รายงานวิจัยฉบับสมบูรณ์

โครงการ การประมาณโดยวิธีทำซ้ำเพื่อหาจุดตรึงและผลเฉลยของปัญหาอสมการแปรผันและความเป็นไปได้บนเซตคอนเวกซ์

Iterative approximation of fixed points and solution of variational inequality problems and convex feasibility problems

โดย ผู้ช่วยศาสตราจารย์ ดร.อิสระ อินจันทร์

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โดย

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กิตติกรรมประกาศ

งานวิจัยเรื่อง การประมาณโดยวิธีทำซ้ำเพื่อหาจุดตรึงและผลเฉลยของปัญหาอสมการแปรผันและความเป็นไปได้บนเซตคอนเวกซ์ (MRG5180026) นี้ สำเร็จลุล่วงด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และ สำนักงานคณะกรรมการอุดมศึกษา (สกอ.) ประจำปี 2551-2553 และขอขอบคุณ ศ.ดร.สมยศ พลับเที่ยง ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวรนักวิจัยที่ปรึกษา ที่ได้ให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา

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Abstract

The purposes of this research are to create new knowledge of fixed point theorem and construct several new iterative approximation methods for approximating the fixed point of nonexpansive mappings, and to solve many mathematical problems in Hilbert . We introduce the proof of new convergence theorems of a new iterative approximation method for finding the common element of the set of common fixed points of nonexpansive mappings, the set solutions of the variational inequality problems for nonlinear mappings and the set of solutions of equilibrium problems in Hilbert space. Therefore, by using the previous result, an iterative algorithm for the solution of a optimization problems was obtained.

Keywords: Iterative approximation method/ Variational inequality problem/ Equilibrium problem/ Nonexpansive mapping / Optimization problem

บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้ คือ การสร้างองค์ความรู้ใหม่ของทฤษฎีบทจุดตรึง และการสร้างวิธีการประมาณค่าแบบทำซ้ำชนิดใหม่ต่างๆ เพื่อใช้ในการประมาณค่าจุดตรึงของการส่งแบบไม่ขยาย และเพื่อแก้ไขปัญหาต่างๆ ทางคณิตศาสตร์ในปริภูมิฮิลเบิร์ต เรานำเสนอวิธีการพิสูจน์ทฤษฎีบทการลู่เข้าของวิธีการประมาณค่าแบบทำซ้ำแบบใหม่สำหรับใช้เพื่อการหาสมาชิกร่วมของเซตของจุดตรึงของการส่งแบบไม่ขยาย เซตคำตอบของอสมการเชิงแปรผันสำหรับการส่งแบบไม่เชิงเส้น และเซตคำตอบของปัญหาเชิงดุลยภาพ ในปริภูมิฮิลเบิร์ต ดังนั้นโดยการใช้ผลลัพธ์ที่ได้มาก่อนหน้านี้และขั้นตอนวิธีการแบบทำซ้ำทำให้ได้คำตอบของปัญหาค่าเหมาะสมที่สุด

คำสำคัญ : วิธีการประมาณค่าแบบทำซ้ำ / ปัญหาอสมการเชิงแปรผัน/ปัญหาเชิงดุลยภาพ/ การส่งแบบไม่ขยาย / ปัญหาค่าเหมาะสมที่สุด

บทที่ 1

บทนำ (Introduction)

การศึกษาเกี่ยวกับทฤษฎีการประมาณจุดตรึงและการประยุกต์ในปริภูมิบานาค มีผลงานเกี่ยวข้องอย่างมากมาย นับตั้งแต่ Mann [18] (1953) ได้นิยามการหาสูตรของเมตริกโดยวิธีทำซ้ำ ซึ่งวิธีทำซ้ำแบบมานน์ได้มีการศึกษาเพิ่มเติมโดย Dotson [8] (1970) และ Senter และ Dotson [33] (1974) นอกจากการประมาณจุดตรึงของ การส่งแบบไม่ขยาย (nonexpansive) แล้ววิธีการทำซ้ำแบบมานน์ยังมีประโยชน์ในการประมาณจุดตรึงของการส่งแบบไม่เป็นเชิงเส้นอื่นๆ เช่น การส่งแบบการหดเทียมอย่างเข้ม (strongly pseudo-contractive) ต่อมาจะพบว่าลำดับที่เกิดจากการทำซ้ำแบบมานน์จะลู่เข้าสู่จุดตรึงในกรณีที่ T เป็นการส่งลิฟชิทท์และการหดเทียมอย่างเข้ม อย่างไรก็ตามถ้า T เป็นการส่งแบบการหดเทียมแล้วลำดับที่เกิดจากวิธีทำซ้ำของมานน์อาจจะไม่ลู่เข้าสู่จุดตรึงของ T ดังนั้นจึงเป็นไปได้ที่จะประมาณจุดตรึงของ T ด้วยลำดับที่เกิดจากการทำซ้ำแบบอื่นๆ

วิธีการทำซ้ำแบบ อิชิคาวา จึงได้ถูกนำเสนอโดย Ishikawa [11] (1974) เพื่อประมาณหาจุดตรึงสำหรับการส่งแบบลิฟชิทท์ การหดตัวเทียม ทั้งนี้เพราะว่าในกรณีที่ T เป็นแค่การส่งหดเทียมวิธีทำซ้ำแบบมานน์ไม่สามารถทำให้ลำดับที่เกิดขึ้นลู่เข้าไปยังจุดตรึงของ T ได้

ต่อมา Schu [28] (1991) ได้ปรับการทำซ้ำของมานน์เพื่อประมาณจุดตรึงของการส่งแบบไม่ขยายเชิงเส้นกำกับ (asymptotically nonexpansive) และปรับปรุงวิธีการทำซ้ำของอิชิคาวาเพื่อประมาณจุดตรึงของการส่งแบบการหดเทียมเชิงเส้นกำกับ จากที่กล่าวมาข้างตอนจะพบว่าลำดับที่เกิดจากการทำซ้ำขอ งมานน์จะลู่เข้าสู่จุดตรึงของการส่ง T อย่างอ่อน ในปริภูมิฮิลเบิร์ตด้วย

สำหรับการศึกษาเกี่ยวกับอสมการแปรผัน (variational inequality) เริ่มขึ้นในปี ค.ศ. 1964 โดย G. Stampacchia [32] และหลังจากนั้นเป็นต้นมาได้มีผลงานวิจัยที่เกี่ยวข้องกับหัวข้อดังกล่าวอย่างมากมาย โดยมีการพัฒนาการศึกษาทั้งเกี่ยวกับทฤษฎีบทและวิธีการเชิงตัวเลข (numerical method) เพื่อให้ได้ผลที่ดีขึ้น ในที่นี้จะกล่าวถึงเฉพาะผลงานวิจัยที่สำคัญๆและปัญหาที่ผู้ดำเนินการวิจัยสนใจ ดังนี้

ให้ H เป็นปริภูมิฮิลเบิร์ตโดยที่ $\langle \cdot,\cdot \rangle$ และ $\|\cdot\|$ แทนผลคูณภายใน (inner product) และ นอร์ม (norm) บน H ตามลำดับ สำหรับสับเซตย่อยปิดที่ไม่เป็นเซตว่าง K ของปริภูมิฮิลเบิร์ต H ให้ $T:K\to H$ เป็นการส่งแบบไม่ขยาย (nonexpansive mapping) และจะใช้สัญลักษณ์ P_K แทน การฉาย (projection) ของ H ไปยังเซตนูน K

ในปี 1964 G. Stampacchia [32] ได้ศึกษาเกี่ยวกับการการหาคำตอบของอสมการซึ่งเป็นแบบจำลองที่สามารถนำประยุกต์ใช้ได้ในหลายๆแขนงวิชาทั้งวิทยาศาสตร์บริสุทธิ์และวิทยาศาสตร์ประยุกต์ นั่นคือหาสมาชิก $u\in K$ ซึ่งทำให้

$$\langle Tu, v-u\rangle \geq 0 \qquad \text{สำหรับทุกๆ} \quad v\in K \text{ -------------------------------- (V)}$$

ซึ่งอสมการดังกล่าวเรียกว่า ***อสมการแปรผัน (Variational inequalities)*** และสมาชิก $u\in K$ จะเรียกว่า *คำตอบของอสมการแปรผัน (V)* โดย และได้แสดงความสัมพันธ์ว่า “$u^*\in K$ *จะเป็นคำตอบของอสมการ (V) ก็ต่อเมื่อ* u^* *สอดคล้องความสัมพันธ์* $u^* = P_K[u^* - \rho Tu^*]$ *เมื่อ* $\rho > 0$ *เป็นจำนวนจริง* “ ซึ่ง

จากความสัมพันธ์ดังกล่าวข้างต้นจะเห็นว่าคำตอบของอสมการแปรผัน (V) มีความสัมพันธ์กับการศึกษาเกี่ยวกับการฉาย (projection) และจุดตรึง (fixed point) ของตัวดำเนินการ ตัวอย่างเช่น Noor [24] ได้ศึกษาวิเคราะห์ การกระทำซ้ำสามขั้นตอน (three-step iteration) เพื่อที่จะใช้หาคำตอบให้กับ อสมการการแปรผันสำหรับการส่งแบบไม่เป็นเชิงเส้นต่างๆ กันออกไป ซึ่ง Noor [24] ก็ได้ค้นพบว่า การกระทำซ้ำสามขั้นตอนนั้นสามารถที่จะใช้แก้อสมการได้ดีกว่า การกระทำซ้ำสองขั้นตอน และ การกระทำซ้ำหนึ่งขั้นตอน นอกจากนั้นแล้วคำตอบของอสมการแปรผัน ยังเป็นที่มาของการมีจุดตรึงของการส่งแบบไม่ขยาย เช่น ในปี 2004 Xu [42] ได้แสดงวิธีการประมาณค่าความหนืดสำหรับการส่งแบบไม่ขยายของการทำซ้ำ $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx$ เมื่อ $n \geq 1$ และ $\{\alpha_n\} \subset (0,1)$ ลู่เข้าไปยังจุดตรึงของ T ต่อมาในปี Yao, Liou และ C. Yao [44] ได้เสนอ Extragradient Methods $\{x_n\},\{y_n\}$ โดย

$$y_n = P_C(x_n - \lambda_n Ax_n$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n Ay_n)$$

เมื่อ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ เป็นลำดับใน [0, 1] สำหรับเพื่อหาจุดตรึงร่วมของเซตของจุดตรึงสำหรับการส่งแบบไม่เป็นเชิงเส้นและสำหรับเซตของอสมการแปรผัน สำหรับการส่งทางเดียว ในปี 2007 Yao และ Noor [45] ได้ประมาณผลเฉลยของ การกระทำซ้ำแบบหนืดสำหรับอสมการแปรผัน (New viscosity iterative methods for Variational inequality)

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{r_n} x$$

สำหรับ $f \in \prod_C, x_0 \in C, \{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ เป็นลำดับใน (0, 1) และ $\{r_n\}$ ไม่มีขอบเขต แล้ว $\{x_n\}$ ลู่เข้าไปยังผลเฉลยของสมการแปรผัน

สำหรับการศึกษาเกี่ยวกับปัญหาความเหมาะสมแบบคอนเวกซ์ (The convex feasibility problem (CFP)) นั่นคือ สำหรับปริภูมิบานาค E และ $C_1, C_2, ..., C_N$ ที่เป็นเซตของจุดตรึงของการส่งแบบไม่ขยาย $T_1, T_2, ..., T_N$ ตามลำดับ แล้ว ***ปัญหาความเป็นไปได้แบบคอนเวกซ์ (The convex feasibility problem (CFP))*** คือ ถ้ามี $x \in E$ ซึ่งทำให้

$$x \in \bigcap_{i=1}^{N} C_i$$

เมื่อ $N \geq 1$ ได้เริ่มมีการศึกษาขึ้นในปี ค.ศ. 1967 โดย ฮาลเพิล (Halpern) ต่อมาในปี 1977 Lions [10] ได้พิสูจน์ว่า กระบวนการทำซ้ำ $\{x_n\}$ สำหรับการส่งแบบไม่ขยาย $T_1, T_2, ..., T_N$ กำหนดโดย

$$x_{n+1} = \lambda_{n+1} y + (1-\lambda_{n+1}) T_{n+1} x_n \qquad (1)$$

เมื่อ $x_0 \in E$ และ $y \in E$ เป็นสมาชิกใดๆ และ $T_k = T_{k \bmod N}$ สำหรับ $k \geq N$ และ $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$ แล้ว $\{x_n\}$ ลู่เข้า แต่อย่างไรก็ตามเงื่อนไขของ Lions ยังไม่จริงเมื่อ $\lambda_n = \frac{1}{(n+1)}$ ซึ่งในปี 2006 O' Hara, Pillay และ Xu ได้พิสูจน์ว่าการทำซ้ำใน (1) ลู่เข้าอย่างเข้มไปยังผลเฉลยของ ปัญหาความเป็นไปได้แบบคอนเวกซ์ (CFP) ในปริภูมิบานาคปรับเรียบแบบเอกรูป (หรือ ปริภูมิสะท้อนต่อเนื่องแบบคู่อย่างอ่อน) ภายใต้เงื่อนไข $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ และ ($\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < 0$ หรือ $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+N}}{\lambda_{n+N}} = 0$ หรือ $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$)
ต่อมาในปี 2007 Chang, และ Yao [7] ได้หาเงื่อนไขที่พอเพียงที่จะทำให้ลำดับ $\{x_n\}$ ที่กำหนดโดย

$$x_{n+1} = P[\alpha_{n+1} f(x_n) + (1-\alpha_n) T_{n+1} x_n]$$

สำหรับ $T_1, T_2, ..., T_N$ เป็นการส่งไม่ขยายบนปริภูมิบานาค และการส่งเทียม f ลู่เข้าไปยังผลเฉลยของ CFP นอกจากนั้นแล้ว ก็มีประเด็นอื่นอีกมากมายในเรื่องของบทประยุกต์ของ Fixed point iterations

ในปีเดียวกัน Aoyama, Kimura, Takahashi และ Toyoda [1] ได้ศึกษาลำดับที่เกิดจากกระบวนการทำซ้ำ นั่นคือ สำหรับ $x_1 = x \in C$ นิยามโดย

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T_n x_n \quad (2)$$

สำหรับทุก $n \in N$ เมื่อ C เป็นเซตย่อยปิดแบบนูนที่ไม่เป็นเซตว่างของปริภูมิบานาค และ $\{\alpha_n\} \subset [0,1]$ และ $\{T_n\}$ เป็นการส่งไม่ขยายแบบนับได้ (countable nonexpansive mappings) แล้วลำดับที่นิยามใน (2) ลู่เข้าสู่ Px

จากที่กล่าวข้างต้นทำให้ผู้วิจัยได้เห็นแนวทางในการวิจัยโดยจะได้เสนอกระบวนการทำซ้ำแบบใหม่ และทำการพิสูจน์การลู่เข้าของกระบวนการทำซ้ำดังกล่าวสู่จุดตรึงของการส่งแบบไม่ขยายแบบ เพื่อการนำไปประยุกต์ใช้สำหรับปัญหาอสมการแปรผัน (VIP) และปัญหาความเป็นไปได้แบบคอนเวกซ์ (CFP) นั่นคือ สำหรับ $x_1 = x \in C$ นิยามโดย

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n - \beta_n) T_n x_n \quad (3)$$

สำหรับทุก $n \in N$ เมื่อ C เป็นเซตย่อยปิดแบบนูนที่ไม่เป็นเซตว่างของปริภูมิบานาค และ $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ และ $\{T_n\}$ เป็นการส่งไม่ขยายแบบนับได้ แล้วลำดับที่นิยามใน (3) ลู่เข้าสู่ผลเฉยของอสมการแปรผัน (V)

จากการนิยามกระบวนการทำซ้ำ (3) จะเห็นว่าถ้า $f(y) = x$ สำหรับทุกๆ $x \in C$ และ $\beta_n \equiv 0$ แล้วจะสามารถลดรูปไปเป็นสมการ (2) นั่นจะทำให้เป็นผลงานที่ครอบคลุมผลงานดังกล่าวได้

จากผลงานวิจัยต่างๆ ที่กล่าวมาข้างต้นจะเห็นว่าพัฒนาการในเรื่องวิธีการประมาณค่านั้นได้ถูกคิดค้นอยู่เสมอๆ ในปริภูมิที่แตกต่างกันไป จึงเป็นเหตุผลที่ทำให้ผู้วิจัยต้องการที่จะค้นหาหรือนำเสนอวิธีการประมาณค่าแบบใหม่ๆ เพื่อให้สามารถประยุกต์ใช้กับปัญหาทางคณิตศาสตร์ในรูปแบบต่างๆ หรือบางปัญหาในทาง ฟิสิกส์ และ ทางเศรษฐศาสตร์ ได้มากขึ้น พร้อมทั้งยังเป็นการก่อให้เกิดองค์ความรู้ หรือทฤษฎีใหม่ๆ ในทางการวิเคราะห์เชิงฟังก์ชันหรือสาขาอื่นๆ ที่เกี่ยวข้อง

CHAPTER 2

PRELIMINARIES

In this chapter, we give some defnitions, notations, and some useful results that will be used in the later chapters.

2.1 Basic results.

Definition 2.1.1. Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ denoted for either $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a *norm* on X if it satisfies the following conditions:

(i) $\|x\| \geq 0$, $\forall x \in X$;

(ii) $\|x\| = 0 \Leftrightarrow x = 0$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$;

(iv) $\|\alpha x\| = |\alpha|\|x\|$, $\forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.1.2. Let X be a linear space over the field $\mathbb{K}$. A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ that assigns each ordered pair (x, y) of vectors in X to a scalar $\langle x, y \rangle$ is said to be an *inner product* on X if it satisfies the following conditions:

(i) $\langle x, x \rangle \geq 0, \quad \forall\ x \in X$ and $< x, x >= 0 \Leftrightarrow x = 0$;

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}, \quad \forall\ x, y \in X$;

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \quad \forall\ x, y \in X$ and $\forall \alpha \in \mathbb{K}$;

(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \quad \forall\ x, y, z \in X$.

Definition 2.1.3. A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or convergent in norm) if there exists $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0 \ \text{ denoted by } x_n \to x.$$

Definition 2.1.4. A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there exists an element $x \in X$ such that

$$\lim_{n\to\infty} f(x_n) = f(x),$$

for all $f \in X^*$ where X^* is the dual space. Denoted by $x_n \rightharpoonup x$ or $\omega - \lim_{n\to\infty} x_n = x$.
It is clear that strong convergence implies weak convergence. And in a finite dimension normed space, weak convergence implies strong convergence.

Definition 2.1.5. A norm space X is said to be a *complete norm space* if every Cauchy sequence in X is a convergent sequence in X.

Definition 2.1.6. A complete norm linear space over the field $I\!K$ is called a *Banach space* over $\mathbb{K}$.

Definition 2.1.7. A subset C of a linear space X over the field $I\!K$ is *convex* if for any $x, y \in C$ implies

$$M = \{z \in X : z = \alpha x + (1-\alpha)y, 0 < \alpha < 1\} \subset C.$$

(M is called *closed segment with boundary point* x, y) or a subset C of X is *convex* if every $x, y \in C$ the segment joining x and y is contained in C.

Definition 2.1.8. A subset M of X is said to be *weakly compact* if every sequence $\{x_n\}$ in M contains a subsequence converging weakly to some point in M.

Theorem 2.1.9. Let $\{x_n\}$ be a sequence in extended real numbers and let $b = \limsup_{n\to\infty} x_n$. Then

$$(1) \quad r > b \Rightarrow x_n < r \quad ultimately;$$

$$(2) \quad r < b \Rightarrow x_n > r \quad frequently.$$

Ultimately means from some index onward ; *frequently* means for infinitely many indices.

Theorem 2.1.10. Let $\{x_n\}$ be a sequence in extended real numbers and let $c = \liminf_{n\to\infty} x_n$. Then

$$(1) \quad r < c \Rightarrow x_n > r \quad ultimately;$$

$$(2) \quad r > c \Rightarrow x_n < r \quad frequently.$$

Definition 2.1.11. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

Definition 2.1.12. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if, for each $n \geq 1$, there exists a sequence of positive real numbers $\{k_n\}$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \text{ for all } x, y \in C.$$

Definition 2.1.13. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically quasi−nonexpansive* mapping if there exists $u_n \in [0, +\infty)$, with $\lim_{n\to\infty} u_n = 0$, such that

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\|,$$

for all $x \in C$ and for all $p \in F(T)$, and $n \in \mathbb{N}$.

Definition 2.1.14. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive type* if T^N is continuous for some integer $N \geq 1$ and

$$\limsup_{n\to\infty}[\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}] \leq 0 \text{ for each } x \in C.$$

Definition 2.1.15. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$. A mapping T is called an *asymptotically nonexpansive mapping in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

Definition 2.1.16. Let C be a nonempty subset of a real normed space X. Let $P : X \to C$ be a *nonexpansive retraction* of X onto K i.e.,

$$\|Px - Py\| \leq \|x - y\|$$

for all $x, y \in X$ and $Px = x$ for all $x \in C$, then C is said to be nonexpansive retract.

Definition 2.1.17. Let C be a nonempty subset of a real normed space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called an *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \in \mathbb{N}$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \quad \text{for every } x, y \in C.$$

T is said to be $uniformly\ L - Lipschitzian$ if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \quad \text{for every } x, y \in C.$$

Definition 2.1.18. Let C be a nonempty subset of a real Banach space X. A mapping $T : C \to X$ is called $asymptotically\ nonexpansive\ in\ the\ intermediate\ sense\ nonself - mapping$ provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \leq 0,$$

where P is a nonexpansive retraction of X onto C.

Definition 2.1.19. A mapping $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{2.1.1}$$

Definition 2.1.20. A mapping $T : C \to C$ is an asymptotically k-strict pseudo-contractive mapping if there exists a constant $0 \leq k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{2.1.2}$$

for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\gamma_n \geq 0$ for all n such that $\lim_{n\to\infty} k_n = 1$.

Definition 2.1.21. Let X be a Banach space. An element $x \in X$ is said to be a *fixed point* of a mapping $T : X \to X$ if $Tx = x$.

Definition 2.1.22. A mapping $f : C \to C$ is *demiclosed* at y if for each $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $f(x_n) \to y$, then $f(x) = y$.

