreasing.
- (viii) By using the same argument as in (vii), we obtain the desired result. $\square$

Lemma 3.3. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function. Let $\{x_n\}_{n=1}^\infty$ be the sequence defined by (1.1)–(1.3) or (1.4), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ are sequences in $[0, 1)$. Then the following are satisfied:

- (i) If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.
 (ii) If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.

Proof. We shall prove only the case that $\{x_n\}_{n=1}^\infty$ is defined by (1.4) because other cases can be proved similarly.

- (i) Suppose that $p \in F(f)$ and $x_1 > p$. Since f is nondecreasing, we have $f(x_1) \geq f(p) = p$. By (1.4), we have $z_1 \geq p$. Thus $f(z_1) \geq p$. This implies by (1.4) that $y_1 \geq p$. Thus $f(y_1) \geq p$, it follows that $x_2 \geq p$. Assume that $x_k \geq p$. Thus $f(x_k) \geq f(p) = p$. By (1.4), we have $z_k \geq p$. Thus $f(z_k) \geq p$. This implies that $y_k \geq p$. Thus $f(y_k) \geq p$, it follows that $x_{k+1} \geq p$. By induction, we can conclude that $x_n \geq p$ for all $n \geq 1$.
 (ii) Suppose that $p \in F(f)$ and $x_1 < p$. By using the same argument as in (i), we can show that $x_n \leq p$ for all $n \geq 1$. $\square$

Lemma 3.4. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^\infty, \{s_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty, \{x_n\}_{n=1}^\infty$ be the sequences defined by (1.1)–(1.4), respectively. Then the following are satisfied:

- (i) If $f(u_1) < u_1$, then $x_n \leq w_n \leq s_n \leq u_n$ for all $n \geq 1$.
 (ii) If $f(u_1) > u_1$, then $x_n \geq w_n \geq s_n \geq u_n$ for all $n \geq 1$.

Proof. (i) Let $f(u_1) < u_1$. Since $u_1 = s_1 = w_1 = x_1$, we get $f(s_1) < s_1, f(w_1) < w_1$ and $f(x_1) < x_1$. First, we show that $x_n \leq w_n$ for all $n \geq 1$. By (1.4), we have $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1$. This implies $f(z_1) < y_1 \leq z_1 \leq x_1$. Using (1.3) and (1.4), we have $z_1 - r_1 = (1 - \gamma_1)(x_1 - w_1) + \gamma_1(f(x_1) - f(w_1)) = 0$, that is $z_1 = r_1$, and we get $y_1 - q_1 = (1 - \beta_1)(z_1 - w_1) + \beta_1(f(z_1) - f(r_1)) \leq 0$. Since f is nondecreasing, we have $f(y_1) \leq f(q_1)$. This implies

$$\begin{aligned} x_2 - w_2 &= (1 - \alpha_1)(y_1 - w_1) + \alpha_1(f(y_1) - f(q_1)) \\ &= (1 - \alpha_1)(y_1 - x_1) + \alpha_1(f(y_1) - f(q_1)) \\ &\leq 0, \end{aligned}$$

that is $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. By Lemma 3.2(v) and (vii), we have $f(w_k) < w_k$ and $f(x_k) < x_k$. This implies $f(x_k) < z_k \leq x_k \leq w_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus $y_k - w_k = (z_k - w_k) + \beta_k(f(z_k) - z_k) \leq 0$ and $z_k - r_k = (1 - \gamma_k)(x_k - w_k) + \gamma_k(f(x_k) - f(w_k)) \leq 0$. That is $y_k \leq w_k$ and $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have $y_k - q_k = (1 - \beta_k)(z_k - w_k) + \beta_k(f(z_k) - f(r_k)) \leq 0$, so $y_k \leq q_k$, which implies $f(y_k) \leq f(q_k)$. It follows that

$$\begin{aligned} x_{k+1} - w_{k+1} &= (1 - \alpha_k)(y_k - w_k) + \alpha_k(f(y_k) - f(q_k)) \\ &\leq 0, \end{aligned}$$

that is $x_{k+1} \leq w_{k+1}$. By mathematical induction, we obtain $x_n \leq w_n$ for all $n \geq 1$.

Next, we show that $w_n \leq s_n$ for all $n \geq 1$. Using (1.2) and (1.3), we have $r_1 - s_1 = (w_1 - s_1) + \gamma_1(f(w_1) - w_1) \leq 0$, that is $r_1 \leq s_1$. Since $f(r_1) \leq f(s_1)$, we obtain $q_1 - t_1 = (1 - \beta_1)(w_1 - s_1) + \beta_1(f(r_1) - f(s_1)) \leq 0$, so $q_1 \leq t_1$, which implies $f(q_1) \leq f(t_1)$. It follows that

$$\begin{aligned} w_2 - s_2 &= (1 - \alpha_1)(w_1 - s_1) + \alpha_1(f(q_1) - f(t_1)) \\ &\leq 0, \end{aligned}$$

that is $w_2 \leq s_2$. Assume that $w_k \leq s_k$. Thus $f(w_k) \leq f(s_k)$. By Lemma 3.2(iii) and (v), we have $f(s_k) < s_k$ and $f(w_k) < w_k$. This implies $r_k - s_k = (w_k - s_k) + \gamma_k(f(w_k) - w_k) \leq 0$, thus $r_k \leq s_k$. Since $f(r_k) \leq f(s_k)$, we have $q_k - t_k = (1 - \beta_k)(w_k - s_k) + \beta_k(f(r_k) - f(s_k)) \leq 0$, so $q_k \leq t_k$, which implies $f(q_k) \leq f(t_k)$. It follows that

$$\begin{aligned} w_{k+1} - s_{k+1} &= (1 - \alpha_k)(w_k - s_k) + \alpha_k(f(q_k) - f(t_k)) \\ &\leq 0, \end{aligned}$$

that is $w_{k+1} \leq s_{k+1}$. By mathematical induction, we obtain $w_n \leq s_n$ for all $n \geq 1$.

Finally, we show that $s_n \leq u_n$ for all $n \geq 1$. Using (1.1) and (1.2), we have $t_1 - u_1 = (s_1 - u_1) + \beta_1(f(s_1) - s_1) \leq 0$, thus $t_1 \leq u_1$. Since f is nondecreasing, we have $f(t_1) \leq f(u_1)$. This implies

$$\begin{aligned} s_2 - u_2 &= (1 - \alpha_1)(s_1 - u_1) + \alpha_1(f(t_1) - f(u_1)) \\ &\leq 0, \end{aligned}$$

that is $s_2 \leq u_2$. Assume that $s_k \leq u_k$. Thus $f(s_k) \leq f(u_k)$. By Lemma 3.2(i) and (iii), we have $f(u_k) < u_k$ and $f(s_k) < s_k$. This implies $t_k - u_k = (s_k - u_k) + \beta_k(f(s_k) - s_k) \leq 0$, so $t_k \leq u_k$, which implies $f(t_k) \leq f(u_k)$. It follows that

$$\begin{aligned} s_{k+1} - u_{k+1} &= (1 - \alpha_k)(s_k - u_k) + \alpha_k(f(t_k) - f(u_k)) \\ &\leq 0, \end{aligned}$$

that is $s_{k+1} \leq u_{k+1}$. By mathematical induction, we obtain $s_n \leq u_n$ for all $n \geq 1$.

(ii) By using Lemma 3.2(ii, iv, vi, viii) and the same argument as in (i), we can show that $x_n \geq w_n \geq s_n \geq u_n$ for all $n \geq 1$. $\square$

Proposition 3.5. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. If $f(x_1) > x_1$, then the sequence $\{x_n\}_{n=1}^\infty$ defined by one of the following iteration methods: $M(x_1, \alpha_n, f)$, $I(x_1, \alpha_n, \beta_n, f)$, $N(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ does not converge to a fixed point of f .

Proof. By Lemma 3.2(ii, iv, vi, viii), we have that $\{x_n\}_{n=1}^\infty$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^\infty$ does not converge to a fixed point of f .

Proposition 3.6. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. If $f(x_1) < x_1$, then the sequence $\{x_n\}_{n=1}^\infty$ defined by one of the following iteration method: $M(x_1, \alpha_n, f)$, $I(x_1, \alpha_n, \beta_n, f)$, $N(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ does not converge to a fixed point of f .

Proof. By Lemma 3.2(i, iii, v, vii), we have that $\{x_n\}_{n=1}^\infty$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^\infty$ does not converge to a fixed point of f . $\square$

Theorem 3.7. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^\infty, \{s_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty, \{x_n\}_{n=1}^\infty$ be the sequences defined by (1.1)–(1.4), respectively. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty$ be sequences in $[0, 1)$. Then the following are satisfied:

- (i) The Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to $p \in F(f)$ if and only if the Mann iteration $\{u_n\}_{n=1}^\infty$ converges to p . Moreover, the Ishikawa iteration converges faster than the Mann iteration.
- (ii) The Noor iteration $\{w_n\}_{n=1}^\infty$ converges to $p \in F(f)$ if and only if the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to p . Moreover, the Noor iteration converges faster than the Ishikawa iteration.
- (iii) The SP-iteration $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$ if and only if the Noor iteration $\{w_n\}_{n=1}^\infty$ converges to p . Moreover, the SP-iteration converges faster than the Noor iteration.

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$.

(i) $(\Rightarrow)$ If the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then set $\beta_n = 0$ for all $n \geq 1$ in (1.2), we can get the convergence of the Mann iteration.

$(\Leftarrow)$ Suppose that the Mann iteration $\{u_n\}_{n=1}^\infty$ converges to $p \in F(f)$. We divide our proof into the following three cases: Case 1: $u_1 = s_1 > U$, Case 2: $u_1 = s_1 < L$, Case 3: $L \leq u_1 = s_1 \leq U$.

Case 1: $u_1 = s_1 > U$. By Proposition 3.5, we get $f(u_1) < u_1$ and $f(s_1) < s_1$. This implies by Lemma 3.4(i) that $s_n \leq u_n$ for all $n \geq 1$. We note that $U < s_1$ and by using (1.2) and mathematical induction, we can show that $U \leq s_n$ for all $n \geq 1$. Then, we have $0 \leq s_n - p \leq u_n - p$, so

$$|s_n - p| \leq |u_n - p| \quad \text{for all } n \geq 1. \quad (3.1)$$

It follows that $s_n \rightarrow p$. That is the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to the same fixed point p . Moreover, by (3.1), we see that the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$.

Case 2: $u_1 = s_1 < L$. By Proposition 3.6, we get $f(u_1) > u_1$ and $f(s_1) > s_1$. This implies by Lemma 3.4(ii) that $s_n \geq u_n$ for all $n \geq 1$. We note that $s_1 < L$ and by using (1.2) and mathematical induction, we can show that $s_n \leq L$ for all $n \geq 1$. Then, we have $|s_n - p| \leq |u_n - p|$ for all $n \geq 1$. It follows that $s_n \rightarrow p$ and the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$.

Case 3: $L \leq u_1 = s_1 \leq U$. Suppose that $f(u_1) \neq u_1$. If $f(u_1) < u_1$, we have by Lemma 3.2(i) that $\{u_n\}_{n=1}^\infty$ is nonincreasing with limit p . By Lemmas 3.3(i) and 3.4(i), we have $p \leq s_n \leq u_n$ for all $n \geq 1$. It follows that $|s_n - p| \leq |u_n - p|$ for all $n \geq 1$. Hence, we have that $s_n \rightarrow p$ and the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$. If $f(u_1) > u_1$, we have by Lemma 3.2(ii) that $\{u_n\}_{n=1}^\infty$ is nondecreasing with limit p . By Lemmas 3.3(ii) and 3.4(ii), we have $p \geq s_n \geq u_n$ for all $n \geq 1$. It follows that $|s_n - p| \leq |u_n - p|$ for all $n \geq 1$. Hence, we have that $s_n \rightarrow p$ and the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges faster than the Mann iteration $\{u_n\}_{n=1}^\infty$.

(ii) $(\Rightarrow)$ If the Noor iteration $\{w_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then set $\gamma_n = 0$ for all $n \geq 1$ in (1.3), we can get the convergence of the Ishikawa iteration.

$(\Leftarrow)$ Suppose that the Ishikawa iteration $\{s_n\}_{n=1}^\infty$ converges to $p \in F(f)$. By using Lemmas 3.2(iii, iv), 3.3 and 3.4, Propositions 3.5, 3.6 and the same proof as in (i), we obtain the desired result.

(iii) $(\Rightarrow)$ If the SP-iteration $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then set $\beta_n = 0$ and $\gamma_n = 0$ for all $n \geq 1$ in (1.4), we can get the convergence of the Mann iteration. Then, the result is obtained directly by (i) and (ii).

$(\Leftarrow)$ Suppose that the Noor iteration $\{w_n\}_{n=1}^\infty$ converges to $p \in F(f)$. By using Lemma 3.2(v, vi), Lemmas 3.3 and 3.4, Propositions 3.5 and 3.6 and the same proof as in (i), we obtain the desired result. $\square$

The speed of convergence for the Mann, Ishikawa, Noor and SP-iterations also depends on the choice of $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ in the interval $[0, 1)$. We present such result only for the SP-iteration. The others are very similar.

Theorem 3.8. Let E be a closed interval on the real line and $f : E \rightarrow E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$, $\{\alpha_n^*\}_{n=1}^\infty$, $\{\beta_n^*\}_{n=1}^\infty$, $\{\gamma_n^*\}_{n=1}^\infty$ be sequences in $[0, 1)$ such that $\alpha_n \leq \alpha_n^*$, $\beta_n \leq \beta_n^*$ and $\gamma_n \leq \gamma_n^*$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ be defined by $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$, respectively. If $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$, then $\{x_n^*\}_{n=1}^\infty$ converges to p . Moreover, $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$, provided that $x_1^* = x_1 \in E$.

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$. Suppose that $\{x_n\}_{n=1}^\infty$ converges to $p \in F(f)$. We divide our proof into the following three cases:

Case 1: $x_1 = x_1^* > U$. By Proposition 3.5, we have $f(x_1) < x_1$. This implies by Lemma 3.2(vii) that $f(x_n) < x_n$ for all $n \geq 1$. It follows by (1.4), we can show that $f(z_n) < z_n$ and $f(y_n) < y_n$ for all $n \geq 1$. Using (1.4), we have

$$\begin{aligned} z_1^* - z_1 &= (x_1^* - x_1) + \gamma_1^*(f(x_1^*) - x_1^*) + \gamma_1(x_1 - f(x_1)) \\ &= (\gamma_1^* - \gamma_1)(f(x_1) - x_1) \\ &\leq 0, \end{aligned}$$

that is $z_1^* \leq z_1$. Since f is nondecreasing, we have $f(z_1^*) \leq f(z_1)$. By $z_1 - f(z_1) > 0$, it follows that

$$\begin{aligned} y_1^* - y_1 &= (z_1^* - z_1) + \beta_1^*(f(z_1^*) - z_1^*) + \beta_1(z_1 - f(z_1)) \\ &\leq (z_1^* - z_1) + \beta_1^*(f(z_1^*) - z_1^*) + \beta_1(z_1 - f(z_1)) \\ &= (1 - \beta_1^*)(z_1^* - z_1) + \beta_1^*(f(z_1^*) - f(z_1)) \\ &\leq 0, \end{aligned}$$

that is $y_1^* \leq y_1$. Since $f(y_1^*) \leq f(y_1)$ and $y_1 - f(y_1) > 0$, we have

$$\begin{aligned} x_2^* - x_2 &= (y_1^* - y_1) + \alpha_1^*(f(y_1^*) - y_1^*) + \alpha_1(y_1 - f(y_1)) \\ &\leq (y_1^* - y_1) + \alpha_1^*(f(y_1^*) - y_1^*) + \alpha_1(y_1 - f(y_1)) \\ &= (1 - \alpha_1^*)(y_1^* - y_1) + \alpha_1^*(f(y_1^*) - f(y_1)) \\ &\leq 0, \end{aligned}$$

that is $x_2^* \leq x_2$. Assume that $x_k^* \leq x_k$. Since $f(x_k^*) \leq f(x_k) < x_k$, we have $z_k^* - z_k \leq (1 - \gamma_k^*)(x_k^* - x_k) + \gamma_k^*(f(x_k^*) - f(x_k)) \leq 0$, that is $z_k^* \leq z_k$. Since $f(z_k^*) \leq f(z_k) < z_k$, we have $y_k^* - y_k \leq (1 - \beta_k^*)(z_k^* - z_k) + \beta_k^*(f(z_k^*) - f(z_k)) \leq 0$, so $y_k^* \leq y_k$, then $f(y_k^*) \leq f(y_k) < y_k$. It follows that

$$\begin{aligned} x_{k+1}^* - x_{k+1} &\leq (1 - \alpha_k^*)(y_k^* - y_k) + \alpha_k^*(f(y_k^*) - f(y_k)) \\ &\leq 0, \end{aligned}$$

that is $x_{k+1}^* \leq x_{k+1}$. By mathematical induction, we obtain $x_n^* \leq x_n$ for all $n \geq 1$. We note that $U < x_1^*$ and by using (1.4) and mathematical induction, we can show that $U \leq x_n^*$ for all $n \geq 1$. Hence, we have $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. It follows that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$.

Case 2: $x_1 = x_1^* < L$. By Proposition 3.6, we get $f(x_1) > x_1$. In the same way as Case 1, we can show that $x_n^* \geq x_n$ for all $n \geq 1$. We note that $x_1^* < L$ and by using (1.4) and mathematical induction, we can show that $x_n^* \leq L$ for all $n \geq 1$. This implies $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. It follows that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$.

Case 3: $L \leq x_1 = x_1^* \leq U$. Suppose that $f(x_1) \neq x_1$. If $f(x_1) < x_1$, we have by Lemma 3.2(vii) that $\{x_n\}_{n=1}^\infty$ is nonincreasing with limit p . By Lemma 3.3(i), we have $p \leq x_n^*$ for all $n \geq 1$. By using the same argument as in Case 1, we can show that $x_n^* \leq x_n$ for all $n \geq 1$, so $p \leq x_n^* \leq x_n$ for all $n \geq 1$. It follows that $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. Hence, we have that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$. If $f(x_1) > x_1$, we have by Lemma 3.2(viii) that $\{x_n\}_{n=1}^\infty$ nondecreasing with limit p . By Lemma 3.3(ii), we have $p \geq x_n^*$ for all $n \geq 1$. By using the same argument as in Case 1, we can show that $x_n^* \geq x_n$ for all $n \geq 1$, so $p \geq x_n^* \geq x_n$ for all $n \geq 1$. It follows that $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. Hence, we obtain that $x_n^* \rightarrow p$ and $\{x_n^*\}_{n=1}^\infty$ converges faster than $\{x_n\}_{n=1}^\infty$.

Table 1

Comparison of rate of convergence of the Mann, Ishikawa, Noor and SP-iterations for the given function in Example 3.10.

n	Mann	Ishikawa	Noor	SP-iteration		
	u_n	s_n	w_n	x_n	$ f(x_n) - x_n $	$ \frac{x_n - p}{x_{n-1} - p} $
10	1.13516510	1.10003535	1.09340527	1.02246989	1.7925E–02	6.7448E–01
⋮	⋮	⋮	⋮	⋮	⋮	⋮
33	1.00006355	1.00004644	1.00004331	1.00000919	7.3542E–06	7.3219E–01
34	1.00004667	1.00003410	1.00003181	1.00000674	5.3932E–06	7.3335E–01
35	1.00003433	1.00002508	1.00002339	1.00000495	3.9611E–06	7.3447E–01

Table 2

Comparison of rate of convergence of the SP-iteration for the given function in Example 3.11.

n	$\alpha_n = \frac{1}{n^{0.2}+1}$	$\alpha_n^* = 0.7$				
	x_n	$ f(x_n) - x_n $	$ \frac{x_n - x_{n-1}}{x_n} $	x_n^*	$ f(x_n^*) - x_n^* $	$ \frac{x_n^* - x_{n-1}^*}{x_n^*} $
10	3.01562105	1.3019E-02	2.6712E-03	3.00031983	2.6652E-04	1.5452E-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	3.00073037	6.0865E-04	1.0716E-04	3.00000027	2.2357E-07	1.2644E-07
19	3.00050902	4.2418E-04	7.3772E-05	3.00000011	9.2676E-08	5.2357E-08
20	3.00035599	2.9666E-04	5.1004E-05	3.00000005	3.8437E-08	2.1695E-08

Remark 3.9. From Theorem 3.8, the control sequences $\{\alpha_n^*\}_{n=1}^\infty$, $\{\beta_n^*\}_{n=1}^\infty$ and $\{\gamma_n^*\}_{n=1}^\infty$ in $[0, 1)$ can be chosen such that $SP(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$ converges to a fixed point of f but those sequences may not satisfy the control condition in Theorem 2.1.

Next, we will present two numerical examples. The first example shows that our iteration converges faster than the Mann, Ishikawa and Noor iterations for continuous and nondecreasing functions on an arbitrary interval and the second example compares the speed of convergence of the SP-iteration with respect to various control conditions. Throughout this section, we use $\alpha_n = \frac{1}{n^{0.2}+1}$, $\beta_n = \frac{1}{n^2+1}$ and $\gamma_n = \frac{1}{n^2+1}$.

Example 3.10. Let $f : [0, 8] \rightarrow [0, 8]$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is a continuous and nondecreasing function. The comparison of the convergences of the Mann, Ishikawa, Noor and SP-iterations to the exact fixed point $p = 1$ are given in Table 1, with the initial point $u_1 = s_1 = w_1 = x_1 = 4$.

From Table 1, we see that the SP-iteration converges faster than the Mann, Ishikawa and Noor iterations. The value $|\frac{x_n - p}{x_{n-1} - p}|$ is called the *order of convergence*. If $|\frac{x_n - p}{x_{n-1} - p}| \rightarrow K$ with $0 < K < 1$, the sequence $\{x_n\}_{n=1}^\infty$ is said to be *linear convergent*. From Table 1, the sequence $\{x_n\}_{n=1}^\infty$ is seen to be linear convergent.

Example 3.11. Let $f : [-6, \infty) \rightarrow [-6, \infty)$ be defined by $f(x) = \sqrt{x+6}$. Then f is a continuous and nondecreasing function. The comparison of the convergence for SP-iterations with new control conditions to the exact fixed point $p = 3$ is given in Table 2, with initial point $x_1 = x_1^* = 9$.

From Table 2, we see that $SP(x_1^*, \alpha_n^*, \beta_n, \gamma_n, f)$ converges faster than $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$. It clear that $\{\alpha_n^*\}_{n=1}^\infty$ does not satisfy the control condition in Theorem 2.1.

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Convergence theorems from monotone hybrid methods for an infinitely countable family of Lipschitz asymptotically quasi-nonexpansive mappings

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ABSTRACT

In this paper, we prove a weak convergence theorem for the modified Mann iteration process for a uniformly Lipschitzian and asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space. We also introduce two new kinds of monotone hybrid methods and obtain strong convergence theorems for an infinitely countable family of uniformly Lipschitzian and asymptotically quasi-nonexpansive mappings in a Hilbert space. The results of this paper improve on and extend corresponding ones announced by many authors.

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1. Introduction

Let C be a nonempty closed and convex subset of a real Banach space X and let $T : C \rightarrow C$. We denote as $F(T)$ the set of fixed points of T , that is, $F(T) = \{x \in C : x = Tx\}$. Then T is said to be:

- (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
- (ii) *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$;
- (iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n \rightarrow \infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$;

- (iv) *asymptotically quasi-nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n \rightarrow \infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\|,$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

- (v) *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$.

From the above definitions, it is clear that:

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- (i) a nonexpansive mapping is an asymptotically quasi-nonexpansive mapping;
- (ii) a quasi-nonexpansive mapping is an asymptotically quasi-nonexpansive mapping;
- (iii) an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping.

However, the converse of each of above statements may be not true.

The map $T : C \rightarrow C$ is said to be *demi-closed* at 0 if for each sequence $\{x_n\}$ in C converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

A Banach space X is said to satisfy *Opial's property* (see [1]) if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x , the following condition holds for all $x \neq y$:

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

In 1953, Mann [2] introduced the following iterative procedure for approximating a fixed point of a nonexpansive mapping T in a Hilbert space H :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \quad (1.1)$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see [3,4].

In 2003, Nakajo and Takahashi [5] proposed the following modification of the Mann iteration for a single nonexpansive mapping T in a Hilbert space H . They proved the following theorem:

Theorem 1.1. *Let C be a closed and convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1)$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C using the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \quad (1.2)$$

Then $\{x_n\}$ defined by (1.2) converges strongly to $P_{F(T)}x_0$.

Recently, Kim and Xu [6] extended the result of Nakajo and Takahashi [5] from a single nonexpansive mapping to a single asymptotically nonexpansive mapping in a Hilbert space. They proved the following theorem:

Theorem 1.2. *Let C be a nonempty bounded, closed and convex subset of a Hilbert space H and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \rightarrow 1$ as $n \rightarrow \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C using the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (1.3)$$

where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\text{diam}C)^2 \rightarrow 0$, as $n \rightarrow \infty$. Then $\{x_n\}$ defined by (1.3) converges strongly to $P_{F(T)}x_0$.

Since 2003, the strong convergence problems of the CQ method for fixed point iteration processes in a Hilbert space or a Banach space have been studied by many authors; see [7,8,6,9–11,5,12–16].

Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of uniformly L_i -Lipschitzian and asymptotically quasi-nonexpansive mappings and let $F := \bigcap_{i=1}^{\infty} F(T_i)$. In this paper, motivated by Kim and Xu [6], Nakajo and Takahashi [5], we introduce two kinds of new algorithms for finding a common fixed point of a countable family of uniformly Lipschitzian and asymptotically quasi-nonexpansive mappings which are modifications of the normal Mann iterative scheme. Our iterative schemes are defined as follows:

Algorithm 1.3. For an initial point $x_0 \in C$, compute the sequence $\{x_n\}$ by means of the iterative process

$$\begin{cases} y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n})T_i^n x_n, \\ C_{i,n} = \{z \in C : \|y_{i,n} - z\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1 - \alpha_{i,n})\|T_i^n x_n - x_n\|^2 + \theta_{i,n}\}, \\ C_n = \bigcap_{i=1}^{\infty} C_{i,n}, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle z - x_n, x_0 - x_n \rangle \leq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_{i,n} = (1 - \alpha_{i,n})(k_{i,n}^2 - 1)\nabla_n^2$, $\nabla_n = \sup_{n \in \mathbb{N}} \{\|x_n - z\| : z \in F\} < \infty$.

Algorithm 1.4. For an initial point $x_0 \in H$, compute the sequence $\{x_n\}$ by means of the iterative process

$$\begin{cases} C_{i,1} = C, & C_1 = \bigcap_{i=1}^{\infty} C_{i,1}, & x_1 = P_{C_1}x_0, \\ y_{i,n} = \alpha_{i,n}x_n + (1 - \alpha_{i,n})T_i^n x_n, \\ C_{i,n+1} = \{z \in C_{i,n} : \|y_{i,n} - z\|^2 \leq \|x_n - z\|^2 - \alpha_{i,n}(1 - \alpha_{i,n})\|T_i^n x_n - x_n\|^2 + \theta_{i,n}\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, & n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0, & n \geq 0, \end{cases}$$

where $\theta_{i,n} = (1 - \alpha_{i,n})(k_{i,n}^2 - 1)\nabla_n^2$, $\nabla_n = \sup_{z \in F} \{\|x_n - z\| : z \in F\} < \infty$.

2. Preliminaries

In this section, we collect and give some useful lemmas which will be used in our main results.

Lemma 2.1 ([17]). Let the sequences $\{a_n\}$ and $\{u_n\}$ of real numbers satisfy:

$$a_{n+1} \leq (1 + u_n)a_n, \quad \forall n \geq 1,$$

where $a_n \geq 0$, $u_n \geq 0$ and $\sum_{n=1}^{\infty} u_n < \infty$. Then:

- (i) $\lim_{n \rightarrow \infty} a_n$ exists;
- (ii) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2 ([18]). Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \omega_p(\lambda)g(\|x - y\|),$$

for all $x, y \in B_r(0) = \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0, 1]$ where $\omega_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 2.3. Let C be a nonempty subset of a Banach space X and $T : C \rightarrow C$ be uniformly L -Lipschitzian. Let $\{x_n\}$ be a sequence in C such that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$. Then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Proof. Observe that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_n - x_{n+1}\| + L\|T^n x_n - x_n\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$, it follows that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. $\square$

The following lemmas give some characterizations and a useful property of the metric projection P_C in a Hilbert space.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H . For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C . We know that P_C is a nonexpansive mapping of H onto C .

Lemma 2.4 ([9]). Let C be a closed and convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C . Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the following holds:

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.5 ([5]). Let C be a nonempty closed and convex subset of a real Hilbert space H and $P_C : H \rightarrow C$ be the metric projection from H onto C . Then the following inequality holds:

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in C.$$

Lemma 2.6 ([9]). Let H be a real Hilbert space. Then the following equations hold:

- (i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$, $\forall x, y \in H$;
- (ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, $\forall t \in [0, 1]$ and $x, y \in H$.

Lemma 2.7 ([19]). Let C be a nonempty closed and convex subset of a real Hilbert space H . Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex and closed.

Lemma 2.8. Let C be a closed and convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian and asymptotically quasi-nonexpansive mapping. Then $F(T)$ is a closed and convex subset of C .

Proof. Since T is continuous, $F(T)$ is closed. Next, we show that $F(T)$ is convex. Let $p = tp_1 + (1 - t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0, 1)$. By Lemma 2.6, we have

$$\begin{aligned} \|p - T^n p\|^2 &= \|t(T^n p - p_1) + (1 - t)(T^n p - p_2)\|^2 \\ &= t\|T^n p - p_1\|^2 + (1 - t)\|T^n p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &\leq tk_n^2\|p - p_1\|^2 + (1 - t)k_n^2\|p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &= t(1 - t)^2k_n^2\|p_1 - p_2\|^2 + (1 - t)t^2k_n^2\|p_1 - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &= ((1 - t)t(k_n^2 - 1))\|p_1 - p_2\|^2. \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \|p - T^n p\| = 0$. Since T is continuous, we have $\lim_{n \rightarrow \infty} \|Tp - T^{n+1}p\| = 0$, so $p = Tp$. Hence $p \in F(T)$. $\square$

3. Main results

First, we prove a weak convergence theorem for a uniformly Lipschitzian and asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space.

Theorem 3.1. Let X be a uniformly convex Banach space which satisfies Opial's property. Let C be a nonempty, closed and convex subset of X , and $T : C \rightarrow C$ a uniformly L -Lipschitzian and asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $\{k_n\} \subset [1, \infty)$ for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} k_n = 1$. Assume that $I - T$ is demi-closed at 0, where I is the identity mapping of C , $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C generated by the modified Mann iteration process:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Proof. Let $p \in F(T)$; we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|T^n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) k_n \|x_n - p\| \\ &\leq (1 + (k_n - 1)(1 - \alpha_n)) \|x_n - p\|. \end{aligned} \tag{3.1}$$

Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, then by Lemma 2.1 and (3.1), we have that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded. It follows that

$$\|T^n x_n - p\| \leq k_n \|x_n - p\|.$$

Hence $\{T^n x_n - p\}$ is bounded. Put $r = \sup_{n \in \mathbb{N}} \{\|T^n x_n - p\|, \|x_n - p\|\}$. By Lemma 2.2, there is a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(T^n x_n - p)\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|T^n x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|T^n x_n - x_n\|) \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)k_n^2 \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|T^n x_n - x_n\|). \end{aligned}$$

It follows that

$$\alpha_n(1 - \alpha_n)g(\|T^n x_n - x_n\|) \leq (k_n^2 - \alpha_n(k_n^2 - 1))\|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

We get that $\lim_{n \rightarrow \infty} g(\|T^n x_n - x_n\|) = 0$. Since g is continuous strictly increasing with $g(0) = 0$, we can conclude that $\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0$. Observe that $\|x_{n+1} - x_n\| \leq \alpha_n \|T^n x_n - x_n\| \rightarrow 0$. Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0$ and T is uniformly L -Lipschitzian, we obtain from Lemma 2.3 that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $q_1 \in C$. From $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $I - T$ being demi-closed at 0, we obtain that $Tq_1 = q_1$. That is $q_1 \in F$. Next, we show that $\{x_n\}$ converges weakly to q_1 .

Take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ converging weakly to some $q_2 \in C$. Again, as above, we can conclude that $q_2 \in F$. Finally, we show that $q_1 = q_2$. Assume $q_1 \neq q_2$. Then by the Opial's property of X , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q_1\| &= \lim_{k \rightarrow \infty} \|x_{n_k} - q_1\| < \lim_{k \rightarrow \infty} \|x_{n_k} - q_2\| = \lim_{n \rightarrow \infty} \|x_n - q_2\| \\ &= \lim_{k \rightarrow \infty} \|x_{m_k} - q_2\| < \lim_{k \rightarrow \infty} \|x_{m_k} - q_1\| = \lim_{n \rightarrow \infty} \|x_n - q_1\|, \end{aligned}$$

which is a contradiction. Therefore $q_1 = q_2$. This shows that $\{x_n\}$ converges weakly to $q_1 \in F(T)$. $\square$

Next, we prove strong convergence theorems for a countable family of uniformly Lipschitzian and asymptotically quasi-nonexpansive mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of uniformly L_i -Lipschitzian and asymptotically quasi-nonexpansive mappings of C into itself with $\{k_{i,n}\} \subset [1, \infty)$ such that $\lim_{n \rightarrow \infty} k_{i,n} = 1$ for all $i \in \mathbb{N}$. Assume that $F := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and the sequence $\{\alpha_{i,n}\} \subset [0, 1]$ is such that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.3 converges strongly to $P_F x_0$.*

Proof. We split the proof into seven steps.

Step 1. Show that $P_F x_0$ is well defined for every $x_0 \in C$.

By Lemma 2.8, we obtain that $F(T_i)$ is a closed convex subset of C for every $i \in \mathbb{N}$. Hence, $F := \bigcap_{i=1}^\infty F(T_i)$ is a nonempty closed convex subset of C ; consequently, $P_F x_0$ is well defined for every $x_0 \in C$.

Step 2. Show that both C_n and Q_n are closed and convex for all $n \geq 0$.

From the definition of C_n and Q_n , it is obvious that Q_n is closed and convex for each $n \in \mathbb{N}$. It follows from Lemma 2.7 that C_n is closed and convex for each $n \in \mathbb{N}$, so $P_{C_n \cap Q_n} x_0$ is well defined.

Step 3. Show that $F \subset C_n \cap Q_n$, for all $n \geq 0$.

First, we prove that $F \subset C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. Since T_i is an asymptotically quasi-nonexpansive mapping for all $i \in \mathbb{N}$, we obtain for any $p \in F$ that

$$\begin{aligned} \|y_{i,n} - p\|^2 &= \|\alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n - p\|^2 \\ &= \alpha_{i,n} \|x_n - p\|^2 + (1 - \alpha_{i,n}) \|T_i^n x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\ &\leq \alpha_{i,n} \|x_n - p\|^2 + (1 - \alpha_{i,n}) k_{i,n}^2 \|x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + (1 - \alpha_{i,n}) (k_{i,n}^2 - 1) \|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + \theta_{i,n}. \end{aligned} \quad (3.2)$$

We have $p \in C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. This proves that $F \subset C_{i,n}$ for all $n \geq 0$ and $i \in \mathbb{N}$. Hence, $F \subset C_n$ for all $n \geq 0$.

