รายงานวิจัยฉบับสมบูรณ์

โครงการ “การศึกษาฟังชันก์แนวกั้นสำหรับ copositive programming”

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มกราคม ปี 2553

สัญญาเลขที่ MRG5080192

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โครงการ การศึกษาฟังชันก์แนวกั้นสำหรับ copositive programming

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

i

กิติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณสำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้การสนับสนุนทุนวิจัยมาอย่างต่อเนื่อง ขอขอบพระคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้ให้การสนับสนุนการทำวิจัยอย่างเต็มที่

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สารบัญ

หน้า

สัญญาเลขที่: MRG5080192

ชื่อโครงการ: การศึกษาฟังชันก์แนวกั้นสำหรับ copositive programming

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บทคัดย่อ

งานวิจัยนี้ได้ทำการศึกษาคุณสมบัติของฟังชันก์แนวกั้นสำหรับ copositive programming ซึ่งได้วิธีการคำนวนและประมาณค่า ค่าของฟังชันก์ Gradient และ Hessian ของฟังก์ชันดังกล่าว เรายังได้ทำการประมาณ Cone of copositive matrices โดย LMI ซึ่งสามารถนำมาประยุกต์ใช้ในการประมาณค่าของ copositive programming ได้

Abstract

In this research, we study properties of copositive barrier. We are able of compute and approximate value, gradient and hessian of the function accurately. Moreover, we approximate cone of copositive matrices via. linear matrix inequalities (LMI) or system of inequalities which can be used for approximating copositive programming.

Keywords: Copositive programming, Semidefinite programming, Linear programming, Monte Carlo methods

Executive Summary

1. โครงการ การศึกษาฟังชันก์แนวกั้นสำหรับ copositive programming

2. วัตถุประสงค์

- เพื่อศึกษาคุณสมบัติของ the self-concordant barrier ของ cone K_n เพื่อที่จะนำมาพัฒนา a polynomial small-step path-following algorithm สำหรับแก้ไขปัญหา copositive programming
- เพื่อประมาณค่า integral $\int_{\Re_+^n} e^{-x^T Qx} dx_1 ... dx_n = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{n}} \int_{\Sigma^{n-1}} \frac{d_{n-1}y}{(y^T Qy)^{n/2}}$ (and its corresponding gradients and Hessians). we propose to use some ideas from the theory of Monte Carlo methods, more precisely, methods based on the theory of Markov chains and random walks (see e.g. [5], chapter 7).
- เพื่อศึกษา acceleration technique developed by Vempala for logconvex densities เพื่อที่จะสามารถคำนวณหา integrals of the type (6) and related integrals for gradients and Hessians with a sufficient accuracy in a reasonable computation time.

3. การดำเนินการ

ทีมวิจัยได้ศึกษาคุณสมบัติของ the self-concordant barrier ของ cone K_n เพื่อที่จะนำมาพัฒนา a polynomial small-step path-following algorithm สำหรับแก้ไขปัญหา copositive programming และได้ประมาณค่า integral $\int_{\Re_+^n} e^{-x^T Qx} dx_1 ... dx_n = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{n}} \int_{\Sigma^{n-1}} \frac{d_{n-1}y}{(y^T Qy)^{n/2}}$ และค่า gradients และ Hessians โดยใช้ทฤษฎี Monte Carlo based on Markov chains and random walks. ในการศึกษาวิธีการ คำนวณ ค่าของ gradients และ Hessians ในกรณีนี้ ผลการศึกษาได้แสดงว่า เทคนิคเสนอโดย Vempala ไม่สามารถทำให้การคำนวณสำเร็จได้ภายในเวลาอันเหมาะสม ซึ่งจากการปรึกษาพูดคุยกับผู้ชำนาญการจึงไม่เหมาะที่จะใช้กระบวนการวิธีดังกล่าว ทีมวิจัยจึงได้ใช้วิธีการประมาณ cone of copositive โดยใช้ second-order approximation และได้ผลลัพธ์ ที่ดีกว่าและเป็นที่น่าพอใจ

เนื้อหางานวิจัย

1. วัตถุประสงค์

- เพื่อศึกษาคุณสมบัติของ the self-concordant barrier ของ cone K_n เพื่อที่จะนำมาพัฒนา a polynomial small-step path-following algorithm สำหรับแก้ไขปัญหา copositive programming
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2. การดำเนินงาน

2.1 การวิจัย

ทีมวิจัยได้ศึกษาคุณสมบัติของ the self-concordant barrier ของ cone K_n เพื่อที่จะนำมาพัฒนา a polynomial small-step path-following algorithm สำหรับแก้ไขปัญหา copositive programming และได้ประมาณค่า integral $\int_{\Re_+^n} e^{-x^T Qx} dx_1 ... dx_n = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{n}} \int_{\Sigma^{n-1}} \frac{d_{n-1}y}{(y^T Qy)^{n/2}}$ และค่า gradients และ Hessians โดยใช้ทฤษฎี Monte Carlo based on Markov chains and random walks. ในการศึกษาวิธีการ คำนวณ ค่าของ gradients และ Hessians ในกรณีนี้ ผลการศึกษาได้แสดงว่า เทคนิคเสนอโดย Vempala ไม่สามารถทำให้การคำนวณสำเร็จได้ภายในเวลาอันเหมาะสม ซึ่งจากการปรึกษาพูดคุยกับผู้ชำนาญการจึงไม่เหมาะที่จะใช้กระบวนการวิธีดังกล่าว ทีมวิจัยจึงได้ใช้วิธีการประมาณ cone of copositive โดยใช้ second-order approximation และได้ผลลัพธ์ ที่ดีกว่าและเป็นที่น่าพอใจ

2.2 การคุมวิทยานิพนธ์

จากการสนับสนุนการทำวิจัยครั้งนี้ ทีมวิจัยได้ผลิตนักศึกษาปริญญาโทจำนวน 1 คน ดังนี้

ชื่อนักศึกษา	ชื่อหัวข้อวิจัย/วิทยานิพนธ์	ปีที่สำเร็จการศึกษา
นางสาวอรุณวรรณ สืบศรีวิชัย (คณิตศาสตร์)	Approximating Cone of Copositive via Second order Approximation	(คาดว่า)2553

