รายงานวิจัยฉบับสมบูรณ์

โครงการ : คุณสมบัติเรขาคณิตของปริภูมิบานาค

Geometric Properties of Banach Spaces

โดย ดร.นรินทร์ เพชร์โรจน์ และคณะ

30 มิถุนายน 2551

สัญญาเลขที่ MRG4980167

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

โครงการวิจัยนี้ได้รับทุนสนับสนุนตามโครงการความร่วมมือระหว่างสำนักงานคณะกรรมการการอุดมศึกษากับสำนักงานกองทุนสนับสนุนการวิจัย เพื่อเป็นการพัฒนาศักยภาพในการทำงานวิจัยอาจารย์รุ่นใหม่ ผู้วิจัยขอขอบพระคุณเจ้าของทุนเป็นอย่างสูงมา ณ โอกาสนี้

ขอขอบพระคุณ ศ.ดร.สุเทพ สวนใต้ เป็นอย่างสูง ที่ให้คำปรึกษาแนะนำแก่ผู้วิจัยอย่างดียิ่ง

ดร.นรินทร์ เพชร์โรจน์

Abstract

Project Code: MRG-4980167

Project Title: Geometric Properties of Banach Spaces

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Project Period: July 1, 2006 –June 30, 2008

In this project, we present some results relate to the geometric properties of a Banach space, namely Calder´on–Lozanovskii space. We also giving some basic properties of the general modular space, and Criteria for strictly monotone points, extreme points and SU-points in such space are focused. Consequently, the sufficient and necessary conditions for the rotundity properties are also given.

Moreover, by considering the three-step Noor iterative process, some results relate to the fixed point theory (the topic which is related to the geometric properties of Banach spaces) are presented, i.e., we proved the strong convergence theorems for the nonlinear operators and showed some method for finding the solution of the nonlinear equation by using the fixed point theory.

Keywords: Generalized Calder´on–Lozanovskii spaces**;** Modular space**;** Geometric properties**;** Fixed point theory**;** Three-step Noor iterative process**.**

บทคัดย่อ

รหัสโครงการ: MRG-4980167

ชื่อโครงการ: คุณสมบัติเรขาคณิตของปริภูมิบานาค

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ระยะเวลาโครงการ: 1 กรกฎาคม 2549 – 30 มิถุนายน 2551

ในงานวิจัยนี้ ผู้วิจัยได้มุ่งที่จะศึกษาเกี่ยวกับสมบัติเรขาคณิตของปริภูมิบานาคที่สำคัญ คือ ปริภูมิ รวมถึงการพิสูจน์สมบัติเบื้องต้นบางประการบนปริภูมิโมดุลาร์ทั่วไป โดยหลังจากนั้น จึงได้พิจารณาเกณฑ์สำหรับการเป็นจุดทางเดียวโดยแท้ จุดสุดขีด และจุดเอส-ยู ในปริภูมิคาร์เดรอน-ลอซานนอฟสกีวางนัยทั่วไป ซึ่งทำให้ได้ผลลัพธ์ที่สำคัญตามมาคือ การทราบถึงเงื่อนไขความจำเป็นและเพียงพอสำหรับการเป็นปริภูมิโค้งเป็นวงของปริภูมิบานาคดังกล่าว

ยิ่งกว่านั้น ผู้วิจัยยังได้ทำการศึกษาเกี่ยวกับทฤษฎีบทจุดตรึง ซึ่งนับเป็นหัวข้อที่มีความสัมพันธ์กับการศึกษาสมบัติเรขาคณิตเป็นอย่างยิ่ง โดยได้พิจารณากระบวนการกระทำซ้ำสามขั้นตอนของนูร์ ซึ่งทำให้ได้ผลลัพธ์เกี่ยวกับทฤษฎีบทการลู่เข้าอย่างเข้มสำหรับการส่งแบบไม่เชิงเส้น ซึ่งสามารถนำไปประยุกต์ใช้ในการหาคำตอบของสมการไม่เชิงเส้นได้

คำหลัก: ปริภูมิคาร์เดรอน-ลอซานนอฟสกีวางนัยทั่วไป; ปริภูมิโมดุลาร์; สมบัติเรขาคณิต; ทฤษฎีบทจุดตรึง; กระบวนการกระทำซ้ำสามขั้นตอนของนูร์

บทนำ

การศึกษาสมบัติทางเรขาคณิตของปริภูมิบานาค คือการศึกษาสมบัติที่ไม่เปลี่ยนแปลงภายใต้การสมมิติ (isometry) ซึ่งได้เริ่มมีการศึกษาอย่างต่อเนื่องนับตั้งแต่ปี ค.ศ. 1936 ซึ่งสามารถแบ่งสมบัติทางเรขาคณิตออกได้เป็น 5 กลุ่ม คือ

1. สมบัติความโค้งมน (rotundity properties)
2. สมบัติความปรับเรียบ (smooth properties)
3. สมบัติทางเดียว (monotonicity properties)
4. สมบัติความโค้งมนเชิงซ้อน (complex rotundity properties)
5. สมบัติเรขาคณิตที่มีความเกี่ยวข้องกับทฤษฎีจุดตรึง (fixed point theory) เช่น nearly uniform rotundity, uniform Kadec-Klee property, nearly uniform smoothness, noncreaseness, uniform noncreaseness, Opial property, uniform Opial property เป็นต้น

การศึกษาคุณสมบัติเรขาคณิตของปริภูมิบานาคในแนวดังกล่าวข้างต้น ทฤษฎีและองค์ความรู้ใหม่ๆที่ได้รับอาจแบ่งเป็นสองแนวทางใหญ่ๆ คือ ทฤษฎีและองค์ความรู้ใหม่ๆเกี่ยวกับสมบัติเรขาคณิตที่เป็นสมบัติเรขาคณิตในปริภูมิบานาคทั่วไป (general Banach spaces) และปริภูมิบานาคเฉพาะเจาะจงที่น่าสนใจ

สมบัติทางเรขาคณิตสามารถนำไปประยุกต์ใช้ได้ในหลายๆแขนงวิชา เช่น approximation theory, fixed point theory, probability theory, ergodic theory, optimization theory, control theory, operator theory เป็นต้น ดังนั้นนักคณิตศาสตร์จึงได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง เพราะในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการ และ การพัฒนาประเทศ ซึ่งเป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ๆที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆนั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (Basic science) ดังตัวอย่างข้างต้น อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

จุดประสงค์ของการวิจัย

1. คิดค้นทฤษฎีและองค์ความรู้ใหม่ๆ เกี่ยวกับ สมบัติเรขาคณิต ทั้งที่เป็นสมบัติเรขาคณิตในปริภูมิบานาคทั่วไป (general Banach spaces) และ ปริภูมิบานาคเฉพาะเจาะจงที่น่าสนใจ)

2. นำทฤษฎีและองค์ความรู้ใหม่ที่คิดค้นในข้อที่ 1 ไปประยุกต์ใช้ในแขนงวิชาอื่นๆ ดังที่กล่าวไว้ข้างต้น

ผลการวิจัย

1. N. Petrot and S. Suantai, *The criteria of strict monotonicity and rotundity points in generalized Calderon-Lozanovskii spaces,*

Theorem 1 Let X_ρ be a modular space generated by a convex modular ρ and $x, y \in B(X_\rho)$. If $\xi(x) < 1$ then $\xi\left(\frac{x+y}{2}\right) < 1$.

Theorem 2 Let X_ρ be the modular space generated by a convex modular ρ and $x \in B\left(X_\rho\right)$ be such that $\xi(x) < 1$. If y is any element in $B\left(X_\rho\right)$ satisfying $\left\|\frac{x+y}{2}\right\|_\rho = 1$, then $\rho\left(\frac{x+y}{2}\right) = 1$.

Theorem 3 For any $x \in E_\varphi$ and any measurable partition $\{T_i\}_{i=1}^n$ of T we have,

$$\xi(x) = \max_{1 \le i \le n}\{\xi(x\chi_{T_i})\}.$$

Theorem 4 A point $x \in S(E_\varphi^+)$ is upper monotone if and only if

(i) $\varrho_\varphi(x) = 1$;

(ii) $\mu(\{t \in T : x(t) < a(t)\}) = 0$;

(iii) $\varphi \circ x$ is an upper monotone point of E.

Theorem 5 A point $x \in S(E_\varphi^+)$ is a lower monotone point if and only if

(i) $\xi(x) < 1$;

(ii) $\mu(\{t \in \operatorname{supp} x : x(t) \le a(t)\}) = 0$;

(iii) $\varphi \circ x$ is a lower monotone point of E.

Theorem 6 A point $x \in S(E_\varphi)$ is an extreme point of $B(E_\varphi)$ if and only if

(i) $\varrho_\varphi(x) = 1$;

(ii) $\mu(\{t \in T : |x(t)| < a(t)\}) = 0$;

(iii) $\varphi \circ |x|$ is an UM-point;

(iv) if $u, v \in S(E)$ satisfy $\frac{u+v}{2} = \varphi \circ |x|$ then either

$$u = v \ \ or \ \ \varphi \circ \left(\frac{y+z}{2}\right) < \frac{1}{2}(\varphi \circ y + \varphi \circ z),$$

where $y(t) = \varphi^{-1}(t, |u(t)|), z(t) = \varphi^{-1}(t, |v(t)|)$ for all $t \in T$.

Theorem 7 Let E be a strictly monotone Köthe space and $x \in S(E_\varphi)$. Then x is an SU-point of $B(E_\varphi)$ if and only if:

(i) $\xi(x) < 1$;

(ii) $\mu(\{t \in \operatorname{supp} x : |x|(t) \leq a(t)\}) = 0$;

(iii) if $u \in S(E^+)$ satisfies $\|u + \varphi \circ |x|\|_E = 2$ then either

$$u = \varphi \circ |x| \ \ \text{or} \ \ \varphi \circ \left(\frac{|x|+y}{2}\right) < \frac{1}{2}(\varphi \circ |x| + \varphi \circ y),$$

where $y(t) = \varphi^{-1}(t, u(t))$ for all $t \in T$.

