# รายงานวิจัยฉบับสมบูรณ์

**โครงการ** การวิเคราะห์สมมาตรของสมการ

$$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\big(u(x,t),u(x,t-\tau)\big)$$

**โดย** ผศ.ดร.เจษฎา ตัณฑนุช

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

## บทคัดย่อ

สมการ

$$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\big(u(x,t),u(x,t-\tau)\big) \tag{1}$$

เป็นสมการ สมการเชิงอนุพันธ์ย่อยชนิดประวิง (delay partial differential equations) ที่มีลักษณะใกล้เคียงกันกับสมการ สมการของเบอร์เจอร์ (Burger equation) และสมการ KdV (Korteweg-de Vries equation) ซึ่งเป็นสมการที่ถูกนำมาศึกษาโดยตรง และ ประยุกต์ เพื่อใช้ในการวิเคราะห์ปรากฏการณ์ธรรมชาติในเชิงฟิสิกส์หลายด้าน เนื่องด้วยฟังก์ชันนัล $G$ ที่ปรากฏในสมการเป็นไปได้อย่างหลากหลาย งานวิจัยชิ้นนี้ได้ทำการหาผลเฉลยวิเคราะห์ (analytical solutions) ของสมการ (1) และจำแนกประเภททั้งหมดที่เป็นไปได้ของฟังก์ชันนัล $G$ แต่เนื่องด้วยความซับซ้อนของสมการ จึงทำได้เฉพาะในกรณีฟังก์ชัน $G$ ขึ้นอยู่กับตัวแปร $u(x,t-\tau)$ เพียงตัวแปรเดียว และกรณี

$$G\left(u(x,t-\tau),u(x,t-\tau)\right)=g(u(x,t-\tau)-u(x,t-\tau))+H(u(x,t-\tau))$$

เมื่อ $g$ เป็นฟังก์ชันนัลใดๆ ของ $u(x,t-\tau)-u(x,t)$ และ $H$ เป็นฟังก์ชันนัลใดๆ ของ $u(x,t)$

**Abstract**

Equation

$$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\big(u(x,t),u(x,t-\tau)\big) \tag{1}$$

is a delay partial differential equation with arbitrary functional $G$. The equation is similar to Burger's equation and KdV (Korteweg-de Vries equation) which are studied in many fields of Physics. By the arbitrariness of he functional $G$, its solutions and the classification of them are presented in this report. However, the complexity of problem restricts to be able to show only the case $G$ depends on only $u(x,t-\tau)$ and the case

$$G\left(u(x,t-\tau),u(x,t-\tau)\right)=g(u(x,t-\tau)-u(x,t-\tau))+H(u(x,t-\tau)),$$

where $g$ is an arbitrary functional of $u(x,t-\tau)-u(x,t)$ and $H$ is an arbitrary functional of $u(x,t)$.

# เนื้อหางานวิจัย

## 1. ที่มาและหลักการ

สมการเชิงอนุพันธ์ย่อยชนิดประวิง (delay partial differential equation - DPDE) ซึ่งมีค่าการประวิง $\tau > 0$ [2]

(1.1) $$\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = G\big(u(x,t-\tau), u(x,t)\big)$$

เป็นสมการที่มีลักษณะใกล้เคียงกันกับสมการของเบอร์เจอร์ (Burger equation) และสมการ KdV (Korteweg-de Vries equation) ซึ่งเป็นสมการที่ใช้ในการอธิบายปรากฏการณ์การขยายตัวของก๊าซ (rarefaction gas) [1] เพื่อความสะดวก เราจะสามารถเขียนสมการ (1.1) ใหม่ได้ในรูป

(1.2) $$u_t + uu_x = G\left(u^\tau, u\right),$$

เมื่อ $u^\tau$ หมายถึง $u(x,t-\tau)$, $u$ หมายถึง $u(x,t)$ และ $u_x$, $u_t$ หมายถึงอนุพันธ์ย่อยของตัวแปรไม่อิสระ $u$ เทียบกับตัวแปรอิสระ $x$ และ $t$ ตามลำดับ

เนื่องด้วยสมการดังกล่าวเป็นสมการที่มีพจน์ประวิง ทำให้เป็นการยากที่จะหาผลเฉลยวิเคราะห์ (analytical solution) แนวทางหนึ่งที่จะสามารถนำมาประยุกต์ใช้ในการหาผลเฉลยวิเคราะห์ของสมการเชิงอนุพันธ์ได้ก็คือ การประยุกต์ใช้ **กลุ่มวิเคราะห์** (group analysis) [8,9,10] โดยแนวคิดที่จะใช้กลุ่มวิเคราะห์หาผลเฉลยของสมการเชิงอนุพันธ์ชนิดประวิงมีมาไม่นาน และถูกรวบรวมแนวคิดและพัฒนาเป็นขั้นตอนวิธีในปี พ.ศ. 2546 [5,6,7] นอกจากนี้กลุ่มวิเคราะห์ยังสามารถถูกนำมาประยุกต์ใช้จำแนกประเภท (classification) ของสมการเชิงอนุพันธ์โดยพิจารณาจากรูปแบบของผลเฉลยได้

## 2. การประยุกต์ใช้กลุ่มวิเคราะห์หาผลเฉลยของสมการเชิงอนุพันธ์

โดยทฤษฎีของกลุ่มวิเคราะห์ เราจะพิจารณา **สมมาตร** (symmetry) ซึ่งก็คือการแปลง (transformation) $\varphi : \Omega \times \Delta \to \Omega$ ที่ส่งผลเฉลยของสมการเชิงอนุพันธ์ไปยังอีกผลเฉลยหนึ่งของสมการเดียวกัน โดย $\Omega$ เป็นปริภูมิของตัวแปร $(x,t,u)$ และ $\Delta \subset \mathbb{R}$ เป็นช่วงสมมาตรรอบจุดศูนย์ ถ้ากำหนดให้ตัวแปร $\varepsilon$ เป็นพารามิเตอร์ของการแปลง $\varphi$ ซึ่งส่งชุดตัวแปร $(x,t,u)$ ไปยังชุดตัวแปรใหม่ $(\overline{x},\overline{t},\overline{u})$ (ตัวแปรใหม่ยังคงอยู่ในเซตเดิม) จะใช้สัญกรณ์ $\varphi(x,t,u;\varepsilon) = (\overline{x},\overline{t},\overline{u})$ หรือ $\varphi_\varepsilon(x,t,u) = (\overline{x},\overline{t},\overline{u})$ แทนความหมายดังกล่าว

เซตของฟังก์ชัน $\varphi_\varepsilon$ จะมีคุณสมบัติเป็น **กลุ่มการแปลงพารามิเตอร์เดี่ยวของปริภูมิ** $\Omega$ (a one-parameter transformation group of space $\Omega$) ถ้าสมาชิกในเซตดังกล่าวมีคุณสมบัติต่อไปนี้ [7,8,9,10]

1.) $\varphi_0(x,t,u) = (x,t,u)$ สำหรับทุกๆ $(x,t,u) \in \Omega$;

2.) $\varphi_{\varepsilon_1}\left(\varphi_{\varepsilon_2}(x,t,u)\right) = \varphi_{\varepsilon_1+\varepsilon_2}(x,t,u)$ สำหรับทุกๆ $\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2 \in \Delta$ และ $(x,t,u) \in \Omega$;

3.) ถ้า $\varphi_\varepsilon(x,t,u) = (x,t,u)$ สำหรับทุกๆ $(x,t,u) \in \Omega$, แล้ว $\varepsilon = 0$.

สัญกรณ์ต่อไปนี้ $\bar{x} = \varphi^{x}(x,t,u;\varepsilon), \bar{t} = \varphi^{t}(x,t,u;\varepsilon), \bar{u} = \varphi^{u}(x,t,u;\varepsilon)$ สามารถนำมาใช้ได้โดยมีความหมายเดียวกันกับ $\varphi_{\varepsilon}(x,t,u) = (\bar{x},\bar{t},\bar{u})$

จากแนวคิดข้างต้น สามารถนิยามการแปลงตัวแปร $u$ ที่มีพจน์ประวิงและอนุพันธ์ ให้อยู่ในรูปแบบของตัวแปรใหม่ได้คือ $\bar{u}^{\tau} = \bar{u}(\bar{x},\bar{t}-\tau)$ และ $\bar{u}_{\bar{x}} = \partial\bar{u} / \partial\bar{x}, \bar{u}_{\bar{t}} = \partial\bar{u} / \partial\bar{t}$ ตามลำดับ

การเชื่อมโยงระหว่างสมมาตร และ สมการเชิงอนุพันธ์ชนิดประวิงกระทำได้ตามแนวคิดต่อไปนี้ พิจารณาสมการเชิงอนุพันธ์ย่อยชนิดประวิงใดๆ

(2.1) $$F(x,t,u,u^{\tau},u_{x},u_{t}) = 0$$

สมการเชิงอนุพันธ์นี้จะเป็นจริงสำหรับชุดตัวแปร $(x,t,u)$ ซึ่งเป็นผลเฉลยของสมการเชิงอนุพันธ์ และสำหรับชุดตัวแปร $(\bar{x},\bar{t},\bar{u})$ ซึ่งได้มาจากการแปลงผลเฉลยดังกล่าวด้วยสมมาตรซึ่งส่งผลเฉลยของสมการเชิงอนุพันธ์ (2.1) ไปยังผลเฉลยของสมการเดียวกัน หรือ เรียกอีกอย่างหนึ่งว่า **สมมาตรซึ่งถูกรองรับโดยสมการเชิงอนุพันธ์ (2.1)** (symmetries admitted by equation (2.1) ) เมื่อแทนค่าชุดตัวแปรดังกล่าวลงในสมการเชิงอนุพันธ์ ก็จะทำให้สมการ (2.1) เป็นจริงด้วย นั่นทำให้อนุพันธ์ของสมการ (2.1) ที่มีตัวแปร $\bar{x},\bar{t},\bar{u}$ และ อนุพันธ์ $\bar{u}_{\bar{x}},\bar{u}_{\bar{t}}$ เทียบกับพารามิเตอร์ $\varepsilon$ ต้องมีค่าเป็นศูนย์

(2.2) $$\left.\frac{\partial F(\bar{x},\bar{t},\bar{u},\bar{u}^{\tau},\bar{u}_{\bar{x}},\bar{u}_{\bar{t}})}{\partial\varepsilon}\right|_{\varepsilon=0,(2.1)} = \left.\widetilde{X}F(x,t,u,u^{\tau},u_{x},u_{t})\right|_{(2.1)} \equiv 0$$