ทั้งนี้หัวหน้าโครงการวิจัยได้ตีพิมพ์ผลงานร่วมกับนักศึกษาท่านนี้อีกด้วย (ดังเอกสารแนบในภาคผนวก 1)

3. การไปเสนอผลงาน (โดย อาจารย์ ดร. ธนะศักดิ์ หมวกทองหลาง)

ได้รับเชิญให้นำเสนอการนำเสนอผลงานวิจัยในหัวข้อ Properties of Copositive Barrier ณ National University of Singapore ในช่วงเวลาที่ไปพบ Prof. Faybusovich นักวิจัยที่เป็นผู้เชี่ยวชาญในสาขาวิชาที่ทำการวิจัย ที่ประเทศสิงค์โปร์ เพื่อขอคำแนะนำ และได้รวบรวมเอกสารที่เกี่ยวข้องกับหัวข้อวิจัยและเอกสารที่จำเป็นเพิ่มเติมโดยการสืบค้นจากฐานข้อมูลต่าง ๆ ที่หาไม่ได้ในประเทศไทย

4. การเชื่อมโยงทางวิชาการกับนักวิชาการอื่น ๆทั้งในและต่างประเทศ

ปัจจุบันทีมวิจัยมีความร่วมมือทางวิชาการกับนักคณิตศาสตร์ในต่างประเทศ อาทิเช่น นักวิจัยจากประเทศ สหรัฐอเมริกา (L. Faybusovich, University of Notre Dame) และญี่ปุ่น (T. Tsuchiya, Institute of Statistical Mathematics) ซึ่งได้มีการพบปะกับท่านในงานการประชุมวิชาการระดับนานาชาติ และมีการปรึกษาความรู้ทางวิชาการกับท่านทาง email อยู่ตลอด

Output ที่ได้รับจากโครงการ

ผลงานวิจัยที่ได้รับการตอบรับเพื่อการตีพิมพ์

1) **Properties of Universal Barrier for Cone of Copositive Matrices,**

ได้งานวิจิยส่งไปตีพิมพ์ในวารสาร Computational Optimization and Applications (IMPACT: 0.886) ซึ่งได้ถูกแนะนำให้แก้ไข และได้ทำการส่งกลับไปแล้ว (Resubmitted)

2) **Approximation Copositive Programming via Semidefinite Programming using Second order Sum of Square**

ได้งานวิจิยส่งไปตีพิมพ์ในวารสาร Journal of Computational and Applied Mathematics (IMPACT: 0.569)

ภาคผนวก 1 : ผลงานที่ได้รับการตอบรับเพื่อการตีพิมพ์ในวารสารระดับนานาชาติ

PROPERTIES OF UNIVERSAL BARRIER FOR CONE OF COPOSITIVE MATRICES

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Abstract. It is shown that using ideas from the theory of Monte Carlo methods, more precisely, methods based on the theory of Markov chains and random walks, we can approximately evaluate integral for universal barrier for cone of copositive matrices, its corresponding gradients and Hessians with a sufficient accuracy in a reasonable computation time.

1. Introduction. Let Q be n by n symmetric matrix. We say that Q is copositive if $x^T Qx \geq 0$ for every $x \in \mathbb{R}^n_+$ (i.e. x is a vector in $\mathbb{R}^n$ such that all its components are nonnegative). The set of all copositive matrices forms the closed convex cone $\mathbb{K}_n$ in the vector subspace S^n of symmetric n by n matrices. S^n is a Euclidean vector space with the scalar product

$$< X, Y >= \mathrm{Tr}(XY). \tag{1.1}$$

The dual cone $\mathbb{K}^*_n$ (with respect to the scalar product (1.1)) consists of the so-called completely positive matrices, i.e. matrices of the form

$$\sum_{i=1}^{n} x^{(i)}[x^{(i)}]^T, m \geq 0 \tag{1.2}$$

where $x^{(i)} \in \mathbb{R}^n_+$, $i = 1, 2, \cdots, n + 1$.

The optimization problem of the form:

$$\mathrm{Tr}(CX) \to \min, \tag{1.3}$$

$$\mathrm{Tr}(A_{(i)}X) = b_i, \tag{1.4}$$

$$X \in \mathbb{K}_n, \tag{1.5}$$

is called a copositive programming problem. Many difficult problems in combinatorial optimization are either reduced or approximated with a high accuracy by copositive programming problems.

For example, let us consider the partitions problem on graphs. Let $G = (V, E)$ be an undirected graph on n vertices associated with adjacency matrix $A \geq 0$ so $a_{ij} > 0$ implies that the edge $(ij) \in E(G)$ with weight a_{ij}. Given m_1, m_2, and m_3 where $m_1 + m_2 + m_3 = n$, find subset S_1, S_2, and S_3 of $V(G)$ with cardinalities m_1, m_2, and m_3 respectively, such that the total weight edges between S_1 and S_2 is minimal. This problem is NP-complete. In [5], J. Povh and F. Rendl show that this problem can be equivalently reformulated as a linear programming over the cone of completely

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positive matrices (the dual cone of copositive matrices). The graph partitions appear in many applications for example in circuit board, microchip design, floor planning and analysis of bottlenecks in communication networks. Many approaches have been developed to solve this class of problems. Semidefinite programming turns out to be a very useful approach to get tractable relaxation for the graph partitioning problem. See, for more examples of copositive programming [1], [2], [3], [5]. Despite the fact that (2) - (4) belongs to the class of convex optimization problems, they are very difficult to solve. Indeed, even the problem of checking that matrix $Q \in S^n$ is on $\mathbb{K}_n$ is NP-hard (see e.g. [4]).

2. Barrier for Cone of Copositive Matrices. By using ideas from the theory of interior-point methods to approximate solutions to (1.3) - (1.5) we need to calculate the values of barrier function and its corresponding gradients and hessians with a sufficient accuracy in a reasonable computation time.

Indeed one can show that the function,

$$F(Q) = \ln \int_{\mathbb{R}_n^+} e^{-x^TQx} dx_1 \cdots dx_n, \tag{2.1}$$

$Q \in \mathbb{K}_n$ is the self-concordant barrier on the cone $\mathbb{K}_n$. For the theory of self-concordant barriers and the corresponding interior-point method see [4]. Knowledge of a self-concordant barrier, its gradient and Hessian allows one to develop a polynomial small-step path-following algorithm for solving an optimization problem of the type (1.3) - (1.5). At first glance, polynomiality of such algorithms contradicts the NP-hardness of (1.3) - (1.5) state above.