Theorem 8 Let E be a Köthe space and φ be a Musielak-Orlicz function. Then $E_\varphi \in (R)$ if and only if

(i) $E \in (SM)$;

(ii) $\varphi \in \Delta_2^E$;

(iii) if $u, v \in S(E^+)$ with $u \neq v$ then either

$$\left\|\frac{u+v}{2}\right\|_E < 1 \ \ \text{or} \ \ \varphi \circ \left(\frac{x+y}{2}\right) < \frac{1}{2}(\varphi \circ x + \varphi \circ y),$$

where $x(t) = \varphi^{-1}(t, u(t))$ and $y(t) = \varphi^{-1}(t, v(t))$ for all $t \in T$.

2. N. Petrot *Modified Noor iterative process by non-Lipschitzian mappings for nonlinear equations in Banach spaces,*

Theorem 1 Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be self maps of C. T_1 is a Φ-hemicontractive uniformly continuous mapping with bounded range and T_2, T_3 are generalized Lipschitzian mapping functions. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the three-step iterative process defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n \end{aligned} \tag{1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfying conditions:

$$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n \ \ and \ \ \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{2}$$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Theorem 2 Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be self maps of C. T_1 is a Φ-hemicontractive uniformly continuous mapping with $T_1(C), T_2(C)$ are bounded sets. Suppose that a sequence $\{x_n\}_{n=0}^{\infty}$ is defined as (1) when $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfying the condition (2). If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Theorem 3 Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T be a Φ-hemicontractive uniformly continuous mapping with bounded range self map of C. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the three-step iterative process defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned} \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfying conditions:

$$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n \ \ and \ \ \sum_{n=0}^{\infty} \alpha_n = \infty,$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Theorem 4 Let E be a real Banach space and $T : E \to E$ be uniformly continuous operator. For a given $f \in E$, let x^* denote the unique solution of the equation $Tx = f$. Define the operator $H : E \to E$ by $Hx = f + x - Tx$, and suppose that the range of H is bounded. For any $x_0 \in E$ let $\{x_n\}_{n=0}^{\infty}$ be the three-step iterative process defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n H y_n \\ y_n &= (1-\beta_n)x_n + \beta_n H z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n H x_n \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ satisfying conditions:

$$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for any $x \in E$, there exists a $j(x - x^*) \in J(x - x^*)$ satisfying

$$\langle Tx - Tx^*, j(x - x^*)\rangle \geq \Phi(\|x - x^*\|)\|x - x^*\|,$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Tx = f$.

Theorem 5 Let E be a real Banach space and $T : E \to E$ be uniformly continuous and Φ-strongly accretive operator. For a given $f \in E$, let x^* denote the unique solution of the equation $Tx = f$. Let $H, \{x_n\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem 4. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution x^* of $Tx = f$.

Theorem 6 Let E be a real Banach space and $T : E \to E$ be uniformly continuous and Φ-strongly quasi-accretive operator. Let x^* denote the unique solution of the equation $Tx = 0$. Let $H, \{x_n\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem 4. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution x^* of $Tx = 0$.

Out Put จากโครงการที่ได้รับจาก สกว.

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 N. Petrot and S. Suantai, *The criteria of strict monotonicity and rotundity points in generalized Calderon-Lozanovskii spaces*, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.120

1.2 N. Petrot, *Modified Noor iterative process by non-Lipschitzian mappings for nonlinear equations in Banach spaces*, J. Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.04.065

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการ และเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอนและมีเครือข่ายความร่วมมือสร้างกระแสความสนใจในวงกว้าง

3. อื่น ๆ: การเสนอผลงานในที่ประชุมวิชาการ

3.1 วันที่ 16 ก.ค.-22 ก.ค. 2550

หัวข้อ: Monotonicity and rotundity structure of generalized Calderon-Lozanovskii spaces

ชื่อการประชุม: THE 8TH INTERNATIONAL CONFERENCE ON FIXED POINT THEORY AND ITS APPLICATIONS, Chiang Mai, Thailand.

3.2 วันที่ 31 พ.ค.-4 มิ.ย. 2550

หัวข้อ: The iteration of non-Lipschitzian mappings for nonlinear Equations in Banach Spaces

ชื่อการประชุม : The Fifth International Conference on Nonlinear Analysis and Convex Analysis (NACA2007), National Tsing-Hua University, Hsinchu, Taiwan

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ภาคผนวก 1

The criteria of strict monotonicity and rotundity points in generalized Calder´on–Lozanovskii spaces

NARIN PETROT and SUTHEP SUANTAI

Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.120

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The criteria of strict monotonicity and rotundity points in generalized Calderón–Lozanovskiĭ spaces[☆]

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Received 20 December 2007; accepted 28 February 2008

Abstract

In this paper, some basic properties of the general modular space are proven. Criteria for strictly monotone points, extreme points and SU-points in generalized Calderón–Lozanovskiĭ spaces are obtained. Consequently, the sufficient and necessary conditions for the rotundity properties of such spaces are given.

MSC: 46A45; 46B20; 46B30; 46C05; 46E30

Keywords: Musielak–Orlicz function; Generalized Calderón–Lozanovskiĭ spaces; Point of lower(upper) monotonicity; Extreme point; SU-point; Rotundity

1. Introduction

Throughout the paper $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the sets of reals, nonnegative reals and natural numbers, respectively. For a real vector space X, a function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\rho(0) = 0$ and $x = 0$ whenever $\rho(\lambda x) = 0$ for any $\lambda > 0$;
(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

If we replace (iii) by

(iii)′ $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$,

then the modular ρ is called convex modular. Moreover, for arbitrary $x \in X$ we define

$$\xi(x) := \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) < \infty\right\}.$$

We put $\inf \emptyset = \infty$ by the definition.

☆ The present study was supported by the Thailand Research Fund (Project No. MRG4980167).
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0362-546X/$ - see front matter
doi:10.1016/j.na.2008.02.120

Please cite this article in press as: N. Petrot, S. Suantai, The criteria of strict monotonicity and rotundity points in generalized Calderón–Lozanovskiĭ spaces, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.120

For any modular ρ on X, the space

$$X_\rho = \left\{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\right\},$$

is called the *modular space*. If ρ is a convex modular, the functional

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\},$$

is a norm on X_ρ, which is called the *Luxemburg norm* (see [35]). A modular ρ is called right-continuous (left-continuous) [continuous] if $\lim_{\lambda\to 1^+} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$ ($\lim_{\lambda\to 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$) [it is both right- and left-continuous].

Remark 1.1. If ρ is a convex modular and $\rho(\lambda_o x) < \infty$ for some $x \in X_\rho$ and $\lambda_o > 0$, then ρ is right-continuous at λx for any $\lambda \in [0, \lambda_o)$ and left-continuous at λx for any $\lambda \in (0, \lambda_o]$. Indeed, this follows from the fact that the function $f(t) = \rho(tx)$ is convex on $\mathbb{R}^+$ and has finite values on the interval $[0, \lambda_o]$ so it is a continuous function on $[0, \lambda_o]$.

A triple (T, Σ, μ) stands for a nonatomic, positive, complete and σ-finite measure space, while $L^0 = L^0(\mu)$ denotes the space of all (equivalence classes of) σ-measurable functions $x : T \to \mathbb{R}$. In what follows we will identify measurable functions which differ only on a set of measure zero. For $x, y \in L^0$, we write $x \leq y$ if $x(t) \leq y(t)$ for μ-a.e. $t \in T$ and the notion $x < y$ is used for $x \leq y$ and $x \neq y$. Moreover, for any $x \in L^0$, we denote by $|x|$ the absolute value of x, i.e. $|x|(t) = |x(t)|$ for μ-a.e. $t \in T$.

By E we denote a *Köthe space* over the measure space (T, Σ, μ), i.e. $E \subset L^0$ which satisfies the following conditions:

(i) if $x \in E$, $y \in L^0$ and $|y| \leq |x|$ for μ-a.e. then $y \in E$ and $\|y\|_E \leq \|x\|_E$,
(ii) there exists a function x in E which is strictly positive on the whole T.

A function $\varphi : T \times \mathbb{R} \to [0, \infty)$ is said to be a *Musielak–Orlicz function* if $\varphi(t, \cdot)$ is a nonzero function, it vanishes at zero, it is convex and even for μ-a.e. $t \in T$ and $\varphi(\cdot, u)$ as well as $\varphi^{-1}(\cdot, u)$ are Σ-measurable functions for any $u \in R^+$, where $\varphi^{-1}(t, \cdot)$ is the generalized inverse function of $\varphi(t, \cdot)$ defined on $[0, \infty)$ by

$$\varphi^{-1}(t, u) = \inf\{v \geq 0 : \varphi(t, v) > u\}$$

for each $t \in T$ (see [35]). For Musielak–Orlicz function φ we define a measurable function with respect to $t \in T$ by

$$a(t) = \sup\{u \geq 0 : \varphi(t, u) = 0\},$$

see [6, page 175].

Remark 1.2. Let $\varphi : T \times \mathbb{R} \to [0, \infty)$ be a Musielak–Orlicz function. Then

(i) $\varphi^{-1}(t, \cdot)$ vanishes only at zero;
(ii) $\varphi(t, \varphi^{-1}(t, u)) = u$ for all $u \in [0, \infty)$ and

$$\varphi^{-1}(t, \varphi(t, u)) = \begin{cases} 0, & \text{if } u \in [0, a(t)],\\ u, & \text{if } u \in (a(t), \infty); \end{cases}$$

for μ-a.e. $t \in T$.

Given any Musielak–Orlicz function φ, we define on L^0 a convex modular ϱ_φ by

$$\varrho_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E,\\ \infty & \text{otherwise}; \end{cases}$$

and *the generalized Calderón–Lozanovskiĭ space* is defined by

$$E_\varphi = \{x \in L^0 : \varphi \circ \lambda x \in E \text{ for some } \lambda > 0\}.$$

Then $E_\varphi = (E_\varphi, \|\cdot\|_\varphi)$ becomes a normed space, where $\|\cdot\|_\varphi$ denotes for the Luxemburg norm induced by ϱ_φ (see [4,9]).