ตัวดำเนินการ $\widetilde{X}$ ถูกนิยามดังนี้ $\widetilde{X} = (\zeta - u_{x}\xi - u_{t}\eta)\partial_{u} + (\zeta^{\tau} - u_{x}^{\tau}\xi^{\tau} - u_{t}^{\tau}\eta^{\tau})\partial_{u^{\tau}} + \zeta^{u_{x}}\partial_{u_{x}} + \zeta^{u_{t}}\partial_{u_{t}}$,
เมื่อ

$$\xi(x,t,u) = \frac{\partial\varphi^{x}}{\partial\varepsilon}(x,t,u;0), \eta(x,t,u) = \frac{\partial\varphi^{t}}{\partial\varepsilon}(x,t,u;0), \zeta(x,t,u) = \frac{\partial\varphi^{u}}{\partial\varepsilon}(x,t,u;0),$$
$$\xi^{\tau} = \xi(x,t-\tau,u^{\tau}), \eta^{\tau} = \eta(x,t-\tau,u^{\tau}), \zeta^{\tau} = \zeta(x,t-\tau,u^{\tau}),$$
$$\zeta^{u_{x}} = D_{x}(\zeta - u_{x}\xi - u_{t}\eta), \zeta^{u_{t}} = D_{t}(\zeta - u_{x}\xi - u_{t}\eta),$$
$$D_{x} = \partial_{x} + u_{x}\partial_{u} + u_{x}^{\tau}\partial_{u^{\tau}} + u_{xx}\partial_{u_{x}} + u_{xt}\partial_{u_{t}} + \ldots,$$
$$D_{t} = \partial_{t} + u_{t}\partial_{u} + u_{t}^{\tau}\partial_{u^{\tau}} + u_{xt}\partial_{u_{t}} + u_{tt}\partial_{u_{t}} + \ldots,$$

เราเรียกตัวดำเนินการ $\widetilde{X}$ ว่า **ตัวก่อกำเนิดกณิกนันต์ของลี-แบ็กกลันด์แบบบัญญัติ** (*a canonical Lie-Bäcklund infinitesimal generator*) [3] และเรียกสมการ (2.2) ว่า**สมการกำหนด** (*a determining equation* (DME)) เนื่องด้วย $\left.\widetilde{X}F(x,t,u,u^{\tau},u_{x},u_{t})\right|_{(2.1)} \equiv 0$, ดังนั้นเราจะกล่าวว่าตัวก่อกำหนด $\widetilde{X}$ ถูกรองรับ (admitted) โดยสมการ (2.1) หรือ สมการ (2.1) รองรับ (admits) ตัวดำเนินการ $\widetilde{X}$

ทฤษฎีของลี [8,9,10] กล่าวไว้ว่าตัวดำเนินการดังกล่าวจะสมนัยแบบหนึ่งต่อหนึ่ง (one-to-one correspondence) กับสมมาตร นั่นคือถ้าหาตัวดำเนินการ $\widetilde{X}$ ได้ เราก็จะสามารถหาสมมาตรได้ และในทำนองเดียวกันถ้าเราสามารถหาสมมาตรได้ เราก็จะสามารถหาตัวดำเนินการ $\widetilde{X}$ ได้เช่นกัน

แนวความคิดของกลุ่มวิเคราะห์ในการหาผลเฉลยของสมการเชิงอนุพันธ์ (2.1) จะไม่ได้หาผลเฉลยของสมการเชิงอนุพันธ์ดังกล่าวโดยตรง แต่จะหาค่าของฟังก์ชันไม่ทราบค่า $\xi,\eta$ และ $\zeta$ ที่ปรากฏในสมการกำหนด (2.2) และเนื่องด้วย ตัวดำเนินการ $\widetilde{X}$ สมมูลกับตัวดำเนินการแบบฉบับ (*a classical infinitesimal generator*)[9]

$$X = \xi\partial_x + \eta\partial_t + \zeta\partial_u$$

ทำให้สามารถประยุกต์ใช้ทฤษฎีของสมการเชิงอนุพันธ์ และ การวิเคราะห์ (analysis) หาผลเฉลยของสมการเชิงอนุพันธ์ (2.1) โดยการหาผลเฉลยของสมการแคแรกเทอริสติก

(2.3) $$\frac{dx}{\xi} = \frac{dt}{\eta} = \frac{du}{\zeta}$$

## 3. ผลเฉลยของสมการเชิงอนุพันธ์ $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau)\right)$

โดยการประยุกต์ใช้กลุ่มวิเคราะห์หาผลเฉลยของสมการเชิงอนุพันธ์ ทำให้สามารถหาสมการกำหนดของสมการ

(3.1) $$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau)\right)$$

ได้โดยพิจารณาจาก $\widetilde{X}\left[u_t+uu_x-G(u^\tau,u)\right]_{u_t=G(u^\tau,u)-uu_x}\equiv 0$ และสมการกำหนดที่ได้คือ

(3.2) $$\begin{aligned}&u_x^\tau G'\left(u^\tau\left[\eta-\eta^\tau\right]-\xi+\xi^\tau\right)+u_x\left(u^2\eta_x+u\left[\eta_t+\eta_u G-\xi_x\right]-\xi_u G-\xi_t+\zeta\right)\\&+G'\left(G^\tau\left[\eta^\tau-\eta\right]-\zeta^\tau\right)-G^2\eta_u-G\left(u\eta_x+\eta_t-\zeta_u\right)+u\zeta_x+\zeta_t\equiv 0\end{aligned},$$

เมื่อ

$\xi=\xi(x,t,u),\xi^\tau=\xi(x,t-\tau,u^\tau),\eta=\eta(x,t,u),\eta^\tau=\eta(x,t-\tau,u^\tau),\zeta=\zeta(x,t,u),\zeta^\tau=\zeta(x,t-\tau,u^\tau),$
$\xi_x=\frac{\partial\xi}{\partial x}(x,t,u),\xi_t=\frac{\partial\xi}{\partial t}(x,t,u),\xi_u=\frac{\partial\xi}{\partial u}(x,t,u),\eta_x=\frac{\partial\eta}{\partial x}(x,t,u),\eta_t=\frac{\partial\eta}{\partial t}(x,t,u),\eta_u=\frac{\partial\eta}{\partial u}(x,t,u),$
$\zeta_x=\frac{\partial\zeta}{\partial x}(x,t,u),\zeta_t=\frac{\partial\zeta}{\partial t}(x,t,u),\zeta_u=\frac{\partial\zeta}{\partial u}(x,t,u)$ และ $G^\tau=G\left(u(x,t-2\tau)\right)$

ในการหาค่าฟังก์ชันไม่ทราบค่า $\xi,\eta$ และ $\zeta$ เราจะพิจารณาว่า $x,t,u,u^\tau,u_x,u_x^\tau$ เป็นตัวแปรอิสระใดๆ ทำให้สามารถลดทอนรูปของสมการ (3.2) เหลือเพียง

(3.3) $$\xi_1\left(G'u^\tau-G\right)\equiv 0,$$

และได้ว่า $\xi=\xi_1 x+\xi_2,\eta=\eta_1,\zeta=\xi_1 u$ และ $\xi_1,\xi_2,\eta_1$ เป็นค่าคงตัวใดๆ

### 3.1 เคอร์เนล (kernel)

เคอร์เนล เป็นเซตของสมมาตรสำหรับสมการเชิงอนุพันธ์เมื่อ $G$ ฟังก์ชันนัลใดๆ ซึ่งในกรณีนี้ สมการ (3.3) จะเป็นจริงได้เมื่อค่าคงตัว $\xi_1$ มีค่าเป็นศูนย์ ซึ่งทำให้ได้ว่า $\xi$ และ $\eta$ เป็นค่าคงตัวใดๆ และ $\zeta$ มีค่าเป็นศูนย์

กำหนดให้ $\xi = C_1$ และ $\eta = C_2$ เมื่อพิจารณาสมการแคแรกเทอริสติกทำให้ได้ว่า $\dfrac{dx}{C_1} = \dfrac{dt}{C_2} = \dfrac{du}{0}$ โดยทฤษฎีบทของกลุ่มวิเคราะห์ทำให้ได้ว่า ฟังก์ชันนัลใดๆ ของ $\left(u, C_2x - C_1t\right)$ จะมีคุณสมบัติเป็นค่าคงตัวภายใต้แมนิโฟล์ด (manifold) ซึ่งถูกกำหนดโดยสมการเชิงอนุพันธ์ (3.1) และ โดยทฤษฎีบทของการวิเคราะห์ทำให้เราสามารถเขียนตัวแปร $u$ ให้อยู่ในรูปของฟังก์ชันใดๆ ของ $C_2x - C_1t$ ได้ นั่นคือ $u = f\left(C_2x - C_1t\right)$ และค่าที่ได้เป็นรูปแบบของผลเฉลยของสมการ (3.1) ซึ่งเมื่อแทนลงในสมการดังกล่าว จะทำให้แปลงสมการเชิงอนุพันธ์ย่อยชนิดประวิง เป็นสมการเชิงอนุพันธ์สามัญฟังก์ชันนัล (Functional Ordinary Differential Equation - FODE)

$$f'(\theta) = \frac{G\left(f(\theta + C_1\tau)\right)}{C_2 f(\theta) - C_1}$$

เมื่อ $\theta = C_2x - C_1t$

**3.2 ภาคขยายของเคอร์เนล (Extension of Kernel)**

ภาคขยายของเคอร์เนล หมายถึงเซตของสมมาตรสำหรับสมการเชิงอนุพันธ์เมื่อ $G$ เป็นฟังก์ชันนัลเฉพาะบางฟังก์ชันนัล

พบว่าถ้า $G(u^{\tau}) = ku^{\tau}$ โดย $k$ เป็นค่าคงตัวใดๆ จะทำให้สมการ (3.3) เป็นจริง สำหรับกรณีนี้ ทำให้เราหาฟังก์ชันไม่ทราบค่าได้คือ $\xi = \xi_1 x + \xi_2, \eta = \eta_1, \zeta = \xi_1 u$ เมื่อ $\xi_1, \xi_2, \eta_1$ เป็นค่าคงตัวใดๆ สำหรับกรณีนี้ เราจะสามารถหาผลเฉลยของสมการเชิงอนุพันธ์ได้ 2 รูปแบบคือ

**3.2.1 กรณี $\eta_1 = 0$**

สำหรับกรณีนี้เมื่อหาผลเฉลยของสมการแคแรกเทอริสติกและจัดรูปสมการ จะได้ว่า $u = (x + C)f(t)$ เมื่อ $C$ เป็นค่าคงตัวใดๆ และ $f$ เป็นฟังก์ชันใดๆ ของตัวแปร $t$ เป็นผลเฉลยของสมการ (3.1) และแปลงสมการ (3.1) ให้อยู่ในรูปของสมการเชิงอนุพันธ์สามัญประวิง (Delay Ordinary Differential Equation - DODE)

$$f'(t) = kf(t - \tau) - \left[f(t)\right]^2$$

**3.2.2 กรณี $\eta_1 \neq 0$**

สำหรับกรณีนี้เมื่อหาผลเฉลยของสมการแคแรกเทอริสติกและจัดรูปสมการ จะได้ว่า $u = e^{C_1 t} f\left((x + C_2)e^{-C_1 t}\right)$ เมื่อ $C_1, C_2$ เป็นค่าคงตัวใดๆ และ $f$ เป็นฟังก์ชันใดๆ เป็นผลเฉลยของสมการ (3.1) และแปลงสมการ (3.1) ให้อยู่ในรูปของสมการเชิงอนุพันธ์สามัญฟังก์ชันนัล

$$f'(\phi) = \frac{C_1 f(\phi) - kf\left(e^{C_1\tau}\phi\right)}{f(\phi) - C_1\phi}$$