The difficulty of the realization of this program is in obtaining computable version of (2.1) and its gradient and the Hessian (it also resolve a seeming contradiction mentioned above).

The following proposition reduces the integral (2.1) to an integral over a standard simplex in $\mathbb{R}^n$

THEOREM 1. *Let*

$$\Sigma^{n-1} = \{x \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 1, x_i \geq 0, i = 1, 2, \cdots, n\}$$

be a standard simplex in $\mathbb{R}^n$, Γ *be the gamma function and* $d_{n-1}y$ *be the standard surface measure on* Σ^{n-1}. *Then*

$$\int_{\mathbb{R}_n^+} e^{-x^TQx} dx_1 \cdots dx_n = \frac{\Gamma(\frac{n}{2})}{2\sqrt{n}} \int_{\Sigma^{n-1}} \frac{d_{n-1}y}{(y^TQy)^{n/2}}. \tag{2.2}$$

Proof Consider the well-known general integral formula

$$\int_{\mathbb{R}_+^n} f(x)dx = \frac{1}{\sqrt{n}} \int_{\mathbb{R}_+} c^{n-1} \int_{\Sigma^{n-1}} f(cy) d_{n-1}y dc$$

Take $f(x) = e^{-x^TQx}$ then $f(cx) = e^{-c^2x^TQx}$. Consider

$$\begin{aligned}\int_{\mathbb{R}_+} c^{n-1} \int_{\Sigma^{n-1}} f(cy) d_{n-1}y dc &= \int_{\mathbb{R}_+} \int_{\Sigma^{n-1}} f(cy) c^{n-1} d_{n-1}y dc \\ &= \int_{\Sigma^{n-1}} \int_{\mathbb{R}_+} e^{-c^2\gamma(y)} c^{n-1} dc d_{n-1}y\end{aligned}$$

where $\gamma(y) = y^T Q y$.
Consider the following change of variables:

Let $t = c^2\gamma(y)$. Then we have $c = \sqrt{\dfrac{t}{\gamma(y)}}$ where we assume strict positivity of $\gamma(y)$. Hence $c^{n-1} = \dfrac{t^{(n-1)/2}}{\gamma(y)^{(n-1)/2}}$ and $dc = \dfrac{1}{2\sqrt{\gamma(y)}} t^{-1/2} dt$.

$$\text{Therefore, } \int_{\mathbb{R}_+} e^{-c^2\gamma(y)} c^{n-1} dc = \int_0^{+\infty} \frac{e^{-t} t^{(n-1)/2}}{\gamma(y)^{(n-1)/2}} \cdot \frac{1}{2\sqrt{\gamma(y)}} t^{-1/2} dt$$
$$= \frac{1}{2\gamma(y)^{n/2}} \int_0^{+\infty} e^{-t} t^{n/2-1} dt$$
$$= \frac{1}{2\gamma(y)^{n/2}} \Gamma(n/2).$$

Hence
$$\int_{\mathbb{R}_n^+} e^{-x^T Q x} dx_1 \cdots dx_n = \frac{\Gamma(\frac{n}{2})}{2\sqrt{n}} \int_{\Sigma^{n-1}} \frac{d_{n-1}y}{(y^T Q y)^{n/2}}. \ \blacksquare$$

For gradients and hessians for our barrier, we need the following simple calculations. Let $\Gamma(Q) = \displaystyle\int_{\mathbb{R}_+^n} e^{\mathrm{Tr}(Qxx^T)} dx$. Then $F(Q) = \ln \Gamma(Q)$.

We have $F'(Q) \cdot H = \dfrac{\Gamma'(Q) \cdot H}{\Gamma(Q)}$.

Then $F''(Q)(K, H) = \dfrac{\Gamma''(Q)(K, H)}{\Gamma(Q)} - \dfrac{(\Gamma'(Q) \cdot H)((\Gamma'(Q) \cdot K))}{\Gamma(Q)^2}$.

Since $\Gamma(Q) = \int_{\mathbb{R}_+^n} e^{\mathrm{Tr}(Qxx^T)} dx$, we get

$$\Gamma'(Q) \cdot H = -\int_{\mathbb{R}_+^n} \mathrm{Tr}(Hxx^T) e^{\mathrm{Tr}(Qxx^T)} dx$$

and

$$\Gamma''(Q)(K, H) = \int_{\mathbb{R}_+^n} \mathrm{Tr}(Hxx^T)\mathrm{Tr}(Kxx^T) e^{\mathrm{Tr}(Qxx^T)} dx \tag{2.3}$$

which leads to the following proposition.

Proposition 1. *Following notations introduced from above we have*

$$F''(Q)(H, K) = < H_F(Q)H, K >$$
$$= \frac{\int_{\mathbb{R}_+^n} Tr(Hxx^T)\, Tr(Kxx^T) e^{Tr(Qxx^T)} dx}{\int_{\mathbb{R}_+^n} e^{Tr(Qxx^T)} dx}$$
$$- \frac{\int_{\mathbb{R}_+^n} Tr(Hxx^T) e^{Tr(Qxx^T)} dx \int_{\mathbb{R}_+^n} Tr(Kxx^T) e^{Tr(Qxx^T)} dx}{[\int_{\mathbb{R}_+^n} e^{Tr(Qxx^T)} dx]^2}.$$

Proof Direct computation. $\blacksquare$

From (2.3) we have, $\Gamma''(Q)(H, H) = \int_{\mathbb{R}_+^n} \mathrm{Tr} e^{\mathrm{Tr}(Qxx^T)} (Hxx^T)^2 dx$ and $< H_F(x)h, h >= F''(x)(h, h)$. Let $H_{ij} = \dfrac{E_{ij} + E_{ji}}{2}$, then $\Gamma''(Q)(H_{ij}, H_{ij}) = \displaystyle\int_{\mathbb{R}_+^n} e^{-x^T Q x} x_i^2 x_j^2 dx$ and$\nabla\Gamma(Q) =$

$-\int_{\mathbb{R}^n_+} e^{-x^TQx}xx^Tdx$. Therefore to approximate corresponding hessians we need to approximate integral in the following types

$$\int_{\mathbb{R}^n_+} e^{-x^TQx}dx, \int_{\mathbb{R}^n_+} e^{-x^TQx}x_ix_jdx \text{ and } \int_{\mathbb{R}^n_+} e^{-x^TQx}x_i^2x_j^2dx.$$

3. Markov Chains. We use techniques from the theory of Monte Carlo methods, namely, methods based on the theory of Markov chains and random walks (see e.g. [4], chapter 7).