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As for the investigations of generalized Calderón–Lozanovskiĭ space we refer to [8–10,27].

In the case when φ is an Orlicz function, i.e. there is a set $A \in \Sigma$ with $\mu(A) = 0$ such that $\varphi(t_1, \cdot) = \varphi(t_2, \cdot)$ for all $t_1, t_2 \in T \setminus A$, these Calderón–Lozanovskiĭ spaces were investigated in [3,4,30] and the investigations were continued in the papers [5,11,15,17,20,26,28,29,32–34,36,37].

We say a Musielak–Orlicz function φ satisfies the condition Δ_2^E if there exist a set $A \in \Sigma$ with $\mu(A) = 0$, a constant $K > 0$ and a nonnegative function $h \in E$ such that the inequality

$$\varphi(t, 2u) \leq K\varphi(t, u) + h(t)$$

holds for all $t \in T \setminus A$ and $u \in \mathbb{R}$ (see [35] when $E = L^1$ and [9] in general).

Lemma 1.3 ([9, Lemma 5]). *The property that $\|x\|_\varphi = 1$ if and only if $\varrho_\varphi(x) = 1$ holds true for any $x \in E_\varphi$ if and only if $\varphi \in \Delta_2^E$.*

Lemma 1.4 ([19, Lemma 1]). *For any Musielak–Orlicz function φ the inequality*

$$\varphi(t, u + v) \geq \varphi(t, u) + \varphi(t, a(t) + v)$$

holds for μ-a.e. $t \in T$ and any $u \geq a(t)$, $v \geq 0$.

Lemma 1.5 ([9, Corollary 7]). *If $\varphi \in \Delta_2^E$ then $\mu(\{t \in T : a(t) > 0\}) = 0$.*

By $S(E)$, $B(E)$ and $E^+(= \{x \in E : x \geq 0\})$ we denote the unit sphere, the closed unit ball and the positive cone of the Köthe space E. For any $x \in E$, define $\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$.

A point $x \in E^+$ is called a point of *upper monotonicity* (*UM*-point for short) if for every $y \in E^+ \setminus \{0\}$ we have $\|x\|_E < \|x + y\|_E$. A point $x \in E^+ \setminus \{0\}$ is called a point of *lower monotonicity* (*LM*-point for short) if for every $y \in E^+ \setminus \{0\}$, such that $y < x$, we have $\|x - y\|_E < \|x\|_E$. If every point of $S(E^+)$ is a *UM*-point (or an *LM*-point), then we say that the space E is *strictly monotone*. It is easy to see that $x \in E^+ \setminus \{0\}$ in any Köthe space E is a *UM*-point (*LM*-point) if and only if $x/\|x\|$ is a *UM*-point (*LM*-point). Therefore, it is enough to formulate the criteria of monotonicity for points in $S(E^+)$ only.

A point $x \in S(E)$ is said to be an *extreme point* of $B(E)$ ($x \in \operatorname{ext} B(E)$ for short) if for any $y, z \in B(E)$ such that $2x = y + z$ we have $y = z$. If any point of $S(E)$ is an extreme point of $B(E)$, we say that the space E is rotund ($X \in (R)$).

A point $x \in S(E)$ is called a *strong U-point* (*SU*-point for short) of $B(E)$ if for any $y \in S(E)$ with $\|x + y\|_E = 2$, we have $x = y$. It is obvious that a Banach space E is rotund if and only if any $x \in S(E)$ is an *SU*-point, but the notions of an extreme point and an *SU*-point are different (see [7]).

It is well known that rotundity properties of Banach spaces have applications in various branches of mathematics, such as, Fixed point Theory, Approximation Theory, Ergodic Theory, and many others. Moreover, if the focus of the study is Banach lattices, then there are strong relationships between rotundity properties and monotonicity properties (see [2,13,14,16,18,21,24,25]). Specially, in [17,20] the local rotundity and local monotonicity structures of a certain Banach lattice, namely Calderón–Lozanovskiĭ spaces, were studied. The results of our paper will be a generalization of two such excellent papers [17,20] by considering Orlicz function with parameter called Musielak–Orlicz function instead of Orlicz function. Of course, some ideas from those papers are also applied in our paper. However, because of the different properties among functions, in many parts of the proofs of our results new methods and techniques are developed.

Let us note that if E has the Fatou property, i.e. for any $x \in L^0$ and $(x_n)_{n=1}^\infty$ in E such that $0 \leq x_n \nearrow x$ μ-a.e. and $\sup_n \|x_n\|_E < \infty$ we have that $x \in E$ and $\|x\|_E = \lim_{n\to\infty} \|x_n\|_E$ (see [1,23,31]), then E_φ also has this property, and moreover, the modular ϱ_φ is left-continuous (see [9, Theorem 12]). Consequently, E_φ is a Banach space. So, in the whole paper we will assume that E is a Köthe space with the Fatou property. Moreover, we will denote $(\varphi \circ x)(t) = \varphi(t, x(t))$ for each $t \in T$.

The paper is organized as follows. In Section 2 we give some basic auxiliary results of general modular space and E_φ. Section 3 is devoted to the strictly monotone points of E_φ. We study rotundity points of E_φ in Section 4. Finally, in Section 5 we give a characterization of rotundity structure in E_φ.

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2. Auxiliary lemmas

We start by proving some facts in any modular space.

Lemma 2.1. *Let X_ρ be a modular space generated by a convex modular ρ and $x, y \in B(X_\rho)$. If $\xi(x) < 1$ then $\xi\left(\frac{x+y}{2}\right) < 1$.*

Proof. Since $\xi(x) < 1$, we take a real number $a \in (\xi(x), 1)$ and put $\varepsilon = \frac{1-a}{1+a}$. Then $\varepsilon > 0$ and $\frac{(1+\varepsilon)a}{2} + \frac{1+\varepsilon}{2} = 1$. Thus,

$$\begin{aligned}\rho\left((1+\varepsilon)\left(\frac{x+y}{2}\right)\right) &= \rho\left(\frac{1+\varepsilon}{2}\cdot x + \frac{1+\varepsilon}{2}\cdot y\right)\\ &= \rho\left(\frac{(1+\varepsilon)a}{2}\cdot\frac{x}{a} + \frac{1+\varepsilon}{2}\cdot y\right)\\ &\le \frac{(1+\varepsilon)a}{2}\rho\left(\frac{x}{a}\right) + \frac{1+\varepsilon}{2}\rho(y) < \infty,\end{aligned}$$

which implies that $\xi\left(\frac{x+y}{2}\right) < 1$. This completes the proof. □

Lemma 2.2. *Let X_ρ be the modular space generated by a convex modular ρ and $x \in B\left(X_\rho\right)$ be such that $\xi(x) < 1$. If y is any element in $B\left(X_\rho\right)$ satisfying $\left\|\frac{x+y}{2}\right\|_\rho = 1$, then $\rho\left(\frac{x+y}{2}\right) = 1$.*

Proof. By $\xi(x) < 1$ and Lemma 2.1, we have $\xi\left(\frac{x+y}{2}\right) < 1$. Put $I = \left[0, \frac{1}{\xi\left(\frac{x+y}{2}\right)}\right)$ and define a function $f : I \to \mathbb{R}$ by $f(t) = \rho\left(t\frac{x+y}{2}\right)$. Then f is a convex function and has finite values on I, which imply that f is a continuous function on I. Assuming that $\rho\left(\frac{x+y}{2}\right) < 1$, there exists a $\lambda > 1$ such that $\rho\left(\lambda\frac{x+y}{2}\right) < 1$ whence $\left\|\frac{x+y}{2}\right\|_\rho \le \frac{1}{\lambda} < 1$, a contradiction. □

We close this section by giving a basic result on the generalized Calderón–Lozanovskiĭ space as follows:

Lemma 2.3. *For any $x \in E_\varphi$ and any measurable partition $\{T_i\}_{i=1}^n$ of T we have,*

$$\xi(x) = \max_{1\le i\le n}\{\xi(x\chi_{T_i})\}.$$

Proof. Put $\alpha = \max_{1\le i\le n}\{\xi(x\chi_{T_i})\}$, then it is obvious that $\alpha \le \xi(x)$. We now show that the converse inequality holds. If not, then a real number $\beta \in (\alpha, \xi(x))$ can be found and consequently,

$$\varrho_\varphi\left(\frac{x}{\beta}\right) = \left\|\varphi\circ\left(\frac{x}{\beta}\right)\right\|_E = \left\|\sum_{i=1}^n \varphi\circ\left(\frac{x}{\beta}\chi_{T_i}\right)\right\|_E \le \sum_{i=1}^n\left\|\varphi\circ\left(\frac{x}{\beta}\chi_{T_i}\right)\right\|_E = \sum_{i=1}^n \varrho_\varphi\left(\frac{x}{\beta}\chi_{T_i}\right) < \infty,$$

which contradicts the definition of the number $\xi(x)$. □

3. Points of monotonicity in E_φ

In this section, we give some criteria for upper and lower monotonicity points in E_φ.

