Since N is a subgroup of G containing B, by Theorem 2.7.9, we may assume that N is imprimitive. Let $\mathcal{B} = \{\Delta_1, \Delta_2, \ldots, \Delta_k\}$ be a non-trivial block system invariant under N. Since N is transitive on Ω, it follows that N acts transitively on $\mathcal{B}$. Set $t = |\Omega|/k$. Then $t = |\Delta_i|$ for $i = 1, \ldots, k$. By Theorem 2.7.11, t is a power of p. Set $M = \mathrm{Stab}_G(\mathcal{B})$. Then

$$T \leqslant B \leqslant N \leqslant M \cong \mathrm{Sym}(t) \wr \mathrm{Sym}(k).$$

For $i = 1, \ldots, k$, put $K_i = \mathrm{Sym}(\Delta_i)$ and $K = K_1 \times K_2 \times \cdots \times K_k$. Then for $i = 1, \ldots, k$, as $K_i \trianglelefteq K \trianglelefteq M$, $1 \neq R_i = T \cap K_i \in \mathrm{Syl}_p(K_i)$, $T \cap K = R_1 \times R_2 \times \cdots \times R_k \in \mathrm{Syl}_p(K)$ and $B_i = B \cap K_i = N_{K_i}(R_i)$. Since t is a power of p, R_i is transitive on Δ_i for all i. Suppose that $O_p(N) \cap K = 1$. Since $[O_p(N), N \cap K] \leqslant O_p(N) \cap K$, this gives $[O_p(N), N \cap K] = 1$. As $R_i \leqslant N \cap K$, for all i, $O_p(N)$ centralizes R_i and, because of the structure of $\mathrm{Sym}(t) \wr \mathrm{Sym}(k)$, this forces $O_p(N) \leqslant K$. But now $O_p(N) \cap K = O_p(N) \neq 1$, a contradiction. Therefore $O_p(N) \cap K \neq 1$.

Let $\varphi_i : K \longrightarrow K_i$ be the projection map of K onto K_i and set $L_i = \varphi_i(N \cap K)$. We see that $R_i \leqslant B_i \leqslant L_i \leqslant K_i$ and that L_i is transitive on Δ_i. If $O_p(L_i) = 1$, then $O_p(N \cap K) \leqslant \prod_{j \neq i} K_j$. For all $n \in N$, as $O_p(N \cap K) \trianglelefteq N$, we then have $O_p(N \cap K) = O_p(N \cap K)^n \leqslant (\prod_{j \neq i} K_j)^n$. Let $l \in \{1, \ldots, k\}$. We may choose an $n \in N$ so as $\Delta_i = \Delta_l^n$. Therefore $(\prod_{j \neq i} K_j)^n = \prod_{j \neq l} K_j$, whence it follows that $O_p(N \cap K) \leqslant \bigcap_{i=1}^k (\prod_{j \neq i} K_j) = 1$, a contradiction. Hence $O_p(L_i) \neq 1$. So $L_i \in \mathcal{N}(K_i, B_i)$ for all $i = 1, \ldots, k$. Let $H_1 \in \mathcal{N}_{max}(K_1, B_1)$ be such that $H_1 \geqslant L_1$. Since H_1 is transitive on Δ_1, by induction H_1 leaves invariant a block system with blocks of size p. Then H_1 contains E_1, a normal elementary abelian p-subgroup of order $p^{|\Delta_1|/p} = p^{t/p}$. Hence $E_1 \trianglelefteq L_1$ and it follows that $E_1 \trianglelefteq N \cap K$. Put $E = \langle E_1^N \rangle$. By the Frattini argument, $N = N_N(T \cap K)(N \cap K)$. So $E = \langle E_1^{N_N(T \cap K)} \rangle \leqslant N \cap K$. Since N is transitive on $\mathcal{B}$, $N_N(T \cap K)$ is transitive on $\mathcal{B}$. Let $g \in N_N(T \cap K)$ be such that $R_1^g = R_j$ for some j. Since $E_1 \trianglelefteq R_1$, E_1^g is an elementary abelian normal p-subgroup of R_j of order $p^{t/p}$. Therefore, E is an elementary abelian normal p-subgroup of T of order $p^{kt/p} = p^{p^{m-1}}$. Thus, using Theorem 2.6.14, up to conjugacy we see that

$$E = \langle (1, 2, \ldots, p), (p+1, p+2, \ldots, 2p), \ldots, (p(p^{m-1}-1)+1, p(p^{m-1}-1)+2, \ldots, p^m) \rangle.$$

By Theorem 2.6.15, we have that $B \leqslant N_G(E)$. Thus, as $N_G(E) \geqslant N$ and $N \in \mathcal{N}_{max}(G, T)$, $N_G(E) = N$. Therefore N leaves invariant a block system with blocks of size p. This complete the proof of Lemma.

3.2 $\mathrm{Sym}(kp^m)$

Theorem 3.2.1 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = kp$, where p is a prime and $k \in \mathbb{N}$ such that $1 < k < p$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{max}(G, B)$, then N is isomorphic to $B = N_{\mathrm{Sym}(p)}(\bar{T}) \wr \mathrm{Sym}(k)$, where $\bar{T} \in \mathrm{Syl}_p(\mathrm{Sym}(p))$.*

Proof. Let $P = \mathrm{Sym}(p) \wr \mathrm{Sym}(k)$. By Theorem 2.7.4, $B \leqslant P$ and B is transitive on Ω and so, by Theorem 2.8.2, P is a maximal subgroup of G. Since N is a subgroup of G containing B and $O_p(N) \neq 1$, using Theorem 2.7.9, so we may assume that N is imprimitive. By the transitivity of B, it follows that every subgroup containing B can only have blocks of length $1, p$ and kp. So every nontrivial block of N must have length p. By Corollary 2.5.17, N is isomorphic to a subgroup of P. Since $O_p(B) \neq 1$, so B is a p-local subgroup of G with respect to B. Using Theorem 2.7.4, $B \cong N_{\mathrm{Sym}(p)}(\bar{T}) \wr \mathrm{Sym}(k)$, where $\bar{T} \in \mathrm{Syl}_p(\mathrm{Sym}(p))$. Moreover, by Lemma 2.4.18, B is a maximal subgroup of P. Thus $N \leqslant B$ and hence $N = B$. Then $N \cong N_{\mathrm{Sym}(p)}(\bar{T}) \wr \mathrm{Sym}(k)$ and this complete the proof.

Theorem 3.2.2 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = kp^m$, where p is a prime such that $p > 2$ and $m, k \in \mathbb{N}$ such that $k < p$ and $m > 1$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{max}(G, B)$, then N is isomorphic to $(\mathrm{Sym}(p) \wr \mathrm{Sym}(p^{m-1})) \wr \mathrm{Sym}(k)$.*

Proof. Since N is a subgroup of G containing B and, by Theorem 2.7.4, $B = N_{\mathrm{Sym}(p^m)}(\bar{T}) \wr \mathrm{Sym}(k)$ where $\bar{T} \in \mathrm{Syl}_p(\mathrm{Sym}(p^m))$, so we have, using Corollary 2.7.12, $N \leqslant \mathrm{Sym}(p^m) \wr \mathrm{Sym}(k)$. By Lemma 3.1.3 and Theorem 2.8.2, $\bar{N} = \mathrm{Sym}(p) \wr \mathrm{Sym}(p^{m-1}) \in \mathcal{N}_{max}(\mathrm{Sym}(p^m), N_{\mathrm{Sym}(p^m)}(\bar{T}))$ and $\bar{N}$ is a maximal subgroup of $\mathrm{Sym}(p^m)$. Thus, by Lemma 2.4.18, $\bar{N} \wr \mathrm{Sym}(k)$ is a maximal subgroup of $\mathrm{Sym}(p^m) \wr \mathrm{Sym}(k)$. It follows that $N \leqslant \bar{N} \wr \mathrm{Sym}(k)$. As $O_p(\bar{N}) \neq 1$, $O_p(\bar{N} \wr \mathrm{Sym}(k)) \neq 1$ and hence $\bar{N} \wr \mathrm{Sym}(k) \in \mathcal{N}(G, B)$. Therefore, $N = \bar{N} \wr \mathrm{Sym}(k)$.

3.3 $\mathrm{Sym}(k_1p^m + k_0)$

Lemma 3.3.1 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p^m + 1$, where p is a prime and $m \in \mathbb{N}$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then every proper subgroup of G containing B is contained in $\mathrm{Stab}_G(\omega) \cong \mathrm{Sym}(n-1)$, fixes $\omega \in \Omega$.*

Proof. We know that T and B fix a unique point $\omega \in \Omega$ and operates transitively on $\Omega \setminus \{\omega\}$. Suppose that $L \nleqslant \mathrm{Stab}_G(\omega)$ and $G \geqslant L \geqslant B$. Then L is 2-transitive on Ω, and, as B contains a transpositions, Lemma 2.5.14 (i) implies that $L = G$. Thus all proper subgroups of G which contain B are contained in $\mathrm{Stab}_G(\omega) \cong \mathrm{Sym}(n-1)$.

Lemma 3.3.2 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p^m + 1$, where p is a prime and $m \in \mathbb{N}$. Suppose that $T \in \mathrm{Syl}_p(G)$, $B = N_G(T)$ and put $H = \mathrm{Stab}_G(\omega)$, fixed $\omega \in \Omega$. Then $\mathcal{N}_{max}(G, B) = \mathcal{N}_{max}(H, B)$.*

Proof. Let $N \in \mathcal{N}_{max}(G, B)$. Since B is transitive on $\Omega \setminus \{\omega\}$, Lemma 3.3.1 implies that N is contained in $H \cong \mathrm{Sym}(n-1)$. It follows that $\mathcal{N}_{max}(G, B) \subseteq \mathcal{N}_{max}(H, B)$. But $H \leqslant G$, so that $\mathcal{N}_{max}(H, B) \subseteq \mathcal{N}_{max}(G, B)$ and the lemma is complete.

Theorem 3.3.3 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p + 1$, where p is a prime. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then $\mathcal{N}_{max}(G, B) = \{B\}$.*

Proof. For $p = 2$, $T = B \cong \mathrm{Sym}(2)$ and $O_p(B) \neq 1$. Also, for $p = 3$, $T \cong \mathrm{Alt}(3)$, $B \cong \mathrm{Sym}(3)$ and $O_p(B) \neq 1$. So, in both cases, B is a maximal subgroup of G and, thus, B is the unique maximal p-local subgroup of G with respect to B.

Suppose that $p \neq 2, 3$ and let $H \cong \mathrm{Sym}(p)$ be the stabilizer in G of a point in Ω, say $H = G_\sigma$, for some $\sigma \in \Omega$. By order we may assume that $T \in \mathrm{Syl}_p(H)$. Then, by Theorem 2.7.1, $N_H(T) = B$ and so, since $O_p(N_H(T)) \neq 1$, $N_H(T) = B$ is a p-local subgroup of G. It remains to prove that B is the only maximal p-local subgroup of G with respect to B. So let L be a maximal p-local subgroup of G, that is, $B \leqslant L$ and $O_p(L) \neq 1$. Using Lemma 3.3.1, we get that $L \leqslant H$ and so, by Theorem 2.7.14, B is a maximal subgroup of H implies that $L = B$.

Lemma 3.3.4 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = k_1 p^m + k_0$, where p is a prime and $m, k_0, k_1 \in \mathbb{N}$ such that $k_1 < p$, is the p-adic decomposition of n. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $n \geq p + 2$ and $N \in \mathcal{N}_{max}(G, B)$, then $N \leqslant \mathrm{Sym}(k_1 p^m) \times \mathrm{Sym}(k_0)$.*

Proof. Let $U = \mathrm{Sym}(k_1 p^m) \times \mathrm{Sym}(k_0)$. By Theorem 2.7.1, U contains B and from Theorem 2.8.1 we know that U is a maximal subgroup Assume that $N \nleqslant U$. Then

N fuses the two orbits of U on Ω and so is transitive on Ω. Thus, by Corollary 2.7.15, N is primitive on Ω. Then Theorem 2.7.9 implies that $N = G$. Hence $N \leqslant U$.

Theorem 3.3.5 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = k_1p^m + k_0$, where p is a prime and $m, k_0, k_1 \in \mathbb{N}$ such that $k_0 < p$ and $k_1 < p$, is the p-adic decomposition of n. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $n \geq p+2$ and $N \in \mathcal{N}_{max}(G, B)$, then $N = \bar{N} \times \mathrm{Sym}(k_0)$ where $\bar{N}$ is a maximal p-local subgroup of $\mathrm{Sym}(k_1p^m)$ with respect to $B \cap \mathrm{Sym}(k_1p^m)$.*

Proof. By Lemma 3.3.4, $N \leqslant U \times V$ where $U = \mathrm{Sym}(k_1p^m)$ and $V = \mathrm{Sym}(k_0)$. Using Proposition 2.1.16, $T = (T \cap U) \times (T \cap V)$ with $T \cap U \in \mathrm{Syl}_p(U)$, $T \cap V \in \mathrm{Syl}_p(V)$ and $B = (B \cap U) \times (B \cap V)$ with $B \cap U = N_U(T \cap U)$, $B \cap V = N_V(T \cap V)$. As $T \cap V = 1$ and $1 \neq O_p(N) \leqslant T$, we have $1 \neq O_p(N) \cap (T \cap U) \leqslant O_p(N) \cap U$. Since $N \leqslant U \times V$, $O_p(N) \cap U \trianglelefteq N$ and so $1 \neq O_p(N) \cap U \leqslant O_p(NV)$. Therefore $NV \in \mathcal{N}(G, B)$ and hence, as $N \in \mathcal{N}_{max}(G, B)$, $N = NV$. So $V \leqslant N$ which implies, using Dedekinds' Modular Law, that $N = (N \cap U)V$. Now, as $N \cap U \in \mathcal{N}(U, B \cap U)$, we may choose $\bar{N} \in \mathcal{N}_{max}(U, B \cap U)$ with $N \cap U \leqslant \bar{N}$. Since $1 \neq O_p(\bar{N}) \leqslant O_p(\bar{N}V)$ and $B \leqslant \bar{N}V$, $\bar{N}V \in \mathcal{N}(G, B)$ and so, as $N = (N \cap U)V \leqslant \bar{N}V$, $N = \bar{N}V$.

