รายงานวิจัยฉบับสมบูรณ์

โครงการ การวิเคราะห์เชิงค่าเซตและเงื่อนไขทางเรขาคณิต

SET-VALUED ANALYSIS AND GEOMETRIC CONDITIONS

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และ
สำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

คำนำ

เอกสารฉบับนี้เป็นการรายงานผลการวิจัยเรื่อง การวิเคราะห์เชิงค่าเซตและเงื่อนไขทางเรขาคณิต ซึ่งได้เริ่มตั้งแต่วันที่ 1 กรกฎาคม 2549 – 30 มิถุนายน 2551 เป็นระยะเวลาทั้งสิ้น 2 ปี เป็นโครงการวิจัยทุนพัฒนาศักยภาพอาจารย์รุ่นใหม่ ปี 2549 โดยได้รับการสนับสนุนจากสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย เป็นเงินอุดหนุนทั้งสิ้น 346,000 บาท

ผู้วิจัยขอขอบพระคุณสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย ที่ได้ให้การสนับสนุนการทำวิจัยในครั้งนี้ ขอขอบพระคุณ ศาสตราจารย์ ดร.สมพงษ์ ธรรมพงษา นักวิจัยที่ปรึกษา สำหรับคำแนะนำที่มีคุณค่าจนนำไปสู่ความสำเร็จของโครงการตามจุดประสงค์ที่วางไว้ทุกประการ ผู้วิจัยขอขอบพระคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยบูรพาที่ให้ความสนับสนุนทางอ้อม (in kind) ต่อโครงการวิจัยนี้ และสุดท้ายนี้ผู้วิจัยขอขอบพระคุณบุคคลในครอบครัวที่ให้ความรักและเป็นกำลังใจให้ผู้วิจัยเสมอมา

อรรถพล แก้วขาว

สารบัญ

Abstract

We give some sufficient conditions for the Dominguez-Lorenzo condition for a Banach space X in terms of the James constant, the Jordan-von Neumann constant, and the coefficient of weak orthogonality. As a consequence, we obtain fixed point theorems for multivalued nonexpansive mappings. For a uniformly convex Banach space X, let E be a nonempty closed bounded convex subset of X, and $f: E \to E$ and $T: E \to KC(E)$ is an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping respectively. Assume in addition f and T are commuting. Then f and T have a common fixed point, i.e., there exists a point x in E such that $x = f(x) \in T(x)$.

Keywords : James constant; Jordan-von Neumann constant; Weak orthogonality; Normal structure; Fixed point; Multivalued nonexpansive mapping; Asymptotically nonexpansive mapping

บทคัดย่อ

ในการศึกษาวิจัยนี้ ได้สร้างเงื่อนไขที่เพียงพอสำหรับเงื่อนไขโดมิงเกซและลอเรนโซ สำหรับปริภูมิบานาค X เงื่อนไขดังกล่าวอยู่ในรูปของค่าคงที่เจมส์ ค่าคงที่จอร์แดน ฟอนนอยมันน์และสัมประสิทธิ์การตั้งฉากแบบอ่อน ทำให้ได้ทฤษฎีบทจุดตรึงสำหรับการส่งค่าเซตที่ไม่ขยาย สำหรับปริภูมิบานาค X ที่เป็นปริภูมินูนแบบเอกรูป เมื่อ E เป็นย่อยปิด นูนและมีขอบเขต $f: E \to E$ และ $T: E \to KC(E)$ เป็นการส่งแบบไม่ขยายและการส่งค่าเซตแบบไม่ขยายตามลำดับ ถ้า f และ T มีสมบัติสลับที่แล้ว f และ T จะมีจุดตรึงร่วม นั่นคือจะมีจุด $x \in E$ ที่ $x = f(x) \in T(x)$

คำสำคัญ : ค่าคงที่เจมส์; ค่าคงที่จอร์แดน ฟอนนอยมันน์; การตั้งฉากแบบอ่อน; โครงสร้างปกติ; จุดตรึง; การส่งค่าเซตแบบไม่ขยาย; การส่งไม่ขยายแบบเชิงเส้นกำกับ

เนื้อหาโครงการโดยสรุป
การวิเคราะห์เชิงค่าเซตและเงื่อนไขทางเรขาคณิต
Executive Summary
SET-VALUED ANALYSIS AND GEOMETRIC CONDITIONS

1. ความสำคัญและที่มาของปัญหา

ให้ E เป็นเซตย่อยปิด นูนและมีขอบเขต (bounded closed convex subset) ของปริภูมิบานาค $(X,\|\cdot\|)$, $KC(E)$ เป็นเซตของเซตย่อยของ E ที่กระชับ (compact) และนูน (convex), $K(E)$ เป็นเซตของเซตย่อยของ E ที่กระชับ (compact), และ $T:E\to K(E)$ เป็นการส่งค่าเซตที่ไม่ขยาย (non-expansive multivalued mapping) นั่นคือ T สอดคล้องกับอสมการ

$$H(T(x),T(y))\le\|x-y\| \quad \text{สำหรับทุกๆ} \quad x,y\in X$$

เมื่อ $H(\cdot,\cdot)$ คือเมตริกของ Hausdorff (the Hausdorff metric) กำหนดโดย

$$H(A,B)=\max\left\{\sup_{a\in A}\text{dist (a,B)},\sup_{b\in B}\text{dist (b,A)}\right\}$$

เมื่อ A,B เป็นเซตปิดใน X เราสนใจเงื่อนไขทางเรขาคณิต (geometric condition) บนปริภูมิ X ที่เพียงพอต่อการมีจุดตรึง (fixed point) ของการส่ง T นั่นคือมีจุด x ใน E ที่ $x\in T(x)$ การศึกษาการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยายเป็นที่น่าสนใจมาแล้วกว่า 40 ปี และพิสูจน์แล้วว่ามีบทประยุกต์ที่มีคุณค่าต่อทั้งคณิตศาสตร์เองและต่อวิชาอื่นเช่น Control Theory, Economics, Approximation Theory และอื่นๆอีกมากมาย แต่งานวิจัยและระเบียบวิธีที่เกี่ยวข้องกับเรื่องนี้มีไม่มากและยังมีปัญหาที่ยังหาคำตอบไม่ได้อีกไม่น้อย งานวิจัยนี้จะศึกษาการมีจุดตรึงของการส่งที่ไม่ขยายและหา(หรือสร้าง)เงื่อนไขทางเรขาคณิตที่เพียงพอต่อการมีจุดตรึงของการส่งที่ไม่ขยาย ซึ่งเป็นการขยายวงความรู้และสร้างระเบียบวิธีใหม่ในเรื่องนี้

2. วัตถุประสงค์ของโครงการ

1. หาผลสรุป (result) ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่เกี่ยวข้องกับสมบัติทางเรขาคณิต
2. หาเงื่อนไขทางเรขาคณิตที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T
3. เพื่อขยายวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T

3. ระเบียบวิธีวิจัย

1. รวบรวมความรู้พื้นฐานที่เกี่ยวกับทฤษฎีจุดตรึง (fixed point theory) ของการส่งค่าเซต โดยเฉพาะอย่างยิ่งสมบัติทางเรขาคณิต (geometric property) ต่าง ๆของปริภูมิที่อาจมีผลต่อการเกิดจุดตรึงของการส่งค่าเซตที่ไม่ขยาย
2. สร้าง conjecture ที่คาดว่าจะเป็นจริงเพื่อหาบทพิสูจน์ หรือหาตัวอย่างแย้งเพื่อนำไปสู่การปรับปรุง conjecture เพื่อสร้างเป็นทฤษฎีบทต่อไป
3. ส่งผลงานให้นักวิจัยที่ปรึกษาตรวจสอบและให้คำแนะนำ
4. เขียน paper และส่งให้ International Journal ทางคณิตศาสตร์พิจารณาเพื่อตีพิมพ์ต่อไป