Let (Ω, A, P) be a probability model and $X_i : (\Omega, A) \to (R^k, B^k)$ for $i = 0, 9, \cdots$ where B^k is the Borel class on R^k. Then the stochastic process $\{X_i; i = 0, 1, \cdots\}$ is a Makov chain if

$$P(X_n \in B|X_0, \cdots, X_{n-1}) = P(X_n \in B|X_{n-1})$$

for every n and every $B \in B^k$. Another word, the conditional distribution of the state of the process at the future time n depends only on the present state. Suppose that $X_1, X_2, \cdots$ are random variables with density g with respect to Lebesgue measure on R^k statistically independent of $X_0 \sim k_0$. Let

$$W_n = \sum_{i=0}^{n} X_i$$

Then it is well-known that $\{W_n; n = 0, 1, \cdots\}$ is a Markov Chain with $K(x, B) = \int_B g(y - x)dy$. This Markov Chain is known as a random walk.

To approximate the integrals mentioned in the previous section, we need to generate uniform random points. In this paper, we use hit-and-run chains

The hit-and-run algorithm is then

(1) we start the chain by selecting $x_0 \sim k_0$.

(2) the hit: generate a unite-length direction vector $d_i \sim U(S^{k-1})$.

(3) the run: generate $\lambda_i \in R$ according to the density proportional $p(X_{i-1} + \lambda_i d_i)$ and let $X_i = X_{i-1} + \lambda_i d_i$

From (2.2),

$$\int_{\mathbb{R}_n^+} e^{-x^TQx}dx = \frac{\Gamma(\frac{n}{2})}{2\sqrt{n}} \int_{\Sigma^{n-1}} \frac{d_{n-1}y}{(y^TQy)^{n/2}}$$

which can be approximate by the following quality:

$$\int_{\mathbb{R}_n^+} e^{-x^TQx}dx \approx \frac{\Gamma(\frac{n}{2})}{2\sqrt{n}} \frac{\left(\sum_{k=1}^{N} \frac{1}{(y(k)^TQy(k))^{(n+1)/2}}\right)^{\frac{1}{n!}}}{N}$$

where $y(k)$'s are random points and $y(k) = [x^{(1)}(k), \cdots, x^{(n)}(k), 1 - \sum_{i=1}^{n} x_i(k)]^T$

The integrals $\int_{\mathbb{R}^n_+} e^{-x^TQx}x_ix_jdx$ and $\int_{\mathbb{R}^n_+} e^{-x^TQx}x_i^2x_j^2dx$ can also be approximated using similar technique.

4. Concluding and Remarks. We approximate the values of barrier function and its corresponding gradients and hessians with a sufficient accuracy in a reasonable computation time. For n less than 10 our calculation for hessians is reasonability fast and more importantly, our hessians is positive definite. However, for a larger n the calculation are quite slow. Most of the time we obtain positive definite hessians.

Knowledge of a approximation of self-concordant barrier, its gradient and Hessian allows one to develop a polynomial small-step path-following algorithm for solving an optimization problem of the type (1.3) - (1.5).

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Approximation Copositive Programming via Semidefinite Programming using Second order Sum of Square

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Abstract

In this paper, we approximate cone of copositive matrices by second order sum of square. The estimation is done by is using a set of linear matrix inequalities or a set of linear inequalities which can be used to approximate copositive programming via semidefinite programming or linear programming, respectively.

1 Introduction

Let Q be n by n symmetric matrix. We say that Q is copositive if $x^T Qx \geq 0$ for every $x \in R^n_+$ (i.e. x is a vector in R^n such that all its components are nonnegative). The set of all copositive matrices forms the closed convex cone C_n in the vector subspace S_n of symmetric n by n matrices. S_n is a Euclidean vector space with the scalar product

$$< X, Y >= tr(X^T Y). \tag{1.1}$$

The dual cone C^*_n (with respect to the scalar product (1.1)) consists of the so-called completely positive matrices, i.e. matrices of the form

$$\sum_{i=1}^{n} x^{(i)}[x^{(i)}]^T, m \geq 0 \tag{1.2}$$

where $x^{(i)} \in R^n_+$, $i = 1, 2, \cdots, n + 1$.

The optimization problem of the form:

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$$
\begin{array}{ll}
\min & tr(C^T X) \\
s.t. & AX = b \\
 & X \in C_n
\end{array}
\tag{1.3}
$$

is called a copositive programming problem. Many difficult problems in combinatorial optimization are either reduced or approximated with a high accuracy by copositive programming problems.

For example, let us consider the partitions problem on graphs. Let $G = (V, E)$ be an undirected graph on n vertices associated with adjacency matrix $A \geq 0$ so $a_{ij} > 0$ implies that the edge $(ij) \in E(G)$ with weight a_{ij}. Given m_1, m_2, and m_3 where $m_1 + m_2 + m_3 = n$, find subset S_1, S_2, and S_3 of $V(G)$ with cardinalities m_1, m_2, and m_3 respectively, such that the total weight edges between S_1 and S_2 is minimal. This problem is NP-complete. In [6], J. Povh and F. Rendl show that this problem can be equivalently reformulated as a linear programming over the cone of completely positive matrices (the dual cone of copositive matrices). The graph partitions appear in many applications for example in circuit board, microchip design, floor planning and analysis of bottlenecks in communication networks. Many approaches have been developed to solve this class of problems. Semidefinite programming turns out to be a very useful approach to get tractable relaxation for the graph partitioning problem. See, for more examples of copositive programming [1], [3], [4], [6]. Despite the fact that (1.3) belongs to the class of convex optimization problems, they are very difficult to solve. Indeed, even the problem of checking that matrix $Q \in S_n$ is on C_n is already NP-hard (see e.g. [5]).

The following are notations of a few important convex cones which we use in the paper.