เมื่อ $\phi = (x + C_2)e^{-C_1 t}$

## 4. ผลเฉลยของสมการเชิงอนุพันธ์ $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$

สมการกำหนดของสมการ

(4.1) $$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$$

สามารถหาได้โดยพิจารณาจาก $\widetilde{X}\left(u_t+uu_x-G\left(u^\tau,u\right)\right)\Big|_{u_t=G(u^\tau,u)-uu_x}\equiv 0$ และเพื่อลดความกำกวมในการคำนวณ จะกำหนดให้ $u_t=G-uu_x$ ซึ่งส่งผลให้ $u_{xt}=u_xG_u+u_x^\tau G_{u^\tau}-\left(u_x\right)^2-uu_{xx}$, $u_{tt}=u_tG_u+u_t^\tau G_{u^\tau}-u_xu_t-uu_{xt}$ และ $u_t^\tau=G^\tau-u^\tau u_x^\tau$, เมื่อ $G^\tau=G(u^\tau,u^{\tau\tau})$, $u^{\tau\tau}=u(x,t-2\tau)$, $u_t^\tau=u_t(x,t-\tau)$, $u_x^\tau=u_x(x,t-\tau)$

สมการกำหนดที่ได้จากการคำนวณ คือ

(4.2) $$\begin{aligned}&u_x^\tau G_{u^\tau}\left[u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\right]+u_x\left[u(\eta_t+\eta_xu+\eta_uG)-(\xi_t+\xi_xu+\xi_uG)+\zeta\right]\\&-G(\eta_t+\eta_xu+\eta_uG)+G_{u^\tau}\left[G^\tau(\eta^\tau-\eta)-\zeta^\tau\right]-G_u\zeta+\zeta_t+\zeta_xu+\zeta_uG\equiv 0\end{aligned}$$

จากการคำนวณหาผลเฉลยโดยหลักการเดียวกับตัวอย่างข้างต้น ทำให้ได้

### 4.1 เคอร์เนล

สำหรับกรณีนี้ ผลเฉลยยังคงเป็น $u=f\left(C_2x-C_1t\right)$ เมื่อ $C_1$ และ $C_2$ เป็นค่าคงตัวใดๆ และเมื่อแทนลงในสมการ (4.1) ทำให้แปลงสมการดังกล่าวให้อยู่ในรูปสมการเชิงอนุพันธ์สามัญฟังก์ชันนัล

$$f'(\theta)=\frac{G\left(f(\theta+C_1\tau),f(\theta)\right)}{C_2f(\theta)-C_1},$$

เมื่อ $\theta=C_2x-C_1t$.

### 4.2 ภาคขยายของเคอร์เนล

เนื่องด้วยความซับซ้อนของสมการ ทำให้สามารถวิเคราะห์หาผลเฉลยได้เฉพาะในกรณีของฟังก์ชันนัล $G\left(u^\tau,u\right)=g(u-u^\tau)+H(u)$ เมื่อ $g$ เป็นฟังก์ชันนัลใดๆ ของ $u-u^\tau$ และ $H$ เป็นฟังก์ชันนัลใดๆ ของ $u$

#### 4.2.1 กรณี $G\left(u^\tau,u\right)=g(u-u^\tau)+H(u)$

สำหรับกรณีนี้ $u=f\left(C_2x-C_1t\right)$ เมื่อ $C_1$ และ $C_2$ เป็นค่าคงตัวใดๆ เป็นผลเฉลยของสมการ (4.1) และแปลงสมการ (4.1) ให้อยู่ในรูป

$$f'(\theta)=\frac{g\left(f(\theta)-f(\theta+C_1\tau)\right)+H\left(f(\theta)\right)}{C_2f(\theta)-C_1},$$

เมื่อ $\theta=C_2x-C_1t$.

**4.2.2 กรณี** $G\left(u^{\tau},u\right)=C_{1}(u-u^{\tau})^{2}+H_{1}u^{2}+H_{2}u+H_{3}$ เมื่อ $C_{1},H_{1},H_{2}$ และ $H_{3}$ เป็นค่าคงตัวใดๆ
ผลเฉลยของสมการอยู่ในรูป

$$u=e^{-\int p\ dt}\left[\int qe^{\int p\ dt}\ dt+\mathfrak{F}(x-\psi(t))\right]$$

เมื่อ $p=\lambda\dfrac{C_{5}\cos\lambda t-C_{4}\sin\lambda t}{C_{3}+C_{4}\cos\lambda t+C_{5}\sin\lambda t},q=\lambda\dfrac{C_{8}\cos\lambda t-C_{7}\sin\lambda t}{C_{3}+C_{4}\cos\lambda t+C_{5}\sin\lambda t},\psi(t)=\int\dfrac{C_{6}+C_{7}\cos\lambda t+C_{8}\sin\lambda t}{C_{3}+C_{4}\cos\lambda t+C_{5}\sin\lambda t}dt,$
$\mathfrak{F}$ เป็นฟังก์ชันใดๆ $C_{1},C_{3},C_{4},C_{5},C_{6},H_{1},H_{2},H_{3}$ เป็นค่าคงตัวใดๆ, $H_{1}\neq 0$,
$\lambda=4H_{1}(C_{1}+H_{3})-\left(H_{2}\right)^{2}$ $C_{7}=-\dfrac{H_{2}C_{4}+\lambda C_{5}}{2H_{1}},C_{8}=-\dfrac{H_{2}C_{5}+\lambda C_{4}}{2H_{1}}$

**4.2.3 กรณี** $G\left(u^{\tau},u\right)=C_{1}u+C_{2}u^{\tau}$ เมื่อ $C_{1}$ และ $C_{2}$ เป็นค่าคงตัวใดๆ
ผลเฉลยของสมการอยู่ในรูป $u=\left(\xi_{1}x+\xi_{2}\right)\mathfrak{F}\left(\eta\ln(\xi_{1}x+\xi_{2})-\xi_{1}t\right)$ เมื่อ $\mathfrak{F}$ เป็นฟังก์ชันใดๆ $\xi_{1},\xi_{2},\eta$ เป็นค่าคงตัวใดๆ และแปลงสมการ (4.1) ให้อยู่ในรูป

$$\mathfrak{F}'(\chi)\left[\eta\mathfrak{F}(\chi)-1\right]+\left[\mathfrak{F}(\chi)\right]^{2}=C_{1}\mathfrak{F}(\chi)+C_{2}\mathfrak{F}(\chi+\xi_{1}\tau)$$

เมื่อ $\chi=\eta\ln(\xi_{1}x+\xi_{2})-\xi_{1}t$

**4.2.4 กรณี** $G\left(u^{\tau},u\right)=C_{1}(u-u^{\tau})+H_{3}$ เมื่อ $C_{1}$ และ $H_{3}\neq 0$ เป็นค่าคงตัวใดๆ
สามารถหาผลเฉลยได้จากสมการแคแรกเทอริสติก

$$\frac{dx}{\xi_{1}x+C_{3}\cos\rho t+C_{4}\sin\rho t}=\frac{dt}{\eta}=\frac{du}{\xi_{1}u-C_{3}\rho\sin\rho t+C_{4}\rho\cos\rho t}$$

เมื่อ $\xi_{1},C_{3},C_{4}$ เป็นค่าคงตัวใดๆ และ $\rho=\sqrt{H_{3}\xi_{1}}$

**4.2.5 กรณี** $G\left(u^{\tau},u\right)=C_{1}(u-u^{\tau})+H_{1}u^{2}$ เมื่อ $C_{1}$ และ $H_{1}\neq 0$ เป็นค่าคงตัวใดๆ
มีผลเฉลยคือ

$$u=\left(\xi_{1}e^{H_{1}x}+\xi_{2}\right)\mathfrak{F}\left(\frac{\eta}{\xi_{2}H_{1}}\ln\left(\frac{\xi_{1}e^{H_{1}x}}{\xi_{1}e^{H_{1}x}+\xi_{2}}\right)-t\right)$$

เมื่อ $\mathfrak{F}$ เป็นฟังก์ชันใดๆ $\xi_{1},\xi_{2},\eta$ เป็นค่าคงตัวใดๆ และแปลงสมการ (4.1) ให้อยู่ในรูป

$$\mathfrak{F}'(\Theta)=C_{1}\frac{\mathfrak{F}(\Theta)-\mathfrak{F}(\Theta+\tau)+H_{1}\xi_{2}\left[\mathfrak{F}(\Theta)\right]^{2}}{\eta\mathfrak{F}(\Theta)-1}$$

เมื่อ $\Theta=\dfrac{\eta}{\xi_{2}H_{1}}\ln\left(\dfrac{\xi_{1}e^{H_{1}x}}{\xi_{1}e^{H_{1}x}+\xi_{2}}\right)-t$

**4.2.6 กรณี** $G\left(u^{\tau},u\right)=C_1(u-u^{\tau})+H_1u^2+H_3$ เมื่อ $C_1$ และ $H_1\neq 0, H_3\neq 0$ เป็นค่าคงตัวใดๆ สามารถหาผลเฉลยได้จากสมการแคแรกเทอริสติก

$$\frac{dx}{e^{H_1x}\left(C_3\cos\rho t+C_4\sin\rho t\right)+\xi_2}=\frac{dt}{\eta}=\frac{du}{e^{H_1x}\left[H_1u\left(C_3\cos\rho t+C_4\sin\rho t\right)+\rho\left(-C_3\sin\rho t+C_4\cos\rho t\right)\right]}$$

เมื่อ $\xi_2, C_3, C_4$ เป็นค่าคงตัวใดๆ และ $\rho=\sqrt{H_1H_3}$

**4.2.7 กรณีอื่นๆ**

สำหรับค่า $G\left(u^{\tau},u\right)=g(u-u^{\tau})+H(u)$ ในกรณีนอกเหนือจากกรณี 4.2.2-4.2.7 สมการ (4.1) จะมีผลเฉลยในรูป $u=f\left(C_2x-C_1t\right)$ เมื่อ $C_1$ และ $C_2$ เป็นค่าคงตัวใดๆ

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# ภาคผนวก

ภาคผนวก ก

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# SYMMETRY ANALYSIS ON $\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = G\left(u(x, t-\tau)\right)$

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**Abstract.** Equation $\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = G\left(u(x, t-\tau)\right)$ is a delay partial differential equation with an arbitrary functional $G$. Group analysis method is applied to find symmetries of the equation and to make group classification. Representations of analytical solutions and reduced equations are obtained from the symmetries.

## 1. Introduction

Consider delay partial differential equation with delay $\tau > 0$

$$\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = G\left(u(x, t-\tau)\right). \tag{1.1}$$

For simplicity, notation $u^\tau$ will be used to denote $u(x, t-\tau)$, $u$ denote $u(x,t)$ and $u_x$, $u_t$ mean first partial derivatives of $u$ with respect to $x$ and $t$, respectively. Equation (1.1) can be simply written as

$$u_t + uu_x = G(u^\tau). \tag{1.2}$$

Equation (1.2) is similar to Hopf or inviscid Burgers' equation [1]. However, (1.2) has a delay term, which makes the equation difficult to be solved [2]. Applications of delay differential equations can be found in [2, 3, 4, 5].

One of the powerful methods for finding analytical solutions of differential equations is *group analysis*. Group analysis was introduced by Shopus Lie in 1895 [6, 7, 8]. Group analysis is applied for finding analytical solutions of many types of ODEs and PDEs [8]. Later, it was developed to apply to integro-differential equations [8], delay differential equations [3], functional differential equations [4, 5] and stochastic differential equations [9].

In this manuscript, group analysis is applied to find symmetries of equation (1.2). Classification of (1.2) with respect to groups of symmetries admitted by the equation is done. Representations of analytical solutions and reduced equations are also presented.