Lemma 3.3.6 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p+k$, where p is a prime and $k \in \mathbb{N}$ such that $1 < k < p$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then $\mathcal{N}_{max}(G, B) = \{B = N_{\mathrm{Sym}(p)}(T) \times \mathrm{Sym}(k)\}$.*

Proof. Let $U = \mathrm{Sym}(p) \times \mathrm{Sym}(k)$. By Theorem 2.7.1, U contains B and from Theorem 2.8.1 we know that U is a maximal subgroup of G. Assume that $N \in \mathcal{N}_{max}(G, B)$ and $N \nleqslant U$. Then, since N is a subgroup of G containing B, N fuses the two orbits of U on Ω and so is transitive on Ω. Thus, by Lemma 2.7.15, N is primitive on Ω. Therefore Theorem 2.7.9 implies that $N = G$. Hence $N \leqslant U$. Since $O_p(B) \neq 1$, so B is a p-local subgroup of G with respect to B. But, as $k < p$, $T \in \mathrm{Syl}_p(\mathrm{Sym}(p))$ and $B = N_{\mathrm{Sym}(p)}(T) \times \mathrm{Sym}(k)$, where, by Theorem 2.7.14, $N_{\mathrm{Sym}(p)}(T)$ is a maximal subgroup of $\mathrm{Sym}(p)$. Therefore B is a maximal subgroup of U. It follows that, as $B \leqslant N$, $N = B$ and we have the result.

We now want to show that those examined in the previous sections are the only B, the normalizer of a Sylow p-subgroup of $\mathrm{Sym}(\Omega)$, which are maximal p-local subgroups with respect to B.

Theorem 3.3.7 *Let $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$, where p is a prime. Then B is a maximal p-local subgroup with respect to B if one of the following occurs:*

(i) $n = p$

(ii) $n = kp$ *with* $1 < k < p$

(iii) $n = p + k$ *with* $1 \leq k < p$.

Proof. Follows from Theorems 3.1.1, 3.2.1, 3.3.3 and Lemma 3.3.6.

3.4 An overview of the problem

Our next result concerns subgroups in $\mathcal{N}_{max}(G, B)$ which do not act transitively on Ω. Recall that if $n = k_t p^t + k_{t-1}p^{t-1} + \cdots + k_1 p + k_0$, with $0 \leq k_j < p$, for all $j = 0, 1, \ldots, t$, is the p-adic decomposition of n, where p is a prime and k_j is an integer, then T has $t+1$ orbits on Ω. Let $\Omega_0, \Omega_1, \ldots, \Omega_t$ denote these orbits where $|\Omega_i| = k_i p^i$ for $i \in \{0, 1, \ldots, t\}$. Note that $T = T_0 \times T_1 \times \cdots \times T_t$ where, $T_i \in \mathrm{Syl}_p(\mathrm{Sym}(\Omega_i))$, $i \in \{0, 1, \ldots, t\}$ and, moreover, each T_i is the direct product of k_i factors, each isomorphic to a Sylow p-subgroup of $\mathrm{Sym}(\Delta)$, with $|\Delta| = p^i$ (see Findlay [11]).

Theorem 3.4.1 *Let $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Let $n = k_t p^t + k_{t-1}p^{t-1} + \cdots + k_1 p + k_0$, where p is a prime, with $0 \leq k_j < p$, for all $j = 0, \ldots, t$, be the p-adic decomposition of n and $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_t$, with $|\Omega_j| = k_j p^j$, for all $j = 0, \ldots, t$, be the corresponding partition of Ω into B-orbits. Let J be a proper subset of $I = \{0, 1, \ldots, t\}$. Set $\Delta = \bigcup_{i \in J} \Omega_i$, $U = \mathrm{Sym}(\Delta)$ and $V = \mathrm{Sym}(\Omega \setminus \Delta)$. Suppose that $N \in \mathcal{N}_{max}(G, B)$ and $N \leqslant U \times V$.*

(i) If $O_p(N) \cap U \neq 1$, then $N = N_U \times V$ where $N_U \in \mathcal{N}_{max}(U, B \cap U)$.

(ii) If $O_p(N) \cap V \neq 1$, then $N = U \times N_V$ where $N_V \in \mathcal{N}_{max}(V, B \cap V)$.

Proof. First we examine the case when $O_p(N) \cap U \neq 1$. Since $N \leqslant U \times V$, $O_p(N) \cap U \trianglelefteq N$ and so $1 \neq O_p(N) \cap U \leqslant O_p(NV)$. Therefore $NV \in \mathcal{N}(G,B)$ and hence, as $N \in \mathcal{N}_{max}(G,B)$, $N = NV$. So $V \leqslant N$ which implies, using Dedekinds' Modular Law, that $N = (N \cap U)V$. Now, as $N \cap U \in \mathcal{N}(U, B \cap U)$, we may choose $N_U \in \mathcal{N}_{max}(U, B \cap U)$ with $N \cap U \leqslant N_U$. Since $1 \neq O_p(N_U) \leqslant O_p(N_U V)$ and $B \leqslant N_U V$, $N_U V \in \mathcal{N}(G,B)$ and so, as $N = (N \cap U)V \leqslant N_U V$, $N = N_U V$. If we have $O_p(N) \cap V \neq 1$, the same argument yields $N = U \times N_V$ for some $N_V \in \mathcal{N}_{max}(V, B \cap V)$.

Theorem 3.4.2 *Let the hypothesis of Theorem 3.4.1 holds. Suppose that $0 \leq k_j \leq 1$, for all $j = 0, \dots, t$. Then either $N = N_U \times V$, where $N_U \in \mathcal{N}_{max}(U, B \cap U)$ and N_U is transitive on Δ, or $N = U \times N_V$, where $N_V \in \mathcal{N}_{max}(V, B \cap V)$ and N_V is transitive on $\Omega \setminus \Delta$.*

Proof. Thanks to the study carried out in Theorem 3.4.1, we only need to eliminate the situation $O_p(N) \cap U = 1 = O_p(N) \cap V$. From

$$[O_p(N), T_U] \leqslant O_p(N) \cap T_U \leqslant O_p(N) \cap U = 1$$

and

$$[O_p(N), T_V] \leqslant O_p(N) \cap T_V \leqslant O_p(N) \cap V = 1$$

where $T_U \in \mathrm{Syl}_p(U), T_V \in \mathrm{Syl}_p(V)$, we deduce that $O_p(N) \leqslant Z(T)$. Therefore, $C_G(Z(T)) \leqslant C_G(O_p(N)) \leqslant N_G(O_p(N)) = N$.

Let $1 \neq \sigma \in O_p(N)$, so $\sigma \in Z(T)$. For any $g \in N$, $\sigma^g \in O_p(N) \trianglelefteq N$ and hence $\sigma^g \in Z(T)$. Since $T = \prod_{i \in I} T_i$ where, for $i \in I$, $T_i \in \mathrm{Syl}_p(\mathrm{Sym}(\Omega_i))$, $Z(T) = \prod_{i \in I} Z(T_i)$. By Lemma 2.6.7, $Z(T_i) = \langle \sigma_i \rangle$ where σ_i has order p and cycle type $p^{p^{i-1}}$. Now let $1 \neq \mu \in Z(T)$ with $\mu \neq \sigma$. So $\sigma = \prod_{k \in K} \sigma_k$ and $\mu = \prod_{k \in K'} \sigma_k$, where $K, K' \subseteq I$ with $K \neq K'$ and consequently, as $t > t-1 > \cdots > 1$, σ and μ have different cycle types. Therefore $\sigma^g = \sigma$ and then $N \leqslant C_G(\sigma)$. Since $\langle \sigma \rangle \subseteq Z(C_G(\sigma)) \leqslant O_p(C_G(\sigma))$, $C_G(\sigma) \in \mathcal{N}(G,B)$. This implies that $N = C_G(\sigma)$ for all $1 \neq \sigma \in O_p(N)$, as $N \in \mathcal{N}_{max}(G,B)$. We see that

$$C_G(\sigma) = \prod_{k \in K} C_{\mathrm{Sym}(\Omega_k)}(\sigma_k) \times \mathrm{Sym}(\bigcup_{i \in I \setminus K} \Omega_i)$$

and so $\langle \sigma_k \mid k \in K \rangle \leqslant Z(C_G(\sigma))$. In particular, $\langle \sigma_k \mid k \in K \rangle \leqslant O_p(C_G(\sigma)) = O_p(N)$. Now either $\langle \sigma_k \mid k \in K \rangle \cap T_U \neq 1$ or $\langle \sigma_k \mid k \in K \rangle \cap T_V \neq 1$ because $O_p(N) \trianglelefteq T = T_U \times T_V$, a contradiction.

Aiming for a contradiction we assume N_U is not transitive on Δ. Thus $N_U \leqslant X \times Y \leqslant U$ where $\Theta = \bigcup_{i\in K} \Omega_i$, $X = \mathrm{Sym}(\Theta)$ and $Y = \mathrm{Sym}(\Delta \setminus \Theta)$ for some $K \subset J$. Applying the previous part to $N_U \in \mathcal{N}_{max}(U, B \cap U)$ we deduce that either $N_U = N_X \times Y$ where $N_X \in \mathcal{N}_{max}(X, B \cap X)$ or $N_U = X \times N_Y$ where $N_Y \in \mathcal{N}_{max}(Y, B \cap Y)$. Without loss of generally we assume the former to hold. Since $O_p(N_X) \neq 1$ and $T \leqslant N_X \times \mathrm{Sym}(\Omega \setminus \Theta)$, clearly $N_G(O_p(N_X)) \in \mathcal{N}(G, B)$. However we have that

$$\begin{aligned} N = N_U \times V &= N_X \times Y \times V \\ &< N_X \times \mathrm{Sym}(\Omega \setminus \Theta) \leqslant N_G(O_p(N_X)), \end{aligned}$$

a contradiction. Therefore we conclude that N_U is transitive on Δ and hence the proof of the theorem is complete.

3.5 $\mathcal{N}_{max}(G^*, T^*)$ for $G^* \cong \mathrm{Alt}(\Omega)$

We now use G^* to denote $\mathrm{Alt}(\Omega)$, the alternating group on Ω, and also $\mathrm{Alt}(m)$ to denote the alternating group of degree m. Put $T^* = G^* \cap T$ and $B^* = N_{G^*}(T^*)$. Recall that $T^* \in \mathrm{Syl}_p(G^*)$. Here we look at the relationship between $\mathcal{N}_{max}(G, B)$ and $\mathcal{N}_{max}(G^*, B^*)$. In order to do this we study some specific cases.

Lemma 3.5.1 *Let $G^* = \mathrm{Alt}(\Omega)$ with $|\Omega| = p$, where p is a prime and $p \neq 2$. Suppose that $T^* \in \mathrm{Syl}_p(G^*)$, and $B^* = N_{G^*}(T^*)$. Then $B^* \in \mathcal{N}_{max}(G^*, B^*)$.*

Proof. If $p = 2$, then, as $B^* = 1$, $\mathcal{N}_{max}(G^*, B^*) = \emptyset$. Now assume that $p \neq 2$. Thus, $T^* = \langle (1, 2, 3, \dots, p) \rangle$ with $|T| = p$. Since T^* is a normal p-subgroup of B^*, so $O_p(B^*) \neq 1$. Therefore, using Theorem 2.7.14, $B^* \in \mathcal{N}_{max}(G^*, B^*)$.

Lemma 3.5.2 *Let $G^* = \mathrm{Alt}(\Omega)$ with $|\Omega| = p + 1$, where p is a prime and $p \neq 2$. Suppose that $T^* \in \mathrm{Syl}_p(G^*)$, and $B^* = N_{G^*}(T^*)$. Then $B^* \in \mathcal{N}(G^*, B^*)$.*

Proof. If $p = 2$, then $T^* = 1$ and $B^* = G^*$. As $O_p(B^*) = 1$, $\mathcal{N}_{max}(G^*, B^*) = \emptyset$. Now assume that $p \neq 2$ and let $H \cong \mathrm{Sym}(p)$. Thus, $T^* = \langle (1, 2, 3, \dots, p) \rangle \in \mathrm{Syl}_p(H)$ with $|T| = p$ and $N_H(T^*) = B^*$. Since $O_p(B^*) \neq 1$, hence $B^* \in \mathcal{N}(G^*, B^*)$.