• $n \times n$ symmetric matrices $S_n = \{X \in R^{n \times n}, X = X^T\}$

• $n \times n$ symmetric positive semidefinite matrices $S_n^+ = \{X \in S_n, \mathbf{y}^T X \mathbf{y} \geq 0 \ , \forall \mathbf{y} \in R^n\}$

• $n \times n$ symmetric copositive matrices $C_n = \{X \in S_n, \mathbf{y}^T X \mathbf{y} \geq 0 \ , \forall \mathbf{y} \in R_+^n\}$

• $n \times n$ symmetric completely positive matrices $C_n^* = \{x = \sum_{i=1}^k \mathbf{y}_i \mathbf{y}_i^T \ , \mathbf{y}_i \in R_+^n (i = 1, 2, ..., k)\}$

• $n \times n$ symmetrical nonnegative matrices $N_n = \{X \in S_n, X_{ij} \geq 0 (i, j = 1, 2, ..., n)\}$

• $n \times n$ symmetric doubly nonnegative matrices $D_n = S_n^+ \cap N_n$

Given $\mathbf{x} = [x_1 \ , \ x_2 \ , \ x_3 \ , \ ... \ , \ x_n]^T$ where $x_i \in R$, then The Hadamard product$\mathbf{x} \circ \mathbf{x}$ is defined as follows $\mathbf{x} \circ \mathbf{x} = [x_1^2 \ , \ x_2^2 \ , \ x_3^2 \ , \ ... \ , \ x_n^2]^T$.

Semidefinite programming problem
Consider a class of well know optimization problems known as semidefinite programming problem and its dual

$$\begin{array}{llll} (P) & \min & C \bullet X & \\ & s.t. & Ai \bullet X = b_i, \quad i = 1, 2, ..., m & \\ & & X \succcurlyeq 0 & \\ (D) & \min & b^T y & \\ & s.t. & \sum_{i=1}^{m} y_i A_i + Z = C & \\ & & Z \succcurlyeq 0 & \end{array} \tag{1.4}$$

where $C \in S_n, A_i \in S_n, i = 1, 2, \ldots, m,\ b = (b_1, b_2, \ldots, b_m)^T \in R^m$
and $X \in S_n^+, (y, Z) \in R^m \times S_n^+$ is primal and dual feasible solution respectively. Here $X \succcurlyeq (\succ) 0$ refers to X as a symmetric positive semidefinite (positive definite, respectively)matrices. $X \succcurlyeq Y$ means that $X - Y \succcurlyeq 0$.

We can rewrite constrain conditions (D) as the form

$$\sum_{i=1}^{m} y_i A_i \preceq C \quad \text{or} \quad \sum_{i=1}^{m} y_i A_i - C \preceq 0$$

In recent years, there are a lot of development in theories for solving the problems of this type. A lot of practical packages both commercial and free ware have been developed to handle a large size of the problems very efficiently such as Mosek, Sedumi etc.

2 Approximation of the Cone of Copositive Matrices

Since any $\mathbf{y} \in R^n_+$ can be written as $\mathbf{y} = \mathbf{x} \circ \mathbf{x}$ for some $\mathbf{x} \in R^n$, we can represent the copositivity requirement for an $n \times n$ symmetric matrix M as

$$P(\mathbf{x}) := (\mathbf{x} \circ \mathbf{x})^T M (\mathbf{x} \circ \mathbf{x}) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \geq 0 \quad for\ all\ \ \mathbf{x} \in R^n \tag{2.5}$$

We can represent the polynomial P as a homogeneous polynomial of degree four, where the coefficients of $(x_i x_j)(x_k x_l)$ are nonzero for $i \neq j \neq k \neq l$.

$$P(\mathbf{x}) = \tilde{\mathbf{x}}^T \tilde{M} \tilde{\mathbf{x}} \tag{2.6}$$

where $\tilde{\mathbf{x}} = [x_1^2, ..., x_n^2, x_1 x_2, x_1 x_3, ..., x_{n-1} x_n]^T$ and $\tilde{M}$ is a symmetric matrix of order $n + \frac{1}{2} n(n-1)$. In fact, $\tilde{M}$ is not uniquely determined. The non-uniqueness follows from the identities:

$$\begin{array}{l} (x_i x_j)^2 = (x_i)^2 (x_j)^2 \\ (x_i x_j)(x_i x_k) = (x_i)^2 (x_j x_k) \\ (x_i x_j)(x_k x_l) = (x_i x_k)(x_j x_l) = (x_i x_l)(x_j x_k) \end{array}$$

Sum of squares decompositions : S.O.S
Polynomial $P(\mathbf{x})$ is called the sum of squares decompositions (S.O.S) if and only if $P(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x})^2$ for some polynomial functions $f_i(\mathbf{x})$, $i = 1, ..., n$

Note For any $\mathbf{x} \in R^n$ and any multi-index $\mathbf{m} \in N_0^n$ (where $N_0 = \{0, 1, 2, ...\}$)we define $|\mathbf{m}| = \sum_i m_i$ and $\mathbf{x}^{\mathbf{m}} = \prod_i x_i^{m_i}$ the corresponding monomial of degree $|\mathbf{m}|$. And $I^n(s) = \{\mathbf{m} \in N_0^n : |\mathbf{m}| = s\}$ refers to the set of all possible exponents of monomials of degree s (there are $d = \left(\begin{array}{c} n+s-1 \\ s \end{array} \right)$) and $2I^n(s) = \{2\mathbf{m} : \mathbf{m} \in I^n(s)\}$. Finally, given a set of multi-indices I and a vector $\mathbf{x} \in R^n$, we define $[\mathbf{x}^{\mathbf{m}}]_{\mathbf{m} \in I}$ as the vector with components $\mathbf{x}^{\mathbf{m}} = \prod\limits_i x_i^{m_i}$ for each $\mathbf{m} \in I$.