## 2. Applications of group analysis to delay differential equations

Let $\varphi : \Omega \times \triangle \to \Omega$ be a transformation where $\Omega$ is a set of variables $(x,t,u)$ and $\triangle \subset \mathbb{R}$ is a symmetric interval with respect to zero. Variable $\varepsilon$ is considered as a parameter of transformation $\varphi$, which transforms variable $(x,t,u)$ to $(\bar{x}, \bar{t}, \bar{u})$ of the same space. Let $\varphi(x,t,u;\varepsilon)$ be denoted by $\varphi_\varepsilon(x,t,u)$. The set of functions $\varphi_\varepsilon$ forms *a one-parameter transformation group of space* $\Omega$ if the following properties hold [6, 7, 8]:

(1) $\varphi_0(x,t,u) = (x,t,u)$ for any $(x,t,u) \in \Omega$;
(2) $\varphi_{\varepsilon_1}\left(\varphi_{\varepsilon_2}(x,t,u)\right) = \varphi_{\varepsilon_1+\varepsilon_2}(x,t,u)$ for any $\varepsilon_1, \varepsilon_2, \varepsilon_1+\varepsilon_2 \in \triangle$ and $(x,t,u) \in \Omega$;
(3) if $\varphi_\varepsilon(x,t,u) = (x,t,u)$ for any $(x,t,u) \in \Omega$, then $\varepsilon = 0$.

The other notations $\bar{x} = \varphi^x(x,t,u;\varepsilon)$, $\bar{t} = \varphi^t(x,t,u;\varepsilon)$, $\bar{u} = \varphi^u(x,t,u;\varepsilon)$ are used as the same meaning as $\varphi_\varepsilon(x,t,u) = (\bar{x}, \bar{t}, \bar{u})$. The transformed variable $u$ with delay term and it's derivatives are defined by $\bar{u}^\tau = \bar{u}(\bar{x}, \bar{t}-\tau)$ and $\bar{u}_{\bar{x}} = \partial\bar{u}/\partial\bar{x}$, $\bar{u}_{\bar{t}} = \partial\bar{u}/\partial\bar{t}$, respectively. Suppose that the transformations map a solution $u(x,t)$ of differential equation

$$F(x,t,u,u^\tau,u_x,u_t) = 0 \tag{2.1}$$

into a solution of the same equation. These transformations are called *symmetries*. In [5], it is shown that for a symmetry

$$\frac{\partial F(\bar{x}, \bar{t}, \bar{u}, \bar{u}^\tau, \bar{u}_{\bar{x}}, \bar{u}_{\bar{t}})}{\partial \varepsilon}\bigg|_{\varepsilon=0,\ (2.1)} = \tilde{X}F(x,t,u,u^\tau,u_x,u_t)\Big|_{(2.1)} \equiv 0. \tag{2.2}$$

The operator $\tilde{X}$ is defined by

$$\tilde{X} = (\zeta - u_x\xi - u_t\eta)\partial_u + (\zeta^\tau - u_x^\tau\xi^\tau - u^\tau u_t^\tau\eta)\partial_{u^\tau} + \zeta^{u_x}\partial_{u_x} + \zeta^{u_t}\partial_{u_t},$$

where

$$\xi(x,t,u) = \frac{\partial\varphi^x}{\partial\varepsilon}(x,t,u;0), \quad \eta(x,t,u) = \frac{\partial\varphi^t}{\partial\varepsilon}(x,t,u;0),$$

$$\zeta(x,t,u) = \frac{\partial\varphi^u}{\partial\varepsilon}(x,t,u;0), \quad \xi^\tau = \xi(x,t-r,u^\tau),$$

$$\eta^\tau = \eta(x,t-r,u^\tau), \quad \zeta^\tau = \zeta(x,t-r,u^\tau),$$

$$\zeta^{u_x} = D_x(\zeta - u_x\xi - u_t\eta), \quad \zeta^{u_t} = D_t(\zeta - u_x\xi - u_t\eta),$$

$$\begin{aligned} D_x &= \partial_x + u_x\partial_u + u_x^\tau\partial_u^\tau + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \dots, \\ D_t &= \partial_t + u_t\partial_u + u_t^\tau\partial_u^\tau + u_{xt}\partial_{u_x} + u_{tt}\partial_{u_t} + \dots. \end{aligned}$$

The operator $\tilde{X}$ is called *a canonical Lie–Bäcklund infinitesimal generator* of a symmetry. Equation (2.2) is called *a determining equation.* Lie's theory [6, 7, 8] shows that there is a one-to-one correspondence between the generator and a symmetry. This generator is also equivalent to *an infinitesimal generator* [7]

$$X = \xi\partial_x + \eta\partial_t + \zeta\partial_u. \tag{2.3}$$

By the theory of existence of a solution of a delay differential equation, the initial value problem has a particular solution corresponding to a particular initial value. Because initial values are arbitrary, variables $u$, $u^\tau$ and their derivatives can be considered as arbitrary elements. Since every transformed-solution $\bar{u}(\bar{x},\bar{t})$ is a solution of equation (2.1), the determining equation must be identical to zero. Thus, if determining equation (2.2) is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish. In order to solve a determining equation, one solves the several equations of these coefficients. This method is called *splitting the determining equation.* Unknown functions $\xi, \eta$ and $\zeta$ can be obtained from this process.

## 3. Symmetries of (1.2)

We define determining equation for $u_t + uu_x = G(u^\tau)$ by letting $F = u_t + uu_x - G(u^\tau)$, then

$$\tilde{X}(u_t + uu_x - G(u^\tau))\Big|_{u_t = G - uu_x} \equiv 0. \tag{3.1}$$

Splitting determining equation (3.1) with respect to $u_x^\tau, u_x$ and later with respect to $u^\tau, u$, the equation is simplified to

$$\xi_1\left(G'u^\tau - G\right) = 0, \tag{3.2}$$

where the unknown function $\xi, \eta$ and $\zeta$ are

$$\xi = \xi_1 x + \xi_2, \quad \eta = \eta_1, \quad \zeta = \xi_1 u.$$

Here, $\xi_1, \xi_2, \eta_1$ are constants.

**3.1. Kernel.** The set of symmetries, which are admitted for any functional appeared in the equation is called *a kernel* of admitted generators. In this case, $G'u^\tau$ and $G$ are arbitrary. This implies that coefficients of $G'u^\tau$ and $G$ vanish, $\xi_1 = 0$. Unknown functions $\xi, \eta, \zeta$ are

$$\xi = \xi_2, \ \eta = \eta_1, \ \zeta = 0.$$

For the sake of convenience, let arbitrary constants $\xi_2$, $\eta_1$ be denoted by $C_1$, $C_2$, respectively. The obtained infinitesimal generator is

$$X = C_1\partial_x + C_2\partial_t. \tag{3.3}$$

This generator is admitted for any functional $G$. By Lie's theory, symmetry is derived from the infinitesimal generator [7, 8]:

$$\bar{x} = x + C_1\varepsilon, \ \bar{t} = t + C_2\varepsilon, \ \bar{u} = u. \tag{3.4}$$

**3.2. Extensions of the kernel.** *Extensions* are symmetries for the particular functional $G$ only. In this case, there exists $G(u^\tau)$ satisfying equation (3.2). Here, the extension of kernel (3.3) will be considered. Since $\xi = \xi_2$, $\eta = \eta_1$, $\zeta = 0$ are considered in the case of kernel, then functions $\xi, \eta$ and $\zeta$ for this case are

$$\xi = \xi_1 x, \quad \eta = 0, \quad \zeta = \xi_1 u.$$

For the nontrivial case, $\xi_1 \neq 0$ and a solution of equation (3.2) is

$$G(u^\tau) = ku^\tau,$$

where $k$ is a nonzero arbitrary constant. For the sake of convenience, let $\xi_1$ be denoted by $C_3$. The extension of kernel (3.3) is

$$X = C_3\left(x\partial_x + u\partial u\right). \tag{3.5}$$

The symmetry derived from $X$ is

$$\bar{x} = xe^{C_3\varepsilon}, \quad \bar{t} = t, \quad \bar{u} = ue^{C_3\varepsilon}. \tag{3.6}$$

## 4. Representations of solutions

*Invariants* are functions such that their values do not change by symmetries [6, 7, 8], i.e.

$$\Psi(x,t,u) = \Psi(\bar{x},\bar{t},\bar{u}),$$

where $\Psi$ is an invariant for a symmetry $\varphi_\varepsilon(x,t,u) = (\bar{x},\bar{t},\bar{u})$. If $X = \xi\partial_x + \eta\partial_t + \zeta\partial_u$ is an infinitesimal generator for a symmetry $\varphi_\varepsilon$, then

$$X\Psi(x,t,u) = 0. \tag{4.1}$$

Invariants of symmetries are found by solving differential equation (4.1) [7]. The system of characteristic equations for the infinitesimal generator (2.3) is

$$\frac{dx}{\xi} = \frac{dt}{\eta} = \frac{du}{\zeta}.$$

Representations of solutions are obtained from the invariants.

**4.1. Representations of solutions for equation (1.2) with arbitrary functional $G$.** For infinitesimal generator (3.3), the system of characteristic equations is

$$\frac{dx}{C_1} = \frac{dt}{C_2} = \frac{du}{0}.$$

Solving the system of equations, the invariants are $u$ and $C_2x - C_1t$. For constructing a representation of solution [6, 7], the relation between these two invariants is

$$u = f_1(C_2x - C_1t), \tag{4.2}$$

where $f_1$ is an arbitrary function. We call $u$ in equation (4.2) *a representation of solution of equation (1.2) for the infinitesimal generator (3.3).*

**4.2. Representations of solutions for $G = ku^\tau$.** The infinitesimal generator for equation

$$u_t + uu_x = ku^\tau \tag{4.3}$$

is the linear combination of kernel (3.3) and extension (3.5) :

$$X = \left(C_1 + C_3x\right)\partial_x + C_2\partial_t + C_3u\partial_u. \tag{4.4}$$

Thus, the system of characteristic equations for infinitesimal generator (4.4) is

$$\frac{dx}{C_1 + C_3x} = \frac{dt}{C_2} = \frac{du}{C_3u}.$$

Let $C_2 = 0$. In this case, the invariants are $t$ and $\dfrac{u}{x + C_1/C_3}$.

Since $C_1$ and $C_3$ are arbitrary and $C_3 \neq 0$, for the sake of convenience, we denote $C_4 = C_1/C_3$. *The representation of a solution for equation (1.2) with the functional* $G = ku^\tau$ is

$$u = (x + C_4)f_2(t), \tag{4.5}$$

where $f_2$ is an arbitrary function and $C_4$ is an arbitrary constant.

Let $C_2 \neq 0$. In this case, the invariants are $(x + C_4)\, e^{-(C_3/C_2)t}$ and $ue^{-(C_3/C_2)t}$. *The representation of a solution for equation (1.2) with the functional* $G = ku^\tau$ is

$$u = e^{(C_3/C_2)t} f_3 \left( (x + C_4)\, e^{-(C_3/C_2)t} \right), \tag{4.6}$$

where $f_3$ is an arbitrary function. Let $C_5 = C_3/C_2$, equation (4.6) is simply written as

$$u = e^{C_5 t} f_3 \left( (x + C_4)\, e^{-C_5 t} \right).$$

## 5. Reduced equations

Representations of solutions obtained in section 4 simplify equation (1.2). They reduce the number of independent variables appearing in the equation. Substituting the representations into the equation, equation (1.2) is reduced to an ordinary differential equation, which is called *a reduced equation*.

