รายงานวิจัยฉบับสมบูรณ์

โครงการ การประมาณค่าจุดตรึงร่วมสำหรับวงค์นับได้ ของการส่งแบบไม่ขยายและปัญหาดุลยภาพ

Approximation of common fixed points for a countable family of nonexpansive mappings and an equilibrium problems

โดย ผศ.ดร.ภูมิ คำเอม

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โดย

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Abstract

The purposes of this research are to create new knowledge of fixed point theorem and construct several new iterative approximation methods for approximating the fixed point of nonexpansive mappings, and to solve many mathematical problems in Hilbert spaces and Banach spaces. We introduce the proof of new convergence theorems of a new iterative approximation method for finding the common element of the set of common fixed points of nonexpansive mappings, the set solutions of the variational inequality problems for nonlinear mappings and the set of solutions of equilibrium problems in Hilbert space and Banach spaces. Therefore, by using the previous result, an iterative algorithm for the solution of a optimization problems was obtained.

Keywords: Iterative approximation method/ Variational inequality problem/ Equilibrium problem/ Nonexpansive mapping / Optimization problem

บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้ คือ การสร้างองค์ความรู้ใหม่ของทฤษฏีบทจุดตรึง และการสร้างวิธีการประมาณค่าแบบทำซ้ำชนิดใหม่ต่างๆ เพื่อใช้ในการประมาณค่าจุดตรึงของการส่งแบบไม่ขยาย และเพื่อแก้ไขปัญหาต่างๆ ทางคณิตศาสตร์ในปริภูมิฮิลเบิร์ต และปริภูมิบานาค เรานำเสนอวิธีการพิสูจน์ทฤษฏีบทการลู่เข้าของวิธีการประมาณค่าแบบทำซ้ำแบบใหม่สำหรับใช้เพื่อการหาสมาชิกร่วมของเซตของจุดตรึงของการส่งแบบไม่ขยาย เซตคำตอบของอสมการเชิงแปรผันสำหรับการส่งแบบไม่เชิงเส้น และเซตคำตอบของปัญหาเชิงดุลยภาพ ในปริภูมิฮิลเบิร์ต และปริภูมิบานาค ดังนั้นโดยการใช้ผลลัพธ์ที่ได้มาก่อนหน้านี้และขั้นตอนวิธีการแบบทำซ้ำทำให้ได้คำตอบของปัญหาค่าเหมาะสมที่สุด

คำสำคัญ : วิธีการประมาณค่าแบบทำซ้ำ / ปัญหาอสมการเชิงแปรผัน/ปัญหาเชิงดุลยภาพ/ การส่งแบบไม่ขยาย / ปัญหาค่าเหมาะสมที่สุด

บทที่ 1

บทนำ (Introduction)

การศึกษาคณิตศาสตร์ทางด้านการวิเคราะห์เชิงฟังก์ชัน (Functional Analysis) นั้น เรื่องของจุดตรึง (Fixed points) มักจะถูกหยิบยกขึ้นมาศึกษาอยู่เสมอ ซึ่งพอจะแบ่งแยกออกเป็นสองประเด็นใหญ่ๆ ด้วยกัน ดังนี้

ประเด็นแรก คือประเด็นในเรื่องของการหาเงื่อนไขที่เพียงพอให้กับปริภูมิต่างๆ หรือฟังก์ชันต่างๆ เพื่อที่จะทำฟังก์ชันนั้นมีจุดตรึง เมื่อศึกษาการมีคำตอบของสมการต่างๆ แล้ว ปัญหาที่น่าสนใจต่อไปก็คือ

ประเด็นที่สอง เราจะหาคำตอบของสมการต่างๆ นั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้นระเบียบวิธีการทำซ้ำของจุดตรึง (fixed-point iterations) ต่างๆที่ใช้ในการหาคำตอบโดยวิธีประมาณค่า เพื่อนำไปประยุกต์ใช้ในการแก้ปัญหาของระบบสมการที่มีตัวดำเนินการไม่เชิงเส้น (nonlinear operator equations) ในเรื่องของแก้ปัญหาอสมการแปรผัน (variational inequality problem (VIP)) และแก้สมการหาคำตอบของปัญหาดุลภาพ(equilibrium problems (EP)) ปัญหาที่ดีที่สุด(optimizations problems) ปัญหาน้อยที่สุด (minimizations problems) ทั้งในปริภูมิฮิลเบิร์ตและปริภูมิบานาค ซึ่งปัญหาดังกล่าวเป็นปัญหาที่สำคัญที่มีประโยชน์มากมายในสาขาวิชาต่างๆ เช่น สาขาวิชาฟิสิกส์ คณิตศาสตร์ประยุกต์ วิศวกรรม สาขาทางเศรษฐศาสตร์ ระบบโลจิสติกส์ และพัฒนาโครงสร้างพื้นฐานระบบการขนส่งและระบบบริหารจัดการที่เหมาะสม เป็นต้น

กระบวนการสร้างลำดับด้วยวิธีทำซ้ำ ในปริภูมิเวกเตอร์ต่างๆ และการศึกษาหาเงื่อนไขที่จำเป็น และเพียงพอที่จะทำให้ลำดับเหล่านั้นลู่เข้าสู่จุดตรึงจุดหนึ่งของฟังก์ชันต่างๆ เป็นเรื่องที่จำเป็นต้องคิดค้นและศึกษา เพื่อให้ขอบเขตการใช้งานกว้างขึ้นและง่ายขึ้น กระบวนการสร้างลำดับด้วยวิธีทำซ้ำมีหลากหลายวิธีด้วยกัน เช่น Mann Iteration process, Ishikawa Iteration process, Noor Iteration process เป็นต้น ซึ่งเราสามารถนำเอาเทคโนโลยีการคำนวณด้วยคอมพิวเตอร์มาช่วยในการหาจุดตรึงโดยกระบวนการสร้างลำดับ ที่กล่าวมาแล้วข้างต้น

นอกจากนี้ ยังมีกระบวนการสร้างลำดับอีกแบบหนึ่งคือ กระบวนการสร้างลำดับด้วยวิธีทำซ้ำโดยปริยาย (Implicit Iteration process) ซึ่งเป็นกระบวนการสร้างลำดับ ที่ลู่เข้าสู่จุดตรึงได้รวดเร็วกว่ากระบวนการสร้างลำดับที่กล่าวมาแล้วข้างต้น และเป็นกระบวนการที่ผู้วิจัย สนใจที่จะปรับปรุง พัฒนา และศึกษาถึงเงื่อนไขที่จะทำให้ ลำดับดังกล่าวลู่เข้าสู่จุดตรึง ของการส่งแบบไม่ขยาย

การพัฒนาทฤษฎีบทเกี่ยวกับการลู่เข้าสู่จุดตรึงสุ่มของ กระบวนการสร้างลำดับด้วยวิธีทำซ้ำโดยปริยาย การประมาณค่าการลู่เข้าของการกระทำซ้ำของจุดตรึงร่วมสำหรับวงค์นับได้ และการประยุกต์หาคำตอบของปัญหาดุลภาพ และ ปัญหาอสมการแปรผัน เป็นเรื่องที่ควรจะศึกษาเพื่อที่จะสร้างทฤษฎีใหม่ทางคณิตศาสตร์บริสุทธิ์ และ คณิตศาสตร์ประยุกต์เพื่อใช้ในการแก้ปัญหาไม่เชิงเส้นต่อไป ในการคิดค้นทฤษฎีเพื่อหาองค์

ความรู้ใหม่ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการและการพัฒนาประเทศ ซึ่งทฤษฎีและองค์ความรู้ใหม่ๆที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆนั้นแล้ว ยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน (basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

หากเราสามารถแก้ปัญหาไม่เชิงเส้นภายใต้เงื่อนไขต่างๆ ด้วยกระบวนการที่คิดค้นเองได้ เราก็สามารถอธิบายปรากฏการณ์ต่างๆได้ใกล้เคียงความเป็นจริงมากขึ้น รวมถึงลดการพึ่งพาและการนำเข้าเทคโนโลยีจากต่างชาติ

ทฤษฎีบทจุดตรึง (Fixed point theorem) ถือว่าเป็นทฤษฎีที่สำคัญมากในการนำไปประยุกต์ทั้งในสาขาคณิตศาสตร์และในสาขาอื่น ๆ การศึกษาวิจัยในเรื่องของทฤษฎีบทจุดตรึงบนปริภูมิบานาค เป็นการหาเงื่อนไขที่เพียงพอที่ทำให้การส่ง T ที่ส่งจากเซตย่อย K ของปริภูมิบานาค E ไปยัง K ที่มีจุดตรึง (นั่นคือจะมีจุด a ใน K ซึ่งทำให้ $T(a)=a$)

หลังจากนั้นเป็นต้นมาได้มีนักคณิตศาสตร์จำนวนมากที่ทำการศึกษาวิจัยเพื่อหาเงื่อนไขที่เพียงพอที่เป็นสมบัติทางเรขาคณิตของปริภูมิบานาคเพื่อใช้พิสูจน์เกี่ยวกับทฤษฎีบทจุดตรึง
ต่อมาได้มีนักคณิตศาสตร์จำนวนหนึ่งที่ทำการศึกษาวิจัยหาสมบัติทางเรขาคณิตที่เป็นเงื่อนไขเพียงพอที่จะพิสูจน์ทฤษฎีบทจุดตรึงสำหรับนัยทั่วไปของการส่งแบบไม่ขยาย(นั่นคือ $\|T(x)-T(y)\| \le \|x-y\|$ ทุก $x, y \in K$)

ใน ปี ค.ศ. 1922 นั้น Banach ได้ค้นพบทฤษฎีบทที่มีชื่อเสียงซึ่งถูกเรียกว่า Banach Contraction Principle Theorem ดังนี้ กำหนดให้ (M, d) เป็นปริภูมิเมทริกซ์บริบูรณ์ และ $T: C \to C$ เป็นการส่งแบบหด (contraction mapping) นั่นคือถ้า $d(Tx,Ty) \le kd(x,y)$ สำหรับบางจำนวน $k \in (0,1)$) แล้ว จะมีจุด $x \in M$ ซึ่งเป็นจุดตรึงของ ฟังก์ชัน T (นั่นคือ $Tx = x$)

สำหรับกระบวนการสร้างลำดับ เพื่อให้ลู่เข้าสู่จุดตรึงนั้น มีความสำคัญไม่น้อย กระบวนการสร้างลำดับที่สำคัญ และมีชื่อเสียงซึ่งนักวิจัยทางคณิตศาสตร์ มักจะหยิบยกมาอ้างอิงในการทำวิจัยในด้านนี้เสมอๆ คือในปี ค.ศ. 1953 W.R. Mann ได้สร้างลำดับ $\{x_n\}$ ในปริภูมิฮิลเบิร์ต H ดังต่อไปนี้ กล่าวคือ สำหรับ $\varnothing \ne C \subset H$ ซึ่งเป็นเซตนูนปิด นิยามลำดับ

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \end{cases}$$

เมื่อ $T: C \to C$ เป็นการส่งแบบไม่ขยาย และ $\{\alpha_n\} \subset [0,1]$ เขาได้พิสูจน์ว่า ถ้า $\{\alpha_n\}$ สอดคล้องกับคุณสมบัติที่เหมาะสมแล้ว $\{x_n\}$ จะลู่เข้าแบบอ่อนไปยังจุดตรึงของ T

ต่อมาในปี ค.ศ. 1967 B. Halpern ได้เสนอลำดับที่คล้ายคลึงกับลำดับของ Mannกล่าวคือ

$$\begin{cases} u \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n \end{cases}$$

เมื่อ $T : C \to C$ เป็นการส่งแบบไม่ขยาย และ $\{\alpha_n\} \subset [0,1]$ เขาได้พิสูจน์ว่า ถ้า $\{\alpha_n\}$ สอดคล้องกับคุณสมบัติที่เหมาะสมแล้ว $\{x_n\}$ จะลู่เข้าแบบเข้มไปยังจุดตรึงของ T

และในปี ค.ศ. 1974 S. Ishikawa ได้เสนอลำดับแบบใหม่ ที่เป็นรูปทั่วไปมากกว่าลำดับของ Mann กล่าวคือ

$$\begin{cases} x_0 \in C \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n \end{cases}$$

เมื่อ $T : C \to C$ เป็นการส่งแบบไม่ขยาย และ $\{\alpha_n\} \subset [0,1]$ เขาได้พิสูจน์ว่า ถ้า $\{\alpha_n\}$ สอดคล้องกับคุณสมบัติที่เหมาะสมแล้ว $\{x_n\}$ จะลู่เข้าแบบเข้มไปยังจุดตรึงของ T

และต่อมาในปี ค.ศ. 2001 Xu และ Ori ได้สร้างลำดับ $\{x_n\}$ ใน Hilbert Spaces H ดังต่อไปนี้ สำหรับ $x_0 \in C \subset H$ จะได้

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 \\ x_3 &= \alpha_3 x_2 + (1-\alpha_3)T_3 x_3 \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N \\ &\vdots \end{aligned}$$

หรือเขียนในรูปทั่วได้เป็น $x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n$ เมื่อ $T_n = T_{n \bmod N}$ และ $T_1, T_2, \ldots, T_N$ เป็นการส่งแบบไม่ขยาย จาก C ไป C ซึ่ง $F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$ และได้พิสูจน์ว่าถ้า $\lim_{n\to\infty} \alpha_n = 0$ แล้วลำดับ $\{x_n\}$ จะลู่เข้าอย่างอ่อน (Weakly Converges) ไปหาจุดตรึงร่วมจุดหนึ่งของ $T_1, T_2, \ldots, T_N$

อย่างไรก็ตาม สำหรับลำดับของ Mann ยังคงเป็นลำดับที่ลู่เข้าแบบอ่อนไปยังจุดตรึง แม้ว่าจะเป็นลำดับในปริภูมิเบิร์ต ความพยายามที่จะทำให้ลำดับของ Mann ลู่เข้าแบบเข้มไปยังจุดตรึง ถูกสร้างขึ้นโดยนักคณิตศาสตร์หลายท่านด้วยกัน และในปี ค.ศ. 2005 T. W. Kim และ H. K. Xu ได้ศึกษาลำดับของ Mann และปรับปรุง ลำดับนี้ บนปริภูมิบานาคเอกรูปราบเรียบ (uniformly smooth Banach) X โดยการใช้จุดที่ได้ตรึงไว้ แล้วเพิ่มกระบวนการของ ลำดับ เพื่อให้ลำดับที่เกิดขึ้นมาใหม่ลู่เข้าแบบเข้มไปยังจุดตรึงของการส่งแบบไม่ขยาย $T : C \to C$ กล่าวคือ ให้ $\varnothing \neq C \subset X$ เป็นเซตนูนปิด และ $F(T) \neq \varnothing$ เลือก $u \in C$ แบบไม่เจาะจง และให้ $\{\alpha_n\}$ และ $\{\beta_n\}$ เป็นลำดับในช่วง (0,1) ซึ่งสอดคล้องกับเงื่อนไขที่เหมาะสม นิยามลำดับ $\{x_n\}$ ใน C โดย

$$\begin{cases} x_0 \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ x_{n+1} = \beta_n u + (1-\beta_n) y_n \end{cases}$$

แล้ว $\{x_n\}$ จะลู่เข้าแบบเข้มไปยังจุดตรึงของ T

ซึ่งการวิจัยเกี่ยวกับการกระทำซ้ำของจุดตรึงและการประมาณค่าจุดตรึงที่สำคัญนั้นสามารถนำมาแก้สมการหาคำตอบของปัญหาดุลภาพ เช่นใน ปี 1997 Combettes และ Hirstoaga ได้เริ่มต้นศึกษาและใช้วิธีการทำซ้ำในการหาการประมาณค่าที่ดีที่สุดเพื่อแก้ปัญหาดุลภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้ม (strong convergence theorems) และมีนักคณิตศาสตร์อีกมากมาย นำทฤษฎีบทการทำซ้ำดังกล่าวมาประยุกต์ใช้ในการแก้อสมการแปรผัน

ดังความสำคัญที่ได้กล่าวมาแล้วข้างต้น ผู้วิจัยจะกล่าวถึงที่มาของการทำวิจัยนี้ โดยเริ่มต้นจากในปี ค.ศ. 1994 Stampacchaia [46] ได้เป็นผู้คิดค้นใช้วิธีการประมาณค่าแบบซ้ำเพื่อแก้ไขปัญหาอสมการการแปรผันภายใต้ตัวดำเนินการทางเดียวอย่างเข้ม และต่อเนื่องแบบลิฟซิทพ์ ต่อมา Korpelevich [27] เห็นว่าวิธีการประมาณค่าแบบซ้ำดังกล่าวมีข้อจำกัดมากมายเพื่อให้ได้มาซึ่งคำตอบของอสมการการแปรผัน จึงได้คิดค้นวิธีการประมาณค่าแบบซ้ำขึ้นมาใหม่ซึ่งเรียกว่า วิธีเอ็กซ์ตร้ากราเดียน (Extragradient method) และพบว่าการแก้ปัญหาอสมการการแปรผันดังกล่าวนั้น ตัวดำเนินการไม่จำเป็นต้องเป็น **ตัวดำเนินการทางเดียวอย่างเข้ม** และต่อเนื่องแบบลิฟซิทพ์ แต่ขอให้เป็น **ตัวดำเนินการทางเดียว** และต่อเนื่องแบบลิฟซิทพ์ ก็เพียงพอแล้ว นอกจากวิธีการประมาณค่าแบบซ้ำจะสามารถแก้ไขปัญหาอสมการการแปรผันแล้ว ยังสามารถประยุกต์ใช้ในการค้นหาจุดตรึงของการส่งแบบไม่ขยายดังรายละเอียดดังนี้

กำหนดให้ H เป็นปริภูมิฮิลเบิร์ตบนเซตของจำนวนจริง และ C เป็นเซตย่อยปิด (closed) นูน (convex) ของ H กำหนดการส่ง $A: C \to H$ จะเรียกการส่ง A ว่า การส่งทางเดียว (monotone mapping) ถ้า

$$\langle Au, Av \rangle \geq 0, \quad \forall u, v \in C.$$

ปัญหาอสมการการแปรผัน (variational inequality problem(VIP)) คือการหา $u_0 \in C$ ซึ่งทำให้อสมการต่อไปนี้เป็นจริง

$$\langle Au_0, u - u_0 \rangle \geq 0, \quad \forall u \in C \tag{1}$$

เซตคำตอบของปัญหาอสมการการแปรผันจะถูกเขียนแทนด้วย $VI(C, A)$ นั่นคือ $VI(C, A) = \{u \in C : \langle Au, v - u \rangle \geq 0\}$ ปัญหาอสมการการแปรผันนั่นได้ถูกศึกษากันอย่างกว้างขวางโดยดูได้จากเอกสารอ้างอิง [58] และ [59] สำหรับการส่ง $A: C \to H$ จะเรียกว่าเป็นการส่งแบบ α-ทางเดียวอย่างผกผัน(α -inverse-strongly monotone mapping) ถ้ามีจำนวนจริง $\alpha > 0$ ซึ่งทำให้

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2,$$

สำหรับทุกๆ $u, v \in C$ และเรียก $T: C \to C$ ว่าการส่งแบบไม่ขยาย (nonexpansive mapping) ถ้า

$$\|Tx - Ty\| \leq \|x - y\|$$

สำหรับทุกๆ $x, y \in C$ และกำหนดให้ $F(T)$ แทนเซตของจุดตรึงทั้งหมดของการส่ง T นั่นคือ $F(T) = \{x \in C : Tx = x\}$ จากนิยามดังกล่าวจะเห็นได้ว่า u เป็นคำตอบของอสมการการแปรผันในสมการที่ (1) ก็ต่อเมื่อ $u = P_C(u - \lambda Au)$ เมื่อ $\lambda > 0$ และ P_C เป็นภาพฉายระยะทาง(metric projection) นั่นแสดงให้เห็นว่าปัญหาอสมการการแปรผันเป็นความสัมพันธ์สมมูลกับปัญหาจุดตรึง (Fixed point problems) จึงเป็นผลให้ในปี ค.ศ. 2003 Takahashi และ Toyoda [52] ได้คิดค้นวิธีการทำซ้ำต่อไปนี้ เพื่อค้นหาคำตอบร่วมระหว่างจุดตรึงของการส่งแบบไม่ขยายและผลเฉลยของอสมการการแปรผันนั่นคือเพื่อค้นหาคำตอบของ $F(S) \cap V(C, A)$ โดยกำหนดให้ $x_0 \in C$ และนิยาม ลำดับ $\{x_n\}$ โดย

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n) \qquad (2)$$

สำหรับทุกๆ $n = 0, 1, 2, ...,$ เมื่อ $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0, 2\alpha)$ และ $S : C \to C$ เป็นการส่งแบบไม่ขยายบน C, $P_C : H \to C$ เป็นภาพฉายระยะทางและ $A : C \to H$ เป็น α-ทางเดียวอย่างผกผัน จากนั้น Takahashi และ Toyoda [52] ก็ได้พิสูจน์ว่าลำดับ $\{x_n\}$ ซึ่งนิยามโดยสมการ (2) ลู่เข้าอย่างอ่อนสู่สมาชิกร่วมของ $F(T) \cap VI(C, A)$ ในปริภูมิฮิลเบิร์ต

ต่อมา Iiduka and Takahashi [15] ต้องการทฤษฎีบทการลู่เข้าอย่างเข้มสู่สมาชิกร่วมของ $F(T) \cap VI(C, A)$ จึงได้กำหนดวิธีการทำซ้ำแบบใหม่ดังนี้ กำหนดให้ $x_0 = u \in C$ และนิยาม ลำดับ $\{x_n\}$ โดย

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n) \qquad (3)$$

สำหรับทุกๆ $n = 0, 1, 2, ...,$ เมื่อ $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0, 2\alpha)$ และ $S : C \to C$ เป็นการส่งแบบไม่ขยายบน C, $P_C : H \to C$ เป็นภาพฉายระยะทางและ $A : C \to H$ เป็น α-ทางเดียวอย่างผกผัน นอกจากนั้นแล้วโดยใช้วิธีการเอ็กซ์ตร้ากราเดียน (Extragradeint) ซึ่งทั้ง Nadezhkina และ Takahashi [34] และ Zeng และ Yao [66] ก็ได้ใช้การประมาณค่าแบบซ้ำชนิดใหม่เพื่อค้นหาผลเฉลยร่วมของ $F(T) \cap VI(C, A)$ ต่อมา Yao และ Yao [62] ได้แนะนำวิธีการประมาณค่าแบบซ้ำเพื่อหาสมาชิกของ $F(T) \cap VI(C, A)$ ดังนี้ กำหนดให้ $A : C \to H$ เป็น α-ทางเดียวอย่างผกผัน และ $S : C \to C$ เป็นการส่งแบบไม่ขยายซึ่ง $F(T) \cap VI(A, C) \neq \varnothing$

กำหนดให้ $x_0 = u \in C$ และนิยาม ลำดับ $\{x_n\}$ และ $\{y_n\}$ โดย

$$\begin{cases} y_n = P_C(x_n - \lambda_n x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n A y_n) \end{cases} \qquad (4)$$

เมื่อ $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด [0,1] และ $\{\lambda_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0, 2\alpha]$ Yao และ Yao [64] ได้พิสูจน์ว่าถ้าลำดับ $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ สอดคล้องเงื่อนไขบางอย่างแล้วลำดับ $\{x_n\}$ และ $\{y_n\}$ นิยามโดย (4) ลู่เข้าอย่างเข้มสู่จุดตรึงร่วมของเซตคำตอบของจุดตรึงของการส่งแบบไม่ขยายและเซตของผลเฉลยของอสมการการแปรผัน นั่นคือ $\{x_n\}$ และ $\{y_n\}$ ลู่เข้าอย่างเข้มสู่สมาชิกของ $F(T) \cap VI(A, C)$

อีกหนึ่งปัญหาทางคณิตศาสตร์ที่เป็นที่สนใจของนักคณิตศาสตร์หลายคนในปัจจุบันคือ **ปัญหาเชิงดุลยภาพ** (equilibrium problems (EP)) ซึ่งหมายถึง การหาค่าของ $x \in C$ ซึ่งสอดคล้องกับอสมการต่อไปนี้

$$F(x, y) \geq 0, \quad \forall y \in C \tag{5}$$

เมื่อกำหนดฟังก์ชันโดเมนเชิงคู่ $F : C \times C \to \mathbb{R}$ และเซตคำตอบของปัญหาเชิงดุลยภาพ (5) ข้างบนนี้เราจะเขียนแทนด้วย $EP(F)$ ถ้ากำหนดการส่ง $T : C \to H$ ให้ $F(x, y) = \langle Tx, y - x \rangle$ สำหรับทุกๆ $x, y \in C$ แล้วจะได้ว่า $z \in EP(F)$ ก็ต่อเมื่อ $\langle Tz, y - z \rangle \geq 0, \forall y \in C$ นั่นแสดงให้เห็นว่าคำตอบของปัญหาเชิงดุลภาพสามารถแก้ไขบางปัญหาอสมการการแปร นอกจากแล้วจะเห็นว่าปัญหาต่างๆ ในทางฟิสิกส์ หรือในทางเศรษฐศาสตร์ บางอย่างสามารถแปลงเป็นสมการหรืออสมการ ให้อยู่ในรูปอสมการ (5) ดังนั้นการหาผลเฉลยหรือการประมาณค่าของปัญหาเชิงดุลยภาพ (5) ถือเป็นการแก้ไขปัญหาในทางฟิสิกส์ หรือในทางเศรษฐศาสตร์ได้อีกทางหนึ่ง ซึ่งสามารถดูได้จากเอกสารอ้างอิง [4] , [12] และ [33] โดยในปี 1997 Combettes และ Hirstoaga [10] ได้เริ่มต้นศึกษาและใช้วิธีการประมาณค่าแบบซ้ำในการประมาณค่าที่ดีที่สุดเพื่อไปหาคำตอบ (solutions) ให้กับปัญหาเชิงดุลยภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้ม เพื่อค้นหาคำตอบร่วมของ $EP(F) \cap F(T)$ และเมื่อเร็วๆนี้ Takahashi และ Takahashi [53] จึงได้ศึกษาวิธีการประมาณค่าแบบหนืด (viscosity approximation method) ในปริภูมิฮิลเบิร์ต โดยกำหนดให้ $S : C \to C$ เป็นการส่งแบบไม่ขยาย และให้ $x_1 \in C$ และนิยามลำดับ $\{x_n\}$ และ $\{u_n\}$ โดย

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n \end{cases}$$

สำหรับทุกๆ $n \in \mathbb{N}$ เมื่อ $\{\alpha_n\} \subset [0,1]$ และ $\{r_n\} \subset (0, \infty)$ ภายใต้เงื่อนไขที่เหมาะสมบางอย่างสำหรับลำดับ $\{\alpha_n\}, \{\gamma_n\}$ Takahashi และ Takahashi [53] ได้พิสูจน์ว่า $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มสู่จุดตรึงร่วม $z \in F(T) \cap EP(F)$ เมื่อ $z = P_{F(T) \cap EP(F)} f(z)$ โดยใช้แนวคิดดังกล่าว ต่อมาในปี ค.ศ. 2007 Su, Shang และ Qin [47] ได้สร้างวิธีการประมาณค่าแบบซ้ำแบบใหม่เพื่อหาคำตอบร่วมระหว่าง เซตของจุดตรึงของการส่งแบบไม่ขยาย $F(S)$ เซตคำตอบของอสมการการแปรฝัน $VIP(A, C)$ (เมื่อ A เป็นการส่งแบบ $\alpha -$inverse-strongly monotone) และ เซตคำตอบของปัญหาเชิงดุลยภาพ $EP(F)$ ในปริภูมิฮิลเบิร์ต โดยกำหนดให้ $x_1 \in C$ และนิยามลำดับโดย

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_c (u_n - \lambda_n A u_n), \ n \geq 1 \end{cases}$$

และ Su, Shang และ Qin [13] ได้พิสูจน์ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มสู่จุดตรึงร่วม $z \in F(T) \cap EP(F) \cap VI(A, C)$ เมื่อ $z = P_{F(T) \cap EP(F) \cap VI(A,C)} f(z)$ และเมื่อเร็วๆ นี้ Plubtieng และ Punpaeng [40] ได้นำเสนอวิธีการแก้ไขปัญหาเชิงดุลยภาพโดยใช้วิธีการเอ็กซ์ตร้ากราเดียนแบบใหม่ซึ่งแตกต่างจาก Su, Shang และ Qin [47] โดยผสมแนวคิดของ Takahashi และ Takahashi [53] และ Yao และ Yao [65] ดังนี้ กำหนด $x_1 \in C$ และนิยามลำดับ $\{x_n\}, \{y_n\}$ และ $\{u_n\}$ ดังนี้

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n) \end{cases} \qquad (6)$$

เมื่อ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0,1]$ และ $\{\lambda_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0,2\alpha]$ ซึ่ง Plubtieng และ Punpaeng [41] ได้พิสูจน์ว่าถ้าลำดับ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ สอดคล้องเงื่อนไขบางอย่าง แล้วลำดับ $\{x_n\}$ และ $\{y_n\}$ นิยามโดย (6) ลู่เข้าอย่างเข้มสู่จุดตรึงร่วมของเซตคำตอบของจุดตรึงของการส่งแบบไม่ขยาย เซตของผลเฉลยของอสมการการแปรผัน และ เซตคำตอบของปัญหาเชิงดุลยภาพ ในอีกด้านหนึ่งของปัญหาจุดตรึงที่เป็นที่สนใจของนักคณิตศาสตร์ คือ การใช้วิธีการประมาณค่าเพื่อประยุกต์ใช้ในการแก้ **ปัญหาค่าน้อยสุด (minimization problem)** ซึ่งเป็นที่ทราบกันดีว่าปัญหาค่าน้อยสุดนี้มีบทบาทสำคัญในการพัฒนาทั้งทางด้านวิทยาศาสตร์บริสุทธิ์ และ วิทยาศาสตร์ประยุกต์ ปัญหาค่าน้อยสุดหมายถึง การหาค่าของ $x \in C$ ซึ่ง ทำให้ $\min\limits_{x\in F(S)} \frac{1}{2}\langle Ax,x\rangle - \langle x,b\rangle$ มีจริง เมื่อ $b \in H$ S เป็นการส่งแบบไม่ขยายบนปริภูมิฮิลเบิร์ต H และ A เป็นตัวดำเนินการเชิงเส้นที่มีขอบเขตและมีค่าเป็นบวกอย่างเข้มบน H (strongly positive bounded linear operator on H) ในปีค.ศ. 2003 Xu [58] ได้นำเสนอวิธีการประมาณค่าคำตอบของปัญหาค่าน้อยสุด โดยกำหนดค่าเริ่มต้น $x_0 = u \in H$ และนิยามลำดับ $\{x_n\}$ โดยความสัมพันธ์ดังนี้

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n u, \ n \geq 0 \qquad (7)$$

เมื่อ $\{\alpha_n\}$ เป็นลำดับในช่วง $(0,1)$ A และ S ดังนิยามข้างบน จากนั้น Xu [18] ก็ได้พิสูจน์ว่าลำดับ $\{x_n\}$ ที่นิยามโดย (7) ลู่เข้าอย่างเข้มไปยังผลเฉลยเดี่ยว (unique solution) ของปัญหาค่าน้อยสุดที่ได้กล่าวไปแล้ว จากนั้น Marino และ Xu [30] ได้ใช้ความคิดจากวิธีการประมาณค่าแบบหนืด(viscosity approximation methods) ของ Moudafi [32] และ Xu [58] โดยนำเสนอวิธีการประมาณค่าแบบใหม่ดังนี้ นิยามลำดับ $\{x_n\}$ โดยความสัมพันธ์

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \ n \geq 0 \qquad (8)$$

เมื่อ $\{\alpha_n\}$ เป็นลำดับในช่วง $(0,1)$ A และ S ดังนิยามข้างบน จากนั้น Marino และ Xu [30] ก็ได้พิสูจน์ว่าลำดับ $\{x_n\}$ ที่นิยามโดย (8) ลู่เข้าอย่างเข้มไปยังผลเฉลยเดี่ยวของอสมการแปรผัน

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C \qquad (9)$$

ซึ่งเป็นเงื่อนไขที่สำคัญในการแก้ปัญหาค่าน้อยสุด $\min\limits_{x\in C} \frac{1}{2}\langle Ax,x\rangle - h(x)$ เมื่อ h เป็นฟังก์ชันโพเทนเชียล (potential function) สำหรับ γf นั่นคือ $h'(x) = \gamma f(x), \ \forall x \in H$ และเมื่อเร็วๆ นี้ โดยอาศัยแนวความคิดจาก Takahashi และ Takahashi [53] และ Marino และ Xu [30] เป็นผลให้ Plubtieng และ Punpaeng [41] ได้นำเสนอวิธีการประมาณค่าแบบใหม่ขึ้นมาใหม่ในปริภูมิฮิลเบิร์ต H ดังนี้ กำหนดให้ C เป็นเซตย่อยปิดนูนของ H และให้ค่าเริ่มต้น $x_1 \in C$ นิยามลำดับ $\{x_n\}$ และ $\{u_n\}$ โดย

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n A)Su_n, \ n \geq 1 \end{cases} \quad (10)$$

และ Plubtieng และ Punpaeng [41] ได้พิสูจน์ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มสู่จุดตรึงร่วม $z \in F(T) \cap EP(F)$ เมื่อ $z = P_{F(T)\cap EP(F)} f(z)$

จากแนวแนวที่กล่าวมาทั้งหมดจะเห็นได้เพียงแค่การประมาณค่าจุดตรึงของหนึ่งการส่งแบบไม่ขยาย แต่ปัญหาต่างๆ ในทางคณิตศาสตร์นั้นมักจะพบกับหลายๆการส่งแบบไม่ยาย จำเป็นอย่างยิ่งที่จะพัฒนานาวิธีการประมาณค่าแบบซ้ำเพื่อใช้ในการค้นหาคำตอบร่วมของลำดับของการส่งแบบไม่ขยาย หรือลำดับของการส่งแบบไม่เชิงเส้นชนิดอื่นๆ จากความสำคัญตรงนี้เป็นผลให้ ในปี 2008 Yao, Liou และ Yao [65] ได้นำเสนอวิธีการประมาณค่าแบบซ้ำสำหรับการประมาณค่าของคำตอบร่วมของลำดับของการส่งแบบไม่ขยาย โดยกำหนดให้ $\{W_n\}$ เป็นลำดับของการส่งแบบไม่ขยายที่ก่อกำเนิดโดยลำดับของการส่งแบบไม่ขยาย $\{S_n\}$ และนิยามลำดับ $\{x_n\}$ และ $\{u_n\}$ โดย

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n u_n, \ n \geq 1 \end{cases} \quad (11)$$

เมื่อ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด (0,1) ซึ่ง Yao, Liou และ Yao [64] ได้พิสูจน์ว่าถ้าลำดับ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ และ $\{r_n\}$ สอดคล้องเงื่อนไขบางอย่าง แล้วลำดับ $\{x_n\}$ และ $\{u_n\}$ นิยามโดย (11) ลู่เข้าอย่างเข้มสู่สมาชิกร่วมระหว่าง เซตคำตอบของจุดตรึงร่วมของลำดับของการส่งแบบไม่ขยาย $\{S_n\}$ และ เซตคำตอบของปัญหาเชิงดุลยภาพ นอกจากนั้นแล้ว Yao, Liou และ Chen [61] ยังประยุกต์ใช้วิธีการประมาณค่าแบบซ้ำสำหรับการประมาณค่าของคำตอบร่วมของลำดับของการส่งแบบไม่ขยาย $\{S_n\}$ ที่สามารถแก้ไขปัญหาค่าน้อยสุดได้ ดังนี้

$$x_1 \in H, \ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n, \ n \geq 1 \quad (12)$$

และ Yao, Liou และ Chen [61] ได้พิสูจน์ว่าลำดับ $\{x_n\}$ ที่นิยามโดย (12) ลู่เข้าอย่างเข้มไปยังผลเฉลยเดี่ยวของบางอสมการแปรผัน

จากผลงานวิจัยต่างๆ ที่กล่าวมาข้างต้นจะเห็นว่าพัฒนาการในเรื่องวิธีการประมาณค่านั้นได้ถูกคิดค้นอยู่เสมอๆ ในปริภูมิที่แตกต่างกันไป จึงเป็นเหตุผลที่ทำให้ผู้วิจัยต้องการที่จะค้นหาหรือนำเสนอวิธีการประมาณค่าแบบใหม่ๆ เพื่อให้สามารถประยุกต์ใช้กับปัญหาทางคณิตศาสตร์ในรูปแบบต่างๆ หรือบางปัญหาในทาง ฟิสิกส์ และ ทางเศรษฐศาสตร์ ได้มากขึ้น พร้อมทั้งยังเป็นการก่อให้เกิดองค์ความรู้ หรือทฤษฎีใหม่ๆ ในทางการวิเคราะห์เชิงฟังก์ชันหรือสาขาอื่นๆ ที่เกี่ยวข้อง

CHAPTER 2

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

2.1 Basic results

Throughout this thesis, we let $\mathbb{R}$ stand for the set of all real numbers and $\mathbb{N}$ the set of all natural numbers.

Definition 2.1. *Let X be a nonempty set. A mapping $d : X \times X \longrightarrow \mathbb{R}$, satisfying the following condition for all x, y and z in X:*

(M1) $d(x, y) = 0 \Longleftrightarrow x = y$;

(M2) $d(x, y) = d(y, x)$;

(M2) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d assigns to each pair (x, y) of element of X a nonnegative real number $d(x, y)$, which does not on the order of the elements; $d(x, y)$ is called the *distance* between x and y. The set X together with a metric, denoted by (X, d), is called a *metric space.* The conditions (M1)-(M3) are usually called the *metric axioms.*

Definition 2.2. *Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be a norm on X if it satisfies the following conditions:*

(N1) $\|x\| \geq 0, \forall x \in X$;

(N2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N3) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X$;

(N4) $\|\alpha x\| = |\alpha|\|x\|, \quad \forall x \in X$ *and* $\forall \alpha \in \mathbb{K}$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\|, \quad (x, y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space.*

Definition 2.3. *Let $(X, \|\cdot\|)$ be a normed space.*

(1) A sequence $\{x_n\} \subset X$ is said to converge strongly in X if there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is a Cauchy sequence *in X if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.*

Definition 2.4. *A normed space X is called complete if every Cauchy sequence in X converges to an element in X.*

Definition 2.5. *A complete normed linear space over field $\mathbb{K}$ is called a Banach space over $\mathbb{K}$.*

Definition 2.6. *Let F and X be linear spaces over the field $\mathbb{K}$.*

(1) A mapping $T : F \longrightarrow X$ is called a linear operator if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called a linear functional on F if T is a linear operator.

Definition 2.7. *A sequence $\{x_n\}$ in a normed spaces is said to converge weakly to some vector x if $\lim_{n\longrightarrow\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converges weakly to x.*

Definition 2.8. *Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \longrightarrow F$ a linear operator. T is said to be bounded on X if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.*

Definition 2.9. *Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be a bounded sequence if there exists $M > 0$ such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.*

Definition 2.10. *Let F and X be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow X$ an operator and $c \in F$. We say that T is continuous at c if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If T is continuous at each $x \in F$, then T is said to be continuous on F.*

Definition 2.11. *Let X and Y be normed spaces. The mapping $T : X \longrightarrow Y$ is said to be completely continuous if $T(C)$ is a compact subset of Y for every bounded subset C of X.*

Definition 2.12. *A subset C of a normed linear space X is said to be convex subset in X if $\lambda x+(1-\lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.*

Definition 2.13. *The real-value function of two variables $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{R}$ is called inner product on a real vector space X if for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$ the following conditions are satisfied:*

(I1) $\langle\alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(I2) $\langle x, y\rangle = \langle y, x\rangle$;

(I3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. A real inner product space is a real vector space equipped with an inner product.

Definition 2.14. *A* **Hilbert spaces** *is an inner product space which is complete under the norm induced by its inner product.*

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.15. [51]**(The Schwarz inequality)**
If x and y are any two vector in an inner product space X, then

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Lemma 2.16. [52] *Let H be a real Hilbert space. Then the following inequalities hold:*

(i) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$,

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(iii)$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$,

(iv) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad \forall\lambda \in [0, 1]$.

Definition 2.17. *A sequence of points x_n in a Hilbert space H is said to converge weakly to a point x in H if $\lim_{n\longrightarrow\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.*

Definition 2.18. *The metric (or* nearest point*) projection from H onto C is the mapping $P_C : H \longrightarrow C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property*

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C)$$

Lemma 2.19. [52] *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then*

(i) $z = P_C x \Longleftrightarrow \langle z - x, y - x\rangle \geq 0, \quad \forall y \in C,$

(ii) $\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H,$

(iii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H,$

(iv) $\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C,$

(v) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C.$

2.2 The Classical of Fixed Point Theory

Definition 2.20. *An element $x \in C$ is said to be a fixed point of a mapping $S : C \longrightarrow C$ proved $Sx = x$. The set of all fixed points of S is denoted by $F(S) = \{x \in C : Sx = x\}$.*

Definition 2.21. *Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $S : C \longrightarrow C$ is called* nonexpansive *on C if*

$$\|Sx - Sy\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.22. [52] *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let S be a nonexpansive mapping of C into itself. Then, $F(S) \neq \emptyset$.*

Definition 2.23. *Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $f : C \longrightarrow C$ is called* a contraction *on C if there exists a constant $\alpha \in [0, 1)$ such that*

$$\|f(x) - f(y)\| \leqslant \alpha\|x - y\|, \quad \forall x, y \in C.$$

Theorem 2.24. [52] **(The Banach Contraction Principle)**
Let X be a complete metric space and f a contraction of X into itself. Then f has a unique fixed point, in the sense that $f(u) = u$ for some $u \in X$.

Let (X, d) be a metric space, $C \subset X$ a closed subset of X and $S : C \longrightarrow C$ a selfmap possessing at least one fixed point $p \in F(S)$. For a given $x_0 \in X$, we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iterative method

$$x_{n+1} = Sx_n = S^n x_0, \quad \forall n \geq 0. \tag{2.1}$$

We are interested in obtaining(additional) conditions on S, C and X, as general as possible, and which should guarantee the (strong) convergence of the $\{x_n\}_{n=0}^{\infty}$ to a fixed point of S in X.

As we already mentioned, the sequence defined by (2.1) is known as the sequence of successive approximations or, simply, *Picard iteration*.

2.3 Some Nonlinear Mappings

Definition 2.25. *A mapping $S : C \longrightarrow C$ is called* strictly pseudo-contractive *if there exists a constant $0 \le k < 1$ such that*

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Remark 2.26. *If $k = 0$, then S is nonexpansive.*

In this case, we say that $S : C \longrightarrow C$ is a k-strictly pseudo-contraction.

Putting $B = I - S$. Then, we have

$$\|(I - B)x - (I - B)y\|^2 \le \|x - y\|^2 + k\|Bx - By\|^2, \quad \forall x, y \in C.$$

Observe that

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By\rangle, \quad \forall x, y \in C.$$

Hence, we obtain

$$\langle x - y, Bx - By\rangle \ge \frac{1 - k}{2}\|Bx - By\|^2, \quad \forall x, y \in C.$$

Then, B is $\frac{1-k}{2}$-inverse-strongly monotone mapping.

Lemma 2.27. [32] *Assume that C is a closed convex subset of Hilbert space H, and let $S : C \to C$ be a self-mapping of C*

(i) If S is a k-strict pseudo-contraction, then S satisfies the Lipscchitz condition

$$\|Sx - Sy\| \le \frac{1 + k}{1 - k}\|x - y\| \qquad \forall x, y \in C.$$

(ii) If S is a k-strict pseudo-contraction, then the mapping $I - S$ is demiclosed(at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then$(I - S)\tilde{x} = 0$.

(iii) If S is a k-strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.

Lemma 2.28. [69] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \longrightarrow C$ be a k-strict pseudo-contraction mapping with a fixed point. Then $F(S)$ is closed and convex. Define $S_k : C \longrightarrow C$ by $S_k = kx + (1 - k)Sx$ for each $x \in C$. Then S_k is nonexpansive such that $F(S_k) = F(S)$.*

Let C be a subset of a Banach space E and let $\{T_n\}$ be a family of mappings from C into E. For a subset B of C, we say that

(a) $(\{T_n\}, B)$ satisfies condition AKTT if

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

(b) $(\{T_n\}, B)$ satisfies condition *AKTT if

$$\sum_{n=1}^{\infty} \sup\{\|JT_{n+1}z - JT_n z\| : z \in B\} < \infty.$$

For more information, see Aoyama et al. [3].

Lemma 2.29. (*Aoyama et al.* [3, Lemma 3.2]). *Let C be a nonempty closed convex subset of E. Suppose that $\sum_{n=1}^{\infty} \sup\{(\|T_{n+1}z - T_n z\|) : z \in C\} < \infty$. Then, for each $y \in C, \{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\to\infty} T_n y \quad \textit{for all} \quad y \in C.$$

Then $\lim_{n\to\infty} sup\{\|Tz - T_n z\| : z \in C\} = 0$.

Inspired by Lemma 2.29, Nilsrakoo and Saejung [36] proved the following results.

Lemma 2.30. (*Nilsrakoo and Saejung* [36]). *Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of a Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfies condition *AKTT, then there exists a mapping $\hat{T} : B \to E$ such that*

$$\hat{T}x = \lim_{n\longrightarrow\infty} T_n x, \quad \textit{for all} \quad x \in B$$

and $\limsup_{n\longrightarrow\infty}\{\|J\hat{T}z - JT_n z\| : z \in B\} = 0$.

Lemma 2.31. (*Nilsrakoo and Saejung* [36]). *Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Suppose that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Then there exists a mapping $T : B \to E$ such that*

$$Tx = \lim_{n\longrightarrow\infty} T_n x, \quad \textit{for all} \quad x \in C.$$

Lemma 2.32. (*Kamimura and Takahashi* [25]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.33. (*Alber* [2]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \qquad \forall y \in C. \tag{2.2}$$

Lemma 2.34. (*Alber* [2]). *Let E be a reflexive, strictly convex, and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C. \tag{2.3}$$

Lemma 2.35. (*Matsushita and Takahashi* [30]). *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.36. (*Cho et al.* [9]). *Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.37. (*Kamimura and Takahashi* [25]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

We make use of the following mapping V studied in Alber [2]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x, y)$.

Lemma 2.38. (*Kohsaka and Takahashi* [27, Lemma 3.2]). *Let E be a reflexive, strictly convex and smooth Banach space and let V be as in (2.4). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

2.4 Variational inequality problem

Definition 2.39. *Let $B : C \longrightarrow H$ be a nonlinear mapping. The* variational inequality problem *is to find $x \in C$ such that*

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

We denote by $VI(C, B)$ the set of solutions of the variational inequality problem, that is,

$$VI(C, B) = \{x \in C : \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C\}. \tag{2.6}$$

Lemma 2.40. [52] *Let H be Hilbert space, let C be a nonempty closed convex subset of H and let B be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu),$$

where P_C is the metric projection of H onto C.

Remark 2.41. *It is clear from Lemma 2.40 that the variational inequality and fixed point problem are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.*

Definition 2.42. *Let $B : C \longrightarrow H$ be a nonlinear mapping. Then, B is called*

(1) monotone *if $\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C$,*

(2) v-strongly monotone *if there exists a positive real number v such that*

$$\langle Bx - By, x - y \rangle \geq v\|x - y\|^2, \quad \forall x, y \in C,$$

for constant $v > 0$. This implies that $\|Bx - By\| \geq v\|x - y\|$, that is, A is v-expansive and when $v = 1$, it is expansive.

(3) L-Lipschitz continuous *if there exists a positive real number L such that*

$$\|Bx - By\| \leq L\|x - y\|, \quad \forall x, y \in C,$$

(4) ξ-inverse-strongly monotone *if there exists a positive real number* ξ *such that*

$$\langle Bx - By, x - y\rangle \geq u\|Bx - By\|^2, \quad \forall x, y \in C.$$

Clearly, every ξ*-inverse-strongly monotone map* B *is* $\frac{1}{\xi}$*-Lipschitz continuous,*

(5) relaxed (u, v)-cocoercive, *if there exists a positive real number* u, v *such that*

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2, \quad \forall x, y \in C,$$

for $u = 0$*,* A *is* v*-strongly monotone. This class of maps is more general that the class of strongly monotone maps. It is easy to see that we have the following implication:* v*-strongly monotonicity implying relaxed* (u, v)*-cocoercivity.*

Lemma 2.43. [52] *Let* H *be a Hilbert space and let* C *be a nonempty bounded closed convex subset of. Let* $\xi > 0$ *and let* $B : C \longrightarrow B$ *be* ξ*-inverse strongly monotone. Then,* $VI(C, B) \neq \emptyset$*.*

Definition 2.44. *A set-valued mapping* $T : H \longrightarrow 2^H$ *is called* monotone *if for all* $x, y \in H$*,* $f \in Tx$ *and* $g \in Ty$ *imply* $\langle x - y, f - g\rangle \geq 0$*.*

Definition 2.45. *A monotone mapping* $T : H \longrightarrow 2^H$ *is* maximal *if the graph of* $G(T)$ *of* T *is not properly contained in the graph of any other monotone mapping.*

Remark 2.46. *It is known that a monotone mapping* T *is maximal if and only if for* $(x, f) \in H \times H$*,* $\langle x - y, f - g\rangle \geq 0$ *for every* $(y, g) \in G(T)$ *implies* $f \in Tx$*.*

Lemma 2.47. [45] *Let* B *be a monotone mapping of* C *into* H *and let* $N_C w_1$ *be the* normal cone *to* C *at* $w_1 \in C$*, i.e.,*

$$N_C w_1 = \{w \in H : \langle w_1 - w_2, w\rangle \geq 0, \quad \forall w_2 \in C\}$$

and define a mapping T *on* C *by*

$$Tw_1 = \begin{cases} Bw_1 + N_C w_1, & w_1 \in C; \\ \emptyset, & w_1 \notin C. \end{cases}$$

Then, T *is the maximal monotone and* $0 \in Tw_1$ *if and only if* $w_1 \in VI(C, B)$*.*

2.5 Equilibrium Problems

Definition 2.48. *Let* F *be a bifunction of* $C \times C$ *into* $\mathbb{R}$*, where* $\mathbb{R}$ *is the set of real numbers. The* equilibrium problem *for* $F : C \times C \longrightarrow \mathbb{R}$ *is to find* $x \in C$ *such that*

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.7}$$

The set of solutions of (2.7) is denoted by $EP(F)$*, that is,*

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}.$$

Given a mapping $B : C \longrightarrow H$*, let* $F(x, y) = \langle Bx, y - x\rangle$ *for all* $x, y \in C$*. Then,* $z \in EP(F)$ *if and only if* $\langle Bz, y - z\rangle \geq 0$ *for all* $y \in C$*, i.e.,* z *is a solution of the variational inequality.*

Let $\Im = \{F_k\}_{k\in\Lambda}$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\Im = \{F_k\}_{k\in\Lambda}$ is to determine common equilibrium points for $\Im = \{F_k\}_{k\in\Lambda}$ such that

$$F_k(x, y) \geq 0, \quad \forall k \in \Lambda \quad \forall y \in C, \tag{2.8}$$

where Λ is an arbitrary index set. The set of solutions of (2.8) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \left\{ \ x \in C : F_k(x,y) \geq 0, \quad \forall k \in \Lambda \quad \forall y \in C \ \right\}. \tag{2.9}$$

If Λ is a singleton, then the problem (2.8) is reduced to the problem (2.7).

Definition 2.49. *For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [4]):*

(A1) $F(x,x) = 0$ *for all* $x \in C$;

(A2) F *is monotone, i.e.,* $F(x,y) + F(y,x) \leq 0$ *for any* $x, y \in C$;

(A3) F *is upper-hemicontinuous, i.e., for each* $x, y, z \in C$,

$$\limsup_{t \longrightarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) $F(x,\cdot)$ *is convex and lower semicontinuous for each* $x \in C$.

Definition 2.50. *Let* $B : C \longrightarrow H$ *be a nonlinear mapping and* F *be a bifunction of* $C \times C$ *into* $\mathbb{R}$. *The* generalized equilibrium problem *is to find* $x \in C$ *such that*

$$F(x,y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

The set of solution of (2.10) is denoted by $GEP(F,B)$, *that is,*

$$GEP(F,B) = \big\{ x \in C : F(x,y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C \big\}.$$

In the case of $B \equiv 0$ (:the zero mapping), then the problem (2.10) is reduced to the problem (2.7). In the case of $F \equiv 0$, the problem (2.10) is reduced to the classical variational inequality problem (2.5).

Definition 2.51. *The domain of the function* $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is the set*

$$dom\varphi = \{x \in C : \varphi(x) < +\infty\}.$$

Let $\varphi : C \longrightarrow \mathbb{R}$ *be a real-valued function and* $F : C \times C \longrightarrow \mathbb{R}$ *be an equilibrium bifunction, i.e.,* $F(x,x) = 0$ *for each* $x \in C$. *The* mixed equilibrium problem *which is to find* $x \in C$ *such that*

$$F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.11}$$

The set of solution of (2.11) is denoted by $MEP(F,\varphi)$, *that is,*

$$MEP(F,\varphi) = \big\{ x \in C : F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C \big\}.$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problems (2.7).

Definition 2.52. *For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function φ and the set C:*

(A1)-(A4) (in Definition 2.49);

(A5) for each $y \in C, x \mapsto F(x,y)$ *is weakly upper semicontinuous;*

(B1) for each $x \in H$ *and* $r > 0$, *there exist a bounded subset* $D_x \subseteq C$ *and* $y_x \in C$ *such that for any* $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle < 0; \tag{2.12}$$

(B2) C *is a bounded set.*

Definition 2.53. *Let* $\varphi : C \longrightarrow \mathbb{R}\cup\{+\infty\}$ *be a proper extended real-valued function and let* F *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *such that* $C \cap \ dom\varphi \neq \emptyset$, *where* $\mathbb{R}$ *is the set of real numbers and* $dom\varphi = \{x \in C : \varphi(x) < +\infty\}$.
The generalized mixed equilibrium problem *for finding* $x \in C$ *such that*

$$F(x,y) + \langle Bx, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.13}$$

The set of solutions of (2.13) *is denoted by* $GMEP(F,\varphi,B)$, *that is,*

$$\begin{aligned} GMEP(F,\varphi,B) \quad &= \quad \big\{x \in C : F(x,y) + \langle Bx, y-x\rangle \\ &\qquad + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\big\}. \end{aligned}$$

We see that x is a solution of a problem (2.13) implies that $x \in \mathrm{dom}\varphi = \{x \in C : \varphi(x) < +\infty\}$. If $B \equiv 0$, the problem (2.13) is reduced into the *mixed equilibrium problem.* If $\varphi \equiv 0$, the problem (2.13) is reduced into the *generalized equilibrium problem.* If $B \equiv 0$ and $\varphi \equiv 0$, the problem (2.13) is reduced into the *equilibrium problem.* If $F \equiv 0$ and $\varphi \equiv 0$, the problem (2.13) is reduced into the *variational inequality problem.*

The generalized mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems and the equilibrium problem as special cases. Numerous problems in physics, optimization and economics reduce to find a solution of (2.13). In 1997, Combettes and Hirstoaga [10] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Many authors have proposed some useful methods for solving the $GMEP(F,\varphi,B)$, $GEP(F,B)$, $MEP(F,\varphi)$ and $EP(F)$; see, for instance [7], [10], [12], [16], [18], [19], [20], [21], [22], [23], [24], [29], [37], [44], [49], [54], [55], [58], [65].

Definition 2.54. *Let* A *be a* strongly positive linear bounded operator *on* H *if there is a constant* $\bar{\gamma} > 0$ *with property*

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Lemma 2.55. [32]. *Let* C *be a nonempty closed convex subset of* H *and let* f *be a contraction of* H *into itself with* $\alpha \in (0,1)$, *and* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$. *Then , for* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\Big\langle x-y, (A-\gamma f)x - (A-\gamma f)y\Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x-y\|^2, \quad x,y \in H.$$

That is, $A - \gamma f$ *is strongly monotone with coefficient* $\bar{\gamma} - \alpha\gamma$.

Lemma 2.56. [32]. *Assume* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$. *Then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.57. [1]. *Let* C *be a closed convex subset of* H. *Let* $\{x_n\}$ *be a bounded sequence in* H. *Assume that*

(i) The weak ω*-limit set* $\omega_w(x_n) \subset C$,

(ii) For each $z \in C$, $\lim_{n\longrightarrow\infty}\|x_n - z\|$ *exists. Then,* $\{x_n\}$ *is weakly convergent to a point in* C.

Lemma 2.58. [38]. *Each Hilbert space* H *satisfies* ***Opial's condition***, *i.e., for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality* $\liminf_{n\longrightarrow\infty}\|x_n - x\| < \liminf_{n\longrightarrow\infty}\|x_n - y\|$, *holds for each* $y \in H$ *with* $y \neq x$.

Lemma 2.59. [52]. *Each Hilbert space* H, *satisfies the* ***Kadec-Klee property***, *that is, for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$ *and* $\|x_n\| \longrightarrow \|x\|$ *together imply* $\|x_n - x\| \longrightarrow 0$.

Lemma 2.60. [50] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.

Lemma 2.61. [13]**(Demiclosedness Principle)**
Let H *be a Hilbert space,* C *a closed convex subset of* H*, and* $S : C \longrightarrow C$ *a nonexpansive mapping with* $F(S) \neq \emptyset$. *If* $\{x_n\}$ *is a sequence in* C *weakly converging to* $x \in C$ *and if* $\{(I-S)x_n\}$ *converges strongly to* y*, then* $(I-S)x = y$.

Lemma 2.62. [60] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ \ n \geq 1,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$. *Then,* $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.63. [39] *Let* $(H, \langle ., . \rangle)$ *be an inner product space. Then, for all* $x, y, z \in H$ *and* $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha + \beta + \gamma = 1$*, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

CHAPTER 3

MAIN RESULTS

3.1 Equilibrium Problems in Hilbert spaces

3.1.1 Weak Convergence Theorems

In this section, we prove a weak convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of a variational inequality problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Before proving our theorem, we need the following lemmas.

Lemma 3.1. [4] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{3.1}$$

Lemma 3.2. [10] *Let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Lemma 3.3. [53] *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, ...,$ and let $\{v_n\}$ and $\{w_n\}$ be sequences of H such that* $\limsup_{n\longrightarrow\infty} \|v_n\| \leq c$, $\limsup_{n\longrightarrow\infty} \|w_n\| \leq c$ *and* $\lim_{n\longrightarrow\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = c$, *for some* $c \geq 0$, *then* $\lim_{n\longrightarrow\infty} \| v_n - w_n\| = 0$.

Lemma 3.4. [53] *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in C$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|$$

for every $n = 0, 1, 2, \ldots$. Then, the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$, where P_C stands for the metric projection of H onto C.

Now, we show the following weak convergence theorem which solves the problem of finding a common element of the set of solutions of an equilibrium problem, the set of solutions of a variational inequality problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4), let A be a monotone, k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) P_C(u_n - \lambda_n A y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \longrightarrow \infty} r_n > 0$. Then, $\{x_n\}$ converges weakly to $p \in F(S) \cap VI(C, A) \cap EP(F)$, where $p = \lim_{n \to \infty} P_{F(S) \cap VI(C,A) \cap EP(F)} x_n$.

Proof. By Lemma 3.26, $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are well-defined. We divide the proof into five steps.
Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, let $x^* \in F(S) \cap VI(C, A) \cap EP(F)$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 3.2. Then, $x^* = P_C(x^* - \lambda_n A x^*) = T_{r_n} x^*$ and $u_n = T_{r_n} x_n$. So, we have

$$\|u_n - x^*\|^2 = \|T_{r_n} x_n - T_{r_n} x^*\|^2 \leq \|x_n - x^*\|^2. \tag{3.3}$$

Put $v_n = P_C(u_n - \lambda_n A y_n)$. Then, from Lemma 2.19 (v) and the monotonicity of A, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|u_n - \lambda_n A y_n - x^*\|^2 - \|u_n - \lambda_n A y_n - v_n\|^2 \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n \langle A y_n, x^* - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 \\ &\quad + 2\lambda_n \big(\langle A y_n - A x^*, x^* - y_n\rangle + \langle A x^*, x^* - y_n\rangle + \langle A y_n, y_n - v_n\rangle\big) \\ &\leq \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n \langle A y_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle. \end{aligned}$$

Moreover, since $y_n = P_C(u_n - \lambda_n A u_n)$ and Lemma 2.19 (iv), we have

$$\langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle \leq 0. \tag{3.4}$$

Since A is k-Lipschitz continuous, from (3.4) we obtain that

$$\begin{aligned} &\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle \\ &= \langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle + \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\ &\leq \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\ &\leq \lambda_n \|A u_n - A y_n\| \|v_n - y_n\| \\ &\leq \lambda_n k \|u_n - y_n\| \|v_n - y_n\|. \end{aligned}$$

Thus, we have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +\lambda_n^2 k^2\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.5) \\
&\leq \|u_n - x^*\|^2,
\end{aligned}$$

and hence

$$\|v_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\|. \tag{3.6}$$

Thus, by (3.3) and (3.5), we can calculate

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(Sx_n - x^*) + (1-\alpha_n)(v_n - x^*)\|^2 \\
&\leq \alpha_n\|Sx_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\Big\{\|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2\Big\} \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.7) \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Since the sequence $\{\|x_n - x^*\|\}$ is a bounded and nonincreasing sequence, there exists

$$c = \lim_{n\to\infty}\|x_n - x^*\| \tag{3.8}$$

and hence $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$ and $\{v_n\}$ are also bounded.

Step 2. We claim that $\lim_{n\longrightarrow\infty}\|Sx_n - x_n\| = 0$.

Indeed, let $x^* \in F(S) \cap VI(C, A) \cap EP(F)$. Then, we obtain that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\
&\leq \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle = \langle u_n - x^*, x_n - x^*\rangle \\
&= \frac{1}{2}\big(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2\big).
\end{aligned}$$

Therefore, $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. Thus, we can calculate

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(Sx_n - x^*) + (1-\alpha_n)(v_n - x^*)\|^2 \\
&\leq \alpha_n\|Sx_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\Big\{\|x_n - x^*\|^2 - \|x_n - u_n\|^2\Big\} \\
&= \|x_n - x^*\|^2 - (1-\alpha_n)\|x_n - u_n\|^2,
\end{aligned}$$

and hence

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - (1-\alpha_n)\|u_n - x_n\|^2.$$

Since $0 < c \leq \alpha_n \leq d < 1$, it follows that

$$\begin{aligned}
(1-d)\|x_n - u_n\|^2 &\leq (1-\alpha_n)\|x_n - u_n\|^2 \\
&= \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \qquad (3.9)
\end{aligned}$$

From (3.8) and (3.9), we have

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0. \tag{3.10}$$

Since $\liminf_{n\longrightarrow\infty} r_n>0$, we obtain

$$\lim_{n\longrightarrow\infty}\left\|\frac{x_n-u_n}{r_n}\right\|=0. \tag{3.11}$$

From (3.7), we also have

$$\|x_{n+1}-x^*\|^2\leq\|x_n-x^*\|^2+(1-\alpha_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2.$$

It follows that

$$(1-\alpha_n)(1-\lambda_n^2k^2)\|u_n-y_n\|\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2.$$

So, we have

$$\|u_n-y_n\|\leq\frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2). \tag{3.12}$$

Hence,

$$\lim_{n\longrightarrow\infty}\|u_n-y_n\|=0. \tag{3.13}$$

From (3.10) and (3.13), we also have

$$\|x_n-y_n\|\leq\|x_n-u_n\|+\|u_n-y_n\|\longrightarrow 0 \quad as \quad n\longrightarrow\infty. \tag{3.14}$$

Note that

$$\begin{aligned}\|y_n-v_n\|^2 &= \|P_C(u_n-\lambda_nAu_n)-P_C(u_n-\lambda_nAy_n)\|^2\\ &\leq \|(u_n-\lambda_nAu_n)-(u_n-\lambda_nAy_n)\|^2\\ &= \lambda_n^2\|Au_n-Ay_n\|^2\leq\lambda_n^2k^2\|u_n-y_n\|^2\leq\|u_n-y_n\|^2.\end{aligned}$$

Hence,

$$\lim_{n\longrightarrow\infty}\|y_n-v_n\|=0.$$

From

$$\|x_n-v_n\|\leq\|x_n-y_n\|+\|y_n-v_n\|,$$

we also have

$$\lim_{n\longrightarrow\infty}\|x_n-v_n\|=0.$$

Since $\|Sx_n-x^*\|\leq\|x_n-x^*\|$, from (3.6) and (3.8) we have

$$\limsup_{n\longrightarrow\infty}\|Sx_n-x^*\|\leq c \quad\text{and}\quad \limsup_{n\longrightarrow\infty}\|v_n-x^*\|\leq c.$$

Furthermore, we have

$$\lim_{n\longrightarrow\infty}\|\alpha_n(Sx_n-x^*)+(1-\alpha_n)(v_n-x^*)\|=\lim_{n\longrightarrow\infty}\|x_{n+1}-x^*\|=c.$$

By Lemma 3.3, we obtain

$$\lim_{n\longrightarrow\infty}\|Sx_n-v_n\|=0.$$

Also, we have

$$\|Sx_n-x_n\|\leq\|Sx_n-v_n\|+\|v_n-x_n\|.$$

Therefore, we get

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0.$$

Step 3. We claim that, for a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup p$, $p\in VI(C,A)\cap EP(F)\cap F(S)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some $p \in H$. We first prove $p \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}).$$

From (3.10), we get $u_{n_i} \rightharpoonup p$. Moreover, we have $p \in C$. In fact, if C is closed and convex, then C is weakly closed. Therefore, from $\{x_{n_i}\} \subset C$, we obtain $p \in C$. From (3.11) and (A4), it follows that $0 \geq F(y,p)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. From $y \in C$ and $p \in C$, we have $y_t \in C$ and hence $F(y_t, p) \leq 0$. So, by (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, p) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(p, y)$ for all $y \in C$ and hence $p \in EP(F)$. Next, we prove that $p \in F(S)$. Let $x^* \in F(S) \cap VI(C,A) \cap EP(F)$. From the demiclosedness (Lemma 2.61) of $I - S$, we know that $x_{n_i} \rightharpoonup p$ and $\lim_{n\longrightarrow\infty} \|Sx_n - x_n\| = 0$ mean $p \in F(S)$. Finally, we show that $p \in VI(C,A)$. By using the same argument in the proof of Theorem 3.5, we can get $p \in VI(C,A)$. So, we have $p \in VI(C,A) \cap EP(F) \cap F(S)$.

Step 4. We claim that $x_n \rightharpoonup p$ as $n \longrightarrow \infty$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p'$. Then, we have $p' \in F(S) \cap VI(C,A) \cap EP(F)$. We may show that $p = p'$. Assume that $p \neq p'$. From Opial's condition, we get $p = p'$. This implies that

$$x_n \rightharpoonup p \in F(S) \cap VI(C,A) \cap EP(F).$$

Step 5. Finally, we prove $p = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)\cap EP(F)} x_n$.

Let $z_n = P_{F(S)\cap VI(C,A)\cap EP(F)} x_n$ and $z \in F(S) \cap VI(C,A) \cap EP(F)$. Then, we have from (3.6)

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(Sx_n - z) + (1-\alpha_n)[P_C(u_n - \lambda_n Ay_n) - z]\| \\ &\leq \alpha_n\|Sx_n - z\| + (1-\alpha_n)\|v_n - z\| \\ &\leq \alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned}$$

By Lemma 3.4, we obtain that $\{z_n\}$ converges strongly to some $p_0 \in F(S) \cap VI(C,A) \cap EP(F)$. Since $\langle p - z_n, z_n - x_n\rangle \geq 0$, we have $\langle p - p_0, p_0 - p\rangle \geq 0$, and hence $p = p_0$. So, we have that $p = \lim_{n\longrightarrow\infty} P_{F(S)\cap VI(C,A)\cap EP(F)} x_n$. This completes the proof. □

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4) and let A be a monotone, k-Lipschitz continuous mapping of C into H such that $VI(C,A) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C(u_n - \lambda_n Ay_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\longrightarrow\infty} r_n > 0$. Then, $\{x_n\}$ converges weakly to $p \in VI(C,A) \cap EP(F)$, where $p = \lim_{n\longrightarrow\infty} P_{VI(C,A)\cap EP(F)} x_n$.

Proof. Putting $S = I$, by Theorem 3.5, we obtain the desired result. $\square$

3.1.2 Strong Convergence Theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem for a monotone, k-Lipschitz continuous mapping in a Hilbert space by using the hybrid extragradient method.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and let A be a monotone, k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ w_n = \alpha_n S x_n + (1 - \alpha_n) P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\longrightarrow\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)\cap EP(F)}x_0$.

Proof. We first show that the sequence $\{x_n\}$ is well-defined. From the definitions of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. Since $C_n = \{z \in C : \|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - z\rangle \leq 0\}$, we deduce that C_n is convex for all $n \in \mathbb{N} \cup \{0\}$. So, $C_n \cap Q_n$ is closed and convex for any $n \in \mathbb{N} \cup \{0\}$. Let $x^* \in F(S) \cap VI(C, A) \cap EP(F)$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 3.2. Then, $x^* = P_C(x^* - \lambda_n A x^*) = T_{r_n} x^*$ and $u_n = T_{r_n} x_n$. So, we have

$$\|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|. \tag{3.15}$$

Put $v_n = P_C(u_n - \lambda_n A y_n)$. From Lemma 2.19 (v) and the monotonicity of A, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|u_n - \lambda_n A y_n - x^*\|^2 - \|u_n - \lambda_n A y_n - v_n\|^2 \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle A y_n, x^* - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 \\ &\quad +2\lambda_n\big(\langle A y_n - A x^*, x^* - y_n\rangle + \langle A x^*, x^* - y_n\rangle + \langle A y_n, y_n - v_n\rangle\big) \\ &\leq \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\ &\qquad +2\lambda_n\langle A y_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\qquad +2\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle. \end{aligned}$$

Moreover, from $y_n = P_C(u_n - \lambda_n A u_n)$ and Lemma 2.19 (iv), we have

$$\langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle \leq 0. \tag{3.16}$$

Since A is k-Lipschitz continuous, from (3.16) we obtain that

$$\begin{aligned}
&\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle \\
&\quad = \langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle + \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\
&\quad \leq \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \leq \lambda_n k \|u_n - y_n\| \|v_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +2\lambda_n k \|u_n - y_n\| \|v_n - y_n\| \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +\lambda_n^2 k^2 \|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.17) \\
&\leq \|u_n - x^*\|^2,
\end{aligned}$$

and hence

$$\|v_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\|. \tag{3.18}$$

Thus, from (3.17) we have

$$\begin{aligned}
&\|w_n - x^*\|^2 \\
&\quad = \|\alpha_n (S x_n - x^*) + (1-\alpha_n)(v_n - x^*)\|^2 \\
&\quad \leq \alpha_n \|S x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\quad \leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\Big\{\|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2\Big\} \\
&\quad \leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\quad = \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \leq \|x_n - x^*\|^2. \qquad (3.19)
\end{aligned}$$

So, we have $x^* \in C_n$ and hence

$$F(S) \cap VI(C, A) \cap EP(F) \subset C_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.20}$$

Next, we show that

$$F(S) \cap VI(C, A) \cap EP(F) \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.21}$$

We prove this by induction. For $n = 0$, we have $F(S) \cap VI(C, A) \cap EP(F) \subset C_0$ and $Q_0 = C$. So, we get

$$F(S) \cap VI(C, A) \cap EP(F) \subset C_0 \cap Q_0.$$

Suppose that $F(S) \cap VI(C, A) \cap EP(F) \subset C_k \cap Q_k$ for $k \in \mathbb{N} \cup \{0\}$. Then, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x_0$. Therefore, for each $z \in C_k \cap Q_k$, we also have

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0.$$

Since $F(S) \cap VI(C, A) \cap EP(F) \subset C_k \cap Q_k$ for any $z \in F(S) \cap VI(C, A) \cap EP(F)$ we obtain

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0$$

and hence $z \in Q_{k+1}$. It follows that $F(S) \cap VI(C, A) \cap EP(F) \subset Q_{k+1}$. This together with (3.20) gives

$$F(S) \cap VI(C, A) \cap EP(F) \subset C_{k+1} \cap Q_{k+1}.$$

This implies that $\{x_n\}$ is well-defined. From Lemma 3.26, the sequence $\{u_n\}$ is also well-defined.

Since $F(S) \cap VI(C, A) \cap EP(F)$ is a nonempty closed convex subset of H, there exists a unique $z' \in F(S) \cap VI(C, A) \cap EP(F)$ such that

$$z' = P_{F(S)\cap VI(C,A)\cap EP(F)}x_0.$$

From $x_{n+1} = P_{C_n \cap Q_n}x_0$, we obtain

$$\|x_{n+1} - x_0\| \leq \|z - x_0\| \quad \text{for all } z \in C_n \cap Q_n \text{ and all } n \in \mathbb{N} \cup \{0\}.$$

Since $z' \in F(S) \cap VI(C, A) \cap EP(F) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|z' - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.22}$$

Therefore, $\{x_n\}$ is bounded. From (3.18) and (3.19), $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also bounded. Since $x_n = P_{Q_n}x_0$ and $x_{n+1} \in Q_n$, we have

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$$

for every $n \in \mathbb{N}\cup\{0\}$. Hence, the sequence $\{\|x_0 - x_n\|\}$ is bounded and nondecreasing. So, $\lim_{n\longrightarrow\infty} \|x_0 - x_n\|$ exists. Since $x_n = P_{Q_n}x_0, x_{n+1} \in Q_n$ and $\frac{x_n+x_{n+1}}{2} \in Q_n$, we have

$$\begin{aligned}
\|x_0 - x_n\|^2 &\leq \left\|x_0 - \frac{x_n + x_{n+1}}{2}\right\|^2 \\
&= \left\|\frac{1}{2}(x_0 - x_n) + \frac{1}{2}(x_0 - x_{n+1})\right\|^2 \\
&= \frac{1}{2}\|x_0 - x_n\|^2 + \frac{1}{2}\|x_0 - x_{n+1}\|^2 - \frac{1}{4}\|x_n - x_{n+1}\|^2.
\end{aligned}$$

So, we obtain

$$\frac{1}{4}\|x_n - x_{n+1}\|^2 \leq \frac{1}{2}\|x_0 - x_{n+1}\|^2 - \frac{1}{2}\|x_0 - x_n\|.$$

Since $\lim_{n\longrightarrow\infty} \|x_0 - x_n\|$ exists, this implies that

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n+1}\| = 0. \tag{3.23}$$

From $x_{n+1} \in C_n$, we have

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (3.23), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - w_n\| = 0. \tag{3.24}$$

Next, let $x^* \in F(S) \cap VI(C, A) \cap EP(F)$. Then, as in the proof of Theorem 3.5, we obtain that

$$\|w_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \alpha_n)\|u_n - x_n\|^2.$$

Since $0 < c \leq \alpha_n \leq d < 1$, it follows that

$$\begin{aligned}
(1 - d)\|x_n - u_n\|^2 &\leq (1 - \alpha_n)\|x_n - u_n\|^2 \\
&= \|x_n - x^*\|^2 - \|w_n - x^*\|^2 \\
&= (\|x_n - x^*\| + \|w_n - x^*\|)(\|x_n - x^*\| - \|w_n - x^*\|) \\
&\leq \|x_n - w_n\|(\|x_n - x^*\| + \|w_n - x^*\|).
\end{aligned}$$

Since $\{x_n\}$ and $\{w_n\}$ are bounded, from (3.24) we have that

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \tag{3.25}$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, we obtain

$$\lim_{n\longrightarrow\infty} \left\| \frac{x_n - u_n}{r_n} \right\| = 0.$$

For $x^* \in F(S) \cap VI(C, A) \cap EP(F)$, from (3.19) we obtain

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 + (1 - \alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2.$$

Therefore, we have

$$\begin{aligned}
&\|u_n - y_n\| \\
&\le \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|w_n - x^*\|^2) \\
&= \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\| + \|w_n - x^*\|)(\|x_n - x^*\| - \|w_n - x^*\|) \\
&\le \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}\|x_n - w_n\|(\|x_n - x^*\| + \|w_n - x^*\|).
\end{aligned}$$

So, by (3.24) we obtain

$$\lim_{n\longrightarrow\infty} \|u_n - y_n\| = 0. \tag{3.26}$$

Since $\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - y_n\|$, from (3.25) and (3.26), we also have

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{3.27}$$

From $\|v_n - y_n\| \le \|u_n - y_n\|$ and (3.26), we obtain

$$\lim_{n\longrightarrow\infty} \|v_n - y_n\| = 0. \tag{3.28}$$

From (3.26) and (3.28), we also have

$$\lim_{n\longrightarrow\infty} \|u_n - v_n\| = 0. \tag{3.29}$$

From (3.27) and (3.28), we also have

$$\lim_{n\longrightarrow\infty} \|x_n - v_n\| = 0. \tag{3.30}$$

From (3.24) and (3.30), we also have

$$\lim_{n\longrightarrow\infty} \|w_n - v_n\| = 0. \tag{3.31}$$

Since $\alpha_n(Sx_n - x_n) = \alpha_n(v_n - x_n) + (w_n - v_n)$ and $0 < c \le \alpha_n \le d < 1$, it follows that

$$\begin{aligned}
c\|Sx_n - x_n\| &\le \alpha_n\|Sx_n - x_n\| \\
&\le \alpha_n\|v_n - x_n\| + \|w_n - v_n\|.
\end{aligned}$$

From (3.30) and (3.31), we obtain

$$\lim_{n\longrightarrow\infty} \|Sx_n - x_n\| = 0. \tag{3.32}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z. From (3.25), we obtain also that $u_{n_i} \rightharpoonup z$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, C is weakly closed and hence $z \in C$. We will show that $z \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq F(y,u_n)$$

and hence

$$\left\langle y-u_n, \frac{u_n-x_n}{r_n}\right\rangle \geq F(y,u_n).$$

From $\frac{u_n-x_n}{r_n} \longrightarrow 0$, $u_{n_i} \rightharpoonup z$ and (A4), we have

$$0 \geq F(y,z), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So, by (A1) and (A4), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(z, y)$ for all $y \in C$ and hence $z \in EP(F)$. Let us show that $z \in F(S)$. Assume $z \notin F(S)$. From Opial's condition (Lemma 2.58) and (3.32), we have $z \in F(S)$. On the other hand, as in the proof of Theorem 3.5, we obtain $z \in VI(C,A)$. So, we have $z \in F(S) \cap VI(C,A) \cap EP(F)$.

Finally, we show that $x_n \longrightarrow z'$, where $z' = P_{F(S)\cap VI(C,A)EP(F)}x_0$. From $x_{n_i} \rightharpoonup z$, the lower semicontinuity of the norm and (3.22), we have that

$$\|z'-x_0\| \leq \|z-x_0\| \leq \liminf_{i\longrightarrow\infty} \|x_{n_i}-x_0\| \leq \limsup_{i\longrightarrow\infty} \|x_{n_i}-x_0\| \leq \|z'-x_0\|.$$

Thus, we obtain $\lim_{i\longrightarrow\infty} \|x_{n_i}-x_0\| = \|z-x_0\| = \|z'-x_0\|$. Using the Kadec-Klee property (Lemma 2.59) of H, we obtain that

$$\lim_{i\longrightarrow\infty} x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z', where $z' = P_{F(T)\cap VI(C,A)\cap EP(F)}x_0$. $\square$

Corollary 3.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and let A be a monotone, k-Lipschitz continuous mapping of C into H such that $VI(C,A) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and let*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ w_n = \alpha_n x_n + (1-\alpha_n)P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\longrightarrow\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $P_{VI(C,A)\cap EP(F)}x_0$.

Proof. Putting $S = I$, by Theorem 3.7 we obtain the desired result. $\square$

We first prove that the following Lemmas.

Lemma 3.9. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $A : C \longrightarrow H$ be α-inverse-strongly monotone. If $0 \leq \lambda_n \leq 2\alpha$, then $I - \lambda_n A$ is a nonexpansive mapping in H.*

Proof. See [16, 20] □

Now, we prove the following main Theorem.

Theorem 3.10. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4) and $A : C \longrightarrow H$ be an α-inverse-strongly monotone mapping. Let $f : C \longrightarrow C$ be a contraction with coefficient k $(0 \leq k < 1)$ and S be a nonexpansive mappings of C into itself such that $F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Suppose $x_1 \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n f(x_n) + (1 - \beta_n) SP_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad \forall n \geq 1, \end{cases} \tag{3.33}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequence in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a \leq \lambda_n \leq b < 2\alpha$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions:

(C1) $\lim_{n\longrightarrow\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$,

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C4) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F(S) \cap VI(C, A) \cap EP(F)$, where $q = P_{F(S)\cap VI(C,A)\cap EP(F)} f(q)$.

Proof. See [16]. □

3.1.3 An Infinite Family of Nonexpansive Mappings

In this section, we prove the strong convergence theorem of an iterative algorithm based on extragradient method which solves the problem of finding a common element of the set of fixed point of an infinite family of nonexpansive mappings, the set of solution of a equilibrium problems and the set of solution of a variational inequality problem for a inverse-strongly monotone mapping in a real Hilbert space.

Definition 3.11. *[8]. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\mu_n\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of C into itself* *as follows:*

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\
&\vdots \\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\
&\vdots \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I.
\end{aligned} \tag{3.34}$$

Such a mappings W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (3.34). Then we can have the following crucial conclusions concerning W_n. You can find them in [46]. Now we only need the following similar version in Hilbert spaces.

Lemma 3.12. [46]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\longrightarrow\infty} U_{n,k}x$ exists.*

Using Lemma 3.12, one can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\longrightarrow\infty} W_n x = \lim_{n\longrightarrow\infty} U_{n,1}x \tag{3.35}$$

for every $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, ...$ and $\mu_1, \mu_2, ...$ Throughout this thesis, we will assume that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, we have the following results.