4. แผนการดำเนินงานวิจัยตลอดโครงการในแต่ละช่วง 6 เดือน

<u>ปีที่1</u>

กิจกรรมและแผนการดำเนินงาน	วัตถุประสงค์	ช่วงเวลา	ผลที่คาดว่าจะได้ (out put)	ความก้าวหน้า
1-6 เดือน				
1. รวบรวมเอกสารที่มีอยู่ที่เกี่ยวข้องกับหัวข้อวิจัย	รวบรวมข้อมูลที่จำเป็น	เดือนที่ 1-2	ได้ทบทวนความรู้ที่จำเป็นและเป็นปัจจุบันทั้งหมดที่เกี่ยวข้องกับหัวข้อวิจัย	2%
3. พบนักวิจัยที่ปรึกษา	เพื่อนำเสนอความรู้ที่เกี่ยวข้องกับหัวข้อวิจัยและเสนอแผนวิจัยต่อนักวิจัยที่ปรึกษา	เดือนที่ 3	ได้รับคำแนะนำที่มีประโยชน์และได้แลกเปลี่ยนความคิดเห็นเกี่ยวกับแผนการดำเนินการวิจัยเพื่อความสมบูรณ์แผนงานวิจัย	5%
4. ศึกษาระเบียบวิธีที่มีอยู่เดิมและหาระเบียบวิธีใหม่ที่เกี่ยวข้องกับการมีจุดตรึงการส่งค่าเซตที่ไม่ขยาย T	สร้างระเบียบวิธีเพื่อศึกษาการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T	เดือนที่ 4-6	ได้ระเบียบวิธีและข้อสรุปเพื่อศึกษาการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T	30%
7-12 เดือน				
5. ศึกษาเงื่อนไขทางเรขาคณิตที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึง	เพื่อทราบแนวคิดและวงความรู้ที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T	เดือนที่ 7-9	ได้แนวคิดและวงความรู้ที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย	40%

ของการส่งค่าเซตที่ไม่ขยาย T			T	
6. หา(หรือสร้าง)เงื่อนไขทางเรขาคณิตที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T	เพื่อได้เงื่อนไขทางเรขาคณิตที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T	เดือนที่ 10-12	ได้เงื่อนไขทางเรขาคณิตที่เกี่ยวข้องกับค่าคงที่ทางเรขาคณิต (geometric constant) ของปริภูมิ X ที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T	55%

<u>ปีที่2</u>

กิจกรรม	วัตถุประสงค์	ช่วงเวลา	ผลที่คาดว่าจะได้ (out put)	ความก้าวหน้า
		เดือนที่ 13-18		
1. พบนักวิจัยที่ปรึกษา	รายงานความก้าวหน้าและปรึกษาหารือเกี่ยวกับผลงานที่ได้และแนวทางการดำเนินงานต่อไป	เดือนที่ 1	1. ได้รับคำแนะนำทั้งจากนักวิจัยที่ปรึกษาและผู้เชี่ยวชาญ 2. ได้ทราบข้อคิดเห็นต่างๆ เพื่อนำไปปรับปรุงผลงานให้ดีขึ้น 3. ทราบแนวทางการดำเนินการวิจัยต่อไป	60%
2. แลกเปลี่ยนแนวความคิดกับผู้เชี่ยวชาญทั้งในและต่างประเทศทางโทรศัพท์และe-mail	เพื่อทราบความคิดเห็นของผู้เชี่ยวชาญเกี่ยวกับผลงาน			
3. ศึกษาวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่มีอยู่เดิม	เพื่อทราบวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่มีอยู่เดิม	เดือนที่ 2-3	ทราบวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่มีอยู่เดิม	65%
4. สร้างระเบียบวิธีเพื่อขยายวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่มีอยู่เดิม	เพื่อได้ระเบียบวิธีเพื่อขยายวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่มีอยู่เดิม	เดือนที่ 4-6	ได้ระเบียบวิธีเพื่อขยายวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย T ที่มีอยู่เดิม	85%
		เดือนที่ 19-24		
5. พบนักวิจัยที่ปรึกษา	1. รายงานความก้าวหน้า	เดือนที่7- 8	ได้แนวทางการพัฒนาหรือ	90%

6. แลกเปลี่ยนแนวความคิดกับผู้เชี่ยวชาญทั้งในและต่างประเทศทางโทรศัพท์และe-mail	ของผลงานวิจัยทั้งหมด 2. รายงานปัญหาต่าง ๆและเพื่อได้รับการแนะนำเกี่ยวกับปัญหา 3. แลกเปลี่ยนแนวคิดเพื่อปรับปรุงผลงานให้ดีขึ้น		ปรับปรุงผลงานวิจัยให้ดีขึ้น	
7. ปรับปรุงผลงานวิจัยและพิมพ์ผลงานวิจัยที่ได้	เพื่อปรับปรุงผลงานวิจัย	เดือนที่ 9-11	ได้ผลงานที่ดีขึ้นและสมบูรณ์มากขึ้นที่คาดว่าน่าจะตีพิมพ์ได้	95%
8. พบนักวิจัยที่ปรึกษา	1. รายงานผลงานวิจัยที่สมบูรณ์ 2. ส่งผลงานวิจัยไปตีพิมพ์	เดือนที่10-12	ได้ผลงานวิจัยที่หาเงื่อนไขทางเรขาคณิตที่เพียงพอต่อการมีจุดตรึงของการส่งค่าเซตแบบไม่ขยายและขยายวงความรู้ของการมีจุดตรึงของการส่งค่าเซตที่ไม่ขยาย	100%

5. ผลงาน/หัวข้อเรื่องที่คาดว่าจะตีพิมพ์ในวารสารวิชาการระดับนานาชาติในแต่ละปี

ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : "The geometric conditions that imply the existence of fixed point for nonexpansive multivalued mappings"

ชื่อวารสารที่คาดว่าจะตีพิมพ์ : Journal of Mathematical Analysis and Applications (impact factor = 2007: 0.872)

เนื้อหางานวิจัย

Set-valued Analysis and Geometric Conditions

1 Introduction

In 1969, Nadler [27] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then some classical fixed point theorems for single valued nonexpansive mappings have been extended to multivalued nonexpansive mappings. Let X be a Banach space and let E be a nonempty bounded closed and convex subset of X. In 1974, Lim [24], using Edelstein's method of asymptotic centers, proved the existence of a fixed point for a nonempty compact-valued nonexpansive self-mapping $T : E \to K(E)$ where X is uniformly convex. Kirk and Massa [23] in 1990 extended Lim's theorem by proving that every multivalued nonexpansive self-mapping $T : E \to K(E)$ has a fixed point for a space X on which every asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, Xu [37] extended Kirk-Massa's theorem to a nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

In 2004, Domínguez Benavides and Lorenzo [12] obtained a certain relationship between the Chebyshev radius of the asymptotic center of a bounded sequence and the modulus of noncompactness. With this result and a modification of the proof in Xu [37], they were able to solve an open problem in [36] by proving that every nonempty compact and convex valued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where X is a nearly uniformly convex Banach space. Their method was generalized by Dhompongsa, Kaewcharoen, and Kaewkhao [7], and by Dhompongsa et al. [6]. In [7] the authors defined the Domínguez-Lorenzo condition ((DL)-condition, in short) and proved the existence of a fixed point for a multivalued nonexpansive and $1-\chi-$contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set and T satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of a reflexive Banach space X which satisfies the (DL)-condition. Very recently, the (DL)-condition has been studied by Wiśnicki and Wośko [32], Domínguez Benavides and Gavira [8], and Seajung [29]. It is worth to mention the main results of the first two of these papers. Wiśnicki and Wośko [32] introduced an ultrafilter version of the (DL)-condition. Their approach enables them to drop the separability condition in [7]. Domínguez Benavides and Gavira [8] proved that every uniformly smooth Banach space satisfies the (DL)-condition and hence has the weak multivalued fixed point property (see [8, Theorem 2]).

