รายงานวิจัยฉบับสมบูรณ์

โครงการ : สมบัติเชิงพีชคณิตของกึ่งกรุปการแปลงบนปริภูมิเวกเตอร์

Algebraic Properties of Linear Transformation Semigroups

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

โดย

รองศาสตราจารย์ ดร.ธวัช ช่างผัส
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยขอนแก่น

24 สิงหาคม 2553

สัญญาเลขที่ MRG5180122

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หัวหน้าโครงการวิจัยผู้รับทุน : รองศาสตราจารย์ ดร.ธวัช ช่างผัส
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยขอนแก่น

นักวิจัยที่ปรึกษา : ศาสตราจารย์ ดร. ณรงค์ ปั้นนิ่ม
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยศรีนครินทรวิโรฒ

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ในการทำงานวิจัยค้นคว้าครั้งนี้

ศาสตราจารย์ ดร. ณรงค์ ปั้นนิ่ม นักวิจัยที่ปรึกษาให้กับโครงการนี้ที่อบรมสั่งสอน ถ่ายทอด ความรู้ด้านต่าง ๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

รองศาสตราจารย์ ดร.ธวัช ช่างผัส
หัวหน้าโครงการวิจัย

รหัสโครงการ: MRG5180122
ชื่อโครงการ: สมบัติเชิงพีชคณิตของกึ่งกรุปการแปลงบนปริภูมิเวกเตอร์
ชื่อนักวิจัย: ผู้ช่วยศาสตราจารย์ ดร.ธวัช ช่างผัส
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยขอนแก่น
ชื่อนักวิจัยที่ปรึกษา: ศาสตราจารย์ ดร. ณรงค์ ปั้นนิ่ม
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยศรีนครินทรวิโรฒ
E-mail Address: `thacha@kku.ac.th` (ธวัช ช่างผัส)
`narongp@swu.ac.th` (ณรงค์ ปั้นนิ่ม)
ระยะเวลาโครงการ: พฤษภาคม 2551–เมษายน 2553

บทคัดย่อ: ในงานวิจัยนี้ ศึกษาสมบัติเชิงพีชคณิตของกึ่งกรุปของ hypersubstitutions ประกอบด้วย order ของ hypersubstitutions และ regular hypersubstitutions การวิจัยเน้นที่ hypersubstitutions type $(1,2)$, type (n) และ type $(2,2)$ ผลการวิจัยมีทั้งที่ขยายผลการวิจัยเดิม และเป็นผลที่เกิดขึ้นใหม่

คำหลัก: กึ่งกรุป, hypersubstitutions, regular elements, order

Project Code: MRG5180122

Project Title: Algebraic Properties of Linear Transformation Semigroups

Investigator: Asst. Prof. Dr. Thawhat Changphas
Department of Mathematics, Faculty of Science
Khon Kaen University, Thailand

Mentor: Professor Dr. Narong Punnim
Department of Mathematics, Faculty of Science
Srinakarinwirot, Thailand

E-mail Address: thacha@kku.ac.th (Thawhat Changphas)
narongp@swu.ac.th (Narong Punnim)

Project Period: May 2008–April 2010

Abstract: In this project, we study algebraic properties of a semigroup consisting hypersubstitutions of a given type. These include order of hypersubstitutions and regular hypersubstitutions. We focus on hypersubstitutions of type $(2, 1)$, type (n) and type $(2, 2)$. The result generalize known results and some of them are new results.

Keywords: hypersubstitution, order, regular element, semigroup

Contents

Chapter 1

Executive Summary

Hypersubstitutions were introduced to make precise the concept of hyperidentities. An identity $s \approx t$ of type τ of a variety V of type τ is called a `hyperidentity` of V if, for every substitution of terms of appropriate arity for the operation symbols in $s \approx t$, the resulting identity holds in V. This leads to the definition of a map $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ such that $\sigma(f_i)$ is an n_i-ary term of type τ. Any such mapping σ is called a `hypersubstitution` of type τ.

It is known that the collection of all varieties of a type τ, denoted by $\mathcal{L}(\tau)$, forms a complete lattice under the usual inclusion. Moreover, this lattice is dually isomorphic to the lattice of all equational theories of type τ. It is of interest to know what the lattices $\mathcal{L}(\tau)$ look like, but it has become clear that they are very complicated. Even for type $\tau = (2)$, $\mathcal{L}(2)$ is uncountably infinite. Denecke and M. Reichel have described a method of studying the lattice of all varieties of a given type by using monoids of hypersubstitutions.

Let $\tau = (n_i)_{i \in I}$, $n_i \in \mathbb{N}$, be a type with an operation symbol f_i for each $i \in I$. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables which is disjoint from $\{f_i \mid i \in I\}$. For $n \in \mathbb{N}$, let $X_n = \{x_1, \ldots, x_n\}$ be an n-element alphabet of variables. For each $n \in \mathbb{N}$, the n`-ary terms` of type τ are inductively defined as follows:

(i) every variable $x_i \in X_n$ is an n-ary term of type τ,

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X_n)$ be the smallest set containing $x_1, \ldots, x_n$ which is closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined as the union

$$W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n).$$

The hypersubstitution σ_{id} of type τ is defined by

$$\forall\ i \in I,\ \sigma_{id}(f_i) := f_i(x_1, \ldots, x_{n_i}).$$

Any hypersubstitution σ of type τ uniquely determines a map $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ on $W_\tau(X)$, inductively defined as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$,

(ii) $\hat{\sigma}[t] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

In (ii), if t is an m-ary term of type τ, then $\sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ means $S^{n_i}_m(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$. Using the induced maps $\hat{\sigma}$, a binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$.
For any hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$, $\sigma_1 \circ_h \sigma_2$ is defined by

$$\forall\ i \in I,\ (\sigma_1 \circ_h \sigma_2)(f_i) := \hat{\sigma}_1[\sigma_2(f_i)].$$

Theorem. Let τ be a type. The following hold:

(i) for any $\sigma_1, \sigma_2 \in Hyp(\tau)$, $(\sigma_1 \circ_h \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$,

(ii) the structure $(Hyp(\tau); \circ_h, \sigma_{id})$ forms a monoid. The hypersubstitution σ_{id} acts as the identity of $Hyp(\tau)$.

In this work, we are interested in semigroup properties of hypersubstitutions of a given type. These include regular elements and order of hypersubstitutions. Work has focused on type $\tau = (2, 2)$.

Chapter 2

Main Results

2.1 The order of hypersubstitutions of type (n)

We submitted one paper on this topic. The order of a hypersubstitution σ is defined in the usual way, that is, the order of the cyclic subsemigroup $\langle\sigma\rangle = \{\sigma^n \mid n \in \mathbb{N}\}$ generated by σ. In [1], the auther proved that the order of hypersubstitutions of type (3) is $1, 2, 3$ or infinite.

For any $n \in \mathbb{N}$, the set of all mappings on $\{1, 2, 3, \ldots, n\}$ is denoted by T_n. It is known that T_n forms a semigroup under the usual composition of functions, the so-called a *transformation semigroup*. The semigroup T_n has n^n elements. For $\alpha \in T_n$, let

$$fix(\alpha) = \{x \in \{1, 2, 3, \ldots, n\} : x\alpha = x\}.$$

We give an easy observation that for any $\alpha \in T_n$ if $fix(\alpha) \subset \{1, 2, 3, \ldots, n\}$, then there is $\alpha_0 \in T_{n_0}$ with $n_0 = |\{1, 2, 3, \ldots, n\} \setminus fix(\alpha)|$ which has the same order with α.

Our main result proved that the order of a hypersubstitution of type (n) corresponds to the order of transformation $\alpha \in T_n$ or infinite.

Main Theorem.

(1) The order of a hypersubstitution in $P(n)$ is 1.

(2) The order of a hypersubstitution in $Short(n)$ is equal to the order of a mapping α for some $\alpha \in T_n$.

In [1], we determine all order of hypersubstitutions of type (n). This extend the result [2].

2.2 The order of hypersubstitutions of type $(2,1)$

I is known that the order of hypersubstitutions of type (3) is 1, 2, 3 or infinite and that the order of hypersubstitutions of type $(2,2)$ is 1, 2, 3, 4 or infinite. In this paper, we ask for the order of hypersubstitutions of type $(2,1)$ and of type $(1,2)$. The main result is

Main Theorem. The order of hypersubstitutions of type $(2,1)$ is $1,2,3$ or infinite.

2.3 Regular weak projection hypersubstitutions

We published one paper on this topic. We begin with the notations used in this paper.

Notation. Let $t \in W_{(2,2)}(X_2)$. Since t can be represented by a tree, we address each node in the usual way: the root is labeled by 0, the first node on the left branch starting from the root is labeled by 00, the first node on the right branch starting from the root is labeled by 01, etc. Then we label the different occurrences of the operation symbols f and g as follow: for instant, f_{011111} means the operation symbol position at 011111 is f. We abbreviate $01110\underbrace{11\ldots1}_{k}\underbrace{0\ldots0}_{l}$ by $01^301^k0^l$. For an example see Figure 1:

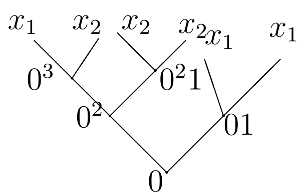

Figure 1.

Assume that $firstop(t) = f$. If g_{01^k} exists for some $k > 1$, we let $Rp'_g(t)$ and $Rp''_g(t)$ denote the parts:

$$f_0f_{01}f_{01^2}\ldots f_{01^{k-1}}g_{01^k}F_{01^k0}F_{01^k01}\ldots F_{01^k01^m}\ldots$$

for some $m \geq 1$ and

$$f_0f_{01}f_{01^2}\ldots f_{01^{k-1}}g_{01^k}F_{01^k0}F_{01^k0^2}\ldots,$$

respectively, $F \in \{f, g\}$. Dually, assume that $firstop(t) = g$. If f_{01^j} exists for some $j > 1$, we let $Lp'_f(t)$ and $Lp''_f(t)$ denote the parts:

$$g_0g_{01}g_{01^2}\ldots g_{01^{j-1}}f_{01^j}F_{01^j0}F_{01^j01}\ldots F_{01^j01^n}\ldots$$

for some $n \geq 1$ and

$$g_0g_{01}g_{01^2}\ldots g_{01^{j-1}}f_{01^j}F_{01^j0}F_{01^j0^2}\ldots,$$

respectively.

Similarly, assume that $firstop(t) = f$. If g_{0^k} exists for some $k > 1$, we let $Lp'_g(t)$ and $Lp''_g(t)$ denote the parts:

$$f_0 f_{0^2} f_{0^3} \dots f_{0^{k-1}} g_{0^k} F_{0^k 1} F_{0^k 10} \dots F_{0^k 10^q} \dots$$

for some $q \geq 1$ and

$$f_0 f_{0^2} f_{0^3} \dots f_{0^{k-1}} g_{0^k} F_{0^k 1} F_{0^k 11} \dots F_{0^k 1^q} \dots$$

respectively, $F \in \{f, g\}$. Assume that $firstop(t) = g$. If f_{0^j} exists for some $j > 1$, we let $Lp'_f(t)$ and $Lp''_f(t)$ denote the parts:

$$g_0 g_{0^2} g_{0^3} \dots g_{0^{j-1}} f_{0^j} F_{0^j 1} F_{0^j 10} \dots F_{0^j 10^r} \dots$$

for some $r \geq 1$ and

$$g_0 g_{0^2} g_{0^3} \dots g_{0^{j-1}} f_{0^j} F_{0^j 1} F_{0^j 11} \dots F_{0^j 1^r} \dots$$

respectively.

Let $FRp'_g(t)$ and $GRp'_g(t)$ denote the number of occurrences of f in $Rp'_g(t)$ and the number of occurrences of g in $Rp'_g(t)$, respectively. $FRp''_g(t)$, $GRp''_g(t)$, $FLp'_g(t)$, $GLp'_g(t)$, $FLp''_g(t)$ and $GLp''_g(t)$ are similarly defined. We define

$$rightmost'_f(t) := rightmost(Rp'_f(t)),\ rightmost''_f(t) := rightmost(Rp''_f(t)),$$

and $leftmost(Lp'_f(t))$, $leftmost(Lp''_f(t))$, $rightmost(Rp'_g(t))$, $leftmost(Lp'_g(t))$, $rightmost(Rp''_g(t))$, $leftmost(Lp''_g(t))$ are similarly defined. For an example, consider the following term

$$t = f_0(g_{00}(x_1, x_2), f_{01}(x_1, g_{01^2}(f_{01^20}(g_{01^20^2}(x_2, x_1), f_{01^201}(x_2, x_1)), x_2)))$$

given by the tree diagram in Figure 2:

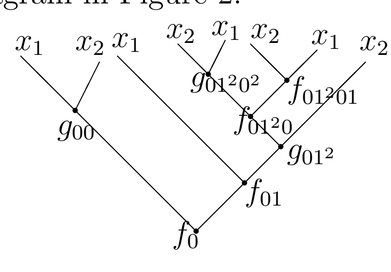

Figure 2.

We have $Rp'_g(t) := f_0 f_{01} g_{01^2} f_{01^20} f_{01^201}$ and $Rp''_g(t) := f_0 f_{01} g_{01^2} f_{01^20} g_{01^20^2}$. Then $GRp''_g(t) = 2$, $FRp''_g(t) = 3$, $rightmost'_g(t) = x_1$, $rightmost''_g(t) = x_2$ and $leftmost'_g(t) = x_2 = leftmost''_g(t)$.

This is one of the main results.