Lemma 3.13. [46]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 3.14. [63]. *If $\{x_n\}$ is a bounded sequence in C, then $\lim_{n\longrightarrow\infty} \|Wx_n - W_n x_n\| = 0$.*

Theorem 3.15. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive of C into itself and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C, B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases}
x_1 = x \in C \text{ chosen arbitrary}, \\
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \lambda_n B u_n), \\
k_n = \alpha_n u_n + (1-\alpha_n) P_C(u_n - \lambda_n B y_n), \\
x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big) W_n k_n, \quad \forall n \ge 1,
\end{cases} \tag{3.36}$$

where $\{W_n\}$ *is the sequence generated by (3.34)and* $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *are three sequences in* $(0,1)$ *and* $\{r_n\}$ *is a real sequence in* $(0,\infty)$ *satisfy the following conditions:*

(i) $\lim_{n\longrightarrow\infty}\epsilon_n = 0, \sum_{n=1}^{\infty}\epsilon_n = \infty$,

(ii) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty}\beta_n \leq \limsup_{n\longrightarrow\infty}\beta_n < 1$,

(iv) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty}|r_{n+1} - r_n| = 0$,

(v) $\{\frac{\lambda_n}{\beta}\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\Big\langle (A-\gamma f)z, x-z \Big\rangle \geq 0, \ \ \forall x \in \Theta. \tag{3.37}$$

Equivalently, we have $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. From [16, 29] it follows that $P_{\Theta}(I - A + \gamma f)$ is a contraction of H into itself. Therefore, by the Banach Contraction Mapping Principle, which implies that there exists a unique element $z \in H$ such that $z = P_{\Theta}(I - A + \gamma f)(z)$.

We will divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded. Indeed, pick any $p \in \Theta$. From the definition of T_r, we note that $u_n = T_{r_n}x_n$. If follows that

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|.$$

Since $I - \lambda_n B$ is nonexpansive and $p = P_C(p - \lambda_n Bp)$ from Lemma 2.40, we have

$$\begin{aligned}
\|y_n - p\| &= \|P_C(u_n - \lambda_n Bu_n) - P_C(p - \lambda_n Bp)\| \\
&\leq \|(u_n - \lambda_n Au_n) - (p - \lambda_n Bp)\| \\
&= \|(I - \lambda_n A)u_n - (I - \lambda_n B)p\| \\
&\leq \|u_n - p\| \leq \|x_n - p\|.
\end{aligned}$$

Put $v_n = P_C(u_n - \lambda_n By_n)$. Since $p \in VI(C,B)$, we have $p = P_C(p - \lambda_n Bp)$. Substituting $x = u_n - \lambda_n Ay_n$ and $y = p$ in Lemma 2.19 (v), we can write

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|u_n - \lambda_n By_n - p\|^2 - \|u_n - \lambda_n By_n - v_n\|^2 \\
&= \|u_n - p\|^2 - 2\lambda_n\langle By_n, u_n - p\rangle + \lambda_n^2\|By_n\|^2 \\
&\quad - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, u_n - v_n\rangle - \lambda_n^2\|By_n\|^2 \\
&= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, p - v_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n - Bp, p - y_n\rangle \\
&\quad + 2\lambda_n\langle Bp, p - y_n\rangle + 2\lambda_n\langle By_n, y_n - v_n\rangle.
\end{aligned} \tag{3.38}$$

Using the fact that B is β-inverse-strongly monotone mapping and p is a solution of the variational inequality problem $VI(C,B)$, we also have

$$\langle By_n - Bp, p - y_n\rangle \leq 0 \quad \text{and} \quad \langle Bp, p - y_n\rangle \leq 0. \tag{3.39}$$

It follows from (3.38) and (3.39) that

$$\begin{aligned}
&\|v_n - p\|^2 \\
\leq\ & \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, y_n - v_n\rangle \\
=\ & \|u_n - p\|^2 - \|(u_n - y_n) + (y_n - v_n)\|^2 + 2\lambda_n\langle By_n, y_n - v_n\rangle \\
\leq\ & \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle \\
& + 2\lambda_n\langle By_n, y_n - v_n\rangle \\
=\ & \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
& +2\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle. \qquad (3.40)
\end{aligned}$$

Substituting x by $u_n - \lambda_n Bu_n$ and $y = v_n$ in Lemma 2.19 (iv), we obtain

$$\langle u_n - \lambda_n Bu_n - y_n, v_n - y_n\rangle \quad \leq \quad 0.$$

It follows that

$$\begin{aligned}
\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle \quad &= \quad \langle u_n - \lambda_n Bu_n - y_n, v_n - y_n\rangle \\
&\quad\ + \langle \lambda_n Bu_n - \lambda_n By_n, v_n - y_n\rangle \\
&\leq \quad \langle \lambda_n Bu_n - \lambda_n By_n, v_n - y_n\rangle \\
&\leq \quad \lambda_n\|Bu_n - By_n\|\|v_n - y_n\| \\
&\leq \quad \frac{\lambda_n}{\beta}\|u_n - y_n\|\|v_n - y_n\|. \qquad (3.41)
\end{aligned}$$

Substituting (3.41) into (3.40), we have

$$\begin{aligned}
\|v_n - p\|^2 \quad &\leq \quad \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle \\
&\leq \quad \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\frac{\lambda_n}{\beta}\|u_n - y_n\|\|v_n - y_n\| \\
&\leq \quad \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \frac{\lambda_n^2}{\beta^2}\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\
&= \quad \|u_n - p\|^2 - \|u_n - y_n\|^2 + \frac{\lambda_n^2}{\beta^2}\|u_n - y_n\|^2 \\
&= \quad \|u_n - p\|^2 + \Big(\frac{\lambda_n^2}{\beta^2} - 1\Big)\|u_n - y_n\|^2 \qquad (3.42) \\
&\leq \quad \|u_n - p\|^2 \leq \|x_n - p\|^2.
\end{aligned}$$

Setting $k_n = \alpha_n u_n + (1 - \alpha_n)v_n$, we can calculate

$$\begin{aligned}
&\|x_{n+1} - p\| \\
=\ & \Big\|\epsilon_n\big(\gamma f(x_n) - Ap\big) + \beta_n(x_n - p) + \big((1 - \beta_n)I - \epsilon_n A\big)(W_n k_n - p)\Big\| \\
\leq\ & (1 - \beta_n - \epsilon_n\bar{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
\leq\ & (1 - \beta_n - \epsilon_n\bar{\gamma})\big\{\alpha_n\|u_n - p\| + (1 - \alpha_n)\|v_n - p\|\big\} \\
& +\beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
\leq\ & (1 - \beta_n - \epsilon_n\bar{\gamma})\big\{\alpha_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|\big\} \\
& +\beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
=\ & (1 - \beta_n - \epsilon_n\bar{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
=\ & (1 - \epsilon_n\bar{\gamma})\|x_n - p\| + \epsilon_n\gamma\|f(x_n) - f(p)\| + \epsilon_n\|\gamma f(p) - Ap\| \\
\leq\ & (1 - \epsilon_n\bar{\gamma})\|x_n - p\| + \epsilon_n\gamma\alpha\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| \\
=\ & (1 - (\bar{\gamma} - \gamma\alpha)\epsilon_n)\|x_n - p\| + (\bar{\gamma} - \gamma\alpha)\epsilon_n\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}.
\end{aligned}$$

By induction that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad n \in \mathbb{N}.$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{W_n k_n\}$, $\{f(x_n)\}$, $\{Bu_n\}$ and $\{y_n\}$.

Step2. We claim that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.
Observing that $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we get

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in H \tag{3.43}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \text{for all } y \in H. \tag{3.44}$$

Putting $y = u_{n+1}$ in (3.43) and $y = u_n$ in (3.44), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\
&\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|,
\end{aligned} \tag{3.45}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Note that

$$\begin{aligned}
\|v_{n+1} - v_n\| &\leq \|P_C(u_{n+1} - \lambda_{n+1}By_{n+1}) - P_C(u_n - \lambda_n By_n)\| \\
&\leq \|u_{n+1} - \lambda_{n+1}By_{n+1} - (u_n - \lambda_n By_n)\| \\
&= \|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1}Bu_n) \\
&\quad + \lambda_{n+1}(Bu_{n+1} - By_{n+1} - Bu_n) + \lambda_n By_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1}Bu_n)\| \\
&\quad + \lambda_{n+1}(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|) + \lambda_n\|By_n\| \\
&\leq \|u_{n+1} - u_n\| + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) + \lambda_n\|By_n\|
\end{aligned}$$

and

$$
\begin{aligned}
& \|k_{n+1}-k_n\| \\
= \;& \|\alpha_{n+1}u_{n+1}+(1-\alpha_{n+1})v_{n+1}-\alpha_n u_n-(1-\alpha_n)v_n\| \\
= \;& \|\alpha_{n+1}(u_{n+1}-u_n)+(\alpha_{n+1}-\alpha_n)u_n+(1-\alpha_{n+1})(v_{n+1}-v_n)+(\alpha_n-\alpha_{n+1})v_n\| \\
\leq \;& \alpha_{n+1}\|u_{n+1}-u_n\|+(1-\alpha_{n+1})\|v_{n+1}-v_n\|+|\alpha_n-\alpha_{n+1}|\|u_n+v_n\| \\
= \;& \alpha_{n+1}\|u_{n+1}-u_n\|+(1-\alpha_{n+1})\Big\{\|u_{n+1}-u_n\|+\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\| \\
& +\|Bu_n\|\big)+\lambda_n\|By_n\|\Big\}+|\alpha_n-\alpha_{n+1}|\|u_n+v_n\| \\
= \;& \|u_{n+1}-u_n\|+(1-\alpha_{n+1})\lambda_{n+1}(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|) \\
& +(1-\alpha_{n+1})\lambda_n\|By_n\|+|\alpha_n-\alpha_{n+1}|\|u_n+v_n\| \\
\leq \;& \|x_{n+1}-x_n\|+\frac{M_1}{c}|r_{n+1}-r_n|+(1-\alpha_{n+1})\lambda_{n+1}(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|) \\
& +(1-\alpha_{n+1})\lambda_n\|By_n\|+|\alpha_n-\alpha_{n+1}|\|u_n+v_n\|.
\end{aligned}
\tag{3.46}
$$

Setting

$$
z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}=\frac{\epsilon_n\gamma f(x_n)+\big((1-\beta_n)I-\epsilon_n A\big)W_n k_n}{1-\beta_n},
$$

we have $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$, $n\geq 1$. It follows that

$$
\begin{aligned}
z_{n+1}-z_n \;=\;& \frac{\epsilon_{n+1}\gamma f(x_{n+1})+\big((1-\beta_{n+1})I-\epsilon_{n+1}A\big)W_{n+1}k_{n+1}}{1-\beta_{n+1}} \\
& -\frac{\epsilon_n\gamma f(x_n)+\big((1-\beta_n)I-\epsilon_n A\big)W_n k_n}{1-\beta_n} \\
=\;& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\epsilon_n}{1-\beta_n}\gamma f(x_n)+W_{n+1}k_{n+1}-W_n k_n \\
& +\frac{\epsilon_n}{1-\beta_n}AW_n k_n-\frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}k_{n+1} \\
=\;& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\gamma f(x_{n+1})-AW_{n+1}k_{n+1}\big)+\frac{\epsilon_n}{1-\beta_n}\big(AW_n k_n-\gamma f(x_n)\big) \\
& +W_{n+1}k_{n+1}-W_{n+1}k_n+W_{n+1}k_n-W_n k_n.
\end{aligned}
\tag{3.47}
$$

It follows from (3.46) and (3.47) that

$$
\begin{aligned}
& \|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\
\leq \;& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\|+\|AW_{n+1}k_{n+1}\|\big) \\
& +\frac{\epsilon_n}{1-\beta_n}\big(\|AW_n k_n\|+\|\gamma f(x_n)\|\big)+\|W_{n+1}k_{n+1}-W_{n+1}k_n\| \\
& +\|W_{n+1}k_n-W_n k_n\|-\|x_{n+1}-x_n\| \\
\leq \;& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\|+\|AW_{n+1}k_{n+1}\|\big) \\
& +\frac{\epsilon_n}{1-\beta_n}\big(\|AW_n k_n\|+\|\gamma f(x_n)\|\big)+\|k_{n+1}-k_n\| \\
& +\|W_{n+1}k_n-W_n k_n\|-\|x_{n+1}-x_n\| \\
\leq \;& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\|+\|AW_{n+1}k_{n+1}\|\big) \\
& +\frac{\epsilon_n}{1-\beta_n}\big(\|AW_n k_n\|+\|\gamma f(x_n)\|\big)+\frac{M_1}{c}|r_{n+1}-r_n| \\
& +(1-\alpha_{n+1})\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big) \\
& +(1-\alpha_{n+1})\lambda_n\|By_n\|+|\alpha_n-\alpha_{n+1}|\|u_n+v_n\| \\
& +\|W_{n+1}k_n-W_n k_n\|.
\end{aligned}
\tag{3.48}
$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}k_n - W_nk_n\| &= \|\mu_1 T_1 U_{n+1,2}k_n - \mu_1 T_1 U_{n,2}k_n\| \\
&\le \mu_1\|U_{n+1,2}k_n - U_{n,2}k_n\| \\
&= \mu_1\|\mu_2 T_2 U_{n+1,3}k_n - \mu_2 T_2 U_{n,3}k_n\| \\
&\le \mu_1\mu_2\|U_{n+1,3}k_n - U_{n,3}k_n\| \\
&\vdots \\
&\le \mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}k_n - U_{n,n+1}k_n\| \\
&\le M_2\prod_{i=1}^{n}\mu_i, \qquad (3.49)
\end{aligned}$$

where $M_2 \ge 0$ is a constant such that $\|U_{n+1,n+1}k_n - U_{n,n+1}k_n\| \le M_2$ for all $n \ge 0$. Combining (3.48) and (3.49), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le\ & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|AW_{n+1}k_{n+1}\|\big) \\
& + \frac{\epsilon_n}{1-\beta_n}\big(\|AW_nk_n\| + \|\gamma f(x_n)\|\big) + \frac{M_1}{c}|r_{n+1} - r_n| \\
& + (1-\alpha_{n+1})\lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\
& + (1-\alpha_{n+1})\lambda_n\|By_n\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
& + M_2\prod_{i=1}^{n}\mu_i,
\end{aligned}$$

which implies that (noting that (i), (ii), (iii), (iv), (v) and $0 < \mu_i \le b < 1, \forall i \ge 1$)

$$\limsup_{n\longrightarrow\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

Hence, by Lemma 2.60, we obtain

$$\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|z_n - x_n\| = 0. \qquad (3.50)$$

Applying (3.50) and (ii),(iv), (v) to (3.45) and (3.46), we obtain that

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = \lim_{n\longrightarrow\infty}\|k_{n+1} - k_n\| = 0. \qquad (3.51)$$

Since $x_{n+1} = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_nk_n$, we have

$$\begin{aligned}
&\|x_n - W_nk_n\| \\
\le\ & \|x_n - x_{n+1}\| + \|x_{n+1} - W_nk_n\| \\
=\ & \|x_n - x_{n+1}\| + \Big\|\epsilon_n\gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)W_nk_n - W_nk_n\Big\| \\
=\ & \|x_n - x_{n+1}\| + \Big\|\epsilon_n\big(\gamma f(x_n) - AW_nk_n\big) + \beta_n(x_n - W_nk_n)\Big\| \\
\le\ & \|x_n - x_{n+1}\| + \epsilon_n\big(\|\gamma f(x_n)\| + \|AW_nk_n\|\big) + \beta_n\|x_n - W_nk_n\|
\end{aligned}$$

that is,

$$\|x_n - W_nk_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\big(\|\gamma f(x_n)\| + \|AW_nk_n\|\big).$$

By (i), (iii) and (3.50) it follows that

$$\lim_{n\longrightarrow\infty}\|W_nk_n - x_n\| = 0. \qquad (3.52)$$

Step 3. We claim that the following statements hold:

$\lim_{n\longrightarrow\infty}\|u_n-k_n\|=0$ and $\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0$.

For any $p\in\Theta:=\cap_{n=1}^{\infty}F(T_n)\cap EP(F)\cap VI(C,B)$ and (3.42), we have

$$
\begin{aligned}
\|k_n-p\|^2 &= \|\alpha_n(u_n-p)+(1-\alpha_n)(v_n-p)\|^2\\
&\le \alpha_n\|u_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2\\
&\le \alpha_n\|u_n-p\|^2+(1-\alpha_n)\Big\{\|u_n-p\|^2+\Big(\frac{\lambda_n^2}{\beta^2}-1\Big)\|u_n-y_n\|^2\Big\}\\
&= \|u_n-p\|^2+(1-\alpha_n)\Big(\frac{\lambda_n^2}{\beta^2}-1\Big)\|u_n-y_n\|^2\\
&\le \|x_n-p\|^2+(1-\alpha_n)\Big(\frac{\lambda_n^2}{\beta^2}-1\Big)\|u_n-y_n\|^2.\\
&\quad \|x_{n+1}-p\|^2\\
&= \Big\|\big((1-\beta_n)I-\epsilon_nA\big)(W_nk_n-p)+\beta_n(x_n-p)+\epsilon_n\big(\gamma f(x_n)-Ap\big)\Big\|^2\\
&= \|\big((1-\beta_n)I-\epsilon_nA\big)(W_nk_n-p)+\beta_n(x_n-p)\|^2+\epsilon_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad +2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad +2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_nA\big)(W_nk_n-p),\gamma f(x_n)-Ap\Big\rangle\\
&\le \big[(1-\beta_n-\epsilon_n\bar{\gamma})\|W_nk_n-p\|+\beta_n\|x_n-p\|\big]^2+\epsilon_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad +2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad +2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_nA\big)(W_nk_n-p),\gamma f(x_n)-Ap\Big\rangle\\
&\le \big[(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n-p\|+\beta_n\|x_n-p\|\big]^2+c_n\\
&= (1-\beta_n-\epsilon_n\bar{\gamma})^2\|k_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad +2(1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\|k_n-p\|\|x_n-p\|+c_n\\
&\le (1-\beta_n-\epsilon_n\bar{\gamma})^2\|k_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad +(1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\big(\|k_n-p\|^2+\|x_n-p\|^2\big)+c_n\\
&= \big[(1-\epsilon_n\bar{\gamma})^2-2(1-\epsilon_n\bar{\gamma})\beta_n+\beta_n^2\big]\|k_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad +\big((1-\epsilon_n\bar{\gamma})\beta_n-\beta_n^2\big)\big(\|k_n-p\|^2+\|x_n-p\|^2\big)+c_n\\
&= (1-\epsilon_n\bar{\gamma})^2\|k_n-p\|^2-(1-\epsilon_n\bar{\gamma})\beta_n\|k_n-p\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n-p\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n, \qquad (3.53)
\end{aligned}
$$

where

$$
\begin{aligned}
c_n &= \epsilon_n^2\|\gamma f(x_n)-Ap\|^2+2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad +2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_nA\big)(W_nk_n-p),\gamma f(x_n)-Ap\Big\rangle.
\end{aligned}
$$

It follows from condition (i) that

$$\lim_{n\longrightarrow\infty}c_n=0. \qquad (3.54)$$

By (3.53), and using (v), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n-p\|^2+(1-\alpha_n)\Big(\frac{\lambda_n^2}{\beta^2}-1\Big)\|u_n-y_n\|^2\Big\}\\
& +(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n\bar{\gamma})^2\|x_n-p\|^2\\
& +(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\Big(\frac{\lambda_n^2}{\beta^2}-1\Big)\|u_n-y_n\|^2+c_n\\
\le\ & \|x_n-p\|^2+(1-\alpha_n)\Big(\frac{\lambda_n^2}{\beta^2}-1\Big)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)\delta\|u_n-y_n\|^2 &\le (1-\alpha_n)\Big(1-\frac{\lambda_n^2}{\beta^2}\Big)\|u_n-y_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\
&= (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}c_n=0$ and from (3.50), we obtain

$$\lim_{n\longrightarrow\infty}\|u_n-y_n\|=0. \tag{3.55}$$

Note that

$$k_n-v_n=\alpha_n(u_n-v_n).$$

Since $\lim_{n\longrightarrow\infty}\alpha_n=0$, we have

$$\lim_{n\longrightarrow\infty}\|k_n-v_n\|=0. \tag{3.56}$$

From B is $\frac{1}{\beta}$-Lipschitz continuous, we obtain

$$\begin{aligned}
\|v_n-y_n\| &= \|P_C(u_n-\lambda_nBy_n)-P_C(u_n-\lambda_nBu_n)\|\\
&\le \|(u_n-\lambda_nBy_n)-(u_n-\lambda_nBu_n)\|\\
&= \lambda_n\|Bu_n-By_n\|\\
&\le \frac{\lambda_n}{\beta}\|u_n-y_n\|
\end{aligned}$$

then we get

$$\lim_{n\longrightarrow\infty}\|v_n-y_n\|=0. \tag{3.57}$$

From

$$\|u_n-k_n\| \le \|u_n-y_n\|+\|y_n-v_n\|+\|v_n-k_n\|.$$

Applying (3.55), (3.56) and (3.57), we have

$$\lim_{n\longrightarrow\infty}\|u_n-k_n\|=0. \tag{3.58}$$

For any $p\in\Theta$, note that T_r is firmly nonexpansive (Lemma 3.2), then we have

$$\begin{aligned}
\|u_n-p\|^2 &= \|T_{r_n}x_n-T_{r_n}p\|^2\\
&\le \langle T_{r_n}x_n-T_{r_n}p, x_n-p\rangle\\
&= \langle u_n-p, x_n-p\rangle\\
&= \frac{1}{2}(\|u_n-p\|^2+\|x_n-p\|^2-\|x_n-u_n\|^2)
\end{aligned}$$

and hence

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

This together with (3.53) gives

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
\leq\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n - p\|^2 + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
=\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\Big\{\|k_n - u_n\|^2 + \|u_n - p\|^2 + 2\langle k_n - u_n, u_n - p\rangle\Big\} \\
& + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
\leq\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|u_n - p\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
\leq\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2\Big\} \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
\leq\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n - p\|^2 - (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
=\ & (1-\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 - (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n - u_n\|^2 \\
& + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + c_n \\
=\ & \big[1 - 2\epsilon_n\bar{\gamma} + (\epsilon_n\bar{\gamma})^2\big]\|x_n - p\|^2 - (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n - u_n\|^2 \\
& + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + c_n \\
\leq\ & \|x_n - p\|^2 + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& - (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + c_n.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
& (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n - u_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 \\
& + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + c_n \\
=\ & (\|x_n - p\| - \|x_{n+1} - p\|)(\|x_n - p\| + \|x_{n+1} - p\|) \\
& + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + c_n \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 \\
& + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|^2 \\
& + 2(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|k_n - u_n\|\|u_n - p\| + c_n.
\end{aligned}$$

Using $\epsilon_n \longrightarrow 0$, $c_n \longrightarrow 0$ as $n \longrightarrow \infty$, (3.50) and (3.58), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0 \tag{3.59}$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|\frac{x_n-u_n}{r_n}\| = \lim_{n\longrightarrow\infty}\frac{1}{r_n}\|x_n-u_n\| = 0. \tag{3.60}$$

Observe that

$$\begin{aligned}\|W_nu_n-u_n\| &\leq \|W_nu_n-W_nk_n\|+\|W_nk_n-x_n\|+\|x_n-u_n\|\\ &\leq \|u_n-k_n\|+\|W_nk_n-x_n\|+\|x_n-u_n\|.\end{aligned}$$

Applying (3.52), (3.58) and (3.59) to the last inequality, we obtain

$$\lim_{n\longrightarrow\infty}\|W_nu_n-u_n\| = 0. \tag{3.61}$$

Let W be the mapping defined by (3.35). Since $\{u_n\}$ is bounded, applying Lemma 3.14 and (3.61), we have

$$\|Wu_n-u_n\| \leq \|Wu_n-W_nu_n\|+\|W_nu_n-u_n\| \longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.62}$$

Step 4. We claim that $\limsup_{n\longrightarrow\infty}\langle(A-\gamma f)z, z-x_n\rangle \leq 0$, where z is the unique solution of the variational inequality $\langle(A-\gamma f)z, x-z\rangle \geq 0,\ \ \forall x\in\Theta$.

Since $z = P_\Theta(I-A+\gamma f)(z)$ is a unique solution of the variational inequality (3.37). To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\longrightarrow\infty}\langle(A-\gamma f)z, z-u_{n_i}\rangle = \limsup_{n\longrightarrow\infty}\langle(A-\gamma f)z, z-u_n\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w\in C$. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup w$. From $\|Wu_n-u_n\|\longrightarrow 0$, we obtain $Wu_{n_i}\rightharpoonup w$. Next, we show that $w\in\Theta$, where $\Theta := \cap_{n=1}^{\infty}F(T_n)\cap EP(F)\cap VI(C,B)$. First, we show that $w\in EP(F)$. It follow form Theorem 3.5, we obtain $w\in EP(F)$. Next, we show that $w\in\cap_{n=1}^{\infty}F(T_n)$. By Lemma 3.13, we have $F(W) = \cap_{n=1}^{\infty}F(T_n)$. Assume $w\notin F(W)$. Since $u_{n_i}\rightharpoonup w$ and $w\neq Ww$, it follows by the Opial's condition (Lemma 2.58) then $w\in F(W) = \cap_{n=1}^{\infty}F(T_n)$. By the same argument as that in the proof of [43, Theorem 2.1, p. 10–11], we can show that $w\in VI(C,B)$. Hence, $w\in\Theta$. Since $z = P_\Theta(I-A+\gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle(A-\gamma f)z, z-x_n\rangle &= \limsup_{n\longrightarrow\infty}\langle(A-\gamma f)z, z-u_n\rangle\\ &= \lim_{i\longrightarrow\infty}\langle(A-\gamma f)z, z-u_{n_i}\rangle\\ &= \langle(A-\gamma f)z, z-w\rangle \leq 0.\end{aligned} \tag{3.63}$$

It follows from the last inequality, (3.52) and (3.59) that

$$\limsup_{n\longrightarrow\infty}\langle\gamma f(z)-Az, W_nk_n-z\rangle \leq 0. \tag{3.64}$$

Step 5. Finally, we show that $\{x_n\}$ converges strongly to $z = P_\Theta(I-A+\gamma f)(z)$.

Indeed, from (3.36) , we have

$$
\begin{aligned}
& \|x_{n+1}-z\|^2 \\
= \; & \big\|\epsilon_n\gamma f(x_n)+\beta_n x_n+\big((1-\beta_n)I-\epsilon_n A\big)W_nk_n-z\big\|^2 \\
= \; & \|\big((1-\beta_n)I-\epsilon_n A\big)(W_nk_n-z)+\beta_n(x_n-z)+\epsilon_n\big(\gamma f(x_n)-Az\big)\|^2 \\
= \; & \|\big((1-\beta_n)I-\epsilon_n A\big)(W_nk_n-z)+\beta_n(x_n-z)\|^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(x_n)-Az\big\rangle \\
& +2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_n A\big)(W_nk_n-z),\gamma f(x_n)-Az\Big\rangle \\
\leq \; & \Big[(1-\beta_n-\epsilon_n\bar{\gamma})\|W_nk_n-z\|+\beta_n\|x_n-z\|\Big]^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\gamma\big\langle x_n-z,f(x_n)-f(z)\big\rangle+2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle \\
& +2(1-\beta_n)\gamma\epsilon_n\big\langle W_nk_n-z,f(x_n)-f(z)\big\rangle+2(1-\beta_n)\epsilon_n\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle \\
& -2\epsilon_n^2\big\langle A(W_nk_n-z),\gamma f(z)-Az\big\rangle, \\
\leq \; & \Big[(1-\beta_n-\epsilon_n\bar{\gamma})\|W_nk_n-z\|+\beta_n\|x_n-z\|\Big]^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\gamma\|x_n-z\|\|f(x_n)-f(z)\|+2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle \\
& +2(1-\beta_n)\gamma\epsilon_n\|W_nk_n-z\|\|f(x_n)-f(z)\|+2(1-\beta_n)\epsilon_n\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle \\
& -2\epsilon_n^2\big\langle A(W_nk_n-z),\gamma f(z)-Az\big\rangle, \\
\leq \; & \Big[(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\Big]^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\gamma\alpha\|x_n-z\|^2+2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle \\
& +2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n-z\|^2+2(1-\beta_n)\epsilon_n\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle \\
& -2\epsilon_n^2\big\langle A(W_nk_n-z),\gamma f(z)-Az\big\rangle
\end{aligned}
\tag{3.65}
$$

$$
\begin{aligned}
= \; & \Big[(1-\epsilon_n\bar{\gamma})^2+2\beta_n\epsilon_n\gamma\alpha+2(1-\beta_n)\gamma\epsilon_n\alpha\Big]\|x_n-z\|^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle+2(1-\beta_n)\epsilon_n\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle \\
& -2\epsilon_n^2\big\langle A(W_nk_n-z),\gamma f(z)-Az\big\rangle \\
\leq \; & \big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\big]\|x_n-z\|^2+\bar{\gamma}^2\epsilon_n^2\|x_n-z\|^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle+2(1-\beta_n)\epsilon_n\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle \\
& +2\epsilon_n^2\|A(W_nk_n-z)\|\|\gamma f(z)-Az\| \\
= \; & \big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\big]\|x_n-z\|^2+\epsilon_n\Big\{\epsilon_n\Big[\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2 \\
& +2\|A(W_nk_n-z)\|\|\gamma f(z)-Az\|\Big]+2\beta_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle \\
& +2(1-\beta_n)\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle\Big\}
\end{aligned}
$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{W_nk_n\}$ are bounded, we can take a constant $M>0$ such that

$$\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2+2\|A(W_nk_n-z)\|\|\gamma f(z)-Az\|\leq M,$$

for all $n\geq 0$. It then follows that

$$\|x_{n+1}-z\|^2\leq\big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\big]\|x_n-z\|^2+\epsilon_n\sigma_n, \tag{3.66}$$

where

$$\sigma_n=2\beta_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle+2(1-\beta_n)\big\langle W_nk_n-z,\gamma f(z)-Az\big\rangle+\epsilon_nM.$$

Using (i), (3.63) and (3.64), we get $\limsup_{n\longrightarrow\infty}\sigma_n \leq 0$. Applying Lemma 2.62 to (3.66), we conclude that $x_n \longrightarrow z$ in norm. Finally, noticing

$$\|u_n - z\| = \|T_{r_n}x_n - T_{r_n}z\| \leq \|x_n - z\|.$$

We also conclude that $u_n \longrightarrow z$ in norm. This completes the proof. □

Corollary 3.16. *Let C be nonempty closed convex subset of a real Hilbert space H, let $\{T_n\}$ be an infinitely many nonexpansive of C into itself and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta := \cap_{n=1}^{\infty}F(T_n)\cap VI(C,B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$, and $\{k_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ y_n = P_C(x_n - \lambda_n B x_n), \\ k_n = \alpha_n x_n + (1-\alpha_n)P_C(x_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n k_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequences generated by (3.34) and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are three sequences in $(0,1)$ satisfy the following conditions:

(i) $\lim_{n\longrightarrow\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$,

(ii) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty}\beta_n \leq \limsup_{n\longrightarrow\infty}\beta_n < 1$,

(iv) $\{\frac{\lambda_n}{\beta}\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ converges strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

Proof. Put $F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.15. Then, we have $u_n = P_C x_n = x_n$. So, by Theorem 3.15, we can conclude the desired conclusion easily. □

3.1.4 A Finite Family of Nonexpansive Mappings

In this section, we show a strong convergence theorem of an iterative algorithm based on shrinking relaxed extragradient method which solves the problem of finding a common element of the set of fixed point of a finite family of nonexpansive mappings, the set of solution of a generalized equilibrium problems and the set of solution of a variational inequality problem for a inverse-strongly monotone mapping in a real Hilbert space.

Definition 3.17. [26] *Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and sequence $\{\lambda_{n,i}\}_{i=1}^N$ in [0,1], define the mapping $K_n : C \longrightarrow C$ as follows:*

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})U_{n,1}, \\ U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1-\lambda_{n,3})U_{n,2}, \\ &. \\ &. \\ &. \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})U_{n,N-2}, \\ K_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}, \end{aligned} \tag{3.67}$$

Definition 3.18. [26] *Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself, and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N$. Define a mapping $K : C \to C$ as follows:*

$$\begin{aligned} U_1 &= \lambda_1T_1 + (1-\lambda_1)I, \\ U_2 &= \lambda_2T_2U_1 + (1-\lambda_2)U_1, \\ U_3 &= \lambda_3T_3U_2 + (1-\lambda_3)U_2, \\ &. \\ &. \\ &. \\ U_{N-1} &= \lambda_{N-1}T_{N-1}U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\ K &= U_N = \lambda_NT_NU_{N-1} + (1-\lambda_N)U_{N-1}. \end{aligned} \tag{3.68}$$

Such a mapping K is called the *K-mapping* generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$.

Lemma 3.19. [26] *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then, $F(K) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 3.20. [26] *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in [0,1] such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1, 2, ..., N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$, and $T_1, T_2, ..., T_N$ and $\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,N}$ respectively. Then, for every $x \in C$, we have* $\lim_{n\to\infty} \|K_nx - Kx\| = 0$.

Remark 3.21. *Replacing x with $x - rAx \in H$ in (3.1), then there exists $z \in C$, such that*

$$F(z, y) + \langle Ax, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Theorem 3.22. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself, let A be an β-inverse-strongly monotone mapping of C into H and let B be a ξ-inverse-strongly monotone mapping of C into H such that $\Omega := \cap_{i=1}^N F(T_i) \cap GEP(F, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$*

and $\{u_n\}$ *be sequences generated by* $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ *and let*

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \delta_n Bu_n), \\ v_n = \epsilon_n x_n + (1-\epsilon_n)P_C(y_n - \lambda_n By_n), \\ z_n = \alpha_n x_n + (1-\alpha_n)K_n v_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.69}$$

where $\{K_n\}$ *is the sequence generated by (3.67)and* $\{\alpha_n\} \subset (0,1)$ *satisfy the following conditions:*

(i) $\{\epsilon_n\} \subset [0, e]$ *for some* e *with* $0 \leq e < 1$ *and* $\lim_{n\longrightarrow\infty} \alpha_n = 0$,

(ii) $\{\delta_n\}$, $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < 2\xi$,

(iii) $\{r_n\} \subset [c, d]$ *for some* c, d *with* $0 < c < d < 2\beta$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{\cap_{i=1}^N F(T_i)\cap GEP(F,A)\cap VI(C,B)}x_0$.

Proof. In the light of the definition of the resolvent, u_n can be rewritten as $u_n = T_{r_n}(x_n - r_nAx_n)$. Let $p \in \Omega := \cap_{i=1}^N F(T_i) \cap GEP(F, A) \cap VI(C, B)$ and using the fact $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 3.2, A be an β-inverse-strongly monotone and that $p = T_{r_n}(p - r_nAp)$, where $\{r_n\} \subset [c, d]$ for some c, d with $0 < c < d < 2\beta$, we can write

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\ &\leq \|(x_n - r_nAx_n) - (p - r_nAp)\|^2 \\ &= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\ &= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\beta\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.70}$$

Next, we will divide the proof into six steps.

Step 1. We show that $\{x_n\}$ is well-defined and C_n is closed and convex for any $n \in \mathbb{N}$.

From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. Next, we show that C_{k+1} is closed and convex for some k. for any $p \in C_k$, we obtain

$$\|z_k - p\| \leq \|x_k - p\|$$

is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - p\rangle \leq 0. \tag{3.71}$$

Thus, C_{k+1} is closed and convex. Then, C_n is closed and Convex for any $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined.

Step 2. We show that $\Omega \subset C_n$ for each $n \geq 1$.

From the assumption, we see that $\Omega \subset C = C_1$. Suppose $\Omega \subset C_k$ for some $k \geq 1$. For any $p \in \Omega \subset C_k$, we have

$$\begin{aligned}\|y_k - p\| &= \|P_C(u_k - \delta_k B u_k) - P_C(p - \delta_k B p)\| \\ &\le \|(u_k - \delta_k B u_k) - (p - \delta_k B p)\| \\ &\le \|(I - \delta_k B)u_k - (I - \delta_k B)p\| \\ &\le \|u_k - p\| \le \|x_k - p\|\end{aligned}$$

and

$$\begin{aligned}\|v_k - p\| &= \|\epsilon_k(x_k - p) + (1-\epsilon_k)(P_C(y_k - \lambda_k B y_k) - p)\| \\ &\le \epsilon_k\|x_k - p\| + (1-\epsilon_k)\|y_k - p\| \\ &\le \epsilon_k\|x_k - p\| + (1-\epsilon_k)\|x_k - p\| = \|x_k - p\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|z_k - p\| &= \|\alpha_k(x_k - p) + (1-\alpha_k)(K_k v_k - p)\| \\ &\le \alpha_k\|x_k - p\| + (1-\alpha_k)\|v_k - p\| \\ &\le \alpha_k\|x_k - p\| + (1-\alpha_k)\|x_k - p\| = \|x_k - p\|.\end{aligned}$$

It follows that $p \in C_{k+1}$. This implies that $\Omega \subset C_n$ for each $n \ge 1$.
Step 3. We show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$.
From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0$$

for each $y \in C_n$. Using $\Omega \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - p\rangle \ge 0, \quad \forall p \in \Omega \quad \text{and} \quad n \in \mathbb{N}.$$

So, for $p \in \Omega$, we have

$$\begin{aligned}0 &\le \langle x_0 - x_n, x_n - p\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|.\end{aligned}$$

This implies that

$$\|x_0 - x_n\| \le \|x_0 - p\|, \quad \forall p \in \Omega \quad \text{and} \quad n \in \mathbb{N}.$$

From $x_n = P_{C_n}x_0$, and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0. \tag{3.72}$$

From (3.72), we have, for $n \in \mathbb{N}$,

$$\begin{aligned}0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.\end{aligned}$$

It follows that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|.$$

Thus the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence,
so $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists, and then there exists m such that

$$\lim_{n\longrightarrow\infty}\|x_n - x_0\| = m. \tag{3.73}$$

Indeed, from (3.72) we get

$$\begin{aligned}
& \|x_n - x_{n+1}\|^2 \\
= \; & \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
= \; & \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
= \; & \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
= \; & \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
= \; & -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
\le \; & -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

From (3.73), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - x_{n+1}\| = 0. \tag{3.74}$$

Since $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_n - x_{n+1}\|.$$

By (3.74), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0. \tag{3.75}$$

Step 4. We show that $\lim_{n\longrightarrow\infty} \|K_n\phi_n - \phi_n\| = 0$, where $\phi_n = P_C(y_n - \lambda_n By_n)$.

Since B is a ξ-inverse-strongly monotone by the assumptions imposed on $\{\lambda_n\}$ for any $p \in \Omega$, we have

$$\begin{aligned}
\|\phi_n - p\|^2 &= \|P_C(y_n - \lambda_n By_n) - P_C(p - \lambda_n Bp)\|^2 \\
&\le \|(y_n - \lambda_n By_n) - (p - \lambda_n Bp)\|^2 \\
&= \|(y_n - p) - \lambda_n(By_n - Bp)\|^2 \\
&\le \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, By_n - Bp\rangle + \lambda_n^2\|By_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - 2\lambda_n\langle y_n - p, By_n - Bp\rangle + \lambda_n^2\|By_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2.
\end{aligned}$$

Similarly, we can prove

$$\|y_n - p\|^2 \le \|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2. \tag{3.76}$$

Observe that

$$\begin{aligned}
\|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(K_n v_n - p)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|K_n v_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|v_n - p\|^2 \qquad (3.77)
\end{aligned}$$

and

$$\begin{aligned}
&\|v_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|\phi_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\Big\{\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2\Big\} \\
=\ & \|x_n - p\|^2 + (1-\epsilon_n)\lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2. \qquad (3.78)
\end{aligned}$$

Substituting (3.78) into (3.77), and using condition (i) and (ii), we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
\leq\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
\leq\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 + (1-\epsilon_n)\lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2\Big\} \\
=\ & \|x_n - p\|^2 + (1-\alpha_n)(1-\epsilon_n)\lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n)(1-e)a(2\xi - b)\|By_n - Bp\|^2 \\
\leq\ & (1-\alpha_n)(1-\epsilon_n)\lambda_n(2\xi - \lambda_n)\|By_n - Bp\|^2 \\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 \\
=\ & (\|x_n - p\| - \|z_n - p\|)(\|x_n - p\| + \|z_n - p\|) \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|By_n - Bp\| = 0. \qquad (3.79)$$

By Lemma 2.19 (iii), we also have

$$\begin{aligned}
\|\phi_n - p\|^2 &= \|P_C(y_n - \lambda_n By_n) - P_C(p - \lambda_n Bp)\|^2 \\
&= \|P_C(I - \lambda_n B)y_n - P_C(I - \lambda_n B)p\|^2 \\
&\leq \Big\langle (I - \lambda_n B)y_n - (I - \lambda_n B)p, \phi_n - p\Big\rangle \\
&= \frac{1}{2}\Big\{\|(I - \lambda_n B)y_n - (I - \lambda_n B)p\|^2 + \|\phi_n - p\|^2 \\
&\quad - \|(I - \lambda_n B)y_n - (I - \lambda_n B)p - (\phi_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|y_n - p\|^2 + \|\phi_n - p\|^2 - \|(y_n - \phi_n) - \lambda_n(By_n - Bp)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|\phi_n - p\|^2 - \|y_n - \phi_n\|^2 \\
&\quad - \lambda_n^2\|By_n - Bp\|^2 + 2\lambda_n\langle y_n - \phi_n, By_n - Bp\rangle\Big\},
\end{aligned}$$

which yields that

$$\|\phi_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - \phi_n\|^2 + 2\lambda_n\|y_n - \phi_n\|\|By_n - Bp\|. \qquad (3.80)$$

Similarly, we have

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta_n\|u_n - y_n\|\|Bu_n - Bp\|. \qquad (3.81)$$

Using (3.78) again and (3.80), we have

$$\begin{aligned}
& \|v_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|\phi_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\Big\{\|x_n - p\|^2 - \|y_n - \phi_n\|^2 + 2\lambda_n\|y_n - \phi_n\|\|By_n - Bp\|\Big\} \\
=\ & \|x_n - p\|^2 - (1-\epsilon_n)\|y_n - \phi_n\|^2 + 2(1-\epsilon_n)\lambda_n\|y_n - \phi_n\|\|By_n - Bp\|. \qquad (3.82)
\end{aligned}$$

Substituting (3.82) into (3.77), and using condition (i), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - (1-\epsilon_n)\|y_n - \phi_n\|^2 \\
&\quad + 2(1-\epsilon_n)\lambda_n\|y_n - \phi_n\|\|By_n - Bp\|\Big\} \\
&= \|x_n - p\|^2 - (1-\alpha_n)(1-\epsilon_n)\|y_n - \phi_n\|^2 \\
&\quad + 2(1-\alpha_n)(1-\epsilon_n)\lambda_n\|y_n - \phi_n\|\|By_n - Bp\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
& (1-\alpha_n)(1-e)\|y_n - \phi_n\|^2 \qquad (3.83) \\
\leq\ & (1-\alpha_n)(1-\epsilon_n)\|y_n - \phi_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)(1-\epsilon_n)\lambda_n\|y_n - \phi_n\|\|By_n - Bp\| \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)(1-\epsilon_n)\lambda_n\|y_n - \phi_n\|\|By_n - Bp\|.
\end{aligned}$$

Applying $\|x_n - z_n\| \longrightarrow 0$ and $\|By_n - Bp\| \longrightarrow 0$ as $n \longrightarrow \infty$ to the last inequality, we get

$$\lim_{n\longrightarrow\infty}\|y_n - \phi_n\| = 0. \qquad (3.84)$$

Note that

$$\begin{aligned}
\|v_n - p\|^2 &\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|\phi_n - p\|^2 \\
&\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|y_n - p\|^2. \qquad (3.85)
\end{aligned}$$

Substituting (3.76) into (3.85), we have

$$\begin{aligned}
\|v_n - p\|^2 &\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|y_n - p\|^2 \\
&\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\Big\{\|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2\Big\} \\
&= \|x_n - p\|^2 + (1-\epsilon_n)\delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2. \qquad (3.86)
\end{aligned}$$

Substituting (3.86) into (3.77), and using condition (i) and (ii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 \\
&\quad + (1-\epsilon_n)\delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2\Big\} \\
&= \|x_n - p\|^2 + (1-\alpha_n)(1-\epsilon_n)\delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)(1-e)a(2\xi - b)\|Bu_n - Bp\|^2 &\leq (1-\alpha_n)(1-\epsilon_n)\delta_n(2\xi - \delta_n)\|Bu_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|Bu_n - Bp\| = 0. \tag{3.87}$$

Using (3.85) again and (3.81), we have

$$\begin{aligned}
& \|v_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|y_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\Big\{\|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta_n\|u_n - y_n\|\|Bu_n - Bp\|\Big\} \\
=\ & \|x_n - p\|^2 - (1-\epsilon_n)\|u_n - y_n\|^2 + 2(1-\epsilon_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.
\end{aligned} \tag{3.88}$$

Substituting (3.88) into (3.77), and using condition (i) and (ii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - (1-\epsilon_n)\|u_n - y_n\|^2 \\
&\quad + 2(1-\epsilon_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|\Big\} \\
&= \|x_n - p\|^2 - (1-\alpha_n)(1-\epsilon_n)\|u_n - y_n\|^2 \\
&\quad + 2(1-\alpha_n)(1-\epsilon_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
& (1-\alpha_n)(1-e)\|u_n - y_n\|^2 \\
\leq\ & (1-\alpha_n)(1-\epsilon_n)\|u_n - y_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)(1-\epsilon_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\| \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)(1-\epsilon_n)\delta_n\|u_n - y_n\|\|Bu_n - Bp\|.
\end{aligned}$$

Applying $\|x_n - z_n\| \longrightarrow 0$ and $\|Bu_n - Bp\| \longrightarrow 0$ as $n \longrightarrow \infty$ to the last inequality, we get

$$\lim_{n\longrightarrow\infty}\|u_n - y_n\| = 0. \tag{3.89}$$

From (3.84) and (3.89), we have

$$\lim_{n\longrightarrow\infty}\|u_n - \phi_n\| = 0. \tag{3.90}$$

Using (3.85) again and (3.70), we have

$$\begin{aligned}
\|v_n - p\|^2 &\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|y_n - p\|^2 \\
&\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\|u_n - p\|^2 \\
&\leq \epsilon_n\|x_n - p\|^2 + (1-\epsilon_n)\Big\{\|x_n - p\|^2 + r_n(r_n - 2\beta)\|Ax_n - Ap\|^2\Big\} \\
&= \|x_n - p\|^2 + (1-\epsilon_n)r_n(r_n - 2\beta)\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.91}$$

Substituting (3.91) into (3.77), and using condition (i) and (ii), we have

$$\begin{aligned}
& \|z_n - p\|^2 \\
\leq\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
\leq\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 + (1-\epsilon_n)r_n(r_n - 2\beta)\|Ax_n - Ap\|^2\Big\} \\
=\ & \|x_n - p\|^2 + (1-\alpha_n)(1-\epsilon_n)r_n(r_n - 2\beta)\|Ax_n - Ap\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)(1-e)c(2\beta - d)\|Ax_n - Ap\|^2 &\leq (1-\alpha_n)(1-\epsilon_n)r_n(\beta - r_n)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|Ax_n - Ap\| = 0. \tag{3.92}$$

On the other hand, in the light of Lemma 3.2 T_{r_n} is firmly nonexpansive, so we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\leq \langle T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp), u_n - p\rangle \\
&= \langle x_n - r_nAx_n - (p - r_nAp), u_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - r_nAx_n) - (p - r_nAp)\|^2 + \|u_n - p\|^2 \\
&\quad -\|(x_n - r_nAx_n) - (p - r_nAp) - (u_n - p)\|^2\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad +2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2\Big\}.
\end{aligned}$$

So, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|. \tag{3.93}$$

Using (3.85) again and (3.93), we have

$$\begin{aligned}
&\|v_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1 - \epsilon_n)\|y_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1 - \epsilon_n)\|u_n - p\|^2 \\
\leq\ & \epsilon_n\|x_n - p\|^2 + (1 - \epsilon_n)\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|\Big\} \\
=\ & \|x_n - p\|^2 - (1 - \epsilon_n)\|x_n - u_n\|^2 + 2(1 - \epsilon_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned} \tag{3.94}$$

Therefore, from (3.77) and (3.94), we arrive at

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big\{\|x_n - p\|^2 - (1 - \epsilon_n)\|x_n - u_n\|^2 \\
&\quad + 2(1 - \epsilon_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|\Big\} \\
&= \|x_n - p\|^2 - (1 - \alpha_n)(1 - \epsilon_n)\|x_n - u_n\|^2 \\
&\quad + 2(1 - \alpha_n)(1 - \epsilon_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned}$$

It follows that and condition (i), we have

$$\begin{aligned}
&(1 - \alpha_n)(1 - e)\|x_n - u_n\|^2 \\
\leq\ & (1 - \alpha_n)(1 - \epsilon_n)\|x_n - u_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1 - \alpha_n)(1 - \epsilon_n)r_n\|x_n - u_n\|\|Ax_n - Ap\| \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1 - \alpha_n)(1 - \epsilon_n)r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$, $\lim_{n\longrightarrow\infty}\|Ax_n - Ap\| = 0$ implies that

$$\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0. \tag{3.95}$$

From (3.89) and (3.95), we have

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0. \tag{3.96}$$

From (3.84) and (3.96), we have

$$\lim_{n\longrightarrow\infty}\|x_n - \phi_n\| = 0. \tag{3.97}$$

By (3.69), we have

$$z_n - K_n v_n = \alpha_n(x_n - K_n v_n)$$

and

$$v_n - \phi_n = \epsilon_n(x_n - \phi_n).$$

Since $\alpha_n \longrightarrow 0$ and $\|x_n - \phi_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we also have

$$\lim_{n\longrightarrow\infty}\|z_n - K_n v_n\| = 0 \tag{3.98}$$

and

$$\lim_{n\longrightarrow\infty}\|v_n - \phi_n\| = 0. \tag{3.99}$$

From (3.84) and (3.99), we have

$$\lim_{n\longrightarrow\infty}\|v_n - y_n\| = 0. \tag{3.100}$$

On the other hand, we observe that

$$\begin{aligned}\|x_n - K_n\phi_n\| &\leq \|x_n - z_n\| + \|z_n - K_n v_n\| + \|K_n v_n - K_n\phi_n\| \\ &\leq \|x_n - z_n\| + \|z_n - K_n v_n\| + \|v_n - \phi_n\|.\end{aligned}$$

Applying (3.75), (3.98) and (3.99), we have

$$\lim_{n\longrightarrow\infty}\|x_n - K_n\phi_n\| = 0. \tag{3.101}$$

Furthermore, by the triangular inequality we also have

$$\|K_n\phi_n - \phi_n\| \leq \|K_n\phi_n - x_n\| + \|x_n - u_n\| + \|u_n - \phi_n\|.$$

Applying (3.90), (3.95) and (3.101), we obtain

$$\lim_{n\longrightarrow\infty}\|K_n\phi_n - \phi_n\| = 0. \tag{3.102}$$

Let K be the mapping defined by (3.68). Since $\{\phi_n\}$ is bounded, applying Lemma 3.20 and (3.102), we have

$$\|K\phi_n - \phi_n\| \leq \|K\phi_n - K_n\phi_n\| + \|K_n\phi_n - \phi_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Step 5. We show that there exists a subsequence $\{\phi_{n_i}\}$ of $\{\phi_n\}$ which converges weakly to z, where $z \in \cap_{i=1}^N F(T_i) \cap GEP(F,A) \cap VI(C,B)$.

Since $\{\phi_n\}$ is bounded, there exists a subsequence $\{\phi_{n_i}\}$ of $\{\phi_n\}$ which converges weakly to z. Without loss of generality, we can assume that $\phi_{n_i} \rightharpoonup z$. Since $\phi_{n_i} \subset C$ and C is closed and convex, C is weakly closed and hence $z \in C$. From $\|K\phi_n - \phi_n\| \longrightarrow 0$, we obtain $K\phi_{n_i} \rightharpoonup z$.

Next, we show that $z \in \cap_{i=1}^N F(T_i) \cap GEP(F,A) \cap VI(C,B)$.

(a) First, we prove that $z \in VI(C,B)$.

In fact, let T be the maximal monotone mapping defined:

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Let $(v,u)\in G(T)$. Since $u-Bv\in N_Cv$ and $\phi_n\in C$, we have

$$\langle v-\phi_n, u-Bv\rangle \geq 0.$$

On the other hand, from $\phi_n = P_C(y_n-\lambda_n By_n)$, we have $\langle v-\phi_n, \phi_n-(y_n-\lambda_n By_n)\rangle \geq 0$, and hence, $\left\langle v-\phi_n, \frac{\phi_n-y_n}{\lambda_n}+By_n\right\rangle \geq 0$. Therefore, we have

$$\begin{aligned}
\langle v-\phi_{n_i}, u\rangle &\geq \langle v-\phi_{n_i}, Bv\rangle \\
&\geq \langle v-\phi_{n_i}, Bv\rangle - \left\langle v-\phi_{n_i}, \frac{\phi_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle \\
&= \left\langle v-\phi_{n_i}, Bv-By_{n_i}-\frac{\phi_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle v-\phi_{n_i}, Bv-B\phi_{n_i}\rangle + \langle v-\phi_{n_i}, B\phi_{n_i}-By_{n_i}\rangle \\
&\quad -\left\langle v-\phi_{n_i}, \frac{\phi_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v-\phi_{n_i}, B\phi_{n_i}-By_{n_i}\rangle - \left\langle v-\phi_{n_i}, \frac{\phi_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned}$$

Since $\lim_{i\longrightarrow\infty}\|\phi_{n_i}-y_{n_i}\|=0$, $\phi_{n_i}\rightharpoonup z$ and B is Lipschitz continuous, we obtain that $\lim_{i\longrightarrow\infty}\|B\phi_{n_i}-By_{n_i}\|=0$ and $\phi_{n_i}\rightharpoonup z$. From $\liminf_{n\longrightarrow\infty}\lambda_n>0$, we obtain

$$\lim_{i\longrightarrow\infty}\langle v-\phi_{n_i}, u\rangle = \langle v-z, u\rangle \geq 0.$$

Since T is maximal monotone, we have $z\in T^{-1}0$ and hence $z\in VI(C,B)$.

(b) Next, we prove that $z\in GEP(F,A)$.

Since $u_n=T_{r_n}(x_n-r_nAx_n)$ for any $y\in C$, we can write

$$F(u_n,y)+\langle Ax_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in C.$$

From (A2), we also have

$$\left\langle Ax_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n, u_n-x_n\right\rangle \geq -F(u_n,y)\geq F(y,u_n).$$

Replacing n by n_i, we have

$$\langle Ax_{n_i}, y-u_{n_i}\rangle+\left\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle \geq F(y,u_{n_i}). \tag{3.103}$$

For any t with $0<t\leq 1$ and $y\in C$, let $\varphi_t=ty+(1-t)z$. Since $y\in C$ and $z\in C$, we have $\varphi_t\in C$. So, from (3.103) we have

$$\begin{aligned}
\langle \varphi_t-u_{n_i}, A\varphi_t\rangle &\geq \langle \varphi_t-u_{n_i}, A\varphi_t\rangle - \langle Ax_{n_i}, \varphi_t-u_{n_i}\rangle \\
&\quad -\left\langle \varphi_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle + F(\varphi_t,u_{n_i}) \\
&\geq \langle \varphi_t-u_{n_i}, A\varphi_t-Au_{n_i}\rangle + \langle \varphi_t-u_{n_i}, Au_{n_i}-Ax_{n_i}\rangle \\
&\quad -\left\langle \varphi_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle + F(\varphi_t,u_{n_i}).
\end{aligned} \tag{3.104}$$

Since A is Lipschitz continuous, from (3.95), we have $\|Au_{n_i}-Ax_{n_i}\|\longrightarrow 0$ as $i\longrightarrow\infty$. Further, from the monotonicity of A, we get that

$$\langle \varphi_t-u_{n_i}, A\varphi_t-Au_{n_i}\rangle \geq 0.$$

It follows from (A4) and (3.104) that

$$\langle \varphi_t - z, A\varphi_t \rangle \geq F(\varphi_t, z). \tag{3.105}$$

From (A1), (A4) and (3.105), we also have

$$\begin{aligned} 0 = F(\varphi_t, \varphi_t) &\leq tF(\varphi_t, y) + (1-t)F(\varphi_t, z) \\ &\leq tF(\varphi_t, y) + (1-t)\langle \varphi_t - z, A\varphi_t \rangle \\ &= tF(\varphi_t, y) + (1-t)t\langle y - z, A\varphi_t \rangle \end{aligned}$$

and hence

$$F(\varphi_t, y) + (1-t)\langle y - z, A\varphi_t \rangle \geq 0.$$

Letting $t \longrightarrow \infty$ in the above inequality, we have, for each $y \in C$,

$$F(z, y) + \langle y - z, Az \rangle \geq 0.$$

Thus, $z \in GEP(F, A)$.

(c) Now, we prove that $z \in F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Assume $z \notin F(K)$. Since $\|x_n - \phi_n\| \longrightarrow 0$ we know that $\phi_{n_i} \rightharpoonup z$ $(i \longrightarrow \infty)$ and $z \neq Kz$, it follows from the Opial's condition (Lemma 2.58), we get $z \in F(K) = \bigcap_{i=1}^{N} F(T_i)$. The conclusion is $z \in \cap_{i=1}^{N} F(T_i) \cap GEP(F, A) \cap VI(C, B)$.

Step 6. Finally, we show that $x_n \longrightarrow z$ and $u_n \longrightarrow z$, where $z = P_\Omega x_0$. Since Ω is nonempty closed convex subset of H, there exists a unique $z' \in \Omega$ such that $z' = P_\Omega x_0$. Since $z' \in \Omega \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_0 - x_n\| = \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z'\| \tag{3.106}$$

for all $n \in \mathbb{N}$. From (3.106), $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$. By the weak lower semi-continuity of the norm, we have

$$\|x_0 - z\| \leq \liminf_{i \longrightarrow \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z'\|. \tag{3.107}$$

However, Since $z \in \omega_w(x_n) \subset \Omega$, we have

$$\|x_0 - z'\| = \|x_0 - P_\Omega x_0\| \leq \|x_0 - z\|.$$

Using (3.106) and (3.107), we obtain $z' = z$. Thus, $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So, we have

$$\|x_0 - z'\| \leq \|x_0 - z\| \leq \liminf_{n \longrightarrow \infty} \|x_0 - x_n\| \leq \limsup_{n \longrightarrow \infty} \|x_0 - x_n\| \leq \|x_0 - z'\|.$$

Thus, we obtain that

$$\|x_0 - z\| = \lim_{n \longrightarrow \infty} \|x_0 - x_n\| = \|x_0 - z'\|.$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using the Kadec-Klee property (Lemma 2.59) of H, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

and hence $x_n \longrightarrow z$ in norm. Finally, noticing $\|u_n - z\| = \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(z - r_n Az)\| \leq \|x_n - z\|$. We also conclude that $u_n \longrightarrow z$ in norm. This completes the proof. $\square$

Corollary 3.23. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself, let A be an β-inverse-strongly monotone mapping of C into H and let B be a ξ-inverse-strongly monotone mapping of C into H such that $\Omega := \cap_{i=1}^N F(T_i) \cap VI(C,A) \cap VI(C,B) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$ and let*

$$\begin{cases} u_n = P_C(x_n - r_n A x_n), \\ y_n = P_C(u_n - \delta_n B u_n), \\ v_n = \epsilon_n x_n + (1-\epsilon_n) P_C(y_n - \lambda_n B y_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) K_n v_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\{K_n\}$ is the sequence generated by (3.67)and $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\{\epsilon_n\} \subset [0,e]$ *for some a with* $0 \leq e < 1$ *and* $\lim_{n\longrightarrow\infty} \alpha_n = 0$,

(ii) $\{\delta_n\}$, $\{\lambda_n\} \subset [a,b]$ *for some a, b with* $0 < a < b < 2\xi$,

(iii) $\{r_n\} \subset [c,d]$ *for some c, d with* $0 < c < d < 2\beta$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{\cap_{i=1}^N F(T_i) \cap VI(C,A) \cap VI(C,B)} x_0$.

Proof. Put $F(x,y) = 0$ for all $x, y \in C$ in Theorem 3.22. From

$$\langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0,$$

we have

$$\langle y - u_n, x_n - u_n - r_n A x_n \rangle \geq 0, \quad \forall y \in C.$$

This implies that

$$u_n = P_C(x_n - r_n A x_n).$$

From the proof of Theorem 3.22, we can obtain the desired conclusion easily. □

3.1.5 Systems of equilibrium problems

In this section, we deal with the strong convergence of extragradient approximation method for finding a common element of the set of solutions of the system equilibrium problems, the set of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of variational inequality for be a monotone and ζ-Lipschitz continuous mapping in a real Hilbert space.

Theorem 3.24. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1,2,3,\ldots,M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive mappings of C into itself and let B be a monotone and ζ-Lipschitz continuous mapping of C into H such that*

$$\Upsilon := \cap_{n=1}^\infty F(T_n) \cap \left(\cap_{k=1}^M SEP(F_k)\right) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \dots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \epsilon_n \gamma f(W_n x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big) W_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases} \tag{3.108}$$

where $\{W_n\}$ is the sequence generated by (3.34) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{\zeta})$ and $\{r_{k,n}\}, k \in \{1,2,3,\dots,M\}$ are a real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\liminf_{n\longrightarrow\infty} r_{k,n} > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1,2,3,\dots,M\}$,

(C4) $\lim_{n\to\infty} \lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Upsilon$ which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \quad \forall x \in \Upsilon. \tag{3.109}$$

Equivalently, we have $z = P_\Upsilon(I - A + \gamma f)(z)$.

Proof. See [18] □

Corollary 3.25. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1,2,3,\dots,M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be a monotone and ζ-Lipschitz continuous mapping of C into H such that*

$$\Upsilon := \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \dots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big) P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{\zeta})$ and $\{r_{k,n}\}, k \in \{1,2,3,\dots,M\}$ are real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\liminf_{n\longrightarrow\infty} r_{k,n} > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{k,n+1} - r_{k,n}| = 0$,

(C4) $\lim_{n\to\infty} \lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Upsilon$ which is the unique solution of the variational inequality

$$\Big\langle (A-\gamma f)z, x-z \Big\rangle \geq 0, \;\; \forall x \in \Upsilon.$$

Equivalently, we have $z = P_\Upsilon(I - A + \gamma f)(z)$.

Proof. Put $T_n = I$ for all $n \in \mathbb{N}$ and for all $x \in C$. Then $W_n = I$ for all $x \in C$. The conclusion follows from Theorem 3.24. This completes the proof. □

3.2 Equilibrium Problems in Banach spaces

Lemma 3.26. (*Blum and Oettli* [4]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \textit{ for all } y \in C.$$

Lemma 3.27. (*Takahashi and Zembayashi* [56]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \longrightarrow C$ as follows:*

$$T_r x = \{z \in C : \Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}, \tag{3.110}$$

for all $x \in E$. Then, the followings hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = EP(\Theta)$;*

(4) *$EP(\Theta)$ is closed and convex.*

Lemma 3.28. (*Takahashi and Zembayashi* [56]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Theorem 3.29. *Let E a uniformly convex and uniformly smooth real Banach space, let C be a nonempty and closed convex subset of a E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $T, S : C \to C$ are closed hemi-relatively nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C, \quad C_0 = C \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n), \\ u_n \in C \textit{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0) \end{cases} \tag{3.111}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\longrightarrow\infty} \alpha_n < 1, \lim_{n\longrightarrow\infty} \beta_n = 1, \liminf_{n\longrightarrow\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If S is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto $F := F(T) \cap F(S) \cap EP(\Theta)$.

***Proof*.** First, we show that C_{n+1} is closed and convex for each $n \geq 0$. From the definition of C_{n+1}, it is obvious that C_n is closed. Therefore,

$$\phi(z,u_n) \leq \phi(z,x_n) \Leftrightarrow 2(\langle z, Jx_n\rangle - 2\langle z, Ju_n\rangle) \leq \|x_n\|^2 - \|u_n\|^2.$$

It is easy to see that C_{n+1} is convex. Then, for all $n \geq 0$, C_n is closed and convex. This shows that $F \subset C_n$ for all $n \geq 0$. Let $p \in F$. Putting $u_n = T_{r_n} y_n$ for all $n \geq 0$. On the other hand, from Lemma 3.27, one has T_{r_n} is hemi-relatively nonexpansive mapping. Next, we prove $F \subset C_n$ for all $n \geq 0$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Then, for $\forall p \in F \subset C_k$, one has

$$\begin{aligned}
\phi(p,u_k) &= \phi(p, T_{r_k} y_k) \\
&\leq \phi(p, y_k) \\
&= \phi(p, J^{-1}(\alpha_k Jx_k + (1-\alpha_k)JSz_k)) \\
&= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1-\alpha_k)JSz_k\rangle \\
&\quad + \|\alpha_k Jx_k + (1-\alpha_k)JSz_k\|^2 \\
&\leq \|p\|^2 - 2\alpha_k\langle p, Jx_k\rangle - 2(1-\alpha_k)\langle p, JSz_k\rangle + \alpha_k\|Jx_k\|^2 \\
&\quad + (1-\alpha_k)\|JSz_k\|^2 \\
&= \alpha_k\phi(p,x_k) + (1-\alpha_k)\phi(p,Sz_k) \\
&\leq \alpha_k\phi(p,x_k) + (1-\alpha_k)\phi(p,z_k),
\end{aligned} \tag{3.112}$$

and then

$$\begin{aligned}
\phi(p,z_k) &= \phi(p, J^{-1}(\beta_k Jx_k + (1-\beta_k)JTx_k)) \\
&= \|p\|^2 - 2\langle p, \beta_k Jx_k + (1-\beta_k)JTx_k\rangle \\
&\quad + \|\beta_k Jx_k + (1-\beta_k)Tx_k\|^2 \\
&\leq \|p\|^2 - 2\beta_k\langle p, Jx_k\rangle - 2(1-\beta_k)\langle w, JTx_k\rangle + \beta_k\|x_k\|^2 \\
&\quad + (1-\beta_k)\|Tx_k\|^2 \\
&= \beta_k(\|p\|^2 - 2\langle p, Jx_k\rangle + \|x_k\|^2) + (1-\beta_k) \\
&\quad (\|p\|^2 - 2\langle p, JTx_k\rangle + \|Tx_k\|^2) \\
&= \beta_k\phi(p,x_k) + (1-\beta_k)\phi(p,Tx_k) \\
&\leq \beta_k\phi(p,x_k) + (1-\beta_k)\phi(p,x_k) \\
&= \phi(p,x_k).
\end{aligned} \tag{3.113}$$

Substituting (3.113) into (3.112), we have

$$\begin{aligned}
\phi(p,u_k) &\leq \alpha_k\phi(p,x_k) + (1-\alpha_k)\phi(p,x_k) \\
&\leq \phi(p,x_k),
\end{aligned} \tag{3.114}$$

that is $p \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \geq 0$. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n \tag{3.115}$$

and

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geq 0, \quad \forall p \in F. \tag{3.116}$$

From Lemma 2.34, one has

$$\phi(x_n,x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p,x_0) - \phi(p,x_n) \leq \phi(p,x_0)$$

for each $p \in F \subset C_n$ and $n \geq 1$. Then, the sequence $\{\phi(x_n, x_0)\}$ is bounded. Since $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.117}$$

Letting $m, n \longrightarrow \infty$ in (3.117), one has $\phi(x_m, x_n) \longrightarrow 0$. It follows from Lemma 2.32, that $x_m - x_n \longrightarrow 0$ as $m, n \longrightarrow \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \longrightarrow \tilde{x} \in C$ as $n \longrightarrow \infty$. Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0)$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0$. From Lemma 2.33, we get $\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we also have

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

Since $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = \lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0$ and E is uniformly convex and smooth. Previously, we know from Lemma 2.32 that

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \longrightarrow \infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0$, we have

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_n\| = 0.$$

Since E is uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach apace. Let $r = \sup_{n \in \mathbb{N} \cup \{0\}} \{\|x_n\|, \|Tx_n\|, \|Sz_n\|\}$. From Lemma 2.36, we have

$$\begin{aligned} \phi(p, z_n) &= \phi(p, J^{-1}(\beta_n J x_n + (1 - \beta_n) J T x_n)) \\ &= \|p\|^2 - 2 \langle p, \beta_n J x_n + (1 - \beta_n) J T x_n \rangle + \|\beta_n J x_n + (1 - \beta_n) J T x_n\|^2 \\ &\leq \|p\|^2 - 2\beta \langle p, J x_n \rangle - 2(1 - \beta_n) \langle p, J T x_n \rangle \\ &\quad + \beta_n \|J x_n\|^2 + (1 - \beta_n) \|J T x_n\|^2 - \beta_n (1 - \beta_n) g(\|J x_n - J T x_n\|) \\ &= \beta_n \phi(p, x_n) + (1 - \beta_n) \phi(p, T x_n) - \beta_n (1 - \beta_n) g(\|J x_n - J T x_n\|) \\ &= \phi(p, x_n) - \beta_n (1 - \beta_n) g(\|J x_n - J T x_n\|), \end{aligned} \tag{3.118}$$

and

$$\begin{aligned} \phi(p, u_n) &= \phi(p, T_{r_n} y_n) \leq \phi(p, y_n) \\ &\leq \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, z_n). \end{aligned} \tag{3.119}$$

Substituting (3.118) into (3.119), we have

$$\begin{aligned} \phi(p, u_n) &\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)(\phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|)) \\ &\leq \phi(p, x_n) - (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|). \end{aligned} \tag{3.120}$$

It follows that

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|) \leq \phi(p, x_n) - \phi(p, u_n).$$

On the other hand, we have

$$\begin{aligned} \phi(p, x_n) - \phi(p, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n \rangle \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\|. \end{aligned}$$

It follows from $\|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$ that

$$\phi(p, x_n) - \phi(p, u_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.121}$$

By assumption $\liminf_{n\longrightarrow\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$ and by Lemma 2.37, we also get

$$g(\|Jx_n - JTx_n\|) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

From the property of g,

$$\|Jx_n - JTx_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0.$$

Since T is a closed operator and $x_n \longrightarrow \tilde{x}$, the $\tilde{x}$ is a fixed point of T. Since $\lim_{n\longrightarrow\infty}\beta_n = 1$ and $\{x_n\}$ is bounded, we obtain

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n)) \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n Jx_n + (1-\beta_n)JT_n x_n \rangle \\ &\quad + \|\beta_n Jx_n + (1-\beta_n)JTx_n\|^2 \\ &\leq \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, Jx_n\rangle - 2(1-\beta_n)\langle x_{n+1}, JTx_n\rangle \\ &\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|Tx_n\|^2 \\ &= \beta_n\phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, Tx_n) \\ &\leq \phi(x_{n+1}, x_n). \end{aligned} \tag{3.122}$$

Since $\phi(x_{n+1}, x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, $\phi(x_{n+1}, z_n) \rightarrow 0$ as $n \rightarrow \infty$.
Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, from (3.112) and (3.113), we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$$

for all $n \geq 0$. Thus,

$$\phi(x_{n+1}, y_n) \rightarrow 0, \text{as} \quad n \rightarrow \infty.$$

By using Lemma 2.32, we also have

$$\lim_{n\rightarrow\infty} \|x_{n+1} - y_n\| = \lim_{n\rightarrow\infty} \|x_{n+1} - x_n\| = \lim_{n\rightarrow\infty} \|x_{n+1} - z_n\| = 0. \tag{3.123}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\rightarrow\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\rightarrow\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\rightarrow\infty} \|Jx_{n+1} - Jz_n\| = 0. \tag{3.124}$$

For each $n \in \mathbb{N} \cup \{0\}$, we observe that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JSz_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JSz_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JSz_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JSz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|. \end{aligned}$$

It follows that

$$\|Jx_{n+1} - JSz_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

By (3.124) and $\limsup_{n\longrightarrow\infty}\alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JSz_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - Sz_n\| = 0. \tag{3.125}$$

Since

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

By (3.123), we obtain

$$\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0. \tag{3.126}$$

By using the triangle inequality, we get

$$\|x_n - Sx_n\| \quad \leq \quad \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| + \|Sz_n - Sx_n\|.$$

Since, S is uniformly continuous, it follows from (3.123), (3.125) and (3.126) that $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0.$

Again using J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty} \|Jx_n - JSx_n\| = 0.$$

Since S is closed operator and $x_n \longrightarrow \tilde{x}$, $\tilde{x}$ is a fixed point of S. Hence $\tilde{x} \in F(T) \cap F(S)$. Next, we show $\tilde{x} \in EF(\Theta) = F(T_r)$. From $u_n = T_{r_n}y_n$ and Lemma 3.28, we obtain

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n}y_n, y_n) \\ &\leq \phi(\tilde{x}, y_n) - \phi(\tilde{x}, T_{r_n}y_n) \\ &\leq \phi(\tilde{x}, x_n) - \phi(\tilde{x}, T_{r_n}y_n) \\ &= \phi(\tilde{x}, x_n) - \phi(\tilde{x}, u_n). \end{aligned}$$

It follows from (3.121) that

$$\phi(u_n, y_n) \to 0 \quad as\ n \to \infty.$$

Noticing that Lemma 2.32, we get

$$\|u_n - y_n\| \to 0 \quad as\ n \to \infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the (A2), we note that

$$\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -\Theta(u_n, y) \geq \Theta(y, u_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_n \longrightarrow \tilde{x}$, we have $\Theta(y, \tilde{x}) \le 0, \quad \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\tilde{x}$. Noticing that $y, \tilde{x} \in C$, we obtain $y_t \in C$ which yield that $\Theta(y_t, \tilde{x}) \le 0$. It follows from (A1) that

$$0 = \Theta(y_t, y_t) \le t\Theta(y_t, y) + (1-t)\Theta(y_t, \tilde{x}) \le t\Theta(y_t, y).$$

That is, $\Theta(y_t, y) \ge 0$.

Let $t \downarrow 0$, from (A3), we obtain $\Theta(\tilde{x}, y) \ge 0$, for $\forall y \in C$. This implies that $\tilde{x} \in EP(\Theta)$. This shows that $\tilde{x} \in F$.

Finally, we prove $\tilde{x} = \Pi_F x_0$. By taking limit in (3.115), one has

$$\langle \tilde{x} - p, Jx_0 - Jp \rangle \ge 0, \quad \forall p \in F.$$

At this point, in view of Lemma 2.33, one sees that $\tilde{x} = \Pi_F x_0$. This completes the proof.

Theorem 3.30. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.127}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $0 \le \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n \longrightarrow \infty} \alpha_n < 1$;

(c) $\liminf_{n \to \infty} \alpha_n \beta_n > 0$, $\liminf_{n \to \infty} \alpha_n \gamma_n > 0$;

(d) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then, $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. First, we show that C_n is closed and convex for all $n \ge 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, one obtains that

$$\phi(z, y_k) \le \phi(z, x_k)$$

is equivalent to

$$2(\langle z, Jx_k \rangle - 2\langle z, Jy_k \rangle) \le \|x_k\|^2 - \|y_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 0$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_0$ is well-defined. Notice that $u_n = T_{r_n} z_n$ for all $n \ge 0$.

On the other hand, from Lemma 3.27, one has T_{r_n} is relatively quasi-nonexpansive mapping. Next, we prove $F \subset C_n$ for all $n \geq 0$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Then, for $\forall w \in F \subset C_k$, one has

$$\begin{aligned}
\phi(w, u_k) &= \phi(w, T_{r_k} z_k) \\
&\leq \phi(w, z_k) \\
&= \phi(w, J^{-1}(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k)) \\
&= \|w\|^2 - 2\alpha_k \langle w, Jx_k \rangle - 2\beta_k \langle w, JTx_k \rangle - 2\gamma_k \langle w, JSx_k \rangle \\
&\quad + \|\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k\|^2 \\
&\leq \|w\|^2 - 2\alpha_k \langle w, Jx_k \rangle - 2\beta_k \langle w, JTx_k \rangle - 2\gamma_k \langle w, JSx_k \rangle \\
&\quad + \alpha_k \|Jx_k\|^2 + \beta_k \|JTx_k\|^2 + \gamma_k \|JSx_k\|^2 \\
&= \alpha_k \phi(w, x_k) + \beta_k \phi(w, Tx_k) + \gamma_k \phi(w, Sx_k) \\
&\leq \phi(w, x_k),
\end{aligned} \tag{3.128}$$

and then

$$\begin{aligned}
\phi(w, y_k) &= \phi(w, J^{-1}(\delta_k Jx_k + (1-\delta_k) Jz_k)) \\
&= \|w\|^2 - 2\left\langle w, \delta_k Jx_k + (1-\delta_k) Jz_k \right\rangle \\
&\quad + \|\delta_k Jx_k + (1-\delta_k) Jz_k\|^2 \\
&\leq \|w\|^2 - 2\delta_k \left\langle w, Jx_k \right\rangle - 2(1-\delta_k) \left\langle w, Jz_k \right\rangle + \delta_k \|x_k\|^2 \\
&\quad + (1-\delta_k)\|z_k\|^2 \\
&= \delta_k (\|w\|^2 - 2\left\langle w, Jx_k \right\rangle + \|x_k\|^2) \\
&\quad + (1-\delta_k)(\|w\|^2 - 2\left\langle w, Jz_k \right\rangle + \|z_k\|^2) \\
&= \delta_k \phi(w, x_k) + (1-\delta_k)\phi(w, z_k) \\
&\leq \delta_k \phi(w, x_k) + (1-\delta_k)\phi(w, x_k) \\
&= \phi(w, x_k),
\end{aligned} \tag{3.129}$$

which show that $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \geq 0$. From $x_n = \Pi_{C_n} x_n$, one see

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n \tag{3.130}$$

and

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in F. \tag{3.131}$$

From Lemma 2.34, one has

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} a_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in F \subset C_n$ and $x_n = \Pi_{C_n} x$, we have

$$\phi(x_n, x) \leq \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\{\phi(x_n, x)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\
&\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned} \tag{3.132}$$

Letting $m, n \longrightarrow \infty$ in (3.132), one has $\phi(x_m, x_n) \longrightarrow 0$. It follows from Lemma 2.32, that $x_m - x_n \longrightarrow 0$ as $m, n \longrightarrow \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \longrightarrow p \in C$ as $n \longrightarrow \infty$. Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n}) \leq \phi(x_{n+1}, x) - \phi(\Pi_{C_n} x, x) = \phi(x_{n+1}, x) - \phi(x_n, x)$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_{n+1}} x \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we also have

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

Since $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = \lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0$, E is uniformly convex and smooth and from Lemma 2.32, we have

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \longrightarrow \infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0$, we have

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_n\| = 0.$$

Since E is uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r = \sup_{n \in \mathbb{N} \cup \{0\}} \{\|x_n\|, \|Tx_n\|, \|Sx_n\|\}$. From Lemma 2.36, we have

$$\begin{aligned}
\phi(w, u_n) &= \phi(w, T_{r_n} z_n) \leq \phi(w, z_n) \\
&= \phi(w, J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n)) \\
&= \|w\|^2 - 2\alpha_n \langle w, Jx_n \rangle - 2\beta_n \langle w, JTx_n \rangle - 2\gamma_n \langle w, JSx_n \rangle \\
&\leq \|w\|^2 - 2\alpha \langle w, Jx_n \rangle - 2\beta_n \langle w, JTx_n \rangle - 2\gamma_n \langle w, JSx_n \rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + \beta_n \|JTx_n\|^2 + \gamma_n \|JSx_n\|^2 - \alpha_n \beta_n g(\|JTx_n - Jx_n\|) \\
&= \alpha_n \phi(w, x_n) + \beta_n \phi(w, Tx_n) + \gamma_n \phi(w, Sx_n) - \alpha_n \beta_n g(\|JTx_n - Jx_n\|) \\
&\leq \phi(w, x_n) - \alpha_n \beta_n g(\|JTx_n - Jx_n\|).
\end{aligned} \tag{3.133}$$

It follows that

$$\alpha_n \beta_n g(\|JTx_n - Jx_n\|) \leq \phi(w, x_n) - \phi(w, u_n).$$

On the other hand, one has

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2 \langle w, Jx_n - Ju_n \rangle \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|w\|\|Jx_n - Ju_n\|.
\end{aligned}$$

It follows from $\|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$ that

$$\phi(w, x_n) - \phi(w, u_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.134}$$

Observing that assumption $\liminf_{n \longrightarrow \infty} \alpha_n \beta_n > 0$ and by Lemma 2.37, we also have

$$g\|Jx_n - JTx_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

It follows from the property of g that

$$\|Jx_n - JTx_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0.$$

Similarly, one can obtain

$$\lim_{n \longrightarrow \infty} \|x_n - Sx_n\| = 0.$$

Next, we show $p \in EF(\Theta) = F(T_r)$. On the other hand, from (3.129), we arrive at

$$\phi(u, y_n) \leq \phi(u, x_n). \tag{3.135}$$

From $u_n = T_{r_n} z_n$ and Lemma 3.28, we obtain

$$\begin{aligned} \phi(u_n, z_n) &= \phi(T_{r_n} z_n, z_n) \\ &\leq \phi(w, z_n) - \phi(w, T_{r_n} z_n) \\ &\leq \phi(w, x_n) - \phi(w, T_{r_n} z_n) \\ &= \phi(w, x_n) - \phi(w, u_n). \end{aligned}$$

It follows from (3.134) that

$$\phi(u_n, z_n) \to 0, \quad as\ n \to \infty.$$

Noticing that Lemma 2.32, we get

$$\|u_n - z_n\| \to 0, \quad as\ n \to \infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_n - Jz_n\| = 0.$$

From the assumption $r_n \geq a$, one sees

$$\lim_{n\to\infty} \frac{\|Ju_n - Jz_n\|}{r_n} = 0.$$

Noticing that $u_n = T_{r_n} z_n$, one obtains

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C.$$

From the (A2), we note that

$$\|y - u_n\| \frac{\|Ju_n - Jz_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq -\Theta(u_n, y) \geq \Theta(y, u_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_n \longrightarrow p$, we obtain $\Theta(y, p) \leq 0, \quad \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, we obtain $y_t \in C$ which yield that $\Theta(y_t, p) \leq 0$. It follows from (A1) that

$$0 = \Theta(y_t, y_t) \leq t\Theta(y_t, y) + (1-t)\Theta(y_t, p) \leq t\Theta(y_t, y).$$

That is, $\Theta(y_t, y) \geq 0$.