As shown in Marino and Xu [9], by induction, we can show that $F \subset Q_n$ for all $n \geq 0$. Hence $F \subset C_n \cap Q_n$, for all $n \geq 0$.

Step 4. Show that $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

From $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0. \quad (3.3)$$

On the other hand, as $F \subset Q_n$, we obtain

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad \forall n \geq 0, \forall z \in F. \quad (3.4)$$

Combining (3.3) and (3.4), the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

Step 5. Show that $x_n \rightarrow q$, where $q \in C$.

For $m > n$, by the definition of Q_n , we see that $Q_m \subset Q_n$. Noting that $x_m = P_{Q_m} x_0$ and $x_n = P_{Q_n} x_0$, by Lemma 2.5, we conclude that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

From Step 4, we obtain that $x_m - x_n \rightarrow 0$ as $m, n \rightarrow \infty$, which means that $\{x_n\}$ is Cauchy. Since H is complete and C is closed, we can assume that $x_n \rightarrow q \in C$ as $n \rightarrow \infty$. In particular, we have that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Step 6. Show that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$.

Let $i, n \in \mathbb{N}$. Since $x_{n+1} \in C_n$, it follows that

$$\|y_{i,n} - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2 + \theta_{i,n}. \quad (3.5)$$

Moreover, since $y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n$ and by Lemma 2.6, we have

$$\begin{aligned} \|y_{i,n} - x_{n+1}\|^2 &= \|\alpha_{i,n} x_n + (1 - \alpha_{i,n}) T_i^n x_n - x_{n+1}\|^2 \\ &= \alpha_{i,n} \|x_n - x_{n+1}\|^2 + (1 - \alpha_{i,n}) \|T_i^n x_n - x_{n+1}\|^2 - \alpha_{i,n} (1 - \alpha_{i,n}) \|T_i^n x_n - x_n\|^2. \end{aligned} \quad (3.6)$$

Substitute (3.6) into (3.5) to get

$$(1 - \alpha_{i,n}) \|T_i^n x_n - x_{n+1}\|^2 \leq (1 - \alpha_{i,n}) \|x_n - x_{n+1}\|^2 + \theta_{i,n}. \quad (3.7)$$

From (3.7) and our assumptions, we have

$$\lim_{n \rightarrow \infty} \|T_i^n x_n - x_{n+1}\| = 0.$$

It follows that

$$\|T_i^n x_n - x_n\| \leq \|T_i^n x_n + x_{n+1}\| + \|x_{n+1} - x_n\| \rightarrow 0$$

for all $i \in \mathbb{N}$. From Lemma 2.3, we get that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$.

Step 7. Show that $q = P_F x_0$.

Since $x_n \rightarrow q$ and $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$, we have $q = T_i q$, $\forall i \in \mathbb{N}$. Hence $q \in F$. By Lemma 2.4, we obtain

$$\langle z - x_n, x_0 - x_n \rangle \leq 0$$

for all $z \in Q_n$. This leads to

$$\langle z - q, x_0 - q \rangle \leq 0$$

for all $z \in F$. By Lemma 2.4, we conclude that $q = P_F x_0$. This completes the proof. $\square$

Theorem 3.3. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of uniformly L_i -Lipschitzian and asymptotically quasi-nonexpansive mappings of C into itself with $\{k_{i,n}\} \subset [1, \infty)$ such that $\lim_{n \rightarrow \infty} k_{i,n} = 1$ for all $i \in \mathbb{N}$. Assume that $F := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and the sequence $\{\alpha_{i,n}\} \subset [0, 1]$ is such that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.4 converges strongly to $P_F x_0$.

Proof. We divide our proof into four steps.

Step 1. Show that C_n is closed and convex and $F \subset C_n$ for all $n \geq 1$.

It suffices to show that $\forall i \in \mathbb{N}$, $C_{i,n}$ is closed and convex and $F \subset C_{i,n}$ for every $n \geq 1$. We prove this by induction on n . In fact, for $n = 1$, $C_{i,1} = C$ is closed and convex. Assume that $C_{i,n}$ is closed and convex for some $n \geq 1$; from the definition $C_{i,n+1}$, we obtain that $C_{i,n+1}$ is also closed and convex for some $n \geq 1$, and hence $C_{i,n}$ is closed and convex for all $n \geq 1$. For $n = 1$, $F \subset C_{i,1} = C$. Assume that $F \subset C_{i,n}$ for some $n \geq 1$; it follows from the induction assumption, (3.2) and the definition of $C_{i,n+1}$ that we have that $F \subset C_{i,n+1}$; hence $F \subset C_{i,n}$.

Step 2. Show that $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

From $x_n = P_{C_n} x_0$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \geq 0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0. \quad (3.8)$$

On the other hand, as $F \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad \forall n \geq 0, \forall z \in F. \quad (3.9)$$

Combining (3.8) and (3.9), the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

Step 3. Show that $x_n \rightarrow q$, where $q \in C$.

For $m > n$, by the definition of C_n , we see that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By Lemma 2.5, we obtain that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence $x_n \rightarrow q \in C$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. By using the same proof as in Step 6 of Theorem 3.2, we can show that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$.

Step 4. Show that $q = P_F x_0$.

Since $x_n \rightarrow q$ and $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, $\forall i \in \mathbb{N}$, we have $q = T_i q$ $\forall i \in \mathbb{N}$. Hence $q \in F \subset C_n$ and since $x_n = P_{C_n} x_0$, by Lemma 2.4, we have

$$\langle z - x_n, x_0 - x_n \rangle \leq 0,$$

for all $z \in C_n$, and hence,

$$\langle z - q, x_0 - q \rangle \leq 0,$$

for all $z \in F$. This shows that $q = P_F x_0$, which completes the proof. $\square$

Remark 3.4. Theorems 3.2 and 3.3 extend and improve the main result of [6,5].

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Strong convergence of composite iterative schemes for a countable family of nonexpansive mappings in Banach spaces

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ABSTRACT

In this paper we propose a new modified viscosity approximation method for approximating common fixed points for a countable family of nonexpansive mappings in a Banach space. We prove strong convergence theorems for a countable family nonexpansive mappings in a reflexive Banach space with uniformly Gateaux differentiable norm under some control conditions. These results improve and extend the results of Jong Soo Jung [J.S. Jung, Convergence on composite iterative schemes for nonexpansive mappings in Banach spaces, Fixed Point Theory and Appl. 2008 (2008) 14 pp., Article ID 167535]. Further, we apply our result to the problem of finding a zero of an accretive operator and extend the results of Kim and Xu [T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 51–60], Ceng, et al. [L.-C. Ceng, A.R. Khan, Q.H. Ansari, J.-C. Yao, Strong convergence of composite iterative schemes for zeros of m -accretive operators in Banach space, Nonlinear Anal. 70 (2009) 1830–1840] and Chen and Zhu [R. Chen, Z. Zhu, Viscosity approximation methods for accretive operator in Banach space, Nonlinear Anal. 69 (2008) 1356–1363].

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E . Recall that a mapping $T : C \rightarrow C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C | x = Tx\}$, and a self-mapping $f : C \rightarrow C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C , that is $\Pi_C = \{f : C \rightarrow C | f \text{ is a contraction with a constant } \alpha\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C , and, for any fixed element $x_0 \in C$, Picard's iteration $x_{n+1} = f^n(x_0)$ converges strongly to a unique fixed point of f . However, a simple example shows that Picard's iteration cannot be used in the case of nonexpansive mappings. One method used for nonexpansive mappings is to employ a Halpern-type iterative scheme which produces a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \end{aligned} \tag{1.1}$$

where $u \in C$ is arbitrary (but fixed) and $\{\alpha_n\} \subset [0, 1]$.

In 1967, Halpern [1] was the first to study the convergence of the iterative scheme (1.1) in a framework of Hilbert space. Lions [2] and Wittmann [3] improved the result of Halpern. Reich [4] and Shioji and Takahashi [5] extended the result of

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Wittmann [3] to the case of Banach space. They proved the strong convergence of the sequence $\{x_n\}$ defined by (1.1) under certain control conditions.

In 2000, Moudafi [6] introduced a viscosity approximation method for a nonexpansive mapping as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \end{aligned} \quad (1.2)$$

where $f \in \Pi_C$ and $\{\alpha_n\} \subset [0, 1]$. He proved that if E is a real Hilbert space and $\{\alpha_n\}$ satisfies certain control conditions then the sequence $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T which is the unique solution to the variational inequality $\langle (I - f)\bar{x}, x - \bar{x} \rangle \geq 0$ for all $x \in F(T)$. In 2004, Xu [7] extended the results of Moudafi [6] for the iterative scheme (1.2) to a Banach space setting.

In 2005, Kim and Xu [8] studied the sequence $\{x_n\}$ generated as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n u + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \quad (1.3)$$

where $u \in C$ is arbitrary (but fixed) and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. They proved the strong convergence of iterative scheme (1.3) in the framework of a uniformly smooth Banach space. In 2008, Yao et al. [9] proposed a new modified iteration by combining the Mann iteration and the viscosity approximation method introduced by Moudafi [6]. This iterative scheme is defined as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \quad (1.4)$$

where $f \in \Pi_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. They proved that under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ the iterative scheme (1.4) strongly converges to a fixed point of T .

Recently, Jung [10] introduced a new composite iterative scheme for a nonexpansive mapping T as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n Ty_n, \quad n \geq 0 \end{aligned} \quad (1.5)$$

where $f \in \Pi_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. He proved the strong convergence of the sequence $\{x_n\}$ defined by (1.5) under suitable conditions of the control parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a uniformly Gateaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

On the other hand, Aoyama et al. [11] introduced a Halpern-type iterative scheme for finding a common fixed point of a countable family of nonexpansive mappings as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)T_n x_n, \quad n \geq 0 \end{aligned} \quad (1.6)$$

where $u \in C$ is arbitrary (but fixed), $\{\alpha_n\} \subset [0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved that the sequence $\{x_n\}$ defined by (1.6) converges strongly to a common fixed point of $\{T_n\}$.

Recently, Takahashi [12] studied the iterative scheme (1.2) for finite countable families of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n, \quad n \geq 0 \end{aligned} \quad (1.7)$$

where $f \in \Pi_C$, $\{\alpha_n\} \subset [0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. He proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of $\{T_n\}$.

Recently, Plubtieng and Wangkeeree [13] modified the iterative scheme (1.4) for a countable families of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)T_n x_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad n \geq 0 \end{aligned} \quad (1.8)$$

where $f \in \Pi_C$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved, under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that the sequence $\{x_n\}$ defined by (1.8) converges strongly to a common fixed point of $\{T_n\}$.

Inspired and motivated by these facts, a new iterative scheme for countable family nonexpansive mappings is introduced and studied in this paper. This iterative scheme can be viewed as an extension of the iterative scheme (1.5). We introduce the following iterative scheme:

$$\begin{aligned} x_0 &= x \in C, \quad \text{arbitrarily;} \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0 \end{aligned} \quad (1.9)$$

where C is a nonempty closed subset of a Banach space, $f \in \Pi_C$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. If $T_n = T$ for all $n \geq 0$, then (1.9) reduces to (1.5) and if $T_n = T$ for all $n \geq 0$ and $\beta_n = 0$, then (1.9) reduces to (1.2); finally, if $\beta_n = 0$, then (1.9) reduces to (1.7). The main propose of this paper is to prove the strong convergence of the sequence $\{x_n\}$ defined by (1.9) to a common fixed point of $\{T_n\}$ under certain suitable conditions of the control parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and the asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a uniformly Gateaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Further, we apply our result to the problem of finding a zero of an accretive operator.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E . The value of $f \in E^*$ at $x \in E$ is denoted by $\langle x, f \rangle$. Recall that the duality map J from E into 2^{E^*} is defined by $J(x) = \{f \in E^* | \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$ for every $x \in E$. Let $S = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gateaux differentiable if, for each $x, y \in S$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists. In this case, the Banach space E is said to be smooth. The norm of E is said to be uniformly Gateaux differentiable if, for each $y \in S$, the limit (2.1) is attained uniformly for $x \in S$. We know that, if E is smooth, then the duality mapping J is single valued. The norm of E is said to be uniformly Frechet differentiable if the limit (2.1) is attained uniformly for $x, y \in S$, and in this case E said to be uniformly smooth. It is known that E is uniformly smooth if and only if the duality mapping J is single valued and norm-to-norm uniformly continuous on each bounded subset of E . It also known that, if E has a uniformly Gateaux differentiable norm, J is uniformly norm to weak continuous on each bounded subsets of E . We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and write $x_n \rightarrow x$ to indicate that $\{x_n\}$ converges strongly to x .

Recall that, if C is nonempty closed convex subset of E , then C is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D . We need the following lemmas for the proof of our main results.

Lemma 2.1 ([7]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, $n \geq 0$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$, (ii) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2. Let J be the duality map of a Banach space E . Suppose that X is smooth. Then, for all $x, y \in X$, there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle.$$

Lemma 2.3 ([11]). Let C be a nonempty closed convex subset of E . Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C . Moreover, let T be a mapping of C into itself defined by $Tz = \lim_n T_n z$ for all $z \in C$. Then $\lim_{n \rightarrow \infty} \sup\{\|Tz - T_nz\| : z \in C\} = 0$.

Lemma 2.4 ([14]). Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C a nonempty closed convex subset of E , $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then the unique fixed point $x_t \in C$ of the contraction $x \mapsto tf(x) + (1 - t)Tx$ converges strongly to a fixed point of T as $t \rightarrow 0^+$. If we define $Q : \Pi_C \rightarrow F(T)$ by $Q(f) = \lim_{t \rightarrow 0^+} x_t$, then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, \quad p \in F(T).$$

Finally, a sequence $\{x_n\}$ in E is said to be asymptotically regular if $\lim_n \|x_{n+1} - x_n\| = 0$.

3. Main results

In this section, we prove several strong convergence theorems of the composite iterative scheme (1.9) to a common fixed point of a countable family of nonexpansive mappings.

Theorem 3.1. Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E , and $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ which satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0. \end{aligned}$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C . Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, take an arbitrary fixed $p \in \bigcap_{n=1}^{\infty} F(T_n)$. By the definition of $\{y_n\}$ and $\{x_n\}$, we have

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1 - \alpha_n) T_n x_n - p\| \\ &= \|\alpha_n (f(x_n) - f(p)) + \alpha_n (f(p) - p) + (1 - \alpha_n) (T_n x_n - p)\| \\ &\leq \alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &= (1 - (1 - \alpha) \alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned} \quad (3.1)$$

We obtain that $\|y_n - p\| \leq \max\{\|x_n - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\}$.

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \beta_n) y_n + \beta_n T_n y_n - p\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|T_n y_n - p\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|y_n - p\| \\ &= \|y_n - p\| \leq \max\left\{\|x_n - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\right\}. \end{aligned}$$

By induction on n , we obtain that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\}$ for all $n \geq 0$ and all $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence, the sequence $\{x_n\}$ is bounded and so $\{y_n\}$, $\{T_n x_n\}$ and $\{f(x_n)\}$ are bounded sequences. Moreover, it follows from condition (i) that

$$\|y_n - T_n x_n\| = \alpha_n \|f(x_n) - T_n x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

Observing that

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|(1 - \beta_n) y_n + \beta_n T_n y_n - y_n\| \\ &= \beta_n \|T_n y_n - y_n\| \\ &\leq \beta_n (\|T_n y_n - T_n x_n\| + \|T_n x_n - y_n\|) \\ &\leq b (\|y_n - x_n\| + \|T_n x_n - y_n\|) \\ &\leq b (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|T_n x_n - y_n\|), \end{aligned}$$

we obtain

$$\|x_{n+1} - y_n\| \leq \frac{b}{1-b} (\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|). \quad (3.3)$$

So, by (3.2) and the asymptotic regularity of $\{x_n\}$, we have $\|x_{n+1} - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.4)$$

Next, we show that $\|T x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

By the asymptotic regularity of $\{x_n\}$ and Lemma 2.3, we have

$$\lim_n \|x_{n+1} - x_n\| = 0 = \lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in \{x_k\}\}. \quad (3.5)$$

By (3.3), we have

$$\begin{aligned}\|T_n x_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \frac{b}{1-b}(\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|) + \|y_n - T_n x_n\| \\ &\leq \frac{1}{1-b}(\|x_{n+1} - x_n\| + \|T_n x_n - y_n\|).\end{aligned}$$

It follows by (3.2) that

$$\|T_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

Therefore, by (3.5) and (3.6), we have

$$\begin{aligned}\|T x_n - x_n\| &\leq \|T x_n - T_n x_n\| + \|T_n x_n - x_n\| \\ &\leq \sup\{\|T z - T_n z\| : z \in \{x_k\}\} + \|T_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Now, observe that

$$\begin{aligned}\|y_n - T_n y_n\| &\leq \|y_n - x_n\| + \|x_n - T_n x_n\| + \|T_n x_n - T_n y_n\| \\ &\leq \|y_n - x_n\| + \|x_n - T_n x_n\| + \|x_n - y_n\| \\ &= 2\|x_n - y_n\| + \|x_n - T_n x_n\|.\end{aligned} \quad (3.7)$$

It follows by (3.4) and (3.6) that $\|y_n - T_n y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we have

$$\begin{aligned}\|T y_n - y_n\| &\leq \|T y_n - T_n y_n\| + \|T_n y_n - y_n\| \\ &\leq \sup\{\|T z - T_n z\| : z \in \{y_k\}\} + \|T_n x_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned} \quad (3.8)$$

Next, we claim that $\limsup_{n \rightarrow \infty} \langle f(q) - q, J(y_n - q) \rangle \leq 0$, where $q = Q(f)$ is defined as in Lemma 2.4. To prove this, let a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ be such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(y_n - q) \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, J(y_{n_k} - q) \rangle \quad (3.9)$$

and $y_{n_k} \rightharpoonup p$ for some $p \in C$. By the demiclosed principle and the fact that $\|T y_{n_k} - y_{n_k}\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain $p \in F(T)$. By Lemma 2.4 we have $x_t \rightarrow q = Q(f) \in F(T)$, where x_t is the fixed point of the contraction $x \mapsto t f(x) + (1-t)Tx$. By Lemma 2.2, we obtain

$$\begin{aligned}\|x_t - y_{n_k}\|^2 &= \|(1-t)(T x_t - y_{n_k}) + t(f(x_t) - y_{n_k})\|^2 \\ &\leq (1-t)^2 \|T x_t - y_{n_k}\|^2 + 2t \langle f(x_t) - y_{n_k}, J(x_t - y_{n_k}) \rangle \\ &\leq (1-t)^2 (\|T x_t - T y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|)^2 + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + 2t \langle x_t - y_{n_k}, J(x_t - y_{n_k}) \rangle \\ &\leq (1-t)^2 (\|x_t - y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|)^2 + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + 2t \|x_t - y_{n_k}\|^2 \\ &\leq (1-t)^2 \|x_t - y_{n_k}\|^2 + \|T y_{n_k} - y_{n_k}\| (2\|x_t - y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|) \\ &\quad + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + 2t \|x_t - y_{n_k}\|^2 \\ &= (1+t^2) \|x_t - y_{n_k}\|^2 + 2t \langle f(x_t) - x_t, J(x_t - y_{n_k}) \rangle + a_k(t)\end{aligned} \quad (3.10)$$

where $a_k(t) = \|T y_{n_k} - y_{n_k}\| (2\|x_t - y_{n_k}\| + \|T y_{n_k} - y_{n_k}\|)$. It follows by (3.8) that $a_k(t) \rightarrow 0$ as $k \rightarrow \infty$. By (3.10), we obtain

$$\langle f(x_t) - x_t, J(y_{n_k} - x_t) \rangle \leq \frac{t}{2} \|x_t - y_{n_k}\|^2 + \frac{1}{2t} a_k(t). \quad (3.11)$$

Letting $k \rightarrow \infty$ in (3.11), we obtain

$$\limsup_{k \rightarrow \infty} \langle f(x_t) - x_t, J(x_{n_k} - x_t) \rangle \leq \frac{t}{2} M, \quad (3.12)$$

where $M > 0$ is a constant such that $M \geq \|x_t - y_{n_k}\|^2$ for all $t \in (0, 1)$ and $k \geq 0$. Taking the limsup as $t \rightarrow 0$ in (3.12), we immediately get

$$\limsup_{t \rightarrow 0} \limsup_{k \rightarrow \infty} \langle f(x_t) - x_t, J(y_{n_k} - x_t) \rangle \leq 0.$$

Since E is a reflexive Banach space with uniformly Gateaux differentiable norm, then the duality map is norm-to-weak uniformly continuous on bounded subsets of C . By the same argument as in the proof of [10, Theorem 3.2], we obtain

$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(y_n - q) \rangle \leq 0$. Finally, we claim that $x_n \rightarrow q$ as $n \rightarrow \infty$. Indeed, using Lemma 2.2 and (3.1) again, we obtain that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 = \|\alpha_n(f(x_n) - q) + (1 - \alpha_n)(T_n x_n - q)\|^2 \\ &\leq (1 - \alpha_n)^2 \|T_n x_n - q\|^2 + 2\alpha_n \langle f(x_n) - q, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q), J(y_n - q) \rangle + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \|x_n - q\| \|y_n - q\| + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &\quad + 2\alpha_n \|x_n - q\| ((1 - (1 - \alpha)\alpha_n) \|x_n - q\| + \alpha_n \|f(q) - q\|) \\ &= (1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n(1 - (1 - \alpha)\alpha_n)) \|x_n - q\|^2 \\ &\quad + 2\alpha_n^2 \|x_n - q\| \|f(q) - q\| + 2\alpha_n \langle f(q) - q, J(y_n - q) \rangle \\ &= (1 - \gamma_n) \|x_n - q\|^2 + \delta_n, \end{aligned}$$

where $\gamma_n = \alpha_n(2 - \alpha_n - 2\alpha(1 - (1 - \alpha)\alpha_n))$ and $\delta_n = \alpha_n(2\alpha\alpha_n \|x_n - q\| \|f(q) - q\| + 2\langle f(q) - q, J(y_n - q) \rangle)$. It is easily seen that $\lim_{n \rightarrow \infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$. By Lemma 2.1, we conclude that $x_n \rightarrow q$. $\square$

Theorem 3.2. Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E . Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ which satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0. \end{aligned}$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C . Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. Since E is a uniformly smooth Banach space, E is reflexive and the norm is uniformly Gateaux differentiable norm and its every nonempty weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Thus, the conclusion of the theorem follows from Theorem 3.1 immediately. $\square$

Theorem 3.3. Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E , and $\{T_n\}$ a sequence nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C . Let T be a mapping of C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0. \end{aligned}$$

If $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ which satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$,
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Proof. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ which satisfy conditions (i)–(iii). We shall show that $\{x_n\}$ is asymptotic regular. By the definition of $\{y_n\}$ and $\{x_n\}$, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|(\alpha_n f(x_n) + (1 - \alpha_n) T_n x_n) - (\alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1}) T_{n-1} x_{n-1})\| \\ &\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + (1 - \alpha_n) \|T_n x_n - T_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - T_{n-1} x_{n-1}\| \\ &\leq \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n) \|T_n x_n - T_{n-1} x_{n-1}\| \\ &\quad + (1 - \alpha_n) \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - T_{n-1} x_{n-1}\| \\ &= (1 - (1 - \alpha)\alpha_n) \|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - T_{n-1} x_{n-1}\|, \end{aligned}$$

and

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|((1 - \beta_n)y_n + \beta_n T_n y_n) - ((1 - \beta_{n-1})y_{n-1} + \beta_{n-1} T_{n-1} y_{n-1})\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|T_n y_n - T_{n-1} y_{n-1}\| + |\beta_n - \beta_{n-1}|\|T_{n-1} y_{n-1} - y_{n-1}\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|T_n y_n - T_n y_{n-1}\| + \beta_n\|T_n y_{n-1} - T_{n-1} y_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}|\|T_{n-1} y_{n-1} - y_{n-1}\| \\
 &\leq \|y_n - y_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\| + |\beta_n - \beta_{n-1}|\|T_{n-1} y_{n-1} - y_{n-1}\| \\
 &\leq (1 - (1 - \alpha)\alpha_n)\|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\
 &\quad + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\|
 \end{aligned}$$

where $M = \max\{\sup_n \|f(x_{n-1}) - T_{n-1} x_{n-1}\|, \sup_n \|T_{n-1} y_{n-1} - y_{n-1}\|\}$.

Put

$$\mu_n = M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| + \|T_n y_{n-1} - T_{n-1} y_{n-1}\|, \quad n \geq 2.$$

We have

$$\begin{aligned}
 \sum_{n=2}^{\infty} \mu_n &\leq M \sum_{n=2}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) + \sum_{n=2}^{\infty} \sup\{\|T_n z - T_{n-1} z\| : z \in \{x_k\}\} \\
 &\quad + \sum_{n=2}^{\infty} \sup\{\|T_n z - T_{n-1} z\| : z \in \{y_k\}\} < \infty.
 \end{aligned}$$

It follows by Lemma 2.1 that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$; hence $\{x_n\}$ is asymptotic regular. By Theorem 3.1, we obtain that $\{x_n\}$ strongly converges to a common fixed point of $\{T_n\}$. $\square$

If $T_n = T$ for all $n \in \mathbb{N}$, then the following result is obtained directly from Theorem 3.1, and we get the result of Jung [10].

Corollary 3.4 ([10], Theorem 3.2). *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E , and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ which satisfy the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (ii) $\beta_n \in [0, b)$ for some $b \in (0, 1)$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
 x_0 &= x \in C, \\
 y_n &= \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 0 \\
 x_{n+1} &= (1 - \beta_n) y_n + \beta_n T y_n, \quad n \geq 0.
 \end{aligned}$$

If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to a fixed point of T .

4. Applications

In this section, we apply Theorem 3.1 for finding a zero of an accretive operator. Let E be a real Banach space. Recall that an operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ in E is called accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ ($i = 1, 2$), there exists a $j \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j \rangle \geq 0$, where $J : E \rightarrow 2^{E^*}$ is the duality mapping. An accretive operator A is m -accretive if $R(I + rA) = E$ for each $r \geq 0$. Throughout this section we always assume that A is m -accretive and has a zero. The set of zeros of A is denoted by $A^{-1}0$, that is $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$. For each $r \geq 0$, we denote by J_r the resolvent of A , that is $J_r = (I + rA)^{-1}$. Note that, if A is m -accretive, then $J_r : E \rightarrow E$ is a nonexpansive mapping and $F(J_r) = A^{-1}0$ for all $r \geq 0$. It is well known that, for each $r, s \geq 0$ and $x \in R(I + rA) \cap R(I + sA)$, it holds that

$$\|J_r x - J_s x\| \leq \frac{|r - s|}{r} \|x - J_r x\|. \quad (4.1)$$

From the resolvent identity, we have the following lemma:

Lemma 4.1 ([11]). *Let E be a Banach space and C a nonempty closed convex subset of E . Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$. Suppose that $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\inf\{r_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$. Then*

- (i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| : z \in B\} < \infty$ for any bounded subset B of C .
- (ii) $\lim_{n \rightarrow \infty} J_{r_n} z = J_r z$ for all $z \in C$ and $F(J_r) = \bigcap_{n=1}^{\infty} F(J_{r_n})$, where $r_n \rightarrow r$ as $n \rightarrow \infty$.

From Lemma 4.1 and Theorem 3.3, we obtain the following results.

Theorem 4.2. Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E , and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Given sequences $\{\alpha_n\}$ in $(0, 1)$, $\{\beta_n\}$ in $[0, 1]$ and $\{r_n\}$ in $(0, \infty)$, suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\beta_n \in [0, b)$ for some $b \in (0, 1)$,
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $f \in \Pi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n J_{r_n} y_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

If $f(x) = u$ for all $x \in C$ in Theorem 4.2, we obtain the result of Ceng et al. [15].

Theorem 4.3 ([15], Theorem 2.1). Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Given sequences $\{\alpha_n\}$ in $(0, 1)$, $\{\beta_n\}$ in $[0, 1]$ and $\{r_n\}$ in $(0, \infty)$, suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\beta_n \in [0, b)$ for some $b \in (0, 1)$,
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n J_{r_n} y_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

If $\beta_n = 0$ in Theorem 4.2, we obtain the result of Chen and Zhu [16].

Theorem 4.4 ([16], Theorem 3.2). Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ and $\{r_n\}$ in $(0, \infty)$ which satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $f \in \Pi_C$ and x_0 be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

If $\beta_n = 0$ and $f(x) = u$ for all $x \in C$ in Theorem 4.2, we obtain the result of Kim and Xu [8].

Theorem 4.5 ([8], Theorem 2). Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and let A be an m -accretive operator in E such that $A^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ and $\{r_n\}$ in $(0, \infty)$ which satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
- (ii) $r_n \geq \epsilon$ for all n and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a zero of A .

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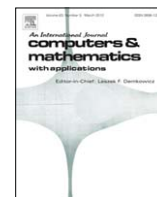
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Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces

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ABSTRACT

We investigate strong convergence for Bregman strongly nonexpansive mappings by Halpern's iteration in the framework of reflexive Banach spaces. Using the obtained result, convergence for a family of Bregman strongly nonexpansive mappings is also discussed. As applications, we apply our main result to problems of finding zeros of maximal monotone operators and equilibrium problems in reflexive Banach spaces.

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1. Introduction

Let E be a real reflexive Banach space and C a nonempty, closed and convex subset of E . Let $T : C \rightarrow C$ be a nonlinear mapping. The fixed points set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. A mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In recent years, several types of iterative schemes have been constructed and proposed in order to get strong convergence results for nonexpansive mappings in various settings. One classical and effective iteration process is defined as follows: $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (1.1)$$

where $\{\alpha_n\} \subset (0, 1)$. Such a method was introduced, in 1967, by Halpern [1] and is often called *Halpern's iteration*. In fact, he proved, in a real Hilbert space, strong convergence of $\{x_n\}$ to a fixed point of a nonexpansive mapping T , where $\alpha_n = n^{-a}$, $a \in (0, 1)$.

Now, because of a simple construction, Halpern's iteration is widely used to approximate a solution of fixed points for nonexpansive mappings and other classes of nonlinear mappings by mathematicians in different styles.

In 1977, Lions [2] obtained a strong convergence provided the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1: \lim_{n \rightarrow \infty} \alpha_n = 0; \quad C2: \sum_{n=1}^{\infty} \alpha_n = \infty; \quad C3: \lim_{n \rightarrow \infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}^2} = 0.$$

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Reich [3] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lions' conditions imposed on the real sequence $\{\alpha_n\}$ exclude the canonical choice $\alpha_n = 1/n$ for all $n \in \mathbb{N}$.

Subsequently, Wittmann [4] overcome the problem mentioned above by proving strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the conditions C1, C2 and C4:

$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

In 1997, Shioji–Takahashi [5] extended Wittmann's result from Hilbert spaces to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings.

In 2002, Xu [6] introduced another control condition C5: $\frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}} \rightarrow 0$ instead of the conditions C3 or C4 and proved strong convergence of the sequence $\{x_n\}$.

In 2005, Cho–Kang–Zhou [7] pointed out that the control conditions C4 and C5 are not comparable, in general. They gave some examples which satisfy the conditions C1, C2, C3, C4 and C5, and also presented the control condition C6:

$$|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$. This includes the conditions C3, C4 and C5 as special cases.

One question arises in literature naturally: Is it possible to get strong convergence of (1.1) when the sequence $\{\alpha_n\}$ satisfies only the conditions C1 and C2?

Recently, Chidume–Chidume [8] and Suzuki [9] independently gave an affirmative answer to the above question. To be more precise, they introduced the following Halpern-type iteration: $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n), \quad \forall n \geq 1, \quad (1.2)$$

and obtained strong convergence results for the sequence $\{x_n\}$ generated by (1.2) when only the conditions C1 and C2 are imposed on the sequence $\{\alpha_n\}$.

Very recently, Saejung [10] focused in studying Halpern's iteration for an important subclass of nonexpansive mappings which is the so-called *strongly nonexpansive* [11], i.e., a mapping $T : C \rightarrow C$ satisfying T is nonexpansive and

$$x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0.$$

He proved that the sequence generated by Halpern's iteration converges strongly to a fixed point of T if the sequence $\{\alpha_n\} \subset (0, 1)$ just satisfies the conditions C1 and C2. This shows that a class of strongly nonexpansive mappings works for Halpern's iteration with the conditions C1 and C2.

The purpose of this work is to consider strong convergence results for Bregman strongly nonexpansive mappings in reflexive Banach spaces by Halpern's iteration with only the conditions C1 and C2. We note that there are many examples which are Bregman strongly nonexpansive such as the Bregman projection, the resolvents of maximal monotone operators, the resolvents of equilibrium problems, the resolvents of variational inequality problems and others (see, for example, [12–15]). Furthermore, using the main result, some convergence theorems for a family of Bregman strongly nonexpansive are also established. Finally, we give some applications concerning the problems of finding zeros of maximal monotone operators and equilibrium problems.

2. Preliminaries and lemmas

In this section, we begin by recalling some preliminaries and lemmas which will be used in the proof.

Let E be a real reflexive Banach space with the norm $\|\cdot\|$ and E^* the dual space of E . Throughout this paper, $f : E \rightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\text{dom } f$ the domain of f , that is, the set $\{x \in E : f(x) < +\infty\}$.

Let $x \in \text{int dom } f$. The *subdifferential* of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y) \quad \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \rightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup \{\langle x^*, x \rangle - f(x) : x \in E\}.$$

We know that the Young–Fenchel inequality holds:

$$\langle x^*, x \rangle \leq f(x) + f^*(x^*) \quad \forall x \in E, x^* \in E^*.$$

Furthermore, equality holds if $x^* \in \partial f(x)$ (see also [16, Theorem 23.5]).

The set $\text{lev}_{\leq}^f(r) = \{x \in E : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a *sublevel* of f .

A function f on E is *coercive* [17] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty.$$

A function f on E is said to be *strongly coercive* [18] if

$$\lim_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \text{int dom } f$ and $y \in E$, the *right-hand derivative* of f at x in the direction y is defined by

$$f^o(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x if $\lim_{t \rightarrow 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y . In this case, $f^o(x, y)$ coincides with $\nabla f(x)$, the value of the *gradient* ∇f of f at x . The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \text{int dom } f$. The function f is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int dom } f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $\text{int dom } f$ (see also [19,20]).

Definition 2.1 ([21, Definition 5.2]). The function f is said to be:

- (i) *essentially smooth*, if ∂f is both locally bounded and single-valued on its domain.
- (ii) *essentially strictly convex*, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\text{dom } \partial f$.
- (iii) *Legendre*, if it is both essentially smooth and essentially strictly convex.