Definition 2.1.23. Let M be the set a mapping $f : M \to \mathbb{R}$ is $weak \;\; lower \;\; semi-continuous$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \text{ whenever } x_n \rightharpoonup x \text{ in } M.$$

Recall also that a one-parameter family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) $Lipschitian\ semigroup$ on C (see, e. g., [41]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$,

(ii) $T(t+s)x = T(t)T(s)x,\ t, s \geq 0, x \in C$,

(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,

(iv) there exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|,\ x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called $nonexpansive$ (or $a\ contraction$
$semigroup$) if $L_t = 1$ for all $t > 0$, and $asymptotically\ nonexpansive$ if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

2.2 Useful lemmas.

Lemma 2.2.1. Let H be a real Hilbert space. Then for any $x, y \in H$ we have

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$

(ii) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$

(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2,\ \forall t \in [0, 1]$.

Lemma 2.2.2. [40] Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,

$$a_{n+1} \leq (1 - \gamma_n)a_n + b_n,\ n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that:

i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2.3. [27] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2.4. [20] Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$,

$$\langle x - y, (A - \gamma f)x - A(A - \gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.2.5. [20] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.2.6. [31] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.2.7. [47] Let H be a Hilbert space, C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.

Lemma 2.2.8. [47] Let H be a Hilbert space, C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.

Lemma 2.2.9. [47] Let H be a Hilbert space, C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping. Define a mapping $S : C \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.

Lemma 2.2.10. [20] Let H be a Hilbert space, C a nonempty closed convex subset of H. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with fixed point x_t of contraction $C \ni x \mapsto t\gamma f(x) + (1 - tA)Tx$. Then $\{x_t\}$ converges strongly to fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the following variational inequality:

$$\langle(\gamma f - A)\widetilde{x}, z - \widetilde{x}\rangle \leq 0, \quad \forall z \in F(T).$$

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, ...) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, ...) \in l^\infty$. If μ is a Banach limit, then we have the following:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,

(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r,

(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $s = (a_0, a_1, ...) \in l^\infty$.

Lemma 2.2.11. [42] Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \subset l^\infty$ satisfying the condition $\mu_n(a_n) \leq a$ for all Banach limits μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.

Lemma 2.2.12. [46] Let H be a Hilbert space, C a nonempty closed convex subset of H. For any integer $N \geq 1$, assume that, for each $1 \leq i \leq N$, $T_i : C \to H$ be k_i-strictly pseudo-contractive mappings for some $0 \leq k_i < 1$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\Sigma_{i=1}^N \eta_i = 1$. Then $\Sigma_{i=1}^N \eta_i T_i$ is a non-self-k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.

Lemma 2.2.13. [46] Let $\{T_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ be given as in Lemma 2.2.12. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\Sigma_{i=1}^N \eta_i T_i) = \cap_{i=1}^{\infty} F(T_i)$.

Lemma 2.2.14. [14] Let T be an asymptotically k-strictly pseudo-contractive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties

(i) $x_n \rightharpoonup z$ and

(ii) $Tx_n - x_n \to 0$.

Then $(I - T)z = 0$.

Lemma 2.2.15. [27] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2.16. [14] Assume that C is a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically k-strictly pseudo-contraction. Then for each $n \geq 1$, T^n satisfies the Lipschitz condition:

$$\|T^n x - T^y\| \leq L_n \|x - y\|$$

for all $x, y \in C$, where $L_n = \frac{k + \sqrt{1 + \gamma_n (1-k)}}{1-k}$.

Lemma 2.2.17. [13] Let C be a nonempty bounded closed convex subset of a Hilbert spaces H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties

a) $x_n \rightharpoonup z$; and

b) $\limsup_{t \to \infty} \limsup_{n \to \infty} \|T(t)x_n - x_n\| = 0$,

then $z \in F(\Im)$.

Lemma 2.2.18. [13] Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that

$$\limsup_{s \to \infty} \limsup_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(u)x du - T(s) \left(\frac{1}{t} \int_0^t T(u)x du \right) \right\| = 0.$$

CHAPTER 3

MAIN RESULTS

3.1 Strong convergence theorems for modified Mann for iteration method for asymptotically non-expansive mapping

In this section, we prove strong convergence theorems by hybrid methods for asymptotically nonexpansive mappings in Hilbert spaces. Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \ n \in \mathbf{N}, \end{cases} \tag{3.1.1}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbf{N}$.

Theorem 3.1.1. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, Then $\{x_n\}$ generated by (3.1.1) converges strongly to $z_0 = P_{F(T)} x_0$.

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbf{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_k$ for some $k \in \mathbf{N}$. Then we have, for $u \in F(T) \subset C_k$

$$\begin{aligned} \|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k x_k - u\|^2 \\ &= \|\alpha_k(x_k - u) + (1-\alpha_k)(T^k x_k - u)\|^2 \\ &= \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k x_k\|^2 \\ &\le \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 \\ &\le \alpha_k\|x_k - u\|^2 + (1-\alpha_k)k_k^2\|x_k - u\|^2 \\ &= \|x_k - u\|^2 + (\alpha_k + (1-\alpha_k)k_k^2 - 1)\|x_k - u\|^2 \\ &= \|x_k - u\|^2 + (1-\alpha_k)(k_k^2 - 1)\|x_k - u\|^2 \\ &\le \|x_k - u\|^2 + (1-\alpha_k)(k_k^2 - 1)(diamC)^2 \\ &= \|x_k - u\|^2 + \theta_k \text{ with } \theta_k \to 0. \end{aligned}$$

It follows that $u \in C_{k+1}$ and $F(T) \subset C_{k+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbf{N}$. Let $z_m \in C_{k+1} \subset C_k$ with $z_m \to z$. Since C_k is closed, $z \in C_k$ and $\|y_k - z_m\|^2 \le \|z_m - x_k\|^2 + \theta_k$. Then

$$\|y_k - z\|^2 = \|y_k - z_m + z_m - z\|^2$$

$$= \|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle$$
$$\leq \|z_m - x_k\|^2 + \theta_k + \|z_m - z\|^2 + 2\|y_k - z_m\|\|z_m - z\|.$$

Taking $m \to \infty$,

$$\|y_k - z\|^2 \leq \|z - x_k\|^2 + \theta_k.$$

Hence $z \in C_{k+1}$. Let $x, y \in C_{k+1} \subset C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$ and $\|y_k - x\|^2 \leq \|x - x_k\|^2 + \theta_k, \|y_k - y\|^2 \leq \|y - x_k\|^2 + \theta_k$, we have

$$\|y_k - z\|^2 = \|y_k - (\alpha x + (1-\alpha)y)\|^2$$
$$= \|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2$$
$$= \alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|(y_k - x) - (y_k - y)\|^2$$
$$\leq \alpha(\|x - x_k\|^2 + \theta_k) + (1-\alpha)(\|y - x_k\|^2 + \theta_k) - \alpha(1-\alpha)\|y - x\|^2$$
$$= \alpha\|x - x_k\|^2 + (1-\alpha)\|y - x_k\|^2 - \alpha(1-\alpha)\|(x_k - x) - (x_k - y)\|^2 + \theta_k$$
$$= \|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_k$$
$$= \|x_k - z\|^2 + \theta_k.$$

Then $z \in C_{k+1}$, it follows that C_{k+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \text{ for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for all } u \in F(T) \text{ and } n \in \mathbf{N}. \tag{3.1.2}$$

So, for $u \in F(T)$, we have

$$0 \leq \langle x_0 - x_n, x_n - u\rangle$$
$$= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle$$
$$= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle$$
$$\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F(T) \text{ and } n \in \mathbf{N}. \tag{3.1.3}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \text{for all } n \in \mathbf{N}. \tag{3.1.4}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbf{N}$

$$\begin{aligned}0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\|x_0 - x_{n+1}\|$$

hence

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbf{N}. \tag{3.1.5}$$

From (3.3.3) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.3.4) we have

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n, \tag{3.1.6}$$

which implies that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

Further, we have

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\ &= (1-\alpha_n)\|T^n x_n - x_n\|.\end{aligned}$$

From (3.3.7), we have

$$\begin{aligned}\|T^n x_n - x_n\| &= \tfrac{1}{(1-\alpha_n)}\|y_n - x_n\| \\ &\le \tfrac{1}{(1-a)}\|y_n - x_n\| \\ &= \tfrac{1}{(1-a)}\|y_n - x_{n+1} + x_{n+1} - x_n\| \\ &\le \tfrac{1}{(1-a)}\|y_n - x_{n+1}\| + \tfrac{1}{(1-a)}\|x_{n+1} - x_n\| \\ &\le \tfrac{1}{(1-a)}(\|x_n - x_{n+1}\| + \sqrt{\theta_n}) + \tfrac{1}{(1-a)}\|x_{n+1} - x_n\| \\ &= \tfrac{2}{(1-a)}\|x_n - x_{n+1}\| + \tfrac{1}{(1-a)}\sqrt{\theta_n}.\end{aligned}$$

Hence

$$\|T^n x_n - x_n\| \leq \frac{2}{(1-a)}\|x_n - x_{n+1}\| + \frac{1}{(1-a)}\sqrt{\theta_n} \to 0.$$

Putting

$$k_\infty = \sup\{k_n : n \geq 1\} < \infty,$$

we deduce that

$$\|Tx_n - x_n\| \leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\|$$

$$+\|x_{n+1} - x_n\|$$

$$\leq k_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+k_\infty)\|x_n - x_{n+1}\| \to 0. \quad (3.1.7)$$

By (3.3.10), Lemma 2.2.14 and boundedness of $\{x_n\}$ we obtain $\emptyset \neq \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\|x_0 - z_0\| \leq \|x_0 - w\| \leq \liminf_{n\to\infty} \|x_0 - x_n\|$$

$$\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\|x_n - z_0\|^2 = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2$$

$$\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.$$

This complete the proof. ◇

Now, we present the strong convergence theorem of asymptotically nonexpansive semigroups on C in a Hilbert space.

Suppose that $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ is an asymptotically nonexpansive semigroup defined on a nonempty closed convex bounded subset C of a Hilbert space H. Recall that we use L_t^T to denote the Lipschitzian constant of the mapping $T(t)$. In the rest of this section, we put $L_\infty = \sup\{L_t^T\}$ and we use $Fix(\mathcal{T})$ to denote the fixed point set of $\mathcal{T}$. Furthermore, we use $\mathcal{F} := Fix(\mathcal{T})$ to denote the set of fixed points of asymptotically nonexpansive semigroups. Note that the boundedness of C implies that $Fix(\mathcal{T})$ is nonempty (see [40]) and we assume throughout in this theorem that the set of fixed point F is nonempty. Let C be a closed bounded convex subset of a Hilbert space H, $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be asymptotically nonexpansive semigroup of self mappings of a nonempty closed convex sunset C of a Hilbert space such that $\mathcal{F} \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta}_n\}, \quad (1.8) \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N} \end{cases} \quad (3.1.8)$$

where $\widetilde{\theta}_n = (1-\alpha_n)\left[\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2 - 1\right](diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$ and $\lambda_n \to \infty$.

Theorem 3.1.2. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter asymptotically nonexpansive of C into itself such that $\mathcal{F} := Fix(\mathcal{T}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then $\{x_n\}$ generated by (3.1.8) converges strongly to $z_0 = P_{\mathcal{F}}x_0$.

Proof. First, we observe that $F(\Im) \subset C_n$ for all $n \in \mathbf{N}$. Since $F(\Im) \subset C = C_1$. Let $F(\Im) \subset C_k$ for some $k \in \mathbf{N}$. For all $z \in F(\Im) \subset C_k$ we have

$$\begin{aligned}
\|y_k - z\|^2 &= \left\|\alpha_k x_k + (1-\alpha_k)\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z\right\|^2 \\
&= \left\|\alpha_k(x_k - z) + (1-\alpha_k)(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z)\right\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left\|\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z\right\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} \|T(s)x_k - z\| ds\right)^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s\|x_k - z\| ds\right)^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s ds\right)^2 \|x_k - z\|^2 \\
&\leq \|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s ds\right)^2 (diamC)^2 \\
&= \|x_k - z\|^2 + \widetilde{\theta}_k
\end{aligned}$$

So, $z \in C_{k+1}$. Hence $F(\Im) \subset C_n$ for all $n \in \mathbf{N}$. By the same argument as in the proof of Theorem 3.1, C_n is closed and convex, $\{x_n\}$ is well-defined. Also, similar to the proof of Theorem 3.1, we can show that

$$\|x_n - x_{n+1}\| \to 0. \tag{3.1.9}$$

We can deduce that for all $0 \leq t < \infty$,

$$\begin{aligned}
\|T(t)x_n - x_n\| &= \left\|T(t)x_n - T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right)\right\| \\
&\quad + \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\| \\
&\quad + \left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&\leq (L_\infty + 1)\left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&\quad + \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\| \\
&:= (L_\infty + 1)A_n + B_n(t),
\end{aligned} \tag{8}$$

where $A_n := \left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x_n\right\|$ and

$$B_n := \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\|.$$

We claim that

(i) $\lim_{n\to\infty} A_n = 0$; and

(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} B_n(t) = 0$.

By Lemma 2.2.17, we have that (ii) is true, while (i) is verified by the following argument. By the definition of y_n we have

$$\begin{aligned} A_n &= \left\| \tfrac{1}{\lambda_n} \int_0^{\lambda_n} T(s)x_n ds - x_n \right\| \\ &= \tfrac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\leq \tfrac{1}{1-a}\|y_n - x_n\| \\ &\leq \tfrac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|). \end{aligned} \tag{9}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \widetilde{\theta}_n$$

which in turn implies that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\widetilde{\theta}_n}.$$

It follows from (9) that

$$A_n \leq \frac{1}{1-a}\left(2\|x_{n+1} - x_n\| + \sqrt{\widetilde{\theta}_n}\right) \to 0.$$

We thus conclude from (8) that

$$\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0.$$

We note that by Lemma 2.2.17 that every weak limit point of $\{x_n\}$ is a number of $F(\Im)$. Repeating the last of the proof of Theorem 2.2 [14], we can prove that $\omega_w(x_n) = \{P_{F(\Im)}\}$. Hence $\{x_n\}$ weakly converges to $P_{F(\Im)}$, and therefore the convergence is strong. This complete the proof.

3.2 Strong convergence theorems of hybrid method for two asymptotically nonexpansive mappings

Let C be a closed bounded convex subset of a Hilbert space H, S and T be two asymptotically nonexpansive mappings of C into H and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \;\; n \in \mathbb{N} \end{cases} \tag{3.2.1}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$.

Theorem 3.2.1. Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $S, T : C \to H$ be two asymptotically nonexpansive mappings with sequence $\{s_n\}$ and $\{t_n\}$ respectively, and $F = F(S) \cap F(T) \neq \emptyset$. Let $x_0 \in H$ and $\{x_n\}$ be a sequence generated by (3.2.1). Then $\{x_n\}$ converges strongly to $z_0 = P_F x_0$.

Proof. Putting $t_\infty = \sup\{t_n : n \geq 1\} < \infty$ and $s_\infty = \sup\{s_n : n \geq 1\} < \infty$. We first show by induction that $F \subset C_n$ for all $n \in \mathbb{N}$. For $F \subset C_1$ is obvious. Suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. Let $u \in F \subset C_k$. Then, we have

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k z_k - u\|^2 \\
&= \|\alpha_k (x_k - u) + (1-\alpha_k)(T^k z_k - u)\|^2 \\
&= \alpha_k \|x_k - u\|^2 + (1-\alpha_k)\|T^k z_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k z_k\|^2 \\
&\leq \alpha_k \|x_k - u\|^2 + (1-\alpha_k)\|T^k z_k - u\|^2 \\
&\leq \alpha_k \|x_k - u\|^2 + (1-\alpha_k)t_k^2 \|z_k - u\|^2. \qquad (3.1)
\end{aligned}$$

Similarly, we note that

$$\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k x_k + (1-\beta_k)S^k x_k - u\|^2 \\
&= \|\beta_k (x_k - u) + (1-\beta_k)(S^k x_k - u)\|^2 \\
&= \beta_k \|x_k - u\|^2 + (1-\beta_k)\|S^k x_k - u\|^2 - \beta_k(1-\beta_k)\|x_k - S^k x_k\|^2 \\
&\leq \beta_k \|x_k - u\|^2 + (1-\beta_k)s_k^2\|x_k - u\|^2 - \beta_k(1-\beta_k)\|x_k - S^k x_k\|^2 \\
&\leq \|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2. \qquad (3.2)
\end{aligned}$$

From (3.1) and (3.2), we have

$$\begin{aligned}
\|y_k - u\|^2 &\leq \alpha_k \|x_k - u\|^2 + (1-\alpha_k)t_k^2[\|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2] \\
&\leq \alpha_k \|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 + (1-\alpha_k)t_k^2(1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - \|x_k - u\|^2 + \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 + (1-\alpha_k)t_k^2(1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)t_k^2(1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)[(t_k^2 - 1) + t_k^2(1-\beta_k)(s_k^2 - 1)]\|x_k - u\|^2 \\
&\leq \|x_k - u\|^2 + (1-\alpha_k)[(t_k^2 - 1) + t_k^2(1-\beta_k)(s_k^2 - 1)](diamC)^2 \\
&= \|x_k - u\|^2 + \theta_k
\end{aligned}$$

It follows that $u \in C_{k+1}$ and $F \subset C_{k+1}$. Hence $F \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty} \subseteq C_{k+1} \subset C_k$ with $z_m \to z$ as $m \to \infty$. Since C_k is closed and $z_m \in C_{k+1}$, we have $z \in C_k$ and $\|y_k - z_m\|^2 \leq \|z_m - x_k\|^2 + \theta_k$. Then

$$\|y_k - z\|^2 = \|y_k - z_m + z_m - z\|^2$$

$$= \|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle$$

$$\leq \|z_m - x_k\|^2 + \theta_k + \|z_m - z\|^2 + 2\|y_k - z_m\|\|z_m - z\|.$$

Taking $m \to \infty$,

$$\|y_k - z\|^2 \leq \|z - x_k\|^2 + \theta_k.$$

Then $z \in C_{k+1}$ and hence C_{k+1} is closed. Let $x, y \in C_{k+1} \subset C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$. Thus, we have $\|y_k - x\|^2 \leq \|x - x_k\|^2 + \theta_k$ and $\|y_k - y\|^2 \leq \|y - x_k\|^2 + \theta_k$. Hence

$$\|y_k - z\|^2 = \|y_k - (\alpha x + (1-\alpha)y)\|^2$$

$$= \|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2$$

$$= \alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|(y_k - x) - (y_k - x)\|^2$$

$$\leq \alpha(\|x - x_k\|^2 + \theta_k) + (1-\alpha)(\|y - x_k\|^2 + \theta_k) - \alpha(1-\alpha)\|y - x\|^2$$

$$= \alpha\|x - x_k\|^2 + (1-\alpha)\|y - x_k\|^2 - \alpha(1-\alpha)\|(x_k - x) - (x_k - y)\|^2 + \theta_k$$

$$= \|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_k$$

$$= \|x_k - z\|^2 + \theta_k.$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is closed and convex. Therefore C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \tag{3.3}$$

for all $y \in F \subset C_n$ and $\forall n \in \mathbb{N}$. So, for $u \in F$, we have

$$0 \leq \langle x_0 - x_n, x_n - u\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle$$

$$\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F \text{ and } n \in \mathbb{N}. \tag{3.4}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$0 \leq \langle x_0 - x_n, x_n - x_{n+1}\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle$$

$$\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.5), we have

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. We now claim that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0 = \lim_{n\to\infty} \|Sx_n - x_n\|$. Indeed, by definition of y_n, we have

$$\|y_n - x_n\| = \|\alpha_n x_n - (1-\alpha_n)T^n z_n - x_n\| = (1-\alpha_n)\|T^n z_n - x_n\|,$$

it follows that

$$\|T^n z_n - x_n\| = \tfrac{1}{1-\alpha_n}\|y_n - x_n\| \leq \tfrac{1}{1-\alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

Since $x_{n+1} \in C_n, \|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n \to 0$ as $n \to \infty$, this implies that $\|T^n z_n - x_n\| \to 0$ as $n \to \infty$. We now show that $\|S^n x_n - x_n\| \to 0$. Let $\{\|S^{n_k} x_{n_k} - x_{n_k}\|\}$ be any subsequence of $\{\|S^n x_n - x_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\lim_{j\to\infty} \|x_{n_{k_j}} - u\| = \limsup_{k\to\infty} \|x_{n_k} - u\| := a$. We note that $\|x_{n_{k_j}} - u\| \leq \|x_{n_{k_j}} - T^{n_{k_j}} z_{n_{k_j}}\| + \|T^{n_{k_j}} z_{n_{k_j}} - u\| \leq \|x_{n_{k_j}} - T^{n_{k_j}} z_{n_{k_j}}\| + k_{n_{k_j}} \|z_{n_{k_j}} - u\|, \forall j \geq 1$. This implies that

$$a = \liminf_{j\to\infty} \|x_{n_{k_j}} - u\| \leq \liminf_{j\to\infty} \|z_{n_{k_j}} - u\|. \tag{3.7}$$

By (3.2), we note that

$$\|z_{n_{k_j}} - u\| \leq \|x_{n_{k_j}} - u\| + ((1-\beta_{n_{k_j}})(s^2_{n_{k_j}} - 1))^{\frac{1}{2}} \|x_{n_{k_j}} - u\|$$

and hence

$$\limsup_{j\to\infty} \|z_{n_{k_j}} - u\| \leq \limsup_{j\to\infty} \|x_{n_{k_j}} - u\| := a. \tag{3.8}$$

Therefore

$$\lim_{j\to\infty} \|z_{n_{k_j}} - u\| = a = \lim_{j\to\infty} \|x_{n_{k_j}} - u\|.$$

Furthermore by (3.2) again, we observe that

$$\begin{aligned}
\beta_{n_{k_j}}(1-\beta_{n_{k_j}})\|S^{n_{k_j}} x_{n_{k_j}} - x_{n_{k_j}}\|^2 &\leq \|x_{n_{k_j}} - u\|^2 - \|z_{n_{k_j}} - u\|^2 \\
&\quad + (1-\beta_{n_{k_j}})(s^2_{n_{k_j}} - 1)\|x_{n_{k_j}} - u\|^2 \\
&\quad \to 0 \text{ as } j \to \infty.
\end{aligned}$$

This implies that $\lim_{j\to\infty} \|S^{n_{k_j}} x_{n_{k_j}} - x_{n_{k_j}}\| = 0$ and hence

$$\lim_{j\to\infty} \|S^n x_n - x_n\| = 0. \tag{3.9}$$

Next, we note that

$$\|x_n - T^n x_n\| \leq \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \leq \|x_n - T^n z_n\| + k_n\|z_n - x_n\|. \tag{3.10}$$

Since

$$\|z_n - x_n\| = \|\beta_n x_n + (1-\beta_n)S^n x_n - x_n\| = (1-\beta_n)\|S^n x_n - x_n\| \ \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$, we have

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.11}$$

It follows that

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\leq t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+t_\infty)\|x_n - x_{n+1}\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.12}$$

Similarly, we have

$$\|Sx_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.13}$$

By (3.12), (3.13), Lemma 2.2 and the boundedness of $\{x_n\}$, we have $\emptyset \neq \omega_w(x_n) \subset F$. Since $z_0 = P_F x_0, z_0 \in F \subset C_n$ and $x_n = P_{C_n}x_0$ by the definition of P, we obtain

$$\|x_0 - x_n\| = \|x_0 - P_{C_n}x_0\| \leq \|x_0 - z_0\| \text{ for all } n \geq 0. \tag{3.14}$$

Let $w \in \omega_w(x_n)$, by weak lower semi continuous of the norm, we have

$$\|w - x_0\| \leq \liminf_n \|x_n - x_0\| \leq \|z_0 - x_0\|. \tag{3.15}$$

Similarly, for $z_0 = P_F x_0$ and $w \in \omega_w(x_n) \subset F$, it follows that

$$\|x_0 - z_0\| = \|x_0 - P_F x_0\| \leq \|x_0 - w\|, \text{ for } w \in F. \tag{3.16}$$

From (3.15) and (3.16), this implies that $z_0 = w$ thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$, and we note that

$$\begin{aligned}\|x_n - z_0\|^2 &= \|x_n - x_0 + x_0 - z_0\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &\leq \|z_0 - x_0\|^2 - 2\langle x_0 - x_n, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &= 2\|z_0 - x_0\|^2 - 2\langle x_0 - x_n, x_0 - z_0\rangle \rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

Hence, $x_n \rightarrow z_0 = P_F x_0$. This complete the proof. □

If $S \equiv T$, then Theorem 3.1 reduces to corollary.