Let $t \downarrow 0$, from (A3), we obtain $\Theta(p, y) \geq 0$, for $\forall y \in C$. This implies that $p \in EP(\Theta)$. This shows that $p \in F$.

Finally, we prove $p = \Pi_F x_0$. By taking limit in (3.130), one has

$$\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in F.$$

At this point, in view of Lemma 2.33, one sees that $p = \Pi_F x_0$. This completes the proof. □

Corollary 3.31. (*Qin et al.* [44], *Theorem 3.1*) *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying*

(A1)-(A4) and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.136}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n \longrightarrow \infty} \alpha_n < 1$;

(c) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;

(d) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then, $\{x_n\}$ converges strongly to $\Pi_F x_0$.

***Proof*.** Setting $\delta_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, then (3.127) reduced to (3.136) and putting $u_n = T_{r_n} y_n$ for $z \in F$, we have $\phi(z, u_n) = \phi(z, T_{r_n} y_n) \leq \phi(z, y_n)$. Therefore, the conclusion follows immediately from Theorem 3.30. □

Corollary 3.32. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(S) \cap EP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{3.137}$$

where J is the duality mapping on E, $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\liminf_{n \longrightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $P_F x_0$.

Remark 3.33. *Corollary 3.32 improves Theorem 3.1 of Takahashi and Zembayashi [56] in the following senses:*

(1) *from relatively nonexpansive mappings to more general relatively quasi-nonexpansive mappings; that is, we relax the strong restriction: $\widetilde{F(T)} = F(T)$;*

(2) *the algorithm in Theorem 3.30 is also more general than the one given by Qin et al. [44] and Takahashi and Zembayashi [57].*

Lemma 3.34. (*Zhang* [68]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \longrightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : C \longrightarrow C$ as follows:

$$\begin{aligned} K_r(x) = \{u \in C : \Theta(u, y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) \\ + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\} \end{aligned} \tag{3.138}$$

for all $x \in E$. Then, the followings hold:

1. *K_r is single-valued;*
2. *K_r is firmly nonexpansive, i.e., for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;*
3. *$F(K_r) = \Omega$;*
4. *Ω is closed and convex.*
5. *$\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$ $\forall p \in F(K_r)$, $z \in E$.*

Remark 3.35. *(*Zhang [68]*). It follows from Lemma 3.27 that the mapping $K_r : C \longrightarrow C$ defined by (3.138) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.*

In this section, using the CQ hybrid method, we prove a strong convergence theorem for finding a common element of the set of solutions of a mixed equilibrium problem, the set of solutions of the variational inequality problem and the set of fixed points of quasi-ϕ-nonexpansive mappings in a Banach space.

Theorem 3.36. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let A be an α-inverse-strongly monotone operator of C into E^* and let $T, S : C \to C$ be closed quasi-ϕ-nonexpansive mappings such that $F := F(T) \cap F(S) \cap VI(A, C) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ u_n \in C \ \text{ such that } \ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) \\ \qquad\qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases} \tag{3.139}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying the restrictions:

(i) $\limsup_{n \longrightarrow \infty} \alpha_n < 1$;

(ii) $\beta_n + \gamma_n + \delta_n = 1$;

(iii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \beta_n\delta_n > 0$;

(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(v) $\{\lambda_n\} \subset [b, c] \in (0, 2\alpha)$, *for some* $b, c \in \mathbb{N}$.

Then, $\{x_n\}$ *converges strongly to* $p \in F$*, where* $p = \Pi_F x$.

Proof. We first show that $C_n \cap Q_n$ is closed and convex for each $n \geq 0$. It is obvious that C_n is closed and Q_n is closed and convex. Since

$$\phi(z, u_n) \leq \phi(z, x_n)$$

is equivalent to

$$2\langle z, Ju_n\rangle - 2\langle z, Jx_n\rangle \leq \|u_n\|^2 - \|x_n\|^2,$$

C_n is convex. So, $C_n \cap Q_n$ is closed and convex subset of E for all $n \in \mathbb{N} \cup \{0\}$.
Put $v_n = J^{-1}(Jx_n - \lambda_n Ax_n)$. We observe that $u_n = K_{r_n} y_n$ for all $n \geq 1$ and let $p \in F$, it follows from the definition of quasi-ϕ-nonexpansive that

$$\begin{aligned}
\phi(p, u_n) &= \phi(p, K_{r_n} y_n) \\
&\leq \phi(p, y_n) \\
&= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)Jz_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, Jz_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n), \qquad (3.140)
\end{aligned}$$

and then

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n)) \\
&= \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2\gamma_n\langle p, JTx_n\rangle - 2\delta_n\langle p, Jwx_n\rangle \\
&\quad + \|\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n\|^2 \\
&\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2\gamma_n\langle p, JTx_n\rangle - 2\delta_n\langle p, JSw_n\rangle \\
&\quad + \beta_n\|Jx_n\|^2 + \gamma_n\|JTx_n\|^2 + \delta_n\|JSw_n\|^2 \\
&= \beta_n\phi(p, x_n) + \gamma_n\phi(p, Tx_n) + \delta_n\phi(p, Sw_n) \\
&\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, x_n) + \delta_n\phi(p, w_n). \qquad (3.141)
\end{aligned}$$

From Lemma 2.34 and Lemma 2.38,

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, \Pi_C v_n) \\
&\leq \phi(p, v_n) = \phi(p, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq V(p, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\rangle \\
&= V(p, Jx_n) - 2\lambda_n\langle v_n - p, Ax_n\rangle \\
&= \phi(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n\rangle + 2\langle v_n - x_n, -\lambda_n Ax_n\rangle. \qquad (3.142)
\end{aligned}$$

Since $p \in VI(A, C)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned}
-2\lambda_n\langle x_n - p, Ax_n\rangle &= -2\lambda_n\langle x_n - p, Ax_n - Ap\rangle - 2\lambda_n\langle x_n - p, Ap\rangle \\
&\leq -2\alpha\lambda_n\|Ax_n - Ap\|^2, \qquad (3.143)
\end{aligned}$$

and we obtain

$$\begin{aligned} 2\langle v_n - x_n, -\lambda_n A x_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n A x_n) - x_n, -\lambda_n A x_n\rangle \\ &\le 2\|J^{-1}(Jx_n - \lambda_n A x_n) - x_n\|\|\lambda_n A x_n\| \\ &= 2\|Jx_n - \lambda_n A x_n - Jx_n\|\|\lambda_n A x_n\| \\ &= 2\lambda_n^2\|Ax_n\|^2 \\ &\le 2\lambda_n^2\|Ax_n - Ap\|^2. \end{aligned} \tag{3.144}$$

Replacing (3.143) and (3.144) into (3.142), we get

$$\begin{aligned} \phi(p, w_n) &\le \phi(p, x_n) - 2\lambda_n(\alpha - \lambda_n)\|Ax_n - Ap\|^2 \\ &\le \phi(p, x_n). \end{aligned} \tag{3.145}$$

From (3.140), (3.141) and (3.145), we have

$$\phi(p, u_n) \le \phi(p, x_n). \tag{3.146}$$

Hence, we have $p \in C_n$. This implies that

$$F \subset C_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.147}$$

Next, we show by induction that $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $Q_0 = C$, we have $F \subset C_0 \cap Q_0$. Suppose that $F \subset C_k \cap Q_k$ for some $k \in \mathbb{N} \cup \{0\}$. Then, there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k \cap Q_k} x$. From the definition of x_{k+1}, we have

$$\langle x_{k+1} - z, Jx - Jx_{k+1}\rangle \ge 0 \ \text{ for all } \ z \in C_k \cap Q_k. \tag{3.148}$$

Since $F \subset C_k \cap Q_k$, we have

$$\langle x_{k+1} - p, Jx_0 - Jx_{k+1}\rangle \ge 0, \quad \forall p \in F, \tag{3.149}$$

and hence $p \in Q_{k+1}$. So, we have

$$F \subset Q_{k+1}. \tag{3.150}$$

Hence by (3.147) and (3.150) we obtain

$$F \subset C_{k+1} \cap Q_{k+1}.$$

Threrfore, we have that $F \subset C_k \cap Q_k$ for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well-defined.

Using $x_n = \Pi_{Q_n} x$, from Lemma 2.30, one has

$$\phi(x_n, x) = \phi(\Pi_{Q_n} x, x) \le \phi(p, x) - \phi(p, x_n) \le \phi(p, x)$$

for each $p \in F \subset Q_n$ and $x_n = \Pi_{Q_n} x$. Thus, $\phi(x_n, x)$ is bounded. Then, $\{x_n\}$, $\{Sw_n\}$ and $\{Tx_n\}$ are bounded.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n \cap Q_n$ and $x_n = \Pi_{Q_n} x$, we have

$$\phi(x_n, x) \le \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\{\phi(x_n, x)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x)\}$ exists. By the construction of Q_n, we have $Q_m \subset Q_n$ and $x_m = \Pi_{Q_m} x \in Q_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{Q_n} x) \\ &\le \phi(x_m, x) - \phi(\Pi_{Q_n} x, x) \\ &= \phi(x_m, x) - \phi(x_n, x). \end{aligned} \tag{3.151}$$

Letting $m, n \longrightarrow \infty$ in (3.151), we have $\phi(x_m, x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. It follows that, from Lemma 2.32, $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \longrightarrow \hat{x} \in C$ as $n \longrightarrow \infty$. Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x) \leq \phi(x_{n+1}, x) - \phi(\Pi_{Q_n} x, x) = \phi(x_{n+1}, x) - \phi(x_n, x)$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we also have

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

Since E is uniformly convex and smooth, we have from Lemma 2.32 that

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \longrightarrow \infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_n\| = 0.$$

Since E is uniformly smooth Banach spaces, one knows that E^* is a uniformly convex Banach apace. Let $r = \sup_{n \in \mathbb{N} \cup \{0\}} \{\|x_n\|, \|Tx_n\|, \|Sw_n\|\}$. From Lemma 2.36 and (3.145), we have

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n)) \\
&= \|p\|^2 - 2\beta_n \langle p, Jx_n \rangle - 2\gamma_n \langle p, JTx_n \rangle - 2\delta_n \langle p, JSw_n \rangle \\
&\quad + \|\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n\|^2 \\
&\leq \|p\|^2 - 2\beta \langle p, Jx_n \rangle - 2\gamma_n \langle p, JTx_n \rangle - 2\delta_n \langle p, JSw_n \rangle \\
&\quad + \beta_n \|x_n\|^2 + \gamma_n \|Tx_n\|^2 + \delta_n \|Sw_n\|^2 \\
&\quad - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\
&= \beta_n \phi(p, x_n) + \gamma_n \phi(p, Tx_n) + \delta_n \phi(p, Sw_n) \\
&\quad - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\
&\leq \phi(p, x_n) - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\
&\quad - 2\lambda_n (\alpha - \lambda_n) \delta_n \|Ax_n - Ap\|^2.
\end{aligned} \tag{3.152}$$

Substituting (3.152) into (3.140), we have

$$\begin{aligned}
\phi(p, u_n) &\leq \alpha_n \phi(p, x_n) + (1 - \alpha_n)[\phi(p, x_n) - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\
&\quad - 2\lambda_n (\alpha - \lambda_n) \delta_n \|Ax_n - Ap\|^2] \\
&\leq \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, x_n) \\
&\quad - (1 - \alpha_n) \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\
&\quad - 2\lambda_n (1 - \alpha_n)(\alpha - \lambda_n) \delta_n \|Ax_n - Ap\|^2.
\end{aligned} \tag{3.153}$$

Therefore, we have

$$(1 - \alpha_n) \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \leq \phi(p, x_n) - \phi(p, u_n).$$

On the other hand, we have

$$\begin{aligned}
\phi(p, x_n) - \phi(p, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2 \langle p, Jx_n - Ju_n \rangle \\
&\leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|p\| \|Jx_n - Ju_n\|.
\end{aligned}$$

It follows from $\|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$ that

$$\lim_{n\longrightarrow\infty}(\phi(p, x_n) - \phi(p, u_n)) = 0. \tag{3.154}$$

Observing that assumption $\liminf_{n\longrightarrow\infty}\beta_n\gamma_n > 0$ and by Lemma 2.37, we also

$$\lim_{n\longrightarrow\infty} g\|Jx_n - JTx_n\| = 0.$$

It follows from the property of g that

$$\lim_{n\longrightarrow\infty}\|Jx_n - JTx_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0. \tag{3.155}$$

Similarly, one can obtain

$$\lim_{n\longrightarrow\infty}\|x_n - Sw_n\| = 0. \tag{3.156}$$

By (3.153), we have

$$2\lambda_n(\alpha - \lambda_n)\delta_n\|Ax_n - Ap\|^2 \leq \phi(p, x_n) - \phi(p, u_n),$$

which yield that

$$\lim_{n\longrightarrow\infty}\|Ax_n - Ap\| = 0. \tag{3.157}$$

From Lemma 2.34, Lemma 2.38, and (3.144), we have

$$\begin{aligned}
\phi(x_n, w_n) = \phi(x_n, \Pi_C v_n) &\leq \phi(x_n, v_n)\\
&= \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n))\\
&= V(x_n, Jx_n - \lambda_n Ax_n)\\
&\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n)\\
&\quad -2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle\\
&= \phi(x_n, x_n) + 2\langle v_n - x_n, \lambda_n Ax_n\rangle\\
&= 2\langle v_n - x_n, \lambda_n Ax_n\rangle\\
&\leq 2\lambda_n^2\|Ax_n - Ap\|^2.
\end{aligned}$$

From Lemma 2.32 and (3.157), we have

$$\lim_{n\longrightarrow\infty}\|x_n - w_n\| = 0. \tag{3.158}$$

Since J is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\longrightarrow\infty}\|Jx_n - Jw_n\| = 0. \tag{3.159}$$

By (3.156) and (3.158), we obtain

$$\lim_{n\longrightarrow\infty}\|Sw_n - w_n\| = 0. \tag{3.160}$$

From (3.158), we have

$$\lim_{n\longrightarrow\infty}\|Sx_n - x_n\| = 0.$$

Since S and T are closed operators and $x_n \longrightarrow \hat{x}$, $\hat{x}$ is a common fixed point of S and T, i.e., $\hat{x} \in F(T)\cap F(S)$. Next, we show that $\hat{x} \in MEP(\Theta, \varphi)$. Since $u_n = K_{r_n}y_n$. From Lemma 3.34, we have

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(K_{r_n}y_n, y_n)\\
&\leq \phi(\hat{x}, y_n) - \phi(\hat{x}, K_{r_n}y_n)\\
&\leq \phi(\hat{x}, x_n) - \phi(\hat{x}, K_{r_n}y_n)\\
&= \phi(\hat{x}, x_n) - \phi(\hat{x}, u_n)\\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle\hat{x}, Jx_n - Ju_n\rangle\\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|\hat{x}\|\|Jx_n - Ju_n\|.
\end{aligned}$$

It follows from $\|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$ that

$$\phi(u_n, y_n) \to 0 \quad as \ n \to \infty.$$

and so

$$\lim_{n \longrightarrow \infty} \|u_n - y_n\| = 0. \tag{3.161}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n \longrightarrow \infty} \|Ju_n - Jy_n\| = 0. \tag{3.162}$$

From (3.139) and (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq \Theta(y, u_n), \quad \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \langle y - u_{n_i}, \frac{Ju_{n_i} - Jy_{n_i}}{r_{n_i}}\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C.$$

From $\|x_n - u_n\| \longrightarrow 0$, we get $u_{n_i} \longrightarrow \hat{x}$. Since $\dfrac{Ju_{n_i} - Jy_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) and the weakly lower semicontinuous of φ that

$$\Theta(y, \hat{x}) + \varphi(\hat{x}) - \varphi(y) \leq 0, \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)u$. Since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and hence $\Theta(y_t, \hat{x}) + \varphi(\hat{x}) - \varphi(y_t) \leq 0$. So, from (A1), (A4) and the convexity of φ, we have

$$\begin{aligned}
0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, u) + t\varphi(y) + (1-t)\varphi(y) - \varphi(y_t) \\
&\leq t(\Theta(y_t, y) + \varphi(y) - \varphi(y_t)).
\end{aligned}$$

Dividing by t, we get $\Theta(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0$. From (A3) and the weakly lower semicontinuity of φ, we have $\Theta(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0$ for all $y \in C$ implies $\hat{x} \in MEP(\Theta, \varphi)$. Next, we show that $\hat{x} \in VI(A, C)$. Define $T \subset E \times E^*$ be as in Theorem 2.47. Thus by Theorem 2.47, T is maximal monotone and $T^{-1}0 = VI(A, C)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Av\rangle \geq 0. \tag{3.163}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, by Lemma 2.33, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0;$$

thus,

$$\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\rangle \leq 0. \tag{3.164}$$

It follows from (3.163) and (3.164) that

$$\begin{aligned}
\langle v - w_n, w\rangle &\geq \langle v - w_n, Av\rangle \\
&\geq \langle v - w_n, Av\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\rangle \\
&= \langle v - w_n, Av - Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\rangle \\
&= \langle v - w_n, Av - Aw_n\rangle + \langle v - w_n, Aw_n - Ax_n\rangle \\
&\quad + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\rangle \\
&\geq -\|v - w_n\|\tfrac{\|w_n - x_n\|}{\alpha} - \|v - w_n\|\frac{\|Jx_n - Jw_n\|}{b} \\
&\geq -M(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{b}),
\end{aligned}$$

where $M = \sup_{n\geq 1}\{\|v - w_n\|\}$. From (3.158) and (3.159), we obtain $\langle v - \hat{x}, w\rangle \geq 0$. By the maximality of T, we have $\hat{x} \in T^{-1}0$ and hence $\hat{x} \in VI(A, C)$. Hence, $\hat{x} \in F := VI(C, A) \cap T^{-1}(0) \cap MEP(\Theta, \varphi)$.

Finally, we prove that $\hat{x} = \Pi_F x_0$. From $x_n = \Pi_{C_n \cap Q_n} x$, we have

$$\langle Jx - Jx_n, x_n - z\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

Since $F \subset C_n \cap Q_n$, we also have

$$\langle Jx - Jx_n, x_n - p\rangle \geq 0, \quad \forall p \in F. \tag{3.165}$$

By taking limit in (3.165), one has

$$\langle Jx - J\hat{x}, \hat{x} - p\rangle \geq 0, \quad \forall p \in F.$$

At this point, in view of Lemma 2.33, one sees that $\hat{x} = \Pi_F x_0$. This completes the proof.

3.3 Optimization Problems

Let H be a real Hilbert space and C be a closed convex subset of H.

Now, we consider the following ***Optimization Problem***:

$$\min_{x\in\widehat{F}}\left\{\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)\right\}, \tag{3.166}$$

where $\widehat{F} = \cap_{n=1}^{\infty} C_n$, $C_1, C_2, \cdots$ are infinitely many closed convex subsets of H such that $\cap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For solving the mixed equilibrium problem for an equilibrium bifunction $\Theta : C \times C \longrightarrow \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(H1) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0, \quad \forall x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto \Theta(x, y)$ is convex and upper semicontinuous;

(H3) for each $x \in C, y \mapsto \Theta(x, y)$ is convex.

Definition 3.37. *Let $\eta : C \times C \longrightarrow H$, which is called* Lipschitz continuous *if there exists a constant $\delta > 0$ such that*

$$\|\eta(x, y)\| \leq \delta\|x - y\|, \quad \forall x, y \in C.$$

Definition 3.38. *Let $K : C \longrightarrow \mathbb{R}$ be a differentiable functional on a convex set C, which is called:*

(D1) η-convex [7] if

$$K(y) - K(x) \geq \Big\langle K'(x), \eta(y, x)\Big\rangle, \ \forall x, y \in C,$$

where $K'(x)$ is the Fréchet derivative at x;

(D2) η-strongly convex [61] if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \Big\langle K'(x), \eta(y, x)\Big\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \ \forall x, y \in C.$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in C$, then K is said to be strongly convex.

Definition 3.39. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\varphi : C \longrightarrow \mathbb{R}$ be a real-valued function and $\Theta : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction. Let r be a positive parameter. For a given point $x \in C$, the auxiliary problem for mixed equilibrium problem (2.11) consists of finding $y \in C$ such that*

$$\Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z,y)\Big\rangle \geq 0, \quad \forall z \in C.$$

Definition 3.40. *Let $S_r : C \longrightarrow C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of the auxiliary problem the mixed equilibrium problem (2.11), that is,*

$$\begin{aligned} S_r(x) \quad &= \quad \Big\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) \\ &\qquad + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z,y)\Big\rangle \geq 0, \quad \forall z \in E \Big\}, \quad \forall x \in C. \end{aligned}$$

Lemma 3.41. [7]. *Let C be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from C to $\mathbb{R}$. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H3). Assume that*

(i) $\eta : C \times C \longrightarrow H$ *is Lipschitz continuous with constant $\lambda > 0$ such that;*

- *(a)* $\eta(x,y) + \eta(y,x) = 0, \quad \forall x, y \in C,$
- *(b)* $\eta(\cdot,\cdot)$ *is affine in the first variable,*
- *(c) for each fixed $y \in C$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) $K : E \longrightarrow \mathbb{R}$ *is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;*

(iii) for each $x \in C$, there exist a bounded subset $D_x \subset E$ and $z_x \in C$ such that for any $y \in \backslash D_x$,

$$\Theta(y,z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z_x,y)\Big\rangle < 0.$$

Then, there exists $y \in C$ such that

$$\Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z,y)\Big\rangle \geq 0, \quad \forall z \in C.$$

Lemma 3.42. [7]. *Assume that Θ satisfies the same assumptions as Lemma 3.41 for $r > 0$ and $x \in C$, the mapping $S_r : C \longrightarrow C$ can be defined as follows:*

$$S_r(x) = \Big\{ y \in E : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z,y)\Big\rangle \geq 0, \quad \forall z \in C \Big\}.$$

Then, the following hold:

(i) S_r *is single-valued;*

(ii) *(a)* $\Big\langle K'(x_1) - K'(x_2), \eta(u_1,u_2)\Big\rangle \geq \Big\langle K'(u_1) - K'(u_2), \eta(u_1,u_2)\Big\rangle, \quad \forall (x_1,x_2) \in C \times C$, *where* $u_i = S_r(x_i)$, $i = 1, 2$;

- *(b)* S_r *is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\sigma > \lambda\nu$;*

(iii) $F(S_r) = MEP(\Theta, \varphi)$;

(iv) $MEP(\Theta, \varphi)$ *is closed and convex.*

Remark 3.43. *From Lemma 3.42 in particular, whenever* $K(x) = \frac{\|x\|^2}{2}$ *and* $\eta(x,y) = x - y$ *for each* $(x,y) \in C \times C$*, then* S_r *is firmly nonexpansive, that is,*

$$\|S_r(x_1) - S_r(x_2)\|^2 \leq \Big\langle x_1 - x_2, S_r(x_1) - S_r(x_2) \Big\rangle.$$

Lemma 3.44. [64]. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*, and* $g : C \longrightarrow \mathbb{R} \cup \{\infty\}$ *be a proper lower-semicontinuous differentiable convex function. If* z *is a solution to the minimization problem*

$$g(z) = \inf_{x \in C} g(x),$$

then

$$\Big\langle g'(x), x - z \Big\rangle \geq 0, \quad x \in C.$$

In particular, if z *solves problem* Optimization Problem, *then*

$$\Big\langle u + \big[\gamma f - (I + \mu A)\big] z, x - z \Big\rangle \leq 0.$$

We prove a strong convergence theorem of a new iterative method to compute the approximate solutions of the mixed equilibrium problems and optimization problems in a real Hilbert space.

We first prove that the following Lemmas.

Lemma 3.45. *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* $B : C \longrightarrow H$ *be a relaxed* (m,v)*-cocoercive and* μ*-Lipschitz continuous. It* $0 \leq \tau_n \leq \frac{2(v - m\mu^2)}{\mu^2}$*,* $v > m\mu^2$*, then* $I - \tau_n B$ *is a nonexpansive mapping in* H*.*

Proof. See [20]. □

Now, we prove the following main Theorem.

Theorem 3.46. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* φ *be a lower semicontinuous and convex functional from* C *to* $\mathbb{R}$*. Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (H1)-(H3), let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of* C *into itself and let* B *be a* ξ*-Lipschitz continuous and relaxed* (m,v)*-cocoercive map* C *into* H *such that*

$$\Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP(\Theta, \varphi) \cap VI(C, B) \neq \emptyset.$$

Let $\mu > 0$*,* $\gamma > 0$ *and* $r > 0$*, which are three constants. Let* f *be a contraction of* E *into itself with* $\alpha \in (0,1)$ *and let* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$*. For given* $x_1 \in H$ *arbitrarily and fixed* $u \in H$*, suppose the* $\{x_n\}$*,* $\{y_n\}$ *and* $\{z_n\}$ *are generated iteratively by*

$$\begin{cases} \Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\Big\langle K'(z_n) - K'(x_n), \eta(x, z_n) \Big\rangle \geq 0, \quad \forall x \in C, \\ y_n = \alpha_n z_n + (1 - \alpha_n) W_n P_C(z_n - \lambda_n B z_n), \\ x_{n+1} = \epsilon_n\big(u + \gamma f(W_n x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big) W_n P_C(y_n - \tau_n B y_n), \end{cases} \tag{3.167}$$

for all $n \in \mathbb{N}$*, where* W_n *be the* W*-mapping defined by (3.34) and* $\{\epsilon_n\}$*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are three sequences in* $(0,1)$*. Assume the following conditions are satisfied:*

(C1) $\eta : C \times C \longrightarrow H$ *is Lipschitz continuous with constant* $\lambda > 0$ *such that;*

(a) $\eta(x,y) + \eta(y,x) = 0, \quad \forall x, y \in C$

(b) $\eta(\cdot,\cdot)$ *is affine in the first variable,*

(c) *for each fixed* $y \in C$, $x \mapsto \eta(y,x)$ *is sequentially continuous from the weak topology to the weak topology;*

(C2) $K : C \longrightarrow \mathbb{R}$ *is* η*-strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma > \lambda\nu$;

(C3) *for each* $x \in C$, *there exist a bounded subset* $D_x \subset E$ *and* $z_x \in C$ *such that for any* $y \in C\backslash D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z_x, y)\Big\rangle < 0;$$

(C4) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(C5) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;

(C6) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$;

(C7) $\{\tau_n\}$, $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(v-m\xi^2)}{\xi^2}$.

Then, $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* $z \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP(\Theta, \varphi) \cap VI(C, B)$ *provided that* S_r *is firmly nonexpansive, which solves the following Optimization Problem:*

$$\min_{x\in\Gamma}\bigg\{\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)\bigg\}. \tag{3.168}$$

Proof. See [16]. □

Corollary 3.47. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of* E *into itself and let* B *be a* ξ*-Lipschitz continuous and relaxed* (m, v)*-cocoercive map* C *into* H *such that*

$$\Gamma := \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset.$$

Let $\mu > 0$ *and* $\gamma > 0$, *which are two constants. Let* f *be a contraction of* C *into itself with* $\alpha \in (0, 1)$ *and let* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$. *For given* $x_1 \in H$ *arbitrarily and fixed* $u \in H$, *suppose the* $\{x_n\}$ *and* $\{y_n\}$ *are generated iteratively by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)W_n P_C(x_n - \lambda_n B x_n), \\ x_{n+1} = \epsilon_n\big(u + \gamma f(W_n x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)W_n P_C(y_n - \tau_n B y_n) \end{cases}$$

for all $n \in \mathbb{N}$, *where* W_n *be the* W*-mapping defined by (3.34) and* $\{\epsilon_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are three sequences in* $(0, 1)$. *Assume the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;

(C3) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$;

(C4) $\{\tau_n\}$, $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(v-m\xi^2)}{\xi^2}$.

Then, $\{x_n\}$ *converge strongly to* $z \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B)$, *which solves the following Optimization Problem:*

$$\min_{x\in\Gamma}\bigg\{\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)\bigg\}.$$

CHAPTER 4

CONCLUSIONS

4.1 Outputs 15 papers (Supported by TRF: MRG5180034)

(ISI 10 papers)

1. Kumam P. and Plubtieng S., Weak Convergence Theorem for MonotoneMappings and a Countable Family of Nonexpansive Mappings, Journal Computational and Applied Mathematics, 224 (2009), 614-621. **(Cover by ISI, 2008 Impact factor 1.048)**

2. Sintunavarat W. and Kumam P., "Coincidence and common fixed points for hybrid strict contractions without weakly commuting condition", Applied Mathematics Letters 22 (2009) 1877-1881. **(ISI 2008: impact factor 0.948)**

3. Jaiboon C., Kumam P. and Humphries U. W., Weak convergence theorem by an extragradient method for variaotinal inequality, equilibrium and fixed point problems, BULLETIN of the Malaysian Mathematical Sciences Society, 2009; 32(2): 173-185. Cover by ISI Science Citation Index Expanded (SCIE) 2008. **(Impact factor: soon)**

4. Kumam P., A relaxed extragradient approximation method of two inverse-strongly monotone mappings for a general system of variational inequalities, fixed point and equilibrium problems, accepted in Bulletin of the Iranian Mathematical Society. Cover by ISI Science Citation Index Expanded (SCIE) 2008. (**Impact factor: soon)**

5. Jaiboon C. and Kumam P., A hybrid extragradient viscosity approximation method for solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings, Fixed Point Theory and Applications. volume 2009, Article ID 374815, 32 pages **(Cover by ISI, 2008: Impact Factor 0.728)**

6. Kumam P. Strong Convergence Theorems by an Extragradient Method for Solving Variational Inequality and Equilibrium Problems in a Hilbert space, Turkish Journal of Mathematics, Cover by ISI Science Citation Index Expanded ? (SCIE) 2008. (Impact factor: soon)

7. Saewan S. Kumam P. and Wattanawitoon K., "Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces," Abstract and Applied Analysis, Volume 2010 (2010), Article ID 734126, 25 pages. **(ISI 2008 Impact Factor: 0.644)**

8. Katchang P. and Kumam P., Strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces, Computers and Mathematics with Applications, 59 (2010), 1473-1483. **(ISI, 2008 Impact factor 0.997)**

9. Jaiboon C. and Kumam P., "Strong Convergence for generalized equilibrium problems and relaxed cocoercive variational inequalities," Journal of Inequalities and Applications, Volume 2010, Article ID 728028, 43 pages, doi:10.1155/2010/728028 **(ISI, 2008 Impact Factor 0.764)**

10. Jaiboon C. and Kumam P., Strong convergence theorems for solving equilibrium problems and fixed point problems of ξ-strict pseudo-contraction mappings by two hybrid projection methods, Journal of

Computational and Applied Mathematics, (2010). In press doi:10.1016/j.cam.2010.01.012 **(ISI, 2008 Impact factor 1.048)**

(Peer-Referee Journals 4 papers)

1. Jaiboon C. and Kumam P., A new hybrid iterative method for mixed equilibrium problems and variational inequality problem for relaxed cocoercive mappings with application to optimization problems, Nonlinear Analysis: Hybrid Systems, 3 (2009) 510-530.

2. Kumam P, A new hybrid iterative method for solution of Equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, Journal of Applied Mathematics and Computing, 2009; 29 (1): 263-280.

3. Jaiboon C, Kumam P and Humphries U. W, Convergence theorems by the viscosity approximation method for equilibrium prolems and variational inequality problems, Journal of Computational Mathematics and Optimization, 2009; 5 (1): 29-56.

4. Wattanawitoon K. and Kumam P., Strong convergence theorems by a new hybrid projection algorithm for fixed point problems and equilibrium problems of two relatively quasi-nonexpansive mappings, Nonlinear Analysis: Hybrid Systems, 3 (2009) 11-20.

(Proceeding International Conference peer-review 1 paper)

1. Kriengsak Wattanawitoon, Poom Kumam, and Usa Wannasingha Humphries, "Strong convergence theorems of modified Ishikawa iterations for two family of relatively quasi-nonexpansive mappings in Banach spaces", to appear in : Proceedings of the Asian Conference on Nonlinear Analysis and Optimization (NAO-Asia2008).

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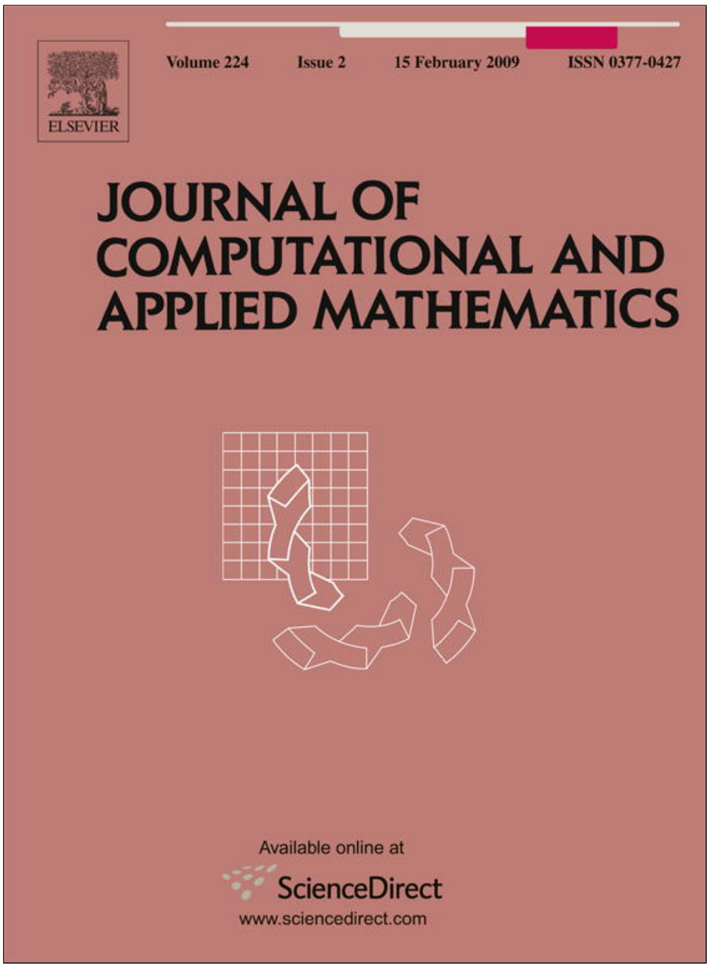
APPENDIX A

Publications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Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings[☆]

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ABSTRACT

In this paper, we introduce an iterative process for finding the common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for an α-inverse-strongly-monotone mapping. We obtain a weak convergence theorem for a sequence generated by this process. Moreover, we apply our result to the problem for finding a common element of the set of equilibrium problems and the set of solutions of the variational inequality problem of a monotone mapping.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \tag{1}$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \ge 0, \tag{2}$$

for all $u, v \in C$. A is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \ge \alpha\|Au - Av\|^2, \tag{3}$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au\rangle \ge 0$ for all $v \in C$. We denoted by $VI(A, C)$ the set of solutions of this variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g. [13,14] and the references therein.

Construction of fixed points of nonexpansive mapping is an important subject in the theory of nonexpansive mappings. However, the sequence $\{S^n x\}_{n=0}^{\infty}$ of iterates of the mapping S at a point $x \in C$ may not converge even in weak topology. More precisely, Mann's iterated procedure is a sequence $\{x_n\}$ which is generated in the following recursive way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \quad n \ge 0, \tag{4}$$

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where the initial guess $x_0 \in C$ is chosen arbitrarily. However, we note that Mann's iterations have only weak convergence even in a Hilbert space [6]. For finding an element of $F(S) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly-monotone. Takahashi and Toyoda [12] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{5}$$

for every $n \in \mathbb{N} \cup \{0\}$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(C, A) \neq \emptyset$, then such a sequence $\{x_n\}$ converges weakly to some $z \in F(S) \cap VI(C, A)$.

On the other hand, Aoyama et al. [1] introduced an iterative sequence $\{x_n\}$ of C defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S_n x_n, \quad n \in \mathbb{N}, \tag{6}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, C is a closed convex subset of H and $\{S_n\}$ is a sequence of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. They also proved that such a sequence converges strongly to a common fixed point of nonexpansive mappings.

In this paper, we introduce the following iteration process of α-inverse-strongly-monotone mappings A of C into H and a countable family of nonexpansive mappings $\{S_n\}$ of C into itself, where C is a closed convex subset of a Hilbert space H. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S_n P_C(x_n - \lambda_n Ax_n), \tag{7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\} \subset (a, b) \subset (0, 2\alpha)$. We will prove that if the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ of parameters satisfy the appropriate condition, then the sequence $\{x_n\}$ generated by (7) converges weakly to the point $z \in F(S) \cap VI(C, A)$. Moreover, we apply our result to the problem for finding a common element of the set of equilibrium problems and the set of common fixed points of a countable family of nonexpansive mappings.

2. Preliminaries

Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{8}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{9}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfy

(1) the *Opial condition* [9], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

(2) the *Kadec-Klee property* [5,10], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ implies $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{10}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{11}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{12}$$

for all $x \in H, y \in C$.

In the context of the variational inequality problem, this implies that

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \quad \text{for all } \lambda > 0. \tag{13}$$

We have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{14}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 1 ([1, Lemma 3.2]). *Let C be a nonempty closed subset of a Banach space and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself. Suppose that $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\to\infty} T_n y \quad \text{for all } y \in C.$$

Then $\lim_{n\to\infty}\sup\{\|T_n z - Tz\| : z \in C\} = 0$.

Lemma 2 ([12, Lemma 3.1]). *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \le \alpha_n \le b < 1$ for all $n = 0, 1, 2, \ldots$, and let $\{v_n\}$ and $\{w_n\}$ be sequences of H such that*

$$\limsup_{n\to\infty}\|v_n\| \le c, \qquad \limsup_{n\to\infty}\|w_n\| \le c,$$

and

$$\lim_{n\to\infty}\|\alpha_n v_n + (1-\alpha_n)w_n\| = c, \quad \text{for some } c > 0.$$

Then,

$$\lim_{n\to\infty}\|v_n - w_n\| = 0.$$

Lemma 3 ([12, Lemma 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H such that*

$$\|x_{n+1} - y\| \le \|x_n - y\| \quad \text{for all } y \in C \text{ and } n \in \mathbb{N}.$$

Then the sequence $\{P_C(x_n)\}$ converges strongly to some point in C.

3. Weak convergence theorems

In this section, we prove some weak convergence theorems for monotone mappings and a countable family of nonexpansive mappings.

Theorem 4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly-monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty}\mathrm{F}(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_n P_C(x_n - \lambda_n A x_n),$$

for all $n = 0, 1, 2, \ldots$, where $0 < a < \lambda_n < b < 2\alpha$, $0 < c < \alpha_n < d < 1$ and $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. Suppose that $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty}\mathrm{F}(S_n)$. Then $\{x_n\}$ converges weakly to $z \in \mathrm{F}(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{\mathrm{F}(S)\cap VI(C,A)} x_n$.

Proof. Put $y_n = P_C(x_n - \lambda_n A x_n)$, for every $n = 0, 1, 2, \ldots$. Let $u \in F(S) \cap VI(C, A)$.

Since $I - \lambda_n A$ is nonexpansive and

$$u = P_C(u - \lambda_n A u),$$

we have

$$\begin{aligned}
\|y_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\
&\le \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \\
&\le \|(I - \lambda_n A)x_n - (I - \lambda_n A)u\| \\
&\le \|x_n - u\|
\end{aligned}$$

for every $n = 0, 1, 2, \ldots$. From (14), we note that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)(S_n y_n - u)\|^2 \\
&\le \alpha_n\|(x_n - u)\|^2 + (1-\alpha_n)\|(S_n y_n - u)\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2\}
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - u\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2 \\
&\le \|x_n - u\|^2 + (1-d)a(b-2\alpha)\|Ax_n - Au\|^2 \\
&\le \|x_n - u\|^2
\end{aligned}
$$

for all $n \in \mathbb{N}$. This implies that

$$\|x_{n+1} - u\| \le \|x_n - u\| \tag{15}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \|x_n - u\|$ exists and so $Ax_n - Au \to 0$. Then $\{x_n\}$ and $\{y_n\}$ are bounded. From (10), we have

$$
\begin{aligned}
\|y_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \langle y_n - u, (x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\rangle \\
&= (1/2)\{\|y_n - u\|^2 + \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 - \|(y_n - u) - [(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)]\|^2\} \\
&\le (1/2)\{\|y_n - u\|^2 + \|x_n - u\|^2 - \|(y_n - x_n) + \lambda_n(Ax_n - Au)\|^2\} \\
&= (1/2)\{\|y_n - u\|^2 + \|x_n - u\|^2 - \|(y_n - x_n)\|^2 - 2\lambda_n\langle y_n - x_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2\}.
\end{aligned}
$$

So, we obtain

$$\|y_n - u\|^2 \le \|x_n - u\|^2 - \|y_n - x_n\|^2 - 2\lambda_n\langle y_n - x_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2$$

and hence

$$
\begin{aligned}
\|x_{n+1} - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|S_n y_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\
&\le \|x_n - u\|^2 - (1-\alpha_n)\|y_n - x_n\|^2 - 2\lambda_n(1-\alpha_n)\langle y_n - x_n, Ax_n - Au\rangle - \lambda_n^2(1-\alpha_n)\|Ax_n - Au\|^2 \\
&\le \|x_n - u\|^2 - (1-d)\|y_n - x_n\|^2 - 2\lambda_n(1-\alpha_n)\langle y_n - x_n, Ax_n - Au\rangle - \lambda_n^2(1-\alpha_n)\|Ax_n - Au\|^2.
\end{aligned}
$$

Since

$$\lim_{n\to\infty} \|x_n - u\|^2 = \lim_{n\to\infty} \|x_{n+1} - u\|^2$$

and

$$Ax_n - Au \to 0,$$

we obtain

$$y_n - x_n \to 0. \tag{16}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to z. Finally, we shall show that

$$z \in F(S) \cap VI(C, A).$$

First, we show that $z \in VI(C, A)$. By the same argument as in the proof of Theorem 3.1 in [12, pp. 423–424], we note that $z \in VI(C, A)$.

Next, we show that $z \in F(S)$. Let

$$u \in F(S) \cap VI(C, A).$$

Since

$$\|S_n y_n - u\| \le \|y_n - u\| \le \|x_n - u\|,$$

we have

$$\limsup_{n\to\infty} \|S_n y_n - u\| \le c,$$

where

$$c = \lim_{n\to\infty} \|x_n - u\|.$$

Furthermore, we have

$$\lim_{n\to\infty} \|\alpha_n(x_n - u) + (1-\alpha_n)(S_n y_n - u)\| = \lim_{n\to\infty} \|x_{n+1} - u\| = c.$$

By Lemma 2, we have

$$\lim_{n\to\infty} \|S_n y_n - x_n\| = 0. \tag{17}$$

Since $\{y_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty}\{\|S_n z - S_{n+1} z\| : z \in \{y_n\}\} < \infty.$$

By applying (16) and (17) and Lemma 1, we note that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - Sy_n\| + \|Sy_n - S_n y_n\| + \|S_n y_n - x_n\| \\ &\le \|x_n - y_n\| + \sup\{\|S_n z - Sz\| : z \in \{y_n\}\} + \|S_n y_n - x_n\| \to 0.\end{aligned}$$

Hence $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$. Since $x_{n_i} \rightharpoonup z$, it follows by the demiclosedness principle of S that $z \in \mathrm{F}(S)$. Hence $z \in F(S) \cap VI(C, A)$.

We will prove that $x_n \rightharpoonup z$. Suppose that there exist $\{x_{m_j}\} \subset \{x_n\}$ and $z' \neq z$ such that $x_{m_j} \rightharpoonup z'$. So, we have $z' \in \mathrm{F}(S) \cap VI(C, A)$. From Opial's condition, it follows that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z\| \lim_{i\to\infty}\|x_{n_i} - z\| &< \lim_{i\to\infty}\|x_{n_i} - z'\| \\ &= \lim_{j\to\infty}\|x_{m_j} - z'\| < \lim_{j\to\infty}\|x_{m_j} - z\| \\ &= \lim_{n\to\infty}\|x_n - z\|,\end{aligned}$$

which leads to a contradiction. Hence $x_n \rightharpoonup z \in \mathrm{F}(S) \cap VI(C, A)$. Finally we prove that $\lim_{n\to\infty} z_n = z$, where $z_n = P_{\mathrm{F}(S)\cap VI(C,A)} x_n$ for each $n \in \mathbb{N}$. By (15) and Lemma 3, there is $z_0 \in \mathrm{F}(T)$ such that $z_n \to z_0$. From $z_n = P_{\mathrm{F}(S)\cap VI(C,A)} x_n$ and $z \in \mathrm{F}(S) \cap VI(C, A)$, we have

$$\langle x_n - z_n, z_n - z\rangle \ge 0, \quad \text{for all } n \in \mathbb{N}.$$

It follows from $z_n \to z_0$ and $x_n \rightharpoonup z$ that

$$\langle z - z_0, z_0 - z\rangle \ge 0$$

and then $z_0 = z$. This completes the proof. □

Setting $S_n \equiv S$ in Theorem 4, we immediately obtain the following result.

Corollary 5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly-monotone mapping of C into H. Let S be a nonexpansive mapping from C into itself such that $\mathrm{F}(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n),$$

for all $n = 0, 1, 2, \dots$, where $0 < a < \lambda_n < b < 2\alpha$, $0 < c < \alpha_n < d < 1$ and $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to $z \in \mathrm{F}(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{\mathrm{F}(S)\cap VI(C,A)} x_n$.

By using the same argument presented in the proof of Theorem 4, we have the following theorem.

Theorem 6 ([8, Corollary 6]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty}\mathrm{F}(S_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n x_n,$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. Suppose that $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty}\mathrm{F}(S_n)$. Then $\{x_n\}$ converges weakly to $z \in \mathrm{F}(S)$, where $z = \lim_{n\to\infty} P_{\mathrm{F}(S)} x_n$.

Proof. Putting $y_n = x_n$ in the proof of Theorem 4. Then, by using the same argument as in the proof of Theorem 4, we can show that $\{x_n\}$ converges weakly to a point $z \in F(S)$, where $z = \lim_{n\to\infty} P_{F(S)} x_n$. □

4. Applications

4.1. Equilibrium problems

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \ge 0 \quad \text{for all } y \in C. \tag{18}$$

The set of solutions of (18) is denoted by EP(F). Numerous problems in physics, optimization, and economics can be reduced to find a solution of (18). Some methods have been proposed to solve the equilibrium problem (see [2,4,7,11]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [2]):

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for any $x, y \in C$;
(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1 - t)x, y) \le F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [2, Corollary 1] and [3, Lemma 2.12], we have the following lemma.

Lemma 7. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *and let $r > 0$ and $x \in H$. Then there exists unique $x^* \in C$ such that*

$$F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \ge 0 \quad \text{for all } y \in C.$$

Moreover, let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(ii) $\mathrm{F}(T_r) = \mathrm{EP}(F)$;
(iii) $\mathrm{EP}(F)$ *is closed and convex.*

Using [8, Theorem 16], we have the following lemma.

Lemma 8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let $\{r_n\}$ be a sequence of positive integers and T_{r_n} be the mapping defined as in Lemma 7. If* $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *then the following hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} < \infty$ *for any bounded subset B of C,*
(i) $F(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_{r_n})$ *where T is a mapping defined by* $Tx = \lim_{n\to\infty} T_{r_n} x$ *for all $x \in C$.*

Using Theorem 4 and Lemma 8, we have the following theorem.

Theorem 9. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let A be an α-inverse-strongly-monotone mapping of C into H such that* $VI(C, A) \cap \mathrm{EP}(F) \ne \emptyset$. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) u_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ *and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges weakly to $w \in VI(C, A) \cap \mathrm{EP}(F)$. Moreover, $w = \lim_{n\to\infty} P_{VI(C,A)\cap \mathrm{EP}(F)} x_n$.*

Using Theorem 6 and Lemma 8, we have the following theorem.

Theorem 10 (*Nilsrakoo and Saejung [8, Theorem 16]*). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *with* $\mathrm{EP}(F) \ne \emptyset$. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) u_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ *and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges weakly to $w \in \mathrm{EP}(F)$. Moreover, $w = \lim_{n\to\infty} P_{\mathrm{EP}(F)} x_n$.*

4.2. Accretive operator

In this section, we consider the problem of finding a zero of an accretive operator. An operator $T \subset H \times H$ is said to be accretive if for each (x_1, y_1) and $(x_2, y_2) \in T$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_2 - y_1, j \rangle \geq 0$. An accretive operator T is *m-accretive* if $R(I + rT) = H$ for each $r > 0$. An accretive operator T is said to satisfy the range condition if $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$, where $D(T)$ is the domain of T, I is the identity mapping on H, $R(I + rT)$ is the range of $I + rT$, and $\overline{D(T)}$ is the closure of $D(T)$. If T is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$, a mapping $J_r : R(I + rT) \to D(T)$ by $J_r = (I + rT)^{-1}$, which is called the *resolvent* of T. We know that J_r is nonexpansive and $F(J_r) = T^{-1}0$ for all $r > 0$ (see [1]).

Lemma 11. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T \subset C \times C$ be an accretive operator such that $T^{-1}0 \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I+rT)$, and $\{r_n\}$ be a sequence in $(0, \infty)$. If $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then the following hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}}z - J_{r_n}z\| : z \in B\} < \infty$ *for any bounded subset B of C,*
(ii) $F(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n})$, *where S is a mapping defined by $Sx = \lim_{n\to\infty} J_{r_n}x$ for all $x \in C$.*

Using Theorem 4 and Lemma 11, we have the following theorem.

Theorem 12. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly-monotone mapping of C into H. Let $T \subset C \times C$ be an accretive operator such that $T^{-1}0 \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} P_C(x_n - \lambda_n A x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a nonexpansive mapping defined by $Sx = \lim_{n\to\infty} J_{r_n}x$ for all $x \in C$. If $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges weakly to $z \in T^{-1}(0) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{T^{-1}(0)\cap VI(C,A)} x_n$.

Proof. By Lemma 11, we have the following

$$\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n}) = T^{-1}(0) \neq \emptyset.$$

Therefore, by Theorem 4, we obtain that $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} P_{T^{-1}(0)\cap VI(C,A)} x_n$. □

4.3. Monotone mappings

A mapping $T : C \to C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \quad \text{for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. It is known that, A is $\frac{1-k}{2}$-inverse-strongly-monotone and $A^{-1}(0) = F(T)$.

Now, using Theorem 4 we state a strong convergence theorem for a pair of nonexpansive mapping and strictly pseudocontractive mapping as follows.

Theorem 13. *Let C be a closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself and let f be a contraction of H into itself. Let T be a strictly pseudocontractive mapping with constant k of C into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n P_C((1 - \lambda_n)x_n + \lambda_n T x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{r_n\}$ is a sequence in $(0, \infty)$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a nonexpansive mapping defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$. If $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges weakly to $z \in \mathrm{F}(S) \cap F(T)$, where $z = \lim_{n\to\infty} P_{\mathrm{F}(S)\cap F(T)} x_n$.

Proof. Put $A = I - T$. Then A is $\frac{1-k}{2}$-inverse-strongly-monotone. We have that $F(T)$ is the solution set of $VI(A, C)$ i.e., $F(T) = VI(A, C)$ and

$$P_C(x_n - \lambda_n A x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n.$$

Therefore, by Theorem 4, the conclusion follows. □

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Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition ☆

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ABSTRACT

In this work, we establish new coincidence and common fixed point theorems for hybrid strict contraction maps by dropping the assumption "f is T-weakly commuting" for a hybrid pair (f, T) of multivalued maps in Theorem 3.10 of T. Kamran (2004) [8]. As an application, an invariant approximation result is derived.

1. Introduction

The theory of nonlinear analysis has emerged as a momentous mathematical discipline during the last 50 years. The fixed point theorem, generally known as the Banach Contraction Mapping Principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis.

In 1969, the Banach Contraction Mapping Principle was extended nicely to set-valued or multivalued mappings, a fact first noticed by Nadler [1]. Sessa [2] introduced the concept of weakly commuting maps. Jungck [3] defined the notion of compatible maps to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true [3]. In recent years, a number of fixed point theorems have been obtained by various authors utilizing this notion. Jungck further weakens the notion of compatibility by introducing the notion of weak compatibility and in [4] Jungck and Rhoades further extended weak compatibility to the setting of single-valued and multivalued maps. In [5] Singh and Mishra also introduced the notion of (I, T)-commutativity for a hybrid pair of single-valued and multivalued maps. In 2002, Aamri and El Moutawakil [6] defined the property $(E.A.)$ for self-maps and obtained some fixed point theorems for such mappings under strict contractive conditions. The class of maps satisfying property $(E.A.)$ contains the class of noncompatible maps. In 2003, Hussain and Khan [7] proved more general invariant approximation results for 1-subcommuting maps. Recently, Kamran [8] defined the property "f is T-weakly commuting" as follows:

Definition 1.1 (*[8]*). Assume that (X, d) is a metric space and $x \in X$. Let $f : X \to X$ and $T : X \to CB(X)$. The map f is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

Theorem 1.2 (*Theorem 3.10 [8]*). *Let f be a self-mapping of a metric space (X, d) and T be a map from X into $CB(X)$ such that:*
(i) *f and T satisfy the property $(E.A.)$.*

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(ii) *For all* $x \neq y \in X$

$$H(Tx, Ty) \leq \max\left\{d(fx, fy), \frac{1}{2}[d(fx, Tx) + d(fy, Ty)], \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right\}. \tag{1}$$

(iii) f *is* T*-weakly commuting at* v *and* $ffv = fv$ *for* $v \in C(f, T)$.

If fX *is a closed subset of* X*, then* f *and* T *have a common fixed point.*

The aim of this work is to drop the assumption of "f is T-weakly commuting" in the above theorem, establish some new coincidence and common fixed point theorems for hybrid strict contraction maps and derive an invariant approximation result.

2. Preliminaries

Throughout the work, X denotes a metric space with metric d.

Definition 2.1. We denote by $CB(X)$, the families of all nonempty bounded closed subsets of X. The Hausdorff metric induced by d on $CB(X)$ is given by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for $A, B \in CB(X)$, where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from a to $B \subseteq X$.

Definition 2.2. Let $f : X \to X$ and $T : X \to CB(X)$.

1. A point $x \in X$ is a fixed point of f (resp. T) iff $x = fx$ (resp. $x \in Tx$).
   The set of all fixed points of f (resp. T) is denoted by $F(f)$ (resp. $F(T)$).
2. A point $x \in X$ is a coincidence point of f and T iff $fx \in Tx$.
   The set of all coincidence points of f and T is denoted by $C(f, T)$.
3. A point $x \in X$ is a common fixed point of f and T iff $x = fx \in Tx$.
   The set of all common fixed points of f and T is denoted by $F(f, T)$.

Definition 2.3 ([6,8]). Let f be a self-mapping of a metric space (X, d) and T be a map from X into $CB(X)$. We say that f and T satisfy the property $(E.A.)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n, \tag{2}$$

for some $t \in X$ and $A \in CB(X)$.

For examples of the property $(E.A.)$ see [6,8].

Definition 2.4 ([9]). The maps $f : X \to X$ and $T : X \to CB(X)$ are said to be compatible if $fTx \in CB(X)$ for all $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to A \in CB(X)$ and $fx_n \to t \in A$.

The maps $f : X \to X$ and $T : X \to CB(X)$ are noncompatible if $fTx \in CB(X)$ for all $x \in X$ and there exists at least one sequence $\{x_n\}$ in X such that $Tx_n \to A \in CB(X)$ and $fx_n \to t \in A$ but $\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0$ or is nonexistent.

Remark 2.5. The noncompatible hybrid pair (f, T) satisfy the property $(E.A.)$.

3. Main results

Theorem 3.1. *Let* f *be self-mapping of a metric space* (X, d) *and* T *be a mapping from* X *into* $CB(X)$ *such that the following conditions are satisfied:*

(i) f *and* T *satisfy the property* $(E.A.)$ *and* (1) *holds.*
(ii) $ffv = fv$ *for* $v \in C(f, T)$.

If fX *is a closed subset of* X*, then* f *and* T *have a common fixed point.*

Proof. By virtue of (2) there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in CB(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Since fX is closed, we have $\lim_{n\to\infty} fx_n = fx$ for some $x \in X$. Thus $t = fx \in A$. We claim that $fx \in Tx$. If not, then condition (1) implies

$$H(Tx_n, Tx) \le \max\left\{d(fx_n, fx), \frac{1}{2}[d(fx_n, Tx_n) + d(fx, Tx)], \frac{1}{2}[d(fx_n, Tx) + d(fx, Tx_n)]\right\}.$$

Letting $n \to \infty$, we have

$$\begin{aligned} H(A, Tx) &\le \max\left\{d(fx, fx), \frac{1}{2}[d(fx, A) + d(fx, Tx)], \frac{1}{2}[d(fx, Tx) + d(fx, A)]\right\} \\ &= \frac{1}{2}d(fx, Tx). \end{aligned}$$

Since $fx \in A$, it follows from the definition of the Hausdorff metric that

$$d(fx, Tx) \le H(A, Tx) \le \frac{1}{2}d(fx, Tx),$$

which is a contradiction. Hence $fx \in Tx$ and $C(f, T) \neq \emptyset$. From this, and $ffv = fv$ for some $v \in C(f, T)$, we let $z := fv$. This implies that

$$z = fv = ffv = fz \in Tv.$$

We claim that $v = z$. Assume not; then condition (1) implies

$$\begin{aligned} H(Tv, Tz) &\le \max\left\{d(fv, fz), \frac{1}{2}[d(fv, Tv) + d(fz, Tz)], \frac{1}{2}[d(fv, Tz) + d(fz, Tv)]\right\} \\ &= \max\left\{d(fv, fz), \frac{1}{2}[d(fv, Tv) + d(fz, Tz)], \frac{1}{2}[d(fz, Tz) + d(fz, Tv)]\right\} \\ &= \frac{1}{2}d(fz, Tz). \end{aligned}$$

Since $fv \in Tv$, it follows from the definition of the Hausdorff metric that

$$d(fz, Tz) \le H(Tv, Tz) \le \frac{1}{2}d(fz, Tz),$$

which is a contradiction. Hence $v = z$. Thus $z = fz \in Tz$. □

Remark 3.2. Theorem 3.1 extends and improves the Banach Contraction Principle, Nadler's Contraction Principle [1], Theorem 3.10 of Kamran [8], and results of many authors.

Now we give an example to support our result.

Example 3.3. Let $X = [0, 1]$ with the usual metric. Define $f : X \to X$ and $T : X \to CB(X)$ by $fx = \frac{x}{2}$ and $Tx = [0, \frac{x}{4}]$ for all $x \in X$. Then:

(1) f and T satisfy the property $(E.A.)$ for the sequence $x_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$ and (1) holds.
(2) $ff0 = f0$ for $0 \in C(f, T)$.

Thus the conditions (i) and (ii) of Theorem 3.1 are satisfied and $0 = f0 \in T0$, i.e., 0 is a common fixed point of f and T.

Remark 3.4. (i) If $C(f, T)$ is singleton set, then the common fixed point of f and T is the limit of the sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n,$$

for some $t \in X$ and $A \in CB(X)$.

(ii) If $C(f, T)$ is not a singleton set, then the common fixed point of f and T is $v \in C(f, T)$ for $ffv = fv$.

Corollary 3.5. *Let T be a map from a metric space (X, d) into $CB(X)$ such that the following conditions hold:*

(i) *There exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} x_n \in A = \lim_{n\to\infty} Tx_n,$$

for some $A \in CB(X)$.

(ii) *For all $x \neq y \in X$*

$$H(Tx, Ty) \le \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

Then T has a fixed point.

Proof. Let $f : X \to X$ be the identity mapping. It follows from Theorem 3.1 that T has a fixed point. $\square$

Theorem 3.6. *Let f be a self-mapping of a metric space (X, d) and T be a map from X into $CB(X)$ such that the following conditions hold:*

(i) *f and T satisfy the property $(E.A.)$ and (1) holds.*
(ii) *$ffv = fv$ for $v \in C(f, T)$.*

If TX is a closed subset of X and $TX \subseteq fX$, then f and T have a common fixed point.

Proof. By virtue of (2) there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in CB(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Since TX is closed, we have $\lim_{n\to\infty} Tx_n = Ta$ for some $a \in X$. It follows from $TX \subseteq fX$ that $t \in Ta \subseteq TX \subseteq fX$. Thus $t = fx \in A$. Now the results follow from Theorem 3.1. $\square$

Since a noncompatible hybrid pair (f, T) satisfy the property $(E.A.)$, we get the following results:

Corollary 3.7. *Let f be a self-mapping of a metric space (X, d) and T be a map from X into $CB(X)$ such that the following conditions hold:*

(i) *f and T are noncompatible and (1) holds.*
(ii) *$ffv = fv$ for $v \in C(f, T)$.*

If fX is a closed subset of X, then f and T have a common fixed point.

Corollary 3.8. *Let f be a self-mapping of a metric space (X, d) and T be a map from X into $CB(X)$ such that the following conditions hold:*

(i) *f and T are noncompatible and (1) holds.*
(ii) *$ffv = fv$ for $v \in C(f, T)$.*

If TX is a closed subset of X and $TX \subseteq fX$, then f and T have a common fixed point.

Invariant approximation for non-commuting maps was considered for the first time by Shahzad [10,11]. Let M be a subset of a normed space X and $p \in X$. The set $B_M(p) := \{x \in M : \|x - p\| = d(p, M)\}$ is called the set of best M-approximates to $p \in X$ out of M.

Theorem 3.9. *Let f be self-mapping of a normed space X and T be a map from X into $CB(X)$ such that the following conditions are satisfied:*

(i) *f and T satisfy the property $(E.A.)$ and (1) holds on $B_M(p)$.*
(ii) *$ffv = fv$ for $v \in C(f, T) \cap B_M(p)$.*
(iii) *$f(B_M(p)) = B_M(p)$.*
(iv) *$\sup_{y\in Tx} \|y - p\| \le \|fx - p\|$ for all $x \in B_M(p)$.*

If $f(B_M(p))$ is a closed subset of $B_M(p)$, then $F(f, T) \cap B_M(p) \neq \emptyset$.

Proof. Let $x \in B_M(p)$ and $z \in Tx$. Since $f(B_M(p)) = B_M(p)$, so $fx \in B_M(p)$ for all $x \in B_M(p)$. It follows from the definition of $B_M(p)$ that $\|fx - p\| = d(p, M)$. Since

$$\|z - p\| \le \sup_{y\in Tx} \|y - p\| \le \|fx - p\| = d(p, M),$$

so $z \in B_M(p)$. Thus $Tx \subseteq B_M(p)$ for all $x \in B_M(p)$. Since Tx is closed for all $x \in X$, so Tx is closed for all $x \in B_M(p)$. Therefore $f|_{B_M(p)} : B_M(p) \to B_M(p), T|_{B_M(p)} : B_M(p) \to CB(B_M(p))$. Clearly, $F(f|_{B_M(p)}, T|_{B_M(p)}) = F(I, T) \cap B_M(p)$. Now the result follows from Theorem 3.1 with $X = B_M(p)$. $\square$

Remark 3.10. Theorem 3.9 extends and improves theorems of Khan, Domlo and Hussain [12], and results of many authors.

Corollary 3.11. *Let X be a normed space and T be a map from X into $CB(X)$ such that the following conditions hold:*

(i) *There exists a sequence $\{x_n\}$ in $B_M(p)$ such that*

$$\lim_{n\to\infty} x_n \in A = \lim_{n\to\infty} Tx_n,$$

for some $A \in CB(B_M(p))$.

(ii) *For all $x \neq y \in B_M(p)$*

$$H(Tx, Ty) \le \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

(iii) *$\sup_{y\in Tx} \|y - p\| \le \|x - p\|$ for all $x \in B_M(p)$.*

Then $F(T) \cap B_M(p) \neq \emptyset$.

Proof. Take f as the identity mapping from X into X in Theorem 3.9 to get $F(T) \cap B_M(p) \neq \emptyset$. $\square$

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Weak Convergence Theorem by an Extragradient Method for Variational Inequality, Equilibrium and Fixed Point Problems

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Abstract. In this paper, we introduce a new iterative scheme for finding the common element of the set of: fixed points; equilibrium; and the variational inequality problems for monotone and k-Lipschitz continuous mappings. The iterative process is based on the so-called extragradient method. We show that the sequence converges weakly to a common element of the above three sets under some parameter controlling conditions. This main theorem extends a recent result of Nadezhkiha and Takahashi [7].

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \text{for all} \quad y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \longrightarrow H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality is denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0,$$

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for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [12, 14] and the references therein. A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0,$$

for all $u, v \in C$ and A is called α-inverse-strongly monotone [2, 6] if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2,$$

for all $u, v \in C$. A mapping A of C into H is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Au - Av\| \leq k\|u - v\|, \quad \text{for all} \quad u, v \in C.$$

It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|,$$

for all $u, v \in C$. We denote the set of fixed points of S by $F(S)$. To find an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [11] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{1.2}$$

for every $n = 0, 1, 2, ...,$ where $x_0 = x \in C, \alpha_n$ is a sequence in (0, 1), and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [7] and Zeng and Yao [15] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$.

In 1976, Korpelevič [5] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda A\bar{x}_n) \end{cases} \tag{1.3}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(A, C)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.3), converge to the same point $z \in VI(A, C)$. Recently, motivated by the idea of Korpelevičs extragradient method [5], Nadezhkina and Takahashi [7] introduced the following iterative scheme for finding an element of $F(S) \cap VI(A, C)$ and proved the following weak convergence theorem.

Theorem 1.1. [7, Theorem 3.1] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and a k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \ \forall n \geq 0, \end{cases} \tag{1.4}$$

where $\alpha_n \subset [c,d]$ *for some* $c, d \in (0,1)$ *and* $\lambda_n \subset [a,b]$ *for some* $a, b \in (o, \frac{1}{k})$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to the same point* $z \in F(S) \cap VI(A,C)$ *where* $z = \lim_{n \longrightarrow \infty} P_{F(S) \cap VI(A,C)} x_n$.

In 2007, Plubtieng and Punpaeng [8] introduced a new iterative scheme for finding a common element of the set of solutions of: an equilibrium; the variational inequality; and the set of fixed points problems of a nonexpansive mapping in a Hilbert space.

In this paper motivated by the iterative schemes considered in [7, 11], we will introduce a new iterative process (3.1) which is different from the Plubtieng and Punpaeng iterative schemes [8], the aim is to find a common element of the set of solutions, the fixed points of a nonexpansive mapping, an equilibrium, and the variational inequality problems of a k-Lipschitz continous mapping in a real Hilbert space. Then, we prove a weak convergence theorem, which is connected with Korpelevičs extragradient method [5] and Nadezhkina and Takahashi's result [7].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{2.2}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.3}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A,C) \Leftrightarrow u = P_C(u - \lambda A u), \lambda > 0. \tag{2.4}$$

It is also known that H satisfies the Opial condition; for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|, \tag{2.5}$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas will be useful for proving the convergent result of this paper.

Lemma 2.1. [9] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n \longrightarrow \infty} \beta_n \le \limsup_{n \longrightarrow \infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n) y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n \longrightarrow \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n \longrightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.2. [4] (Demiclosedness Principle) Let H be a Hilbert space, C a closed convex subset of H, and $T : C \longrightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. Here, I is the identity operator of H.

For solving the equilibrium problem for a bifunction $F : C \times C \longrightarrow R$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \longrightarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.3. [1] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into R satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \textit{ for all } y \in C.$$

The following lemma was also given in [3].

Lemma 2.4. [3] *Assume that $F : C \times C \longrightarrow R$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\},$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(3) *$F(T_r) = EP(F)$; and*
(4) *$EP(F)$ is closed and convex.*

3. Weak convergence theorems

In this section, we prove a weak convergence theorem for finding a common element of the set of solutions for an equilibrium problem, the set variational inequality and the set of fixed points of a nonexpansive mapping in a Hilbert space. Before proving the theorem, we need the following lemmas.

Lemma 3.1. [11] *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, ...,$ and let $\{v_n\}$ and $\{w_n\}$ be sequences of H such that $\limsup_{n \longrightarrow \infty} \|v_n\| \leq c$, $\limsup_{n \longrightarrow \infty} \|w_n\| \leq c$, and $\lim_{n \longrightarrow \infty} \|\alpha_n v_n + (1 - \alpha_n)w_n\| = c$, for some $c \geq 0$, then $\lim_{n \longrightarrow \infty} \| v_n - w_n\| = 0$.*

Lemma 3.2. [11] *Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|,$$

for every $n = 0, 1, 2, ...,$ then the sequence $\{P_D(x_n)\}$ converges strongly to some $z \in D$, where P_D stands for the metric projection of H onto D.

Now, we show the following weak convergence theorem, which solves the problem of finding a common element of the set solutions of: an equilibrium; variational inequality; and fixed point problems of a nonexpansive mapping in Hilbert spaces.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow R$ satisfying (A1)–(A4) and let A be a monotone k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$(3.1) \qquad \begin{cases} u_n \in C; F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset (a, b) \subset (0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 1/k)$ and $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:

(i) $\liminf_{n\longrightarrow\infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$ *and*

(ii) $\lim_{n\longrightarrow\infty} \lambda_n = 0,$ *then $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converges weakly to the same point $p \in F(S) \cap VI(A, C) \cap EP(F)$,*
where $p = \lim_{n\to\infty} P_{F(S)\cap VI(A,C)\cap EP(F)} x_n$.

Proof. We divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded. Indeed, let $x^* \in F(S) \cap VI(A, C) \cap EP(F)$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.4. Then $x^* = P_C(x^* - \lambda_n A x^*) = T_{r_n} x^*$ and $u_n = T_{r_n} x_n$. Putting $v_n = P_C(x_n - \lambda_n A y_n)$, we note that

$$(3.2) \qquad \|u_n - x^*\|^2 \;=\; \|T_{r_n} x_n - T_{r_n} x^*\|^2 \leq \|x_n - x^*\|^2.$$

From (2.3) and the monotonicity of A, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|x_n - \lambda_n A y_n - x^*\|^2 - \|x_n - \lambda_n A y_n - v_n\|^2 \\ &= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n \langle A y_n, x^* - v_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 \\ &\quad + 2\lambda_n(\langle A y_n - A x^*, x^* - y_n\rangle + \langle A x^*, x^* - y_n\rangle) + \langle A y_n, y_n - v_n\rangle \\ &\leq \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n \langle A y_n, y_n - v_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, y_n - v_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\langle x_n - \lambda_n A y_n - y_n, v_n - y_n\rangle. \end{aligned}$$

Moreover, since $y_n = P_C(u_n - \lambda_n A u_n)$ and from (2.2), we have

$$(3.3) \qquad \langle x_n - \lambda_n A x_n - y_n, v_n - y_n\rangle \leq 0.$$

Where A is k-Lipschitz continuous, so that

$$\begin{aligned}\langle x_n - \lambda_n A y_n - y_n, v_n - y_n \rangle &= \langle x_n - \lambda_n A x_n - y_n, v_n - y_n \rangle \\ &\quad + \langle \lambda_n A x_n - \lambda_n A y_n, v_n - y_n \rangle \\ &\leq \langle \lambda_n A x_n - \lambda_n A y_n, v_n - y_n \rangle \\ &\leq \lambda_n \|A x_n - A y_n\| \|v_n - y_n\| \\ &\leq \lambda_n k \|x_n - y_n\| \|v_n - y_n\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n k \|x_n - y_n\| \|v_n - y_n\| \\ &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + \lambda_n^2 k^2 \|x_n - y_n\|^2 + \|v_n - y_n\|^2 \\ &= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \qquad (3.4) \\ &\leq \|x_n - x^*\|^2,\end{aligned}$$

and we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n (x_n - x^*) + (1-\alpha_n)(S v_n - x^*)\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|S v_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2.\end{aligned}$$

Since the sequence $\{\|x^* - x_n\|\}$ is a bounded and nonincreasing sequence, $\lim_{n\to\infty} \|x_0 - x_n\|$ exists, that is there exists

$$c = \lim_{n\to\infty} \|x_n - x^*\|. \qquad (3.5)$$

Hence $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$ and $\{v_n\}$ are also bounded.

Step 2. We claim that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0$. Indeed, since P_C is a nonexpansive mapping, we obtain

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_C(u_{n+1} - \lambda_{n+1} A u_{n+1}) - P_C(u_n - \lambda_n A u_n)\| \\ &\leq \|(u_{n+1} - \lambda_{n+1} A u_{n+1}) - (u_n - \lambda_n A u_n)\| \\ &= \|(u_{n+1} - \lambda_{n+1} A u_{n+1}) - (u_n - \lambda_{n+1} A u_n) + (\lambda_n - \lambda_{n+1}) A u_n\| \\ &= \|(u_{n+1} - u_n) - \lambda_{n+1}(A u_{n+1} - A u_n) - (\lambda_{n+1} - \lambda_n) A u_n\| \\ &\leq \|u_{n+1} - u_n\| + \lambda_{n+1}\|A u_{n+1} - A u_n\| + (\lambda_{n+1} - \lambda_n)\|A u_n\| \\ &\leq \|u_{n+1} - u_n\| + k\lambda_{n+1}\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n| \|A u_n\|,\end{aligned}$$

by (ii), it follows that

$$\|y_{n+1} - y_n\| \leq \|u_{n+1} - u_n\|. \qquad (3.6)$$

Hence, we observe that

$$\|v_{n+1} - v_n\| = \|P_C(x_{n+1} - \lambda_{n+1} A y_{n+1}) - P_C(x_n - \lambda_n A y_n)\|$$

$$
\begin{aligned}
&\leq \|(x_{n+1} - \lambda_{n+1} A y_{n+1}) - (x_n - \lambda_n A y_n)\| \\
&= \|(x_{n+1} - \lambda_{n+1} A y_{n+1}) - (x_n - \lambda_{n+1} A y_n) + (\lambda_n - \lambda_{n+1}) A y_n\| \\
&= \|(x_{n+1} - x_n) - \lambda_{n+1}(A y_{n+1} - A y_n) - (\lambda_{n+1} - \lambda_n) A y_n\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_{n+1}\|A y_{n+1} - A y_n\| + (\lambda_{n+1} - \lambda_n)\|A y_n\| \\
&\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\|y_{n+1} - y_n\| + |\lambda_{n+1} - \lambda_n|\|A y_n\| \\
&\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|A y_n\|.
\end{aligned}
$$

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we note that

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \tag{3.7}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \text{ for all } y \in C. \tag{3.8}$$

Substituting $y = u_{n+1}$ into (3.7) and $y = u_n$ into (3.8), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\
&\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|, \qquad (3.9)
\end{aligned}
$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Hence, we have

$$
\begin{aligned}
\|v_{n+1} - v_n\| &\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|A y_n\| \\
&\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\{\|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|\} \\
&\quad + |\lambda_{n+1} - \lambda_n|\|A y_n\|
\end{aligned}
$$

$$\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\|x_{n+1} - x_n\| + k\lambda_{n+1}\frac{L}{c}|r_{n+1} - r_n| + |\lambda_{n+1} - \lambda_n|\|Ay_n\|. \tag{3.10}$$

Let $x_{n+1} = (1-\alpha_n)z_n + \alpha_n x_n$. Thus, we note that $z_n = Sv_n$ and we have

$$\|z_{n+1} - z_n\| = \|Sv_{n+1} - Sv_n\| \leq \|v_{n+1} - v_n\|. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| \\ &\leq k\lambda_{n+1}\|x_{n+1} - x_n\| + k\lambda_{n+1}\frac{L}{c}|r_{n+1} - r_n| \\ &\quad + |\lambda_{n+1} - \lambda_n|\|Ay_n\|. \end{aligned}$$

This inequality together with (i) and (ii) imply that

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.1, we obtain $\|z_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently,

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}(1-\alpha_n)\|z_n - x_n\| = 0. \tag{3.12}$$

From (i), (ii), (3.9) and (3.10), we also have
$\|v_{n+1} - v_n\| \longrightarrow 0, \|u_{n+1} - u_n\| \longrightarrow 0$ and $\|y_{n+1} - y_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently,

$$x_{n+1} - x_n = \alpha_n x_n + (1-\alpha_n)Sv_n - x_n = (1-\alpha_n)(Sv_n - x_n),$$

and from (3.12) we also have

$$\lim_{n\longrightarrow\infty}\|Sv_n - x_n\| = 0. \tag{3.13}$$

By (3.4), we obtain

$$\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2,$$

then, we also have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sv_n - x^*\|^2 \\ &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)S(v_n - x^*)\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|Sv_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 \\ &\quad + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2\} \\ &= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2, \end{aligned}$$

and hence

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2.$$

It follows that

$$(1-\alpha_n)(1-\lambda_n^2 k^2)\|x_n - y_n\| \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

So we have

$$(3.14) \qquad \|x_n - y_n\| \quad \leq \quad \frac{1}{(1-\alpha_n)(\lambda_n^2 k^2 - 1)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2).$$

Since $\{\alpha_n\} \subset (a,b) \subset (0,1)$, $\lambda_n \in (0, 1/k)$ and

$$\lim_{n\to\infty}\|x_{n+1} - x^*\| = \lim_{n\to\infty}\|x_n - x^*\| = c,$$

we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0.$$

We note that

$$\begin{aligned}
\|y_n - v_n\| &= \|P_C(u_n - \lambda_n A u_n) - P_C(x_n - \lambda_n A y_n)\| \\
&\leq \|(u_n - \lambda_n A u_n) - (x_n - \lambda_n A y_n)\| \\
&\leq \|u_n - x_n\| + \lambda_n \|A u_n - A y_n\| \\
&\leq \|u_n - x_n\| + k\lambda_n \|u_n - y_n\|.
\end{aligned}$$

Since $\lambda_n \longrightarrow 0\,(n \longrightarrow \infty)$, we get

$$(3.15) \qquad \|y_n - v_n\| \quad \leq \quad \|u_n - x_n\|.$$

From (3.15), we also have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + 2\lambda_n k \|x_n - y_n\| \|v_n - y_n\| \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + \|x_n - y_n\|^2 + \lambda_n^2 k^2 \|v_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|v_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2,
\end{aligned}$$

then, we also have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|S v_n - x^*\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2\} \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2,
\end{aligned}$$

and hence

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2.$$

It follows that

$$(1-\alpha_n)(1-\lambda_n^2 k^2)\|x_n - u_n\| \quad \leq \quad \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

So we have

$$(3.16) \qquad \|x_n - u_n\| \quad \leq \quad \frac{1}{(1-\alpha_n)(\lambda_n^2 k^2 - 1)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2).$$

Since $\{\alpha_n\} \subset (a,b) \subset (0,1)$, $\lambda_n \subset (0, \frac{1}{k})$ and

$$\lim_{n\to\infty}\|x_{n+1} - x^*\|^2 = \lim_{n\to\infty}\|x_n - x^*\|^2 = c^2,$$

thus

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0.$$

Since $\liminf_{n\longrightarrow\infty} r_n>0$, we obtain

$$\lim_{n\longrightarrow\infty}\|\frac{x_n-u_n}{r_n}\|=0.$$

From (3.15) we also have

$$\lim_{n\to\infty}\|y_n-v_n\|=0.$$

From

$$\|x_n-v_n\| \quad \leq \quad \|x_n-y_n\|+\|y_n-v_n\|.$$

we also have

$$\lim_{n\longrightarrow\infty}\|x_n-v_n\|=0.$$

And

$$\|Sv_n-v_n\|\leq\|Sv_n-x_n\|+\|x_n-y_n\|+\|y_n-v_n\|$$

and hence

$$\lim_{n\longrightarrow\infty}\|Sv_n-v_n\|=0.$$

Step 3. We claim that $p\in VI(A,C)\cap EP(F)\cap F(S)$.
First, we prove $p\in EP(F)$. Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to p. Without loss of generality, we can assume that $v_{n_i}\rightharpoonup p$. From $\|Sv_n-v_n\|\longrightarrow 0$, we obtain $Sv_{n_i}\rightharpoonup p$. Let us show $p\in EP(F)$. Since $u_n=T_{r_n}x_n$, we have

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq F(y,u_n)$$

and hence

$$\left\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle\geq F(y,u_{n_i}).$$

From $\|u_n-x_n\|\longrightarrow 0, \|x_n-Sv_n\|\longrightarrow 0$, and $\|Sv_n-v_n\|\longrightarrow 0$, we get $u_{n_i}\rightharpoonup p$. Since $\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\longrightarrow 0$, it follows by (A4) that $0\geq F(y,p)$ for all $y\in C$. For t with $0<t\leq 1$ and $y\in C$, let $y_t=ty+(1-t)p$. Since $y\in C$ and $p\in C$, we have $y_t\in C$ and hence $F(y_t,p)\leq 0$. So, from (A1) and (A4) we have

$$0=F(y_t,y_t)\leq tF(y_t,y)+(1-t)F(y_t,p)\leq tF(y_t,y)$$

and hence $0\leq F(y_t,y)$. From (A3), we have $0\leq F(p,y)$ for all $y\in C$ and hence $p\in EP(F)$.
Next, we prove that $p\in F(S)$. Let $x^*\in F(S)\cap VI(A,C)\cap EP(F)$. Since $\|Sv_n-x^*\|\leq\|v_n-x^*\|\leq\|x_n-x^*\|$, from (3.5), we have

$$\limsup_{n\longrightarrow\infty}\|Sv_n-x^*\|\leq c.$$

Furthermore, we have

$$\lim_{n\longrightarrow\infty}\|\alpha_n(x_n-x^*)+(1-\alpha_n)(Sv_n-x^*)\|=\lim_{n\longrightarrow\infty}\|x_{n+1}-x^*\|=c.$$

By Lemma 3.1, we obtain

$$\lim_{n\longrightarrow\infty}\|Sv_n-x_n\|=0.$$

Also, we have

$$\begin{aligned}\|Sx_n-x_n\| &\le \|Sx_n-Sv_n\|+\|Sv_n-x_n\|\\ &\le \|x_n-v_n\|+\|Sv_n-x_n\|.\end{aligned}$$

Therefore, we get

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0.$$

From the demiclosedness (Lemma 2.2) of $I-S$, we know that $x_{n_i}\rightharpoonup p$, and $\lim_{n\longrightarrow\infty}\|Sx_n-x_n\|=0$. We obtain $p\in F(S)$. Finally, by the same argument as in the proof of [7, Theorem 3.1, pp.197–198], we prove that $p\in VI(A,C)$. Hence $p\in VI(A,C)\cap EP(F)\cap F(S)$.