Theorem 3.1. *A point $x \in S(E_\varphi^+)$ is upper monotone if and only if*

(i) $\varrho_\varphi(x) = 1$;
(ii) $\mu(\{t \in T : x(t) < a(t)\}) = 0$;
(iii) *$\varphi\circ x$ is an upper monotone point of E.*

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Proof. *Necessity*. If condition (i) does not hold, then $\varrho_\varphi(x) =: r < 1$. Let D be a subset of A such that $\mu(D) > 0$ and $\chi_D \in E$. Let u be a nonnegative measurable function defined by

$$u(t) = \varphi^{-1}\left(t, \frac{1-r}{\|\chi_D\|_E}\right)\chi_D(t).$$

Then $\varphi \circ u = \frac{1-r}{\|\chi_D\|_E}\chi_D$ which implies $\varphi \circ u \in E$, and moreover,

$$\|\varphi \circ u\|_E = \left\|\frac{(1-r)}{\|\chi_D\|_E}\chi_D\right\|_E = 1 - r.$$

Since $u > 0$, there exist a real number $\lambda > 0$ and a measurable function $y > 0$ with supp $y = D$ satisfying

$$\varphi(t, x(t) + y(t)) \le \varphi(t, x(t)) + \varphi(t, u(t)), \quad y(t) \le \lambda$$

for μ-a.e. $t \in T$. On the other hand, an ascending sequence $(T_n)_{n=1}^{\infty}$ such that $\bigcup_n T_n = T$ and $\sup_{t \in T_n} \varphi(t, u) < \infty$ for each $n \in \mathbb{N}$ and $u \in \mathbb{R}^+$ can be found (see [22]), which allows us to obtain a nonnegative real number d_λ such that,

$$d_\lambda = \sup\{\varphi(t, \lambda) : t \in D\}.$$

Consequently, $\varphi \circ y \le d_\lambda \chi_D$ which implies that $y \in E_\varphi$. Moreover,

$$\begin{aligned} \varrho_\varphi(x + y) &= \|\varphi \circ x\chi_{T\setminus D} + \varphi \circ (x + y)\chi_D\|_E \le \|\varphi \circ x\chi_{T\setminus D} + \varphi \circ x\chi_D + \varphi \circ u\|_E \\ &= \|\varphi \circ x + \varphi \circ u\|_E \le \|\varphi \circ x\|_E + \|\varphi \circ u\|_E = r + (1 - r) = 1. \end{aligned}$$

Hence, $1 = \|x\|_\varphi \le \|x + y\|_\varphi \le 1$ and therefore, x is not an upper monotone point.

Suppose that (ii) is not satisfied. Then the set $A = \{t \in T : x(t) < a(t)\}$ has a positive measure. Let us define $y(t) = (a - x)(t)\chi_A(t)$ for all $t \in T$. We see that $y \in E_\varphi^+ \setminus \{0\}$ and

$$\begin{aligned} \varrho_\varphi(x + y) = \|\varphi \circ (x + y)\|_E &= \|\varphi \circ x\chi_{T\setminus A} + \varphi \circ (x + y)\chi_A\|_E \\ &= \|\varphi \circ x\chi_{T\setminus A} + \varphi \circ a\chi_A\|_E \\ &= \|\varphi \circ x\chi_{T\setminus A}\|_E \le \varrho_\varphi(x) \le 1. \end{aligned}$$

Hence, $\|x + y\|_\varphi \le 1$. But, since $y \in E_\varphi^+ \setminus \{0\}$ the fact that $\|x + y\|_\varphi \ge \|x\|_\varphi = 1$ is always true, we obtain $\|x + y\|_\varphi = 1$. This means that x is not an upper monotone point.

It remains to show the necessity of condition (iii). Let us assume that $x \in S(E_\varphi^+)$ is an upper monotone point. Since the necessity of (i) has been proved, we may assume that $\varphi \circ x \in S(E)$ and suppose that condition (iii) is not satisfied, i.e. there exists $y \in E^+ \setminus \{0\}$ such that $\|\varphi \circ x + y\|_E = 1$. Let us define $z \in E_\varphi^+ \setminus \{0\}$ by $z(t) = \varphi^{-1}(t, y(t))$ for all $t \in T$. Hence there exists a nonnegative measurable function h such that supp $h \subset$ supp z and

$$\varphi(t, x(t) + h(t)) \le \varphi(t, x(t)) + \varphi(t, z(t)), \quad h(t) \le \lambda$$

for all $t \in T$. Thus $h \in E_\varphi$ and

$$\varrho_\varphi(x + h) = \|\varphi \circ (x + h)\|_E \le \|\varphi \circ x + \varphi \circ z\|_E = \|\varphi \circ x + y\|_E = 1,$$

which implies that $\|x + h\|_\varphi = 1$. This contradicts the upper monotonicity of x and the proof is completed.

Sufficiency. Let $x \in S(E_\varphi^+)$ and assume that conditions (i)–(iii) are satisfied. Let $y \in E^+ \setminus \{0\}$ be given. In view of Lemma 1.4, condition (ii) gives

$$\varphi(t, x(t) + y(t)) \ge \varphi(t, x(t)) + \varphi(t, a(t) + y(t))$$

for μ-a.e. $t \in T$. Since $\mu(\{t \in T : \varphi(t, a(t) + y(t)) > 0\}) > 0$ and $\varphi \circ x$ is an upper monotone point in E, we have

$$\varrho_\varphi(x + y) = \|\varphi \circ (x + y)\|_E \ge \|\varphi \circ x + \varphi \circ (a + y)\|_E > \|\varphi \circ x\|_E = \varrho_\varphi(x) = 1,$$

that is, $\|x + y\|_\varphi > 1$. This completes the proof. □

Theorem 3.2. *A point $x \in S(E_\varphi^+)$ is a lower monotone point if and only if*

(i) $\xi(x) < 1$;

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(ii) $\mu(\{t \in \operatorname{supp} x : x(t) \le a(t)\}) = 0$;
(iii) $\varphi \circ x$ *is a lower monotone point of* E.

Proof. *Necessity*. Let $x \in S(E^+)$ be a lower monotone point. Suppose that condition (i) is not satisfied, i.e. $\xi(x) = 1$. Take $A, B \in \Sigma$, both of positive measure, such that $A \cap B = \emptyset$ and $A \cup B = \operatorname{supp} x$. Thus by Lemma 2.3 we obtain $\xi(x\chi_A) = 1$ or $\xi(x\chi_B) = 1$. Without loss of generality we may assume that $\xi(x\chi_A) = 1$, and it would be $\xi(x - x\chi_B) = \xi(x\chi_A) = 1$. This implies $\|x - x\chi_B\|_\varphi \ge 1$, a contradiction.

If condition (ii) does not hold, then the set $A = \{t \in \operatorname{supp} x : x(t) \le a(t)\}$ has positive measure. By (i), the necessity of which has been already proved, we have $\xi(x) < 1$, and consequently $\varrho_\varphi(x) = 1$ by Lemma 2.2. Define $y(t) = x(t)\chi_A(t)$, then we have $0 < y < x$, and

$$\varrho_\varphi(x - y) = \|\varphi \circ x\chi_{T\setminus A}\|_E = \|\varphi \circ x\|_E = \varrho_\varphi(x) = 1.$$

This implies that $\|x - y\|_\varphi = 1$, a contradiction.

Now we will show that condition (iii) holds. By (i), we have $\varphi \circ x \in S(E)$. Let us take $y \in E$ such that $0 < y < \varphi \circ x$ and choose a measurable function z such that $0 < z < x$ with $\varphi \circ x - y \le \varphi \circ (x - z)$. Since x is a lower monotone point, we have

$$\|\varphi \circ x - y\|_E \le \|\varphi \circ (x - z)\|_E = \varrho_\varphi(x - z) \le \|x - z\|_\varphi < 1.$$

This shows that $\varphi \circ x$ is then a lower monotone point of E.

Sufficiency. Let $x \in S(E_\varphi^+)$, $y \in E^+ \setminus \{0\}$ be such that $y < x$ and conditions (i)–(iii) are satisfied. Obviously, supp $y \subset \operatorname{supp} x$ which together with condition (ii) imply that for $z = \varphi \circ x - \varphi \circ (x - y)$ we have $z > 0$. Moreover, by condition (i), we have $\varrho_\varphi(x) = 1$. Since $\varphi \circ x$ is a lower monotone point of E and $z \le \varphi \circ x$, so

$$\varrho_\varphi(x - y) = \|\varphi \circ (x - y)\|_E = \|\varphi \circ x - z\|_E < \|\varphi \circ x\|_E = \varrho_\varphi(x) = 1. \tag{3.1}$$

Using Eq. (3.1) together with $\xi(x - y) < 1$ (by condition (i)) and the continuity of ϱ_φ, in light of Lemma 2.2, we have $\|x - y\|_\varphi < 1$. This completes the proof. □

4. Points of rotundity in E_φ

We will study the points of rotundity, such as extreme point and SU-point in this Section. We begin with the following definition:

A point $x \in S(E^+)$ is said to be an extreme point of $B(E^+)$ ($x \in \operatorname{ext} B(E^+)$ for short) if for any $x, y \in S(E^+)$ such that $x = (y + z)/2$, we have $y = z = x$.

Lemma 4.1 ([17, Lemma 4]). *In any Köthe space* E, $x \in S(E)$ *is an extreme point of* $B(E)$ *if and only if* $|x|$ *is a UM-point of* E *and* $|x| \in \operatorname{ext} B(E^+)$.

Theorem 4.2. *A point* $x \in S(E_\varphi)$ *is an extreme point of* $B(E_\varphi)$ *if and only if*

(i) $\varrho_\varphi(x) = 1$;
(ii) $\mu(\{t \in T : |x(t)| < a(t)\}) = 0$;
(iii) $\varphi \circ |x|$ *is a UM-point;*
(iv) *if* $u, v \in S(E)$ *satisfy* $\frac{u+v}{2} = \varphi \circ |x|$ *then either*

$$u = v \quad or \quad \varphi \circ \left(\frac{y + z}{2}\right) < \frac{1}{2}(\varphi \circ y + \varphi \circ z),$$

where $y(t) = \varphi^{-1}(t, |u(t)|)$, $z(t) = \varphi^{-1}(t, |v(t)|)$ *for all* $t \in T$.