Lemma 3.5.3 *Let* $G = \mathrm{Sym}(\Omega)$ *and* $G^* = \mathrm{Alt}(\Omega)$ *with* $|\Omega| = p^m$*, where* p *is a prime such that* $p > 2$ *and* $m \in \mathbb{N}$ *such that* $m > 1$*. Suppose that* $T \in \mathrm{Syl}_p(G)$*,* $T^* = G^* \cap T$*,* $B = N_G(T)$ *and* $B^* = N_{G^*}(T^*)$*. If* $N \in \mathcal{N}_{max}(G, B)$*, then* $N \cap G^* \in \mathcal{N}(G^*, B^*)$*.*

Proof. The assumption on N means that $|O_p(N)| \geq p^2$. Hence $1 \neq O_p(N) \cap G^* \lhd N \cap G^*$. Using Proposition 2.1.16, $B^* = B \cap G^* \leqslant N \cap G^*$ and so $N \cap G^* \in \mathcal{N}(G^*, B^*)$.

Appendix A

Examples of the maximal p-local subgroups of G

In Appendix we shows some examples related to the main results achieved. We maintain the notation introduced in Chapter 1.

The definition of maximal p-local subgroup in terms of the normalizer of a Sylow p-subgroup is given in the works of Rowley and Saninta [22], in which they study all the maximal p-local subgroups for the symmetric groups, with respect to the prime $p = 2$. This case is relatively easy to study and an example illustrating their result is presented next.

Example 1: Sym(12)
Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 12$. Suppose that $p = 2$ and let T be a Sylow p-subgroup of G. Recall that $N_G(T) = T$. Consider the p-adic decomposition of n, $n = 2^3 + 2^2$ with $\Omega = \Omega_1 \cup \Omega_2$ where $|\Omega_1| = 8$ and $|\Omega_2| = 4$. Also $T = T_1 \times T_2 \cong (C_2 \wr C_2 \wr C_2) \times (C_2 \wr C_2)$, with T_i Sylow 2-subgroup of $\mathrm{Sym}(\Omega_i)$, for $i = 1, 2$, and C_2 cyclic group of order 2. Also the Ω_i's are the orbits of T on Ω. We begin by listing the subgroups in $\mathcal{N}_{max}(G, T)$, using Theorem 3.4 of [22].

$$\begin{aligned} N_1 &\cong \mathrm{Sym}(8) \times \mathrm{Sym}(4) \\ N_2 &\cong \mathrm{Sym}(4) \wr \mathrm{Sym}(3) \\ N_3 &\cong \mathrm{Sym}(2) \wr \mathrm{Sym}(6). \end{aligned}$$

Therefore,

$$N_0 = N_1 \cap N_2 \cap N_3 \cong ((\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \wr \mathrm{Sym}(2)) \times (\mathrm{Sym}(2) \wr \mathrm{Sym}(2))$$

$$
\begin{aligned}
N_{12} = N_1 \cap N_2 &\cong \mathrm{Sym}(4) \times (\mathrm{Sym}(4) \wr \mathrm{Sym}(2)) \\
N_{13} = N_1 \cap N_3 &\cong (\mathrm{Sym}(2) \wr \mathrm{Sym}(4)) \times (\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \\
N_{23} = N_2 \cap N_3 &\cong (\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \wr \mathrm{Sym}(3).
\end{aligned}
$$

Furthermore, $\langle N_1, N_2 \rangle = \langle N_1, N_3 \rangle = \langle N_2, N_3 \rangle = G$.

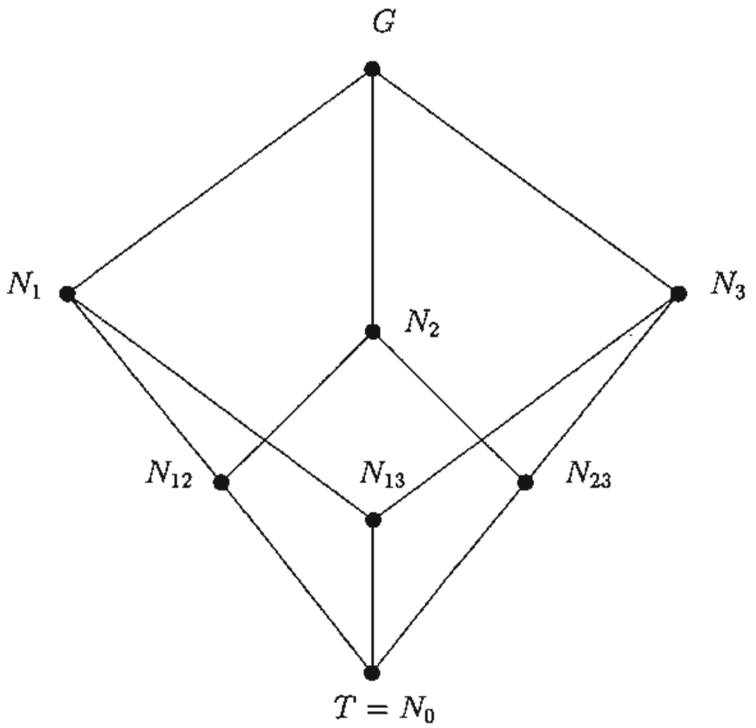

Figure A.1: The lattice of the maximal 2-local subgroups of Sym(12).

Example 2: Sym(3)
Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 3$. Suppose that $T_i \in \mathrm{Syl}_i(G)$ and $B_i = N_G(T_i)$ for $i = 2, 3$. Then $T_2 = B_2 \cong \mathrm{Sym}(2)$, $T_3 = \langle (1,2,3) \rangle \cong \mathrm{Alt}(3)$ and $B_3 = \langle (1,2,3),(2,3) \rangle \cong \mathrm{Sym}(3)$ Therefore, as $T_i \trianglelefteq B_i$, $O_i(B_i) \neq 1$ for $i = 2, 3$. Then for $i = 2, 3$, by Theorems 3.1.1 and 3.3.1, G has a unique maximal i-local subgroup with respect to B_i, which is B_i.

Example 3: Sym(6)
Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 6$. Suppose that T be a Sylow 2-subgroup of G. Recall that $B = N_G(T) = T$. Consider the 2-adic decomposition of n, $n = 2^3 + 2^2$.

By Theorem 3.4 of [22], the subgroups in $\mathcal{N}_{max}(G, B)$ are

$$\begin{aligned} N_1 &= N(\{1\};2) \cong \mathrm{Sym}(4) \times \mathrm{Sym}(2) \\ N_2 &= N(\emptyset;2) \cong \mathrm{Sym}(2) \wr \mathrm{Sym}(3). \end{aligned}$$

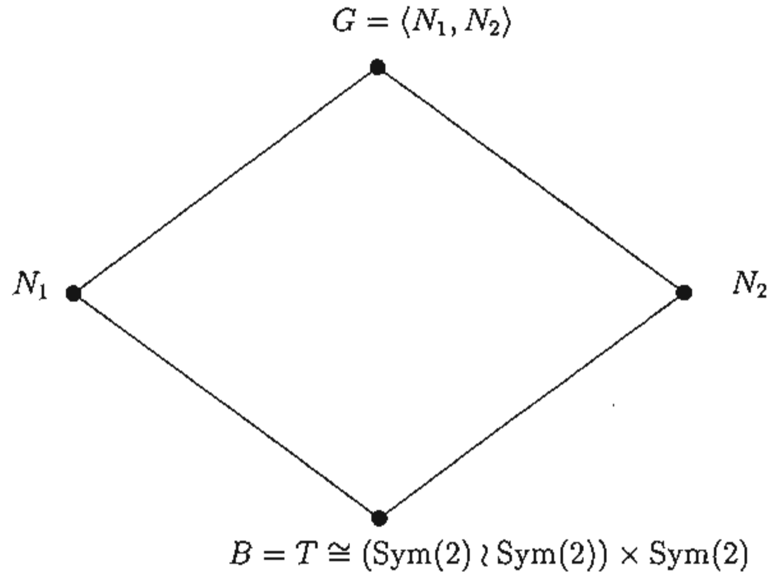

Figure A.2: The lattice of the maximal 2-local subgroups of Sym(6).

Suppose now that T_1 be a Sylow 3-subgroup of G and $B_1 = N_G(T_1)$. Thus, $T_1 = \langle (1,2;3),(4,5,6) \rangle$ with $|T_1| = 9$ and $B_1 = \langle (4,5,6),(1,2,3),(4,5),(2,3)(4,6),(1,4,3,6)(2,5) \rangle$ with $|B_1| = 72$. Consider the 3-adic decomposition of n, $n = 2(3)$. By Theorem 3.2.1, the subgroups in $\mathcal{N}_{max}(G, B_1)$ is $B_1 \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$.

We now consider the 5-adic decomposition of n, $n = 5 + 1$. A Sylow 5-subgroup T_2 of G can be generated by the element $(1,2,3,4,5)$. Then, by Theorem 3.3.1, G has a unique maximal 5-local subgroup with respect to $B = N_G(T_2)$, which is $B = \langle (1,2,3,4,5),(2,5)(3,4),(2,4,5,3) \rangle$.

Example 4: Sym(9)

Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 9$. Suppose that T be a Sylow 2-subgroup of G. Recall that $B = N_G(T) = T$. Consider the 2-adic decomposition of n, $n = 2^3 + 1$. By Theorem 3.4 of [22], the subgroups in $\mathcal{N}_{max}(G, T)$ are

$$N_1 = N(\{2\};4) \cong \mathrm{Sym}(4) \wr \mathrm{Sym}(2)$$

$$N_2 \;=\; N(\{2\};2) \cong \mathrm{Sym}(2) \wr \mathrm{Sym}(4).$$

Therefore, the simplicial set of $\mathcal{N}_{max}(G,T)$ is $\mathcal{N}_{max}(G,T)$.

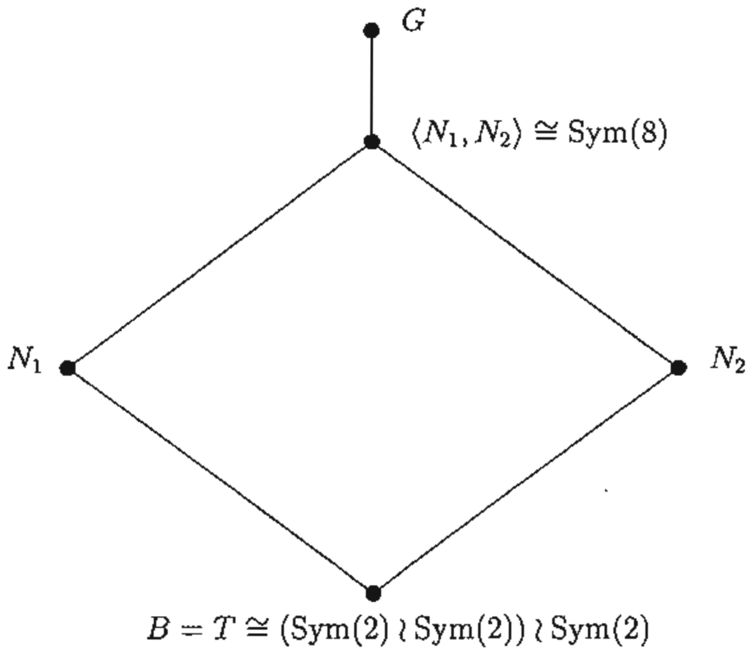

Figure A.3: The lattice of the maximal 2-local subgroups of Sym(9).

Suppose now that $\bar{T}$ be a Sylow 3-subgroup of G and $\bar{B} = N_G(\bar{T})$. Consider the 3-adic decomposition of n, $n = 3^2$. Then $\bar{T} = \langle (1,2,3),(4,5,6),(7,8,9),$ $(1,4,7)(2,5,8)(3,6,9)\rangle$ with $|\bar{T}| = 81$ and $\bar{B} = N_G(\bar{T}) = \langle (1,5,9)(2,6,7)(3,4,8),$ $(7,8,9),(4,5,6),(1,2,3),(4,9)(5,7)(6,8),(2,3)(4,8,5,7,6,9)\rangle$ with $|\bar{B}| = 324$. Then, by Theorem 3.1.3, G has a unique maximal 3-local subgroup with respect to $\bar{B}$, which is $N = \mathrm{Sym}(3) \wr \mathrm{Sym}(3) \cong N_G(E)$, where $E = \langle (1,2,3),(4,5,6),(7,8,9)\rangle$. That is, $N = \langle (1,2,3),(1,2),(4,5,6),(4,5),(7,8,9),(7,8),(1,4,7)(2,5,8)(3,6,8),$ $(1,4)(2,5)(3,6)\rangle$ with $|N| = 1296$.

Example 5: $\mathrm{Sym}(3^3)$

Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 27$, $T \in \mathrm{Syl}_3(G)$, and let

$$\begin{aligned} x_1 &\cong (1,2,3),\\ x_2 &\cong (1,4,7)(2,5,8)(3,6,9),\\ x_3 &\cong (1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)\\ &\quad (8,17,26)(9,18,27). \end{aligned}$$

be its generators. The normalizer B of T in G can be described as $B = T \rtimes \langle h_1, h_2, h_3 \rangle$, with

$$h_1 \cong (2,3)(5,6)(8,9)(11,12)(14,15)(17,18)(20,21)(23,24)(26,27),$$
$$h_2 \cong (4,7)(5,8)(6,9)(13,16)(14,17)(15,18)(22,25)(23,26)(24,27),$$
$$h_3 \cong (10,19)(11,20)(12,21)(13,22)(14,23)(15,24)(16,25)(17,26)(18,27).$$

By Theorem 3.1.3, the subgroups in $\mathcal{N}_{max}(G,B)$ is $N = \mathrm{Sym}(3) \wr \mathrm{Sym}(9)$.