Step 4. We claim that $x_n\rightharpoonup p'(n\longrightarrow\infty)$.
Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup p'$, then we have $p'\in VI(A,C)\cap EP(F)\cap F(S)$. We may show that $p=p'$. Assume that $p\neq p'$, from the Opial condition, we get

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n-p\| &= \liminf_{n\longrightarrow\infty}\|x_{n_i}-p\|<\liminf_{n\longrightarrow\infty}\|x_{n_i}-p'\|\\ &= \lim_{n\longrightarrow\infty}\|x_n-p'\|=\liminf_{j\longrightarrow\infty}\|x_{n_j}-p'\|\\ &< \liminf_{j\longrightarrow\infty}\|x_{n_j}-p\|=\lim_{n\longrightarrow\infty}\|x_n-p\|.\end{aligned}$$

This is a contradiction. So, we have $p=p'$. This implies that

$$x_n\rightharpoonup p\in VI(A,C)\cap EP(F)\cap F(S).$$

Step 5. Finally we prove $p=\lim_{n\to\infty}P_{F(S)\cap VI(A,C)\cap EP(F)}x_n$.
Let $z_n=P_{VI(A,C)\cap EP(F)\cap F(S)}x_n$, so that for all $p\in VI(A,C)\cap EP(F)\cap F(S)$, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(x_n-p)+(1-\alpha_n)[SP_C(x_n-\lambda_n y_n)-p]\|\\ &\le \|x_n-p\|.\end{aligned}$$

By Lemma 3.2, we have that $\{z_n\}$ converges strongly to some $p_0\in VI(A,C)\cap EP(F)\cap F(S)$.
Since $\langle p-z_n,z_n-x_n\rangle\ge 0$ then, we have $\langle p-p_0,p_0-p\rangle\ge 0$, and hence $p=p_0=\lim_{n\to\infty}P_{VI(A,C)\cap EP(F)\cap F(S)}x_n$. This completes the proof. ∎

Using Theorem 3.1, we prove two corollaries in Hilbert spaces.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and a k-Lipschitz continuous mapping of C into H. Let S*

be a nonexpansive mapping from C into itself such that $F(S) \cap VI(A,C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0, \end{cases} \tag{3.17}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (a,b) \subset (0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,1/k)$. Then $\{x_n\}$ and $\{y_n\}$ converges weakly to the same point $z \in F(S) \cap VI(A,C)$ where $z = \lim_{n\to\infty} P_{F(S)\cap VI(A,C)} x_n$.

Proof. Put $F(x,y) = 0$ for all $x,y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1. Then, we have $u_n = P_C x_n = x_n$. So, from Theorem 3.1 the sequence $\{x_n\}$ generated in Corollary 3.1 converges weakly to $P_{F(S)\cap VI(A,C)} x_n$. ∎

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and a k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping from C into itself such that $VI(A,C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \ \forall n \geq 0, \end{cases} \tag{3.18}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (a,b) \subset (0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,1/k)$. Then $\{x_n\}$ and $\{y_n\}$ converges weakly to the same point $z \in VI(A,C)$ where $z = \lim_{n\to\infty} P_{VI(A,C)} x_n$.

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A RELAXED EXTRAGRADIENT APPROXIMATION METHOD OF TWO INVERSE-STRONGLY MONOTONE MAPPINGS FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES, FIXED POINT AND EQUILIBRIUM PROBLEMS†

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Abstract. We introduce and study an iterative sequence for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the solutions of the general system of variational inequality for two inverse-strongly monotone mappings. Under suitable conditions, some strong convergence theorems for approximating a common element of the above three sets are obtained. Moreover, using the above theorem, we also apply to finding solutions of a general system of variational inequalities and a zero of a maximal monotone operator in a real Hilbert space. As applications, at the end of paper we utilize our results to study the zeros of the maximal monotone and some convergence problem for strictly pseudocontractive mappings. Our results include the previous results as special cases extend and improve the results of Ceng et al. [4], Yao and Yao [18] and some others.

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1. **Introduction**

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in H$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Let f be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to H$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(f)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.1). In 1997 Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to initial data when $EP(f)$ is nonempty and proved a strong convergence theorem.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [1][6][17] [19] [20] and the references therein. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \tag{1.3}$$

for all $u, v \in C$. A mapping A of C into H is called α*-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.4}$$

for all $u, v \in C$. It is obvious that any α -inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. For finding an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [12] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.5}$$

for every $n = 0, 1, 2, ...,$ where $x_0 = x \in C, \alpha_n$ is a sequence in (0, 1), and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [7] and Zeng and Yao [20] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$. In 1976, Korpelevich [2] introduced the following so-called extragradient method:

$$(1.6) \qquad \begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A \bar{x}_n) \end{cases}$$

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C in to $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.6), converge to the same point $z \in VI(C, A)$.

Motivated by the idea of Korpelevichs extragradient method Zeng and Yao [20] introduced a new extragradient method for finding an element of $F(S) \cap VI(C, A)$ and obtained the following strong convergence theorem under some suitable conditions . Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:

$$(1.7) \qquad \begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n u + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \ \ \forall n \geq 0, \end{cases}$$

Then the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to the same point $P_{\mathrm{F}(S) \cap VI(C,A)} x_0$ proved that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. Later, Nadezhkina and Takahashi [7] and Zeng and Yao [20] proposed some new iterative schemes for finding elements in $F(S) \cap VI(C, A)$. In the same year, Yao and Yao [18] introduced the following iterative scheme: Let C be a closed convex subset of real Hilbert space H. Let A be an $\alpha-$ inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by (1.7) where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings under some parameter controlling conditions. After that, Plubtieng and Punpaeng [9] introduced an iterative scheme:

$$(1.8) \qquad \begin{cases} f(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n \rangle \geq 0, & \forall u \in C; \\ y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n), \end{cases}$$

for approximating a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

Let C be a closed convex subset of real Hilbert space H. Let $A, B : C \to H$ be two mappings. We consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.9}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.9) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.10}$$

which is defined by Verma [14] and Verma [15], and is called the new system of variational inequalities. Further, if $x^* = y^*$, then problem (1.10) reduces to the classical variational inequality $VI(C, A)$.

Recently, Ceng et al. [4], introduced the following iterative scheme by a relaxed extragradient method. Let the mappings $A, B : C \longrightarrow H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : C \longrightarrow C$ be a nonexpansive mapping and suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + S P_C(y_n - \lambda A y_n), \ \ n \geq 1, \end{cases} \tag{1.11}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$. Then, they proved that the iterative sequence $\{x_n\}$ converges strongly to some point $x_0 \in C$.

In this paper, motivated and inspired by the above results, we will introduce a new iterative scheme (3.1) below for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solutions of a general system of variational inequality problem for two inverse-strongly-monotone mappings in a Hilbert space. Then we prove some strong convergence theorems which are connected with Ceng et al. [4] Takahashi and Takahashi's result [13] and Zeng and Yao's result [20]. Our results of this paper extended and improved the corresponding results of Ceng et al. [4], Plubtieng and Punpaeng [9], Su et al. [10] and many others.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(C,A) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{2.6}$$

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. (Osilike and Igbokwe [8]) *Let $(E, \langle .,.\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.2. (Suzuki [11]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. (Goebel and Kirk [3]) *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.4. (Xu [16]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \; n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} f(tz + (1 - t)x, y) \leq f(x, y)$;
(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1]

Lemma 2.5. (Blum and Oettli [1]) *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [5].

Lemma 2.6. (Combettes and Hirstoaga [5]) *Assume that $f : C \times C \to \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single- valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.7. (Ceng et al. [4, Lemma 2.1]) *For given $x^*, y^* \in C \times C, (x^*, y^*)$ is a solution of problem (1.9) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$ λ, μ are positive constants and $A, B : C \to H$ are two mappings.

Remark 2.8. *Let $\mathcal{A} : C \to H$ be an α-inverse-strongly-monotone. For each $u, v \in C$ and $\lambda > 0$, we have*

$$\begin{aligned} \|(I - \lambda\mathcal{A})u - (I - \lambda\mathcal{A})v\|^2 &= \|(u - v) - \lambda(\mathcal{A}u - \mathcal{A}v)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, \mathcal{A}u - \mathcal{A}v\rangle \\ &\quad + \lambda^2\|\mathcal{A}u - \mathcal{A}v\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|\mathcal{A}u - \mathcal{A}v\|^2. \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda\mathcal{A}$ is a nonexpansive mapping from C to H.

We note that the mapping $G : C \to C$ is a nonexpansive mapping provided $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$.
Throughout this paper, the set of fixed points of the mapping G is denote by Ω.

3. Main results

In this section, we introduce an iterative scheme by the relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the general system of variational inequality problem for two inverse-strongly monotone mappings in a real Hilbert space. We prove that the iterative sequences converge strongly to a common element of the above three sets.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and $A, B : C \longrightarrow H$ be α and β-inverse-strongly monotone mappings, respectively. Let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \cap EP(f) \neq \emptyset$ and given $x_1 = u \in H$ arbitrarily. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$(3.1) \qquad \begin{array}{ll} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ y_n = P_C(u_n - \mu B u_n) & \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \forall n \in \mathbf{N}, & \end{array}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap \Omega \cap EP(f)$, where $z = P_{F(S)\cap\Omega\cap EP(f)}u$ and (z, y) is a solution of problem (1.9), where $y = P_C(z - \mu B z)$.

Proof. Let $x^* \in F(S) \cap \Omega \cap EP(f)$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.6 and $u_n = T_{r_n} x_n$. Then $x^* = Sx^*$, $x^* = T_{r_n} x^*$ and

$$x^* = P_C[P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)],$$

where put $y^* = P_C(x^* - \mu B x^*)$ and $v_n = P_C(y_n - \lambda A y_n)$. Then $x^* = P_C(y^* - \lambda A y^*)$ and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C v_n.$$

For any $n \in \mathbb{N}$, we have

$$(3.2) \qquad \|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|.$$

Since P_C is nonexpansive and from remark 2.8, we obtain that $I-\lambda A$ and $I-\mu B$ are nonexpansive. Then, it follows that

$$\begin{aligned}
\|v_n - x^*\|^2 &= \|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\
&\leq \|(I-\lambda A)y_n - (I-\lambda A)y^*\|^2 \\
&\leq \|y_n - y^*\|^2 \\
&= \|P_C(u_n - \mu B u_n) - P_C(x^* - \mu B x^*)\|^2 \\
&\leq \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\
&\leq \|u_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Thus, we also have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n S v_n - x^*\| \\
&\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|v_n - x^*\| \\
&\leq \alpha_n (\|u - x^*\|) + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\
&\leq \alpha_n (\|u - x^*\|) + (1-\alpha_n)\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} \\
&= \|u - x^*\|.
\end{aligned}$$

Therefore the sequence $\{x_n\}$ is bounded. Hence, we also that the sets $\{u_n\}$, $\{v_n\}$ $\{Ay_n\}$, $\{Bx_n\}$ and $\{Sv_n\}$ are bounded. Moreover, by nonexpansiveness of $I-\lambda A$, $I-\mu B$ and P_C, we get

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|P_C(y_{n+1} - \lambda A y_{n+1}) - P_C(y_n - \lambda A y_n)\| \\
&\leq \|(y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n)\| \\
&\leq \|(I-\lambda A)y_{n+1} - (I-\lambda A)y_n\| \\
&\leq \|y_{n+1} - y_n\| \\
&= \|P_C(u_{n+1} - \mu B u_{n+1}) - P_C(u_n - \mu B u_n)\| \\
&\leq \|(I-\mu B)u_{n+1} - (I-\mu B)u_n\| \\
&\leq \|u_{n+1} - u_n\|. \qquad (3.3)
\end{aligned}$$

On the other hand, from $u_j = T_{r_j} x_j$, where $j = n, n+1$, we have

$$f(u_j, y) + \frac{1}{r_j}\langle y - u_j, u_j - x_j\rangle \geq 0 \text{ for all } y \in C. \qquad (3.4)$$

Putting $y = u_{n+1}$ and $y = u_n$ in (3.4), we get

$$f(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$f(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

From (A2) that

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle \\ &\leq \|u_{n+1} - u_n\| \{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{L}{c} |r_{n+1} - r_n|, \end{aligned} \tag{3.5}$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting (3.5) into (3.3), we obtain

$$\|v_{n+1} - v_n\| \leq \|x_{n+1} - x_n\| + \frac{L}{c} |r_{n+1} - r_n|. \tag{3.6}$$

Let $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$. Thus, we get

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n S P_C(y_n - \lambda A y_n)}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n S v_n}{1 - \beta_n}$$

it follows that

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1} u + \gamma_{n+1} S v_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n S v_n}{1 - \beta_n} \\ &= (\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}) u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (S v_{n+1} - S v_n) \\ &\quad + (\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}) S v_n. \end{aligned}$$

Combining (3.6) and (3.7), we obtain

$$
\begin{aligned}
& \|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \\
\leq\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\| \\
& +|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|Sv_n\| - \|x_{n+1}-x_n\| \\
\leq\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\
& +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\frac{L}{c}|r_{n+1}-r_n| \\
& +|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|Sv_n\| - \|x_{n+1}-x_n\| \\
\leq\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\,(\|u\|+\|Sv_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\frac{L}{c}|r_{n+1}-r_n|.
\end{aligned}
$$

From (ii), (iv) and (v) imply that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.7}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0. \tag{3.8}$$

By (iv), (v), (3.3) and (3.5), we also have
$\|v_{n+1}-v_n\| \to 0, \|u_{n+1}-u_n\| \to 0$ and $\|y_{n+1}-y_n\| \to 0$ as $n\to\infty$. Since

$$x_{n+1}-x_n = \alpha_n u + \beta_n x_n + \gamma_n S v_n - x_n = \alpha_n(u-x_n) + \gamma_n(Sv_n - x_n),$$

it follows from (ii) and (3.8) that

$$\lim_{n\to\infty}\|x_n - Sv_n\| = 0. \tag{3.9}$$

Since $x^* \in F(S)\cap\Omega\cap EP(f)$, we observe that

$$
\begin{aligned}
\|v_n - x^*\| &= \|P_C(y_n-\lambda A y_n) - P_C(y^*-\lambda A y^*)\| \\
&\leq \|(y_n-\lambda A y_n) - (y^*-\lambda A y^*)\| \\
&\leq \|y_n - y^*\| = \|P_C(u_n-\lambda A u_n) - P_C(x^*-\lambda A x^*)\| \\
&\leq \|(u_n-\lambda A u_n) - (x^*-\lambda A x^*)\| \\
&\leq \|u_n - x^*\| \qquad (3.10)
\end{aligned}
$$

and

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle \\ &= \langle u_n - x^*, x_n - x^*\rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2)\end{aligned}$$

then $\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. From (3.10), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + \beta_n x_n + \gamma_n S v_n - x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|Sv_n - x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + \gamma_n(\|x_n - x^*\|^2 - \|x_n - u_n\|^2) \\ &= \alpha_n\|u - x^*\|^2 + (\beta_n + \gamma_n)\|x_n - x^*\|^2 - \gamma_n\|x_n - u_n\|^2 \\ &= \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \gamma_n\|x_n - u_n\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|x_n - u_n\|^2 \qquad (3.11)\end{aligned}$$

and hence

$$\begin{aligned}\gamma_n\|x_n - u_n\|^2 &\le \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \qquad (3.12)\end{aligned}$$

by (ii) and (3.8) imply that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \qquad (3.13)$$

Since $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty}\|\frac{x_n - u_n}{r_n}\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \qquad (3.14)$$

Again since $\alpha_n \to 0$ and (3.8) imply that $\|u_n - x_n\| \to 0$ as $n \to \infty$.
From (3.2), (3.10) and Lemma 2.1, we get

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
\leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\
\leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2\} \\
\leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|y_n - y^*\|^2 \\
& + \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2\} \\
=\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 \\
& + \gamma_n \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2 \\
=\ & \alpha_n \|u - x^*\|^2 + (\beta_n + \gamma_n)\|x_n - x^*\|^2 \\
& + \gamma_n \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2 \\
=\ & \alpha_n \|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\
\leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|y_n - x^*\|^2 \\
\leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|(u_n - \mu B u_n) \\
& - (x^* - \mu B x^*)\|^2\} \\
\leq\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|u_n - x^*\|^2 \\
& + \mu(\mu - 2\beta)\|Bu_n - Bx^*\|^2\} \\
=\ & \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\
& + \gamma_n \mu(\mu - 2\beta)\|Bu_n - Bx^*\|^2 \\
\leq\ & \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n \mu(\mu - 2\beta)\|Bu_n - Bx^*\|^2.
\end{aligned}$$

Hence, by (3.15) and (3.15), we obtain

$$\begin{aligned}
& \gamma_n \lambda(2\alpha - \lambda)\|Ay_n - Ay^*\|^2 \\
\leq\ & \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
=\ & \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
\leq\ & \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|
\end{aligned}$$

and

$$
\begin{aligned}
& \gamma_n \mu(2\beta - \mu)\|Bu_n - Bx^*\|^2 \\
\leq \ & \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
= \ & \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
\leq \ & \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|.
\end{aligned}
$$

From (ii), (iii), (3.8), (3.15) and (3.15), respectively, we also have

$$
\|Ay_n - Ay^*\| \to 0 \text{ and } \|Bu_n - Bx^*\| \to 0 \text{ as } n \to \infty. \tag{3.15}
$$

By (2.3), we obtain

$$
\begin{aligned}
\|y_n - y^*\|^2 &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2 \\
&\leq \langle (u_n - \mu Bu_n) - (x^* - \mu Bx^*), y_n - y^* \rangle \\
&= \frac{1}{2}\{\|(u_n - \mu Bu_n) - (x^* - \mu Bx^*)\|^2 + \|y_n - y^*\|^2 \\
&\quad - \|(u_n - \mu Bu_n) - (x^* - \mu Bx^*) - (y_n - y^*)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - x^*\|^2 + \|y_n - y^*\|^2 - \|(u_n - y_n) - \mu(Bu_n - Bx^*) \\
&\quad - (x^* - y^*)\|^2\} \\
&= \frac{1}{2}\{\|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|(u_n - y_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu\langle (u_n - y_n) - (x^* - y^*), Bu_n - Bx^* \rangle - \mu^2\|Bu_n - Bx^*\|^2\}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|y_n - y^*\|^2 &\leq \|u_n - x^*\|^2 - \|(u_n - y_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu\langle (u_n - y_n) - (x^* - y^*), Bu_n - Bx^* \rangle - \mu^2\|Bu_n - Bx^*\|^2.
\end{aligned}
$$

Thus, we observe that

$$
\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
\leq\ & \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\
\leq\ & \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - y^*\|^2 \\
\leq\ & \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
& +\gamma_n\{\|u_n - x^*\|^2 - \|(u_n - y_n) - (x^* - y^*)\|^2 \\
& +2\mu\langle (u_n - y_n) - (x^* - y^*), Bu_n - Bx^*\rangle - \mu^2\|Bu_n - Bx^*\|^2\} \\
\leq\ & \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
& +\gamma_n\|x_n - x^*\|^2 - \gamma_n\|(u_n - y_n) - (x^* - y^*)\|^2 \\
& +2\gamma_n\mu\|(u_n - y_n) - (x^* - y^*)\|\|Bu_n - Bx^*\| - \gamma_n\mu^2\|Bu_n - Bx^*\|^2 \\
\leq\ & \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \gamma_n\|(u_n - y_n) - (x^* - y^*)\|^2 \\
& +2\gamma_n\mu\|(u_n - y_n) - (x^* - y^*)\|\|Bu_n - Bx^*\|
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& \gamma_n\|(u_n - y_n) - (x^* - y^*)\|^2 \\
\leq\ & \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& +2\gamma_n\mu\|(u_n - y_n) - (x^* - y^*)\|\|Bu_n - Bx^*\| \\
\leq\ & \alpha_n\|u - x^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|^2) \\
& +2\gamma_n\mu\|(u_n - y_n) - (x^* - y^*)\|\|Bu_n - Bx^*\|.
\end{aligned}
\tag{3.16}
$$

From (ii), (3.15), (3.8) and $\|Bu_n - Bx^*\| \to 0$ as $n \to \infty$, we have

$$
\|(u_n - y_n) - (x^* - y^*)\| \to 0 \text{ as } n \to \infty. \tag{3.17}
$$

We observe that

$$
\begin{aligned}
& \|(y_n - v_n) + (x^* - y^*)\|^2 \\
= \;& \|(y_n - y^*) - [P_C(y_n - \lambda A y_n) - x^*]\|^2 \\
= \;& \|y_n - \lambda A y_n - (y^* - \lambda A y^*) \\
& -[P_C(y_n - \lambda A y_n) - x^*] + \lambda(A y_n - A y^*)\|^2 \\
\leq \;& \|y_n - \lambda A y_n - (y^* - \lambda A y^*)\|^2 \\
& -\|P_C(y_n - \lambda A y_n) - P_C(y^* - \lambda A y^*)\|^2 \\
& +2\lambda\langle A y_n - A y^*, (y_n - v_n) + (x^* - y^*)\rangle \\
\leq \;& \|y_n - \lambda A y_n - (y^* - \lambda A y^*)\|^2 \\
& -\|SP_C(y_n - \lambda A y_n) - SP_C(y^* - \lambda A y^*)\|^2 \\
& +2\lambda\|A y_n - A y^*\|\|(y_n - v_n) + (x^* - y^*)\| \\
\leq \;& \|y_n - \lambda A y_n - (y^* - \lambda A y^*)\|^2 - \|S v_n - S x^*\|^2 \\
& +2\lambda\|A y_n - A y^*\|\|(y_n - v_n) + (x^* - y^*)\| \\
\leq \;& \|y_n - \lambda A y_n - (y^* - \lambda A y^*) - S v_n - S x^*\| \\
& \times(\|y_n - \lambda A y_n - (y^* - \lambda A y^*)\| + \|S v_n - S x^*\|) \\
& +2\lambda\|A y_n - A y^*\|\|(y_n - v_n) + (x^* - y^*)\| \\
\leq \;& \|x_n - S v_n + x^* - y^* - (x_n - y_n) - \lambda(A y_n - A y^*)\| \\
& \times(\|y_n - \lambda A y_n - (y^* - \lambda A y^*)\|^2\| + \|S v_n - S x^*\|) \\
& +2\lambda\|A y_n - A y^*\|\|(y_n - v_n) + (x^* - y^*)\|.
\end{aligned}
$$

From (3.9), (3.17) and $\|A y_n - A y^*\| \to 0,$ as $n \to \infty$, it follows that

$$\|(y_n - v_n) + (x^* - y^*)\| \to 0, \quad (n \to \infty).$$

We note that

$$
\begin{aligned}
\|S v_n - v_n\| \;\leq\; & \|S v_n - x_n\| + \|x_n - u_n\| + \|(u_n - y_n) - (x^* - y^*)\| \\
& +\|(y_n - v_n) + (x^* - y^*)\|,
\end{aligned}
$$

from above, we obtain

$$\lim_{n\to\infty} \|S v_n - v_n\| = 0. \tag{3.18}$$

Next, we show that

$$\limsup_{n\to\infty} \langle u - z_0, x_n - z_0\rangle \leq 0,$$

where $z_0 = P_{F(S)\cap\Omega\cap EP(f)}u$. To show this inequality, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, Sv_n - z_0\rangle = \lim_{i\to\infty}\langle u - z_0, Sv_{n_i} - z_0\rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup z$. From $\|Sv_n - v_n\| \to 0$, we obtain $Sv_{n_i} \rightharpoonup z$. Let us show $z \in EP(f)$. Since $u_n = T_{r_n}x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

From (A2), it follow that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n)$$

and hence $\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i})$. From $\|u_n - x_n\| \to 0, \|x_n - Sv_n\| \to 0$, and $\|Sv_n - v_n\| \to 0$, we get $u_{n_i} \rightharpoonup z$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, it follows by (A4) that $0 \geq f(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $f(y_t, z) \leq 0$. So, from (A1) and (A4) we have $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, z) \leq tf(y_t, y)$ and hence $0 \leq f(y_t, y)$. From (A3), we have $f(z, y) \geq 0, \quad$ for all $y \in C$ and hence $z \in EP(f)$. By Opial's condition, we obtain $z \in F(S)$. Finally, by the same argument as that in the proof of [4, Theorem 3.1, p. 384-385] , we can show that $z \in \Omega$. Hence $z \in F(S) \cap \Omega \cap EP(f)$. Now from (2.4), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle u - z_0, Sv_n - z_0\rangle \\
&= \lim_{i\to\infty}\langle u - z_0, Sv_{n_i} - z_0\rangle \\
&= \langle u - z_0, z - z_0\rangle \leq 0. \qquad (3.19)
\end{aligned}$$

Finally, we show that $x_n \to z_0$, where $z_0 = P_{F(S)\cap VI(A,C)\cap EP(f)}u$. We observe that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \langle \alpha_n u + \beta_n x_n \\
&\quad + \gamma_n S v_n - z_0, x_{n+1} - z_0 \rangle \\
&= \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle + \beta_n \langle x_n - z_0, x_{n+1} - z_0 \rangle \\
&\quad + \gamma_n \langle S v_n - z_0, x_{n+1} - z_0 \rangle \\
&\leq \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\quad + \frac{1}{2}\gamma_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\leq \frac{1}{2}\{(1 - \alpha_n)\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\} \\
&\quad + \alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle
\end{aligned}$$

which implies that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle.$$

Finally by (3.19) and Lemma 2.4, we get that $\{x_n\}$ converges to z_0, where $z_0 = P_{F(S)\cap\Omega\cap EP(f)}u$. This completes the proof. □

By Theorem 3.1, we obtain the following corollaries:

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and $A : C \longrightarrow H$ be α-inverse-strongly monotone. Let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \cap EP(f) \neq \emptyset$. Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C; \\ y_n = P_C(u_n - \lambda A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \forall n \in \mathbf{N}, \end{cases} \tag{3.20}$$

where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\lambda \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$ and $\{r_n\} \subset (0,\infty)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $u = P_{F(S)\cap\Omega\cap EP(f)}u$.

Proof. Put $A = B$ and $\lambda = \mu$ for $n \in \mathbb{N}$ in Theorem 3.1, we obtain the desired easily. $\square$

Setting $P_H = I$, we obtain the following corollary:

Corollary 3.3. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* f *be a bifunction from* $C \times C \to \mathbf{R}$ *satisfying (A1)-(A4) and* $A : C \longrightarrow H$ *be* α*-inverse-strongly monotone. Let* S *be a nonexpansive mapping of* C *into itself such that* $F(S) \cap VI(A, C) \cap EP(f) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and given* $x_0 \in H$ *arbitrarily. Suppose* $x_1 = u \in C$ *and* $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *are given by*

$$(3.21)\quad \begin{array}{ll} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(u_n - \lambda A u_n), \forall n \in \mathbf{N}, & \end{array}$$

where $\lambda \in (0, 2\alpha)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$. *If* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\lambda \in [a, b]$ *for some* a, b *with* $0 < a < b < 2\alpha$ *and* $\{r_n\} \subset (0, \infty)$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $z \in F(S) \cap VI(C, A) \cap EP(f)$, *where* $z = P_{F(S)\cap VI(C,A)\cap EP(f)}u$.

Using Theorem 3.1, we obtain the following two corollaries in Hilbert space:

Corollary 3.4. (Ceng et. al [4, Theorem 3.1]) *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* A, B *be* α*and* β*-inverse-strongly monotone mappings of* C *into* H, *respectively and let* S *be a nonexpansive mapping of* C *into itself such that* $F(S)\cap\Omega \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}, \{y_n\}$ *are given by*

$$\begin{array}{l} y_n = P_C(x_n - \mu B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda A y_n), \end{array}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ *converges strongly to* $P_{F(S)\cap\Omega}u$ *and* (x^*, y^*) *is a solution of problem (1.9), where* $y^* = P_C(x^* - \mu Bx^*)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1 . Then, we have $u_n = P_C x_n = x_n$. So, from Theorem 3.1 the sequence $\{x_n\}$ converges strongly to $P_{F(S)\cap\Omega}u$. □

Corollary 3.5. *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* A *be an* $\alpha-$*inverse-strongly monotone mapping of* C *into* H *and let* S *be a nonexpansive mapping of* C *into itself such that* $F(S) \cap VI(A, C) \neq \emptyset$*. Suppose* $x_1 = u \in C$ *and* $\{x_n\}, \{y_n\}$ *are given by*

$$\begin{aligned} y_n &= P_C(x_n - \lambda Ax_n) \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{aligned}$$

where $\lambda \in [0, 2\alpha]$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$ *satisfying*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ *converges strongly to* $P_{F(S)\cap\Omega}u$*. Moreover, we also have* (x^*, y^*) *is a solution of problem (1.10), where* $y^* = P_C(x^* - \lambda Ax^*)$.

Proof. Take $A = B$ and $\lambda = \mu$ in Corollary 3.4, we can get the desired conclusion easily. □

4. **Applications**

A mapping $T : C \to C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we have

$$\|(I-A)x-(I-A)y\|^2 = \|x-y\|^2 - 2\langle x-y, Ax-Ay\rangle + \|Ax-Ay\|^2.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-k}{2}\|Ax-Ay\|^2.$$

Then, A is $\frac{1-k}{2}-$ inverse strongly monotone.

Now, using Theorem 3.1, we state a strong convergence theorem for a pair of nonexpansive mappings and strictly pseudocontractive mappings.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)-(A4) and let S be a nonexpansive mappings of C into itself and let T, V be strictly pseudocontractive mapping with constant k of C into itself such that $F(S) \cap F(T) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{aligned} &f(u_n, y) + \tfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \qquad \forall y \in C;\\ &y_n = (1-\mu)u_n + \mu V u_n\\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n), \end{aligned}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, $\lambda \in [0, 1-k]$ and $\mu \in [0, 1-l]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ $\{r_n\} \subset (0,\infty)$ satisfying

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iv) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

then $\{x_n\}$ converges strongly to $z = P_{F(S)\cap F(T)\cap EP(f)} u$.

Proof. Put $A = I - T$ and $B = I - V$. Then A is $\frac{1-k}{2}-$inverse-strongly monotone and B is $\frac{1-l}{2}-$inverse-strongly monotone. We have that $F(T)$ is the solution set of $VI(A, C)$ and Ω i.e., $F(T) = VI(A, C) \Leftrightarrow$ problem (1.9) $\Leftrightarrow$ problem (1.10) (see cf. Ceng et al. [4, Theorem 4.1 pp. 388–389]) and

$$P_C(u_n - \mu B u_n) = (1-\mu)u_n + \mu V u_n \text{ and } P_C(y_n - \lambda A y_n) = (1-\lambda)y_n + \lambda T y_n.$$

Therefore, by Theorem 3.1, the conclusion follows. □

Therefore, the conclusion follows immediately from Theorem 4.1.

Corollary 4.2. (Ceng et al. [4, Corollary 3.3]) *Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T, V be strictly pseudocontractive mappings with constant k of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$\begin{aligned} y_n &= (1-\mu)x_n + \mu V x_n \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n), \end{aligned}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, $\lambda \in [0, 1-k]$ and $\mu \in [0, 1-l]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to $x^* = P_{F(S)\cap\Omega} u$ and (x^*, y^*) is a solution of problem (1.10), where $y^* = (1-\mu)x^* - \mu V x^*)$.

Corollary 4.3. *Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping with constant k of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$\begin{aligned} y_n &= (1-\lambda)x_n + \lambda T x_n \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n), \end{aligned}$$

for all $n \in \mathbf{N}$, where $\lambda \in [0, 1-k]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and satisfying

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)} u$.

The following three theorems are connected with the problem of obtaining of a common element of the sets of zeroes of a maximal monotone operator and an $\alpha-$inverse-strongly monotone operator.

Theorem 4.4. *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying $(A1) - (A4)$ and let A be an $\alpha-$inverse-strongly monotone operator in H and $B : H \to 2^H$ be a maximal monotone operator such that $A^{-1}(0) \cap B^{-1}(0) \cap EP(f) \neq$. Let*

J_r^B be the resolvent of B for each $r > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = u \in H$ and

$$\tag{4.1} \begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H; \\ y_n = (u_n - \lambda A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_r^B(y_n - \lambda A y_n), \end{cases}$$

where $\{\lambda\} \subset [c, d]$ for some $[c, d] \subset (0, 2\alpha)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(*ii*) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$;
(*iii*) $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,
(*iv*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in A^{-1}(0) \cap B^{-1}(0) \cap EP(f)$, where $z = P_{A^{-1}(0)\cap B^{-1}(0)EP(f)}x_1$.

Proof. Since $A^{-1}0 = V(I, A)$ and $F(J_r^B) = B^{-1}(0)$. Putting $P_H = I$ then, by Theorem 3.1, we obtain the desired result easily. $\square$

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Research Article

A Hybrid Extragradient Viscosity Approximation Method for Solving Equilibrium Problems and Fixed Point Problems of Infinitely Many Nonexpansive Mappings

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We introduce a new hybrid extragradient viscosity approximation method for finding the common element of the set of equilibrium problems, the set of solutions of fixed points of an infinitely many nonexpansive mappings, and the set of solutions of the variational inequality problems for β-inverse-strongly monotone mapping in Hilbert spaces. Then, we prove the strong convergence of the proposed iterative scheme to the unique solution of variational inequality, which is the optimality condition for a minimization problem. Results obtained in this paper improve the previously known results in this area.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive (see [1]) if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T. Recall also that a self-mapping $f : H \to H$ is a *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in H$. In addition, let $B : C \to H$ be a nonlinear mapping. Let P_C be the projection of H onto C. The classical variational inequality which is denoted by $VI(C,B)$ is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \tag{1.2}$$

if and only if $u = P_C z$. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.3}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{1.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{1.5}$$

It is easy to see that the following is true:

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{1.6}$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. The variational inequality has been extensively studied in literature; see, for instance, [2–6]. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. Recall the following.

(1) A mapping B of C into H is called monotone if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{1.7}$$

(2) A mapping B is called β-strongly monotone (see [7, 8]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C. \tag{1.8}$$

(3) A mapping B is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Bx - By\| \leq k \|x - y\|, \quad \forall x, y \in C. \tag{1.9}$$

(4) A mapping B is called β-inverse-strongly monotone (see [7, 8]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2, \quad \forall x, y \in C. \tag{1.10}$$

Remark 1.1. It is obvious that any β-inverse-strongly monotone mapping B is monotone and $1/\beta$-Lipschitz continuous.

(5) An operator A is strongly positive on H if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.11}$$

(6) A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply $\langle x-y, f-g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H, and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w \rangle \geq 0,\ \text{for all } u \in C\}$, .

$$Tv = \begin{cases} Bv + N_C v, & v \in \ C, \\ \emptyset, & v \notin \ C. \end{cases} \tag{1.12}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [9].

(7) Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.13}$$

The set of solutions of (1.13) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$. Numerous problems in physics, saddle point problem, fixed point problem, variational inequality problems, optimization, and economics are reduced to find a solution of (1.13). Some methods have been proposed to solve the equilibrium problem; see, for instance, [10–16]. Recently, Combettes and Hirstoaga [17] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

In 1976, Korpelevich [18] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda B x_n), \\ x_{n+1} &= P_C(x_n - \lambda B y_n) \end{aligned} \tag{1.14}$$

for all $n \geq 0$, where $\lambda \in (0, 1/k), C$ is a closed convex subset of $\mathbb{R}^n$, and B is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.14), converge to the same point $z \in VI(C, B)$. For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solution of variational inequalities for β-inverse-strongly monotone, Takahashi and Toyoda [19] introduced the following iterative scheme:

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrary}, \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C (x_n - \lambda_n B x_n), \quad \forall n \geq 0, \end{aligned} \tag{1.15}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.15) converges weakly to some $z \in F(S) \cap VI(C, B)$. Recently, Iiduka and Takahashi [20] proposed a new iterative scheme as follows:

$$\begin{aligned} &x_0 = x \in C \quad \text{chosen arbitrary}, \\ &x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C (x_n - \lambda_n B x_n), \quad \forall n \geq 0, \end{aligned} \tag{1.16}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.16) converges strongly to some $z \in F(S) \cap VI(C, B)$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [21–24] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.17}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S on H, and b is a given point in H. Moreover, it is shown in [25] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) S x_n \tag{1.18}$$

converges strongly to $z = P_{F(S)}(I - A + \gamma f)(z)$. Recently, Plubtieng and Punpaeng [26] proposed the following iterative algorithm:

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S u_n. \end{gathered} \tag{1.19}$$

They prove that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle (A - \gamma f) q, q - p \rangle \geq 0, \quad p \in F(S) \cap EP(F), \tag{1.20}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.21}$$

where h is a potential function for γf (i.e., $h\prime(x) = \gamma f(x)$ for $x \in H$).

Furthermore, for finding approximate common fixed points of an infinite countable family of nonexpansive mappings $\{T_n\}$ under very mild conditions on the parameters. Wangkeeree [27] introduced an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.13) and the set of common fixed points of a countable family of nonexpansive mappings on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{gathered} \tag{1.22}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$. It is proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.22) strongly converges to the unique solution $q \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C,B) \cap EP(F)$, where $p = P_{\cap_{n=1}^{\infty} F(S_n)\cap VI(C,B)\cap EP(F)} f(q)$ which extend and improve the result of Kumam [14].

Definition 1.2 (see [21]). Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself, and let $\{\mu_n\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\ U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\ &\vdots \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\ U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\ &\vdots \\ U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\ W_n &= U_{n,1} = \mu_1 T_1 U_{n,2} + (1-\mu_1)I. \end{aligned} \tag{1.23}$$

Such a mapping W_n is nonexpansive from C to C, and it is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

On the other hand, Colao et al. [28] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.13) and the set of common fixed points of infinitely many nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$, define a sequence $\{x_n\}$ recursively by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \left((1-\beta)I - \epsilon_n A\right)W_n u_n, \end{gathered} \tag{1.24}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$. It is proved [28] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.24) strongly converges to $z \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and is the unique solution of the variational inequality (1.20), that is, $z = P_{\cap_{n=1}^{\infty} F(T_n) \cap EP(F)}(I - (A - \gamma f))z$.

In this paper, motivated by Wangkeeree [27], Plubtieng and Punpaeng [26], Marino and Xu [25], and Colao, et al. [28], we introduce a new iterative scheme in a Hilbert space H which is mixed by the iterative schemes of (1.18), (1.19), (1.22), and (1.24) as follows.

Let f be a contraction of H into itself, A a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$, and B a β-inverse-strongly monotone mapping of C into H; define sequences $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ recursively by

$$\begin{gathered} x_1 = x \in C \quad \text{chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ k_n = \alpha_n u_n + (1-\alpha_n) P_C(u_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \epsilon_n A\right)W_n k_n, \quad \forall n \geq 1, \end{gathered} \tag{1.25}$$

where $\{W_n\}$ is the sequence generated by (1.23), $\{\epsilon_n\}, \{\alpha_n\}$, and $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfying appropriate conditions. We prove that the sequences $\{x_n\}$, $\{y_n\}$, $\{k_n\}$ and $\{u_n\}$ generated by the above iterative scheme (1.25) converge strongly to a common element of the set of solutions of the equilibrium problem (1.13), the set of common fixed points of infinitely family nonexpansive mappings, and the set of solutions of variational inequality (1.1) for a β-inverse-strongly monotone mapping in Hilbert spaces. The results obtained in this paper improve and extend the recent ones announced by Wangkeeree [27], Plubtieng and Punpaeng [26], Marino and Xu [25], Colao, et al. [28], and many others.

2. Preliminaries

We now recall some well-known concepts and results.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively.

A space H is said to satisfy Opial's condition [29] if for each sequence $\{x_n\}$ in H which converges weakly to point $x \in H$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in H, \quad y \neq x. \tag{2.1}$$

Lemma 2.1 (see [25]). *Let C be a nonempty closed convex subset of H, let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$. Then , for $0 < \gamma < \overline{\gamma}/\alpha$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\overline{\gamma} - \alpha\gamma) \|x - y\|^2, \quad x, y \in H. \tag{2.2}$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.2 (see [25]). *Assume that A is a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to R$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C, \lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [30].

Lemma 2.3 (see [30]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \tag{2.3}$$

The following lemma was also given in [17].

Lemma 2.4 (see [17]). *Assume that $F : C \times C \to R$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\} \tag{2.4}$$

for all $z \in H$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \tag{2.5}$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (1.23). Then we can have the following crucial conclusions concerning W_n. You can find them in [31]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.5 (see [31]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.5, one can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.6}$$

for every $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$. Throughout this paper, we will assume that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, we have the following results.

Lemma 2.6 (see [31]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.7 (see [32]). *If $\{x_n\}$ is a bounded sequence in C, then $\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0$.*

Lemma 2.8 (see [33]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n) z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.9 (see [34]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n) a_n + \sigma_n, \quad n \geq 0, \tag{2.7}$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$;

(2) $\limsup_{n\to\infty} \sigma_n / l_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

3. Main Results

In this section, we prove the strong convergence theorem for infinitely many nonexpansive mappings in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\{T_n\}$ be an infinitely many nonexpansive of C into itself, and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by* (1.25), *where $\{W_n\}$ is the sequence generated by* (1.23), *$\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$, $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(v) $\{\lambda_n/\beta\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.1}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Note that from the condition (i), we may assume, without loss of generality, that $\epsilon_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.2, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$. We will assume that $\|I - A\| \leq 1 - \overline{\gamma}$. First, we show that $I - \lambda_n B$ is nonexpansive. Indeed, from the β-inverse-strongly monotone mapping definition on B and condition (v), we have

$$\begin{aligned}
\left\|(I-\lambda_n B)x - (I-\lambda_n B)y\right\|^2 &= \left\|(x-y) - \lambda_n(Bx - By)\right\|^2 \\
&= \left\|x-y\right\|^2 - 2\lambda_n\langle x-y, Bx-By\rangle + \lambda_n^2\left\|Bx-By\right\|^2 \\
&\leq \left\|x-y\right\|^2 - 2\lambda_n\beta\left\|Bx-By\right\|^2 + \lambda_n^2\left\|Bx-By\right\|^2 \\
&= \left\|x-y\right\|^2 + \lambda_n(\lambda_n - 2\beta)\left\|Bx-By\right\|^2 \\
&\leq \left\|x-y\right\|^2,
\end{aligned} \tag{3.2}$$

which implies that the mapping $I - \lambda_n B$ is nonexpansive. On the other hand, since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.3}$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle &= 1-\beta_n-\epsilon_n\langle Ax,x\rangle\\ &\geq 1-\beta_n-\epsilon_n\|A\|\\ &\geq 0,\end{aligned} \tag{3.4}$$

and this show that $(1-\beta_n)I-\epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n A\| &= \sup\{|\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle| : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\epsilon_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\leq 1-\beta_n-\epsilon_n\overline{\gamma}.\end{aligned} \tag{3.5}$$

Let $Q=P_\Theta$, where $\Theta := \cap_{n=1}^{\infty}F(T_n)\cap EP(F)\cap VI(C,B)$. Note that f is a contraction of H into itself with $\alpha\in(0,1)$. Then, we have

$$\begin{aligned}\|Q(I-A+\gamma f)(x)-Q(I-A+\gamma f)(y)\| &= \|P_\Theta(I-A+\gamma f)(x)-P_\Theta(I-A+\gamma f)(y)\|\\ &\leq \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\ &\leq \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\|\\ &\leq (1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\ &= (1-\overline{\gamma}+\gamma\alpha)\|x-y\|\\ &= (1-(\overline{\gamma}-\gamma\alpha))\|x-y\|, \quad \forall x,y\in H.\end{aligned} \tag{3.6}$$

Since $0 < 1-(\overline{\gamma}-\gamma\alpha) < 1$, it follows that $Q(I-A+\gamma f)$ is a contraction of H into itself. Therefore by the Banach Contraction Mapping Principle, which implies that there exists a unique element $z\in H$ such that $z=Q(I-A+\gamma f)(z)=P_\Theta(I-A+\gamma f)(z)$.

We will divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded. Indeed, pick any $p\in\Theta$. From the definition of T_r, we note that $u_n=T_{r_n}x_n$. If follows that

$$\|u_n-p\| = \|T_{r_n}x_n-T_{r_n}p\| \leq \|x_n-p\|. \tag{3.7}$$

Since $I-\lambda_n B$ is nonexpansive and $p=P_C(p-\lambda_n Bp)$ from (1.6), we have

$$\begin{aligned}\|y_n-p\| &= \|P_C(u_n-\lambda_n Bu_n)-P_C(p-\lambda_n Bp)\|\\ &\leq \|(u_n-\lambda_n Au_n)-(p-\lambda_n Bp)\|\\ &= \|(I-\lambda_n A)u_n-(I-\lambda_n B)p\|\\ &\leq \|u_n-p\| \leq \|x_n-p\|.\end{aligned} \tag{3.8}$$

Put $v_n = P_C(u_n - \lambda_n B y_n)$. Since $p \in VI(C, B)$, we have $p = P_C(p - \lambda_n Bp)$. Substituting $x = u_n - \lambda_n A y_n$ and $y = p$ in (1.5), we can write

$$\begin{aligned}
\left\|v_n - p\right\|^2 &\leq \left\|u_n - \lambda_n B y_n - p\right\|^2 - \left\|u_n - \lambda_n B y_n - v_n\right\|^2 \\
&= \left\|u_n - p\right\|^2 - 2\lambda_n \langle B y_n, u_n - p\rangle + \lambda_n^2 \left\|B y_n\right\|^2 \\
&\quad - \left\|u_n - v_n\right\|^2 + 2\lambda_n \langle B y_n, u_n - v_n\rangle - \lambda_n^2 \left\|B y_n\right\|^2 \\
&= \left\|u_n - p\right\|^2 - \left\|u_n - v_n\right\|^2 + 2\lambda_n \langle B y_n, p - v_n\rangle \\
&= \left\|u_n - p\right\|^2 - \left\|u_n - v_n\right\|^2 + 2\lambda_n \langle B y_n - Bp, p - y_n\rangle \\
&\quad + 2\lambda_n \langle Bp, p - y_n\rangle + 2\lambda_n \langle B y_n, y_n - v_n\rangle.
\end{aligned} \tag{3.9}$$

Using the fact that B is β-inverse-strongly monotone mapping, and p is a solution of the variational inequality problem $VI(C, B)$, we also have

$$\langle B y_n - Bp, p - y_n\rangle \leq 0, \qquad \langle Bp, p - y_n\rangle \leq 0. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}
\left\|v_n - p\right\|^2 &\leq \left\|u_n - p\right\|^2 - \left\|u_n - v_n\right\|^2 + 2\lambda_n \langle B y_n, y_n - v_n\rangle \\
&= \left\|u_n - p\right\|^2 - \left\|(u_n - y_n) + (y_n - v_n)\right\|^2 + 2\lambda_n \langle B y_n, y_n - v_n\rangle \\
&\leq \left\|u_n - p\right\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - v_n\right\|^2 \\
&\quad - 2\langle u_n - y_n, y_n - v_n\rangle + 2\lambda_n \langle B y_n, y_n - v_n\rangle \\
&= \left\|u_n - p\right\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - v_n\right\|^2 + 2\langle u_n - \lambda_n B y_n - y_n, v_n - y_n\rangle.
\end{aligned} \tag{3.11}$$

Substituting x by $u_n - \lambda_n B u_n$ and $y = v_n$ in (1.4), we obtain

$$\langle u_n - \lambda_n B u_n - y_n, v_n - y_n\rangle \leq 0. \tag{3.12}$$

It follows that

$$\begin{aligned}
\langle u_n - \lambda_n B y_n - y_n, v_n - y_n\rangle &= \langle u_n - \lambda_n B u_n - y_n, v_n - y_n\rangle \\
&\quad + \langle \lambda_n B u_n - \lambda_n B y_n, v_n - y_n\rangle \\
&\leq \langle \lambda_n B u_n - \lambda_n B y_n, v_n - y_n\rangle \\
&\leq \lambda_n \left\|B u_n - B y_n\right\| \left\|v_n - y_n\right\| \\
&\leq \frac{\lambda_n}{\beta} \left\|u_n - y_n\right\| \left\|v_n - y_n\right\|.
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.11), we have

$$\begin{aligned}
\|v_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle u_n - \lambda_n B y_n - y_n, v_n - y_n\rangle \\
&\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\frac{\lambda_n}{\beta}\|u_n - y_n\|\|v_n - y_n\| \\
&\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \frac{\lambda_n^2}{\beta^2}\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\
&= \|u_n - p\|^2 - \|u_n - y_n\|^2 + \frac{\lambda_n^2}{\beta^2}\|u_n - y_n\|^2 \\
&= \|u_n - p\|^2 + \left(\frac{\lambda_n^2}{\beta^2} - 1\right)\|u_n - y_n\|^2 \\
&\le \|u_n - p\|^2 \le \|x_n - p\|^2.
\end{aligned} \tag{3.14}$$

Setting $k_n = \alpha_n u_n + (1-\alpha_n)v_n$, we can calculate

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \epsilon_n A)(W_n k_n - p)\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\{\alpha_n\|u_n - p\| + (1-\alpha_n)\|v_n - p\|\} \\
&\quad + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\{\alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\|\} \\
&\quad + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&= (1 - \beta_n - \epsilon_n\overline{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&= (1 - \epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\|f(x_n) - f(p)\| + \epsilon_n\|\gamma f(p) - Ap\| \\
&\le (1 - \epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\alpha\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\epsilon_n)\|x_n - p\| + (\overline{\gamma} - \gamma\alpha)\epsilon_n\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}.
\end{aligned} \tag{3.15}$$

By induction,

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \in \mathbb{N}. \tag{3.16}$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{W_n k_n\}$, $\{f(x_n)\}$, $\{Bu_n\}$, $\{y_n\}$, and $\{By_n\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Observing that $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we get

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in H \tag{3.17}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \forall y \in H. \tag{3.18}$$

Putting $y = u_{n+1}$ in (3.17) and $y = u_n$ in (3.18), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle &\geq 0 \\ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle &\geq 0. \end{aligned} \tag{3.19}$$

So, from (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.20}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.21}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.22}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|, \end{aligned} \tag{3.23}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Note that

$$
\begin{aligned}
\|v_{n+1} - v_n\| &\le \left\|P_C\left(u_{n+1} - \lambda_{n+1} B y_{n+1}\right) - P_C\left(u_n - \lambda_n B y_n\right)\right\| \\
&\le \left\|u_{n+1} - \lambda_{n+1} B y_{n+1} - \left(u_n - \lambda_n B y_n\right)\right\| \\
&= \|(u_{n+1} - \lambda_{n+1} B u_{n+1}) - (u_n - \lambda_{n+1} B u_n) \\
&\quad + \lambda_{n+1}\left(B u_{n+1} - B y_{n+1} - B u_n\right) + \lambda_n B y_n\| \\
&\le \|(u_{n+1} - \lambda_{n+1} B u_{n+1}) - (u_n - \lambda_{n+1} B u_n)\| \\
&\quad + \lambda_{n+1}\left(\|B u_{n+1}\| + \left\|B y_{n+1}\right\| + \|B u_n\|\right) + \lambda_n \|B y_n\| \\
&\le \|u_{n+1} - u_n\| + \lambda_{n+1}\left(\|B u_{n+1}\| + \left\|B y_{n+1}\right\| + \|B u_n\|\right) + \lambda_n \|B y_n\|, \\
\|k_{n+1} - k_n\| &= \|\alpha_{n+1} u_{n+1} + (1 - \alpha_{n+1}) v_{n+1} - \alpha_n u_n - (1 - \alpha_n) v_n\| \\
&= \|\alpha_{n+1}(u_{n+1} - u_n) + (\alpha_{n+1} - \alpha_n) u_n \\
&\qquad + (1 - \alpha_{n+1})(v_{n+1} - v_n) + (\alpha_n - \alpha_{n+1}) v_n\| \\
&\le \alpha_{n+1}\|u_{n+1} - u_n\| + (1 - \alpha_{n+1})\|v_{n+1} - v_n\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&= \alpha_{n+1}\|u_{n+1} - u_n\| + (1 - \alpha_{n+1}) \\
&\quad \times \left\{\|u_{n+1} - u_n\| + \lambda_{n+1}\left(\|B u_{n+1}\| + \left\|B y_{n+1}\right\| + \|B u_n\|\right)\right. \\
&\qquad\qquad \left. + \lambda_n \|B y_n\|\right\} + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&= \|u_{n+1} - u_n\| + (1 - \alpha_{n+1})\lambda_{n+1}\left(\|B u_{n+1}\| + \left\|B y_{n+1}\right\| + \|B u_n\|\right) \\
&\quad + (1 - \alpha_{n+1})\lambda_n \left\|B y_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&\le \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n| + (1 - \alpha_{n+1})\lambda_{n+1} \\
&\quad \times \left(\|B u_{n+1}\| + \left\|B y_{n+1}\right\| + \|B u_n\|\right) \\
&\quad + (1 - \alpha_{n+1})\lambda_n \left\|B y_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\|.
\end{aligned}
\tag{3.24}
$$

Setting

$$
z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\epsilon_n \gamma f(x_n) + \left((1 - \beta_n) I - \epsilon_n A\right) W_n k_n}{1 - \beta_n},
\tag{3.25}
$$

we have $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, $n \geq 1$. It follows that

$$\begin{aligned}
z_{n+1} - z_n &= \frac{\epsilon_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \epsilon_{n+1}A)W_{n+1}k_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\epsilon_n\gamma f(x_n) + ((1-\beta_n)I - \epsilon_n A)W_n k_n}{1-\beta_n} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(x_n) + W_{n+1}k_{n+1} - W_n k_n \\
&\quad + \frac{\epsilon_n}{1-\beta_n}AW_n k_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}k_{n+1} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - AW_{n+1}k_{n+1}\right) + \frac{\epsilon_n}{1-\beta_n}\left(AW_n k_n - \gamma f(x_n)\right) \\
&\quad + W_{n+1}k_{n+1} - W_{n+1}k_n + W_{n+1}k_n - W_n k_n.
\end{aligned} \tag{3.26}$$

It follows from (3.24) and (3.26) that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\|AW_n k_n\| + \left\|\gamma f(x_n)\right\|\right) + \|W_{n+1}k_{n+1} - W_{n+1}k_n\| \\
&\quad + \|W_{n+1}k_n - W_n k_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\|AW_n k_n\| + \left\|\gamma f(x_n)\right\|\right) + \|k_{n+1} - k_n\| \\
&\quad + \|W_{n+1}k_n - W_n k_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\|AW_n k_n\| + \left\|\gamma f(x_n)\right\|\right) + \frac{M_1}{c}|r_{n+1} - r_n| \\
&\quad + (1-\alpha_{n+1})\lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) \\
&\quad + (1-\alpha_{n+1})\lambda_n\left\|By_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&\quad + \|W_{n+1}k_n - W_n k_n\|.
\end{aligned} \tag{3.27}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}k_n - W_nk_n\| &= \left\|\mu_1 T_1 U_{n+1,2}k_n - \mu_1 T_1 U_{n,2}k_n\right\| \\
&\le \mu_1\|U_{n+1,2}k_n - U_{n,2}k_n\| \\
&= \mu_1\left\|\mu_2 T_2 U_{n+1,3}k_n - \mu_2 T_2 U_{n,3}k_n\right\| \\
&\le \mu_1\mu_2\|U_{n+1,3}k_n - U_{n,3}k_n\| \\
&\ \ \vdots \\
&\le \mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}k_n - U_{n,n+1}k_n\| \\
&\le M_2\prod_{i=1}^{n}\mu_i,
\end{aligned} \tag{3.28}$$

where $M_2 \ge 0$ is a constant such that $\|U_{n+1,n+1}k_n - U_{n,n+1}k_n\| \le M_2$ for all $n \ge 0$.

Combining (3.27) and (3.28), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le{}& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&+ \frac{\epsilon_n}{1-\beta_n}\left(\|AW_nk_n\| + \left\|\gamma f(x_n)\right\|\right) + \frac{M_1}{c}|r_{n+1} - r_n| \\
&+ (1-\alpha_{n+1})\lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) \\
&+ (1-\alpha_{n+1})\lambda_n\left\|By_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&+ M_2\prod_{i=1}^{n}\mu_i,
\end{aligned} \tag{3.29}$$

which implies that (noting that (i), (ii), (iii), (iv), (v), and $0 < \mu_i \le b < 1$, for all $i \ge 1$)

$$\limsup_{n\to\infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.30}$$

Hence, by Lemma 2.8, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.31}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.32}$$

Applying (3.32) and (ii), (iv), and (v) to (3.23) and (3.24), we obtain that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = \lim_{n\to\infty}\|k_{n+1}-k_n\| = 0. \tag{3.33}$$

Since $x_{n+1} = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n k_n$, we have

$$\begin{aligned}
\|x_n - W_n k_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n k_n\| \\
&= \|x_n - x_{n+1}\| + \left\|\epsilon_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \epsilon_n A\right)W_n k_n - W_n k_n\right\| \\
&= \|x_n - x_{n+1}\| + \left\|\epsilon_n\left(\gamma f(x_n) - AW_n k_n\right) + \beta_n(x_n - W_n k_n)\right\| \\
&\le \|x_n - x_{n+1}\| + \epsilon_n\left(\left\|\gamma f(x_n)\right\| + \|AW_n k_n\|\right) + \beta_n\|x_n - W_n k_n\|,
\end{aligned} \tag{3.34}$$

that is

$$\|x_n - W_n k_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\left(\left\|\gamma f(x_n)\right\| + \|AW_n k_n\|\right). \tag{3.35}$$

By (i), (iii), and (3.32) it follows that

$$\lim_{n\to\infty}\|W_n k_n - x_n\| = 0. \tag{3.36}$$

Step 3. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|u_n - k_n\| = 0$;

(ii) $\lim_{n\to\infty}\|x_n - u_n\| = 0$.

For any $p \in \Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B)$ and (3.14), we have

$$\begin{aligned}
\left\|k_n - p\right\|^2 &= \left\|\alpha_n(u_n - p) + (1-\alpha_n)(v_n - p)\right\|^2 \\
&\le \alpha_n\left\|u_n - p\right\|^2 + (1-\alpha_n)\left\|v_n - p\right\|^2 \\
&\le \alpha_n\left\|u_n - p\right\|^2 + (1-\alpha_n)\left\{\left\|u_n - p\right\|^2 + \left(\frac{\lambda_n^2}{\beta^2} - 1\right)\left\|u_n - y_n\right\|^2\right\} \\
&= \left\|u_n - p\right\|^2 + (1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2} - 1\right)\left\|u_n - y_n\right\|^2 \\
&\le \left\|x_n - p\right\|^2 + (1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2} - 1\right)\left\|u_n - y_n\right\|^2.
\end{aligned} \tag{3.37}$$

Observe that

$$
\begin{aligned}
\left\|x_{n+1}-p\right\|^2 &= \left\|((1-\beta_n)I-\epsilon_n A)(W_n k_n-p)+\beta_n(x_n-p)+\epsilon_n(\gamma f(x_n)-Ap)\right\|^2 \\
&= \left\|((1-\beta_n)I-\epsilon_n A)(W_n k_n-p)+\beta_n(x_n-p)\right\|^2+\epsilon_n^2\left\|\gamma f(x_n)-Ap\right\|^2 \\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle \\
&\quad+2\epsilon_n\big\langle\big((1-\beta_n)I-\epsilon_n A\big)(W_n k_n-p),\gamma f(x_n)-Ap\big\rangle \\
&\le \left[(1-\beta_n-\epsilon_n\overline{\gamma})\left\|W_n k_n-p\right\|+\beta_n\left\|x_n-p\right\|\right]^2 \\
&\quad+\epsilon_n^2\left\|\gamma f(x_n)-Ap\right\|^2 \\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle \\
&\quad+2\epsilon_n\big\langle\big((1-\beta_n)I-\epsilon_n A\big)(W_n k_n-p),\gamma f(x_n)-Ap\big\rangle \\
&\le \left[(1-\beta_n-\epsilon_n\overline{\gamma})\left\|k_n-p\right\|+\beta_n\left\|x_n-p\right\|\right]^2+c_n \\
&\le (1-\beta_n-\epsilon_n\overline{\gamma})^2\left\|k_n-p\right\|^2+\beta_n^2\left\|x_n-p\right\|^2 \\
&\quad+2(1-\beta_n-\epsilon_n\overline{\gamma})\beta_n\left\|k_n-p\right\|\left\|x_n-p\right\|+c_n \\
&\le (1-\beta_n-\epsilon_n\overline{\gamma})^2\left\|k_n-p\right\|^2+\beta_n^2\left\|x_n-p\right\|^2 \\
&\quad+(1-\beta_n-\epsilon_n\overline{\gamma})\beta_n\Big(\left\|k_n-p\right\|^2+\left\|x_n-p\right\|^2\Big)+c_n \\
&= \Big[(1-\epsilon_n\overline{\gamma})^2-2(1-\epsilon_n\overline{\gamma})\beta_n+\beta_n^2\Big]\left\|k_n-p\right\|^2+\beta_n^2\left\|x_n-p\right\|^2 \\
&\quad+\ \Big((1-\epsilon_n\overline{\gamma})\beta_n-\beta_n^2\Big)\Big(\left\|k_n-p\right\|^2+\left\|x_n-p\right\|^2\Big)+c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\left\|k_n-p\right\|^2-(1-\epsilon_n\overline{\gamma})\beta_n\left\|k_n-p\right\|^2 \\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\|k_n-p\right\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n,
\end{aligned}
\tag{3.38}
$$

where

$$
\begin{aligned}
c_n = \epsilon_n^2\left\|\gamma f(x_n)-Ap\right\|^2+2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle \\
+\ 2\epsilon_n\big\langle\big((1-\beta_n)I-\epsilon_n A\big)(W_n k_n-p),\gamma f(x_n)-Ap\big\rangle.
\end{aligned}
\tag{3.39}
$$

It follows from condition (i) that

$$
\lim_{n\to\infty} c_n = 0. \tag{3.40}
$$

Substituting (3.37) into (3.38), and using (v), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n-p\|^2+(1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2+c_n \\
&\leq \|x_n-p\|^2+(1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2+c_n.
\end{aligned} \tag{3.41}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)\delta\|u_n-y_n\|^2 &\leq (1-\alpha_n)\left(1-\frac{\lambda_n^2}{\beta^2}\right)\|u_n-y_n\|^2 \\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n \\
&= (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n \\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned} \tag{3.42}$$

Since $\lim_{n\to\infty}c_n=0$ and from (3.32), we obtain

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.43}$$

Note that

$$k_n-v_n=\alpha_n(u_n-v_n). \tag{3.44}$$

Since $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|k_n-v_n\|=0. \tag{3.45}$$

As B is $1/\beta$-Lipschitz continuous, we obtain

$$\begin{aligned}\|v_n - y_n\| &= \|P_C(u_n - \lambda_n B y_n) - P_C(u_n - \lambda_n B u_n)\| \\ &\le \|(u_n - \lambda_n B y_n) - (u_n - \lambda_n B u_n)\| \\ &= \lambda_n \|B u_n - B y_n\| \\ &\le \frac{\lambda_n}{\beta} \|u_n - y_n\|,\end{aligned} \tag{3.46}$$

then, we get

$$\lim_{n \to \infty} \|v_n - y_n\| = 0. \tag{3.47}$$

from

$$\|u_n - k_n\| \le \|u_n - y_n\| + \|y_n - v_n\| + \|v_n - k_n\|. \tag{3.48}$$

Applying (3.43), (3.45), and (3.47), we have

$$\lim_{n \to \infty} \|u_n - k_n\| = 0. \tag{3.49}$$

For any $p \in \Theta$, note that T_r is firmly nonexpansive (Lemma 2.4), then we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \\ &\le \langle T_{r_n} x_n - T_{r_n} p, x_n - p \rangle \\ &= \langle u_n - p, x_n - p \rangle \\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right),\end{aligned} \tag{3.50}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2, \tag{3.51}$$

which together with (3.38) gives

$$
\begin{aligned}
\left\|x_{n+1}-p\right\|^2 \leq{} & (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\|k_n-p\right\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n \\
={} & (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n) \\
& \times\left\{\|k_n-u_n\|^2+\left\|u_n-p\right\|^2+2\langle k_n-u_n,u_n-p\rangle\right\} \\
& + (1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n \\
\leq{} & (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
& +(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\left\|u_n-p\right\|^2 \\
& + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\left\|u_n-p\right\| \\
& +(1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n \\
\leq{} & (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
& + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\left\{\left\|x_n-p\right\|^2-\|x_n-u_n\|^2\right\} \\
& + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\left\|u_n-p\right\| \\
& +(1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n \\
\leq{} & (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
& + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\left\|x_n-p\right\|^2 \\
& -(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2 \\
& + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\left\|u_n-p\right\| \\
& +(1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n-p\right\|^2+c_n \\
={} & (1-\epsilon_n\overline{\gamma})^2\left\|x_n-p\right\|^2-(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2 \\
& + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
& + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\left\|u_n-p\right\|+c_n \\
={} & \left[1-2\epsilon_n\overline{\gamma}+(\epsilon_n\overline{\gamma})^2\right]\left\|x_n-p\right\|^2 \\
& -(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2 \\
& + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
& +2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\left\|u_n-p\right\|+c_n \\
\leq{} & \left\|x_n-p\right\|^2+(\epsilon_n\overline{\gamma})^2\left\|x_n-p\right\|^2 \\
& +(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2 \\
&+ 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\| + c_n.
\end{aligned}
\tag{3.52}
$$

So

$$
\begin{aligned}
&(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2 \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\| + c_n \\
&= (\|x_n-p\| - \|x_{n+1}-p\|)(\|x_n-p\| + \|x_{n+1}-p\|) \\
&\quad + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\| + c_n \\
&\le \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2 \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\| + c_n.
\end{aligned}
\tag{3.53}
$$

Using $\epsilon_n \to 0$, $c_n \to 0$ as $n \to \infty$, (3.32), and (3.49), we obtain

$$
\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.54}
$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$
\lim_{n\to\infty}\left\|\frac{x_n-u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n-u_n\| = 0. \tag{3.55}
$$

Observe that

$$
\begin{aligned}
\|W_nu_n-u_n\| &\le \|W_nu_n-W_nk_n\| + \|W_nk_n-x_n\| + \|x_n-u_n\| \\
&\le \|u_n-k_n\| + \|W_nk_n-x_n\| + \|x_n-u_n\|.
\end{aligned}
\tag{3.56}
$$

Applying (3.36), (3.49), and (3.54) to the last inequality, we obtain

$$
\lim_{n\to\infty}\|W_nu_n-u_n\| = 0. \tag{3.57}
$$

Let W be the mapping defined by (2.6). Since $\{u_n\}$ is bounded, applying Lemma 2.7 and (3.57), we have

$$\|Wu_n - u_n\| \leq \|Wu_n - W_n u_n\| + \|W_n u_n - u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.58}$$

Step 4. We claim that $\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle \leq 0$, where z is the unique solution of the variational inequality $\langle (A-\gamma f)z, z-x\rangle \geq 0$, for all $x \in \Theta$.

Since $z = P_\Theta(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.1), to show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle (A-\gamma f)z, z-u_{n_i}\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z-u_n\rangle. \tag{3.59}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|Wu_n - u_n\| \to 0$, we obtain $Wu_{n_i} \rightharpoonup w$. Next, We show that $w \in \Theta$, where $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B)$. First, we show that $w \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.60}$$

If follows from (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n), \tag{3.61}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}). \tag{3.62}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup w$, it follows by (A4) that $F(y,w) \leq 0$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)w$. Since $y \in H$ and $w \in H$, we have $y_t \in H$ and hence $F(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y), \tag{3.63}$$

and hence $F(y_t, y) \geq 0$. From (A3), we have $F(w, y) \geq 0$ for all $y \in H$ and hence $w \in EP(F)$.

Next, we show that $w \in \cap_{n=1}^{\infty} F(T_n)$. By Lemma 2.6, we have $F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume $w \notin F(W)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Ww$, it follows by the Opial's condition that

$$\begin{aligned}
\liminf_{i\to\infty}\|u_{n_i} - w\| &< \liminf_{i\to\infty}\|u_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty}\{\|u_{n_i} - Wu_{n_i}\| + \|Wu_{n_i} - Ww\|\} \\
&\leq \liminf_{i\to\infty}\|u_{n_i} - w\|,
\end{aligned} \tag{3.64}$$

which derives a contradiction. Thus, we have $w \in F(W) = \cap_{n=1}^{\infty} F(T_n)$. By the same argument as that in the proof of [35, Theorem 2.1, Pages 10–11], we can show that $w \in VI(C, B)$. Hence $w \in \Theta$. Since $z = P_\Theta(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle (A - \gamma f)z, z - x_n \rangle &= \limsup_{n\to\infty} \langle (A - \gamma f)z, z - u_n \rangle \\
&= \lim_{i\to\infty} \langle (A - \gamma f)z, z - u_{n_i} \rangle \\
&= \langle (A - \gamma f)z, z - w \rangle \leq 0.
\end{aligned} \tag{3.65}$$

It follows from the last inequality, (3.36), and (3.54) that

$$\limsup_{n\to\infty} \langle \gamma f(z) - Az, W_n k_n - z \rangle \leq 0. \tag{3.66}$$

Step 5. Finally, we show that $\{x_n\}$and $\{u_n\}$ converge strongly to $z = P_\Theta(I - A + \gamma f)(z)$. Indeed, from (1.25) , we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \left\| \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \epsilon_n A)W_n k_n - z \right\|^2 \\
&\quad = \left\| ((1 - \beta_n)I - \epsilon_n A)(W_n k_n - z) + \beta_n(x_n - z) + \epsilon_n(\gamma f(x_n) - Az) \right\|^2 \\
&\quad = \left\| ((1 - \beta_n)I - \epsilon_n A)(W_n k_n - z) + \beta_n(x_n - z) \right\|^2 + \epsilon_n^2 \left\| \gamma f(x_n) - Az \right\|^2 \\
&\qquad + 2\beta_n \epsilon_n \langle x_n - z, \gamma f(x_n) - Az \rangle \\
&\qquad + 2\epsilon_n \langle ((1 - \beta_n)I - \epsilon_n A)(W_n k_n - z), \gamma f(x_n) - Az \rangle \\
&\quad \leq \left[(1 - \beta_n - \epsilon_n \overline{\gamma}) \|W_n k_n - z\| + \beta_n \|x_n - z\| \right]^2 + \epsilon_n^2 \left\| \gamma f(x_n) - Az \right\|^2 \\
&\qquad + 2\beta_n \epsilon_n \gamma \langle x_n - z, f(x_n) - f(z) \rangle + 2\beta_n \epsilon_n \langle x_n - z, \gamma f(z) - Az \rangle \\
&\qquad + 2(1 - \beta_n)\gamma \epsilon_n \langle W_n k_n - z, f(x_n) - f(z) \rangle + 2(1 - \beta_n)\epsilon_n \langle W_n k_n - z, \gamma f(z) - Az \rangle \\
&\qquad - 2\epsilon_n^2 \langle A(W_n k_n - z), \gamma f(z) - Az \rangle \\
&\quad \leq \left[(1 - \beta_n - \epsilon_n \overline{\gamma}) \|W_n k_n - z\| + \beta_n \|x_n - z\| \right]^2 + \epsilon_n^2 \left\| \gamma f(x_n) - Az \right\|^2 \\
&\qquad + 2\beta_n \epsilon_n \gamma \|x_n - z\| \left\| f(x_n) - f(z) \right\| + 2\beta_n \epsilon_n \langle x_n - z, \gamma f(z) - Az \rangle \\
&\qquad + 2(1 - \beta_n)\gamma \epsilon_n \|W_n k_n - z\| \left\| f(x_n) - f(z) \right\| + 2(1 - \beta_n)\epsilon_n \langle W_n k_n - z, \gamma f(z) - Az \rangle \\
&\qquad - 2\epsilon_n^2 \langle A(W_n k_n - z), \gamma f(z) - Az \rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\right]^2+\epsilon_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+\ 2\beta_n\epsilon_n\gamma\alpha\|x_n-z\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+\ 2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n-z\|^2+2(1-\beta_n)\epsilon_n\langle W_nk_n-z,\gamma f(z)-Az\rangle\\
&\quad-\ 2\epsilon_n^2\langle A(W_nk_n-z),\gamma f(z)-Az\rangle\\
&=\left[(1-\epsilon_n\overline{\gamma})^2+2\beta_n\epsilon_n\gamma\alpha+2(1-\beta_n)\gamma\epsilon_n\alpha\right]\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_nk_n-z,\gamma f(z)-Az\rangle\\
&\quad-\ 2\epsilon_n^2\langle A(W_nk_n-z),\gamma f(z)-Az\rangle\\
&\leq\left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\overline{\gamma}^2\epsilon_n^2\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_nk_n-z,\gamma f(z)-Az\rangle\\
&\quad+\ 2\epsilon_n^2\|A(W_nk_n-z)\|\left\|\gamma f(z)-Az\right\|\\
&=\left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\epsilon_n\\
&\quad\times\Big\{\epsilon_n\Big[\overline{\gamma}^2\|x_n-z\|^2+\left\|\gamma f(x_n)-Az\right\|^2\\
&\qquad+\ 2\|A(W_nk_n-z)\|\left\|\gamma f(z)-Az\right\|\Big]+2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\qquad+\ 2(1-\beta_n)\langle W_nk_n-z,\gamma f(z)-Az\rangle\Big\}.
\end{aligned}
\tag{3.67}
$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{W_nk_n\}$ are bounded, we can take a constant $M>0$ such that

$$\overline{\gamma}^2\|x_n-z\|^2+\left\|\gamma f(x_n)-Az\right\|^2+2\|A(W_nk_n-z)\|\left\|\gamma f(z)-Az\right\|\leq M \tag{3.68}$$

for all $n\geq 0$. It then follows that

$$\|x_{n+1}-z\|^2\leq\left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\epsilon_n\sigma_n, \tag{3.69}$$

where

$$\sigma_n=2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\langle W_nk_n-z,\gamma f(z)-Az\rangle+\epsilon_nM. \tag{3.70}$$

Using (i), (3.65), and (3.66), we get $\limsup_{n\to\infty}\sigma_n\leq 0$. Applying Lemma 2.9 to (3.69), we conclude that $x_n\to z$ in norm. Finally, noticing

$$\|u_n-z\|=\|T_{r_n}x_n-T_{r_n}z\|\leq\|x_n-z\|, \tag{3.71}$$

we also conclude that $u_n\to z$ in norm. This completes the proof. □

Corollary 3.2 ([28, Theorem 3.1]). *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\{T_n\}$ be an infinitely many nonexpansive of C into itself such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$ and $\{u_n\}$ are the sequences generated by*

$$\begin{gathered} x_1 = x \in C \quad \text{chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)W_n u_n, \quad \forall n \geq 1, \end{gathered} \tag{3.72}$$

where $\{W_n\}$ is the sequence generated by (1.23), *$\beta \in (0,1)$, $\{\epsilon_n\}$ is a sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$;

(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.73}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $B = 0$, $\{\beta_n\} = \beta$, and $\{\alpha_n\} = 0$ in Theorem 3.1., then $y_n = k_n = u_n$. The conclusion of Corollary 3.2 can obtain the desired result easily. □

Corollary 3.3. *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta := EP(F) \cap VI(C,B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by*

$$\begin{aligned} &x_1 = x \in C \quad \text{chosen arbitrary}, \\ &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &y_n = P_C(u_n - \lambda_n B u_n), \\ &k_n = \alpha_n u_n + (1-\alpha_n)P_C(u_n - \lambda_n B y_n), \\ &x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)k_n, \quad \forall n \geq 1, \end{aligned} \tag{3.74}$$

where $\{\epsilon_n\}$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *are three sequences in* $(0,1)$, *and* $\{r_n\}$ *is a real sequence in* $(0,\infty)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(v) $\{\lambda_n/\beta\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \ge 0, \quad \forall x \in \Theta. \tag{3.75}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $T_n x = x$ for all $n \in \mathbb{N}$ and for all $x \in C$. Then $W_n x = x$ for all $x \in C$. The conclusion follows from Theorem 3.1. □

Corollary 3.4. *Let* C *be nonempty closed convex subset of a real Hilbert space* H, *let* $\{T_n\}$ *be an infinitely many nonexpansive of* C *into itself, and let* B *be a* β*-inverse-strongly monotone mapping of* C *into* H *such that* $\Theta := \cap_{n=1}^{\infty}F(T_n) \cap VI(C,B) \neq \emptyset$. *Let* f *be a contraction of* H *into itself with* $\alpha \in (0,1)$ *and let* A *be a strongly positive linear bounded operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let* $\{x_n\}$, $\{y_n\}$, *and* $\{k_n\}$ *be sequences generated by*

$$\begin{aligned}
&x_1 = x \in C \quad \textit{chosen arbitrary},\\
&y_n = P_C(x_n - \lambda_n B x_n),\\
&k_n = \alpha_n x_n + (1-\alpha_n)P_C\left(x_n - \lambda_n B y_n\right),\\
&x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \epsilon_n A\right)W_n k_n, \quad \forall n \ge 1,
\end{aligned} \tag{3.76}$$

where $\{W_n\}$ *is the sequences generated by* (1.23), *and* $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *are three sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\{\lambda_n/\beta\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \ge 0, \quad \forall x \in \Theta. \tag{3.77}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $F(x,y)=0$ for all $x,y\in C$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 3.1. Then, we have $u_n=P_Cx_n=x_n$. So, by Theorem 3.1, we can conclude the desired conclusion easily. □

If $A=I,\gamma\equiv 1$ and $\gamma_n=1-\epsilon_n-\beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.5. *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\{T_n\}$ be an infinitely many nonexpansive of C into itself, and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta:=\cap_{n=1}^{\infty}F(T_n)\cap EP(F)\cap VI(C,B)\neq\emptyset$. Let f be a contraction of H into itself with $\alpha\in(0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by*

$$\begin{aligned}
&x_1=x\in C \quad \text{chosen arbitrary},\\
&F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\
&y_n=P_C(u_n-\lambda_nBu_n),\\
&k_n=\alpha_nu_n+(1-\alpha_n)P_C(u_n-\lambda_nBy_n),\\
&x_{n+1}=\epsilon_nf(x_n)+\beta_nx_n+\gamma_nW_nk_n,\quad \forall n\geq 1,
\end{aligned}\tag{3.78}$$

where $\{W_n\}$ is the sequences generated by (1.23), $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences in $(0,1)$ and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:

(i) $\epsilon_n+\beta_n+\gamma_n=1$;

(ii) $\lim_{n\to\infty}\epsilon_n=0$ *and* $\sum_{n=1}^{\infty}\epsilon_n=\infty$;

(iii) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(iv) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(v) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n|=0$,

(vi) $\{\lambda_n/\beta\}\subset(\tau,1-\delta)$ *for some* $\tau,\delta\in(0,1)$ *and* $\lim_{n\to\infty}\lambda_n=0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z\in\Theta$ which is the unique solution of the variational inequality

$$\langle z-f(z),z-x\rangle\geq 0,\quad \forall x\in\Theta.\tag{3.79}$$

Equivalently, one has $z=P_{\Theta}f(z)$.

Corollary 3.6. *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $\{T_n\}$ be an infinite family of nonexpansive of C into itself such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{gathered} x_1 = x \in C \quad \text{chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n W_n u_n, \quad \forall n \geq 1, \end{gathered} \tag{3.80}$$

where $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:

(i) $\epsilon_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.81}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $B = 0$ and $\{\alpha_n\} = 0$ in Corollary 3.5. then $y_n = k_n = u_n$. The conclusion of Corollary 3.6 can obtain the desired result easily. □

4. Application for Optimization Problem

In this section, we shall utilize the results presented in the paper to study the following optimization problem:

$$\begin{gathered} \min \quad h(x), \\ x \in C. \end{gathered} \tag{4.1}$$

where $h(x)$ is a convex and lower semicontinuous functional defined on a closed subset C of a Hilbert space H. We denote by T the set of solution of (4.1). Let F be a bifunction from $C \times C$ to $\mathbb{R}$ defined by $F(x,y) = h(y) - h(x)$. We consider the following equilibrium problem, that is, to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

It is obvious that $EP(F) = T$, where $EP(F)$ denotes the set of solution of equilibrium problem (4.2). In addition, it is easy to see that $F(x, y)$ satisfies the conditions $(A1)$–$(A4)$ in Section 1. Therefore, from the Corollary 3.6, we know the following iterative sequence $\{x_n\}$ defined by

$$x_1 \in C \quad \text{chosen arbitrary,}$$
$$h(y) - h(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \tag{4.3}$$
$$x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n u_n,$$

where $\{\epsilon_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:

(i) $\epsilon_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty}\epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ converges strongly to a point $z = P_T f(z)$ of optimization problem (4.1).

In special case, we pick $f(x) = 0$ for all $x \in H$ and $\beta_n = 0$, $r_n = 1$, $\epsilon_n = 1/2$ for all $n \in \mathbb{N}$, then $x_{n+1} = 1/2u_n$, and from (4.3) we obtain a special iterative scheme

$$\begin{gathered} h(y) - h(u_n) + \left\langle y - u_n, u_n - \frac{1}{2}u_{n-1}\right\rangle \geq 0, \quad \forall y \in C, \quad n \geq 2, \\ h(y) - h(u_1) + \langle y - u_1, u_1 - x_1\rangle \geq 0, \quad \forall y \in C. \end{gathered} \tag{4.4}$$

Then, $\{u_n\}$ converges strongly to a solution $z = P_T 0$ of optimization problem (4.1). In fact, the z is the minimum norm point on the T.