Proof. *Sufficiency*. Assume that conditions (i)–(iv) are satisfied. Let $x \in S(E_\varphi)$ and $y, z \in B(E_\varphi)$ be such that $2x = y + z$. We shall show that $y = z$. First, we will show that

$$\varphi \circ |x|(t) = \varphi \circ \frac{|y + z|}{2}(t) = \varphi \circ \left[\frac{|y| + |z|}{2}\right](t) = \frac{1}{2}[\varphi \circ |y|(t) + \varphi \circ |z|(t)] \tag{4.1}$$

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for μ-a.e. $t \in T$. Note that, we always have

$$\varphi \circ |x|(t) = \varphi \circ \frac{|y+z|}{2}(t) \le \varphi \circ \left[\frac{|y|+|z|}{2}\right](t) \le \frac{1}{2}\left[\varphi \circ |y|(t) + \varphi \circ |z|(t)\right]$$

for μ-a.e. $t \in T$. Let $A = \{t \in T : \varphi \circ |x|(t) < \frac{1}{2}[\varphi \circ |y|(t) + \varphi \circ |z|(t)]\}$. If $\mu(A) > 0$ then by conditions (i) and (iii) we have

$$1 = \varrho_\varphi(x) = \|\varphi \circ |x|\|_E < \left\| \frac{1}{2}\varphi \circ |y| + \frac{1}{2}\varphi \circ |z| \right\|_E \le \frac{1}{2}\left(\|\varphi \circ |y|\|_E + \|\varphi \circ |z|\|_E\right) \le 1,$$

which is a contradiction. Consequently, Eq. (4.1) holds.

Let $C_\varphi = \{t \in T : \varphi(t, \cdot)$ is a convex and even function$\}$. It is clear that $\mu(T \setminus C_\varphi) = 0$. Next for each $t \in T$ we define $\hat{y}(t) = \varphi^{-1}(t, \varphi(t, |y(t)|))$ and $\hat{z}(t) = \varphi^{-1}(t, (\varphi(t, |z(t)|)))$. Using condition (ii) together with Eq. (4.1), in light of Remark 1.2(ii), we have $\hat{y}(t) = |y(t)|$ and $\hat{z}(t) = |z(t)|$ for μ-a.e. $t \in C_\varphi$. Consequently, by Eq. (4.1) and condition (iv) we conclude that $\varphi \circ |y|(t) = \varphi \circ |z|(t)$ for μ-a.e. $t \in C_\varphi$. We claim that $|y| = |z|$. Put $B = \{t \in C_\varphi : |y|(t) \ne |z|(t)\}$ and suppose that $\mu(B) > 0$. Thus, since $\varphi(t, \cdot)$ is an injective function on the set $[a(t), \infty)$ for all $t \in C_\varphi$ we should have

$$|y(t)| \vee |z(t)| \le a(t) \quad \text{and} \quad |y(t)| \wedge |z(t)| < a(t) \tag{4.2}$$

for all $t \in B \subset C_\varphi$. So

$$\varphi \circ |x|(t) = \frac{1}{2}[\varphi \circ |y|(t) + \varphi \circ |z|(t)] = 0$$

for all $t \in B$. Combining this equation with Eq. (4.2) and the assumption that $2x = y + z$ we obtain $|x(t)| < |a(t)|$ for all $t \in B$, which contradicts condition (ii). Hence, we have the claim. Finally, by condition (ii) and the fact that $\varphi(t, \cdot)$ is an injective function on $[a(t), \infty)$ for all $t \in C_\varphi$, in view of Eq. (4.1), we obtain that $|y(t) + z(t)| = |y(t)| + |z(t)|$ for μ-a.e. $t \in T$. This together with $|y(t)| = |z(t)|$ for μ-a.e. $t \in T$ implies that $y = z$.

Necessity. Let $x \in S(E_\varphi)$ be an extreme point of $B(E_\varphi)$. By, Lemma 4.1 we obtain that $|x|$ is a UM-point in E_φ. Thus by Theorem 3.1 we have $x(t) \ge a(t)$ for μ-a.e. $t \in T$, $\varrho_\varphi(x) = 1$ and $\varphi \circ x$ is an upper monotone point of E. Therefore, it remains only to prove that if $x \in \text{ext}\, B(E_\varphi)$ then condition (iv) holds. If not, there are $u, v \in S(E)$ such that

$$u(t) \ne v(t) \quad \text{and} \quad \varphi \circ \left[\frac{y+z}{2}\right](t) = \frac{1}{2}\left[\varphi \circ y(t) + \varphi \circ z(t)\right] = \frac{u(t)+v(t)}{2} = \varphi \circ |x|(t),$$

for μ-a.e. $t \in T$, where $y(t), z(t)$ are defined in condition (iv). Clearly, $y, z \in S(E_\varphi)$ with $y \ne z$. Consequently, $|x| \notin \text{ext}\, B(E_\varphi^+)$. Finally, Lemma 4.1 yields that $x \notin \text{ext}\, B(E_\varphi)$. □

Recall that a point $x \in S(E^+)$ is called a *strong U-point*(an SU-point for short) of $B(E^+)$ if for any $y \in S(E^+)$ with $\|x + y\|_E = 2$, we have $x = y$.

Remark 4.3 (*[17, page 387]*). If a point $x \in S(E^+)$ is an SU-point of $B(E^+)$, then x is a LM-point of E and x is an UM-point of E.

Lemma 4.4 ([17, *Lemma 7]*). *A point $x \in S(E)$ is an SU-point of $B(E)$ if and only if $|x|$ is an SU-point of $B(E^+)$.*

Theorem 4.5. *Let E be a strictly monotone Köthe space and $x \in S(E_\varphi)$. Then x is an SU-point of $B(E_\varphi)$ if and only if:*

(i) $\xi(x) < 1$;
(ii) $\mu(\{t \in \text{supp}\, x : |x|(t) \le a(t)\}) = 0$;
(iii) *if $u \in S(E^+)$ satisfies $\|u + \varphi \circ |x|\|_E = 2$ then either*

$$u = \varphi \circ |x| \quad or \quad \varphi \circ \left(\frac{|x| + y}{2}\right) < \frac{1}{2}(\varphi \circ |x| + \varphi \circ y),$$

where $y(t) = \varphi^{-1}(t, u(t))$ for all $t \in T$.

Please cite this article in press as: N. Petrot, S. Suantai, The criteria of strict monotonicity and rotundity points in generalized Calderón–Lozanovskiĭ spaces, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.120

Proof. *Necessity*. Assume that x is an SU-point of $B(E_\varphi)$. Applying Lemma 4.4, Remark 4.3 and Theorem 3.2 we see that the remainder is condition (iii). Suppose the converse, that is, there are $u \in S(E^+)$ such that $\|u + \varphi \circ |x|\|_E = 2$, $u \neq \varphi \circ |x|$ and $\varphi \circ \left(\frac{|x|+y}{2}\right) = \frac{1}{2}[\varphi \circ |x| + \varphi \circ y]$, where $y(t)$ is defined as in condition (iii). Then,

$$\varrho_\varphi(y) = \|\varphi \circ y\|_E = \|u\|_E = 1,$$

and consequently,

$$\begin{aligned} 2 = \|u + \varphi \circ |x|\|_E &= \|\varphi \circ y + \varphi \circ |x|\|_E \\ &\le \|\varphi \circ y\|_E + \|\varphi \circ |x|\|_E \\ &\le \varrho_\varphi(y) + \varrho_\varphi(x) \le 2. \end{aligned}$$

This implies that

$$\begin{aligned} \varrho_\varphi\left(\frac{|x|+y}{2}\right) &= \left\|\varphi \circ \left(\frac{x+y}{2}\right)\right\|_E \\ &= \frac{1}{2}\left[\|\varphi \circ |x| + \varphi \circ y\|_E\right] \\ &= \frac{1}{2}\left[\|\varphi \circ |x|\|_E + \|\varphi \circ y\|_E\right] \\ &= \frac{1}{2}[\varrho_\varphi(|x|) + \varrho_\varphi(y)] = 1, \end{aligned}$$

so $\left\|\frac{|x|+y}{2}\right\|_\varphi = 1$. Since $u \neq \varphi \circ |x|$, we have $|x| \neq y$, which implies that $|x|$ is not an SU-point of $B(E_\varphi^+)$. Thus, Lemma 4.4 finishes the proof of the necessity.

Sufficiency. Let $y \in S(E_\varphi)$ be such that

$$\left\|\frac{x+y}{2}\right\|_\varphi = 1. \tag{4.3}$$

We shall show that $x = y$. Combining Eq. (4.3) with condition (i), and applying Lemma 2.2, we get $\varrho_\varphi\left(\frac{x+y}{2}\right) = 1$. This gives

$$\begin{aligned} 1 = \varrho_\varphi\left(\frac{x+y}{2}\right) &= \left\|\varphi \circ \left(\frac{x+y}{2}\right)\right\|_E \\ &\le \frac{1}{2}\|\varphi \circ x + \varphi \circ y\|_E \\ &\le \frac{1}{2}\left[\varrho_\varphi(x) + \varrho_\varphi(y)\right] \\ &\le 1, \end{aligned} \tag{4.4}$$

whence

$$\|\varphi \circ x + \varphi \circ y\|_E = 2. \tag{4.5}$$

Using this equation together with the strict monotonicity of E, the fact $\varrho_\varphi\left(\frac{x+y}{2}\right) = 1$ and the convexity of $\varphi(t, \cdot)$ on $\mathbb{R}$ for all $t \in C_\varphi$, where C_φ defined as in Theorem 4.2 it is easy to see that

$$\varphi \circ \left(\frac{|x|+|y|}{2}\right)(t) = \frac{\varphi \circ |x|(t) + \varphi \circ |y|(t)}{2} \tag{4.6}$$

for μ-a.e. $t \in C_\varphi$. Put $u(t) = \varphi \circ |y|(t)$ for all $t \in T$. Then $u \in E^+$ and $\|u\|_E = \|\varphi \circ y\|_E = \varrho_\varphi(y) = 1$, by Eq. (4.4). Moreover, by virtue of condition (iii), Eqs. (4.5) and (4.6) imply that $\varphi \circ |x|(t) = \varphi \circ |y|(t)$ for μ-a.e. $t \in C_\varphi$. Since $\mu(\{t \in \operatorname{supp} x : |x|(t) \le a(t)\}) = 0$ and $\varphi(t, \cdot)$ is an injective function on the interval $[a(t), \infty)$ for μ-a.e. $t \in C_\varphi$ we get $|x|(t) = |y|(t)$ for μ-a.e. $t \in T$. Then $|x + y| \le |x| + |y| = 2|x|$. If $|x + y| < |x| + |y| = 2|x|$,

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then $\|(x+y)/2\|_\varphi < 1$ (since $|x|$ is an LM-point of E_φ by Theorem 3.2). This contradicts Eq. (4.3) and proves that $|x+y| = |x| + |y|$. Combining this equality with $|x| = |y|$, we get $x = y$. □

5. Rotundity of E_φ

In this final section we present a result concerning the rotundity structure of E_φ.