Lemma 2.1. (Bomze [2]) *If $\overline{P}(x)$ is a homogeneous polynomial of degree $2s$ in n variables $\boldsymbol{x} = [x_1, ..., x_n]^T$, which has a representation*

$$\overline{P}(x) = \sum_{i=1}^{l} f_i(\boldsymbol{x})^2$$

for some polynomials $f_i(\boldsymbol{x})(i = 1, ..., n)$,then there are polynomials $h_i(\boldsymbol{x})$ which are homogeneous of degree s for all i such that $\overline{P}(x) = \sum_{i=1}^{t} h_i(\boldsymbol{x})^2$ with $1 \leqslant t \leqslant l$. Further, $\overline{P}$ has a s.o.s representation as above if and only if there is a symmetric positive-semidefinite matrix $d \times d$ matrix $\tilde{M} \in S_d^+$ such that

$$\overline{P}(\boldsymbol{x}) = \tilde{\boldsymbol{x}}^T \tilde{M} \tilde{\boldsymbol{x}}$$

where $d = \left(\begin{array}{c} n+s-1 \\ s \end{array} \right)$ and $\tilde{\boldsymbol{x}} = [\boldsymbol{x}^k]_{k \in I^n(s)} \in R^d$.

Lemma 2.2. (Bomze [2]) *Let $\overline{P}(x) = \sum_{\boldsymbol{m} \in I^n(s)} A_{\boldsymbol{m}} \boldsymbol{x}^{2m}$ be a homogeneous polynomial of degree $2s$ in n variables $\boldsymbol{x} = [x_1, ..., x_n]^T$ and define $\tilde{M} \in S_d$ and $\tilde{\boldsymbol{x}} \in R^d$ as in Lemma (2.1). Then $\overline{P}(x) = \tilde{\boldsymbol{x}}^T \tilde{M} \tilde{\boldsymbol{x}}$ if and only if*

$$\sum_{(\boldsymbol{j},\boldsymbol{k}) \in [I^n(s)]^2 : \boldsymbol{j}+\boldsymbol{k}=2\boldsymbol{m}} \tilde{M}_{\boldsymbol{j},\boldsymbol{k}} = A_{\boldsymbol{m}} \quad forall \quad \boldsymbol{m} \in I^n(s) \tag{2.7}$$

$$\sum_{(\boldsymbol{j},\boldsymbol{k}) \in [I^n(s)]^2 : \boldsymbol{j}+\boldsymbol{k}=\boldsymbol{n}} \tilde{M}_{\boldsymbol{j},\boldsymbol{k}} = 0 \quad forall \quad \boldsymbol{n} \in I^n(2s) \backslash 2I^n(s). \tag{2.8}$$

We define the cone $K_n^0 := S_n^+ + N_n = D_n^*$, the cone dual dual to that of all doubly nonnegative matrices.

Theorem 2.3. (Parrilo [7]). *$P(\boldsymbol{x}) = (\boldsymbol{x} \circ \boldsymbol{x})^T M (\boldsymbol{x} \circ \boldsymbol{x})$ allows for a polynomial s.o.s if and only if $M \in K_n^0$, i.e., if and only if $M = S + T$ for matrices $S \in S_n^+$ and $T \in N_n$.*

Higher order sufficient conditions can be derived by the polynomial:

$$P^{(r)}(\mathbf{x}) = P(\mathbf{x}) (\sum_{k=1}^{n} x_k^2)^r = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 (\sum_{k=1}^{n} x_k^2)^r \tag{2.9}$$

and we can consider $P^{(r)}(\mathbf{x})$ has a sum of squares decomposition (S.O.S) from Lemma (2.1).

Definition 2.4. (De Klerk and Pasechnik [4]) *The convex cone K_n^r consists of the matrices for which $P^{(r)}(\boldsymbol{x})$ in (2.9) allows a polynomial sum of squares decomposition.*

Obviously, these cones are contained in each other: $K_n^r \subseteq K_n^{r+1}$ for all r. This follows from

$$P^{(r+1)}(\mathbf{x}) = \sum_k x_k^2 P^{(r)}(\mathbf{x}) = \sum_{i,k} [f_i(\mathbf{x})x_k]^2$$

By explicitly calculating the coefficients $A_{\mathbf{m}}(M)$ of the homogeneous polynomial $P^{(r)}(\mathbf{x})$ of degree $2(r+2)$ and summarizing the above auxiliary results, The characterization of K_n^r is obtained [2].

Theorem 2.5. (Bomze [2]) *Let $n, r \in N$, $d = \begin{pmatrix} n+r+1 \\ r+2 \end{pmatrix}$, $\boldsymbol{m}(i,j) = \boldsymbol{m} - \boldsymbol{e}^i - \boldsymbol{e}^j$ for any $\boldsymbol{m} \in R^n$ and introduce the multinomial coefficients*

$$\begin{aligned} c(\boldsymbol{m}) &= |\boldsymbol{m}|!/\prod_i (m_i)! \quad \text{if} \quad \boldsymbol{m} \in N_0^n, \\ c(\boldsymbol{m}) &= 0 \qquad\qquad\qquad \text{if} \quad \boldsymbol{m} \in R^n \setminus N_0^n. \end{aligned} \tag{2.10}$$

For a symmetric matrix $M \in S_n$, define

$$A_{\boldsymbol{m}}(M) = \sum_{i,j} c(\boldsymbol{m}(i,j))M_{ij}. \tag{2.11}$$

Then $M \in K_n^r$ if and only if there is a symmetric positive-semidefinite $d \times d$ $\tilde{M} \in S_d^+$ such that

$$\begin{aligned} \sum_{(\boldsymbol{j},\boldsymbol{k}) \in [I^n(r+2)]^2 : \boldsymbol{j}+\boldsymbol{k} = 2\boldsymbol{m}} \tilde{M}_{\boldsymbol{j},\boldsymbol{k}} &= A_{\boldsymbol{m}}(M) \quad \text{for all} \quad \boldsymbol{m} \in I^n(r+2) \\ \sum_{(\boldsymbol{j},\boldsymbol{k}) \in [I^n(r+2)]^2 : \boldsymbol{j}+\boldsymbol{k} = \boldsymbol{n}} \tilde{M}_{\boldsymbol{j},\boldsymbol{k}} &= 0 \qquad\quad \text{for all} \quad \boldsymbol{n} \in I^n(2r+4) \setminus 2I^n(r+2) \end{aligned} \tag{2.12}$$

Lemma 2.6. (Bomze [2]) *Let M be an arbitrary $n \times n$ matrix and denote by $diagM^{(i)} = [M_{ii}]_i \in R^n$ the vector obtained by extracting the diagonal elements of M. If $A_{\boldsymbol{m}}(M)$ is defined as in (2.11), then*

$$A_{\boldsymbol{m}}(M) = \frac{c(\boldsymbol{m})}{s(s-1)}[\boldsymbol{m}^T M \boldsymbol{m} - \boldsymbol{m}^T diagM] \quad \text{for all} \quad \boldsymbol{m} \in I^n(s), \quad s \in N. \tag{2.13}$$