Therefore, we consider a special from of optimization problem (4.1) which is as follows: (i.e., is taking $h(x) = \|x\|$)

$$\begin{gathered} \min \ \|x\|, \\ x \in C. \end{gathered} \tag{4.5}$$

In fact, the solution of optimization problem (4.4) is named the minimum norm point on the closed convex set C. From iterative algorithm (4.4) we obtain the following iterative algorithm (4.5), and $\{u_n\}$ is defined by

$$\begin{gathered} \|y\| - \|u_n\| + \left\langle y - u_n, u_n - \frac{1}{2}u_{n-1}\right\rangle \geq 0, \quad \forall y \in C, \quad n \geq 2, \\ \|y\| - \|u_1\| + \langle y - u_1, u_1 - x_1\rangle \geq 0, \quad \forall y \in C \end{gathered} \tag{4.6}$$

for any initial guess $x_1 \in H$. Then, $\{u_n\}$ converges strongly to a minimum norm point on the closed convex set C.

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Strong Convergence Theorems by an Extragradient Method for Solving Variational Inequalities and Equilibrium Problems in a Hilbert Space*

Poom Kumam

Abstract

In this paper, we introduce an iterative process for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for monotone, Lipschitz-continuous mappings. The iterative process is based on the so-called extragradient method. We show that the sequence converges strongly to a common element of the above three sets under some parametric controlling conditions. This main theorem extends a recent result of Yao, Liou and Yao [Y. Yao, Y. C. Liou and J.-C. Yao, "An Extragradient Method for Fixed Point Problems and Variational Inequality Problems," *Journal of Inequalities and Applications* Volume 2007, Article ID 38752, 12 pages doi:10.1155/2007/38752] and many others.

Key Words: Nonexpansive mapping; Equilibrium problem; Fixed point; Lipschitz-continuous mappings; Variational inequality; Extragradient method.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \longrightarrow H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). In

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1997 Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that $\langle Ax^*, v - x^*\rangle \geq 0$ for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [12, 15] and the references therein. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \tag{1.2}$$

for all $u, v \in C$. A is called *k-Lipschitz-continuous* if there exists a positive constant k such that for all $u, v \in C$

$$\|Au - Av\| \leq k\|u - v\|. \tag{1.3}$$

We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [9] introduced the iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.4}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C, \alpha_n$ is a sequence in (0, 1), and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [6] and Zeng and Yao [16] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$.

The algorithm suggested by Takahashi and Toyoda [9] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968; see [3] for more details.

In 1976, Korpelevich [4] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = P_C(x_n - \lambda_n A\bar{x}_n) \end{cases} \tag{1.5}$$

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C in to $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.5), converge to the same point $z \in VI(C, A)$.

Motivated by the idea of Korpelevichs extragradient method Zeng and Yao [16] introduced a new extragradient method for finding an element of $F(S) \cap VI(C, A)$ and proved the following strong convergence theorem.

Theorem 1.1 ([16, Theorem 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and k-Lipschitz-continous mapping of C into H. Let S be a nonexpansive mappings from C into itself such that* $\mathrm{F}(S) \cap VI(C, A) \neq \emptyset$. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \ \ \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the conditions*

(i) $\lambda_n k \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$*;*

(ii) $\alpha_n \subset (0,1), \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\longrightarrow\infty} \alpha_n = 0$*,*

Then the sequence $\{x_n\}$ *and* $\{y_n\}$ *converges strongly to the same point* $P_{\mathrm{F}(S)\cap VI(C,A)}x_0$ *provied that* $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

In 2007, Yao, Liou and Yao [14] introduced the following iterative scheme: Let C be a closed convex subset of real Hilbert space H. Let A be a monotone k-Lipschitz-continous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A,C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone k-Lipschitz-continous mapping under some parameters controlling conditions.

Recently, Takahashi and Takahashi [10] introduced an iterative scheme:

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T y_n, \ \ n \geq 1 \end{cases}$$

for approximatiing a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

In this paper, motivated and inspired by the above results, we introduce a new iterative scheme by the extragradient method as follows: For $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ y_n = P_C(u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \ \ n \geq 1, \end{cases} \tag{1.8}$$

for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone k-Lipschitz-continous mapping in a real Hilbert space. Moreover, we obtain a strong convergence theorem which is connected with Yao, Liou and Yao's result [14], Takahashi and Tada's result [9] and Zeng and Yao's result [16].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A,C) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{2.6}$$

We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)u - (I-\lambda A)v\|^2 &= \|(u-v) - \lambda(Au - Av)\|^2 \\ &= \|u-v\|^2 - 2\lambda\langle u-v, Au-Av\rangle + \lambda^2\|Au-Av\|^2 \\ &\leq \|u-v\|^2 + \lambda(\lambda - 2\alpha)\|Au-Av\|^2. \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (See Osilike and Igbokwe [7].) *Let $(E, \langle .,.\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.2 (See Suzuki [8]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty}\|y_n - x_n\| = 0$.*

Lemma 2.3 (**Demiclosedness Principle**; cf. Goebel and Kirk [5].) *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \longrightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.4 (See Xu [11]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that:

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \longrightarrow \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\longrightarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.5 (See Blum and Oettli [1]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [2].

Lemma 2.6 (See Combettes and Hirstoaga [2].) *Assume that $F : C \times C \longrightarrow \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single- valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

3. Main Results

In this section, we introduce an iterative process by the extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone k-Lipschitz-continous mapping in a real Hilbert space. We prove that the iterative sequences converges strongly to a common element of the above three sets.

Theorem 3.1 *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow \mathbf{R}$ satisfying (A1)–(A4) and let A be a monotone k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A,C) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ y_n = P_C(u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{3.1}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{r_n\} \subset (0,\infty)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(iv) $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(v) $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(A,C)\cap EP(F)}u$.

Proof. For all $x, y \in C$, we note that

$$\begin{aligned} \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda_n\langle x-y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda_n^2 k^2\|x-y\|^2 = (1+\lambda_n^2k^2)\|x-y\|^2, \end{aligned} \tag{3.2}$$

which implies that

$$\|(I - \lambda_n A)x - (I - \lambda_n A)y\| \leq (1 + \lambda_n k)\|x - y\|. \tag{3.3}$$

Let $x^* \in F(S) \cap VI(A,C) \cap EP(F)$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.6 and $u_n = T_{r_n}x_n$. Then $x^* = P_C(x^* - \lambda_n Ax^*) = T_{r_n}x^*$. Put $v_n = P_C(x_n - \lambda_n Ay_n)$. For any $n \in \mathbb{N}$, we get

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|.$$

From (2.5) and the monotonicity of A, we have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|x_n - \lambda_n Ay_n - x^*\|^2 - \|x_n - \lambda_n Ay_n - v_n\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n \langle Ay_n, u - v_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n(\langle Ay_n - Au, x^* - y_n\rangle + \langle Au, x^* - y_n\rangle) + \langle Ay_n, y_n - v_n\rangle \\
&\leq \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n \langle Ay_n, y_n - v_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 + 2\lambda_n \langle Ay_n, y_n - v_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle.
\end{aligned}$$

Since A is k-Lipschitz-continuous, it follows that

$$\begin{aligned}
\langle x_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle &= \langle x_n - \lambda_n Ax_n - y_n, v_n - y_n\rangle + \langle \lambda_n Ax_n - \lambda_n Ay_n, v_n - y_n\rangle \\
&\leq \langle \lambda_n Ax_n - \lambda_n Ay_n, v_n - y_n\rangle \\
&\leq \lambda_n k\|x_n - y_n\|\|v_n - y_n\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|v_n - y_n\| \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2 k^2(\|x_n - y_n\|^2 + \|v_n - y_n\|^2) \\
&= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \qquad (3.4) \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Then, we have also

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n Sv_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|v_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\|, \|x_0 - x^*\|\}
\end{aligned}$$

Therefore $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$ and $\{v_n\}$ are also bounded. Moreover, we observe that

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|P_C(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(x_n - \lambda_n Ay_n)\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - (x_n - \lambda_n Ay_n)\| \\
&= \|(x_{n+1} - x_n) - \lambda_{n+1}(Ay_{n+1} - Ay_n) - (\lambda_{n+1} - \lambda_n)Ay_n\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_{n+1}k\|y_{n+1} - y_n\| + |\lambda_{n+1} - \lambda_n|\|Ay_n\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_{n+1}k\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Ay_n\|. \qquad (3.5)
\end{aligned}$$

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \qquad (3.6)$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \text{ for all } y \in C. \tag{3.7}$$

Putting $y = u_{n+1}$ in (3.6) and $y = u_n$ in (3.7), we obtain

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

It follows from (A2) that

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|, \end{aligned} \tag{3.8}$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting (3.8) into (3.5), we have

$$\begin{aligned} \|v_{n+1} - v_n\| &\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\{\|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|\} + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &\leq (1 + k\lambda_{n+1})\|x_{n+1} - x_n\| + k\lambda_{n+1}\frac{L}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|. \end{aligned} \tag{3.9}$$

Let $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$. Thus, we get

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n SP_C(x_n - \lambda_n Ay_n)}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n Sv_n}{1 - \beta_n}$$

and hence we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n Sv_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}u + \gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}u + \gamma_{n+1}Sv_n}{1-\beta_{n+1}} + \frac{\alpha_{n+1}u + \gamma_{n+1}Sv_n}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n Sv_n}{1-\beta_n} \\ &= (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(Sv_{n+1} - Sv_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})Sv_n. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1} - v_n\| \\ &\quad + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|Sv_n\| - \|x_{n+1} - x_n\| \\ &\leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(1 + \lambda_{n+1}k)\|x_{n+1} - x_n\| \\ &\quad + \frac{\gamma_{n+1}}{(1-\beta_{n+1})}\frac{L}{c}\lambda_{n+1}k|r_{n+1} - r_n| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &\quad + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|Sv_n\| - \|x_{n+1} - x_n\| \\ &\leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|u\| + \|Sv_n\|) + \frac{\gamma_{n+1}\lambda_{n+1}k - \alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\{\lambda_{n+1}k\frac{L}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|\}. \end{aligned}$$

This together with (ii), (iv) and (v) imply that

$$\limsup_{n \longrightarrow \infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.2, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.11}$$

Consequently,

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \longrightarrow \infty} (1 - \beta_n)\|z_n - x_n\| = 0. \tag{3.12}$$

From (iv), (v), (3.5) and (3.8), we also have $\|v_{n+1} - v_n\| \longrightarrow 0, \|u_{n+1} - u_n\| \longrightarrow 0$ and $\|y_{n+1} - y_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since

$$x_{n+1} - x_n = \alpha_n u + \beta_n x_n + \gamma_n Sv_n - x_n = \alpha_n(u - x_n) + \gamma_n(Sv_n - x_n),$$

it follows by (ii) and (3.12) that

$$\lim_{n\to\infty} \|x_n - Sv_n\| = 0. \tag{3.13}$$

We note that

$$\begin{aligned}
\|y_n - v_n\| &\leq \|P_C(u_n - \lambda_n Au_n) - P_C(x_n - \lambda_n Ay_n)\| \\
&\leq \|(u_n - \lambda_n Au_n) - (x_n - \lambda_n Ay_n)\| \\
&\leq \|u_n - x_n\| + \lambda_n \|Au_n - Ay_n\| \\
&\leq \|u_n - x_n\| + \lambda_n k \|u_n - y_n\| \\
&\leq \|u_n - x_n\|,
\end{aligned}$$

since $\lambda_n \leq 1$, hence we also have

$$\|y_n - v_n\|^2 \leq \|u_n - x_n\|^2. \tag{3.14}$$

From this and by (3.4) and (3.14) we obtain when $n \geq N$ that

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \\
&\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \\
&\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2.
\end{aligned}$$

So, from this, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + \beta_n x_n + \gamma_n S v_n - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|S v_n - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2\} \\
&= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + \gamma_n (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1 - \lambda_n^2 k^2)\|x_n - u_n\|^2 &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| - \|x_{n+1} - x^*\|).
\end{aligned}$$

Since $\alpha_n \longrightarrow 0$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{k})$ and $\|x_{n+1} - x_n\| \longrightarrow 0$, imply that

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0. \tag{3.15}$$

Since $\liminf_{n \longrightarrow \infty} r_n > 0$, we get

$$\lim_{n \longrightarrow \infty} \|\frac{x_n - u_n}{r_n}\| = \lim_{n \longrightarrow \infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.16}$$

By (3.4), we note that

$$\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2. \tag{3.17}$$

Thus, from Lemma 2.1 and (3.17), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|Sv_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\{\|x_n-x^*\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2\}\\ &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2.\end{aligned} \tag{3.18}$$

Therefore, we have

$$\begin{aligned}(1-\lambda_n^2k^2)\|x_n-y_n\|^2 &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &= \alpha_n\|u-x^*\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|).\end{aligned} \tag{3.19}$$

Since $\alpha_n \longrightarrow 0$ and $\|x_n-x_{n+1}\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain

$$\lim_{n\longrightarrow\infty}\|x_n-y_n\|=0. \tag{3.20}$$

We note that

$$\begin{aligned}\|v_n-y_n\| &= \|P_C(x_n-\lambda_nAy_n)-P_C(u_n-\lambda_nAu_n)\|\\ &\le \|(x_n-\lambda_nAy_n)-(u_n-\lambda_nAu_n)\|\\ &\le \|x_n-u_n\|+\lambda_n\|Au_n)-Ay_n)\|\\ &\le \|x_n-u_n\|+\lambda_nk\|u_n-y_n\|\\ &\le \|x_n-u_n\|+\lambda_nk\{\|u_n-x_n\|+\|x_n-y_n\|\}\\ &\le (1+\lambda_nk)\|u_n-x_n\|+\lambda_nk\|x_n-y_n\|\end{aligned}$$

since (3.15) and (3.20), we have

$$\lim_{n\longrightarrow\infty}\|v_n-y_n\|=0. \tag{3.21}$$

Since

$$\|Sv_n-v_n\|\le\|Sv_n-x_n\|+\|x_n-y_n\|+\|y_n-v_n\|,$$

and hence

$$\lim_{n\longrightarrow\infty}\|Sv_n-v_n\|=0. \tag{3.22}$$

Next, we show that

$$\limsup_{n\longrightarrow\infty}\langle u-z_0,x_n-z_0\rangle\le 0,$$

where $z_0=P_{F(S)\cap VI(A,C)\cap EP(F)}(u)$. To show this inequality, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle u-z_0,Sv_n-z_0\rangle=\lim_{i\longrightarrow\infty}\langle u-z_0,Sv_{n_i}-z_0\rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup z$. From $\|Sv_n - v_n\| \longrightarrow 0$, we obtain $Sv_{n_i} \rightharpoonup z$. Let us show $z \in EP(F)$. Since $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

From $\|u_n - x_n\| \longrightarrow 0, \|x_n - Sv_n\| \longrightarrow 0$, and $\|Sv_n - v_n\| \longrightarrow 0$, we get $u_{n_i} \rightharpoonup z$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) that $0 \geq F(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(z, y)$ for all $y \in C$ and hence $z \in EP(F)$. By the opial's condition, we obtain $z \in F(S)$. Finally, by the same argument as that in the proof of [9, Theorem 3.1, p. 197-198] , we can show that $z \in VI(A, C)$. Hence $z \in F(S) \cap VI(A, C) \cap EP(F)$.

Now from (2.4), we have

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle u - z_0, x_n - z_0\rangle &= \limsup_{n\longrightarrow\infty}\langle u - z_0, Sv_n - z_0\rangle = \lim_{i\longrightarrow\infty}\langle u - z_0, Sv_{n_i} - z_0\rangle \\
&= \langle u - z_0, z - z_0\rangle \leq 0. \qquad (3.23)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n Sv_n - z_0, x_{n+1} - z_0\rangle \\
&= \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle + \beta_n\langle x_n - z_0, x_{n+1} - z_0\rangle + \gamma_n\langle Sv_n - z_0, x_{n+1} - z_0\rangle \\
&\leq \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle + \frac{1}{2}\gamma_n(\|v_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\leq \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle + \frac{1}{2}\gamma_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&= \frac{1}{2}(1 - \alpha_n)(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\leq \frac{1}{2}\{(1 - \alpha_n)\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\} + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle
\end{aligned}$$

which implies that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.$$

Finally by (3.23) and Lemma 2.4, we get that $\{x_n\}$ converges to z_0, where $z_0 = P_{F(S)\cap VI(A,C)\cap EP(F)}(u)$. This completes the proof. □

Using Theorem 3.1, we can prove the following result.

Theorem 3.2 (Yao Liou and Yao [14, Theorem 3.1]) *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone k-Lipschitz-continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S)\cap VI(A,C) \neq \emptyset$. For fixed $u \in H$ and give $x_0 \in H$ arbitrary, let the sequence $\{x_n\}, \{y_n\}$ be generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{3.24}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, \frac{1}{k}]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < \frac{1}{k}$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(iv) $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(A,C)} x_0$.

Proof. Put $F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1 .
Then, we have $u_n = P_C x_n = x_n$. So, from Theorem 3.1 the sequence $\{x_n\}$ generated in Theorem 3.2 converges strongly to $P_{F(S)\cap VI(A,C)} u$. □

Remark 3.3 *In Theorem 3.2, we also obtain Yao et al.'s theorem [14].*

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Research Article

Convergence Theorem Based on a New Hybrid Projection Method for Finding a Common Solution of Generalized Equilibrium and Variational Inequality Problems in Banach Spaces

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The purpose of this paper is to introduce a new hybrid projection method for finding a common element of the set of common fixed points of two relatively quasi-nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone, and the set of solutions of the generalized equilibrium problem in the framework of a real Banach space. We obtain a strong convergence theorem for the sequences generated by this process in a 2-uniformly convex and uniformly smooth Banach space. Base on this result, we also get some new and interesting results. The results in this paper generalize, extend, and unify some well-known strong convergence results in the literature.

1. Introduction

Let E be a real Banach space, E^* the dual space of E. A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$, with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function

$\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}. \tag{1.2}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$; see [1, 2] for more details. Observe that every p-uniform convex is uniformly convex. One should note that no Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}\right\} \tag{1.3}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a real Banach space with norm $\|\cdot\|$ and E^* denotes the dual space of E. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x, y \in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) reduces to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x); \tag{1.5}$$

existence and uniqueness of the mapping Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [3–7]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.6}$$

Remark 1.1. If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.13), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [5, 7] for more details.

Next, we give some examples which are closed relatively quasi-nonexpansive (see [8]).

Example 1.2. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then, Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Let E be a real Banach space and let C be a nonempty closed and convex subset of E, and $A : C \to E^*$ be a mapping. *The classical variational inequality problem* for a mapping A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.4) is denoted by $\mathrm{VI}(A, C)$. Recall that A is called

(i) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C, \tag{1.8}$$

(ii) an *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C. \tag{1.9}$$

Such a problem is connected with the convex minimization problem, the complementary problem, and the problem of finding a point $x^* \in E$ satisfying $Ax^* = 0$.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. *The equilibrium problem* (for short, EP) is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.10}$$

The set of solutions of (1.10) is denoted by $\mathrm{EP}(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $x^* \in \mathrm{EP}(f)$ if and only if $\langle Tx^*, y - x^* \rangle \geq 0$ for all $y \in C$; that is, x^* is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.10). Some methods have been proposed to solve the equilibrium problem; see, for instance, [9–11].

Let C be a closed convex subset of E; a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.11}$$

A point $x \in C$ is a *fixed point* of T provided that $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a point p in C is said to be an *asymptotic fixed point* of T [12] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F(T)}$. A mapping T from C into itself is said to be *relatively nonexpansive* [13–15] if $\widehat{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [16–18]. T is said to be *ϕ-nonexpansive*, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be *relatively quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$. A mapping T in a Banach space E is *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Remark 1.3. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [16–19] which requires the strong restriction $F(T) = \widehat{F(T)}$.

In Hilbert spaces H, Iiduka et al. [20] proved that the sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n A x_n), \tag{1.12}$$

where P_C is the metric projection of H onto C, and $\{\lambda_n\}$ is a sequence of positive real numbers, and converges weakly to some element of $\mathrm{VI}(A, C)$.

It is well know that if C is a nonempty closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [4] recently introduced a generalized projection mapping Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

In 2008, Iiduka and Takahashi [21] introduced the following iterative scheme for finding a solution of the variational inequality problem for inverse-strongly monotone A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(J x_n - \lambda_n A x_n) \tag{1.13}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^*, and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.13) converges weakly to some element of $\mathrm{VI}(A, C)$.

Matsushita and Takahashi [22] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \tag{1.14}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2005, Matsushita and Takahashi [19] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned} x_0 &\in C \quad \text{chosen arbitrarily}, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\ x_{n+1} &= \prod_{C_n \cap Q_n} x_0. \end{aligned} \tag{1.15}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Recently, Takahashi and Zembayashi [23] proposed the following modification of iteration (1.15) for a relatively nonexpansive mapping:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n), \\ u_n \in C \quad &\text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C, \\ H_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, J x - J x_n \rangle \ge 0\}, \\ x_{n+1} &= \prod_{H_n \cap W_n} x, \end{aligned} \tag{1.16}$$

where J is the duality mapping on E. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(f)} x$, where $\Pi_{F(S)\cap \mathrm{EP}(f)}$ is the generalized projection of E onto $F(S) \cap \mathrm{EP}(f)$. Also, Takahashi and Zembayashi [24] proved the following iteration for a relatively nonexpansive mapping:

$$\begin{aligned} y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n), \\ u_n \in C \quad &\text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} &= \prod_{C_{n+1}} x, \end{aligned} \tag{1.17}$$

where J is the duality mapping on E. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(f)}x$, where $\Pi_{F(S)\cap \mathrm{EP}(f)}$ is the generalized projection of E onto $F(S)\cap \mathrm{EP}(f)$. Qin and Su [25] proved the following iteration for relatively nonexpansive mappings T in a Banach space E:

$$
\begin{gathered}
x_0 \in C, \quad \text{chosen arbitrarily}, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T x_n), \\
C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\
Q_n = \{v \in C : \langle J x_0 - J x_n, x_n - v\rangle \ge 0\}, \\
x_{n+1} = \prod_{C_n\cap Q_n} x_0,
\end{gathered} \tag{1.18}
$$

the sequence $\{x_n\}$ generated by (1.18) converges strongly to $\Pi_{F(T)}x_0$.

In 2009, Wei et al. [26] proved the following iteration for two relatively nonexpansive mappings in a Banach space E:

$$
\begin{gathered}
x_0 \in C, \\
J z_n = \alpha_n J x_n + (1-\alpha_n) J T x_n, \\
J u_n = (\beta_n J x_n + (1-\beta_n) J S z_n), \\
H_n = \{v \in C : \phi(v, u_n) \le \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \le \phi(v, x_n)\}, \\
W_n = \{z \in C : \langle z - x_n, J x_0 - J x_n\rangle \le 0\}, \\
x_{n+1} = Q_{H_n\cap W_n} x_0;
\end{gathered} \tag{1.19}
$$

if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1)$ such that $\alpha_n \le 1-\delta_1$ and $\beta_n \le 1-\delta_2$ for some $\delta_1, \delta_2 \in (0,1)$, then $\{x_n\}$ generated by (1.19) converges strongly to a point $Q_{F(T)\cap F(S)}x_0$, where the mapping Q_C of E onto C is the generalized projection. Very recently, Cholamjiak [27] proved the following iteration:

$$
\begin{gathered}
z_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\
y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S z_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0,
\end{gathered} \tag{1.20}
$$

where J is the duality mapping on E. Assume that α_n, β_n, and γ_n are sequences in $[0,1]$. Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$, where $F := F(T)\cap F(S)\cap \mathrm{EP}(f)\cap \mathrm{VI}(A, C)$.

Motivated and inspired by Iiduka and Takahashi [21], Takahashi and Zembayashi [23, 24], Wei et al. [26], Cholamjiak [27], and Kumam and Wattanawitoon [28], we introduce a new hybrid projection iterative scheme which is difference from the algorithm (1.20) of Cholamjiak in [27, Theorem 3.1] for two relatively quasi-nonexpansive mappings in a Banach space. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$\begin{gathered} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) JTw_n\right), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JSz_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \left\{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\right\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{gathered} \tag{1.21}$$

where J is the duality mapping on E. Then, we prove that under certain appropriate conditions on the parameters, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.21) converge strongly to $\Pi_{F(S)\cap F(T)\cap \mathrm{EP}(f)\cap \mathrm{VI}(A,C)}$.

The results presented in this paper improve and extend the corresponding results announced by Iiduka and Takahashi [21], Wei et al. [26], Kumam and Wattanawitoon [28], and many other authors in the literature.

2. Preliminaries

We also need the following lemmas for the proof of our main results.

Lemma 2.1 (Beauzamy [29] and Xu [30]). *If E is a 2-uniformly convex Banach space, then, for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{2.1}$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $1/c$ in the Lemma is called the *p-uniformly convex constant* of E.

Lemma 2.2 (Beauzamy [29] and Zălinescu [31]). *If E is a p-uniformly convex Banach space and p is a given real number with $p \geq 2$, then for all $x, y \in E$, $J_x \in J_p(x)$, and $J_y \in J_p(y)$,*

$$\langle x - y, Jx - Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p, \tag{2.2}$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Lemma 2.3 (Kamimura and Takahashi [6]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.4 (Alber [4]). *Let C be a nonempty closed and convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Lemma 2.5 (Alber [4]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed and convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.4}$$

Lemma 2.6 (Qin et al. [8]). *Let E be a uniformly convex and smooth Banach space, let C be a closed and convex subset of E, and let T be a closed relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed and convex subset of C.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y); \tag{2.5}$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 2.7 (Blum and Oettli [9]). *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.6}$$

Lemma 2.8 (Combettes and Hirstoaga [10]). *Let C be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}, \tag{2.7}$$

for all $x \in C$. *Then the following holds:*

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive-type mapping, for all* $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \tag{2.8}$$

(3) $F(T_r) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex.*

Lemma 2.9 (Takahashi and Zembayashi [24]). *Let* C *be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space* E, *let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4), *and let* $r > 0$. *Then, for* $x \in E$ *and* $q \in F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.9}$$

Let E be a reflexive, strictly convex, and smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} is also single value, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [4]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.10}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.10 (Alber [4]). *Let* E *be a reflexive, strictly convex, and smooth Banach space and let* V *be as in* (2.10) *. Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*) \tag{2.11}$$

for all $x \in E$ *and* $x^*, y^* \in E^*$.

Let A be an inverse-strongly monotone of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^*, defined by $F(t) = A(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$*the normal cone* for C at a point $v \in C$; that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.12}$$

Theorem 2.11 (Rockafellar [32]). *Let* C *be a nonempty, closed and convex subset of a Banach space* E, *and* A *a monotone, hemicontinuous mapping of* C *into* E^*. *Let* $T \subset E \times E^*$ *be a mapping defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.13}$$

Then T *is maximal monotone and* $T^{-1}0 = \mathrm{VI}(A, C)$.

3. Main Results

In this section, we establish a new hybrid iterative scheme for finding a common element of the set of solutions of an equilibrium problems, the common fixed point set of two relatively quasi-nonexpansive mappings, and the solution set of variational inequalities for α-inverse strongly monotone in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$. Let $T, S : C \to C$ be closed relatively quasi-nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered}
w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
z_n = J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JTw_n\right), \\
y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JSz_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \left\{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\right\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{3.1}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1 - \delta_1$ and $\beta_n \leq 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Proof. We have several steps to prove this theorem as follows:

Step 1. We show that C_{n+1} is closed and convex.

Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know that

$$\phi(z, u_n) \leq \phi(z, x_n) \tag{3.2}$$

is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2. \tag{3.3}$$

So, C_{n+1} is closed and convex. Then, by induction, C_n is closed and convex for all $n \geq 1$.

Step 2. We show that $\{x_n\}$ is well defined.

Put $u_n = T_{r_n} y_n$ for all $n \geq 0$. On the other hand, from Lemma 2.8 one has T_{r_n} is relatively quasi-nonexpansive mappings and $\Omega \subset C_1 = C$. Supposing $\Omega \subset C_k$ for $k \in \mathbb{N}$, by the convexity of $\|\cdot\|^2$, for each $q \in \Omega \subset C_k$, we have

$$\begin{aligned}
\phi(q, u_k) &= \phi(q, T_{r_k} y_k) \\
&\leq \phi(q, y_k) \\
&= \phi\Big(q, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J S z_k)\Big) \\
&= \|q\|^2 - 2\langle q, \alpha_k J x_k + (1-\alpha_k) J S z_k \rangle + \|\alpha_k J x_k + (1-\alpha_k) J S z_k\|^2 \\
&\leq \|q\|^2 - 2\alpha_k \langle q, J x_k \rangle - 2(1-\alpha_k)\langle q, J S z_k \rangle + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|S z_k\|^2 \\
&= \alpha_k \phi(q, x_k) + (1-\alpha_k)\phi(q, S z_k) \\
&\leq \alpha_k \phi(q, x_k) + (1-\alpha_k)\phi(q, z_k),
\end{aligned} \tag{3.4}$$

and so

$$\begin{aligned}
\phi(q, z_k) &= \phi\Big(q, J^{-1}(\beta_k J x_k + (1-\beta_k) J T w_k)\Big) \\
&= \|q\|^2 - 2\langle q, \beta_k J x_k + (1-\beta_k) J T w_k \rangle + \|\beta_k J x_k + (1-\beta_k) J T w_k\|^2 \\
&\leq \|q\|^2 - 2\beta_k \langle q, J x_k \rangle - 2(1-\beta_k)\langle q, J T w_k \rangle + \beta_k \|J x_k\|^2 + (1-\beta_k)\|J T w_k\|^2 \\
&= \beta_k \phi(q, x_k) + (1-\beta_k)\phi(q, T w_k) \\
&\leq \beta_k \phi(q, x_k) + (1-\beta_k)\phi(q, w_k).
\end{aligned} \tag{3.5}$$

For all $q \in \Omega$, we know from Lemma 2.10, that

$$\begin{aligned}
\phi(q, w_k) &= \phi\Big(q, \Pi_C J^{-1}(J x_k - \lambda_k A x_k)\Big) \\
&\leq \phi\Big(q, J^{-1}(J x_k - \lambda_k A x_k)\Big) \\
&= V(q, J x_k - \lambda_k A x_k) \\
&\leq V(q, (J x_k - \lambda_k A x_k) + \lambda_k A x_k) - 2\Big\langle J^{-1}(J x_k - \lambda_k A x_k) - q, \lambda_k A x_k \Big\rangle \\
&= V(q, J x_k) - 2\lambda_k \Big\langle J^{-1}(J x_k - \lambda_k A x_k) - q, A x_k \Big\rangle \\
&= \phi(q, x_k) - 2\lambda_k \langle x_k - q, A x_k \rangle + 2\Big\langle J^{-1}(J x_k - \lambda_k A x_k) - x_k, -\lambda_k A x_k \Big\rangle.
\end{aligned} \tag{3.6}$$

Since $q \in \mathrm{VI}(A, C)$ and from A being an α-inverse-strongly monotone, we get

$$\begin{aligned} -2\lambda_k \langle x_k - q, Ax_k \rangle &= -2\lambda_k \langle x_k - q, Ax_k - Aq \rangle - 2\lambda_k \langle x_k - q, Aq \rangle \\ &\leq -2\lambda_k \langle x_k - q, Ax_k - Aq \rangle \\ &= -2\alpha\lambda_k \left\| Ax_k - Aq \right\|^2. \end{aligned} \tag{3.7}$$

From Lemma 2.1 and A being an α-inverse-strongly monotone, we obtain

$$\begin{aligned} 2\left\langle J^{-1}(Jx_k - \lambda_k Ax_k) - x_k, -\lambda_k Ax_k \right\rangle &= 2\left\langle J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k), -\lambda_k Ax_k \right\rangle \\ &\leq 2\left\| J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k) \right\| \|\lambda_k Ax_k\| \\ &\leq \frac{4}{c^2} \left\| JJ^{-1}(Jx_k - \lambda_k Ax_k) - JJ^{-1}(Jx_k) \right\| \|\lambda_k Ax_k\| \\ &= \frac{4}{c^2} \|Jx_k - \lambda_k Ax_k - Jx_k\| \|\lambda_k Ax_k\| \\ &= \frac{4}{c^2} \|\lambda_k Ax_k\|^2 \\ &= \frac{4}{c^2} \lambda_k^2 \|Ax_k\|^2 \\ &\leq \frac{4}{c^2} \lambda_k^2 \left\| Ax_k - Aq \right\|^2. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we have

$$\begin{aligned} \phi(q, w_k) &\leq \phi(q, x_k) - 2\alpha\lambda_k \left\| Ax_k - Aq \right\|^2 + \frac{4}{c^2} \lambda_k^2 \left\| Ax_k - Aq \right\|^2 \\ &= \phi(q, x_k) + 2\lambda_k \left(\frac{2}{c^2} \lambda_k - \alpha \right) \left\| Ax_k - Aq \right\|^2 \\ &\leq \phi(q, x_k). \end{aligned} \tag{3.9}$$

Replacing (3.9) into (3.5), we get

$$\phi(q, z_k) \leq \phi(q, x_k). \tag{3.10}$$

Substituting (3.10) into (3.4), we also have

$$\begin{aligned} \phi(q, u_k) &\leq \alpha_k \phi(q, x_k) + (1 - \alpha_k)\phi(q, x_k), \\ &= \phi(q, x_k). \end{aligned} \tag{3.11}$$

This shows that $q \in C_{k+1}$ and hence, $\Omega \subset C_{k+1}$. Hence, $\Omega \subset C_n$ for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists and $\{x_n\}$ is bounded.

From $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1\ , \tag{3.12}$$

and from Lemma 2.5, we have

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n}(x_0), x_0) \\ &\leq \phi(p, x_0) - \phi(p, x_n) \\ &\leq \phi(p, x_0), \quad \forall p \in \Omega.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. In particular, by (1.6), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies that $\{x_n\}$ is also bounded.

Step 4. We show that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$, for $m > n$, by Lemma 2.5, we have

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.14}$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. We have $\lim_{n\to\infty}\phi(x_{n+1}, x_0) = 0$. From Lemma 2.3, we get $\lim_{n\to\infty}\|x_{n+1} - x_0\| = 0$. Thus $\{x_n\}$ is a Cauchy sequence.

Step 5. We cliam that $\|Ju_n - Jx_n\| \to 0$, as $n \to \infty$.

By the completeness of E, the closedness of C and $\{x_n\}$ is a Cauchy sequence (from Step 4); we can assume that there exists $p \in C$ such that $\{x_n\} \to p$ as $n \to \infty$.

By definition of $\Pi_{C_n}x_0$, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.15}$$

Since $\lim_{n\to\infty}\phi(x_n, x_0)$ exists, we get

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{3.16}$$

It follow form Lemma 2.3, that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and from the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n), \quad \forall n \ge 1 \tag{3.18}$$

and so

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.19}$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.20}$$

By using the triangle inequality, we obtain

$$\begin{aligned} \|u_n - x_n\| &= \|u_n - x_{n+1} + x_{n+1} - x_n\| \\ &\le \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.21}$$

By (3.17), (3.20), we get

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.23}$$

Step 6. Show that $x_n \to p \in \mathrm{EP}(f)$.

Applying (3.4) and (3.11), we get $\phi(p, y_n) \le \phi(p, x_n)$. From Lemma 2.9 and $u_n = T_{r_n} y_n$, we observe that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\le \phi(p, y_n) - \phi(p, T_{r_n} y_n) \\ &\le \phi(p, x_n) - \phi(p, T_{r_n} y_n) \\ &= \phi(p, x_n) - \phi(p, u_n) \\ &= \|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2 - \left(\|p\|^2 - 2\langle p, Ju_n\rangle + \|u_n\|^2\right) \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|(\|x_n + u_n\|) + 2\|p\|\|Jx_n - Ju_n\|. \end{aligned} \tag{3.24}$$

From (3.22), (3.23) and Lemma 2.3, we get

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.25}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{3.26}$$

From $r_n > 0$, we have $\|Ju_n - Jy_n\|/r_n \to 0$ as $n \to \infty$ and

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C. \tag{3.27}$$

By (A2), that

$$\begin{aligned} \|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -f(u_n, y) \\ &\geq f(y, u_n), \quad \forall y \in C \end{aligned} \tag{3.28}$$

and $u_n \to p$, we get $f(y,p) \leq 0$ for all $y \in C$. For $0 < t < 1$, define $y_t = ty + (1-t)p$. Then $y_t \in C$ which implies that $f(y_t, p) \leq 0$. From (A1), we obtain that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y). \tag{3.29}$$

Thus $f(y_t, y) \geq 0$. From (A3), we have $f(p, y) \geq 0$ for all $y \in C$. Hence $p \in \mathrm{EP}(f)$.

Step 7. We show that $x_n \to p \in F(T) \cap F(S)$.

From definition of C_n, we have

$$\alpha_n\phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n) \Longleftrightarrow \phi(z, z_n) \leq \phi(z, x_n). \tag{3.30}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n). \tag{3.31}$$

It follows from (3.16) that

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0, \tag{3.32}$$

again from Lemma 2.3, we get

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.33}$$

By using the triangle inequality, we get

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{3.34}$$

Again by (3.17) and (3.33), we also have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.35}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0. \tag{3.36}$$

Since

$$\left\|y_n - z_n\right\| \le \left\|y_n - u_n\right\| + \|u_n - x_n\| + \|x_n - z_n\|, \tag{3.37}$$

from (3.22), (3.25), and (3.35), we have

$$\lim_{n\to\infty} \left\|y_n - z_n\right\| = 0. \tag{3.38}$$

Since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n\to\infty} \left\|Jy_n - Jz_n\right\| = 0. \tag{3.39}$$

From (3.1), we get

$$\begin{aligned} \left\|Jy_n - Jz_n\right\| &= \|\alpha_n(Jx_n - Jz_n) + (1-\alpha_n)(JSz_n - Jz_n)\| \\ &= \|(1-\alpha_n)(JSz_n - Jz_n) - \alpha_n(Jz_n - Jx_n)\| \\ &\ge (1-\alpha_n)\|JSz_n - Jz_n\| - \alpha_n\|Jz_n - Jx_n\|; \end{aligned} \tag{3.40}$$

it follows that

$$(1-\alpha_n)\|JSz_n - Jz_n\| \le \left\|Jy_n - Jz_n\right\| + \alpha_n\|Jz_n - Jx_n\|, \tag{3.41}$$

and hence

$$\|JSz_n - Jz_n\| \le \frac{1}{1-\alpha_n}\left(\left\|Jy_n - Jz_n\right\| + \alpha_n\|Jz_n - Jx_n\|\right). \tag{3.42}$$

Since $\alpha_n \leq 1 - \delta_1$ for some $\delta_1 \in (0,1)$, (3.36), and (3.39), one has $\lim_{n\to\infty}\|JSz_n - Jz_n\| = 0$. Since J^{-1} is uniformly norm-to-norm continuous, we get

$$\lim_{n\to\infty}\|Sz_n - z_n\| = 0. \tag{3.43}$$

Since

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - Sz_n\| + \|Sz_n - z_n\| + \|z_n - x_n\| \\ &\leq \|x_n - z_n\| + \|Sz_n - z_n\| + \|z_n - x_n\|,\end{aligned} \tag{3.44}$$

from (3.35) and (3.43), we obtain

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{3.45}$$

Since S is closed and $x_n \to p$, we have $p \in F(S)$.

On the other hand, we note that

$$\begin{aligned}\phi(q, x_n) - \phi(q, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle \\ &\leq \|x_n - u_n\|(\|x_n + u_n\|) + 2\|q\|\|Jx_n - Ju_n\|.\end{aligned} \tag{3.46}$$

It follows from $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$, that

$$\phi(q, x_n) - \phi(q, u_n) \longrightarrow 0. \tag{3.47}$$

Furthermore, from (3.4) and (3.5),

$$\begin{aligned}\phi(q, u_n) &\leq \phi(q, y_n) \\ &\leq \alpha_n\phi(q, x_n) + (1-\alpha_n)\phi(q, z_n) \\ &\leq \alpha_n\phi(q, x_n) + (1-\alpha_n)\left[\beta_n\phi(q, x_n) + (1-\beta_n)\phi(q, w_n)\right] \\ &= \alpha_n\phi(q, x_n) + (1-\alpha_n)\beta_n\phi(q, x_n) + (1-\alpha_n)(1-\beta_n)\phi(q, w_n) \\ &\leq \alpha_n\phi(q, x_n) + (1-\alpha_n)\beta_n\phi(q, x_n) + (1-\alpha_n)(1-\beta_n) \\ &\quad \times \left[\phi(q, x_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Aq\|^2\right] \\ &= \alpha_n\phi(q, x_n) + (1-\alpha_n)\beta_n\phi(q, x_n) + (1-\alpha_n)(1-\beta_n)\phi(q, x_n) \\ &\quad - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Aq\|^2 \\ &= \phi(q, x_n) - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Aq\|^2,\end{aligned} \tag{3.48}$$

and hence

$$\begin{aligned}\delta_1\delta_2 2a\left(\alpha-\frac{2a}{c^2}\right)\|Ax_n-Aq\|^2 &\le (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha-\frac{2}{c^2}\lambda_n\right)\|Ax_n-Aq\|^2\\ &\le \phi(q,x_n)-\phi(q,u_n).\end{aligned} \tag{3.49}$$

From (3.47) and (3.49), we have

$$\|Ax_n-Aq\| \longrightarrow 0. \tag{3.50}$$

From Lemma 2.5, Lemma 2.10, and (3.8), we compute

$$\begin{aligned}\phi(x_n,w_n) &= \phi\left(x_n,\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n)\right)\\ &\le \phi\left(x_n,J^{-1}(Jx_n-\lambda_n Ax_n)\right)\\ &= V(x_n,Jx_n-\lambda_n Ax_n)\\ &\le V(x_n,(Jx_n-\lambda_n Ax_n)+\lambda_n Ax_n)-2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,\lambda_n Ax_n\right\rangle\\ &= \phi(x_n,x_n)+2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\right\rangle\\ &= 2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\right\rangle\\ &\le \frac{4\lambda_n^2}{c^2}\|Ax_n-Aq\|^2\\ &\le \frac{4b^2}{c^2}\|Ax_n-Aq\|^2.\end{aligned} \tag{3.51}$$

Applying Lemmas 2.3 and (3.50), we obtain that

$$\|x_n-w_n\| \longrightarrow 0. \tag{3.52}$$

Again since J is uniformly norm-to-norm continuous on bounded set, we have

$$\|Jx_n-Jw_n\| \longrightarrow 0. \tag{3.53}$$

Since

$$\|z_n-w_n\| \le \|z_n-x_n\|+\|x_n-w_n\|, \tag{3.54}$$

by (3.35) and (3.52), we have

$$\lim_{n\to\infty}\|z_n-w_n\|=0, \tag{3.55}$$

and hence

$$\lim_{n\to\infty}\|Jz_n - Jw_n\| = 0. \tag{3.56}$$

From (3.1) we obtain that

$$\begin{aligned}\|Jz_n - Jw_n\| &= \left\|\beta_n Jx_n + (1-\beta_n)JTw_n - Jw_n\right\| \\ &\geq (1-\beta_n)\|JTw_n - Jw_n\| - \beta_n\|Jw_n - Jx_n\|,\end{aligned} \tag{3.57}$$

and hence

$$(1-\beta_n)\|JTw_n - Jw_n\| \leq \|Jz_n - Jw_n\| + \beta_n\|Jw_n - Jx_n\|, \tag{3.58}$$

so

$$\|JTw_n - Jw_n\| \leq \frac{1}{1-\beta_n}\|Jz_n - Jw_n\| + \beta_n\|Jw_n - Jx_n\|. \tag{3.59}$$

By (3.53), (3.56) and condition $\beta_n \leq 1-\delta_2$ for some $\delta_2 \in (0,1)$, we obtain

$$\|JTw_n - Jw_n\| \longrightarrow 0. \tag{3.60}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded set, we obtain

$$\|Tw_n - w_n\| \longrightarrow 0. \tag{3.61}$$

Since $x_n \to w_n$, then $\|Tx_n - x_n\| \to 0$. Thus by the closedness of T and $x_n \to p$, we get $p \in F(T)$. Hence $p \in F(T) \cap F(S)$.

Step 8. We show that $x_n \to p \in \mathrm{VI}(A, C)$.

Define $T \subset E \times E^*$ by Theorem 2.11; T is maximal monotone and $T^{-1}0 = \mathrm{VI}(A, C)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.

From $w_n \in C$, we have

$$\langle v - w_n, w - Av\rangle \geq 0. \tag{3.62}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$, then by Lemma 2.4, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0, \tag{3.63}$$

and hence

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\right\rangle \leq 0. \tag{3.64}$$

It follows from (3.62) and (3.64), that

$$\begin{aligned}
\langle v - w_n, w \rangle &\geq \langle v - w_n, Av \rangle \\
&\geq \langle v - w_n, Av \rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \right\rangle \\
&= \langle v - w_n, Av - Ax_n \rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \right\rangle \\
&= \langle v - w_n, Av - Aw_n \rangle + \langle v - w_n, Aw_n - Ax_n \rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \right\rangle \\
&\geq -\|v - w_n\| \frac{\|w_n - x_n\|}{\alpha} - \|v - w_n\| \frac{\|Jx_n - Jw_n\|}{a} \\
&\geq -M\left(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{a} \right).
\end{aligned} \tag{3.65}$$

Where $M = \sup_{n\geq 1}\|v - w_n\|$. Taking the limit as $n \to \infty$ and (3.53), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of T, we have $p \in T^{-1}0$; that is, $p \in \mathrm{VI}(A, C)$.

Step 9. We show that $p = \Pi_\Omega x_0$.

From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0, \forall z \in C_n$. Since $\Omega \subset C_n$, we also have

$$\left\langle Jx_0 - Jx_n, x_n - y \right\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.66}$$

By taking limit $n \to \infty$, we obtain that

$$\langle Jx_0 - Jp, p - y \rangle \geq 0, \quad \forall y \in \Omega. \tag{3.67}$$

By Lemma 2.4, we can conclude that $p = \Pi_\Omega x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

Setting $S \equiv T$ in Theorem 3.1., so, we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$. Let $T : C \to C$ be closed relatively quasi-nonexpansive mappings such that*

$\Omega := F(T) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$
\begin{gathered}
w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T w_n), \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n) \}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered}
\tag{3.68}
$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1 - \delta_1$ and $\beta_n \leq 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$, and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

If $A \equiv 0$ in Theorem 3.1, then we obtain the following corollary.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $T, S : C \to C$ is closed relatively quasi-nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(f) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:*

$$
\begin{gathered}
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T w_n), \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S z_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n) \}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered}
\tag{3.69}
$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1-\delta_1$ and $\beta_n \leq 1-\delta_2$, for some $\delta_1, \delta_2 \in (0,1)$ and $\{r_n\} \subseteq (0, 2\alpha)$. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

4. Application

4.1. Complementarity Problem

Let K be a nonempty, closed and convex cone E, A a mapping of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0,\ \forall x \in K\}. \tag{4.1}$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \langle u, Au \rangle = 0. \tag{4.2}$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Theorem 4.1. *Let K be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be an α-inverse-strongly monotone of E into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in K$ and $u \in C(K, A) \neq \emptyset$. Let $T, S : K \to K$ be closed relatively quasi-nonexpansive mappings and $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(f) \cap C(K, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{K_1}$ and $K_1 = K$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered}
w_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\
z_n = J^{-1}\left(\beta_n Jx_n + (1 - \beta_n) JTw_n\right), \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSz_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in K, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{4.3}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\alpha_n \leq 1 - \delta_1$ and $\beta_n \leq 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0, 1)$, $\{r_n\} \subseteq (0, 2\alpha)$, and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Proof. As in the proof of Takahashi in [7, Lemma 7.11], we get that $\mathrm{VI}(K, A) = C(K, A)$. So, we obtain the result. □

4.2. Approximation of a Zero of a Maximal Monotone Operator

Let B be a multivalued mapping from E to E^* with domain $D(B) = \{z \in E : Az \neq \emptyset\}$ and range $R(B) = \cup\{Bz : z \in D(B)\}$. A mapping B is said to be a *monotone operator* if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(B)$ and $y_i \in Ax_i, i = 1, 2$. A monotone operator B is said to be *maximal* if its graph $G(B) = \{(x,y) : y \in Ax\}$ is not property contained in the graph of any other monotone operator. We know that if B is a maximal monotone operator, then $B^{-1}(0)$ is closed and convex. Let E be a reflexive, strictly convex, and smooth Banach space, and let B be a monotone operator from E to E^*, we know that B is maximal if and only if $R(J + rB) = E^*$ for all $r > 0$. Let $J_r : E \to D(B)$be defined by $J_r = (J + rB)^{-1}J$and such a J_r is called the resolvent of B . We know that J_r is a relatively nonexpansive (closed relatively quasi-nonexpansive for example; see [8]), and $B^{-1}(0) = F(J_r)$ for all $r > 0$ (see [7, 33–35] for more details).

Theorem 4.2. *Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be α-inverse-strongly monotone of E into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A,C) \neq \emptyset$. Let B be a maximal monotone operator of E into E^* and let J_r be a resolvent of B and a closed mapping such that $\Omega := B^{-1}(0) \cap F(S) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J J_r w_n\right), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \left\{z \in C_n : \phi(z,u_n) \leq \alpha_n \phi(z,x_n) + (1-\alpha_n)\phi(z,z_n) \leq \phi(z,x_n)\right\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{gathered} \tag{4.4}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1 - \delta_1$ and $\beta_n \leq 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Proof. Since J_r is a closed relatively nonexpansive mapping and $B^{-1}0 = F(J_r)$. So, we obtain the result. □

Corollary 4.3. *Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be α-inverse-strongly monotone of E into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A,C) \neq \emptyset$. Let B be a maximal monotone operator of E into E^* and let J_r be a resolvent of B and closed such that $\Omega := B^{-1}(0) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J J_r w_n\right), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J J_r z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \left\{z \in C_n : \phi(z,u_n) \leq \alpha_n \phi(z,x_n) + (1-\alpha_n)\phi(z,z_n) \leq \phi(z,x_n)\right\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{gathered} \tag{4.5}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1-\delta_1$ and $\beta_n \leq 1-\delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0,2\alpha)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

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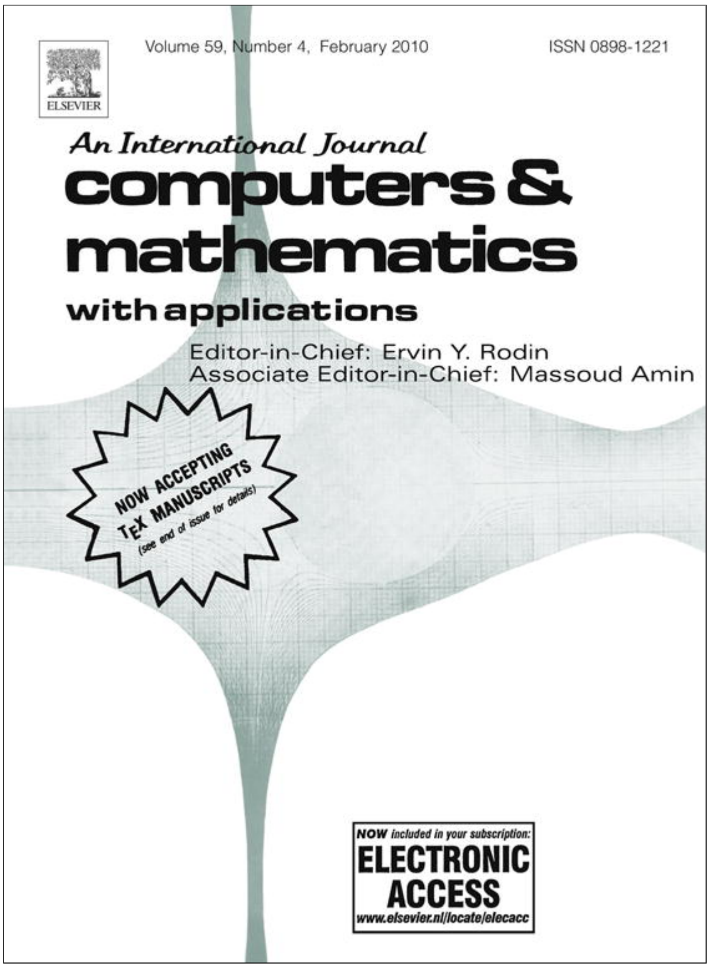
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Strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces[☆]

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ABSTRACT

In this paper, we introduce a new modified Ishikawa iterative process for computing fixed points of an infinite family nonexpansive mapping in the framework of Banach spaces. Then, we establish the strong convergence theorem of the proposed iterative scheme under some mild conditions which solves a variational inequality. The results obtained in this paper extend and improve on the recent results of Qin et al. [Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces, Journal of Computational and Applied Mathematics 230 (1) (2009) 121–127], Cho et al. [Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces, Computers and Mathematics with Applications 56 (2008) 2058–2064] and many others.

1. Introduction

In recent years, the existence of fixed points for finitely or infinitely many nonexpansive mappings has been considered by many authors (see also [1–8]). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [9,10]). The problem of finding an optimal point that minimizes a given cost function over a common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, e.g., [9,11,12]. A simple algorithmic solution to the problem of minimizing a quadratic function over a common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [13,12]). It is an interesting topic for investigating the approximation of fixed points of a family of nonexpansive mappings.

Let E be a real Banach space, C be a closed convex subset of E and $T : C \to C$ be a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [14–16]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in C, \tag{1.2}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [14] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [15] extended

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Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$. Xu [16] proved that Reich's results hold in reflexive Banach spaces which have a weakly continuous duality mapping.

Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|. \tag{1.3}$$

We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f | f : C \to C \textit{ a contraction}\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C. Throughout the paper we assume that $F(T) \neq \emptyset$. Given a real number $t \in (0, 1)$ and a contraction $f \in \Pi_C$, define another mapping $T_t^f : C \to C$ by

$$T_t^f x = tf(x) + (1 - t)Tx, \quad x \in C.$$

For simplicity we will write T_t for T_t^f provided no confusion occurs.

It is not hard to see that T_t is a contraction on C. Indeed, for $x, y \in C$ we have

$$\begin{aligned} \|T_t x - T_t y\| &= \|t(f(x) - f(y)) + (1 - t)(Tx - Ty)\| \\ &\le \alpha t\|x - y\| + (1 - t)\|x - y\| \\ &= (1 - t(1 - \alpha))\|x - y\| \\ &\le \|x - y\|. \end{aligned}$$

Let $x_t := x_t^f \in C$ be the unique fixed point of T_t. Thus x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t.$$

Let A be a strongly positive bounded linear operator on the Hilbert space H [17] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.4}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where S is a nonexpansive mapping and b is a given point in H.

In this paper, we consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \end{cases} \tag{1.6}$$

where $T_1, T_2, \ldots$ is an infinite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, \ldots$ are real numbers such that $0 \le \lambda_n \le 1$ for every $n \in \mathbb{N}$.

Recently, Qin et al. [6] proved that the sequences $\{x_n\}$ converge strongly to a common fixed point of the infinite family nonexpansive mappings in Banach spaces under certain appropriate assumptions on the sequences α_n and β_n. Let the sequences $\{x_n\}$ be generated by

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad \forall n \ge 0. \end{cases} \tag{1.7}$$

Cho et al. [1] also modified the iterative algorithm (1.7) to have strong convergence by using the viscosity approximation method. They considered the following iterative algorithm:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad \forall n \ge 0. \end{cases} \tag{1.8}$$

On the other hand, Shang et al. [18] introduced the following new iterative algorithms for a nonexpansive mapping in Hilbert spaces; let the sequences $\{x_n\}$ be generated by

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ z_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n \gamma f(x) + (I - \alpha_n A)y_n, \quad \forall n \geq 0. \end{cases} \tag{1.9}$$

They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T under some mild assumptions.

In this paper, motivated by (1.7)–(1.9), we extend the algorithm (1.9) to an infinite family of nonexpansive mappings in Banach spaces and introduce a composite iterative algorithm as follows:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ z_n = \gamma_n x_n + (1-\gamma_n)W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n)W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.10}$$

where W_n is defined by (1.6), f is a contraction and A is a strongly positive linear bounded self-adjoint operator. Then, we prove that the sequence $\{x_n\}$ generated by (1.10) converges strongly to a common fixed point.

Next, we consider some special cases of the iterative scheme. If $\{\gamma_n\} = 1$ for all $n \geq 0$ in (1.10), then (1.10) reduces to

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad \forall n \geq 0. \end{cases} \tag{1.11}$$

If $\{\gamma_n\} = \gamma = 1$ for all $n \geq 0$ and $A = I$ (the identity mapping) in (1.10), then (1.10) reduces to (1.8) of Cho et al. [1]. If $f(x_n) = u$ for all $n \in \mathbb{N}$ in (1.8), then (1.8) reduces to (1.7) of Qin et al. [6]. If $\{\beta_n\} = 0$ and $\{\gamma_n\} = 1$ for all $n \geq 0$ in (1.10), then (1.10) reduces to

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n x_n, \quad \forall n \geq 0. \end{cases} \tag{1.12}$$

If $\{\gamma_n\} = \gamma = 1$ and $\{\beta_n\} = 0$ for all $n \geq 0$ and $A = I$ in (1.10), then (1.10) reduces to

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)W_n x_n, \quad \forall n \geq 0. \end{cases} \tag{1.13}$$

Our results presented in this paper introduce the composite iterative scheme for approximating a fixed point of an infinite family nonexpansive mapping. We also establish the strong convergence of the composite iterative sequences $\{x_n\}$ defined by (1.10), which solves a variational inequality. With an appropriate setting, we obtain the corresponding results due to Qin et al. [6], Cho et al. [1] and many others.

2. Preliminaries

Recall that we let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists a $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [19]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

Let E^* be the dual space of E. Let $\varphi : [0, \infty) := \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a *gauge function*. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. In the case where $\varphi(t) = t$, we write J for J_φ and call J the *normalized duality mapping*.

In a smooth Banach space, we define an operator A as strongly positive [20] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J(x) \rangle \geq \bar{\gamma}\|x\|^2, \qquad \|aI - bA\| = \sup_{\|x\|\leq 1} |\langle (aI - bA)x, J(x) \rangle| \quad a \in [0, 1], b \in [-1, 1],$$

where I is the identity mapping and J is the normalized duality mapping.

If C and D are nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny [21,22] provided that $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [21,22]: if E is a smooth Banach space, then $Q : C \to D$ is a *sunny nonexpansive retraction* if and only if the following inequality holds:

$$\langle x - Qx, J(y - Qx)\rangle \le 0, \quad \forall x \in C, y \in D. \tag{2.1}$$

Following Browder [23], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x)$ converges weakly to J_φ). It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set $\Phi(t) = \int_0^t \varphi(t)\mathrm{d}t, \forall t \ge 0$; then $J_\varphi(x) = \partial\Phi(\|x\|), \forall x \in E$, where ∂ denotes the sub-differential in the sense of convex analysis.

We need the following lemmas for proving our main results.

Lemma 2.1 (*[24]*). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (*[7]*). *Let C be a nonempty closed and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \le b < 1$ for any $n \ge 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.1 (*See [4, Remark 3.2]*). It can be found from Lemma 2.2 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \ge k$ such that for all $n > n_0$,

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \le \epsilon.$$

Remark 2.2 (*See [4, Remark 3.3]*). Using Lemma 2.2, we define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, then $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \le \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \ge 0\}$. Hence, it is clear from Remark 2.1 that for an arbitrary $\epsilon > 0$ there exists $N_0 \ge 1$ such that for all $n > N_0$,

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \le \sup_{x\in D} \|U_{n,1}x - U_1 x\| \le \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.3 (*[7]*). *Let C be a nonempty closed and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \le b < 1$ for any $n \ge 1$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

In 2006, Xu [16] proved that, if E is a reflexive Banach space and has a weakly continuous duality, then there is a sunny nonexpansive retraction from C onto $F(T)$ and it can be constructed as follows.

Lemma 2.4 (*[6, Lemma 1.3.]*). *Let E be a reflexive Banach space that has a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be a closed convex subset of E and let $T : C \longrightarrow C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C of Eq. (1.2). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T.*

Under the condition of Lemma 2.4, we define a mapping $Q : C \to F(T)$ by

$$Q(u) := \lim_{t\to 0} x_t \quad \forall u \in C.$$

From Xu ([16], Theorem 3.2), we know that Q is the sunny nonexpansive retraction from C onto $F(T)$.

Lemma 2.5 ([25]). *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let us have $f \in \Pi_C$. Then the sequence $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1-t)Tx_t,$$

converges strongly to a point in $F(T)$. Suppose we define a mapping $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Pi_C.$$

Then $Q(f)$ solves the following variational inequality:

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad \forall f \in \Pi_C, p \in F(T). \tag{2.2}$$

In particular, if $f = u \in C$ is a constant, then (2.2) is reduced to the sunny nonexpansive retraction of Reich [15] from C onto $F(T)$,

$$\langle Q(u) - u, J(Q(u)-p)\rangle \le 0, \quad u \in C, p \in F(T). \tag{2.3}$$

Lemma 2.6 ([26]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7 ([27,6]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*

Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 2.8 ([20]). *Assume that A is a strong positive linear bounded operator on a smooth Banach space E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

3. Main results

Let E be a Banach space, C a closed convex subset of E, A a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. As previously, let Π_C be the set of all contractions on C. For $t \in (0,1)$ and $f \in \Pi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Tx$ on C; that is

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t.$$

In this section, we prove a strong convergence theorem.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a reflexive, smooth and strictly convex Banach space E which also has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $T_1, T_2, \ldots$ be a nonexpansive mapping from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$ and let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, the initial guess $x_0 \in C$ is chosen arbitrarily and the given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$, the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, where $x^* = Q(f)$ and Q is a unique sunny nonexpansive retraction from Π_C onto $\bigcap_{n=1}^{\infty} F(T_n)$. If we define $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$ by

$$Q(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (\gamma f - A)Q(f), J_\varphi(p - Q(f))\rangle \le 0, \quad \forall f \in \Pi_C, p \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.1}$$

In particular, if $f = u \in C$ is a constant, then (3.1) is reduced to the sunny nonexpansive retraction from Π_C onto $\bigcap_{n=1}^{\infty} F(T_n)$,

$$\langle \gamma u - AQ(u), J_\varphi(p - Q(u))\rangle \le 0, \quad u \in C, p \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.2}$$

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with out loss of generality, that $\alpha_n < (1-\delta_n)\|A\|^{-1}$ for all n. From Lemma 2.8, we know that if $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$. First we show that $\{x_n\}$ is bounded. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. By the definition of $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$, we have

$$\begin{aligned}
\|z_n - p\| &= \|\gamma_n x_n + (1-\gamma_n)W_n x_n - p\| \\
&\le \gamma_n\|x_n - p\| + (1-\gamma_n)\|W_n x_n - p\| \\
&\le \gamma_n\|x_n - p\| + (1-\gamma_n)\|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned}$$

and from this, we have

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)W_n z_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|W_n z_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|z_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n - p\| \\
&= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\| \\
&\le \alpha_n\|\gamma f(x) - Ap\| + (1 - \alpha_n\bar{\gamma})\|y_n - p\| \\
&\le \alpha_n\|\gamma f(x_n) - \gamma f(p) + \gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\
&\le \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\
&\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n(\bar{\gamma} - \gamma\alpha)\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}.
\end{aligned}$$

By induction on n, we obtain $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\}$ for every $n \ge 0$ and $x_0 \in C$; then $\{x_n\}$ is bounded. So, $\{y_n\}$, $\{z_n\}$, $\{W_n x_n\}$, and $\{f(x_n)\}$ are also bounded.

Next, we claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since T_n and $U_{n,n}$ are nonexpansive, from (1.6), we have

$$\begin{aligned}
\|W_{n+1}x_n - W_n x_n\| &= \|\lambda_1 T_1 U_{n+1,2}x_n - \lambda_1 T_1 U_{n,2}x_n\| \\
&\le \lambda_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \lambda_1\|\lambda_2 T_2 U_{n+1,3}x_n - \lambda_2 T_2 U_{n,3}x_n\| \\
&\le \lambda_1\lambda_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\
&\ \ \vdots \\
&\le \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&\le M_1\prod_{i=1}^{n}\lambda_i,
\end{aligned}$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ for all $n \geq 0$. Similarly, we also have $\|W_{n+1}z_n - W_nz_n\| \leq M_2 \prod_{i=1}^{n} \lambda_i$, where $M_2 \geq 0$ such that $\|U_{n+1,n+1}z_n - U_{n,n+1}z_n\| \leq M_2$ for all $n \geq 0$. It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|(\gamma_{n+1}x_{n+1} + (1-\gamma_{n+1})W_{n+1}x_{n+1}) - (\gamma_n x_n + (1-\gamma_n)W_nx_n)\| \\
&\leq (1-\gamma_{n+1})\|W_{n+1}x_{n+1} - W_{n+1}x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n - W_{n+1}x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\gamma_n)\|W_{n+1}x_n - W_nx_n\| \\
&\leq (1-\gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\gamma_n)\|W_{n+1}x_n - W_nx_n\| \\
&= \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + (1-\gamma_n)\|W_{n+1}x_n - W_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + \|W_{n+1}x_n - W_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M_1 \prod_{i=1}^{n} \lambda_i.
\end{aligned}$$

Observe that, on putting $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we have

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 0. \tag{3.3}$$

Now, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| &= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\gamma f(x_n) + (I-\alpha_nA)y_n - \beta_nx_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1-\beta_{n+1}} + \frac{y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1-\beta_n} - \frac{y_n - \beta_nx_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1-\beta_{n+1}} + W_{n+1}z_{n+1} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1-\beta_n} - W_nz_n \right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|W_{n+1}z_{n+1} - W_nz_n\| \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|W_{n+1}z_{n+1} - W_{n+1}z_n + W_{n+1}z_n - W_nz_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|W_{n+1}z_{n+1} - W_{n+1}z_n\| + \|W_{n+1}z_n - W_nz_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|z_{n+1} - z_n\| + \|W_{n+1}z_n - W_nz_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M_1 \prod_{i=1}^{n} \lambda_i + M_2 \prod_{i=1}^{n} \lambda_i \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M \prod_{i=1}^{n} \lambda_i,
\end{aligned}$$

where $M = M_1 + M_2$. Therefore, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&+ |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M \prod_{i=1}^{n} \lambda_i.
\end{aligned}$$

From the conditions (i), (ii), (iii), and $0 < \lambda_n \leq b < 1$, we obtain

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 2.6, that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. Noting (3.3), we see that

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|l_n - x_n\| \to 0$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Observing that

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n - y_n\| \\ &= \alpha_n\|\gamma f(x_n) - Ay_n\|,\end{aligned}$$

and the condition (i), we get

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.5}$$

On the other hand, we have

$$\|y_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

Combining (3.4) with (3.5), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.6}$$

Consider

$$\|y_n - W_n z_n\| = \|\beta_n x_n + (1-\beta_n)W_n z_n - W_n z_n\| = \beta_n\|x_n - W_n z_n\|$$

and

$$\begin{aligned}\|z_n - x_n\| &= \|\gamma_n x_n + (1-\gamma_n)W_n x_n - x_n\| \\ &= \|\gamma_n x_n + W_n x_n - \gamma_n W_n x_n - x_n\| \\ &= \|(W_n x_n - x_n) - \gamma_n(W_n x_n - x_n)\| \\ &= \|(1-\gamma_n)(W_n x_n - x_n)\| \\ &= (1-\gamma_n)\|W_n x_n - x_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|W_n x_n - x_n\| &\le \|x_n - y_n\| + \|y_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \\ &\le \|x_n - y_n\| + \beta_n\|x_n - W_n z_n\| + \|z_n - x_n\| \\ &= \|x_n - y_n\| + \beta_n\|x_n - W_n z_n\| + (1-\gamma_n)\|W_n x_n - x_n\|.\end{aligned}$$

This implies that

$$\gamma_n\|W_n x_n - x_n\| \le \|x_n - y_n\| + \beta_n\|x_n - W_n z_n\|.$$

From the condition (ii) and (3.6), we get

$$\lim_{n\to\infty}\|W_n x_n - x_n\| = 0.$$

On the other hand, we obtain

$$\|Wx_n - x_n\| \le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|.$$

From Remark 2.2 (see also Remark 3.3 of [28]), we have that $\|Wx_n - W_n x_n\| \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty}\|Wx_n - x_n\| = 0. \tag{3.7}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle \gamma f(x_n) - AQ(f), J_\varphi(x_n - Q(f))\rangle \le 0. \tag{3.8}$$

By Lemma 2.5, we have the sunny nonexpansive retraction $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x_n) - AQ(f), J_\varphi(x_n - Q(f))\rangle = \limsup_{k\to\infty}\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n_k} - Q(f))\rangle. \tag{3.9}$$

Since E is reflexive, we may assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since J_φ is weakly continuous, from Lemma 2.7, we have

$$\limsup_{k\to\infty} \Phi(\|x_{n_k} - x\|) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

Put

$$g(x) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - x\|), \quad \forall x \in E.$$

It follows that

$$g(x) = g(\bar{x}) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

From (3.7), we have

$$\begin{aligned} g(W\bar{x}) &= \limsup_{k\to\infty} \Phi(\|x_{n_k} - W\bar{x}\|) = \limsup_{k\to\infty} \Phi(\|Wx_{n_k} - W\bar{x}\|) \\ &\leq \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) = g(\bar{x}). \end{aligned} \tag{3.10}$$

On the other hand, we note that

$$g(W\bar{x}) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|W\bar{x} - \bar{x}\|) = g(\bar{x}) + \Phi(\|W\bar{x} - \bar{x}\|). \tag{3.11}$$

Combining (3.10) with (3.11), we obtain

$$\Phi(\|W\bar{x} - \bar{x}\|) \leq 0.$$

Hence $W\bar{x} = \bar{x}$ and $\bar{x} \in F(W)$. That is, $\bar{x} \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence, by (3.9) and the sunny nonexpansive retraction from Π_C onto $\bigcap_{n=1}^{\infty} F(T_n)$, we get

$$\limsup_{n\to\infty} \langle \gamma f(x_n) - AQ(f), J_\varphi(x_n - Q(f))\rangle = \langle \gamma f(x_n) - AQ(f), J_\varphi(\bar{x} - Q(f))\rangle \leq 0. \tag{3.12}$$

Therefore, we obtain that (3.8) holds.

Finally, we prove that $x_n \to Q(f)$ as $n \to \infty$. Now from Lemma 2.7, we have

$$\begin{aligned} &\Phi(\|x_{n+1} - Q(f)\|) = \Phi(\|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - Q(f)\|) \\ &\quad = \Phi(\|\alpha_n \gamma f(x_n) + (1-\delta_n)y_n - \alpha_n A y_n + \delta_n(y_n - Q(f)) - (1-\delta_n)Q(f) + \alpha_n AQ(f) - \alpha_n AQ(f)\|) \\ &\quad = \Phi(\|\alpha_n \gamma f(x_n) + ((1-\delta_n) - \alpha_n A)y_n + \delta_n(y_n - Q(f)) - ((1-\delta_n) - \alpha_n A)Q(f) - \alpha_n AQ(f)\|) \\ &\quad = \Phi(\|[((1-\delta_n)I - \alpha_n A)(y_n - Q(f)) + \delta_n(y_n - Q(f))] + \alpha_n(\gamma f(x_n) - AQ(f))\|) \\ &\quad \leq \Phi(\|((1-\delta_n)I - \alpha_n A)(y_n - Q(f)) + \delta_n(y_n - Q(f))\|) + \alpha_n \langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\quad \leq \Phi((1 - \delta_n - \alpha_n \bar{\gamma})\|y_n - Q(f)\| + \delta_n \|y_n - Q(f)\|) + \alpha_n \langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\quad \leq \Phi((1 - \delta_n - \alpha_n \bar{\gamma})\|y_n - Q(f)\| + \delta_n \|y_n - Q(f)\|) + \alpha_n \langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\quad = (1 - \alpha_n \bar{\gamma})\Phi(\|y_n - Q(f)\|) + \alpha_n \langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\quad \leq (1 - \alpha_n \bar{\gamma})\Phi(\|x_n - Q(f)\|) + \sigma_n, \end{aligned} \tag{3.13}$$

where $\sigma_n = \alpha_n \langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle$. By (3.12) and (i), using Lemma 2.1, we see that $\Phi(\|x_n - Q(f)\|) \to 0$ as $n \to \infty$. This implies that $\|x_n - Q(f)\| \to 0$ as $n \to \infty$. This completes the proof. $\square$

Corollary 3.2. *Let C be a nonempty closed convex subset of a reflexive, smooth and strictly convex Banach space E which also has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $T_1, T_2, \ldots$ be a nonexpansive mapping from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$ and let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, the initial guess $x_0 \in C$ is chosen arbitrarily and the given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. Let $\{x_n\}$ be the composite iterative process defined by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad \forall n \geq 0. \end{cases} \tag{3.14}$$

The following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;

Then the composite process $\{x_n\}$ *converges strongly to* $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, *where* $x^* = Q(f)$ *and* $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$ *is the unique sunny nonexpansive retraction from* C *onto* $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Taking $\{\gamma_n\} = 1$ in (1.10), we can reach the desired conclusion easily. □

Corollary 3.3 ([5, Theorem 3.2,]). *Let* C *be a nonempty closed convex subset of a reflexive and strictly convex Banach space* E *which also has a weakly continuous duality map* $J_\varphi(x)$ *with the gauge* φ. *Let* $T_1, T_2, \ldots$ *be a nonexpansive mapping from* C *into itself such that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, *and let* $\lambda_1, \lambda_2, \ldots$ *be real numbers. Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0, 1)$. *The initial guess* $x_0 \in C$ *is chosen arbitrarily and the given sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are in* $(0, 1)$. *Let* $\{x_n\}$ *be the composite iterative process defined by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad \forall n \geq 0. \end{cases} \tag{3.15}$$

The following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;

Then the composite process $\{x_n\}$ *converges strongly to* $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, *where* $x^* = Q(f)$ *and* $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$ *is the unique sunny nonexpansive retraction from* C *onto* $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Taking $\gamma = 1$ and $A = I$ in (3.14), we can reach the desired conclusion easily. □

Corollary 3.4 ([6, Theorem 2.1,]). *Let* C *be a nonempty closed convex subset of a reflexive and strictly convex Banach space* E *which also has a weakly continuous duality map* $J_\varphi(x)$ *with the gauge* φ. *Let* $T_1, T_2, \ldots$ *be a nonexpansive mapping from* C *into itself such that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, *and let* $\lambda_1, \lambda_2, \ldots$ *be real numbers. The initial guess* $x_0 \in C$ *is chosen arbitrarily and the given sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are in* $(0, 1)$. *Let* $\{x_n\}$ *be the composite iterative process defined by*

$$\begin{cases} x_0 = u \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad \forall n \geq 0. \end{cases} \tag{3.16}$$

The following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;

Then the composite process $\{x_n\}$ *converges strongly to* $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, *where* $x^* = Q(u)$ *and* $Q : C \to \bigcap_{n=1}^{\infty} F(T_n)$ *is the unique sunny nonexpansive retraction from* C *onto* $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Taking $f(x) = u \in C$ for all $x \in C$ in (3.15), we can reach the desired conclusion easily. □

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Research Article

Strong Convergence for Generalized Equilibrium Problems, Fixed Point Problems and Relaxed Cocoercive Variational Inequalities

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We introduce a new iterative scheme for finding the common element of the set of solutions of the generalized equilibrium problems, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of the variational inequality problems for a relaxed (u, v)-cocoercive and ξ-Lipschitz continuous mapping in a real Hilbert space. Then, we prove the strong convergence of a common element of the above three sets under some suitable conditions. Our result can be considered as an improvement and refinement of the previously known results.

1. Introduction

Variational inequalities introduced by Stampacchia [1] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences. It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation has been used to suggest and analyze in variational inequalities. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. It is well known that the convergence of a projection method requires the operator to be strongly monotone and Lipschitz continuous. Gabay [2] has shown that the convergence of a projection method can be proved for cocoercive operators. Note that cocoercivity is a weaker condition than strong monotonicity.

Equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization which has been extended and generalized in many directions using novel and innovative technique; see [3, 4]. Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems and a set of the fixed points of infinitely (finitely) many nonexpansive mappings; see [5–7] and the references therein. In this paper, we suggest and analyze a new iterative method for finding a common element of a set of the solutions of generalized equilibrium problems and a set of fixed points of an infinite family of nonexpansive mappings and the set solution of the variational inequality problems for a relaxed (u,v)-cocoercive mapping in a real Hilbert space.

Let H be a real Hilbert space and let E be a nonempty closed convex subset of H and P_E is the metric projection of H onto E. Recall that a mapping $f : E \to E$ is contraction on E if there exists a constant $\alpha \in (0,1)$ such that $\|f(x)-f(y)\| \le \alpha\|x-y\|$ for all $x,y \in E$. A mapping S of E into itself is called nonexpansive if $\|Sx - Sy\| \le \|x-y\|$ for all $x,y \in E$. We denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in E : Sx = x\}$. If $E \subset H$ is nonempty, bounded, closed, and convex and S is a nonexpansive mapping of E into itself, then $F(S)$ is nonempty; see, for example, [8]. We recalled some definitions as follows.

Definition 1.1. Let $B : E \to H$ be a mapping. Then one has the following.

(1) B is called *monotone* if $\langle Bx - By, x-y\rangle \ge 0$, for all $x,y \in E$.

(2) B is called *v-strongly monotone* if there exists a positive real number v such that

$$\langle Bx - By, x-y\rangle \ge v\|x-y\|^2, \quad \forall x,y \in E. \tag{1.1}$$

(3) B is called *ξ-Lipschitz continuous* if there exists a positive real number ξ such that

$$\|Bx - By\| \le \xi\|x-y\|, \quad \forall x,y \in E. \tag{1.2}$$

(4) B is called *η-inverse-strongly monotone*, [9, 10] if there exists a positive real number η such that

$$\langle Bx - By, x-y\rangle \ge \eta\|Bx-By\|^2, \quad \forall x,y \in E. \tag{1.3}$$

If $\eta = 1$, we say that B is firmly nonexpansive. It is obvious that any η-inverse-strongly monotone mapping B is monotone and $(1/\eta)$-Lipschitz continuous.

(5) B is called *relaxed (u,v)-cocoercive* if there exists a positive real number u,v such that

$$\langle Bx - By, x-y\rangle \ge (-u)\|Bx-By\|^2 + v\|x-y\|^2, \quad \forall x,y \in E. \tag{1.4}$$

For $u = 0$, B is v-strongly monotone. This class of maps is more general than the class of strongly monotone maps. It is easy to see that we have the following implication: v-strongly monotonicity $\Rightarrow$ relaxed (u,v)-cocoercivity.

(6) A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone mapping of E into H and let $N_E w_1$ be the *normal cone* to E at $w_1 \in E$, that is,

$$N_E w_1 = \{w \in H : \langle \vartheta - w_1, w\rangle \geq 0, \ \forall \vartheta \in E\}. \tag{1.5}$$

Define

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & \text{if } w_1 \in E, \\ \emptyset, & \text{if } w_1 \notin E. \end{cases} \tag{1.6}$$

Then T is the maximal monotone and $0 \in Tw_1$ if and only if $w_1 \in \mathrm{VI}(E, B)$; see [11, 12]

In addition, let $D : E \to H$ be a inverse-strongly monotone mapping. Let F be a bifunction of $E \times$ E into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem for $F : E \times E \to \mathbb{R}$ is to find $x \in E$ such that

$$F(x, y) + \langle Dx, y - x\rangle \geq 0, \quad \forall y \in E. \tag{1.7}$$

The set of such $x \in E$ is denoted by $\mathrm{EP}(F, D)$, that is,

$$\mathrm{EP}(F, D) = \{x \in E : F(x, y) + \langle Dx, y - x\rangle \geq 0, \ \forall y \in E\}. \tag{1.8}$$

Special Cases

(I) If $D \equiv 0$ (:the zero mapping), then the problem (1.7) is reduced to the equilibrium problem:

$$\text{Find } x \in E \text{ such that } F(x, y) \geq 0, \quad \forall y \in E. \tag{1.9}$$

The set of solutions of (1.9) is denoted by $\mathrm{EP}(F)$, that is,

$$\mathrm{EP}(F) = \{x \in E : F(x, y) \geq 0, \ \forall y \in E\}. \tag{1.10}$$

(II) If $F \equiv 0$, the problem (1.7) is reduced to the variational inequality problem:

$$\text{Find } x \in E \text{ such that } \langle Dx, y - x\rangle \geq 0, \quad \forall y \in E. \tag{1.11}$$

The set of solutions of (1.11) is denoted by $\mathrm{VI}(E, D)$, that is,

$$\mathrm{VI}(E, D) = \{x \in E : \langle Dx, y - x\rangle \geq 0, \ \forall y \in E\}. \tag{1.12}$$

The generalized equilibrium problem (1.7) is very general in the sense that it includes, as special case, some optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, economics, and others (see, e.g., [4, 13]). Some methods have been proposed to solve the equilibrium problem and the generalized equilibrium problem; see, for instance, [5, 14–28]. Recently, Combettes and Hirstoaga [29] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem. Very recently, Moudafi [24] introduced an itertive method for finding an element of $\mathrm{EP}(F,D) \cap F(S)$, where $D : E \to H$ is an inverse-strongly monotone mapping and then proved a weak convergence theorem.

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem for an η-inverse-strongly monotone, Takahashi and Toyoda [30] introduced the following iterative scheme:

$$\begin{gathered} x_0 \in E \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\mathrm{SP}_E(x_n - \tau_n B x_n), \quad \forall n \geq 0, \end{gathered} \tag{1.13}$$

where B is an η-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\tau_n\}$ is a sequence in $(0, 2\eta)$. They showed that if $F(S) \cap \mathrm{VI}(E,B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.13) converges weakly to some $z \in F(S) \cap \mathrm{VI}(E,B)$. On the other hand, Shang et al. [31] introduced a new iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a relaxed (u,v)-cocoercive mapping in a real Hilbert space. Let $S : E \to E$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in E$, defined sequences $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \mathrm{SP}_E(I - \tau_n B)x_n, \quad \forall n \geq 1. \tag{1.14}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n\}$, the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap \mathrm{VI}(E,B)$, where $z = P_{F(S)\cap \mathrm{VI}(E,B)} f(z)$.