Theorem 5.1. *Let E be a Köthe space and φ be a Musielak–Orlicz function. Then $E_\varphi \in (R)$ if and only if*

(i) $E \in (SM)$;
(ii) $\varphi \in \Delta_2^E$;
(iii) *if $u, v \in S(E^+)$ with $u \neq v$ then either*

$$\left\| \frac{u+v}{2} \right\|_E < 1 \quad or \quad \varphi \circ \left(\frac{x+y}{2} \right) < \frac{1}{2}(\varphi \circ x + \varphi \circ y),$$

where $x(t) = \varphi^{-1}(t, u(t))$ and $y(t) = \varphi^{-1}(t, v(t))$ for all $t \in T$.

Proof. *Necessity*. Suppose on the contrary that $E_\varphi \in (R)$ and $E \notin (SM)$. Then an element $u \in S(E^+)$ which is not a UM-point can be found. Put $x(t) = \varphi^{-1}(t, u(t))$. Then $\varrho_\varphi(x) = \|\varphi \circ x\|_E = \|u\|_E = 1$, so $x \in S(E_\varphi)$ and hence $x \in \text{ext } B(E_\varphi)$. However, $\varphi \circ x$ is not a UM-point in E, thus Theorem 4.2 yields a contradiction.

Suppose that $E_\varphi \in (R)$ and $\varphi \notin \Delta_2^E$. By Lemma 1.3, there exists $x \in S(E_\varphi)$ with $\varrho_\varphi(x) < 1$. By $E_\varphi \in (R)$, $x \in \text{ext } B(E_\varphi)$ and Theorem 4.2 yields a contradiction.

Suppose that condition (iii) is not satisfied. Then there are $u, v \in S(E^+)$ with $u \neq v$ such that $\|u+v\|_E = 2$ and $\varphi \circ \left(\frac{x+y}{2}\right) = \frac{1}{2}(\varphi \circ x + \varphi \circ y) = \frac{u+v}{2}$, where $x(t)$, $y(t)$ are defined in condition (iii). Putting $z = \frac{x+y}{2}$, we have $\varrho_\varphi(z) = 1$, thus $z \in \text{ext } B(E_\varphi)$. Since $x \in \text{ext } B(E_\varphi)$, Theorem 4.2 yields a contradiction.

Sufficiency. Let $x \in S(E_\varphi)$ be arbitrary. We shall show that $x \in \text{ext } B(E_\varphi)$, by proving that conditions (i)–(iv) in Theorem 4.2 are satisfied. First, by $\varphi \in \Delta_2^E$ we have $\varrho_\varphi(x) = 1$ and $|x(t)| \geq a(t)$ for μ-a.e. $t \in T$ by Lemmas 1.3 and 1.5, respectively. Next, $\varphi \circ |x|$ is a UM-point in E, because $E \in (SM)$. Finally, we will show that condition (iv) in Theorem 4.2 holds. Let $u, v \in S(E)$ be such that $\frac{u+v}{2} = \varphi \circ |x|$. By condition (iii) in our assumptions, we get $\varphi \circ \left(\frac{y+z}{2}\right) < \frac{1}{2}(\varphi \circ y + \varphi \circ z)$, where $\varphi \circ y = u$ and $\varphi \circ z = v$, which means that condition (iv) from Theorem 4.2 is satisfied. Hence, our theorem is proved. □

Note that, if $E = L^1$ then $E_\varphi = \{x \in L^0 : \int_T \varphi(t, \lambda x(t)) \mathrm{d}\mu < \infty \text{ for some } \lambda > 0\} =: L^\varphi$, which is called the Musielak–Orlicz space. Therefore, a direct consequence of Theorem 5.1, we have the following result.

Corollary 5.2. *Let φ be a Musielak–Orlicz function and L^φ be the Musielak–Orlicz space generated by φ. Then $L^\varphi \in (R)$ if and only if*

(i) $\varphi \in \Delta_2^{L_1}$;
(ii) *if $u, v \in S(L_1^+)$ with $u \neq v$ then*

$$\varphi \circ \left(\frac{x+y}{2} \right) < \frac{1}{2}(\varphi \circ x + \varphi \circ y),$$

where $x(t) = \varphi^{-1}(t, u(t))$ and $y(t) = \varphi^{-1}(t, v(t))$ for all $t \in T$.

Proof. Since $L^1 \in (SM)$ and for any $u, v \in S(L_1^+)$ we must have $\|\frac{u+v}{2}\|_{L_1} = 1$, thus, the conclusion of Corollary 5.2 follows exactly from Theorem 5.1. This completes the proof. □

Remark 5.3. Rotundity properties of Musielak–Orlicz space, L^φ, equipped with the Luxemburg norm were given by Hudzik [12], in terms of the strict convexity of Musielak–Orlicz function φ. Since condition (ii) in Corollary 5.2 means that $\varphi(t, \cdot)$ is a strictly convex Musielak–Orlicz function for μ-a.e. $t \in T$, therefore, Corollary 5.2 gives a result from [12].

Please cite this article in press as: N. Petrot, S. Suantai, The criteria of strict monotonicity and rotundity points in generalized Calderón–Lozanovskiĭ spaces, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.120

Acknowledgement

The authors are thankful to the referees for their valuable suggestions that helped to improve the presentation, specially Lemma 2.2 and Theorem 5.1.

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Please cite this article in press as: N. Petrot, S. Suantai, The criteria of strict monotonicity and rotundity points in generalized Calderón–Lozanovskiĭ spaces, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.120

ภาคผนวก 2

Modified Noor iterative process by non-Lipschitzian mappings for nonlinear equations in Banach spaces

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J. Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.04.065

ARTICLE IN PRESS YJMAA:12442

Available online at www.sciencedirect.com
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J. Math. Anal. Appl. ••• (••••) •••–•••

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Modified Noor iterative process by non-Lipschitzian mappings for nonlinear equations in Banach spaces ☆

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Received 17 January 2007

Submitted by M.A. Noor

Abstract

Let E be an arbitrary real Banach space, C be a nonempty closed convex subset of E and T_1, T_2, T_3 be self maps of C. This paper proves that, the three-step Noor iterative process converges strongly to the common fixed point of T_1, T_2, T_3 when T_1 is a Φ-hemicontractive uniformly continuous mapping with bounded range and T_2, T_3 are generalized Lipschitzian mapping functions. The related result deals with the strong convergence of these sequences to the unique solution of the equation $Tx = f$ when $T: E \to E$ is uniformly continuous and Φ-strongly accretive operator. Such results improve and generalize recent known results in the literature.

Keywords: Three-step Noor iterative process; Φ-hemicontractive mapping; Generalized Lipschitzian mapping; Φ-strongly accretive operator

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and E^* be the dual space of E. Let J be the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{x^* \in E^*: \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

for all $x \in E$, where $\langle \cdot,\cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

An operator T with domain $D(T)$ and range $R(T)$ in E is said to be *generalized Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leqslant L(1 + \|x - y\|),$$

for every $x, y \in D(T)$. Without loss of generality, we may assume that $L \geqslant 1$.

☆ The present studies were supported by The Thailand Research Fund (Project No. MRG4980167) and Faculty of Science, Naresuan University, Thailand.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.04.065

We remark immediately that, if T either is Lipschitzian or has bounded range, then it is generalized Lipschitzian. On the other hand, in general, every generalized Lipschitzian operator neither is Lipschitzian nor has the bounded range. For example, let $E = (-\infty, \infty)$ and $T : E \to E$ be defined by

$$Tx = \begin{cases} x - 1, & \text{if } x \in (-\infty, -1), \\ x - \sqrt{1 - (x+1)^2}, & \text{if } x \in [-1, 0), \\ x + \sqrt{1 - (x-1)^2}, & \text{if } x \in [0, 1], \\ x + 1, & \text{if } x \in (1, +\infty). \end{cases}$$

Clearly T is generalized Lipschitzian, but T is not Lipschitzian and its range is not bounded.

An operator $T : E \to E$ is said to be *strongly pseudocontractive* if there exists a constant $k \in (0, 1)$ such that for any $x, y \in E$, there exists a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leqslant k\|x-y\|^2.$$

An operator $T : E \to E$ is said to be *Φ-strongly pseudocontractive* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for any $x, y \in E$, there exists a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leqslant \|x-y\|^2 - \Phi(\|x-y\|)\|x-y\|.$$

An operator $T : E \to E$ is said to be *strongly accretive* if there exists a constant $k \in (0, 1)$ such that for any $x, y \in E$, there exists a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geqslant (1-k)\|x-y\|^2. \tag{1.1}$$

An operator T is said to be *Φ-strongly accretive* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for any $x, y \in E$, there exists a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geqslant \Phi(\|x-y\|)(\|x-y\|). \tag{1.2}$$

Let $N(T) = \{x \in E\colon Tx = 0\}$. If $N(T) \neq \emptyset$ and inequalities (1.1)–(1.2) hold for any $x \in D(T)$ and $y \in N(T)$, then the corresponding operator T is called *strongly quasi-accretive* and *Φ-strongly quasi-accretive*, respectively.

Let $F(T) = \{x \in D(T)\colon Tx = x\}$. A mapping $T : E \to E$ is said to be a *Φ-hemicontractive* if $(I - T)$ is Φ-strongly quasi-accretive, where I is the identity mapping on E. It is very clear that, if T is hemicontractive, then $F(T) \neq \emptyset$ and there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for any $x \in D(T)$, $y \in F(T)$, there exists a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leqslant \|x-y\|^2 - \Phi(\|x-y\|)\|x-y\|. \tag{1.3}$$

The class of strongly Φ-pseudocontractive mappings includes the class of strongly pseudocontractive mappings by setting $\Phi(s) = ks$ for all $s \in [0, \infty)$. However, the converse is not true. An example by Hirano and Huang (see [7, Example 1, p. 1462]) showed that a strongly pseudocontractive operator T is not always a strongly Φ-pseudocontractive operator.