**5.1.** $u = f_1(C_2 x - C_1 t)$**.** Substituting $u$ into equation (1.2), the equation is transformed to

$$-C_1 f_1'(\theta) + C_2 f_1(\theta) f_1'(\theta) = G(f_1(\theta + C_1\tau)),$$

where $\theta = C_2 x - C_1 t$. This equation may be written in the other form,

$$f_1'(\theta) = \frac{G(f_1(\theta + C_1\tau))}{C_2 f_1(\theta) - C_1}. \tag{5.1}$$

**5.2.** $u = (x + C_4) f_2(t)$**.** Substituting $u$ into equation (4.3), the equation is transformed to

$$(x + C_4) f_2'(t) + (x + C_4)\left[f_2(t)\right]^2 = k(x + C_4) f_2(t - \tau).$$

It can be simplified to

$$f_2'(t) = k f_2(t - \tau) - \left[f_2(t)\right]^2. \tag{5.2}$$

**5.3.** $u = e^{C_5 t} f_3\left((x + C_4)\, e^{-C_5 t}\right)$**.** Substitute $u$ into equation (4.3), the equation is transform to

$$C_5 f_3(\phi) - C_5 \phi f_3'(\phi) + f_3(\phi) f_3'(\phi) = k e^{-C_5 \tau} f_3\left(e^{C_5 \tau}\phi\right),$$

where $\phi = (x + C_4)\, e^{-C_5 t}$. The other form of the equation is

$$f_3'(\phi) = \frac{C_5 f_3(\phi) - k e^{-C_5\tau} f_3\left(e^{C_5\tau}\phi\right)}{f_3(\phi) - C_5 \phi}. \tag{5.3}$$

Note that equation (5.1), (5.2) and (5.3) are not typical ODEs, they are functional ODEs [5].

## 6. Conclusion

Symmetries, representation of solutions of equation (1.2) and reduced equations are presented in section 3, 4 and 5, respectively. Equation (1.2) is classified with respect to the symmetries into the case of $G(u^\tau) = ku^\tau$ (symmetry is (3.6)) and otherwise (symmetry is (3.4)). By the review literature, there are not many examples of applications of group analysis to delay differential equations. This manuscript presents another example.

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**ภาคผนวก ข**

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# ANALYSIS ON $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$

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**Abstract.** *Equation* $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$ *is a delay partial differential equation with an arbitrary functional* $G$. *This delay partial differential equation is more general than* $\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau)\right)$ *which has been applied group analysis to find representations of analytical solutions* [3]. *Application of group analysis to the equation and group classification of representations of solutions where* $G=g\left(u(x,t)-u(x,t-\tau)\right)+H(u)$, $g$ *and* $H$ *are arbitrary functions, are presented in the article.*

## 1 Introduction

Consider delay partial differential equation (DPDE) with delay $\tau>0$

(1.1) $$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=G\left(u(x,t-\tau),u(x,t)\right)$$

For the simplicity, notation $u^\tau$ will be used to denote $u(x,t-\tau)$, $u$ denotes $u(x,t)$ and $u_x$, $u_t$ mean first partial derivatives of $u$ with respect to $x$ and $t$, respectively. Equation (1.1) can be simply written as

(1.2) $$u_t+uu_x=G\left(u^\tau,u\right).$$

Equation (1.2) is similar to Hopf or inviscid Burgers' equation [1]. However, equation (1.2) has a delay term, which makes the equation difficult to be solved [2]. Applications of delay differential equations can be found in [2,3,4,5,6]. The representations of solutions for the particular case of equation (1.2),

(1.3) $$u_t+uu_x=G\left(u^\tau\right).$$

has been found [3]. These solutions were obtained by applying group analysis method [7,8,9] to the equation. Group analysis also classifies equation (1.3) w.r.t. symmetries into two cases, arbitrary functional $G$ and $G=ku^\tau$ :

- **For arbitrary functional** $G(u^\tau)$

The solution is $u=f_1(C_2x-C_1t)$, where $f_1$ is an arbitrary function, $C_1,C_2$ are arbitrary constants. The solution reduces equation (1.3) into a functional ordinary differential equation (FODE) $f_1{}'(\theta)=\frac{G\left(f_1(\theta+C_1\tau)\right)}{C_2f_1(\theta)-C_1}$, where $\theta=C_2x-C_1t$.

- **For particular functional** $G(u^\tau)=ku^\tau$, where $k$ is an arbitrary constant.

For this case, equation (1.3) has two possible forms of representation of solutions, i.e.

1. $u=(x+C_3)f_2(t)$, where $f_2$ is an arbitrary function, $C_3$ is an arbitrary constant.

This solution reduces the equation into delay ordinary differential equation (DODE) $f_2{}'(t)=kf_2(t-\tau)-\left[f_2(t)\right]^2$.

2. $u=e^{C_5t}f_3\left((x+C_4)e^{-C_5t}\right)$, where $f_2$ is an arbitrary function, $C_4,C_5$ are arbitrary constants.

By this solution, equation (1.3) can be simplified to FODE $f_3{}'(\phi)=\frac{C_5f_3(\phi)-kf_3\left(e^{C_5\tau}\phi\right)}{f_3(\phi)-C_5\phi}$, where $\phi=(x+C_4)e^{-C_5t}$.

In this article, group analysis is applied to find symmetries of equation (1.2) which is more general than (1.3). However, for the sake of simplicity, equation (1.2) is considered for the case $G=g\left(u(x,t)-u(x,t-\tau)\right)+H(u)$ only, where $g$ and $H$ are arbitrary functions. Classification of the equation with respect to groups of symmetries admitted by the equation are presented in the following sections.

## 2 Applications of group analysis to delay differential equations

By the theory of group analysis, a *symmetry* of equation (1.2) is defined as the transformation $\varphi:\Omega\times\Delta\to\Omega$ which transforms a solution of the differential equation to a solution of the same equation, where $\Omega$ is a set of variables $(x,t,u)$ and $\Delta\subset\mathbb{R}$ is a symmetric interval with respect to zero. Variable $\varepsilon$ is considered as a parameter of transformation $\varphi$, which transforms variable $(x,t,u)$ to new variable $(\overline{x},\overline{t},\overline{u})$ of the same space. Let $\varphi(x,t,u;\varepsilon)$ be denoted by $\varphi_\varepsilon(x,t,u)$. The set of functions $\varphi_\varepsilon$ forms *a one-parameter transformation group of space* $\Omega$ if the following properties hold [7,8,9]:

(1) $\varphi_0(x,t,u)=(x,t,u)$ for any $(x,t,u)\in\Omega$;

(2) $\varphi_{\varepsilon_1}\left(\varphi_{\varepsilon_2}(x,t,u)\right)=\varphi_{\varepsilon_1+\varepsilon_2}(x,t,u)$ for any $\varepsilon_1,\varepsilon_2,\varepsilon_1+\varepsilon_2\in\Delta$ and $(x,t,u)\in\Omega$;

(3) if $\varphi_\varepsilon(x,t,u)=(x,t,u)$ for any $(x,t,u)\in\Omega$, then $\varepsilon=0$.

The other notations $\overline{x}=\varphi^x(x,t,u;\varepsilon),\overline{t}=\varphi^t(x,t,u;\varepsilon),\overline{u}=\varphi^u(x,t,u;\varepsilon)$ are used as the same meaning as $\varphi_\varepsilon(x,t,u)=(\overline{x},\overline{t},\overline{u})$. The transformed variable $u$ with delay term and its derivatives are defined by $\overline{u}^\tau=\overline{u}(\overline{x},\overline{t}-\tau)$ and $\overline{u}_{\overline{x}}=\partial\overline{u}/\partial\overline{x},\overline{u}_{\overline{t}}=\partial\overline{u}/\partial\overline{t}$, respectively. Consider DPDE

(2.1) $$F(x,t,u,u^\tau,u_x,u_t)=0.$$

[6] shows the derivative of an equation, with the transformed variables $\overline{x},\overline{t},\overline{u}$ and derivatives $\overline{u}_{\overline{x}},\overline{u}_{\overline{t}}$, with respect to parameter $\varepsilon$ vanishes if the transformation is symmetry :

(2.2) $$\left.\frac{\partial F(\overline{x},\overline{t},\overline{u},\overline{u}^\tau,\overline{u}_{\overline{x}},\overline{u}_{\overline{t}})}{\partial\varepsilon}\right|_{\varepsilon=0,(2.1)}=\left.\widetilde{X}F(x,t,u,u^\tau,u_x,u_t)\right|_{(2.1)}\equiv 0.$$

The operator $\widetilde{X}$ is defined by $\widetilde{X}=(\zeta-u_x\xi-u_t\eta)\partial_u+(\zeta^\tau-u_x^\tau\xi^\tau-u_t^\tau\eta^\tau)\partial_{u^\tau}+\zeta^{u_x}\partial_{u_x}+\zeta^{u_t}\partial_{u_t}$, where

$$\xi(x,t,u)=\frac{\partial\varphi^x}{\partial\varepsilon}(x,t,u;0),\eta(x,t,u)=\frac{\partial\varphi^t}{\partial\varepsilon}(x,t,u;0),\zeta(x,t,u)=\frac{\partial\varphi^u}{\partial\varepsilon}(x,t,u;0),$$

$$\xi^\tau=\xi(x,t-\tau,u^\tau),\eta^\tau=\eta(x,t-\tau,u^\tau),\zeta^\tau=\zeta(x,t-\tau,u^\tau),$$

$$\zeta^{u_x}=D_x(\zeta-u_x\xi-u_t\eta),\zeta^{u_t}=D_t(\zeta-u_x\xi-u_t\eta),$$

$$D_x=\partial_x+u_x\partial_u+u_x^\tau\partial_{u^\tau}+u_{xx}\partial_{u_x}+u_{xt}\partial_{u_t}+\dots,$$

$$D_t=\partial_t+u_t\partial_u+u_t^\tau\partial_{u^\tau}+u_{xt}\partial_{u_x}+u_{tt}\partial_{u_t}+\dots,$$

The operator $\widetilde{X}$ is is called *a canonical Lie-Bäcklund infinitesimal generator* of a symmetry. Equation (2.2) is called *a determining equation* (DME). Since $\left.\widetilde{X}F(x,t,u,u^\tau,u_x,u_t)\right|_{(2.1)}\equiv 0$, we say that the operator $\widetilde{X}$ is *admitted* by equation (2.1) or equation (2.1) *admits* the operator $\widetilde{X}$. Lie's theory [7,8,9] shows the generator is one-to-one correspondent to the symmetry. This generator is also equivalent to *an infinitesimal generator* [8]

(2.3) $$X=\xi\partial_x+\eta\partial_t+\zeta\partial_u.$$

## 3 Finding and solving the determining equation

The DME for $u_t+uu_x=G(u^\tau,u)$ can be found by letting $F=u_t+uu_x-G(u^\tau,u)$ and substitute it into equation (2.2),

(3.1) $$\left.\widetilde{X}\left(u_t+uu_x-G(u^\tau,u)\right)\right|_{u_t=G(u^\tau,u)-uu_x}\equiv 0.$$

By letting $u_t=G-uu_x$ so $u_{xt}=u_xG_u+u_x^\tau G_{u^\tau}-(u_x)^2-uu_{xx}$, $u_{tt}=u_tG_u+u_t^\tau G_{u^\tau}-u_xu_t-uu_{xt}$ and $u_t^\tau=G^\tau-u^\tau u_x^\tau$, where

where $G^\tau=G(u^\tau,u^{\tau\tau})$, $u^{\tau\tau}=u(x,t-2\tau)$, $u_t^\tau=u_t(x,t-\tau)$, $u_x^\tau=u_x(x,t-\tau)$. Thus DME (3.1) becomes

(3.2) $$\begin{aligned}&u_x^\tau G_{u^\tau}\left[u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\right]+u_x\left[u(\eta_t+\eta_xu+\eta_uG)-(\xi_t+\xi_xu+\xi_uG)+\zeta\right]\\&-G(\eta_t+\eta_xu+\eta_uG)+G_{u^\tau}\left[G^\tau(\eta^\tau-\eta)-\zeta^\tau\right]-G_u\zeta+\zeta_t+\zeta_xu+\zeta_uG\equiv 0\end{aligned}.$$

By the theory of existence of a solution of a delay differential equation, the initial value problem has a particular solution corresponding to a particular initial value. Because initial values are arbitrary, variables $u$, $u^\tau$ and their derivatives can be considered as arbitrary elements. Since every transformed-solution $\overline{u}(\overline{x},\overline{t})$ is a solution of equation (2.1), the DME must be identical to zero. Thus, if DME (2.2) is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish. In order to solve a DME, one solves the several equations of these coefficients. This method is called *splitting the DME*. Unknown functions $\xi$, $\eta$ and $\zeta$ can be obtained from this process.