In 2008, S. Takahashi and W. Takahashi [27] introduced the following iterative scheme for finding an element of $F(S) \cap \mathrm{EF}(F,D)$ under some mild conditions. Let E be a nonempty closed convex subset of a real Hilbert space H. Let D be an η-inverse-strongly monotone mapping of E into H and let S be a nonexpansive mapping of E into itself such that $F(S) \cap \mathrm{EP}(F,D) \neq \emptyset$. Suppose $x_1 = \sigma \in E$ and let $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ by sequences generated by

$$\begin{gathered} F(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n \sigma + (1-\alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \end{gathered} \tag{1.15}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [0,2\eta]$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.15) converges strongly to a common element of $F(S) \cap \mathrm{EF}(F,D)$.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [32–35] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping in a real Hilbert space H:

$$\min_{x\in E}\left\{\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle\right\}, \tag{1.16}$$

where E is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Assume that A is a *strongly positive bounded linear operator* on H; that is, there exists a constant $\overline{\gamma}>0$ such that

$$\langle Ax,x\rangle\geq\overline{\gamma}\|x\|^2,\quad \forall x\in H. \tag{1.17}$$

In 2006, Marino and Xu [36] considered the following iterative method:

$$x_{n+1}=\epsilon_n\gamma f(x_n)+(1-\epsilon_nA)Sx_n,\quad \forall n\geq 0. \tag{1.18}$$

They proved that if the sequence $\{\epsilon_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.18) converges strongly to the unique of the variational inequality

$$\langle(A-\gamma f)z,x-z\rangle\geq 0,\quad \forall x\in F(S), \tag{1.19}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)}\left\{\frac{1}{2}\langle Ax,x\rangle-h(x)\right\}, \tag{1.20}$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

In 2008, Qin et al. [26] proposed the following iterative algorithm:

$$\begin{gathered}F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in H,\\ x_{n+1}=\epsilon_n\gamma f(x_n)+(I-\epsilon_nA)SP_E(I-\tau_nB)u_n,\end{gathered} \tag{1.21}$$

where A is a strongly positive linear bounded operator and B is a relaxed cocoercive mapping of E into H. They prove that if the sequences $\{\epsilon_n\}$, $\{\tau_n\}$, and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle(A-\gamma f)z,x-z\rangle\geq 0,\quad \forall x\in F(S)\cap \mathrm{VI}(E,B)\cap \mathrm{EP}(F), \tag{1.22}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap \mathrm{VI}(E,B)\cap \mathrm{EP}(F)} \left\{ \frac{1}{2}\langle Ax, x\rangle - h(x) \right\}, \tag{1.23}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Furthermore, for finding approximate common fixed points of an infinite family of nonexpansive mappings $\{T_n\}$ under very mild conditions on the parameters, we need the following definition.

Definition 1.2 (see [37]). Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of E into itself and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For each $n \geq 1$, define a mapping W_n of E into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n) I, \\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k) I, \\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2) I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1) I.
\end{aligned} \tag{1.24}$$

Such a mappings W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$. It is obvious that W_n is nonexpansive, and if $x = T_n x$, then $x = U_{n,k} = W_n x$.

On the other hand, Yao et al. [38] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of an infinite family of nonexpansive mappings on E. Starting with an arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{gathered}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\
x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \epsilon_n A) W_n u_n, \quad \forall n \geq 1,
\end{gathered} \tag{1.25}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$. It is proved [38] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.25) converges strongly to $z = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F)}(I - A + \gamma f)z$. Very recently, Qin et al. [6] introduced an iterative scheme for finding a common fixed points of a finite family of nonexpansive mappings, the set of solutions of the variational inequality problem for a relaxed cocoercive mapping, and the set of solutions of the equilibrium problems in a real Hilbert space. Starting with an arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ &x_{n+1} = \epsilon_n \gamma f(W_n x_n) + (I - \epsilon_n A) W_n P_E (I - \tau_n B) u_n, \quad \forall n \geq 1, \end{aligned} \tag{1.26}$$

where B is a relaxed (u, v)-cocoercive mapping and A is a strongly positive linear bounded operator. They proved that under certain appropriate conditions imposed on $\{\epsilon_n\}, \{\tau_n\}$, and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.26) converge strongly to some point $z \in \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \cap \mathrm{VI}(E, B)$, which is a unique solution of the variation inequality:

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \cap \mathrm{VI}(E, B) \tag{1.27}$$

and is also the optimality for some minimization problems.

In this paper, motivated by iterative schemes considered in (1.15), (1.25), and (1.26) we will introduce a new iterative process (3.4) below for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the generalized equilibrium problem, and the set of solutions of variational inequality problem for a relaxed (u, v)-cocoercive mapping in a real Hilbert space. The results obtained in this paper improve and extend the recent ones announced by Yao et al. [38], S. Takahashi and W. Takahashi [27], and Qin et al. [6] and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let E be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively. Recall that the (nearest point) projection P_E from H to E assigns each $x \in H$ the unique point in $P_E x \in E$ satisfying the property

$$\|x - P_E x\| = \min_{y \in E} \|x - y\|. \tag{2.1}$$

The following characterizes the projection P_E.

We need some facts tools in a real Hilbert space H which are listed as follows.

Lemma 2.1. *For any $x \in H$, $z \in E$,*

$$z = P_E x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in E. \tag{2.2}$$

It is well known that P_E is a firmly nonexpansive mapping of H onto E and satisfies

$$\|P_E x - P_E y\|^2 \leq \langle P_E x - P_E y, x - y\rangle, \quad \forall x, y \in H. \tag{2.3}$$

Moreover, $P_E x$ is characterized by the following properties: $P_E x \in E$ and for all $x \in H, y \in E$,

$$\langle x - P_E x, y - P_E x\rangle \leq 0. \tag{2.4}$$

Lemma 2.2 (see [39]). *Let H be a Hilbert space, let E be a nonempty closed convex subset of H, and let B be a mapping of E into H. Let $p \in E$. Then for $\lambda > 0$,*

$$p \in \mathrm{VI}(E, B) \Longleftrightarrow p = P_E(p - \lambda Bp), \tag{2.5}$$

where P_E is the metric projection of H onto E.

It is clear from Lemma 2.2 that variational inequality and fixed point problem are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Lemma 2.3 (see [40]). *Each Hilbert space H satisfies Opials condition; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.6}$$

holds for each $y \in H$ with $y \neq x$.

Lemma 2.4 (see [36]). *Assume that A is a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : E \times E \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in E$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in E$;

(A3) $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$, for all $x, y, z \in E$;

(A4) for each $x \in E, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [4].

Lemma 2.5 (see [4]). *Let E be a nonempty closed convex subset of H and let F be a bifunction of $E \times E$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in E$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in E. \tag{2.7}$$

The following lemma was also given in [5].

Lemma 2.6 (see [5]). *Assume that $F : E \times E \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to E$ as follows:*

$$T_r(x) = \left\{ z \in E : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in E \right\}, \tag{2.8}$$

for all $z \in H$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r x - T_r y \right\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.9}$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is closed and convex.*

Remark 2.7. Replacing x with $x - rDx \in H$ in (2.7), then there exists $z \in E$, such that

$$F(z, y) + \langle Dx, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in E. \tag{2.10}$$

Lemma 2.8 (see [41]). *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in E$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Using Lemma 2.8, one can define a mapping W of E into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \tag{2.11}$$

for every $x \in E$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$. Throughout this paper, we will assume that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, we have the following results.

Lemma 2.9 (see [41]). *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.10 (see [7]). *If $\{x_n\}$ is a bounded sequence in E, then $\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0$.*

Lemma 2.11 (see [42]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.12. *Let H be a real Hilbert space. Then the following inequality holds:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$

for all $x, y \in H$.

Lemma 2.13 (see [43]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \quad \forall n \geq 0, \tag{2.12}$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$,

(2) $\limsup_{n\to\infty}(\sigma_n/l_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we prove a strong convergence theorem of a new iterative method (3.4) for an infinite family of nonexpansive mappings and relaxed (u,v)-cocoercive mappings in a real Hilbert space.

We first prove the following lemmas.

Lemma 3.1. *Let H be a real Hilbert space, let E be a nonempty closed convex subset of H, and let $D : E \to H$ be η-inverse-strongly monotone. It $0 \leq r_n \leq 2\eta$, then $I - r_n D$ is a nonexpansive mapping in H.*

Proof. For all $x, y \in E$ and $0 \leq r_n \leq 2\eta$, we have

$$\begin{aligned} \|(I - r_nD)x - (I - r_nD)y\|^2 &= \|(x - y) - r_n(Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Dx - Dy\rangle + r_n^2\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - 2r_n\eta\|Dx - Dy\| + r_n^2\|Dx - Dy\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\eta)\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{3.1}$$

So, $I - r_nD$ is a nonexpansive mapping of E into H. □

Lemma 3.2. *Let H be a real Hilbert space, let E be a nonempty closed convex subset of H, and let $B : E \to H$ be a relaxed (u,v)-cocoercive and ξ-Lipschitz continuous. If $0 \leq \tau_n \leq 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$, then $I - \tau_nB$ is a nonexpansive mapping in H.*

Proof. For any $x, y \in E$ and $\tau_n \leq 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$.

Putting $r = 1 + 2\tau_n u\xi^2 - 2\tau_n v + \tau_n^2\xi^2$, we obtain

$$\begin{aligned} \tau_n \leq \frac{2(v - u\xi^2)}{\xi^2} &\Longleftrightarrow \tau_n\xi^2 + 2u\xi^2 - 2v \leq 0 \\ &\Longleftrightarrow \tau_n^2\xi^2 + 2\tau_n u\xi^2 - 2\tau_n v \leq 0 \\ &\Longleftrightarrow 1 + \tau_n^2\xi^2 + 2\tau_n u\xi^2 - 2\tau_n v \leq 1, \end{aligned} \tag{3.2}$$

that is, $r \leq 1$. It follows that

$$\begin{aligned}
\left\|(I-\tau_n B)x-(I-\tau_n B)y\right\|^2 &= \left\|(x-y)-\tau_n(Bx-By)\right\|^2 \\
&= \|x-y\|^2 - 2\tau_n\langle x-y, Bx-By\rangle + \tau_n^2\|Bx-By\|^2 \\
&\leq \|x-y\|^2 - 2\tau_n\left\{-u\|Bx-By\|^2 + v\|x-y\|^2\right\} + \tau_n^2\|Bx-By\|^2 \\
&\leq \|x-y\|^2 + 2\tau_n u\xi^2\|x-y\|^2 - 2\tau_n v\|x-y\|^2 + \tau_n^2\xi^2\|x-y\|^2 \\
&= \left(1+2\tau_n u\xi^2 - 2\tau_n v + \tau_n^2\xi^2\right)\|x-y\|^2 \\
&= r\|x-y\|^2 \\
&\leq \|x-y\|^2,
\end{aligned} \tag{3.3}$$

for all $x, y \in E$. Thus $\|(I-\tau_n B)x-(I-\tau_n B)y\| \leq \|x-y\|$.

So, $I-\tau_n B$ is a nonexpansive mapping of E into H. □

Now, we prove the following main theorem.

Theorem 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H, and let $F : E\times E \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let*

(1) *$\{T_n\}$ be an infinite family of nonexpansive mappings of E into E;*

(2) *D be an η-inverse strongly monotone mappings of E into H;*

(3) *B be relaxed (u,v)-cocoercive and ξ-Lipschitz continuous mappings of E into H.*

Assume that $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F,D) \cap \mathrm{VI}(E,B) \neq \emptyset$. Let $f : E \to E$ be a contraction mapping with $0<\alpha<1$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma}>0$ and $0<\gamma<\overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by

$$\begin{gathered}
x_1 \in E \text{ chosen arbitrary}, \\
F(u_n,y) + \langle Dx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in E, \\
y_n = \varphi_n u_n + (1-\varphi_n)W_n P_E(u_n - \delta_n B u_n), \\
k_n = \alpha_n x_n + (1-\alpha_n)W_n P_E(y_n - \lambda_n B y_n), \\
x_{n+1} = \epsilon_n \gamma f(W_n x_n) + \beta_n x_n + \left((1-\beta_n)I - \epsilon_n A\right)W_n P_E(k_n - \tau_n B k_n), \quad \forall n \geq 1,
\end{gathered} \tag{3.4}$$

where $\{W_n\}$ is the sequence generated by (1.24) and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$, and $\{\beta_n\}$ are sequences in $(0,1)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\epsilon_n = 0$, $\sum_{n=1}^{\infty}\epsilon_n = \infty$,

(C2) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\varphi_n = 0$, (C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C5) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\delta_{n+1} - \delta_n| = \lim_{n\to\infty}|\tau_{n+1} - \tau_n| = 0$,

(C6) $\{\tau_n\}$, $\{\lambda_n\}$, $\{\delta_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$,

(C7) $\{r_n\} \subset [c,d]$ *for some* c, d *with* $0 < c < d < 2\eta$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$, *where* $z = P_\Theta(I - A + \gamma f)(z)$, *which solves the variational inequality*

$$\langle (A - \gamma f)z, x - z\rangle \ge 0, \quad \forall x \in \Theta, \tag{3.5}$$

which is the optimality condition fot the minimization problem

$$\min_{x\in\Theta}\left\{\frac{1}{2}\langle Ax, x\rangle - h(x)\right\}, \tag{3.6}$$

where h *is a potential function for* γf *(i.e.,* $h'(x) = \gamma f(x)$ *for* $x \in H$*).*

Proof. Since $\lim_{n\to\infty}\epsilon_n = 0$ by the condition (C1) and $\limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\epsilon_n \le (1 - \beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H, then

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.7}$$

Observe that

$$\begin{aligned}\langle ((1 - \beta_n)I - \epsilon_n A)x, x\rangle &= 1 - \beta_n - \epsilon_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \epsilon_n\|A\| \\ &\ge 0,\end{aligned} \tag{3.8}$$

that is to say $(1 - \beta_n)I - \epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \epsilon_n A\| &= \sup\{|\langle ((1 - \beta_n)I - \epsilon_n A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \epsilon_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\le 1 - \beta_n - \epsilon_n\overline{\gamma}.\end{aligned} \tag{3.9}$$

We will divide the proof of Theorem 3.3 into six steps.

Step 1. We prove that there exists $z \in E$ such that $z = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)}(I - A + \gamma f)z$.

Let $\Im = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)}$. Note that f is a contraction mapping of E into itself with coefficient $\alpha \in (0,1)$. Then, we have

$$\begin{aligned}
\left\|\Im(I-A+\gamma f)(x)-\Im(I-A+\gamma f)(y)\right\| &\leq \left\|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\right\| \\
&\leq \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\| \\
&\leq (1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\| \\
&= (1-(\overline{\gamma}-\alpha\gamma))\|x-y\|, \quad \forall x,y\in H.
\end{aligned} \tag{3.10}$$

Therefore, $\Im(I-A+\gamma f)$ is a contraction mapping of E into itself. Therefore by the Banach Contraction Mapping Principle guarantee that $\Im(I-A+\gamma f)$ has a unique fixed point, say $z\in E$. That is, $z=\Im(I-A+\gamma f)(z)=P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)}(I-A+\gamma f)(z)$.

Step 2. We prove that $\{x_n\}$ is bounded.

Since

$$F(u_n,y)+\langle Dx_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in E, \tag{3.11}$$

we obtain

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-(I-r_nD)x_n\rangle \geq 0, \quad \forall y\in E. \tag{3.12}$$

From Lemma 2.6, we have $u_n=T_{r_n}(x_n-r_nDx_n)$, for all $n\in\mathbb{N}$.

For any $p\in\Theta := \bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)$, it follows from $p\in \mathrm{EP}(F,D)$ that

$$F(p,y)+\langle y-p, Dp\rangle \geq 0, \quad \forall y\in E. \tag{3.13}$$

So, we have

$$F(p,y)+\frac{1}{r_n}\langle y-p, p-(p-r_nDp)\rangle \geq 0, \quad \forall y\in E. \tag{3.14}$$

By Lemma 2.6 again, we have $p=T_{r_n}(p-r_nDp)$, for all $n\in\mathbb{N}$. If follows that

$$\begin{aligned}
\|u_n-p\| &= \left\|T_{r_n}(x_n-r_nDx_n)-T_{r_n}(p-r_nDp)\right\| \\
&\leq \left\|(x_n-r_nDx_n)-(p-r_nDp)\right\| \\
&= \left\|(I-r_nD)x_n-(I-r_nD)p\right\| \leq \|x_n-p\|.
\end{aligned} \tag{3.15}$$

If we applied Lemma 3.2, we get $I - \lambda_n B$ and $I - \delta_n B$ are nonexpansive. Since $p \in \mathrm{VI}(E, B)$ and W_n is a nonexpansive, we have $p = W_n P_E(p - \lambda_n Bp) = W_n P_E(p - \delta_n Bp)$, and we have

$$\begin{aligned}
\|y_n - p\| &\le \varphi_n \|u_n - p\| + (1 - \varphi_n)\|W_n P_E(u_n - \delta_n Bu_n) - W_n P_E(p - \delta_n Bp)\| \\
&\le \varphi_n \|u_n - p\| + (1 - \varphi_n)\|(u_n - \delta_n Bu_n) - (p - \delta_n Bp)\| \\
&= \varphi_n \|u_n - p\| + (1 - \varphi_n)\|(I - \delta_n B)u_n - (I - \delta_n B)p\| \\
&\le \varphi_n \|u_n - p\| + (1 - \varphi_n)\|u_n - p\| \\
&\le \|u_n - p\| \le \|x_n - p\|.
\end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned}
\|k_n - p\| &\le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|W_n P_E(y_n - \lambda_n By_n) - W_n P_E(p - \lambda_n Bp)\| \\
&\le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|(y_n - \lambda_n By_n) - (p - \lambda_n Bp)\| \\
&= \alpha_n \|x_n - p\| + (1 - \alpha_n)\|(I - \lambda_n B)y_n - (I - \lambda_n B)p\| \\
&\le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\
&\le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|x_n - p\| = \|x_n - p\|,
\end{aligned} \tag{3.17}$$

which yields that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \epsilon_n A)(W_n P_E(k_n - \tau_n Bk_n) - p)\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\|P_E(I - \tau_n B)k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\le (1 - \epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\|f(x_n) - f(p)\| + \epsilon_n\|\gamma f(p) - Ap\| \\
&\le (1 - \epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\alpha\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| \\
&= (1 - (\overline{\gamma} - \alpha\gamma)\epsilon_n)\|x_n - p\| + (\overline{\gamma} - \alpha\gamma)\epsilon_n\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma}.
\end{aligned} \tag{3.18}$$

This in turn implies that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma}\right\}, \quad n \in \mathbb{N}. \tag{3.19}$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{k_n\}$, $\{y_n\}$, $\{Bu_n\}$, $\{Bk_n\}$, $\{By_n\}$, $\{W_nu_n\}$, $\{W_nk_n\}$, $\{W_ny_n\}$, and $\{f(W_nx_n)\}$ are all bounded.

Step 3. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\|W_n\theta_n - x_n\| = 0$.

From Lemma 2.6, we have $u_n = T_{r_n}(x_n - r_nDx_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}Dx_{n+1})$. Let $\varpi_n = x_n - r_nDx_n$, we get $u_n = T_{r_n}\varpi_n$, $u_{n+1} = T_{r_{n+1}}\varpi_{n+1}$, and so

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - \varpi_n\rangle \geq 0, \quad \forall y \in E, \tag{3.20}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - \varpi_{n+1}\rangle \geq 0, \quad \forall y \in E. \tag{3.21}$$

Putting $y = u_{n+1}$ in (3.20) and $y = u_n$ in (3.21), we have

$$\begin{gathered} F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - \varpi_n\rangle \geq 0, \\ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - \varpi_{n+1}\rangle \geq 0. \end{gathered} \tag{3.22}$$

So, from the monotonicity of F, we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - \varpi_n}{r_n} - \frac{u_{n+1} - \varpi_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.23}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - \varpi_n - \frac{r_n}{r_{n+1}}(u_{n+1} - \varpi_{n+1})\right\rangle \geq 0. \tag{3.24}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, \varpi_{n+1} - \varpi_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - \varpi_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\|\left\{\|\varpi_{n+1} - \varpi_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - \varpi_{n+1}\|\right\}, \end{aligned} \tag{3.25}$$

and hence

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \|\varpi_{n+1}-\varpi_n\| + \frac{1}{c}|r_{n+1}-r_n|\|u_{n+1}-\varpi_{n+1}\| \\
&= \|x_{n+1}-r_{n+1}Dx_{n+1}-(x_n-r_nDx_n)\| + \frac{1}{c}|r_{n+1}-r_n|\|u_{n+1}-\varpi_{n+1}\| \\
&\le \|x_{n+1}-r_{n+1}Dx_{n+1}-(x_n-r_{n+1}Dx_n)\| + |r_{n+1}-r_n|\|Dx_n\| \\
&\quad + \frac{1}{c}|r_{n+1}-r_n|\|u_{n+1}-\varpi_{n+1}\| \\
&\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|Dx_n\| + \frac{1}{c}|r_{n+1}-r_n|\|u_{n+1}-\varpi_{n+1}\| \\
&\le \|x_{n+1}-x_n\| + M_1|r_{n+1}-r_n|,
\end{aligned} \tag{3.26}$$

where $M_1 = \sup\{\|Dx_n\| + (1/c)\|u_{n+1}-\varpi_{n+1}\| : n \in \mathbb{N}\}$.

Put $\theta_n = P_E(k_n - \tau_n Bk_n)$, $\phi_n = P_E(y_n - \lambda_n By_n)$, and $\varphi_n = P_E(u_n - \delta_n Bu_n)$. Since $I-\tau_n B$, $\mathrm{I}-\lambda_n B$, and $I-\delta_n B$ are nonexpansive, then we have the following some estimates:

$$\begin{aligned}
\|\varphi_{n+1}-\varphi_n\| &\le \|P_E(u_{n+1}-\delta_{n+1}Bu_{n+1}) - P_E(u_n-\delta_nBu_n)\| \\
&\le \|(u_{n+1}-\delta_{n+1}Bu_{n+1}) - (u_n-\delta_nBu_n)\| \\
&= \|(u_{n+1}-\delta_{n+1}Bu_{n+1}) - (u_n-\delta_{n+1}Bu_n) + (\delta_n-\delta_{n+1})Bu_n\| \\
&\le \|(u_{n+1}-\delta_{n+1}Bu_{n+1}) - (u_n-\delta_{n+1}Bu_n)\| + |\delta_n-\delta_{n+1}|\|Bu_n\| \\
&= \|(I-\delta_{n+1}B)u_{n+1} - (I-\delta_{n+1}B)u_n\| + |\delta_n-\delta_{n+1}|\|Bu_n\| \\
&\le \|u_{n+1}-u_n\| + |\delta_n-\delta_{n+1}|\|Bu_n\|.
\end{aligned} \tag{3.27}$$

Similarly, we can prove that

$$\|\phi_{n+1}-\phi_n\| \le \|y_{n+1}-y_n\| + |\lambda_n-\lambda_{n+1}|\|By_n\|, \tag{3.28}$$

$$\|\theta_{n+1}-\theta_n\| \le \|k_{n+1}-k_n\| + |\tau_n-\tau_{n+1}|\|Bk_n\|. \tag{3.29}$$

Since T_i and $U_{n,i}$ are nonexpansive, we deduce that, for each $n \le 1$,

$$\begin{aligned}
\|W_{n+1}\varphi_n - W_n\varphi_n\| &= \|\mu_1T_1U_{n+1,2}\varphi_n - \mu_1T_1U_{n,2}\varphi_n\| \\
&\le \mu_1\|U_{n+1,2}\varphi_n - U_{n,2}\varphi_n\| \\
&= \mu_1\|\mu_2T_2U_{n+1,3}\varphi_n - \mu_2T_2U_{n,3}\varphi_n\| \\
&\le \mu_1\mu_2\|U_{n+1,3}\varphi_n - U_{n,3}\varphi_n\|
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\leq \prod_{i=1}^{n}\mu_i \left\| U_{n+1,n+1}\varphi_n - U_{n,n+1}\varphi_n \right\|\\
&\leq M_2 \prod_{i=1}^{n}\mu_i,
\end{aligned}
\tag{3.30}
$$

where $M_2 \geq 0$ is a constant such that $\|U_{n+1,n+1}\varphi_n - U_{n,n+1}\varphi_n\| \leq M_2$ for all $n \geq 0$.

Similarly, we can obtain that there exist nonnegative numbers M_3, M_4 such that

$$
\left\| U_{n+1,n+1}\varphi_n - U_{n,n+1}\varphi_n \right\| \leq M_3, \qquad \|U_{n+1,n+1}\theta_n - U_{n,n+1}\theta_n\| \leq M_4, \tag{3.31}
$$

and so are

$$
\left\| W_{n+1}\phi_n - W_n\phi_n \right\| \leq M_3 \prod_{i=1}^{n}\mu_i, \qquad \|W_{n+1}\theta_n - W_n\theta_n\| \leq M_4 \prod_{i=1}^{n}\mu_i. \tag{3.32}
$$

Observing that

$$
\begin{aligned}
y_n &= \varphi_n u_n + (1-\varphi_n) W_n \psi_n,\\
y_{n+1} &= \varphi_{n+1} u_{n+1} + (1-\varphi_{n+1}) W_{n+1} \psi_{n+1},
\end{aligned}
\tag{3.33}
$$

we obtain

$$
y_n - y_{n+1} = \varphi_n (u_n - u_{n+1}) + (1-\varphi_n)(W_n\psi_n - W_{n+1}\psi_{n+1}) + (W_{n+1}\psi_{n+1} - u_{n+1})(\varphi_{n+1} - \varphi_n), \tag{3.34}
$$

which yields that

$$
\begin{aligned}
\left\| y_n - y_{n+1} \right\| &\leq \varphi_n \|u_n - u_{n+1}\| + (1-\varphi_n) \left\| W_{n+1}\psi_{n+1} - W_n\psi_n \right\| + \left|\varphi_{n+1} - \varphi_n\right| \left\| W_{n+1}\psi_{n+1} - u_{n+1} \right\|\\
&\leq \varphi_n \|u_n - u_{n+1}\| + (1-\varphi_n) \left\{ \left\| W_{n+1}\psi_{n+1} - W_{n+1}\psi_n \right\| + \left\| W_{n+1}\psi_n - W_n\psi_n \right\| \right\}\\
&\quad + \left|\varphi_{n+1} - \varphi_n\right| \left\| W_{n+1}\psi_{n+1} - u_{n+1} \right\|\\
&\leq \varphi_n \|u_n - u_{n+1}\| + (1-\varphi_n) \left\{ \left\| \psi_{n+1} - \psi_n \right\| + \left\| W_{n+1}\psi_n - W_n\psi_n \right\| \right\}\\
&\quad + \left|\varphi_{n+1} - \varphi_n\right| \left\| W_{n+1}\psi_{n+1} - u_{n+1} \right\|\\
&\leq \varphi_n \|u_n - u_{n+1}\| + (1-\varphi_n) \left\| \psi_{n+1} - \psi_n \right\| + \left\| W_{n+1}\psi_n - W_n\psi_n \right\|\\
&\quad + \left|\varphi_{n+1} - \varphi_n\right| \left\| W_{n+1}\psi_{n+1} - u_{n+1} \right\|.
\end{aligned}
\tag{3.35}
$$

Substitution of (3.27) and (3.30) into (3.35) yields that

$$
\begin{aligned}
\|y_n - y_{n+1}\| &\leq \varphi_n \|u_n - u_{n+1}\| + (1-\varphi_n)\{\|u_{n+1} - u_n\| + |\delta_n - \delta_{n+1}|\|Bu_n\|\} \\
&\quad + M_2 \prod_{i=1}^{n} \mu_i + |\varphi_{n+1} - \varphi_n| \|W_{n+1}\varphi_{n+1} - u_{n+1}\| \\
&= \|u_n - u_{n+1}\| + (1-\varphi_n)|\delta_n - \delta_{n+1}|\|Bu_n\| \\
&\quad + M_2 \prod_{i=1}^{n} \mu_i + \|W_{n+1}\varphi_{n+1} - u_{n+1}\| |\varphi_{n+1} - \varphi_n| \\
&\leq \|u_n - u_{n+1}\| + M_5(|\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n|) + M_2 \prod_{i=1}^{n} \mu_i,
\end{aligned}
\tag{3.36}
$$

where M_5 is an appropriate constant such that $M_5 = \max\{\sup_{n\geq 1}\|Bu_n\|, \sup_{n\geq 1}\|W_n\varphi_n - u_n\|\}$.

Observing that

$$
\begin{aligned}
k_n &= \alpha_n x_n + (1-\alpha_n) W_n \phi_n, \\
k_{n+1} &= \alpha_{n+1} x_{n+1} + (1-\alpha_{n+1}) W_n \phi_{n+1},
\end{aligned}
\tag{3.37}
$$

we obtain

$$
k_n - k_{n+1} = \alpha_n (x_n - x_{n+1}) + (1-\alpha_n)(W_n\phi_n - W_{n+1}\phi_{n+1}) + (W_{n+1}\phi_{n+1} - x_{n+1})(\alpha_{n+1} - \alpha_n),
\tag{3.38}
$$

which yields that

$$
\begin{aligned}
\|k_n - k_{n+1}\| &\leq \alpha_n \|x_n - x_{n+1}\| + (1-\alpha_n)\|W_n\phi_n - W_{n+1}\phi_{n+1}\| + |\alpha_{n+1} - \alpha_n| \|W_{n+1}\phi_{n+1} - x_{n+1}\| \\
&\leq \alpha_n \|x_n - x_{n+1}\| + (1-\alpha_n)\{\|W_{n+1}\phi_{n+1} - W_{n+1}\phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\|\} \\
&\quad + |\alpha_{n+1} - \alpha_n| \|W_{n+1}\phi_{n+1} - x_{n+1}\| \\
&\leq \alpha_n \|x_n - x_{n+1}\| + (1-\alpha_n)\|\phi_{n+1} - \phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n| \|W_{n+1}\phi_{n+1} - x_{n+1}\|.
\end{aligned}
\tag{3.39}
$$

Substitution of (3.28) and (3.32) into (3.39) yields that

$$\begin{aligned}
\|k_n - k_{n+1}\| &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\{\left\|y_{n+1} - y_n\right\| + |\lambda_n - \lambda_{n+1}|\left\|By_n\right\|\right\} \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\left\|W_{n+1}\phi_{n+1} - x_{n+1}\right\| \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\|y_{n+1} - y_n\right\| + (1-\alpha_n)|\lambda_n - \lambda_{n+1}|\left\|By_n\right\| \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\left\|W_{n+1}\phi_{n+1} - x_{n+1}\right\| \\
&\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\|y_{n+1} - y_n\right\| + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|),
\end{aligned} \tag{3.40}$$

where M_6 is an appropriate constant such that $M_6 = \max\{\sup_{n\geq 1}\|By_n\|, \sup_{n\geq 1}\|W_n\phi_n - x_n\|\}$.

Substituting (3.26) and (3.36) into (3.40), we obtain

$$\begin{aligned}
\|k_n - k_{n+1}\| &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\{\|u_n - u_{n+1}\| + M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + M_2\prod_{i=1}^{n}\mu_i\right\} \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|u_n - u_{n+1}\| + (1-\alpha_n)M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) \\
&\quad + (1-\alpha_n)M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\{\|x_{n+1} - x_n\| + M_1|r_{n+1} - r_n|\} \\
&\quad + (1-\alpha_n)M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + (1-\alpha_n)M_2\prod_{i=1}^{n}\mu_i \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|x_{n+1} - x_n\| + (1-\alpha_n)M_1|r_{n+1} - r_n| \\
&\quad + (1-\alpha_n)M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + (1-\alpha_n)M_2\prod_{i=1}^{n}\mu_i \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\leq \|x_n - x_{n+1}\| + M_1|r_{n+1} - r_n| + M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\leq \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + K_1\left(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right| + |\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|\right),
\end{aligned} \tag{3.41}$$

where K_1 is an appropriate constant such that $K_1 = \max\{M_1, M_5, M_6\}$.

Substituting (3.41) into (3.29), we obtain

$$\begin{aligned}
\|\theta_{n+1}-\theta_n\| &\le \|k_{n+1}-k_n\| + |\tau_n-\tau_{n+1}|\|Bk_n\| \\
&\le \|x_n-x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + K_1\left(|r_{n+1}-r_n| + |\delta_n-\delta_{n+1}| + \left|\varphi_{n+1}-\varphi_n\right| + |\lambda_n-\lambda_{n+1}| + |\alpha_{n+1}-\alpha_n|\right) \\
&\quad + |\tau_n-\tau_{n+1}|\|Bk_n\| \\
&\le \|x_n-x_{n+1}\| + \mathrm{M}_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + K_2\left(|r_{n+1}-r_n| + |\delta_n-\delta_{n+1}| + \left|\varphi_{n+1}-\varphi_n\right| + |\lambda_n-\lambda_{n+1}| + |\alpha_{n+1}-\alpha_n| + |\tau_n-\tau_{n+1}|\right),
\end{aligned} \tag{3.42}$$

where K_2 is an appropriate constant such that $K_2 = \max\{\sup_{n\ge1}\|Bk_n\|, K_1\}$.

Define

$$x_{n+1} = (1-\beta_n)z_n + \beta_n x_n, \quad n \ge 1. \tag{3.43}$$

Observe that from the definition z_n, we obtain

$$\begin{aligned}
z_{n+1}-z_n &= \frac{\epsilon_{n+1}\gamma f(W_{n+1}x_{n+1}) + \left((1-\beta_{n+1})I - \epsilon_{n+1}A\right)W_{n+1}\theta_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\epsilon_n\gamma f(W_nx_n) + \left((1-\beta_n)I - \epsilon_nA\right)W_n\theta_n}{1-\beta_n} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(W_{n+1}x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(W_nx_n) + W_{n+1}\theta_{n+1} - W_n\theta_n \\
&\quad + \frac{\epsilon_n}{1-\beta_n}AW_n\theta_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}\theta_{n+1} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\gamma f(W_{n+1}x_{n+1}) - AW_{n+1}\theta_{n+1}\right) + \frac{\epsilon_n}{1-\beta_n}\left(AW_n\theta_n - \gamma f(W_nx_n)\right) \\
&\quad + W_{n+1}\theta_{n+1} - W_{n+1}\theta_n + W_{n+1}\theta_n - W_n\theta_n.
\end{aligned} \tag{3.44}$$

It follows from (3.32), (3.42), and (3.44) that

$$
\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+\|W_{n+1}\theta_{n+1}-W_{n+1}\theta_n\|+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+\|\theta_{n+1}-\theta_n\|+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+M_2\prod_{i=1}^{n}\mu_i+M_3\prod_{i=1}^{n}\mu_i+M_4\prod_{i=1}^{n}\mu_i \\
&\qquad+K_2\left(|r_{n+1}-r_n|+|\delta_n-\delta_{n+1}|+\left|\varphi_{n+1}-\varphi_n\right|+|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\tau_n-\tau_{n+1}|\right) \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+3K\prod_{i=1}^{n}\mu_i \\
&\qquad+K_2\left(|r_{n+1}-r_n|+|\delta_n-\delta_{n+1}|+\left|\varphi_{n+1}-\varphi_n\right|+|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\tau_n-\tau_{n+1}|\right),
\end{aligned}
\tag{3.45}
$$

where K is an appropriate constant such that $K=\max\{M_2,M_3,M_4\}$.

It follows from conditions (C1), (C2), (C3), (C4), (C5), and $0<\mu_i\le b<1$, for all $i\ge 1$

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.46}$$

Hence, by Lemma 2.11, we obtain

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.47}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{3.48}$$

Applying (3.48) and conditions in Theorem 3.3 to (3.26), (3.41), and (3.42), we obtain that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = \lim_{n\to\infty}\|k_{n+1}-k_n\| = \lim_{n\to\infty}\|\theta_{n+1}-\theta_n\| = 0. \tag{3.49}$$

From (3.49), (C2), (C5), and $0 < \mu_i \le b < 1$, for all $i \ge 1$, we also have

$$\lim_{n\to\infty}\left\|y_{n+1}-y_n\right\| = 0. \tag{3.50}$$

Since $x_{n+1} = \epsilon_n\gamma f(W_nx_n) + \beta_nx_n + ((1-\beta_n)I-\epsilon_nA)W_n\theta_n$, we have

$$\begin{aligned}
\|x_n - W_n\theta_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n\theta_n\| \\
&= \|x_n - x_{n+1}\| + \left\|\epsilon_n\gamma f(W_nx_n) + \beta_nx_n + \left((1-\beta_n)I-\epsilon_nA\right)W_n\theta_n - W_n\theta_n\right\| \\
&= \|x_n - x_{n+1}\| + \left\|\epsilon_n\left(\gamma f(W_nx_n) - AW_n\theta_n\right) + \beta_n(x_n - W_n\theta_n)\right\| \\
&\le \|x_n - x_{n+1}\| + \epsilon_n\left(\left\|\gamma f(W_nx_n)\right\| + \|AW_n\theta_n\|\right) + \beta_n\|x_n - W_n\theta_n\|,
\end{aligned} \tag{3.51}$$

that is,

$$\|x_n - W_n\theta_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\left(\left\|\gamma f(W_nx_n)\right\| + \|AW_n\theta_n\|\right). \tag{3.52}$$

By (C1), (C3), and (3.48) it follows that

$$\lim_{n\to\infty}\|W_n\theta_n - x_n\| = 0. \tag{3.53}$$

Step 4. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|u_n-\theta_n\| = 0$;

(ii) $\lim_{n\to\infty}\|x_n-u_n\| = 0$;

(iii) $\lim_{n\to\infty}\|W_n\theta_n-\theta_n\| = 0$.

Since B is relaxed (u,v)-cocoercive and ξ-Lipschitz continuous mappings, by the assumptions imposed on $\{\tau_n\}$ for any $p \in \Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F,D) \cap \mathrm{VI}(E,B)$, we have

$$
\begin{aligned}
\|W_n\theta_n - p\|^2 &\le \|P_E(k_n - \tau_n B k_n) - P_E(p - \tau_n B p)\|^2 \\
&\le \|(k_n - \tau_n B k_n) - (p - \tau_n B p)\|^2 \\
&= \|(k_n - p) - \tau_n(Bk_n - Bp)\|^2 \\
&\le \|k_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - 2\tau_n\left\{-u\|Bk_n - Bp\|^2 + v\|k_n - p\|^2\right\} + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + 2\tau_n u\|Bk_n - Bp\|^2 - 2\tau_n v\|k_n - p\|^2 + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + 2\tau_n u\|Bk_n - Bp\|^2 - \frac{2\tau_n v}{\xi^2}\|Bk_n - Bp\|^2 + \tau_n^2\|Bk_n - Bp\|^2 \\
&= \|x_n - p\|^2 + \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\|Bk_n - Bp\|^2.
\end{aligned}
\tag{3.54}
$$

Similarly, we have

$$
\begin{aligned}
\|W_n\phi_n - p\|^2 &\le \|x_n - p\|^2 + \left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2, \\
\|W_n\psi_n - p\|^2 &\le \|x_n - p\|^2 + \left(2\delta_n u + \delta_n^2 - \frac{2\delta_n v}{\xi^2}\right)\|Bu_n - Bp\|^2.
\end{aligned}
\tag{3.55}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p) + \beta_n(x_n - p) + \epsilon_n(\gamma f(W_n x_n) - Ap)\|^2 \\
&= \|((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p) + \beta_n(x_n - p)\|^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(W_n x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle ((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(W_n x_n) - Ap\rangle \\
&\le ((1 - \beta_n - \epsilon_n\overline{\gamma})\|W_n\theta_n - p\| + \beta_n\|x_n - p\|)^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(W_n x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle ((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(W_n x_n) - Ap\rangle \\
&\le ((1 - \beta_n - \epsilon_n\overline{\gamma})\|W_n\theta_n - p\| + \beta_n\|x_n - p\|)^2 + c_n \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})^2\|W_n\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + 2(1 - \beta_n - \epsilon_n\overline{\gamma})\beta_n\|W_n\theta_n - p\|\|x_n - p\| + c_n \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})^2\|W_n\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + (1 - \beta_n - \epsilon_n\overline{\gamma})\beta_n\left(\|W_n\theta_n - p\|^2 + \|x_n - p\|^2\right) + c_n
\end{aligned}
$$

$$\begin{aligned}
&= \left[(1-\epsilon_n\overline{\gamma})^2 - 2(1-\epsilon_n\overline{\gamma})\beta_n + \beta_n^2\right]\|W_n\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + \left((1-\epsilon_n\overline{\gamma})\beta_n - \beta_n^2\right)\left(\|W_n\theta_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\
&= \left[(1-\epsilon_n\overline{\gamma})^2 - (1-\epsilon_n\overline{\gamma})\beta_n\right]\|W_n\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n,
\end{aligned} \tag{3.56}$$

where

$$\begin{aligned}
c_n &= \epsilon_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(x_n) - Ap\rangle.
\end{aligned} \tag{3.57}$$

It follows from condition (C1) that

$$\lim_{n\to\infty} c_n = 0. \tag{3.58}$$

Substituting (3.54) into (3.56), and using condition (C6), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n - p\|^2 + \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\|Bk_n - Bp\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n - p\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma}) \\
&\quad \times \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\|Bk_n - Bp\|^2 + c_n \\
&\le \|x_n - p\|^2 + \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\|Bk_n - Bp\|^2 + c_n.
\end{aligned} \tag{3.59}$$

It follows that

$$\begin{aligned}
\left(\frac{2av}{\xi^2} - 2bu - b^2\right)\|Bk_n - Bp\|^2 &\le \left(\frac{2\tau_n v}{\xi^2} - 2\tau_n u - \tau_n^2\right)\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + c_n \\
&= (\|x_n - p\| - \|x_{n+1} - p\|)(\|x_n - p\| + \|x_{n+1} - p\|) + c_n \\
&\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + c_n.
\end{aligned} \tag{3.60}$$

Since $c_n \to 0$ as $n \to \infty$ and (3.48), we obtain

$$\lim_{n\to\infty} \|Bk_n - Bp\| = 0. \tag{3.61}$$

Note that

$$\begin{aligned}
\|k_n - p\|^2 &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|W_n\phi_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{ \|x_n - p\|^2 + \left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2 \right\} \\
&\le \|x_n - p\|^2 + (1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2,
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
\|y_n - p\|^2 &\le \varphi_n \|u_n - p\| + (1-\varphi_n)\|W_n\psi_n - p\|^2 \\
&\le \varphi_n \|x_n - p\|^2 + (1-\varphi_n)\left\{ \|x_n - p\|^2 + \left(2\delta_n u + \delta_n^2 - \frac{2\delta_n v}{\xi^2}\right)\|Bu_n - Bp\|^2 \right\} \\
&\le \|x_n - p\|^2 + (1-\varphi_n)\left(2\delta_n u + \delta_n^2 - \frac{2\delta_n v}{\xi^2}\right)\|Bu_n - Bp\|^2.
\end{aligned} \tag{3.63}$$

Using (3.56) again, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n.
\end{aligned} \tag{3.64}$$

Substituting (3.62) into (3.64) and using condition (C2) and (C6), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n - p\|^2 + (1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})^2\|x_n - p\|^2 + c_n \\
&\le \|x_n - p\|^2 + (1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2 + c_n.
\end{aligned} \tag{3.65}$$

It follows that

$$(1-\alpha_n)\left(\frac{2av}{\xi^2}-2bu-b^2\right)\left\|By_n-Bp\right\|^2 \le (1-\alpha_n)\left(\frac{2\tau_n v}{\xi^2}-2\tau_n u-\tau_n^2\right)\left\|By_n-Bp\right\|^2$$
$$\le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2+c_n$$
$$\le \|x_n-x_{n+1}\|\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right)+c_n. \tag{3.66}$$

Since $c_n \to 0$ as $n \to \infty$ and (3.48), we obtain

$$\lim_{n\to\infty}\left\|By_n-Bp\right\|=0. \tag{3.67}$$

In a similar way, we can prove

$$\lim_{n\to\infty}\left\|Bu_n-Bp\right\|=0. \tag{3.68}$$

By (2.3), we also have

$$\begin{aligned}
\left\|\theta_n-p\right\|^2 &= \left\|P_E(k_n-\tau_nBk_n)-P_E(p-\tau_nBp)\right\|^2\\
&= \left\|P_E(I-\tau_nB)k_n-P_E(I-\tau_nB)p\right\|^2\\
&\le \langle (I-\tau_nB)k_n-(I-\tau_nB)p,\theta_n-p\rangle\\
&= \frac{1}{2}\Big\{\left\|(I-\tau_nB)k_n-(I-\tau_nB)p\right\|^2+\left\|\theta_n-p\right\|^2\\
&\qquad -\left\|(I-\tau_nB)k_n-(I-\tau_nB)p-(\theta_n-p)\right\|^2\Big\}\\
&\le \frac{1}{2}\left\|k_n-p\right\|^2+\left\|\theta_n-p\right\|^2-\left\|(k_n-\theta_n)-\tau_n(Bk_n-Bp)\right\|^2\\
&\le \frac{1}{2}\Big\{\left\|x_n-p\right\|^2+\left\|\theta_n-p\right\|^2-\|k_n-\theta_n\|^2-\tau_n^2\left\|Bk_n-Bp\right\|^2+2\tau_n\langle k_n-\theta_n,Bk_n-Bp\rangle\Big\},
\end{aligned} \tag{3.69}$$

which yields that

$$\left\|\theta_n-p\right\|^2 \le \left\|x_n-p\right\|^2-\|k_n-\theta_n\|^2+2\tau_n\|k_n-\theta_n\|\left\|Bk_n-Bp\right\|. \tag{3.70}$$

Substituting (3.70) into (3.56), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n-p\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-p\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n-p\|^2-\|k_n-\theta_n\|^2+2\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|\right\} \\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-\theta_n\|^2 \\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n \\
&\le \|x_n-p\|^2-(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-\theta_n\|^2 \\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n.
\end{aligned} \tag{3.71}$$

It follows that

$$\begin{aligned}
(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-\theta_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n \\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n.
\end{aligned} \tag{3.72}$$

Applying $\|x_{n+1}-x_n\| \to 0$, $\|Bk_n-Bp\| \to 0$ and $c_n \to 0$ as $n \to \infty$ to the last inequality, we have

$$\lim_{n\to\infty}\|k_n-\theta_n\|=0. \tag{3.73}$$

On the other hand, we have

$$\begin{aligned}
\|W_n\theta_n-p\|^2 &\le \|P_E(k_n-\tau_nBk_n)-P_E(p-\tau_nBp)\|^2 \\
&= \|P_E(I-\tau_nB)k_n-P_E(I-\tau_nB)p\|^2 \\
&\le \langle (I-\tau_nB)k_n-(I-\tau_nB)p, W_n\theta_n-p\rangle \\
&= \frac{1}{2}\Big\{\|(I-\tau_nB)k_n-(I-\tau_nB)p\|^2+\|W_n\theta_n-p\|^2 \\
&\qquad -\|(I-\tau_nB)k_n-(I-\tau_nB)p-(W_n\theta_n-p)\|^2\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\left\|k_n - p\right\|^2 + \left\|W_n\theta_n - p\right\|^2 - \left\|(k_n - W_n\theta_n) - \tau_n(Bk_n - Bp)\right\|^2 \\
&\leq \frac{1}{2}\Big\{\left\|x_n - p\right\|^2 + \left\|W_n\theta_n - p\right\|^2 - \|k_n - W_n\theta_n\|^2 \\
&\qquad -\tau_n^2\left\|Bk_n - Bp\right\|^2 + 2\tau_n\left\langle k_n - W_n\theta_n, Bk_n - Bp\right\rangle\Big\},
\end{aligned} \tag{3.74}$$

which yields that

$$\left\|W_n\theta_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \|k_n - W_n\theta_n\|^2 + 2\tau_n\|k_n - W_n\theta_n\|\left\|Bk_n - Bp\right\|. \tag{3.75}$$

Similarly, we can prove

$$\left\|W_n\phi_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \left\|y_n - W_n\phi_n\right\|^2 + 2\lambda_n\left\|y_n - W_n\phi_n\right\|\left\|By_n - Bp\right\|, \tag{3.76}$$

$$\left\|W_n\psi_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \left\|u_n - W_n\psi_n\right\|^2 + 2\delta_n\left\|u_n - W_n\psi_n\right\|\left\|Bu_n - Bp\right\|. \tag{3.77}$$

Substituting (3.75) into (3.56), we have

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\left\|W_n\theta_n - p\right\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\left\|x_n - p\right\|^2 + c_n \\
&\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma}) \\
&\quad \times \left\{\left\|x_n - p\right\|^2 - \|k_n - W_n\theta_n\|^2 + 2\tau_n\|k_n - W_n\theta_n\|\left\|Bk_n - Bp\right\|\right\} \\
&\quad + (1 - \epsilon_n\overline{\gamma})\beta_n\left\|x_n - p\right\|^2 + c_n \\
&= (1 - \epsilon_n\overline{\gamma})^2\left\|x_n - p\right\|^2 - (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad + 2(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\left\|Bk_n - Bp\right\| + c_n \\
&\leq \left\|x_n - p\right\|^2 - (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad + 2(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\left\|Bk_n - Bp\right\| + c_n,
\end{aligned} \tag{3.78}$$

which yields that

$$\begin{aligned}
&(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad \leq \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 + 2(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\left\|Bk_n - Bp\right\| + c_n \\
&\quad \leq \|x_n - x_{n+1}\|\left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right) \\
&\qquad + 2(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\left\|Bk_n - Bp\right\| + c_n.
\end{aligned} \tag{3.79}$$

Applying (3.48) and (3.61) to the last inequality, we have

$$\lim_{n\to\infty}\|k_n - W_n\theta_n\| = 0. \tag{3.80}$$

Using (3.64) again, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|\alpha_n(x_n-p)+(1-\alpha_n)(W_n\phi_n-p)\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|W_n\phi_n-p\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|W_n\phi_n-p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|x_n-p\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n) \\
&\quad \times \left\{\|x_n-p\|^2 - \|y_n-W_n\phi_n\|^2 + 2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|x_n-p\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-p\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| + c_n \\
&\le \|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| + c_n,
\end{aligned} \tag{3.81}$$

which implies that

$$\begin{aligned}
&(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|+c_n \\
&\quad\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\qquad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|+c_n.
\end{aligned} \tag{3.82}$$

From (3.48) and (3.67), we obtain

$$\lim_{n\to\infty}\|y_n-W_n\phi_n\|=0. \tag{3.83}$$

By using the same argument, we can prove that

$$\lim_{n\to\infty}\|u_n-W_n\psi_n\|=0. \tag{3.84}$$

Note that

$$\begin{aligned}
k_n-W_n\phi_n&=\alpha_n(x_n-W_n\phi_n),\\
y_n-W_n\psi_n&=\varphi_n(u_n-W_n\psi_n).
\end{aligned} \tag{3.85}$$

Since $\alpha_n \to 0$ and $\varphi_n \to 0$ as $n \to \infty$, respectively, we also have

$$\lim_{n\to\infty}\|k_n-W_n\phi_n\|=\lim_{n\to\infty}\|y_n-W_n\psi_n\|=0. \tag{3.86}$$

On the other hand, we observe

$$\|u_n-\theta_n\|\le\|u_n-W_n\psi_n\|+\|W_n\psi_n-y_n\|+\|y_n-W_n\phi_n\|+\|W_n\phi_n-k_n\|+\|k_n-\theta_n\|. \tag{3.87}$$

Applying (3.73), (3.83), (3.84), and (3.86), we have

$$\lim_{n\to\infty}\|u_n-\theta_n\|=0. \tag{3.88}$$

On the other hand, we have

$$
\begin{aligned}
\|k_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|W_n\phi_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|\phi_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\varphi_n\|u_n - p\|^2 + (1-\varphi_n)\|W_n\psi_n - p\|\right\} \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\varphi_n\|u_n - p\|^2 + (1-\varphi_n)\|\psi_n - p\|\right\} \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\varphi_n\|u_n - p\|^2 + (1-\varphi_n)\|u_n - p\|\right\} \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|T_{r_n}(I - r_nD)x_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|(I - r_nD)x_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 - r_n(2\eta - r_n)\|Dx_n - Dp\|^2\right\} \\
&= \|x_n - p\|^2 - (1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|^2.
\end{aligned}
\tag{3.89}
$$

Substituting (3.89) into (3.64) and using conditions (C2) and (C7), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n - p\|^2 - (1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})^2\|x_n - p\|^2 + c_n \\
&\le \|x_n - p\|^2 - (1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|.
\end{aligned}
\tag{3.90}
$$

This implies that

$$
(1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\| \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + c_n. \tag{3.91}
$$

In view of the restrictions (C2) and (C7), we obtain that

$$
\lim_{n\to\infty}\|Dx_n - Dp\| = 0. \tag{3.92}
$$

Let $p \in \Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F, D) \cap \mathrm{VI}(E, B)$. Since $u_n = T_{r_n}(x_n - r_n D x_n)$ and T_{r_n} is firmly nonexpansive (Lemma 2.6), then we obtain

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(p - r_n D p)\|^2 \\
&\le \langle T_{r_n}(x_n - r_n D x_n) - T_{r_n}(p - r_n D p), u_n - p\rangle \\
&= \langle x_n - r_n D x_n - (p - r_n D p), u_n - p\rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - r_n D x_n) - (p - r_n D p)\|^2 + \|u_n - p\|^2 \\
&\qquad - \|(x_n - r_n D x_n) - (p - r_n D p) - (u_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(D x_n - D p)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2 r_n \langle x_n - u_n, D x_n - D p\rangle \\
&\qquad - r_n^2 \|D x_n - D p\|^2 \Big\}.
\end{aligned} \tag{3.93}$$

So, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2 r_n \|x_n - u_n\| \|D x_n - D p\|. \tag{3.94}$$

Therefore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|W_n \theta_n - p\|^2 + (1 - \epsilon_n \overline{\gamma}) \beta_n \|x_n - p\|^2 + c_n \\
&\le (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|\theta_n - p\|^2 + (1 - \epsilon_n \overline{\gamma}) \beta_n \|x_n - p\|^2 + c_n \\
&= (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|(\theta_n - u_n) + (u_n - p)\|^2 + (1 - \epsilon_n \overline{\gamma}) \beta_n \|x_n - p\|^2 + c_n \\
&\le (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \Big\{ \|\theta_n - u_n\|^2 + \|u_n - p\|^2 + 2\langle \theta_n - u_n, u_n - p\rangle \Big\} \\
&\quad + (1 - \epsilon_n \overline{\gamma}) \beta_n \|x_n - p\|^2 + c_n \\
&\le (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|\theta_n - u_n\|^2 + (1 - \epsilon_n \overline{\gamma})(1 - \epsilon_n \overline{\gamma} - \beta_n) \|u_n - p\|^2 \\
&\quad + 2(1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|\theta_n - u_n\| \|u_n - p\| + (1 - \epsilon_n \overline{\gamma}) \beta_n \|x_n - p\|^2 + c_n \\
&\le (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|\theta_n - u_n\|^2 \\
&\quad + (1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \Big\{ \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2 r_n \|x_n - u_n\| \|D x_n - D p\| \Big\} \\
&\quad + 2(1 - \epsilon_n \overline{\gamma})(1 - \beta_n - \epsilon_n \overline{\gamma}) \|\theta_n - u_n\| \|u_n - p\| + (1 - \epsilon_n \overline{\gamma}) \beta_n \|x_n - p\|^2 + c_n
\end{aligned}$$

$$
\begin{aligned}
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-p\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n \\
&= \left(1-2\epsilon_n\overline{\gamma} + (\epsilon_n\overline{\gamma})^2\right)\|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n \\
&\le \|x_n-p\|^2 + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n.
\end{aligned} \tag{3.95}
$$

It follows that

$$
\begin{aligned}
&(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\qquad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\qquad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n \\
&\quad\le \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\qquad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\qquad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n.
\end{aligned} \tag{3.96}
$$

Using $\epsilon_n \to 0$, $c_n \to 0$ as $n \to \infty$, (3.48), (3.88), and (3.92), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.97}$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n-u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n-u_n\| = 0. \tag{3.98}$$

Note that

$$\|x_n-\theta_n\| \le \|x_n-u_n\| + \|u_n-\theta_n\|, \tag{3.99}$$

and thus from (3.88) and (3.97), we have

$$\lim_{n\to\infty}\|x_n-\theta_n\| = 0. \tag{3.100}$$

Observe that

$$\|W_n\theta_n-\theta_n\| \le \|W_n\theta_n-x_n\| + \|x_n-\theta_n\|. \tag{3.101}$$

Applying (3.53) and (3.100), we obtain

$$\lim_{n\to\infty}\|W_n\theta_n-\theta_n\| = 0. \tag{3.102}$$

Let W be the mapping defined by (2.11). Since $\{\theta_n\}$ is bounded, applying Lemma 2.10 and (3.102), we have

$$\|W\theta_n-\theta_n\| \le \|W\theta_n-W_n\theta_n\| + \|W_n\theta_n-\theta_n\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.103}$$

Step 5. We claim that $\limsup_{n\to\infty}\langle(A-\gamma f)z, z-x_n\rangle \le 0$, where z is the unique solution of the variational inequality $\langle(A-\gamma f)z, x-z\rangle \ge 0$, for all $x\in\Theta$.

Since $z = P_\Theta(I-A+\gamma f)(z)$ is a unique solution of the variational inequality (3.5), to show this inequality, we choose a subsequence $\{\theta_{n_i}\}$ of $\{\theta_n\}$ such that

$$\lim_{i\to\infty}\langle(A-\gamma f)z, z-\theta_{n_i}\rangle = \limsup_{n\to\infty}\langle(A-\gamma f)z, z-\theta_n\rangle. \tag{3.104}$$

Since $\{\theta_{n_i}\}$ is bounded, there exists a subsequence $\{\theta_{n_{i_j}}\}$ of $\{\theta_{n_i}\}$ which converges weakly to $w\in E$. Without loss of generality, we can assume that $\theta_{n_i} \rightharpoonup w$. From $\|W\theta_n-\theta_n\| \to 0$, we obtain $W\theta_{n_i} \rightharpoonup w$. Next, We show that $w\in\Theta$, where $\Theta := \bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)$.

(a) First, we prove $w\in \mathrm{EP}(F,D)$.

Since $u_n = T_{r_n}(x_n - r_nDx_n)$, we know that

$$F(u_n,y) + \langle Dx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y\in E. \tag{3.105}$$

From (A2), we also have

$$\langle Dx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq -F(u_n, y) \geq F(y, u_n). \tag{3.106}$$

Replacing n by n_i, we have

$$\langle Dx_{n_i}, y - u_{n_i} \rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}). \tag{3.107}$$

For any t with $0 < t \leq 1$ and $y \in E$, let $\varphi_t = ty + (1-t)z$. Since $y \in E$ and $z \in E$, we have $\varphi_t \in E$. So, from (3.107) we have

$$\begin{aligned} \langle \varphi_t - u_{n_i}, D\varphi_t \rangle &\geq \langle \varphi_t - u_{n_i}, D\varphi_t \rangle - \langle Dx_{n_i}, \varphi_t - u_{n_i} \rangle - \left\langle \varphi_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(\varphi_t, u_{n_i}) \\ &\geq \langle \varphi_t - u_{n_i}, D\varphi_t - Du_{n_i} \rangle + \langle \varphi_t - u_{n_i}, Du_{n_i} - Dx_{n_i} \rangle \\ &\quad - \langle \varphi_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + F(\varphi_t, u_{n_i}). \end{aligned} \tag{3.108}$$

Since D is Lipschitz continuous, from (3.97), we have $\|Du_{n_i} - Dx_{n_i}\| \to 0$ as $i \to \infty$.

Further, from the monotonicity of D, we get that

$$\langle \varphi_t - u_{n_i}, D\varphi_t - Du_{n_i} \rangle \geq 0. \tag{3.109}$$

It follows from (A4) and (3.108) that

$$\langle \varphi_t - z, D\varphi_t \rangle \geq F(\varphi_t, z). \tag{3.110}$$

From (A1), (A4), and (3.110), we also have

$$\begin{aligned} 0 = F(\varphi_t, \varphi_t) &\leq tF(\varphi_t, y) + (1-t)F(\varphi_t, z) \\ &\leq tF(\varphi_t, y) + (1-t)\langle \varphi_t - z, D\varphi_t \rangle \\ &= tF(\varphi_t, y) + (1-t)t\langle y - z, D\varphi_t \rangle, \end{aligned} \tag{3.111}$$

and hence

$$F(\varphi_t, y) + (1-t)\langle y - z, D\varphi_t \rangle \geq 0. \tag{3.112}$$

Letting $t \to \infty$ in the above inequality, we have, for each $y \in E$,

$$F(z, y) + \langle y - z, Dz \rangle \geq 0. \tag{3.113}$$

Thus $z \in \mathrm{EP}(F, D)$.

(b) Next, we show that $w \in \bigcap_{n=1}^{\infty} F(T_n)$.

By Lemma 2.9, we have $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. Assume $w \notin F(W)$. Since $\|x_n - \theta_n\| \to 0$, we know that $\theta_{n_i} \rightharpoonup w$ $(i \to \infty)$ and $w \neq Ww$, and it follows by the Opial's condition (Lemma 2.3) that

$$\begin{aligned}
\liminf_{i \to \infty} \|\theta_{n_i} - w\| &< \liminf_{i \to \infty} \|\theta_{n_i} - Ww\| \\
&\leq \liminf_{i \to \infty} (\|\theta_{n_i} - W\theta_{n_i}\| + \|W\theta_{n_i} - Ww\|) \\
&< \liminf_{i \to \infty} \|\theta_{n_i} - w\|,
\end{aligned} \tag{3.114}$$

that is a contradiction. Thus, we have $w \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.

(c) Finally, Now we prove that $w \in \mathrm{VI}(E, B)$. Define,

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & \text{if } w_1 \in E, \\ \emptyset, & \text{if } w_1 \notin E. \end{cases} \tag{3.115}$$

Since B is relaxed (u, v)-cocoercive and condition (C6), we have

$$\langle Bx - By, x - y \rangle \geq (-u) \|Bx - By\|^2 + v \|x - y\|^2 \geq \left(v - u\xi^2 \right) \|x - y\|^2 \geq 0, \tag{3.116}$$

which yields that B is monotone. Then, T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_E w_1$ and $\theta_n \in E$, we have $\langle w_1 - \theta_n, w_2 - Bw_1 \rangle \geq 0$. On the other hand, from $\theta_n = P_E(k_n - \tau_n B k_n)$, we have

$$\langle w_1 - \theta_n, \theta_n - (k_n - \tau_n B k_n) \rangle \geq 0, \tag{3.117}$$

and hence

$$\left\langle w_1 - \theta_n, \frac{(\theta_n - k_n)}{\tau_n} + Bk_n \right\rangle \geq 0. \tag{3.118}$$

Therefore, we have

$$\begin{aligned}
\langle w_1-\theta_{n_i}, w\rangle &\geq \langle w_1-\theta_{n_i}, Bw_1\rangle \\
&\geq \langle w_1-\theta_{n_i}, Bw_1\rangle - \left\langle w_1-\theta_{n_i}, \frac{(\theta_{n_i}-k_{n_i})}{\tau_{n_i}} + Bk_{n_i}\right\rangle \\
&= \left\langle w_1-\theta_{n_i}, Bw_1 - Bk_{n_i} - \frac{(\theta_{n_i}-k_{n_i})}{\tau_{n_i}}\right\rangle \\
&= \langle w_1-\theta_{n_i}, Bv - B\theta_{n_i}\rangle + \langle w_1-\theta_{n_i}, B\theta_{n_i} - Bk_{n_i}\rangle - \left\langle w_1-\theta_{n_i}, \frac{(\theta_{n_i}-k_{n_i})}{\tau_{n_i}}\right\rangle \\
&\geq \langle w_1-\theta_{n_i}, B\theta_{n_i} - Bk_{n_i}\rangle - \left\langle w_1-\theta_{n_i}, \frac{(\theta_{n_i}-k_{n_i})}{\tau_{n_i}}\right\rangle,
\end{aligned} \tag{3.119}$$

which implies that

$$\langle w_1 - w, w_2\rangle \geq 0. \tag{3.120}$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in \mathrm{VI}(E,B)$. That is, $w \in \Theta$, where $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F,D) \cap \mathrm{VI}(E,B)$. Since $z = P_\Theta(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle &= \limsup_{n\to\infty}\langle (A-\gamma f)z, z-\theta_n\rangle \\
&= \lim_{i\to\infty}\langle (A-\gamma f)z, z-\theta_{n_i}\rangle \\
&= \langle (A-\gamma f)z, z-w\rangle \leq 0.
\end{aligned} \tag{3.121}$$

On the other hand, we have

$$\begin{aligned}
\langle (A-\gamma f)z, z - W_n\theta_n\rangle &= \langle (A-\gamma f)z, x_n - W_n\theta_n\rangle + \langle (A-\gamma f)z, z - x_n\rangle \\
&\leq \|(A-\gamma f)z\|\|x_n - W_n\theta_n\| + \langle (A-\gamma f)z, z-x_n\rangle.
\end{aligned} \tag{3.122}$$

From (3.53) and (3.121), we obtain that

$$\limsup_{n\to\infty}\langle \gamma f(z) - Az, W_n\theta_n - z\rangle \leq 0. \tag{3.123}$$

Step 6. Finally, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta(I - A + \gamma f)(z)$.

Indeed, from (3.4) and Lemma 2.4, we obtain

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\|\epsilon_n\gamma f(W_nx_n)+\beta_nx_n+((1-\beta_n)I-\epsilon_nA)W_n\theta_n-z\right\|^2 \\
&= \left\|((1-\beta_n)I-\epsilon_nA)(W_n\theta_n-z)+\beta_n(x_n-z)+\epsilon_n(\gamma f(W_nx_n)-Az)\right\|^2 \\
&= \left\|((1-\beta_n)I-\epsilon_nA)(W_n\theta_n-z)+\beta_n(x_n-z)\right\|^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\langle x_n-z,\gamma f(W_nx_n)-Az\rangle \\
&\quad +2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(W_n\theta_n-z),\gamma f(W_nx_n)-Az\rangle \\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\gamma\langle x_n-z,f(W_nx_n)-f(z)\rangle+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle \\
&\quad +2(1-\beta_n)\gamma\epsilon_n\langle W_n\theta_n-z,f(W_nx_n)-f(z)\rangle \\
&\quad +2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle, \\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\gamma\|x_n-z\|\left\|f(W_nx_n)-f(z)\right\|+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle \\
&\quad +2(1-\beta_n)\gamma\epsilon_n\|W_n\theta_n-z\|\left\|f(W_nx_n)-f(z)\right\| \\
&\quad +2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle, \\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\gamma\|x_n-z\|\left\|f(W_nx_n)-f(z)\right\|+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle \\
&\quad +2(1-\beta_n)\gamma\epsilon_n\|\theta_n-z\|\left\|f(W_nx_n)-f(z)\right\|+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle \\
&\quad -2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle \\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\gamma\alpha\|x_n-z\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle \\
&\quad +2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n-z\|^2 \\
&\quad +2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle \\
&= \left[(1-\epsilon_n\overline{\gamma})^2+2\beta_n\epsilon_n\gamma\alpha+2(1-\beta_n)\gamma\epsilon_n\alpha\right]\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle \\
&\quad -2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle \\
&\le [1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n]\|x_n-z\|^2+\overline{\gamma}^2\epsilon_n^2\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2 \\
&\quad +2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle \\
&\quad +2\epsilon_n^2\|A(W_n\theta_n-z)\|\left\|\gamma f(z)-Az\right\|
\end{aligned}
$$

$$= \left[1 - 2(\overline{\gamma} - \alpha\gamma)\epsilon_n\right]\|x_n - z\|^2$$
$$+ \epsilon_n\Big\{\epsilon_n\Big(\overline{\gamma}^2\|x_n - z\|^2 + \left\|\gamma f(W_n x_n) - Az\right\|^2 + 2\|A(W_n\theta_n - z)\|\left\|\gamma f(z) - Az\right\|\Big)$$
$$+2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle W_n\theta_n - z, \gamma f(z) - Az\rangle\Big\}. \tag{3.124}$$

Since $\{x_n\}$, $\{f(W_n x_n)\}$, and $\{W_n\theta_n\}$ are bounded, we can take a constant $M > 0$ such that

$$\overline{\gamma}^2\|x_n - z\|^2 + \left\|\gamma f(W_n x_n) - Az\right\|^2 + 2\|A(W_n\theta_n - z)\|\left\|\gamma f(z) - Az\right\| \le M \tag{3.125}$$

for all $n \ge 0$. It then follows that

$$\|x_{n+1} - z\|^2 \le (1 - l_n)\|x_n - z\|^2 + \epsilon_n\sigma_n, \tag{3.126}$$

where

$$\begin{gathered} l_n = 2(\overline{\gamma} - \alpha\gamma)\epsilon_n, \\ \sigma_n = \epsilon_n M + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle W_n\theta_n - z, \gamma f(z) - Az\rangle. \end{gathered} \tag{3.127}$$

Using (C1), (3.121), and (3.123), we get $l_n \to 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\to\infty}(\sigma_n/l_n) \le 0$. Applying Lemma 2.13 to (3.126), we conclude that $x_n \to z$ in norm. Finally, noticing $\|u_n - z\| = \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(z - r_n D z)\| \le \|x_n - z\|$, we also conclude that $u_n \to z$ in norm. This completes the proof. □

Corollary 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $F : E\times E \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4), let $B : E \to H$ be relaxed (u,v)-cocoercive and ξ-Lipschitz continuous mappings, and let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself such that $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \cap \mathrm{VI}(E,B) \ne \emptyset$. Let f be a contraction mapping of E into itself with $\alpha \in (0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by*

$$\begin{gathered} x_1 \in E \text{ chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in E, \\ y_n = \varphi_n u_n + (1-\varphi_n) W_n P_E(u_n - \delta_n B u_n), \\ k_n = \alpha_n x_n + (1-\alpha_n) W_n P_E(y_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n P_E(k_n - \tau_n B k_n), \quad \forall n \ge 1, \end{gathered} \tag{3.128}$$

where $\{W_n\}$ is the sequence generated by (1.24) *and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$, and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:*

(C1) $\epsilon_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty}\epsilon_n = 0$, $\sum_{n=1}^{\infty}\epsilon_n = \infty$, (C4) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\varphi_n = 0$,

(C5) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C6) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\delta_{n+1} - \delta_n| = \lim_{n\to\infty}|\tau_{n+1} - \tau_n| = 0$,

(C7) $\{\tau_n\}$, $\{\lambda_n\}$, $\{\delta_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$*, where* $z = P_\Theta f(z)$.

Proof. Put $A = I, \gamma \equiv 1, \gamma_n = 1-\epsilon_n-\beta_n, D = 0$ (:the zero mapping) and $\{\epsilon_n\} = 0$ in Theorem 3.3. Then $y_n = v_n = u_n$, and for any $\eta > 0$, we see that

$$\langle Dx - Dy, x - y\rangle \ge \eta\|Dx - Dy\|^2, \quad \forall x, y \in E. \tag{3.129}$$

Let $\{r_n\}$ be a sequence satisfying the restriction: $c \le r_n \le d$, where $c, d \in (0,\infty)$. Then we can obtain the desired conclusion easily from Theorem 3.3. □

Corollary 3.5. *Let* E *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of* E *into itself and let* $B : E \to H$ *be relaxed* (u,v)*-cocoercive and* ξ*-Lipschitz continuous mappings such that* $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(E,B) \ne \emptyset$*. Let* $f : E \to E$ *be a contraction mapping with* $0 < \alpha < 1$ *and let* A *be a strongly positive linear bounded operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$*. Let* $\{x_n\}$*,*$\{y_n\}$*, and* $\{k_n\}$ *be sequences generated by*

$$\begin{gathered}
x_1 \in E \text{ chosen arbitrary}, \\
y_n = \varphi_n x_n + (1-\varphi_n) W_n P_E(x_n - \delta_n B x_n), \\
k_n = \alpha_n x_n + (1-\alpha_n) W_n P_E(y_n - \lambda_n B y_n), \\
x_{n+1} = \epsilon_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A) W_n P_E(k_n - \tau_n B k_n), \quad \forall n \ge 1,
\end{gathered} \tag{3.130}$$

where $\{W_n\}$ *is the sequence generated by* (1.24) *and* $\{\epsilon_n\}$*,* $\{\alpha_n\}$*,* $\{\varphi_n\}$*, and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\epsilon_n = 0, \sum_{n=1}^{\infty}\epsilon_n = \infty$,

(C2) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\varphi_n = 0$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C4) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\delta_{n+1} - \delta_n| = \lim_{n\to\infty}|\tau_{n+1} - \tau_n| = 0$,

(C5) $\{\tau_n\}$, $\{\lambda_n\}$, $\{\delta_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Theta$*, where* $z = P_\Theta(I - A + \gamma f)(z)$*, which solves the variational inequality*

$$\langle (A - \gamma f)z, x - z\rangle \ge 0, \quad \forall x \in \Theta, \tag{3.131}$$

which is the optimality condition fot the minimization problem

$$\min_{x\in\Theta}\left\{\frac{1}{2}\langle Ax,x\rangle - h(x)\right\}, \tag{3.132}$$

where h *is a potential function for* γf *(i.e.,* $h'(x)=\gamma f(x)$ *for* $x\in H$*).*

Proof. Put $D=0$, $F(x,y)=0$ for all $x,y\in E$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 3.3. Then, we have $u_n=P_C x_n = x_n$. So, by Theorem 3.3, we can conclude the desired conclusion easily. □

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A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping

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Abstract In this paper, we introduce an iterative scheme by a new hybrid method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a real Hilbert space. We show that the iterative sequence converges strongly to a common element of the above three sets under some parametric controlling conditions by the new hybrid method which is introduced by Takahashi et al. (J. Math. Anal. Appl., doi: 10.1016/j.jmaa.2007.09.062, 2007). The results are connected with Tada and Takahashi's result [A. Tada and W. Takahashi, Weak and strong convergence theorems for a nonexpansive mappings and an equilibrium problem, J. Optim. Theory Appl. 133, 359–370, 2007]. Moreover, our result is applicable to a wide class of mappings.

Keywords Nonexpansive mapping · Monotone mapping · Equilibrium problem · Variational inequality

Mathematics Subject Classification (2000) 46C05 · 47D03 · 47H09 · 47H10 · 47H20

1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a closed convex subset of H and let P_C be the metric projection of H onto C. Let F

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be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of such solutions is denoted by EP(F). This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization, and economics. Some methods have been proposed to solve the equilibrium problem, please consult [3, 4, 21]. Let A of C in to H be a nonlinear mapping. The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au\rangle \geq 0$ for all $v \in C$. We denoted by $VI(A, C)$ the set of solutions of this variational inequality problem. The variational inequality has been extensively studied in the literature; see [27, 28] and the references therein.

The above formulation (1.1) was shown in [1] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP(F). In other words, the EP(F) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP(F); see, for example [1, 4, 10, 21] and references therein. Some solution methods have been proposed to solve the EP(F); see, for example, [3, 4, 18, 19, 21] and references therein. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty and they also proved a strong convergence theorem. They also studied the strong convergence of the sequences generated by their algorithm to a solution of EP(F) which is also a fixed point of a nonexpansive mapping on a closed convex subset of a Hilbert space.

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by F(S) the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then F(S) is nonempty (see [8]). We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0.$$

A mapping A of C into H is called α-*inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.2}$$

for all $u, v \in C$. It is obvious that any α -inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

In 1953, Mann [9] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n, \tag{1.3}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [12]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [5]. Attempts to modify the Mann iteration method (1.3) so that strong convergence is guaranteed have recently been made. Generally speaking, the algorithm suggested by Takahashi and Toyoda [22] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968; see [2] for more details.

Recently, for finding an element of $\mathrm{EP}(F) \cap \mathrm{F}(S)$, Tada and Takahashi [19] introduced the following iterative scheme by the hybrid method in a Hilbert space: $x_0 = x \in H$ and let

$$\begin{cases} u_n \in C \quad \text{such that} \quad F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Further, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathrm{F}(S) \cap \mathrm{EP}(F)$, where $z = P_{\mathrm{F}(S)\cap \mathrm{EP}(F)} x_0$.

On the other hand, for finding an element of $\mathrm{F}(S) \cap VI(C,A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda [22] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C (x_n - \lambda_n A x_n) \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They shown that, if $\mathrm{F}(S) \cap VI(C,A) \neq \emptyset$, then such a sequence $\{x_n\}$ converges weakly to some $z \in P_{\mathrm{F}(S)\cap VI(C,A)} x$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see [24–26] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

Very recently, Takahashi, Takeuchi and Kubota [23] proved the following strong convergence theorem by using the hybrid method in mathematical programming. For

$C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Sx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $0 \le \alpha_n < \alpha < 1$ for all $n \in \mathbb{N}$. They proved a strongly convergence theorem in a Hilbert space.

In this paper, motivated and inspired by the above results, we introduce a new following iterative scheme: For $C_1 = C$, $x_1 = P_{C_1}x_0$, and let

$$\begin{cases} u_n \in C \quad \text{such that} \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)SP_C(u_n - \lambda_n Au_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a Hilbert space. Consequently, we prove a strong convergence theorem by the new hybrid iterative algorithm method in the mathematical programming which solves some fixed point problems, variational inequality problems and equilibrium problems. Using this theorem, we can apply to a wide class of mappings. Our results are connected with Tada and Takahashi's result [19] and Takahashi's et al. result [23].

2 Preliminaries

Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies the *Opial's condition* [11], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$$

holds for every $y \in H$ with $y \ne x$. Hilbert space H satisfies the *Kadec-Klee property* [6, 20], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, P_Cx is characterized by the following properties: $P_Cx \in C$ and

$$\langle x-P_Cx, y-P_Cx\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x-P_Cx\|^2 + \|y-P_Cx\|^2 \tag{2.5}$$

for all $x \in H$, $y \in C$.

In the context of the variational inequality problem, this implies that

$$u \in VI(A,C) \Leftrightarrow u = P_C(u-\lambda Au), \quad \text{for all } \lambda > 0. \tag{2.6}$$

We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)u-(I-\lambda A)v\|^2 &= \|(u-v)-\lambda(Au-Av)\|^2\\ &= \|u-v\|^2 - 2\lambda\langle u-v, Au-Av\rangle\\ &\quad + \lambda^2\|Au-Av\|^2\\ &\leq \|u-v\|^2 + \lambda(\lambda-2\alpha)\|Au-Av\|^2. \end{aligned}\tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping from C to H.

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-h\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let N_Cv be the normal cone to C at $v \in C$, i.e.,

$$N_Cv = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_Cv, & v \in C,\\ \emptyset, & v \notin C.\end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [13, 14].

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [1]):

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [1]

Lemma 2.1 ([1]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The following lemma was also given in [3].

Lemma 2.2 ([3]) *Assume that $F : C \times C \to \mathbf{R}$ satisfies* (A1)–(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
3. $F(T_r) = \mathrm{EP}(F)$;
4. $\mathrm{EP}(F)$ *is closed and convex.*

3 Strong convergence theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality of an α-inverse-strongly monotone mapping in a Hilbert space by the hybrid method in the mathematical programming.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be an α-inverse-strongly monotone mapping of C into H. Let S be a nonexpansive mappings from C into H such that* $\mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \neq \emptyset$. *For $C_1 = C$, $x_1 = P_{C_1} x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows:*

$$\begin{cases} u_n \in C \quad \text{such that} \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ *and* $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(S)\cap VI(C,A)\cap \mathrm{EP}(F)}x_0$.

Proof We show first that the sequence $\{x_n\}$ is well defined. By induction we can show that

$$\mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \subset C_n \quad \text{for all } n \in \mathbb{N}.$$

It is obvious that $\mathrm{F}(S)\cap VI(C, A)\cap\mathrm{EP}(F) \subset C = C_1$. Suppose that $\mathrm{F}(S)\cap VI(C, A)\cap \mathrm{EP}(F) \subset C_k$ for each $k \in \mathbb{N}$. Hence, for $v \in \mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \subset C_k$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. Then $v = P_C(v-\lambda_n Av) = T_{r_n}v$. Put $v_n = P_C(u_n - \lambda_n Au_n)$ and from $u_n = T_{r_n}x_n$, we have

$$\begin{aligned}
\|v_n - v\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\| \\
&\le \|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\| \\
&\le \|u_n - v\| \\
&= \|T_{r_n}x_n - T_{r_n}v\| \\
&\le \|x_n - v\|
\end{aligned} \tag{3.1}$$

for every $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned}
\|y_n - v\| &= \|\alpha_n x_n + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n) - v\| \\
&\le \alpha_n\|x_n - v\| + (1 - \alpha_n)\|Sv_n - v\| \\
&\le \alpha_n\|x_n - v\| + (1 - \alpha_n)\|v_n - v\| \\
&\le \alpha_n\|x_n - v\| + (1 - \alpha_n)\|x_n - v\| \\
&= \|x_n - v\|.
\end{aligned} \tag{3.2}$$

So, we have $v \in C_n$. This implies that

$$\mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \subset C_n \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Next, we prove that C_n is closed and convex for all $n \in \mathbb{N}$. From the definition of C_n, it is obvious that C_n is closed for all $n \ge 0$. Since $C_n = \{z \in C : \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \le 0\}$, we deduce that C_n is convex for all $n \ge 0$. This implies that $\{x_n\}$ is well-defined. From Lemma 2.1, the sequence $\{u_n\}$ is also well defined. From $x_n = P_{C_n}x_0$, we obtain

$$\langle x_0 - x_n, x_n - y\rangle \ge 0$$

for each $y \in C_n$. Using $\mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \ge 0 \quad \text{for each } u \in \mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \quad \text{and} \quad n \in \mathbb{N}.$$

Hence, for $u \in \mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F)$, we obtain

$$0 \le \langle x_0 - x_n, x_n - u\rangle$$

$$
\begin{aligned}
&= \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\
&= -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - u\|.
\end{aligned}
$$

This implies that

$$
\|x_0 - x_n\| \le \|x_0 - u\| \quad \text{for all } u \in \mathrm{F}(S) \cap VI(C, A) \cap \mathrm{EP}(F) \quad \text{and } n \in \mathbb{N}.
$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we obtain

$$
\langle x_0 - x_n, x_n - x_{n+1} \rangle \ge 0. \tag{3.4}
$$

It follow that, for $n \in \mathbb{N}$,

$$
\begin{aligned}
0 &\le \langle x_0 - x_n, x_n - x_{n+1} \rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\
&\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|
\end{aligned}
$$

and hence

$$
\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|.
$$

Since $\{\|x_n - x_0\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we can show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. In deed, from (3.4) we have

$$
\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\
&\le -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}
$$

Since $\lim_{n\to\infty} \|x_0 - x_n\|$ exists, this implies that

$$
\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.5}
$$

Since $x_{n+1} \in C_n$, we obtain that

$$
\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\|.
$$

By (3.5), we have

$$
\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.6}
$$

For $v \in \mathrm{F}(S) \cap VI(A, C) \cap \mathrm{EP}(F)$ and from Lemma 2.2, we get

$$
\begin{aligned}
\|v_n - v\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(v - \lambda_n A v)\|^2 \\
&\le \|(u_n - \lambda_n A u_n) - (v - \lambda_n A v)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|(I-\lambda_n A)u_n - (I-\lambda_n A)v\|^2 \\
&\le \|u_n - v\|^2 = \|T_{r_n}x_n - T_{r_n}v\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\
&= \langle u_n - v, x_n - v\rangle \\
&= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2)
\end{aligned}
$$

and hence $\|v_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2$.