For $M = E_n$, we have, from $\mathbf{m}^T E_n \mathbf{m} = (\mathbf{e}^T \mathbf{m})^2 = |\mathbf{m}|^2$, thus

$$A_{\mathbf{m}}(E_n) = \frac{c(\mathbf{m})}{s(s-1)}[s^2 - s] = c(\mathbf{m}) \quad \text{for all} \quad \mathbf{m} \in I^n(s), \quad s \in N. \tag{2.14}$$

Parrilo [7] showed that $M \in K_n^1$ if the following system of linear matrix inequalities has a solution.

$$\begin{aligned} M - M^{(i)} &\in S_n^+ && ,i = 1, ..., n, \\ M_{ii}^{(i)} &= 0 && ,i = 1, ..., n, \\ M_{ii}^{(j)} + 2M_{ij}^{(i)} &= 0 && ,i \neq j, \\ M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} &\geq 0 && ,i < j < k \end{aligned} \tag{2.15}$$

where $M^{(i)} \in S_n$ for $i = 1, ..., n$. Then, Bomze and De Klerk have proven the system of linear matrix inequality is necessary and sufficient [2].

3 Second Order Sum of Squares Decomposition

By directly calculating coefficients $A_m(M)$ of the homogeneous polynomial $P^{(2)}(x)$, we obtain the characterizations of the cone $K^{(2)}$. Further, we obtain a necessary and sufficient condition for a matrix M belongs in $K^{(2)}$.

Theorem 3.1. *$M \in K_n^2$ if and only if there are n symmetric $n \times n$ matrices $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$ such that the following system of linear inequalities has a solution:*

$$\begin{aligned} M - M^{(ii)} &\in S_n^+ && ,i = 1, ..., n, \\ M_{ii}^{(ii)} &\geqslant 0 && ,i = 1, ..., n, \\ 2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} &\geqslant 0 && i \neq j \\ M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} &\geqslant 0 && i \neq j \\ 2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) &\geqslant 0 && i \neq j, j \neq k, i \neq k \\ 4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) &\geqslant 0 && i < j < k < l \end{aligned} \tag{3.16}$$

where $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$

Proof. By(2.13) for $r = 2$,we have

$$P^2(x) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \Big(\sum_{k=1}^{n} x_k^2 \Big)^2 = \tilde{x}^T \tilde{M} \tilde{x}$$

where $\tilde{x} = [x^k]_{k \in I_4^n} \in R^d$ and $d = \begin{pmatrix} n+3 \\ 4 \end{pmatrix}$

Assume that $M \in K_n^2$. By the Theorem 2.5 and (2.13) there exists a $\tilde{M} \in S_d^+$ satisfying (2.12) such that the left-hand side of (2.12) in case $n = 2m$; are satisfied as follows

(1) $\tilde{M}_{iiii,iiii}$,if $n = 8e_i$

(2) $\tilde{M}_{iiij,iiij} + 2\tilde{M}_{iiii,iijj}$,if $n = 6e_i + 2e_j$

(3) $\tilde{M}_{iijj,iijj} + 2(\tilde{M}_{iiii,jjjj} + \tilde{M}_{iiij,ijjj})$,if $n = 4e_i + 4e_j$

(4) $\tilde{M}_{iijk,iijk} + 2(\tilde{M}_{iiii,jjkk} + \tilde{M}_{iiij,ijkk} + \tilde{M}_{iiik,ijjk}) + \tilde{M}_{iijj,iikk}$,if $n = 4e_i + 2e_j + 2e_k$

(5) $\tilde{M}_{ijkl,ijkl} + 2(\tilde{M}_{iijj,kkll} + \tilde{M}_{iikk,jjll} + \tilde{M}_{iill,jjkk} + \tilde{M}_{iijk,jkll} + \tilde{M}_{iijl,jkkl}$ $+\tilde{M}_{iikl,jjkl} + \tilde{M}_{ijjk,ikll} + \tilde{M}_{ijjl,ikkl} + \tilde{M}_{ijll,ijkk})$,if $n = 2e_i + 2e_j + 2e_k + 2e_l$

Similarly, the equation (2.13) is equivalent to

$$
\begin{array}{llcl}
(6) & A_{iiii}(M) & = & M_{ii} \\
(7) & A_{iiij}(M) & = & 2(M_{ii}+M_{ij}) \\
(8) & A_{iijj}(M) & = & M_{ii}+M_{jj}+4M_{ij} \\
(9) & A_{iijk}(M) & = & 2(M_{ii}+2M_{ij}+2M_{ik}+M_{jk}) \\
(10) & A_{ijkl}(M) & = & 4(M_{ij}+M_{ik}+M_{il}+M_{jk}+M_{jl}+M_{kl})
\end{array}
$$

Letting $S_{kl}^{(ij)} = \tilde{M}_{ijkk,ijll}$ for all $(ijkl)$ and setting $M^{(ij)} = M - \frac{S^{(ij)}}{2}$. Now consider $S^{(ii)} = M - M^{(ii)}$ since $S^{(ii)} \in S_n^+$, then $M - M^{(ii)} \in S_n^+$ for $i = 1, ..., n$. Using the fact that $\tilde{M}$ is positive semidefinite, we have to following inequalities.