Corollary 3.2.2. [39] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{t_n\}$. Assume that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N} \end{cases}$$

where $\theta_n = (1-\alpha_n)(t_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

By the same as argument in the proof of Theorem 3.1, we obtain the following theorem.

Theorem 3.2.3. Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $S, T : C \to H$ be two nonexpansive mappings and $F = F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Sx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N} \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 = P_F x_0$.

If $S = T$, then Theorem 3.3 reduces to Theorem 1.1. totically nonexpansive semigroup on C in a Hilbert spaces.

Suppose that $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ are two asymptotically nonexpansive semigroups defined on a nonempty closed convex bounded subset C of a Hilbert space H. Recall that we use L_t^T and L_t^S to denote the Lipschitzian constant of the mapping $T(t)$ and $S(t)$, respectively. In the rest of this section, we put $L_\infty = \sup\{L_t^T, L_t^S : 0 < t < \infty\}$ and we use $Fix(\mathcal{T})$ and $Fix(\mathcal{S})$ to denote the common fixed point set of $\mathcal{T}$ and $\mathcal{S}$, respectively. Furthermore we use $\mathcal{F} := Fix(\mathcal{T}) \cap Fix(\mathcal{S})$ to denote the set of common fixed points of two asymptotically nonexpansive semigroups $\mathcal{T}$ and $\mathcal{S}$. Note that the boundedness of C implies that $Fix(\mathcal{T})$ and $Fix(\mathcal{S})$ are nonempty (see [40]) and we assume throughout in this theorem that the set of two common fixed point F in nonempty.

Theorem 3.2.4. Let H be a Hilbert space and let C be a nonempty closed bounded convex subset of H. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and let $x_0 \in H$. Let $C_1 = C$, $x_1 = P_{C_1}x_0$ and $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt, \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $\widetilde{\theta}_n = (1-\alpha_n)[(\widetilde{t}_n^2-1)+(1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2-1)](diamC)^2 \to 0$ as $n \to \infty$ ($here\ \widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt$ and $\widetilde{s}_n = \frac{1}{s_n}\int_0^{s_n} L_t^S dt$), $0 \le \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $t_n \to \infty, s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}x_0$.

Proof. First observe that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. For $\mathcal{F} \subset C = C_1$ is obvious. Suppose that $\mathcal{F} \subset C_k$ for some $k \in \mathbb{N}$. Let $z \in \mathcal{F} \subset C_k$. Then we have

$$\begin{aligned}
\|y_k - z\|^2 &= \left\|\alpha_k(x_k - z) + (1-\alpha_k)(\tfrac{1}{t_k}\textstyle\int_0^{t_k} T(t)z_k dt - z)\right\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left\|\tfrac{1}{t_k}\textstyle\int_0^{t_k} T(t)z_k dt - z\right\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{t_k}\textstyle\int_0^{t_k} \|T(t)z_k - z\|dt\right)^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{t_k}\textstyle\int_0^{t_k} L_t^T\|z_k - z\|dt\right)^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{t_k}\textstyle\int_0^{t_k} L_t^T dt\right)^2\|z_k - z\|^2 \\
&\le \|x_k - z\|^2 + (1-\alpha_k)(\widetilde{t}_n^2\|z_k - z\|^2 - \|x_k - z\|^2). \qquad (3.17)
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
\|z_k - z\|^2 &= \beta_k\|x_k - z\|^2 + (1-\beta_k)\|\tfrac{1}{s_k}\textstyle\int_0^{s_k} S(t)x_k dt - z\|^2 \\
&\quad -\beta_k(1-\beta_k)\|x_k - \tfrac{1}{s_k}\textstyle\int_0^{s_k} S(t)x_k dt\|^2 \\
&\le \beta_n\|x_k - z\|^2 + (1-\beta_k)(\tfrac{1}{s_k}\textstyle\int_0^{s_k} \|S(t)x_k - z\|dt)^2 \\
&\quad -\beta_k(1-\beta_k)\|x_k - \tfrac{1}{s_k}\textstyle\int_0^{s_k} S(t)x_k dt\|^2 \\
&\le \beta_n\|x_k - z\|^2 + (1-\beta_k)(\tfrac{1}{s_k}\textstyle\int_0^{s_k} L_t^S dt)^2\|x_k - z\|^2 \\
&\quad -\beta_k(1-\beta_k)\|x_k - \tfrac{1}{s_k}\textstyle\int_0^{s_k} S(t)x_k dt\|^2 \\
&\le \|x_k - z\|^2 + (1-\beta_k)(\widetilde{s}_k^2 - 1)\|x_k - z\|^2. \qquad (3.18)
\end{aligned}$$

Substituting (3.18) in (3.17) yields,

$$\begin{aligned}
\|y_k - z\|^2 &\le \|x_k - z\|^2 + (1-\alpha_k)(\widetilde{t}_n^2[\|x_k - z\|^2 \\
&\quad +(1-\beta_k)(\widetilde{s}_k^2 - 1)\|x_k - z\|^2] - \|x_k - z\|^2) \\
&\le \|x_k - z\|^2 + [(1-\alpha_k)(\widetilde{t}_n^2 - 1) + (1-\alpha_k)(1-\beta_k)\widetilde{t}_k^2(\widetilde{s}_k^2 - 1)]\|x_k - z\|^2 \\
&\le \|x_k - z\|^2 + [(1-\alpha_k)(\widetilde{t}_n^2 - 1) + (1-\alpha_k)(1-\beta_k)\widetilde{t}_k^2(\widetilde{s}_k^2 - 1)](diamC)^2 \\
&\le \|x_k - z\|^2 + \widetilde{\theta}_k^2.
\end{aligned}$$

It follows that $z \in C_{k+1}$. Hence $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. Again, by using the same argument in the proof of Theorem 3.1, we have C_n is closed and convex for all $n \in \mathbb{N}$ and

$$\|x_n - x_{n+1}\| \to 0. \qquad (3.19)$$

We now claim that

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n-x_n\|=0=\limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n-x_n\|.$$

Indeed, by definition of y_n and $x_{n+1}\subset C_n$ we have

$$\begin{aligned}\left\|\tfrac{1}{t_n}\int_0^{t_n}T(t)z_ndt-x_n\right\| &= \tfrac{1}{1-\alpha_n}\|y_n-x_n\|\\ &\le \tfrac{1}{1-\alpha_n}(\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|)\\ &\le \tfrac{1}{1-\alpha_n}(2\|x_{n+1}-x_{n+1}\|+\sqrt{\widetilde{\theta}_n})\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.20}$$

We now show that $\limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n-x_n\|=0$.

Let $\left\{\|\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)x_{n_k}dt-x_{n_k}\|\right\}$ be any subsequence of $\left\{\|\frac{1}{s_n}\int_0^{s_n}S(t)x_ndt-x_n\|\right\}$. Since $\{x_{n_k}\}$ is bounded, there is a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\|x_{n_{k_j}}-z\|=\limsup_{k\to\infty}\|x_{n_k}-z\|:=a.$$

We observe that

$$\begin{aligned}\|x_{n_{k_j}}-z\| &\le \left\|x_{n_{k_j}}-\tfrac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}}T(t)z_{n_{k_j}}dt\right\|+\left\|\tfrac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}}T(t)z_{n_{k_j}}dt-z\right\|\\ &\le \left\|x_{n_{k_j}}-\tfrac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}}T(t)z_{n_{k_j}}dt\right\|+\tfrac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}}\|T(t)z_{n_{k_j}}-z\|dt\\ &\le \left\|x_{n_{k_j}}-\tfrac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}}T(t)z_{n_{k_j}}dt\right\|+\widetilde{t}_n\|z_{n_{k_j}}-z\|.\end{aligned}$$

This implies that $a=\liminf_{j\to\infty}\|x_{n_{k_j}}-z\|\le\liminf_{j\to\infty}\|z_{n_{k_j}}-z\|$. By (3.18) we note that $\|z_{n_{k_j}}-z\|\le\|x_{n_{k_j}}-z\|+((1-\beta_{n_{k_j}})(\widetilde{s}_n^2-1))^{\frac{1}{2}}\|x_{n_{k_j}}-z\|$ and hence

$$\limsup_{j\to\infty}\|z_{n_{k_j}}-z\|\le\limsup_{j\to\infty}\|x_{n_{k_j}}-z\|=a.$$

Therefore $\lim_{j\to\infty}\|z_{n_{k_j}}-z\|=a=\lim_{j\to\infty}\|x_{n_{k_j}}-z\|$. Furthermore, by (3.18) again, we observe that

$$\begin{aligned}\beta_{n_{k_j}}(1-\beta_{n_{k_j}})\|x_{n_{k_j}}-\tfrac{1}{s_{n_{k_j}}}\int_0^{s_{n_{k_j}}}S(t)x_{n_{k_j}}dt\|^2 &\le \|x_{n_{k_j}}-z\|^2-\|z_{n_{k_j}}-z\|^2\\ &\quad +(1-\beta_{n_{k_j}})(\widetilde{s}^2_{n_{k_j}}-1)\|x_{n_{k_j}}-z\|^2\\ &\quad \to 0 \text{ as } j\to\infty.\end{aligned}$$

This implies that $\lim_{j\to\infty}\|x_{n_{k_j}}-\frac{1}{s_{n_{k_j}}}\int_0^{s_{n_{k_j}}}S(t)x_{n_{k_j}}dt\|=0$ and hence

$$\lim_{n\to\infty}\|\frac{1}{s_n}\int_0^{s_n}S(t)x_ndt-x_n\|=0. \tag{3.21}$$

For all $0\le r<\infty$, we note that

$$\begin{aligned}\|S(r)x_n-x_n\| &\le \left\|S(r)x_n-S(r)(\tfrac{1}{s_n}\int_0^{s_n}S(t)x_ndt)\right\|+\left\|\tfrac{1}{s_n}\int_0^{s_n}S(t)x_ndt-x_n\right\|\\ &\quad +\left\|S(r)(\tfrac{1}{s_n}\int_0^{s_n}S(t)x_ndt)-\tfrac{1}{s_n}\int_0^{s_n}S(t)x_ndt\right\|\\ &\le (L_\infty+1)\left\|\tfrac{1}{s_n}\int_0^{s_n}S(t)x_ndt-x_n\right\|\end{aligned}$$

$$+\left\|S(r)(\tfrac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \tfrac{1}{s_n}\int_0^{s_n} S(t)x_n dt\right\|$$
$$:= (L_\infty + 1)A_n^S + B_n^S(r), \tag{3.22}$$

where $A_n^S := \left\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\right\|$ and
$B_n^S(r) := \left\|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\right\|$. By (3.21) and Lemma 2.5, we have $\lim_{n\to\infty} A_n^S = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty} B_n^S(r)$. Moreover, we observe that

$$\begin{aligned}\left\|x_n - \tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right\| &\le \left\|x_n - \tfrac{1}{t_n}\int_0^{t_n} T(t)z_n dt\right\| \\ &\quad + \left\|\tfrac{1}{t_n}\int_0^{t_n} T(t)z_n dt - \tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right\| \\ &\le \left\|x_n - \tfrac{1}{t_n}\int_0^{t_n} T(t)z_n dt\right\| + \tfrac{1}{t_n}\int_0^{t_n}\|T(t)z_n - T_t x_n\|dt \\ &\le \left\|x_n - \tfrac{1}{t_n}\int_0^{t_n} T(t)z_n dt\right\| + \widetilde{t}_n\|z_n - x_n\|.\end{aligned}$$

Since $\|z_n - x_n\| = (1-\beta_n)\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| \to 0$ and (3.20) we obtain

$$\lim_{n\to\infty}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right\| = 0. \tag{3.23}$$

We can deduce that for all $0 \le r < \infty$,

$$\begin{aligned}\|T(r)x_n - x_n\| &\le \left\|T(r)x_n - T(r)\left(\tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right)\right\| + \left\|\tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\right\| \\ &\quad + \left\|T(r)\left(\tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right) - \tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right\| \\ &\le (L_\infty + 1)\left\|\tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\right\| \\ &\quad + \left\|T(r)\left(\tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right) - \tfrac{1}{t_n}\int_0^{t_n} T(t)x_n dt\right\| \\ &:= (L_\infty + 1)A_n^T + B_n^T(r).\end{aligned}$$

By (3.23) and Lemma 2.5, we have

$$\lim_{n\to\infty} A_n^T = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty} B_n^T(r). \tag{3.24}$$

From (3.22) and (3.24), we obtain

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

We note by Lemma 2.5 that every weak limit point of $\{x_n\}$ is a member of $\mathcal{F}$. From $x_n \rightharpoonup z_0 = P_{\mathcal{F}}x_0$, we have $x_0 - x_n \rightharpoonup x_0 - z_0$ from H satisfies the Kadec-Klee property, it follows that

$$x_0 - x_n \to x_0 - z_0.$$

So, we have

$$\|x_n - z_0\| = \|x_n - x_0 - (z_0 - x_0)\| \to 0 \text{ as } n \to \infty.$$

Hence $x_n \to z_0$. This complete the proof. □

If $\mathcal{S} \equiv \mathcal{T}$, then $S(t)x_n = x_n$ for all $n \in \mathbb{N}$ and for all $t > 0$. Hence $\frac{1}{s_n}\int_0^{s_n} S(u)x_n du = x_n, z_n = x_n$ for all $n \in \mathbb{N}$ and therefore theorem 3.3 reduces to the following corollary.

Corollary 3.2.5. [39] Let H be a Hilbert space and let C be a nonempty closed bounded convex subset of H. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and be an asymptotically nonexpansive semigroup on C such that $F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N} \end{cases}$$

where $\widetilde{\theta}_n = (1-\alpha_n)(\widetilde{t}_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ (here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt$, $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t_n \to \infty, s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

By the same as argument in the proof of Theorem 3.4, we obtain the following theorem.

Theorem 3.2.6. Let H be a Hilbert space and let C be a nonempty closed bounded convex subset of H. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ be two nonexpansive semigroups on C such that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and let $x_0 \in H$. Let $C_1 = C$, $x_1 = P_{C_1}x_0$ define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt, \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N} \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$ and $t_n \to \infty, s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_F x_0$.

If $S = T$, then Theorem 3.6 reduces to Theorem 1.2.

3.3 Strong convergence theorems of hybrid method for asymptotically k-strictly pseudo-contractive mapping

In this section, we prove strong convergence theorems by hybrid methods for asymptotically k-strict pseudo-contractive mappings in Hilbert spaces.

Theorem 3.3.1. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically k-strictly pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\| + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.1}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0, \ (n \to \infty)$. Assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (3.3.1) converges strongly to $z_0 = P_{F(T)}x_0$.

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. Then we have, for $u \in F(T) \subset C_m$

$$\|y_m - u\|^2 = \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2$$
$$= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2$$
$$= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2$$
$$\leq \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 + k\|x_m - T^m x_m\|^2] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2$$
$$= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2$$
$$\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2$$
$$\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$. Then

$$\|y_m - z\|^2 = \|y_m - z_j + z_j - z\|^2$$
$$= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle$$
$$\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, we have

$$\|y_m - z\|^2 = \|y_m - (\alpha x + (1-\alpha)y)\|^2$$
$$= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2$$
$$= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2$$
$$\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m)$$
$$+(1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m) - \alpha(1-\alpha)\|y - x\|^2$$
$$= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2$$
$$+[k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$$
$$= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$$
$$= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \text{ for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.3.2}$$

So, for $u \in F(T)$, we have

$$\begin{aligned}
0 \leq \langle x_0 - x_n, x_n - u\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\
&= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.3.3}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.3.4}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned}
0 \leq \langle x_0 - x_n, x_n - x_{n+1}\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\
&= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.3.5}$$

From (3.3.3) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.3.4) we have

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2
\end{aligned}$$

$$= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3.6}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n, \tag{3.3.7}$$

By the definition of y_n, we have

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\ &= (1-\alpha_n)\|T^n x_n - x_n\|.\end{aligned}$$

From (3.3.7), we have

$$\begin{aligned}(1-\alpha_n)^2\|T^n x_n - x_n\|^2 &= \|y_n - x_n\|^2 \\ &= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \\ &\le \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &\le \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + \|x_{n+1} - x_n\|^2 \\ &\quad +2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n.\end{aligned}$$

It follows that

$$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \le 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n.$$

Hence

$$(1 - k - \alpha_n)\|T^n x_n - x_n\| \le 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n. \tag{3.3.8}$$

From $\limsup_{n\to\infty} \alpha_n < 1 - k$, we can chosen $\epsilon > 0$ such that $\alpha_n \le 1 - k - \epsilon$ for large enough n. From (3.4.4) and (3.3.8), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.3.9}$$

Next, we show that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. From Lemma 2.2.16, we have

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le L_1\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + L_{n+1})\|x_n - x_{n+1}\|. \end{aligned} \tag{3.3.10}$$

From (3.4.4) and (3.4.6), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.3.11}$$

By (3.3.10), Lemma 2.2.14 and boundedness of $\{x_n\}$ we obtain $\emptyset \ne \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \le \|z_0 - x_0\|$ for all $n \ge 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\|x_0 - z_0\| \leq \|x_0 - w\| \leq \liminf_{n\to\infty} \|x_0 - x_n\|$$
$$\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\|x_n - z_0\|^2 = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2$$
$$\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.$$

This complete the proof. □

Using this Theorem 3.4.1, we have the following corollaries.

Corollary 3.3.2. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be k-strictly pseudo-contractive mapping of C into itself for some $0 \leq k < 1$ such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{3.3.12}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$. Then $\{x_n\}$ generated by (3.4.9) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3.3. [12] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.13}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (3.4.8) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3.4. ([39] Theorem 4.1) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.14}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

3.4 Strong convergence theorems for a new iterative method of k-strictly pseudo-contractive mappings

In this section, first we show that a mapping $S : C \to H$ defined by $Sx = kx + (1-k)Tx$ is a nonexpansive mapping, where C a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ be a k-strictly pseudo contractive mapping with a fixed point for some $0 \le k < 1$. Let $x, y \in C$ and from Lemma 2.2.1 (iv), we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|kx + (1-k)Tx - (ky + (1-k)Ty)\|^2 \\
&= \|k(x-y) + (1-k)(Tx - Ty)\|^2 \\
&= k\|x-y\|^2 + (1-k)\|Tx - Ty\|^2 - k(1-k)\|(x-y)x - (Tx - Ty)\|^2 \\
&= k\|x-y\|^2 + (1-k)(\|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2) - k(1-k)\|(x-y)x - (Tx-Ty)\|^2 \\
&= \|x-y\|^2 + (1-k)k\|(I-T)x - (I-T)y\|^2) - k(1-k)\|(I-T)x - (I-T)y\|^2 \\
&\le \|x-y\|^2.
\end{aligned}$$

Hence $\|Sx - Sy\| \le \|x - y\|$. Then S is a nonexpansive mapping and we have $P_C S$ is also nonexpansive where P_C is a metrics projection on C. For any $j \in \mathbb{N}$, defined a mapping $S_j : C \to C$ by $S_j x = \frac{1}{j}\gamma f(x) + (I - \frac{1}{j}A)P_C Sx$. Let us show that S_j is contraction, let $x, y \in C$, we have

$$\begin{aligned}
\|S_j x - S_j y\| &= \|\tfrac{1}{j}\gamma f(x) + (I - \tfrac{1}{j}A)P_C Sx - (\tfrac{1}{j}\gamma f(y) + (I - \tfrac{1}{j}A)P_C Sy)\| \\
&\le \tfrac{1}{j}\gamma\alpha\|x-y\| + (1 - \tfrac{1}{j}\overline{\gamma})\|P_C Sx - P_C Sy\| \\
&\le \tfrac{1}{j}\gamma\alpha\|x-y\| + (1 - \tfrac{1}{j}\overline{\gamma})\|x-y\| \\
&\le (1 - \tfrac{1}{j}(\overline{\gamma} - \gamma\alpha))(\|x-y\|.
\end{aligned}$$

Hence, S_j is a contraction. By Banach's contraction principle there exists a unique fixed point $u_j \in C$ such that

$$u_j = \frac{1}{j}\gamma f(u_j) + (1 - \frac{1}{j}A)P_C S u_j. \tag{3.4.1}$$

Next, we prove the main results.