From this, we obtain

$$
\begin{aligned}
\|y_n - v\|^2 &\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|Sv_n - v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\{\|x_n - v\|^2 - \|x_n - u_n\|^2\} \\
&= \|x_n - v\|^2 - (1-\alpha_n)\|x_n - u_n\|^2.
\end{aligned}
$$

Since $\{\alpha_n\} \subset (0,1)$, we get

$$
\begin{aligned}
\|x_n - u_n\|^2 &\le \|x_n - v\|^2 - \|y_n - v\|^2 \\
&\le \|x_n - y_n\|\{\|x_n - v\| - \|y_n - v\|\}.
\end{aligned}
$$

From this and (3.6), implies

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.7}$$

Since $\liminf_{n\to\infty} r_n > 0$, we also have

$$\lim_{n\to\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.8}$$

Next, we show that $\|u_n - v_n\| \to 0$. For $v \in \mathrm{F}(S) \cap VI(A,C) \cap \mathrm{EP}(F)$, from (2.7), we compute that

$$
\begin{aligned}
\|y_n - v\|^2 &\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|Sv_n - v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\
&= \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|(I-\lambda_n A)u_n - (I-\lambda_n A)v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)(\|u_n - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Av\|^2) \\
&\le \alpha_n\|x_n - v\|^2 + \|u_n - v\|^2 + (1-\alpha_n)a(b-2\alpha)\|Au_n - Av\|^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&-(1-\alpha_n)a(b-2\alpha)\|Au_n - Av\|^2 \\
&\quad \le \alpha_n\|x_n - v\|^2 + \|u_n - v\|^2 - \|y_n - v\|^2 \\
&\quad \le \alpha_n\|x_n - v\|^2 + (\|u_n - v\| + \|y_n - v\|)\|u_n - y_n\|. \qquad (3.9)
\end{aligned}
$$

From (3.6) and (3.7), we get $\|u_n - y_n\| \le \|u_n - x_n\| + \|x_n - y_n\| \to 0$. Since $\alpha_n \to 0, a, b \in (0, 2\alpha)$ and $\|u_n - y_n\| \to 0$, we obtain $\|Au_n - Av\| \to 0$. From (2.3), we have

$$
\begin{aligned}
\|v_n - v\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (v - \lambda_n Av), v_n - v\rangle \\
&= (1/2)\{\|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\|^2 + \|v_n - v\|^2 \\
&\quad - \|[(u_n - \lambda_n Au_n) - (v - \lambda_n Av)] - (v_n - v)\|^2\} \\
&\le (1/2)\{\|u_n - v\|^2 + \|v_n - v\|^2 - \|(u_n - v_n) - \lambda_n(Au_n - Av)\|^2\} \\
&= (1/2)\{\|u_n - v\|^2 + \|v_n - v\|^2 - \|(u_n - v_n)\|^2 \\
&\quad + 2\lambda_n\langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2\}.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
\|v_n - v\|^2 \le \|u_n - v\|^2 - \|u_n - v_n\|^2& \\
+ 2\lambda_n\langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2&
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|y_n - v\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sv_n - v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + \|u_n - v\|^2 - \|u_n - v_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\|u_n - v_n\|^2 &\le \alpha_n\|x_n - v\|^2 + \|u_n - v\|^2 - \|y_n - v\|^2 \\
&\quad + 2\lambda_n\langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (\|u_n - v\|^2 - \|y_n - v\|^2) \\
&\quad + 2\lambda_n\langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (\|u_n - v\| - \|y_n - v\|)\|u_n - y_n\| \\
&\quad + 2\lambda_n\langle u_n - v_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2.
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|u_n - y_n\| \to 0$ and $\|Au_n - Av\| \to 0$, we obtain

$$
\|u_n - v_n\| \to 0. \qquad (3.10)
$$

From (3.7) and (3.10), we also have

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\| \to 0. \tag{3.11}$$

Since $y_n - u_n = \alpha_n x_n + (1-\alpha_n)Sv_n - u_n$, we obtain

$$\begin{aligned}(\alpha_n - 1)\|Sv_n - u_n\| &\le \alpha_n\|x_n - u_n\| + \|y_n - u_n\| \\ &\le \alpha_n\|x_n - u_n\| + \|y_n - u_n\|.\end{aligned}$$

From $\alpha_n \to 0$ and $\|y_n - u_n\| \to 0$, we get $\|Sv_n - u_n\| \to 0$ as $n \to \infty$. Since $\|Sv_n - v_n\| \le \|Sv_n - u_n\| + \|u_n - v_n\|$ and from (3.10) that

$$\lim_{n\to\infty} \|Sv_n - v_n\| = 0. \tag{3.12}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z. From (3.11), we obtain also that $v_{n_i} \rightharpoonup z$. Since $v_{n_i} \subset C$ and C is closed and convex, we obtain $z \in C$. From $\|Sv_n - v_n\| \to 0$, we obtain $Sv_{n_i} \rightharpoonup z$. Let us show that $z \in \mathrm{EP}(F)$. Since $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F(y, u_{n_i}).$$

From $\|u_n - x_n\| \to 0$, $\|u_n - Sv_n\| \to 0$, and $\|Sv_n - v_n\| \to 0$, we get $u_{n_i} \rightharpoonup z$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, it follows by (A4) that $0 \ge F(y, z)$ for all $y \in C$. For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \le 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, z) \le tF(y_t, y)$$

and hence $0 \le F(y_t, y)$. From (A3), we have $0 \le F(z, y)$ for all $y \in C$ and hence $z \in \mathrm{EP}(F)$. Let us show that $z \in \mathrm{F}(S)$. Assume $z \notin \mathrm{F}(S)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{n\to\infty} \|v_{n_i} - z\| &< \liminf_{i\to\infty} \|v_{n_i} - Sz\| \\ &= \liminf_{i\to\infty} \|v_{n_i} - Sv_{n_i} + Sv_{n_i} - Sz\| \\ &= \liminf_{i\to\infty} \|Sv_{n_i} - Sz\| \\ &\le \liminf_{i\to\infty} \|v_{n_i} - z\|.\end{aligned}$$

This is a contradiction. Thus, we obtain $z \in \mathrm{F}(S)$. Finally, we can show that $z \in VI(A, C)$. Defined

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(u, v) \in G(T)$. Since $u - Av \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, u - Av \rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n A u_n)$, we have

$$\langle v - v_n, v_n - (u_n - \lambda_n A u_n) \rangle \geq 0,$$

and hence,

$$\left\langle v - v_n, \frac{(v_n - u_n)}{\lambda_n} + Au_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} \langle v - v_{n_i}, u) \rangle &\geq \langle v - v_{n_i}, Av \rangle \\ &\geq \langle v - v_{n_i}, Av \rangle - \left\langle v - v_{n_i}, \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} + Au_{n_i} \right\rangle \\ &= \left\langle v - v_{n_i}, Av - Au_{n_i} - \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} \right\rangle \\ &= \langle v - v_{n_i}, Av - Av_{n_i} \rangle + \langle v - v_{n_i}, Av_{n_i} - Au_{n_i} \rangle \\ &\quad - \left\langle v - v_{n_i}, \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - v_{n_i}, Av_{n_i} - Au_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{(v_{n_i} - u_{n_i})}{\lambda_{n_i}} \right\rangle, \end{aligned}$$

which together with $\|v_n - u_n\| \to 0$ and A is Lipschitz continuous implies that

$$\langle v - z, u \rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Hence $z \in \mathrm{F}(S) \cap VI(A, C) \cap \mathrm{EP}(F)$.

Finally, we show that $x_n \to z$, where $z = P_{\mathrm{F}(S) \cap VI(C,A) \mathrm{EP}(F)} x_0$. Since $x_n = P_{C_n} x_0$ and $z \in \mathrm{F}(S) \cap VI(C, A) \cap \mathrm{FP}(F) \subset C_n$, we have

$$\|x_n - x_0\| \leq \|z - x_0\|.$$

It follows from $z' = P_{\mathrm{F}(S) \cap VI(C,A) \cap \mathrm{EP}(F)} x_0$ and the lower semicontinuity of the norm that

$$\|z' - x_0\| \leq \|z - x_0\| \leq \liminf_{i \to \infty} \|x_{n_i} - x_0\| \leq \limsup_{i \to \infty} \|x_{n_i} - x_0\| \leq \|z' - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty}\|x_{n_i}-x_0\|=\|z-x_0\|=\|z'-x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{k\to\infty} x_{n_i}=z=z'.$$

Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z=P_{\mathrm{F}(T)\cap VI(C,A)\cap \mathrm{EP}(F)}x_0$. □

As direct consequences of Theorem 3.1, we can obtain the following results.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let S be a nonexpansive mappings from C into H such that* $\mathrm{F}(S)\cap\mathrm{EP}(F)\neq\emptyset$. *For $C_1=C$, $x_1=P_{C_1}x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows*:

$$\begin{cases} u_n\in C \quad \text{such that} \quad F(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C,\\ y_n=\alpha_n x_n+(1-\alpha_n)Su_n,\\ C_{n+1}=\{z\in C_n:\|y_n-z\|\leq\|x_n-z\|\},\\ x_{n+1}=P_{C_{n+1}}x_0, \quad n\in\mathbb{N}, \end{cases}$$

for every $n\in\mathbb{N}$, where $\{\alpha_n\}\subset(0,1)$ such that $\lim_{n\to\infty}\alpha_n=0$ *and $\{r_n\}\subset(0,\infty)$ satisfies* $\liminf_{n\to\infty}r_n>0$. *Then $\{x_n\}$ converges strongly to* $P_{\mathrm{F}(S)\cap\mathrm{EP}(F)}x_0$.

Proof Putting $P_C(I-\lambda_n A)=I$, by Theorem 3.1, we obtain the desired result easily. □

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *such that* $\mathrm{EP}(F)\neq\emptyset$. *For $C_1=C$, $x_1=P_{C_1}x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows*:

$$\begin{cases} u_n\in C \quad \text{such that} \quad F(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C,\\ C_{n+1}=\{z\in C_n:\|y_n-z\|\leq\|x_n-z\|\},\\ x_{n+1}=P_{C_{n+1}}x_0, \quad n\in\mathbb{N}, \end{cases}$$

for every $n\in\mathbb{N}$, where $\{\alpha_n\}\subset(0,1)$ such that $\lim_{n\to\infty}\alpha_n=0$ *and $\{r_n\}\subset(0,\infty)$ satisfies* $\liminf_{n\to\infty}r_n>0$. *Then $\{x_n\}$ converges strongly to* $P_{\mathrm{EP}(F)}x_0$.

Proof Putting $S=I$, and $\alpha_n=0$ in Theorem 3.2, we obtain the Theorem 3.3. □

Theorem 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be an α-inverse-strongly monotone mapping of C into H. Let S be a nonexpansive mappings from C into H such that* $\mathrm{F}(S)\cap VI(C,A)\neq\emptyset$. *For $C_1=C$, $x_1=P_{C_1}x_0$, define* *sequences* $\{x_n\}$ *and* $\{u_n\}$ *of* C *as follows*:

$$\begin{cases} u_n \in C \quad \text{such that} \quad \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S P_C(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(S)\cap VI(C,A)} x$.

Proof Putting $F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, the conclusion follows. □

Remark 3.5 (i) Our results generally improve and extend the results of Tada and Takahashi [19] by the new hybrid method in the mathematical programming.

(ii) The hybrid iterative algorithm in Theorem 3.1 solves some variational inequality problems and equilibrium problems. It follows that from Theorem 3.2 and 3.3, we can replace by using the CQ-hybrid method (see [16, 17, 19]) for finding a common element of fixed point problems, equilibrium problems. For example to solving a linear system of two equations in two unknowns by hybrid methods (see an example in [17]). More example for projection algorithm for solving variational inequality problems by MATLAB see [15, pp. 772–774].

4 Applications

4.1 Maximal monotone operators

Let H be a Hilbert space and T be a maximal monotone operator on H. Consider the problem find $x \in H$ such that $0 \in T(x)$. We shall denote the solution set of this problem by

$$S = \{x \in H : 0 \in T(x)\} = T^{-1}(0).$$

Then S is a closed convex nonempty subset of H and thus the projection P_S from H onto $\mathrm{F}(S)$ is well-defined. One of the fundamental problems in the theory of maximal monotone operators is to find a solution of T, that is, a point is S. Rockafellar's proximal point algorithm [14] provides us with a powerful numerical tool to find a point in S.

The following theorem is connected with the problem of obtaining of a common element of the sets of zeroes of a maximal monotone operator and an α-inverse-strongly monotone operator.

Theorem 4.1 *Let* C *be a nonempty closed convex subset of* H. *Let* F *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying* (A1)–(A4) *and let* A *be an* α*-inverse-strongly monotone operator in* H *and* $B : H \to 2^H$ *be a maximal monotone operator such that* $A^{-1}(0) \cap$

$B^{-1}(0) \cap \mathrm{EP}(F) \neq \emptyset$. *Let* J_r^B *be the resolvent of* B *for each* $r > 0$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_0 = u \in H$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_r^B(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$ *and* $\{\lambda_n\} \subset [c,d]$ *for some* $[c,d] \subset (0, 2\alpha)$;
(ii) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$

then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in A^{-1}(0) \cap B^{-1}(0) \cap \mathrm{EP}(F)$, *where* $z = P_{A^{-1}(0)\cap B^{-1}(0)\mathrm{EP}(F)} x_0$.

Proof Since J_r^B is nonexpansive, we have the following

$$A^{-1}0 = V(I,A) \quad \text{and} \quad F(J_r^B) = B^{-1}(0).$$

Putting $P_H = I$ then, by Theorem 3.1, we obtain the desired result. □

Motivated by above results, we consider the following algorithm:

Algorithm 1 *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_0 = u \in H$ *and*

$$\begin{aligned} & u_n \in C \quad \text{such that} \quad \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ & y_n = u_n - \lambda_n A u_n, \\ & x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_r^B y_n, \\ & C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ & x_{n+1} = P_{C_{n+1}} x_0, \end{aligned}$$

where J_r^B *be the resolvent of* B *for each* $r > 0$.

Theorem 4.2 *Let* C *be a nonempty closed convex subset of* H. *Let* A *be an* α*-inverse-strongly monotone operator in* H *and* $B : H \to 2^H$ *be a maximal monotone operator such that* $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. *Let* J_r^B *be the resolvent of* B *for each* $r > 0$. *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 1 *suppose that* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy the conditions* (i) *and* (iii) *in Theorem* 4.1. *Then*, $\{x_n\}$ *converges strongly to* $z \in A^{-1}(0) \cap B^{-1}(0)$, *where* $z = P_{A^{-1}(0)\cap B^{-1}(0)} x_0$.

4.2 Strictly pseudocontractive maps

A mapping $T : C \to C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I-T)x + (I-T)y\|^2, \quad \text{for all } x, y \in C.$$

If $k=0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then we have, for all $x, y \in C$,

$$\|(I-A)x - (I-A)y\|^2 \le \|x-y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we have

$$\|(I-A)x - (I-A)y\|^2 = \|x-y\|^2 - 2\langle x-y, Ax - Ay\rangle + \|Ax - Ay\|^2.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \ge \frac{1-k}{2}\|Ax-Ay\|^2.$$

Then, A is $\frac{1-k}{2}$-inverse strongly monotone and $A^{-1}(0) = \mathrm{F}(T)$ (see [7]).

Now, using Theorem 3.1, we state a strong convergence theorem for a pair of a nonexpansive mapping and strictly pseudocontractive mapping.

Theorem 4.3 *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbf{R}$ satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping with constant k of C into itself such that $\mathrm{F}(S) \cap \mathrm{F}(T) \cap \mathrm{EP}(F) \neq \emptyset$. For $C_1 = C$, $x_1 = P_{C_1}x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows*:

$$\begin{cases} u_n \in C \quad \textit{such that} \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C,\\ y_n = (1-\alpha_n)x_n + \alpha_n S((1-\lambda_n)u_n + \lambda_n T u_n),\\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(S)\cap \mathrm{F}(T)\cap \mathrm{EP}(F)}x_0$.

Proof Put $A = I - T$. Then A is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $\mathrm{F}(T)$ is the solution set of $VI(A, C)$ i.e., $\mathrm{F}(T) = VI(A, C)$ and

$$P_C(u_n - \lambda_n A u_n) = (1-\lambda_n)u_n + \lambda_n T u_n.$$

Therefore, by Theorem 3.1, the conclusion follows. □

Corollary 4.4 *Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping with constant k of C into itself such that $\mathrm{F}(S) \cap \mathrm{F}(T) \neq \emptyset$. For $C_1 = C$,*

$x_1 = P_{C_1}x_0$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated*

$$\begin{cases} u_n \in C \quad \text{such that} \quad \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\lambda_n)x_n + \lambda_n T x_n \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sy_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha)$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(S)\cap\mathrm{F}(T)}x_0$.

Proof Put $F(x,y) = 0$ for all $x, y \in C$, $r_n = 1$ for all $n \in \mathbf{N}$ and $A = I - T$, we get $u_n = P_C x_n = x_n$ and A is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $\mathrm{F}(T)$ is the solution set of $VI(A,C)$ i.e., $\mathrm{F}(T) = VI(A,C)$ and

$$P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n.$$

Therefore, by Theorem 3.2, the conclusion follows. □

Theorem 4.5 *Let* H *be a real Hilbert space. Let* F *be a bifunction from* $H \times H \to \mathbf{R}$ *satisfying* (A1)–(A4) *and let* A *be an* α*-inverse-strongly monotone mapping of* H *into itself and let* S *be a nonexpansive mapping of* H *into itself such that* $\mathrm{F}(S) \cap A^{-1}(0) \cap \mathrm{EP}(F) \neq \emptyset$. *Suppose* $C_1 = C$, $x_1 = P_{C_1}x_0$, *and* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are given by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in; \\ y_n = (u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sy_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$ *and* $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha)$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(S)\cap A^{-1}(0)\cap \mathrm{EP}(F)}u$.

Proof Since $A^{-1}(0)$ is the solution set of $VI(A,H)$, we can obtain the conclusion by Theorem 3.1 and by noting that P_H is the identity mapping on H. This completes the proof. □

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A new hybrid iterative method for mixed equilibrium problems and variational inequality problem for relaxed cocoercive mappings with application to optimization problems[☆]

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ABSTRACT

In this paper, we introduce and study a new hybrid iterative method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of variational inequalities for a ξ-Lipschitz continuous and relaxed (m, v)-cocoercive mappings in Hilbert spaces. Then, we prove a strong convergence theorem of the iterative sequence generated by the proposed iterative algorithm which solves some optimization problems under some suitable conditions. Our results extend and improve the recent results of Yao et al. [Y. Yao, M.A. Noor, S. Zainab and Y.C. Liou, Mixed equilibrium problems and optimization problems, J. Math. Anal. Appl (2009). doi:10.1016/j.jmaa.2008.12.005] and Gao and Guo [X. Gao and Y. Guo, Strong convergence theorem of a modified iterative algorithm for Mixed equilibrium problems in Hilbert spaces, J. Inequal. Appl. (2008). doi:10.1155/2008/454181] and many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let E be a nonempty closed convex subset of H. Let A be a strongly positive linear bounded operator on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.1}$$

Now, we consider the following optimization problem (for short, OP):

$$\mathrm{OP} : \min_{x \in \widehat{F}} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \tag{1.2}$$

where $\widehat{F} = \cap_{n=1}^{\infty} E_n$, $E_1, E_2, \ldots$ are infinitely many closed convex subsets of H such that $\cap_{n=1}^{\infty} E_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). This kind of optimization problem has been studied extensively by many authors, see, for example, [1–4] when $\widehat{F} = \cap_{n=1}^{\infty} E_n$ and $h(x) = \langle x, b \rangle$, where b is a given point in H.

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Let $\varphi : E \longrightarrow \mathbb{R}$ be a real-valued function and $\Theta : E \times E \longrightarrow \mathbb{R}$ be an equilibrium bifunction, i.e., $\Theta(u, u) = 0$ for each $u \in E$. The mixed equilibrium problem (for short, MEP) which is to find $z \in E$ such that

$$\text{MEP} : \Theta(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in E. \tag{1.3}$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problems (for short, EP), which is to find $z \in E$ such that

$$\text{EP} : \Theta(z, y) \geq 0, \quad \forall y \in E. \tag{1.4}$$

The set of solutions of MEP (1.3) is denoted by Ω. The mixed equilibrium problems include Nash equilibrium problems, fixed point problems, saddle point problems, variational inequality problems, optimization problems and the equilibrium problems as special cases; see, for example, [5–13]. Some methods have been proposed to solve the MEP and EP; see, for instance [14–27]. Recently, Combettes and Hirstoaga [15] introduced an iterative scheme of finding the best approximation to the initial data when EP is nonempty and proved a strong convergence theorem. On the other hand, Gao and Guo [17] studied strong convergence by using the hybrid iterative scheme for finding a common element of the set of solutions of MEP, the set of common fixed points of finitely many nonexpansive mappings and the set of solution of variational inequalities for relaxed cocoercive mappings in a real Hilbert space. In 2009, Yao et al. [27] introduced and analyzed a new hybrid iterative algorithm for finding a common element of the set of solutions of MEP and the set of fixed points of an infinite family of nonexpansive mappings. Furthermore, they proved that the sequences generated by the new hybrid iterative scheme converge strongly to a common element of the set of solutions of MEP and the set of common fixed points of finitely many nonexpansive mappings, which solves optimization problem (OP).

Let $B : E \longrightarrow H$ be a nonlinear mapping. The classical variational inequality problem is to find $u \in E$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in E. \tag{1.5}$$

We denoted by $VI(E, B)$ the set of solutions of the variational inequality problem. The variational inequality problem has been extensively studied in literature, see, e.g. [28,29] and reference therein.

We now recall some well-known concepts and results as follows.

Lemma 1.1. *The function $u \in E$ is a solution of the variational inequality (1.5) if and only if $u \in E$ satisfies the relation $u = P_E(u - \lambda Bu)$, where $\lambda > 0$ is a constant.*

It is clear from Lemma 1.1 that the variational inequality and fixed point problem are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Definition 1.2. Let $B : E \longrightarrow H$ be a mapping. Then B is called

(1) monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in E,$$

(2) v-strongly monotone if there exists a positive real number v such that

$$\langle Bx - By, x - y\rangle \geq v\|x - y\|^2, \quad \forall x, y \in E,$$

for constant $v > 0$. This implies that

$$\|Bx - By\| \geq v\|x - y\|,$$

that is, B is v-expansive and when $v = 1$, it is expansive.

(3) ξ-Lipschitz continuous if there exists a positive real number ξ such that

$$\|Bx - By\| \leq \xi\|x - y\|, \quad \forall x, y \in E,$$

(4) m-cocoercive, [30,13] if there exists a positive real number m such that

$$\langle Bx - By, x - y\rangle \geq m\|Bx - By\|^2, \quad \forall x, y \in E.$$

Clearly, every m-cocoercive map A is $\frac{1}{m}$-Lipschitz continuous.

(5) Relaxed m-cocoercive, if there exists a positive real number m such that

$$\langle Bx - By, x - y\rangle \geq (-m)\|Bx - By\|^2, \quad \forall x, y \in E.$$

(6) Relaxed (m, v)-cocoercive, if there exists a positive real number m, v such that

$$\langle Bx - By, x - y\rangle \geq (-m)\|Bx - By\|^2 + v\|x - y\|^2, \quad \forall x, y \in E,$$

for $m = 0$, B is v-strongly monotone. This class of maps is more general that the class of strongly monotone maps. It is easy to see that we have the following implication: v-strongly monotonicity implying relaxed (m, v)-cocoercivity.

(7) A mapping T of E into itself is called nonexpansive (see [31]) if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in E.$$

We denote by $F(T) = \{x \in E : Tx = x\}$ the set of fixed points of T.

(8) Let $f : E \longrightarrow E$ is said to be a α-contraction if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in E.$$

(9) A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of E into H and let $N_E w_1$ be the *normal cone* to E at $w_1 \in E$, i.e.,

$$N_E w_1 = \{w \in H : \langle u - w_1, w\rangle \ge 0, \forall u \in E\}.$$

Define

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & w_1 \in E; \\ \emptyset, & w_1 \notin E. \end{cases}$$

Then T is the maximal monotone and $0 \in Tw_1$ if and only if $w_1 \in VI(E, B)$; see [32,33]

This paper is inspired and motivated by ongoing research in this field, so we will introduce an iterative scheme by the new hybrid iterative method (3.1) for finding a common element of the set of solutions of (1.3), the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of variational inequalities for a ξ-Lipschitz continuous and relaxed (m, v)-cocoercive mappings in Hilbert spaces. The results shown in this paper improve and extend the recent ones announced by Gao and Guo [17], Yao et al., [27] and many others.

2. Preliminaries

Let H be a real Hilbert space and let E be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\longrightarrow$, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x - (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. Recall that the (nearest point) projection P_E from H onto E assigns to each $x \in H$ the unique point in $P_E \in E$ satisfying the property

$$\|x - P_E x\| = \min_{y \in E} \|x - y\|.$$

The following characterizes the projection P_E.

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *For a given $z \in H, u \in E$,*

$$u = P_E z \Leftrightarrow \langle u - z, v - u\rangle \ge 0, \quad \forall v \in E.$$

It is well known that P_E is a firmly nonexpansive mapping of H onto E and satisfies

$$\|P_E x - P_E y\|^2 \le \langle P_E x - P_E y, x - y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_E x$ is characterized by the following properties: $P_E x \in E$ and for all $x \in H, y \in E$,

$$\langle x - P_E x, y - P_E x\rangle \le 0. \tag{2.2}$$

Using Lemma 2.1 one can see that the variational inequality (1.5) is equivalent to a fixed point problem.

Lemma 2.2 ([34]). *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|,$$

hold for each $y \in H$ with $y \neq x$.

Lemma 2.3 ([35]). *Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

For solving the mixed equilibrium problem for an equilibrium bifunction $\Theta : E \times E \longrightarrow \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(H1) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \le 0, \forall x, y \in E$;
(H2) for each fixed $y \in E, x \mapsto \Theta(x, y)$ is convex and upper semicontinuous;
(H3) for each $x \in E, y \mapsto \Theta(x, y)$ is convex.

Definition 2.4. Let $\eta : E \times E \longrightarrow H$, which is called Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|\eta(x, y)\| \leq \delta\|x - y\|, \quad \forall x, y \in E.$$

Definition 2.5. Let $K : E \longrightarrow \mathbb{R}$ be a differentiable functional on a convex set E, which is called:

(K1) η-convex [16] if

$$K(y) - K(x) \geq \left\langle K'(x), \eta(y, x)\right\rangle, \quad \forall x, y \in E,$$

where $K'(x)$ is the Fréchet derivative at x;

(K2) η-strongly convex [36] if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \left\langle K'(x), \eta(y, x)\right\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in E.$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in E$, then K is said to be strongly convex.

Definition 2.6. Let E be a nonempty closed convex subset of a real Hilbert space H, let $\varphi : E \longrightarrow \mathbb{R}$ be a real-valued function and $\Theta : E \times E \longrightarrow \mathbb{R}$ be an equilibrium bifunction. Let r be a positive parameter. For a given point $x \in E$, the auxiliary problem for MEP consists of finding $y \in E$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), \eta(z, y)\right\rangle \geq 0, \quad \forall z \in E.$$

Definition 2.7. Let $S_r : E \longrightarrow E$ be the mapping such that for each $x \in E$, $S_r(x)$ is the solution set of the auxiliary problem MEP, that is,

$$S_r(x) = \left\{y \in E : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), \eta(z, y)\right\rangle \geq 0,\ \forall z \in E\right\}, \quad \forall x \in E.$$

The following lemma appears implicitly in [16].

Lemma 2.8 ([16]). *Let E be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from E to $\mathbb{R}$. Let Θ be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (H1)–(H3). *Assume that*

(i) *$\eta : E \times E \longrightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that;*
  (a) *$\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in E$,*
  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in E$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : E \longrightarrow \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each $x \in E$, there exist a bounded subset $D_x \subset E$ and $z_x \in E$ such that for any $y \in E \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), \eta(z_x, y)\right\rangle < 0.$$

Then, there exists $y \in E$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), \eta(z, y)\right\rangle \geq 0, \quad \forall z \in E.$$

Lemma 2.9 ([16]). *Assume that Θ satisfies the same assumptions as Lemma 2.8 for $r > 0$ and $x \in E$, the mapping $S_r : E \longrightarrow E$ can be defined as follows:*

$$S_r(x) = \left\{y \in E : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), \eta(z, y)\right\rangle \geq 0,\ \forall z \in E\right\}.$$

Then, the following hold:

(i) *S_r is single-valued;*

(ii) (a) *$\left\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\right\rangle \geq \left\langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\right\rangle$, $\forall (x_1, x_2) \in E \times E$, where $u_i = S_r(x_i)$, $i = 1, 2$;*
  (b) *S_r is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\sigma > \lambda\nu$;*

(iii) *$F(S_r) = \Omega$; and*

(iv) *Ω is closed and convex.*

Remark 2.10. From Lemma 2.9 in particular, whenever $K(x) = \frac{\|x\|^2}{2}$ and $\eta(x, y) = x - y$ for each $(x, y) \in E \times E$, then S_r is firmly nonexpansive, that is,

$$\|S_r(x_1) - S_r(x_2)\|^2 \leq \Big\langle x_1 - x_2, S_r(x_1) - S_r(x_2)\Big\rangle.$$

Definition 2.11 (*[8]*). Let $\{T_n\}$ be a sequence of nonexpansive mappings of E into itself and let $\{\mu_n\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of E into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \mu_n T_n U_{n,n+1} + (1 - \mu_n)I,\\
&U_{n,n-1} = \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1})I,\\
&\vdots\\
&U_{n,k} = \mu_k T_k U_{n,k+1} + (1 - \mu_k)I,\\
&U_{n,k-1} = \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1})I,\\
&\vdots\\
&U_{n,2} = \mu_2 T_2 U_{n,3} + (1 - \mu_2)I,\\
&W_n = U_{n,1} = \mu_1 T_1 U_{n,2} + (1 - \mu_1)I.
\end{aligned} \tag{2.3}$$

Such a mapping W_n is nonexpansive from E to E and it is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (2.3). Then we can have the following crucial conclusions concerning W_n. You can find them in [37]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.12 (*[37]*). *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in E$ and $k \in \mathbb{N}$, the limit $\lim_{n\longrightarrow\infty} U_{n,k}x$ exists.*

Using Lemma 2.12, one can define a mapping W of E into itself as follows:

$$Wx = \lim_{n\longrightarrow\infty} W_n x = \lim_{n\longrightarrow\infty} U_{n,1}x, \tag{2.4}$$

for every $x \in E$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$. Throughout this paper, we will assume that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, we have the following results.

Lemma 2.13 (*[37]*). *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.14 (*[24]*). *If $\{x_n\}$ is a bounded sequence in E, then $\lim_{n\longrightarrow\infty} \|Wx_n - W_n x_n\| = 0$.*

Lemma 2.15 (*[38]*). *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)v_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|v_n - x_n\| = 0$.*

Lemma 2.16. *Let H be a real Hilbert space. Then the following inequalities hold:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$;

for all $x, y \in H$.

Lemma 2.17 (*[39]*). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \quad \forall n \geq 0,$$

where $\{l_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$
(2) $\limsup_{n\longrightarrow\infty} \frac{\sigma_n}{l_n} \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.18 ([27]). *Let E be a nonempty closed convex subset of a real Hilbert space H, and $g : E \longrightarrow \mathbb{R} \cup \{\infty\}$ be a proper lower-semicontinuous differentiable convex function. If z is a solution to the minimization problem*

$$g(z) = \inf_{x \in E} g(x),$$

then

$$\Big\langle g'(x), x - z \Big\rangle \geq 0, \quad x \in E.$$

In particular, if z solves problem OP, then

$$\Big\langle u + \big[\gamma f - (I + \mu A)\big] z, x - z \Big\rangle \leq 0.$$

3. Main results

In this section, we prove a strong convergence theorem of a new hybrid iterative method (3.1) to compute the approximate solutions of the mixed equilibrium problems and optimization problems in a real Hilbert space.

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from E to $\mathbb{R}$. Let Θ be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying (H1)–(H3), let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself and let B be a ξ-Lipschitz continuous and relaxed (m, υ)-cocoercive map E into H such that*

$$\Gamma := \cap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(E, B) \neq \emptyset.$$

Let $\mu > 0$, $\gamma > 0$ and $r > 0$, which are three constants. Let f be a contraction of E into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$. For given $x_1 \in H$ arbitrarily and fixed $u \in H$, suppose the $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated iteratively by

$$\begin{cases} \Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \dfrac{1}{r}\Big\langle K'(z_n) - K'(x_n), \eta(x, z_n)\Big\rangle \geq 0, \quad \forall x \in E, \\ y_n = \alpha_n z_n + (1 - \alpha_n) W_n P_E(z_n - \lambda_n B z_n), \\ x_{n+1} = \epsilon_n\big(u + \gamma f(W_n x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big) W_n P_E(y_n - \tau_n B y_n), \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where W_n be the W-mapping defined by (2.3) and $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are three sequences in $(0, 1)$. Assume the following conditions are satisfied:

(C1) *$\eta : E \times E \longrightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that;*
 (a) *$\eta(x, y) + \eta(y, x) = 0, \quad \forall x, y \in E$*
 (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
 (c) *for each fixed $y \in E$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(C2) *$K : E \longrightarrow \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma > \lambda\nu$;*

(C3) *for each $x \in E$, there exist a bounded subset $D_x \subset E$ and $z_x \in E$ such that for any $y \in E \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z_x, y)\Big\rangle < 0;$$

(C4) *$\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$;*

(C5) *$0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;*

(C6) *$\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$;*

(C7) *$\{\tau_n\}$, $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 \leq a \leq b \leq \frac{2(\upsilon - m\xi^2)}{\xi^2}$.*

Then, $\{x_n\}$ and $\{z_n\}$ converge strongly to $z \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(E, B)$ provided that S_r is firmly nonexpansive, which solves the following optimization problem:

$$OP : \min_{x \in \Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{3.2}$$

Proof. Since $\epsilon_n \longrightarrow 0$ by the condition (C4) and (C5), we may assume, without loss of generality, that $\epsilon_n \leq (1 - \beta_n)(1 + \mu\|A\|)^{-1}$ for all $n \in \mathbb{N}$. First, we show that $I - \tau_n B$ is nonexpansive. Indeed, $B : E \longrightarrow H$ be a ξ-Lipschitz continuous and relaxed (m, υ)-cocoercive mappings, we note that

$$\begin{aligned}\|(I-\tau_nB)x-(I-\tau_nB)y\|^2 &= \|(x-y)-\tau_n(Bx-By)\|^2\\ &= \|x-y\|^2-2\tau_n\langle x-y,Bx-By\rangle+\tau_n^2\|Bx-By\|^2\\ &\le \|x-y\|^2-2\tau_n\big\{-m\|Bx-By\|^2+\upsilon\|x-y\|^2\big\}+\tau_n^2\|Bx-By\|^2\\ &\le \|x-y\|^2+2\tau_n m\xi^2\|x-y\|^2-2\tau_n\upsilon\|x-y\|^2+\tau_n^2\xi^2\|x-y\|^2\\ &= (1+2\tau_n m\xi^2-2\tau_n\upsilon+\tau_n^2\xi^2)\|x-y\|^2\\ &= \theta^2\|x-y\|^2,\end{aligned}$$

where

$$\theta=\sqrt{1+2\tau_n m\xi^2-2\tau_n\upsilon+\tau_n^2\xi^2}.$$

It follows (C7) that $\theta<1$. Hence

$$\|(I-\tau_nB)x-(I-\tau_nB)y\|\le\|x-y\|,$$

which implies that the mapping $I-\tau_nB$ is nonexpansive, and so is $I-\lambda_nB$.

Since A is a strongly positive bounded linear operator on H, we have

$$\|A\|=\sup\{|\langle Ax,x\rangle|:x\in H,\ \|x\|=1\}.$$

Observe that

$$\begin{aligned}\Big\langle\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)x,x\Big\rangle &= 1-\beta_n-\epsilon_n-\epsilon_n\mu\langle Ax,x\rangle\\ &\ge 1-\beta_n-\epsilon_n-\epsilon_n\mu\|A\|\\ &\ge 0,\end{aligned}$$

so this shows that $(1-\beta_n)I-\epsilon_n(I+\mu A)$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n(I+\mu A)\| &= \sup\Big\{\Big|\Big\langle\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)x,x\Big\rangle\Big|:x\in H,\ \|x\|=1\Big\}\\ &= \sup\{1-\beta_n-\epsilon_n-\epsilon_n\mu\langle Ax,x\rangle:x\in H,\ \|x\|=1\}\\ &\le 1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma}.\end{aligned}$$

We shall divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, let $p\in\Gamma:=\cap_{n=1}^{\infty}F(T_n)\cap\Omega\cap VI(E,B)$ and let $\{S_r\}$ be a sequence of mappings defined as in Lemma 2.9. Then $z_n=S_rx_n$. So, we have

$$\|z_n-p\|=\|S_rx_n-S_rp\|\le\|x_n-p\|.$$

Because $I-\lambda_nB$, $I-\delta_nB$, P_E and W_n are nonexpansive mappings and $p=W_nP_E(p-\lambda_nBp)$, we have

$$\begin{aligned}\|y_n-p\| &= \|\alpha_n(z_n-p)+(1-\alpha_n)(W_nP_E(z_n-\lambda_nBz_n)-p)\|\\ &\le \alpha_n\|z_n-p\|+(1-\alpha_n)\|P_E(z_n-\lambda_nBz_n)-P_E(p-\lambda_nBp)\|\\ &\le \alpha_n\|z_n-p\|+(1-\alpha_n)\|(z_n-\lambda_nBz_n)-(p-\lambda_nBp)\|\\ &= \alpha_n\|z_n-p\|+(1-\alpha_n)\|(I-\lambda_nB)z_n-(I-\lambda_nB)p\|\\ &\le \alpha_n\|z_n-p\|+(1-\alpha_n)\|z_n-p\|=\|z_n-p\|\le\|x_n-p\|,\end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\epsilon_nu+\epsilon_n\big(\gamma f(W_nx_n)-(I+\mu A)p\big)+\beta_n(x_n-p)\\ &\quad+\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)(W_nP_E(k_n-\tau_nBk_n)-p)\|\\ &\le \big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|P_E(I-\tau_nB)k_n-p\|+\beta_n\|x_n-p\|+\epsilon_n\|u\|+\epsilon_n\|\gamma f(W_nx_n)-(I+\mu A)p\|\\ &\le \big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|k_n-p\|+\beta_n\|x_n-p\|+\epsilon_n\|u\|+\epsilon_n\|\gamma f(W_nx_n)-(I+\mu A)p\|\\ &\le \big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|k_n-p\|+\beta_n\|x_n-p\|+\epsilon_n\|u\|\\ &\quad+\epsilon_n\gamma\|f(W_nx_n)-f(p)\|+\epsilon_n\|\gamma f(p)-(I+\mu A)p\|\\ &\le \big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|x_n-p\|+\beta_n\|x_n-p\|+\epsilon_n\|u\|\\ &\quad+\epsilon_n\gamma\alpha\|x_n-p\|+\epsilon_n\|\gamma f(p)-(I+\mu A)p\|\\ &\le \big(1-\epsilon_n(1+\mu)\bar{\gamma}+\epsilon_n\gamma\alpha\big)\|x_n-p\|+\epsilon_n\big(\|\gamma f(p)-(I+\mu A)p\|+\|u\|\big)\end{aligned}$$

$$= \Big(1-\big((1+\mu)\bar{\gamma}-\gamma\alpha\big)\epsilon_n\Big)\|x_n-p\|+\epsilon_n\big(\|\gamma f(p)-(I+\mu A)p\|+\|u\|\big)$$
$$= \Big(1-\big((1+\mu)\bar{\gamma}-\gamma\alpha\big)\epsilon_n\Big)\|x_n-p\|+\big((1+\mu)\bar{\gamma}-\gamma\alpha\big)\epsilon_n\frac{\|\gamma f(p)-(I+\mu A)p\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma\alpha}. \tag{3.3}$$

It follows that (3.3) and induction that

$$\|x_n-p\|\le \max\left\{\|x_1-p\|, \frac{\|\gamma f(p)-(I+\mu A)p\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma\alpha}\right\},\quad n\ge 1. \tag{3.4}$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Step 2. We claim that $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\longrightarrow\infty}\|W_n\theta_n-x_n\|=0$.

Observing that $z_n=S_rx_n$ and $z_{n+1}=S_rx_{n+1}$, by the nonexpansiveness of S_r, we get

$$\|z_{n+1}-z_n\|=\|S_rx_{n+1}-S_rx_n\|\le\|x_{n+1}-x_n\|. \tag{3.5}$$

Put $\theta_n=P_E(y_n-\tau_nBy_n)$ and $\phi_n=P_E(z_n-\lambda_nBz_n)$. Since $I-\tau_nB$ and $I-\lambda_nB$ are nonexpansive mappings, we have the following estimates:

$$\begin{aligned}
\|\phi_{n+1}-\phi_n\| &\le \|P_E(z_{n+1}-\lambda_{n+1}Bz_{n+1})-P_E(z_n-\lambda_nBz_n)\|\\
&\le \|(z_{n+1}-\lambda_{n+1}Bz_{n+1})-(z_n-\lambda_nBz_n)\|\\
&= \|(z_{n+1}-\lambda_{n+1}Bz_{n+1})-(z_n-\lambda_{n+1}Bz_n)+(\lambda_n-\lambda_{n+1})Bz_n\|\\
&\le \|(z_{n+1}-\lambda_{n+1}Bz_{n+1})-(z_n-\lambda_{n+1}Bz_n)\|+|\lambda_n-\lambda_{n+1}|\|Bz_n\|\\
&= \|(I-\lambda_{n+1}B)z_{n+1}-(I-\lambda_{n+1}B)z_n\|+|\lambda_n-\lambda_{n+1}|\|Bz_n\|\\
&\le \|z_{n+1}-z_n\|+|\lambda_n-\lambda_{n+1}|\|Bz_n\|\le\|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\|Bz_n\|
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\|\theta_{n+1}-\theta_n\| &\le \|P_E(y_{n+1}-\tau_{n+1}By_{n+1})-P_E(y_n-\tau_nBy_n)\|\\
&\le \|(y_{n+1}-\tau_{n+1}By_{n+1})-(y_n-\tau_nBy_n)\|\\
&\le \|(y_{n+1}-\tau_{n+1}By_{n+1})-(y_n-\tau_{n+1}By_n)\|+|\tau_n-\tau_{n+1}|\|By_n\|\\
&= \|(I-\tau_{n+1}B)y_{n+1}-(I-\tau_{n+1}B)y_n\|+|\tau_n-\tau_{n+1}|\|By_n\|\\
&\le \|y_{n+1}-y_n\|+|\tau_n-\tau_{n+1}|\|By_n\|.
\end{aligned} \tag{3.7}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}\phi_n-W_n\phi_n\| &= \|\mu_1T_1U_{n+1,2}\phi_n-\mu_1T_1U_{n,2}\phi_n\|\\
&\le \mu_1\|U_{n+1,2}\phi_n-U_{n,2}\phi_n\|\\
&= \mu_1\|\mu_2T_2U_{n+1,3}\phi_n-\mu_2T_2U_{n,3}\phi_n\|\\
&\le \mu_1\mu_2\|U_{n+1,3}\phi_n-U_{n,3}\phi_n\|\\
&\ \ \vdots\\
&\le \mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}\phi_n-U_{n,n+1}\phi_n\|\\
&\le M_2\prod_{i=1}^{n}\mu_i,
\end{aligned} \tag{3.8}$$

where $M_2\ge 0$ is a constant such that $\|U_{n+1,n+1}\phi_n-U_{n,n+1}\phi_n\|\le M_2$ for all $n\ge 0$.

Similarly, we obtain that there exist nonnegative numbers M_3 such that

$$\|U_{n+1,n+1}\theta_n-U_{n,n+1}\theta_n\|\le M_3,$$

and so is

$$\|W_{n+1}\theta_n-W_n\theta_n\|\le M_3\prod_{i=1}^{n}\mu_i. \tag{3.9}$$

Observing that

$$\begin{cases} y_n=\alpha_nz_n+(1-\alpha_n)W_n\phi_n\\ y_{n+1}=\alpha_{n+1}z_{n+1}+(1-\alpha_{n+1})W_n\phi_{n+1},\end{cases}$$

we obtain

$$y_n - y_{n+1} = \alpha_n(z_n - z_{n+1}) + (1-\alpha_n)(W_n\phi_n - W_{n+1}\phi_{n+1}) + (W_{n+1}\phi_{n+1} - z_{n+1})(\alpha_{n+1} - \alpha_n),$$

which yields that

$$\begin{aligned}\|y_n - y_{n+1}\| &\le \alpha_n\|z_n - z_{n+1}\| + (1-\alpha_n)\|W_n\phi_n - W_{n+1}\phi_{n+1}\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - z_{n+1}\| \\ &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\Big\{\|W_{n+1}\phi_{n+1} - W_{n+1}\phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\|\Big\} \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - z_{n+1}\| \\ &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|\phi_{n+1} - \phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - z_{n+1}\|. \end{aligned} \tag{3.10}$$

Substitution of (3.6) and (3.8) into (3.10) yields that

$$\begin{aligned}\|y_n - y_{n+1}\| &= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\Big\{\|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Bz_n\|\Big\} \\ &\quad + M_2\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - z_{n+1}\| \\ &= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|x_{n+1} - x_n\| + (1-\alpha_n)|\lambda_n - \lambda_{n+1}|\|Bz_n\| \\ &\quad + M_2\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - z_{n+1}\| \\ &\le \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_4\big(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|\big), \end{aligned} \tag{3.11}$$

where M_4 is an appropriate constant such that $M_4 = \max\Big\{\sup_{n\ge1}\|Bz_n\|, \sup_{n\ge1}\|W_n\phi_n - z_n\|\Big\}$.

Substitution of (3.11) into (3.7), we obtain

$$\begin{aligned}\|\theta_{n+1} - \theta_n\| &\le \|y_{n+1} - y_n\| + |\tau_n - \tau_{n+1}|\|By_n\| \\ &\le \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_4\big(||\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n||\big) + |\tau_n - \tau_{n+1}|\|By_n\| \\ &\le \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_5\big(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\tau_n - \tau_{n+1}|\big), \end{aligned} \tag{3.12}$$

where M_5 is an appropriate constant such that $M_5 = \max\big\{\sup_{n\ge1}\|By_n\|, M_4\big\}$.

Let $x_{n+1} = (1-\beta_n)v_n + \beta_n x_n$, $n \ge 1$; therefore,

$$v_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\epsilon_n\big(u + \gamma f(W_n x_n)\big) + \big((1-\beta_n)I - \epsilon_n(I + \mu A)\big)W_n\theta_n}{1-\beta_n}.$$

Then we have

$$\begin{aligned} v_{n+1} - v_n &= \frac{\epsilon_{n+1}\big(u + \gamma f(W_{n+1}x_{n+1})\big) + \big((1-\beta_{n+1})I - \epsilon_{n+1}(I + \mu A)\big)W_{n+1}\theta_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\epsilon_n\big(u + \gamma f(W_n x_n)\big) + \big((1-\beta_n)I - \epsilon_n(I + \mu A)\big)W_n\theta_n}{1-\beta_n} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(u + \gamma f(W_{n+1}x_{n+1})\big) - \frac{\epsilon_n}{1-\beta_n}\big(u + \gamma f(W_n x_n)\big) + W_{n+1}\theta_{n+1} - W_n\theta_n \\ &\quad + \frac{\epsilon_n}{1-\beta_n}(I + \mu A)W_n\theta_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(I + \mu A)W_{n+1}\theta_{n+1} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\big(u + \gamma f(W_{n+1}x_{n+1})\big) - (I + \mu A)W_{n+1}\theta_{n+1}\Big) + \frac{\epsilon_n}{1-\beta_n}\big((I + \mu A)W_n\theta_n - u - \gamma f(W_n x_n)\big) \\ &\quad + W_{n+1}\theta_{n+1} - W_{n+1}\theta_n + W_{n+1}\theta_n - W_n\theta_n. \end{aligned} \tag{3.13}$$

It follows from (3.9), (3.12) and (3.13) that

$$\begin{aligned}
&\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\| \qquad (3.14)\\
&\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|\Big)\\
&\quad+\frac{\epsilon_n}{1-\beta_n}\Big(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|\Big)\\
&\quad+\|W_{n+1}\theta_{n+1}-W_{n+1}\theta_n\|+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|\Big)\\
&\quad+\frac{\epsilon_n}{1-\beta_n}\Big(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|\Big)+\|\theta_{n+1}-\theta_n\|\\
&\quad+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|\Big)\\
&\quad+\frac{\epsilon_n}{1-\beta_n}\Big(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|\Big)+M_3\prod_{i=1}^{n}\mu_i\\
&\quad+M_2\prod_{i=1}^{n}\mu_i+M_5\big(|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\tau_n-\tau_{n+1}|\big)\\
&\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|\Big)\\
&\quad+\frac{\epsilon_n}{1-\beta_n}\Big(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|\Big)+2L\prod_{i=1}^{n}\mu_i\\
&\quad+M_5\big(|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\tau_n-\tau_{n+1}|\big), \qquad (3.15)
\end{aligned}$$

where L is an appropriate constant such that $L=\max\{M_2,M_3\}$.

It follows from condition (C4), (C5), (C6) and $0<\mu_i\le b<1$, $\forall i\ge 1$, we have

$$\limsup_{n\longrightarrow\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 2.15, we obtain

$$\lim_{n\longrightarrow\infty}\|v_n-x_n\|=0.$$

It follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=\lim_{n\longrightarrow\infty}(1-\beta_n)\|v_n-x_n\|=0. \tag{3.16}$$

Applying (3.16) and condition in Theorem 3.1 to (3.5), (3.6), (3.11) and (3.12), we obtain that

$$\lim_{n\longrightarrow\infty}\|z_{n+1}-z_n\|=\lim_{n\longrightarrow\infty}\|y_{n+1}-y_n\|=\lim_{n\longrightarrow\infty}\|\phi_{n+1}-\phi_n\|=\lim_{n\longrightarrow\infty}\|\theta_{n+1}-\theta_n\|=0.$$

Since $x_{n+1}=\epsilon_n\big(u+\gamma f(W_nx_n)\big)+\beta_nx_n+\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)W_n\theta_n$, we have

$$\begin{aligned}
\|x_n-W_n\theta_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-W_n\theta_n\|\\
&=\|x_n-x_{n+1}\|+\|\epsilon_n\big(u+\gamma f(W_nx_n)\big)+\beta_nx_n+\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)W_n\theta_n-W_n\theta_n\|\\
&=\|x_n-x_{n+1}\|+\Big\|\epsilon_n\Big(\big(u+\gamma f(W_nx_n)\big)-(I+\mu A)W_n\theta_n\Big)+\beta_n(x_n-W_n\theta_n)\Big\|\\
&\le \|x_n-x_{n+1}\|+\epsilon_n\big(\|u\|+\|\gamma f(W_nx_n)\|+\|(I+\mu A)W_n\theta_n\|\big)+\beta_n\|x_n-W_n\theta_n\|,
\end{aligned}$$

that is

$$\|x_n-W_n\theta_n\|\le\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\epsilon_n}{1-\beta_n}\big(\|u\|+\|\gamma f(W_nx_n)\|+\|(I+\mu A)W_n\theta_n\|\big).$$

By (C4), (C5) and (3.16) it follows that

$$\lim_{n\longrightarrow\infty}\|W_n\theta_n-x_n\|=0. \tag{3.17}$$

Step 3. We claim that $\lim_{n\longrightarrow\infty}\|x_n-\theta_n\|=0$ and $\lim_{n\longrightarrow\infty}\|W_n\theta_n-\theta_n\|=0$.

Since B is a ξ-Lipschitz continuous and relaxed (m,v)-cocoercive mappings by the assumptions imposed on $\{\tau_n\}$ for any $p\in\Gamma:=\cap_{n=1}^{\infty}F(T_n)\cap\Omega\cap VI(E,B)$, we have

$$\begin{aligned}
\|W_n\theta_n-p\|^2 &\le \|P_E(y_n-\tau_nBy_n)-P_E(p-\tau_nBp)\|^2\\
&\le \|(y_n-\tau_nBy_n)-(p-\tau_nBp)\|^2\\
&= \|(y_n-p)-\tau_n(By_n-Bp)\|^2\\
&\le \|y_n-p\|^2-2\tau_n\langle y_n-p,By_n-Bp\rangle+\tau_n^2\|By_n-Bp\|^2\\
&\le \|x_n-p\|^2-2\tau_n\langle y_n-p,By_n-Bp\rangle+\tau_n^2\|By_n-Bp\|^2\\
&\le \|x_n-p\|^2-2\tau_n\{-m\|By_n-Bp\|^2+v\|y_n-p\|^2\}+\tau_n^2\|By_n-Bp\|^2\\
&\le \|x_n-p\|^2+2\tau_nm\|By_n-Bp\|^2-2\tau_nv\|y_n-p\|^2+\tau_n^2\|By_n-Bp\|^2\\
&\le \|x_n-p\|^2+2\tau_nm\|By_n-Bp\|^2-\frac{2\tau_nv}{\xi^2}\|By_n-Bp\|^2+\tau_n^2\|By_n-Bp\|^2\\
&= \|x_n-p\|^2+\left(2\tau_nm+\tau_n^2-\frac{2\tau_nv}{\xi^2}\right)\|By_n-Bp\|^2.
\end{aligned}\tag{3.18}$$

Similarly, we have

$$\|W_n\phi_n-p\|^2\le\|x_n-p\|^2+\left(2\lambda_nm+\lambda_n^2-\frac{2\lambda_nv}{\xi^2}\right)\|Bz_n-Bp\|^2.\tag{3.19}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)(W_n\theta_n-p)+\beta_n(x_n-p)+\epsilon_n\big(u+\gamma f(W_nx_n)-(I+\mu A)p\big)\|^2\\
&= \|\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)(W_n\theta_n-p)+\beta_n(x_n-p)\|^2+\epsilon_n^2\|u+\gamma f(W_nx_n)-(I+\mu A)p\|^2\\
&\quad+2\beta_n\epsilon_n\Big\langle x_n-p,u+\gamma f(W_nx_n)-(I+\mu A)p\Big\rangle\\
&\quad+2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)(W_n\theta_n-p),u+\gamma f(W_nx_n)-(I+\mu A)p\Big\rangle\\
&\le \Big[\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)\|W_n\theta_n-p\|+\beta_n\|x_n-p\|\Big]^2+\epsilon_n^2\|u+\gamma f(W_nx_n)-(I+\mu A)p\|^2\\
&\quad+2\beta_n\epsilon_n\Big\langle x_n-p,u+\gamma f(W_nx_n)-(I+\mu A)p\Big\rangle\\
&\quad+2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)(W_n\theta_n-p),u+\gamma f(W_nx_n)-(I+\mu A)p\Big\rangle\\
&\le \Big[(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|W_n\theta_n-p\|+\beta_n\|x_n-p\|\Big]^2+c_n\\
&= (1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|W_n\theta_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad+2(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|W_n\theta_n-p\|\|x_n-p\|+c_n\\
&\le (1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|W_n\theta_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad+(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n(\|W_n\theta_n-p\|^2+\|x_n-p\|^2)+c_n\\
&= \Big[(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2-2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n+\beta_n^2\Big]\|W_n\theta_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad+((1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n-\beta_n^2)(\|W_n\theta_n-p\|^2+\|x_n-p\|^2)+c_n\\
&= \Big[(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\Big]\|W_n\theta_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|W_n\theta_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n,
\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}
c_n &= \epsilon_n^2\|u+\gamma f(x_n)-(I+\mu A)p\|^2+2\beta_n\epsilon_n\Big\langle x_n-p,u+\gamma f(W_nx_n)-(I+\mu A)p\Big\rangle\\
&\quad+2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_n(I+\mu A)\big)(W_n\theta_n-p),u+\gamma f(W_nx_n)-(I+\mu A)p\Big\rangle.
\end{aligned}$$

It follows from condition (C4) that

$$\lim_{n\longrightarrow\infty}c_n=0.\tag{3.21}$$

Substituting (3.18) into (3.20), and using condition (C7), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\left\{\|x_n-p\|^2+\left(2\tau_n m+\tau_n^2-\frac{2\tau_n v}{\xi^2}\right)\|By_n-Bp\|^2\right\}\\
&\quad+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2\\
&\quad+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\left(2\tau_n m+\tau_n^2-\frac{2\tau_n v}{\xi^2}\right)\|By_n-Bp\|^2+c_n\\
&\le \|x_n-p\|^2+\left(2\tau_n m+\tau_n^2-\frac{2\tau_n v}{\xi^2}\right)\|By_n-Bp\|^2+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(\frac{2av}{\xi^2}-2bm-b^2\right)\|By_n-Bp\|^2 &\le \left(\frac{2\tau_n v}{\xi^2}-2\tau_n m-\tau_n^2\right)\|By_n-Bp\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\
&=(\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned}$$

Since $c_n \longrightarrow 0$ as $n \longrightarrow \infty$ and (3.16), we obtain

$$\lim_{n\longrightarrow\infty}\|By_n-Bp\|=0. \tag{3.22}$$

Note that

$$\begin{aligned}
\|y_n-p\|^2 &\le \alpha_n\|z_n-p\|^2+(1-\alpha_n)\|W_n\phi_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\left\{\|x_n-p\|^2+\left(2\lambda_n m+\lambda_n^2-\frac{2\lambda_n v}{\xi^2}\right)\|Bz_n-Bp\|^2\right\}\\
&=\|x_n-p\|^2+(1-\alpha_n)\left(2\lambda_n m+\lambda_n^2-\frac{2\lambda_n v}{\xi^2}\right)\|Bz_n-Bp\|^2.
\end{aligned} \tag{3.23}$$

Using (3.20) again, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|W_n\theta_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|y_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n.
\end{aligned} \tag{3.24}$$

Substituting (3.23) into (3.24), and using condition (C4) and (C7), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\left\{\|x_n-p\|^2+(1-\alpha_n)\left(2\lambda_n m+\lambda_n^2-\frac{2\lambda_n v}{\xi^2}\right)\|Bz_n-Bp\|^2\right\}\\
&\quad+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\left(2\lambda_n m+\lambda_n^2-\frac{2\lambda_n v}{\xi^2}\right)\|Bz_n-Bp\|^2\\
&\quad+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2+c_n\\
&\le \|x_n-p\|^2+(1-\alpha_n)\left(2\lambda_n m+\lambda_n^2-\frac{2\lambda_n v}{\xi^2}\right)\|Bz_n-Bp\|^2+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)\left(\frac{2av}{\xi^2}-2bm-b^2\right)\|Bz_n-Bp\|^2 &\le (1-\alpha_n)\left(\frac{2\lambda_n v}{\xi^2}-2\lambda_n m-\lambda_n^2\right)\|Bz_n-Bp\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned}$$

Since $c_n \longrightarrow 0$ as $n \longrightarrow \infty$ and (3.16), we obtain

$$\lim_{n\longrightarrow\infty}\|Bz_n-Bp\|=0. \tag{3.25}$$

By (2.1), we also have

$$\begin{aligned}
\|\theta_n - p\|^2 &= \|P_E(y_n - \tau_n By_n) - P_E(p - \tau_n Bp)\|^2 \\
&= \|P_E(I - \tau_n B)y_n - P_E(I - \tau_n B)p\|^2 \\
&\le \Big\langle (I - \tau_n B)y_n - (I - \tau_n B)p, \theta_n - p \Big\rangle \\
&= \frac{1}{2}\Big\{ \|(I - \tau_n B)y_n - (I - \tau_n B)p\|^2 + \|\theta_n - p\|^2 - \|(I - \tau_n B)y_n - (I - \tau_n B)p - (\theta_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|y_n - p\|^2 + \|\theta_n - p\|^2 - \|(y_n - \theta_n) - \tau_n(By_n - Bp)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|\theta_n - p\|^2 - \|y_n - \theta_n\|^2 - \tau_n^2\|By_n - Bp\|^2 + 2\tau_n\langle y_n - \theta_n, By_n - Bp\rangle \Big\},
\end{aligned}$$

which yields that

$$\|\theta_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - \theta_n\|^2 + 2\tau_n\|y_n - \theta_n\|\|By_n - Bp\|. \tag{3.26}$$

Substituting (3.26) into (3.20), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|W_n\theta_n - p\|^2 + (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|\theta_n - p\|^2 + (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\Big\{\|x_n - p\|^2 - \|y_n - \theta_n\|^2 \\
&\quad + 2\tau_n\|y_n - \theta_n\|\|By_n - Bp\|\Big\} + (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})^2\|x_n - p\|^2 - (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|y_n - \theta_n\|^2 \\
&\quad + 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|y_n - \theta_n\|\|By_n - Bp\| + c_n \\
&\le \|x_n - p\|^2 - (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|y_n - \theta_n\|^2 \\
&\quad + 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|y_n - \theta_n\|\|By_n - Bp\| + c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|y_n - \theta_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|y_n - \theta_n\|\|By_n - Bp\| + c_n \\
&\quad \le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad + 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|y_n - \theta_n\|\|By_n - Bp\| + c_n.
\end{aligned}$$

Applying $\|x_{n+1} - x_n\| \longrightarrow 0$, $\|By_n - Bp\| \longrightarrow 0$ and $c_n \longrightarrow \infty$ as $n \longrightarrow \infty$ to the last inequality, we get

$$\lim_{n\longrightarrow\infty} \|y_n - \theta_n\| = 0. \tag{3.27}$$

On the other hand, we have

$$\begin{aligned}
\|W_n\phi_n - p\|^2 &\le \|P_E(z_n - \lambda_n Bz_n) - P_E(p - \lambda_n Bp)\|^2 \\
&= \|P_E(I - \lambda_n B)z_n - P_E(I - \lambda_n B)p\|^2 \\
&\le \Big\langle (I - \lambda_n B)z_n - (I - \lambda_n B)p, W_n\phi_n - p \Big\rangle \\
&= \frac{1}{2}\Big\{ \|(I - \lambda_n B)z_n - (I - \lambda_n B)p\|^2 + \|W_n\phi_n - p\|^2 - \|(I - \lambda_n B)z_n - (I - \lambda_n B)p - (W_n\phi_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|z_n - p\|^2 + \|W_n\phi_n - p\|^2 - \|(z_n - W_n\phi_n) - \lambda_n(Bz_n - Bp)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|W_n\phi_n - p\|^2 - \|z_n - W_n\phi_n\|^2 - \lambda_n^2\|Bz_n - Bp\|^2 + 2\lambda_n\langle z_n - W_n\phi_n, Bz_n - Bp\rangle \Big\},
\end{aligned}$$

which yields that

$$\|W_n\phi_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - W_n\phi_n\|^2 + 2\lambda_n\|z_n - W_n\phi_n\|\|Bz_n - Bp\|. \tag{3.28}$$

Using (3.24) again, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|y_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\Big\{\alpha_n\|z_n-p\|^2+(1-\alpha_n)\|W_n\phi_n-p\|^2\Big\}\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\Big\{\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|W_n\phi_n-p\|^2\Big\}\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|W_n\phi_n-p\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\Big\{\|x_n-p\|^2-\|z_n-W_n\phi_n\|^2\\
&\quad +2\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|\Big\}+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|x_n-p\|^2\\
&\quad -(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|z_n-W_n\phi_n\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-p\|^2\\
&\quad -(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|z_n-W_n\phi_n\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|z_n-W_n\phi_n\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|+c_n\\
&\le \|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|z_n-W_n\phi_n\|^2\\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|+c_n
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|z_n-W_n\phi_n\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|+c_n\\
&\quad \le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\qquad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\lambda_n\|z_n-W_n\phi_n\|\|Bz_n-Bp\|+c_n.
\end{aligned}$$

From (3.16) and (3.25), we obtain

$$\lim_{n\longrightarrow\infty}\|z_n-W_n\phi_n\|=0. \tag{3.29}$$

Note that

$$y_n-W_n\phi_n=\alpha_n(z_n-W_n\phi_n).$$

Since $\alpha_n\longrightarrow\infty$ as $n\longrightarrow\infty$, we also have

$$\lim_{n\longrightarrow\infty}\|y_n-W_n\phi_n\|=0. \tag{3.30}$$

From (3.29) and (3.30), we have

$$\lim_{n\longrightarrow\infty}\|y_n-z_n\|=0. \tag{3.31}$$

On the other hand, we observe that

$$\|z_n-\theta_n\|\le\|z_n-y_n\|+\|y_n-\theta_n\|.$$

Applying (3.27) and (3.31), we have

$$\lim_{n\longrightarrow\infty}\|z_n-\theta_n\|=0. \tag{3.32}$$

Let $p \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(E,B)$. Since $z_n = S_r x_n$ and S_r is firmly nonexpansive (Remark 2.10), we obtain

$$\begin{aligned}\|z_n-p\|^2 &= \|S_r x_n - S_r p\|^2\\ &\le \langle S_r x_n - S_r p, x_n - p\rangle\\ &= \langle z_n - p, x_n - p\rangle\\ &= \frac{1}{2}(\|z_n-p\|^2 + \|x_n-p\|^2 - \|x_n-z_n\|^2).\end{aligned}$$

So, we have

$$\|z_n-p\|^2 \le \|x_n-p\|^2 - \|x_n-z_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-p\|^2 + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2 + c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|(\theta_n-z_n)+(z_n-p)\|^2 + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2 + c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\Big\{\|\theta_n-z_n\|^2 + \|z_n-p\|^2 + 2\langle\theta_n-z_n, z_n-p\rangle\Big\}\\
&\quad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2 + c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2 + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|z_n-p\|^2\\
&\quad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2 + c_n\\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\quad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\Big\{\|x_n-p\|^2 - \|x_n-z_n\|^2\Big\}\\
&\quad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2 + c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\quad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-p\|^2\\
&\quad - (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&\quad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2 + c_n\\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2 - (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&\quad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\quad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + c_n\\
&= \big(1-2\epsilon_n(1+\mu)\bar{\gamma} + \epsilon_n^2(1+\mu)^2\bar{\gamma}^2\big)\|x_n-p\|^2 - (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&\quad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\quad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + c_n\\
&\le \|x_n-p\|^2 + \epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2 - (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&\quad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\quad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + c_n.\end{aligned}$$

It follows that

$$\begin{aligned}&(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2 + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\qquad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + c_n\\
&\quad\le \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + \epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2\\
&\qquad + (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&\qquad + 2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\| + c_n.\end{aligned}$$

Using $\epsilon_n \longrightarrow 0$, $c_n \longrightarrow 0$ as $n \longrightarrow \infty$, (3.16) and (3.32), we obtain

$$\lim_{n\longrightarrow\infty}\|z_n-x_n\|=0. \tag{3.33}$$

Note that

$$\|x_n - \theta_n\| \le \|x_n - z_n\| + \|z_n - \theta_n\|$$

from (3.32) and (3.33), and we have

$$\lim_{n\longrightarrow\infty} \|x_n - \theta_n\| = 0. \tag{3.34}$$

Observe that

$$\|W_n\theta_n - \theta_n\| \le \|W_n\theta_n - x_n\| + \|x_n - \theta_n\|.$$

Applying (3.17) and (3.34), we obtain

$$\lim_{n\longrightarrow\infty} \|W_n\theta_n - \theta_n\| = 0. \tag{3.35}$$

Let W be the mapping defined by (2.4). Since $\{\theta_n\}$ is bounded, applying Lemma 2.14 and (3.35), we have

$$\|W\theta_n - \theta_n\| \le \|W\theta_n - W_n\theta_n\| + \|W_n\theta_n - \theta_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.36}$$

Step 4. We claim that

$$\limsup_{n\longrightarrow\infty}\Big\langle u + \big[\gamma f - (I + \mu A)\big]z, x_n - z\Big\rangle \le 0,$$

where z is a solution of the optimization problem:

$$\text{OP} : \min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x).$$

To show this inequality, we can choose a subsequence $\{\theta_{n_i}\}$ of $\{\theta_n\}$ such that

$$\lim_{i\longrightarrow\infty}\Big\langle u + \big[\gamma f - (I + \mu A)\big]z, \theta_{n_i} - z\Big\rangle = \limsup_{n\longrightarrow\infty}\Big\langle u + \big[\gamma f - (I + \mu A)\big]z, \theta_n - z\Big\rangle. \tag{3.37}$$

Since $\{\theta_{n_i}\}$ is bounded, there exists a subsequence $\{\theta_{n_{i_j}}\}$ of $\{\theta_{n_i}\}$ which converges weakly to $w \in E$. Without loss of generality, we can assume that $\theta_{n_i} \rightharpoonup w$. From $\|W\theta_n - \theta_n\| \longrightarrow 0$, we obtain $W\theta_{n_i} \rightharpoonup w$. Next, we show that $w \in \Gamma$ where $\Gamma := \cap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(E, B)$. First, we prove $w \in \Omega$. Since $z_n = S_r x_n$, we derive

$$\Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\Big\langle K'(z_n) - K'(x_n), \eta(x, z_n)\Big\rangle \ge 0, \quad \forall x \in E.$$

From (H1), we also have

$$\frac{1}{r}\Big\langle K'(z_n) - K'(x_n), \eta(x, z_n)\Big\rangle + \varphi(x) - \varphi(z_n) \ge -\Theta(z_n, x) \ge \Theta(x, z_n),$$

and hence

$$\Big\langle \frac{K'(z_{n_i}) - K'(x_{n_i})}{r}, \eta(x, z_{n_i})\Big\rangle + \varphi(x) - \varphi(z_{n_i}) \ge \Theta(x, z_{n_i}).$$

Since $\frac{K'(z_{n_i}) - K'(x_{n_i})}{r} \longrightarrow 0$ and $z_{n_i} \rightharpoonup w$, from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we also have $\Theta(x, w) + \varphi(w) - \varphi(x) \le 0$ for all $x \in E$. For t with $0 < t \le 1$ and $x \in E$, let $x_t = tx + (1 - t)w$. Since $x \in E$ and $w \in E$, we have $x_t \in E$ and hence $\Theta(x_t, w) + \varphi(w) - \varphi(x_t) \le 0$. From the convexity of equilibrium bifunction $\Theta(x, y)$ in the second variable y, we have

$$\begin{aligned} 0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\ &\le t\Theta(x_t, x) + (1 - t)\Theta(x_t, w) + t\varphi(x) + (1 - t)\varphi(w) - \varphi(x_t) \\ &\le t\big[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)\big], \end{aligned}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \ge 0$. Then, we have $\Theta(w, x) + \varphi(x) - \varphi(w) \ge 0$ for all $x \in E$ and hence $w \in \Omega$.

Next, we show that $w \in \cap_{n=1}^{\infty} F(T_n)$. By Lemma 2.13, we have $F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume that $w \notin F(W)$. Since $\|x_n - \theta_n\| \longrightarrow 0$, we know that $\theta_{n_i} \rightharpoonup w$ $(i \longrightarrow \infty)$ and $w \ne Ww$, it follows by the Opial's condition (Lemma 2.2) that

$$\begin{aligned} \liminf_{i\longrightarrow\infty} \|\theta_{n_i} - w\| &< \liminf_{i\longrightarrow\infty} \|\theta_{n_i} - Ww\| \\ &\le \liminf_{i\longrightarrow\infty}(\|\theta_{n_i} - W\theta_{n_i}\| + \|W\theta_{n_i} - Ww\|) \\ &< \liminf_{i\longrightarrow\infty} \|\theta_{n_i} - w\|, \end{aligned}$$

which is a contradiction. Thus, we get $w \in F(W) = \cap_{n=1}^{\infty} F(T_n)$.

Finally, we show that $w \in VI(E, B)$.
Define

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & w_1 \in E, \\ \emptyset, & w_1 \notin E. \end{cases}$$

Since B is relaxed (m, v)-cocoercive and condition (C7), we have

$$\langle Bx - By, x - y\rangle \geq (-m)\|Bx - By\|^2 + v\|x - y\|^2 \geq (v - m\xi^2)\|x - y\|^2 \geq 0,$$

which yields that B is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_E w_1$ and $\theta_n \in E$, we have

$$\langle w_1 - \theta_n, w_2 - Bw_1\rangle \geq 0.$$

On the other hand, from $\theta_n = P_C(y_n - \tau_n B y_n)$, we have

$$\big\langle w_1 - \theta_n, \theta_n - (y_n - \tau_n B y_n)\big\rangle \geq 0,$$

and hence

$$\left\langle w_1 - \theta_n, \frac{\theta_n - y_n}{\tau_n} + By_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} \langle w_1 - \theta_{n_i}, w\rangle &\geq \langle w_1 - \theta_{n_i}, Bw_1\rangle \\ &\geq \langle w_1 - \theta_{n_i}, Bw_1\rangle - \left\langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - y_{n_i}}{\tau_{n_i}} + By_{n_i} \right\rangle \\ &= \left\langle w_1 - \theta_{n_i}, Bw_1 - By_{n_i} - \frac{\theta_{n_i} - y_{n_i}}{\tau_{n_i}} \right\rangle \\ &= \langle w_1 - \theta_{n_i}, Bw_1 - B\theta_{n_i}\rangle + \langle w_1 - \theta_{n_i}, B\theta_{n_i} - By_{n_i}\rangle - \left\langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - y_{n_i}}{\tau_{n_i}} \right\rangle \\ &\geq \langle w_1 - \theta_{n_i}, B\theta_{n_i} - By_{n_i}\rangle - \left\langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - y_{n_i}}{\tau_{n_i}} \right\rangle. \end{aligned} \tag{3.38}$$

Noting that $\|\theta_{n_i} - y_{n_i}\| \longrightarrow 0$ as $n \longrightarrow \infty$ and B is Lipschitz continuous, hence from (3.38), we obtain

$$\langle w_1 - w, w_2\rangle \geq 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(E, B)$. That is $w \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(E, B)$. Therefore, from Lemma 2.18, $\|x_n - \theta_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ and (3.37), we have

$$\begin{aligned} \limsup_{n\longrightarrow\infty} \big\langle u + [\gamma f - (I + \mu A)]z, x_n - z\big\rangle &= \limsup_{n\longrightarrow\infty} \big\langle u + [\gamma f - (I + \mu A)]z, \theta_n - z\big\rangle \\ &= \lim_{i\longrightarrow\infty} \big\langle u + [\gamma f - (I + \mu A)]z, \theta_{n_i} - z\big\rangle \\ &= \big\langle u + [\gamma f - (I + \mu A)]z, w - z\big\rangle \leq 0. \end{aligned} \tag{3.39}$$

By using (3.17), (3.34) and (3.39), we obtain

$$\limsup_{n\longrightarrow\infty} \big\langle u + [\gamma f - (I + \mu A)]z, W_n\theta_n - z\big\rangle \leq 0. \tag{3.40}$$

Step 5. Finally, we prove that $\{x_n\}$ and $\{z_n\}$ converge strongly to $z \in \Gamma$.
From (3.1), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\epsilon_n\big(u + \gamma f(W_n x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)W_n\theta_n - z\|^2 \\ &= \|\big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)(W_n\theta_n - z) + \beta_n(x_n - z) + \epsilon_n\big(u + \gamma f(W_n x_n) - (I + \mu A)z\big)\|^2 \\ &= \|\big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)(W_n\theta_n - z) + \beta_n(x_n - z)\|^2 + \epsilon_n^2\|u + \gamma f(W_n x_n) - (I + \mu A)z\|^2 \end{aligned}$$

$$
\begin{aligned}
&+2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(W_nx_n) - (I+\mu A)z\Big\rangle \\
&+2\epsilon_n\Big\langle\big((1-\beta_n)I - \epsilon_n(I+\mu A)\big)(W_n\theta_n - z), u + \gamma f(W_nx_n) - (I+\mu A)z\Big\rangle \\
\le\ & \Big[\big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|W_n\theta_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
&+2\beta_n\epsilon_n\gamma\Big\langle x_n - z, f(W_nx_n) - f(z)\Big\rangle + 2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&+2(1-\beta_n)\gamma\epsilon_n\Big\langle W_n\theta_n - z, f(W_nx_n) - f(z)\Big\rangle + 2(1-\beta_n)\epsilon_n\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&-2\epsilon_n^2\Big\langle(I+\mu A)(W_n\theta_n - z), u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
\le\ & \Big[\big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|W_n\theta_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
&+2\beta_n\epsilon_n\gamma\|x_n - z\|\|f(W_nx_n) - f(z)\| + 2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&+2(1-\beta_n)\gamma\epsilon_n\|W_n\theta_n - z\|\|f(W_nx_n) - f(z)\| + 2(1-\beta_n)\epsilon_n\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&-2\epsilon_n^2\Big\langle(I+\mu A)(W_n\theta_n - z), u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
\le\ & \Big[\big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|\theta_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
&+2\beta_n\epsilon_n\gamma\|x_n - z\|\|f(W_nx_n) - f(z)\| + 2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&+2(1-\beta_n)\gamma\epsilon_n\|\theta_n - z\|\|f(W_nx_n) - f(z)\| + 2(1-\beta_n)\epsilon_n\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&-2\epsilon_n^2\Big\langle(I+\mu A)(W_n\theta_n - z), u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
\le\ & \Big[\big(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma}\big)\|x_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
&+2\beta_n\epsilon_n\gamma\alpha\|x_n - z\|^2 + 2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&+2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n - z\|^2 + 2(1-\beta_n)\epsilon_n\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&-2\epsilon_n^2\Big\langle(I+\mu A)(W_n\theta_n - z), u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
=\ & \Big[\big(1-\epsilon_n(1+\mu)\bar{\gamma}\big)^2 + 2\beta_n\epsilon_n\gamma\alpha + 2(1-\beta_n)\gamma\epsilon_n\alpha\Big]\|x_n - z\|^2 + \epsilon_n^2\|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
&+2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle + 2(1-\beta_n)\epsilon_n\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&-2\epsilon_n^2\Big\langle(I+\mu A)(W_n\theta_n - z), u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
\le\ & \Big[1 - 2\big((1+\mu)\bar{\gamma} - \alpha\gamma\big)\epsilon_n]\|x_n - z\|^2 + \epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n - z\|^2 + \epsilon_n^2\|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
&+2\beta_n\epsilon_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle + 2(1-\beta_n)\epsilon_n\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle \\
&+2\epsilon_n^2\|(I+\mu A)(W_n\theta_n - z)\|\|u + \gamma f(z) - (I+\mu A)z\| \\
=\ & \Big[1 - 2\big((1+\mu)\bar{\gamma} - \alpha\gamma\big)\epsilon_n\Big]\|x_n - z\|^2 + \epsilon_n\Bigg\{\epsilon_n\Big[(1+\mu)^2\bar{\gamma}^2\|x_n - z\|^2 + \|u + \gamma f(W_nx_n) \\
&-(I+\mu A)z\|^2 + 2\|(I+\mu A)(W_n\theta_n - z)\|\|u + \gamma f(z) - (I+\mu A)z\|\Big] \\
&+2\beta_n\Big\langle x_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle + 2(1-\beta_n)\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I+\mu A)z\Big\rangle\Bigg\}.
\end{aligned}
$$

Since $\{x_n\}$, $\big\{f(W_nx_n)\big\}$ and $\{W_n\theta_n\}$ are bounded, we can take a constant $M > 0$ such that

$$(1+\mu)^2\bar{\gamma}^2\|x_n - z\|^2 + \|u + \gamma f(W_nx_n) - (I+\mu A)z\|^2 + 2\|(I+\mu A)(W_n\theta_n - z)\|\|u + \gamma f(z) - (I+\mu A)z\| \le M,$$

for all $n \geq 0$. It follows that

$$\|x_{n+1} - z\|^2 \leq (1 - l_n)\|x_n - z\|^2 + \epsilon_n \sigma_n, \tag{3.41}$$

where

$$\begin{aligned} l_n &= 2\big((1+\mu)\bar{\gamma} - \alpha\gamma\big)\epsilon_n, \\ \sigma_n &= \epsilon_n M + 2\beta_n \Big\langle x_n - z, u + \gamma f(z) - (I + \mu A)z \Big\rangle + 2(1 - \beta_n)\Big\langle W_n\theta_n - z, u + \gamma f(z) - (I + \mu A)z \Big\rangle. \end{aligned}$$

By using (C4), (3.39) and (3.40), we get $l_n \longrightarrow 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\longrightarrow\infty} \frac{\sigma_n}{l_n} \leq 0$. Applying Lemma 2.17 and (3.39) to (3.41), we conclude that $x_n \longrightarrow z$ in norm. Finally, noticing $\|z_n - z\| = \|S_r x_n - S_r z\| \leq \|x_n - z\|$. We also conclude that $z_n \longrightarrow z$ in norm. This completes the proof. $\square$

Corollary 3.2. *Let E be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from E to $\mathbb{R}$. Let Θ be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (H1)–(H3) *and let B be a ξ-Lipschitz continuous and relaxed (m, υ)-cocoercive map E into H such that*

$$\Gamma := \Omega \cap VI(E, B) \neq \emptyset.$$

Let $\mu > 0$, $\gamma > 0$ and $r > 0$, which are three constants. Let f be a contraction of E into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$. For given $x_1 \in H$ arbitrarily and fixed $u \in H$, suppose the $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated iteratively by

$$\begin{cases} \Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \dfrac{1}{r}\Big\langle K'(z_n) - K'(x_n), \eta(x, z_n)\Big\rangle \geq 0, \quad \forall x \in E, \\ y_n = \alpha_n z_n + (1 - \alpha_n)P_E(z_n - \lambda_n B z_n), \\ x_{n+1} = \epsilon_n\big(u + \gamma f(x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)P_E(y_n - \tau_n B y_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are three sequences in $(0, 1)$. Assume the following conditions are satisfied:

(C1) *$\eta : E \times E \longrightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that;*
  (a) *$\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in E$,*
  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in E$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(C2) *$K : E \longrightarrow \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma > \lambda\nu$;*

(C3) *for each $x \in E$, there exist a bounded subset $D_x \subset E$ and $z_x \in E$ such that for any $y \in E \setminus D_x$*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\Big\langle K'(y) - K'(x), \eta(z_x, y)\Big\rangle < 0;$$

(C4) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(C5) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;

(C6) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$;

(C7) $\{\tau_n\}, \{\lambda_n\} \subset [a, b]$ *for some a, b with $0 \leq a \leq b \leq \frac{2(\upsilon - m\xi^2)}{\xi^2}$.*

Then, $\{x_n\}$ and $\{z_n\}$ converge strongly to $z \in \Gamma := \Omega \cap VI(E, B)$ provided that S_r is firmly nonexpansive, which solves the following optimization problem:

$$OP : \min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x).$$

Proof. Put $T_n = I$ for all $n \in \mathbb{N}$ and for all $x \in E$. Then $W_n = I$ for all $x \in E$. The conclusion follows from Theorem 3.1. This completes the proof. $\square$

Corollary 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H, let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself and let B be a ξ-Lipschitz continuous and relaxed (m, υ)-cocoercive map E into H such that*

$$\Gamma := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(E, B) \neq \emptyset.$$

Let $\mu > 0$ and $\gamma > 0$, which are two constants. Let f be a contraction of E into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$. For given $x_1 \in H$ arbitrarily and fixed $u \in H$, suppose the $\{x_n\}$ and $\{y_n\}$ are generated iteratively by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)W_n P_E(x_n - \lambda_n B x_n), \\ x_{n+1} = \epsilon_n\big(u + \gamma f(W_n x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)W_n P_E(y_n - \tau_n B y_n), \end{cases}$$

for all $n \in \mathbb{N}$, where W_n be the W-mapping defined by (2.3) and $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are three sequences in $(0, 1)$. Assume the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;
(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(C3) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$;
(C4) $\{\tau_n\}, \{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(\upsilon - m\xi^2)}{\xi^2}$.