Remark 2.2. Let E be a reflexive Banach space. Then we have

- (a) f is essentially smooth if and only if f^* is essentially strictly convex (see [21, Theorem 5.4]).
- (b) $(\partial f)^{-1} = \partial f^*$ (see [20]).
- (c) f is Legendre if and only if f^* is Legendre. (see [21, Corollary 5.5]).
- (d) If f is Legendre, then ∇f is a bijection satisfying

$\nabla f = (\nabla f^*)^{-1}$, $\text{ran } \nabla f = \text{dom } \nabla f^* = \text{int dom } f^*$ and $\text{ran } \nabla f^* = \text{dom } \nabla f = \text{int dom } f$ (see [21, Theorem 5.10] and [14]).

Examples of Legendre functions were given in [22,21]. One important and interesting Legendre function is $\frac{1}{p} \|\cdot\|^p$ ($1 < p < \infty$) when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E , i.e., $\nabla f = J_p$ ($1 < p < \infty$). In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \rightarrow (-\infty, +\infty]$ is Legendre.

The following crucial lemma was proved by Reich–Sabach [23] (see also [18, Proposition 3.6.3]).

Lemma 2.3 ([23, Proposition 2.1]). If $f : E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E , then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .

Let $f : E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom } f \times \text{int dom } f \rightarrow [0, +\infty)$ defined as follows:

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance* with respect to f [24].

Recall that the *Bregman projection* [25] of $x \in \text{int dom } f$ onto the nonempty closed and convex set $C \subset \text{dom } f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}.$$

Let $f : E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The *modulus of total convexity* of f at $x \in \text{int dom } f$ is the function $v_f(x, \cdot) : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$v_f(x, t) := \inf \{D_f(y, x) : y \in \text{dom } f, \|y - x\| = t\}.$$

The function f is called *totally convex* at x if $v_f(x, t) > 0$ whenever $t > 0$. The function f is called *totally convex* if it is totally convex at any point $x \in \text{int dom } f$ and is said to be *totally convex on bounded sets* if $v_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$v_f(B, t) := \inf \{v_f(x, t) : x \in B \cap \text{dom } f\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [26, Theorem 2.10]).

Recall that the function f is said to be *sequentially consistent* [26] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded,

$$\lim_{n \rightarrow \infty} D_f(y_n, x_n) = 0 \implies \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

The following crucial lemma was proved by Butnariu–Iusem [27].

Lemma 2.4 ([27, Lemma 2.1.2]). *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Let C be a convex subset of $\text{int dom } f$ and let T be a self-mapping of C . A point $p \in C$ is called an *asymptotic fixed point* of T [28,29] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T .

Definition 2.5. A mapping T with a nonempty fixed point set is said to be:

(i) *Bregman strongly nonexpansive* [11,14] with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in \widehat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{n \rightarrow \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n \rightarrow \infty} D_f(x_n, Tx_n) = 0.$$

(ii) *Bregman firmly nonexpansive* [12,30,15] if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

The existence and approximation of Bregman firmly nonexpansive mappings was studied in [15]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E , then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [15]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Let C be a nonempty, closed and convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [26] that $z = P_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \quad \forall y \in C.$$

We also know the following:

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall x \in E, y \in C.$$

Let $f : E \rightarrow \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [31] and [24], we make use of the function $V_f : E \times E^* \rightarrow [0, +\infty)$ associated with f , which is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x \rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*) \quad (2.1)$$

for all $x \in E$ and $x^*, y^* \in E^*$ (see also [32, Lemmas 3.2 and 3.3]). In addition, if $f : E \rightarrow (-\infty, +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \rightarrow (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function (see [33]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i), \quad (2.2)$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

The properties of the Bregman projection and the relative projection operators were studied in [26,34].

The following lemmas give us some nice properties of real sequences.

Lemma 2.6 ([35]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

- (a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
 (b) $\limsup_{n \rightarrow \infty} \frac{b_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < +\infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7 ([36]). Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$\gamma_{m_k} \leq \gamma_{m_k+1} \quad \text{and} \quad \gamma'_k \leq \gamma_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. Main results

In this section, we now prove our main result in this paper. To this end, we first prove the following proposition.

Proposition 3.1. Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $T : C \rightarrow C$ be a mapping such that $\widehat{F}(T) = F(T)$ is nonempty, closed and convex. Suppose that $u \in C$ and $\{x_n\}$ is a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. Then

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle \leq 0,$$

where $p = P_{F(T)}^f(u)$ and $P_{F(T)}^f$ is the Bregman projection of C onto $F(T)$.

Proof. Let $u \in C$ and put $p = P_{F(T)}^f(u)$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$ and

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle = \langle \nabla f(u) - \nabla f(p), v - p \rangle.$$

On the other hand, since $\|x_{n_k} - Tx_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$, we have $v \in \widehat{F}(T) = F(T)$. It follows from definition of the Bregman projection that

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle = \langle \nabla f(u) - \nabla f(p), v - p \rangle \leq 0.$$

This completes the proof. $\square$

Theorem 3.2. Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let T be a Bregman strongly nonexpansive mapping on E such that $F(T) = \widehat{F}(T) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}^f(u)$, where $P_{F(T)}^f$ is the Bregman projection of E onto $F(T)$.

Proof. We note, by Reich and Sabach [15], that $F(T)$ is closed and convex. Let $p = P_{F(T)}^f(u) \in F(T) = \widehat{F}(T)$. Using (2.2), we have

$$\begin{aligned} D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n) \\ &\leq \max \{D_f(p, u), D_f(p, x_n)\}. \end{aligned}$$

By induction, the sequence $\{D_f(p, x_n)\}$ is bounded.

We next show that the sequence $\{x_n\}$ is also bounded. We follow the proof line as in [15]. Since $\{D_f(p, x_n)\}$ is bounded, there exists $M > 0$ such that

$$f(p) - \langle \nabla f(x_n), p \rangle + f^*(\nabla f(x_n)) = V_f(p, \nabla f(x_n)) = D_f(p, x_n) \leq M.$$

Hence $\{\nabla f(x_n)\}$ is contained in the sublevel set $\text{lev}_{\leq}^{\psi}(M - f(p))$, where $\psi = f^* - \langle \cdot, p \rangle$. Since f is lower semicontinuous, f^* is weak* lower semicontinuous. Hence the function ψ is coercive by Moreau–Rockafellar Theorem (see [37, Theorem 7A])

and [38]). This shows that $\{\nabla f(x_n)\}$ is bounded. Since f is strongly coercive, f^* is bounded on bounded sets (see [18, Lemma 3.6.1] and [21, Theorem 3.3]). Hence ∇f^* is also bounded on bounded subsets of E^* (see [27, Proposition 1.1.11]). Since f is a Legendre function, it follows that $x_n = \nabla f^*(\nabla f(x_n))$ is bounded for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded. So are $\{Tx_n\}$ and $\{\nabla f(Tx_n)\}$. Indeed, since f is bounded on bounded subsets of E , ∇f is also bounded on bounded subsets of E (see [27, Proposition 1.1.11]). Therefore $\{\nabla f(Tx_n)\}$ is bounded.

We next show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \rightarrow \infty} (D_f(p, x_{n_k+1}) - D_f(p, x_{n_k})) = 0,$$

then

$$\lim_{k \rightarrow \infty} (D_f(p, Tx_{n_k}) - D_f(p, x_{n_k})) = 0.$$

Since $\{\nabla f(Tx_{n_k})\}$ is bounded and $\alpha_{n_k} \rightarrow 0$, we obtain

$$\lim_{k \rightarrow \infty} \|\nabla f(x_{n_k+1}) - \nabla f(Tx_{n_k})\| = \lim_{k \rightarrow \infty} \alpha_{n_k} \|\nabla f(u) - \nabla f(Tx_{n_k})\| = 0. \quad (3.1)$$

Since f is strongly coercive and uniformly convex on bounded subsets of E , f^* is uniformly Fréchet differentiable on bounded subsets of E^* (see [18, Proposition 3.6.2]). Moreover, f^* is bounded on bounded sets (see [18, Lemma 3.6.1] and [21, Theorem 3.3]). Since f is Legendre, applying Lemma 2.3, we have

$$\lim_{k \rightarrow \infty} \|x_{n_k+1} - Tx_{n_k}\| = \lim_{k \rightarrow \infty} \|\nabla f^*(\nabla f(x_{n_k+1})) - \nabla f^*(\nabla f(Tx_{n_k}))\| = 0. \quad (3.2)$$

On the other hand, if f is uniformly Fréchet differentiable on bounded subsets of E , then f is uniformly continuous on bounded subsets of E (see [39, Theorem 1.8]). It follows that

$$\lim_{k \rightarrow \infty} |f(x_{n_k+1}) - f(Tx_{n_k})| = 0. \quad (3.3)$$

We now consider the following equality.

$$\begin{aligned} D_f(p, Tx_{n_k}) - D_f(p, x_{n_k}) &= f(p) - f(Tx_{n_k}) - \langle \nabla f(Tx_{n_k}), p - Tx_{n_k} \rangle - D_f(p, x_{n_k}) \\ &= f(p) - f(x_{n_k+1}) + f(x_{n_k+1}) - f(Tx_{n_k}) - \langle \nabla f(x_{n_k+1}), p - x_{n_k+1} \rangle \\ &\quad + \langle \nabla f(x_{n_k+1}), p - x_{n_k+1} \rangle - \langle \nabla f(Tx_{n_k}), p - Tx_{n_k} \rangle - D_f(p, x_{n_k}) \\ &= D_f(p, x_{n_k+1}) + (f(x_{n_k+1}) - f(Tx_{n_k})) + \langle \nabla f(x_{n_k+1}), p - x_{n_k+1} \rangle \\ &\quad - \langle \nabla f(Tx_{n_k}), p - Tx_{n_k} \rangle - D_f(p, x_{n_k}) \\ &= (D_f(p, x_{n_k+1}) - D_f(p, x_{n_k})) + (f(x_{n_k+1}) - f(Tx_{n_k})) \\ &\quad + \langle \nabla f(x_{n_k+1}) - \nabla f(Tx_{n_k}), p - x_{n_k+1} \rangle - \langle \nabla f(Tx_{n_k}), x_{n_k+1} - Tx_{n_k} \rangle. \end{aligned}$$

It follows from (3.1) to (3.3) that

$$\lim_{k \rightarrow \infty} (D_f(p, Tx_{n_k}) - D_f(p, x_{n_k})) = 0.$$

We next consider the following two cases.

Case 1. $D_f(p, x_{n+1}) \leq D_f(p, x_n)$ for all sufficiently large n . Hence the sequence $\{D_f(p, x_n)\}$ is bounded and nonincreasing. So we have $\lim_{n \rightarrow \infty} D_f(p, x_n)$ exists. This shows that $\lim_{n \rightarrow \infty} (D_f(p, x_{n+1}) - D_f(p, x_n)) = 0$ and hence

$$\lim_{n \rightarrow \infty} (D_f(p, Tx_n) - D_f(p, x_n)) = 0.$$

Since T is Bregman strongly nonexpansive,

$$\lim_{n \rightarrow \infty} D_f(x_n, Tx_n) = 0.$$

Since f is totally convex on bounded subsets of E , by Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

Proposition 3.1 yields that

$$\limsup_{n \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_n - p \rangle \leq 0.$$

Finally, we show that $x_n \rightarrow p$. Using (2.1), we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n)) \\ &\leq V_f(p, \alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n) - \alpha_n (\nabla f(u) - \nabla f(p))) \\ &\quad + \langle \alpha_n (\nabla f(u) - \nabla f(p)), x_{n+1} - p \rangle \\ &= V_f(p, \alpha_n \nabla f(p) + (1 - \alpha_n) \nabla f(Tx_n)) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &\leq \alpha_n V_f(p, \nabla f(p)) + (1 - \alpha_n) V_f(p, \nabla f(Tx_n)) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &= (1 - \alpha_n) D_f(p, Tx_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &\leq (1 - \alpha_n) D_f(p, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle. \end{aligned}$$

By Lemma 2.6, we can conclude that $\lim_{n \rightarrow \infty} D_f(p, x_n) = 0$. Therefore, by Lemma 2.4, $x_n \rightarrow p$ since f is totally convex on bounded subsets of E .

Case 2. There exists a subsequence $\{D_f(p, x_{n_j})\}$ of $\{D_f(p, x_n)\}$ such that $D_f(p, x_{n_j}) < D_f(p, x_{n_j+1})$ for all $j \in \mathbb{N}$. By Lemma 2.7, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1}) \quad \text{and} \quad D_f(p, x_k) \leq D_f(p, x_{m_k+1}).$$

So we have

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} (D_f(p, x_{m_k+1}) - D_f(p, x_{m_k})) \\ &\leq \limsup_{n \rightarrow \infty} (D_f(p, x_{n+1}) - D_f(p, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, Tx_n) - D_f(p, x_n)) \\ &= \limsup_{n \rightarrow \infty} (\alpha_n (D_f(p, u) - D_f(p, Tx_n)) + (D_f(p, Tx_n) - D_f(p, x_n))) \\ &\leq \limsup_{n \rightarrow \infty} \alpha_n (D_f(p, u) - D_f(p, Tx_n)) = 0. \end{aligned}$$

This implies that

$$\lim_{k \rightarrow \infty} (D_f(p, x_{m_k+1}) - D_f(p, x_{m_k})) = 0. \quad (3.4)$$

Following the proof line in Case 1, we can verify

$$\limsup_{k \rightarrow \infty} \langle \nabla f(u) - \nabla f(p), x_{m_k} - p \rangle \leq 0$$

and

$$D_f(p, x_{m_k+1}) \leq (1 - \alpha_{m_k}) D_f(p, x_{m_k}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p \rangle.$$

This implies

$$\begin{aligned} \alpha_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p \rangle \\ &\leq \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p \rangle. \end{aligned}$$

Hence $\lim_{k \rightarrow \infty} D_f(p, x_{m_k}) = 0$. Using this and (3.4) together, we conclude that

$$\limsup_{k \rightarrow \infty} D_f(p, x_k) \leq \lim_{k \rightarrow \infty} D_f(p, x_{m_k+1}) = 0.$$

This completes the proof. $\square$

As a direct consequence of Theorem 3.2, we obtain the convergence result concerning strongly relatively nonexpansive mappings [40,29] in a uniformly smooth and uniformly convex Banach space.

Corollary 3.3. Let E be a uniformly smooth and uniformly convex Banach space. Let T be a strongly relatively nonexpansive mapping on E such that $F(T) = \widehat{F}(T) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n J(u) + (1 - \alpha_n) J(Tx_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}(u)$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

4. Convergence theorems for a family of mappings

In this section, using Theorem 3.2, we present two strong convergence theorems for a family of Bregman strongly nonexpansive mappings.

Firstly, we consider the following condition: Let C be a subset of a real Banach space E , $f : E \rightarrow \mathbb{R}$ a convex and Gâteaux differentiable function and $\{T_n\}_{n=1}^\infty$ a sequence of mappings of C such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^\infty$ is said to satisfy the AKTT-condition [41] if, for any bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} < \infty.$$

We follow [41,42] and give details of proof here.

Proposition 4.1. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a Legendre and Fréchet differentiable function. Let $\{T_n\}_{n=1}^\infty$ be a sequence of mappings from C into E such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^\infty$ satisfies the AKTT-condition. Then there exists the mapping $T : B \rightarrow E$ such that*

$$Tx = \lim_{n \rightarrow \infty} T_n x, \quad \forall x \in B \quad (4.1)$$

and $\lim_{n \rightarrow \infty} \sup_{z \in B} \|\nabla f(Tz) - \nabla f(T_n z)\| = 0$.

Proof. To complete this, we show that $\{\nabla f(T_n x)\}$ is a Cauchy sequence in E^* for all $x \in B$. Let $\epsilon > 0$ be given. By the AKTT-condition, there exists $\ell_0 \in \mathbb{N}$ such that

$$\sum_{n=\ell_0}^{\infty} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} < \epsilon.$$

In particular, if $k > \ell \geq \ell_0$, then

$$\begin{aligned} \|\nabla f(T_k x) - \nabla f(T_\ell x)\| &\leq \sum_{n=\ell}^{k-1} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} \\ &\leq \sum_{n=\ell_0}^{\infty} \sup \{ \|\nabla f(T_{n+1}z) - \nabla f(T_n z)\| : z \in B \} < \epsilon. \end{aligned}$$

This shows that $\{\nabla f(T_n x)\}$ is a Cauchy sequence in E^* and $\lim_{n \rightarrow \infty} \nabla f(T_n x)$ exists for all $x \in B$. It is easily seen that the convergence is uniform on B . Since E is reflexive and f is Legendre, ∇f is bijective and $\nabla f^* = (\nabla f)^{-1}$. So we can define the mapping $T : B \rightarrow E$ as follows:

$$Tx = \nabla f^* \left(\lim_{n \rightarrow \infty} \nabla f(T_n x) \right) \quad \forall x \in B.$$

Since f is Fréchet differentiable, ∇f is norm-to-norm continuous. Hence

$$Tx = \nabla f^* \nabla f \left(\lim_{n \rightarrow \infty} T_n x \right) = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in B.$$

This completes that proof. $\square$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^\infty$ satisfies the AKTT-condition and T is defined by (4.1) with $\bigcap_{n=1}^\infty F(T_n) = F(T)$.

Theorem 4.2. *Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $\{T_n\}_{n=1}^\infty$ be a sequence of Bregman strongly nonexpansive mappings on E such that $F(T_n) = \widehat{F}(T_n)$ for all $n \in \mathbb{N}$ and $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^* (\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(T_n x_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ converges strongly to $P_F^f(u)$, where P_F^f is the Bregman projection of E onto F .

Proof. As in Theorem 3.2, we can prove that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. Since f is uniformly Fréchet differentiable and bounded on bounded subsets of E , it follows that $\lim_{n \rightarrow \infty} \|\nabla f(x_n) - \nabla f(T_n x_n)\| = 0$ by Lemma 2.3.

To this end, it suffices to show that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. By Proposition 4.1, we see that

$$\begin{aligned} \|\nabla f(x_n) - \nabla f(Tx_n)\| &\leq \|\nabla f(x_n) - \nabla f(T_n x_n)\| + \|\nabla f(T_n x_n) - \nabla f(Tx_n)\| \\ &\leq \|\nabla f(x_n) - \nabla f(T_n x_n)\| + \sup_{z \in \{x_n\}} \|\nabla f(T_n z) - \nabla f(Tz)\| \\ &\rightarrow 0. \end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. This completes the proof. $\square$

Next, employing the idea in [29], we consider the mapping $T : E \rightarrow E$ defined by $T = T_N T_{N-1} \cdots T_1$, where T_i ($i = 1, 2, \dots, N$) are Bregman strongly nonexpansive mappings on E .

Theorem 4.3. Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let T_i ($i = 1, 2, \dots, N$) be Bregman strongly nonexpansive mappings on E such that $F(T_i) = \widehat{F}(T_i)$ ($i = 1, 2, \dots, N$) and $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(T_N T_{N-1} \cdots T_1 x_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_F^f(u)$, where P_F^f is the Bregman projection of E onto F .

Proof. Put $T = T_N T_{N-1} \cdots T_1$. We see that $\emptyset \neq F(T) \subset \widehat{F}(T)$ since $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. By Lemma 1 in [29], we know that $\widehat{F}(T) \subset \bigcap_{i=1}^N \widehat{F}(T_i) = \bigcap_{i=1}^N F(T_i)$. It follows that

$$F(T) \subset \widehat{F}(T) \subset \bigcap_{i=1}^N \widehat{F}(T_i) = \bigcap_{i=1}^N F(T_i) \subset F(T).$$

Therefore $F(T) = \widehat{F}(T)$. On the other hand, by Lemma 2 in [29], we also know that T is Bregman strongly nonexpansive. Using Theorem 3.2, we deduce that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$. $\square$

5. Applications

In this section, we give some applications of Theorem 3.2 in the framework of reflexive Banach spaces.

5.1. Zeros of maximal monotone operators

Let $A : E \rightarrow 2^{E^*}$ be a set-valued mapping. The domain of A is denoted by $\text{dom } A = \{x \in E : Ax \neq \emptyset\}$ and also the graph of A is denote by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A is said to be *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for each $(x, x^*), (y, y^*) \in G(A)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operators on E . It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$ is closed and convex. Now, we apply Theorem 3.2 to the problem of finding $x \in E$ such that $0^* \in Ax$ which strongly relates to the convex minimization problems in optimization, economics and applied sciences.

The resolvent of A , denoted by $\text{Res}_A^f : E \rightarrow 2^E$, is defined as follows [12]:

$$\text{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(\text{Res}_A^f) = A^{-1}(0^*)$, and Res_A^f is single-valued and Bregman firmly nonexpansive (see [12]). If f is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E , then $\widehat{F}(\text{Res}_A^f) = F(\text{Res}_A^f)$ (see [15]). The Yosida approximation $A_\lambda : E \rightarrow E$, $\lambda > 0$, is also defined by

$$A_\lambda(x) = \frac{1}{\lambda} \left(\nabla f(x) - \nabla f(\text{Res}_{\lambda A}^f(x)) \right)$$

for all $x \in E$. From Proposition 2.7 in [43], we know that $(\text{Res}_{\lambda A}^f(x), A_\lambda(x)) \in G(A)$, and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$. Using these facts, we obtain the following result by replacing $T = \text{Res}_{\lambda A}^f$, $\lambda > 0$ in Theorem 3.2.

Theorem 5.1. Let E be a real reflexive Banach space and $f : E \rightarrow \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $A : E \rightarrow 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(\text{Res}_{\lambda A}^f x_n) \right), \quad \forall n \geq 1,$$

where $\lambda > 0$ and $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{A^{-1}(0^*)}^f(u)$, where $P_{A^{-1}(0^*)}^f$ is the Bregman projection of E onto $A^{-1}(0^*)$.

5.2. Equilibrium problems

Let C be a nonempty, closed and convex subset of E and $G : C \times C \rightarrow \mathbb{R}$ a bifunction. Finally, we apply Theorem 3.2 to the problem of finding $x \in C$ such that $G(x, y) \geq 0$ for all $y \in C$. Such a problem is called *equilibrium problem* and the solutions set is denoted by $EP(G)$. Numerous problems in economics, physics and applied sciences can be reduced to find solutions of equilibrium problems.

In order to solve the equilibrium problem, let us assume that a bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions [44]:

- (A1) $G(x, x) = 0$ for all $x \in C$;
- (A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;
- (A4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semi-continuous.

The resolvent of a bifunction G [45] is the operator $\text{Res}_G^f : E \rightarrow 2^C$ defined by

$$\text{Res}_G^f(x) = \{z \in C : G(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \forall y \in C\}.$$

From Lemma 1 in [14], if $f : E \rightarrow (-\infty, +\infty]$ is a strongly coercive and Gâteaux differentiable function, and G satisfies conditions (A1)–(A4), then $\text{dom}(\text{Res}_G^f) = E$. We also know the following lemma which give us some characterizations of the resolvent Res_G^f .

Lemma 5.2 ([14, Lemma 2]). *Let $f : E \rightarrow (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of E . If the bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfies conditions (A1)–(A4), then*

- (i) Res_G^f is single-valued;
- (ii) Res_G^f is a Bregman firmly nonexpansive operator;
- (iii) $F(\text{Res}_G^f) = EP(G)$;
- (iv) $EP(G)$ is a closed and convex subset of C ;
- (v) for all $x \in E$ and for all $p \in F(\text{Res}_G^f)$, we have

$$D_f(p, \text{Res}_G^f(x)) + D_f(\text{Res}_G^f(x), x) \leq D_f(p, x).$$

In addition, by Reich and Sabach [15], if f is uniformly Fréchet differentiable and bounded on bounded subsets of E , then we have from Lemma 5.2 that $F(\text{Res}_G^f) = \widehat{F}(\text{Res}_G^f) = EP(G)$ is closed and convex. Also, by replacing $T = \text{Res}_G^f$ in Theorem 3.2, we obtain the following result.

Theorem 5.3. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $G : C \times C \rightarrow \mathbb{R}$ be a bifunction which satisfies the conditions (A1)–(A4) such that $EP(G) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(\text{Res}_G^f x_n) \right), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{EP(G)}^f(u)$, where $P_{EP(G)}^f$ is the Bregman projection of E onto $EP(G)$.

Remark 5.4. Theorem 3.2 can be applied to convex feasibility problems, variational inequality problems and the problems of finding zeros of Bregman inverse strongly monotone operators in reflexive Banach spaces (see Sections 4, 7, 8 in [14]).

Remark 5.5. Theorem 4.3 can be applied to a system of variational inequality problems and a system of equilibrium problems.

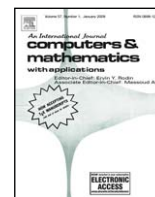
Remark 5.6. Our results are new even if the convex function f is chosen so that $f(\cdot) = \frac{1}{p} \|\cdot\|^p$ ($1 < p < \infty$) in uniformly smooth and uniformly convex Banach spaces.

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Strong convergence for a countable family of strict pseudocontractions in q -uniformly smooth Banach spaces

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ABSTRACT

We introduce a new iterative algorithm for finding a common fixed point of a countable family of strict pseudocontractions in q -uniformly smooth and uniformly convex Banach spaces. We then prove that the sequence generated by the proposed algorithm converges strongly to a common fixed point of an infinite family of strict pseudocontractions. Our results mainly improve and extend the results announced by Yao et al. [Y. Yao, Y.-C. Liou, G. Marino, Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2009 (2009) 7 pages. doi:10.1155/2009/279058. Art. ID 279058].

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1. Introduction

Let E and E^* be a real Banach space and the space dual to E , respectively. Let J_q ($q > 1$) denote the generalized duality mapping from E into 2^{E^*} given by $J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$, $\forall x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^* . It is well known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \neq 0$. If E is smooth then J_q is single-valued, which is denoted by j_q . The duality mapping J from a smooth Banach space E into E^* is said to be weakly sequentially continuous if x_n weak convergent to x implies Jx_n weak* convergent to Jx . We denote the fixed points set of a nonlinear mapping $T : C \rightarrow E$ by $F(T) = \{x \in C : Tx = x\}$ and write $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called:

- (i) λ -strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q - \lambda \|(I - T)x - (I - T)y\|^q,$$

or equivalently

$$\langle (I - T)x - (I - T)y, j_q(x - y) \rangle \geq \lambda \|(I - T)x - (I - T)y\|^q;$$

- (ii) L -Lipschitzian if for all $x, y \in D(T)$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping. By the definition, we know that every λ -strictly pseudocontractive mapping is $(\frac{1+\lambda}{\lambda})$ -Lipschitzian (see [2]).

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Remark 1.2. Let C be a nonempty subset of a real Hilbert space and $T : C \rightarrow C$ be a mapping. Then T is said to be κ -strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2. \quad (1.1)$$

It is well known that (1.1) is equivalent to the following:

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2.$$

It is worth mentioning that the class of strict pseudocontractions includes properly the class of nonexpansive mappings. In 1953, Mann [3] introduced an iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \quad (1.2)$$

where $\alpha_n \in [0, 1]$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [4]).

In 1967, Browder and Petryshyn [1] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using the Mann iterative algorithm (1.2) with a constant sequence $\alpha_n = \alpha$ for all $n \geq 0$. Recently, Marino and Xu [5] and Zhou [6] extended the results of Browder and Petryshyn [1] to Mann's iteration process (1.2). Zhou [7] also investigated the weak convergence in a 2-uniformly smooth Banach space. In a much more general setting, Osilike and Udomene [8], Zhang and Su [9], Zhang and Guo [10] and Zhou [11] investigated the weak convergence in a q -uniformly smooth Banach space. Since 1967, the construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors (see, e.g., [12–15]).

Due to the weak convergence of the Mann iterative algorithm, in 1967, Halpern [16] introduced a modified Mann's iteration (1.2) which is the so called *Halpern iterative scheme* as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \quad (1.3)$$

where $\alpha_n \in [0, 1]$ and $u \in C$. He proved, in a real Hilbert space, the convergence of $\{x_n\}$ to a fixed point of T where $\alpha_n := n^{-a}$, $a \in (0, 1)$.

In 1977, Lions [17] obtained a strong convergence provided the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0; \quad C2 : \sum_{n=0}^{\infty} \alpha_n = \infty; \quad C3 : \lim_{n \rightarrow \infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$$

Reich [18] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lions' conditions imposed on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$.

In 1992, Wittmann [19] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0; \quad C2 : \sum_{n=0}^{\infty} \alpha_n = \infty; \quad C3 : \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Later, Shioji and Takahashi [20] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of the Halpern iterative scheme has been widely used to approximate the fixed points for nonexpansive mappings (see, e.g., [21–26] and the references cited therein).

In 2003, Nakajo and Takahashi [27] introduced another modified Mann iterative algorithm for a nonexpansive mapping T in a real Hilbert space as follows: $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 1, \end{cases} \quad (1.4)$$

where $\{\alpha_n\} \subset [0, 1]$ and P_K is a metric projection from H into a nonempty, closed and convex subset K . Such an algorithm is called the *CQ method*. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T under suitable conditions.

Recently, Takahashi et al. [28] introduced a new modification of the Mann iterative scheme called the *shrinking projection method* for a nonexpansive mapping T in a Hilbert space as follows: for $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad n \geq 1, \end{cases} \quad (1.5)$$

where $\{\alpha_n\} \subset [0, 1]$. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T under suitable conditions.

The concepts of the CQ method and the shrinking projection method have also been extensively investigated in order to get strong convergence results by many authors (see, e.g., [29–33]). A natural question arises:

Question. Besides the concepts mentioned before, could one construct a new modification of the Mann iterative algorithm in order to get strong convergence?

In 2009, Yao et al. [34] introduced a new modified Mann iterative algorithm which is different from those in the literature for a nonexpansive mapping in a real Hilbert space. To be more precise, they proved the following theorem.

Theorem 1.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. For $x_0 \in C$ given arbitrarily, let the sequence $\{x_n\}$, $n \geq 0$, be generated iteratively by*

$$\begin{aligned} y_n &= P_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T y_n. \end{aligned} \quad (1.6)$$

Suppose that the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;

then the sequence $\{x_n\}$ generated by (1.6) strongly converges to a fixed point of T .

Let C be a nonempty, closed and convex subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty} : C \rightarrow C$ a countable family of strict pseudocontractions.

Motivated and inspired by Marino and Xu [5] and Yao et al. [34], we consider the following modified Mann-type iteration: $x_1 \in C$ and

$$\begin{aligned} y_n &= Q_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n[(1 - \gamma_n)y_n + \gamma_n T_n y_n], \quad n \geq 1, \end{aligned} \quad (1.7)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ and Q_C is a sunny, nonexpansive retraction of E onto C .

In this paper, first we prove path convergence for a nonexpansive mapping in uniformly convex and q -uniformly smooth Banach spaces. Then we prove a strong convergence of the modified Mann-type iteration process (1.7) for a countable family of strict pseudocontractions. Using the result obtained, the strong convergence is also established for an infinite strict pseudocontractions. Our results extend the main result of Yao et al. [34] in several aspects.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The *modulus of convexity* of E is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\},$$

for all $\epsilon \in [0, 2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case E is called *smooth*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. The norm of E is called *uniformly Fréchet differentiable* if the limit is attained uniformly for $x, y \in S(E)$. It is well known that (uniform) *Fréchet differentiability* of the norm of E implies (uniform) *Gâteaux differentiability* of the norm of E .

Let $\rho_E : [0, \infty) \rightarrow [0, \infty)$ be the *modulus of smoothness* of E defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t \right\}.$$

A Banach space E is said to be *uniformly smooth* if $\frac{\rho_E(t)}{t} \rightarrow 0$ as $t \rightarrow 0$. Suppose that $q > 1$; then E is said to be *q-uniformly smooth* if there exists $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q -uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable, and hence the norm of E is Fréchet differentiable. For more details, we refer the reader to [35,36].

In 1972, Gossez and Lami [37] gave some geometric properties related to the fixed point theory for nonexpansive mappings. They proved that a space with a weakly continuous duality map satisfies Opial's condition [38]. Conversely, if a space satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous duality mapping at zero.

Let E be a real Banach space, C a nonempty, closed and convex subset of E , and K a nonempty subset of C . Let $Q : C \rightarrow K$. Then Q is said to be:

(1) *sunny* if for each $x \in C$ and $t \in [0, 1]$, we have

$$Q(tx + (1-t)Qx) = Qx;$$

(2) *a retraction of C onto K* if

$$Qx = x, \quad \forall x \in K;$$

(3) *a sunny nonexpansive retraction* if Q is sunny, nonexpansive and a retraction onto K (see also Bruck [39], Goebel and Reich [40] and Reich [24]).

In the sequel, we need the following crucial lemmas.

Lemma 2.1 ([41]). *Let E be a real q -uniformly smooth Banach space. Then the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + C_q \|y\|^q,$$

for all $x, y \in E$ and for some $C_q > 0$.

Lemma 2.2 ([42]). *Suppose that $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b .

Lemma 2.3 ([43]). *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space E such that*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n, \quad n \geq 1,$$

where $\{\beta_n\}$ satisfies the following condition: $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. If $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Lemma 2.4 ([44,45]). *Let E be a real uniformly convex Banach space and let C be a nonempty, closed and convex subset of E and $S : C \rightarrow C$ a nonexpansive mapping such that $F(S) \neq \emptyset$; then $I - S$ is demiclosed at zero.*

Lemma 2.5 ([26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$(a) \sum_{n=1}^{\infty} \gamma_n = \infty; \quad (b) \limsup_{n \rightarrow \infty} \delta_n \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty.$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

To deal with a family of mappings, the following conditions are introduced. Let C be a subset of a real Banach space E and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [21] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Lemma 2.6 ([21]). *Let C be a nonempty and closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to a point in C . Moreover, let the mapping T be defined by*

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in C.$$

Then for each bounded subset B of C ,

$$\limsup_{n \rightarrow \infty} \{\|Tz - T_n z\| : z \in B\} = 0.$$

The following results can be found in [46,47].

Lemma 2.7 ([46,47]). Let C be a closed and convex subset of a smooth Banach space E . Suppose that $\{T_n\}_{n=1}^\infty$ is a family of λ -strictly pseudocontractive mappings from C into E with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\mu_n\}_{n=1}^\infty$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^\infty \mu_n = 1$. Then the following conclusions hold:

- (1) $G := \sum_{n=1}^\infty \mu_n T_n : C \rightarrow E$ is a λ -strictly pseudocontractive mapping;
- (2) $F(G) = \bigcap_{n=1}^\infty F(T_n)$.

Lemma 2.8 ([47]). Let C be a closed and convex subset of a smooth Banach space E . Suppose that $\{S_k\}_{k=1}^\infty$ is a countable family of λ -strictly pseudocontractive mappings of C into itself with $\bigcap_{k=1}^\infty F(S_k) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \rightarrow C$ by

$$T_n x = \sum_{k=1}^n \mu_n^k S_k x, \quad x \in C,$$

where $\{\mu_n^k\}$ is a family of nonnegative numbers satisfying:

- (i) $\sum_{k=1}^n \mu_n^k = 1$ for all $n \in \mathbb{N}$; (ii) $\mu^k := \lim_{n \rightarrow \infty} \mu_n^k > 0$ for all $k \in \mathbb{N}$;
- (iii) $\sum_{n=1}^\infty \sum_{k=1}^n |\mu_{n+1}^k - \mu_n^k| < \infty$.