Theorem 3.4.1. Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ and $T : C \to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0 \le k < 1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be sequence generated by;

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n, \end{cases} \tag{3.4.2}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0$,

(ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converge strongly to a fixed point p of T, which solves the following solution of the variational inequalities (3.3.1).

Proof. Note that from the condition $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H, then

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0,\end{aligned}$$

that is to say $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \alpha_n A\| &= \sup\{\langle((1 - \beta_n)I - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned}$$

We now observe that $\{x_n\}$ is bounded. Indeed pick any $p \in F(T) \cap EP(F)$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)P_C Sx_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(P_C Sx_n - p)\| \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + \|((1 - \beta_n)I - \alpha_n A)\|\|P_C Sx_n - p\| \\ &\leq \alpha_n\|\gamma f(x_n) - \gamma f(p) + \gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|\|x_n - p\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|\|x_n - p\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|\|x_n - p\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n(\overline{\gamma} - \gamma\alpha)\tfrac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}.\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}\}, n \geq 0,$$

and hence $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{P_C Sx_n\}$ are bounded. From (3.4.1), we have for any $n, j \in \mathbb{N}$

$$\|x_{n+1} - P_C S u_j\| = \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n - P_C S u_j\|$$
$$= \|\alpha_n(\gamma f(x_n) - AP_C S u_j) + \beta_n(x_n - P_C S u_j)$$
$$+((1-\beta_n)I - \alpha_n A)(P_C S x_n - P_C S u_j)\|$$
$$\leq \alpha_n\|\gamma f(x_n) - AP_C S u_j\| + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|P_C S x_n - P_C S u_j\|$$
$$\leq \alpha_n\|\gamma f(x_n) - AP_C S u_j\| + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - u_j\|$$
$$= \alpha_n(\|\gamma f(x_n) - AP_C S u_j\| - \overline{\gamma}\|x_n - u_j\|) + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|$$
$$= \delta_n + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|$$

where $\delta_n = \alpha_n(\|\gamma f(x_n) - AP_C S u_j\| - \overline{\gamma}\|x_n - u_j\|)$, from $\lim_{n\to\infty}\alpha_n = 0$, we have $\delta_n \to 0$ as $n \to \infty$. It follows that

$$\|x_{n+1} - P_C S u_j\|^2 = (\delta_n + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)^2$$
$$= (\beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)^2 + 2(\beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2$$
$$= \beta_n^2\|x_n - P_C S u_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + 2\beta_n(1-\beta_n)\|x_n - P_C S u_j\|\|x_n - u_j\| + \sigma_n$$

where $\sigma_n = 2(\beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \to 0$ as $n \to \infty$, and hence

$$\|x_{n+1} - P_C S u_j\|^2 \leq \beta_n^2\|x_n - P_C S u_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + \beta_n(1-\beta_n)(\|x_n - P_C S u_j\|^2$$
$$+\|x_n - u_j\|^2) + \sigma_n$$
$$= \beta_n\|x_n - P_C S u_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n.$$

For any Banach limit μ and $\beta_n \to 0$, we have

$$\mu_n\|x_n - P_C S u_j\|^2 = \mu_n\|x_{n+1} - P_C S u_j\|^2 \leq \mu_n\|x_n - u_j\|^2. \tag{3.4.3}$$

Since $u_j - x_n = \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n) + (1-\frac{1}{j})(P_C S u_j - x_n)$, thus we have

$$(1-\frac{1}{j})(x_n - P_C S u_j) = (x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n).$$

It follows from Lemma 2.2.1 (ii), that

$$(1-\tfrac{1}{j})^2\|x_n - P_C S u_j\|^2 = \|(x_n - u_j) + \tfrac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n)\|^2$$
$$\geq \|x_n - u_j\|^2 + \tfrac{2}{j}\langle(\gamma f(u_j) + (I-A)P_C S u_j - x_n), x_n - u_j\rangle$$
$$= \|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f u_j) + (I-A)P_C S u_j - u_j - (x_n - u_j), x_n - u_j\rangle$$
$$= \|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle - \tfrac{2}{j}\langle x_n - u_j, x_n - u_j\rangle$$
$$= \|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle - \tfrac{2}{j}\|x_n - u_j\|^2$$
$$= (1-\frac{2}{j})\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle. \tag{3.4.4}$$

So by (3.4.3) and (3.4.4), we have

$$(1-\tfrac{1}{j})^2\|x_n-u_j\|^2 \geq (1-\tfrac{1}{j})^2\|P_CSu_j-x_n\|^2$$
$$\geq (1-\tfrac{2}{j})\|x_n-u_j\|^2+\tfrac{2}{j}\langle\gamma f(u_j)+(I-A)P_CSu_j-u_j,x_n-u_j\rangle$$

and hence

$$\tfrac{1}{j^2}\|x_n-u_j\|^2 \geq \tfrac{2}{j}\langle\gamma f(u_j)+(I-A)P_CSu_j-u_j,x_n-u_j\rangle.$$

This implies that

$$\tfrac{2}{j}\mu_n\|x_n-u_j\|^2 \geq \mu_n\langle\gamma f(u_j)+(I-A)P_CSu_j-u_j,x_n-u_j\rangle.$$

From Lemma 2.2.8 and 2.2.10, $u_j \to p \in F(T)=F(P_CS) \quad as\ j\to\infty$, we get

$$\mu_n\langle\gamma f(p)-Ap,x_n-p\rangle \leq 0, \tag{3.4.5}$$

and p which the solution of variational inequality (3.3.1). Since, $\{x_n\}$, $\{f(x_n)\}$ and $\{P_CSx_n\}$ are bounded, we choose

$$M=\sup\{\|f(x_n)\|+\|x_n\|+\|P_CSx_n\|+\|AP_CSx_n\|: n\in\mathbb{N}\}.$$

On the other hand,

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}x_{n+1}+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}\\
&\quad -(\alpha_{n+1}\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_nA)P_CSx_n)\|\\
&= \|\alpha_{n+1}\gamma f(x_{n+1})-\alpha_{n+1}\gamma f(x_n)+\alpha_{n+1}\gamma f(x_n)-\alpha_n\gamma f(x_n)+\beta_{n+1}x_{n+1}-\beta_{n+1}x_n\\
&\quad +\beta_{n+1}x_n-\beta_nx_n+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}-((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n\\
&\quad +((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n-((1-\beta_n)I-\alpha_nA)P_CSx_n\|\\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})\|P_CSx_{n+1}-P_CSx_n\|\\
&\quad +|[((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_nA)]|\|P_CSx_n\|\\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})\|x_{n+1}-x_n\|+[|\beta_{n+1}-\beta_n|)\|P_CSx_n\|+|\alpha_{n+1}-\alpha_n|\|AP_CSx_n\|\\
&\leq (1-\alpha_{n+1}(\overline{\gamma}-\gamma\alpha))\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\gamma M+|\beta_{n+1}-\beta_n|M\\
&\quad +[|\beta_{n+1}-\beta_n|)M+|\alpha_{n+1}-\alpha_n|M.
\end{aligned}$$

From (ii), (iii) and Lemma 2.2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4.6}$$

Next, we show that $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$. We consider

$$\begin{aligned}
\|x_n-P_CSx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-P_CSx_n\|\\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n-Ap\|+\beta_n\|x_n-P_CSx_n\|.
\end{aligned}$$

From, $\alpha_n \to 0, \beta_n \to 0$ and (3.4.6), it follows that $\lim_{n\to\infty} \|x_n - P_C S x_n\| = 0$.

Next, we show that

$\limsup_{n\to\infty} \langle \gamma f(p) - Ap, x_n - p\rangle \leq 0,$

where $p \in F(T)$, which p the solution of variational inequality (??). From (3.4.6), we have

$$\limsup_{n\to\infty} |\langle \gamma f(p) - Ap, x_{n+1} - p\rangle - \langle \gamma f(p) - Ap, x_n - p\rangle| = 0. \tag{3.4.7}$$

Hence it follows from (3.4.5), (3.4.7) and Lemma 2.2.11 that

$$\limsup_{n\to\infty} \langle \gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.4.8}$$

and from $\lim_{n\to\infty} \|x_n - P_C S x_n\| = 0$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(p) - Ap, P_C S x_n - p\rangle &= \limsup_{n\to\infty} \langle \gamma f(p) - Ap, (P_C S x_n - x_n) + (x_n - p)\rangle \\ &= \limsup_{\to\infty} \langle \gamma f(p) - Ap, x_n - p\rangle \leq 0. \end{aligned} \tag{3.4.9}$$

Finally, we prove that $x_n \to p$ as $n \to \infty$, we note that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) P_C S x_n - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(P_C S x_n - p)\|^2 \\
&= \|\beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(P_C S x_n - p)\|^2 + \alpha_n^2 \|\gamma f(x_n) - Ap\|^2 \\
&\quad +2\langle \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(P_C S x_n - p), \alpha_n(\gamma f(x_n) - Ap)\rangle \\
&\leq (\beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\overline{\gamma})\|P_C S x_n - p\|)^2 + 2\beta_n\alpha_n\langle x_n - p, (\gamma f(x_n) - Ap)\rangle \\
&\quad +\alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\
&\quad +2(1-\beta_n)\alpha_n\langle (P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle - 2\alpha_n^2\langle A(P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle \\
&\leq (\beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\overline{\gamma})\|x_n - p\|)^2 + 2\beta_n\alpha_n\alpha\gamma\|x_n - p\|^2 + 2\beta_n\alpha_n\langle x_n - p, \gamma f(p) - Ap\rangle \\
&\quad +2(1-\beta_n)\alpha_n\langle (P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle - 2\alpha_n^2\langle A(P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle \\
&\quad +\alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\
&\leq (1-\alpha_n\overline{\gamma})^2\|x_n - p\|^2 + 2\beta_n\alpha_n\alpha\gamma\|x_n - p\|^2 + 2\beta_n\alpha_n\langle x_n - p, \gamma f(p) - Ap\rangle \\
&\quad +2(1-\beta_n)\alpha_n\langle (P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle + 3\alpha_n^2 M \\
&= (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\|^2 + (\alpha_n\overline{\gamma})^2 M + 2\beta_n\alpha_n\alpha\gamma\|x_n - p\|^2 \\
&\quad +2\beta_n\alpha_n\langle x_n - p, \gamma f(p) - Ap\rangle \\
&\quad +2(1-\beta_n)\alpha_n\langle (P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle + 3\alpha_n^2 M \\
&= (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\|^2 + \alpha_n[2\beta_n\langle x_n - p, \gamma f(p) - Ap\rangle \\
&\quad +2(1-\beta_n)\langle (P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle + 3\alpha_n M + \alpha_n\overline{\gamma}^2 M]
\end{aligned}$$

$$=: (1-\gamma_n)\|x_n - p\|^2 + b_n$$

where $\gamma_n = 2(\overline{\gamma} - \gamma\alpha)\alpha_n$ and $b_n = \alpha_n[2\beta_n\langle x_n - p, \gamma f(p) - Ap)\rangle + 2(1-\beta_n)\langle (P_C S x_n - p), (\gamma f(x_n) - Ap)\rangle + 3\alpha_n M + \alpha_n \overline{\gamma}^2 M]$. From $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, (3.4.8), (3.4.9), we have $\Sigma_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\frac{b_n}{\gamma_n} \leq 0$. by Lemma 2.2.2, we have a sequence $\{x_n\}$ converges strongly to a fixed point p of T which the solution of variational inequality (3.3.1). This completes the proof. □

If $\beta_n \equiv 0$, in Theorem 3.4.1, we obtain the following corollary.

Corollary 3.4.2. [46] Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ and $T : C \to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k < 1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be sequence generated by;

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.4.10}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$. If the control sequence $\{\alpha_n\} \subset (0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converge strongly to a fixed point p of T, which solves the following solution of the variational inequalities (3.3.1).

□

Theorem 3.4.3. Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ and $T : C \to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k < 1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be sequence generated by;

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.4.11}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converge strongly to a fixed point p of T, which solves the following solution of the variational inequalities (3.3.1).

Proof. In the proof of theorem 3.4.1 we have, $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{P_C Sx_n\}$ are bounded. Next, we show that $\|x_{n+1} - x_n\| \to 0$. Define the sequence $z_n = \frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_n}$, such that $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$, $n \geq 0$. Observe that from the definition of z_n we obtain

$$\begin{aligned}
z_{n+1} - z_n &= \tfrac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)P_C Sx_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_n}\\
&= \tfrac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} - \tfrac{\alpha_{n+1}\gamma f(x_n)}{1-\beta_{n+1}} + \tfrac{\alpha_{n+1}\gamma f(x_n)}{1-\beta_{n+1}} - \tfrac{\alpha_n\gamma f(x_n)}{1-\beta_n}\\
&\quad + \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_C Sx_{n+1}}{1-\beta_{n+1}} - \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_C Sx_n}{1-\beta_{n+1}} + \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_C Sx_n}{1-\beta_{n+1}}\\
&\quad - \tfrac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_{n+1}} + \tfrac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_{n+1}} - \tfrac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_n}\\
&= \tfrac{\alpha_{n+1}\gamma\alpha(x_{n+1}-x_n)}{1-\beta_{n+1}} + (\alpha_{n+1}-\alpha_n)\tfrac{\|\gamma f(x_{n+1})\|}{1-\beta_{n+1}} + \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)}{1-\beta_{n+1}}(P_C Sx_{n+1} - P_C Sx_n)\\
&\quad + [((1-\beta_{n+1})I - \alpha_{n+1}A) - ((1-\beta_n)I - \alpha_n A)](P_C Sx_n)\\
&\quad + ((1-\beta_n)I - \alpha_n A)(\tfrac{1}{1-\beta_{n+1}} - \tfrac{1}{1-\beta_n})(P_C Sx_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq |\tfrac{\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|}{1-\beta_{n+1}} + |\alpha_{n+1}-\alpha_n|\tfrac{\|\gamma f(x_{n+1})\|}{1-\beta_{n+1}} + \tfrac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|P_C Sx_{n+1} - P_C Sx_n\|\\
&\quad + [(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma}) - (1-\beta_n-\alpha_n\overline{\gamma})]\|P_C Sx_n\| + ((1-\beta_n-\alpha_n\overline{\gamma})|\tfrac{1}{1-\beta_{n+1}}\\
&\quad - \tfrac{1}{1-\beta_n}|\|P_C Sx_n\|\\
&\leq \tfrac{\alpha_{n+1}\gamma\alpha}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \tfrac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M + \tfrac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\
&\quad + [|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n|\overline{\gamma}]\|AP_C Sx_n\| + ((1-\beta_n-\alpha_n\overline{\gamma})|\tfrac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}\|P_C Sx_n\|\\
&= \tfrac{\alpha_{n+1}\gamma\alpha}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \tfrac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M + \|x_{n+1}-x_n\| - \tfrac{\alpha_{n+1}\overline{\gamma}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\
&\quad + [|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n|\overline{\gamma}]M + ((1-\beta_n-\alpha_n\overline{\gamma})|\tfrac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}M
\end{aligned}$$

where $M = \sup\{\|f(x_n)\| + \|P_C Sx_n\| + \|AP_C Sx_n\| + \|x_{n+1}-x_n\| : n \in \mathbb{N}\}$. It follows that

$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq |\frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M + [|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n|\overline{\gamma}]M + ((1-\beta_n - \alpha_n\overline{\gamma})|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}M$.

Since $\Sigma_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty, \Sigma_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0. \tag{3.4.12}$$

From $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, (3.4.12) and Lemma 2.2.6, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.4.13}$$

We consider

$$\|x_{n+1}-x_n\| = \|(1-\beta_n)z_n - \beta_n x_n - x_n\|$$
$$= (1-\beta_n)\|z_n - x_n\|$$

then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0$$

Next, we show that $\lim_{n\to\infty}\|x_n - P_C Sx_n\| = 0$. We note that

$$\|x_n - P_C Sx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - P_C Sx_n\|$$
$$\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - AP_C Sx_n\| + \beta_n\|x_n - P_C Sx_n\|, \qquad (3.4.14)$$

and hence

$$(1-\beta_n)\|x_n - P_C Sx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - AP_C Sx_n\|,$$

from $\alpha_n \to 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, it follows that $\lim_{n\to\infty}\|x_n - P_C Sx_n\| = 0$. From (3.4.1), we have for any $n, j \in \mathbb{N}$

$$\|x_{n+1} - P_C Su_j\| = \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C Sx_n - P_C Su_j\|$$
$$= \|\alpha_n(\gamma f(x_n) - AP_C Su_j) + \beta_n(x_n - P_C Su_j) + ((1-\beta_n)I - \alpha_n A)(P_C Sx_n - P_C Su_j)\|$$
$$\le \alpha_n\|\gamma f(x_n) - AP_C Su_j\| + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|P_C Sx_n - P_C Su_j\|$$
$$\le \alpha_n\|\gamma f(x_n) - AP_C Su_j\| + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - u_j\|$$
$$= \alpha_n(\|\gamma f(x_n) - AP_C Su_j\| - \overline{\gamma}\|x_n - u_j\|) + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|$$
$$= \delta_n + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|$$

where $\delta_n = \alpha_n(\|\gamma f(x_n) - AP_C Su_j\| - \overline{\gamma}\|x_n - u_j\|)$, from $\lim_{n\to\infty}\alpha_n = 0$, we have $\delta_n \to 0$ as $n \to \infty$. It follows that

$$\|x_{n+1} - P_C Su_j\|^2 = (\delta_n + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)^2$$
$$= (\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)^2 + 2(\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2$$
$$= \beta_n^2\|x_n - P_C Su_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + 2\beta_n(1-\beta_n)\|x_n - P_C Su_j\|\|x_n - u_j\| + \sigma_n$$

where $\sigma_n = 2(\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \to 0$ as $n \to \infty$, and hence

$$\|x_{n+1} - P_C Su_j\|^2 \le \beta_n^2\|x_n - P_C Su_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2$$
$$+\beta_n(1-\beta_n)(\|x_n - P_C Su_j\|^2 + \|x_n - u_j\|^2) + \sigma_n$$
$$= \beta_n\|x_n - P_C Su_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n. \qquad (3.4.15)$$

From (3.4.25), we have

$$\|x_n - P_C S u_j\|^2 = \|(x_n - x_{n+1}) + (x_{n+1} - P_C S u_j)\|^2$$
$$= \|x_{n+1} - P_C S u_j\|^2 + 2\langle x_{n+1} - P_C S u_j, x_n - x_{n+1}\rangle + \|x_n - x_{n+1}\|^2$$
$$= \|x_{n+1} - P_C S u_j\|^2 + 2\|x_{n+1} - P_C S u_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2,$$
$$\leq \beta_n\|x_n - P_C S u_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n + 2\|x_{n+1} - P_C S u_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2$$

and hence

$$(1-\beta_n)\|x_n - P_C S u_j\|^2 \leq (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n + 2\|x_{n+1} - P_C S u_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2.$$

For any Banach limit μ and $\sigma_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, we have

$$\mu_n\|x_n - P_C S u_j\|^2 \leq \mu_n\|x_n - u_j\|^2. \tag{3.4.16}$$

Since $u_j - x_n = \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n) + (1 - \frac{1}{j})(P_C S u_j - x_n)$, thus we have

$$(1 - \frac{1}{j})(x_n - P_C S u_j) = (x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n).$$

It follows from Lemma 2.2.1 (ii), that

$$(1 - \tfrac{1}{j})^2\|x_n - P_C S u_j\|^2 = \|(x_n - u_j) + \tfrac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n)\|^2$$
$$\geq \|x_n - u_j\|^2 + \tfrac{2}{j}\langle(\gamma f(u_j) + (I-A)P_C S u_j - x_n), x_n - u_j\rangle$$
$$= \|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f u_j) + (I-A)P_C S u_j - u_j - (x_n - u_j), x_n - u_j\rangle$$
$$= \|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle - \tfrac{2}{j}\langle x_n - u_j, x_n - u_j\rangle$$
$$= \|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle - \tfrac{2}{j}\|x_n - u_j\|^2$$
$$= (1 - \frac{2}{j})\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle. \tag{3.4.17}$$

So by (3.4.23) and (3.4.17), we have

$$(1 - \tfrac{1}{j})^2\|x_n - u_j\|^2 \geq (1 - \tfrac{1}{j})^2\|P_C S u_j - x_n\|^2$$
$$\geq (1 - \tfrac{2}{j})\|x_n - u_j\|^2 + \tfrac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle$$

and hence

$$\tfrac{1}{j^2}\|x_n - u_j\|^2 \geq \tfrac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle.$$

This implies that

$$\tfrac{2}{j}\mu_n\|x_n - u_j\|^2 \geq \mu_n\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle.$$

From Lemma 2.2.8 and Lemma 2.2.10, $u_k \to p \in F(T) = F(P_C S)\ \ as\ j \to \infty$, we get

$$\mu_n\langle\gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.4.18}$$

and p which the solution of variational inequality (3.3.1). Next, we show that

$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0,$

where $p \in F(T)$, which p the solution of variational inequality (3.3.1). From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have

$$\limsup_{n\to\infty} |\langle \gamma f(p) - Ap, x_{n+1} - p\rangle - \langle \gamma f(p) - Ap, x_n - p\rangle| = 0. \tag{3.4.19}$$

Hence it follows from (3.4.18), (3.4.19) and Lemma 2.2.11 that

$$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.4.20}$$

and from (3.4.14), we have

$$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, P_C S x_n - p\rangle = \limsup_{n\to\infty}\langle \gamma f(p) - Ap, (P_C S x_n - x_n) + (x_n - p)\rangle$$

$$= \limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0. \tag{3.4.21}$$

By the same argument of final in Theorem 3.4.1, we have a sequence $\{x_n\}$ converges strongly to a fixed point p of T which the solution of variational inequality (3.3.1). This completes the proof. □

Theorem 3.4.4. Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ and $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be sequence generated by;

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n, \end{cases} \tag{3.4.22}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)\Sigma_{i=1}^N \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, ..., N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\beta_n = 0,$

(ii) $\Sigma_{n=1}^\infty \alpha_n = \infty,$

(iii) $\Sigma_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty,\ \Sigma_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converge strongly to a common fixed point p of $\{T_i\}_{i=1}^N$, which solves the following solution of the variational inequalities:

$$\langle (A - \gamma f)p, p - x\rangle \leq 0, \quad \forall x \in \cap_{i=1}^N F(T_i). \tag{3.4.23}$$

Proof. Define a mapping $T : C \to H$ by $Tx = \Sigma_{i=1}^N \eta_i T_i x$. By Lemma 2.2.12 and 2.2.13, we conclude that $: C \to H$ is a $k-$strictly pseudo-contractive mapping with $k = \max\{k_i : i = 1, 2, ..., N\}$ and $F(T) = F(\Sigma_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$. From Theorem 3.4.1, we can obtain desired conclusion easily. This completes the proof. □

If $\beta_n \equiv 0$, Theorem 3.4.4 reduced to the following corollary.