Theorem. Let $b \in W_{(2,2)}(X_2)$ be such that $op(b) > 1$ and $var(b) = \{x_1\}$. Then $\sigma_{x_1,b}$ is regular (in $Hyp(2,2)$) if and only if

(i) $FLp(b) = 1$; or

(ii) $FLp(b) > 1$ and $GLp(b) = 1$; or

(iii) If $FLp(b) = 0$, then $GLp(b) = 1$ or $GRp(b) = 1$ or $FRp(b) = 1$ or $FRp'_f(b) = 1$ or $GRp''_f(b) = 1$; or

(iv) If $FLp(b) > 1$ and $GLp(b) = 0$, then $GRp(b) = 1$ or $FRp(b) = 1$ or $FRp''_g(b) = 1$; or

(v) If $FLp(b) > 1$ and $GLp(b) > 1$, then we have one of the following cases: $GRp(b) = 1$, $FRp(b) = 1$, $GLp'_g(b) = 1$, $FLp'_f(b) = 1$, $FLp''_g(b) = 1$, $GRp'_g(b) = 1$, $FRp''_g(b) = 1$, $GLp''_f(b) = 1$.

Appendix

1 **T. Changphas** and Wonlop Hemvong, Regular weak projection hypersubstitutions, Accepted in Asian-European Journal of Mathematics.

2 **T. Changphas** and Wonlop Hemvong, The order of hypersubstitutions of type $(2,1)$, Supmitted to International Journal of Algebra and Computation (IJAC)

3 **T. Changphas** and Wonlop Hemvong, The order of hypersubstitutions of type (n), Supmitted to the journal Computers and Mathematics with Applications.

Regular Weak Projection Hypersubstitutions

Tawhat Changphas[1] and Wonlop Hemvong

Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand.
email: thacha@kku.ac.th and hwonlop@hotmail.com

Abstract: It is known that there is a Galois connection between submonoids of the monoid consisting of all hypersubstitutions and complete sublattices of the lattice of all varieties of the same type. It is of interest to know which semigroup properties of hypersubstitutions can be transferred by this Galois connection. In this paper, we characterize regular weak projection hypersubstitutions of all mappings from the set of operation symbols to the set of terms which preserve arities type (2,2).

Keywords: Hypersubstitutions, Regular elements, Semigroups
2000 Mathematics Subject Classification: 20M07

1 Preliminaries

In 1991, K. Denecke, D. Lau, R. Pöschel and D. Schweigert [1] defined the concept of a hypersubstitution to make precise the concept of a hyperidentity. It is known that there is a Galois connection between submonoids of the monoid consisting of all hypersubstitutions and complete sublattices of the lattice of all varieties of the same type. It is of interest to know which semigroup properties of hypersubstitutions can be transferred by this Galois connection. Semigroup properties of hypersubstitutions have been widely studied see, for example, [2], [3], [4] and [5]. Properties of monoids of generalized hypersubstitution, i.e. non-arity preserving ones are studied in [6] and [7].

Let $\tau = \{(f_i, n_i) \mid i \in I\}$ be a type. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables such that the sequence of the operation symbols $(f_i)_{i\in I}$ is disjoint with X, and let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an n-element alphabet where $n \in \mathbb{N}$. Here f_i is n_i-ary for a natural number $n_i \geq 1$. An n-ary ($n \geq 1$) `term` of type τ is inductively defined as follows:

(i) every variable $x_j \in X_n$ is an n-ary term,

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the smallest set containing $x_1, \ldots, x_n$ and being closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined as $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$.

Any mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ is called a `hypersubstitution` of type τ if $\sigma(f_i)$ is an n_i-ary term of type τ for every $i \in I$. Any hypersubstitution σ of type τ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$,

(ii) $\hat{\sigma}[t] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

A binary operation $\circ_h$ is defined on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, by

$$(\sigma_1 \circ_h \sigma_2)(f_i) := \hat{\sigma_1}[\sigma_2(f_i)]$$

for all n_i-ary operation symbols f_i. Together with this binary associative operation $Hyp(\tau)$ forms a monoid since the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$

[1]Research Supported by the Thai Research Fund.

is an identity element. Throughout, we will write $\sigma_1\sigma_2$ instead of $\sigma_1 \circ_h \sigma_2$. For notions concerning semigroup properties we refer the reader to [8]. An element e of a semigroup S is said to be `idempotent` if $ee = e$; and an element a of a semigroup S is called `regular` if there exists an $x \in S$ such that $a = axa$. Clearly, every idempotent element is regular. A hypersubstitution mapping every operation symbol to variables is called a `projection hypersubstitution`. Since every projection hypersubstitution is idempotent, we have the following remark.

Remark. Every projection hypersubstitution is regular.

2 Weak Projection Hypersubstitutions

From now on, let f and g be the binary operation symbols of the type $\tau = (2, 2)$. For binary terms a and b of type τ, the hypersubstitution which maps the operation symbol f to the term a and the operation symbol g to the term b will be denoted by $\sigma_{a,b}$. A hypersubstitution $\sigma_{a,b}$ such that $a \in X_2$ or $b \in X_2$ is called a `weak projection hypersubstitution`. Therefore, the concept of a weak projection hypersubstitution generalizes that of a projection hypersubstitution.

Notation. For $t \in W_{(2,2)}(X_2)$, the first variable (from the left) which occurs in t, the last variable which occurs in t, the set of all variables occurring in t, the first operation symbol occurring in t and the number of occurrence of all operation symbols in t are denoted by $leftmost(t)$, $rightmost(t)$, $var(t)$, $firstop(t)$ and $op(t)$, respectively. For an example, if $t = f(f(x_2, x_1), g(g(x_2, x_2), f(x_2, x_1)))$, then $leftmost(t) = x_2$, $rightmost(t) = x_1$, $var(t) = \{x_1, x_2\}$, $firstop(t) = f$ and $op(t) = 5$.

The following result extends the remark above.

Theorem 2.1. *If $\sigma_{a,b} \in Hyp\ (2, 2)$ such that $op(a) \leq 1$ and $op(b) \leq 1$, then $\sigma_{a,b}$ is regular.*

Proof. If $a \in X_2$ and $b \in X_2$, by the remark above, we have $\sigma_{a,b}$ is regular. Assume that $a \in X_2$ and $op(b) = 1$. Then $b \in \{f(x_i, x_j), g(x_i, x_j)\}$, $i, j \in \{1, 2\}$. If $b = f(x_i, x_j)$, we let $u = g(x_i, x_j), v = x_2$. By calculation, we have $\sigma_{a,b}\sigma_{u,v}\sigma_{a,b} = \sigma_{a,b}$. If $b = g(x_i, x_j)$, let $u = x_1, v = g(x_i, x_j)$ and we have $\sigma_{a,b}\sigma_{u,v}\sigma_{a,b} = \sigma_{a,b}$. □

The following results reduce our work.

Theorem 2.2. *Let $b \in W_{(2,2)}(X_2)$.*

(1) *$\sigma_{x_1,b}$ is regular if and only if $\sigma_{x_2,b}$ is regular.*

(2) *$\sigma_{x_1,b}$ is regular if and only if σ_{b,x_1} is regular.*

(3) *$\sigma_{x_2,b}$ is regular if and only if σ_{b,x_2} is regular.*

Proof. (1) Assume that $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2, 2)$ such that $\sigma_{x_1,b} = \sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b}$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]$. Since $\sigma_{x_2,b}(g) = b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = (\sigma_{x_2,b}(\sigma_{x_1,g(x_1,x_2)} \sigma_{u,v})\sigma_{x_2,b})(g)$ we have that $\sigma_{x_2,b}$ is regular. Conversely, assume that $\sigma_{x_2,b}$ is regular. Then there exists $\sigma_{u',v'} \in Hyp(2, 2)$ such that $\sigma_{x_2,b} = \sigma_{x_2,b}\sigma_{u',v'}\sigma_{x_2,b}$. Thus $b = \hat{\sigma}_{x_2,b}[\hat{\sigma}_{u',v'}[b]]$. Since $\sigma_{x_1,b}(g) = b = \hat{\sigma}_{x_2,b}[\hat{\sigma}_{u',v'}[b]] = (\sigma_{x_1,b}(\sigma_{x_2,g(x_1,x_2)}\sigma_{u',v'})\sigma_{x_1,b})(g)$ we have that $\sigma_{x_1,b}$ is regular.

(2) Assume that $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2, 2)$ such that $\sigma_{x_1,b} = \sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b}$. Thus $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]$. Since $\sigma_{b,x_1}(f) = b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = (\sigma_{b,x_1}(\sigma_{x_1,f(x_1,x_2)} \sigma_{u,v})\sigma_{b,x_1})(f)$ we have that σ_{b,x_1} is regular. Conversely, assume that σ_{b,x_1} is regular. Then there exists $\sigma_{u',v'} \in Hyp(2, 2)$ such that $\sigma_{b,x_1} = \sigma_{b,x_1}\sigma_{u',v'}\sigma_{b,x_1}$. Thus $b = \hat{\sigma}_{b,x_1}[\hat{\sigma}_{u',v'}[b]]$. Since $\sigma_{x_1,b}(g) = b = \hat{\sigma}_{b,x_1}[\hat{\sigma}_{u',v'}[b]] = (\sigma_{x_1,b}(\sigma_{g(x_1,x_2),x_1}\sigma_{u',v'})\sigma_{x_1,b})(g)$ we have that $\sigma_{x_1,b}$ is regular.

(3) Assume that $\sigma_{x_2,b}$ is regular. Let $\sigma_{u',v'} \in Hyp(2,2)$ such that $\sigma_{x_2,b} = \sigma_{x_2,b}\sigma_{u',v'}\sigma_{x_2,b}$. Thus $b = \hat{\sigma}_{x_2,b}[\hat{\sigma}_{u',v'}[b]]$. Since $\sigma_{b,x_2}(f) = b = \hat{\sigma}_{x_2,b}[\hat{\sigma}_{u',v'}[b]] = (\sigma_{b,x_2}(\sigma_{x_2,f(x_1,x_2)}\sigma_{u',v'})\sigma_{b,x_2})(f)$ we have that σ_{b,x_2} is regular. Conversely, assume that σ_{b,x_2} is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{b,x_2} = \sigma_{b,x_2}\sigma_{u,v}\sigma_{b,x_2}$. Thus $b = \hat{\sigma}_{b,x_2}[\hat{\sigma}_{u,v}[b]]$. Since $\sigma_{x_2,b}(g) = b = \hat{\sigma}_{b,x_2}[\hat{\sigma}_{u,v}[b]] = (\sigma_{x_2,b}(\sigma_{g(x_1,x_2),x_2}\sigma_{u,v})\sigma_{x_2,b})(g)$ we have that $\sigma_{x_2,b}$ is regular.

As a consequence, $\sigma_{x_1,b}$ is regular if and only if σ_{b,x_2} is regular, $\sigma_{x_2,b}$ is regular if and only if σ_{b,x_1} is regular and σ_{b,x_1} is regular if and only if σ_{b,x_2} is regular. □

Let $b \in W_{(2,2)}(X_2)$. Then we will write $b(x_i, x_j), i, j \in \{1, 2\}$ if x_i and x_j occur in the term b.

Theorem 2.3. *Let $b \in W_{(2,2)}(X_2)$. Then $\sigma_{x_1,b(x_1,x_1)}$ is regular if and only if $\sigma_{x_1,b(x_2,x_2)}$ is regular.*

Proof. Assume first that $\sigma_{x_1,b(x_1,x_1)}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{x_1,b(x_1,x_1)} = \sigma_{x_1,b(x_1,x_1)}\sigma_{u,v}\sigma_{x_1,b(x_1,x_1)}$. Since $b(x_1,x_1) = \hat{\sigma}_{x_1,b(x_1,x_1)}[\hat{\sigma}_{u,v}[b(x_1,x_1)]]$, we get $b(x_2,x_2) = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b(x_2,x_2)]]$. Since

$$\begin{aligned}\hat{\sigma}_{x_1,b(x_2,x_2)}[\hat{\sigma}_{x_1,g(x_1,x_1)}[\hat{\sigma}_{u,v}[b(x_2,x_2)]]] &= \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b(x_2,x_2)]] \\ &= \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]](x_2,x_2) \\ &= b(x_2,x_2),\end{aligned}$$

we have that $\sigma_{x_1,b(x_2,x_2)}$ is regular.

Conversely, assume that $\sigma_{x_1,b(x_2,x_2)}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{x_1,b(x_2,x_2)} = \sigma_{x_1,b(x_2,x_2)}\sigma_{u,v}\sigma_{x_1,b(x_2,x_2)}$. Since $b(x_2,x_2) = \hat{\sigma}_{x_1,b(x_2,x_2)}[\hat{\sigma}_{u,v}[b(x_2,x_2)]]$, $b(x_1,x_1) = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b(x_1,x_1)]]$. Since

$$\begin{aligned}\hat{\sigma}_{x_1,b(x_1,x_1)}[\hat{\sigma}_{x_1,g(x_2,x_2)}[\hat{\sigma}_{u,v}[b(x_1,x_1)]]] &= \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b(x_1,x_1)]] \\ &= \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]](x_1,x_1) \\ &= b(x_1,x_1),\end{aligned}$$

we have that $\sigma_{x_1,b(x_1,x_1)}$ is regular. □

Using Theorem 2.1, Theorem 2.2 and Theorem 2.3, there are two cases to consider:

(1) the regularity of $\sigma_{x_1,b}$ where $b \notin X_2$ and $var(b) = \{x_1\}$,

(2) the regularity of $\sigma_{x_1,b}$ where $b \notin X_2$ and $var(b) = \{x_1, x_2\}$.

In the present paper, we consider the regularity of $\sigma_{x_1,b}$ where $var(b) = \{x_1\}$, i.e. the case (1).