Asymptotic fixed point theorems are those theorems from which the existence of fixed points of a mapping $f : X \to X$ are derived from the behavior of the iterates f^n for large n. A mapping $f : E \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $\lim_n k_n = 1$ such that

$$\|f^n x - f^n y\| \leq k_n \|x - y\| \qquad \text{for } x, y \in E \text{ and } n = 1, 2, 3, ...$$

In 1972, Goebel and Kirk [17] proved the following theorem.

Theorem 1.1. *(Goebel and Kirk [17]). Let X be a uniformly convex Banach space, E a nonempty closed bounded convex subset of X, and $f : E \to E$ an asymptotically nonexpansive mapping. Then f has a fixed point. Moreover, the set of fixed points of f is closed convex.*

Some generalizations of this result were proved by Yu and Dai [38] when X is 2-uniformly rotund, by Martínez Yañez [26] and Xu [33] when X is $k-$uniformly rotund for some $k \geq 1$, by Xu [35] when X is nearly uniformly convex, by Lim, Tan and Xu [25] when X satisfies the uniform Opial condition and by Kim, and Xu [22] when X has uniform normal structure.

2 Preliminaries

In this section we are going to recall some concepts and results which will be used in the following sections. For more details the reader may consult, for instance,[2] and [18].

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E, by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \qquad A,B\in FB(X).$$

A multivalued mapping $T : E \to FB(X)$ is said to be a contraction if there exists a constant $k < 1$ such that

$$H(Tx,Ty) \le k\|x-y\|, \qquad x,y \in E,$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\|, \qquad x,y \in E.$$

Let

$$\chi(A) = \inf\{d > 0 : A \text{ can be covered by finitely many balls of radii} \le d\}$$

denote the Hausdorff measure of noncompactness of a bounded set A.

A multivalued mapping $F : E \to 2^X$ is said to be $1-\chi-$contraction if, for each bounded subset A of E with $\chi(A) > 0$, $F(A)$ is bounded and

$$\chi(F(A)) \le \chi(A).$$

Here $F(A) = \bigcup_{x\in A} Fx$.

The inward set of E at $x \in E$ is defined by

$$I_E(x) = \{x + \lambda(y-x) : \lambda \le 1, y \in E\}.$$

Throughout the paper we let B_X and S_X denote, respectively, the closed unit ball and the unit sphere of X. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y\in A}\|x-y\| : x \in A\}$ is called the Chebyshev radius of A. The number $\text{diam}(A) = \sup\{\|x-y\| : x,y \in A\}$ is called the diameter of A. A Banach space X has normal structure (resp. weak normal structure) if

$$r(A) < \text{diam}(A)$$

for every bounded closed (resp. weakly compact) convex subset A of X with $\text{diam}(A) > 0$.

The property WORTH was introduced by B. Sims in [30] as follows : X is said to satisfy property WORTH if for any $x \in X$ and any weakly null sequence $\{x_n\}$ in X,

$$\limsup_{n\to\infty}\|x_n - x\| = \limsup_{n\to\infty}\|x_n + x\|.$$

In [19], A. Jiménez-Melado and E. Llorens-Fuster defined the coefficient of weak orthogonality $\mu(X)$, which is defined as the infimum of the set of the real numbers $r > 0$ such that

$$\limsup_{n\to\infty}\|x + x_n\| \le r \limsup_{n\to\infty}\|x - x_n\|$$

for all $x \in X$ and for all weakly null sequences $\{x_n\}$ in X. It is known that X satisfies property WORTH if and only if $\mu(X) = 1$.

For a Banach space X, the James constant, or the nonsquare constant was defined by Gao and Lau [15] as

$$J(X) = \sup\{\|x+y\| \wedge \|x-y\| : x, y \in B_X\}.$$

The Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of X, introduced by Clarkson [4], is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero}\right\}.$$

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded subset of X and $\{x_n\}$ be a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r(E, \{x_n\}) = \inf\left\{\limsup_{n\to\infty} \|x_n - x\| : x \in E\right\},$$

$$A(E, \{x_n\}) = \left\{x \in E : \limsup_{n\to\infty} \|x_n - x\| = r(E, \{x_n\})\right\}.$$

It is known that $A(E, \{x_n\})$ is a nonempty weakly compact convex set whenever E is [18]. Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{x_n\}$ is called asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform relative to E if $\{x_n\}$ is regular and asymptotically uniform relative to E. There always exists a subsequence of $\{x_n\}$ which is regular relative to E (see [16] and [24]).

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\left\{\sup_{y\in C} \|x - y\| : x \in E\right\}.$$

The Domínguez-Lorenzo condition introduced in [7] is defined as follows :

Definition 2.1. [7, Definition 3.1] *A Banach space X is said to satisfy the Domínguez-Lorenzo condition ((DL)-condition, in short) if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,*

$$r_E(A(E, \{x_n\})) \le \lambda r(E, \{x_n\}).$$

Theorem 2.2. [7, Theorem 3.3] *Let X be a reflexive Banach space satisfying the (DL)-condition and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi-$contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition :*

$$Tx \subset I_E(x) \text{ for all } x \in E,$$

then T has a fixed point.

In 2001, Dominguez and Lorenzo proved a very interesting theorem which is a generalization of the famous theorem of Buck [3]. Before stating the theorem, we need the following concepts.

We say that a nonempty closed convex subset D of E satisfies property (ω) with respect to a mapping $f : E \to E$ if $\omega_f(x) \subseteq D$ for every $x \in D$ where

$$\omega_f(x) = \{y \in E : y = w - \lim_k f^{n_k} x \text{ for some } n_k \to \infty\}.$$

Definition 2.3. *[9] A mapping $f : E \to E$ is said to satisfy the $(\omega)-$fixed point property ($(\omega)-$fpp) if f has a fixed point in every nonempty closed convex subset D of E which satisfies (ω).*

In their theorem they concerned with the class of mappings that is larger than the class of asymptotically nonexpansive mappings.

Definition 2.4. *A mapping $f : E \to E$ is said to be weakly asymptotically nonexpansive if it satisfies the condition*

$$\limsup_n \|f^n x - f^n y\| \leq \|x - y\| \text{ for each } x, y \in E.$$

Theorem 2.5. *[9] Let X be a Banach space, E a nonempty weakly compact convex subset of X, and $f : E \to E$ a weakly asymptotically nonexpansive mapping satisfying $(\omega)-$fpp. Then there exists a nonexpansive retraction R from E onto $Fix(f)$ which satisfies :*

(i) $R \circ f = R$,

(i) *every closed convex $f-$invariant subset of E is also $R-$invariant.*

In connection with Definition 2.3 we can restate Theorem 1.1 as follows :

Theorem 2.6. *(Goebel and Kirk [17]). Let X be a uniformly convex Banach space, E a nonempty closed bounded convex subset of X, and $f : E \to E$ an asymptotically nonexpansive mapping. Then f satisfies the $(\omega)-$fpp. Moreover, the set of fixed points of f is closed convex.*

We now present a formulation of an ultrapower of Banach spaces. Let $\mathcal{U}$ be a free ultrafilter on the set of natural numbers. Consider the closed linear subspace of $l_\infty(X)$:

$$\mathcal{N} = \Big\{ \{x_n\} \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0 \Big\}.$$

The ultrapower $\widetilde{X}$ of the space X is defined as the quotient space $l_\infty(X)/\mathcal{N}$. Given an element $x = \{x_n\} \in l_\infty(X)$, $\widetilde{x}$ stands for the equivalence class of x. The quotient norm in $\widetilde{X}$ satisfies $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$. For more details about the construction of an ultrapower of a Banach space X, see [1, Aksoy and Khamsi] and [29, Sims]. Since the ultrapower $\widetilde{X}$ is finitely representable in X, $\widetilde{X}$ inherits all finite dimensional geometrical properties of X. In particular we obtain the following result.