In recent year, much attention has been given to solve the nonlinear operator equations in Banach spaces by using the two-step and the one-step iterative schemes, see [2–4,8–10,17] for examples. Noor [12,13] has suggested and analyzed three-step iterative methods for finding the approximate solutions of the variational inclusions (inequalities) in a Hilbert space by using the techniques of updating the solution and the auxiliary principle. These three-step schemes are similar to those of the so-called θ-schemes of Glowinski and Le Tallec [5] for finding a zero of the sum of two (more) maximal monotone operators, which they have suggested by using the Lagrange multiplier method. Glowinski and Le Tallec [5] used these three-step iterative schemes for solving elastoviscoplasticity, liquid crystal and eigenvalue problems. They have shown that the three-step approximations perform better than the two-step and one-step iterative methods. Haubruge et al. [6] have studied the convergence analysis of the three-step schemes of Glowinski and Le Tallec [5] and applied these three-step iterations to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They have also proved that three-step iterations lead also to highly parallelized algorithms under certain conditions. It has been shown in [6,12,13] that three-step schemes are a natural generalization of the splitting methods for solving partial differential equations (inclusions). For the applications of the splitting and decomposition methods, see [1,5,6,12–14] and the references therein. Thus we conclude that three-step schemes play an important and significant part in solving various problems, which arise in pure and applied sciences.

In 2002, Noor et al. [15] suggested the following three-step iteration process for solving the nonlinear equations $Tu = 0$.

Let E is a real normed space and C be a nonempty closed convex subset of E.

Algorithm (NRH). Let $T : C \to C$ be a mapping. For given $x_0 \in C$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geqslant 0, \end{aligned} \tag{1.4}$$

which is called the three-step iterative process, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0, 1]$ satisfying some certain conditions.

If $\gamma_n = 0$ Algorithm (NRH) becomes:

Algorithm (Is). For given $x_0 \in C$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geqslant 0, \end{aligned} \tag{1.5}$$

which is called the two-step Ishikawa iterative process, and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0, 1]$ satisfying some certain conditions.

If $\gamma_n = 0$ and $\beta_n = 0$, then Algorithm (NRH) reduces to:

Algorithm (Ma). For given $x_0 \in C$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geqslant 0, \tag{1.6}$$

which is called the Mann iterative process and $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequences in $[0, 1]$ satisfying some certain conditions.

Recently, A. Rafiq [16] has proved the following theorem which is an extension of the result in [15] as following:

Theorem 1.1. *(See [16, Theorem 2].) Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be strongly pseudocontractive self maps of C with $T_1(C)$ bounded and T_1, T_3 be uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0, 1]$ satisfying the conditions:

$$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Remark 1.2. It has been observed that Theorem 1.1 contains an error. The proof of such theorem at line 15, p. 593, presented as:

$$\|y_n - x_{n+1}\| = \big\|-\beta_n(x_n - T_2 z_n) + \alpha_n(x_n - T_1 y_n)\big\| \leqslant \beta_n\|x_n - T_2 z_n\| + \alpha_n\|x_n - T_1 y_n\| \leqslant 2M(\alpha_n + \beta_n),$$

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where $M = \sup_{n\geqslant 0}\|x_n - p\| + \sup_{n\geqslant 0}\|T_1 y_n - p\| < \infty$. Since we have no assumption on operator T_2, we see that the last inequality is not assuredly hold and this is a point which may break down the conclusion of Theorem 1.1. Because, if the last inequality is not true, then the equation

$$\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0 \tag{$*$}$$

would be failed, but it is an important tool in the proof of such theorem.

On the other hand, let us observe that if $T_2(C)$ is a bounded set then Eq. $(*)$ must hold. In fact, by the boundness of $T_1(C)$, $T_2(C)$ and $\{x_n - p\}_{n=0}^{\infty}$ we have

$$\begin{aligned}\|y_n - x_{n+1}\| &\leqslant \beta_n\|x_n - T_2 z_n\| + \alpha_n\|x_n - T_1 y_n\| \\ &\leqslant \beta_n\big[\|x_n - p\| + \|T_2 z_n - p\|\big] + \alpha_n\big[\|x_n - p\| + \|T_1 y_n - p\|\big] \leqslant 2M'(\alpha_n + \beta_n),\end{aligned}$$

where $M' = M + \sup_{n\geqslant 0}\|T_2 z_n - p\|$, and consequently $(*)$ is obtained. Evidently, the boundness of $T_2(C)$ should be added in the hypothesis of Theorem 1.1.

In this paper, we study the strong convergence of three-step Noor iterative scheme for Φ-hemicontractive mapping under some suitable conditions and this is the main motivation of this paper.

2. Main results

For the purpose we need the following lemmas.

Lemma 2.1. *(See [3, Lemma 2.1].) Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\big\langle y, j(x + y)\big\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.2. *(See [11, Lemma 2.1].) Let $\Psi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\Psi(0) = 0$ and $\{a_n\}, \{b_n\}, \{c_n\}$ be nonnegative real sequences such that*

$$\lim_{n\to\infty} b_n = 0, \qquad c_n = o(b_n), \qquad \sum_{n=1}^{\infty} b_n = \infty.$$

Suppose that for all $n \geqslant 1$,

$$a_{n+1}^2 \leqslant a_n^2 - \Psi(a_{n+1})b_n + c_n,$$

then $\lim_{n\to\infty} a_n = 0$.

Now we are in position to prove our main results.

Theorem 2.3. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be self maps of C. T_1 is a Φ-hemicontractive uniformly continuous mapping with bounded range and T_2, T_3 are generalized Lipschitzian mapping functions. For any $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the three-step iterative process defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ satisfy conditions:

$$\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n \quad \text{and} \quad \sum_{n=0}^{\infty}\alpha_n = \infty. \tag{2.2}$$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Proof. Firstly, we show that $F(T_1)$ is a singleton set. Otherwise, there exist two distinct elements, say $p, q \in F(T_1)$, then by T_1 is Φ-hemicontractive mapping, there exist a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and a $j(p-q) \in J(p-q)$ such that

$$\|p-q\|^2 = \langle T_1 p - T_1 q, j(p-q)\rangle \leqslant \|p-q\|^2 - \Phi(\|p-q\|)\|p-q\| < \|p-q\|^2,$$

which is a contradiction, so we get a result. Also, if $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then it must be a singleton. Let p be the unique common fixed point of T_1, T_2, T_3.

Now, we will show that $\{x_n - p\}_{n=0}^{\infty}$ is a bounded sequence. Since T_1 has bounded range, we set

$$K = \|x_0 - p\| + \sup_{n \geqslant 0} \|T_1 y_n - p\|.$$

We will prove by induction that $\|x_n - p\| \leqslant K$ for all $n \in \mathbb{N}$. Suppose $\|x_n - p\| \leqslant K$, we consider

$$\|x_{n+1} - p\| \leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n \|T_1 y_n - p\| \leqslant (1-\alpha_n)K + \alpha_n K = K,$$

this show that $\{x_n - p\}_{n=0}^{\infty}$ is a bounded sequence. As a consequence of this one it may easily show, by using T_3 is a generalized Lipschitzian mapping, that the sequence $\{z_n - p\}_{n=0}^{\infty}$ is also bounded.

Next, observe that

$$x_{n+1} - p = (1-\alpha_n)(x_n - p) + \alpha_n(T_1 y_n - T_1 x_{n+1}) + \alpha_n(T_1 x_{n+1} - T_1 p) \tag{2.3}$$

then it follows from (1.3), (2.3) and Lemma 2.1 that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leqslant \big[(1-\alpha_n)\|x_n - p\| + \alpha_n\|T_1 y_n - T_1 x_{n+1}\|\big]^2 + 2\alpha_n\langle T_1 x_{n+1} - T_1 p, j(x_{n+1} - p)\rangle \\
&\leqslant (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n(1-\alpha_n)\|x_n - p\| \cdot \|T_1 y_n - T_1 x_{n+1}\| \\
&\quad + \alpha_n^2\|T_1 y_n - T_1 x_{n+1}\|^2 + 2\alpha_n\big[\|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|) \cdot \|x_{n+1} - p\|\big],
\end{aligned}$$

then

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leqslant \frac{(1-\alpha_n)^2}{1-2\alpha_n}\|x_n - p\|^2 - \frac{2\alpha_n}{1-2\alpha_n}\Phi(\|x_{n+1} - p\|) \cdot \|x_{n+1} - p\| \\
&\quad + \frac{2\alpha_n(1-\alpha_n)}{1-2\alpha_n}\|x_n - p\| \cdot \|T_1 y_n - T_1 x_{n+1}\| + \frac{\alpha_n^2}{1-2\alpha_n}\|T_1 y_n - T_1 x_{n+1}\|^2 \\
&\leqslant \|x_n - p\|^2 - \frac{2\alpha_n}{1-2\alpha_n}\Psi(\|x_{n+1} - p\|) + \frac{2K\alpha_n(1-\alpha_n)}{1-2\alpha_n}\|T_1 y_n - T_1 x_{n+1}\| \\
&\quad + \frac{\alpha_n^2}{1-2\alpha_n}\big[K^2 + \|T_1 y_n - T_1 x_{n+1}\|^2\big],
\end{aligned} \tag{2.4}$$

where $\Psi : [0,\infty) \to [0,\infty)$ is defined by $\Psi(t) = t\Phi(t)$, moreover, Ψ is a strictly increasing function with $\Psi(0) = 0$. Let us denote

$$a_n = \|x_n - p\|, \qquad b_n = \frac{2\alpha_n}{1-2\alpha_n},$$

and

$$c_n = \frac{2K\alpha_n(1-\alpha_n)}{1-2\alpha_n}\|T_1 y_n - T_1 x_{n+1}\| + \frac{\alpha_n^2}{1-2\alpha_n}\big[K^2 + \|T_1 y_n - T_1 x_{n+1}\|^2\big].$$