$$
\begin{aligned}
M_{ii}^{(ii)} &= M_{ii} - \frac{S_{ii}^{(ii)}}{2} \\
&= \tilde{M}_{iiii,iiii} - \frac{\tilde{M}_{iiii,iiii}}{2} \\
&= \frac{\tilde{M}_{iiii,iiii}}{2} \quad \geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} &= 2(M_{ii} - \frac{S_{ii}^{(ij)}}{2}) + 2(M_{ij} - \frac{S_{ij}^{(ii)}}{2}) \\
&= A_{iiij}(M) - \tilde{M}_{iiij,iiij} - \tilde{M}_{iiii,iijj} \\
&= \tilde{M}_{iiij,iiij} + 2\tilde{M}_{iiii,iijj} - \tilde{M}_{iiij,iiij} - \tilde{M}_{iiii,iijj} \\
&= \tilde{M}_{iiii,iijj} \quad \geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} &= M_{ii} - \frac{S_{ii}^{(jj)}}{2} + M_{jj} - \frac{S_{jj}^{(ii)}}{2} + 4(M_{ij} - \frac{S_{ij}^{(ij)}}{2}) \\
&= A_{iijj}(M) - \frac{\tilde{M}_{iijj,iijj}}{2} - \frac{\tilde{M}_{iijj,iijj}}{2} - 2\tilde{M}_{iiij,ijjj} \\
&= \tilde{M}_{iiii,jjjj} \quad \geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
&2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) \\
&\quad = A_{iijk}(M) - \tilde{M}_{iijk,iijk} - 2\tilde{M}_{iiik,ijjk} - 2\tilde{M}_{iiij,ijkk} - \tilde{M}_{iijj,iikk} \\
&\quad = 2\tilde{M}_{iiii,jjkk} + \tilde{M}_{iijj,iikk} \quad \geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
&4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) \\
&\quad = 4(M_{ij} + M_{ik} + M_{il} + M_{jk} + M_{jl} + M_{kl}) - 2\big(S_{ij}^{(kl)} + S_{ik}^{(jl)} + S_{il}^{(jk)} + S_{jk}^{(il)} + S_{jl}^{(ik)} + S_{kl}^{(ij)}\big) \\
&\quad = A_{ijkl}(M) + 2(\tilde{M}_{iikl,jjkl} + \tilde{M}_{iijl,jkkl} + \tilde{M}_{iijk,jkll} + \tilde{M}_{ijjl,ikkl} + \tilde{M}_{ijjk,ikll} + \tilde{M}_{ijkk,ijll}) \\
&\quad = \tilde{M}_{ijkl,ijkl} + 2\tilde{M}_{iijj,kkll} + 2\tilde{M}_{iikk,jjll} + 2\tilde{M}_{iill,jjkk} \geqslant 0
\end{aligned}
$$

Thus we have satisfied the system of LMI's (3.16). Conversely, assuming that a solution to (3.16) is given. Then we have,

$$
\begin{aligned}
P^2(x) &= \big(\sum_{i=1}^{n} x_i^2\big)^2 (x \circ x)^T M (x \circ x) \\
&= \big(\sum_{i=1}^{n} x_i^2\big)^2 (x \circ x)^T (M - M^{(ij)})(x \circ x) + \big(\sum_{i=1}^{n} x_i^2\big)^2 (x \circ x)^T (M^{(ij)})(x \circ x)
\end{aligned}
$$

Since $M - M^{(ij)} \in S_n^+$ for every i, the first sum is a s.o.s. Let now consider the second sum,

$$
\begin{aligned}
&\Big(\sum_{i=1}^{n} x_i^2\Big)^2 (x \circ x)^T (M^{(ij)})(x \circ x) \\
&\quad = \sum_{i,j,k,l} M_{kl}^{(ij)} x_i^2 x_j^2 x_k^2 x_l^2 \\
&\quad = \sum_i M_{ii}^{(ii)} x_i^8 + \sum_{i \neq j} (2M_{ii}^{(ij)} + 2M_{ij}^{(ii)}) x_i^6 x_j^2 \\
&\qquad + \sum_{i \neq j} (M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)}) x_i^4 x_j^4 \\
&\qquad + \sum_{i \neq j, j \neq k, i \neq k} 2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) x_i^4 x_j^2 x_k^2 \\
&\qquad + \sum_{i<j<k<l} 4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) x_i^2 x_j^2 x_k^2 x_l^2 \\
&\quad = \sum_i \Big(\sqrt{M_{ii}^{(ii)}} x_i^4\Big)^2 + \sum_{i \neq j} \Big(\sqrt{2M_{ii}^{(ij)} + 2M_{ij}^{(ii)}} x_i^3 x_j\Big)^2 \\
&\qquad + \sum_{i \neq j} \Big(\sqrt{M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)}} x_i^2 x_j^2\Big)^2 \\
&\qquad + \sum_{i \neq j, j \neq k, i \neq k} \Big(\sqrt{2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)})} x_i^2 x_j x_k\Big)^2 \\
&\qquad + \sum_{i<j<k<l} \Big(\sqrt{4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)})} x_i x_j x_k x_l\Big)^2
\end{aligned}
$$

Hence the second sum is also S.O.S. Therefore the proof is completed. □

A similar technique can be used to obtain a similar result for cone $C_n^{(2)}$. We obtain a system of linear inequality which can be use for approximating copositive programming via linear programming. (For more properties of $C_n^{(2)}$, see e.g. [2].)

Theorem 3.2. *$M \in C_n^2$ if and only if there are n symmetric $n \times n$ matrices $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$ such that the following system of linear inequalities has a solution:*

$$
\begin{array}{rcll}
M - M^{(ii)} & \in & N_n^+ & , i = 1, ..., n, \\
M_{ii}^{(ii)} & \geqslant & 0 & , i = 1, ..., n, \\
2M_{ii}^{(ij)} + 2M_{ij}^{(ii)} & \geqslant & 0 & i \neq j \\
M_{ii}^{(jj)} + M_{jj}^{(ii)} + 4M_{ij}^{(ij)} & \geqslant & 0 & i \neq j \\
2(M_{ii}^{(jk)} + 2M_{ij}^{(ik)} + 2M_{ik}^{(ij)} + M_{jk}^{(ii)}) & \geqslant & 0 & i \neq j, j \neq k, i \neq k \\
4(M_{ij}^{(kl)} + M_{ik}^{(jl)} + M_{il}^{(jk)} + M_{jk}^{(il)} + M_{jl}^{(ik)} + M_{kl}^{(ij)}) & \geqslant & 0 & i < j < k < l
\end{array}
\tag{3.17}
$$

where $M^{(ij)} \in S_n$ for $i = 1, ..., n$ and $j = 1, ..., n$

4 Conclusion

By directly calculating coefficients $A_m(M)$ of the homogeneous polynomial $P^{(2)}(x)$, we obtain the characterizations of the cone $K^{(2)}$. We then obtain a necessary and sufficient condition for a matrix M belongs in $K^{(2)}$ which can be used for approximating copositive programming via semidefinite programming. Moreover, we extend our approximation via linear programming by using the cone $C_n^{(2)}$.

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