Then:

- (1) Each T_n is a λ -strictly pseudocontractive mapping.
- (2) $\{T_n\}$ satisfies the AKTT-condition.
- (3) If $T : C \rightarrow C$ is defined by

$$Tx = \sum_{k=1}^\infty \mu^k S_k x, \quad x \in C,$$

then $Tx = \lim_{n \rightarrow \infty} T_n x$ and $F(T) = \bigcap_{n=1}^\infty F(T_n) = \bigcap_{k=1}^\infty F(S_k)$.

In the sequel, we will write that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.6 with $F(T) = \bigcap_{n=1}^\infty F(T_n)$.

3. The path convergence theorem

Let C be a nonempty, closed and convex subset of a real Banach space E . Let $S : C \rightarrow C$ be a nonexpansive mapping. For each $t \in (0, 1)$, we define the mapping $S_t : C \rightarrow C$ by

$$S_t x = SQ_C[(1-t)x], \quad \forall x \in C. \quad (3.1)$$

It is easy to see that S_t is a contraction. By the Banach contraction principle, there exists a unique fixed point x_t of S_t in C , that is,

$$x_t = SQ_C[(1-t)x_t]. \quad (3.2)$$

Theorem 3.1. Let E be a real q -uniformly smooth and uniformly convex Banach space which admits the weakly sequentially continuous duality mapping j_q and C be a nonempty, closed and convex subset of E . Let $S : C \rightarrow C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. For each $t \in (0, 1)$, let $\{x_t\}$ be generated by (3.2), then as $t \rightarrow 0$, $\{x_t\}$ converges strongly to a fixed point of S .

Proof. First, we prove that $\{x_t\}$ is bounded. Let $x^* \in F(S)$. From (3.2) we have

$$\begin{aligned} \|x_t - x^*\| &= \|SQ_C[(1-t)x_t] - SQ_C x^*\| \\ &\leq (1-t)\|x_t - x^*\| + t\|x^*\|. \end{aligned}$$

This implies that $\|x_t - x^*\| \leq \|x^*\|$ and hence $\{x_t\}$ is bounded.

Next, we prove that $\|x_t - Sx_t\| \rightarrow 0$ as $t \rightarrow 0$. From (3.2) we also have

$$\begin{aligned} \|x_t - Sx_t\| &= \|SQ_C[(1-t)x_t] - Sx_t\| \\ &\leq t\|x_t\| \rightarrow 0, \end{aligned} \quad (3.3)$$

as $t \rightarrow 0$.

Next, we prove that $\{x_t\}$ is relatively norm compact as $t \rightarrow 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Put $x_n := x_{t_n}$. From (3.3) we get that

$$\|x_n - Sx_n\| \rightarrow 0, \quad (3.4)$$

as $n \rightarrow \infty$. Using Lemma 2.1, we obtain for each $u \in F(S)$ that

$$\begin{aligned}\|x_t - u\|^q &= \|SQ_C[(1-t)x_t] - SQ_C u\|^q \\ &\leq \|(x_t - u) - tx_t\|^q \\ &\leq \|x_t - u\|^q - qt \langle x_t, j_q(x_t - u) \rangle + C_q t^q \|x_t\|^q \\ &= \|x_t - u\|^q - qt \langle x_t - u, j_q(x_t - u) \rangle - qt \langle u, j_q(x_t - u) \rangle + C_q t^q \|x_t\|^q \\ &= \|x_t - u\|^q - qt \|x_t - u\|^q - qt \langle u, j_q(x_t - u) \rangle + C_q t^q \|x_t\|^q.\end{aligned}$$

This shows that

$$\|x_t - u\|^q \leq \langle u, j_q(u - x_t) \rangle + \frac{C_q t^{q-1}}{q} \|x_t\|^q.$$

In particular, we obtain that

$$\|x_n - u\|^q \leq \langle u, j_q(u - x_n) \rangle + \frac{C_q t_n^{q-1}}{q} \|x_n\|^q.$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n \rightarrow z^* \in C$. By Lemma 2.4 and (3.4), we know that $z^* \in F(S)$. So we have

$$\|x_n - z^*\|^q \leq \langle z^*, j_q(z^* - x_n) \rangle + \frac{C_q t_n^{q-1}}{q} \|x_n\|^q.$$

Since j_q is weakly sequential continuous, we obtain $x_n \rightarrow z^*$ as $n \rightarrow \infty$. Next, we prove that $x_t \rightarrow z^*$ as $t \rightarrow 0$. Assume that $x_{t_m} \rightarrow x^*$, where $t_m \rightarrow 0$. Put $x_m := x_{t_m}$. Using the same argument as above, we can show that $x^* \in F(S)$ and

$$\|x_m - z^*\|^q \leq \langle z^*, j_q(z^* - x_m) \rangle + \frac{C_q t_m^{q-1}}{q} \|x_m\|^q.$$

Therefore,

$$\|x^* - z^*\|^q \leq \langle z^*, j_q(z^* - x^*) \rangle. \quad (3.5)$$

Interchanging x^* and z^* , we have

$$\|z^* - x^*\|^q \leq \langle x^*, j_q(x^* - z^*) \rangle. \quad (3.6)$$

Combining (3.5) and (3.6) we obtain

$$\begin{aligned}2\|x^* - z^*\|^q &\leq \langle z^*, j_q(z^* - x^*) \rangle + \langle x^*, j_q(x^* - z^*) \rangle \\ &= \|x^* - z^*\|^q.\end{aligned}$$

This implies that $x^* = z^*$ and hence the proof is complete. $\square$

4. The main results

Theorem 4.1. Let E be a real q -uniformly smooth and uniformly convex Banach space which admits the weakly sequentially continuous duality mapping j_q and C be a nonempty, closed and convex subset of E . Let $\{T_n\}_{n=1}^\infty : C \rightarrow C$ be a family of λ -strict pseudocontractions, $0 < \lambda < 1$, such that $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$ which satisfy the conditions:

- (C1) $\sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C3) $\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty$;
- (C4) $0 < a \leq \gamma_n \leq \mu$, $\mu = \min \left\{ 1, \left(\frac{q\lambda}{C_q} \right)^{\frac{1}{q-1}} \right\}$.

Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a common fixed point of $\{T_n\}_{n=1}^\infty$.

Proof. First, we prove that $\{x_n\}$ is bounded. Put $z_n = (1 - \gamma_n)y_n + \gamma_n T_n y_n$ for all $n \geq 1$. For each $u \in F$, it follows from Lemma 2.1 that

$$\begin{aligned} \|z_n - u\|^q &= \|(y_n - u) + \gamma_n(T_n y_n - y_n)\|^q \\ &\leq \|y_n - u\|^q + q\gamma_n \langle T_n y_n - y_n, j_q(y_n - u) \rangle + C_q \gamma_n^q \|y_n - T_n y_n\|^q \\ &\leq \|y_n - u\|^q - q\lambda \gamma_n \|y_n - T_n y_n\|^q + C_q \gamma_n^q \|y_n - T_n y_n\|^q \\ &= \|y_n - u\|^q - \gamma_n(q\lambda - C_q \gamma_n^{q-1}) \|y_n - T_n y_n\|^q. \end{aligned}$$

This implies by (C4) that

$$\|z_n - u\| \leq \|y_n - u\|.$$

Hence

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1 - \beta_n)(x_n - u) + \beta_n(z_n - u)\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n\|z_n - u\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n\|y_n - u\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n((1 - \alpha_n)\|x_n - u\| + \alpha_n\|u\|) \\ &= (1 - \alpha_n\beta_n)\|x_n - u\| + \alpha_n\beta_n\|u\| \\ &\leq \max\{\|x_n - u\|, \|u\|\}. \end{aligned}$$

By induction, we get that $\{x_n\}$ is bounded, and so is $\{y_n\}$. First, we observe that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|(1 - \alpha_{n+1})x_{n+1} - (1 - \alpha_n)x_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\|. \end{aligned} \quad (4.1)$$

For each $n \geq 1$, define $S_n x = (1 - \gamma_n)x + \gamma_n T_n x$ for all $x \in C$. By (C4), it is easy to verify that S_n is nonexpansive and $F(T_n) = F(S_n)$ for all $n \geq 1$. Moreover, $\{S_n\}$ satisfies the AKTT-condition. In fact, for any bounded subset B of C ,

$$\begin{aligned} \sup_{z \in B} \|S_{n+1}z - S_n z\| &= \sup_{z \in B} \|((1 - \gamma_{n+1})z + \gamma_{n+1}T_{n+1}z) - ((1 - \gamma_n)z + \gamma_n T_n z)\| \\ &\leq |\gamma_{n+1} - \gamma_n| \sup_{z \in B} \|z\| + \gamma_{n+1} \sup_{z \in B} \|T_{n+1}z - T_n z\| + |\gamma_{n+1} - \gamma_n| \sup_{z \in B} \|T_n z\| \\ &\leq |\gamma_{n+1} - \gamma_n| \left(\sup_{z \in B} \|z\| + \sup_{z \in B} \|T_n z\| \right) + \sup_{z \in B} \|T_{n+1}z - T_n z\|. \end{aligned}$$

From condition (C3) and $\{T_n\}$ satisfying the AKTT-condition, we get that

$$\sum_{n=1}^{\infty} \sup_{z \in B} \|S_{n+1}z - S_n z\| < \infty. \quad (4.2)$$

Next, we show that $\|x_{n+1} - x_n\| \rightarrow 0$ and $\|y_n - T_n y_n\| \rightarrow 0$ as $n \rightarrow \infty$. We observe that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|S_{n+1}y_{n+1} - S_n y_n\| \\ &\leq \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_n y_n\| \\ &\leq \|y_{n+1} - y_n\| + \sup_{z \in \{y_n\}} \|S_{n+1}z - S_n z\|. \end{aligned} \quad (4.3)$$

It follows from (4.1) and (4.3) that

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{z \in \{y_n\}} \|S_{n+1}z - S_n z\|.$$

From (C1) and (4.2) we obtain

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.3 and (C2), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (4.4)$$

Hence

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \beta_n \|z_n - x_n\| = 0. \quad (4.5)$$

From (1.7) and (C1) we see that

$$\begin{aligned}\|y_n - x_n\| &= \|Q_C[(1 - \alpha_n)x_n] - Q_Cx_n\| \\ &\leq \alpha_n \|x_n\| \rightarrow 0,\end{aligned}\tag{4.6}$$

as $n \rightarrow \infty$. So from (4.4) and (4.6) we have $\|z_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Observe that

$$\|T_n y_n - y_n\| = \frac{1}{\gamma_n} \|z_n - y_n\|,$$

which implies by condition (C4) that

$$\lim_{n \rightarrow \infty} \|T_n y_n - y_n\| = 0.\tag{4.7}$$

Since $\{S_n\}$ satisfies the AKTT-condition, we can define $S : C \rightarrow C$ by $Sx = \lim_{n \rightarrow \infty} S_n x$ for all $x \in C$. It is easy to see that S is nonexpansive. Since $\{\gamma_n\}$ is bounded, there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} \rightarrow \gamma$ as $j \rightarrow \infty$. It follows that

$$Sx = \lim_{j \rightarrow \infty} S_{n_j} x = \lim_{j \rightarrow \infty} [(1 - \gamma_{n_j})x + \gamma_{n_j} T_{n_j} x] = (1 - \gamma)x + \gamma T x, \quad \forall x \in C.$$

This shows that $F(S) = F(T) = F$. We observe that

$$\begin{aligned}\|y_n - S y_n\| &\leq \|y_n - S_n y_n\| + \|S_n y_n - S y_n\| \\ &\leq \|y_n - z_n\| + \sup_{z \in \{y_n\}} \|S_n z - S z\| \rightarrow 0,\end{aligned}$$

as $n \rightarrow \infty$. On the other hand, we see that

$$\begin{aligned}\|x_n - S x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_n y_n\| + \|S_n y_n - S_n x_n\| + \|S_n x_n - S x_n\| \\ &\leq \|x_n - x_{n+1}\| + (1 - \beta_n) \|x_n - z_n\| + \|y_n - x_n\| + \sup_{z \in \{x_n\}} \|S_n z - S z\|.\end{aligned}$$

From (4.4)–(4.6), we have

$$\lim_{n \rightarrow \infty} \|S x_n - x_n\| = 0.\tag{4.8}$$

Let $\{x_t\}$ be defined by (3.2). By Theorem 3.1, we have $x_t \rightarrow x^* \in F(S)$ as $t \rightarrow 0$. Next, we prove that

$$\limsup_{n \rightarrow \infty} \langle x^*, j_q(x^* - x_n) \rangle \leq 0.$$

From Lemma 2.1 and (3.2), we obtain that

$$\begin{aligned}\|x_t - x_n\|^q &= \|x_t - S x_n + S x_n - x_n\|^q \\ &\leq \|x_t - S x_n\|^q + q \langle S x_n - x_n, j_q(x_t - S x_n) \rangle + C_q \|S x_n - x_n\|^q \\ &\leq \|x_t - S x_n\|^q + q \|S x_n - x_n\| \|x_t - S x_n\|^{q-1} + C_q \|S x_n - x_n\|^q \\ &\leq \|S Q_C(1 - t)x_t - S Q_C x_n\|^q + M \|S x_n - x_n\| \\ &\leq \|x_t - x_n - t x_t\|^q + M \|S x_n - x_n\| \\ &\leq \|x_t - x_n\|^q - q t \langle x_t, j_q(x_t - x_n) \rangle + C_q t^q \|x_t\|^q + M \|S x_n - x_n\|\end{aligned}$$

for some $M > 0$. It follows that

$$\langle x_t, j_q(x_t - x_n) \rangle \leq \frac{C_q t^{q-1}}{q} \|x_t\|^q + \frac{M}{q t} \|S x_n - x_n\|.$$

From (4.8) we obtain that

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle x_t, j_q(x_t - x_n) \rangle \leq 0.\tag{4.9}$$

On the other hand, we observe that

$$\begin{aligned}\langle x^*, j_q(x^* - x_n) \rangle &= \langle x^*, j_q(x^* - x_n) - j_q(x_t - x_n) \rangle + \langle x^* - x_t, j_q(x_t - x_n) \rangle + \langle x_t, j_q(x_t - x_n) \rangle \\ &\leq \langle x^*, j_q(x^* - x_n) - j_q(x_t - x_n) \rangle + \|x^* - x_t\| \|x_t - x_n\|^{q-1} + \langle x_t, j_q(x_t - x_n) \rangle.\end{aligned}\tag{4.10}$$

Since E is q -uniformly smooth, j_q is uniformly continuous on bounded sets of E . So the limits $\limsup_{n \rightarrow \infty}$ and $\limsup_{t \rightarrow 0}$ are interchangeable. Hence, from (4.9) and (4.10), we have

$$\limsup_{n \rightarrow \infty} \langle x^*, j_q(x^* - x_n) \rangle \leq 0.\tag{4.11}$$

From (1.7), we have

$$\begin{aligned} \|y_n - x^*\|^q &\leq \|(1 - \alpha_n)(x_n - x^*) - \alpha_n x^*\|^q \\ &\leq (1 - \alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - q \alpha_n \langle x^*, j_q((1 - \alpha_n)(x_n - x^*)) \rangle \\ &= (1 - \alpha_n)^q \|x_n - x^*\|^q + C_q \alpha_n^q \|x^*\|^q - q \alpha_n (1 - \alpha_n)^{q-1} \langle x^*, j_q(x_n - x^*) \rangle \\ &\leq (1 - \alpha_n) \|x_n - x^*\|^q + \alpha_n (C_q \alpha_n^{q-1} \|x^*\|^q - q (1 - \alpha_n)^{q-1} \langle x^*, j_q(x_n - x^*) \rangle) \\ &= (1 - \alpha_n) \|x_n - x^*\|^q + \alpha_n \delta_n, \end{aligned} \quad (4.12)$$

where $\delta_n = C_q \alpha_n^{q-1} \|x^*\|^q + q(1 - \alpha_n)^{q-1} \langle x^*, j_q(x^* - x_n) \rangle$. From (C1) and (4.11), it is easy to see that $\limsup_{n \rightarrow \infty} \delta_n \leq 0$.

Finally, we prove that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. By Lemma 2.2 and (1.7), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^q &= (1 - \beta_n) \langle x_n - x^*, j_q(x_{n+1} - x^*) \rangle + \beta_n \langle z_n - x^*, j_q(x_{n+1} - x^*) \rangle \\ &\leq (1 - \beta_n) \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \beta_n \|z_n - x^*\| \|x_{n+1} - x^*\|^{q-1} \\ &\leq (1 - \beta_n) \left(\frac{1}{q} \|x_n - x^*\|^q + \frac{q-1}{q} \|x_{n+1} - x^*\|^q \right) + \beta_n \left(\frac{1}{q} \|y_n - x^*\|^q + \frac{q-1}{q} \|x_{n+1} - x^*\|^q \right) \\ &= \frac{(1 - \beta_n)}{q} \|x_n - x^*\|^q + \frac{q-1}{q} \|x_{n+1} - x^*\|^q + \frac{\beta_n}{q} \|y_n - x^*\|^q. \end{aligned} \quad (4.13)$$

It follows from (4.12) and (4.13) that

$$\begin{aligned} \|x_{n+1} - x^*\|^q &\leq (1 - \beta_n) \|x_n - x^*\|^q + \beta_n \|y_n - x^*\|^q \\ &\leq (1 - \beta_n) \|x_n - x^*\|^q + \beta_n ((1 - \alpha_n) \|x_n - x^*\|^q + \alpha_n \delta_n) \\ &= (1 - \alpha_n \beta_n) \|x_n - x^*\|^q + \alpha_n \beta_n \delta_n. \end{aligned} \quad (4.14)$$

Applying Lemma 2.5 to (4.14), we conclude that $x_n \rightarrow x^* \in F$ as $n \rightarrow \infty$. This completes the proof. $\square$

Remark 4.2. Theorem 4.1 extends the main result of Yao et al. in the following senses:

- (i) from real Hilbert spaces to q -uniformly smooth and uniformly convex Banach spaces;
- (ii) from a single nonexpansive mapping to an infinite family of strict pseudocontractions.

As a direct consequence of Lemmas 2.7 and 2.8 and Theorem 4.1, we obtain the following result.

Theorem 4.3. Let E be a real q -uniformly smooth and uniformly convex Banach space which admits the weakly sequentially continuous duality mapping j_q and C be a nonempty, closed and convex subset of E . Let $\{S_k\}_{k=1}^\infty$ be a sequence of λ_k -strict pseudocontractions of C into itself such that $\bigcap_{k=1}^\infty F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define the sequence $\{x_n\}$ by $x_1 \in C$,

$$\begin{aligned} y_n &= Q_C[(1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n \left((1 - \gamma_n)y_n + \gamma_n \sum_{k=1}^n \mu_n^k S_k y_n \right), \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$ which satisfy (C1)–(C4) of Theorem 4.1 and $\{\mu_n^k\}$ is a real sequence which satisfies (i)–(iii) of Lemma 2.8. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{S_k\}_{k=1}^\infty$.

Remark 4.4. By Gossez and Lami [37], we know that if E satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then E has a weakly continuous duality mapping at zero. Thus, Theorems 4.1 and 4.3 hold in uniformly convex and q -uniformly smooth Banach spaces which satisfy Opial's condition and also hold in real Hilbert spaces.

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RESEARCH

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Composite iterative schemes for maximal monotone operators in reflexive Banach spaces

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Abstract

In this article, we introduce composite iterative schemes for finding a zero point of a finite family of maximal monotone operators in a reflexive Banach space. Then, we prove strong convergence theorems by using a shrinking projection method. Moreover, we also apply our results to a system of convex minimization problems in reflexive Banach spaces.

AMS Subject Classification: 47H09, 47H10

Keywords: Maximal monotone operator, Shrinking projection method, Proximal point algorithm, Bregman projection, Totally convex function, Legendre function

Introduction

Let E be a real Banach space and C a nonempty subset of E . Let E^* be the dual space of E . We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x \rangle$. Let $A : E \rightarrow 2^{E^*}$ be a set-valued mapping. We denote $\text{dom } A$ by *domain* of A , that is, $\text{dom } A = \{x \in E : Ax \neq \emptyset\}$ and also denote $G(A)$ by the *graph* of A , that is, $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A set-valued mapping A is said to be *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $(x, x^*) \in G(A)$; $(y, y^*) \in G(A)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operator on E . It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$ is closed and convex.

The problem of finding zero points for maximal monotone operators plays an important role in optimizations. This is because it can be reduced to a convex minimization problem and a variational inequality problem. Many authors have studied the convergence of such problems in several settings, (see [1-6]). Initiated by Martinet [7], in a Hilbert space, Rockafellar [8] introduced the following iterative schemes:

$$\begin{cases} x_1 = x \in E, \\ x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \geq 1, \end{cases} \quad (1.1)$$

where $\{\lambda_n\} \subset (0, \infty)$ and J_λ is the resolvent of A defined by $J_\lambda = (I + \lambda A)^{-1}$ for all $\lambda > 0$, and A is a maximal monotone operator on E . Such an algorithm is called the *proximal point algorithm*. He proved that the sequence $\{x_n\}$ generated by (1.1) converges weakly to an element in $A^{-1}(0)$ provided $\liminf_{n \rightarrow \infty} \lambda_n > 0$. Later, Kamimura and Takahashi [9] introduced the following iteration in a Hilbert space:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 1, \quad (1.2)$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$. The weak convergence theorems are also established in a Hilbert space under suitable conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2005, Kohsaka and Takahashi [10] studied the above iteration process in a more general setting, reflexive Banach spaces. In fact, those authors proposed the following algorithm:

$$x_{n+1} = \nabla f^* (\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(J_{\lambda_n} x_n)), \quad \forall n \geq 1, \quad (1.3)$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, \infty)$, $f: E \rightarrow \mathbb{R}$ is a Bregman function and $J_\lambda = (\nabla f + \lambda A)^{-1} \nabla f$ for all $\lambda > 0$. They also proved a weak convergence theorem of the proposed algorithm.

Very recently, in 2010, Reich and Sabach [11] proposed an algorithm for finding a zero point of maximal monotone operators $A_i: E \rightarrow 2^{E^*}$ ($i = 1, 2, \dots, N$) in a general reflexive Banach space E as follows:

$$\begin{cases} x_0 \in E, \\ \gamma_n^i = \text{Res}_{\lambda_n^i A_i}^f(x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, \gamma_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}^f(x_0), \quad \forall n \geq 0, \end{cases} \quad (1.4)$$

where $\{\lambda_n^i\}_{i=1}^N \subset (0, \infty)$, $\{e_n^i\}_{i=1}^N$ is an error sequence in E with $e_n^i \rightarrow 0$ and P_K^f the Bregman projection with respect to f from E onto a closed and convex subset K of E . Those authors showed that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common element in $\bigcap_{i=1}^N A_i^{-1}(0^*)$ under some mild conditions.

Motivated by the previous ones, we first introduce a composite iterative scheme which is different from (1.4) for finding a zero point of maximal monotone operators $A_i: E \rightarrow 2^{E^*}$ ($i = 1, 2, \dots, N$) in reflexive Banach spaces. Using the shrinking projection technique, introduced by Takahashi et al. [12], we then prove that a sequence generated by the proposed algorithm converges strongly to an element in $\bigcap_{i=1}^N A_i^{-1}(0^*)$ under some appropriate control conditions. Finally, we also apply our result to a system of convex minimization problems.

Preliminaries and lemmas

Let E be a real reflexive Banach space with a norm $\|\cdot\|$ and E^* be the dual space of E . Throughout this article, $f: E \rightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous, and convex function, and the Fenchel conjugate of f is the function $f^*: E^* \rightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup \{ \langle x^*, x \rangle - f(x) : x \in E \}.$$

We denote by $\text{dom } f$ the domain of f , that is, the set $\{x \in E : f(x) < +\infty\}$. For any $x \in \text{int dom } f$ and $y \in E$, the *right-hand derivative* of f at x in the direction y is defined by

$$f^o(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x if $\lim_{t \rightarrow 0^+} \frac{f(x+ty) - f(x)}{t}$ exists for any y . In this case, $f'(x, y)$ coincides with $\nabla f(x)$, the value of the *gradient* ∇f of f at x . The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \text{int dom } f$. The function f is said to be *Fréchet differentiable at x* if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let E be a reflexive Banach space. The Legendre function is defined from a general Banach space E into $(-\infty, +\infty]$ (see [13]). According to [13], the function f is *Legendre* if and only if it satisfies the following conditions:

(L1) The interior of the domain of f (denoted by $\text{int dom } f$) is nonempty, f^* is Gâteaux differentiable on $\text{int dom } f$, and $\text{dom } \nabla f = \text{int dom } f$;

(L2) The interior of the domain f^* (denoted by $\text{int dom } f^*$) is nonempty, f^* is Gâteaux differentiable on $\text{int dom } f^*$, and $\text{dom } \nabla f^* = \text{int dom } f^*$.

Since E is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [14]). This fact, when combined with the conditions (L1) and (L2), implies the following equalities [15]:

$$\begin{aligned}\nabla f &= (\nabla f^*)^{-1}, \\ \text{ran } \nabla f &= \text{dom } \nabla f^* = \text{int dom } f^*, \\ \text{ran } \nabla f^* &= \text{dom } \nabla f = \text{int dom } f.\end{aligned}$$

Also, the conditions (L1) and (L2), in conjunction with [13], imply that the functions f and f^* are strictly convex on the interior of their respective domains. Several interesting examples of the Legendre functions are presented in [13,16]. Especially, the functions $\frac{1}{s} \|\cdot\|^s$ with $s \in (1, \infty)$ are Legendre, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space. Throughout this article, we assume that the convex function $f: E \rightarrow (-\infty, +\infty]$ is Legendre.

Lemma 2.1. [17] *If $f: E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E , then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .*

Let $f: E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f: \text{dom } f \times \text{int dom } f \rightarrow [0, +\infty)$ is defined as follows:

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance* with respect to f [18].

Recall that the *Bregman projection* [19] of $x \in \text{int dom } f$ onto the nonempty, closed, and convex set $C \subset \text{dom } f$ is necessarily the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}.$$

Let $f: E \rightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function f is said to be *totally convex* at $x \in \text{int dom } f$ if its modulus of total convexity at x , that is, the function $v_f: \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty]$ defined by

$$v_f(x, t) := \inf \{D_f(y, x) : y \in \text{dom } f, \|y - x\| = t\}$$

is positive, whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in \text{int dom } f$. In addition, the function f is said to be

totally convex on bounded sets if $v_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f: \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty]$ defined by

$$v_f(B, t) := \inf \{v_f(x, t) : x \in B \cap \text{dom } f\}.$$

Let C be a nonempty, closed, and convex subset of E . Let $f: E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [20] that $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0$ for all $y \in C$. We also have

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall x \in E, y \in C. \quad (2.1)$$

Recall that the function f is said to be *sequentially consistent* [20] if, for any two sequences, $\{x_n\}$ and $\{y_n\}$, in E such that the first is bounded:

$$\lim_{n \rightarrow \infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

The following lemmas were proved by Reich and Sabach [11].

Lemma 2.2. [11] *Let $f: E \rightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}_{n=1}^\infty$ is also bounded.*

We know that the resolvent of A , denoted by $\text{Res}_A^f: E \rightarrow 2^E$, is defined as follows [21]:

$$\text{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(\text{Res}_A^f) = A^{-1}(0^*)$, and Res_A^f is single-valued (see [21]). If f is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded, subsets of E , then $\hat{F}(\text{Res}_A^f) = F(\text{Res}_A^f)$ (see [22]). The *Yosida approximation* $A_\lambda: E \rightarrow E$, $\lambda > 0$, is also defined by

$$A_\lambda(x) = \frac{1}{\lambda} \left(\nabla f(x) - \nabla f(\text{Res}_{\lambda A}^f(x)) \right)$$

for all $x \in E$. From Proposition 2.7 in [11], we know that $(\text{Res}_{\lambda A}^f(x), A_\lambda(x)) \in G(A)$ and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$.

Lemma 2.3. [11] *Let $A: E \rightarrow 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Then,*

$$D_f(p, \text{Res}_{\lambda A}^f(x)) + D_f(\text{Res}_{\lambda A}^f(x), x) \leq D_f(p, x)$$

for all $\lambda > 0$, $p \in A^{-1}(0^*)$ and $x \in E$.

Strong convergence theorems

Now, in this section, we prove our main results of this article.

Theorem 3.1. *Let E be a real reflexive Banach space and $f: E \rightarrow \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E . Let $A_i: E \rightarrow 2^{E^*}$ ($i = 1, 2, \dots, N$) be maximal monotone operators such that $F := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset E$ be such that $\lim_{n \rightarrow \infty} e_n = 0$. Define a sequence $\{x_n\}_{n=1}^\infty$ in E as follows:*

$$\begin{cases} x_1 \in E, \\ C_1 = E, \\ \gamma_n = \text{Res}_{\lambda_n^N A_N}^f \circ \text{Res}_{\lambda_n^{N-1} A_{N-1}}^f \circ \cdots \circ \text{Res}_{\lambda_n^1 A_1}^f (x_n + e_n), \\ C_{n+1} = \{z \in C_n : D_f(z, \gamma_n) \leq D_f(z, x_n + e_n)\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \quad \forall n \geq 1. \end{cases} \quad (3.1)$$

If $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$ for each $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ converges strongly to a point $P_F^f(x_1)$

Proof. We divide our proof into six steps as follows:

Step 1. $F \subset C_n$ for all $n \geq 1$.

Since $A_i^{-1}(0^*)$ is closed and convex for each $i = 1, 2, \dots, N$, we get that $F := \bigcap_{i=1}^N A_i^{-1}(0^*)$ is a nonempty, closed and convex subset of E . It is easy to see that C_n is closed and convex for all $n \geq 1$. Indeed, for each $z \in C_n$, it follows that $D_f(z, \gamma_n) \leq D_f(z, x_n + e_n)$ is equivalent to

$$\langle \nabla f(x_n + e_n) - \nabla f(\gamma_n), z \rangle \leq f(\gamma_n) - f(x_n + e_n) + \langle \nabla f(x_n + e_n), x_n + e_n \rangle - \langle \nabla f(\gamma_n), \gamma_n \rangle.$$

This shows that C_n is closed and convex for all $n \geq 1$. It is obvious that $F \subset C_1 = E$. Now, suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. For any $p \in F$, by Lemma 2.3, we have

$$\begin{aligned} D_f(p, \gamma_k) &= D_f\left(p, \text{Res}_{\lambda_k^N A_N}^f \circ \text{Res}_{\lambda_k^{N-1} A_{N-1}}^f \circ \cdots \circ \text{Res}_{\lambda_k^1 A_1}^f (x_k + e_k)\right) \\ &\leq D_f\left(p, \text{Res}_{\lambda_k^{N-1} A_{N-1}}^f \circ \text{Res}_{\lambda_k^{N-2} A_{N-2}}^f \circ \cdots \circ \text{Res}_{\lambda_k^1 A_1}^f (x_k + e_k)\right) \\ &\quad \dots \\ &\leq D_f\left(p, \text{Res}_{\lambda_k^2 A_2}^f \circ \text{Res}_{\lambda_k^1 A_1}^f (x_k + e_k)\right) \\ &\leq D_f\left(p, \text{Res}_{\lambda_k^1 A_1}^f (x_k + e_k)\right) \\ &\leq D_f(p, x_k + e_k). \end{aligned} \quad (3.2)$$

This implies that $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$.

Step 2. $\lim_{n \rightarrow \infty} D_f(x_n, x_0)$ exists.

From $x_n = P_{C_n}^f(x_1)$ and $x_{n+1} = P_{C_{n+1}}^f(x_1) \in C_{n+1} \subset C_n$ we have

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1), \quad \forall n \geq 1. \quad (3.3)$$

By (2.1), for any $p \in F \subset C_m$ we have

$$D_f(x_n, x_1) = D_f(P_{C_n}^f(x_1), x_1) \leq D_f(p, x_1) - D_f(p, x_n) \leq D_f(p, x_1). \quad (3.4)$$

Combining (3.3) and (3.4), we know that $\lim_{n \rightarrow \infty} D_f(x_n, x_1)$ exists.