Corollary 3.4.5. [46] Let H be a Hilbert space, K a nonempty closed convex subset of H such that $K \pm K \subset K$ and $T_i : K \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let A be strongly positive bounded linear operator on K with coefficient $\overline{\gamma} > 0$ and $f : K \to K$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be sequence generated by;

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.4.24}$$

where $S : K \to H$ is a mapping defined by $Sx = kx + (1-k)\Sigma_{i=1}^{N} \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, ..., N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then $\{x_n\}$ converge strongly to a common fixed point p of $\{T_i\}_{i=1}^{N}$, which solves the following solution of the variational inequalities:

$$\langle (A - \gamma f)p, p - x \rangle \leq 0, \quad \forall x \in \cap_{i=1}^{N} F(T_i).$$

From the proved of Theorem 3.4.3, we can obtain the following Theorem.

Theorem 3.4.6. Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ and $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be sequence generated by;

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.4.25}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)\Sigma_{i=1}^{N} \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, ..., N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\Sigma_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converge strongly to a common fixed point p of $\{T_i\}_{i=1}^{N}$, which solves the following solution of the variational inequalities (3.4.23).

CHAPTER 4

CONCLUSIONS

3.1 Outputs 4 papers (Supported by TRF: MRG5180026)

1. Strong Convergence Theorems of Modified Mann Iteration Methods for Asymptotically Nonexpansive Mappings in Hilbert Spaces. Int. Journal of Math. Analysis, Vol. 2 (2008), no. 23. 1135 - 1145.

2. Strong convergence theorems of hybrid methods for two asymptotically nonexpansive mappings in Hilbert spaces. Nonlinear Analysis: Hybrid Systems 2 (2008) 1125-1135.

3. Strong convergence theorems by hybrid method for asymptotically k-strict pseudo-contractive mapping in Hilbert space. Nonlinear Analysis: Hybrid Systems 3 (2009) 380-385.

4. Strong convergence theorems for a new iterative method of k-strictly pseudo-contractive mappings in Hilbert spaces. Computers and Mathematics with Applications 58 (2009) 1397-1407.

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Strong Convergence Theorems of Modified Mann Iteration Methods for Asymptotically Nonexpansive Mappings in Hilbert Spaces

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Abstract

In this paper, we introduce the iterative sequence for an asymptotically nonexpansive mapping and an asymptotically nonexpansive semigroup. Then we prove that such a sequence converges strongly to $P_{F(T)}x_0$ and $P_{\mathcal{F}}x_0$, respectively. This main theorem concern result of Takahashi, Takeuchi and Kubota [Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. V.341, 2008, 276-286], and many others.

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1 Introduction

Let X be a real Banach Space, C a nonempty closed convex subset of X and $T : C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of *fixed points* of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. We know that a Hilbert space H satisfies Opial's condition [8], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. We also know that H has Kadec-Klee property, that is, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Infact, from

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2,$$

we get that a Hilbert space has the Kadec-Klee property.

Recall also that a one-parameter family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) *Lipschitian semigroup* on C (see, e. g., [12]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$,

(ii) $T(t+s)x = T(t)T(s)x,\ t, s \geq 0, x \in C$,

(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,

(iv) there exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|,\ x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called *nonexpansive* (or *a contraction semigroup*) if $L_t = 1$ for all $t > 0$, and *asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [2, 5, 9, 10]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [3, 10].

In 2003, Nakajo and Takahashi [7] introduced the following modification of the Mann iteration method for a nonexpansive mapping T of C into itself in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily },\\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\},\\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (1.1) converges strongly to $P_{Fix(T)}(x_0)$. Moreover they introduced and studied an iteration process of a nonexpansive semigroup $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self mappings of a nonempty closed convex subset C of a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily }, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \quad (1.2)$$

Recently, Kim and Xu [3] adapted the iteration (1.1) to an asymptotically nonexpansive mappings T of C into itself in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily }, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \quad (1.3)$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$. They prove that if $\alpha_n \leq a$ for all n and for some $0 < a < 1$, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{Fix(T)}(x_0)$. They also modified an iterative method (1.2) to the case of an asymptotically nonexpansive semigroup $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self mappings of a nonempty closed convex sunset C of a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily }, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \quad (1.4)$$

where $\theta_n = (1-\alpha_n)\left[(\frac{1}{t_n}\int_0^{t_n} L_u du)^2 - 1\right](diamC)^2 \to 0$ as $n \to \infty$.

In 2007, Takahashi, Takeuchi and Kubota [10] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$ and nonexpansive semigroup $\Im = \{T(t) : 0 \leq t < \infty\}$ in a Hilbert space H. They prove the following theorem;

Theorem 1.1 *([10] Theorem 4.1) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \quad (1.5)$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.2 *([10] Theorem 4.4) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $\mathcal{T} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$ define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{1.6}$$

where $0 \leq \alpha_n \leq a < 1, 0 < \lambda_n < \infty$ for all $n \in \mathbf{N}$ and $\lambda_n \to \infty$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

Inspired and motivated by these fact, it is the purpose of this paper to introduce the modified Ishikawa iteration processes for an asymptotically nonexpansive mapping by idear in (1.5). Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{1.7}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$.

The second purpose of this paper is to study the modified Ishikawa iteration process for an asymptotically nonexpansive semigroup. Let C be a closed bounded convex subset of a Hilbert space H, $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be asymptotically nonexpansive semigroup of self mappings of a nonempty closed convex sunset C of a Hilbert space such that $\mathcal{F} \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta}_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N} \end{cases} \tag{1.8}$$

where $\widetilde{\theta}_n = (1-\alpha_n)\left[\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2 - 1\right](diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$ and $\lambda_n \to \infty$.

We shall prove that both iteration processes (1.7) and (1.8) converge strongly to a fixed point of T and a common fixed point of $\mathcal{T}$, respectively, provided the sequence $\{\alpha_n\}$ is bounded from above.

2 Preliminary

In this section, we collect some lemmas which will be used in the proof for the main result in next section.

Lemma 2.1 *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x||^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.2 *[4] Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties*
(i) $x_n \rightharpoonup z$ and
(ii) $Tx_n - x_n \to 0$.
Then $z \in Fix(T)$.

Lemma 2.3 *[3] Let C be a nonempty bounded closed convex subset of a Hilbert spaces H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties*
a) $x_n \rightharpoonup z$; and
b) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,
then $z \in F(\Im)$.

Lemma 2.4 *[3] Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that*

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(u)x du - T(s)\left(\frac{1}{t}\int_0^t T(u)x du\right)\right\| = 0.$$

3 Main Results

In this section, we prove strong convergence theorems by hybrid methods for asymptotically nonexpansive mappings in Hilbert spaces.

Theorem 3.1 *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, Then $\{x_n\}$ generated by (1.7) converges strongly to $z_0 = P_{F(T)}x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbf{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_k$ for some $k \in \mathbf{N}$. Then we have, for $u \in F(T) \subset C_k$

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k x_k - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(T^k x_k - u)\|^2 \\
&= \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k x_k\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)k_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (\alpha_k + (1-\alpha_k)k_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(k_k^2 - 1)\|x_k - u\|^2 \\
&\leq \|x_k - u\|^2 + (1-\alpha_k)(k_k^2 - 1)(diam C)^2 \\
&= \|x_k - u\|^2 + \theta_k \text{ with } \theta_k \to 0.
\end{aligned}$$

It follows that $u \in C_{k+1}$ and $F(T) \subset C_{k+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbf{N}$. Let $z_m \in C_{k+1} \subset C_k$ with $z_m \to z$. Since C_k is closed, $z \in C_k$ and $\|y_k - z_m\|^2 \leq \|z_m - x_k\|^2 + \theta_k$. Then

$$\begin{aligned}
\|y_k - z\|^2 &= \|y_k - z_m + z_m - z\|^2 \\
&= \|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle \\
&\leq \|z_m - x_k\|^2 + \theta_k + \|z_m - z\|^2 + 2\|y_k - z_m\|\|z_m - z\|.
\end{aligned}$$

Taking $m \to \infty$,

$$\|y_k - z\|^2 \leq \|z - x_k\|^2 + \theta_k.$$

Hence $z \in C_{k+1}$. Let $x, y \in C_{k+1} \subset C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$ and $\|y_k - x\|^2 \leq \|x - x_k\|^2 + \theta_k, \|y_k - y\|^2 \leq \|y - x_k\|^2 + \theta_k$, we have

$$\begin{aligned}
\|y_k - z\|^2 &= \|y_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2 \\
&= \alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|(y_k - x) - (y_k - y)\|^2 \\
&\leq \alpha(\|x - x_k\|^2 + \theta_k) + (1-\alpha)(\|y - x_k\|^2 + \theta_k) - \alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_k\|^2 + (1-\alpha)\|y - x_k\|^2 - \alpha(1-\alpha)\|(x_k - x) - (x_k - y)\|^2 + \theta_k \\
&= \|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_k \\
&= \|x_k - z\|^2 + \theta_k.
\end{aligned}$$

Then $z \in C_{k+1}$, it follows that C_{k+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \text{ for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for all } u \in F(T) \text{ and } n \in \mathbf{N}. \tag{1}$$

So, for $u \in F(T)$, we have

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - u\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\| \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\|x_0 - u\|$$

hence

$$\|x_0 - x_n\| \le \|x_0 - u\| \text{ for all } u \in F(T) \text{ and } n \in \mathbf{N}. \tag{2}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0 \text{ for all } n \in \mathbf{N}. \tag{3}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbf{N}$

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\| \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\|x_0 - x_{n+1}\|$$

hence

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\| \text{ for all } n \in \mathbf{N}. \tag{4}$$

From (2) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n, \tag{5}$$

which implies that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

Further, we have

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n)T^n x_n - x_n\| \\ &= (1 - \alpha_n)\|T^n x_n - x_n\|. \end{aligned}$$

From (5), we have

$$\begin{aligned}\|T^n x_n - x_n\| &= \tfrac{1}{(1-\alpha_n)}\|y_n - x_n\| \\ &\leq \tfrac{1}{(1-a)}\|y_n - x_n\| \\ &= \tfrac{1}{(1-a)}\|y_n - x_{n+1} + x_{n+1} - x_n\| \\ &\leq \tfrac{1}{(1-a)}\|y_n - x_{n+1}\| + \tfrac{1}{(1-a)}\|x_{n+1} - x_n\| \\ &\leq \tfrac{1}{(1-a)}(\|x_n - x_{n+1}\| + \sqrt{\theta_n}) + \tfrac{1}{(1-a)}\|x_{n+1} - x_n\| \\ &= \tfrac{2}{(1-a)}\|x_n - x_{n+1}\| + \tfrac{1}{(1-a)}\sqrt{\theta_n}.\end{aligned}$$

Hence

$$\|T^n x_n - x_n\| \leq \frac{2}{(1-a)}\|x_n - x_{n+1}\| + \frac{1}{(1-a)}\sqrt{\theta_n} \to 0.$$

Putting

$$k_\infty = \sup\{k_n : n \geq 1\} < \infty,$$

we deduce that

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\leq k_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+k_\infty)\|x_n - x_{n+1}\| \to 0. \quad (6)\end{aligned}$$

By (6), Lemma 2.2 and boundedness of $\{x_n\}$ we obtain $\emptyset \neq \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| \leq \|x_0 - w\| &\leq \liminf_{n\to\infty}\|x_0 - x_n\| \\ &\leq \limsup_{n\to\infty}\|x_0 - x_n\| \leq \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\begin{aligned}\|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.\end{aligned}$$

This complete the proof. ◇

Now, we present the strong convergence theorem of asymptotically nonexpansive semigroups on C in a Hilbert space.

Suppose that $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ is an asymptotically nonexpansive semigroup defined on a nonempty closed convex bounded subset C of a Hilbert space H. Recall that we use L_t^T to denote the Lipschitzian constant of the mapping $T(t)$. In the rest of this section, we put $L_\infty = \sup\{L_t^T\}$ and we use $Fix(\mathcal{T})$ to denote the fixed point set of $\mathcal{T}$. Furthermore, we use $\mathcal{F} := Fix(\mathcal{T})$ to denote the set of fixed points of asymptotically nonexpansive semigroups. Note that the boundedness of C implies that $Fix(\mathcal{T})$ is nonempty (see [11]) and we assume throughout in this theorem that the set of fixed point F is nonempty.

Theorem 3.2 *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ be a one-parameter asymptotically nonexpansive of C into itself such that $\mathcal{F} := Fix(\mathcal{T}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then $\{x_n\}$ generated by (1.8) converges strongly to $z_0 = P_{\mathcal{F}}x_0$.*

Proof. First, we observe that $F(\Im) \subset C_n$ for all $n \in \mathbf{N}$. Since $F(\Im) \subset C = C_1$. Let $F(\Im) \subset C_k$ for some $k \in \mathbf{N}$. For all $z \in F(\Im) \subset C_k$ we have

$$\begin{aligned}
\|y_k - z\|^2 &= \left\|\alpha_k x_k + (1-\alpha_k)\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z\right\|^2 \\
&= \left\|\alpha_k(x_k - z) + (1-\alpha_k)(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z)\right\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left\|\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z\right\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} \|T(s)x_k - z\| ds\right)^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s\|x_k - z\| ds\right)^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s ds\right)^2\|x_k - z\|^2 \\
&\le \|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s ds\right)^2 (diamC)^2 \\
&= \|x_k - z\|^2 + \widetilde{\theta}_k
\end{aligned}$$

So, $z \in C_{k+1}$. Hence $F(\Im) \subset C_n$ for all $n \in \mathbf{N}$. By the same argument as in the proof of Theorem 3.1, C_n is closed and convex, $\{x_n\}$ is well-defined. Also, similar to the proof of Theorem 3.1, we can show that

$$\|x_n - x_{n+1}\| \to 0. \tag{7}$$

We can deduce that for all $0 \le t < \infty$,

$$\begin{aligned}
\|T(t)x_n - x_n\| &= \left\|T(t)x_n - T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right)\right\| \\
&\quad + \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\| \\
&\quad + \left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&\le (L_\infty + 1)\left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&\quad + \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\| \\
&:= (L_\infty + 1)A_n + B_n(t),
\end{aligned} \tag{8}$$

where $A_n := \left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x_n\right\|$ and

$$B_n := \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\|.$$

We claim that

(i) $\lim_{n\to\infty} A_n = 0$; and

(ii) $\limsup_{t\to\infty}\limsup_{n\to\infty} B_n(t) = 0$.

By Lemma 2.3, we have that (ii) is true, while (i) is verified by the following argument. By the definition of y_n we have

$$\begin{aligned}
A_n &= \left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&= \tfrac{1}{1-\alpha_n}\|y_n - x_n\|
\end{aligned}$$

$$\leq \tfrac{1}{1-a}\|y_n - x_n\|$$
$$\leq \tfrac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|). \quad (9)$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \widetilde{\theta}_n$$

which in turn implies that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\widetilde{\theta}_n}.$$

It follows from (9) that

$$A_n \leq \frac{1}{1-a}\left(2\|x_{n+1} - x_n\| + \sqrt{\widetilde{\theta}_n}\right) \to 0.$$

We thus conclude from (8) that

$$\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0.$$

We note that by Lemma 2.3 that every weak limit point of $\{x_n\}$ is a number of $F(\Im)$. Repeating the last of the proof of Theorem 2.2 [4], we can prove that $\omega_w(x_n) = \{P_{F(\Im)}\}$. Hence $\{x_n\}$ weakly converges to $P_{F(\Im)}$, and therefore the convergence is strong. This complete the proof. ◇

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Strong convergence theorems of hybrid methods for two asymptotically nonexpansive mappings in Hilbert spaces

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In this paper, we introduce two modifications of the Ishikawa iteration, by using the hybrid methods, for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups in a Hilbert space. Then, we prove that such two sequences converge strongly to common fixed points of two symptotically nonexpansive mappings and asymptotically nonexpansive semigroups, respectively. Our main result is connected with the results of Plubtieng and Ungchittrakool [S. Plubtieng, K. Ungchittrakool, Strong convergence of modified Ishikawa iteration for two asymptotically nonexpansive mappings and semigroups, Nonlinear. Anal. 67(2007) 2306-2315], Martinez-Yanes and Xu [C. Martinez-Yanes, H.K. Xu, Strong convergence of CQ method for fixed point iteration processes, Nonlinear. Anal. 64 (2006) 2400-2411] and many others.

1. Introduction

Let X be a real Banach Space, C a nonempty closed convex subset of X and $T : C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of *fixed points* of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. If S and T are two nonexpansive (asymptotically nonexpansive) mappings, then the point $x \in Fix(S) \cap Fix(T)$ is called the *common fixed point* of S and T.

Recall also that a one-parameter family $\mathscr{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian *semigroup* on C (see, e. g., [13]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$,
(ii) $T(t+s)x = T(t)T(s)x,\ t, s \geq 0, x \in C$,
(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,
(iv) there exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$, $\|T(t)x - T(t)y\| \leq L_t \|x - y\|,\ x, y \in C$.

A Lipschitzian semigroup $\mathscr{T}$ is called *nonexpansive* (or *a contraction semigroup*) if $L_t = 1$ for all $t > 0$, and *asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $F(\mathscr{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathscr{T}) = \{x \in C : T(t)x = x, t > 0\}$.

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Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [2,5,8,10,11]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [3,11].

In 2003, Nakajo and Takahashi [7] introduced the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (1.1) converges strongly to $P_{Fix(T)}(x_0)$. Moreover they introduced and studied an iteration process of a nonexpansive semigroup $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily ,} \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(u)x_n \mathrm{d}u, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.2}$$

Recently, Kim and Xu [3] adapted the iteration (1.1) to asymptotically nonexpansive mappings in Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily ,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$. They proved that if $\alpha_n \le a$ for all n and for some $0 < a < 1$, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{Fix(T)}(x_0)$. They also modified an iterative method (1.2) to the case of an asymptotically nonexpansive semigroup $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ in Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily ,} \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(u)x_n \mathrm{d}u, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.4}$$

where $\theta_n = (1-\alpha_n)\left[\left(\frac{1}{t_n}\int_0^{t_n} L_u \mathrm{d}u\right)^2 - 1\right](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$.

In 2007, Plubtieng and Ungchitrakool [9], introduced the modified Ishikawa iteration processes for two asymptotically nonexpansive mappings S and T, and two asymptotically nonexpansive semigroups $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ and $\mathscr{S} = \{S(t) : 0 \le t < \infty\}$, with C a closed convex bounded subset of a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily ,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \ge 0, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.5}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ (here $\{t_n\}$ and $\{s_n\}$ are two sequences from T and S, respectively.) and

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily },\\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(u)z_n\mathrm{d}u,\\ z_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} S(u)x_n\mathrm{d}u,\\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \widetilde{\theta}_n\},\\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0,\\ x_{n+1} = P_{C_n\cap Q_n}x_0, \end{cases} \tag{1.6}$$

where $\widetilde{\theta}_n = (1-\alpha_n)[\widetilde{t}_n^{\,2} - 1) + (1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2 - 1)](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$(*here* $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T \mathrm{d}t$ and $\widetilde{s}_n = \frac{1}{s_n}\int_0^{s_n} L_t^S \mathrm{d}t$). They also proved that both iterations (1.5) and (1.6) converge strongly to a common fixed point of two asymptotically nonexpansive mappings S and T, and two asymptotically nonexpansive semigroups $\mathscr{T}$ and $\mathscr{S}$, respectively.

Very recently, Takahashi, Takeuchi and Kubota [11] proved the following strong convergence theorems by using the new hybrid method for nonexpansive mappings and nonexpansive semigroups in Hilbert spaces.

Theorem 1.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n,\\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\},\\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.2. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $\mathscr{T} = \{T(s) : 0 \le s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathscr{T}) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$ define a sequence $\{u_n\}$ of C as follows: space H:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)\dfrac{1}{\lambda_n}\displaystyle\int_0^{\lambda_n} T(s)u_n\mathrm{d}s,\\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\},\\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N} \end{cases}$$

where and $0 \le \alpha_n \le a < 1$, $0 < \lambda_n < \infty$ for all $n \in \mathbb{N}$ and $\lambda_n \to \infty$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathscr{T})}x_0$.

Inspired and motivated by these facts, it is the purpose of this paper to introduce the modified Ishikawa iteration processes for two asymptotically nonexpansive mappings by in idea in (1.3). Let C be a closed bounded convex subset of a Hilbert space H, S and T be two asymptotically nonexpansive mappings of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n,\\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n,\\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\},\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N} \end{cases} \tag{1.7}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ and $0 \le \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N}$.

Our second modification Ishikawa iteration processes for two asymptotically nonexpansive semigroups. Let C be a closed bounded convex subset of a Hilbert space H, $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ and $\mathscr{S} = \{S(t) : 0 \le t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathscr{F} = F(\mathscr{T}) \cap F(\mathscr{S}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(t)z_n\mathrm{d}t,\\ z_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} S(t)x_n\mathrm{d}t,\\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \widetilde{\theta}_n\},\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\widetilde{\theta}_n = (1-\alpha_n)[\widetilde{t}_n^{\,2} - 1) + (1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2 - 1)](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$(*here* $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T \mathrm{d}t$ and $\widetilde{s}_n = \frac{1}{s_n}\int_0^{s_n} L_t^S \mathrm{d}t$), $0 \le \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t_n \to \infty$, $s_n \to \infty$.