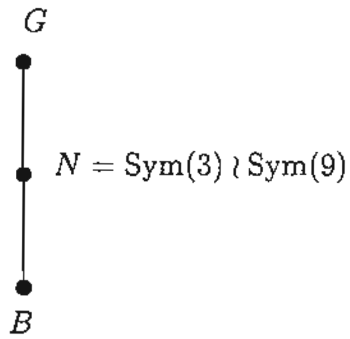

Figure A.4: The lattice of the maximal 3-local subgroups of Sym(27).

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Output จากโครงการวิจัยที่ได้รับทุนจาก สกอ. และ สกว.

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1. T. Phatthanangkul and S. Dhompongsa, Maximal p-local subgroups of the symmetric groups for some critical cases, *Algebra Colloquium*. (submitted for publication)
2. T. Phatthanangkul and S. Dhompongsa, The intransitive maximal p-local subgroups of the symmetric groups, *The International Journal of Mathematics and Mathematical Sciences*. (submitted for publication)

2. การนำผลงานวิจัยไปใช้ประโยชน์

- **เชิงสาธารณะ**

การทำโครงการวิจัยนี้นอกจากจะได้ติดต่อและขอคำปรึกษาจากนักวิจัยที่ปรึกษาแล้ว ก็ยังได้ขอคำแนะนำจาก Professor P.J. Rowley, University of Manchester Institute of Science and Technology ประเทศสหราชอาณาจักร อีกด้วย ซึ่งถือว่าเป็นการเชื่อมโยงทางวิชาการกับนักวิชาการทั้งในและต่างประเทศ และเป็นการสร้างเครือข่ายทางการวิจัยทางด้านคณิตศาสตร์บริสุทธิ์อีกด้วย

- **เชิงวิชาการ**

โครงการวิจัยนี้ทำให้เกิดผลงานวิจัยในสาขาคณิตศาสตร์บริสุทธิ์เพิ่มขึ้น ทำให้เกิดแนวความคิดใหม่ๆ เพื่อนำไปสู่การพัฒนาทางคณิตศาสตร์บริสุทธิ์ต่อไป นอกจากนี้ยังสามารถนำไปใช้เป็นสื่อในการเรียนการสอนในระดับบัณฑิตศึกษา สาขาคณิตศาสตร์บริสุทธิ์ อีกทั้งแนวคิดของการทำโครงการวิจัยนี้ยังเป็นแนวทางในการทำวิจัยต่อเนื่องสำหรับผู้ที่สนใจ และเป็นการพัฒนาไปสู่งานวิจัยทางด้านคณิตศาสตร์บริสุทธิ์ขั้นสูงต่อไป

ภาคผนวก

Manuscript และบทความโครงการวิจัยสำหรับการเผยแพร่

เรื่อง

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Manuscript / บทความโครงการวิจัยสำหรับการเผยแพร่

เรื่อง

Maximal p-local subgroups of the symmetric groups for some critical cases

Maximal p-local subgroups of the symmetric groups for some critical cases

Tipaval Phatthanangkul*and Sompong Dhompongsa

June 27, 2006

Abstract: The subgroups in the set $\mathcal{N}_{max}(G, B)$ consisting of all maximal p-local subgroups of $G = \text{Sym}(n)$ with respect to B, the normalizer of a Sylow p-subgroup of G in G, is investigated for some critical cases.

Keywords: Symmetric group, Sylow p-subgroup, Normalizer, Maximal p-local subgroup.
2000 Mathematics Subject Classification: 20B30, 20B35, 20D20, 20E28

1 Introduction

Maximal 2-local geometries for certain sporadic simple groups were firstly introduced by Ronan and Smith (1980). These geometries were inspired by the theory of buildings for the groups of Lie type which was developed by Tits (1956, 1974) in the fifties. For each finite simple group of Lie type, there is a natural geometry associated with it called its building. For G a group of Lie type of characteristic p, its building is a geometric structure whose vertex stabilizers are the maximal parabolic subgroups which are also p-local subgroups of G containing a Sylow p-subgroup. As is well-known, each building has a Coxeter diagram associated with it. Buekenhout (1979) generalized these concepts to obtain diagrams for many geometries related to sporadic simple groups. Ronan and Smith (1980)pursued these ideas further and introduced the maximal 2-local geometries. Other in variants on buildings for the

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sporadic simple groups have been defined, notably the minimal parabolic geometries as described by Ronan and Stroth (1984).

We now define what we mean, generally, by a minimal parabolic subgroup. Suppose that H is a finite group and p is a prime dividing the order of H. Let S be a Sylow p-subgroup of H and B the normalizer of S in H. A subgroup P of H properly containing B is said to be a *minimal parabolic subgroup* of H with respect to B if B lies in exactly one maximal subgroup of P.

The definition of minimal parabolic subgroups in terms of the normalizer of a Sylow p-subgroup is given in the works of Ronan and Smith (1980) and Ronan and Stroth (1984), in which they study minimal parabolic geometries for the 26 sporadic finite simple groups. The connection between minimal parabolic subgroups and group geometries is the best illustrated in the case of groups of Lie type in their defining characteristic. For a group of Lie type, its minimal parabolic system is always geometric. This is not always the case in general (see Ronan and Stroth, 1984). Many studies on the minimal parabolic system of special subgroups have been done over the years. For example, Lempken, Parker and Rowley (1998) determined all the minimal parabolic subgroups and system for the symmetric and alternating groups, with respect to the prime $p = 2$. Later, Covello (2000) has studied minimal parabolic subgroups and systems for the symmetric group with respect to an odd prime p dividing the order of the group. The main results are about the symmetric groups of degree p^r, she also establishes some more general results. More recently, Rowley and Saninta (2004) investigated the maximal 2-local geometries for the symmetric groups. Furthermore, Saninta (2004) considered the relationship between the maximal 2-local subgroups and the minimal parabolic subgroups for the symmetric groups. In this paper we shall investigate maximal p-local subgroups for the symmetric groups.

Let H, p, S and B be defined as above. Define

$$\mathcal{N}(H,B) = \{K \mid B \leqslant K \leqslant H \text{ and } O_p(K) \neq 1\}$$

where $O_p(K)$ is a unique maximal normal p-subgroup of K. A subgroup in $\mathcal{N}(H,B)$ is said to be a **p-local subgroup** of H with respect to B and a subgroup in $\mathcal{N}(H,B)$ which is maximal under inclusion is said to be a **maximal p-local subgroup** of

H with respect to B. We denoted the collection of maximal p-local subgroups of H with respect to B by $\mathcal{N}_{max}(H, B)$.

Throughout all groups considered, and in particular all our sets, will be finite. Let Ω be a set of cardinality $n > 1$. Set $G = \mathrm{Sym}(\Omega)$, the symmetric group on the finite set Ω. We also use $\mathrm{Sym}(m)$ to denote the symmetric group of degree m. Now let T be a fixed Sylow p-subgroup of G, where p is a prime, and B be the normalizer of T in G.

The main purpose of this paper is to study the structure of subgroups in $\mathcal{N}_{max}(G, B)$ for some critical cases.

However, the general case looks, already from the first approach, more complicated. In fact, for $p \neq 2$, a Sylow p-subgroup of the symmetric group is not selfnormalized and so much more work needs to be done in understanding the structure of the normalizer. Moreover, since $p-1 \neq 1$, the prime divisors of $p-1$ play a certain role in the investigation of the overgroups of the normalizer. For instance, in the case of $\mathrm{Sym}(p^2)$, there is an isomorphism between the lattice of subgroups of a cyclic group of order $p-1$ and the lattice of certain overgroups of the normalizer and a similar correspondence holds also for the case $\mathrm{Sym}(p^m)$, with $m > 2$.

2 Preliminary Results

This section gathers together results that will be used. Now we let Ω be a finite set with $|\Omega| > 1$ and let G, T, B and n be defined as in Section 1.

Proposition 2.1 *Let H be a group and suppose that $H = A \times B$. Let $S \in \mathrm{Syl}_p(H)$. Then $S = (S \cap A) \times (S \cap B)$ and*

$$N_H(S) = (N_H(S) \cap A) \times (N_H(S) \cap B),$$

with $N_H(S) \cap A = N_A(S \cap A)$ and $N_H(S) \cap B = N_B(S \cap B)$.

Proof. See Covello [6] (Proposition 1.1.10).

Lemma 2.2 *Suppose that $H = X \times Y$ is a direct product of groups X and Y and suppose that $S \in \mathrm{Syl}_p(H)$ where p is a prime which divides the order of both X*

and Y. Assume that L is a subgroup of H which contains $B := N_H(S)$. Then $L = (L \cap X) \times (L \cap Y)$, with $L \cap X = (B \cap X)^L$ and $L \cap Y = (B \cap Y)^L$.

Proof. See Lempken, Parker and Rowley [10] (Lemma 2.5).

Lemma 2.3 *Suppose that R is a transitive permutation group of degree n. Let $H = L \wr R$ and $P = K \wr R$, with L maximal subgroup of K, and let p be a prime dividing $|K|$. If L contains the normalizer of a Sylow p-subgroup of K, then H is a maximal subgroup of P.*

Proof. See Covello [6] (Lemma 2.6.8).

Lemma 2.4 (Jordan, Marggraf) *Suppose that Σ is a finite set and L is a primitive subgroup of $\mathrm{Sym}(\Sigma)$.*

- *(i) If L contains a transposition, then $L = \mathrm{Sym}(\Sigma)$.*
- *(ii) Suppose L contains a fours group which is transitive on 4 points and fixes all the other points of Σ. If $|\Sigma| > 9$, then $L \geqslant \mathrm{Alt}(\Sigma)$.*

Proof. See Wielandt [19] (Theorems 13.3 and 13.5).

Proposition 2.5 *Let Ω be a set and $H = \mathrm{Sym}(\Omega)$. Let $\mathcal{B} = \{\Omega_1, \ldots, \Omega_m\}$ be a partition of Ω into m subsets of the same cardinality. Then the stabilizer L of $\mathcal{B}$ in H is isomorphic to*

$$\mathrm{Sym}(\Omega_1) \wr \mathrm{Sym}(\mathcal{B}).$$

In particular, L is imprimitive and $\mathcal{B}$ is a complete block system of L.

Proof. See Covello [6] (Theorem 3.5.1).

Corollary 2.6 *Let Ω be a set and $H = \mathrm{Sym}(\Omega)$. Let $K \leqslant H$ be imprimitive and Γ be a block of K. Then the stabilizer in H of the complete block system $\mathcal{B}_\Gamma = \{\Gamma^k \mid k \in K\}$ is isomorphic to $\mathrm{Sym}(\Gamma) \wr \mathrm{Sym}(\mathcal{B}_\Gamma)$. In particular, K is isomorphic to a subgroup of $\mathrm{Sym}(\Gamma) \wr \mathrm{Sym}(\mathcal{B}_\Gamma)$.*

Proof. See Covello [6] (Corollary 3.5.2).

Lemma 2.7 *Suppose p is a prime, n is a positive integer and $T_{p^n} \in \mathrm{Syl}_p(\mathrm{Sym}(p^n))$. Then $|Z(T_{p^n})| = p$.*

Proof. See Saninta [15] (Lemma 2.3.5).

Theorem 2.8 *Let S be a Sylow p-subgroup of $\mathrm{Sym}(p^n)$, where p is a prime and $n \in \mathbb{N}$. If $p > 2$, S has a unique abelian normal subgroup of order $p^{p^{n-1}}$, which is $C_p \wr T_{n-1}$, where T_{n-1} trivial permutation group on p^{n-1} letters, and this is an elementary abelian p-group.*

Proof. See Covello [6] (Theorem 4.4.6).

Theorem 2.9 *Let S be a Sylow p-subgroup of $H = \mathrm{Sym}(p^n)$, where p is a prime and $n \in \mathbb{N}$. Then the normalizer in H of S is contained in the normalizer in H of every abelian normal subgroup of S of order $p^{p^{n-1}}$.*

Proof. See Covello [5] (Theorem 4.4.11).

Theorem 2.10 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, and let $S \in \mathrm{Syl}_p(H)$. Let*

$$n = k_t p^t + k_{t-1} p^{t-1} + \cdots + k_1 p + k_0,$$

with $0 \leq k_j < p$, for all $j = 0, \ldots, t$, be the p-adic decomposition of n. Then the normalizer B of S in H is given by

$$B = B_0 \times \cdots \times B_t,$$

where, for $j = 0, \ldots, t$, B_j is the normalizer of a Sylow p-subgroup of $\mathrm{Sym}(\Omega_j)$, with $\Omega_j \subseteq \Omega$ and $|\Omega_j| = k_j p^j$. In particular,

$$|B| = |S| \prod_{j=0}^{t} k_j!(p-1)^{k_j j}$$

and the sets $\Omega_0, \Omega_1, \ldots, \Omega_t$ are the orbits of B on Ω.

Proof. See Covello [6] (Theorem 5.4.1).

Theorem 2.11 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = p^n$. Let $S \in \mathrm{Syl}_p(H)$ and set $B = N_H(S)$. Then B is transitive on Ω and every block of B has length a power of p. Furthermore, for $i = 1, \dots, n-1$, B has a unique complete block system of blocks of length p^i and, in particular, B has blocks of length p^r, for all $r = 1, \dots, n$.*

Proof. See Covello [6] (Theorem 5.2.9).

Theorem 2.12 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = kp^n$ and $1 \leq k < p$. Let $S \in \mathrm{Syl}_p(H)$ and set $B = N_H(S)$. Then B is isomorphic to the wreath product of $\bar{B}$ by $\mathrm{Sym}(k)$, where $\bar{B}$ is the normalizer in $\mathrm{Sym}(p^n)$ of a Sylow p-subgroup of $\mathrm{Sym}(p^n)$. In particular,*

$$|B| = |S|k!(p-1)^{nk}$$

and B is transtive on Ω.