Theorem 2.7. *A Banach space X is uniformly convex if and only if $\widetilde{X}$ is uniformly convex.*

Another property of a uniformly convex Banach space we will use is the following :

Proposition 2.8. *[34] A Banach space X is uniformly convex if and only if, for each fixed number $r > 0$, there exists a continuous function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(t) = 0 \Leftrightarrow t = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x - y\|)$$

for all $\lambda \in [0, 1]$ and all $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

3 The James constant

We are going to give a sufficient condition for the (DL)-condition in terms of the James constant and the coefficient of weak orthogonality. It is an easy consequence of the following important inequality.

Theorem 3.1. *Let X be a Banach space and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E,\{x_n\})) \leq \left(\frac{J(X)}{1+\frac{1}{\mu(X)}}\right) r(E,\{x_n\}).$$

Proof. See [21]. □

From the above theorem we immediately have the following

Corollary 3.2. *If X is a Banach space with $J(X) < 1+\dfrac{1}{\mu(X)}$, then X satisfies the (DL)-condition.*

By applying Theorem 2.2, we obtain

Corollary 3.3. *Let X be a Banach space with $J(X) < 1+\dfrac{1}{\mu(X)}$ and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi-$contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition :*

$$Tx \subset I_E(x) \text{ for all } x \in E,$$

then T has a fixed point.

Proof. See [21]. □

Remark 3.4. *Corollary 3.2 and Corollary 3.3 cover Corollary 3.5 and Corollary 3.6 of Dhompongsa, Kaewcharoen, and Kaewkhao [7], respectively. To see this, we point that the condition of being uniformly nonsquare and having property WORTH of X imply the condition $J(X) < 1+\dfrac{1}{\mu(X)}$.*

Remark 3.5. *In [20, Theorem 2], Jiménez-Melado, Llorens-Fuster, and Saejung proved that if X is a Banach space with $J(X) < 1+\dfrac{1}{\mu(X)}$, then X has normal structure, and it is proved in [7, Theorem 3.2] that the (DL)-condition implies the weak normal structure. Thus our Corollary 3.2 is stronger than Theorem 2 of [20].*

4 The Jordan-von Neumann constant

In this section, we are going to give a sufficient condition for the (DL)-condition in terms of the Jordan-von Neumann constant and the coefficient of weak orthogonality. Again, as in section 3, we need a corresponding inequality.

Theorem 4.1. *Let X be a Banach space and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E(A(E,\{x_n\})) \leq \left(\sqrt{\frac{2\mu(X)^2 C_{NJ}(X)}{\mu(X)^2+1}}-1\right) r(E,\{x_n\}).$$

Proof. See [21]. □

As a consequence of Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Let X be a Banach space. If $C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2}$, then X satisfies the (DL)-condition.*

Apply Theorem 2.2 and Corollary 4.2 to obtain the following corollary.

Corollary 4.3. *Let X be a Banach space with $C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2}$ and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $1-\chi-$contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition :*

$$Tx \subset I_E(x) \text{ for all } x \in E,$$

then T has a fixed point.

Remark 4.4. *It is shown in [29, Theorem 5] that if $C_{\mathrm{NJ}}(X) < \frac{4}{1+\mu(X)^2}$, then X satisfies the (DL)-condition. Clearly, $\frac{4}{1+\mu(X)^2} \leq 1 + \frac{1}{\mu(X)^2}$. Thus our Corollary 4.2 is better than Theorem 5 of [29].*

Remark 4.5. *Dhompongsa et al. proved in [6] that a Banach space X satisfies property (D), which is implied by the (DL)-condition, whenever $C_{\mathrm{NJ}}(X) < c_0 = 1.273....$ If we compare this result with Corollary 4.2, we observe that for those spaces X with $\mu(X)$ close to 1, the result in [6] does not apply but our Corollary 4.2 still gives information on the (DL)-condition of X.*

Remark 4.6. *As in Remark 3.5, Corollary 4.2 covers Theorem 1 of [20].*

5 A common fixed point

Before stating our theorem we introduce the following concept.

Definition 5.1. *Let E be a nonempty bounded closed convex subset of a Banach space X, $f : E \to X$, and $T : E \to FB(X)$. Then f and T are said to be commuting if for every $x, y \in E$ such that $x \in Ty$ and $fy \in E$, there holds*

$$fx \in Tfy.$$

Theorem 5.2. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $f : E \to E$, $T : E \to KC(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping respectively. Assume that f and T are commuting. Then f and T have a common fixed point, i.e., there exists a point x in E such that $x = fx \in Tx$.*

Proof. See [31]. □

Remark 5.3. *Theorem 5.2 is a generalization of Theorem 4.2 of [7].*

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Output

ได้เผยแพร่งานวิจัยโดยการติพิมพ์แล้ว 1 เรื่อง และอยู่ในระหว่างดำเนินการส่งให้วารสารพิจารณาเพื่อตีพิมพ์ 1 เรื่อง ดังนี้

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The James constant, the Jordan–von Neumann constant, weak orthogonality, and fixed points for multivalued mappings ☆

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Abstract

We give some sufficient conditions for the Domínguez–Lorenzo condition in terms of the James constant, the Jordan–von Neumann constant, and the coefficient of weak orthogonality. As a consequence, we obtain fixed point theorems for multivalued nonexpansive mappings.
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Keywords: James constant; Jordan–von Neumann constant; Weak orthogonality; Normal structure; Fixed point; Multivalued nonexpansive mapping

1. Introduction

In 1969, Nadler [14] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then some classical fixed point theorems for single valued nonexpansive mappings have been extended to multivalued nonexpansive mappings. Let X be a Banach space and let E be a nonempty bounded closed and convex subset of X. In 1974, Lim [13], using Edelstein's method of asymptotic centers, proved the existence of a fixed point for a nonempty compact-valued nonexpansive self-mapping $T : E \to K(E)$ where X is uniformly convex. Kirk and Massa [12] in 1990 extended Lim's theorem by proving that every

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multivalued nonexpansive self-mapping $T : E \to K(E)$ has a fixed point for a space X on which every asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, Xu [19] extended Kirk–Massa's theorem to a nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

In 2004, Domínguez Benavides and Lorenzo [6] obtained a certain relationship between the Chebyshev radius of the asymptotic center of a bounded sequence and the modulus of noncompactness. With this result and a modification of the proof in [19], they were able to solve an open problem in [18] by proving that every nonempty compact and convex valued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where X is a nearly uniformly convex Banach space. Their method was generalized by Dhompongsa, Kaewcharoen, and Kaewkhao [4], and by Dhompongsa et al. [3]. In [4] the authors defined the Domínguez–Lorenzo condition ((DL)-condition, in short) and proved the existence of a fixed point for a multivalued nonexpansive and $(1-\chi)$-contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set and T satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of a reflexive Banach space X which satisfies the (DL)-condition. Very recently, the (DL)-condition has been studied by Wiśnicki and Wośko [17], Domínguez Benavides and Gavira [5], and Seajung [15]. It is worth to mention the main results of the first two of these papers. Wiśnicki and Wośko [17] introduced an ultrafilter version of the (DL)-condition. Their approach enables them to drop the separability condition in [4]. Domínguez Benavides and Gavira [5] proved that every uniformly smooth Banach space satisfies the (DL)-condition and hence has the weak multivalued fixed point property (see [5, Theorem 2]).

In this paper we give two sufficient conditions for the (DL)-condition in terms of the James constant, the Jordan–von Neumann constant, and the weak orthogonality coefficient. Consequently, we obtain two fixed point theorems for multivalued nonexpansive mappings.

2. Preliminaries

In this section we are going to recall some concepts and results which will be used in the following sections. For more details the reader may consult, for instance, [1,9].

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E, by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \quad A, B \in FB(X).$$

A multivalued mapping $T : E \to FB(X)$ is said to be a contraction if there exists a constant $k < 1$ such that

$$H(Tx, Ty) \leqslant k\|x-y\|, \quad x, y \in E,$$

and T is said to be nonexpansive if

$$H(Tx, Ty) \leqslant \|x-y\|, \quad x, y \in E.$$

Let

$$\chi(A) = \inf\{d > 0\colon\ A \text{ can be covered by finitely many balls of radii} \leqslant d\}$$

denote the Hausdorff measure of noncompactness of a bounded set A.