Notation. Let $t \in W_{(2,2)}(X_2)$. Since t can be represented by a tree, we address each node in the usual way: the root is labeled by 0, the first node on the left branch starting from the root is labeled by 00, the first node on the right branch starting from the root is labeled by 01, etc. Then we label the different occurrences of the operation symbols f and g as follow:

for instant, f_{011111} means the operation symbol position at 011111 is f. We abbreviate $01110\underbrace{11\ldots1}_{k}\underbrace{0\ldots0}_{l}$ by $01^301^k0^l$. For an example see Figure 1:

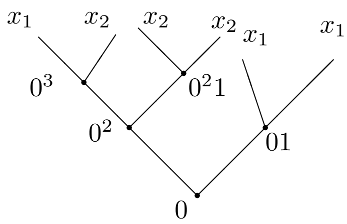

Figure 1: A tree diagram

Assume that $firstop(t) = f$. If g_{01^k} exists for some $k > 1$, we let $Rp'_g(t)$ and $Rp''_g(t)$ denote the parts:

$$f_0 f_{01} f_{01^2} \dots f_{01^{k-1}} g_{01^k} F_{01^k 0} F_{01^k 01} \dots F_{01^k 01^m} \dots$$

for some $m \geq 1$ and

$$f_0 f_{01} f_{01^2} \dots f_{01^{k-1}} g_{01^k} F_{01^k 0} F_{01^k 0^2} \dots,$$

respectively, $F \in \{f, g\}$. Dually, assume that $firstop(t) = g$. If f_{01^j} exists for some $j > 1$, we let $Lp'_f(t)$ and $Lp''_f(t)$ denote the parts:

$$g_0 g_{01} g_{01^2} \dots g_{01^{j-1}} f_{01^j} F_{01^j 0} F_{01^j 01} \dots F_{01^j 01^n} \dots$$

for some $n \geq 1$ and

$$g_0 g_{01} g_{01^2} \dots g_{01^{j-1}} f_{01^j} F_{01^j 0} F_{01^j 0^2} \dots,$$

respectively.

Similarly, assume that $firstop(t) = f$. If g_{0^k} exists for some $k > 1$, we let $Lp'_g(t)$ and $Lp''_g(t)$ denote the parts:

$$f_0 f_{0^2} f_{0^3} \dots f_{0^{k-1}} g_{0^k} F_{0^k 1} F_{0^k 10} \dots F_{0^k 10^q} \dots$$

for some $q \geq 1$ and

$$f_0 f_{0^2} f_{0^3} \dots f_{0^{k-1}} g_{0^k} F_{0^k 1} F_{0^k 11} \dots F_{0^k 1^q} \dots$$

respectively, $F \in \{f, g\}$. Assume that $firstop(t) = g$. If f_{0^j} exists for some $j > 1$, we let $Lp'_f(t)$ and $Lp''_f(t)$ denote the parts:

$$g_0 g_{0^2} g_{0^3} \dots g_{0^{j-1}} f_{0^j} F_{0^j 1} F_{0^j 10} \dots F_{0^j 10^r} \dots$$

for some $r \geq 1$ and

$$g_0 g_{0^2} g_{0^3} \dots g_{0^{j-1}} f_{0^j} F_{0^j 1} F_{0^j 11} \dots F_{0^j 1^r} \dots$$

respectively.

Let $FRp'_g(t)$ and $GRp'_g(t)$ denote the number of occurrences of f in $Rp'_g(t)$ and the number of occurrences of g in $Rp'_g(t)$, respectively. For an example, consider the following term

$$t = f_0(g_{00}(x_1, x_2), f_{01}(x_1, g_{01^2}(f_{01^2 0}(g_{01^2 0^2}(x_2, x_1), f_{01^2 01}(x_2, x_1)), x_2))),$$

given by the tree diagram in Figure 2:

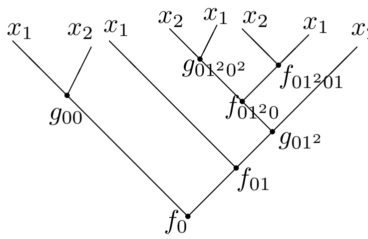

Figure 2: Tree diagram of t

We have $Rp'_g(t) := f_0f_{01}g_{01^2}f_{01^20}f_{01^201}$, $Rp''_g(t) := f_0f_{01}g_{01^2}f_{01^20}g_{01^20^2}$ and $Lp'_g(t) := f_0g_{00} := Lp''_g(t)$. Then $GRp''_g(t) = 2, FRp''_g(t) = 3, GLp'_g(t) = 1 = GLp''_g(t), rightmost(Rp'_g(t)) = x_1, rightmost(Rp''_g(t)) = x_2$ and $lefttmost(Lp'_g(t)) = x_2 = lefttmost(Lp''_g(t))$.

Lemma 2.4. *Let $b, t \in W_{(2,2)}(X_2)$ be such that $op(b) > 1$ and $var(b) = \{x_1\}$. Then $\hat{\sigma}_{x_1,b}[t] = b$ if and only if $GLp(t) = 1$ and $leftmost(t) = x_1$.*

Proof. Assume that $\hat{\sigma}_{x_1,b}[t] = b$. If $GLp(t) = 0$, then $b = \hat{\sigma}_{x_1,b}[t] \in X_2$, this contradicts to $op(b) > 1$. If $GLp(t) > 1$, then $\hat{\sigma}_{x_1,b}[t] = b(b(t_1, t_2), t_3)$ for some $t_1, t_2, t_3 \in W_{(2,2)}(X_2)$. Since $var(b) = \{x_1\}$, $op(\hat{\sigma}_{x_1,b}[t]) = op(b(b(t_1, t_2), t_3)) > op(b)$, we have a contradiction. Then $GLp(t) = 1$. If $leftmost(t) = x_2$, then $b = \hat{\sigma}_{x_1,b}[t] = b(x_2, x_2)$, a contradiction. Hence $leftmost(t) = x_1$.

Conversely, assume that $GLp(t) = 1$ and $leftmost(t) = x_1$. Since $var(b) = \{x_1\}$, we get $\hat{\sigma}_{x_1,b}[t] = b$. □

For convenience, if $\sigma_{x_1,b}$ is regular, we let $\sigma_{u,v} \in Hyp(2, 2)$ such that $\sigma_{x_1,b} = \sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b}$. Thus $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = b$. By Lemma 2.4,

$$GLp(\hat{\sigma}_{u,v}[b]) = 1 \text{ and } leftmost(\hat{\sigma}_{u,v}[b]) = x_1. \tag{2.1}$$

This notion will be used in the proof of Lemma 2.5 - Lemma 2.9

Lemma 2.5. *Let $b \in W_{(2,2)}(X_2)$ be such that $FLp(b) = 0$, $GLp(b) > 1$, $GRp(b) > 1$, $FRp(b) > 1$, $FRp'_f(b) > 1$ and $GRp''_f(b) > 1$. Then $\sigma_{x_1,b}$ is not regular.*

Proof. Suppose that $leftmost(v) = x_1$. If $GLp(v) = 0$, by $FLp(b) = 0$, then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction. If $GLp(v) \geq 1$, by $FLp(b) = 0$ and $GLp(b) > 1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$. This contradicts to (2.1). Therefore, $leftmost(v) = x_2$. Suppose that $leftmost(u) = x_2$. Since $FRp(b) > 1$ and $GRp(b) > 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$ or $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$. In both cases we get a contradiction. Then $leftmost(u) = x_1$. Now, there are two cases to consider.

Case 1: $b = g(t_1, f(g(t_2, f(t_3, t_4)), t_5))$ for some $t_1, t_2, t_3, t_4, t_5 \in W_{(2,2)}(X_2)$. Then $\hat{\sigma}_{u,v}(b) = v(t'_1, u(v(t'_2, v(t'_3, t'_4)), t'_5))$ for some $t'_1, t'_2, t'_3, t'_4, t'_5 \in W_{(2,2)}(X_2)$. If $GLp(u) = 0$ and $GLp(v) = 0$, then $b = \hat{\sigma}_{x_1,b}[t] \in X_2$, a contradiction. If $GLp(u) \geq 1$ or $GLp(v) \geq 1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction.

Case 2: $b = g(s_1, f(f((g(s_2, s_3), s_4)), s_5))$ for some $s_1, s_2, s_3, s_4, s_5 \in W_{(2,2)}(X_2)$. Then $\hat{\sigma}_{u,v}(b) = v(s'_1, u(u((v(s'_2, s'_3), s'_4)), s'_5))$ for some $s'_1, s'_2, s'_3, s'_4, s'_5 \in W_{(2,2)}(X_2)$. If $GLp(u) = 0$ and $GLp(v) = 0$, then $b \in X_2$, a contradiction. If $GLp(u) \geq 1$ or $GLp(v) \geq 1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. □

Lemma 2.6. *Let $b \in W_{(2,2)}(X_2)$ be such that $FLp(b) > 1$, $GLp(b) = 0$, $FRp(b) > 1$, $GRp(b) > 1$ and $FRp''_g(b) > 1$. Then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. If $GLp(u) = 0$, then $GLp(\hat{\sigma}_{u,v}[b]) = 0$ because $GLp(b) = 0$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction. Thus $GLp(u) \geq 1$. If $leftmost(u) = x_1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$ (since $FLp(b) > 1$). This contradicts to (2.1). Therefore, $leftmost(u) = x_2$. If $b = f(f(s_1, s_2), f(s_3, s_4))$ for some $s_1, s_2, s_3, s_4 \in W_{(2,2)}(X_2)$, then $\hat{\sigma}_{u,v}(b) = u(u(s'_1, s'_2), u(s'_3, s'_4))$ for some $s'_1, s'_2, s'_3, s'_4 \in W_{(2,2)}(X_2)$. Since $GLp(u) \geq 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. If $b = f(f(t_1, t_2), g(f(t_3, t_4), f(t_5, t_6)))$ for some $t_1, t_2, t_3, t_4, t_5, t_6 \in W_{(2,2)}(X_2)$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. There are three cases to consider.

Case 1: $b = f(f(t_1, t_2), g(f(t_3, t_4), g(t_5, t_6)))$ for some $t_1, t_2, t_3, t_4, t_5, t_6 \in W_{(2,2)}(X_2)$. If $leftmost(v) = x_1$, then, since $GLp(u) \geq 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$. We get a contradiction. Thus $leftmost(v) = x_2$. Since $FRp(b) > 1, leftmost(u) = x_2$ and $leftmost(v) = x_1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$. We get a contradiction.

Case 2: $b = f(f(t_1, t_2), g(g(t_3, t_4), f(t_5, t_6)))$ for some $t_1, t_2, t_3, t_4, t_5, t_6 \in W_{(2,2)}(X_2)$. This can be considered in the same manner as Case 1.

Case 3: $b = f(f(t_1, t_2), g(g(t_3, t_4), g(t_5, t_6)))$ for some $t_1, t_2, t_3, t_4, t_5, t_6 \in W_{(2,2)}(X_2)$. If $leftmost(v) = x_2$, since $FRp(b) > 1$ and $GRp(b) > 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$. We get a contradiction. Thus $leftmost(v) = x_1$. Since $FRp''_g(b) > 1$, we get $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. □

Lemma 2.7. *Let $b \in W_{(2,2)}(X_2)$ be such that $FLp(b) > 1$, $GLp(b) > 1$, $FRp(b) > 1$, $GRp(b) = 0$, $GLp'_g(b) > 1$ and $FLp''_g(b) > 1$ Then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. We consider into four cases.

Case 1: $GLp(u) = 0$ and $GLp(v) = 0$. Then $GLp(\hat{\sigma}_{u,v}[b]) = 0$. Thus $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 2: $GLp(u) \geq 1$ and $GLp(v) \geq 1$. Then $leftmost(u) = x_1$ and $leftmost(v) = x_2$. Since $GLp'_g(b) > 1$ and $FLp''_g(b) > 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) \neq 1$. This contradicts to (2.1).

Case 3: $GLp(u) = 0$ and $GLp(v) \geq 1$. Since $FRp(b) > 1$ and $GRp(b) = 0$, we have $leftmost(u) = x_1$. Since $GLp(b) > 1$, we get $leftmost(v) = x_2$. Since $GLp'_g(b) > 1$ we have $GLp(\hat{\sigma}_{u,v}[b]) \neq 1$. This contradicts to (2.1).

Case 4: $GLp(u) \geq 1$ and $GLp(v) = 0$. Then $leftmost(u) = x_1$. This gives $leftmost(u) = x_2$. Since $FLp''_g(b) > 1$ and $GLp(u) \geq 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) \neq 1$. This contradicts (2.1). □

Lemma 2.8. *Let $b \in W_{(2,2)}(X_2)$ be such that $firstop(b) = f$, $FLp(b) > 1$, $GLp(b) > 1$, $GRp(b) > 1$, $FRp(b) > 1$, $GRp'_g(b) > 1$, $GLp'_g(b) > 1$, $FRp''_g(b) > 1$ and $FLp''_g(b) > 1$. Then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. We consider into four cases.

Case 1: $GLp(u) = 0$ and $GLp(v) = 0$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 2: $GLp(u) \geq 1$ and $GLp(v) \geq 1$. Then $GLp(\hat{\sigma}_{u,v}[b]) \neq 1$, this contradicts to (2.1).

Case 3: $GLp(u) = 0$ and $GLp(v) \geq 1$. If $leftmost(u) = x_1$ and $leftmost(v) = x_1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. Similarly, if $leftmost(u) = x_1$ and $leftmost(v) = x_2$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. If $leftmost(u) = x_2$ and $leftmost(v) = x_2$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. If $leftmost(u) = x_2$ and $leftmost(v) = x_1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction.