Of course, we will complete our work by using Lemma 2.2. By the way, in light of condition (2.2), we see that the remainder is

$$\lim_{n\to\infty} \|T_1 y_n - T_1 x_{n+1}\| = 0. \tag{2.5}$$

In fact, by uniform continuity of T_1, in order to obtain (2.5) we need only show that $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$. Since T_2 is a generalized Lipschitzian mapping, there exists $L_2 > 0$ such that $\|T_2 u - T_2 v\| \leqslant L_2(1 + \|u - v\|)$ for all $u, v \in C$. Put

$$L = K + L_2 + \sup_{n \geqslant 0} \|z_n - p\|$$

we have

$$\|y_n - x_{n+1}\| \leqslant \alpha_n\|x_n - T_1 y_n\| + \beta_n\|x_n - T_2 z_n\| \leqslant 2L\alpha_n + \beta_n\big[L + L(1 + \|z_n - p\|)\big]$$
$$\leqslant 2L\alpha_n + \beta_n\big(L^2 + 2L\big).$$

Combining this with condition (2.2), we deduce $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$ and consequently (2.5) holds. Finally, in view of (2.4), by using Lemma 2.2 we have

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty} a_n = 0,$$

which completes the proof. □

Theorem 2.4. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be self maps of C. T_1 is a Φ-hemicontractive uniformly continuous mapping with $T_1(C), T_2(C)$ are bounded sets. Suppose that a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as* (2.1) *when $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfy the condition* (2.2). *If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.*

Proof. As following the arguments in Theorem 2.3, we infer that the boundness of $\{x_n - p\}_{n=0}^{\infty}$ and Eq. (2.4) still obtain. Now, we show

$$\lim_{n\to\infty}\|T_1 y_n - T_1 x_{n+1}\| = 0. \tag{2.6}$$

Since, T_1 is uniformly continuous to obtain (2.6) we show $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$ is enough. Since $T_1(C), T_2(C)$ are bounded sets we have

$$\|y_n - x_{n+1}\| \leqslant \alpha_n\|x_n - T_1 y_n\| + \beta_n\|x_n - T_2 z_n\| \leqslant (\alpha_n + \beta_n)\|x_n - p\| + \alpha_n\|T_1 y_n - p\| + \beta_n\|T_2 z_n - p\|$$
$$\leqslant (\alpha_n + \beta_n)\|x_n - p\| + \alpha_n K_1 + \beta_n K_1,$$

where $K_1 = \sup_{n\geqslant 0}\|T_1 y_n - p\| + \sup_{n\geqslant 0}\|T_2 z_n - p\|$. Combining this with condition (2.2) we get (2.6), as required. Finally, the result

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty} a_n = 0$$

now follows as in the proof of Theorem 2.3. □

Remark 2.5. It is clear that every strongly pseudocontractive operator is Φ-strongly pseudocontractive, and every Φ-strongly pseudocontractive operator with a nonempty fixed point set is Φ-hemicontractive. Therefore, Theorem 2.4 contains Theorem 1.1 as a special case, but it is worth noting that the uniform continuity assumption on operator T_3 is not imposed here.

We also get the following result immediately.

Corollary 2.6. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let T be a Φ-hemicontractive uniformly continuous mapping with bounded range self map of C. For any $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the three-step iterative process defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n,\\ y_n &= (1-\beta_n)x_n + \beta_n T z_n,\\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{2.7}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfy conditions:

$$\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n \quad \text{and} \quad \sum_{n=0}^{\infty}\alpha_n = \infty,$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of T.

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Example 2.7. Let $E=\mathbb{R}$ with the usual norm and let $a,b,d\in\mathbb{R}^+$ with $a-d\geqslant 0$, put $C=[\frac{a-d}{b},\infty)$. Define $T:C\to C$ by $Tx=\frac{ax}{bx+d}$ for all $x\in C$. Observe that $(I-T)x=\frac{bx^2+x(d-a)}{bx+d}$ and $N(I-T)=\{\frac{a-d}{b}\}$. Define $\Phi:[0,\infty)\to[0,\infty)$ by $\Phi(t)=\frac{t(bt+a-d)}{bt+a}$. Hence, Φ is strictly increasing and $\Phi(0)=0$. Now, for all $x\in C$ and $y=\frac{a-d}{b}$ we have

$$\begin{aligned}\big\langle (I-T)x-(I-T)y,x-y\big\rangle &= \left(\frac{bx^2+x(d-a)}{bx+d}\right)\left(x-\frac{a-d}{b}\right)=\Phi\left(\left|x-\frac{a-d}{b}\right|\right)\cdot\left|x-\frac{a-d}{b}\right|\\ &=\Phi\big(|x-y|\big)|x-y|,\end{aligned}$$

which implies that T is a Φ-hemicontractive operator. For each $n\in\mathbb{N}$, put $\alpha_n=\frac{1}{3n}$, $\beta_n=\frac{1}{n^r}$ for some $r>1$ and γ_n is any sequence in $[0,1]$ then we have $\{x_n\}_{n=0}^{\infty}$ as define in (2.7) converges strongly to the unique fixed point $\frac{a-d}{b}\in C$.

3. Applications

Theorem 3.1. *Let E be a real Banach space and $T:E\to E$ be uniformly continuous operator. For a given $f\in E$, let x^* denote the unique solution of the equation $Tx=f$. Define the operator $H:E\to E$ by $Hx=f+x-Tx$, and suppose that the range of H is bounded. For any $x_0\in E$ let $\{x_n\}_{n=1}^{\infty}$ be the three-step iterative process defined by*

$$\begin{aligned}x_{n+1}&=(1-\alpha_n)x_n+\alpha_n Hy_n,\\ y_n&=(1-\beta_n)x_n+\beta_n Hz_n,\\ z_n&=(1-\gamma_n)x_n+\gamma_n Hx_n,\end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}\subset[0,\frac{1}{2})$ and $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}\subset[0,1]$ satisfy conditions:

$$\lim_{n\to\infty}\alpha_n=0=\lim_{n\to\infty}\beta_n\quad\text{and}\quad\sum_{n=0}^{\infty}\alpha_n=\infty.$$

If there exists a strictly increasing function $\Phi:[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$ such that for any $x\in E$, there exists a $j(x-x^*)\in J(x-x^*)$ satisfying

$$\big\langle Tx-Tx^*,j(x-x^*)\big\rangle\geqslant\Phi\big(\|x-x^*\|\big)\|x-x^*\|,$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique solution of $Tx=f$.

Proof. Observe that the operator H is uniformly continuous with bounded range and x^* is the unique fixed point of H. Now, we show H is a Φ-hemicontractive operator. For this purpose, we define an operator $S:E\to E$ by $Sx=Tx-f$. Then it is easy to see that $N(S)$ contains exactly one element, such as x^*, and S is Φ-strongly quasi-accretive operator. Indeed, by assumption on T, for each $x\in E$ we have

$$\big\langle Sx-Sx^*,j(x-x^*)\big\rangle=\big\langle Tx-Tx^*,j(x-x^*)\big\rangle\geqslant\Phi\big(\|x-x^*\|\big)\big(\|x-x^*\|\big).$$

Hence, for each $x\in E$ we have

$$\begin{aligned}\big\langle Hx-Hx^*,j(x-x^*)\big\rangle&=\big\langle f+x-Tx-x^*,j(x-x^*)\big\rangle=\big\langle x-x^*,j(x-x^*)\big\rangle-\big\langle Sx,j(x-x^*)\big\rangle\\ &\leqslant\|x-x^*\|^2-\Phi\big(\|x-x^*\|\big)\big(\|x-x^*\|\big),\end{aligned}$$

showing that H is actually a Φ-hemicontractive operator with $x^*\in F(H)$. Therefore, the conclusion of Theorem 3.1 follows exactly from Corollary 2.6. This completes the proof. □

A direct consequence of Theorem 3.1, we have the following result.

Corollary 3.2. *Let E be a real Banach space and $T:E\to E$ be uniformly continuous and Φ-strongly accretive operator. For a given $f\in E$, let x^* denote the unique solution of the equation $Tx=f$. Let H, $\{x_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem* 3.1. *Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique solution x^* of $Tx=f$.*

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Furthermore, if $f = 0$ then a condition on operator T is relaxed.

Corollary 3.3. *Let E be a real Banach space and $T : E \to E$ be uniformly continuous and Φ-strongly quasi-accretive operator. Let x^* denote the unique solution of the equation $Tx = 0$. Let H, $\{x_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem* 3.1. *Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution x^* of $Tx = 0$.*

Proof. Now, let us observe that, the operator S which defined in Theorem 3.1 is nothing but a Φ-strongly quasi-accretive operator T. Therefore, the conclusion now follows as in the proof of Theorem 3.1. □

Remark 3.4. Corollaries 2.6 and 3.2 extend and improve, excellent results, as in [4] in its four aspects:

(1) extended to the slightly more general Φ-hemicontractive and Φ-strongly accretive operators;
(2) from the Ishikawa and Mann iteration schemes to the Three-step Noor iterative process introduced in [15];
(3) abolish the boundedness of subset C in E which imposed in [4, Theorems 3.4 and 4.2];
(4) abolish the uniform smoothness of E and the Lipschitz condition on the operator T which imposed in [4, Theorem 5.2].

Remark 3.5. This paper extends and improves all related papers appeared, such as [15,16] and references therein in the following senses:

(1) form strongly pseudocontractive and strongly accretive operators to slightly more general Φ-hemicontractive and Φ-strongly accretive operators;
(2) the boundedness assumption on $T(C)$ is relaxed to the generalized Lipschitzian condition of operator.

Acknowledgment

The author is grateful to the referee for making good suggestions which improved this paper.

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Please cite this article in press as: N. Petrot, Modified Noor iterative process by non-Lipschitzian mappings for nonlinear equations in Banach spaces, J. Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.04.065