A multivalued mapping $F: E \to 2^X$ is said to be $(1-\chi)$-contraction if, for each bounded subset A of E with $\chi(A) > 0$, $F(A)$ is bounded and

$$\chi\big(F(A)\big) \leqslant \chi(A).$$

Here $F(A) = \bigcup_{x \in A} Fx$.

The inward set of E at $x \in E$ is defined by

$$I_E(x) = \big\{x + \lambda(y - x):\ \lambda \leqslant 1,\ y \in E\big\}.$$

Throughout the paper we let B_X and S_X denote, respectively, the closed unit ball and the unit sphere of X. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y \in A} \|x - y\|:\ x \in A\}$ is called the Chebyshev radius of A. The number $\operatorname{diam}(A) = \sup\{\|x - y\|:\ x, y \in A\}$ is called the diameter of A. A Banach space X has normal structure (respectively weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

for every bounded closed (respectively weakly compact) convex subset A of X with $\operatorname{diam}(A) > 0$.

The property WORTH was introduced by B. Sims in [16] as follows: X is said to satisfy property WORTH if for any $x \in X$ and any weakly null sequence $\{x_n\}$ in X,

$$\limsup_{n\to\infty} \|x_n - x\| = \limsup_{n\to\infty} \|x_n + x\|.$$

In [10], A. Jiménez-Melado and E. Llorens-Fuster defined the coefficient of weak orthogonality $\mu(X)$, which is defined as the infimum of the set of the real numbers $r > 0$ such that

$$\limsup_{n\to\infty} \|x + x_n\| \leqslant r \limsup_{n\to\infty} \|x - x_n\|$$

for all $x \in X$ and for all weakly null sequences $\{x_n\}$ in X. It is known that X satisfies property WORTH if and only if $\mu(X) = 1$.

For a Banach space X, the James constant, or the nonsquare constant was defined by Gao and Lau [7] as

$$J(X) = \sup\big\{\|x + y\| \wedge \|x - y\|:\ x, y \in B_X\big\}.$$

The Jordan–von Neumann constant $C_{\mathrm{NJ}}(X)$ of X, introduced by Clarkson [2], is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)}:\ x, y \in X \text{ not both zero}\right\}.$$

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded subset of X and $\{x_n\}$ be a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r\big(E, \{x_n\}\big) = \inf\Big\{\limsup_{n\to\infty} \|x_n - x\|:\ x \in E\Big\},$$

$$A\big(E, \{x_n\}\big) = \Big\{x \in E:\ \limsup_{n\to\infty} \|x_n - x\| = r\big(E, \{x_n\}\big)\Big\}.$$

It is known that $A(E, \{x_n\})$ is a nonempty weakly compact convex set whenever E is [9].

Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{x_n\}$ is called asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform relative to E if $\{x_n\}$ is regular and asymptotically uniform relative to E. There always exists a subsequence of $\{x_n\}$ which is regular relative to E (see [8,13]).

If C is a bounded subset of X, the Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\Big\{\sup_{y\in C} \|x - y\|: x \in E\Big\}.$$

The Domínguez–Lorenzo condition introduced in [4] is defined as follows:

Definition 2.1. (See [4, Definition 3.1].) A Banach space X is said to satisfy the Domínguez–Lorenzo condition ((DL)-condition, in short) if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leqslant \lambda r\big(E, \{x_n\}\big).$$

Theorem 2.2. *(See [4, Theorem 3.3].) Let X be a reflexive Banach space satisfying the (DL)-condition and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $(1 - \chi)$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition*:

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then T has a fixed point.

3. The James constant

We are going to give a sufficient condition for the (DL)-condition in terms of the James constant and the coefficient of weak orthogonality. It is an easy consequence of the following important inequality.

Theorem 3.1. *Let X be a Banach space and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leqslant \left(\frac{J(X)}{1 + \frac{1}{\mu(X)}}\right) r\big(E, \{x_n\}\big).$$

Proof. Denote $r = r(E, \{x_n\})$ and $A = A(E, \{x_n\})$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can assume, by passing through a subsequence if necessary, that x_n converges weakly to some element in E, say x. We note that since $\{x_n\}$ is regular, $r(E, \{x_n\}) = r(E, \{y_n\})$ for any subsequence $\{y_n\}$ of $\{x_n\}$. Let $z \in A$. Then we have

$$\limsup_n \|x_n - z\| = r. \tag{3.1}$$

Since $(x_n - x) \stackrel{w}{\to} 0$ and by the definition of $\mu(X)$ (for short $\mu = \mu(X)$), we have the following

$$\begin{aligned} \limsup_n \|x_n - 2x + z\| &= \limsup_n \big\|(x_n - x) + (z - x)\big\| \\ &\leqslant \mu \limsup_n \big\|(x_n - x) - (z - x)\big\| \\ &= \mu r. \end{aligned} \tag{3.2}$$

Convexity of E implies that $\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \in E$ and thus we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z\right)\right\| \geqslant r. \tag{3.3}$$

On the other hand, by the weak lower semicontinuity of the $\|\cdot\|$,

$$\liminf_n \left\| \left(1-\frac{1}{\mu}\right)(x_n - x) - \left(1+\frac{1}{\mu}\right)(z-x)\right\| \geqslant \left(1+\frac{1}{\mu}\right)\|z-x\|. \tag{3.4}$$

Fix $\varepsilon > 0$ sufficiently small. Then, using (3.1)–(3.4), we obtain an integer N such that

1. $\|x_N - z\| \leqslant r + \varepsilon$.
2. $\|x_N - 2x + z\| \leqslant \mu(r+\varepsilon)$.
3. $\|x_N - (\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z)\| \geqslant r - \varepsilon$.
4. $\|(1-\frac{1}{\mu})(x_N - x) - (1+\frac{1}{\mu})(z-x)\| \geqslant (1+\frac{1}{\mu})\|z-x\|(\frac{r-\varepsilon}{r})$.

Now, put $u = \frac{1}{r+\varepsilon}(x_N - z)$ and $v = \frac{1}{\mu(r+\varepsilon)}(x_N - 2x + z)$ and use the above estimates to conclude that $u, v \in B_X$, and so that

$$\begin{aligned}
\|u+v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z-x}{r+\varepsilon} + \frac{x_N - x}{\mu(r+\varepsilon)} + \frac{z-x}{\mu(r+\varepsilon)} \right\| \\
&= \left\| \left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)}\right)(x_N - x) - \left(\frac{1}{r+\varepsilon} - \frac{1}{\mu(r+\varepsilon)}\right)(z-x) \right\| \\
&= \left(\frac{1}{r+\varepsilon}\right)\left(1+\frac{1}{\mu}\right)\left\| (x_N - x) - \left(\frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}}\right)(z-x) \right\| \\
&= \left(\frac{1}{r+\varepsilon}\right)\left(1+\frac{1}{\mu}\right)\left\| x_N - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z\right) \right\| \\
&\geqslant \left(1+\frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right), \\
\|u-v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z-x}{r+\varepsilon} - \frac{x_N - x}{\mu(r+\varepsilon)} - \frac{z-x}{\mu(r+\varepsilon)} \right\| \\
&= \left(\frac{1}{r+\varepsilon}\right)\left\| \left(1-\frac{1}{\mu}\right)(x_N - x) - \left(1+\frac{1}{\mu}\right)(z-x) \right\| \\
&\geqslant \left(1+\frac{1}{\mu}\right)\left(\frac{\|z-x\|}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
J(X) &\geqslant \| u+v\| \wedge \|u-v\| \\
&\geqslant \left(1+\frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right) \wedge \left(1+\frac{1}{\mu}\right)\left(\frac{\|z-x\|}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).
\end{aligned}$$