Proof. See Covello [6] (Theorem 5.3.1).

Theorem 2.13 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, and $S \in \mathrm{Syl}_p(H)$. Suppose that M is a primitive subgroup of G containing the normalizer in H of S. If $n \geq p+2$, then $M = G$.*

Proof. See Covello [6] (Theorem 5.5.2).

Corollary 2.14 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, $S \in \mathrm{Syl}_p(H)$ and $B = N_H(S)$. Let $n = k_t p^t + k_{t-1} p^{t-1} + \dots + k_1 p + k_0$ be the p-adic decomposition of n. Suppose that M is an imprimitive subgroup of H containing B. Then there exists $1 \leq r \leq t$ such that $p^r | n$ and M is isomorphic to a subgroup of $\mathrm{Sym}(p^r) \wr \mathrm{Sym}(n/p^r)$. In particular, $k_0 = k_1 = \dots = k_{r-1} = 0$.*

Proof. See Covello [6] (Corollary 5.5.5).

Theorem 2.15 *Let p be a prime, $p \neq 2, 3$, and $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = p$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then B is a maximal subgroup of G.*

Proof. See Covello [6] (Theorem 6.1.2).

Lemma 2.16 *Let $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Suppose that $n = k_1p^m + k_0$, with $a \geq 1$ and $1 \leq k_0, k_1 < p$, is the p-adic decomposition of n. Then every transitive subgroup of G containing B is 2-transitive on Ω, such subgroups are primitive on Ω.*

Proof. See Covello [6] (Lemma 6.5.1).

3 Main Results

We maintain the notation introduced in Section 1. The aim of this section is to reduce the investigation of maximal p-local subgroups to some critical cases. We start examining some specific cases. When we come to consider the symmetric groups $\mathrm{Sym}(p)$ and $\mathrm{Sym}(p+1)$ some fact about the normalizer of a Sylow p-subgroup, for which the reader can refer to [6], are used.

3.1 $\mathrm{Sym}(p^m)$

Recall that the normalizer of a Sylow p-subgroup of $\mathrm{Sym}(p)$ is a maximal subgroup of $\mathrm{Sym}(p)$.

Theorem 3.1.1 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p$, where p is a prime. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then $\mathcal{N}_{max}(G, B) = \{B\}$.*

Proof. If $p = 2, 3$, then $B = G$, $O_p(G) \neq 1$ and there is nothing to prove. So assume that $p \neq 2, 3$. We know that $T \cong C_p$, where C_p is a cyclic group of order p. Since T is a normal p-subgroup of B, we have that $O_p(B) \neq 1$ and Theorem 2.15 implies that B is a maximal p-local subgroup of G. Let N be a maximal p-local subgroup of G with respect to B such that $N \neq B$. Then $B < N \leqslant G$ and $O_p(N) \neq 1$. Using Theorem 2.15, $N = G$, which contradicts the fact that $O_p(G) = 1$. Thus B is a unique maximal p-local subgroup of G with respect to B, which completes the proof.

We now look at those subgroups in $\mathcal{N}_{max}(G,T)$ which act transitively on Ω. Recall that if $G = \mathrm{Sym}(2^2)$, then $\mathcal{N}_{max}(G,B) = \{\mathrm{Sym}(4)\}$ because $\mathrm{Sym}(4) = N_G(A)$ where $A = \langle(1,2)(3,4),(1,3)(2,4)\rangle$. Our next result concerns subgroup in $\mathcal{N}_{max}(G,B)$, where $G = \mathrm{Sym}(p^2)$ with $p > 2$.

Lemma 3.1.2 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p^2$, where p is a prime such that $p > 2$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{max}(G,B)$, then N is isomorphic to $\mathrm{Sym}(p) \wr \mathrm{Sym}(p)$.*

Proof. Let $L = \mathrm{Sym}(p) \wr \mathrm{Sym}(p)$. By Theorem 2.11, using Corollary 2.6, we know that $B \leqslant L$ and so L is a maximal subgroup of G. Since N is a subgroup of G containing B, by Theorem 2.13, we may assume that N is imprimitive. By the transitivity of B, it follows that every subgroup containing B can only have blocks of length $1, p$ and p^2. So every nontrivial block of N must have length p and , by Corollary 2.6, N is isomorphic to a subgroup of L. Since, by Proposition 2.5, L is isomorphic to the stabilizer of $\mathrm{Sym}(p)$ acting on

$$\{\{1,2,\ldots,p\},\{p+1,p+2,\ldots,2p\},\ldots,\{p(p-1)+1,p(p-1)+2,\ldots,p^2\}\}$$

in G. Therefore, $\mathrm{Sym}(p) \wr \mathrm{Sym}(p) \cong N_G(E)$, where

$$E = \langle(1,2,\ldots,p),(p+1,p+2,\ldots,2p),\ldots,(p(p-1)+1,p(p-1)+2,\ldots,p^2)\rangle.$$

Using Theorem 2.8, E is a unique elementary abelian normal p-subgroup of order p^p of T. As $E \trianglelefteq N_G(E)$, we have that $O_p(N_G(E)) \neq 1$. It follow that $\mathrm{Sym}(p) \wr \mathrm{Sym}(p) \cong N_G(E) \in \mathcal{N}(G,T)$ and hence $N \cong \mathrm{Sym}(p) \wr \mathrm{Sym}(p)$.

Theorem 3.1.3 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p^m$, where p is a prime such that $p > 2$ and $m \in \mathbb{N}$ such that $m > 1$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{max}(G,B)$, then N leaves invariant a block system with blocks of size p. In particular, N is isomorphic to $\mathrm{Sym}(p) \wr \mathrm{Sym}(p^{m-1})$.*

Proof. We have that N is transitive on Ω. We argue by induction on m starting with the case $m = 2$. For $m = 2$, the lemma clearly holds. Since N is a subgroup of G containing B, by Theorem 2.13, we may assume that N is imprimitive. Let $\mathcal{B} = \{\Delta_1, \Delta_2, \ldots, \Delta_k\}$ be a non-trivial block system invariant under N. Since N

is transitive on Ω, it follows that N acts transitively on $\mathcal{B}$. Set $t = |\Omega|/k$. Then $t = |\Delta_i|$ for $i = 1, \ldots, k$ and so t is a power of p. Set $M = \mathrm{Stab}_G(\mathcal{B})$. Then

$$T \leqslant B \leqslant N \leqslant M \cong \mathrm{Sym}(t) \wr \mathrm{Sym}(k).$$

For $i = 1, \ldots, k$, put $K_i = \mathrm{Sym}(\Delta_i)$ and $K = K_1 \times K_2 \times \cdots \times K_k$. Then for $i = 1, \ldots, k$, as $K_i \trianglelefteq K \trianglelefteq M$, $1 \neq R_i = T \cap K_i \in \mathrm{Syl}_p(K_i)$, $T \cap K = R_1 \times R_2 \times \cdots \times R_k \in \mathrm{Syl}_p(K)$ and $B_i = B \cap K_i = N_{K_i}(R_i)$. Since t is a power of p, R_i is transitive on Δ_i for all i. Suppose that $O_p(N) \cap K = 1$. Since $[O_p(N), N \cap K] \leqslant O_p(N) \cap K$, this gives $[O_p(N), N \cap K] = 1$. As $R_i \leqslant N \cap K$, for all i, $O_p(N)$ centralizes R_i and, because of the structure of $\mathrm{Sym}(t) \wr \mathrm{Sym}(k)$, this forces $O_p(N) \leqslant K$. But now $O_p(N) \cap K = O_p(N) \neq 1$, a contradiction. Therefore $O_p(N) \cap K \neq 1$.

Let $\varphi_i : K \longrightarrow K_i$ be the projection map of K onto K_i and set $L_i = \varphi_i(N \cap K)$. We see that $R_i \leqslant B_i \leqslant L_i \leqslant K_i$ and that L_i is transitive on Δ_i. If $O_p(L_i) = 1$, then $O_p(N \cap K) \leqslant \prod_{j \neq i} K_j$. For all $n \in N$, as $O_p(N \cap K) \trianglelefteq N$, we then have $O_p(N \cap K) = O_p(N \cap K)^n \leqslant (\prod_{j \neq i} K_j)^n$. Let $l \in \{1, \ldots, k\}$. We may choose an $n \in N$ so as $\Delta_i = \Delta_l^n$. Therefore $(\prod_{j \neq i} K_j)^n = \prod_{j \neq l} K_j$, whence it follows that $O_p(N \cap K) \leqslant \bigcap_{i=1}^k (\prod_{j \neq i} K_j) = 1$, a contradiction. Hence $O_p(L_i) \neq 1$. So $L_i \in \mathcal{N}(K_i, B_i)$ for all $i = 1, \ldots, k$. Let $H_1 \in \mathcal{N}_{max}(K_1, B_1)$ be such that $H_1 \geqslant L_1$. Since H_1 is transitive on Δ_1, by induction H_1 leaves invariant a block system with blocks of size p. Then H_1 contains E_1, a normal elementary abelian p-subgroup of order $p^{|\Delta_1|/p} = p^{t/p}$. Hence $E_1 \trianglelefteq L_1$ and it follows that $E_1 \trianglelefteq N \cap K$. Put $E = \langle E_1^N \rangle$. By the Frattini argument, $N = N_N(T \cap K)(N \cap K)$. So $E = \langle E_1^{N_N(T \cap K)} \rangle \leqslant N \cap K$. Since N is transitive on $\mathcal{B}$, $N_N(T \cap K)$ is transitive on $\mathcal{B}$. Let $g \in N_N(T \cap K)$ be such that $R_1^g = R_j$ for some j. Since $E_1 \trianglelefteq R_1$, E_1^g is an elementary abelian normal p-subgroup of R_j of order $p^{t/p}$. Therefore, E is an elementary abelian normal p-subgroup of T of order $p^{kt/p} = p^{p^{m-1}}$. Thus, using Theorem 2.8, up to conjugacy we see that

$$E = \langle (1, 2, \ldots, p), (p+1, p+2, \ldots, 2p), \ldots, (p(p^{m-1}-1)+1, p(p^{m-1}-1)+2, \ldots, p^m) \rangle.$$

By Theorem 2.9, we have that $B \leqslant N_G(E)$. Thus, as $N_G(E) \geqslant N$ and $N \in \mathcal{N}_{max}(G, T)$, $N_G(E) = N$. Therefore N leaves invariant a block system with blocks of size p. This complete the proof of Lemma.

3.2 $\mathrm{Sym}(kp^m)$

Theorem 3.2.1 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = kp$, where p is a prime and $k \in \mathbb{N}$ such that $1 < k < p$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{max}(G, B)$, then N is isomorphic to $B = N_{\mathrm{Sym}(p)}(\bar{T}) \wr \mathrm{Sym}(k)$, where $\bar{T} \in \mathrm{Syl}_p(\mathrm{Sym}(p))$.*

Proof. Let $P = \mathrm{Sym}(p) \wr \mathrm{Sym}(k)$. By Theorem 2.12, $B \leqslant P$ and B is transitive on Ω and so P is a maximal subgroup of G. Since N is a subgroup of G containing B and $O_p(N) \neq 1$, using Theorem 2.13, so we may assume that N is imprimitive. By the transitivity of B, it follows that every subgroup containing B can only have blocks of length $1, p$ and kp. So every nontrivial block of N must have length p. By Corollary 2.6, N is isomorphic to a subgroup of P. Since $O_p(B) \neq 1$, so B is a p-local subgroup of G with respect to B. Using Theorem 2.12, $B \cong N_{\mathrm{Sym}(p)}(\bar{T}) \wr \mathrm{Sym}(k)$, where $\bar{T} \in \mathrm{Syl}_p(\mathrm{Sym}(p))$. Moreover, by Lemma 2.3, B is a maximal subgroup of P. Thus $N \leqslant B$ and hence $N = B$. Then $N \cong N_{\mathrm{Sym}(p)}(\bar{T}) \wr \mathrm{Sym}(k)$ and this complete the proof.

Theorem 3.2.2 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = kp^m$, where p is a prime such that $p > 2$ and $m, k \in \mathbb{N}$ such that $k < p$ and $m > 1$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{max}(G, B)$, then N is isomorphic to $(\mathrm{Sym}(p) \wr \mathrm{Sym}(p^{m-1})) \wr \mathrm{Sym}(k)$.*

Proof. Since N is a subgroup of G containing B and, by Theorem 2.12, $B = N_{\mathrm{Sym}(p^m)}(\bar{T}) \wr \mathrm{Sym}(k)$ where $\bar{T} \in \mathrm{Syl}_p(\mathrm{Sym}(p^m))$, so we have that, using Corollary 2.14, $N \leqslant \mathrm{Sym}(p^m) \wr \mathrm{Sym}(k)$. Therefore, by Lemma 3.1.3, $\bar{N} = \mathrm{Sym}(p) \wr \mathrm{Sym}(p^{m-1}) \in \mathcal{N}_{max}(\mathrm{Sym}(p^m), N_{\mathrm{Sym}(p^m)}(\bar{T}))$ and $\bar{N}$ is a maximal subgroup of $\mathrm{Sym}(p^m)$. Thus, by Lemma 2.3, $\bar{N} \wr \mathrm{Sym}(k)$ is a maximal subgroup of $\mathrm{Sym}(p^m) \wr \mathrm{Sym}(k)$. It follows that $N \leqslant \bar{N} \wr \mathrm{Sym}(k)$. As $O_p(\bar{N}) \neq 1$, $O_p(\bar{N} \wr \mathrm{Sym}(k)) \neq 1$ and hence $\bar{N} \wr \mathrm{Sym}(k) \in \mathcal{N}(G, B)$. Therefore, $N = \bar{N} \wr \mathrm{Sym}(k)$.