Step 3. $\lim_{n \rightarrow \infty} \|\nabla f(\gamma_n) - \nabla f(x_n + e_n)\| = 0$

Since $x_m = P_{C_m}^f(x_1) \in C_m \subset C_n$ for $m > n \geq 1$, by (2.1), it follows that

$$\begin{aligned} D_f(x_m, x_n) &= D_f\left(x_m, P_{C_n}^f(x_1)\right) \leq D_f\left(x_m, x_1\right) - D_f\left(P_{C_n}^f(x_1), x_1\right) \\ &= D_f(x_m, x_1) - D_f(x_n, x_1). \end{aligned}$$

Letting $m, n \rightarrow \infty$, we have $D_f(x_m, x_n) \rightarrow 0$. Since f is totally convex on bounded subsets of E , f is sequentially consistent by Butnariu and Resmerita [20]. It follows that $\|x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. By the

completeness of the space E , we can assume that $x_n \rightarrow q \in E$ as $n \rightarrow \infty$. In particular, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Since $e_n \rightarrow 0$, we also obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - (x_n + e_n)\| = 0. \quad (3.5)$$

Since $x_{n+1} = P_{C_{n+1}}^f(x_1) \in C_{n+1}$

$$\begin{aligned} D_f(x_{n+1}, \gamma_n) &\leq D_f(x_{n+1}, x_n + e_n) \\ &= f(x_{n+1}) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), x_{n+1} - (x_n + e_n) \rangle. \end{aligned}$$

We know from [23] that, if f is bounded on bounded subsets of E , then ∇f is also bounded on bounded subsets of E . Moreover, if f is uniformly Fréchet differentiable on bounded subsets of E , then f is uniformly continuous on bounded subsets of E (see [24]). Using (3.5), we have

$$\lim_{n \rightarrow \infty} D_f(x_{n+1}, \gamma_n) = 0.$$

Also, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - \gamma_n\| = 0$$

and hence,

$$\lim_{n \rightarrow \infty} \|\gamma_n - x_n\| = 0$$

and, since $e_n \rightarrow 0$,

$$\lim_{n \rightarrow \infty} \|\gamma_n - (x_n + e_n)\| = 0. \quad (3.6)$$

Since f is uniformly Fréchet differentiable on bounded subsets of E , ∇f is norm-to-norm uniformly continuous on bounded subsets of E by Lemma 2.1. Hence, we have

$$\lim_{n \rightarrow \infty} \|\nabla f(\gamma_n) - \nabla f(x_n + e_n)\| = 0. \quad (3.7)$$

Step 4. $\lim_{n \rightarrow \infty} \|\nabla f(\Theta_n^i(x_n + e_n)) - \nabla f(\Theta_n^{i-1}(x_n + e_n))\| = 0 \quad \forall i = 1, 2, \dots, N.$

Denote $\Theta_n^i = \text{Res}_{\lambda_n^i A_i}^f \circ \text{Res}_{\lambda_n^{i-1} A_{i-1}}^f \circ \dots \circ \text{Res}_{\lambda_n^1 A_1}^f$ for each $i \in \{1, 2, \dots, N\}$ and $\Theta_n^0 = I$ for each $n \geq 1$. We note that $\gamma_n = \Theta_n^N(x_n + e_n)$ for each $n \geq 1$. For any $p \in F$, by (3.2), it follows that

$$\begin{aligned} D_f(p, \Theta_n^{N-1}(x_n + e_n)) &\leq D_f(p, \Theta_n^{N-2}(x_n + e_n)) \\ &\leq D_f(p, \Theta_n^{N-3}(x_n + e_n)) \\ &\dots \\ &\leq D_f(p, (x_n + e_n)). \end{aligned} \quad (3.8)$$

Since $p \in A_N^{-1}(0^*)$, by Lemma 2.3 and (3.8), it follows that

$$\begin{aligned} & D_f(\gamma_n, \Theta_n^{N-1}(x_n + e_n)) \\ & \leq D_f(p, \Theta_n^{N-1}(x_n + e_n)) - D_f(p, \gamma_n) \\ & \leq D_f(p, (x_n + e_n)) - D_f(p, \gamma_n) \\ & = f(\gamma_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), p - (x_n + e_n) \rangle + \langle \nabla f(\gamma_n), p - \gamma_n \rangle \\ & = f(\gamma_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), p - \gamma_n \rangle \\ & \quad + \langle \nabla f(x_n + e_n), (x_n + e_n) - \gamma_n \rangle + \langle \nabla f(\gamma_n), p - \gamma_n \rangle \\ & = f(\gamma_n) - f(x_n + e_n) + \langle \nabla f(\gamma_n) - \nabla f(x_n + e_n), p - \gamma_n \rangle \\ & \quad + \langle \nabla f(x_n + e_n), (x_n + e_n) - \gamma_n \rangle. \end{aligned}$$

From (3.6) and (3.7), we get that $\lim_{n \rightarrow \infty} D_f(\gamma_n, \Theta_n^{N-1}(x_n + e_n)) = 0$. Since f is sequentially consistent,

$$\lim_{n \rightarrow \infty} \|\gamma_n - \Theta_n^{N-1}(x_n + e_n)\| = 0. \quad (3.9)$$

Thus, from (3.6) and (3.9), it follows that

$$\lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^{N-1}(x_n + e_n)\| = 0 \quad (3.10)$$

and hence,

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-1}(x_n + e_n))\| = 0. \quad (3.11)$$

Again, since $p \in A_N^{-1}(0^*)$, by Lemma 2.3 and (3.8), we know that

$$\begin{aligned} & D_f(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)) \\ & \leq D_f(p, \Theta_n^{N-2}(x_n + e_n)) - D_f(p, \Theta_n^{N-1}(x_n + e_n)) \\ & \leq D_f(p, (x_n + e_n)) - D_f(p, \Theta_n^{N-1}(x_n + e_n)). \end{aligned}$$

From (3.10) and (3.11), we have

$$\lim_{n \rightarrow \infty} D_f(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)) = 0.$$

Since f is sequentially consistent, it follows that

$$\lim_{n \rightarrow \infty} \|\Theta_n^{N-1}(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n)\| = 0. \quad (3.12)$$

From (3.10) and (3.12), we have

$$\lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n)\| = 0,$$

and hence,

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-2}(x_n + e_n))\| = 0.$$

In a similar way, we can show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\Theta_n^{N-2}(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n)\| &= \dots = \lim_{n \rightarrow \infty} \|\Theta_n^1(x_n + e_n) - (x_n + e_n)\| = 0, \\ \lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n)\| &= \dots = \lim_{n \rightarrow \infty} \|(x_n + e_n) - \Theta_n^1(x_n + e_n)\| = 0 \text{ and} \end{aligned}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-3}(x_n + e_n))\| \\ &= \lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-4}(x_n + e_n))\| \\ & \quad \dots \\ &= \lim_{n \rightarrow \infty} \|\nabla f(x_n + e_n) - \nabla f(\Theta_n^1(x_n + e_n))\| \\ &= 0. \end{aligned}$$

Hence, we can conclude that

$$\lim_{n \rightarrow \infty} \|\nabla f(\Theta_n^i(x_n + e_n)) - \nabla f(\Theta_n^{i-1}(x_n + e_n))\| = 0 \quad (3.13)$$

for each $i = 1, 2, \dots, N$.

Step 5. $q \in \bigcap_{i=1}^N A_i^{-1}(0^*)$.

For each $i = 1, 2, \dots, N$, we note that $\Theta_n^i(x_n + e_n) = \text{Res}_{\lambda_n^i A_i}^f \Theta_n^{i-1}(x_n + e_n)$ and so

$$\|A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)\| = \frac{1}{\lambda_n^i} \|\nabla f(\Theta_n^{i-1}(x_n + e_n)) - \nabla f(\Theta_n^i(x_n + e_n))\|.$$

From (3.13) and $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$, we have

$$\lim_{n \rightarrow \infty} \|A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)\| = 0. \quad (3.14)$$

We note that $(\Theta_n^i(x_n + e_n), A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)) \in G(A_i)$ for each $i = 1, 2, \dots, N$. If $(w, w^*) \in G(A_i)$ for each $i = 1, 2, \dots, N$, then it follows from the monotonicity of A_i that

$$\langle w^* - A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n), w - \Theta_n^i(x_n + e_n) \rangle \geq 0.$$

Since $x_n \rightarrow q$ and $e_n \rightarrow 0$, $x_n + e_n \rightarrow q$. Therefore, $\Theta_n^i(x_n + e_n) \rightarrow q$ for each $i = 1, 2, \dots, N$. Thus, from (3.14), we have

$$\langle w^*, w - q \rangle \geq 0.$$

By the maximality of A_i , we have $q \in A_i^{-1}(0^*)$ for each $i = 1, 2, \dots, N$. Hence, $q \in F := \bigcap_{i=1}^N A_i^{-1}(0^*)$.

Step 6. $q = P_F^f(x_1)$.

From $x_n = P_{C_n}^f(x_1)$, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), x_n - z \rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\langle \nabla f(x_1) - \nabla f(x_n), x_n - z \rangle \geq 0, \quad \forall z \in F. \quad (3.15)$$

Letting $n \rightarrow \infty$ in (3.15), we obtain

$$\langle \nabla f(x_1) - \nabla f(q), q - z \rangle \geq 0, \quad \forall z \in F.$$

Hence, we have $q = P_F^f(x_1)$. This completes the proof.

As a direct consequence of Theorem 3.1, we also obtain the following result concerning a system of convex minimization problems in reflexive Banach spaces:

Theorem 3.2. Let E be a real reflexive Banach space and $f: E \rightarrow \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E . Let $g_i: E \rightarrow (-\infty, \infty]$ ($i = 1, 2, \dots, N$) be proper lower semi-continuous convex functions such that $F := \bigcap_{i=1}^N (\partial g_i^{-1})(0) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset E$ be a sequence in E such that $\lim_{n \rightarrow \infty} e_n = 0$. Define a sequence $\{x_n\}_{n=1}^\infty$ in E as follows:

$$\begin{cases} x_1 \in E, \\ C_1 = E, \\ z_n^1 = \arg \min_{y \in E} \left\{ g_1(y) + \frac{1}{\lambda_n^1} D_f(y, x_n + e_n) \right\}, \\ \dots \\ z_n^{N-1} = \arg \min_{y \in E} \left\{ g_{N-1}(y) + \frac{1}{\lambda_n^{N-1}} D_f(y, z_n^{N-2}) \right\}, \\ \gamma_n = \arg \min_{y \in E} \left\{ g_N(y) + \frac{1}{\lambda_n^N} D_f(y, z_n^{N-1}) \right\}, \\ C_{n+1} = \{z \in C_n : D_f(z, \gamma_n) \leq D_f(z, x_n + e_n)\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \quad \forall n \geq 1. \end{cases} \quad (3.16)$$

If $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$ for each $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ converges strongly to a point $P_F^f(x_1)$.

Proof. By Rockafellar's theorem [25,26], ∂g_i are maximal monotone operators for each $i = 1, 2, \dots, N$. Let $\lambda^i > 0$ for each $i = 1, 2, \dots, N$. Then $z^i = \text{Res}_{\lambda^i \partial g_i}^f(x)$ if and only if

$$\begin{aligned} 0 &\in \partial g_i(z^i) + \frac{1}{\lambda^i} (\nabla f(z^i) - \nabla f(x)) \\ &= \partial \left(g_i + \frac{1}{\lambda^i} (f - \nabla f(x)) \right) (z^i), \end{aligned}$$

which is equivalent to

$$\begin{aligned} z^i &= \arg \min_{y \in E} \left\{ g_i(y) + \frac{1}{\lambda^i} (f(y) - \langle y, \nabla f(x) \rangle) \right\} \\ &= \arg \min_{y \in E} \left\{ g_i(y) + \frac{1}{\lambda^i} D_f(y, x) \right\}. \end{aligned}$$

Using Theorem 3.1, we can complete the proof.

Remark 3.3. By means of the composite iterative scheme together with the shrinking projection method, we can construct the proximal point algorithms for finding a common element in the set $\bigcap_{i=1}^N A_i^{-1}(0^*)$. Moreover, our algorithm is different from that of Reich and Sabach [11] which is based on a finite intersection of sets.

Remark 3.4. Theorems 3.1 and 3.2 also hold in a uniformly convex and uniformly smooth Banach space with the generalized duality mapping.

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Authors' contributions

PC designed of the study, performed the nonlinear and convex analysis and also wrote the article. YJC participated in the design of the study, carried out the materials and helped to check the manuscript. SS conceived of the study, participated in its design and also helped to draft the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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Common fixed points of an infinite family of nonexpansive mappings in uniformly convex metric spaces

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ABSTRACT

We generalize the notion of K -mapping due to Kangtunyakarn and Suantai (2009) [1] for an infinite family of nonexpansive mappings on a convex metric space and study some of its useful properties. We also establish strong convergence of a Mann type iterative sequence to a common fixed point of an infinite family of nonexpansive mappings on a uniformly convex metric space.

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1. Introduction

Let C be a nonempty closed convex subset of a metric space (X, d) , and let $T : C \rightarrow C$ be a mapping. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. A mapping T is called *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$.

Kangtunyakarn and Suantai [1] introduced the K -mapping in a Banach space as follows. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_i\}_{i=1}^N$ be a real sequence in $[0, 1]$. Let $K : C \rightarrow C$ be a mapping defined by

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3)U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\ K &= U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}, \end{aligned} \tag{1.1}$$

where I is the identity mapping of C . Such a mapping K is called the K -mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$. They proved that if X is a strictly convex Banach space, then $F(K) = \bigcap_{i=1}^N F(T_i)$ where $0 < \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$.

Takahashi [2] introduced a notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \rightarrow X$ is said to be a *convex structure* on X if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$$

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for all $z \in X$. A metric space (X, d) together with a convex structure W is called a *convex metric space* which will be denoted by (X, d, W) . A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. Clearly, a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold.

Motivated by the above results, we consider the K -mapping in a convex metric space as follows: For $\{\lambda_n\}$ a sequence in $[0, 1]$ and $x \in X$, mappings $U_1, U_2, \dots, U_n$ and K_n are defined by

$$\begin{aligned} U_1x &= W(T_1x, x, \lambda_1), \\ U_2x &= W(T_2U_1x, U_1x, \lambda_2), \\ U_3x &= W(T_3U_2x, U_2x, \lambda_3), \\ &\vdots \\ U_{n-1}x &= W(T_{n-1}U_{n-2}x, U_{n-2}x, \lambda_{n-1}), \\ K_nx &= U_nx = W(T_nU_{n-1}x, U_{n-1}x, \lambda_n). \end{aligned}$$

Such a mapping K_n is called the K -mapping generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$.

In this paper, we consider the K -mapping generated by an infinite family of nonexpansive mappings in convex metric spaces. We give some useful properties of the common fixed point set of the family of nonexpansive mappings and then prove a Mann type convergence theorem in a uniformly convex metric space.

2. Preliminaries

A convex metric space (X, d, W) has the following properties, see more in [2,3]. For each $x, y \in X$ and $\lambda \in [0, 1]$,

- (i) $W(x, x, \lambda) = x$, $W(x, y, 0) = y$ and $W(x, y, 1) = x$;
- (ii) $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$ and $d(y, W(x, y, \lambda)) = \lambda d(x, y)$;
- (iii) $d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)$.

In 1996, Shimizu and Takahashi [4] introduced the concept of uniform convexity in convex metric spaces and studied some properties of these spaces. A convex metric space (X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, 1]$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$ imply that $d(z, W(x, y, \frac{1}{2})) \leq (1 - \delta)r$. Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

We say that a convex metric space (X, d, W) has the *property (H)* if

$$d(W(x, y, \lambda), W(x, z, \lambda)) \leq (1 - \lambda)d(y, z)$$

for all $x, y, z \in X$ and $\lambda \in [0, 1]$. It is obvious that the property (H) implies continuity of a convex structure W . By the corresponding results of Aoyama et al. [3], Shimizu [5] and Leuştean [6], we conclude that a uniformly convex metric space with the property (H) satisfies the following inequality:

$$d(W(x, y, \lambda), W(z, w, \lambda)) \leq \lambda d(x, z) + (1 - \lambda)d(y, w)$$

for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$.

The following lemma is important to prove our main results.

Lemma 2.1 ([5, Lemma 1]). *Let (X, d, W) be a uniformly convex metric space with continuous convex structure. Then for arbitrary positive number ε , there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$d(z, W(x, y, \lambda)) \leq (1 - 2 \min\{\lambda, 1 - \lambda\}\eta)r$$

for all $r > 0$ and $x, y, z \in X$, $d(z, x) \leq r$, $d(z, y) \leq r$, $d(x, y) \geq r\varepsilon$ and $\lambda \in [0, 1]$.

3. Main results

Lemma 3.1. *Let C be a nonempty closed convex subset of a complete convex metric space (X, d, W) . Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Then for every $x \in C$, $\lim_{n \rightarrow \infty} K_nx$ exists.*

Proof. Let $z \in \bigcap_{i=1}^{\infty} F(T_i)$ and $x \in C$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(K_nx, z) &= d(W(T_nU_{n-1}x, U_{n-1}x, \lambda_n), z) \\ &\leq \lambda_n d(T_nU_{n-1}x, z) + (1 - \lambda_n)d(U_{n-1}x, z) \\ &\leq d(U_{n-1}x, z) \\ &\vdots \\ &\leq d(U_1x, z) \\ &\leq \lambda_1 d(T_1x, z) + (1 - \lambda_1)d(x, z) \\ &\leq d(x, z), \end{aligned}$$

which implies that $\{K_n x\}$ is bounded, so is $\{T_{n+1} K_n x\}$. It follows by $K_n x = U_n x$ that

$$K_{n+1} x = W(T_{n+1} U_n x, U_n x, \lambda_{n+1}) = W(T_{n+1} K_n x, K_n x, \lambda_{n+1}).$$

Thus

$$d(K_{n+1} x, K_n x) = \lambda_{n+1} d(T_{n+1} K_n x, K_n x). \quad (3.1)$$

For $m > n$, it follows by (3.1) that

$$\begin{aligned} d(K_m x, K_n x) &\leq d(K_m x, K_{m-1} x) + \cdots + d(K_{n+1} x, K_n x) \\ &= \lambda_m d(T_m K_{m-1} x, K_{m-1} x) + \cdots + \lambda_{n+1} d(T_{n+1} K_n x, K_n x) \\ &\leq M \sum_{k=n+1}^m \lambda_k, \end{aligned}$$

where $M = \sup \{d(T_{n+1} K_n x, K_n x) : n \in \mathbb{N}\}$. This implies that $\{K_n x\}$ is a Cauchy sequence. Hence $\lim_{n \rightarrow \infty} K_n x$ exists. $\square$

From the above, we can define a mapping $K : C \rightarrow C$ as follows

$$Kx = \lim_{n \rightarrow \infty} K_n x, \quad x \in C.$$

Such a mapping K is called the K -mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$ and we show in the next theorem that $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.

Theorem 3.2. Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K be the K -mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$. Then, K is a nonexpansive mapping and $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. First, we show that K is a nonexpansive mapping. For $x, y \in C$, we have

$$\begin{aligned} d(K_n x, K_n y) &= d(W(T_n U_{n-1} x, U_{n-1} x, \lambda_n), W(T_n U_{n-1} y, U_{n-1} y, \lambda_n)) \\ &\leq \lambda_n d(T_n U_{n-1} x, T_n U_{n-1} y) + (1 - \lambda_n) d(U_{n-1} x, U_{n-1} y) \\ &\leq d(U_{n-1} x, U_{n-1} y) \\ &\vdots \\ &\leq d(U_1 x, U_1 y) \\ &\leq \lambda_1 d(T_1 x, T_1 y) + (1 - \lambda_1) d(x, y) \\ &\leq d(x, y). \end{aligned}$$

This implies that K_n is a nonexpansive mapping and we have

$$d(Kx, Ky) = \lim_{n \rightarrow \infty} d(K_n x, K_n y) \leq d(x, y).$$

Thus K is also a nonexpansive mapping. Next, we show that $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$. Let $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Then, it is obvious that $U_i p = p$ for all $i \in \mathbb{N}$, so $p \in F(K_n)$ for all $n \in \mathbb{N}$, which implies that $p \in F(K)$. Hence $\bigcap_{i=1}^{\infty} F(T_i) \subseteq F(K)$. We now show that $F(K) \subseteq \bigcap_{i=1}^{\infty} F(T_i)$. Let $k \in \mathbb{N}$ be fixed with $n > k$. For each $x \in F(K)$ and $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we observe that

$$\begin{aligned} d(p, K_n x) &= d(p, W(T_n U_{n-1} x, U_{n-1} x, \lambda_n)) \\ &\leq \lambda_n d(p, T_n U_{n-1} x) + (1 - \lambda_n) d(p, U_{n-1} x) \\ &\leq d(p, U_{n-1} x) \\ &\vdots \\ &\leq d(p, U_{k+1} x) \\ &= d(p, W(T_{k+1} U_k x, U_k x, \lambda_{k+1})) \\ &\leq \lambda_{k+1} d(p, T_{k+1} U_k x) + (1 - \lambda_{k+1}) d(p, U_k x) \\ &\leq d(p, U_k x) \\ &\vdots \\ &\leq d(p, U_1 x) \\ &= d(p, W(T_1 x, x, \lambda_1)) \\ &\leq \lambda_1 d(p, T_1 x) + (1 - \lambda_1) d(p, x) \\ &\leq d(p, x). \end{aligned} \quad (3.2)$$

This implies that $d(p, x) = \lim_{n \rightarrow \infty} d(p, K_n x) \leq d(p, W(T_1 x, x, \lambda_1)) \leq d(p, x)$. Thus, we have

$$d(p, W(T_1 x, x, \lambda_1)) = d(p, x) \quad \text{and} \quad d(p, T_1 x) = d(p, x).$$

Suppose that $x \neq T_1 x$. Then $d(x, T_1 x) > 0$. It follows from Lemma 2.1 that

$$d(p, W(T_1 x, x, \lambda_1)) < d(p, x) \quad \text{for } \lambda_1 \in (0, 1),$$

which is a contradiction. Hence $x = T_1 x$, that is $x \in F(T_1)$. Also we have $U_1 x = x$. By the inequality (3.2), we get $d(p, K_n x) \leq d(p, W(T_2 U_1 x, U_1 x, \lambda_2)) = d(p, W(T_2 x, x, \lambda_2))$. This implies that $d(p, x) = \lim_{n \rightarrow \infty} d(p, K_n x) \leq d(p, W(T_2 x, x, \lambda_2)) \leq d(p, x)$. Thus, we have

$$d(p, W(T_2 x, x, \lambda_2)) = d(p, x) \quad \text{and} \quad d(p, T_2 x) = d(p, x).$$

Suppose that $x \neq T_2 x$. Then $d(x, T_2 x) > 0$. Again, by Lemma 2.1, we have

$$d(p, W(T_2 x, x, \lambda_2)) < d(p, x) \quad \text{for } \lambda_2 \in (0, 1),$$

which is a contradiction. Hence $x = T_2 x$, that is $x \in F(T_2)$. Also we have $U_2 x = x$. By using the same argument, we conclude that $T_i x = x$ and $U_i x = x$ for $i = 1, 2, \dots, k$. By the inequality (3.2), we get $d(p, K_n x) \leq d(p, W(T_{k+1} U_k x, U_k x, \lambda_{k+1})) = d(p, W(T_{k+1} x, x, \lambda_{k+1}))$. It follows that

$$\begin{aligned} d(p, x) &= \lim_{n \rightarrow \infty} d(p, K_n x) \leq d(p, W(T_{k+1} x, x, \lambda_{k+1})) \\ &\leq \lambda_{k+1} d(p, T_{k+1} x) + (1 - \lambda_{k+1}) d(p, x) \\ &\leq d(p, x). \end{aligned}$$

Thus, we have

$$d(p, W(T_{k+1} x, x, \lambda_{k+1})) = d(p, x) \quad \text{and} \quad d(p, T_{k+1} x) = d(p, x).$$

Suppose that $x \neq T_{k+1} x$. Then $d(x, T_{k+1} x) > 0$. By Lemma 2.1, we have

$$d(p, W(T_{k+1} x, x, \lambda_{k+1})) < d(p, x) \quad \text{for } \lambda_{k+1} \in (0, 1).$$

This is a contradiction. Hence $x = T_{k+1} x$, that is $x \in F(T_{k+1})$. By induction, we have $x \in F(T_n)$ for all $n \in \mathbb{N}$. Hence $x \in \bigcap_{i=1}^{\infty} F(T_i)$. Therefore, $F(K) \subseteq \bigcap_{i=1}^{\infty} F(T_i)$. $\square$

Next, we prove a convergence theorem for the K -mapping in uniformly convex metric spaces.

Theorem 3.3. Let C be a nonempty compact convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K -mappings generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$. Assume that $x_1 \in C$ and the sequence $\{x_n\}$ generated by

$$x_{n+1} = W(x_n, K_n x_n, \alpha_n),$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges to a common fixed point of $\{T_i\}$.

Proof. Suppose that $\inf_n d(x_n, K_n x_n) = r_1 > 0$. Let $z \in \bigcap_{i=1}^{\infty} F(T_i)$. Since K_n is a nonexpansive mapping, therefore we have

$$\begin{aligned} d(z, x_{n+1}) &= d(z, W(x_n, K_n x_n, \alpha_n)) \\ &\leq \alpha_n d(z, x_n) + (1 - \alpha_n) d(z, K_n x_n) \\ &\leq d(z, x_n). \end{aligned}$$

Hence $\{d(z, x_n)\}$ is nonincreasing, and also $\lim_{n \rightarrow \infty} d(z, x_n)$ exists. Thus, we may assume that $\lim_{n \rightarrow \infty} d(z, x_n) = r_2 > 0$. Since $d(z, K_n x_n) \leq d(z, x_n)$ and

$$d(x_n, K_n x_n) \geq d(z, x_n) \frac{r_1}{d(z, x_n)} \geq d(z, x_n) \frac{r_1}{d(z, x_1)} > 0,$$

for all $n \geq 1$, we have by Lemma 2.1 that there exists $\eta \left(\frac{r_1}{d(z, x_1)} \right) > 0$ such that

$$\begin{aligned} d(z, x_{n+1}) &= d(z, W(x_n, K_n x_n, \alpha_n)) \\ &\leq d(z, x_n) (1 - 2 \min \{\alpha_n, 1 - \alpha_n\} \eta) \\ &\leq d(z, x_n) (1 - 2\alpha_n (1 - \alpha_n) \eta). \end{aligned}$$

Hence, we have

$$2\eta\alpha_n(1 - \alpha_n)r_2 \leq 2\eta\alpha_n(1 - \alpha_n)d(z, x_n) \leq d(z, x_n) - d(z, x_{n-1})$$

for all $n \geq 1$. Therefore, we get

$$2r_2\eta \sum_{n=1}^m \alpha_n(1 - \alpha_n) \leq d(z, x_1) - d(z, x_m)$$

for all $m \geq 1$. By letting $m \rightarrow \infty$ in the above inequality, we have $\infty \leq d(z, x_1) - r_2 < \infty$, which is a contradiction. Hence, $\inf_n d(x_n, K_n x_n) = 0$. Thus there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i \rightarrow \infty} d(x_{n_i}, K_{n_i} x_{n_i}) = 0$. Since C is compact, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ and $p \in C$ such that $\lim_{j \rightarrow \infty} x_{n_{ij}} = p$. Let K be the K -mappings generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$. Thus, we have

$$\begin{aligned} d(p, Kp) &\leq d(p, x_{n_{ij}}) + d(x_{n_{ij}}, K_{n_{ij}} x_{n_{ij}}) + d(K_{n_{ij}} x_{n_{ij}}, K_{n_{ij}} p) + d(K_{n_{ij}} p, Kp) \\ &\leq 2d(p, x_{n_{ij}}) + d(x_{n_{ij}}, K_{n_{ij}} x_{n_{ij}}) + d(K_{n_{ij}} p, Kp) \rightarrow 0. \end{aligned}$$

Hence $p \in F(K)$. By $\lim_{n \rightarrow \infty} d(p, x_n)$ exists, we have $\{x_n\}$ converges to $p \in F(K) = \bigcap_{i=1}^{\infty} F(T_i)$. $\square$

Remark 3.4.

- (i) Uniformly convex hyperbolic spaces and CAT(0) spaces are examples of uniformly convex metric spaces having the property (H); see more details in [6–8]. Therefore, Lemma 3.1, Theorems 3.2 and 3.3 are also true for these two classes of spaces.
- (ii) The reader interested in convergence of Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a complete uniformly convex metric space is referred to Khan et al. [9, Theorem 2.2].

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Research Article

A Hybrid Iterative Scheme for Mixed Equilibrium Problems, General System of Variational Inequality Problems, and Fixed Point Problems in Hilbert Spaces

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We introduce a new iterative scheme for finding a common element of the set of solutions of a general system of variational inequalities, the set of solutions of a mixed equilibrium problem, and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Using the demiclosedness principle for nonexpansive mappings, we prove that the iterative sequence converges strongly to a common element of the above three sets under some control conditions, and we also give some examples for mappings which satisfy conditions of the main result.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H . A mapping $T : C \rightarrow C$ is said to be *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Halpern [1] studied the following iteration formula for approximating a fixed point of T . For an arbitrary $v \in C$, let the sequence $\{x_n\}$ be defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n v + (1 - \alpha_n)Tx_n, \quad n \geq 1, \quad (1.1)$$

where $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$ that satisfies the following conditions: $C1 : \lim_{n \rightarrow \infty} \alpha_n = 0$ and $C2 : \sum_{n=1}^{\infty} \alpha_n = \infty$.

Actually, Halpern studied the special case of (1.1) in which $\alpha_n = n^{-\sigma}$, $\sigma \in (0, 1)$, and $v = 0$ and proved that $\{x_n\}$ converges strongly to a fixed point of T . Under a different restriction on the parameter $\{\alpha_n\}$, Lions [2] improved the result of Halpern, still in Hilbert spaces. He proved strong convergence of $\{x_n\}$ to a fixed point of T , where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0, \quad C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \quad C3 : \lim_{n \rightarrow \infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}^2} = 0. \quad (1.2)$$

Reich [3] proved that the result of Halpern remains true when H is uniformly smooth. It was observed that both Halpern's and Lion's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$. This was overcome by Wittmann [4] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0, \quad C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \quad C3 : \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \quad (1.3)$$

Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (1.4)$$

The variational inequality, nonconvex variational inequality, and mixed variational inequality have been widely studied in the literature; see, for example, Ceng et al. [5–14], Chang et al. [15], Noor [16–18], Peng and Yao [19–21], Plubtieng and Punpaeng [22], Yao et al. [23], Zeng and Yao [24], Zhao and He [25], and the references therein.

For solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k -Lipschitz continuous, and $VI(C, A)$ is nonempty, Korpelevich [26] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda Ax_n), \\ x_{n+1} &= P_C(x_n - \lambda Ay_n), \end{aligned} \quad (1.5)$$

for every $n = 0, 1, 2, \dots$, where $\lambda \in (0, 1/k)$ and P_C is the projection of $\mathbb{R}^n$ onto C . He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in VI(C, A)$. In 2007, Y. Yao and J. C. Yao [27] introduced a new iterative scheme for finding the common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for α -inverse-strongly monotone mappings in a real Hilbert space.

Let C be a nonempty closed convex subset of a real Hilbert space H .

Let $A, B : C \rightarrow H$ be two mappings. In 2008, Ceng et al. [12] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.6)$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then the problem (1.6) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.7)$$

which was defined by Verma [28] and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then the problem (1.7) reduces to the classical variational inequality $VI(C, A)$. Ceng et al. [12] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of the problem (1.6) for the α - and β -inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $x_1 = v \in C$, and $\{x_n\}, \{y_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \mu Bx_n), \\ x_{n+1} &= a_nv + b_nx_n + (1 - a_n - b_n)SP_C(y_n - \lambda Ay_n), \quad n \geq 1, \end{aligned} \quad (1.8)$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{a_n\}, \{b_n\} \subset [0, 1]$. Then, they proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the problem (1.6) under some control conditions.

Let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and F a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [13] considered the following mixed equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (1.9)$$

The set of solutions of (1.9) is denoted by $MEP(F, \varphi)$. It is easy to see that x is a solution of the problem (1.9) implying that $x \in \text{dom } \varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the mixed equilibrium problem (1.9) becomes the following equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (1.10)$$

If $F = 0$, then the mixed equilibrium problem (1.9) reduces to the convex minimization problem

$$\text{find } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (1.11)$$

The set of solutions of (1.10) is denoted by $EP(F)$.

If $\varphi = 0$ and $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$, where A is a mapping from C into H , then (1.9) reduces to the classical variational inequality and $EP(F) = VI(C, A)$.

The equilibrium problems and variational inequality problems can be applied for the problems in science and technology, economics, optimizations, and control theory. The problems of finding a solution of the systems of variational inequalities which is also a solution of the mixed equilibrium problems and the problem of finding a solution of the mixed equilibrium problems under the constraints that it is also a solution of the system of variational inequalities are very useful and important for studying problems in science and applied science.

Inspired and motivated by these facts, we introduce a new iteration process for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a general system of variational inequalities in a real Hilbert space. Start with an arbitrary $v \in C$, and let $x_1 \in C$, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be the sequences generated by

$$\begin{aligned} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= P_C(u_n - \mu B u_n), \\ x_{n+1} &= a_n v + (1 - a_n) T P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{aligned} \quad (1.12)$$

where $\lambda > 0$ and $\mu > 0$ are two constants, $\{r_n\} \subset (0, \infty)$ and $\{a_n\} \subset [0, 1]$. Using the demiclosedness principle for nonexpansive mappings, we will show that the sequence $\{x_n\}$ converges strongly to a common element of the above three sets under some control conditions.

2. Preliminaries

In this section, we recall the well-known results and give some useful lemmas that will be used in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H .

A mapping $A : C \rightarrow H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \quad (2.1)$$

A is called α -strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C. \quad (2.2)$$

This implies that

$$\|Ax - Ay\| \geq \alpha \|x - y\|, \quad (2.3)$$

that is, A is α -expansive and, when $\alpha = 1$, it is *expansive*.

We can see easily that the following implications in monotonicity, strong monotonicity, and expansiveness hold:

$$\begin{aligned} & \text{strong monotonicity} \Rightarrow \text{monotonicity} \\ & \Downarrow \\ & \text{expansiveness.} \end{aligned} \tag{2.4}$$

The mapping A is called *L-Lipschitz continuous* (or *Lipschitzian*) if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

A is called *α -inverse-strongly monotone* (or *α -cocoercive*) if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.6}$$

It is obvious that every α -inverse-strongly monotone mapping A is monotone and Lipschitz continuous. Also, if A is α -strongly and L -Lipschitz continuous, then A is (α/L^2) -inverse-strongly monotone, but inverse-strongly monotone need not be strongly monotone.

A mapping A is called *relaxed c -cocoercive*, if there exists a constant $c > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-c) \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.7}$$

A mapping A is called *relaxed (c, d) -cocoercive*, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-c) \|Ax - Ay\|^2 + d \|x - y\|^2, \quad \forall x, y \in C. \tag{2.8}$$

For $c = 0$, A is d -strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. As a result, we have the following implication: d -strong monotonicity $\Rightarrow$ relaxed (c, d) -cocoercivity.

It is known that if the operator is Lipschitz continuous, then the relaxed cocoercivity is strongly monotone, but the strongly monotone does not imply the cocoercivity as shown in the following example.

Example 2.1. Let $H = \mathbb{R}$, $C = [1, \infty)$, and $A : C \rightarrow H$ be defined by $Ax = x^2$, $x \in C$. For $x, y \in C$, we have

$$\begin{aligned} \langle Ax - Ay, x - y \rangle &= (x^2 - y^2)(x - y) \\ &= (x + y)|x - y|^2 \\ &\geq 2|x - y|^2. \end{aligned} \tag{2.9}$$

Hence, A is 2-strongly monotone. If A is μ -cocoercive for some $\mu > 0$, then $\langle Ax - Ay, x - y \rangle = (x+y)|x-y|^2 \geq \mu|x^2 - y^2|^2$. This implies $x+y \leq 1/\mu$ for all $x, y \in [1, \infty)$ which is a contradiction. Hence, A is not μ -cocoercive for any $\mu > 0$.

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \quad (2.10)$$

P_C is called the *metric projection* of H onto C . It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.11)$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.12)$$

Recall that $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} \langle x - P_C x, y - P_C x \rangle &\leq 0, \\ \|x - y\|^2 &\geq \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{aligned} \quad (2.13)$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [29] for more details.

For solving the mixed equilibrium problem, let us assume the following assumptions for the bifunction F, φ and the set C :

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex;
- (A5) for each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
- (B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z); \quad (2.14)$$

- (B2) C is a bounded set.

In the sequel we will need to use the following lemma.

Lemma 2.2 (see [30]). *Let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \forall y \in C \right\} \quad (2.15)$$

for all $x \in H$. Then the following conclusions hold:

- (1) for each $x \in H$, $T_r(x) \neq \emptyset$;
- (2) T_r is single valued;
- (3) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \quad (2.16)$$

- (4) $F(T_r) = \text{MEP}(F, \varphi)$;
- (5) $\text{MEP}(F, \varphi)$ is closed and convex.

We also need the following lemmas.

Lemma 2.3 (see [31]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + t_n c_n, \quad \forall n \geq 1, \quad (2.17)$$

where $\{t_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions

$$(i) \sum_{n=1}^{\infty} t_n = \infty, \quad (ii) \sum_{n=1}^{\infty} b_n < \infty, \quad (iii) \limsup_{n \rightarrow \infty} c_n \leq 0. \quad (2.18)$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 (see [32]). *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2. \quad (2.19)$$

Lemma 2.5 (see [29] (demiclosedness principle)). *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$) and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \rightarrow y$), it follows that $(I - T)x = y$. Here I is the identity operator of H .*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \quad (2.20)$$

In 2009, Kangtunyakarn and Suantai [33] introduced a new mapping called the S -mapping as follows.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \dots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. Very recently, in 2009, Kangtunyakarn and Suantai [33] introduced the new mapping $S_n : C \rightarrow C$ as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I, \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I, \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I, \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I, \\ S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \end{aligned} \quad (2.21)$$

The mapping S_n is called the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. Nonexpansivity of each T_i ensures the nonexpansivity of S_n . They also showed the following useful fact.