By splitting equation (3.2) with respect to $u_x^\tau$, one obtains $G_{u^\tau}\left[u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\right]\equiv 0$. Since equation (1.2) is considered as a DPDE, it is assumed $G_{u^\tau}\neq 0$. The equation is simplified to

(3.3) $$u^\tau(\eta-\eta^\tau)+\xi^\tau-\xi\equiv 0$$

By the assumption $\xi$ and $\eta$ depend on variables $x,t,u$ while $\xi^\tau$ and $\eta^\tau$ depend on $x,t,u^\tau$, if one differentiates equation (3.3) w.r.t. $u$, the derivative becomes $u^\tau\eta_u-\xi_u\equiv 0$. Splitting the equation w.r.t $u^\tau$ implies $\xi_u=0$ and $\eta_u=0$, which means $\xi$ and $\eta$ do not depend on $u$. By the similar structure of $\xi^\tau$ and $\xi$, and $\eta^\tau$ and $\eta$, both $\xi^\tau$ and $\eta^\tau$ depend on only variables $x$ and $t$. Equation (3.3) can be split again w.r.t. $u^\tau$ which implies $\xi^\tau(x,t)=\xi(x,t)$ and $\eta^\tau(x,t)=\eta(x,t)$. The conditions obtained mean $\xi$ and $\eta$ are periodic functions w.r.t. $t$ with period $\tau$, i.e.

(3.4) $$\xi(x,t-\tau)=\xi(x,t),\quad \eta(x,t-\tau)=\eta(x,t).$$

Again, splitting the DME w.r.t. $u_x$, one gets

(3.5) $$\zeta=\xi_t+\xi_x u-u(\eta_t+\eta_x u),\quad \zeta^\tau=\xi_t+\xi_x u^\tau-u^\tau(\eta_t+\eta_x u^\tau).$$

Substitute $\xi,\eta,\zeta$ and $\zeta^\tau$ into the DME and differentiate it with respect to $u^\tau$,

(3.6) $$\begin{aligned}&\xi_t\left(-\left[G_{uu^\tau}+G_{u^\tau u^\tau}\right]\right)+\xi_x\left(-\left[G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau\right]\right)+\\&\eta_t\left(G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau-G_{u^\tau}\right)+\eta_x\left(G_{uu^\tau}u^2+G_{u^\tau u^\tau}(u^\tau)^2-3G_{u^\tau}u+2G_{u^\tau}u^\tau\right)\equiv 0\end{aligned}$$

Here if we consider equation (3.6) as $\xi_t\mathsf{A}+\xi_x\mathsf{B}+\eta_t\mathsf{C}+\eta_x\mathsf{D}\equiv 0$, which may be written in a vector form as

(3.7) $$\langle\xi_t,\xi_x,\eta_t,\eta_x\rangle\cdot\langle\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\rangle\equiv 0,$$

where $\mathsf{A}=-\left[G_{uu^\tau}+G_{u^\tau u^\tau}\right]$, $\mathsf{B}=-\left[G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau\right]$, $\mathsf{C}=G_{uu^\tau}u+G_{u^\tau u^\tau}u^\tau-G_{u^\tau}$ and $\mathsf{D}=G_{uu^\tau}u^2+G_{u^\tau u^\tau}(u^\tau)^2-3G_{u^\tau}u+2G_{u^\tau}u^\tau$, we are able to classify equation (1.2) as the followings.

### 3.1 The kernel of admitted Lie groups

The set of symmetries, which are admitted for any functional appeared in the equation is called a *kernel* of admitted generators. Assume equation (3.6) is valid for any functional $G$. Since $G$, $G_{u^\tau}$, $G_{uu^\tau}$, $G_{u^\tau u^\tau}$ vary arbitrarily, the set spanned by $\langle\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\rangle$ has dimension 4. Thus $\langle\xi_t,\xi_x,\eta_t,\eta_x\rangle$ must be a zero vector, i.e. all of $\xi_t,\xi_x,\eta_t,\eta_x$ vanish. This implies $\xi$ and $\eta$ are constants and $\zeta$ is zero. Let $\xi$ and $\eta$ be denoted by $C_1$ and $C_2$, respectively. The infinitesimal generator admitted by equation (1.2) is $X=C_1\partial_x+C_2\partial_t$. By the theory from group analysis, the characteristic equations $\dfrac{dx}{\xi}=\dfrac{dt}{\eta}=\dfrac{du}{\zeta}$ imply $u=f(C_2x-C_1t)$ is a representation of a solution. It reduces equation (1.2) into FODE

$$f'(\theta)=\frac{G\left(f(\theta),f(\theta+C_1\tau)\right)}{C_2f(\theta)-C_1},$$

where $\theta=C_2x-C_1t$.

### 3.2 Extension of the kernel

*Extensions* are symmetries for the particular functional $G$. Here, for the sake of simplicity, case $\mathsf{A}=0$ is considered only. For this case, it implies

(3.8) $$G(u,u^\tau)=g(u-u^\tau)+H(u),$$

where $g$ is an arbitrary function of $u-u^\tau$ such that $\dfrac{\partial g}{\partial u^\tau}\neq 0$ (or $g'\neq 0$) and $H$ is an arbitrary function of variable $u$. Equation (3.6) is reduced into the form

(3.9) $$\xi_x(u-u^\tau)g''+\eta_t\left(g'-[u-u^\tau]g''\right)+\eta_x\left(-\left[u^2-(u^\tau)^2\right]g''+[3u-2u^\tau]g'\right)\equiv 0.$$

Equation (3.9) can be considered as a vector form $\langle\xi_x,\eta_t,\eta_x\rangle\cdot\langle A,B,C\rangle\equiv 0$, where $A=(u-u^\tau)g''$, $B=g'-[u-u^\tau]g''$ and $C=-\left[u^2-(u^\tau)^2\right]g''+[3u-2u^\tau]g'$. Let $\mathbb{V}$ be the set spanned by vector $\langle A,B,C\rangle$. All possible cases which make equation (3.9) valid are considered according to the dimension of $\mathbb{V}$.

3.2.1 **dim** $\mathbb{V}$ **=3.** This condition means vector $\langle \xi_x, \eta_t, \eta_x \rangle$ must be a zero vector, i.e. $\xi_x, \eta_t, \eta_x$ vanish. Thus the DME is simplified to $-H'(u)\xi'(t) + \xi''(t) = 0$. The derivative of the DME w.r.t. $u$ is $-H''(u)\xi'(t) = 0$.

**- Case** $H''(u) = 0$, Here $H(u) = H_1 u + H_2$ is a solution of the equation, where $H_1, H_2$ are arbitrary constants. However, by the arbitrariness of function $g$, $H_2$ can be omitted. The DME is $-H_1\xi'(t) + \xi''(t) = 0$, which has $\xi = C_1 e^{H_1 t} + C_2$ as a solution, where $C_1, C_2$ are arbitrary constants. The periodic condition (3.4) of $\xi$ implies $C_1 e^{H_1(t-\tau)} + C_2 = C_1 e^{H_1 t} + C_2$. The condition is valid for $H_1 = 0$ or $C_1 = 0$. For this case $\xi$ must be a constant.

**- Case** $H''(u) \neq 0$. The equation immediately implies $\xi$ is a constant.

Both two cases show equation $G(u, u^\tau) = g(u - u^\tau) + H(u)$ admits $X = \xi\partial_x + \eta\partial_t$, where $\xi$ and $\eta$ are arbitrary constants and $g, H$ are arbitrary functions. For **dim** $\mathbb{V}$ **=3**, it has the same solution with the kernel case.

3.2.2 **dim** $\mathbb{V}$ **=2.** This condition means there exists a constant vector $\langle \alpha, \beta, \gamma \rangle \neq 0$ which is orthogonal to set $\mathbb{V}$, i.e. $\langle \alpha, \beta, \gamma \rangle \cdot \langle A, B, C \rangle = \alpha A + \beta B + \gamma C = 0$. By changing of variable $z = (u - u^\tau)$, the equation is derived to $z(\alpha - \beta + \gamma z)g'' + (\beta + 3\gamma z)g' + u^\tau \gamma(-2zg'' + g') = 0$. Splitting the equation w.r.t. $u^\tau$, we have

(3.10) $$\gamma(-2zg'' + g') = 0,$$

(3.11) $$z(\alpha - \beta + \gamma z)g'' + (\beta + 3\gamma z)g' = 0.$$

**- Case** $\gamma \neq 0$. Solving equation (3.10) makes $g(z) = C_1 z^{3/2} + C_2$, where $C_1, C_2$ are arbitrary constants. Equation (3.11) is simplified to $\frac{3}{4}C_1\left([\alpha + \beta]\sqrt{z} + 5\gamma z^{3/2}\right) = 0$. By the arbitrariness of $z$, $\alpha + \beta$ and $\gamma$ must vanish. This case contradicts to the assumption $\gamma \neq 0$.

**- Case** $\gamma = 0$. Equation (3.15) is reduced to

(3.12) $$z(\alpha - \beta)g'' + \beta g' = 0$$

If $\alpha - \beta = 0$ (or $\alpha = \beta$) it makes $\beta g' = 0$. This case contradicts to the condition $\langle \alpha, \beta, \gamma \rangle \neq 0$ is not zero and $g' \neq 0$. Condition $\alpha - \beta \neq 0$ (or $\alpha \neq \beta$) will be considered only.