3.3 $\mathrm{Sym}(k_1p^m + k_0)$

Lemma 3.3.1 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p^m + 1$, where p is a prime and $m \in \mathbb{N}$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then every proper subgroup of G containing B is contained in $\mathrm{Stab}_G(\omega) \cong \mathrm{Sym}(n-1)$, fixes $\omega \in \Omega$.*

Proof. We know that T and B fix a unique point $\omega \in \Omega$ and operates transitively on $\Omega \setminus \{\omega\}$. Suppose that $L \nleqslant \mathrm{Stab}_G(\omega)$ and $G \geqslant L \geqslant B$. Then L is 2-transitive on Ω, and, as B contains a transpositions, Lemma 2.4 (i) implies that $L = G$. Thus all proper subgroups of G which contain B are contained in $\mathrm{Stab}_G(\omega) \cong \mathrm{Sym}(n-1)$.

Lemma 3.3.2 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p^m + 1$, where p is a prime and $m \in \mathbb{N}$. Suppose that $T \in \mathrm{Syl}_p(G)$, $B = N_G(T)$ and put $H = \mathrm{Stab}_G(\omega)$, fixed $\omega \in \Omega$. Then $\mathcal{N}_{max}(G, B) = \mathcal{N}_{max}(H, B)$.*

Proof. Let $N \in \mathcal{N}_{max}(G, B)$. Since B is transitive on $\Omega \setminus \{\omega\}$, Lemma 3.3.1 implies that N is contained in $H \cong \mathrm{Sym}(n-1)$. It follows that $\mathcal{N}_{max}(G, B) \subseteq \mathcal{N}_{max}(H, B)$. But $H \leqslant G$, so that $\mathcal{N}_{max}(H, B) \subseteq \mathcal{N}_{max}(G, B)$ and the lemma is complete.

Theorem 3.3.3 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p + 1$, where p is a prime. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then $\mathcal{N}_{max}(G, B) = \{B\}$.*

Proof. For $p = 2$, $T = B \cong \mathrm{Sym}(2)$ and $O_p(B) \neq 1$. Also, for $p = 3$, $T \cong \mathrm{Alt}(3)$, $B \cong \mathrm{Sym}(3)$ and $O_p(B) \neq 1$. So, in both cases, B is a maximal subgroup of G and, thus, B is the unique maximal p-local subgroup of G with respect to B.

Suppose that $p \neq 2, 3$ and let $H \cong \mathrm{Sym}(p)$ be the stabilizer in G of a point in Ω, say $H = G_\sigma$, for some $\sigma \in \Omega$. By order we may assume that $T \in \mathrm{Syl}_p(H)$. Then, by Theorem 2.10, $N_H(T) = B$ and so, since $O_p(N_H(T)) \neq 1$, $N_H(T) = B$ is a p-local subgroup of G. It remains to prove that B is the only maximal p-local subgroup of G with respect to B. So let L be a maximal p-local subgroup of G, that is, $B \leqslant L$ and $O_p(L) \neq 1$. Using Lemma 3.3.1, we get that $L \leqslant H$ and so, by Theorem 2.15, B is a maximal subgroup of H implies that $L = B$.

Lemma 3.3.4 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = k_1 p^m + k_0$, where p is a prime and $m, k_0, k_1 \in \mathbb{N}$ such that $k_1 < p$, is the p-adic decomposition of n. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $n \geq p + 2$ and $N \in \mathcal{N}_{max}(G, B)$, then $N \leqslant \mathrm{Sym}(k_1 p^m) \times \mathrm{Sym}(k_0)$.*

Proof. Let $U = \mathrm{Sym}(k_1 p^m) \times \mathrm{Sym}(k_0)$. By Theorem 2.10, U contains B and we know that U is a maximal subgroup Assume that $N \nleqslant U$. Then N fuses the two orbits of U on Ω and so is transitive on Ω. Thus, by Lemma 2.16, N is primitive on Ω. Then Theorem 2.13 implies that $N = G$. Hence $N \leqslant U$.

Theorem 3.3.5 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = k_1p^m + k_0$, where p is a prime and $m, k_0, k_1 \in \mathbb{N}$ such that $k_0 < p$ and $k_1 < p$, is the p-adic decomposition of n. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. If $n \geq p+2$ and $N \in \mathcal{N}_{max}(G, B)$, then $N = \bar{N} \times \mathrm{Sym}(k_0)$ where $\bar{N}$ is a maximal p-local subgroup of $\mathrm{Sym}(k_1p^m)$ with respect to $B \cap \mathrm{Sym}(k_1p^m)$.*

Proof. By Lemma 3.3.4, $N \leqslant U \times V$ where $U = \mathrm{Sym}(k_1p^m)$ and $V = \mathrm{Sym}(k_0)$. Using Proposition 2.1, $T = (T \cap U) \times (T \cap V)$ with $T \cap U \in \mathrm{Syl}_p(U)$, $T \cap V \in \mathrm{Syl}_p(V)$ and $B = (B \cap U) \times (B \cap V)$ with $B \cap U = N_U(T \cap U)$, $B \cap V = N_V(T \cap V)$. As $T \cap V = 1$ and $1 \neq O_p(N) \leqslant T$, we have $1 \neq O_p(N) \cap (T \cap U) \leqslant O_p(N) \cap U$. Since $N \leqslant U \times V$, $O_p(N) \cap U \trianglelefteq N$ and so $1 \neq O_p(N) \cap U \leqslant O_p(NV)$. Therefore $NV \in \mathcal{N}(G, B)$ and hence, as $N \in \mathcal{N}_{max}(G, B)$, $N = NV$. So $V \leqslant N$ which implies, using Dedekinds' Modular Law, that $N = (N \cap U)V$. Now, as $N \cap U \in \mathcal{N}(U, B \cap U)$, we may choose $\bar{N} \in \mathcal{N}_{max}(U, B \cap U)$ with $N \cap U \leqslant \bar{N}$. Since $1 \neq O_p(\bar{N}) \leqslant O_p(\bar{N}V)$ and $B \leqslant \bar{N}V$, $\bar{N}V \in \mathcal{N}(G, B)$ and so, as $N = (N \cap U)V \leqslant \bar{N}V$, $N = \bar{N}V$.

Lemma 3.3.6 *Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = p+k$, where p is a prime and $k \in \mathbb{N}$ such that $1 < k < p$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then $\mathcal{N}_{max}(G, B) = \{B = N_{\mathrm{Sym}(p)}(T) \times \mathrm{Sym}(k)\}$.*

Proof. Let $U = \mathrm{Sym}(p) \times \mathrm{Sym}(k)$. By Theorem 2.10, U contains B and we know that U is a maximal subgroup of G. Assume that $N \in \mathcal{N}_{max}(G, B)$ and $N \nleqslant U$. Then, since N is a subgroup of G containing B, N fuses the two orbits of U on Ω and so is transitive on Ω. Thus, by Lemma 2.16, N is primitive on Ω. Therefore Theorem 2.13 implies that $N = G$. Hence $N \leqslant U$. Since $O_p(B) \neq 1$, so B is a p-local subgroup of G with respect to B. But, as $k < p$, $T \in \mathrm{Syl}_p(\mathrm{Sym}(p))$ and $B = N_{\mathrm{Sym}(p)}(T) \times \mathrm{Sym}(k)$, where, by Theorem 2.15, $N_{\mathrm{Sym}(p)}(T)$ is a maximal subgroup of $\mathrm{Sym}(p)$. Therefore B is a maximal subgroup of U. It follows that, as $B \leqslant N$, $N = B$ and we have the result.

We now want to show that those examined in the previous sections are the only B, the normalizer of a Sylow p-subgroup of $\mathrm{Sym}(\Omega)$, which are maximal p-local subgroups with respect to B.

Theorem 3.3.7 *Let $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$, where p is a prime. Then B is a maximal p-local subgroup with respect to B if one of the following occurs:*

(i) $n = p$

(ii) $n = kp$ *with* $1 < k < p$

(iii) $n = p + k$ *with* $1 \leq k < p$.

Proof. Follows from Theorems 3.1.1, 3.2.1, 3.3.3 and Lemma 3.3.6.

4 Some Examples

This section contains some examples of the subgroups in $\mathcal{N}_{max}(G, B)$ which illustrate some of the results proved earlier. We maintain the notation introduced in Section 1.

The definition of maximal p-local subgroup in terms of the normalizer of a Sylow p-subgroup is given in the works of Rowley and Saninta [14], in which they study all the maximal p-local subgroups for the symmetric groups, with respect to the prime $p = 2$. This case is relatively easy to study and an example illustrating their result is presented next.

Example 1: Sym(12)
Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 12$. Suppose that $p = 2$ and let T be a Sylow p-subgroup of G. Recall that $N_G(T) = T$. Consider the p-adic decomposition of n, $n = 2^3 + 2^2$ with $\Omega = \Omega_1 \cup \Omega_2$ where $|\Omega_1| = 8$ and $|\Omega_2| = 4$. Also $T = T_1 \times T_2 \cong (C_2 \wr C_2 \wr C_2) \times (C_2 \wr C_2)$, with T_i Sylow 2-subgroup of $\mathrm{Sym}(\Omega_i)$, for $i = 1, 2$, and C_2 cyclic group of order 2. Also the Ω_i's are the orbits of T on Ω. We begin by listing the subgroups in $\mathcal{N}_{max}(G, T)$, using Theorem 3.4 of [14].

$$\begin{aligned} N_1 &\cong \mathrm{Sym}(8) \times \mathrm{Sym}(4) \\ N_2 &\cong \mathrm{Sym}(4) \wr \mathrm{Sym}(3) \\ N_3 &\cong \mathrm{Sym}(2) \wr \mathrm{Sym}(6). \end{aligned}$$

Therefore,

$$\begin{aligned}
N_0 = N_1 \cap N_2 \cap N_3 &\cong ((\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \wr \mathrm{Sym}(2)) \times (\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \\
N_{12} = N_1 \cap N_2 &\cong \mathrm{Sym}(4) \times (\mathrm{Sym}(4) \wr \mathrm{Sym}(2)) \\
N_{13} = N_1 \cap N_3 &\cong (\mathrm{Sym}(2) \wr \mathrm{Sym}(4)) \times (\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \\
N_{23} = N_2 \cap N_3 &\cong (\mathrm{Sym}(2) \wr \mathrm{Sym}(2)) \wr \mathrm{Sym}(3).
\end{aligned}$$

Furthermore, $\langle N_1, N_2 \rangle = \langle N_1, N_3 \rangle = \langle N_2, N_3 \rangle = G$.

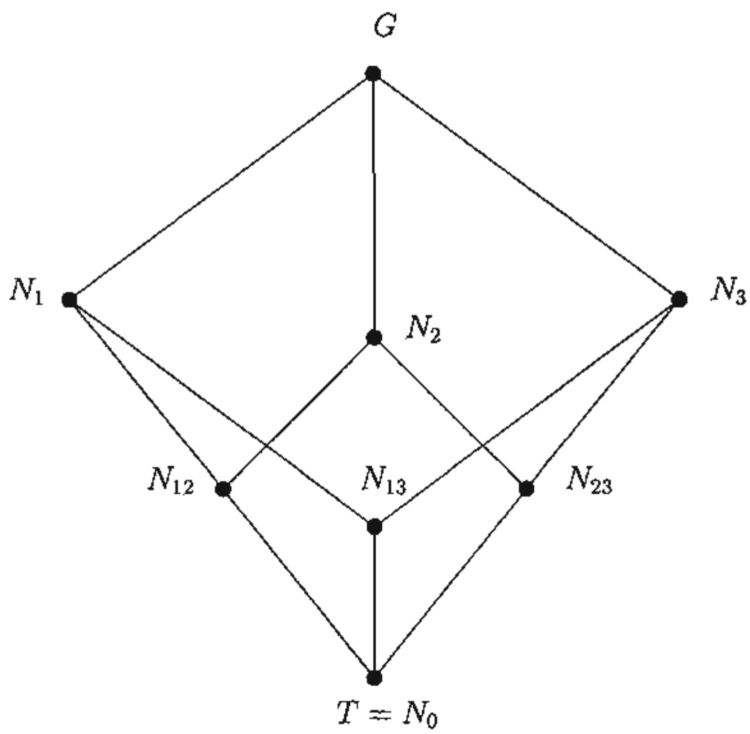

Figure 1: The lattice of the maximal 2-local subgroups of Sym(12).