Lemma 2.7 (see [33]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \dots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$, $\alpha_1^N \in (0, 1]$ and $\alpha_2^j, \alpha_3^j \in [0, 1]$ for all $j = 1, 2, \dots, N$. Let S be the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.8 (see [12]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.6) if and only if x^* is a fixed point of the mapping $G : C \rightarrow C$ defined by*

$$G(x) = P_C [P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C, \quad (2.22)$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $\text{GVI}(C, A, B)$.

Next, we prove a lemma which is very useful for our consideration.

Lemma 2.9. Let $G : C \rightarrow C$ be defined by

$$G(x) = P_C [P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C, \quad (2.23)$$

where $\lambda, \mu > 0$ and $A, B : C \rightarrow H$ are two mappings. If $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings, then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned} \|G(x) - G(y)\| &= \|P_C [P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)] \\ &\quad - P_C [P_C(y - \mu By) - \lambda A P_C(y - \mu By)]\|^2 \\ &\leq \|P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx) - (P_C(y - \mu By) - \lambda A P_C(y - \mu By))\| \\ &= \|(I - \lambda A)P_C(I - \mu B)x - (I - \lambda A)P_C(I - \mu B)y\| \\ &\leq \|x - y\|. \end{aligned} \quad (2.24)$$

This shows that G is a nonexpansive mapping. $\square$

3. Main Results

In this section, we prove strong convergence theorems of the iterative scheme (1.12) to a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a general system of variational inequality in a real Hilbert space.

Theorem 3.1. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper lower semi-continuous and convex function. Let $\lambda, \mu > 0$, and let $A, B : C \rightarrow H$ be such that $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be defined by (1.12), $\{r_n\} \subset (0, \infty)$ and $\{a_n\} \subset [0, 1]$ such that

$$(C1) \lim_{n \rightarrow \infty} a_n = 0, \lim_{n \rightarrow \infty} (a_{n+1} / a_n) = 1, \text{ and } \sum_{n=1}^{\infty} a_n = \infty,$$

$$(C2) \liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

If $\lim_{n \rightarrow \infty} \|Ay_n - Ay^*\| = 0$ and $\lim_{n \rightarrow \infty} \|Bu_n - Bx^*\| = 0$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$, then $\{x_n\}$ converges strongly to $\bar{x} = P_{\Omega}v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ a sequence of mappings defined as in Lemma 2.2. It follows from Lemma 2.8 that

$$x^* = P_C [P_C(x^* - \mu Bx^*) - \lambda A P_C(x^* - \mu Bx^*)]. \quad (3.1)$$

Put $y^* = P_C(x^* - \mu Bx^*)$ and $t_n = P_C(y_n - \lambda Ay_n)$, then $x^* = P_C(y^* - \lambda Ay^*)$ and

$$x_{n+1} = a_nv + (1 - a_n)Tt_n. \quad (3.2)$$

Since $I - \lambda A$, $I - \mu B$, T_{r_n} , and P_C are nonexpansive mappings, we have

$$\begin{aligned} \|t_n - x^*\|^2 &= \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ &\leq \|y_n - y^*\|^2 \\ &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\ &\leq \|x_n - x^*\|^2, \end{aligned} \quad (3.3)$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_nv + (1 - a_n)Tt_n - x^*\| \\ &\leq a_n\|v - x^*\| + (1 - a_n)\|t_n - x^*\| \\ &\leq a_n\|v - x^*\| + (1 - a_n)\|x_n - x^*\| \\ &\leq \max\{\|v - x^*\|, \|x_1 - x^*\|\}. \end{aligned} \quad (3.4)$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bu_n\}$, and $\{Tt_n\}$ are also bounded. Also, observe that

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|P_C(y_{n+1} - \lambda Ay_{n+1}) - P_C(y_n - \lambda Ay_n)\| \\ &\leq \|y_{n+1} - y_n\| \\ &= \|P_C(u_{n+1} - \mu Bu_{n+1}) - P_C(u_n - \mu Bu_n)\| \\ &\leq \|u_{n+1} - u_n\|. \end{aligned} \quad (3.5)$$

On the other hand, from $u_n = T_{r_n}x_n \in \text{dom } \varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \text{dom } \varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (3.6)$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \quad (3.7)$$

Putting $y = u_{n+1}$ in (3.6) and $y = u_n$ in (3.7), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle &\geq 0, \\ F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle &\geq 0. \end{aligned} \quad (3.8)$$

From the monotonicity of F , we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \quad (3.9)$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \quad (3.10)$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right), \end{aligned} \quad (3.11)$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \end{aligned} \quad (3.12)$$

It follows from (3.5) and (3.12) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \quad (3.13)$$

Next, we show that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (3.13), we have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &= \|a_{n+1}v + (1 - a_{n+1})Tt_{n+1} - a_nv - (1 - a_n)Tt_n\| \\ &= \|(a_{n+1} - a_n)v + (a_n - a_{n+1})Tt_{n+1} + (1 - a_n)(Tt_{n+1} - Tt_n)\| \\ &\leq |a_{n+1} - a_n|(\|v\| + \|Tt_{n+1}\|) + (1 - a_n)\|t_{n+1} - t_n\| \end{aligned}$$

$$\begin{aligned}
&\leq |a_{n+1} - a_n|(\|v\| + \|Tt_{n+1}\|) + (1 - a_n)\|x_{n+1} - x_n\| \\
&\quad + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\leq (1 - a_n)\|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n| M + 2M|a_{n+1} - a_n|,
\end{aligned} \tag{3.14}$$

where $M = \max\{\|v\|, \sup_n \|Tt_{n+1}\|, \sup_n \|u_{n+1} - x_{n+1}\|\}$. By the assumptions on $\{a_n\}$ and $\{r_n\}$ and Lemma 2.3, we conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

From (C1), (C2), (3.5), (3.12), and (3.15), we also have $\|u_{n+1} - u_n\| \rightarrow 0$, $\|t_{n+1} - t_n\| \rightarrow 0$, and $\|y_{n+1} - y_n\| \rightarrow 0$, as $n \rightarrow \infty$.

Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n)(Tt_n - x_n), \tag{3.16}$$

we have that

$$\|Tt_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

Next, we prove that $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. From Lemma 2.2(3), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \leq \langle T_{r_n} x_n - T_{r_n} x^*, x_n - x^* \rangle \\
&= \langle u_n - x^*, x_n - x^* \rangle = \frac{1}{2} \left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \right).
\end{aligned} \tag{3.18}$$

Hence,

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2. \tag{3.19}$$

From Lemma 2.4, (3.3), and (3.19), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|u_n - x^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|x_n - x^*\|^2 - \|x_n - u_n\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n) \|x_n - u_n\|^2.
\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned} (1 - a_n)\|x_n - u_n\|^2 &\leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq a_n\|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|. \end{aligned} \quad (3.21)$$

From conditions (C1) and (3.15), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.22)$$

By (3.17) and (3.22), we have

$$\|Tt_n - u_n\| \leq \|Tt_n - x_n\| + \|x_n - u_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.23)$$

Next, we prove that $\|Tt_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (2.11) and nonexpansiveness of $I - \mu B$, we get

$$\begin{aligned} \|y_n - y^*\|^2 &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\leq \langle (u_n - \mu Bu_n) - (x^* - \mu Bx^*), y_n - y^* \rangle \\ &= \frac{1}{2} \left[\|(u_n - \mu Bu_n) - (x^* - \mu Bx^*)\|^2 + \|y_n - y^*\|^2 \right. \\ &\quad \left. - \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*) - (y_n - y^*)\|^2 \right] \\ &\leq \frac{1}{2} \left[\|u_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \right. \\ &\quad \left. + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \right]. \end{aligned} \quad (3.24)$$

By (3.3), we obtain

$$\begin{aligned} \|y_n - y^*\|^2 &\leq \|u_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \\ &\quad + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \\ &\quad + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2. \end{aligned} \quad (3.25)$$

Hence,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|y_n - y^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \\
&\quad \times \left[\|x_n - x^*\|^2 - \|(u_n - x^*) - (y_n - y^*)\|^2 \right. \\
&\quad \left. + 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \|Bu_n - Bx^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n) \|(u_n - x^*) - (y_n - y^*)\|^2 \\
&\quad + (1 - a_n) 2\mu \|(u_n - x^*) - (y_n - y^*)\| \|Bu_n - Bx^*\|,
\end{aligned} \tag{3.26}$$

which implies that

$$\begin{aligned}
(1 - a_n) \|(u_n - x^*) - (y_n - y^*)\|^2 &\leq a_n \|v - x^*\|^2 \\
&\quad + (1 - a_n) 2\mu \|(u_n - x^*) - (y_n - y^*)\| \|Bu_n - Bx^*\| \\
&\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.27}$$

By this together with (C1), (3.15), and $\lim_{n \rightarrow \infty} \|Bu_n - Bx^*\| = 0$, we obtain

$$\|(u_n - x^*) - (y_n - y^*)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.28}$$

From Lemma 2.6 and (2.12), it follows that

$$\begin{aligned}
&\|(y_n - t_n) + (x^* - y^*)\|^2 \\
&= \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)] + \lambda(Ay_n - Ay^*)\|^2 \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)]\|^2 \\
&\quad + 2\lambda \langle Ay_n - Ay^*, (y_n - t_n) + (x^* - y^*) \rangle \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\
&\quad + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|TP_C(y_n - \lambda Ay_n) - TP_C(y^* - \lambda Ay^*)\|^2 \\
&\quad + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\| \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - (Tt_n - x^*)\| \\
&\quad \times [\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|Tt_n - x^*\|] \\
&\quad + 2\lambda \|Ay_n - Ay^*\| \|(y_n - t_n) + (x^* - y^*)\|
\end{aligned}$$

$$\begin{aligned}
 &= \|u_n - Tt_n + x^* - y^* - (u_n - y_n) - \lambda(Ay_n - Ay^*)\| \\
 &\quad \times [\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|Tt_n - x^*\|] \\
 &\quad + 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\|.
 \end{aligned} \tag{3.29}$$

By this together with (3.23), (3.28), and $\lim_{n \rightarrow \infty} \|Ay_n - Ay^*\| = 0$, we obtain $\|(y_n - t_n) + (x^* - y^*)\| \rightarrow 0$ as $n \rightarrow \infty$. This together with (3.17), (3.22), and (3.28), we obtain that

$$\begin{aligned}
 \|Tt_n - t_n\| &\leq \|Tt_n - x_n\| + \|x_n - u_n\| + \|(u_n - y_n) - (x^* - y^*)\| \\
 &\quad + \|(y_n - t_n) + (x^* - y^*)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.30}$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle \leq 0, \tag{3.31}$$

where $\bar{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{Tt_n\}$ are two bounded sequences in C , we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n \rightarrow \infty} \langle v - \bar{x}, Tt_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle v - \bar{x}, Tt_{n_i} - \bar{x} \rangle. \tag{3.32}$$

Since $\lim_{n \rightarrow \infty} \|Tt_n - t_n\| = 0$, we obtain that $Tt_{n_i} \rightharpoonup z$ as $i \rightarrow \infty$.

Next, we show that $z \in \Omega$.

(a) We first show $z \in F(T)$.

Since $t_{n_i} \rightharpoonup z$ and $\|Tt_n - t_n\| \rightarrow 0$, we obtain by Lemma 2.5 that $z \in F(T)$.

(b) Now, we show that $z \in \text{GVI}(C, A, B)$.

From (3.30) and (3.17), we have

$$\|t_n - x_n\| \leq \|Tt_n - t_n\| + \|Tt_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.33}$$

Furthermore, by Lemma 2.9, we have that $G : C \rightarrow C$ is nonexpansive. Then, we have

$$\begin{aligned}
 \|t_n - G(t_n)\| &= \|P_C(y_n - \lambda Ay_n) - G(t_n)\| \\
 &= \|P_C[P(u_n - \mu Bu_n) - \lambda AP(u_n - \mu Bu_n)] - G(t_n)\| \\
 &= \|G(u_n) - G(t_n)\| \leq \|u_n - t_n\| \\
 &\leq \|u_n - x_n\| + \|x_n - t_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.34}$$

Again by Lemma 2.5, we have $z \in \text{GVI}(C, A, B)$.

(c) We show that $z \in \text{MEP}(F, \varphi)$. Since $t_{n_i} \rightarrow z$ and $\|x_n - t_n\| \rightarrow 0$, we obtain that $x_{n_i} \rightarrow z$. From $\|u_n - x_n\| \rightarrow 0$, we also obtain that $u_{n_i} \rightarrow z$. By using the same argument as that in the proof of [30, Theorem 3.1, page 1825], we can show that $z \in \text{MEP}(F, \varphi)$. Therefore, there holds $z \in \Omega$.

On the other hand, it follows from (2.13), (3.17), and $Tt_{n_i} \rightarrow z$ as $i \rightarrow \infty$ that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle &= \limsup_{n \rightarrow \infty} \langle v - \bar{x}, Tt_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle v - \bar{x}, Tt_{n_i} - \bar{x} \rangle \\ &= \langle v - \bar{x}, z - \bar{x} \rangle \leq 0. \end{aligned} \quad (3.35)$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= \langle a_n v + (1 - a_n)Tt_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - a_n) \langle Tt_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2}(1 - a_n) \left(\|t_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2 \right) \\ &\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2}(1 - a_n) \left(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2 \right), \end{aligned} \quad (3.36)$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - a_n) \|x_n - \bar{x}\|^2 + 2a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle. \quad (3.37)$$

By this together with (C1) and (3.35), we have by Lemma 2.3 that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. $\square$

The following examples provide mappings A and B which satisfy those conditions in Theorem 3.1.

Example 3.2. Let $A, B : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. If $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$, then we have

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,
- (2) $\|Ay_n - Ay^*\| \rightarrow 0$ and $\|Bu_n - Bx^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 + \lambda^2 \|Ax - Ay\|^2 - 2\lambda \langle x - y, Ax - Ay \rangle \\ &\leq \|x - y\|^2 + \lambda^2 \|Ax - Ay\|^2 - 2\lambda\alpha \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned} \quad (3.38)$$

hence, $I - \lambda A$ is nonexpansive. Similarly, we can show that $I - \mu B$ is nonexpansive.

(2) Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.3), we have

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
 &= a_n \|v - x^*\|^2 + (1 - a_n) \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \right] \\
 &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2, \\
 \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|y_n - y^*\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|u_n - x^*\|^2 + \mu(\mu - 2\beta) \|B u_n - B x^*\|^2 \right] \\
 &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \mu(\mu - 2\beta) \|B u_n - B x^*\|^2.
 \end{aligned} \tag{3.39}$$

Therefore, we have

$$\begin{aligned}
 -(1 - a_n) \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 &\leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|, \\
 -(1 - a_n) \mu(\mu - 2\beta) \|B u_n - B x^*\|^2 &\leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
 \end{aligned} \tag{3.40}$$

From (3.15) and (C1), we obtain

$$\|A y_n - A y^*\| \longrightarrow 0, \quad \|B u_n - B x^*\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.41}$$

□

Example 3.3. Let $A : C \rightarrow H$ be an L_A -Lipschitzian and relaxed (c, d) -cocoercive mapping and $B : C \rightarrow H$ an L_B -Lipschitzian and relaxed (c', d') -cocoercive mapping. If $0 < \lambda < (2(d - cL_A^2))/L_A^2$ and $0 < \mu < (2(d' - c'L_B^2))/L_B^2$, then we have

(1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,

(2) $\|A y_n - A y^*\| \rightarrow 0$ and $\|B u_n - B x^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu B x^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned}
\|(I - \mu B)x - (I - \mu B)y\|^2 &= \|(x - y) - \mu(Bx - By)\|^2 \\
&= \|x - y\|^2 + \mu^2\|Bx - By\|^2 - 2\mu\langle x - y, Bx - By \rangle \\
&\leq \|x - y\|^2 + \mu^2\|Bx - By\|^2 \\
&\quad - 2\mu\left[-c'\|Bx - By\|^2 + d'\|x - y\|^2\right] \\
&= \|x - y\|^2 + \mu^2\|Bx - By\|^2 \\
&\quad + 2\mu c'\|Bx - By\|^2 - 2\mu d'\|x - y\|^2 \\
&\leq \|x - y\|^2 + \mu^2 L_B^2 \|x - y\|^2 \\
&\quad + 2\mu c' L_B^2 \|x - y\|^2 - 2\mu d' \|x - y\|^2 \\
&= \left(1 + 2\mu c' L_B^2 - 2\mu d' + \mu^2 L_B^2\right) \|x - y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned} \tag{3.42}$$

hence, $I - \mu B$ is nonexpansive. Similarly, we can show that $I - \lambda A$ is nonexpansive.

(2) Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.3), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|y_n - y^*\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \\
&\quad \times \left[\|u_n - x^*\|^2 - 2\mu \langle u_n - x^*, B u_n - B x^* \rangle + \mu^2 \|B u_n - B x^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \\
&\quad \times \left[\|u_n - x^*\|^2 + 2\mu c' \|B u_n - B x^*\|^2 - 2\mu d' \|u_n - x^*\|^2 + \mu^2 \|B u_n - B x^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|x_n - x^*\|^2 + \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|B u_n - B x^*\|^2 \right] \\
&\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|B u_n - B x^*\|^2,
\end{aligned}$$

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|t_n - x^*\|^2 \\
 &= a_n \|v - x^*\|^2 + (1 - a_n) \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2 \\
 &\leq a_n \|v - x^*\|^2 + (1 - a_n) \left[\|x_n - x^*\|^2 + \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|A y_n - A y^*\|^2 \right] \\
 &\leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 + (1 - a_n) \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|A y_n - A y^*\|^2.
 \end{aligned} \tag{3.43}$$

Therefore, we have

$$\begin{aligned}
 & - (1 - a_n) \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2} \right) \|B u_n - B x^*\|^2 \\
 & \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|, \\
 & - (1 - a_n) \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2} \right) \|A y_n - A y^*\|^2 \\
 & \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
 \end{aligned} \tag{3.44}$$

From (3.15) and (C1), we obtain

$$\|A y_n - A y^*\| \rightarrow 0, \quad \|B u_n - B x^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.45}$$

□

By using the same proof as in Examples 3.2 and 3.3, we can obtain the following example.

Example 3.4. Let A be an α -inverse-strongly monotone mapping of C into H and B be a L -Lipschitzian and relaxed (c, d) -cocoercive mapping of C into H . If $\lambda \in (0, 2\alpha)$ and $0 < \mu < (2(d - cL^2))/L^2$, then we have

- (1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,
- (2) $\|A y_n - A y^*\| \rightarrow 0$ and $\|B u_n - B x^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu B x^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Let $\mathcal{A}$ be the class of all α -inverse-strongly monotone mappings from C into H , $\mathcal{B}$ the class of all β -inverse-strongly monotone mappings from C into H , $\mathcal{C}$ the class of all L -Lipschitzian and relaxed (c, d) -cocoercive mappings from C into H , and $\mathcal{D}$ the class of all L_1 -Lipschitzian and relaxed (c', d') -cocoercive mappings from C into H .

Theorem 3.5. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \bigcap_{i=1}^N F(T_i) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \dots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$, $\alpha_1^N \in (0, 1]$ and $\alpha_2^j, \alpha_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let S be the S -mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ the sequences generated by

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$y_n = P_C(u_n - \mu B u_n), \quad (3.46)$$

$$x_{n+1} = a_n v + (1 - a_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1.$$

If one of the following conditions is satisfied:

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, and $\mu \in (0, 2\beta)$,
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, and $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, and $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$ and $\{a_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Proof. By Lemma 2.7, we obtain that S is nonexpansive and $F(S) = \bigcap_{i=1}^N F(T_i)$. Hence, the result is obtained directly from Theorem 3.1 and Examples 3.2–3.4. $\square$

From Theorem 3.1 and Examples 3.2–3.4, we obtain the following result.

Corollary 3.6. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \text{GVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C . If one of the following conditions is satisfied:

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, and $\mu \in (0, 2\beta)$,
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, and $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, and $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$, $\{a_n\}$ are as in Theorem 3.1, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to $\bar{x} = P_\Omega v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Let $\varphi = 0$ in Theorem 3.1. From Theorem 3.1 and Examples 3.2–3.4, we obtain the following result.

Corollary 3.7. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5). Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \text{GVI}(C, A, B) \cap \text{EP}(F) \neq \emptyset$. Let $x_1, v \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= P_C(u_n - \mu B u_n), \\ x_{n+1} &= a_n v + (1 - a_n) T P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{aligned} \tag{3.47}$$

If one of the following conditions is satisfied:

- (1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, and $\mu \in (0, 2\beta)$,
- (2) $A \in \mathcal{C}$, $B \in \mathcal{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, and $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,
- (3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, and $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}, \{a_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\bar{x} = P_{\Omega}v$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.6), where $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$.

Remark 3.8. In Theorem 3.5, if $F \equiv 0$, then the sequence $\{x_n\}$ generated by (3.46) converges strongly to a solution of the minimization problem which is also a solution of a system of variational inequalities.

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Research Article

Strong Convergence to Solutions of Generalized Mixed Equilibrium Problems with Applications

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We introduce a Halpern-type iteration for a generalized mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. Strong convergence theorems are also established in this paper. As applications, we apply our main result to mixed equilibrium, generalized equilibrium, and mixed variational inequality problems in Banach spaces. Finally, examples and numerical results are also given.

1. Introduction

Let E be a real Banach space, C a nonempty, closed, and convex subset of E , and E^* the dual space of E . Let $T : C \rightarrow C$ be a nonlinear mapping. The fixed points set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

One classical way often used to approximate a fixed point of a nonlinear self-mapping T on C was firstly introduced by Halpern [1] which is defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (1.1)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. He proved, in a real Hilbert space, a strong convergence theorem for a nonexpansive mapping T when $\alpha_n = n^{-a}$ for any $a \in (0, 1)$.

Subsequently, motivated by Halpern [1], many mathematicians devoted time to study algorithm (1.1) in different styles. Several strong convergence results for nonlinear mappings were also continuously established in some certain Banach spaces (see also [2–9]).

Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction, $A : C \rightarrow E^*$ a mapping, and $\varphi : C \rightarrow \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.2)$$

The solutions set of (1.2) is denoted by $\text{GMEP}(f, A, \varphi)$ (see Peng and Yao [10]).

If $A \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following mixed equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.3)$$

The solutions set of (1.3) is denoted by $\text{MEP}(f, \varphi)$ (see Ceng and Yao [11]).

If $f \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following mixed variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.4)$$

The solutions set of (1.4) is denoted by $\text{VI}(C, A, \varphi)$ (see Noor [12]).

If $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following generalized equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \quad (1.5)$$

The solutions set of (1.5) is denoted by $\text{GEP}(f, A)$ (see Moudafi [13]).

If $\varphi \equiv 0$, then the mixed equilibrium problem (1.3) reduces to the following equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (1.6)$$

The solutions set of (1.6) is denoted by $\text{EP}(f)$ (see Combettes and Hirstoaga [14]).

If $f \equiv 0$, then the mixed equilibrium problem (1.3) reduces to the following convex minimization problem: finding $\hat{x} \in C$ such that

$$\varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \quad (1.7)$$

The solutions set of (1.7) is denoted by $\text{CMP}(\varphi)$.

If $\varphi \equiv 0$, then the mixed variational inequality problem (1.4) reduces to the following variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \quad (1.8)$$

The solutions set of (1.8) is denoted by $VI(C, A)$ (see Stampacchia [7]).

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others. For more details on these topics, see, for instance, [14–34].

For solving the generalized mixed equilibrium problem, let us assume the following [25]:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The purpose of this paper is to investigate strong convergence of Halpern-type iteration for a generalized mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. As applications, our main result can be deduced to mixed equilibrium, generalized equilibrium, mixed variational inequality problems, and so on. Examples and numerical results are also given in the last section.

2. Preliminaries and Lemmas

In this section, we need the following preliminaries and lemmas which will be used in our main theorem.

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *strictly convex* if, for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1. \quad (2.1)$$

It is also said to be *uniformly convex* if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| < 1 - \delta. \quad (2.2)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0, 2] \rightarrow [0, 1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}. \quad (2.3)$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.4)$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in U$. The *normalized duality mapping* $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (2.5)$$

for all $x \in E$. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E (see [35]).

Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.6)$$

Remark 2.1. We know the following: for any $x, y, z \in E$,

- (1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$;
- (2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$;
- (3) $\phi(x, y) = \|x - y\|^2$ in a real Hilbert space.

Lemma 2.2 (see [36]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, and smooth Banach space and let C be a nonempty closed and convex subset of E . The *generalized projection mapping*, introduced by Alber [37], is a mapping $\Pi_C : E \rightarrow C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}. \quad (2.7)$$

In fact, we have the following result.

Lemma 2.3 (see [37]). *Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

Lemma 2.4 (see [36, 37]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E , $x \in E$, and $z \in C$. Then $z = \Pi_C x$ if and only if*

$$\langle Jx - Jz, y - z \rangle \leq 0, \quad \forall y \in C. \quad (2.8)$$

Lemma 2.5 (see [36, 37]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \quad (2.9)$$

Lemma 2.6 (see [38]). *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty, closed, and convex subset of E . Then Π_C is uniformly norm-to-norm continuous on every bounded set.*

We make use of the following mapping V studied in Alber [37]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \quad (2.10)$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.7 (see [39]). *Let E be a reflexive, strictly convex, smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*) \quad (2.11)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.8 (see [25]). *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq 0, \quad \forall y \in C. \quad (2.12)$$

Following [25, 40], we know the following lemma.

Lemma 2.9 (see [41]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex, and reflexive Banach space E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. For all $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq \varphi(z), \quad \forall y \in C. \quad (2.13)$$

Define the mapping $T_r : E \rightarrow 2^C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq \varphi(z), \quad \forall y \in C \right\}. \quad (2.14)$$

Then, the followings hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive-type mapping [42], that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \quad (2.15)$$

- (3) $F(T_r) = \text{GMEP}(f, A, \varphi)$;
- (4) $\text{GMEP}(f, A, \varphi)$ is closed and convex.

Remark 2.10. It is known that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x) \quad (2.16)$$

for all $x, y \in \text{dom } T$ (see [42]).

The following lemmas give us some nice properties of real sequences.

Lemma 2.11 (see [43]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n, \quad \forall n \geq 1, \quad (2.17)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

- (a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
- (b) $\limsup_{n \rightarrow \infty} b_n / \alpha_n \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < +\infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.12 (see [44]). Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \geq 1$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \geq 1$:

$$\gamma_{m_k} \leq \gamma_{m_k+1}, \quad \gamma_k \leq \gamma_{m_k+1}. \quad (2.18)$$

In fact, m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. Main Results

In this section, we prove our main theorem in this paper. To this end, we need the following proposition.

Proposition 3.1. Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and uniformly smooth Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \rightarrow E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function

from C to $\mathbb{R}$ such that $\text{GMEP}(f, A, \varphi) \neq \emptyset$. Let $\{r_n\} \subset (0, \infty)$ be such that $\liminf_{n \rightarrow \infty} r_n > 0$. For each $n \geq 1$, let T_{r_n} be defined as in Lemma 2.9. Suppose that $x \in C$ and $\{x_n\}$ is a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|x_n - T_{r_n} x_n\| = 0$. Then

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle \leq 0, \quad (3.1)$$

where $p = \Pi_{\text{GMEP}(f, A, \varphi)} x$ and $\Pi_{\text{GMEP}(f, A, \varphi)}$ is the generalized projection of C onto $\text{GMEP}(f, A, \varphi)$.

Proof. Let $x \in C$ and put $p = \Pi_{\text{GMEP}(f, A, \varphi)} x$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$ and

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle = \langle Jx - Jp, v - p \rangle. \quad (3.2)$$

Put $y_n = T_{r_n} x_n$. Since $\lim_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| = 0$, we have $y_{n_k} \rightharpoonup v$. On the other hand, since E is uniformly smooth, J is uniformly norm-to-norm continuous on bounded subsets of E . So we have

$$\lim_{k \rightarrow \infty} \|Jx_{n_k} - Jy_{n_k}\| = 0. \quad (3.3)$$

Since $\liminf_{k \rightarrow \infty} r_{n_k} > 0$,

$$\lim_{k \rightarrow \infty} \frac{\|Jx_{n_k} - Jy_{n_k}\|}{r_{n_k}} = 0. \quad (3.4)$$

By the definition of $T_{r_{n_k}}$, for any $y \in C$, we see that

$$f(y_{n_k}, y) + \langle Ay_{n_k}, y - y_{n_k} \rangle + \varphi(y) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y - y_{n_k} \rangle \geq \varphi(y_{n_k}). \quad (3.5)$$

By (A2), for each $y \in C$, we obtain

$$\begin{aligned} f(y, y_{n_k}) + \varphi(y_{n_k}) &\leq -f(y_{n_k}, y) + \varphi(y_{n_k}) \\ &\leq \langle Ay_{n_k}, y - y_{n_k} \rangle + \varphi(y) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y - y_{n_k} \rangle. \end{aligned} \quad (3.6)$$

For any $t \in (0, 1)$ and $y \in C$, we define $y_t = ty + (1 - t)v$. Then $y_t \in C$. It follows by the monotonicity of A that

$$\begin{aligned} f(y_t, y_{n_k}) + \varphi(y_{n_k}) &\leq \langle Ay_{n_k} - Ay_t, y_t - y_{n_k} \rangle + \langle Ay_t, y_t - y_{n_k} \rangle \\ &\quad + \varphi(y_t) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y_t - y_{n_k} \rangle \\ &\leq \langle Ay_t, y_t - y_{n_k} \rangle + \varphi(y_t) + \frac{1}{r_{n_k}} \langle Jy_{n_k} - Jx_{n_k}, y_t - y_{n_k} \rangle. \end{aligned} \quad (3.7)$$

By (A4), (3.4), and the weakly lower semicontinuity of φ , letting $k \rightarrow \infty$, we obtain

$$f(y_t, v) + \varphi(v) \leq \langle Ay_t, y_t - v \rangle + \varphi(y_t). \quad (3.8)$$

By (A1), (A4), and the convexity of φ , we have

$$\begin{aligned} 0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, v) + t\varphi(y) + (1-t)\varphi(v) - \varphi(y_t) \\ &= t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)(f(y_t, v) + \varphi(v) - \varphi(y_t)) \\ &\leq t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)\langle Ay_t, y_t - v \rangle \\ &= t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)t\langle Ay_t, y - v \rangle. \end{aligned} \quad (3.9)$$

It follows that

$$f(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle Ay_t, y - v \rangle \geq 0. \quad (3.10)$$

By (A3), the weakly lower semicontinuity of φ , and the continuity of A , letting $t \rightarrow 0$, we obtain

$$f(v, y) + \varphi(y) - \varphi(v) + \langle Av, y - v \rangle \geq 0, \quad \forall y \in C. \quad (3.11)$$

This shows that $v \in \text{GMEP}(f, A, \varphi)$. By Lemma 2.4, we have

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle = \langle Jx - Jp, v - p \rangle \leq 0. \quad (3.12)$$

This completes the proof. $\square$

Theorem 3.2. Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \rightarrow E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\text{GMEP}(f, A, \varphi) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$\begin{aligned} f(y_n, y) + \langle Ay_n, y - y_n \rangle + \varphi(y) + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.13)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{GMEP}(f,A,\varphi)}x$, where $\Pi_{\text{GMEP}(f,A,\varphi)}$ is the generalized projection of C onto $\text{GMEP}(f, A, \varphi)$.

Proof. From Lemma 2.9(4), we know that $\text{GMEP}(f, A, \varphi)$ is closed and convex. Let $p = \Pi_{\text{GMEP}(f,A,\varphi)}x$. Put $y_n = T_{r_n}x_n$ and $z_n = J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n)$ for all $n \in \mathbb{N}$. So, by Lemma 2.5, we have

$$\begin{aligned}\phi(p, x_{n+1}) &\leq \phi(p, z_n) \\ &\leq \alpha_n \phi(p, x) + (1 - \alpha_n) \phi(p, y_n) \\ &\leq \alpha_n \phi(p, x) + (1 - \alpha_n) \phi(p, x_n).\end{aligned}\tag{3.14}$$

By induction, we can show that $\phi(p, x_n) \leq \phi(p, x)$ for each $n \in \mathbb{N}$. Hence $\{\phi(p, x_n)\}$ is bounded and thus $\{x_n\}$ is also bounded.

We next show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \rightarrow \infty} (\phi(p, x_{n_{k+1}}) - \phi(p, x_{n_k})) = 0,\tag{3.15}$$

then

$$\lim_{k \rightarrow \infty} (\phi(p, y_{n_k}) - \phi(p, x_{n_k})) = 0.\tag{3.16}$$

Since $\alpha_{n_k} \rightarrow 0$,

$$\lim_{k \rightarrow \infty} \|Jz_{n_k} - Jy_{n_k}\| = \lim_{k \rightarrow \infty} \alpha_{n_k} \|Jx - Jy_{n_k}\| = 0.\tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E , so is J^{-1} . It follows that

$$\lim_{k \rightarrow \infty} \|z_{n_k} - y_{n_k}\| = 0.\tag{3.18}$$

Since E is uniformly smooth and uniformly convex, by Lemma 2.6, Π_C is uniformly norm-to-norm continuous on bounded sets. So we obtain

$$\lim_{k \rightarrow \infty} \|x_{n_{k+1}} - y_{n_k}\| = \lim_{k \rightarrow \infty} \|\Pi_C z_{n_k} - \Pi_C y_{n_k}\| = 0,\tag{3.19}$$

and hence

$$\lim_{k \rightarrow \infty} \|Jx_{n_{k+1}} - Jy_{n_k}\| = 0.\tag{3.20}$$

Furthermore, $\lim_{k \rightarrow \infty} \phi(x_{n_k+1}, y_{n_k}) = 0$. Indeed, by the definition of ϕ , we observe that

$$\begin{aligned} \phi(x_{n_k+1}, y_{n_k}) &= \|x_{n_k+1}\|^2 - 2\langle x_{n_k+1}, Jy_{n_k} \rangle + \|y_{n_k}\|^2 \\ &= \langle x_{n_k+1}, Jx_{n_k+1} - Jy_{n_k} \rangle + \langle y_{n_k} - x_{n_k+1}, Jy_{n_k} \rangle. \end{aligned} \quad (3.21)$$

It follows from (3.19) and (3.20) that $\lim_{k \rightarrow \infty} \phi(x_{n_k+1}, y_{n_k}) = 0$. On the other hand, from Remark 2.1(2), we have

$$\begin{aligned} \phi(p, y_{n_k}) - \phi(p, x_{n_k}) &= (\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})) + (\phi(p, y_{n_k}) - \phi(p, x_{n_k+1})) \\ &= (\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})) \\ &\quad + \phi(x_{n_k+1}, y_{n_k}) + 2\langle p - x_{n_k+1}, Jx_{n_k+1} - Jy_{n_k} \rangle. \end{aligned} \quad (3.22)$$

It follows from (3.20) and (3.21) that $\lim_{k \rightarrow \infty} (\phi(p, y_{n_k}) - \phi(p, x_{n_k})) = 0$.