Case 4: $GLp(u) \geq 1$ and $GLp(v) = 0$. If $leftmost(u) = x_1$ and $leftmost(v) = x_1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. Similarly, if $leftmost(u) = x_1$ and $leftmost(v) = x_2$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. If $leftmost(u) = x_2$ and $leftmost(v) = x_1$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction because $FRp(b) > 1$ and $FRp''_g(b) > 1$. If $leftmost(u) = x_2$ and $leftmost(v) = x_2$, then $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. □

Lemma 2.9. *Let $b \in W_{(2,2)}(X_2)$ be such that $FLp(b) > 1$, $GLp(b) > 1$, $GRp(b) > 1$, $firstop(b) = g$, $FLp'_f(b) > 1$ and $GLp''_f(b) > 1$. Then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. There are four cases to consider.

Case 1: $GLp(u) = 0$ and $GLp(v) = 0$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 2: $GLp(u) \geq 1$ and $GLp(v) \geq 1$. Then $GLp(\hat{\sigma}_{u,v}[b]) \neq 1$, this contradicts to (2.1).

Case 3: $GLp(u) = 0$ and $GLp(v) \geq 1$. Since $GRp(b) > 1$, we get $leftmost(v) = x_1$. This gives $leftmost(u) = x_2$. Since $GLp(v) > 1$ and $GLp''_f(b) > 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction.

Case 4: $GLp(u) \geq 1$ and $GLp(v) = 0$. Then $leftmost(v) = x_1$. Since $GLp(u) \geq 1$, we get $leftmost(u) = x_2$. Since $GLp(u) \geq 1$ and $FLp'_f(b) > 1$, we have $GLp(\hat{\sigma}_{u,v}[b]) > 1$, a contradiction. □

Now, we prove the following main result.

Theorem 2.10. *Let $b \in W_{(2,2)}(X_2)$ be such that $op(b) > 1$ and $var(b) = \{x_1\}$. Then $\sigma_{x_1,b}$ is regular (in $Hyp(2,2)$) if and only if*

(i) *$FLp(b) = 1$; or*

(ii) *$FLp(b) > 1$ and $GLp(b) = 1$; or*

(iii) *if $FLp(b) = 0$, then $GLp(b) = 1$ or $GRp(b) = 1$ or $FRp(b) = 1$ or $FRp'_f(b) = 1$ or $GRp''_f(b) = 1$; or*

(iv) *if $FLp(b) > 1$ and $GLp(b) = 0$, then $GRp(b) = 1$ or $FRp(b) = 1$ or $FRp''_g(b) = 1$; or*

(v) *if $FLp(b) > 1$ and $GLp(b) > 1$, then we have one of the following cases: $GRp(b) = 1$, $FRp(b) = 1$, $GLp'_g(b) = 1$, $FLp'_f(b) = 1$, $FLp''_g(b) = 1$, $GRp'_g(b) = 1$, $FRp''_g(b) = 1$, $GLp''_f(b) = 1$.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$. Since $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = b$, by Lemma 2.4, equation (2.1) holds.

Assume further that $FLp(b) \neq 1$ and $FLp(b) \not> 1$ or $GLp(b) \neq 1$. That is, $FLp(b) \neq 1$ and $FLp(b) \not> 1$ or $FLp(b) \neq 1$ and $GLp(b) \neq 1$. Thus, we have $FLp(b) = 0$ or $FLp(b) > 1$ and $GLp(b) = 0$ or $FLp(b) > 1$ and $GLp(b) > 1$.

We will show that (iii) holds. We suppose that it is not true. Then $FLp(b) = 0$, $GLp(b) \neq 1$, $GRp(b) \neq 1$, $FRp(b) \neq 1$, $FRp'_f(b) \neq 1$ and $GRp''_f(b) \neq 1$. Since $GLp(b) \neq 1$, we have $GLp(b) = 0$ or $GLp(b) > 1$. Since $FLp(b) = 0$, we get $GLp(b) > 1$. Since $GRp(b) \neq 1$, we obtain $GRp(b) = 0$ or $GRp(b) > 1$. Since $FLp(b) = 0$, we have $GRp(b) > 1$. Since $FRp(b) \neq 1$, we get $FRp(b) = 0$ or $FRp(b) > 1$. If $FRp(b) = 0$, then $GLp(\hat{\sigma}_{u,v}[b]) \neq 1$, this contradicts to (2.1). Then $FRp(b) > 1$. Since $FRp'_f(b) \neq 1$, we get $FRp'_f(b) > 1$. Since $GRp''_f(b) \neq 1$, we have $GRp''_f(b) > 1$. By Lemma 2.5, we get a contradiction.

Next, we suppose that (iv) is not true. Then $FLp(b) > 1$, $GLp(b) = 0$, $GRp(b) \neq 1$, $FRp(b) \neq 1$ and $FRp''_g(b) \neq 1$. Since $FRp(b) \neq 1$, we have $FRp(b) = 0$ or $FRp(b) > 1$. Since $GLp(b) = 0$, we get $FRp(b) > 1$. Since $GRp(b) \neq 1$, we obtain $GRp(b) = 0$ or $GRp(b) > 1$. If $GLp(b) = 0$, then $b \in X_2$ or $GLp(\hat{\sigma}_{u,v}[b]) > 1$. We get a contradiction. Thus $GRp(b) > 1$. Since $FRp''_g(b) \neq 1$, we get $FRp''_g(b) > 1$. By Lemma 2.6, $\sigma_{x_1,b}$ is not regular, a contradiction.

Now we show that (v) holds. We suppose that it is not true. Then $FLp(b) > 1$, $GLp(b) > 1$, $GRp(b) \neq 1$, $FRp(b) \neq 1$, $GLp'_g(b) \neq 1$, $FLp'_f(b) \neq 1$, $FLp''_g(b) \neq 1$, $GRp'_g(b) \neq 1$, $FRp''_g(b) \neq 1$ and $GLp''_f(b) \neq 1$. Since $GRp(b) \neq 1$, we have $GRp(b) = 0$ or $GRp(b) > 1$. Now, there are two cases to consider.

Case 1: $GRp(b) = 0$. Since $FRp(b) \neq 1$, we have $FRp(b) > 1$. Since $GLp'_g(b) \neq 1$, we obtain $GLp'_g(b) > 1$. Since $FLp''_g(b) \neq 1$, we get $FLp''_g(b) > 1$. By Lemma 2.7, a contradiction.

Case 2: $GRp(b) > 1$. We consider into two cases: $firstop(b) = f$ or $firstop(b) = g$.

Case 2.1: $firstop(b) = f$. Since $FRp(b) \neq 1$, we have $FRp(b) > 1$. Since $GRp'_g(b) \neq 1$, we get $GRp'_g(b) > 1$. Since $FRp''_g(b) \neq 1$, we obtain $FRp''_g(b) > 1$. Since $GLp'_g(b) \neq 1$, we get $GLp'_g(b) > 1$. Since $FLp''_g(b) \neq 1$, we have $FLp''_g(b) > 1$. By Lemma 2.8, a contradiction.

Case 2.2: $firstop(b) = g$. Since $FLp'_f(b) \neq 1$, we get $FLp'_f(b) > 1$. Since $GLp''_f(b) \neq 1$, we obtain $GLp''_f(b) > 1$. By Lemma 2.9, a contradiction.

Conversely, assume that one of the conditions (i), (ii), (iii), (iv), (v) holds. The table shows that there exist $u, v \in W_{(2,2)}(X_2)$ such that $\sigma_{x_1,b}\ \sigma_{u,v}\ \sigma_{x_1,b} = \sigma_{x_1,b}$.

No.	Cases	u	v
1	$FLp(b) = 1$	$g(x_1, x_1)$	x_1
2	$FLp(b) = 0,\ GLp(b) = 1$	x_1	$g(x_1, x_1)$
3	$FLp(b) = 0,\ GRp(b) = 1$	x_1	$g(x_2, x_2)$
4	$FLp(b) = 0,\ FRp(b) = 1$	$g(x_2, x_2)$	x_2
5	$FLp(b) = 0,\ FRp'_f(b) = 1$	$g(x_1, x_1)$	x_2
6	$FLp(b) = 0,\ GRp''_f(b) = 1$	$g(x_2, x_2)$	x_1
7	$FLp(b) > 1,\ GLp(b) = 1$	x_1	$g(x_1, x_1)$
8	$FLp(b) > 1,\ GLp(b) = 0,\ GRp(b) = 1$	x_2	$g(x_1, x_1)$
9	$FLp(b) > 1,\ GLp(b) = 0,\ FRp(b) = 1$	$g(x_2, x_2)$	x_2
10	$FLp(b) > 1, GLp(b) = 0, GRp'_g(b) = 1$	x_2	$g(x_1, x_1)$
11	$FLp(b) > 1, GLp(b) > 1, GRp(b) = 1$	x_2	$g(x_2, x_2)$
12	$FLp(b) > 1, GLp(b) > 1, FRp(b) = 1$	$g(x_2, x_2)$	x_2
13	$FLp(b) > 1, GLp(b) > 1, GLp'_g(b) = 1$	x_1	$g(x_2, x_2)$
14	$FLp(b) > 1, GLp(b) > 1, FLp''_g(b) = 1$	$g(x_1, x_1)$	x_2
15	$FLp(b) > 1, GLp(b) > 1, FLp'_f(b) = 1$	$g(x_2, x_2)$	x_1
16	$FLp(b) > 1, GLp(b) > 1, GRp'_g(b) = 1$	x_2	$g(x_1, x_1)$
17	$FLp(b) > 1, GLp(b) > 1, FRp''_g(b) = 1$	$g(x_2, x_2)$	x_1
18	$FLp(b) > 1, GLp(b) > 1, GLp''_f(b) = 1$	x_2	$g(x_1, x_1)$

□

Example. A hypersubstitution $\sigma_{x_1,b}$ is regular if b is the term shown in Figure 3:

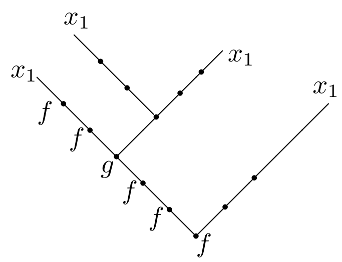

Figure 3: A term b for which $\sigma_{a,b}$ is regular

A hypersubstitution σ_{b,x_2} is not regular if b is the term shown in Figure 4:

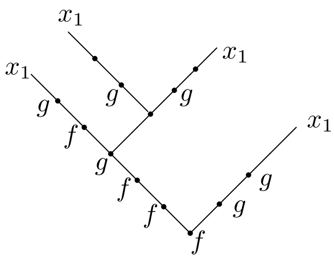

Figure 4: A term b for which $\sigma_{a,b}$ is not regular

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International Journal of Algebra and Computation

THE ORDER OF HYPERSUBSTITUTIONS OF TYPE(2,1)

THAWHAT CHANGPHAS*
Department of Mathematics, Khon Kaen University
Khon Kaen, 40002, Thailand
thacha@kku.ac.th

WONLOP HEMVONG
Department of Mathematics, Khon Kaen University
Khon Kaen, 40002, Thailand
hwonlop@hotmail.com

Received (Day Month Year)
Revised (Day Month Year)

Communicated by [editor]

A hypersubstitution is a mapping which maps operation symbols to terms of the corresponding arities. It is known that the set of all hypersubstitutions of a given type forms a semigroup. In this paper, we determine the order of hypersubstitutions of type $(2, 1)$. We show that the order of hypersubstitutions of type $(2, 1)$ is 1, 2, 3 or infinite.

Keywords: Hypersubstitutions; order; semigroups.

1. Introduction

Semigroups properties of hypersubstitutions of a given type have been studied [1], [2], [3], [6], [9] and [10]. The order of a hypersubstitution σ is defined in the usual way, that is, the order of the cyclic subsemigroup $\langle\sigma\rangle = \{\sigma^n \mid n \in \mathbb{N}\}$ generated by σ. If $\langle\sigma\rangle$ is finite, we say that the order of σ is finite, otherwise the order of σ is infinite. The following are known results concerning the order of hypersubstitutions. Klaus Denecke and Shally Wishmath [6] showed that the order of hypersubstitutions of type (2) is 1, 2 or infinite. Thawhat Changphas and Klaus Denecke [1] showed that the order of hypersubstitutions of type (3) is 1, 2, 3 or infinite. The same authors [3] showed that the order of hypersubstitutions of type $(2, 2)$ is 1, 2, 3, 4 or infinite. In [9], the authors studied the order of generalized hypersubstitutions, they showed that the order of generalized hypersubstitutions of type (3) is 1, 2, 3 or infinite. In this paper, we ask for the order of hypersubstitutions of type $(2, 1)$

*Supported by the Thailand Research Foundation.

and of type $(1, 2)$. The main result is

Main Theorem. The order of hypersubstitutions of type $(2, 1)$ is $1, 2, 3$ or infinite.

2. Preliminaries

Let $\tau = \{(f_i, n_i) : i \in I\}$ be a type and $X = \{x_1, x_2, x_3, \ldots\}$ a countably infinite alphabet of variables such that the sequence of the operation symbols $(f_i)_{i\in I}$ is disjoint with X. Let $X_n = \{x_1, x_2, x_3, \ldots, x_n\}$ be an n-element alphabet. Here f_i is n_i-ary for a natural number $n_i \geq 1$. An n-ary $(n \geq 1)$ term of type τ is inductively defined as follows:

(i) Every variable x_i in X_n is an n-ary term.
(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ denote the set containing $x_1, x_2, x_3, \ldots, x_n$ and being closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined by $W_\tau(X) = \bigcup_{n=1}^{\infty} W_\tau(X_n)$. For $t \in W_{(2,1)}(X_2)$, we introduce the following notations:

$$
\begin{aligned}
leftmost(t) &- \text{the first variable (from the left) occurring in } t\\
rightmost(t) &- \text{the last variable occurring in } t\\
var(t) &- \text{the set of all variable occurring in } t\\
op(t) &- \text{the total number of all operation symbols occurring in } t\\
ops(t) &- \text{the set of all operation symbols occurring in } t\\
firstops(t) &- \text{the first operation symbols (from the left) occurring in } t.
\end{aligned}
$$

For $t \in W_{(2,1)}(X_2)$, let $Lp(t)$ denote the left path from the root to the leaf which is labelled by the leftmost variable in t and $Rp(t)$ denote the right path from the root to the leaf which is labelled by the rightmost variable in t. The operation symbols occurring in $Lp(t)$ and $Rp(t)$ will be denoted by $ops(Lp(t))$ and $ops(Rp(t))$, respectively. If $t \in W_{(2,1)}(X_2)$ such that $var(t) = \{x_1\}$ or $var(t) = \{x_2\}$, we define $t^1 = t$ and

$$t^n = t^{n-1}(t, t) \text{ if } n \geq 1.$$

A mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ is called a hypersubstitution of type τ if $\sigma(f_i)$ is an n_i-ary term of type τ for every $i \in I$. A hypersubstitution σ of type τ can be uniquely extended to a map $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ on $W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] = t$ if $t \in X$.
(ii) $\hat{\sigma}[t] = \sigma(f_i)(\hat{\sigma}[t_1], \hat{\sigma}[t_2], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, t_2, \ldots, t_{n_i})$.