Example 2: Sym(6)
Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 6$. Suppose that T be a Sylow 2-subgroup of G. Recall that $B = N_G(T) = T$. Consider the 2-adic decomposition of n, $n = 2^3 + 2^2$. By Theorem 3.4 of [14], the subgroups in $\mathcal{N}_{max}(G, B)$ are

$$\begin{aligned}
N_1 &= N(\{1\}; 2) \cong \mathrm{Sym}(4) \times \mathrm{Sym}(2) \\
N_2 &= N(\emptyset; 2) \cong \mathrm{Sym}(2) \wr \mathrm{Sym}(3).
\end{aligned}$$

Suppose now that T_1 be a Sylow 3-subgroup of G and $B_1 = N_G(T_1)$. Thus, $T_1 = \langle (1,2,3), (4,5,6) \rangle$ with $|T_1| = 9$ and $B_1 = \langle (4,5,6), (1,2,3), (4,5), (2,3)(4,6),$

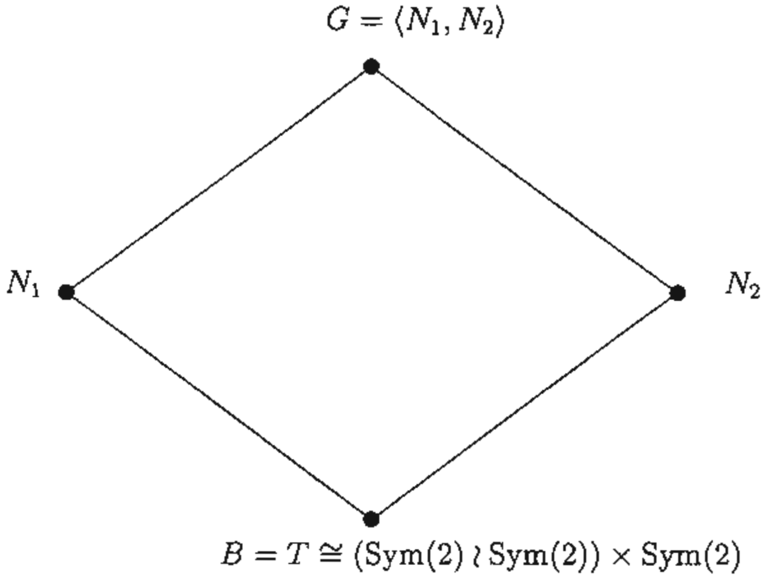

Figure 2: The lattice of the maximal 2-local subgroups of Sym(6).

$(1,4,3,6)(2,5)\rangle$ with $|B_1| = 72$. Consider the 3-adic decomposition of n, $n = 2(3)$. By Theorem 3.2.1, the subgroups in $\mathcal{N}_{max}(G, B_1)$ is $B_1 \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$.

We now consider the 5-adic decomposition of n, $n = 5 + 1$. A Sylow 5-subgroup T_2 of G can be generated by the element $(1,2,3,4,5)$. Then, by Theorem ??, G has a unique maximal 5-local subgroup with respect to $B = N_G(T_2)$, which is $B = \langle (1,2,3,4,5), (2,5)(3,4), (2,4,5,3) \rangle$.

Example 3: Sym(9)
Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 9$. Suppose that T be a Sylow 2-subgroup of G. Recall that $B = N_G(T) = T$. Consider the 2-adic decomposition of n, $n = 2^3 + 1$. By Theorem 3.4 of [14], the subgroups in $\mathcal{N}_{max}(G, T)$ are

$$\begin{aligned} N_1 &= N(\{2\}; 4) \cong \mathrm{Sym}(4) \wr \mathrm{Sym}(2) \\ N_2 &= N(\{2\}; 2) \cong \mathrm{Sym}(2) \wr \mathrm{Sym}(4). \end{aligned}$$

Therefore, the simplicial set of $\mathcal{N}_{max}(G, T)$ is $\mathcal{N}_{max}(G, T)$.

Suppose now that $\bar{T}$ be a Sylow 3-subgroup of G and $\bar{B} = N_G(\bar{T})$. Consider the 3-adic decomposition of n, $n = 3^2$. Then $\bar{T} = \langle (1,2,3), (4,5,6), (7,8,9),$

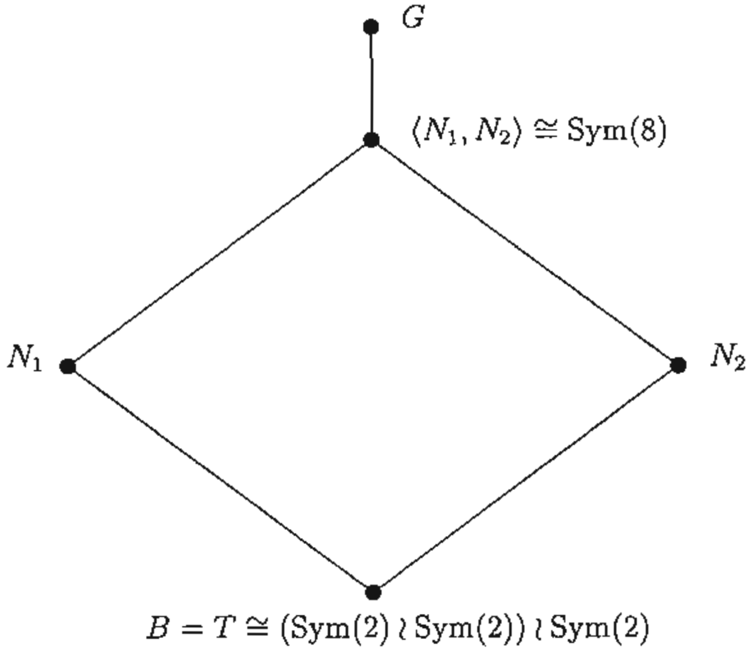

Figure 3: The lattice of the maximal 2-local subgroups of Sym(9).

$(1,4,7)(2,5,8)(3,6,9)\rangle$ with $|\bar{T}| = 81$ and $\bar{B} = N_G(\bar{T}) = \langle(1,5,9)(2,6,7)(3,4,8), (7,8,9), (4,5,6), (1,2,3), (4,9)(5,7)(6,8), (2,3)(4,8,5,7,6,9)\rangle$ with $|\bar{B}| = 324$. Then, by Theorem 3.1.3, G has a unique maximal 3-local subgroup with respect to $\bar{B}$, which is $N = \mathrm{Sym}(3) \wr \mathrm{Sym}(3) \cong N_G(E)$, where $E = \langle(1,2,3), (4,5,6), (7,8,9)\rangle$. That is, $N = \langle(1,2,3), (1,2), (4,5,6), (4,5), (7,8,9), (7,8), (1,4,7)(2,5,8)(3,6,8), (1,4)(2,5)(3,6)\rangle$ with $|N| = 1296$.

Example 4: $\mathrm{Sym}(3^3)$

Let $G = \mathrm{Sym}(\Omega)$ with $|\Omega| = n = 27$, $T \in \mathrm{Syl}_3(G)$, and let

$$\begin{aligned} x_1 &\cong (1,2,3), \\ x_2 &\cong (1,4,7)(2,5,8)(3,6,9), \\ x_3 &\cong (1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25) \\ &\quad (8,17,26)(9,18,27). \end{aligned}$$

be its generators. The normalizer B of T in G can be described as $B = T \rtimes \langle h_1, h_2, h_3\rangle$, with

$$\begin{aligned} h_1 &\cong (2,3)(5,6)(8,9)(11,12)(14,15)(17,18)(20,21)(23,24)(26,27), \\ h_2 &\cong (4,7)(5,8)(6,9)(13,16)(14,17)(15,18)(22,25)(23,26)(24,27), \\ h_3 &\cong (10,19)(11,20)(12,21)(13,22)(14,23)(15,24)(16,25)(17,26)(18,27). \end{aligned}$$

By Theorem 3.1.3, the subgroups in $\mathcal{N}_{max}(G, B)$ is $N = \mathrm{Sym}(3) \wr \mathrm{Sym}(9)$.

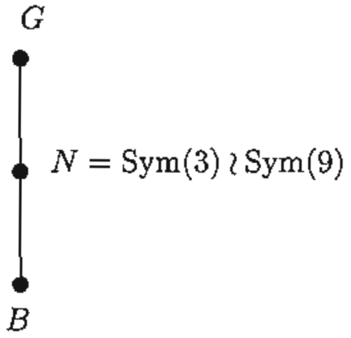

Figure 4: The lattice of the maximal 3-local subgroups of Sym(27).

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เรื่อง

The intransitive maximal p-local subgroups of the symmetric groups

The intransitive maximal p-local subgroups of the symmetric groups

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June 27, 2006

Abstract: The subgroups which do not act transitively on Ω in the set $\mathcal{N}_{max}(G, B)$ consisting of all maximal p-local subgroups of $G = \text{Sym}(\Omega)$ with respect to B, the normalizer of a Sylow p-subgroup of G in G, is investigated.

Keywords: Symmetric group, Sylow p-subgroup, Normalizer, Maximal p-local subgroup.
2000 Mathematics Subject Classification: 20B30, 20B35, 20D20, 20E28

1 Introduction

Maximal 2-local geometries for certain sporadic simple groups were firstly introduced by Ronan and Smith (1980). These geometries were inspired by the theory of buildings for the groups of Lie type which was developed by Tits (1956, 1974) in the fifties. For each finite simple group of Lie type, there is a natural geometry associated with it called its building. For G a group of Lie type of characteristic p, its building is a geometric structure whose vertex stabilizers are the maximal parabolic subgroups which are also p-local subgroups of G containing a Sylow p-subgroup. As is well-known, each building has a Coxeter diagram associated with it. Buekenhout (1979) generalized these concepts to obtain diagrams for many geometries related to sporadic simple groups. Ronan and Smith (1980)pursued these ideas further and introduced the maximal 2-local geometries. Other in variants on buildings for the

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sporadic simple groups have been defined, notably the minimal parabolic geometries as described by Ronan and Stroth (1984).

We now define what we mean, generally, by a minimal parabolic subgroup. Suppose that H is a finite group and p is a prime dividing the order of H. Let S be a Sylow p-subgroup of H and B the normalizer of S in H. A subgroup P of H properly containing B is said to be a *minimal parabolic subgroup* of H with respect to B if B lies in exactly one maximal subgroup of P.

The definition of minimal parabolic subgroups in terms of the normalizer of a Sylow p-subgroup is given in the works of Ronan and Smith (1980) and Ronan and Stroth (1984), in which they study minimal parabolic geometries for the 26 sporadic finite simple groups. The connection between minimal parabolic subgroups and group geometries is the best illustrated in the case of groups of Lie type in their defining characteristic. For a group of Lie type, its minimal parabolic system is always geometric. This is not always the case in general (see Ronan and Stroth, 1984). Many studies on the minimal parabolic system of special subgroups have been done over the years. For example, Lempken, Parker and Rowley (1998) determined all the minimal parabolic subgroups and system for the symmetric and alternating groups, with respect to the prime $p = 2$. Later, Covello (2000) has studied minimal parabolic subgroups and systems for the symmetric group with respect to an odd prime p dividing the order of the group. The main results are about the symmetric groups of degree p^r, she also establishes some more general results. More recently, Rowley and Saninta (2004) investigated the maximal 2-local geometries for the symmetric groups. Furthermore, Saninta (2004) considered the relationship between the maximal 2-local subgroups and the minimal parabolic subgroups for the symmetric groups. In this paper we shall investigate intransitive maximal p-local subgroups for the symmetric groups.

Let H, p, S and B be defined as above. Define

$$\mathcal{N}(H,B) = \{K \mid B \leqslant K \leqslant H \text{ and } O_p(K) \neq 1\}$$

where $O_p(K)$ is a unique maximal normal p-subgroup of K. A subgroup in $\mathcal{N}(H,B)$ is said to be a **p-local subgroup** of H with respect to B and a subgroup in $\mathcal{N}(H,B)$ which is maximal under inclusion is said to be a **maximal p-local subgroup** of

H with respect to B. We denoted the collection of maximal p-local subgroups of H with respect to B by $\mathcal{N}_{max}(H, B)$.

Throughout all groups considered, and in particular all our sets, will be finite. Let Ω be a set of cardinality $n > 1$. Set $G = \mathrm{Sym}(\Omega)$, the symmetric group on the finite set Ω. We also use $\mathrm{Sym}(m)$ to denote the symmetric group of degree m. Now let T be a fixed Sylow p-subgroup of G, where p is a prime, and B be the normalizer of T in G.

The main purpose of this paper is to study the subgroups in $\mathcal{N}_{max}(G, B)$ which do not act transitively on Ω.

2 Preliminary Results

This section gathers together results that will be used. Now we let Ω be a finite set with $|\Omega| > 1$ and let G, T, B and n be defined as in Section 1.

Proposition 2.1 *Let H be a group and suppose that $H = A \times B$. Let $S \in \mathrm{Syl}_p(H)$. Then $S = (S \cap A) \times (S \cap B)$ and*

$$N_H(S) = (N_H(S) \cap A) \times (N_H(S) \cap B),$$

with $N_H(S) \cap A = N_A(S \cap A)$ and $N_H(S) \cap B = N_B(S \cap B)$.

Proof. See Covello [6] (Proposition 1.1.10).

Lemma 2.2 *Suppose that $H = X \times Y$ is a direct product of groups X and Y and suppose that $S \in \mathrm{Syl}_p(H)$ where p is a prime which divides the order of both X and Y. Assume that L is a subgroup of H which contains $B := N_H(S)$. Then $L = (L \cap X) \times (L \cap Y)$, with $L \cap X = (B \cap X)^L$ and $L \cap Y = (B \cap Y)^L$.*

Proof. See Lempken, Parker and Rowley [10] (Lemma 2.5).

Lemma 2.3 *Suppose p is a prime, n is a positive integer and $T_{p^n} \in \mathrm{Syl}_p(\mathrm{Sym}(p^n))$. Then $|Z(T_{p^n})| = p$.*

Proof. See Saninta [15] (Lemma 2.3.5).