We next consider the following two cases.

Case 1. $\phi(p, x_{n+1}) \leq \phi(p, x_n)$ for all sufficiently large n . Hence the sequence $\{\phi(p, x_n)\}$ is bounded and nonincreasing. So $\lim_{n \rightarrow \infty} \phi(p, x_n)$ exists. This shows that $\lim_{n \rightarrow \infty} (\phi(p, x_{n+1}) - \phi(p, x_n)) = 0$ and hence

$$\lim_{n \rightarrow \infty} (\phi(p, y_n) - \phi(p, x_n)) = 0. \quad (3.23)$$

Since T_{r_n} is of firmly nonexpansive type, by Remark 2.10, we have

$$\phi(y_n, p) + \phi(p, y_n) + \phi(y_n, x_n) + \phi(T_{r_n}p, p) \leq \phi(y_n, p) + \phi(p, x_n), \quad (3.24)$$

which implies

$$\phi(p, y_n) + \phi(y_n, x_n) \leq \phi(p, x_n). \quad (3.25)$$

Hence

$$\phi(y_n, x_n) \leq \phi(p, x_n) - \phi(p, y_n) \longrightarrow 0 \quad (3.26)$$

as $n \rightarrow \infty$. By Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.27)$$

Proposition 3.1 yields that

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, x_n - p \rangle \leq 0. \quad (3.28)$$

It also follows that

$$\limsup_{n \rightarrow \infty} \langle Jx - Jp, y_n - p \rangle \leq 0. \quad (3.29)$$

Finally, we show that $x_n \rightarrow p$. Using Lemma 2.7, we see that

$$\begin{aligned} \phi(p, x_{n+1}) &\leq \phi(p, z_n) \\ &= V(p, \alpha_n Jx + (1 - \alpha_n) Jy_n) \\ &\leq V(p, \alpha_n Jx + (1 - \alpha_n) Jy_n - \alpha_n (Jx - Jp)) + \langle \alpha_n (Jx - Jp), z_n - p \rangle \\ &= V(p, \alpha_n Jp + (1 - \alpha_n) Jy_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &\leq \alpha_n V(p, Jp) + (1 - \alpha_n) V(p, Jy_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &= (1 - \alpha_n) \phi(p, y_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &\leq (1 - \alpha_n) \phi(p, x_n) + \alpha_n \langle Jx - Jp, z_n - p \rangle \\ &= (1 - \alpha_n) \phi(p, x_n) + \alpha_n (\langle Jx - Jp, z_n - y_n \rangle + \langle Jx - Jp, y_n - p \rangle). \end{aligned} \quad (3.30)$$

Set $a_n = \phi(p, x_n)$ and $b_n = \alpha_n (\langle Jx - Jp, z_n - y_n \rangle + \langle Jx - Jp, y_n - p \rangle)$. We see that $\limsup_{n \rightarrow \infty} b_n / \alpha_n \leq 0$. By Lemma 2.11, since $\sum_{n=1}^{\infty} \alpha_n = +\infty$, we conclude that $\lim_{n \rightarrow \infty} \phi(p, x_n) = 0$. Hence $x_n \rightarrow p$ as $n \rightarrow \infty$.

Case 2. There exists a subsequence $\{\phi(p, x_{n_j})\}$ of $\{\phi(p, x_n)\}$ such that $\phi(p, x_{n_j}) < \phi(p, x_{n_j+1})$ for all $j \in \mathbb{N}$. By Lemma 2.12, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1}), \quad \phi(p, x_k) \leq \phi(p, x_{m_k+1}). \quad (3.31)$$

So we have

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} (\phi(p, x_{m_k+1}) - \phi(p, x_{m_k})) \\ &\leq \limsup_{n \rightarrow \infty} (\phi(p, x_{n+1}) - \phi(p, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\phi(p, z_n) - \phi(p, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n \phi(p, x) + (1 - \alpha_n) \phi(p, y_n) - \phi(p, x_n)) \\ &= \limsup_{n \rightarrow \infty} (\alpha_n (\phi(p, x) - \phi(p, y_n)) + (\phi(p, y_n) - \phi(p, x_n))) \\ &\leq \limsup_{n \rightarrow \infty} \alpha_n (\phi(p, x) - \phi(p, y_n)) = 0. \end{aligned} \quad (3.32)$$

This shows that

$$\lim_{k \rightarrow \infty} (\phi(p, x_{m_k+1}) - \phi(p, x_{m_k})) = 0. \quad (3.33)$$

Following the proof line in Case 1, we can show that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle Jx - Jp, y_{m_k} - p \rangle &\leq 0, \\ \phi(p, x_{m_k+1}) &\leq (1 - \alpha_{m_k})\phi(p, x_{m_k}) + \alpha_{m_k}(\langle Jx - Jp, z_{m_k} - y_{m_k} \rangle + \langle Jx - Jp, y_{m_k} - p \rangle). \end{aligned} \quad (3.34)$$

This implies

$$\begin{aligned} \alpha_{m_k}\phi(p, x_{m_k}) &\leq \phi(p, x_{m_k}) - \phi(p, x_{m_k+1}) \\ &\quad + \alpha_{m_k}(\langle Jx - Jp, z_{m_k} - y_{m_k} \rangle + \langle Jx - Jp, y_{m_k} - p \rangle) \\ &\leq \alpha_{m_k}(\langle Jx - Jp, z_{m_k} - y_{m_k} \rangle + \langle Jx - Jp, y_{m_k} - p \rangle). \end{aligned} \quad (3.35)$$

Hence $\lim_{k \rightarrow \infty} \phi(p, x_{m_k}) = 0$. Using this and (3.33) together, we conclude that

$$\limsup_{k \rightarrow \infty} \phi(p, x_k) \leq \lim_{k \rightarrow \infty} \phi(p, x_{m_k+1}) = 0. \quad (3.36)$$

This completes the proof. $\square$

As a direct consequence of Theorem 3.2, we obtain the following results.

Corollary 3.3. *Let C be nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\text{MEP}(f, \varphi) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and*

$$\begin{aligned} f(y_n, y) + \varphi(y) + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.37)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{MEP}(f, \varphi)} x$, where $\Pi_{\text{MEP}(f, \varphi)}$ is the generalized projection of C onto $\text{MEP}(f, \varphi)$.

Corollary 3.4. Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and $A : C \rightarrow E^*$ a continuous and monotone mapping such that $\text{GEP}(f, A) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$\begin{aligned} f(y_n, y) + \langle Ay_n, y - y_n \rangle + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.38)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{GEP}(f, A)} x$, where $\Pi_{\text{GEP}(f, A)}$ is the generalized projection of C onto $\text{GEP}(f, A)$.

Corollary 3.5. Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E . Let $A : C \rightarrow E^*$ be a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\text{VI}(C, A, \varphi) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$\begin{aligned} \langle Ay_n, y - y_n \rangle + \varphi(y) + \frac{1}{r_n} \langle Jy_n - Jx_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n), \quad \forall n \geq 1, \end{aligned} \quad (3.39)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (c) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\text{VI}(C, A, \varphi)} x$, where $\Pi_{\text{VI}(C, A, \varphi)}$ is the generalized projection of C onto $\text{VI}(C, A, \varphi)$.

4. Examples and Numerical Results

In this section, we give examples and numerical results for our main theorem.

Example 4.1. Let $E = \mathbb{R}$ and $C = [-1, 1]$. Let $f(x, y) = -9x^2 + xy + 8y^2$, $\varphi(x) = 3x^2$, and $Ax = 2x$. Find $\hat{x} \in [-1, 1]$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in [-1, 1]. \quad (4.1)$$

Solution. It is easy to check that f , φ , and A satisfy all conditions in Theorem 3.2. For each $r > 0$ and $x \in [-1, 1]$, Lemma 2.9 ensures that there exists $z \in [-1, 1]$ such that, for any $y \in [-1, 1]$,

$$\begin{aligned} f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r} \langle z - x, y - z \rangle &\geq \varphi(z) \\ \iff -9z^2 + yz + 8y^2 + 2z(y - z) + 3y^2 + \frac{1}{r}(z - x)(y - z) &\geq 3z^2 \\ \iff 11ry^2 + (3rz + z - x)y - (14rz^2 + z^2 - xz) &\geq 0. \end{aligned} \quad (4.2)$$

Put $G(y) = 11ry^2 + (3rz + z - x)y - (14rz^2 + z^2 - xz)$. Then G is a quadratic function of y with coefficient $a = 11r$, $b = (3rz + z - x)$, and $c = -(14rz^2 + z^2 - xz)$. We next compute the discriminant Δ of G as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= [(3r + 1)z - x]^2 + 44r(14rz^2 + z^2 - xz) \\ &= x^2 - 2(3r + 1)xz + (3r + 1)^2z^2 + 616r^2z^2 + 44rz^2 - 44rxz \\ &= x^2 - 50rxz - 2xz + 625r^2z^2 + 50rz^2 + z^2 \\ &= x^2 - 2(25rz + z)x + (625r^2z^2 + 50rz^2 + z^2) \\ &= [x - (25rz + z)]^2. \end{aligned} \quad (4.3)$$

We know that $G(y) \geq 0$ for all $y \in [-1, 1]$ if it has at most one solution in $[-1, 1]$. So $\Delta \leq 0$ and hence $x = 25rz + z$. Now we have $z = T_r x = x / (25r + 1)$.

Let $\{x_n\}_{n=1}^\infty$ be the sequence generated by $x_1 = x \in [-1, 1]$ and

$$\begin{aligned} f(y_n, y) + \langle Ay_n, y - y_n \rangle + \varphi(y) + \frac{1}{r_n} \langle y_n - x_n, y - y_n \rangle &\geq \varphi(y_n), \quad \forall y \in [-1, 1], \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n)y_n, \quad \forall n \geq 1, \end{aligned} \quad (4.4)$$

and, equivalently,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_{r_n} x_n, \quad \forall n \geq 1. \quad (4.5)$$

We next give two numerical results for algorithm (4.5).

Algorithm 4.2. Let $\alpha_n = 1/80n$ and $r_n = n/(n + 1)$. Choose $x_1 = x = 1$. Then algorithm (4.5) becomes

$$x_{n+1} = \frac{1}{80n} + \left(1 - \frac{1}{80n}\right) \left(\frac{n+1}{26n+1}\right) x_n, \quad \forall n \geq 1. \quad (4.6)$$

Table 1

n	x_n
1	1.0000
2	0.0856
3	0.0111
4	0.0047
5	0.0033
$\vdots$	$\vdots$
261	0.0001
262	0.0000

Table 2

n	x_n
1	-1.0000
2	-0.0481
3	-0.0074
4	-0.0038
5	-0.0027
$\vdots$	$\vdots$
217	-0.0001
218	0.0000

Numerical Result I

See Table 1.

Algorithm 4.3. Let $\alpha_n = 1/100n$ and $r_n = (n+1)/2n$. Choose $x_1 = x = -1$. Then algorithm (4.5) becomes

$$x_{n+1} = -\frac{1}{100n} + \left(1 - \frac{1}{100n}\right) \left(\frac{2n}{27n+25}\right) x_n, \quad \forall n \geq 1. \quad (4.7)$$

Numerical Result II

See Table 2.

5. Conclusion

Tables 1 and 2 show that the sequence $\{x_n\}$ converges to 0 which solves the generalized mixed equilibrium problem. On the other hand, using Lemma 2.9(3), we can check that $\text{GMEP}(f, A, \varphi) = F(T_r) = \{0\}$.

Remark 5.1. In the view of computation, our algorithm is simple in order to get strong convergence for generalized mixed equilibrium problems.

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CONVERGENCE THEOREMS OF HYBRID METHODS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF AN INFINITE FAMILY OF LIPSCHITZIAN QUASI-NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. We use a hybrid iterative method to find a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in real Hilbert spaces. We also show that our main strong convergence theorem for finding that common element can be deduced for nonexpansive mappings and applied for strict pseudo-contraction mappings. Our results extend the work by Cho et al. (2009) [4].

KEYWORDS : Generalized mixed equilibrium problem; Variational inequality; Equilibrium problem; Fixed point; Quasi-nonexpansive mappings; Strict-pseudo contraction.

AMS Subject Classification: 47H05, 47H10.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H . Let $B : C \rightarrow H$ be a nonlinear mapping, $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function and $f : C \times C \rightarrow \mathbb{R}$ be a bifunction. Peng and Yao [10] considered the following *generalized mixed equilibrium problem*:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \quad (1.1)$$

In this paper, we denote the set of solutions of (1.1) by $GMEP(f, \varphi, B)$. It is obvious that if u is a solution of (1.1), it implies that $u \in \text{dom } \varphi = \{u \in C : \varphi(u) < +\infty\}$.

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If $B = 0$ in (1.1), we obtain the following *mixed equilibrium problem* [3]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) \geq \varphi(u), \quad \forall y \in C. \quad (1.2)$$

We denote the set of solutions of (1.2) by $MEP(f, \varphi)$.

If $\varphi = 0$ in (1.1), we obtain the following *generalized equilibrium problem* [16]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \quad (1.3)$$

We denote the set of solutions of (1.3) by $GEP(f, B)$.

If $\varphi = 0$ and $B = 0$ in (1.1), we obtain the following *equilibrium problem* [2]:

$$\text{Finding } u \in C \text{ such that } f(u, y) \geq 0, \quad \forall y \in C. \quad (1.4)$$

We denote the set of solutions of (1.4) by $EP(f)$.

If $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *generalized variational inequality problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \quad (1.5)$$

We denote the set of solutions of (1.5) by $GVI(C, \varphi, B)$.

If $\varphi = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *variational inequality problem* (see also [1, 5]):

$$\text{Finding } u \in C \text{ such that } \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \quad (1.6)$$

We denote the set of solutions of (1.6) by $VI(C, B)$.

If $B = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *minimization problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) \geq \varphi(u), \quad \forall y \in C. \quad (1.7)$$

We denote the set of solutions of (1.7) by $MP(C, \varphi)$.

In 1994, Blum and Oettli showed that the formulation of (1.4) covered monotone inclusive problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems and Nash equilibria in noncooperative games. Several problems in physics, optimization and economics can be reduced to find solutions of (1.4). The existence of equilibrium problems has been discovered by many authors (see, for example, [1, 6, 8, 15] and the references therein). Also, some solution methods have been studied by some authors (see, for example, [6, 15, 13] and the references therein).

In 2003, Takahashi and Toyoda [16] introduced the method for finding an element of $F(S) \cap VI(C, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \rightarrow H$ is a nonexpansive mapping and $A : C \rightarrow H$ is an inverse-strongly monotone mapping. Their iteration is the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), n \geq 0,$$

where $x_0 \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and P_C is the metric projection from H onto C . They proved that, if $F(S) \cap VI(C, A) \neq \emptyset$, $\{x_n\}$ converges weakly to a point $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap VI(C, A)} x_n$.

Later, Takahashi and Takahashi [15] studied the contraction method for finding $F(S) \cap EP(f)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \rightarrow H$

is a nonexpansive mapping, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions. Their algorithm is the following:

$$\begin{aligned} x_1 &\in H, \\ f(y_n, u) + \frac{1}{r_n} \langle u - y_n, y_n - x_n \rangle &\geq 0, \quad \forall u \in C, \\ x_{n+1} &= \alpha_n f_1(x_n) + (1 - \alpha_n) S y_n \quad \forall n \geq 1, \end{aligned}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and some appropriate conditions. They proved that, if $F(S) \cap EP(f) \neq \emptyset$, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in F(S) \cap EP(f)$, where $z = P_{F(S) \cap EP(f)} f(z)$.

Recently, Cho et al. [4] introduced a hybrid projection method for finding $F := F(S) \cap VI(C, B) \cap GEP(f, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \rightarrow C$ is a k -strict pseudo-contraction with a fixed point, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions, $A : C \rightarrow H$ is an α -inverse-strongly monotone mapping and $B : C \rightarrow H$ is an β -inverse-strongly monotone mapping. Their iterative scheme is the following:

$$\begin{aligned} x_1 &\in C, \\ C_1 &= C, \\ f(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ z_n &= P_C(u_n - \lambda_n B u_n), \\ y_n &= \alpha_n x_n + (1 - \alpha_n) S_k z_n, \\ C_{n+1} &= \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{aligned}$$

where $S_k x = kx + (1 - k)Sx$ for all $x \in C$, $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0, 2\alpha)$ and some appropriate conditions. They proved that, if $F \neq \emptyset$, $\{x_n\}$ converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

In this paper, motivated by the above result, we prove a strong convergent theorem of a hybrid projection iterative method defined by (3.1) for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces. Our main result can be deduced for nonexpansive mappings applied for strict pseudo-contraction mappings. It is clear that our result generalizes the work by [4].

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H . Let $B : C \rightarrow H$. $x_n \rightarrow x$ means $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ implies $\{x_n\}$ converges weakly to x . We denote the set of fixed points of T by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.

Recall the following definitions:

- (1) A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(2) A mapping $T : C \rightarrow C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \in F(P).$$

(3) A mapping $T : C \rightarrow C$ is said to be *Lipschitzian* if there is a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

(4) A mapping $T : C \rightarrow C$ is said to be *strictly pseudo-contractive* with the coefficient $k \in [0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

(5) B is said to be *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(6) B is said to be α -*strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(7) B is said to be α -*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

(8) A set-valued mapping $T : H \rightarrow 2^H$ is said to be *monotone* if, for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$.

(9) A monotone mapping $T : H \rightarrow 2^H$ is said to be *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

In the other words, a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $B : C \rightarrow H$ be a monotone mapping and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \quad \forall u \in C\}$. Define a mapping T on C by

$$Tv = \begin{cases} Bv + N_C v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C. \end{cases} \quad (2.1)$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Bv, u - v \rangle \geq 0$ for all $u \in C$ (see [14]).

Let B be a β -inverse-strongly monotone mapping of C into H . It is easy to show that B is $\frac{1}{\beta}$ -Lipschitz. For $\lambda \in (0, 2\beta]$, it is known that $I - \lambda B$ is a nonexpansive mapping of C into H .

Let C be a nonempty closed convex subset of H . Therefore, for any $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H on C . We know that for $x \in H$ and $z \in C, z = P_C x$ is equivalent to $\langle x - z, z - y \rangle \geq 0$ for all $y \in C$. It is also known that a Hilbert space H satisfies the *Opial condition*, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightarrow x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ and $x \neq y$.

Let A be a monotone mapping from C into H . In the context of the variational inequality problem, it is known that

$$u \in \text{VI}(C, A) \Rightarrow u = P_C(u - \lambda Au), \quad \text{for all } \lambda > 0,$$

and

$$u = P_C(u - \lambda Au), \quad \text{for some } \lambda > 0 \Rightarrow u \in \text{VI}(C, A).$$

Let C be a nonempty closed subset of a Hilbert space H . Let $\{T_n\}$ and Γ be two families of nonlinear mappings of C into itself such that $F(\Gamma) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(\Gamma) = \bigcap_{T \in \Gamma} F(T)$. $\{T_n\}$ is said to satisfy the *NST-condition* ([9]) with Γ if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \implies \lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \text{ for all } T \in \Gamma.$$

In the case $\Gamma \in \{T\}$, i.e., Γ consists of one mapping T , $\{T_n\}$ is said to satisfy the NST-condition with T .

For solving the generalized mixed equilibrium problem, we may assume the following conditions for the bifunction f , the function φ and the set C :

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous;
- (A4) for each $x \in C$, $y \mapsto f(x, y)$ is weakly upper semicontinuous;
- (B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \text{dom}(\varphi)$ such that for any $z \in C \setminus D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z);$$

- (B2) C is a bounded set.

The following lemmas are useful for proving some convergence results in next two sections.

Lemma 2.1. ([10, 11, 12]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \quad \forall y \in C\}$$

for all $x \in H$. Then the following conclusions hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle;$$

- (3) $F(T_r) = \text{MEP}(F, \varphi)$;
- (4) $\text{MEP}(F, \varphi)$ is closed and convex.

Lemma 2.2. ([7]). *Let $A : C \rightarrow H$ be α -inverse-strongly monotone. If $r \in (0, 2\alpha)$, then we have $I - rA$ is nonexpansive.*

Lemma 2.3. ([17]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ be a k -strict pseudo-contraction. Define a mapping $S : C \rightarrow C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

We would like to mention the following remark since our result is very interesting. It shows that a monotone mapping maps all points in a generalized mixed equilibrium problem to the same point.

Remark 2.4. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A2) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. Let A be a monotone mapping of C into H . Then $Au = Av$ for all $u, v \in GMEP(f, \varphi, A)$.

Proof. Let $u, v \in GMEP(f, \varphi, A)$. We then get

$$f(u, y) + \varphi(y) + \langle Au, y - u \rangle \geq \varphi(u), \quad \forall y \in C \quad (2.2)$$

and

$$f(v, y) + \varphi(y) + \langle Av, y - v \rangle \geq \varphi(v), \quad \forall y \in C. \quad (2.3)$$

By letting $y = v$ in (2.2) and $y = u$ in (2.3), we get

$$f(u, v) + \varphi(v) + \langle Au, v - u \rangle \geq \varphi(u) \quad (2.4)$$

and

$$f(v, u) + \varphi(u) + \langle Av, u - v \rangle \geq \varphi(v). \quad (2.5)$$

By (2.4), (2.5) and the condition (A2), we have

$$\langle Av - Au, u - v \rangle \geq f(u, v) + f(v, u) + \langle Au, v - u \rangle + \langle Av, u - v \rangle \geq 0. \quad (2.6)$$

From A is a α -inverse-strongly monotone mapping,

$$0 \leq \alpha \|Au - Av\|^2 \leq \langle Au - Av, u - v \rangle \leq 0.$$

That is $Au = Av$. □

By letting $f = 0$ and $\varphi = 0$ in Lemma 2.4, we obtain the following remark.

Remark 2.5. Let C be a closed convex subset of a real Hilbert space H and A be a monotone mapping of C into H . Then $Au = Av$ for all $u, v \in VI(C, A)$.

3. MAIN RESULT

In this section, we show a strong convergent theorem of hybrid methods for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces under some appropriate conditions.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of Lipschitzian quasi-nonexpansive mappings of C into itself such that $\lim_{n \rightarrow \infty} \|S_n x - S_n y\| \leq L_n \|x - y\|$ for all $x, y \in C$, $\sup_n L_n = L$, $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \quad (3.1)$$

where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Proof. We divide our proof into 5 steps.

Step 1: We show that $F \subset C_n$ and C_n is closed and convex for all $n \geq 1$.

From the assumption, $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \geq 1$. Next, we show that C_{m+1} is closed and convex. For any $w \in C_m$, we see that

$$\|y_m - w\| \leq \|x_m - w\|$$

is equivalent to

$$\|x_m\|^2 - \|y_m\|^2 - 2\langle w, x_m - y_m \rangle \geq 0.$$

Therefore, C_{m+1} is closed and convex.

Since A is α -inverse-strongly monotone and B is β -inverse-strongly monotone, by Lemma 2.2, we get that $I - r_n A$ and $I - \lambda_n B$ are nonexpansive.

By nonexpansiveness of T_{r_n} and $I - r_n A$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(p - r_n A p)\|^2 \\ &\leq \|(x_n - r_n A x_n) - (p - r_n A p)\|^2 \\ &= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\ &= \|x_n - p\|^2 - 2r_n \langle x_n - p, Ax_n - Ap \rangle + r_n^2 \|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n \alpha \|Ax_n - Ap\|^2 + r_n^2 \|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 + r_n(r_n - 2\alpha) \|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

We are now ready to show that $F \subset C_n$ for each $n \geq 1$. From the assumption, we have that $F \subset CC_1$. Suppose $F \subset C_m$ for some $m \geq 1$. For any $w \in F \subset C_m$, by nonexpansiveness of $I - \lambda_m B$, we have

$$\begin{aligned} \|y_m - w\| &= \|\alpha_m x_m + (1 - \alpha_m) S_m z_m - w\| \\ &\leq \alpha_m \|x_m - w\| + (1 - \alpha_m) \|z_m - w\| \\ &= \alpha_m \|x_m - w\| + (1 - \alpha_m) \|P_C(I - \lambda_m B)u_m - P_C(I - \lambda_m B)w\| \\ &\leq \alpha_m \|x_m - w\| + (1 - \alpha_m) \|u_m - w\| \\ &\leq \alpha_m \|x_m - w\| + (1 - \alpha_m) \|x_m - w\| \\ &= \|x_m - w\|. \end{aligned}$$

That is $w \in C_{m+1}$. By mathematical induction, we conclude that $F \subset C_n$ for each $n \geq 1$.

Step 2: We show that $\{x_n\}$ is bounded.

Since $x_n = P_{C_n} x_1$ and $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we get

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\| \|x_1 - x_{n+1}\|. \end{aligned} \tag{3.3}$$

Thus

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|. \quad (3.4)$$

Since $x_n = P_{C_n}x_1$, for any $w \in F \subset C_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - w\|. \quad (3.5)$$

In particular, we obtain

$$\|x_1 - x_n\| \leq \|x_1 - P_Fx_1\|. \quad (3.6)$$

By (3.4) and (3.6), we get that $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists. It implies that $\{x_n\}$ is bounded.

Step 3: We show that $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$ for all $S \in \mathcal{S}$.

By using (3.3), we obtain that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2 \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0. \quad (3.7)$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$$

and then

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|.$$

By (3.7), we get

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.8)$$

On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n)S_n z_n\| = (1 - \alpha_n)\|x_n - S_n z_n\|.$$

It follows from (3.8) and the assumption $0 \leq \alpha_n \leq a < 1$ that

$$\lim_{n \rightarrow \infty} \|x_n - S_n z_n\| = 0. \quad (3.9)$$

For any $w \in F$, we have

$$\begin{aligned} \|y_n - w\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)S_n z_n - w\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|S_n z_n - w\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|z_n - w\|^2. \end{aligned} \quad (3.10)$$

From (3.2), we obtain

$$\begin{aligned} \|y_n - w\|^2 &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|P_C(I - \lambda_n B)u_n - w\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|(I - \lambda_n B)u_n - (I - \lambda_n B)w\|^2 \\ &= \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|u_n - w\|^2 + \lambda_n^2 \|Bu_n - Bw\|^2 \\ &\quad - 2\lambda_n \langle u_n - w, Bu_n - Bw \rangle) \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|u_n - w\|^2 + \lambda_n(\lambda_n - 2\beta) \|Bu_n - Bw\|^2) \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|x_n - w\|^2 + \lambda_n(\lambda_n - 2\beta) \|Bu_n - Bw\|^2) \end{aligned}$$

$$\leq \|x_n - w\|^2 + (1 - \alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2.$$

We then have

$$\begin{aligned} (1 - a)b(2\beta - c)\|Bu_n - Bw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|). \end{aligned}$$

By (3.8), we obtain that

$$\lim_{n \rightarrow \infty} \|Bu_n - Bw\| = 0. \quad (3.11)$$

On the other hand, since P_C is firmly nonexpansive and $I - \lambda_n B$ is nonexpansive, we have

$$\begin{aligned} \|z_n - w\|^2 &= \|P_C(I - \lambda_n B)u_n - P_C(I - \lambda_n B)w\|^2 \\ &\leq \langle (I - \lambda_n B)u_n - (I - \lambda_n B)w, z_n - w \rangle \\ &= \frac{1}{2} \{ \|(I - \lambda_n B)u_n - (I - \lambda_n B)w\|^2 + \|z_n - w\|^2 \\ &\quad - \|(I - \lambda_n B)u_n - (I - \lambda_n B)w - (z_n - w)\|^2 \} \\ &\leq \frac{1}{2} \{ \|u_n - w\|^2 + \|z_n - w\|^2 \\ &\quad - \|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2 \} \\ &= \frac{1}{2} \{ \|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2\lambda_n \langle u_n - z_n, Bu_n - Bw \rangle - \lambda_n^2 \|Bu_n - Bw\|^2 \}. \end{aligned} \quad (3.12)$$

From (3.2), it implies that

$$\begin{aligned} \|z_n - w\|^2 &\leq \|u_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle u_n - z_n, Bu_n - Bw \rangle \\ &\quad - \lambda_n^2 \|Bu_n - Bw\|^2 \\ &\leq \|x_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Bu_n - Bw\|. \end{aligned} \quad (3.13)$$

By (3.10) and (3.13), we get

$$\begin{aligned} (1 - \alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\quad + 2(1 - \alpha_n)\lambda_n \|u_n - z_n\| \|Bu_n - Bw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad + 2(1 - \alpha_n)\lambda_n \|u_n - z_n\| \|Bu_n - Bw\|. \end{aligned}$$

By (3.8), (3.11) and the assumption $0 \leq \alpha_n \leq a < 1$, we get

$$\lim_{n \rightarrow \infty} \|u_n - z_n\| = 0. \quad (3.14)$$

Also, by (3.10) and (3.12), we obtain that

$$\begin{aligned} \|y_n - w\|^2 &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|u_n - w\|^2 \\ &= \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(w - r_n Aw)\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|(x_n - r_n Ax_n) - (w - r_n Aw)\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|x_n - w\|^2 - 2r_n \langle x_n - w, Ax_n - Aw \rangle \\ &\quad + r_n^2 \|Ax_n - Aw\|^2) \\ &\leq \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) (\|x_n - w\|^2 \\ &\quad + r_n(r_n - 2\alpha) \|Ax_n - Aw\|^2) \\ &\leq \|x_n - w\|^2 + (1 - \alpha_n) r_n(r_n - 2\alpha) \|Ax_n - Aw\|^2. \end{aligned} \quad (3.15)$$

From the assumptions $0 \leq \alpha_n \leq a < 1$ and $0 < d \leq r_n \leq e < 2\alpha$, we have

$$\begin{aligned} (1-a)d(2\alpha-e)\|Ax_n - Aw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|). \end{aligned}$$

By (3.8), we obtain

$$\lim_{n \rightarrow \infty} \|Ax_n - Aw\| = 0. \quad (3.16)$$

On the other hand, by using Lemma 2.1, we have T_{r_n} is firmly nonexpansive. Then we get

$$\begin{aligned} \|u_n - w\|^2 &= \|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)w\|^2 \\ &\leq \langle (I - r_n A)x_n - (I - r_n A)w, u_n - w \rangle \\ &= \frac{1}{2}(\|(I - r_n A)x_n - (I - r_n A)w\|^2 + \|u_n - w\|^2 \\ &\quad - \|(I - r_n A)x_n - (I - r_n A)w - (u_n - w)\|^2) \\ &\leq \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2) \\ &= \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n \langle x_n - u_n, Ax_n - Aw \rangle - r_n^2 \|Ax_n - Aw\|^2) \end{aligned}$$

and so

$$\begin{aligned} \|u_n - w\|^2 &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n \langle x_n - u_n, Ax_n - Aw \rangle \\ &\quad - r_n^2 \|Ax_n - Aw\|^2 \\ &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Aw\|. \end{aligned} \quad (3.17)$$

By (3.15) and (3.17), we get

$$\begin{aligned} \|y_n - w\|^2 &\leq \|x_n - w\|^2 - (1 - \alpha_n) \|x_n - u_n\|^2 \\ &\quad + 2(1 - \alpha_n)r_n \|x_n - u_n\| \|Ax_n - Aw\|, \end{aligned}$$

which implies that

$$\begin{aligned} (1 - \alpha_n) \|x_n - u_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Aw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad + 2r_n \|x_n - u_n\| \|Ax_n - Aw\|. \end{aligned}$$

From the assumptions $0 \leq \alpha_n \leq a < 1$, $0 < d \leq r_n \leq e < 2\alpha$, (3.8) and (3.16), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.18)$$

On the other hand, we have

$$\begin{aligned} \|x_n - S_n x_n\| &\leq \|S_n x_n - S_n z_n\| + \|S_n z_n - x_n\| \\ &\leq L \|x_n - z_n\| + \|S_n z_n - x_n\| \\ &\leq L \|x_n - u_n\| + L \|u_n - z_n\| + \|S_n z_n - x_n\|. \end{aligned}$$

Using (3.9), (3.14) and (3.18), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \quad (3.19)$$

From the assumption $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$, we have

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0, \quad \forall S \in \mathcal{S}. \quad (3.20)$$

Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ .

Step 4: We show that $\xi \in F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A)$.

First, we show that $\xi \in F(\mathcal{S})$. Suppose that $\xi \neq S\xi$ for some $S \in \mathcal{S}$. From Opial's condition and (3.20), we obtain

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|x_{n_i} - \xi\| &< \liminf_{i \rightarrow \infty} \|x_{n_i} - S\xi\| \\ &= \liminf_{i \rightarrow \infty} \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - S\xi\| \\ &\leq \liminf_{i \rightarrow \infty} \|x_{n_i} - \xi\|, \end{aligned}$$

which give us a contradiction. Hence, $\xi \in F(\mathcal{S})$.

Now, we prove that $\xi \in VI(C, B)$. Let T be the maximal monotone mapping defined by (2.1):

$$Tx = \begin{cases} Bx + N_c x & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, we get $y - Bx \in N_C x$. By $z_n \in C$ and the definition of N_C , we have

$$\langle x - z_n, y - Bx \rangle \geq 0. \quad (3.21)$$

On the other hand, since $z_n = P_C(I - \lambda_n B)u_n$, we obtain

$$\langle x - z_n, z_n - (I - \lambda_n B)u_n \rangle \geq 0$$

and then

$$\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Bu_n \rangle \geq 0. \quad (3.22)$$

Since $u_n - z_n \rightarrow 0$ as $n \rightarrow \infty$, we have that

$$Bu_n - Bz_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.23)$$

From (3.21), (3.22) and the β -inverse monotonicity of B , we obtain

$$\begin{aligned} \langle x - z_{n_i}, y \rangle &\geq \langle x - z_{n_i}, Bx \rangle \\ &\geq \langle x - z_{n_i}, Bx \rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i} \rangle \\ &= \langle x - z_{n_i}, Bx - Bz_{n_i} \rangle + \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i} \rangle \\ &\quad - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i} \rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &= \langle x, Bz_{n_i} - Bu_{n_i} \rangle - \langle z_{n_i}, Bz_{n_i} - Bu_{n_i} \rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle x, Bz_{n_i} - Bu_{n_i} \rangle - \|z_{n_i}\| \|Bz_{n_i} - Bu_{n_i}\| \\ &\quad - \frac{1}{\lambda_{n_i}} \|x - z_{n_i}\| \|z_{n_i} - u_{n_i}\|. \end{aligned} \quad (3.24)$$

By (3.14) and (3.18), we get

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

Since $x_{n_i} \rightharpoonup \xi$, we obtain $z_{n_i} \rightharpoonup \xi$. From (3.14), (3.23) and (3.24), we obtain

$$\langle x - \xi, y \rangle = \lim_{n_i \rightarrow \infty} \langle x - z_{n_i}, y \rangle \geq 0.$$

Since T is maximal monotone, we obtain that $0 \in T\xi$. It follows that $\xi \in VI(C, B)$.