For $\sigma_1, \sigma_2 \in Hyp(\tau)$, define $\sigma_1\sigma_2$ by

$$(\sigma_1\sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)]$$

for all n_i-ary operation symbols f_i. Together with this binary associative operation, $Hyp(\tau)$ forms a monoid with the hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$ is an identity element.

Throughout, let f and g be the operation symbols of the type $(2,1)$. For terms a and b of type $(2,1)$, the hypersubstitution which maps the operation symbol f to the term a and the operation symbol g to the term b will be denoted by $\sigma_{a,b}$, that is $\sigma_{a,b}(f) = a$ and $\sigma_{a,b}(g) = b$.

For the hypersubstitution $\sigma_{a,b}$ of type $(2,1)$, we shall consider six cases:

(i) $op(a) \leq 1$ and $op(b) \leq 1$.
(ii) $op(a) > 1$ and $op(b) > 1$.
(iii) $op(a) > 1$ and $op(b) = 1$.
(iv) $op(a) = 1$ and $op(b) > 1$.
(v) $op(a) > 1$ and $op(b) = 0$.
(vi) $op(a) = 0$ and $op(b) > 1$.

If $op(a) \leq 1$ and $op(b) \leq 1$, there are twenty four cases to consider. The order in each of these cases can be listed in the table shown below. The results can be obtained by simple calculation.

$a \backslash b$	x_1	$g(x_1)$	$f(x_1, x_1)$
x_1	1	1	2
x_2	1	1	2
$f(x_1, x_1)$	1	1	1
$f(x_1, x_2)$	1	1	1
$f(x_2, x_1)$	2	2	2
$f(x_2, x_2)$	1	1	1
$g(x_1)$	1	1	2
$g(x_2)$	1	1	3

3. Case $op(a) > 1$ and $op(b) > 1$

We consider three cases:

(i) $var(a) = \{x_1, x_2\}$ and $var(b) = \{x_1\}$.
(ii) $var(a) = \{x_1\}$ and $var(b) = \{x_1\}$.
(iii) $var(a) = \{x_2\}$ and $var(b) = \{x_1\}$.

Theorem 1. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1, op(b) > 1, var(a) = \{x_1, x_2\}$ and $var(b) = \{x_1\}$. Then the order of $\sigma_{a,b}$ is infinite.*

Proof. It was shown [8] that if $\sigma \in Hyp(\tau)$ is regular (i.e. $var(\sigma(f_i)) = X_{n_i}$ for all $i \in I$), then $op(\hat{\sigma}[t]) > op(t)$ for all $t \in W_\tau(X) \setminus X$. Since $\sigma_{a,b}$ is regular, $op(\sigma^n_{a,b}(f)) < op(\sigma^{n+1}_{a,b}(f))$ for all $n \in \mathbb{N}$. Then the order of $\sigma_{a,b}$ is infinite.

□

Lemma 2. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) \geq 1$ and $op(b) \geq 1$. Then $\hat{\sigma}^n_{a,b}[t]$ is not a variable for all $n \in \mathbb{N}$ and for all $t \in W_{(2,1)}(X_2)\backslash X_2$.*

Proof. This can be argued by induction. □

Lemma 3. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$. If $t \in W_{(2,1)}(X_2)$ such that $var(t) = \{x_i\}$ for some $i \in \{1, 2\}$, then $var(\hat{\sigma}^n_{a,b}[t]) = \{x_i\}$ for all $n \in \mathbb{N}$.*

Proof. Clearly. □

Assume $var(a) = \{x_1\}$ and $var(b) = \{x_1\}$. Let $a = F(a_1, a_2)$ where $F \in \{f, g\}, a_1, a_2 \in W_{(2,1)}(X_2)$. If $F = g$, let $a = g(a_1)$. Let $b = G(b_1, b_2)$ where $G \in \{f, g\}, b_1, b_2 \in W_{(2,1)}(X_1)$. If $G = g$, let $b = g(b_1)$. We consider the following cases and their subcases:

(1) $F = f$ and $G = f$.

- (1.1) $a_1 \notin X_2$.
- (1.2) $a_1 = x_1$ and $b_1 = x_1$.
- (1.3) $a_1 = x_1, b_1 \notin X_1$ and $g \notin ops(Lp(b))$.
- (1.4) $a_1 = x_1, b_1 \notin X_1$ and $g \in ops(Lp(b))$.

(2) $F = g$ and $G = g$.
(3) $F = f$ and $G = g$.
(4) $F = g$ and $G = f$.

- (4.1) $b_1 \notin X_1$.
- (4.2) $b_1 = x_1$ and $f \notin ops(Lp(a))$.
- (4.3) $b_1 = x_1$ and $f \in ops(Lp(a))$.

Using above notations, we have

Theorem 4. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1, op(b) > 1$ and $var(a) = var(b) = \{x_1\}$.*

(i) If a and b satisfy (1.1), (1.4), (2), (3), (4.1) or (4.3), then $\sigma_{a,b}$ has infinite order.
(ii) If a and b satisfy (1.2), (1.3) or (4.2), then the order of $\sigma_{a,b}$ is less than or equal to 3.

Proof. (i) If a and b satisfy (1.1), then $a_1 \notin X_2$. By Lemma 2, $\hat{\sigma}^n_{a,b}[a_1] \notin X_2$ for all $n \in \mathbb{N}$. Since $var(a) = \{x_1\}$, by Lemma 3, we have $var(\hat{\sigma}^n_{a,b}[a]) = \{x_1\}$ for all $n \in \mathbb{N}$. Then $op(\sigma^2_{a,b}(f)) = op(\hat{\sigma}_{a,b}[a]) = op(a(\hat{\sigma}_{a,b}[a_1], \hat{\sigma}_{a,b}[a_2])) > op(a) = op(\sigma_{a,b}(f))$. Let

$k \geq 2$. Therefore,

$$\begin{aligned} op(\sigma_{a,b}^{k+1}(f)) &= op(\hat{\sigma}_{a,b}^{k-1}[a(\hat{\sigma}_{a,b}[a_1], \hat{\sigma}_{a,b}[a_2])]) \\ &= op(\hat{\sigma}_{a,b}^{k-1}[a](\hat{\sigma}_{a,b}^{k}[a_1], \hat{\sigma}_{a,b}^{k}[a_2])) \\ &> op(\hat{\sigma}_{a,b}^{k-1}[a]) \\ &= op(\sigma_{a,b}^{k}(f)) \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite.

Assume a and b satisfy (1.4). Since $g \in ops(Lp(b))$, $\hat{\sigma}_{a,b}[b] = a^l(b(t_1,t_2),t_3)$ where $t_1,t_2,t_3 \in W_{(2,1)}(X_1), l \in \mathbb{N}$. Since $var(a) = \{x_1\}$, $op(\sigma_{a,b}^2(g)) = op(\hat{\sigma}_{a,b}[b]) = op(a^l(b(t_1,t_2),t_3)) > op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. By Lemma 2 and Lemma 3, $\hat{\sigma}_{a,b}^{k-1}[a^l] \notin X_1$ and $var(\hat{\sigma}_{a,b}^{k-1}[a^l]) = \{x_1\}$. Therefore,

$$\begin{aligned} op(\sigma_{a,b}^{k+1}(g)) &= op(\hat{\sigma}_{a,b}^{k-1}[\hat{\sigma}_{a,b}[b]]) \\ &= op(\hat{\sigma}_{a,b}^{k-1}[a^l(b(t_1,t_2),t_3)]) \\ &= op(\hat{\sigma}_{a,b}^{k-1}[a^l](\hat{\sigma}_{a,b}^{k-1}[b](\hat{\sigma}_{a,b}^{k-1}[t_1], \hat{\sigma}_{a,b}^{k-1}[t_2]), \hat{\sigma}_{a,b}^{k-1}[t_3])]) \\ &> op(\hat{\sigma}_{a,b}^{k-1}[b]) \\ &= op(\sigma_{a,b}^{k}(g)) \end{aligned}$$

Thus the order of $\sigma_{a,b}$ is infinite.

Similarly, if a and b satisfy (4.1) or (4.3), then the order of $\sigma_{a,b}$ is infinite.

Assume a and b satisfy (2), then $b_1 \notin X_2$. By Lemma 2, $\hat{\sigma}_{a,b}^{n}[b_1] \notin X_2$ for all $n \in \mathbb{N}$. Since $var(a) = \{x_1\}$, by Lemma 3, we have $var(\hat{\sigma}_{a,b}^{n}[b]) = \{x_1\}$ for all $n \in \mathbb{N}$. Then $op(\sigma_{a,b}^2(g)) = op(\hat{\sigma}_{a,b}[b]) = op(b(\hat{\sigma}_{a,b}[b_1])) > op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. Therefore,

$$\begin{aligned} op(\sigma_{a,b}^{k+1}(g)) &= op(\hat{\sigma}_{a,b}^{k-1}[b(\hat{\sigma}_{a,b}[b_1])]) \\ &= op(\hat{\sigma}_{a,b}^{k-1}[b](\hat{\sigma}_{a,b}^{k}[b_1])) \\ &> op(\hat{\sigma}_{a,b}^{k-1}[b]) \\ &= op(\sigma_{a,b}^{k}(g)) \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite.

Similarly, if a and b satisfy (3), then the order of $\sigma_{a,b}$ is infinite.

(ii) Assume a and b satisfy (1.2). It is easy to see that $\sigma_{a,b}^2(f) = \sigma_{a,b}^3(f)$ and $\sigma_{a,b}^2(g) = \sigma_{a,b}^3(g)$, so the order of $\sigma_{a,b}$ is less than or equal to 2. If a and b satisfy (1.3), then $\sigma_{a,b}^3 = \sigma_{a,b}^4$. Then the order of $\sigma_{a,b}$ is less than or equal to 3. If a and b satisfy (4.2), then $\sigma_{a,b}^3 = \sigma_{a,b}$. It follows that the order of $\sigma_{a,b}$ is less than or equals to 3. □

Assume $var(a) = \{x_2\}$ and $var(b) = \{x_1\}$. Let $a = F(a_1,a_2)$ where $F \in \{f,g\}, a_1,a_2 \in W_{(2,1)}(X_2)$. If $F = g$, let $a = g(a_1)$. Let $b = G(b_1,b_2)$ where $G \in \{f,g\}, b_1,b_2 \in W_{(2,1)}(X_1)$. If $G = g$, let $b = g(b_1)$. We consider the following cases and their subcases:

(1) $F = f$ and $G = f$.

(1.1) $a_2 \notin X_2$.

(1.2) $a_2 = x_2$ and $b_2 = x_1$.

(1.3) $a_2 = x_2, b_2 \notin X_1$ and $g \notin ops(Rp(b))$.

(1.4) $a_2 = x_2, b_2 \notin X_1$ and $g \in ops(Rp(b))$.

(2) $F = g$ and $G = g$.

(3) $F = f$ and $G = g$.

(4) $F = g$ and $G = f$.

Using above notations, we have

Theorem 5. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1, op(b) > 1, var(a) = \{x_2\}$ and $var(b) = \{x_1\}$.*

(i) If a and b satisfy $(1.1), (1.4), (2), (3)$ or (4), then the order of $\sigma_{a,b}$ is infinite.

(ii) If a and b satisfy (1.2) or (1.3), then the order of $\sigma_{a,b}$ is less than or equal to 2.

Proof. (i) Assume a and b satisfy (1.1), then $a_2 \notin X_2$. By Lemma 2, $\hat{\sigma}^n_{a,b}[a_2] \notin X_2$ for all $n \in \mathbb{N}$. Since $var(a) = \{x_2\}$, by Lemma 3, we have $var(\hat{\sigma}^n_{a,b}[a]) = \{x_2\}$ for all $n \in \mathbb{N}$. Then $op(\sigma^2_{a,b}(f)) = op(\hat{\sigma}_{a,b}[a]) = op(a(\hat{\sigma}_{a,b}[a_1], \hat{\sigma}_{a,b}[a_2])) > op(a) = op(\sigma_{a,b}(f))$. Let $k \geq 2$. We obtain

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(f)) &= op(\hat{\sigma}^{k-1}_{a,b}[a(\hat{\sigma}_{a,b}[a_1], \hat{\sigma}_{a,b}[a_2])]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a](\hat{\sigma}^k_{a,b}[a_1], \hat{\sigma}^k_{a,b}[a_2])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[a]) \\ &= op(\sigma^k_{a,b}(f)) \end{aligned}$$

It follows that the order of $\sigma_{a,b}$ is infinite.