Then, $\{x_n\}$ converge strongly to $z \in \Gamma := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(E, B)$, which solves the following optimization problem:

$$OP : \min_{x\in\Gamma} \frac{\mu}{2} \langle Ax, x\rangle + \frac{1}{2} \|x - u\|^2 - h(x).$$

Proof. Put $\Theta(x, y) = \varphi(x) = 0$ for all $x, y \in E$ and $r = 1$. Take $K(x) = \frac{\|x\|^2}{2}$ and $\eta(y, x) = y - x$, for all $x, y \in E$. Then, we get $z_n = P_E x_n = x_n$ in Theorem 3.1. Hence, the conclusion follows. This completes the proof.

Corollary 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself such that*

$$\Gamma := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

Let $\mu > 0$ and $\gamma > 0$, which are two constants. Let f be a contraction of E into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$. For given $x_1 \in H$ arbitrarily and fixed $u \in H$, suppose the $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \epsilon_n\big(u + \gamma f(W_n x_n)\big) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n(I + \mu A)\big)W_n x_n,$$

for all $n \in \mathbb{N}$, where W_n be the W-mapping defined by (2.3) and $\{\epsilon_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Assume the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;
(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then, $\{x_n\}$ converge strongly to $z \in \Gamma := \bigcap_{n=1}^{\infty} F(T_n)$, which solves the following optimization problem:

$$OP : \min_{x\in\Gamma} \frac{\mu}{2} \langle Ax, x\rangle + \frac{1}{2} \|x - u\|^2 - h(x).$$

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Convergence theorems by the viscosity approximation method for equilibrium problems and variational inequality problems[1]

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Abstract. In this paper, we introduce a new iterative scheme for finding the common element of the set of: fixed points of a nonexpansive mapping; equilibrium; and the variational inequality problems for α-inverse-strongly monotone mappings by the viscosity approximation method in a Hilbert spaces. We show that the sequence converges strongly to a common element of the above three sets under some parameter controlling conditions. This main theorem extends a recent result of Chen et al. [5] Su et al. [13] and Yao et al. [19] and many others.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. Let F be a bifunction of $C \times C$

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into R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$. Given a mapping $T : C \longrightarrow H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality problems. Numerous problems in physics, optimization, saddle point problems, complementarity problems, mechanics and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. The classical variational inequality is denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \text{for all} \quad v \in C.$$

The variational inequality has been extensively studied in the literature, see [18, 20] and the references therein.

Recall that the following definitions.

(1) A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \text{for all} \quad u, v \in C. \tag{2}$$

(2) A is called α-inverse-strongly monotone [3, 6] if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \text{for all} \quad u, v \in C. \tag{3}$$

Clearly, every α-inverse-strongly monotone is monotone.

(3) A is said to be β-strongly monotone if there exists a positive real number β such that

$$\langle Au - Av, u - v\rangle \geq \beta\|u - v\|^2, \quad \text{for all} \quad u, v \in C. \tag{4}$$

(4) A is called L-Lipschitz-continuous if there exists a positive real number L such that

$$\|Au - Av\| \leq L\|u - v\|, \quad \text{for all} \quad u, v \in C. \tag{5}$$

It is easy to see that if A is an α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$–Lipschitz continuous.

(5) A mapping $f : C \longrightarrow C$ is said to be a contraction if there exists a coefficient k $(0 < k < 1)$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \quad \text{for all} \quad x, y \in C. \tag{6}$$

To find an element of $F(S) \cap VI(C, A)$, Takahashi and Toyoda [14] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{7}$$

for every $n \geq 0$ where $x_1 = x \in C$, $\{\alpha_n\}$ is a sequence in (0, 1) and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [7] and Zeng and Yao [21] proposed some new iterative schemes for finding elements in $F(S) \cap VI(C, A)$. In 2007, Chen et al. [5] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{8}$$

for every $n \geq 0$, where $x_0 = x \in C, \{\alpha_n\}$ is a sequence in $(0, 1), \{\lambda_n\}$ is a sequence in $(0, 2\alpha), f$ is a contraction on C, S is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. They proved that such a sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping which solves some variational inequality problems. Recently, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see, for instance, [1, 8, 9, 14, 15, 16] and the references therein.

On the other hand, for finding an element of $F(S) \cap VI(C, A) \cap EP(F)$, Su et al. [13] introduced the following iterative scheme by the viscosity approximation method in a Hilbert space: $x_1 \in H$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n), \end{cases} \tag{9}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate conditions. Furthermore, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $z \in F(S) \cap VI(C,A) \cap EP(F)$ where $z = P_{F(S)\cap VI(C,A)\cap EP(F)}f(z)$.

Very recently, Yao et al. [19] also introduced the following iterative scheme:

$$x_{n+1} = \beta x_n + (1-\beta)S[\alpha_n u + (1-\alpha_n)P_C(x_n - \lambda_n A x_n)], \tag{10}$$

for every $n \geq 0$, where $x_1 = u \in C$, $\beta \in (0,1)$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They proved that, if $F(S)\cap VI(C,A) \neq \emptyset$, then the sequence $\{x_n\}$ generate by (10) converges strongly to $F(S) \cap VI(C,A)$.

In this paper motivated by the iterative schemes considered in (8), (9) and (10), we will introduce a new iterative schemes for finding a common element of the set of solutions, the fixed points of a nonexpansive mapping, an equilibrium problem and the variational inequality problem for an α-inverse-strongly monotone mapping by the viscosity approximation method in a real Hilbert space. Then, we prove a strong convergence theorem under the some mild conditions on parameters. The results is connected with Chen et al. [5], Su et al. [13] and Yao et al. [19].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. It is well known that for all $x, y \in H$ and $\lambda \in [0,1]$ there holds

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{11}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{12}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{13}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{14}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{15}$$

We note that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle \\ &\quad +\lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 \\ &\quad +\lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{16}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [10, 11]. It is also known that H satisfies the Opial condition; for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|, \tag{17}$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas will be useful with proving the convergence result of this paper.

Lemma 2.1 [12]. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\longrightarrow\infty} \alpha_n \le \limsup_{n\longrightarrow\infty} \alpha_n < 1$. *Suppose* $x_{n+1} = (1-\alpha_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.

Lemma 2.2 [17]. *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \ n \ge 1,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbf{R}$ *such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(b) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\gamma_n} \le 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \longrightarrow R$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\longrightarrow 0} F(tz + (1-t)x, y) \le F(x,y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2]:

Lemma 2.3 [2]. *Let* C *be a nonempty closed convex subset of* H *and let* F *be a bifunction of* $C \times C$ *into* R *satisfying (A1)-(A4). Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \text{ for all } y \in C.$$

The following lemma was also given in [4].

Lemma 2.4 [4]. *Assume that $F : C \times C \longrightarrow R$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then, the following hold:

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$; *and*

(4) $EP(F)$ *is closed and convex.*

3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions for an equilibrium problem, the set variational inequality and the set of fixed points of a nonexpansive mapping by the viscosity approximation method in Hilbert spaces.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow R$ satisfying (A1)-(A4) and $A : C \longrightarrow H$ be an α-inverse-strongly monotone mapping. Let $f : C \longrightarrow C$ be a contraction with coefficient k $(0 \leq k < 1)$ and S be a nonexpansive mappings of C into itself such that $F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Suppose $x_1 \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \\ y_n = \beta_n f(x_n) + (1 - \beta_n) S P_C(u_n - \lambda_n A u_n), \quad \forall n \geq 1, \end{cases} \tag{18}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequence in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a \leq \lambda_n \leq b < 2\alpha$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions:

(C1) $\lim\limits_{n\longrightarrow\infty}\beta_n = 0$ *and* $\sum\limits_{n=1}^{\infty}\beta_n = \infty$,

(C2) $0 < \liminf\limits_{n\longrightarrow\infty}\alpha_n \leq \limsup\limits_{n\longrightarrow\infty}\alpha_n < 1$,

(C3) $\sum\limits_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$ and $\sum\limits_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(C4) $\liminf\limits_{n\longrightarrow\infty}r_n > 0$ *and* $\sum\limits_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in F(S) \cap VI(C, A) \cap EP(F)$, *where* $q = P_{F(S)\cap VI(C,A)\cap EP(F)}f(q)$.

Proof. Let $Q = P_{F(S)\cap VI(C,A)\cap EP(F)}$. Then Qf is a contraction of H into itself. In fact, there exists $k \in [0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq k\|x - y\|,$$

for all $x, y \in H$. This implies that Qf is a contraction on H into itself. Since H is complete, there exists a unique $q \in H$, such that $q = Qf(q)$. Such a q is an element of C. The unique fixed point of the mapping Qf is denoted by q in the statement of Theorem 3.1. Further, we take $x^* \in F(S)\cap VI(C, A)\cap EP(F)$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lamma 2.4. Then $x^* = P_C(x^* - \lambda_n Ax^*) = T_{r_n}x^*$ and $u_n = T_{r_n}x_n$. Setting $v_n = P_C(u_n - \lambda_n Au_n)$, we have

$$\|u_n - x^*\| \quad = \quad \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|. \tag{19}$$

and hence

$$\begin{aligned}
\|v_n - x^*\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(x^* - \lambda_n Ax^*)\| \\
&\leq \|(u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\| \\
&\leq \|(I - \lambda_n A)u_n - (I - \lambda_n A)x^*\| \\
&\leq \|u_n - x^*\| \leq \|x_n - x^*\|,
\end{aligned}$$

Put M= max $\{\|x_1 - x^*\|, \dfrac{\|f(x^*) - x^*\|}{1-k}\}$. It is obvious that $\|x_1 - x^*\| \leq M$. Suppose $\|x_n - x^*\| \leq M$. Then, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|v_n - x^*\| \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\{\beta_n\|f(x_n) - x^*\| \\
&\quad +(1-\beta_n)\|v_n - x^*\|\} \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\beta_n\|f(x_n) - x^*\| \\
&\quad +(1-\alpha_n)(1-\beta_n)\|v_n - x^*\| \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\beta_n[\|f(x_n) - f(x^*)\| \\
&\quad +\|f(x^*) - x^*\|] + (1-\alpha_n)(1-\beta_n)\|x_n - x^*\| \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\beta_n\|f(x_n) - f(x^*)\| \\
&\quad +(1-\alpha_n)\beta_n\|f(x^*) - x^*\| + (1-\alpha_n)(1-\beta_n)\|x_n - x^*\| \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\beta_n k\|x_n - x^*\| \\
&\quad +(1-\alpha_n)\beta_n\|f(x^*) - x^*\| \\
&\quad +(1-\alpha_n)(1-\beta_n)\|x_n - x^*\| \\
&\leq \{\alpha_n + (1-\alpha_n)\beta_n k + (1-\alpha_n)(1-\beta_n)\}\|x_n - x^*\| \\
&\quad +(1-\alpha_n)\beta_n\|f(x^*) - x^*\| \\
&= \{1-(1-k)(1-\alpha_n)\beta_n\}\|x_n - x^*\| \\
&\quad +(1-k)(1-\alpha_n)\beta_n\frac{\|f(x^*) - x^*\|}{1-k} \\
&\leq \{1-(1-k)(1-\alpha_n)\beta_n\}M + (1-k)(1-\alpha_n)\beta_n M = M.
\end{aligned}$$

So, we have that $\|x_n - x^*\| \leq M$ for all $n \in \mathbf{N}$ and $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{u_n\}$, $\{Sv_n\}$ and $\{f(x_n)\}$ are also bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $I - \lambda_n A$ and P_C are nonexpansive, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|P_C(u_{n+1} - \lambda_{n+1}Au_{n+1}) - P_C(u_n - \lambda_n Au_n)\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_n Au_n)\| \\
&= \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_{n+1}Au_n) \\
&\quad +(\lambda_n - \lambda_{n+1})Au_n\| \\
&= \|(I-\lambda_{n+1}A)u_{n+1} - (I-\lambda_{n+1}A)u_n + (\lambda_n - \lambda_{n+1})Au_n\|
\end{aligned}$$

$$\leq \quad \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|Au_n\|. \tag{20}$$

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \tag{21}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{22}$$

Substituting $y = u_{n+1}$ in to (21) and $y = u_n$ in to (22), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2) we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without the loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|, \end{aligned} \tag{23}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbf{N}\}$. Substituting (20) into (21) we have

$$\begin{aligned} \|v_{n+1} - v_n\| \quad \leq \quad & \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n| \\ & +|\lambda_n - \lambda_{n+1}|\|Au_n\|. \end{aligned} \tag{24}$$

Note that

$$\begin{aligned}
\|y_n - y_{n+1}\| &= \|\beta_n f(x_n) + (1-\beta_n)Sv_n - \beta_{n+1} f(x_{n+1}) \\
&\quad -(1-\beta_{n+1})Sv_{n+1}\| \\
&= \|\beta_n(f(x_n) - f(x_{n+1})) - (1-\beta_n)(Sv_{n+1} - Sv_n) \\
&\quad +(\beta_n - \beta_{n+1})f(x_{n+1}) - (\beta_n - \beta_{n+1})Sv_{n+1}\| \\
&\leq \beta_n\|f(x_n) - f(x_{n+1})\| + (1-\beta_n)\|Sv_{n+1} - Sv_n\| \\
&\quad + |\beta_n - \beta_{n+1}|\|f(x_{n+1})\| + |\beta_n - \beta_{n+1}|\|Sv_{n+1}\| \\
&= \beta_n\|f(x_n) - f(x_{n+1})\| + (1-\beta_n)\|Sv_{n+1} - Sv_n\| \\
&\quad + |\beta_n - \beta_{n+1}|(\|f(x_{n+1})\| + \|Sv_{n+1}\|) \\
&\leq \beta_n k\|x_n - x_{n+1}\| + (1-\beta_n)\|v_{n+1} - v_n\| \\
&\quad +|\beta_n - \beta_{n+1}|(\|f(x_{n+1})\| + \|Sv_{n+1}\|) \\
&\leq \beta_n k\|x_n - x_{n+1}\| + (1-\beta_n)\{\ \|x_{n+1} - x_n\| \\
&\quad +\frac{M_1}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}|\|Au_n\|\} \\
&\quad + |\beta_n - \beta_{n+1}|(\|f(x_{n+1})\| + \|Sy_{n+1}\|) \\
&\leq \beta_n k\|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n| \\
&\quad + |\lambda_n - \lambda_{n+1}|\|Au_n\| + |\beta_n - \beta_{n+1}|(\|f(x_{n+1})\| \\
&\quad +\|Sy_{n+1}\|). \qquad (25)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n - y_{n+1}\| - \|x_{n+1} - x_n\| &\leq \beta_n k\|x_n - x_{n+1}\| + \frac{M_1}{c}|r_{n+1} - r_n| \\
&\quad +|\lambda_n - \lambda_{n+1}|\|Au_n\| \\
&\quad +|\beta_n - \beta_{n+1}|(\|f(x_{n+1})\| + \|Sy_{n+1}\|),
\end{aligned}$$

this together with $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$ and condition (C3), (C4) imply that

$$\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Since $x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n$ and $0 < \liminf_{n\longrightarrow\infty}\alpha_n \leq \limsup_{n\longrightarrow\infty}\alpha_n < 1$. Hence by Lemma 2.1, we obtain

$$\lim_{n\longrightarrow\infty}\|y_n - x_n\| = 0. \qquad (26)$$

It follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = \lim_{n\longrightarrow\infty}(1-\alpha_n)\|y_n-x_n\| = 0. \tag{27}$$

From $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, (23) and (27) we have

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = 0. \tag{28}$$

From $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$ and (27), we obtain

$$\lim_{n\longrightarrow\infty}\|v_{n+1}-v_n\| = 0. \tag{29}$$

Observing condition (C1), (C3), (C4) and $\|x_{n+1}-x_n\|\longrightarrow 0$ as $n\longrightarrow\infty$, we also have

$$\lim_{n\longrightarrow\infty}\|k_{n+1}-k_n\| = 0. \tag{30}$$

Notice that

$$y_n - Sv_n = \beta_n(f(x_n)-Sv_n).$$

we can easily get

$$\lim_{n\longrightarrow\infty}\|y_n-Sv_n\| = 0. \tag{31}$$

On the other hand, we observe

$$\|y_n-Sv_n\| \quad\le\quad \|x_n-y_n\|+\|y_n-Sv_n\|.$$

Applying (26) and (31), we have

$$\lim_{n\longrightarrow\infty}\|x_n-Sv_n\| = 0. \tag{32}$$

Next we show that $\|x_n-u_n\|\longrightarrow 0$, as $n\to\infty$. For each $x^*\in F(S)\cap VI(C,A)\cap EP(F)$, note that T_{r_n} is firmly nonexpansive, then we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|T_{r_n}x_n-T_{r_n}x^*\|^2\le\langle T_{r_n}x_n-T_{r_n}x^*,x_n-x^*\rangle\\ &= \langle u_n-x^*,x_n-x^*\rangle\\ &= \frac{1}{2}(\|u_n-x^*\|^2+\|x_n-x^*\|^2-\|u_n-x_n\|^2). \end{aligned}\tag{33}$$

Then from (33) become

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2. \tag{34}$$

Therefore, from the convexity of $\|\cdot\|^2$, (18) and (34), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|Sv_n - x^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\{\|x_n - x^*\|^2 - \|x_n - u_n\|^2\} \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - (1-\beta_n)\|x_n - u_n\|^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\beta_n\|f(x_n) - x^*\|^2 \\
&\quad +\|x_n - x^*\|^2 - (1-\beta_n)\|x_n - u_n\|^2\} \\
&\leq \|x_n - x^*\|^2 + (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
(1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2 &\leq (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad -\|x_{n+1} - x^*\|^2 \\
&= (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| \\
&\quad +\|x_{n+1} - x^*\|)(\|x_n - x^*\| - \\
&\quad \|x_{n+1} - x^*\|) \\
&\leq (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 \\
&\quad +\|x_n - x_{n+1}\|(\|x_n - x^*\| \\
&\quad +\|x_{n+1} - x^*\|).
\end{aligned}$$

Since $\|x_{n+1} - x_n\| \longrightarrow 0, \alpha_n \longrightarrow 0$ and $\beta_n \longrightarrow 0$, then, we have

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \tag{35}$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, we obtain

$$\lim_{n\longrightarrow\infty} \|\frac{x_n - u_n}{r_n}\| = \lim_{n\longrightarrow\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{36}$$

For $x^* \in F(S) \cap EP(F) \cap VI(A, C)$, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|Sv_n - x^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2 \\
&= \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|P_C(u_n - \lambda_n Au_n) \\
&\quad -P_C(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|(u_n - \lambda_n Au_n) \\
&\quad -(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|(u_n - x^*) \\
&\quad -\lambda_n(Au_n - Ax^*)\|^2 \\
&= \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\{\|u_n - x^*\|^2 \\
&\quad -2\lambda_n\langle u_n - x^*, Au_n - Ax^*\rangle + \lambda_n^2\|Au_n - Ax^*\|^2\} \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\{\|u_n - x^*\|^2 \\
&\quad -2\lambda_n\alpha\|Au_n - Ax^*\|^2 + \lambda_n^2\|Au_n - Ax^*\|^2\} \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\{\|u_n - x^*\|^2 \\
&\quad +\lambda_n(\lambda_n - 2\alpha)\|Au_n - Ax^*\|^2\} \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 \\
&\quad +(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Au_n - Ax^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad +(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Au_n - Ax^*\|^2.
\end{aligned} \tag{37}$$

Using (18) and (38), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\beta_n\|f(x_n) - x^*\|^2 \\
&\quad +\|x_n - x^*\|^2 \\
&\quad +(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Au_n - Ax^*\|^2\} \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2
\end{aligned}$$

$$+(1-\alpha_n)(1-\beta_n)\lambda_n(\lambda_n-2\alpha)\|Au_n-Ax^*\|^2$$

Then we have

$$\begin{aligned}
(1-\alpha_n)(1-\beta_n)a(2\alpha-b)\|Au_n-Ax^*\|^2 &\leq (1-\alpha_n)(1-\beta_n)\\
&\quad \lambda_n(2\alpha-\lambda_n)\\
&\quad \|Au_n-Ax^*\|^2\\
&\leq (1-\alpha_n)\beta_n\|f(x_n)-x^*\|^2\\
&\quad +\|x_n-x^*\|^2\\
&\quad -\|x_{n+1}-x^*\|^2\\
&\leq (1-\alpha_n)\beta_n\|f(x_n)-x^*\|^2\\
&\quad +\|x_{n+1}-x_n\|(\|x_n-x^*\|\\
&\quad +\|x_{n+1}-x^*\|).
\end{aligned}$$

From conditions (C1), (C2), (C3), $\{\lambda_n\}\subset[a,b]\subset(0,2\alpha)$ and $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$, we obtain

$$\lim_{n\longrightarrow\infty}\|Au_n-Ax^*\|=0. \tag{38}$$

From (18) and (12), we have

$$\begin{aligned}
\|v_n-x^*\|^2 &= \|P_C(u_n-\lambda_nAu_n)-P_C(x^*-\lambda_nAx^*)\|^2\\
&\leq \langle(u_n-\lambda_nAu_n)-((x^*-\lambda_nAx^*),v_n-x^*\rangle\\
&= \frac{1}{2}\{\|(u_n-\lambda_nAu_n)-(x^*-\lambda_nA)x^*\|^2+\|v_n-x^*\|^2\\
&\quad -\|(u_n-\lambda_nAu_n)-(x^*-\lambda_nAx^*)-(v_n-x^*)\|^2\}\\
&= \frac{1}{2}\{\|u_n-x^*\|^2++\|v_n-x^*\|^2\\
&\quad -\|(u_n-v_n)-\lambda_n(Au_n-Ax^*)\|^2\}\\
&\leq \frac{1}{2}\{\|u_n-x^*\|^2+\|v_n-x^*\|^2-\|u_n-v_n\|^2\}\\
&\quad +2\lambda_n\langle u_n-y_n,Au_n-Ax^*\rangle-\lambda_n^2\|Au_n-Ax^*\|^2\\
&\leq \frac{1}{2}\{\|x_n-x^*\|^2+\|v_n-x^*\|^2-\|u_n-v_n\|^2\}\\
&\quad +2\lambda_n\langle u_n-v_n,Au_n-Ax^*\rangle-\lambda_n^2\|Au_n-Ax^*\|^2,
\end{aligned}$$

and hence

$$\|v_n-x^*\|^2 \leq \|x_n-x^*\|^2-\|u_n-v_n\|^2$$

$$+2\lambda_n\langle u_n - v_n, Au_n - Ax^*\rangle - \lambda_n^2\|Au_n - Ax^*\|^2. \quad (39)$$

Therefore, from (18) and (39), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|Sv_n - x^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2 \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + (1-\beta_n)\{\|x_n - x^*\|^2 - \|u_n - v_n\|^2 \\
&\quad +2\lambda_n\langle u_n - v_n, Au_n - Ax^*\rangle - \lambda_n^2\|Au_n - Ax^*\|^2\} \\
&\leq \beta_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - (1-\beta_n)\|u_n - v_n\|^2 \\
&\quad +2\lambda_n(1-\beta_n)\langle u_n - v_n, Au_n - Ax^*\rangle \\
&\quad -\lambda_n^2(1-\beta_n)\|Au_n - Ax^*\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\beta_n\|f(x_n) - x^*\|^2 \\
&\quad +\|x_n - x^*\|^2 - (1-\beta_n)\|u_n - y_n\|^2 \\
&\quad +2\lambda_n(1-\beta_n)\langle u_n - v_n, Au_n - Ax^*\rangle \\
&\quad -\lambda_n^2(1-\beta_n)\|Au_n - Ax^*\|^2\} \\
&\leq \|x_n - x^*\|^2 + (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n)\|u_n - v_n\|^2 \\
&\quad +2\lambda_n(1-\beta_n)(1-\alpha_n)\|u_n - v_n\|\|Au_n - Ax^*\| \\
&\quad -\lambda_n^2(1-\beta_n)(1-\alpha_n)\|Au_n - Ax^*\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-\alpha_n)(1-\beta_n)\|u_n - y_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad +(1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 \\
&\quad +2\lambda_n(1-\beta_n)(1-\alpha_n) \\
&\quad \|u_n - v_n\|\|Au_n - Ax^*\| \\
&\leq (1-\alpha_n)\beta_n\|f(x_n) - x^*\|^2 \\
&\quad +\|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad +2\lambda_n(1-\beta_n)(1-\alpha_n)\|u_n - v_n\| \\
&\quad \|Au_n - Ax^*\|.
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$, (C1), (C2) and (38), imply that

$$\lim_{n\longrightarrow\infty}\|u_n-v_n\|=0. \tag{40}$$

Since

$$\|Sv_n-v_n\|\leq\|Sv_n-x_n\|+\|x_n-u_n\|+\|u_n-v_n\|,$$

By (32), (35) and (40) we conclude that

$$\lim_{n\longrightarrow\infty}\|Sv_n-v_n\|=0. \tag{41}$$

Next, we show that

$$\limsup_{n\longrightarrow\infty}\langle f(q)-q,Sv_n-q\rangle\leq 0.$$

Indeed, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle f(q)-q,Sv_n-q\rangle=\lim_{i\longrightarrow\infty}\langle f(q)-q,Sv_{n_i}-q\rangle,$$

where $q=P_{F(S)\cap VI(C,A)\cap EP(F)}f(q)$. Without loss of generality, we may assume that $\{v_{n_i}\}$ converges weakly to $z\in C$. From $\|Sv_n-v_n\|\longrightarrow 0$, we obtain $Sv_{n_i}\rightharpoonup z$. Now, we will show that $z\in F(S)\cap VI(C,A)\cap EP(F)$. Firstly, we will show $z\in F(S)$. Assume that $z\notin F(S)$. Since $v_{n_i}\rightharpoonup z$ and $Sz\neq z$, By the Opial's condition, we obtain

$$\begin{aligned}\liminf_{n\longrightarrow\infty}\|v_{n_i}-z\| &< \liminf_{i\longrightarrow\infty}\|v_{n_i}-Sz\|\\ &= \liminf_{i\longrightarrow\infty}\|v_{n_i}-Sy_{n_i}+Sv_{n_i}-Sz\|\\ &\leq \liminf_{i\longrightarrow\infty}\|v_{n_i}-Sy_{n_i}\|+\|Sv_{n_i}-Sz\|\\ &= \liminf_{i\longrightarrow\infty}\|Sv_{n_i}-Sz\|\\ &\leq \liminf_{i\longrightarrow\infty}\|v_{n_i}-z\|.\end{aligned}$$

This is a contradiction. Thus, we have $z\in F(S)$.

Next, let us show that $z\in VI(C,A)$. Let

$$Tw_1=\begin{cases}Aw_1+N_Cw_1, & w_1\in C;\\ \emptyset, & w_1\notin C.\end{cases}$$

Then T is maximal monotone (see [11]). Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_C(w_1)$ and $v_n \in C$, we have $\langle w_1 - v_n, w_2 - Aw_1\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n Au_n)$, we have

$$\langle w_1 - v_n, v_n - (u_n - \lambda_n Au_n)\rangle \geq 0 \tag{42}$$

that is,

$$\langle w_1 - v_n, \frac{v_n - u_n}{\lambda_n} + Au_n\rangle \geq 0. \tag{43}$$

Therefore, we obtain

$$\begin{aligned}
\langle w_1 - v_{n_i}, w_2\rangle &\geq \langle w_1 - v_{n_i}, Aw_1\rangle \geq \langle w_1 - v_{n_i}, Aw_1\rangle \\
&\quad -\langle w_1 - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\rangle \\
&= \langle w_1 - v_{n_i}, Aw_1 - Au_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&= \langle w_1 - v_{n_i}, Aw_1 - Ay_{n_i}\rangle + \langle w_1 - v_{n_i}, Av_{n_i} - Au_{n_i}\rangle \\
&\quad -\langle w_1 - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&\geq \langle w_1 - v_{n_i}, Av_{n_i}\rangle - \langle w_1 - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\rangle \\
&= \langle w_1 - v_{n_i}, Av_{n_i} - Au_{n_i}\rangle \\
&\quad -\langle w_1 - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle,
\end{aligned} \tag{44}$$

which together $\lim_{n\longrightarrow\infty} \|v_n - u_n\| = 0$ and A is α-inverse-strongly monotone implies that

$$\langle w_1 - z, w_2\rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in VI(C, A)$. Finally, we show that $z \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq F(y, u_{n_i}).$$

From $\|u_n - v_n\| \longrightarrow 0$ and $v_{n_i} \rightharpoonup z$, we have $u_{n_i} \rightharpoonup z$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) that $0 \geq F(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(z, y)$ for all $y \in C$ and hence $z \in EP(F)$. Therefore $z \in F(S) \cap VI(C, A) \cap EP(F)$. Since $q = P_{F(S) \cap VI(C,A) \cap EP(F)} f(q)$, which implies that

$$\begin{aligned} \limsup_{n \longrightarrow \infty} \langle f(q) - q, Sv_n - q \rangle &= \lim_{i \longrightarrow \infty} \langle f(q) - q, Sv_{n_i} - q \rangle \\ &= \langle f(q) - q, z - q \rangle \leq 0. \end{aligned} \tag{45}$$

Finally, we prove that $\{x_n\}$ converge strongly to q. From (18), we also have

$$\begin{aligned} \|y_n - q\|^2 &\leq \|\beta_n(f(x_n) - q) + (1-\beta_n)\|Sv_n - q\|^2 \\ &\leq (1-\beta_n)^2 \|Sv_n - q\|^2 + 2\beta_n \langle f(x_n) - q, y_n - q \rangle \\ &\leq (1-\beta_n)^2 \|v_n - q\|^2 + 2\beta_n [\langle f(x_n) - f(q), y_n - q \rangle \\ &\quad + \langle f(q) - q, y_n - q \rangle] \\ &\leq (1-\beta_n)^2 \|x_n - q\|^2 + 2\beta_n k \|x_n - q\| \|y_n - q\| \\ &\quad + 2\beta_n \langle f(q) - q, y_n - q \rangle \\ &\leq (1-\beta_n)^2 \|x_n - q\|^2 + \beta_n k (\|x_n - q\|^2 + \|y_n - q\|^2) \\ &\quad + 2\beta_n \langle f(q) - q, y_n - q \rangle \\ &= ((1-\beta_n)^2 + \beta_n k) \|x_n - q\|^2 + \beta_n k \|y_n - q\|^2 \\ &\quad + 2\beta_n \langle f(q) - q, y_n - q \rangle, \end{aligned}$$

that is

$$\begin{aligned} \|y_n - q\|^2 &\leq \frac{(1-\beta_n)^2 + \beta_n k}{1 - \beta_n k} \|x_n - q\|^2 \\ &\quad + \frac{2\beta_n}{1 - \beta_n k} \langle f(q) - q, y_n - q \rangle. \end{aligned} \tag{46}$$

From (18) and (47), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \alpha_n\|x_n-q\|^2+(1-\alpha_n)\|y_n-q\|^2\\
&\leq \alpha_n\|x_n-q\|^2+(1-\alpha_n)\{\frac{(1-\beta_n)^2+\beta_nk}{1-\beta_nk}\|x_n-q\|^2\\
&\quad +\frac{2\beta_n}{1-\beta_nk}\langle f(q)-q,y_n-q\rangle\}\\
&= \alpha_n\|x_n-q\|^2+(1-\alpha_n)\frac{(1-\beta_n)^2+\beta_nk}{1-\beta_nk}\|x_n-q\|^2\\
&\quad +(1-\alpha_n)\frac{2\beta_n}{1-\beta_nk}\langle f(q)-q,y_n-q\rangle\\
&\leq \alpha_n\|x_n-q\|^2+\{(1-\alpha_n)-\frac{2(1-k)(1-\alpha_n)\beta_n}{1-\beta_nk}\\
&\quad +\frac{(1-\alpha_n)\beta_n^2}{1-\beta_nk}\}\|x_n-q\|^2\\
&\quad +(1-\alpha_n)\frac{2\beta_n}{1-\beta_nk}\langle f(q)-q,y_n-q\rangle\\
&\leq (1-\frac{2(1-k)(1-\alpha_n)\beta_n}{1-\beta_nk})\|x_n-q\|^2\\
&\quad +\frac{2(1-k)(1-\alpha_n)\beta_n}{1-\beta_nk}\{\frac{\beta_n}{2(1-k)}\|x_n-q\|^2\\
&\quad +\frac{1}{1-k}\langle f(q)-q,y_n-q\rangle\}.
\end{aligned}$$

Put

$$\gamma_n=\frac{2(1-k)(1-\alpha_n)\beta_n}{1-\beta_nk}$$

and

$$\delta_n=\frac{\beta_n}{2(1-k)}\|x_n-q\|^2+\frac{1}{1-k}\langle f(q)-q,y_n-q\rangle.$$

That is

$$\|x_{n+1}-q\|^2 \leq (1-\gamma_n)\|x_n-q\|^2+\gamma_n\delta_n, \tag{47}$$

It is easy to seen that $\gamma_n \longrightarrow 0$, $\sum_{n=1}^{\infty}\gamma_n=\infty$, and $\limsup_{n\longrightarrow\infty}\delta_n\leq 0$. Applying Lemma 2.2 to (47), the sequence $\{x_n\}$ converges strongly to $q=P_{F(S)\cap VI(C,A)\cap EP(F)}f(q)$. Consequently, also $\{u_n\}$ converge strongly to q. This completes the proof. □

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow R$ satisfying (A1)-(A4) and let $f : C \longrightarrow C$ be a contraction with coefficient k $(0 \leq k < 1)$ and S be a nonexpansive mappings of C into itself such that $F(S) \cap EP(F) \neq \emptyset$. Suppose $x_1 \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \\ \\ y_n = \beta_n f(x_n) + (1 - \beta_n) S u_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequence in $[0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions:

(C1) $\lim\limits_{n \longrightarrow \infty} \beta_n = 0$, $\sum\limits_{n=1}^{\infty} \beta_n = \infty$, *and* $\sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C2) $0 < \liminf\limits_{n \longrightarrow \infty} \alpha_n \leq \limsup\limits_{n \longrightarrow \infty} \alpha_n < 1$,

(C3) $\liminf\limits_{n \longrightarrow \infty} r_n > 0$ *and* $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F(S) \cap EP(F)$, where $q = P_{F(S) \cap EP(F)} f(q)$.

Proof. Putting $P_H = I$, by Theorem 3.1, we have the desired result easily.□

4. Applications

In this section, we prove two theorem in Hilbert spaces by using Theorem 3.1. A mapping $T : C \longrightarrow C$ is called strictly pseudo-contraction if there exists a constant $0 \leq \varphi < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \varphi \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

If $\varphi = 0$, then T is nonexpansive. Put $A = I - T$. Then, we have

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \varphi \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Observe that

$$\begin{aligned} \|(I-A)x-(I-A)y\|^2 &= \|x-y\|^2+\|Ax-Ay\|^2 \\ &\quad -2\langle x-y, Ax-Ay\rangle, \quad \forall x,y\in C. \end{aligned}$$

Hence we obtain

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-\varphi}{2}\|Ax-Ay\|^2, \quad \forall x,y\in C. \tag{48}$$

Then, A is $\frac{1-\varphi}{2}$-inverse-strongly monotone.

Now we get the following result.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C \longrightarrow R$ satisfying (A1)-(A4) and $T: C\longrightarrow C$ be a φ-strict pseudo-contraction mapping. Let $f: C\longrightarrow C$ be a contraction with coefficient k $(0\leq k<1)$ and S be a nonexpansive mappings of C into itself such that $F(S)\cap F(T)\cap EP(F)\neq\emptyset$. Suppose $x_1\in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n,y)+\dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in C, \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)y_n, \\ y_n=\beta_n f(x_n)+(1-\beta_n)S((1-\lambda_n)u_n+\lambda_n Tu_n), \quad \forall n\geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequence in [0,1] and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n\in[a,b]$ for some a,b with $0<a\leq\lambda_n\leq b<2\alpha$ and $\{r_n\}\subset(0,\infty)$ satisfying the conditions:

(C1) $\lim_{n\longrightarrow\infty}\beta_n=0$ and $\sum_{n=1}^{\infty}\beta_n=\infty$,

(C2) $0<\liminf_{n\longrightarrow\infty}\alpha_n\leq\limsup_{n\longrightarrow\infty}\alpha_n<1$,

(C3) $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$ and $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,

(C4) $\liminf_{n\longrightarrow\infty} r_n>0$ and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in F(S) \cap F(T) \cap EP(F)$, *where* $q = P_{F(S)\cap F(T)\cap EP(F)} f(q)$.

Proof. Put $A = (I - T) : C \longrightarrow H$, from (48) we know that A is $\dfrac{1-\varphi}{2}$-inverse-strongly monotone mapping. We have $F(T) = VI(C, A)$ and

$$P_C(u_n - \lambda_n A u_n) = P_C((1 - \lambda_n)u_n + \lambda_n T u_n) = (1 - \lambda_n)u_n + \lambda_n T u_n \in C.$$

So, from Theorem 3.1, we obtain the desired result. $\square$

Corollary 4.2. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $T : C \longrightarrow C$ *be a* φ*-strict pseudo-contraction mapping. Let* $f : C \longrightarrow C$ *be a contraction with coefficient* k $(0 \le k < 1)$ *and* S *be a nonexpansive mappings of* C *into itself such that* $F(S) \cap F(T) \neq \emptyset$. *Suppose* $x_1 \in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)[\beta_n f(x_n) + (1-\beta_n)S((1-\lambda_n)x_n + \lambda_n T x_n)], \quad \forall n \ge 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequence in* $[0, 1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *If* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a \le \lambda_n \le b < 2\alpha$ *satisfying the conditions:*

(C1) $\lim\limits_{n\longrightarrow\infty} \beta_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \beta_n = \infty$,

(C2) $0 < \liminf\limits_{n\longrightarrow\infty} \alpha_n \le \limsup\limits_{n\longrightarrow\infty} \alpha_n < 1$,

(C3) $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $q \in F(S) \cap F(T)$, *where* $q = P_{F(S)\cap F(T)} f(q)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$, $r_n = 1$ for all $n \in \mathbf{N}$ and $A = (I - T) : C \longrightarrow H$ in Theorem 3.1. Then, we have $u_n = P_C x_n = x_n$ and A is $\dfrac{1-\varphi}{2}$-inverse-strongly monotone. we have that $F(T)$ is the solution set of $VI(C, A)$ i.e. $F(T) = VI(C, A)$ and

$$P_C(x_n - \lambda_n A x_n) = P_C((1 - \lambda_n)x_n + \lambda_n T x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n \in C.$$

Therefore, by Theorem 3.1, the conclusion follows. $\square$

Theorem 4.3. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow R$ satisfying (A1)-(A4) and $A : C \longrightarrow H$ be an α-inverse-strongly monotone mapping. Let $f : C \longrightarrow C$ be a contraction with coefficient k $(0 \leq k < 1)$ and S be a nonexpansive mappings of C into itself such that $F(S) \cap A^{-1}(0) \cap EP(F) \neq \emptyset$. Suppose $x_1 \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \\ y_n = \beta_n f(x_n) + (1-\beta_n)S(u_n - \lambda_n A u_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a \leq \lambda_n \leq b < 2\alpha$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions:

(C1) $\lim\limits_{n \longrightarrow \infty} \beta_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \beta_n = \infty$,

(C2) $0 < \liminf\limits_{n \longrightarrow \infty} \alpha_n \leq \limsup\limits_{n \longrightarrow \infty} \alpha_n < 1$,

(C3) $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and $\sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C4) $\liminf\limits_{n \longrightarrow \infty} r_n > 0$ *and* $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F(S) \cap A^{-1}(0) \cap EP(F)$, where $q = P_{F(S) \cap A^{-1}(0) \cap EP(F)} f(q)$.

Proof. Since $A^{-1}(0)$ is the solution set of $VI(H, A)$ i.e, $A^{-1}(0) = VI(H, A)$, we can obtain the conclusion by Theorem 3.1 and by noting that $P_H = I$ is the identity on H. It is noted that $F(S) \cap A^{-1}(0) \subset VI(F(S), A)$. This completes the proof. □

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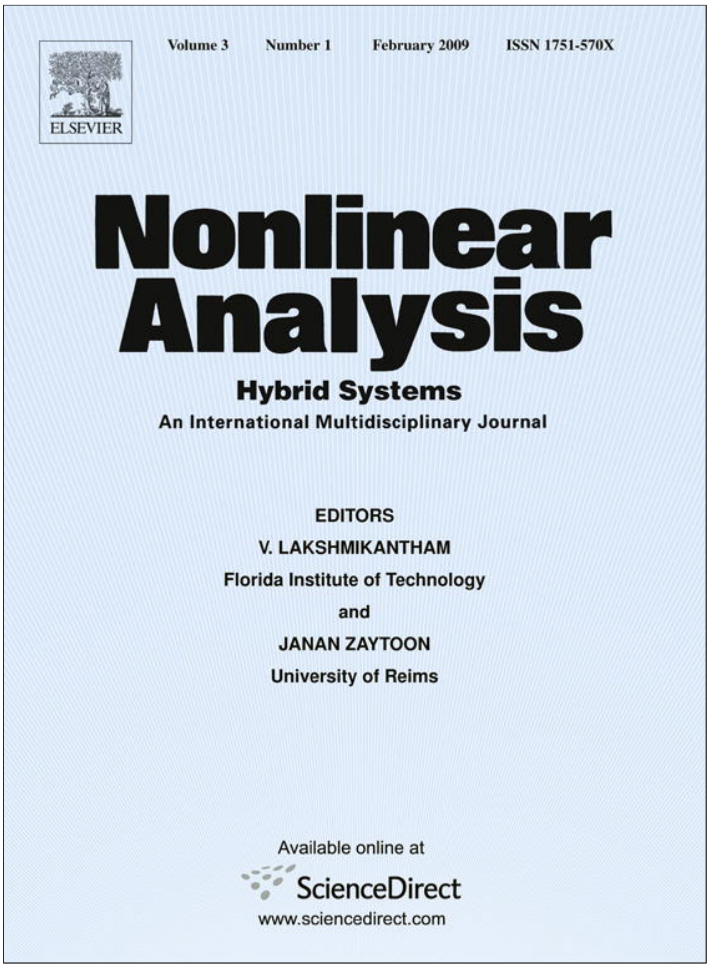
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Strong convergence theorems by a new hybrid projection algorithm for fixed point problems and equilibrium problems of two relatively quasi-nonexpansive mappings ☆

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ABSTRACT

The purpose of this paper is to introduce a new hybrid projection algorithm for finding a common element of the set of common fixed points of two relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Banach spaces. Our results improve and extend the corresponding results announced by Takahashi and Zembayashi [W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. (2008), doi:10.1016/j.na.2007.11.031], Takahashi and Zembayashi [W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory Appl. (2008), doi:10.1155/2008/528476], Qin et al. [X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. (2008), doi:10.1016/j.cam.2008.06.011], Plubtieng and Ungchittrakool [S. Plubtieng, K. Ungchittrakool, Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space, J. Approx. Theory 149 (2007) 103–115].

1. Introduction

Throughout the paper, let E be a real Banach space, E^* the dual space of E. Let C be a nonempty closed convex subset of E. Recall that a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of numbers. The equilibrium problem is to find $p \in C$ such that

$$f(p, y) \ge 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $p \in EP(f)$ if and only if $\langle Tp, y - p\rangle \ge 0$ for all $y \in C$; i.e., p is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [4,9,15]. In 1997 Combettes and Hirstoaga [10] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Recently, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see, for instance, [7,8,14,17,20,21,23,26,28–30] and the references therein. Matsushita and Takahashi [14]

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introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \tag{1.1}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Very recently, Takahashi and Zembayashi [28], proposed the following modification of iteration (1.1) for a relatively nonexpansive mapping:

$$\begin{cases} x_0 = x \in C, \qquad C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSz_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.2}$$

where J is the duality mapping on E, and Π_C is the generalized projection from E onto a closed convex subset C of E and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP(f)} x_0$.

In 2008, Qin et al. [19], introduced the following iterative scheme for two closed relatively quasi-nonexpansive mappings in a Banach space:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{1.3}$$

Under suitable conditions. Some strong convergence theorems are proved which extend and improve the results of Takahashi and Zembayashi [27,28].

Employing the ideas of Qin et al. [19] and Plubtieng and Ungchittrakool [18], we modify iterations (1.2) and (1.3) to obtain strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of two relatively quasi-nonexpansive mappings in the framework Banach spaces. The results obtained in this paper improve and extend the recent ones announced by Qin et al.'s result [19] and Takahashi and Zembayashi's result [28] and many others.

2. Preliminaries

In this section we discuss some results based on the basic properties of a generalized projection, and then we derive some results of relatively quasi-nonexpansive mappings and the equilibrium problem.

Let E be a real Banach space with norm $\|\cdot\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1,2,6,12,25]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that:

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.3}$$

We note that, if E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [6,25] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [22] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be *relatively nonexpansive* [16,24,31] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [3,5]. T is said to be ϕ-nonexpansive, if $\phi(Tx, Ty) \le \phi(x, y)$ for $x, y \in C$. T is said to be *relatively quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for $x \in C$ and $p \in F(T)$ (see also [16]).

Remark 2.1. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [3,5,13] which requires the strong restriction: $F(T) = \widetilde{F(T)}$.

Next, we give some examples which are closed relatively quasi-nonexpansive; (see [19] for more details).

Example 2.2. *Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed relatively quasi-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.*

Example 2.3. *Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.*

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The following lemmas will be needed in proving our main results:

Lemma 2.4 (*Kamimura and Takahashi [12]*). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Concerning the generalized projection, the following are well known.

Lemma 2.5 (*Alber [2]*). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0 \quad \forall y \in C.$$

Lemma 2.6 (*Alber [2]*). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x) \quad \forall y \in C.$$

Lemma 2.7 (*Qin et al. [19, Lemma 2.4.]*). *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed and relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.8 (*Cho et al. [11]*). *Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.9 (*Kamimure and Takahashi [12]*). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \le \phi(x, y)$ for all $x, y \in B_r$.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \le f(x, y);$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 2.10 (*Blum and Oettli [4]*). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying* (A1)–(A4), *and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.11 (*Qin et al. [19, Lemma 2.8.]*). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to R satisfying* (A1)–(A4). *For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \quad \forall y \in C \right\}.$$

Then the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r, JT_r x - JT_r y\rangle \le \langle T_r x - T_r, Jx - Jy\rangle$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.12 (*Takahashi and Zembayashi [28]*). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying* (A1)–(A4), *and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x).$$

3. Main results

In this section, we establish strong convergence theorems for finding a common element of the set of common fixed points of two relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium in the framework of Banach spaces.

Theorem 3.1. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(f) \ne \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ *satisfying the restrictions:*

(a) *$\alpha_n + \beta_n + \gamma_n = 1$;*
(b) *$0 \le \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\limsup_{n \longrightarrow \infty} \alpha_n < 1$;*
(c) *$\liminf_{n \to \infty} \alpha_n \beta_n > 0$, $\liminf_{n \to \infty} \alpha_n \gamma_n > 0$;*
(d) *$\{r_n\} \subset [a, \infty)$ for some $a > 0$.*

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$, where $z = \Pi_F x_0$.

Proof. First, we show that C_n is closed and convex for all $n \ge 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, one obtains that

$$\phi(z, y_k) \le \phi(z, x_k)$$

is equivalent to

$$2(\langle z, Jx_k\rangle - 2\langle z, Jy_k\rangle) \le \|x_k\|^2 - \|y_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \geq 0$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}}x_0$ is well defined. Notice that $u_n = T_{r_n}z_n$ for all $n \geq 0$.

On the other hand, from Lemma 2.11, one has T_{r_n} is relatively quasi-nonexpansive mapping. Next, we prove $F \subset C_n$ for all $n \geq 0$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Then, for $\forall w \in F \subset C_k$, one has:

$$\begin{aligned}\phi(w, z_k) &= \phi(w, J^{-1}(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k)) \\ &= \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2\beta_k\langle w, JTx_k\rangle - 2\gamma_k\langle w, JSx_k\rangle + \|\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k\|^2 \\ &\leq \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2\beta_k\langle w, JTx_k\rangle - 2\gamma_k\langle w, JSx_k\rangle + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\|JSx_k\|^2 \\ &= \alpha_k\phi(w, x_k) + \beta_k\phi(w, Tx_k) + \gamma_k\phi(w, Sx_k) \\ &\leq \phi(w, x_k), \end{aligned} \tag{3.2}$$

and then

$$\begin{aligned}\phi(w, y_k) &= \phi(w, J^{-1}(\delta_k Jx_k + (1-\delta_k)Jz_k)) \\ &= \|w\|^2 - 2\langle w, \delta_k Jx_k + (1-\delta_k)Jz_k\rangle + \|\delta_k Jx_k + (1-\delta_k)Jz_k\|^2 \\ &\leq \|w\|^2 - 2\delta_k\langle w, Jx_k\rangle - 2(1-\delta_k)\langle w, Jz_k\rangle + \delta_k\|x_k\|^2 + (1-\delta_k)\|z_k\|^2 \\ &= \delta_k(\|w\|^2 - 2\langle w, Jx_k\rangle + \|x_k\|^2) + (1-\delta_k)(\|w\|^2 - 2\langle w, Jz_k\rangle + \|z_k\|^2) \\ &= \delta_k\phi(w, x_k) + (1-\delta_k)\phi(w, z_k) \\ &\leq \delta_k\phi(w, x_k) + (1-\delta_k)\phi(w, x_k) \\ &= \phi(w, x_k), \end{aligned} \tag{3.3}$$

that is $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \geq 0$. From $x_n = \Pi_{C_n}x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n \tag{3.4}$$

and

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in F. \tag{3.5}$$

From Lemma 2.6, we obtain

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0),$$

for each $w \in F \subset C_n$ and $n \geq 1$. Then, the sequence $\{\phi(x_n, x_0)\}$ is bounded. Since $x_n = \Pi_{C_n}x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m}x_0 \in C_n$ for any positive integer $m \geq n$. It follows that:

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.6}$$

Letting $m, n \longrightarrow \infty$ in (3.6), one has $\phi(x_m, x_n) \longrightarrow 0$. It follows from Lemma 2.4, that $x_m - x_n \longrightarrow 0$ as $m, n \longrightarrow \infty$. Hence, $\{x_n\}$ is a cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \longrightarrow p \in C$ as $n \longrightarrow \infty$.

Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n}x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0),$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\longrightarrow\infty} \phi(x_{n+1}, x_n) = 0$. From Lemma 2.5, we get $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}}x \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we also have

$$\lim_{n\longrightarrow\infty} \phi(x_{n+1}, u_n) = 0.$$

Since $\lim_{n\longrightarrow\infty} \phi(x_{n+1}, x_n) = \lim_{n\longrightarrow\infty} \phi(x_{n+1}, u_n) = 0$ and E is uniformly convex and smooth, we have from Lemma 2.4 that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \tag{3.7}$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0$, we have

$$\lim_{n\longrightarrow\infty}\|Jx_n-Ju_n\|=0.$$

Since $x_n\longrightarrow p$ as $n\longrightarrow\infty$, we also have $u_n\longrightarrow p$ as $n\longrightarrow\infty$.

Since E is uniformly smooth Banach space, we known that E^* is a uniformly convex Banach apace.

Let $r=\sup_{n\in\mathbb{N}\cup\{0\}}\{\|x_n\|,\|Tx_n\|,\|Sx_n\|\}$. From Lemma 2.8, we have

$$\begin{aligned}\phi(w,u_n) &= \phi(w,T_{r_n}z_n)\le\phi(w,z_n)\\ &= \phi(w,J^{-1}(\alpha_nJx_n+\beta_nJTx_n+\gamma_nJSx_n))\\ &= \|w\|^2-2\alpha_n\langle w,Jx_n\rangle-2\beta_n\langle w,JTx_n\rangle-2\gamma_n\langle w,JSx_n\rangle\\ &\le \|w\|^2-2\alpha\langle w,Jx_n\rangle-2\beta_n\langle w,JTx_n\rangle-2\gamma_n\langle w,JSx_n\rangle\\ &\quad+\alpha_n\|Jx_n\|^2+\beta_n\|JTx_n\|^2+\gamma_n\|JSx_n\|^2-\alpha_n\beta_ng(\|JTx_n-Jx_n\|)\\ &= \alpha_n\phi(w,x_n)+\beta_n\phi(w,Tx_n)+\gamma_n\phi(w,Sx_n)-\alpha_n\beta_ng(\|JTx_n-Jx_n\|)\\ &\le \phi(w,x_n)-\alpha_n\beta_ng(\|JTx_n-Jx_n\|).\end{aligned}\tag{3.8}$$

It follows that

$$\alpha_n\beta_ng(\|JTx_n-Jx_n\|)\le\phi(w,x_n)-\phi(w,u_n).$$

On the other hand, we have

$$\begin{aligned}\phi(w,x_n)-\phi(w,u_n) &= \|x_n\|^2-\|u_n\|^2-2\langle w,Jx_n-Ju_n\rangle\\ &\le \|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|w\|\|Jx_n-Ju_n\|.\end{aligned}$$

It follow from $\|x_n-u_n\|\longrightarrow 0$ and $\|Jx_n-Ju_n\|\longrightarrow 0$ that

$$\phi(w,x_n)-\phi(w,u_n)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{3.9}$$

Observing that assumption $\liminf_{n\longrightarrow\infty}\alpha_n\beta_n>0$ and by Lemma 2.9, we also

$$\lim_{n\longrightarrow\infty}(g\|Jx_n-JTx_n\|)=0.$$

It follows from the property of g that

$$\lim_{n\longrightarrow\infty}\|Jx_n-JTx_n\|=0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\longrightarrow\infty}\|x_n-Tx_n\|=0.$$

Similarly, one can obtain

$$\lim_{n\longrightarrow\infty}\|x_n-Sx_n\|=0.$$

From the closedness of S and T, we have $p\in F(T)\cap F(S)$. Next, we show $p\in EF(f)=F(T_r)$. On the other hand, from (3.3), we have

$$\phi(u,y_n)\le\phi(u,x_n).\tag{3.10}$$

From $u_n=T_{r_n}z_n$ and Lemma 2.12, we obtain

$$\begin{aligned}\phi(u_n,z_n) &= \phi(T_{r_n}z_n,z_n)\\ &\le \phi(w,z_n)-\phi(w,T_{r_n}z_n)\\ &\le \phi(w,x_n)-\phi(w,T_{r_n}z_n)\\ &= \phi(w,x_n)-\phi(w,u_n).\end{aligned}$$

It follows from (3.9) that

$$\phi(u_n,z_n)\to 0,\quad \text{as } n\to\infty.$$

Noticing that Lemma 2.4, we get

$$\|u_n-z_n\|\to 0,\quad \text{as } n\to\infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty}\|Ju_n-Jz_n\|=0.$$

From the assumption $r_n \geq a$, we have

$$\lim_{n\to\infty} \frac{\|Ju_n - Jz_n\|}{r_n} = 0.$$

Noticing that $u_n = T_{r_n} z_n$, we obtain

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C.$$

By condition (A2), we note that

$$\|y - u_n\| \frac{\|Ju_n - Jz_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_n \longrightarrow p$, we have $f(y, p) \leq 0, \quad \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, we obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from (A1) that $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y)$. Hence, $f(y_t, y) \geq 0$. From condition (A3), we obtain $f(p, y) \geq 0$, for $\forall y \in C$. This implies that $p \in EP(f)$. This shows that $p \in F =: F(S) \cap F(T) \cap EP(f)$. Finally, we prove $p = \Pi_F x_0$. By taking limit in (3.4), one has

$$\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in F.$$

At this point, in view of Lemma 2.5, we see that $p = \Pi_F x_0$. By (3.7), $u_n \longrightarrow p$ as $n \longrightarrow \infty$ also. This completes the proof of Theorem 3.1. □

Corollary 3.2 (*Qin et al. [19], Theorem 3.1*). *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.11}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ *satisfying the restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;
(c) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$;
(d) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Setting $\delta_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, then (3.1) reduced to (3.11) and putting $u_n = T_{r_n} y_n$ for $z \in F$, we have $\phi(z, u_n) = \phi(z, T_{r_n} y_n) \leq \phi(z, y_n)$. Therefore, the conclusion follows immediately from Theorem 3.1. □

Corollary 3.3. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $T : C \to C$ be a closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.12}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$, is a sequences in $[0, 1]$ *such that $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\limsup_{n\longrightarrow\infty} \alpha_n < 1$, $\liminf_{n\longrightarrow\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

Proof. In Theorem 3.1 if $S = I$, the identity mapping, then (3.1) reduced to (3.12). $\square$

Corollary 3.4. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{3.13}$$

where J is the duality mapping on E, $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\liminf_{n\longrightarrow\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $P_F x_0$.

Remark 3.5. Corollary 3.4 improves Theorem 3.1 of Takahashi and Zembayashi [28] in the following senses:

(1) from relatively nonexpansive mappings to more general relatively quasi-nonexpansive mappings; that is, we relax the strong restriction: $\widetilde{F(T)} = F(T)$;
(2) the algorithm in Theorem 3.1 is also more general than the one given by Qin et al. [19] and Takahashi and Zembayashi [27].

4. Applications

In Hilbert spaces, we obtain the following results:

Theorem 4.1. *Let C be a nonempty and closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $T, S : C \to C$ be two closed relatively nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \delta_n x_n + (1-\delta_n)z_n, \\ z_n = \alpha_n x_n + \beta_n Tx_n + \gamma_n Sx_n, \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{4.1}$$

where P is a projection from H into its subset. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;
(c) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$;
(d) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_F x_0$.

Proof. Since J is an identity operator, we have

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. Therefore

$$\|Sx - p\| \leq \|x - p\| \Leftrightarrow \phi(p, Sx) \leq \phi(p, x)$$

and

$$\|Tx - p\| \leq \|x - p\| \Leftrightarrow \phi(p, Tx) \leq \phi(p, x)$$

for every $x \in C$ and $p \in F(S)$ and $p \in F(T)$ respectively. Hence, S and T are relatively nonexpansive if and only if S and T are relatively quasi-nonexpansive. Then, by Theorem 3.1, we obtain the result. $\square$

Corollary 4.2. *Let C be a nonempty and closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $T, S : C \to C$ be two closed relatively nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_n + \beta_n T x_n + \gamma_n S x_n, \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{4.2}$$

where P is a projection from H into its subset. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n \longrightarrow \infty} \alpha_n < 1$;
(c) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;
(d) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $P_F x_0$.

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STRONG CONVERGENCE THEOREMS OF MODIFIED ISHIKAWA ITERATIONS FOR TWO FAMILY OF RELATIVELY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove strong convergence theorems of modified Ishikawa iterations for a countable family of two relatively quasi-nonexpansive mappings in Banach spaces. Moreover, we discuss the problem of strong convergence of relatively nonexpansive mappings and we also apply our results to generalize, extend and improve those announced by Qin and Su's result [Strong convergence theorems for relatively nonexpansive mapping in a Banach space, Nonlinear Anal. 67 (2007) 1958–1965.], Nilsrakoo and Saejung's result [Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings, Fixed Point Theory and Appl. (2008), doi:10.1155/2008/312454.] and Su et al.'s result [Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings, Fixed Point Theory and Appl. (2008), doi:10.1155/2008/284613.].

1. Introduction

Let E be a real Banach space, C be a nonempty closed convex subset of E, and $T : C \to C$ be a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all } \ x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : x = Tx\}$. A mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\| \ \text{ for all } \ x \in C \ \text{ and } \ y \in F(T).$$

It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive. Some iteration processes are often used to approximate a fixed

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point of a nonexpansive mapping. The Mann's iterative algorithm was introduced by Mann [9] in 1953. This iteration process is now known as Mann's iteration process, which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

In 1967, Halpern [5] first introduced the following iteration scheme:

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \end{cases} \tag{1.2}$$

see also Browder [3]. He pointed out that the conditions $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sence that, if the iteration 1.2 converges to a fixed point of T, then these conditions must be satisfied.

In 1974, Ishikawa [6] introduced a new iteration scheme, which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0, 1]$.

Matsushita and Takahashi [10] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \tag{1.4}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T. Moreover, Matsushita and Takahashi [11] proposed the following modification of iteration (1.4):

$$\begin{cases} x_0 \in C & \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), & n = 0, 1, 2, \ldots. \end{cases} \tag{1.5}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}(x_0)$.

Many authors have appeared in the literature on Ishikawa's iteration process, see, for example [13, 14, 15] and references therein.

In 2007, Qin and Su [15] proved the following iteration for relatively nonexpansive mappings T in a Banach space E:

$$(1.6) \qquad \begin{cases} x_0 \in C, \qquad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n), \\ Q_n = \{v \in C : \langle Jx_o - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{cases}$$

the sequence $\{x_n\}$ generated by (1.6) converges to $\Pi_{F(T)}x_0$,
and

$$(1.7) \qquad \begin{cases} x_0 \in C, \qquad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n), \\ Q_n = \{v \in C : \langle Jx_o - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{cases}$$

the sequence $\{x_n\}$ generated by (1.7) converges to $\Pi_{F(T)}x_0$.

In 2008, Takahashi et al. [17] proved the following theorem by a hybrid method. We call such a method the shrinking projection method.

Theorem 1.1. (Takahashi et al. [17]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \ne \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$(1.8) \qquad \begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \qquad n \in \mathbb{N}, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Nilsrakoo and Saejung [12] proved the following Mann's iteration process, with C a closed convex bounded subset of Banach spaces:

$$(1.9) \qquad \begin{cases} x_0 \in C, \qquad \text{and } C_{-1} = Q_{-1} = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ C_n = \{v \in C_n : \phi(v, y_n) \le \alpha_n \phi(v, x_n), \\ Q_n = \{v \in C : \langle Jx_o - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x), \quad n = 0, 1, 2, \ldots. \end{cases}$$

converges strongly to a common fixed point of a countable family of relatively nonexpansive mappings.

Recently, Qin and Su [15] and Su et al. [16] extended Takahashi et al.'s theorem [17] to a closed hemi-relatively nonexpansive mapping. They proved a strong convergence theorem by the (CQ) hybrid method. Very recently, Nilsrakoo and Saejung [12], generalized theorem of Su et al. [16, Theorem 3.1]. It is noted that relative quasi-nonexpansiveness considered in the paper and hemi-relative nonexpansiveness of [16] are the same. They do prefer the former name because in a Hilbert space setting, relatively quasi-nonexpansive mappings are just quasi-nonexpansive mappings.

In this paper, motivated by Qin and Su's result [15] and Nilsrakoo and Saejung's result [12] the idea is to modify Ishikawa's iteration process (1.6) and (1.7) for two countable relatively quasi-nonexpansive mappings to have strong convergence theorems in a Banach space by using the shrinking projection method. Our result extends and improves the recent results by Nilsrakoo and Saejung's result [12], Qin and Su [15], Su et al. [16] and Takahashi et al.'s theorem [17] and many other authors.

2. Preliminaries

Let E be a real Banach space with dual E^*. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \tag{2.1}$$

for all $x \in E$.

If C is a nonempty closed convex subset of real Hilbert space H and $P_C : H \to C$ is the metric projection, then P_C is nonexpansive. Alber [1] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue representation of the metric projection in Hilbert spaces.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \qquad \text{for all } x, y \in E. \tag{2.2}$$

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. In the Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\| y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \qquad \text{for all } x, y \in E. \tag{2.3}$$

Remark 2.1. (*Matsushita and Takahashi* [11]). If E is a strictly convex and a smooth Banach space, then for all $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle y, Jx\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$

Lemma 2.2. (*Kamimura and Takahashi* [7]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. The set of fixed points of T is denoted by $F(T)$. A mapping T is said to be relatively quasi-nonexpansive if

$$\phi(p, Tx) \leq \phi(p, x) \qquad \text{for all} \ \ x \in C \quad \text{and} \ \ p \in F(T).$$

A point p in C is said to be an asymptotic fixed point of T [4] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A relatively quasi-nonexpansive mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$. We say that the mapping T is relatively non-expansive if the following conditions are satisfied:
(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x)$ for each $x \in C, p \in F(T)$;
(R3) $F(T) = \hat{F}(T)$.

Lemma 2.3. (*Kamimura and Takahashi* [7]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.4. (*Alber* [1]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \qquad \forall y \in C. \tag{2.4}$$

Lemma 2.5. (*Alber* [1]). *Let E be a reflexive, strict convex, and a smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C. \tag{2.5}$$

Lemma 2.6. (*Nilsrakoo and Saejung* [12, Lemma 2.5]. *Let E be a strictly convex and a smooth real Banach space, let C be a closed convex subset of E, and let T be a relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.7. (*Plutieng and Ungchittrakool* [14]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed convex subset of E. Then, for points $w, x, y, z \in E$ and a real number $a \in \mathbb{R}$, the set $K := \{v \in C : \phi(v, y) \leq \phi(v, x) + \langle v, Jz - Jw\rangle + a\}$ is closed and convex.*

Let C be a subset of Banach space E and let $\{T_n\}$ be a family of mappings from C into E. For a subset B of C, we say that
(i) $(\{T_n\}, B)$ satisfies condition AKTT if

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty;$$

(ii) $(\{T_n\}, B)$ satisfies condition *AKTT if

$$\sum_{n=1}^{\infty} \sup\{\|JT_{n+1}z - JT_n z\| : z \in B\} < \infty;$$

Aoyama et al. [2] prove the following result which is very useful for our main result.

Lemma 2.8. (*Nilsrakoo and Saejung* [12]). *Let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfying condition AKTT, then there exists a mapping $\tilde{T} : B \to E$ such that*

$$\tilde{T}x = \lim_{n \longrightarrow \infty} T_n x \quad \textit{for all} \quad x \in B$$

and $\limsup_{n\longrightarrow\infty}\{\|\tilde{T}z - T_n z\| : z \in B\} = 0$.

Inspired by the preceding Lemma, we have the following result.

Lemma 2.9. (*Nilsrakoo and Saejung* [12]). *Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfies condition *AKTT, then there exists a mapping $\hat{T} : B \to E$ such that*

$$\hat{T}x = \lim_{n \longrightarrow \infty} T_n x \quad \textit{for all} \quad x \in B$$

and $\limsup_{n\longrightarrow\infty}\{\|J\hat{T}z - JT_n z\| : z \in B\} = 0$.

Lemma 2.10. (*Nilsrakoo and Saejung* [12]). *Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Suppose that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Then there exists a mapping $T : C \to E$ such that*

$$Tx = \lim_{n\longrightarrow\infty} T_n x \quad \textit{for all} \quad x \in C$$

Recall that an operator T in a Banach space is closed if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

3. Modify Ishikawa iteration scheme

In this section, we establish a strong convergence theorem for finding common fixed points of a countable family of relatively quasi-nonexpansive mapping in a Banach space. This theorem generalizes recent theorems by Nilsrakoo and Saejung [12] and Su et al. [16].

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty bounded closed convex subset of E. Let $\{S_n\}$ and $\{T_n\}$ be two sequences of relatively quasi-nonexpansive mappings from C into itself, where $F(S) = \bigcap_{n=0}^{\infty} F(S_n)$ and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$; $F(\mathcal{T}) = F(S) \cap F(T)$ is nonempty. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequence in [0,1] such that $\limsup_{n\longrightarrow\infty}\alpha_n < 1$ and $\lim_{n\longrightarrow\infty}\beta_n = 1$, and let $\{x_n\}$ be a sequence in C by the following algorithm:*

$$\left\{\begin{array}{l} x_0 \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n z_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{array}\right. \tag{3.1}$$

*for $n \in \mathbb{N} \cup \{0\}$, where J is the single-valued duality mapping on E. Suppose that for each bounded subset B of C the ordered pair $(\{S_n\}, B)$ and $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Let S and T be two mappings from C into itself defined by $Sv = \lim_{n\longrightarrow\infty} S_n v$ and $Tv = \lim_{n\longrightarrow\infty} T_n v$ for all $v \in C$ and suppose that S and T are closed. If S_n is uniformly continuous for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $\Pi_{F(\mathcal{T})}x_0$, where $\Pi_{F(\mathcal{T})}$ is the generalized projection from C onto $F(\mathcal{T})$.*

Proof. We first show that C_{n+1} is closed and convex for each $n \ge 0$. From the definition of C_{n+1} it is obvious that C_{n+1} is closed for each $n \ge 0$. By Lemma 2.7, C_{n+1} is convex for any $n \ge 0$.

Next, we show that $\bigcap_{n=0}^{\infty} F(T_n) \subset C_n$ for all $n \geq 0$. Indeed, let $p \in \bigcap_{n=0}^{\infty} F(T_n)$, we have

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n z_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JS_n z_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JS_n z_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JS_n z_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|S_n z_n\|^2 \\
&= \alpha_n(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2) \\
&\quad + (1-\alpha_n)(\|p\|^2 - 2\langle p, JS_n z_n\rangle + \|S_n z_n\|^2) \\
&\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, S_n z_n) \\
&\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n).
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_n x_n)) \\
&= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)JT_n x_n\rangle \\
&\quad + \|\beta_n Jx_n + (1-\beta_n)JT_n x_n\|^2 \\
&\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, JT_n x_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|T_n x_n\|^2 \\
&= \beta_n(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2) \\
&\quad + (1-\beta_n)(\|p\|^2 - 2\langle p, JT_n x_n\rangle + \|T_n x_n\|^2) \\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, T_n x_n) \\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) \\
&\leq \phi(p, x_n).
\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\phi(p, y_n) \leq \phi(p, x_n). \tag{3.4}$$

This means that, $p \in C_{n+1}$ for all $n \geq 0$. Thus, $\{x_n\}$ is well defined. Since $x_{n+1} = \Pi_{C_{n+1}} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0),$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing.
By definition of x_n and Lemma 2.5, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, \Pi_{C_n} x_0) \leq \phi(p, x_0), \tag{3.5}$$

for all $p \in F(\mathcal{T}) \subset C_n$. Thus, $\phi(x_n, x_0)$ is bounded. Moreover, by (2.3), we have that $\{x_n\}$ is bounded. So, $\lim_{n\longrightarrow\infty}\phi(x_n, x_0)$ exists. By again Lemma 2.5, we have

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),
\end{aligned}$$

for all $n \geq 0$. Thus, $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Using Lemma 2.2, we have, for m, n such that $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, stricly increasing, and convex function with $g(0) = 0$. then the properties of the function g yield that $\{x_n\}$ is a Cauchy sequence, such that $\{x_n\}$ converges strongly to p for some point p in C. However, since $\lim_{n \longrightarrow \infty} \beta_n = 1$ and $\{x_n\}$ is bounded, we obtain

(3.6)

$$\begin{aligned}
\phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_n J x_n + (1 - \beta_n) J T_n x_n)) \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J x_n + (1 - \beta_n) J T_n x_n \rangle \\
&\quad + \|\beta_n J x_n + (1 - \beta_n) J T_n x_n\|^2 \\
&\leq \|x_{n+1}\|^2 - 2\beta_n \langle x_{n+1}, J x_n \rangle - 2(1 - \beta_n)\langle x_{n+1}, J T_n x_n \rangle \\
&\quad + \beta_n \|x_n\|^2 + (1 - \beta_n)\|T_n x_n\|^2 \\
&= \beta_n \phi(x_{n+1}, x_n) + (1 - \beta_n)\phi(x_{n+1}, T_n x_n).
\end{aligned}$$

Therefore, $\phi(x_{n+1}, z_n) \to 0$ as $n \to \infty$.
Since $x_{n+1} = \Pi_{C_{n+1}}(x_0) \in C_{n+1}$, from the definition of C_n, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n),$$

for all $n \geq 0$. Thus

$$\phi(x_{n+1}, y_n) \to 0, \text{as} \quad n \to \infty.$$

By using Lemma 2.3, we also have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.7}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = 0. \tag{3.8}$$

For each $n \in \mathbb{N} \cup \{0\}$, we observe that

$$\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n J x_n + (1 - \alpha_n) J S_n z_n)\| \\
&= \|\alpha_n (Jx_{n+1} - Jx_n) + (1 - \alpha_n)(Jx_{n+1} - JS_n z_n)\| \\
&= \|(1 - \alpha_n)(Jx_{n+1} - JS_n z_n) - \alpha_n (Jx_n - Jx_{n+1})\| \\
&\geq (1 - \alpha_n)\|Jx_{n+1} - JS_n z_n\| - \alpha_n \|Jx_n - Jx_{n+1}\|.
\end{aligned}$$

It follows that

$$\|Jx_{n+1} - JS_n z_n\| \leq \frac{1}{1 - \alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n \|Jx_n - Jx_{n+1}\|).$$

By (3.8) and $\limsup_{n \longrightarrow \infty} \alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JS_n z_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - S_n z_n\| = 0. \tag{3.9}$$

Since

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

By (3.7), we obtain

$$\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0. \tag{3.10}$$

By using the triangle inequality, we get

$$\|x_n - S_n x_n\| \quad \leq \quad \|x_n - x_{n+1}\| + \|x_{n+1} - S_n z_n\| + \|S_n z_n - S_n x_n\|.$$

Since S_n is uniformly continuous for all $n \in \mathbb{N}$. It follow from (3.7), (3.9) and (3.10) that $\lim_{n\longrightarrow\infty} \|x_n - S_n x_n\| = 0$ and so

$$\lim_{n\longrightarrow\infty} \|Jx_n - JS_n x_n\| = 0.$$

From (3.8), we obtain

$$\begin{aligned}
\|Jx_{n+1} - Jz_n\| &= \|Jx_{n+1} - (\beta_n Jx_n + (1-\beta_n)JT_n x_n)\| \\
&= \|\beta_n(Jx_{n+1} - Jx_n) + (1-\beta_n)(Jx_{n+1} - JT_n x_n)\| \\
&= \|(1-\beta_n)(Jx_{n+1} - JT_n x_n) - \beta_n(Jx_n - Jx_{n+1})\| \\
&\geq (1-\beta_n)\|Jx_{n+1} - JT_n x_n\| - \beta_n\|Jx_n - Jx_{n+1}\|.
\end{aligned}$$

It follows that

$$\|Jx_{n+1} - JT_n x_n\| \leq \frac{1}{1-\beta_n}(\|Jx_{n+1} - Jz_n\| + \beta_n\|Jx_n - Jx_{n+1}\|).$$

By (3.8) and $\beta_n \to 1$, we have

$$\lim_{n\longrightarrow\infty} \|Jx_{n+1} - JT_n x_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - T_n x_n\| = 0.$$

It follows from (3.7) that

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \to 0,$$

and so

$$\lim_{n\longrightarrow\infty} \|Jx_n - JT_n x_n\| = 0.$$

Case 1: $(\{T_n\}, \{x_n\})$ satisfies condition *AKTT. We apply Lemma 2.9 to get

$$\begin{aligned}
\|Jx_n - JTx_n\| &\leq \|Jx_n - JT_n x_n\| + \|JT_n x_n - JTx_n\|, \\
&\leq \|Jx_n - JT_n x_n\| + \sup\{\|JT_n z - JTz\| : z \in \{x_n\}\} \to 0.
\end{aligned}$$

Case 2: $(\{T_n\},\{x_n\})$ satisfies condition AKTT. We apply Lemma 2.8 to get

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\|,\\ &\leq \|x_n - T_nx_n\| + \sup\{\|T_nz - Tz\| : z \in \{x_n\}\} \to 0.\end{aligned}$$

Hence

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = \lim_{n\longrightarrow\infty}\|J^{-1}(Jx_n) - J^{-1}(JTx_n)\| = 0.$$

From both cases, we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0.$$

Similarly, we also have $\lim_{n\longrightarrow\infty}\|x_n - Sx_n\| = 0$.
Since T, S are closed and $x_n \to p$, we have $p \in F(\mathcal{T})$. Moreover, by (3.5), we obtain

$$\phi(p, x_0) = \lim_{n\longrightarrow\infty}\phi(x_n, x_0) \leq \phi(p, x_0),$$

for all $p \in F(\mathcal{T})$. Therefore, $p = \Pi_{F(\mathcal{T})}x_0$. This completes the proof. □

4. Applications in Hilbert spaces

In Hilbert spaces, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same. We obtain the following results:

Theorem 4.1. *Let H be a Hilbert space, and let C be a nonempty bounded closed convex subset of E. Let $\{S_n\}$ and $\{T_n\}$ be two sequences of relatively quasi-nonexpansive mappings from C into itself $F(S) = \bigcap_{n=0}^{\infty} F(S_n)$ and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$; $F(\mathcal{T}) = F(S) \cap F(T)$ is nonempty. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in* [0,1] *such that* $\limsup_{n\longrightarrow\infty}\alpha_n < 1$ *and* $\lim_{n\longrightarrow\infty}\beta_n = 1$*, and let a sequence $\{x_n\}$ in C by the following algorithm:*

$$\tag{4.1} \begin{cases} x_0 \in C, C_0 = C,\\ y_n = \alpha_nx_n + (1-\alpha_n)S_nz_n,\\ z_n = \beta_nx_n + (1-\beta_n)T_nx_n,\\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\},\\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases}$$

for $n \in \mathbb{N}\cup\{0\}$. Suppose that for each bounded subset B of C the ordered pair $(\{S_n\}, B)$ and $(\{T_n\}, B)$ satisfies condition AKTT. Let S and T be two mappings from C into itself defined by $Sv = \lim_{n\longrightarrow\infty}S_nv$ and $Tv = \lim_{n\longrightarrow\infty}T_nv$ for all $v \in C$ and suppose that S and T is closed. Then $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x_0$.

Proof. Since J is an identity operator, we have

$$\phi(x,y) = \|x-y\|^2$$

for every $x, y \in H$. Therefore,

$$\|S_n x - p\| \leq \|x-p\| \Leftrightarrow \phi(p, S_n x) \leq \phi(p,x)$$

and

$$\|T_n x - p\| \leq \|x-p\| \Leftrightarrow \phi(p, T_n x) \leq \phi(p,x)$$

for every $x \in C$ and $p \in F(\mathcal{T})$. Hence, S_n and T_n are quasi-nonexpansive if and only if S_n and T_n are relatively quasi-nonexpansive. Then, by Theorem 3.1, we obtain the result. □

Theorem 4.2. *Let H be a Hilbert space, and let C be a nonempty bounded closed convex subset of E. Let $\{T_n\}$ be a sequence of quasi-nonexpansive mapping from C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in* [0,1] *such that* $\limsup_{n\longrightarrow\infty}\alpha_n < 1$ *and let a sequence $\{x_n\}$ in C be defined by the following algorithm:*

$$\begin{cases} x_0 \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S_n x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases} \tag{4.2}$$

for $n \in \mathbb{N} \cup \{0\}$. Suppose that for each bounded subset B of C the ordered pair $(\{T_n\}, B)$ satisfies condition AKTT. Let T be the mapping from C into itself defined by $Tv = \lim_{n\longrightarrow\infty} T_n v$ for all $v \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. In Theorem 4.1 if $T_n = I$ and $\beta_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$, then (4.1) reduced to (4.2). □

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