By the weak lower semicontinuity of the $\|\cdot\|$ again we conclude that $\|z-x\| \leqslant r$ and hence

$$\left(1+\frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right) \wedge \left(1+\frac{1}{\mu}\right)\left(\frac{\|z-x\|}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right) = \left(1+\frac{1}{\mu}\right)\left(\frac{\|z-x\|}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).$$

Therefore $J(X) \geqslant (1+\frac{1}{\mu})(\frac{\|z-x\|}{r})(\frac{r-\varepsilon}{r+\varepsilon})$. Since ε is arbitrary small, we obtain

$$J(X) \geqslant \left(1+\frac{1}{\mu}\right)\frac{\|z-x\|}{r}.$$

This holds for arbitrary $z \in A$. Hence we have

$$\sup_{z\in A}\|x-z\| \leqslant \left(\frac{J(X)}{1+\frac{1}{\mu}}\right)r,$$

and therefore,

$$r_E(A) \leqslant \left(\frac{J(X)}{1+\frac{1}{\mu}}\right)r. \qquad \square$$

From the above theorem we immediately have the following

Corollary 3.2. *If X is a Banach space with $J(X) < 1+\frac{1}{\mu(X)}$, then X satisfies the* (*DL*)*-condition.*

By applying Theorem 2.2, we obtain

Corollary 3.3. *Let X be a Banach space with $J(X) < 1+\frac{1}{\mu(X)}$ and let E be a nonempty bounded closed convex separable subset of X. If $T: E \to KC(X)$ is a nonexpansive and $(1-\chi)$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition*:

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then T has a fixed point.

Proof. Observe that $J(X) < 2$ since $\mu \geqslant 1$. Thus, X is reflexive, and then every bounded closed convex set is weakly compact. Now Theorem 2.2 and Corollary 3.2 can be applied to obtain a fixed point. $\square$

Remark 3.4. Corollaries 3.2 and 3.3 cover Corollaries 3.5 and 3.6 of Dhompongsa, Kaewcharoen, and Kaewkhao [4], respectively. To see this, we point that the condition of being uniformly nonsquare and having property WORTH of X implies the condition $J(X) < 1+\frac{1}{\mu(X)}$.

Remark 3.5. In [11, Theorem 2], Jiménez-Melado, Llorens-Fuster, and Saejung proved that if X is a Banach space with $J(X) < 1+\frac{1}{\mu(X)}$, then X has normal structure, and it is proved in [4, Theorem 3.2] that the (DL)-condition implies the weak normal structure. Thus our Corollary 3.2 is stronger than Theorem 2 of [11].

4. The Jordan–von Neumann constant

In this section, we are going to give a sufficient condition for the (DL)-condition in terms of the Jordan–von Neumann constant and the coefficient of weak orthogonality. Again, as in Section 3, we need a corresponding inequality.

Theorem 4.1. *Let X be a Banach space and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Then*

$$r_E\big(A\big(E,\{x_n\}\big)\big) \leqslant \left(\sqrt{\frac{2\mu(X)^2 C_{\mathrm{NJ}}(X)}{\mu(X)^2+1}} - 1\right) r\big(E,\{x_n\}\big).$$

Proof. Let r, A, $\{x_n\}$, x, z and μ be as in the proof of the previous theorem. Thus,

$$\limsup_n \|x_n - z\| = r \tag{4.1}$$

and

$$\limsup_n \|x_n - 2x + z\| \leqslant \mu r. \tag{4.2}$$

Since $\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z \in E$ and by the definition of r, we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geqslant r. \tag{4.3}$$

The semicontinuity of the $\|\cdot\|$ yields the following:

$$\liminf_n \big\|\big(\mu^2-1\big)(x_n - x) - \big(\mu^2+1\big)(z-x)\big\| \geqslant \big(\mu^2+1\big)\|z-x\|. \tag{4.4}$$

Now, fix $\varepsilon > 0$ sufficiently small. Then, using (4.1)–(4.4), we obtain an integer N such that

1. $\|x_N - z\| \leqslant r + \varepsilon$.
2. $\|x_N - 2x + z\| \leqslant \mu(r+\varepsilon)$.
3. $\|x_N - (\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z)\| \geqslant r - \varepsilon$.
4. $\|(\mu^2-1)(x_N - x) - (\mu^2+1)(z-x)\| \geqslant (\mu^2+1)\|z-x\|(\frac{r-\varepsilon}{r})$.

Next, put $u = \mu^2(x_N - z)$ and $v = (x_N - 2x + z)$ and use the previous estimates to obtain $\|u\| \leqslant \mu^2(r+\varepsilon)$, $\|v\| \leqslant \mu(r+\varepsilon)$, and so that

$$\begin{aligned}
\|u+v\| &= \big\|\mu^2\big((x_N - x) - (z-x)\big) + (x_N - x) + (z-x)\big\| \\
&= \big(\mu^2+1\big)\left\|(x_N - x) - \frac{\mu^2-1}{\mu^2+1}(z-x)\right\| \\
&= \big(\mu^2+1\big)\left\|x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \\
&\geqslant \big(\mu^2+1\big)(r-\varepsilon), \\
\|u-v\| &= \big\|\mu^2\big((x_N - x) - (z-x)\big) - \big((x_N - x) + (z-x)\big)\big\| \\
&= \big\|\big(\mu^2-1\big)(x_N - x) - \big(\mu^2+1\big)(z-x)\big\| \\
&\geqslant \big(\mu^2+1\big)\|z-x\|\left(\frac{r-\varepsilon}{r}\right).
\end{aligned}$$

By the definition of $C_{\mathrm{NJ}}(X)$ we see that

$$C_{\mathrm{NJ}}(X) \geqslant \frac{\|u+v\|^2+\|u-v\|^2}{2(\|u\|^2+\|v\|^2)}$$
$$\geqslant \left(\frac{\mu^2+1}{2\mu^2}\right)\left(1+\left(\frac{\|z-x\|}{r}\right)^2\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^2.$$

Letting $\varepsilon \to 0^+$ we obtain that $C_{\mathrm{NJ}}(X) \geqslant (\frac{\mu^2+1}{2\mu^2})(1+(\frac{\|z-x\|}{r})^2)$. Then we have

$$\|z-x\| \leqslant \left(\sqrt{\frac{2\mu^2 C_{\mathrm{NJ}}(X)}{\mu^2+1}-1}\right)r.$$

This holds for arbitrary $z \in A$, hence we have

$$r_x(A) \leqslant \left(\sqrt{\frac{2\mu^2 C_{\mathrm{NJ}}(X)}{\mu^2+1}-1}\right)r$$

and therefore,

$$r_E(A) \leqslant \left(\sqrt{\frac{2\mu^2 C_{\mathrm{NJ}}(X)}{\mu^2+1}-1}\right)r. \qquad \square$$

As a consequence of Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Let X be a Banach space. If $C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2}$, then X satisfies the (DL)-condition.*

Apply Theorem 2.2 and Corollary 4.2 to obtain the following corollary.

Corollary 4.3. *Let X be a Banach space with $C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2}$ and let E be a nonempty bounded closed convex separable subset of X. If $T : E \to KC(X)$ is a nonexpansive and $(1-\chi)$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition*:

$$Tx \subset I_E(x) \quad \text{for all } x \in E,$$

then T has a fixed point.

Remark 4.4. It is shown in [15, Theorem 5] that if $C_{\mathrm{NJ}}(X) < \frac{4}{1+\mu(X)^2}$, then X satisfies the (DL)-condition. Clearly, $\frac{4}{1+\mu(X)^2} \leqslant 1 + \frac{1}{\mu(X)^2}$. Thus our Corollary 4.2 is better than Theorem 5 of [15].

Remark 4.5. Dhompongsa et al. proved in [3] that a Banach space X satisfies property (D), which is implied by the (DL)-condition, whenever $C_{\mathrm{NJ}}(X) < c_0 = 1.273\ldots$. If we compare this result with Corollary 4.2, we observe that for those spaces X with $\mu(X)$ close to 1, the result in [3] does not apply but our Corollary 4.2 still gives information on the (DL)-condition of X.