Assume a and b satisfy (1.4). Since $g \in ops(Rp(b))$ and $var(a) = \{x_2\}$, we have $\hat{\sigma}_{a,b}[b] = a^l(t_1, b(t_2, t_3))$ where $t_1, t_2, t_3 \in W_{(2,1)}(X_1), l \in \mathbb{N}$. Since $var(a) = \{x_2\}$, $op(\sigma^2_{a,b}(g)) = op(\hat{\sigma}_{a,b}[b])$
$= op(a^l(t_1, b(t_2, t_3))) > op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. By Lemma 2 and Lemma 3, we have $\hat{\sigma}^{k-1}_{a,b}[a^l] \notin X_1$ and $var(\hat{\sigma}^{k-1}_{a,b}[a^l]) = \{x_2\}$. We obtain

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(g)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[b]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a^l(t_1, b(t_2, t_3))]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a^l](\hat{\sigma}^{k-1}_{a,b}[t_1], \hat{\sigma}^{k-1}_{a,b}[b](\hat{\sigma}^{k-1}_{a,b}[t_2], \hat{\sigma}^{k-1}_{a,b}[t_3]))) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[b]) \\ &= op(\sigma^k_{a,b}(g)) \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite.

Assume a and b satisfy (3), then $b_1 \notin X_2$. By Lemma 2, $\hat{\sigma}^n_{a,b}[b_1] \notin X_2$ for all $n \in \mathbb{N}$. Since $var(b) = \{x_1\}$, by Lemma 3, we have $var(\hat{\sigma}^n_{a,b}[b]) = \{x_1\}$ for all $n \in \mathbb{N}$.

Then $op(\sigma^2_{a,b}(g)) = op(\hat{\sigma}_{a,b}[b]) = op(b(\hat{\sigma}_{a,b}[b_1])) > op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. We obtain

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(g)) &= op(\hat{\sigma}^{k-1}_{a,b}[b(\hat{\sigma}_{a,b}[b_1])]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b](\hat{\sigma}^{k}_{a,b}[b_1])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[b]) \\ &= op(\sigma^{k}_{a,b}(g)) \end{aligned}$$

Therefore, the order of $\sigma_{a,b}$ is infinite.

Similar arguments work if a and b satisfy (2) or (4).

(ii) This is easy to see. □

4. Case $op(a) = 1$ and $op(b) > 1$

We shall consider three cases. Assume $var(a) = \{x_1, x_2\}$ and $var(b) = \{x_1\}$. Let $b = G(b_1, b_2)$ where $G \in \{f, g\}, b_1, b_2 \in W_{(2,1)}(X_1)$. If $G = g$, let $b = g(b_1)$. We consider the following cases:

(1) $a = f(x_1, x_2)$ and $G = f$.
(2) $a = f(x_1, x_2)$ and $G = g$.
(3) $a = f(x_2, x_1)$ and $G = f$.
(4) $a = f(x_2, x_1)$ and $G = g$.

Using above notations, we have

Theorem 6. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) = 1, op(b) > 1, var(a) = \{x_1, x_2\}$ and $var(b) = \{x_1\}$.*

(i) If a and b satisfy (1), then the order of $\sigma_{a,b}$ is 1 or infinite.
(ii) If a and b satisfy (3), then the order of $\sigma_{a,b}$ is less than or equal to 2 or infinite.
(iii) If a and b satisfy (2) or (4), then the order of $\sigma_{a,b}$ is infinite.

Proof. (i) Assume a and b satisfy (1). Then $\hat{\sigma}_{a,b}[a] = a$. If $ops(b) = \{f\}$, by $var(b) = \{x_1\}$, we have $\hat{\sigma}_{a,b}[b] = b$. Then the order of $\sigma_{a,b}$ is 1. Assume $g \in ops(b)$. Note that $op(\hat{\sigma}_{a,b}[t]) \geq op(t)$ for any $t \in W_{(2,1)}(X_1)$. We claim that for $t \in W_{(2,1)}(X_1)$ if $firstop(t) = f$ and $g \in op(t)$, then $op(\hat{\sigma}_{a,b}[t]) > op(t)$. We proceed by induction on $op(t)$, that is, on the total number of all operation symbols occurring in t. If $op(t) = 2$, then $t = f(g(x_1), x_1)$ or $t = f(x_1, g(x_1))$. Each of the cases we have $op(\hat{\sigma}_{a,b}[t]) > op(t)$. Assume the claim holds for any t with $2 \leq op(t) \leq k$. Let $t \in W_{(2,1)}(X_1)$ be such that $firstop(t) = f, g \in op(t)$ and $op(t) = k + 1$. Then $t = f(t_1, t_2)$ for some $t_1, t_2 \in W_{(2,1)}(X_1)$. Therefore,

$$\begin{aligned} op(\hat{\sigma}_{a,b}[t]) = op(\sigma_{a,b}(f)(\hat{\sigma}_{a,b}[t_1], \hat{\sigma}_{a,b}[t_2])) &= 1 + op(\hat{\sigma}_{a,b}[t_1]) + op(\hat{\sigma}_{a,b}[t_2]) \\ &> 1 + op(t_1) + op(t_2) \\ &= op(t) \end{aligned}$$

So we have the claim. Using the claim, the order of $\sigma_{a,b}$ is infinite.

(ii) Assume $ops(b) = \{f\}$. Since $\sigma^3_{a,b}(f) = \sigma_{a,b}(f)$ and $\sigma^2_{a,b}(g) = \sigma_{a,b}(g)$, the order of $\sigma_{a,b}$ is less than or equal to 2. If $g \in ops(b)$, in the same manner as (i), the order of $\sigma_{a,b}$ is infinite.

(iii) Assume a and b satisfy (2). Since $op(b) > 1$, $b = g(G(b_1, b_2))$ for some $b_1, b_2 \in W_{(2,1)}(X_1)$. Then $op(\sigma^2_{a,b}(g)) = op(\hat{\sigma}_{a,b}[g(G(b_1, b_2))]) = op(b(\sigma_{a,b}(G)(\hat{\sigma}_{a,b}[b_1], \hat{\sigma}_{a,b}[b_2]))) > op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. Therefore,

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(g)) &= op(\hat{\sigma}^{k-1}_{a,b}[b(\sigma_{a,b}(G)(\hat{\sigma}_{a,b}[b_1], \hat{\sigma}_{a,b}[b_2]))]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b](\sigma^k_{a,b}(G)(\hat{\sigma}^k_{a,b}[b_1], \hat{\sigma}^k_{a,b}[b_2]))) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[b]) \\ &= op(\sigma^k_{a,b}(g)) \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite. Similar arguments work for (4). □

Assume $var(a) = \{x_1\}$ and $var(b) = \{x_1\}$. Let $b = G(b_1, b_2)$ where $G \in \{f, g\}, b_1, b_2 \in W_{(2,1)}(X_1)$. If $G = g$, let $b = g(b_1)$. We consider the following cases and thire subcases:

(1) $a = f(x_1, x_1)$ and $G = f$.

- (1.1) $b_1 = x_1$.
- (1.2) $b_1 \notin X_1$ and $g \in ops(Lp(b))$.
- (1.3) $b_1 \notin X_1$ and $g \notin ops(Lp(b))$.

(2) $a = g(x_1)$ and $G = g$.
(3) $a = f(x_1, x_1)$ and $G = g$.
(4) $a = g(x_1)$ and $G = f$.

- (4.1) $b_1 = x_1$.
- (4.2) $b_1 \neq x_1$.

Using above notations, we have

Theorem 7. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) = 1, op(b) > 1$ and $var(a) = var(b) = \{x_1\}$.*

(i) If a and b satisfy (1.1), (1.3) or (4.1), then the order of $\sigma_{a,b}$ is less than or equal to 2.
(ii) If a and b satisfy (1.2), (2), (3) or (4.2), then the order of $\sigma_{a,b}$ is infinite.

Proof. (i) This is easy to see.

(ii) Assume a and b satisfy (1.2). Then $\hat{\sigma}_{a,b}[b] = f(x_1, x_1)(b(t_1), t_2)$ for some $t_1, t_2 \in W_{(2,1)}(X_1)$. Since $var(a) = \{x_1\}$, $op(\sigma^2_{a,b}(g)) = op(f(x_1, x_1)(b(t_1), t_2)) >$

$op(b) = op(\sigma_{a,b}(g))$. Let $n \geq 2$. Therefore,

$$\begin{aligned} op(\hat{\sigma}^{n+1}_{a,b}[b]) &= op(\hat{\sigma}^{n}_{a,b}[f(x_1,x_1)(b(t_1),t_2)]) \\ &= op(f(x_1,x_1)(\hat{\sigma}^{n}_{a,b}[b](\hat{\sigma}^{n}_{a,b}[t_1]),\hat{\sigma}^{n}_{a,b}[t_2])) \\ &> op(\hat{\sigma}^{n}_{a,b}[b]) \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite.

Assume a and b satisfy (2). Then $b = g(G(t_1,t_2))$ for some $t_1,t_2 \in W_{(2,1)}(X_1)$. We have $op(\sigma^2_{a,b}(g)) = op(\hat{\sigma}_{a,b}[g(G(t_1,t_2))]) = op(b(\sigma_{a,b}(G)(\hat{\sigma}_{a,b}[t_1],\hat{\sigma}_{a,b}[t_2])) > op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. We obtain

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(g)) &= op(\hat{\sigma}^{k-1}_{a,b}[b(\sigma_{a,b}(G)(\hat{\sigma}_{a,b}[t_1],\hat{\sigma}_{a,b}[t_2])]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b](\sigma^{k}_{a,b}(G)(\hat{\sigma}^{k}_{a,b}[t_1],\hat{\sigma}^{k}_{a,b}[t_2]))) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[b]) \\ &= op(\sigma^{k}_{a,b}(g)) \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite.

Similarly, if a and b satisfy (3), then the order of $\sigma_{a,b}$ is infinite.

Assume a and b satisfy (4.2). Then $a = g(x_1)$ and $b_1 \notin X_1$. If $g \in ops(b_1)$, in the same manner as Theorem 6 (i), the order of $\sigma_{a,b}$ is infinite. Assume $g \notin ops(b_1)$. Since $b \notin X_1, ops(b_1) = \{f\}$. Since $firstops(a) = g$ and $ops(b_1) = \{f\}$, $\hat{\sigma}_{a,b}[b] = \hat{\sigma}^2_{a,b}[a]$. Then $op(\sigma^2_{a,b}(g)) = op(\hat{\sigma}_{a,b}[b]) = op(\hat{\sigma}^2_{a,b}[a]) > op(\hat{\sigma}_{a,b}[a]) = op(b) = op(\sigma_{a,b}(g))$. Let $k \geq 2$. We have

$$op(\sigma^{k+1}_{a,b}(g)) = op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}^2_{a,b}[a]]) > op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[a]]) = op(\hat{\sigma}^{k-1}_{a,b}[b]) = op(\sigma^{k}_{a,b}(g))$$

Therefore, the order of $\sigma_{a,b}$ is infinite. □

Assume $var(a) = \{x_2\}$ and $var(b) = \{x_1\}$. Let $b = G(b_1,b_2)$ where $G \in \{f,g\}, b_1,b_2 \in W_{(2,1)}(X_1)$. If $G = g$, let $b = g(b_1)$. We consider:

(1) $a = f(x_2,x_2)$ and $G = f$.

- (1.1) $b_2 = x_1$.
- (1.2) $b_2 \notin X_1$.

(2) $a = g(x_2)$ and $G = g$.

(3) $a = f(x_2,x_2)$ and $G = g$.

(4) $a = g(x_2)$ and $G = f$.

- (4.1) $b_2 = x_1$.
- (4.2) $b_2 \notin X_1$.

Using above notations, we have the following.

Theorem 8. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) = 1, op(b) > 1, var(a) = \{x_2\}$ and $var(b) = \{x_1\}$.*

(i) If a and b satisfy (1.1) or (4.1), then the order of $\sigma_{a,b}$ is 2.

(ii) If a and b satisfy (1.2), (2), (3) or (4.2), then the order of $\sigma_{a,b}$ is infinite.

Proof. (i) If a and b satisfy (1.1), then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}^2_{a,b}[b] = \hat{\sigma}_{a,b}[b]$. If a and b satisfy (4.1), then $\hat{\sigma}^2_{a,b}[a] = a$ and $\hat{\sigma}^2_{a,b}[b] = b$. Consequently, the order of $\sigma_{a,b}$ is 2.

(ii) Assume a and b satisfy (1.2). Then $b_2 = G(t_1, t_2)$ for some $t_1, t_2 \in W_{(2,1)}(X_1)$. Since $firstops(b) = f$ and $a = f(x_2, x_2)$,

$$\begin{aligned} op(\sigma^2_{a,b}(f)) = op(\hat{\sigma}_{a,b}[f(b_1, G(t_1, t_2))]) &= op(a(\hat{\sigma}_{a,b}[b_1], \sigma_{a,b}(G)(\hat{\sigma}_{a,b}[t_1], \hat{\sigma}_{a,b}[t_2]))) \\ &> op(a) \\ &= op(\sigma_{a,b}(f)). \end{aligned}$$

Let $k \geq 2$. Therefore,

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(f)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[f(b_1, G(t_1, t_2))]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a(\hat{\sigma}_{a,b}[b_1], \sigma_{a,b}(G)(\hat{\sigma}_{a,b}[t_1], \hat{\sigma}_{a,b}[t_2]))]) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[a]) \\ &= op(\sigma^k_{a,b}(f)). \end{aligned}$$

Thus the order of $\sigma_{a,b}$ is infinite.