Next, we show that $\xi \in GMEP(f, \varphi, A)$. For any $y \in C$,

$$f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n).$$

From the condition (A2), we get that

$$\varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq f(y, u_n) + \varphi(u_n).$$

Replacing n by n_i , we obtain

$$\varphi(y) + \langle Ax_{n_i}, y - u_{n_i} \rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq f(y, u_{n_i}) + \varphi(u_{n_i}). \quad (3.25)$$

For any t with $0 < t \leq 1$ and $y \in C$, put $\rho_t = ty + (1 - t)\xi$. Since $y \in C$ and $\xi \in C$, we obtain $\rho_t \in C$. It follows from (3.25) and the monotonicity of A that

$$\begin{aligned} \langle \rho_t - u_{n_i}, A\rho_t \rangle &\geq \langle \rho_t - u_{n_i}, A\rho_t \rangle - \langle Ax_{n_i}, \rho_t - u_{n_i} \rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \\ &\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\ &= \langle \rho_t - u_{n_i}, A\rho_t - Au_{n_i} \rangle + \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle \\ &\quad - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\ &\geq \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \\ &\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t). \end{aligned} \quad (3.26)$$

Since A is Lipschitzian, by (3.18), we get $Au_{n_i} - Ax_{n_i} \rightarrow 0$ as $i \rightarrow \infty$. From (A3) and (3.26), we arrive at

$$\langle \rho_t - \xi, A\rho_t \rangle \geq f(\rho_t, \xi) + \varphi(\xi) - \varphi(\rho_t). \quad (3.27)$$

From (A1), (A3), (3.27) and convexity of φ , we have that

$$\begin{aligned} 0 &= f(\rho_t, \rho_t) \leq tf(\rho_t, y) + (1 - t)f(\rho_t, \xi) \\ &\leq tf(\rho_t, y) + (1 - t)(\langle \rho_t - \xi, A\rho_t \rangle + \varphi(\rho_t) - \varphi(\xi)) \\ &\leq tf(\rho_t, y) + (1 - t)t(\langle y - \xi, A\rho_t \rangle + \varphi(y) - \varphi(\xi)), \end{aligned}$$

which implies that

$$f(\rho_t, y) + (1 - t)(\langle y - \xi, A\rho_t \rangle + \varphi(y) - \varphi(\xi)) \geq 0.$$

Letting $t \rightarrow 0$, by (A4), we arrive at

$$f(\xi, y) + \langle y - \xi, A\xi \rangle + \varphi(y) - \varphi(\xi) \geq 0.$$

This shows that $\xi \in GMEP(f, \varphi, A)$.

Step 5: We show that $x_n \rightarrow P_F x_1$.

Let $\bar{x} = P_F x_1$. Since $\bar{x} = P_F x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_1$, we get

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \bar{x}\|.$$

On the other hand, we have

$$\|x_1 - \bar{x}\| \leq \|x_1 - \xi\|$$

$$\begin{aligned}
 &\leq \liminf_{i \rightarrow \infty} \|x_1 - x_{n_i}\| \\
 &\leq \limsup_{i \rightarrow \infty} \|x_1 - x_{n_i}\| \\
 &\leq \|x_1 - \bar{x}\|.
 \end{aligned}$$

Therefore, we get

$$\|x_1 - \xi\| = \lim_{i \rightarrow \infty} \|x_1 - x_{n_i}\| = \|x_1 - \bar{x}\|.$$

This implies that $\bar{x} = \xi$. Since H has the Kadec-Klee property and $x_1 - x_{n_i} \rightarrow x_1 - \bar{x}$, it follows that $x_{n_i} \rightarrow \bar{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. The proof is now complete. $\square$

4. DEDUCED THEOREMS AND APPLICATIONS

Theorem 3.1 can be reduced to many different results. By putting $S_n = S$ for all $n \geq 1$ in Theorem 3.1, we obtain the following theorem:

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $S : C \rightarrow C$ be a L -Lipschitzian quasi-nonexpansive mapping such that $F = F(S) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases}
 x_1 \in C, \\
 C_1 = C, \\
 f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \forall y \in C, \\
 z_n = P_C(u_n - \lambda_n B u_n), \\
 y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\
 C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\
 x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1,
 \end{cases} \quad (4.1)$$

where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

When $\{S_n\}$ and $\mathcal{S}$ are families of nonexpansive mappings, we get the following theorem:

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Now we show how to apply Theorem 4.2 for families of strict pseudo-contraction mappings.

Theorem 4.3. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $\{R_n\}$ and $\mathcal{R}$ be families of k -strict pseudo-contraction mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(R_n) = F(\mathcal{R})$ and $F = F(\mathcal{R}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S_n : C \rightarrow C$ by $S_n x = kx + (1-k)R_n x$ for all $x \in C$. Suppose that $\{R_n\}$ satisfies the NST-condition with $\mathcal{R}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Proof. By Lemma 2.3, we obtain that S_n is nonexpansive for all positive integer n . We also get that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S} = \{kI + (1-k)T : T \in \mathcal{R}\}$. The proof is now complete because of the direct result of Theorem 4.2. $\square$

By putting $S_n = S$ for all $n \geq 1$ in Theorem 4.3, we obtain the following corollary.

Corollary 4.4. Let C be a closed convex subset of a real Hilbert space H , $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α -inverse-strongly monotone mapping of C into H and B be a β -inverse-strongly monotone mapping of C into H , respectively. Let $R : C \rightarrow C$ be a k -strict pseudo-contraction such that $F = F(R) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S : C \rightarrow C$ by $Sx = kx + (1-k)Rx$ for all $x \in C$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm (4.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\bar{x} = P_F x_1$, where P_F is the metric projection of H onto F .

Remark 4.5. By letting $\varphi = 0$ in Corollary 4.4, we obtain Theorem 2.1 of [4].

Remark 4.6. Since Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 4.4 are for finding a common element of the set of fixed points, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem, we can reduce each theorem or corollary by letting $B = 0$, $A = 0$, $\varphi = 0$ or $f(x, y) = 0$.

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An implicit iteration process for solving a fixed point problem of a finite family of multi-valued mappings in Banach spaces

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ABSTRACT

This work is concerned with the strong convergence of an approximating common fixed point sequence of a finite family of multi-valued mappings in a uniformly convex and smooth Banach space using an implicit iteration scheme introduced by Xu and Ori [H.K. Xu, R.G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Funct. Anal. Optim. 22 (2001) 767–773].

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1. Introduction and preliminaries

In this work, let D be a nonempty convex subset of a Banach spaces X . The metric projection operator (the nearest point projection) P_D defined on X is a mapping from X to 2^D such that

$$P_D x = \{y \in D : \|x - y\| = d(x, D)\}, \quad \forall x \in X.$$

If $P_D x \neq \emptyset$ for every $x \in X$, then D is called a *proximal set*. If $P_D x$ is a singleton, for every $x \in X$, then D is said to be a *Chebyshev set*. Let $CB(D)$, $K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximal bounded subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for $A, B \in CB(D)$. A single-valued mapping $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued mapping $T : D \rightarrow CB(D)$ is called (i) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$ and (ii) *hemicompact* if for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \rightarrow CB(D)$ is *hemicompact*. An element $p \in D$ is called a *fixed point* of $T : D \rightarrow D$ (respectively, $T : D \rightarrow CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$.

A family $\{T_i\}_{i=1}^N : D \rightarrow CB(D)$ is said to satisfy the *condition (A)* if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, T_j x) \geq f \left(d \left(x, \bigcap_{i=1}^N F(T_i) \right) \right)$$

for some $j \in \{1, 2, \dots, N\}$ and for all $x \in D$.

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In 2001, Xu and Ori [1] introduced the following implicit iteration process for solving a fixed point problem of a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \dots, N\}$) in Hilbert spaces. For $x_0 \in D$ and $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{aligned}x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\&\vdots \\x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\&\vdots\end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \quad (1.1)$$

where $T_n = T_{n \bmod N}$ (here the $n \bmod N$ function takes values in I). Xu and Ori proved the weak convergence of this process (1.1) to a common fixed point of a finite family $\{T_i : i \in I\}$.

In 2002, Zhou and Chang [2] studied the theorems of weak and strong convergence of the implicit iteration process to a common fixed point of a finite family of asymptotically nonexpansive mappings in a uniformly convex Banach space with $\alpha_n \in (b, c)$ for some constants $b, c \in (0, 1)$.

In 2005, Chidume and Shahzad [3] generalized Zhou's result, still in the uniformly convex Banach space, with $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$.

However, the studying of nonexpansive multi-valued mappings is harder than that of single-valued nonexpansive mappings in both Hilbert spaces and Banach spaces; see also [4,5].

Let $T : D \rightarrow P(D)$ and $P_T x = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Here P_T is called *the best approximation operator*.

It is remarked that Hussain and Khan [6], in 2003, employed the best approximation operator P_T to study fixed points of a $*$ -nonexpansive multi-valued mapping T and the strong convergence of its iterates to a fixed point of T defined on a closed and convex subset of a Hilbert space.

The following example shows that P_T is quasi-nonexpansive but T is not nonexpansive.

Example 1.1 ([7]). Consider $D = [0, 1]$ with the usual norm. Define $T : D \rightarrow CB(D)$ by

$$Tx = \begin{cases} [0, x], & x \in \left[0, \frac{1}{2}\right], \\ \left\{\frac{1}{2}\right\}, & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

We do indeed have that P_T is nonexpansive.

Case 1: if $x, y \in [0, \frac{1}{2}]$ then $H(P_T x, P_T y) = \|x - y\|$.

Case 2: if $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$ then $H(P_T x, P_T y) = \|x - \frac{1}{2}\| \leq \|x - y\|$.

Case 3: if $x, y \in (\frac{1}{2}, 1]$ then $H(P_T x, P_T y) = 0$.

It would be interesting to study the convergence of the multi-valued mapping T by using the best approximation operator P_T .

Let X be a real Banach space and let J be the normalized duality mapping from X to 2^{X^*} defined by $J(x) = \{f \in X^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}$, $\forall x \in X$, where X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair. It is well known that if X is a smooth Banach space, then J is single-valued. A Banach space X is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. For all $\epsilon \in [0, 2]$, the *modulus of convexity* of X is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x + y\| \leq 1, \|x - y\| \geq \epsilon \right\}.$$

Lemma 1.2 ([8]). Let X be a uniformly convex Banach space with the modulus of uniform convexity δ_X . Then $\delta_X : [0, 2] \rightarrow [0, 1]$ is continuous increasing, $\delta_X(0) = 0$, and $\delta_X(t) > 0$ for $t > 0$, while

$$\|cu + (1 - c)v\| \leq 1 - 2 \min\{c, 1 - c\} \delta_X(\|u - v\|)$$

where $0 \leq c \leq 1$ and $\|u\|, \|v\| \leq 1$.

Lemma 1.3 ([9]). Let $x, y \in X$. Then the following conditions are equivalent:

- (i) $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \geq 0$;
- (ii) there exists $f \in J(x)$ with $\langle y, f \rangle \geq 0$.

2. Strong convergence theorems

In this section, we first introduce an implicit iteration process for finding a common fixed point of a finite family of multi-valued mappings. Then the strong convergence theorems are also established. Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. For $x_0 \in D$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$x_n \in \alpha_n x_{n-1} + (1 - \alpha_n) P_{T_n} x_n, \quad n \geq 1, \quad (2.1)$$

where $T_n = T_{n \bmod N}$ (here the $n \bmod N$ function takes values in $\{1, 2, \dots, N\}$) and P_{T_n} is nonexpansive.

The following well-known result was proved in Nadler [10].

Theorem 2.1 ([10]). *Let D be a nonempty closed subset of a Banach space X and $T : D \rightarrow CB(D)$ be a multi-valued contraction. Then T has a fixed point.*

Let D be a nonempty closed convex subset of a real Banach space X and $T : D \rightarrow CB(D)$ be a nonexpansive multi-valued mapping. For every $u \in D$ and $t \in (0, 1)$, the mapping $G_t : D \rightarrow CB(D)$ defined by $G_t = tu + (1 - t)T$ is a contraction mapping; by Theorem 2.1, there exists $x_t \in D$ (non-unique, in general) such that $x_t \in tu + (1 - t)Tx_t$. This implies that the implicit iteration process (2.1) introduced above can be employed for the approximation of common fixed points of a finite family of multi-valued mappings.

Next, we prove our main results.

Lemma 2.2. *Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. For arbitrary $x_0 \in D$, the sequence $\{x_n\}_{n=1}^\infty$ is generated by (2.1). Assume that the best approximation operator P_{T_i} is nonexpansive for each $i \in \{1, 2, \dots, N\}$. Then:*

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \bigcap_{i=1}^N F(T_i)$,
- (ii) $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for each $i \in \{1, 2, \dots, N\}$.

Proof. (i) By the property of the best approximation operator P_{T_i} and our assumption, we have

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + (1 - \alpha_n) z_n - p\| \quad \text{for some } z_n \in P_{T_n} x_n \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|z_n - p\| \\ &= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) d(z_n, P_{T_n} p) \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) H(P_{T_n} x_n, P_{T_n} p) \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|x_n - p\| \end{aligned}$$

for all $p \in \bigcap_{i=1}^N F(T_i)$. This implies that $\|x_n - p\| \leq \|x_{n-1} - p\|$ and $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \bigcap_{i=1}^N F(T_i)$.

(ii) For each $p \in \bigcap_{i=1}^N F(T_i)$, we have

$$\begin{aligned} \langle z_n - p, j(x_n - p) \rangle &\leq \|z_n - p\| \|x_n - p\| \\ &= d(z_n, P_{T_n} p) \|x_n - p\| \\ &\leq H(P_{T_n} x_n, P_{T_n} p) \|x_n - p\| \\ &\leq \|x_n - p\|^2. \end{aligned}$$

This implies that $\langle x_n - z_n, j(x_n - p) \rangle \geq 0$. From Lemma 1.3, we have

$$\begin{aligned} \|x_n - p\| &\leq \left\| x_n - p + \frac{1 - \alpha_n}{2\alpha_n} (x_n - z_n) \right\| \\ &= \left\| x_n - p + \frac{1 - \alpha_n}{2} (x_{n-1} - z_n) \right\| \\ &= \left\| \alpha_n x_{n-1} + (1 - \alpha_n) z_n - p + \frac{1 - \alpha_n}{2} (x_{n-1} - z_n) \right\| \\ &= \left\| \frac{x_{n-1} + x_n}{2} - p \right\| \\ &\leq \|x_{n-1} - p\| \left[1 - \delta_X \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - p\|} \right) \right]. \end{aligned}$$

By the properties of δ_X , hence $\|x_n - x_{n-1}\| \rightarrow 0$ as $n \rightarrow \infty$. We observe that

$$\|x_{n-1} - z_n\| = \frac{1}{1 - \alpha_n} \|x_n - x_{n-1}\|.$$

Using the assumption that $\limsup_{n \rightarrow \infty} \alpha_n < 1$, we have $\|x_{n-1} - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Hence $\|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. It follows that $d(x_n, P_{T_n} x_n) \leq \|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $\forall i \in \{1, 2, \dots, N\}$,

$$\begin{aligned} d(x_n, P_{T_{n+i}} x_n) &\leq \|x_n - x_{n+i}\| + d(x_{n+i}, P_{T_{n+i}} x_{n+i}) + H(P_{T_{n+i}} x_{n+i}, P_{T_{n+i}} x_n) \\ &\leq 2\|x_n - x_{n+i}\| + d(x_{n+i}, P_{T_{n+i}} x_{n+i}) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, $\forall \ell \in \{1, 2, \dots, N\}$, there exists some $i \in \{1, 2, \dots, N\}$ such that $n + i = \ell \pmod N$. It follows that

$$d(x_n, P_{T_\ell} x_n) = d(x_n, P_{T_{n+i}} x_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Hence

$$d(x_n, T_\ell x_n) \leq d(x_n, P_{T_\ell} x_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This completes the proof. $\square$

Theorem 2.3. Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that the best approximation operator P_{T_i} is nonexpansive for each $i \in \{1, 2, \dots, N\}$. If $\{T_i\}_{i=1}^N$ satisfies the condition (A), then the sequence $\{x_n\}_{n=1}^\infty$ generated by (2.1) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

Proof. Since $\{T_i\}_{i=1}^N$ satisfies the condition (A), we have $d(x_n, \bigcap_{i=1}^N F(T_i)) \rightarrow 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for some $\{p_k\} \subset \bigcap_{i=1}^N F(T_i)$ and all k . So we obtain

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

Next, we shall show that $\{p_k\}$ is a Cauchy sequence in D . Notice that

$$\|p_{k+1} - p_k\| \leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This implies that $\{p_k\}$ is a Cauchy sequence in D and thus converges to $q \in D$. Since

$$d(p_k, T_i q) \leq d(p_k, P_{T_i} q) \leq H(P_{T_i} q, P_{T_i} p_k) \leq \|q - p_k\|$$

for all $i = 1, 2, \dots, N$ and $p_k \rightarrow q$ as $k \rightarrow \infty$, it follows that $d(q, T_i q) = 0$ for all $i = 1, 2, \dots, N$ and thus $q \in \bigcap_{i=1}^N F(T_i)$. Hence $\{x_{n_k}\}$ converges strongly to q . Since $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists, it follows that $\{x_n\}_{n=1}^\infty$ converges strongly to q . This completes the proof. $\square$

Theorem 2.4. Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that the best approximation operator P_{T_i} is nonexpansive for each $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}_{n=1}^\infty$ generated by (2.1) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$.

Proof. Since the necessity is obvious, it suffices to show the sufficiency. Suppose that $\liminf_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$; then by Lemma 2.2(i), we have that $\lim_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i))$ exists. This implies that $\lim_{n \rightarrow \infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$. By using the same argument as in the proof of Theorem 2.3, we obtain that $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of a finite family $\{T_i\}_{i=1}^N$. $\square$

Theorem 2.5. Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space X . Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued mappings of D into $P(D)$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that the best approximation operator P_{T_i} is nonexpansive and hemicompact for each $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}_{n=1}^\infty$ generated by (2.1) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

Proof. Since $\lim_{n \rightarrow \infty} d(x_n, P_{T_i} x_n) = 0$ and P_{T_i} is hemicompact for each $i \in \{1, 2, \dots, N\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in D$. It follows that

$$\begin{aligned} d(p, P_{T_i} p) &\leq \|p - x_{n_k}\| + d(x_{n_k}, P_{T_i} x_{n_k}) + H(P_{T_i} x_{n_k}, P_{T_i} p) \\ &\leq 2\|p - x_{n_k}\| + d(x_{n_k}, P_{T_i} x_{n_k}) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Hence $p \in \bigcap_{i=1}^N F(T_i)$. By Lemma 2.2(i), we obtain $x_n \rightarrow p$ as $n \rightarrow \infty$. This completes the proof. $\square$

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Fixed Point Theorems for Nonspreading-type Multi-valued Mappings in Hilbert Spaces

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Abstract

In this work, we define a new class of nonlinear multi-valued mappings called k -nonspreading in a Hilbert space. We then prove fixed point theorems for $\frac{1}{2}$ -nonspreading multi-valued mappings. Furthermore, we obtain a necessary and sufficient condition for the existence of such mappings in a Hilbert space. Finally, convergence theorems are also discussed.

Keywords: Hilbert space; k -nonspreading multi-valued mapping; Fixed point; Convergence theorem

AMS Subject Classification: 47H10, 54H25.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and D be a nonempty convex subset of H . Let $CB(D)$ and $K(D)$ be the families of nonempty closed bounded subsets and nonempty compact subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for $A, B \in CB(D)$. A single-valued mapping $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued mapping $T : D \rightarrow CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a *fixed point* of $T : D \rightarrow D$ (resp. $T : D \rightarrow CB(D)$) if $p = Tp$ (resp. $p \in Tp$). The fixed points set of T is denoted by $F(T)$. If $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$, then T is called *quasi-nonexpansive*.

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Let $\{x_n\}_{n=1}^\infty$ be a sequence in H . We denote the weak convergence of $\{x_n\}_{n=1}^\infty$ to a point $x \in H$ by $x_n \rightharpoonup x$, and the strong convergence of $\{x_n\}_{n=1}^\infty$ to a point $x \in H$ by $x_n \rightarrow x$.

A multi-valued mapping $T : D \rightarrow CB(D)$ is said to be *demiclosed* at $y \in D$ if $\{x_n\}_{n=1}^\infty \subset D$ such that $x_n \rightharpoonup x$ and $y_n \in Tx_n$ such that $y_n \rightarrow y$ imply $y \in Tx$.

Fixed point theorems of nonlinear mappings have been intensively studied and considered by many authors (see, for examples, [1, 4, 10, 11, 14, 27]).

In 2008, Kohsaka-Takahashi [16, 17] introduced a new class of nonlinear mappings as follows: let H be a Hilbert space and C a nonempty, closed and convex subset of H . Then, a mapping $T : C \rightarrow C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$.

Later, Takahashi [25] defined a class of nonlinear mappings called *hybrid*, i.e.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$.

Recently, Aoyama-Iemoto-Kohsaka-Takahashi [2] introduced a new class of nonlinear mappings containing the class of nonexpansive mappings, nonspreading mappings and hybrid mappings as follows:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda \langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$ and $\lambda \in \mathbb{R}$. Such a mapping is called λ -*hybrid*. They proved fixed point theorems of such mappings in Hilbert spaces.

The fixed point theory of multi-valued mappings is much more complicated and interesting than the corresponding theory of single-valued mappings. However, some classical fixed point theorems for single-valued mappings have already been extended to multi-valued mappings.

In 1968, Markin [20] firstly established the nonexpansive multi-valued convergence results in Hilbert spaces. Banach's Contraction Principle was extended to a multi-valued contraction in 1969 by Nadler [21]. In 1974, one breakthrough was achieved by Lim using Edelstein's method of asymptotic centers [18]. In 1999, Kirk-Massa [15] obtained another important result for the existence fixed point theorem of nonexpansive multi-valued mappings. In 1999, Sahu [24] obtained the strong convergence theorems of the nonexpansive type and nonself multi-valued mappings in uniformly convex Banach spaces. Recently, Xu [30] extended the result of Kirk-Massa [15] to a nonexpansive multi-valued nonself mapping and obtained the fixed point theorem in 2001. The recent fixed point results for nonexpansive multi-valued mappings can be found in [3, 5, 6, 7, 8, 9, 13, 23, 29] and references cited therein.

2 Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In a real Hilbert space H , we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$ (see [26]) and

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$$

for all $x, y, z, w \in H$ (see [12]).

Let D be a closed and convex subset of H . For every point $x \in H$, there exists a unique nearest point in D , denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall y \in D.$$

P_D is called the *metric projection* of H onto D . We know that P_D is a nonexpansive mapping of H onto D .

The following lemmas will be used for the proof of our main results in the sequel.

Lemma 2.1. *Let H be a real Hilbert space. Then the following well known results hold:*

$$(i) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

(ii) *If $\{x_n\}_{n=1}^\infty$ is a sequence in H which converges weakly to $z \in H$ then*

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Lemma 2.2. [19] *Let D be a closed and convex subset of a real Hilbert space H and let P_D be the metric projection from H onto D . Given $x \in H$ and $z \in D$. Then $z = P_D x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D.$$

Lemma 2.3. [28] *Let D be a nonempty closed convex subset of a Hilbert space H , let $\{x_n\}$ be a bounded sequence in H and let μ be a Banach limit. If $g : D \rightarrow \mathbb{R}$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in D,$$

then there exists a unique $z_0 \in D$ such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

3 Main results

Let H be a real Hilbert space, and D a nonempty and convex subset of H , we say that a multi-valued mapping $T : D \rightarrow CB(D)$ is *k-nonspreading* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \leq k(d(Tx, y)^2 + d(x, Ty)^2) \quad (3.1)$$

for all $x, y \in D$.

It is easy to see that if T is $\frac{1}{2}$ -nonspreading, then T is nonspreading in the case of single-valued mappings (see [16, 17]). Moreover, if T is $\frac{1}{2}$ -nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in D$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\leq d(Tx, p)^2 + d(x, Tp)^2 \\ &\leq H(Tx, Tp)^2 + \|x - p\|^2. \end{aligned}$$

It follows that

$$H(Tx, Tp) \leq \|x - p\|. \quad (3.2)$$

We now give an example of a $\frac{1}{2}$ -nonspreading multi-valued mapping which is not a nonexpansive multi-valued mapping.

Example 3.1. Consider $D = [-3, 3]$ with the usual norm. Define $T : D \rightarrow CB(D)$ by

$$Tx = \begin{cases} \{0\} & , x \in [-2, 2], \\ \left[-\frac{|x|}{|x|+1}, \frac{|x|}{|x|+1} \right] & , x \notin [-2, 2]. \end{cases}$$

To see that T is $\frac{1}{2}$ -nonspreading, we observe that

Case 1: if $x, y \in [-2, 2]$, then $H(Tx, Ty) = 0$.

Case 2: if $x \in [-2, 2]$ and $y \notin [-2, 2]$, then $Tx = \{0\}$ and $Ty = \left[-\frac{|y|}{|y|+1}, \frac{|y|}{|y|+1} \right]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|y|}{|y|+1}\right)^2 < 2 < y^2 \leq d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: if $x, y \notin [-2, 2]$, then $Tx = \left[-\frac{|x|}{|x|+1}, \frac{|x|}{|x|+1} \right]$ and $Ty = \left[-\frac{|y|}{|y|+1}, \frac{|y|}{|y|+1} \right]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\right)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

T is not nonexpansive since for $x = 2$ and $y = \frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = \left[-\frac{5}{7}, \frac{5}{7} \right]$. This implies that $H(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = \left| 2 - \frac{5}{2} \right| = \|x - y\|$.

Let D be a nonempty set. We define $T(D) = \cup_{x \in D} Tx$ and $(ST)x = S(Tx)$ for all $x \in D$. We next present some crucial properties of $\frac{1}{2}$ -nonspreading multi-valued mappings.

Lemma 3.2. Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow CB(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. We shall show that $x \in F(T)$. Observe

$$\begin{aligned} d(x, Tx) &\leq \|x - x_n\| + d(x_n, Tx) \\ &\leq \|x - x_n\| + H(Tx_n, Tx) \\ &\leq 2\|x - x_n\|. \end{aligned}$$

It follows that $d(x, Tx) = 0$ and hence $x \in F(T)$. □

Lemma 3.3. *Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow CB(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping with $F(T) \neq \emptyset$. If $Tp = \{p\}$ for all $p \in F(T)$, then $F(T)$ is convex.*

Proof. Let $p = tp_1 + (1 - t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0, 1)$. We shall show that $p \in F(T)$. Let $z \in Tp$. From (3.2) we have

$$\begin{aligned}
 \|p - z\|^2 &= \|t(z - p_1) + (1 - t)(z - p_2)\|^2 \\
 &= t\|z - p_1\|^2 + (1 - t)\|z - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\
 &= td(z, Tp_1)^2 + (1 - t)d(z, Tp_2)^2 - t(1 - t)\|p_1 - p_2\|^2 \\
 &\leq tH(Tp, Tp_1)^2 + (1 - t)H(Tp, Tp_2)^2 - t(1 - t)\|p_1 - p_2\|^2 \\
 &\leq t\|p - p_1\|^2 + (1 - t)\|p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\
 &= t(1 - t)^2\|p_1 - p_2\|^2 + (1 - t)t^2\|p_1 - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\
 &= 0,
 \end{aligned}$$

which yields $p = z$. Therefore $p \in F(T)$. □

Lemma 3.4. *Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. If $x, y \in D$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \leq H(Tx, Ty)^2 \leq \|x - y\|^2 + 2\langle x - a, y - b \rangle.$$

Proof. Let $x, y \in D$ and $a \in Tx$. From Nadler Theorem (see [21]), there exists $b \in Ty$ such that

$$\|a - b\|^2 \leq H(Tx, Ty)^2.$$

It follows that

$$\begin{aligned}
 2H(Tx, Ty)^2 &\leq d(Tx, y)^2 + d(x, Ty)^2 \\
 &\leq \|a - y\|^2 + \|x - b\|^2 \\
 &\leq \|a - x\|^2 + 2\langle a - x, x - y \rangle + \|x - y\|^2 + \|x - a\|^2 + 2\langle x - a, a - b \rangle + \|a - b\|^2 \\
 &= 2\|a - x\|^2 + \|x - y\|^2 + \|a - b\|^2 + 2\langle a - x, x - a - (y - b) \rangle \\
 &\leq 2\|a - x\|^2 + \|x - y\|^2 + H(Tx, Ty)^2 + 2\langle a - x, x - a - (y - b) \rangle.
 \end{aligned}$$

This shows that

$$H(Tx, Ty)^2 \leq \|x - y\|^2 + 2\langle x - a, y - b \rangle.$$

□

Lemma 3.5. *Let D be a closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. Then $(I - T)$ is demiclosed at 0.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence in D which converges weakly to p and $y_n \in Tx_n$ such that $\{x_n - y_n\}_{n=1}^\infty$ converges strongly to 0. We shall show that $p \in F(T)$. By Lemma 3.4, there exists $z_n \in Tp$ such that $\|y_n - z_n\|^2 \leq \|x_n - p\|^2 + 2\langle x_n - y_n, p - z_n \rangle$. Since Tp is compact and $z_n \in Tp$,

there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \rightarrow z \in Tp$. Since $\{x_n\}_{n=1}^\infty$ converges weakly to p , it is bounded. For each $x \in H$, define $f : H \rightarrow [0, \infty)$ by

$$f(x) := \limsup_{i \rightarrow \infty} \|x_{n_i} - x\|^2.$$

Then from Lemma 2.1 (ii), we obtain

$$f(x) = \limsup_{i \rightarrow \infty} \|x_{n_i} - p\|^2 + \|p - x\|^2, \quad \forall x \in H.$$

Thus $f(x) = f(p) + \|p - x\|^2, \forall x \in H$. It also follows that

$$f(z) = f(p) + \|p - z\|^2. \quad (3.3)$$

We observe that

$$f(z) = \limsup_{i \rightarrow \infty} \|x_{n_i} - z\|^2 = \limsup_{i \rightarrow \infty} \|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \leq \limsup_{i \rightarrow \infty} \|y_{n_i} - z\|^2$$

This implies that

$$\begin{aligned} f(z) &\leq \limsup_{i \rightarrow \infty} \|y_{n_i} - z\|^2 \\ &= \limsup_{i \rightarrow \infty} [\|y_{n_i} - z_{n_i} + z_{n_i} - z\|]^2 \\ &\leq \limsup_{i \rightarrow \infty} [\|x_{n_i} - p\|^2 + 2\langle x_{n_i} - y_{n_i}, p - z_{n_i} \rangle] \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \quad (3.4)$$

Hence it follows from (3.3) and (3.4) that $\|p - z\| = 0$. This completes the proof. $\square$

Theorem 3.6. *Let D be a nonempty, closed and convex subset of a real Hilbert space H , let T be a multi-valued mapping of D into $CB(D)$. Suppose that there exists an element $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded and*

$$\mu_n \|z_n - z\|^2 \leq \mu_n \|z_n - y\|^2, \quad \forall y \in D, \forall z \in Ty$$

for some Banach limit μ . Then, T has a fixed point in D .

Proof. Using a Banach limit μ on ℓ^∞ , we can define $g : D \rightarrow \mathbb{R}$ as follows:

$$g(z) := \mu_n \|z_n - z\|^2, \quad \forall z \in D.$$

From Lemma 2.3, there exists a unique $z_0 \in D$ such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

So, for each $a \in Tz_0$, we have

$$g(a) = \mu_n \|z_n - a\|^2 \leq \mu_n \|z_n - z_0\|^2 = g(z_0).$$

Since $a \in D$ and $z_0 \in D$ is unique element such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Hence $z_0 = a \in Tz_0$. This completes the proof. $\square$

Theorem 3.7. *Let D be a nonempty, closed and convex subset of a real Hilbert space H , let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. Then, the following are equivalent:*

- (i) *There exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded.*
- (ii) *$F(T)$ is nonempty.*

Proof. It is obvious that (ii) implies (i). We shall show that (i) implies (ii). Assume that there exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded. Let $x, y \in D$ and $z_{n+1} \in Tz_n$. From Lemma 3.4, there exists $b \in Ty$ such that

$$\begin{aligned} \|z_{n+1} - b\|^2 &\leq \|z_n - y\|^2 + 2\langle z_n - z_{n+1}, y - b \rangle \\ &\leq \|z_n - y\|^2 + (\|z_n - b\|^2 + \|z_{n+1} - y\|^2 - \|z_n - y\|^2 - \|z_{n+1} - b\|^2), \end{aligned}$$

which implies

$$2\|z_{n+1} - b\|^2 - \|z_n - b\|^2 \leq \|z_{n+1} - y\|^2.$$

Let μ be a Banach limit on ℓ^∞ . For any $n \in \mathbb{N}$, we have

$$2\mu_n\|z_{n+1} - b\|^2 - \mu_n\|z_n - b\|^2 \leq \mu_n\|z_{n+1} - y\|^2.$$

This implies that

$$\mu_n\|z_n - b\|^2 \leq \mu_n\|z_n - y\|^2.$$

From Theorem 3.6, T has a fixed point in D . □

We next prove the following weak convergence theorem for a $\frac{1}{2}$ -nonspreading mapping in a Hilbert space.

Theorem 3.8. *Let D be a nonempty, closed and convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a $\frac{1}{2}$ -nonspreading multi-valued mapping. Define the sequence $\{x_n\}$ in D as follows: $x_1 = x \in D$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad z_n \in Tx_n$$

for all $n \in \mathbb{N}$, where $0 \leq \alpha_n < 1$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$.

If $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$, then $\{x_n\}$ converges weakly to $q \in F(T)$.

Proof. Since a $\frac{1}{2}$ -nonexpansive multi-valued mapping T with a fixed point is quasi-nonexpansive,

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|z_n - p\| \\ &= \alpha_n \|x_n - p\| + (1 - \alpha_n) d(z_n, Tp) \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) H(Tx_n, Tp) \\ &\leq \|x_n - p\| \end{aligned}$$

for all $p \in F(T)$. Therefore $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded. For all $p \in F(T)$, we observe that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|\alpha_n(x_n - p) + (1 - \alpha_n)(z_n - p)\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\| \\ &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) d(z_n, Tp)^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\| \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) H(Tx_n, Tp)^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\| \\ &\leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - z_n\|. \end{aligned}$$

This implies that

$$\alpha_n(1 - \alpha_n) \|x_n - z_n\| \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to q . From Lemma 3.5, we obtain $q \in F(T)$. To complete our conclusion, it is sufficient to show that for another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $w \in F(T)$, $q = w$. Suppose that $q \neq w$. We have from Opial's theorem [22] that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - q\| < \lim_{i \rightarrow \infty} \|x_{n_i} - w\| = \lim_{n \rightarrow \infty} \|x_n - w\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - w\| < \lim_{j \rightarrow \infty} \|x_{n_j} - q\| = \lim_{n \rightarrow \infty} \|x_n - q\|. \end{aligned}$$

This is a contradiction. So, $\{x_n\}$ converges weakly to $q \in F(T)$. □

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