We shall prove that both iteration (1.7) and (1.8) converges strongly to a common fixed point of two asymptotically nonexpansive mappings of S and T, and asymptotically nonexpansive semigroups, $\mathscr{T}$ and $\mathscr{S}$, respectively.

2. Preliminaries

In this section, we collect some lemmas which will be used in the proofs for the main result in next section.

Lemma 2.1. *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

Lemma 2.2 ([4]). *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $z \in F(T)$.*

Lemma 2.3 ([6]). *Let H be a real Hilbert space. Given a closed convex subset C of H and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$D := \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

Lemma 2.4 ([3]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroups on C. If $\{x_n\}$ is a sequence in C satisfying the properties*

(a) $x_n \rightharpoonup z$; *and*
(b) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,

then $z \in F(\Im)$.

Lemma 2.5 ([3]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroups on C. Then it holds that*

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(u)x\mathrm{d}u - T(s)\left(\frac{1}{t}\int_0^t T(u)x\mathrm{d}u\right)\right\| = 0.$$

3. Main results

In this section, we prove strong convergence theorems of a common fixed point for two asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups, respectively.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $S, T : C \to C$ be two asymptotically nonexpansive mappings with sequence $\{s_n\}$ and $\{t_n\}$ respectively, and $F = F(S) \cap F(T) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by (1.7). Then $\{x_n\}$ converges strongly to $z_0 = P_F x_0$.*

Proof. Putting $t_\infty = \sup\{t_n : n \geq 1\} < \infty$ and $s_\infty = \sup\{s_n : n \geq 1\} < \infty$. We first show by induction that $F \subset C_n$ for all $n \in \mathbb{N}$. For $F \subset C_1$ is obvious. Suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. Let $u \in F \subset C_k$. Then, we have

$$\begin{aligned}\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k z_k - u\|^2\\ &= \|\alpha_k(x_k - u) + (1-\alpha_k)(T^k z_k - u)\|^2\\ &= \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k z_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k z_k\|^2\\ &\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k z_k - u\|^2\\ &\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|z_k - u\|^2. \end{aligned} \tag{3.1}$$

Similarly, we note that

$$\begin{aligned}\|z_k - u\|^2 &= \|\beta_k x_k + (1-\beta_k)S^k x_k - u\|^2\\ &= \|\beta_k(x_k - u) + (1-\beta_k)(S^k x_k - u)\|^2\\ &= \beta_k\|x_k - u\|^2 + (1-\beta_k)\|S^k x_k - u\|^2 - \beta_k(1-\beta_k)\|x_k - S^k x_k\|^2\\ &\leq \beta_k\|x_k - u\|^2 + (1-\beta_k)s_k^2\|x_k - u\|^2 - \beta_k(1-\beta_k)\|x_k - S^k x_k\|^2\\ &\leq \|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned}\|y_k-u\|^2 &\le \alpha_k\|x_k-u\|^2+(1-\alpha_k)t_k^2[\|x_k-u\|^2+(1-\beta_k)(s_k^2-1)\|x_k-u\|^2]\\ &\le \alpha_k\|x_k-u\|^2+(1-\alpha_k)t_k^2\|x_k-u\|^2+(1-\alpha_k)t_k^2(1-\beta_k)(s_k^2-1)\|x_k-u\|^2\\ &= \|x_k-u\|^2-\|x_k-u\|^2+\alpha_k\|x_k-u\|^2+(1-\alpha_k)t_k^2\|x_k-u\|^2+(1-\alpha_k)t_k^2(1-\beta_k)(s_k^2-1)\|x_k-u\|^2\\ &= \|x_k-u\|^2+(1-\alpha_k)(t_k^2-1)\|x_k-u\|^2+(1-\alpha_k)t_k^2(1-\beta_k)(s_k^2-1)\|x_k-u\|^2\\ &= \|x_k-u\|^2+(1-\alpha_k)[(t_k^2-1)+t_k^2(1-\beta_k)(s_k^2-1)]\|x_k-u\|^2\\ &\le \|x_k-u\|^2+(1-\alpha_k)[(t_k^2-1)+t_k^2(1-\beta_k)(s_k^2-1)](\operatorname{diam} C)^2\\ &= \|x_k-u\|^2+\theta_k.\end{aligned}$$

It follows that $u \in C_{k+1}$ and $F \subset C_{k+1}$. Hence $F \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty} \subseteq C_{k+1} \subset C_k$ with $z_m \to z$ as $m \to \infty$. Since C_k is closed and $z_m \in C_{k+1}$, we have $z \in C_k$ and $\|y_k - z_m\|^2 \le \|z_m - x_k\|^2 + \theta_k$. Then

$$\begin{aligned}\|y_k-z\|^2 &= \|y_k-z_m+z_m-z\|^2\\ &= \|y_k-z_m\|^2+\|z_m-z\|^2+2\langle y_k-z_m, z_m-z\rangle\\ &\le \|z_m-x_k\|^2+\theta_k+\|z_m-z\|^2+2\|y_k-z_m\|\|z_m-z\|.\end{aligned}$$

Taking $m \to \infty$,

$$\|y_k-z\|^2 \le \|z-x_k\|^2+\theta_k.$$

Then $z \in C_{k+1}$ and hence C_{k+1} is closed. Let $x, y \in C_{k+1} \subset C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0, 1]$. Since C_k is convex, $z \in C_k$. Thus, we have $\|y_k - x\|^2 \le \|x - x_k\|^2 + \theta_k$ and $\|y_k - y\|^2 \le \|y - x_k\|^2 + \theta_k$. Hence

$$\begin{aligned}\|y_k-z\|^2 &= \|y_k-(\alpha x+(1-\alpha)y)\|^2\\ &= \|\alpha(y_k-x)+(1-\alpha)(y_k-y)\|^2\\ &= \alpha\|y_k-x\|^2+(1-\alpha)\|y_k-y\|^2-\alpha(1-\alpha)\|(y_k-x)-(y_k-x)\|^2\\ &\le \alpha(\|x-x_k\|^2+\theta_k)+(1-\alpha)(\|y-x_k\|^2+\theta_k)-\alpha(1-\alpha)\|y-x\|^2\\ &= \alpha\|x-x_k\|^2+(1-\alpha)\|y-x_k\|^2-\alpha(1-\alpha)\|(x_k-x)-(x_k-y)\|^2+\theta_k\\ &= \|\alpha(x_k-x)+(1-\alpha)(x_k-y)\|^2+\theta_k\\ &= \|x_k-z\|^2+\theta_k.\end{aligned}$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is closed and convex. Therefore C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0-x_n, x_n-y\rangle \ge 0 \tag{3.3}$$

for all $y \in F \subset C_n$ and $\forall n \in \mathbb{N}$. So, for $u \in F$, we have

$$\begin{aligned}0 &\le \langle x_0-x_n, x_n-u\rangle = -\langle x_n-x_0, x_n-x_0\rangle+\langle x_0-x_n, x_0-u\rangle\\ &\le -\|x_n-x_0\|^2+\|x_0-x_n\|\|x_0-u\|.\end{aligned}$$

This implies that

$$\|x_0-x_n\|^2 \le \|x_0-x_n\|\|x_0-u\|$$

and hence

$$\|x_0-x_n\| \le \|x_0-u\| \quad \text{for all } u \in F \quad \text{and} \quad n \in \mathbb{N}. \tag{3.4}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0-x_n, x_n-x_{n+1}\rangle \ge 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned}0 &\le \langle x_0-x_n, x_n-x_{n+1}\rangle = -\langle x_n-x_0, x_n-x_0\rangle+\langle x_0-x_n, x_0-x_{n+1}\rangle\\ &\le -\|x_n-x_0\|^2+\|x_0-x_n\|\|x_0-x_{n+1}\|.\end{aligned}$$

This implies that

$$\|x_0-x_n\|^2 \le \|x_0-x_n\|\|x_0-x_{n+1}\|$$

and hence

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.5), we have

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. We now claim that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0 = \lim_{n\to\infty} \|Sx_n - x_n\|$. Indeed, by definition of y_n, we have

$$\|y_n - x_n\| = \|\alpha_n x_n - (1-\alpha_n)T^n z_n - x_n\| = (1-\alpha_n)\|T^n z_n - x_n\|,$$

it follows that

$$\|T^n z_n - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \le \frac{1}{1-\alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

Since $x_{n+1} \in C_n$, $\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n \to 0$ as $n \to \infty$, this implies that $\|T^n z_n - x_n\| \to 0$ as $n \to \infty$. We now show that $\|S^n x_n - x_n\| \to 0$. Let $\{\|S^{n_k} x_{n_k} - x_{n_k}\|\}$ be any subsequence of $\{\|S^n x_n - x_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\lim_{j\to\infty} \|x_{n_{k_j}} - u\| = \limsup_{k\to\infty} \|x_{n_k} - u\| := a$. We note that $\|x_{n_{k_j}} - u\| \le \|x_{n_{k_j}} - T^{n_{k_j}} z_{n_{k_j}}\| + \|T^{n_{k_j}} z_{n_{k_j}} - u\| \le \|x_{n_{k_j}} - T^{n_{k_j}} z_{n_{k_j}}\| + k_{n_{k_j}}\|z_{n_{k_j}} - u\|$, $\forall j \ge 1$. This implies that

$$a = \liminf_{j\to\infty} \|x_{n_{k_j}} - u\| \le \liminf_{j\to\infty} \|z_{n_{k_j}} - u\|. \tag{3.7}$$

By (3.2), we note that

$$\|z_{n_{k_j}} - u\| \le \|x_{n_{k_j}} - u\| + ((1-\beta_{n_{k_j}})(s_{n_{k_j}}^2 - 1))^{\frac{1}{2}} \|x_{n_{k_j}} - u\|$$

and hence

$$\limsup_{j\to\infty} \|z_{n_{k_j}} - u\| \le \limsup_{j\to\infty} \|x_{n_{k_j}} - u\| := a. \tag{3.8}$$

Therefore

$$\lim_{j\to\infty} \|z_{n_{k_j}} - u\| = a = \lim_{j\to\infty} \|x_{n_{k_j}} - u\|.$$

Furthermore by (3.2) again, we observe that

$$\begin{aligned}
\beta_{n_{k_j}}(1-\beta_{n_{k_j}})\|S^{n_{k_j}} x_{n_{k_j}} - x_{n_{k_j}}\|^2 \le\ & \|x_{n_{k_j}} - u\|^2 - \|z_{n_{k_j}} - u\|^2 + (1-\beta_{n_{k_j}})(s_{n_{k_j}}^2 - 1)\|x_{n_{k_j}} - u\|^2 \\
&\to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

This implies that $\lim_{j\to\infty} \|S^{n_{k_j}} x_{n_{k_j}} - x_{n_{k_j}}\| = 0$ and hence

$$\lim_{j\to\infty} \|S^n x_n - x_n\| = 0. \tag{3.9}$$

Next, we note that

$$\|x_n - T^n x_n\| \le \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \le \|x_n - T^n z_n\| + k_n\|z_n - x_n\|. \tag{3.10}$$

Since

$$\|z_n - x_n\| = \|\beta_n x_n + (1-\beta_n)S^n x_n - x_n\| = (1-\beta_n)\|S^n x_n - x_n\| \to 0 \quad \text{as } n \to \infty,$$

and $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$, we have

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.11}$$

It follows that

$$\begin{aligned}
\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+t_\infty)\|x_n - x_{n+1}\| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.12}$$

Similarly, we have

$$\|Sx_n - x_n\| \to \infty \quad \text{as } n \to \infty. \tag{3.13}$$

By (3.12) and (3.13), Lemma 2.2 and the boundedness of $\{x_n\}$, we have $\emptyset \neq \omega_w(x_n) \subset F$. Since $z_0 = P_F x_0$, $z_0 \in F \subset C_n$ and $x_n = P_{C_n} x_0$ by the definition of P, we obtain

$$\|x_0 - x_n\| = \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z_0\| \quad \text{for all } n \geq 0. \tag{3.14}$$

Let $w \in \omega_w(x_n)$, by weak lower semi continuous of the norm, we have

$$\|w - x_0\| \leq \liminf_n \|x_n - x_0\| \leq \|z_0 - x_0\|. \tag{3.15}$$

Similarly, for $z_0 = P_F x_0$ and $w \in \omega_w(x_n) \subset F$, it follows that

$$\|x_0 - z_0\| = \|x_0 - P_F x_0\| \leq \|x_0 - w\|, \quad \text{for } w \in F. \tag{3.16}$$

From (3.15) and (3.16), this implies that $z_0 = w$ thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$, and we note that

$$\begin{aligned}
\|x_n - z_0\|^2 &= \|x_n - x_0 + x_0 - z_0\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\
&\leq \|z_0 - x_0\|^2 - 2\langle x_0 - x_n, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\
&= 2\|z_0 - x_0\|^2 - 2\langle x_0 - x_n, x_0 - z_0\rangle \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, $x_n \to z_0 = P_F x_0$. This completes the proof. □

If $S \equiv T$, then Theorem 3.1 reduces to the following corollary:

Corollary 3.2. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{t_n\}$. Assume that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases}
y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\} \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}
\end{cases}$$

where $\theta_n = (1 - \alpha_n)(t_n^2 - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

By the same argument as in the proof of Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $S, T : C \to H$ be two nonexpansive mappings and $F = F(S) \cap F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases}
y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \\
z_n = \beta_n x_n + (1 - \beta_n) S x_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}
\end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 = P_F x_0$.

If $S = T$, then Theorem 3.3 reduces to Theorem 1.1.

Now, we present the strong convergence theorem of two asymptotically nonexpansive semigroups on C in a Hilbert space.

Suppose that $\mathscr{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathscr{S} = \{S(t) : 0 \leq t < \infty\}$ are two asymptotically nonexpansive semigroups defined on a nonempty closed convex bounded subset C of a Hilbert space H. Recall that we use L_t^T and L_t^S to denote the Lipschitzian constant of the mapping $T(t)$ and $S(t)$, respectively. In the rest of this section, we put $L_\infty = \sup\{L_t^T, L_t^S : 0 < t < \infty\}$ and we use $Fix(\mathscr{T})$ and $Fix(\mathscr{S})$ to denote the common fixed point set of $\mathscr{T}$ and $\mathscr{S}$, respectively. Furthermore we use $\mathscr{F} := Fix(\mathscr{T}) \cap Fix(\mathscr{S})$ to denote the set of common fixed points of two asymptotically nonexpansive semigroups $\mathscr{T}$ and $\mathscr{S}$. Note that the boundedness of C implies that $Fix(\mathscr{T})$ and $Fix(\mathscr{S})$ are nonempty (see [12]) and we assume throughout in this theorem that the set of two common fixed point F in nonempty.

Theorem 3.4. *Let H be a Hilbert space and let C be a nonempty closed bounded convex subset of H. Let $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ and $\mathscr{S} = \{S(t) : 0 \le t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathscr{F} = F(\mathscr{T}) \cap F(\mathscr{S}) \ne \emptyset$ and let $x_0 \in C$. Let $C_1 = C$, $x_1 = P_{C_1}x_0$ and $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(t)z_n\mathrm{d}t, \\ z_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} S(t)x_n\mathrm{d}t, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\widetilde{\theta}_n = (1-\alpha_n)[(\widetilde{t}_n^{\,2} - 1) + (1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2 - 1)](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ (here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T \mathrm{d}t$ and $\widetilde{s}_n = \frac{1}{s_n}\int_0^{s_n} L_t^S \mathrm{d}t$), $0 \le \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t_n \to \infty$, $s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathscr{F}}x_0$.

Proof. First observe that $\mathscr{F} \subset C_n$ for all $n \in \mathbb{N}$. For $\mathscr{F} \subset C = C_1$ is obvious. Suppose that $\mathscr{F} \subset C_k$ for some $k \in \mathbb{N}$. Let $z \in \mathscr{F} \subset C_k$. Then we have

$$\begin{aligned} \|y_k - z\|^2 &= \left\|\alpha_k(x_k - z) + (1-\alpha_k)\left(\frac{1}{t_k}\int_0^{t_k} T(t)z_k\mathrm{d}t - z\right)\right\|^2 \\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left\|\frac{1}{t_k}\int_0^{t_k} T(t)z_k\mathrm{d}t - z\right\|^2 \\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\frac{1}{t_k}\int_0^{t_k} \|T(t)z_k - z\|\mathrm{d}t\right)^2 \\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\frac{1}{t_k}\int_0^{t_k} L_t^T\|z_k - z\|\mathrm{d}t\right)^2 \\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\frac{1}{t_k}\int_0^{t_k} L_t^T\mathrm{d}t\right)^2\|z_k - z\|^2 \\ &\le \|x_k - z\|^2 + (1-\alpha_k)(\widetilde{t}_n^{\,2}\|z_k - z\|^2 - \|x_k - z\|^2). \end{aligned} \tag{3.17}$$

By Lemma 2.1, we have

$$\begin{aligned} \|z_k - z\|^2 &= \beta_k\|x_k - z\|^2 + (1-\beta_k)\left\|\frac{1}{s_k}\int_0^{s_k} S(t)x_k\mathrm{d}t - z\right\|^2 - \beta_k(1-\beta_k)\left\|x_k - \frac{1}{s_k}\int_0^{s_k} S(t)x_k\mathrm{d}t\right\|^2 \\ &\le \beta_n\|x_k - z\|^2 + (1-\beta_k)\left(\frac{1}{s_k}\int_0^{s_k} \|S(t)x_k - z\|\mathrm{d}t\right)^2 - \beta_k(1-\beta_k)\left\|x_k - \frac{1}{s_k}\int_0^{s_k} S(t)x_k\mathrm{d}t\right\|^2 \\ &\le \beta_n\|x_k - z\|^2 + (1-\beta_k)\left(\frac{1}{s_k}\int_0^{s_k} Ł_t^S\,\mathrm{d}t\right)^2\|x_k - z\|^2 - \beta_k(1-\beta_k)\left\|x_k - \frac{1}{s_k}\int_0^{s_k} S(t)x_k\mathrm{d}t\right\|^2 \\ &\le \|x_k - z\|^2 + (1-\beta_k)(\widetilde{s}_k^2 - 1)\|x_k - z\|^2. \end{aligned} \tag{3.18}$$

Substituting (3.18) in (3.17) yields,

$$\begin{aligned} \|y_k - z\|^2 &\le \|x_k - z\|^2 + (1-\alpha_k)(\widetilde{t}_n^{\,2}[\|x_k - z\|^2 + (1-\beta_k)(\widetilde{s}_k^2 - 1)\|x_k - z\|^2] - \|x_k - z\|^2) \\ &\le \|x_k - z\|^2 + [(1-\alpha_k)(\widetilde{t}_n^{\,2} - 1) + (1-\alpha_k)(1-\beta_k)\widetilde{t}_k^2(\widetilde{s}_k^2 - 1)]\|x_k - z\|^2 \\ &\le \|x_k - z\|^2 + [(1-\alpha_k)(\widetilde{t}_n^{\,2} - 1) + (1-\alpha_k)(1-\beta_k)\widetilde{t}_k^2(\widetilde{s}_k^2 - 1)](\operatorname{diam} C)^2 \\ &\le \|x_k - z\|^2 + \widetilde{\theta}_k^2. \end{aligned}$$

It follows that $z \in C_{k+1}$. Hence $\mathscr{F} \subset C_n$ for all $n \in \mathbb{N}$. Again, by using the same argument in the proof of Theorem 3.1, we have C_n is closed and convex for all $n \in \mathbb{N}$ and

$$\|x_n - x_{n+1}\| \to 0. \tag{3.19}$$

We now claim that

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

Indeed, by definition of y_n and $x_{n+1} \subset C_n$ we have

$$\begin{aligned}\left\|\frac{1}{t_n}\int_0^{t_n} T(t)z_n\mathrm{d}t - x_n\right\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\le \frac{1}{1-\alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\le \frac{1}{1-\alpha_n}\left(2\|x_{n+1} - x_{n+1}\| + \sqrt{\widetilde{\theta}_n}\right) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

We now show that $\limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\| = 0$. Let $\left\{\|\frac{1}{s_{n_k}}\int_0^{s_{n_k}} S(t)x_{n_k}\mathrm{d}t - x_{n_k}\|\right\}$ be any subsequence of $\left\{\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t - x_n\|\right\}$. Since $\{x_{n_k}\}$ is bounded, there is a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\|x_{n_{k_j}} - z\| = \limsup_{k\to\infty}\|x_{n_k} - z\| := a.$$

We observe that

$$\begin{aligned}\|x_{n_{k_j}} - z\| &\le \left\|x_{n_{k_j}} - \frac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}} T(t)z_{n_{k_j}}\mathrm{d}t\right\| + \left\|\frac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}} T(t)z_{n_{k_j}}\mathrm{d}t - z\right\| \\ &\le \left\|x_{n_{k_j}} - \frac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}} T(t)z_{n_{k_j}}\mathrm{d}t\right\| + \frac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}}\|T(t)z_{n_{k_j}} - z\|\mathrm{d}t \\ &\le \left\|x_{n_{k_j}} - \frac{1}{t_{n_{k_j}}}\int_0^{t_{n_{k_j}}} T(t)z_{n_{k_j}}\mathrm{d}t\right\| + \widetilde{t}_n\|z_{n_{k_j}} - z\|.\end{aligned}$$

This implies that $a = \liminf_{j\to\infty}\|x_{n_{k_j}} - z\| \le \liminf_{j\to\infty}\|z_{n_{k_j}} - z\|$. By (3.18) we note that $\|z_{n_{k_j}} - z\| \le \|x_{n_{k_j}} - z\| + ((1 - \beta_{n_{k_j}})(\widetilde{s}_n^2 - 1))^{\frac{1}{2}}\|x_{n_{k_j}} - z\|$ and hence

$$\limsup_{j\to\infty}\|z_{n_{k_j}} - z\| \le \limsup_{j\to\infty}\|x_{n_{k_j}} - z\| = a.$$

Therefore $\lim_{j\to\infty}\|z_{n_{k_j}} - z\| = a = \lim_{j\to\infty}\|x_{n_{k_j}} - z\|$. Furthermore, by (3.18) again, we observe that

$$\begin{aligned}\beta_{n_{k_j}}(1 - \beta_{n_{k_j}})\|x_{n_{k_j}} - \frac{1}{s_{n_{k_j}}}\int_0^{s_{n_{k_j}}} S(t)x_{n_{k_j}}\mathrm{d}t\|^2 &\le \|x_{n_{k_j}} - z\|^2 - \|z_{n_{k_j}} - z\|^2 + (1 - \beta_{n_{k_j}})(\widetilde{s}^2_{n_{k_j}} - 1)\|x_{n_{k_j}} - z\|^2 \\ &\to 0 \quad \text{as } j \to \infty.\end{aligned}$$