For the conditions $\gamma = 0$, $\alpha \neq \beta$, equation (3.12) is considered into two cases :

**Case** $\frac{\beta}{\alpha - \beta} = 1$, i.e. $\alpha = 2\beta$. The above condition $\alpha \neq \beta$ implies $\alpha \neq 0$. The equation can be reduced to $zg'' + g' = 0$, which has a solution $g(z) = C_1 \ln z + C_2$, where $C_1$ is a nonzero arbitrary constants, $C_2$ is a constant. However, the constant $C_2$ can be omitted because of the arbitrariness of $H$. Substitute $g$ into the DME and differentiate it w.r.t. $u^\tau$, the equation calculated is $\frac{C_1}{u - u^\tau}[2\eta_t - \xi_x + 4\eta_x u + \eta_x u^\tau] = 0$.

Since $C_1 \neq 0$ and unknown functions $\xi$ and $\eta$ depend on $(x, t)$, the equation can be split w.r.t. $u$ and $u^\tau$ which implies $\eta = \eta(t)$ and $\xi(x, t) = 2\eta'(t)x + \xi_2(t)$, where $\xi_2$ is an arbitrary function of $t$. Substitute both obtained functions into the DME :

(3.13) $$2[\eta'''(t) - \eta''(t)H'(u)]x + [3\eta''(t) - \eta'(t)H'(u)]u - C_1\eta'(t) + \xi_2''(t) - H'(u)\xi_2'(t) = 0.$$

Since unknown functions $\xi_2, \eta, H$ do not depend on $x$, then $\eta'''(t) - \eta''(t)H'(u) = 0$. This can be considered into subcases $H'(u) = 0$ and $H'(u) \neq 0$.

(1) $H'(u) = 0$, i.e. $H$ is a constant. Then $\eta'''(t) = 0$. The periodic condition implies $\eta$ is only a constant. The DME is simplified to $\xi_2''(t) = 0$. $\xi_2$ is also a constant by the periodic condition. This subcase shows $u_t + uu_x = C_1 \ln(u - u^\tau) + H$ admits the generator $\xi\partial_x + \eta\partial_t$ where $H, \xi, \eta$ are arbitrary constants.

(2) $H'(u) \neq 0$. The mixed derivative of DME (3.13) w.r.t. $x$ and $u$ shows $-2\eta''(t)H''(u) = 0$. This can be considered into two subcases $H''(u) = 0$ and $H''(u) \neq 0$.

- $H''(u) = 0$. It implies $H = H_1 u + H_2$, where $H_1, H_2$ are arbitrary constants and $H_1 \neq 0$. The derivative of equation (3.13) w.r.t. $x$ is $2(\eta'''(t) - H_1\eta''(t)) = 0$. Thus its solution is $\eta = C_3 + C_4 t + C_5 e^{H_1 t}$. By the periodic condition, $C_4$ and $C_5$ must identical to zero, i.e. $\eta$ is a constant. The DME is reduced to $\xi_2''(t) - H_1\xi_2'(t) = 0$, which has a solution $\xi_2 = C_6 + C_7 e^{H_1 t}$. Also the periodic condition of $\xi$ implies $C_7 = 0$. Then $u_t + uu_x = C_1 \ln(u - u^\tau) + H_1 u + H_2$ admits the generator

- $H''(u) \neq 0$. The equation implies $\eta''(t) = 0$, i.e. with the periodic condition $\eta$ is a constant only. The DME is reduced to $\xi_2''(t) - H'(u)\xi_2'(t) = 0$. Differentiate the equation w.r.t. $u$, $-H''(u)\xi_2'(t) = 0$, it implies $\xi_2$ is a constant.

All above cases shows $u = f(\eta x - \xi t)$, where $\xi, \eta$ are arbitrary constants and $f$ is arbitrary function, is solution of $u_t + uu_x = C_1 \ln(u - u^\tau) + H(u)$, where $C_1$ is a nonzero arbitrary constant and $H$ is an arbitrary function of $u$.

**Case** $\dfrac{\beta}{\alpha - \beta} \neq 1$. Let $\delta = \dfrac{\beta}{\alpha - \beta}$. Hence the solution of equation (3.12) is $g = C_1(u - u^\tau)^{\delta+1} + C_2$, where $C_1$ is a nonzero arbitrary constants, $C_2$ is a constant. However, the constant $C_2$ can be omitted because of the arbitrariness of $H$. Splitting the DME equation w.r.t. $u^\tau$ shows

(3.14) $$C_1(\delta+1)(u - u^\tau)^\delta \left[\delta\xi_x + (1-\delta)\eta_t + \left(3u - 2u^\tau - \delta(u + u^\tau)\right)\eta_t\right] = 0.$$

Since $\xi$ and $\eta$ depend on $(x,t)$ then equation (3.14) can be split w.r.t. $u$ and $u^\tau$. It implies $(3-\delta)\eta_x = 0$ and $-(2+\delta)\eta_x = 0$. The arbitrariness of $\delta$ implies $\eta_x = 0$, i.e. $\eta = \eta(t)$. Equation (3.14) is simplified to

(3.15) $$\delta\xi_x + (1-\delta)\eta'(t) = 0.$$

**Case** $\delta \neq 0$.

(1) If $\delta = 1$, equation (3.15) shows $\xi_x = 0$, i.e. $\xi = \xi(t)$. The DME is reduced to

$$\xi''(t) - \eta''(t)u - 2\eta'(t)H(u) + [\eta'(t)u - \xi'(t)]H'(u) = 0.$$

The second derivative of DME w.r.t. $u$ implies $[\eta'(t)u - \xi'(t)]H'''(u) = 0$.

If $H'''(u) \neq 0$, then $\eta'(t)u - \xi'(t) = 0$. Splitting the equation w.r.t. $u$ shows $\xi$ and $\eta$ are constants. Thus the solution of equation $u_t + uu_x = C_1\ln(u - u^\tau) + H(u)$ is also $u = f(\eta x - \xi t)$.

Suppose $H'''(u) = 0$. This means $H = H_1u^2 + H_2u + H_3$, where $H_1, H_2, H_3$ are arbitrary constants. The derivative of the DME w.r.t. $u$ is

$$-[2H_1\xi'(t) + \eta''(t) + H_2\eta'(t)] = 0.$$

(a) $H_1 = 0$ **and** $H_2 = 0$**.** The periodic condition implies $\eta$ is a constant. The DME is reduced to $\xi''(t) = 0$. So $\xi$ is also a constant.

(b) $H_1 = 0$ **but** $H_2 \neq 0$**.** The equation shows $\eta = C_3 + C_4e^{-H_2t}$. The periodic condition reduces term $C_4e^{-H_2t}$ which makes $\eta$ a constant. The DME is reduced to $\xi''(t) - H_2\xi(t) = 0$, also $\xi$ must be a constant.

(c) $H_1 \neq 0$. Then $\xi'(t) = -\dfrac{\eta''(t) + H_2\eta'(t)}{2H_1}$. The DME is simplified to $\eta'''(t) + \lambda\eta'(t) = 0$, where $\lambda = 4H_1(C_1 + H_3) - (H_2)^2$.

If $\lambda = 0$, this shows $\eta'''(t) = 0$. $\xi$ can be only a constant and $\xi$ is a constant also.

If $\lambda < 0$, $\eta = C_3 + C_4e^{\lambda t} + C_5e^{-\lambda t}$. The periodic property of $\eta$ implies $C_4$ and $C_5$ vanish and $\xi$ must be also a constant.

If $\lambda > 0$, $\eta = C_3 + C_4\cos\lambda t + C_5\sin\lambda t$. By the periodic condition, it is considered into two cases :

- $\tau \neq \dfrac{2\pi}{\lambda}$. In this case, $C_4$ and $C_5$ must vanish and it implies $\xi$ to be a constant.

- $\tau = \dfrac{2\pi}{\lambda}$. Here $\xi$ is equal to $\eta = C_6 + C_7\cos\lambda t + C_8\sin\lambda t$, where $C_6$ is an arbitrary constant, $C_7 = -\dfrac{H_2C_4 + \lambda C_5}{2H_1}, C_8 = -\dfrac{H_2C_5 + \lambda C_4}{2H_1}$. By the condition (3.5),

$$\zeta = \lambda\left[(C_8 - C_5u)\cos\lambda t + (-C_7 + C_4u)\sin\lambda t\right]$$

The solution of equation $u_t + uu_x = C_1(u - u^\tau)^2 + H_1u^2 + H_2u + H_3$ can be found from the characteristic equations, i.e. $u = e^{-\int p\,dt}\left[\int qe^{\int p\,dt}\,dt + \mathfrak{F}(x - \psi(t))\right]$, where

$$p = \lambda\frac{C_5\cos\lambda t - C_4\sin\lambda t}{C_3 + C_4\cos\lambda t + C_5\sin\lambda t}, q = \lambda\frac{C_8\cos\lambda t - C_7\sin\lambda t}{C_3 + C_4\cos\lambda t + C_5\sin\lambda t}, \psi(t) = \int\frac{C_6 + C_7\cos\lambda t + C_8\sin\lambda t}{C_3 + C_4\cos\lambda t + C_5\sin\lambda t}dt,$$

$\mathfrak{F}$ is an arbitrary function, $C_1, C_3, C_4, C_5, C_6, H_1, H_2, H_3$ are arbitrary constants, $H_1 \neq 0$, $\lambda, \tau, C_7, C_8$ are the constants which were defined in this section.

(2) If $\delta \neq 1$, equation (3.15) implies $\xi = \left(\frac{\delta - 1}{\delta}\right)\eta'(t)x + \xi_2(t)$, where $\xi_2(t)$ is an arbitrary function of $t$. Substitute $\xi$ into the DME and differentiate it w.r.t. both $x$ and $u$, we obtain $\left(\frac{\delta-1}{\delta}\right)H''(u)\eta''(t) = 0$.

This may be considered into two subcases.

(a) $H''(u) \neq 0$. This implies $\eta''(t) = 0$. Similar to the previous case, $\eta$ is a constant. The DME is reduced to $\xi_2''(t) - H'(u)\xi_2'(t) = 0$. The derivative of the equation w.r.t. $u$ shows $H''(u)\xi_2'(t) = 0$, which means $\xi_2$ is a constant. This shows both $\xi$ and $\eta$ are constants.

(b) $H''(u) = 0$. Then $H = H_1 u + H_2$, where $H_1, H_2$ are arbitrary constants. Derivative of the DME w.r.t. $u$ is $\left(\frac{\delta-2}{\delta}\right)\eta''(t) - H_1\eta'(t) = 0$.

If $\delta = 2$, the derivative of the DME w.r.t. $x$ gives us $\eta'''(t) - H_1\eta''(t) = 0$. Similar to the previous case, $\eta$ is a constant. The DME is also $\xi_2''(t) - H_1\xi_2'(t) = 0$ and its solution is a constant.

If $\delta \neq 2$. For arbitrary constant $H_1$ and periodic property, $\eta$ must be a constant. The DME is $\xi_2''(t) - H_1\xi_2'(t) = 0$ and its solution is a constant.

Both subcases show equation $u_t + uu_x = C_1(u - u^\tau)^{\delta+1} + H(u)$, for $\delta \neq -1$, has the same solution with the kernel case.

**Case** $\delta = 0$. Equation (3.15) shows $\eta'(t) = 0$, i.e. $\eta$ is a constant. The DME is reduced to $H(u)\xi_x - H'(u)(\xi_t + u\xi_x) + u^2\xi_{xx} + 2u\xi_{xt} + \xi_{tt} = 0$. In order to classify a solution of DPDE, we have to analyze by the following cases :

(1) $\xi_x = 0$. The DME is simplified to $\xi''(t) - H'(u)\xi'(t) = 0$. Its derivative w.r.t. $u$ is $-H''(u)\xi'(t) = 0$.