Remark 4.6. As in Remark 3.5, Corollary 4.2 covers Theorem 1 of [11].

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A common fixed point for an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping *

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Abstract

Let X be a uniformly convex Banach space, E a nonempty closed bounded convex subset of X, and $f : E \to E$ and $T : E \to KC(E)$ is an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping respectively. Assume in addition f and T are commuting. Then f and T have a common fixed point, i.e., there exists a point x in E such that $x = fx \in Tx$.
Keywords: Asymptotically nonexpansive mappings, Fixed point theory, Multivalued nonexpansive mappings.

1 Introduction

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E, by $K(E)$ the family of nonempty compact subsets of E, by $FC(E)$ the family of nonempty closed convex subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\}, \quad A,B \in FB(X),$$

where $\operatorname{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B. A multivalued mapping $T : E \to F(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \le k\|x-y\|, \quad x,y \in E. \tag{1.1}$$

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In this case, we also say that T is $k-$contractive.

If (1.1) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$. Banach's contraction Principle was extended to a multivalued contraction in 1969 by Nadler [23].

One of the most celebrated results about multivalued mappings was given by T. C. Lim in 1974. By using Edelstein's method of asymptotic centers [11].

Theorem 1.1. *(Lim [21]). Let E be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $T : E \to K(E)$ a nonexpansive mapping. Then T has a fixed. point.*

Some results for multivalued nonexpansive mappings were obtained by many authors, for instance [4], [5], [6], [7], [8], [9], [12], [13], [16], [18], [25], [26], and [29].

Asymptotic fixed point theorems are those theorems from which the existence of fixed points of a mapping $f : X \to X$ are derived from the behavior of the iterates f^n for large n. A mapping $f : E \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $\lim_n k_n = 1$ such that

$$\|f^n x - f^n y\| \leq k_n \|x - y\| \qquad \text{for } x, y \in E \text{ and } n = 1, 2, 3, ...$$

In 1972, Goebel and Kirk [14] proved the following theorem.

Theorem 1.2. *(Goebel and Kirk [14]). Let X be a uniformly convex Banach space, E a nonempty closed bounded convex subset of X, and $f : E \to E$ an asymptotically nonexpansive mapping. Then f has a fixed point. Moreover, the set of fixed points of f is closed convex.*

Some generalizations of this result were proved by Yu and Dai [31] when X is 2-uniformly rotund, by Martínez Yañez [22] and Xu [27] when X is $k-$uniformly rotund for some $k \geq 1$, by Xu [28] when X is nearly uniformly convex, by Lim, Tan and Xu [20] when X satisfies the uniform Opial condition and by Kim, and Xu [19] when X has uniform normal structure.

Motivated by Theorem 1.1 and Theorem 1.2, it is the objective of this paper to prove that if E is a nonempty closed bounded convex subset of a uniformly convex Banach space and if $f : E \to E$ and $T : E \to KC(E)$ is an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping respectively. Assume in addition f and T are commuting. Then f and T have a common fixed point, i.e., there exists a point x in E such that $x = fx \in Tx$.

2 Preliminaries

Let X be a Banach space with the norm $\|\cdot\|$ and E be a nonempty subset of X. We shall write $x = w - \lim_n x_n$ when the sequence $\{x_n\}$ converges weakly to x. The Kuratowski, separation, and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the numbers:

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameters} \leq d\},$$

$$\beta(B) = \sup\{\varepsilon > 0 : \text{ there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \geq \varepsilon\},$$

where $sep(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\}$,

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radii} \leq d\}.$$

A multivalued mapping $T : E \to 2^X$ is called $\phi-$condensing (resp. $1-\phi-$contractive) where ϕ is a measure of noncompactness if, for each bounded subset B of E with $\phi(B) > 0$, there holds the inequality

$$\phi(T(B)) < \phi(B) \ \ (\text{resp. } \phi(T(B)) \leq \phi(B)).$$

Here $T(B) = \cup_{x \in B} Tx$.

Definition 2.1. *Let E be a nonempty closed subset of X. The inward set of E at $x \in E$ is given by*

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geq 1, \, y \in E\}.$$

In case E is a nonempty closed convex subset of X, we have

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geq 0, \, y \in E\}.$$

A multivalued mapping $T : E \to 2^X$ is said to be inward (resp. weakly inward) on E if

$$Tx \subset I_E(x) \ \ (\text{resp. } Tx \subset \overline{I_E(x)}) \ \text{ for all } x \in E.$$

In our main theorem, we rely heavily on the following result.

Theorem 2.2. *[3, Deimling] Let E be a nonempty bounded closed convex subset of X and $T : E \to FC(X)$ be an upper semicontinuous $\chi-$condensing mapping. Assume $Tx \cap \overline{I_E(x)} \neq \varnothing$ for all $x \in E$. Then T has a fixed point.*

In 2001, Dominguez and Lorenzo proved a very interesting theorem which is a generalization of the famous theorem of Buck [2]. Before stating the theorem, we need the following concepts.

We say that a nonempty closed convex subset D of E satisfies property (ω) with respect to a mapping $f : E \to E$ if $\omega_f(x) \subseteq D$ for every $x \in D$ where

$$\omega_f(x) = \{y \in E : y = w - \lim_k f^{n_k} x \text{ for some } n_k \to \infty\}.$$

Definition 2.3. *[10] A mapping $f : E \to E$ is said to satisfy the $(\omega)-$fixed point property $((\omega)-$fpp) if f has a fixed point in every nonempty closed convex subset D of E which satisfies (ω).*

In their theorem they concerned with the class of mappings that is larger than the class of asymptotically nonexpansive mappings.

Definition 2.4. *A mapping $f : E \to E$ is said to be weakly asymptotically nonexpansive if it satisfies the condition*

$$\limsup_n \|f^n x - f^n y\| \leq \|x - y\| \ \textit{for each } x, y \in E.$$

Theorem 2.5. *[10] Let X be a Banach space, E a nonempty weakly compact convex subset of X, and $f : E \to E$ a weakly asymptotically nonexpansive mapping satisfying $(\omega)-$fpp. Then there exists a nonexpansive retraction R from E onto $Fix(f)$, the fixed point set of f, which satisfies :*

(i) $R \circ f = R$,

(i) *every closed convex $f-$invariant subset of E is also $R-$invariant.*

In connection with Definition 2.3 we can restate Theorem 1.2 as follows :

Theorem 2.6. *(Goebel and Kirk [14]). Let X be a uniformly convex Banach space, E a nonempty closed bounded convex subset of X, and $f : E \to E$ an asymptotically nonexpansive mapping. Then f satisfies the $(\omega)-$fpp. Moreover, the set of fixed points of f is closed convex.*

We now present a formulation of an ultrapower of Banach spaces.

Let $\mathcal{U}$ be a free ultrafilter on the set of natural numbers. Consider the closed linear subspace of $l_\infty(X)$:

$$\mathcal{N} = \Big\{\{x_n\} \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\Big\}.$$

The ultrapower $\widetilde{X}$ of the space X is defined as the quotient space $l_\infty(X)/\mathcal{N}$. Given an element $x = \{x_n\} \in l_\infty(X)$, $\widetilde{x}$ stands for the equivalence class of x. The quotient norm in $\widetilde{X}$ satisfies $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$. For more details about the construction of an ultrapower of a Banach space X, see [1, Aksoy and Khamsi],[15, Geobel and Kirk], and [24, Sims]. Since the ultrapower $\widetilde{X}$ is finitely representable in X, $\widetilde{X}$ inherits all finite dimensional geometrical properties of X. In particular we obtain the following result.

Theorem 2.7. *A Banach space X is uniformly convex if and only if $\widetilde{X}$ is uniformly convex.*

Another property of a uniformly convex Banach space we will use is the following :

Proposition 2.8. *[30] A Banach space X is uniformly convex if and only if, for each fixed number $r > 0$, there exists a continuous function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(t) = 0 \Leftrightarrow t = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x-y\|)$$

for all $\lambda \in [0,1]$ and all $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$.