This implies that $\lim_{j\to\infty}\|x_{n_{k_j}} - \frac{1}{s_{n_{k_j}}}\int_0^{s_{n_{k_j}}} S(t)x_{n_{k_j}}\mathrm{d}t\| = 0$ and hence

$$\lim_{n\to\infty}\left\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t - x_n\right\| = 0. \tag{3.21}$$

For all $0 \le r < \infty$, we note that

$$\begin{aligned}\|S(r)x_n - x_n\| &\le \left\|S(r)x_n - S(r)\left(\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right)\right\| + \left\|S(r)\left(\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right\| \\ &\quad + \left\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t - x_n\right\| \\ &\le (L_\infty + 1)\left\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t - x_n\right\| + \left\|S(r)\left(\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right\| \\ &:= (L_\infty + 1)A_n^S + B_n^S(r),\end{aligned} \tag{3.22}$$

where $A_n^S := \left\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t - x_n\right\|$ and $B_n^S(r) := \left\|S(r)\left(\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t\right\|$. By (3.21) and Lemma 2.5, we have $\lim_{n\to\infty} A_n^S = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty} B_n^S(r)$. Moreover, we observe that

$$\begin{aligned}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right\| &\le \left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(t)z_n\mathrm{d}t\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(t)z_n\mathrm{d}t - \frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right\| \\ &\le \left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(t)z_n\mathrm{d}t\right\| + \frac{1}{t_n}\int_0^{t_n} \|T(t)z_n - T_t x_n\|\mathrm{d}t \\ &\le \left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(t)z_n\mathrm{d}t\right\| + \widetilde{t}_n\|z_n - x_n\|.\end{aligned}$$

Since $\|z_n - x_n\| = (1-\beta_n)\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n\mathrm{d}t - x_n\| \to 0$ and (3.20) we obtain

$$\lim_{n\to\infty}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right\| = 0. \tag{3.23}$$

We can deduce that for all $0 \le r < \infty$,

$$\begin{aligned}\|T(r)x_n - x_n\| &\le \left\|T(r)x_n - T(r)\left(\frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right)\right\| + \left\|T(r)\left(\frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right\| \\ &\quad + \left\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t - x_n\right\| \\ &\le (L_\infty + 1)\left\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t - x_n\right\| + \left\|T(r)\left(\frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n\mathrm{d}t\right\| \\ &:= (L_\infty + 1)A_n^T + B_n^T(r).\end{aligned}$$

By (3.23) and Lemma 2.5, we have

$$\lim_{n\to\infty} A_n^T = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty} B_n^T(r). \tag{3.24}$$

From (3.22) and (3.24), we obtain

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

We note by Lemma 2.5 that every weak limit point of $\{x_n\}$ is a member of $\mathscr{F}$. From $x_n \rightharpoonup z_0 = P_{\mathscr{F}}x_0$, we have $x_0 - x_n \rightharpoonup x_0 - z_0$ from H satisfies the Kadec–Klee property, it follows that

$$x_0 - x_n \to x_0 - z_0.$$

So, we have

$$\|x_n - z_0\| = \|x_n - x_0 - (z_0 - x_0)\| \to 0 \quad \text{as } n \to \infty.$$

Hence $x_n \to z_0$. This complete the proof. □

If $\mathscr{S} \equiv \mathscr{T}$, then $S(t)x_n = x_n$ for all $n \in \mathbb{N}$ and for all $t > 0$. Hence $\frac{1}{s_n}\int_0^{s_n} S(u)x_n\mathrm{d}u = x_n, z_n = x_n$ for all $n \in \mathbb{N}$ and therefore Theorem 3.3 reduces to the following corollary.

Corollary 3.5. *Let H be a Hilbert space and let C be a nonempty closed bounded convex subset of H. Let $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ and be an asymptotically nonexpansive semigroup on C such that $F(\mathscr{T}) \ne \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(t)x_n\mathrm{d}t, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N} \end{cases}$$

where $\widetilde{\theta}_n = (1-\alpha_n)(\widetilde{t}_n^{\,2} - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$(here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T\mathrm{d}t$, $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $t_n \to \infty$, $s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathscr{T})}x_0$.

By the same argument as in the proof of Theorem 3.4, we obtain the following theorem.

Theorem 3.6. *Let H be a Hilbert space and let C be a nonempty closed bounded convex subset of H. Let $\mathscr{T} = \{T(t) : 0 \le t < \infty\}$ and $\mathscr{S} = \{S(t) : 0 \le t < \infty\}$ be two nonexpansive semigroups on C such that $\mathscr{F} = F(\mathscr{T}) \cap F(\mathscr{S}) \ne \emptyset$ and let $x_0 \in H$. Let*

$C_1 = C$, $x_1 = P_{C_1}x_0$ *define a sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(t)z_n \mathrm{d}t, \\ z_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} S(t)x_n \mathrm{d}t, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N} \end{cases}$$

where $0 \le \alpha_n \le a < 1$ *and* $0 < b \le \beta_n \le c < 1$ *for all* $n \in \mathbb{N}$ *and* $t_n \to \infty, s_n \to \infty$. *Then* $\{x_n\}$ *converges strongly to* $z_0 = P_F x_0$.

If $S = T$, then Theorem 3.6 reduces to Theorem 1.2.

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Strong convergence theorems by hybrid method for asymptotically k-strict pseudo-contractive mapping in Hilbert space

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ABSTRACT

In this paper, we introduce the hybrid method of modified Mann's iteration for an asymptotically k-strict pseudo-contractive mapping. Then we prove that such a sequence converges strongly to $P_{F(T)}x_0$. This main theorem improves the result of Issara Inchan [I. Inchan, Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, Int. J. Math. Anal. 2 (23) (2008) 1135–1145] and concerns the result of Takahashi et al. [W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 341 (2008) 276–286], and many others.

1. Introduction

Let H be a real Hilbert space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. We know that a Hilbert space H satisfies Opial's condition [1], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Recall that a mapping $T : C \to C$ is said to be a strict pseudo-contractive mapping [2] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

for all $x, y \in C$. (If (1.1) holds, we also say that T is a k-strict pseudo-contraction.)

It is known that if T is a 0-strict pseudo-contractive mapping, T is a nonexpansive mapping.

In this paper we will consider an iteration method of modified Mann for asymptotically k-strict pseudo-contractive mapping. We say that $T : C \to C$ is an asymptotically k-strict pseudo-contractive mapping if there exists a constant $0 \leq k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.2}$$

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for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\gamma_n \geq 0$ for all n such that $\lim_{n\to\infty} \gamma_n = 0$. We see that if $k = 0$, then T is an asymptotically nonexpansive mapping. By Goebel and Kirk [3], T is an asymptotically nonexpansive mapping if there exists a sequence $\{\gamma_n\}$ of nonnegative numbers with $\lim_{n\to\infty} \gamma_n = 0$ and such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2, \tag{1.3}$$

for all $x, y \in C$ and all integers $n \geq 1$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [4–7]. However, Mann and Ishikawa iteration processes have only weak convergence even in Hilbert space: see [8,7].

Our iteration method for finding a fixed point of an asymptotically k-strict pseudo-contractive mapping T is the modified Mann's iteration method studied in [9–12] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad n \geq 0, \tag{1.4}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval (0, 1).

In 2007, Takahashi, Takeuchi and Kubota [7] introduced the modification of the Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then we prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2008, Inchan [13], introduced the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then he proves that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these facts, it is the purpose of this paper to introduce the modified Mann iteration processes for an asymptotically k-strict pseudo-contractive mapping by the idea in (1.6). Let C be a closed convex subset of a Hilbert space H, T be an asymptotically k-strict pseudo-contractive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1 - \alpha_n)]\|x_n - T^n x_n\| + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\theta_n = (\operatorname{diam} C)^2(1 - \alpha_n)\gamma_n \to 0$, $(n \to \infty)$.

We shall prove that the iteration generated by (1.7) converges strongly to $z_0 = P_{F(T)}x_0$.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

We collect some lemmas which will be used in the proof of the main result.

Lemma 2.1 (*[14]*). *The following identities hold in a Hilbert space H:*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H$.

(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0, 1]$.

Lemma 2.2 (*[15]*). *Let T be an asymptotically k-strict pseudo-contractive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties*

(i) $x_n \rightharpoonup z$ *and*

(ii) $Tx_n - x_n \to 0$.

Then $(I - T)z = 0$.

Lemma 2.3 ([16]). *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if the following inequality holds*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.4 ([15]). *Assume that C is a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically k-strict pseudo-contraction. Then for each $n \geq 1$, T^n satisfies the Lipschitz condition:*

$$\|T^n x - T^y\| \leq L_n \|x - y\|$$

for all $x, y \in C$, where $L_n = \frac{k+\sqrt{1+\gamma_n(1-k)}}{1-k}$.

3. Main results

In this section, we prove strong convergence theorems by hybrid methods for asymptotically k-strict pseudo-contractive mappings in Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically k-strict pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen such that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (1.7) converges strongly to $z_0 = P_{F(T)} x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. Then we have, for $u \in F(T) \subset C_m$

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\
&= \|\alpha_m (x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m \|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m \|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 + k\|x_m - T^m x_m\|^2] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + (1-\alpha_m)\gamma_m \|x_m - u\|^2 \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.
\end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It obviously follows that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$. Then

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\
&= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\
&\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.
\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0, 1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, we have

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\
&= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\
&\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m) \\
&\quad + (1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m) - \alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m \\
&= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m \\
&= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \quad \text{for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \quad \text{for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.1}$$

So, for $u \in F(T)$, we have

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - u\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.2}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - x_{n+1}\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

From (3.2) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.3) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.5}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1 - \alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n, \tag{3.6}$$

By the definition of y_n, we have

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n)T^n x_n - x_n\| \\ &= (1 - \alpha_n)\|T^n x_n - x_n\|. \end{aligned}$$

From (3.6), we have

$$\begin{aligned} &(1 - \alpha_n)^2\|T^n x_n - x_n\|^2 = \|y_n - x_n\|^2 \\ &\quad = \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\
&\le \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\
&= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n.
\end{aligned}
$$

It follows that

$$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \le 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n.$$

Hence

$$(1 - k - \alpha_n)\|T^n x_n - x_n\| \le 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n. \tag{3.7}$$

From $\limsup_{n\to\infty} \alpha_n < 1 - k$, we can choose $\epsilon > 0$ such that $\alpha_n \le 1 - k - \epsilon$ for large enough n. From (3.5) and (3.7), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.8}$$

Next, we show that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. From Lemma 2.4, we have

$$
\begin{aligned}
\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le L_1\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + L_{n+1})\|x_n - x_{n+1}\|.
\end{aligned} \tag{3.9}
$$

From (3.5) and (3.8), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.10}$$

By (3.9), Lemma 2.2 and boundedness of $\{x_n\}$ we obtain $\emptyset \ne \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \le \|z_0 - x_0\|$ for all $n \ge 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$
\begin{aligned}
\|x_0 - z_0\| &\le \|x_0 - w\| \le \liminf_{n\to\infty} \|x_0 - x_n\| \\
&\le \limsup_{n\to\infty} \|x_0 - x_n\| \le \|x_0 - z_0\|,
\end{aligned}
$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$
\begin{aligned}
\|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\
&\le 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

This completes the proof. □

Using this Theorem 3.1, we have the following corollaries.

Corollary 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a k-strict pseudo-contractive mapping of C into itself for some $0 \le k < 1$ such that $F(T) \ne \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows;*

$$
\begin{cases}
y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2\}, \\
x_{n+1} = P_{C_{n+1}}x_0,
\end{cases} \tag{3.11}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$. Then $\{x_n\}$ generated by (3.11) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3 (*[13]*). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \ne \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows;*

$$
\begin{cases}
y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\
x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N},
\end{cases} \tag{3.12}
$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (3.12) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.4 *([7, Theorem 4.1]). Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.13}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

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Strong convergence theorems for a new iterative method of k-strictly pseudo-contractive mappings in Hilbert spaces

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ABSTRACT

In this paper, we introduce a new iterative method of a k-strictly pseudo-contractive mapping for some $0 \leq k < 1$ and prove that the sequence $\{x_n\}$ converges strongly to a fixed point of T, which solves a variational inequality related to the linear operator A. Our results have extended and improved the corresponding results of Y.J. Cho, S.M. Kang and X. Qin [Some results on k-strictly pseudo-contractive mappings in Hilbert spaces, Nonlinear Anal. 70 (2008) 1956–1964], and many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

Note that the class of k-strictly pseudo-contractive includes strictly the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

This is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings (see [1–3]).

It is clear that, in a real Hilbert space H, (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

The mapping T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

T is strongly pseudo-contractive if and only if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in C. \tag{1.5}$$

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In 2002, Xu [4] studied the following iterative process by the viscosity approximation defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, and then proved that the sequence $\{x_n\}$ converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality:

$$\langle (I-f)q, p-q\rangle \leq 0, \quad \forall p \in F(T). \tag{1.7}$$

Very recently, Marino and Xu [5] introduced and considered the following iterative algorithm:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)Tx_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions and A is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Then they proved that the sequence $\{x_n\}$ converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality:

$$\langle (A-\gamma f)q, q-x\rangle \leq 0, \quad \forall x \in F(T). \tag{1.9}$$

Moreover, Cho, Kang and Qin [6] extended and improved the result of Marino and Xu [5] (see also [2,7–14]) and introduced a general iterative algorithm:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)P_K Sx_n, \quad \forall n \geq 1 \end{cases} \tag{1.10}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$, $\{\alpha_n\}$ of parameters satisfies appropriate conditions, and A is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Then, they proved the strong convergence theorems for T being a k-strictly pseudo-contractive mapping in Hilbert spaces.

In this paper, motivated by Cho et al. [6], we introduce a new iterative scheme generated by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C Sx_n, \end{cases} \tag{1.11}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$ and $T : C \to H$ is a k-strictly pseudo-contractive mapping, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. We will prove in Section 3 that if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.11) converges strongly to the solution of variational inequality (1.9).

2. Preliminary

In this section, we collect some lemmas which will be used in the proof for the main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$
(ii) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$
(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$
(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0, 1]$.

Lemma 2.2 ([12]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + b_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ be a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([15]). *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.4 ([5]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$,*

$$\langle x-y, (A-\gamma f)x - A(A-\gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x-y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.5 ([5]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$. *Then* $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.6 ([16]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.7 ([14]). *Let H be a Hilbert space, and C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set F(T) is closed convex, so that the projection* $P_{F(T)}$ *is well defined.*

Lemma 2.8 ([14]). *Let H be a Hilbert space, and C be a closed convex subset of H. Let* $T : C \to H$ *be a k-strictly pseudo-contractive mapping with* $F(T) \neq \emptyset$. *Then* $F(P_C T) = F(T)$.

Lemma 2.9 ([14]). *Let H be a Hilbert space, and C be a closed convex subset of H. Let* $T : C \to H$ *be a k-strictly pseudo-contractive mapping. Define a mapping* $S : C \to H$ *by* $Sx = \lambda x + (1 - \lambda)Tx$ *for all* $x \in C$. *Then, as* $\lambda \in [k, 1)$, *S is a nonexpansive mapping such that* $F(S) = F(T)$.

Lemma 2.10 ([5]). *Let H be a Hilbert space, and C be a nonempty closed convex subset of H. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient* $\overline{\gamma} > 0$. *Assume that* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $T : C \to C$ *be a nonexpansive mapping with fixed point* x_t *of contraction* $C \ni x \mapsto t\gamma f(x) + (1 - tA)Tx$. *Then* $\{x_t\}$ *converges strongly to fixed point* $\widetilde{x}$ *of T as* $t \to 0$, *which solves the following variational inequality:*

$$\langle(\gamma f - A)\widetilde{x}, z - \widetilde{x}\rangle \leq 0, \quad \forall z \in F(T).$$

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$. If μ is a Banach limit, then we have the following:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,
(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $s = (a_0, a_1, \ldots) \in l^\infty$.

Lemma 2.11 ([13]). *Let* $a \in \mathbb{R}$ *be a real number and a sequence* $\{a_n\} \subset l^\infty$ *satisfying the condition* $\mu_n(a_n) \leq a$ *for all Banach limits* μ. *If* $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, *then* $\limsup_{n\to\infty} a_n \leq a$.

Lemma 2.12 ([17]). *Let H be a Hilbert space, and C be a nonempty closed convex subset of H. For any integer* $N \geq 1$, *assume that, for each* $1 \leq i \leq N$, $T_i : C \to H$ *be* k_i*-strictly pseudo-contractive mappings for some* $0 \leq k_i < 1$. *Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$. *Then* $\sum_{i=1}^N \eta_i T_i$ *is a non-self-k-strictly pseudo-contractive mapping with* $k = \max\{k_i : 1 \leq i \leq N\}$.

Lemma 2.13 ([17]). *Let* $\{T_i\}_{i=1}^N$ *and* $\{\eta_i\}_{i=1}^N$ *be given as in Lemma 2.12. Suppose that* $\{T_i\}_{i=1}^N$ *has a common fixed point in C. Then* $F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^\infty F(T_i)$.

3. Main results

In this section, first we show that a mapping $S : C \to H$ defined by $Sx = kx + (1 - k)Tx$ is a nonexpansive mapping, where C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ is a k-strictly pseudo contractive mapping with a fixed point for some $0 \leq k < 1$. Let $x, y \in C$; then from Lemma 2.1(iv) we have

$$\begin{aligned}\|Sx - Sy\|^2 &= \|kx + (1 - k)Tx - (ky + (1 - k)Ty)\|^2 \\ &= \|k(x - y) + (1 - k)(Tx - Ty)\|^2 \\ &= k\|x - y\|^2 + (1 - k)\|Tx - Ty\|^2 - k(1 - k)\|(x - y)x - (Tx - Ty)\|^2 \\ &= k\|x - y\|^2 + (1 - k)(\|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2) - k(1 - k)\|(x - y)x - (Tx - Ty)\|^2 \\ &= \|x - y\|^2 + (1 - k)k(\|(I - T)x - (I - T)y\|^2) - k(1 - k)\|(I - T)x - (I - T)y\|^2 \\ &\leq \|x - y\|^2.\end{aligned}$$

Hence $\|Sx - Sy\| \leq \|x - y\|$. Then S is a nonexpansive mapping and we have that $P_C S$ is also nonexpansive, where P_C is a metrics projection on C. For any $j \in \mathbb{N}$, define a mapping $S_j : C \to C$ by $S_j x = \frac{1}{j}\gamma f(x) + (I - \frac{1}{j}A)P_C Sx$. Let us show that S_j is a contraction: let $x, y \in C$; we have

$$\begin{aligned}\|S_jx - S_jy\| &= \left\| \frac{1}{j}\gamma f(x) + \left(I - \frac{1}{j}A\right) P_C Sx - \left(\frac{1}{j}\gamma f(y) + \left(I - \frac{1}{j}A\right) P_C Sy\right) \right\| \\ &\le \frac{1}{j}\gamma\alpha\|x - y\| + \left(1 - \frac{1}{j}\overline{\gamma}\right) \|P_C Sx - P_C Sy\| \\ &\le \frac{1}{j}\gamma\alpha\|x - y\| + \left(1 - \frac{1}{j}\overline{\gamma}\right) \|x - y\| \\ &\le \left(1 - \frac{1}{j}(\overline{\gamma} - \gamma\alpha)\right) (\|x - y\|).\end{aligned}$$

Hence, S_j is a contraction. By Banach's contraction principle there exists a unique fixed point $u_j \in C$ such that

$$u_j = \frac{1}{j}\gamma f(u_j) + \left(1 - \frac{1}{j}A\right) P_C Su_j. \tag{3.1}$$

Next, we prove the main results.

Theorem 3.1. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C \pm C \subset C$, and let $T : C \to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0 \le k < 1$. Let A be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)P_C Sx_n, \end{cases} \tag{3.2}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of variational inequality (1.9).