Theorem 2.4 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, and let $S \in \mathrm{Syl}_p(H)$. Let*

$$n = k_t p^t + k_{t-1} p^{t-1} + \cdots + k_1 p + k_0,$$

with $0 \leq k_j < p$, for all $j = 0, \ldots, t$, be the p-adic decomposition of n. Then the normalizer B of S in H is given by

$$B = B_0 \times \cdots \times B_t,$$

where, for $j = 0, \ldots, t$, B_j is the normalizer of a Sylow p-subgroup of $\mathrm{Sym}(\Omega_j)$, with $\Omega_j \subseteq \Omega$ and $|\Omega_j| = k_j p^j$. In particular,

$$|B| = |S| \prod_{j=0}^{t} k_j!(p-1)^{k_j j}$$

and the sets $\Omega_0, \Omega_1, \ldots, \Omega_t$ are the orbits of B on Ω.

Proof. See Covello [6] (Theorem 5.4.1).

Theorem 2.5 *Let p be a prime, $p \neq 2, 3$, and $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = p$. Suppose that $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Then B is a maximal subgroup of G.*

Proof. See Covello [6] (Theorem 6.1.2).

Lemma 2.6 *Let $H = \mathrm{Sym}(\Omega)$, $S \in \mathrm{Syl}_p(H)$ and $B = N_H(S)$. If M is an intransitive subgroup of H containing B, then*

$$M \leqslant \mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\Delta_2),$$

with $\Omega = \Delta_1 \cup \Delta_2$ and the Δ_i's unions of orbits of M on Ω. Moreover

$$M = (M \cap \mathrm{Sym}(\Delta_1)) \times (M \cap \mathrm{Sym}(\Delta_2)).$$

Proof. The first part of the statement is obvious. The second follows from Lemma 2.2.

According to the O'Nan-Scott theorem and the first theorem in [11] we get the following important results:

Theorem 2.7 *Let $H = \mathrm{Sym}(\Omega)$, with $|\Omega| = n > 2$. Then, for all $r \geq 1$ such that $n \neq 2r$, the group*

$$L = \mathrm{Sym}(n-r) \times \mathrm{Sym}(r)$$

is a maximal (intransitive) subgroup of H.

Proof. See Saninta [15] (Lemma 2.4.1).

3 Main Results

We maintain the notation introduced in Section 1. Our next result concerns subgroups in $\mathcal{N}_{max}(G, B)$ which do not act transitively on Ω. We now fix the following notation for p-adic decomposition of n :

$$n = k_t p^t + k_{t-1}p^{t-1} + \cdots + k_1 p + k_0,$$

where p is a prime and k_j is an integer with $0 \leq k_j < p$, for all $j = 0, \ldots, t$. Let $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_t$, with $|\Omega_j| = k_j p^j$, for all $j = 0, \ldots, t$, be the corresponding partition of Ω into B-orbits. Set $I = \{0, 1, \ldots, t\}$. Recall that T has $t+1$ orbits on Ω. Note that $T = T_0 \times T_1 \times \cdots \times T_t$ where, $T_i \in \mathrm{Syl}_p(\mathrm{Sym}(\Omega_i))$, $i \in \{0, 1, \ldots, t\}$ and, moreover, each T_i is the direct product of k_i factors, each isomorphic to a Sylow p-subgroup of $\mathrm{Sym}(\Delta)$, with $|\Delta| = p^i$ (see Findlay [8]).

Lemma 3.1 *Suppose that $G = \mathrm{Sym}(\Omega)$, with $|\Omega| > 1$, $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Let $N \in \mathcal{N}_{max}(G, B)$ and N is not transitive on Ω. Then $N \leqslant \mathrm{Sym}(\Delta) \times \mathrm{Sym}(\Omega \setminus \Delta)$, where $\Delta = \bigcup_{i \in J} \Omega_i$ for some proper subset J of I.*

Proof. It follows from Lemma 2.6.

Theorem 3.2 *Let $G = \mathrm{Sym}(\Omega)$, with $|\Omega| = n$, $T \in \mathrm{Syl}_p(G)$ and $B = N_G(T)$. Let $U = \mathrm{Sym}(\Delta)$ and $V = \mathrm{Sym}(\Omega \setminus \Delta)$ where $\Delta = \bigcup_{i \in J} \Omega_i$ for some proper subset J of I. Suppose that $N \in \mathcal{N}_{max}(G, B)$ and $N \leqslant U \times V$.*

(i) If $O_p(N) \cap U \neq 1$, then $N = N_U \times V$ where $N_U \in \mathcal{N}_{max}(U, B \cap U)$.

(ii) If $O_p(N) \cap V \neq 1$, then $N = U \times N_V$ where $N_V \in \mathcal{N}_{max}(V, B \cap V)$.

Proof. First we examine the case when $O_p(N) \cap U \neq 1$. Since $N \leqslant U \times V$, $O_p(N) \cap U \trianglelefteq N$ and so $1 \neq O_p(N) \cap U \leqslant O_p(NV)$. Therefore $NV \in \mathcal{N}(G, B)$ and hence, as $N \in \mathcal{N}_{max}(G, B)$, $N = NV$. So $V \leqslant N$ which implies, using Dedekinds' Modular Law, that $N = (N \cap U)V$. Now, as $N \cap U \in \mathcal{N}(U, B \cap U)$, we may choose $N_U \in \mathcal{N}_{max}(U, B \cap U)$ with $N \cap U \leqslant N_U$. Since $1 \neq O_p(N_U) \leqslant O_p(N_U V)$ and $B \leqslant N_U V$, $N_U V \in \mathcal{N}(G, B)$ and so, as $N = (N \cap U)V \leqslant N_U V$, $N = N_U V$. If we have $O_p(N) \cap V \neq 1$, the same argument yields $N = U \times N_V$ for some $N_V \in \mathcal{N}_{max}(V, B \cap V)$.

Theorem 3.3 *Let the hypothesis of Theorem 3.2 holds. Suppose that $0 \leq k_j \leq 1$, for all $j = 0, \ldots, t$. Then either $N = N_U \times V$, where $N_U \in \mathcal{N}_{max}(U, B \cap U)$ and N_U is transitive on Δ, or $N = U \times N_V$, where $N_V \in \mathcal{N}_{max}(V, B \cap V)$ and N_V is transitive on $\Omega \setminus \Delta$.*

Proof. Thanks to the study carried out in Theorem 3.2, we only need to eliminate the situation $O_p(N) \cap U = 1 = O_p(N) \cap V$. From

$$[O_p(N), T_U] \leqslant O_p(N) \cap T_U \leqslant O_p(N) \cap U = 1$$

and

$$[O_p(N), T_V] \leqslant O_p(N) \cap T_V \leqslant O_p(N) \cap V = 1$$

where $T_U \in \mathrm{Syl}_p(U), T_V \in \mathrm{Syl}_p(V)$, we deduce that $O_p(N) \leqslant Z(T)$. Therefore, $C_G(Z(T)) \leqslant C_G(O_p(N)) \leqslant N_G(O_p(N)) = N$.

Let $1 \neq \sigma \in O_p(N)$, so $\sigma \in Z(T)$. For any $g \in N$, $\sigma^g \in O_p(N) \trianglelefteq N$ and hence $\sigma^g \in Z(T)$. Since $T = \prod_{i \in I} T_i$ where, for $i \in I$, $T_i \in \mathrm{Syl}_p(\mathrm{Sym}(\Omega_i))$, $Z(T) = \prod_{i \in I} Z(T_i)$. By Lemma 2.3, $Z(T_i) = \langle \sigma_i \rangle$ where σ_i has order p and cycle type $p^{p^{i-1}}$. Now let $1 \neq \mu \in Z(T)$ with $\mu \neq \sigma$. So $\sigma = \prod_{k \in K} \sigma_k$ and $\mu = \prod_{k \in K'} \sigma_k$, where $K, K' \subseteq I$ with $K \neq K'$ and consequently, as $t > t-1 > \cdots > 1$, σ and μ have different cycle types. Therefore $\sigma^g = \sigma$ and then $N \leqslant C_G(\sigma)$. Since $\langle \sigma \rangle \subseteq Z(C_G(\sigma)) \leqslant O_p(C_G(\sigma))$, $C_G(\sigma) \in \mathcal{N}(G, B)$. This implies that $N = C_G(\sigma)$ for all $1 \neq \sigma \in O_p(N)$, as $N \in \mathcal{N}_{max}(G, B)$. We see that

$$C_G(\sigma) = \prod_{k \in K} C_{\mathrm{Sym}(\Omega_k)}(\sigma_k) \times \mathrm{Sym}(\bigcup_{i \in I \setminus K} \Omega_i)$$

and so $\langle \sigma_k \mid k \in K \rangle \leqslant Z(C_G(\sigma))$. In particular, $\langle \sigma_k \mid k \in K \rangle \leqslant O_p(C_G(\sigma)) = O_p(N)$. Now either $\langle \sigma_k \mid k \in K \rangle \cap T_U \neq 1$ or $\langle \sigma_k \mid k \in K \rangle \cap T_V \neq 1$ because $O_p(N) \trianglelefteq T = T_U \times T_V$, a contradiction.

Aiming for a contradiction we assume N_U is not transitive on Δ. Thus $N_U \leqslant X \times Y \leqslant U$ where $\Theta = \bigcup_{i \in K} \Omega_i$, $X = \mathrm{Sym}(\Theta)$ and $Y = \mathrm{Sym}(\Delta \setminus \Theta)$ for some $K \subset J$. Applying the previous part to $N_U \in \mathcal{N}_{max}(U, B \cap U)$ we deduce that either $N_U = N_X \times Y$ where $N_X \in \mathcal{N}_{max}(X, B \cap X)$ or $N_U = X \times N_Y$ where $N_Y \in \mathcal{N}_{max}(Y, B \cap Y)$. Without loss of generally we assume the former to hold. Since $O_p(N_X) \neq 1$ and $T \leqslant N_X \times \mathrm{Sym}(\Omega \setminus \Theta)$, clearly $N_G(O_p(N_X)) \in \mathcal{N}(G, B)$. However we have that

$$\begin{aligned} N = N_U \times V &= N_X \times Y \times V \\ &< N_X \times \mathrm{Sym}(\Omega \setminus \Theta) \leqslant N_G(O_p(N_X)), \end{aligned}$$

a contradiction. Therefore we conclude that N_U is transitive on Δ and hence the proof of the theorem is complete.

4 $\mathcal{N}_{max}(G^*, T^*)$ for $G^* \cong \mathrm{Alt}(\Omega)$

We now use G^* to denote $\mathrm{Alt}(\Omega)$, the alternating group on Ω, and also $\mathrm{Alt}(m)$ to denote the alternating group of degree m. Put $T^* = G^* \cap T$ and $B^* = N_{G^*}(T^*)$. Recall that $T^* \in \mathrm{Syl}_p(G^*)$. Here we look at the relationship between $\mathcal{N}_{max}(G, B)$ and $\mathcal{N}_{max}(G^*, B^*)$. In order to do this we study some specific cases.

Lemma 4.1 *Let $G^* = \mathrm{Alt}(\Omega)$ with $|\Omega| = p$, where p is a prime and $p \neq 2$. Suppose that $T^* \in \mathrm{Syl}_p(G^*)$, and $B^* = N_{G^*}(T^*)$. Then $B^* \in \mathcal{N}_{max}(G^*, B^*)$.*

Proof. If $p = 2$, then, as $B^* = 1$, $\mathcal{N}_{max}(G^*, B^*) = \emptyset$. Now assume that $p \neq 2$. Thus, $T^* = \langle (1, 2, 3, \ldots, p) \rangle$ with $|T| = p$. Since T^* is a normal p-subgroup of B^*, so $O_p(B^*) \neq 1$. Therefore, using Theorem 2.5, $B^* \in \mathcal{N}_{max}(G^*, B^*)$.

Lemma 4.2 *Let $G^* = \mathrm{Alt}(\Omega)$ with $|\Omega| = p + 1$, where p is a prime and $p \neq 2$. Suppose that $T^* \in \mathrm{Syl}_p(G^*)$, and $B^* = N_{G^*}(T^*)$. Then $B^* \in \mathcal{N}(G^*, B^*)$.*

Proof. If $p = 2$, then $T^* = 1$ and $B^* = G^*$. As $O_p(B^*) = 1$, $\mathcal{N}_{max}(G^*, B^*) = \emptyset$. Now assume that $p \neq 2$ and let $H \cong \mathrm{Sym}(p)$. Thus, $T^* = \langle (1, 2, 3, \ldots, p) \rangle \in \mathrm{Syl}_p(H)$ with $|T| = p$ and $N_H(T^*) = B^*$. Since $O_p(B^*) \neq 1$, hence $B^* \in \mathcal{N}(G^*, B^*)$.

Lemma 4.3 *Let* $G = \mathrm{Sym}(\Omega)$ *and* $G^* = \mathrm{Alt}(\Omega)$ *with* $|\Omega| = p^m$, *where* p *is a prime such that* $p > 2$ *and* $m \in \mathbb{N}$ *such that* $m > 1$. *Suppose that* $T \in \mathrm{Syl}_p(G)$, $T^* = G^* \cap T$, $B = N_G(T)$ *and* $B^* = N_{G^*}(T^*)$. *If* $N \in \mathcal{N}_{max}(G, B)$, *then* $N \cap G^* \in \mathcal{N}(G^*, B^*)$.

Proof. The assumption on N means that $|O_p(N)| \geq p^2$. Hence $1 \neq O_p(N) \cap G^* \lhd N \cap G^*$. Using Proposition 2.1, $B^* = B \cap G^* \leqslant N \cap G^*$ and so $N \cap G^* \in \mathcal{N}(G^*, B^*)$.

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