If $H''(u) = 0$, then $H = H_1 u + H_2$, where $H_1, H_2$ are arbitrary constants. The DME is $\xi''(t) - H_1\xi'(t) = 0$. With the periodic condition, $\xi$ can be only a constant.

If $H''(u) \neq 0$, then $\xi'(t) = 0$ which show $\xi$ is also a constant.

(2) $\xi_x \neq 0$. The third derivative of the DME w.r.t. $u$ can be rewritten as

$$\left(\frac{\xi_t}{\xi_x} + u\right)H^{(4)}(u) + 2H'''(u) = 0 \cdot \tag{3.16}$$

The derivatives of equation (3.16) w.r.t. $x$ and $t$ give $\frac{d}{dt}\left(\frac{\xi_t}{\xi_x}\right)H^{(4)}(u) = 0$ and $\frac{d}{dx}\left(\frac{\xi_t}{\xi_x}\right)H^{(4)}(u) = 0$. We consider the problem into two subcases :

- $H^{(4)}(u) = 0$ then $H'''(u) = 0$ which makes $H = H_1 u^2 + H_2 u + H_3$ satisfying equation (3.16). The second derivative of the DME w.r.t. $u$ is $2(\xi_{xx} - H_1\xi_x) = 0$.

**If** $H_1 = 0$ then $\xi(x,t) = \xi_1(t)x + \xi_2(t)$. The derivative of the DME w.r.t. $u$ shows $2\xi_1'(t) = 0$, i.e. $\xi_1$ is a constant. The DME is $\xi_2''(t) - H_2\xi_2'(t) + H_3\xi_1 = 0$. Let $\lambda = (H_2)^2 - 4H_3\xi_1$. With the periodic condition, function $\xi_2$ can be found according to $\lambda$ :

(a) $\lambda \geq 0$. $\xi_2$ must be a constant. Then the DME is $H_3\xi_1 = 0$. If $H_3 \neq 0$ then $\xi_1 = 0$. Thus $u_t + uu_x = C_1(u - u^\tau) + H_3$ admits the same generator and has the same solution with the kernel case. However, if $H_3 = 0$ and $\xi_1 \neq 0$ then $u_t + uu_x = C_1 u + C_2 u^\tau$, where $C_1, C_2$ are arbitrary constants, admits $(\xi_1 x + \xi_2)\partial_x + \eta\partial_t + \xi_1 u\partial_u$ and has a solution $u = (\xi_1 x + \xi_2)\mathfrak{F}(\eta\ln(\xi_1 x + \xi_2) - \xi_1 t)$, where $\mathfrak{F}$ is an arbitrary function. This solution reduces the equation (3.17) into an FODE,

$$\mathfrak{F}'(\chi)[\eta\mathfrak{F}(\chi) - 1] + [\mathfrak{F}(\chi)]^2 = C_1\mathfrak{F}(\chi) + C_2\mathfrak{F}(\chi + \xi_1\tau)$$

where $\xi_1 \neq 0$, $\eta$ are arbitrary constants and $\chi = \eta\ln(\xi_1 x + \xi_2) - \xi_1 t$.

(b) $\lambda < 0$. Let $\rho = \sqrt{-\lambda}/2$. Then $\xi_2 = e^{\frac{H_2}{2}t}(C_3\cos\rho t + C_4\sin\rho t)$. With the periodic condition, $H_2$ must be identical to zero, $H_3\xi_1 > 0$ and $\tau = \frac{2\pi}{\lambda}$. Then equation $u_t + uu_x = C_1(u - u^\tau) + H_3$, $H_3 \neq 0$, admits

$$(\xi_1 x + C_3\cos\rho t + C_4\sin\rho t)\partial_x + \eta\partial_t + (\xi_1 u - C_3\rho\sin\rho t + C_4\rho\cos\rho t)\partial_u,$$

where $\rho = \sqrt{H_3\xi_1}, C_3, C_4$ are arbitrary constants. This case is too complicated to find an exact form of a solution.

**If** $H_1 \neq 0$ then $\xi(x,t) = \xi_1(t)c^{H_1x} + \xi_2(t)$. The derivative of DME w.r.t. $u$ is $-2H_1\xi_2'(t) = 0$ which means $\xi_2'(t) = 0$ or $\xi_2$ is a constant. DME is simplified to $c^{H_1x}\left(\xi_1''(t) - H_2\xi_1'(t) + H_1H_3\xi_1(t)\right) = 0$. $\lambda = (H_2)^2 - 4H_1H_3$. With the periodic condition, function $\xi_1$ can be found according to $\lambda$ :

(a) $\lambda \geq 0$. $\xi_1$ must be a constant.

If $H_2 > 0$, then $\xi_1 = 0$ and DME vanishes, i.e. the solution form is not different to the kernel case.

If $H_2 = 0$, then $\xi_1$ is any constant. The DME is $H_1H_3\xi_1c^{H_1x} = 0$. If $H_3 \neq 0$, the equation has the similar solution with the previous case. On the other hand, $H_3 = 0$ implies $u_t + uu_x = C_1(u - u^\tau) + H_1u^2$ admits $\left(\xi_1c^{H_1x} + \xi_2\right)\partial_x + \eta\partial_t + \xi_1H_1uc^{H_1x}\partial_u$. This means $u = \left(\xi_1c^{H_1x} + \xi_2\right)\mathfrak{F}\left(\frac{\eta}{\xi_2H_1}\ln\left(\frac{\xi_1c^{H_1x}}{\xi_1c^{H_1x} + \xi_2}\right) - t\right)$ is a solution of the equation and reduces the FPDE to $\mathfrak{F}'(\Theta) = C_1\frac{\mathfrak{F}(\Theta) - \mathfrak{F}(\Theta + \tau) + H_1\xi_2\left[\mathfrak{F}(\Theta)\right]^2}{\eta\mathfrak{F}(\Theta) - 1}$, where $\Theta = \frac{\eta}{\xi_2H_1}\ln\left(\frac{\xi_1c^{H_1x}}{\xi_1c^{H_1x} + \xi_2}\right) - t$ and $C_1$ is an arbitrary constant.

(b) Let $\rho = \sqrt{-\lambda}/2$. Then $\xi_2 = c^{\frac{H_2}{2}t}\left(C_3\cos\rho t + C_4\sin\rho t\right)$. With the periodic condition, $H_2$ must be identical to zero, $H_1H_3 > 0$ and $\tau = \frac{2\pi}{\lambda}$. Then equation $u_t + uu_x = C_1(u - u^\tau) + H_1u^2 + H_3$, $H_3 \neq 0$, admits
$\left(c^{H_1x}\left(C_3\cos\rho t + C_4\sin\rho t\right) + \xi_2\right)\partial_x + \eta\partial_t + c^{H_1x}\left[H_1u\left(C_3\cos\rho t + C_4\sin\rho t\right) + \rho\left(-C_3\sin\rho t + C_4\cos\rho t\right)\right]\partial_u$,
where $\rho = \sqrt{H_1H_3}$ and $C_3, C_4$ are arbitrary constants. This case is too complicated to find an exact form of a solution.

- $H^{(4)}(u) \neq 0$.It means $\xi_t/\xi_x$ is a constant which has a solution $\xi = \psi(x + Kt)$, where $K$ is a constant and $\psi$ is an arbitrary function. Substitute $\xi$ into equation (3.16), then $(K + u)H^{(4)}(u) + 2H'''(u) = 0$. The equation has a solution $H(u) = H_1(K + u)\ln(K + u) + H_2u^2 + H_3u + H_4$, where $H_1, H_2, H_3, H_4$ are constants. The DME is simplified to

$$u^2\left[\psi'' - H_2\psi'\right] + u\left[2k\psi'' - (H_1 + 2H_1K)\psi'\right] + K^2\psi'' + \left[H_4 - K(H_1 + H_3)\right]\psi' = 0.$$

Splitting the equation w.r.t. $u^2$ and $u$ shows $\psi$ is a constant. This case has the same solution with the kernel case.

3.2.3 **dim $\mathbb{V}$ =1.** Here $\langle A, B, C\rangle$ can be represented by $\langle A, B, C\rangle = \langle \alpha, \beta, \gamma\rangle\phi(u, u^\tau)$, where $\alpha, \beta, \gamma$ are arbitrary constants which $\langle \alpha, \beta, \gamma\rangle \neq \langle 0,0,0\rangle$ and $\phi$ is a nonconstant function. The system of equations corresponding to the vector is

(3.17) $$(u - u^\tau)g'' = \alpha\phi(u, u^\tau),$$

(3.18) $$g' - (u - u^\tau)g'' = \beta\phi(u, u^\tau),$$

(3.19) $$-\left[u^2 - (u^\tau)^2\right]g'' + 3ug' - 2u^\tau g' = \gamma\phi(u, u^\tau).$$

- Case $\alpha \neq 0$. Equation (3.19) can be derived from equation (3.17) and (3.18) into

$$\left[(2\alpha + 3\beta)u + (-3\alpha - 2\beta)u^\tau - \gamma\right]\phi = 0$$

Since $\phi$ is not identical to zero then its coefficient must vanish and implies $\alpha = \beta = \gamma = 0$. This contradicts to the assumption.

- Case $\alpha = 0$. Here $g'' = 0$, which implies $g = C_1(u - u^\tau) + C_2$, where $C_1, C_2$ are arbitrary constants. Substitute $g$ into equation (3.18), $C_1 = \beta\phi(u, u^\tau)$ is obtained. If $\beta = 0$, it implies $C_1 = 0$ and $g$ is a constant which is invalid. Also if $\beta$ does not vanish, the equation implies $\phi$ is a constant function which also contradicts to the assumption.

This proves that case **dim $\mathbb{V}$ =1** is invalid.

3.2.4 **dim $\mathbb{V}$ =0.** $\langle A, B, C\rangle$ can be consider as a constant vector $\langle \alpha, \beta, \gamma\rangle$ , i.e.

(3.20) $$(u - u^\tau)g'' = \alpha,$$

(3.21) $$g' - (u - u^\tau)g'' = \beta,$$

(3.22) $$-\left[u^2-(u^\tau)^2\right]g''+3ug'-2u^\tau g'=\gamma,$$

where $\alpha,\beta,\gamma$ are arbitrary constants. Substitute equation (3.20) into equation (3.21), it leads to $g'=\alpha+\beta$ and $g''=0$. Substitute both values into equation (3.22), the equation is reduced to

$$3(\alpha+\beta)u-2(\alpha+\beta)u^\tau=\gamma.$$

By the arbitrariness of $u$ and $u^\tau$, $\alpha+\beta$ vanishes which makes $g'=0$. It contradicts to the assumption. This case is invalid also.

## 4 Conclusion

Solutions of equation $u_t+uu_x=C_1(u-u^\tau)^2+H_1u^2+H_2u+H_3$, $u_t+uu_x=C_1u+C_2u^\tau$ $u_t+uu_x=C_1(u-u^\tau)+H_1u^2$ and $u_t+uu_x=C_1(u-u^\tau)+H_1u^2+H_3$ are presented in the article. For other forms of equation $u_t+uu_x=g(u-u^\tau)+H(u)$, where $g,H$ are arbitrary functions, the solution is $u=f(\eta x-\xi t)$ where $f$ is an arbitrary function and $\xi,\eta$ are arbitrary constants.

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