3 Main result

Before stating our main theorem we introduce the following concept.

Definition 3.1. *Let E be a nonempty bounded closed convex subset of a Banach space X, $f : E \to X$, and $T : E \to FB(X)$. Then f and T are said to be commuting if for every $x, y \in E$ such that $x \in Ty$ and $fy \in E$, there holds*

$$fx \in Tfy.$$

Theorem 3.2. *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X, $f : E \to E$, $T : E \to KC(E)$ an asymptotically nonexpansive mapping and a multivalued nonexpansive mapping respectively. Assume that f and T are commuting. Then f and T have a common fixed point, i.e., there exists a point x in E such that $x = fx \in Tx$.*

Proof. Theorem 1.2 guarantees that the fixed point set of f, denoted by $\mathrm{Fix}(f)$, is nonempty, closed, and convex. Let $x \in \mathrm{Fix}(f)$. Since f and T are commuting, we have $fy \in Tx$ for each $y \in Tx$. We see that, for $x \in \mathrm{Fix}(f)$, $Tx \cap \mathrm{Fix}(f) \neq \emptyset$. For a fixed element $x_0 \in \mathrm{Fix}(f)$, define a contraction $T_n : \mathrm{Fix}(f) \to \mathrm{KC}(E)$ by

$$T_n x = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx, \quad x \in \mathrm{Fix}(f).$$

It is easy to see that for each $x \in \mathrm{Fix}(f)$, $T_n x \cap \mathrm{Fix}(f) \neq \emptyset$ as T does.

Theorem 2.6 together with Theorem 2.5 guarantee that $\mathrm{Fix}(f)$ is a nonexpansive retract of E. Then we can show that $T_n : \mathrm{Fix}(f) \to \mathrm{KC}(E)$ is $\chi-$condensing. Indeed, let B be a bounded subset of $\mathrm{Fix}(f)$ and $\chi(B) > 0$. Given $d > 0$ be such that

$$B \subset \cup_{i=1}^{n} B(x_i, d), \ x_i \in E.$$

Let R be a nonexpansive retraction of E onto $\mathrm{Fix}(f)$. For each $a \in B(x_i, d) \cap B$, we have

$$\|Rx_i - a\| = \|Rx_i - Ra\| \leq \|x_i - a\| \leq d.$$

Therefore $B(x_i, d) \cap B \subset B(Rx_i, d)$ for each $i \in \{1, ..., n\}$, and hence

$$B \subset \cup_{i=1}^{n} B(Rx_i, d).$$

Since T_n is $(1 - \frac{1}{n})-$contractive,

$$T_n(B) \subset \cup_{i=1}^{n} (T_n Rx_i + (1 - \frac{1}{n})dB(0, 1)).$$

Thus

$$\chi(T_n(B)) \leq (1 - \frac{1}{n})\chi(B) < \chi(B),$$

and T_n is $\chi-$condensing.

Now we can apply Theorem 2.2 to conclude that T_n has a fixed point, say x_n. Moreover, we can show that

$$\mathrm{dist}(x_n, Tx_n) \to 0.$$

Let $\widetilde{X}$ be a Banach space ultrapower of X and

$$\dot{\mathrm{Fix}}(f) = \{\dot{x} = \widetilde{(x_n)} : x_n \equiv x \in \mathrm{Fix}(f)\}.$$

Then $\dot{\mathrm{Fix}}(f)$ is a nonempty closed convex subset of $\widetilde{X}$. Now, for each $n \in \mathbb{N}$, let y_n be the unique nearest point of x_n in Tx_n, i.e., $\|x_n - y_n\| = \mathrm{dist}(x_n, Tx_n)$. Consequently, $\widetilde{(x_n)} = \widetilde{(y_n)}$. We show now that $y_n \in \mathrm{Fix}(f)$ for each $n \in \mathbb{N}$. Indeed, by Proposition 2.8, we have, for all integers $l, m \geq 1$,

$$\left\| x_n - \frac{f^l y_n + f^m y_n}{2} \right\|^2 \leq \frac{1}{2}\|x_n - f^l y_n\|^2 + \frac{1}{2}\|x_n - f^m y_n\|^2 - \frac{1}{4}\varphi\left(\|f^l y_n - f^m y_n\|\right).$$

Since $x_n \in \text{Fix}(f)$ and $\text{Fix}(f) \subseteq \text{Fix}(f^2) \subseteq \text{Fix}(f^3) \subseteq \cdots$, we have

$$\left\| x_n - \frac{f^l y_n + f^m y_n}{2} \right\|^2 \leq \frac{1}{2} k_l^2 \|x_n - y_n\|^2 + \frac{1}{2} k_m^2 \|x_n - y_n\|^2 - \frac{1}{4} \varphi \left(\|f^l y_n - f^m y_n\| \right). \tag{3.1}$$

However,

$$\|x_n - y_n\|^2 \leq \left\| x_n - \frac{f^l y_n + f^m y_n}{2} \right\|^2$$

for all l, m. So we get from (3.1)

$$\varphi \left(\|f^l y_n - f^m y_n\| \right) \leq 4 \left[\left(\frac{1}{2} \left(k_l^2 + k_m^2 \right) - 1 \right) \|x_n - y_n\|^2 \right] \to 0$$

as $l, m \to \infty$. Hence, $\{f^i y_n\}$ is norm-Cauchy. Let

$$z = \lim_{i \to \infty} f^i y_n.$$

Since f is asymptotically nonexpansive, we have, for all i,

$$\|fz - f^{i+1} y_n\| \leq k_1 \|z - f^i y_n\|.$$

Letting $i \to \infty$ yields $\|fz - z\| \leq 0$; that is $z \in \text{Fix}(f)$. Now letting $l, m \to \infty$ in (3.1) yields

$$\|x_n - z\|^2 \leq \|x_n - y_n\|^2.$$

It follows from the uniqueness of y_n that $y_n = z \in \text{Fix}(f)$.

Since $\dot{\text{Fix}}(f)$ is a closed convex subset of a uniformly convex Banach space $\tilde{X}$, $\widetilde{(x_n)}$ has a unique nearest point $\dot{v} \in \dot{\text{Fix}}(f)$, i.e., $\|\widetilde{(x_n)} - \dot{v}\| = \text{dist}(\widetilde{(x_n)}, \dot{\text{Fix}}(f))$.

As Tv is closed and convex, we can find $v_n \in Tv$ satisfying

$$\|y_n - v_n\| = \text{dist}(y_n, Tv) \leq H(Tx_n, Tv).$$

By the similar idea of above argument, we note here that $v_n \in \text{Fix}(f)$ for each n.

It follows from the nonexpansiveness of T that

$$\|y_n - v_n\| \leq \|x_n - v\|.$$

This means

$$\|\widetilde{(y_n)} - \widetilde{(v_n)}\| \leq \|\widetilde{(x_n)} - \dot{v}\|.$$

Since $\widetilde{(x_n)} = \widetilde{(y_n)}$, we have

$$\|\widetilde{(x_n)} - \widetilde{(v_n)}\| \leq \|\widetilde{(x_n)} - \dot{v}\|. \tag{3.2}$$

Because of the compactness of Tv, there exists $w \in Tv$ such that $w = \lim_{\mathcal{U}} v_n$. It follows that $\widetilde{(v_n)} = \dot{w}$. This fact and (3.2) imply

$$\|\widetilde{(x_n)} - \dot{w}\| \leq \|\widetilde{(x_n)} - \dot{v}\|. \tag{3.3}$$

Moreover, $w \in \text{Fix}(f)$ and then $\dot{w} \in \dot{\text{Fix}}(f)$. Hence $\dot{w} = \dot{v}$ and so $v = fv = fw = w \in Tv$ which then completes the proof. □

Remark 3.3. *1. Theorem 3.2 is a generalization of Theorem 4.2 of [5].*

2. The idea that we use to verify that $y_n \in Fix(f)$ in the above argument comes from the proof of Theorem 3.5 of Kirk and Xu [19].

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