Proof. Note that from the condition $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \le (1 - \beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H,

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \alpha_n\|A\| \\ &\ge 0;\end{aligned}$$

that is to say, $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \alpha_n A\| &= \sup\{\langle((1 - \beta_n)I - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\le 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned}$$

We now observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(T)$; we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)P_C Sx_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(P_C Sx_n - p)\| \\ &\le \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + \|((1 - \beta_n)I - \alpha_n A)\|\|P_C Sx_n - p\| \\ &\le \alpha_n\|\gamma f(x_n) - \gamma f(p) + \gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n(\overline{\gamma} - \gamma\alpha)\frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}.\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}\right\}, \quad n \ge 0,$$

and hence $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{P_C S x_n\}$ are bounded. From (3.1), we have, for any $n, j \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - P_C S u_j\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n - P_C S u_j\| \\
&= \|\alpha_n(\gamma f(x_n) - AP_C S u_j) + \beta_n(x_n - P_C S u_j) + ((1-\beta_n)I - \alpha_n A)(P_C S x_n - P_C S u_j)\| \\
&\le \alpha_n\|\gamma f(x_n) - AP_C S u_j\| + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n - \alpha_n\overline{\gamma})\|P_C S x_n - P_C S u_j\| \\
&\le \alpha_n\|\gamma f(x_n) - AP_C S u_j\| + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n - \alpha_n\overline{\gamma})\|x_n - u_j\| \\
&= \alpha_n(\|\gamma f(x_n) - AP_C S u_j\| - \overline{\gamma}\|x_n - u_j\|) + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\| \\
&= \delta_n + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|
\end{aligned}$$

where $\delta_n = \alpha_n(\|\gamma f(x_n) - AP_C S u_j\| - \overline{\gamma}\|x_n - u_j\|)$, and from $\lim_{n\to\infty}\alpha_n = 0$, we have $\delta_n \to 0$ as $n \to \infty$. It follows that

$$\begin{aligned}
\|x_{n+1} - P_C S u_j\|^2 &= (\delta_n + \beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)^2 \\
&= (\beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)^2 + 2(\beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \\
&= \beta_n^2\|x_n - P_C S u_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + 2\beta_n(1-\beta_n)\|x_n - P_C S u_j\|\|x_n - u_j\| + \sigma_n
\end{aligned}$$

where $\sigma_n = 2(\beta_n\|x_n - P_C S u_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \to 0$ as $n \to \infty$, and hence

$$\begin{aligned}
\|x_{n+1} - P_C S u_j\|^2 &\le \beta_n^2\|x_n - P_C S u_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + \beta_n(1-\beta_n)(\|x_n - P_C S u_j\|^2 + \|x_n - u_j\|^2) + \sigma_n \\
&= \beta_n\|x_n - P_C S u_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n.
\end{aligned}$$

For any Banach limit μ and $\beta_n \to 0$, we have

$$\mu_n\|x_n - P_C S u_j\|^2 = \mu_n\|x_{n+1} - P_C S u_j\|^2 \le \mu_n\|x_n - u_j\|^2. \tag{3.3}$$

Since $u_j - x_n = \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n) + (1 - \frac{1}{j})(P_C S u_j - x_n)$; thus we have

$$\left(1 - \frac{1}{j}\right)(x_n - P_C S u_j) = (x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n).$$

It follows from Lemma 2.1(ii) that

$$\begin{aligned}
\left(1 - \frac{1}{j}\right)^2\|x_n - P_C S u_j\|^2 &= \left\|(x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C S u_j - x_n)\right\|^2 \\
&\ge \|x_n - u_j\|^2 + \frac{2}{j}\langle(\gamma f(u_j) + (I-A)P_C S u_j - x_n), x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j - (x_n - u_j), x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle - \frac{2}{j}\langle x_n - u_j, x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle - \frac{2}{j}\|x_n - u_j\|^2 \\
&= \left(1 - \frac{2}{j}\right)\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle.
\end{aligned} \tag{3.4}$$

So, by (3.3) and (3.4), we have

$$\begin{aligned}
\left(1 - \frac{1}{j}\right)^2\|x_n - u_j\|^2 &\ge \left(1 - \frac{1}{j}\right)^2\|P_C S u_j - x_n\|^2 \\
&\ge \left(1 - \frac{2}{j}\right)\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle
\end{aligned}$$

and hence

$$\frac{1}{j^2}\|x_n - u_j\|^2 \ge \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle.$$

This implies that

$$\frac{2}{j}\mu_n\|x_n-u_j\|^2 \geq \mu_n\langle\gamma f(u_j)+(I-A)P_CSu_j-u_j, x_n-u_j\rangle.$$

From Lemmas 2.8 and 2.10, $u_j \to p \in F(T)=F(P_CS)$ as $j\to\infty$, we get

$$\mu_n\langle\gamma f(p)-Ap, x_n-p\rangle \leq 0, \tag{3.5}$$

where p is the solution of variational inequality (1.9). Since $\{x_n\}$, $\{f(x_n)\}$ and $\{P_CSx_n\}$ are bounded, we choose

$$M=\sup\{\|f(x_n)\|+\|x_n\|+\|P_CSx_n\|+\|AP_CSx_n\| : n\in\mathbb{N}\}.$$

On the other hand,

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}x_{n+1}+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}\\
&\quad -(\alpha_{n+1}\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)P_CSx_n)\|\\
&= \|\alpha_{n+1}\gamma f(x_{n+1})-\alpha_{n+1}\gamma f(x_n)+\alpha_{n+1}\gamma f(x_n)-\alpha_n\gamma f(x_n)+\beta_{n+1}x_{n+1}-\beta_{n+1}x_n\\
&\quad +\beta_{n+1}x_n-\beta_nx_n+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}-((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n\\
&\quad +((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n-((1-\beta_n)I-\alpha_nA)P_CSx_n\|\\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})\|P_CSx_{n+1}-P_CSx_n\|\\
&\quad +\|((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_nA)\|\|P_CSx_n\|\\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|P_CSx_n\|+|\alpha_{n+1}-\alpha_n|\|AP_CSx_n\|\\
&\leq (1-\alpha_{n+1}(\overline{\gamma}-\gamma\alpha))\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\gamma M+|\beta_{n+1}-\beta_n|M\\
&\quad +|\beta_{n+1}-\beta_n|M+|\alpha_{n+1}-\alpha_n|M.
\end{aligned}$$

From (ii), (iii) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

Next, we show that $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$. We consider

$$\begin{aligned}
\|x_n-P_CSx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-P_CSx_n\|\\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n-Ap)\|+\beta_n\|x_n-P_CSx_n\|.
\end{aligned}$$

From $\alpha_n\to 0$, $\beta_n\to 0$ and (3.6), it follows that $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$.

Next, we show that

$$\limsup_{n\to\infty}\langle\gamma f(p)-Ap, x_n-p\rangle\leq 0,$$

where $p\in F(T)$, where p is the solution of variational inequality (1.9). From (3.6), we have

$$\limsup_{n\to\infty}|\langle\gamma f(p)-Ap, x_{n+1}-p\rangle-\langle\gamma f(p)-Ap, x_n-p\rangle|=0. \tag{3.7}$$

Hence it follows from (3.5) and (3.7) and Lemma 2.11 that

$$\limsup_{n\to\infty}\langle\gamma f(p)-Ap, x_n-p\rangle\leq 0, \tag{3.8}$$

and from $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle\gamma f(p)-Ap, P_CSx_n-p\rangle &= \limsup_{n\to\infty}\langle\gamma f(p)-Ap, (P_CSx_n-x_n)+(x_n-p)\rangle\\
&= \limsup_{\to\infty}\langle\gamma f(p)-Ap, x_n-p\rangle\leq 0.
\end{aligned} \tag{3.9}$$

Finally, we prove that $x_n\to p$ as $n\to\infty$. We note that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)P_CSx_n-p\|^2\\
&= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(P_CSx_n-p)\|^2\\
&= \|\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(P_CSx_n-p)\|^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad +2\langle\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(P_CSx_n-p), \alpha_n(\gamma f(x_n)-Ap)\rangle\\
&\leq (\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\overline{\gamma})\|P_CSx_n-p\|)^2+2\beta_n\alpha_n\langle x_n-p, (\gamma f(x_n)-Ap)\rangle+\alpha_n^2\|\gamma f(x_n)-Ap\|^2
\end{aligned}$$

$$
\begin{aligned}
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle-2\alpha_n^2\langle A(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle\\
&\le(\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|)^2+2\beta_n\alpha_n\alpha\gamma\|x_n-p\|^2+2\beta_n\alpha_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle-2\alpha_n^2\langle A(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\
&\le(1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2+2\beta_n\alpha_n\alpha\gamma\|x_n-p\|^2+2\beta_n\alpha_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_n^2M\\
&=(1-2(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|^2+(\alpha_n\overline{\gamma})^2M+2\beta_n\alpha_n\alpha\gamma\|x_n-p\|^2+2\beta_n\alpha_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_n^2M\\
&=(1-2(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_nM+\alpha_n\overline{\gamma}^2M]\\
&=:(1-\gamma_n)\|x_n-p\|^2+b_n
\end{aligned}
$$

where $\gamma_n=2(\overline{\gamma}-\gamma\alpha)\alpha_n$ and $b_n=\alpha_n[2\beta_n\langle x_n-p,(\gamma f(p)-Ap)\rangle+2(1-\beta_n)\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_nM+\alpha_n\overline{\gamma}^2M]$. From $\sum_{n=1}^{\infty}\alpha_n=\infty$, (3.8) and (3.9), we have $\Sigma_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\frac{b_n}{\gamma_n}\le 0$. By Lemma 2.2, we have that the sequence $\{x_n\}$ converges strongly to a fixed point p of T, which is the solution of variational inequality (1.9). This completes the proof. □

If $\beta_n\equiv 0$, in Theorem 3.1, we obtain the following corollary.

Corollary 3.2 ([6]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C\pm C\subset C$, and let $T:C\to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0\le k<1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma}>0$ and $f:C\to C$ be a contraction with the contractive constant $(0<\alpha<1)$ such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}x_1\in C,\\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)P_CSx_n,\end{cases}\tag{3.10}
$$

where $S:C\to H$ is a mapping defined by $Sx=kx+(1-k)Tx$. If the control sequence $\{\alpha_n\}\subset(0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n=0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of variational inequality (1.9). □

Theorem 3.3. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C\pm C\subset C$, and $T:C\to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0\le k<1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma}>0$ and $f:C\to C$ be a contraction with the contractive constant $(0<\alpha<1)$ such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}x_1\in C,\\ x_{n+1}=\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)P_CSx_n,\end{cases}\tag{3.11}
$$

where $S:C\to H$ is a mapping defined by $Sx=kx+(1-k)Tx$. If the control sequence $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n=0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,
(iv) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of variational inequality (1.9).

Proof. In the proof of Theorem 3.1, we have that $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{P_CSx_n\}$ are bounded. Next, we show that $\|x_{n+1}-x_n\|\to 0$. Define the sequence $z_n=\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)P_CSx_n}{1-\beta_n}$, such that $x_{n+1}=\beta_nx_n+(1-\beta_n)z_n$, $n\ge 0$. Observe that from the definition of z_n we obtain

$$
\begin{aligned}
z_{n+1}-z_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)P_CSx_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_{n+1}\gamma f(x_n)}{1-\beta_{n+1}}+\frac{\alpha_{n+1}\gamma f(x_n)}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)}{1-\beta_n}\\
&\quad+\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}}{1-\beta_{n+1}}-\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n}{1-\beta_{n+1}}+\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n}{1-\beta_{n+1}}
\end{aligned}
$$

$$- \frac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_{n+1}} + \frac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_{n+1}} - \frac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_n}$$

$$= \frac{\alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))}{1-\beta_{n+1}} + (\alpha_{n+1}-\alpha_n)\frac{(\gamma f(x_{n+1}))}{1-\beta_{n+1}} + \frac{((1-\beta_{n+1})I-\alpha_{n+1}A)}{1-\beta_{n+1}}(P_C Sx_{n+1}-P_C Sx_n)$$

$$+ \frac{[((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_n A)]}{1-\beta_{n+1}}(P_C Sx_n)$$

$$+ ((1-\beta_n)I-\alpha_n A)\left(\frac{1}{1-\beta_{n+1}} - \frac{1}{1-\beta_n}\right)(P_C Sx_n).$$

Thus,

$$\|z_{n+1}-z_n\| \le \frac{\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|}{1-\beta_{n+1}} + |\alpha_{n+1}-\alpha_n|\frac{\|\gamma f(x_{n+1})\|}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|P_C Sx_{n+1}-P_C Sx_n\|$$

$$+ \frac{\|((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_n A)\|}{1-\beta_{n+1}}\|P_C Sx_n\| + ((1-\beta_n-\alpha_n\overline{\gamma})|\frac{1}{1-\beta_{n+1}}$$

$$- \frac{1}{1-\beta_n}|\|P_C Sx_n\|) \le \frac{\alpha_{n+1}\gamma\alpha}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M + \frac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|x_{n+1}-x_n\|$$

$$+ \frac{[|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n|\overline{\gamma}]}{1-\beta_{n+1}}\|AP_C Sx_n\| + \left((1-\beta_n-\alpha_n\overline{\gamma})\left|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}\right. \|P_C Sx_n\|\right)$$

$$= \frac{\alpha_{n+1}\gamma\alpha}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M + \|x_{n+1}-x_n\| - \frac{\alpha_{n+1}\overline{\gamma}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|$$

$$+ \frac{[|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n|\overline{\gamma}]}{1-\beta_{n+1}}M + \left((1-\beta_n-\alpha_n\overline{\gamma})|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}M\right)$$

where $M = \sup\{\|f(x_n)\| + \|P_C Sx_n\| + \|AP_C Sx_n\| + \|x_{n+1}-x_n\| : n \in \mathbb{N}\}$. It follows that

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le \left|\frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M + \frac{[|\beta_{n+1}-\beta_n| + |\alpha_{n+1}-\alpha_n|\overline{\gamma}]}{1-\beta_{n+1}}M\right.$$

$$+ \left((1-\beta_n-\alpha_n\overline{\gamma})\left|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}M\right.\right).$$

Since $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0. \tag{3.12}$$

From $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, (3.12) and Lemma 2.6, we have

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \tag{3.13}$$

We consider

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\beta_n)z_n - \beta_n x_n - x_n\| \\ &= (1-\beta_n)\|z_n-x_n\|\end{aligned}$$

then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0.$$

Next, we show that $\lim_{n\to\infty}\|x_n-P_C Sx_n\| = 0$. We note that

$$\begin{aligned}\|x_n-P_C Sx_n\| &\le \|x_n-x_{n+1}\| + \|x_{n+1}-P_C Sx_n\| \\ &\le \|x_n-x_{n+1}\| + \alpha_n\|\gamma f(x_n)-AP_C Sx_n\| + \beta_n\|x_n-P_C Sx_n\|,\end{aligned} \tag{3.14}$$

and hence

$$(1-\beta_n)\|x_n-P_C Sx_n\| \le \|x_n-x_{n+1}\| + \alpha_n\|\gamma f(x_n)-AP_C Sx_n\|.$$

From $\alpha_n \to 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, it follows that $\lim_{n\to\infty}\|x_n-P_C Sx_n\| = 0$. From (3.1), we have, for any $n, j \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1}-P_C Su_j\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I-\alpha_n A)P_C Sx_n - P_C Su_j\| \\ &= \|\alpha_n(\gamma f(x_n)-AP_C Su_j) + \beta_n(x_n-P_C Su_j) + ((1-\beta_n)I-\alpha_n A)(P_C Sx_n-P_C Su_j)\|\end{aligned}$$

$$\begin{aligned} &\le \alpha_n\|\gamma f(x_n) - AP_C Su_j\| + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|P_C Sx_n - P_C Su_j\| \\ &\le \alpha_n\|\gamma f(x_n) - AP_C Su_j\| + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - u_j\| \\ &= \alpha_n(\|\gamma f(x_n) - AP_C Su_j\| - \overline{\gamma}\|x_n-u_j\|) + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\| \\ &= \delta_n + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n-u_j\| \end{aligned}$$

where $\delta_n = \alpha_n(\|\gamma f(x_n) - AP_C Su_j\| - \overline{\gamma}\|x_n - u_j\|)$. From $\lim_{n\to\infty}\alpha_n = 0$, we have $\delta_n \to 0$ as $n\to\infty$. It follows that

$$\begin{aligned} \|x_{n+1} - P_C Su_j\|^2 &= (\delta_n + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)^2 \\ &= (\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n-u_j\|)^2 + 2(\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \\ &= \beta_n^2\|x_n - P_C Su_j\|^2 + (1-\beta_n)^2\|x_n-u_j\|^2 + 2\beta_n(1-\beta_n)\|x_n - P_C Su_j\|\|x_n-u_j\| + \sigma_n \end{aligned}$$

where $\sigma_n = 2(\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n-u_j\|)\delta_n + \delta_n^2 \to 0$ as $n\to\infty$, and hence

$$\begin{aligned} \|x_{n+1} - P_C Su_j\|^2 &\le \beta_n^2\|x_n - P_C Su_j\|^2 + (1-\beta_n)^2\|x_n-u_j\|^2 \\ &\quad + \beta_n(1-\beta_n)(\|x_n - P_C Su_j\|^2 + \|x_n-u_j\|^2) + \sigma_n \\ &= \beta_n\|x_n - P_C Su_j\|^2 + (1-\beta_n)\|x_n-u_j\|^2 + \sigma_n. \end{aligned} \tag{3.15}$$

From (3.25), we have

$$\begin{aligned} \|x_n - P_C Su_j\|^2 &= \|(x_n - x_{n+1}) + (x_{n+1} - P_C Su_j)\|^2 \\ &= \|x_{n+1} - P_C Su_j\|^2 + 2\langle x_{n+1} - P_C Su_j, x_n - x_{n+1}\rangle + \|x_n - x_{n+1}\|^2 \\ &= \|x_{n+1} - P_C Su_j\|^2 + 2\|x_{n+1} - P_C Su_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2, \\ &\le \beta_n\|x_n - P_C Su_j\|^2 + (1-\beta_n)\|x_n-u_j\|^2 + \sigma_n + 2\|x_{n+1} - P_C Su_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2 \end{aligned}$$

and hence

$$(1-\beta_n)\|x_n - P_C Su_j\|^2 \le (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n + 2\|x_{n+1} - P_C Su_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2.$$

For any Banach limit μ and $\sigma_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, we have

$$\mu_n\|x_n - P_C Su_j\|^2 \le \mu_n\|x_n - u_j\|^2. \tag{3.16}$$

Since $u_j - x_n = \frac{1}{j}(\gamma f(u_j) + (I-A)P_C Su_j - x_n) + (1-\frac{1}{j})(P_C Su_j - x_n)$, we have

$$\left(1-\frac{1}{j}\right)(x_n - P_C Su_j) = (x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C Su_j - x_n).$$

It follows from Lemma 2.1(ii) that

$$\begin{aligned} \left(1-\frac{1}{j}\right)^2\|x_n - P_C Su_j\|^2 &= \|(x_n-u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C Su_j - x_n)\|^2 \\ &\ge \|x_n - u_j\|^2 + \frac{2}{j}\langle(\gamma f(u_j) + (I-A)P_C Su_j - x_n), x_n - u_j\rangle \\ &= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j - (x_n - u_j), x_n - u_j\rangle \\ &= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle - \frac{2}{j}\langle x_n - u_j, x_n - u_j\rangle \\ &= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle - \frac{2}{j}\|x_n - u_j\|^2 \\ &= \left(1-\frac{2}{j}\right)\|x_n-u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle. \end{aligned} \tag{3.17}$$

So, by (3.16) and (3.17), we have

$$\begin{aligned} \left(1-\frac{1}{j}\right)^2\|x_n - u_j\|^2 &\ge \left(1-\frac{1}{j}\right)^2\|P_C Su_j - x_n\|^2 \\ &\ge \left(1-\frac{2}{j}\right)\|x_n-u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle \end{aligned}$$

and hence

$$\frac{1}{j^2}\|x_n - u_j\|^2 \geq \frac{2}{j}\langle \gamma f(u_j) + (I - A)P_C S u_j - u_j, x_n - u_j\rangle.$$

This implies that

$$\frac{2}{j}\mu_n\|x_n - u_j\|^2 \geq \mu_n\langle \gamma f(u_j) + (I - A)P_C S u_j - u_j, x_n - u_j\rangle.$$

From Lemmas 2.8 and 2.10, $u_j \to p \in F(T) = F(P_C S)$ *as* $j \to \infty$, we get

$$\mu_n\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.18}$$

where p is the solution of variational inequality (1.9). Next, we show that

$$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0,$$

where $p \in F(T)$, where p is the solution of variational inequality (1.9). From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have

$$\limsup_{n\to\infty}|\langle \gamma f(p) - Ap, x_{n+1} - p\rangle - \langle \gamma f(p) - Ap, x_n - p\rangle| = 0. \tag{3.19}$$

Hence it follows from (3.18) and (3.19) and Lemma 2.11 that

$$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.20}$$

and from (3.14), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(p) - Ap, P_C S x_n - p\rangle &= \limsup_{n\to\infty}\langle \gamma f(p) - Ap, (P_C S x_n - x_n) + (x_n - p)\rangle \\ &= \limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0. \end{aligned}\tag{3.21}$$

By the same argument as used in Theorem 3.1, we have that the sequence $\{x_n\}$ converges strongly to a fixed point p of T, which is the solution of variational inequality (1.9). This completes the proof. □

Theorem 3.4. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C \pm C \subset C$, and $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)P_C S x_n, \end{cases} \tag{3.22}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)\Sigma_{i=1}^N \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, \ldots, N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$,
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$,
(iii) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^N$, which solves the following solution of the variational inequalities:

$$\langle (A - \gamma f)p, p - x\rangle \leq 0, \quad \forall x \in \cap_{i=1}^N F(T_i). \tag{3.23}$$

Proof. Define a mapping $T : C \to H$ by $Tx = \sum_{i=1}^N \eta_i T_i x$. By Lemmas 2.12 and 2.13, we conclude that : $C \to H$ is a $k-$strictly pseudo-contractive mapping with $k = \max\{k_i : i = 1, 2, \ldots, N\}$ and $F(T) = F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$. From Theorem 3.1, we can obtain desired conclusion easily. This completes the proof. □

If $\beta_n \equiv 0$, Theorem 3.4 reduces to the following corollary.

Corollary 3.5 (*[6]*). *Let H be a Hilbert space, K be a nonempty closed convex subset of H such that $K \pm K \subset K$, and $T_i : K \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let A be strongly positive* *bounded linear operator on K with coefficient* $\overline{\gamma} > 0$ *and* $f : K \to K$ *be a contraction with the contractive constant* $(0 < \alpha < 1)$ *such that* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.24}$$

where $S : K \to H$ *is a mapping defined by* $Sx = kx + (1-k)\sum_{i=1}^{N} \eta_i T_i x$ *and* $k = \max\{k_i : i = 1, 2, \ldots, N\}$. *If the control sequence* $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then $\{x_n\}$ *converges strongly to a common fixed point* p *of* $\{T_i\}_{i=1}^{N}$, *which solves the following solution of the variational inequalities:*

$$\langle (A - \gamma f)p, p - x \rangle \le 0, \forall x \in \bigcap_{i=1}^{N} F(T_i).$$

From the proof of Theorem 3.3, we can obtain the following theorem.

Theorem 3.6. *Let H be a Hilbert space, C a nonempty closed convex subset of H such that* $C \pm C \subset C$, *and* $T_i : C \to H$ *be a* k_i*-strictly pseudo-contractive mapping with a fixed point for some* $0 \le k_i < 1$ *and* $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let A be strongly positive bounded linear operator on C with coefficient* $\overline{\gamma} > 0$ *and* $f : C \to C$ *be a contraction with the contractive constant* $(0 < \alpha < 1)$ *such that* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.25}$$

where $S : C \to H$ *is a mapping defined by* $Sx = kx + (1-k)\sum_{i=1}^{N} \eta_i T_i x$ *and* $k = \max\{k_i : i = 1, 2, \ldots, N\}$. *If the control sequence* $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to a common fixed point* p *of* $\{T_i\}_{i=1}^{N}$, *which solves the following solution of variational inequalities (3.23).*

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