If a and b satisfy (2) or (3), in the same manner as Theorem 7 (ii), the order of $\sigma_{a,b}$ is infinite. Assume a and b satisfy (4.2). Since $a = g(x_2)$, $op(\sigma^2_{a,b}(f)) = op(\hat{\sigma}_{a,b}[g(x_2)]) = op(b(x_2)) = op(f(b_1, G(t_1, t_2))(x_2)) > op(g(x_2)) = op(a) = op(\sigma_{a,b}(f))$. Let $k \geq 2$. We obtain

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(f)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[a]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[g(x_2)]]) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[g(x_2)]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a]) \\ &= op(\sigma^k_{a,b}(f)) \end{aligned}$$

Thus the order of $\sigma_{a,b}$ is infinite. □

Now, we consider the case $op(a) > 1$ and $op(b) = 1$. The results can be considered similarly as the case $op(a) = 1$ and $op(b) > 1$. We consider three cases:

(1) $var(a) = X_2$ and $var(b) = \{x_1\}$.
(2) $var(a) = \{x_1\}$ and $var(b) = \{x_1\}$.
(3) $var(a) = \{x_2\}$ and $var(b) = \{x_1\}$.

For (1), we consider four cases:

(1.1) $b = f(x_1, x_1)$ and $firstops(a) = f$.
(1.2) $b = f(x_1, x_1)$ and $firstops(a) = g$.
(1.3) $b = g(x_1)$ and $firstops(a) = f$.
(1.4) $b = g(x_1)$ and $firstops(a) = g$.

Using above notation, we have the following.

Theorem 9. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1$, $op(b) = 1, var(a) = X_2$ and $var(b) = \{x_1\}$.*

(i) If a and b satisfy (1.1), (1.2) *or* (1.3)*, then the order of $\sigma_{a,b}$ is infinite.*
(ii) If a and b satisfy (1.4)*, then the order of $\sigma_{a,b}$ is 1 or infinite.*

For (2), let $a = g(a_1, a_2)$ where $a_1, a_2 \in W_{(2,1)}(X_2)$. If $F = g$, let $a = g(a_1)$. We consider two cases:

(2.1) $a_1 \notin X_2$.
(2.2) $a_1 = x_1$.

Theorem 10. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1$ and $var(a) = \{x_1\} = var(b)$.*

(i) If a satisfies (2.1)*, then the order of $\sigma_{a,b}$ is 1 or infinite.*
(ii) If a satisfies (2.2)*, then the order of $\sigma_{a,b}$ is 2.*

For (3), let $a = g(a_1, a_2)$ where $a_1, a_2 \in W_{(2,1)}(X_2)$. If $F = g$, let $a = g(a_2)$. We consider:

(3.1) $a_2 \notin X_2$.
(3.2) $a_2 = x_1$.

Theorem 11. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1$, $var(a) = \{x_2\}$ and $var(b) = \{x_1\}$.*

(i) If a satisfies (3.1)*, then the order of $\sigma_{a,b}$ is 1 or infinite.*
(ii) If a satisfies (3.2)*, then the order of $\sigma_{a,b}$ is less than ro equal to 2.*

5. Case $op(a) > 1$ and $op(b) = 0$

Since $b \in W_{(2,1)}(X_1)$ and $op(b) = 0$, $b = x_1$. We consider three cases. Assume $var(a) = \{x_1, x_2\}$. Let $a = F(a_1, a_2)$ where $F \in \{f, g\}, a_1, a_2 \in W_{(2,1)}(X_2)$. Clearly, if $a_1 \in X_2$ then $a_2 \notin X_2$. If $F = g$, let $a = g(a_1)$. There are six cases to consider:

(1.1) $F = f, a_1 \notin X_2$ and $f \in ops(Lp(a))$.
(1.2) $F = f, a_1 \notin X_2$ and $f \notin ops(Lp(a))$.
(1.3) $F = f, a_1 \in X_2$ and $f \in ops(Rp(a))$.
(1.4) $F = f, a_1 \in X_2$ and $f \notin ops(Rp(a))$.
(1.5) $F = g$ and $f \notin ops(a_1)$.
(1.6) $F = g$ and $f \in ops(a_1)$.

Using above notations, we have Theorem 12.

Theorem 12. *Let $a \in W_{(2,1)}(X_2)$ and $b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1, op(b) = 0$ and $var(a) = \{x_1, x_2\}$.*

(i) If a satisfies (1.1) *or* (1.3)*, then the order of* $\sigma_{a,b}$ *is infinite.*
(ii) If a satisfies (1.2), (1.4), (1.5) *or* (1.6)*, then the order of* $\sigma_{a,b}$ *is less than or equal to* 2.

Proof. (i) Assume a satisfies (1.1). Since $b = x_1, F = f$ and $f \in ops(Lp(a))$, so $\sigma^2_{a,b}(f) = a(a,t)$ for some $t \in W_{(2,1)}(X_2)$. Since $var(a) = \{x_1, x_2\}$, $op(\sigma_{a,b}(f)) < op(\sigma^2_{a,b}(f))$. Since $f \in ops(Lp(a))$, $\hat{\sigma}^n_{a,b}[a] \notin X_2$ for all $n \in \mathbb{N}$. For $k \geq 2$, we have

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(f)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[a]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a(a,t)]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a](\hat{\sigma}^{k-1}_{a,b}[a], \hat{\sigma}^{k-1}_{a,b}[t])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[a] \\ &= op(\sigma^k_{a,b}(f)). \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite. Similarly, if a satisfies (1.3), then the order of $\sigma_{a,b}$ is infinite.

(ii) Assume a satisfies (1.2). If $leftmost(a_1) = x_1$, then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = b$. Thus $\sigma^2_{a,b} = \sigma_{a,b}$. If $leftmost(a_1) = x_2$, then $\hat{\sigma}^2_{a,b}[a] = \hat{\sigma}_{a,b}[a]$ and $\hat{\sigma}_{a,b}[b] = b$. Hence the order of $\sigma_{a,b}$ is less than or equal to 2. Assume a satisfies (1.4). If $rightmost(a_2) = x_2$, then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = b$. Thus $\sigma^2_{a,b} = \sigma_{a,b}$. If $leftmost(a_2) = x_1$, then $\hat{\sigma}^2_{a,b}[a] = \hat{\sigma}_{a,b}[a]$ and $\hat{\sigma}_{a,b}[b] = b$. Hence the order of $\sigma_{a,b}$ is less than or equal to 2. Assume a satisfies (1.5). If $leftmost(a_1) = x_1$, then $\hat{\sigma}_{a,b}[a] = b$ and $\hat{\sigma}_{a,b}[b] = b$. Hence the order of $\sigma_{a,b}$ is is less than or equal to 2. If $leftmost(a_1) = x_2$, then $\hat{\sigma}^2_{a,b}[a] = \hat{\sigma}_{a,b}[a]$ and $\hat{\sigma}_{a,b}[b] = b$. Hence the order of $\sigma_{a,b}$ is less than or equal to 2. Assume a satisfies (1.6). If $leftmost(a_1) = x_1$, then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = b$. Hence $\sigma^2_{a,b} = \sigma_{a,b}$. If $leftmost(a_1) = x_2$, then $\hat{\sigma}^2_{a,b}[a] = \hat{\sigma}_{a,b}[a]$ and $\hat{\sigma}_{a,b}[b] = b$. Hence the order of $\sigma_{a,b}$ is less than or equal to 2.□

Assume $var(a) = \{x_1\}$. Let $a = F(a_1, a_2)$ where $F \in \{f, g\}, a_1, a_2 \in W_{(2,1)}(X_2)$. If $F = g$, let $a = g(a_1)$. There are five cases to consider:

(1) $F = f$ and $a_1 = x_1$.
(2) $F = f, a_1 \notin X_2$ and $f \notin ops(Lp(a))$.
(3) $F = f, a_1 \notin X_2$ and $f \in ops(Lp(a))$.
(4) $F = g$ and $f \notin ops(a_1)$.
(5) $F = g$ and $f \in ops(a_1)$.

Using above notation, we have Theorem 13.

Theorem 13. *Let* $a \in W_{(2,1)}(X_2), b \in W_{(2,1)}(X_1)$ *be such that* $op(a) > 1$ *and* $op(b) = 0$.

(i) If a satisfies (1), (2) *or* (4)*, then the order of* $\sigma_{a,b}$ *is less than or equal to* 2.
(ii) If a satisfies (3)*, then the order of* $\sigma_{a,b}$ *is infinite.*

(iii) If a satisfies (5)*, then the order of $\sigma_{a,b}$ is less than or equal to* 2 *or infinite.*

Proof. (i) This is easy to see.

(ii) Assume a satisfies (3). Since $b = x_1, F = f$ and $f \in ops(Lp(a))$, we have $\sigma^2_{a,b}(f) = a(a,t)$ for some $t \in W_{(2,1)}(X_2)$. Since $var(a) = \{x_1\}$, $op(\sigma_{a,b}(f)) < op(\sigma^2_{a,b}(f))$. Since $f \in ops(Lp(a))$, $\hat{\sigma}^n_{a,b}[a] \notin X_2$ for all $n \in \mathbb{N}$. Let $k \geq 2$. Therefore,

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(f)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[a]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a(a,t)]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a](\hat{\sigma}^{k-1}_{a,b}[a], \hat{\sigma}^{k-1}_{a,b}[t])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[a] \\ &= op(\sigma^k_{a,b}(f)). \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite.

(iii) Assume a satisfies (5). If there is only one $f \in ops(a_1)$, then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = b$. Thus the order of $\sigma_{a,b}$ is less than or equal to 2. Assume there are more than one times $f \in ops(Lp(a))$. Since $b = x_1$ and $F = g$, we have $\sigma^2_{a,b}(f) = a(a,t_1)$ for some $t_1 \in W_{(2,1)}(X_2)$. Since $var(a) = \{x_1\}$, $op(\sigma_{a,b}(f)) < op(\sigma^2_{a,b}(f))$. Since $f \in ops(Lp(a))$, $\hat{\sigma}^n_{a,b}[a] \notin X_2$ for all $n \in \mathbb{N}$. For $k \geq 2$, we have

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(f)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[a]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a(a,t_1)]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[a](\hat{\sigma}^{k-1}_{a,b}[a], \hat{\sigma}^{k-1}_{a,b}[t_1])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[a] \\ &= op(\sigma^k_{a,b}(f)). \end{aligned}$$

Then the order of $\sigma_{a,b}$ is infinite. □

Assume $var(a) = \{x_2\}$. Let $a = F(a_1, a_2)$ where $F \in \{f, g\}$ and $a_1, a_2 \in W_{(2,1)}(X_2)$. If $F = g$, let $a = g(a_1)$. There are five cases to consider:

(1) $F = f$ and $a_2 = x_2$.
(2) $F = f, a_2 \notin X_2$ and $f \notin ops(Rp(a))$.
(3) $F = f, a_2 \notin X_2$ and $f \in ops(Rp(a))$.
(4) $F = g$ and $f \notin ops(a_1)$.
(5) $F = g$ and $f \in ops(a_1)$.

Using above notation, Theorem 14 below can be proved in the same manner as Theorem 13.

Theorem 14. *Let $a \in W_{(2,1)}(X_2), b \in W_{(2,1)}(X_1)$ be such that $op(a) > 1$ and $op(b) = 0$.*

(i) If a satisfies (1), (2) *or* (4)*, then the order of $\sigma_{a,b}$ is less than or equal to* 2.

(ii) If a satisfies (3), then the order of $\sigma_{a,b}$ is infinite.
(iii) If a satisfies (5), then the order of $\sigma_{a,b}$ is less than or equal to 2 or infinite.

6. Case $op(a) = 0$ and $op(b) > 1$

Let $b = F(b_1, b_2)$ where $F \in \{f, g\}, b_1, b_2 \in W_{(2,1)}(X_1)$. If $F = g$, let $b = g(b_1)$. Since $op(a) = 0$, $a = x_1$ or $a = x_2$. There are ten cases to consider:

(1) $a = x_1, F = f$ and $b_1 = x_1$.
(2) $a = x_1, F = f, b_1 \notin X_1$ and $g \notin ops(Lp(b))$.
(3) $a = x_1, F = f, b_1 \notin X_1$ and $g \in ops(Lp(b))$.
(4) $a = x_1, F = g$ and $g \notin ops(b_1)$.
(5) $a = x_1, F = g$ and $g \in ops(b_1)$.
(6) $a = x_2, F = f$ and $b_2 = x_1$.
(7) $a = x_2, F = f, b_2 \notin X_1$ and $g \notin ops(Rp(b))$.
(8) $a = x_2, F = f, b_2 \notin X_1$ and $g \in ops(Rp(b))$.
(9) $a = x_2, F = g$ and $g \notin ops(b_1)$.
(10) $a = x_2, F = g$ and $g \in ops(b_1)$.

Using above notations, we have Theorem 15 and Theorem 16. Theorem 16 can be proved in the same manner as Theorem 15.

Theorem 15. *Let $a \in W_{(2,1)}(X_2), b \in W_{(2,1)}(X_1)$ be such that $op(a) = 0$ and $op(b) > 1$.*

(i) If a and b satisfy (1), (2) or (4), then the order of $\sigma_{a,b}$ is less than or equal to 2.
(ii) If a and b satisfy (3), then the order of $\sigma_{a,b}$ is less than or equal to 2 or infinite.
(iii) If a and b satisfy (5), then the order of $\sigma_{a,b}$ is infinite.

Proof. (i) If a and b satisfy (1), (2) or (4), then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = a$. Thus $\sigma^2_{a,b} = \sigma^3_{a,b}$.

(ii) Assume a and b satisfy (3). If there is only one $g \in ops(Lp(b))$, then $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = b$. Thus $\sigma^2_{a,b} = \sigma^3_{a,b}$. Assume there are more than one times $g \in ops(b_1)$. Since $a = x_1$ and $b \notin X_1$, $\hat{\sigma}^n_{a,b}[b] \notin X_1$ for all $n \in \mathbb{N}$. Since there are more than one times $g \in ops(b_1)$ and $a = x_1$, $\sigma^2_{a,b}(g) = b(b, t)$ for some $t \in W_{(2,1)}(X_1)$. Since $var(b) = \{x_1\}$, $op(\sigma_{a,b}(g)) < op(\sigma^2_{a,b}(g))$. For $k \geq 2$, consider

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(g)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[b]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b(b,t)]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b](\hat{\sigma}^{k-1}_{a,b}[b], \hat{\sigma}^{k-1}_{a,b}[t])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[b] \\ &= op(\sigma^k_{a,b}(g)), \end{aligned}$$

we obtain the order of $\sigma_{a,b}$ is infinite.

(iii) Assume a and b satisfy (5). Then $\hat{\sigma}^n_{a,b}[b]) \notin X_1$ for all $n \in \mathbb{N}$. Since $g \in ops(b_1)$ and $a = x_1$, $\sigma^2_{a,b}(g) = b(b, t_1)$ for some $t_1 \in W_{(2,1)}(X_1)$. For $k \geq 2$, consider

$$\begin{aligned} op(\sigma^{k+1}_{a,b}(g)) &= op(\hat{\sigma}^{k-1}_{a,b}[\hat{\sigma}_{a,b}[b]]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b(b, t_1)]) \\ &= op(\hat{\sigma}^{k-1}_{a,b}[b](\hat{\sigma}^{k-1}_{a,b}[b], \hat{\sigma}^{k-1}_{a,b}[t_1])) \\ &> op(\hat{\sigma}^{k-1}_{a,b}[b] \\ &= op(\sigma^k_{a,b}(g)), \end{aligned}$$

we have the order of $\sigma_{a,b}$ is infinite. □

Theorem 16. *Let $a \in W_{(2,1)}(X_2), b \in W_{(2,1)}(X_1)$ be such that $op(a) = 0$ and $op(b) > 1$.*

(i) If a and b satisfy (6), (7) or (9), then the order of $\sigma_{a,b}$ is less than or equal to 2.
(ii) If a and b satisfy (8), then the order of $\sigma_{a,b}$ is less than or equal to 2 or infinite.
(iii) If a and b satisfy (10), then the order of $\sigma_{a,b}$ is infinite.

Note that the mapping $\varphi : Hyp(2,1) \to Hyp(1,2)$ defined by $\varphi(\sigma_{a,b}) = \sigma_{a,b}\sigma_{g(x_1),f(x_1,x_2)}$ is an isomorphism. Then we can conclude that the order of hypersubstitutions of type $(1,2)$ is $1, 2, 3$ or infinite.

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The Order of Hypersubstitutions of Type (n)

Thawhat Changphas[1] and Wonlop Hemvong

Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand.
email: thacha@kku.ac.th and hwonlop@hotmail.com

Abstract: A hypersubstitution is a mapping which maps operation symbols to terms of the corresponding arities. It is known that the set of all hypersubstitutions of a given type forms a semigroup. In this paper, we determine the order of hypersubstitutions of type (n) for $n \in \mathbb{N}$. The result generalizes the results in [3] and [1].

Keywords: Order of a hypersubstitution, Semigroups.
2000 Mathematics Subject Classification: 20M07.

1 Preliminaries

Let S be a semigroup. The *order* of an element a of S is defined as the order of $\langle a\rangle$, the cyclic subsemigroup of S generated by a. The *index* and the *period* of an element a of S consults [10] (p. 9-11).

Let $\tau = \{(f_i, n_i) \mid i \in I\}$ be a type. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables such that the sequence of the operation symbols $(f_i)_{i\in I}$ is disjoint with X, and let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an n-element alphabet where $n \in \mathbb{N}$. Here f_i is n_i-ary for a natural number $n_i \geq 1$. An n-ary $(n \geq 1)$ *term* of type τ is inductively defined as follows:

(i) every variable $x_j \in X_n$ is an n-ary term,

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the set containing $x_1, \ldots, x_n$ and being closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined by $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$. Any mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ is called a *hypersubstitution* of type τ if $\sigma(f_i)$ is an n_i-ary term of type τ for every $i \in I$. Any hypersubstitution σ of type τ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$,

(ii) $\hat{\sigma}[t] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

A binary operation is defined on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, by

$$(\sigma_1\sigma_2)(f_i) := \hat{\sigma_1}[\sigma_2(f_i)]$$

for all n_i-ary operation symbols f_i. Together with this binary associative operation $Hyp(\tau)$ forms a monoid since the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$ is an identity element. For an n-ary term t of type (n), let

$var(t)$ – the set of all variables occurring in t,
$op(t)$ – the total number of all operation symbols occurring in t.

Several subsemigroups of $Hyp(\tau)$ can be defined: $P(\tau) := \{\sigma : \sigma(f_i) \in X_{n_i}, i \in I\}$, $Short(\tau) := \{\sigma : op(\sigma(f_i)) = 1, i \in I\}$, and $H_k^{op} := \{\sigma : op(\sigma(f_i)) \geq k, i \in I\}$ for any $k \in \mathbb{N}$.

[1]Research Supported by the Thai Research Fund.

Semigroup properties of hypersubstitutions have been widely studied (see [1], [2], [7],[1]). Properties of monoids of generalized hypersubstitution, i.e. non-arity preserving ones are studied in [2], [8] and [9]. Given a type τ. The following problem arise: Describe the order of hypersubstitutions of the type. So far this was done for type (2), (3), and $(2,2)$: It was shown that the order of a hypersubstitutions of type (2) is 1, 2 or infinite ([?]), of type (3) is 1, 2, 3 or infinite ([3]), and of type $(2,2)$ is 1, 2, 3, 4 or infinite ([5]). In this paper we are interested in the order of hypersubstitutions of type (n) for any $n \in \mathbb{N}$.

2 Main Results

For any $n \in \mathbb{N}$, the set of all mappings on $\{1,2,3,\ldots,n\}$ is denoted by T_n. It is known that T_n forms a semigroup under the usual composition of functions, the so-called a *transformation semigroup*. The semigroup T_n has n^n elements (see [10]). For $\alpha \in T_n$, let $fix(\alpha) = \{x \in \{1,2,3,\ldots,n\} : x\alpha = x\}$. We give an easy observation that for any $\alpha \in T_n$ if $fix(\alpha) \subset \{1,2,3,\ldots,n\}$, then there is $\alpha_0 \in T_{n_0}$ with $n_0 = |\{1,2,3,\ldots,n\} \setminus fix(\alpha)|$ which has the same order with α.

For convenience, we let f stand for n-ary operation symbol of type (n) and we denote a hypersubstitution of type (n) which maps the operation symbol f to the n-ary term t by σ_t. Further, let σ_α, for some $\alpha \in T_n$, be a hypersubstitution mapping f to $f(x_{\alpha(1)},\ldots,x_{\alpha(n)})$. We state an easy theorem to make our investigation complete.

Theorem 2.1. (1) *The order of a hypersubstitution in $P(n)$ is 1.*

(2) *The order of a hypersubstitution in $Short(n)$ is equal to the order of a mapping α for some $\alpha \in T_n$.*

Proof. (1) Obvious.

(2) Define a mapping $\varphi : Short(n) \to T_n$ by $\sigma_\alpha \mapsto \alpha$. It can be proved easily that φ is an anti-isomorphism. □

Now, we proceed to the case that hypersubstitutions come from H_2^{op}. In this case if $var(\sigma(f)) = X_n$, then we have the following theorem.

Theorem 2.2. *Let t be a term of type (n) with $op(t) > 1$. If $var(t) = X_n$, then σ_t has infinite order.*

Proof. Claim. If $s \in W_{(n)}(X_n) \setminus X_n$ with $var(s) = X_n$, then $op(s) < op(\hat{\sigma}_t[s])$.
We set $s = f(s_1,\ldots,s_n)$. Since $vb_k(\sigma_t(f)) \geq 1$, $k = 1,\ldots,n$ and $op(\sigma_t(f)) > 1$,

$$\begin{aligned}
op(\hat{\sigma}_t[s]) &= op(\sigma_t(f)(\hat{\sigma}_t[s_1],\ldots,\hat{\sigma}_t[s_n])) \\
&= vb_1(\sigma_t(f))op(\hat{\sigma}_t[s_1]) + \ldots + vb_n(\sigma_t(f))op(\hat{\sigma}_t[s_n]) + op(\sigma_t(f)) \\
&> op(\hat{\sigma}_t[s_1]) + \ldots + op(\hat{\sigma}_t[s_n]) + 1 \\
&\geq op(s_1) + \ldots + op(s_n) + 1 \\
&= op(s).
\end{aligned}$$

So we have the Claim. This gives, for $k \in \mathbb{N}$,

$$op(\sigma_t^{k+1}(f)) = op((\sigma_t\sigma_t^k)(f)) = op((\hat{\sigma}_t[\sigma_t^k(f)]) > op(\sigma_t^k(f)).$$

We conclude that σ_t has infinite order. □

Next, we will investigate the case that $var(t)$ is a proper subset of X_n, i.e. $var(t) \subset X_n$. Hereafter, we let

$$t = f(t_1, \ldots, t_n),\ t_1, \ldots, t_n \in W_{(n)}(X_n)$$

be an n-ary term of type (n) such that $op(t) > 1$ and $var(t) = \{x_{i_1}, \ldots, x_{i_j}\}$. We separate to three cases:

(1) $t_{i_1}, \ldots, t_{i_j} \in var(t)$,

(2) $t_{i_1}, \ldots, t_{i_j} \notin var(t)$,

(3) there is $j' \in \{1, \ldots, j-1\}$ such that $t_{i_1}, \ldots, t_{i_{j'}} \in var(t)$ and $t_{i_{j'+1}}, \ldots, t_{i_j} \notin var(t)$.

For cases (1) and (2) we have Theorem 2.3 and Theorem 2.4, respectively.

Theorem 2.3. *Let $t = f(t_1, \ldots, t_n)$ be an n-ary term of type (n) with $op(t) > 1$ and $var(t) = \{x_{i_1}, \ldots, x_{i_j}\} \subset X_n$. If $t_{i_1} = x_{r_{i_1}}, \ldots, t_{i_j} = x_{r_{i_j}} \in var(t)$, then the order of σ_t is $a + b$ where a and b are, respectively, index and period of some $\alpha_0 \in T_{n_0}$ and some $n_0 < n$.*

Proof. Assume that $t_{i_1} = x_{r_{i_1}}, \ldots, t_{i_j} = x_{r_{i_j}} \in var(t)$. Define $\alpha \in T_n$ by, for $u \in \{1, \ldots, n\}$,

$$\alpha(u) := \begin{cases} r_u & \text{if } u \in \{i_1, \ldots, i_j\}, \\ u & \text{otherwise.} \end{cases}$$

Clearly, the order of α is finite. Assume that α has index a and period b. Then $\alpha^a = \alpha^{a+b}$. Using property of α and assumption, we get

$$t(\hat{\sigma}_t[t_1], \ldots, \hat{\sigma}_t[t_n]) = t(x_{\alpha(1)}, \ldots, x_{\alpha(n)}).$$

It follows that,

$$\hat{\sigma}_t[t] = \hat{\sigma}_t[f(t_1, \ldots, t_n)] = t(\hat{\sigma}_t[t_1], \ldots, \hat{\sigma}_t[t_n]) = t(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$$

We can prove by induction that $(\hat{\sigma}_t)^k[t] = t(x_{\alpha^k(1)}, \ldots, x_{\alpha^k(n)})$ for every $k \in \mathbb{N}$. Hence $(\sigma_t)^{a+1}(f) = (\sigma_t)^{a+b+1}(f)$. This shows that the order of σ_t is $a+b$. Since $var(t) = \{x_{i_1}, \ldots, x_{i_j}\}$ is a proper subset of X_n, then there the set $B \subset \{1, \ldots, n\}$ such that $\alpha(b) = b$ for every $b \in B$ and $\alpha(b') \notin B$ for every $b' \notin B$. Hence, there is a transformation $\alpha_0 \in T_{n_0}$ for some $n_0 < n$ such that its index and period are the same as the index and period of α. □

Theorem 2.4. *Let $t = f(t_1, \ldots, t_n)$ be an n-ary term of type (n) with $op(t) > 1$ and $var(t) = \{x_{i_1}, \ldots, x_{i_j}\} \subset X_n$. If $t_{i_1}, \ldots, t_{i_j} \notin var(t)$, then the order of σ_t is infinite.*

Proof. Let $k \in \mathbb{N}$. We have $var((\sigma_t)^k(f)) \subseteq var(t)$. Since $t_{i_1}, \ldots, t_{i_k} \notin var(t)$, we have $(\hat{\sigma}_t)^k[t_{i_u}] \notin X_n$ for all $u = 1, \ldots, j$. Then,

$$\begin{aligned} op((\sigma_t)^{k+1}(f)) &= op((\hat{\sigma}_t)^k[t]) \\ &= op((\sigma_t)^k(f)[(\hat{\sigma}_t)^k[t_1], \ldots, (\hat{\sigma}_t)^k[t_n]]) \\ &> op((\sigma_t)^k(f)). \end{aligned}$$

This shows that the order of σ_t is infinite. □

To prove (3): there exists $j' \in \{1, \ldots, j-1\}$ such that $t_{i_1}, \ldots, t_{i_{j'}} \in var(t)$ and $t_{i_{j'+1}}, \ldots, t_{i_j} \notin var(t)$, we need the following: It has been known that terms can be represented by tree, i.e., connected graph without cycles having a root. For any term t of type (n), we can label each operation symbol or variable of t by a sequence of numbers from $\{1, \ldots, n\}$, by using the address of the corresponding node in the tree diagram for t; the operation symbol at the root of the tree receives the label 0. For instance, the term $t = f(f(x_1, x_2, x_3), x_3, x_2)$ of type (3) can be written with labels as $L(t) := f^0(f^1(x_1^{11}, x_2^{12}, x_3^{13}), x_3^2, x_2^3)$. For $N \subset \{1, \ldots, n\}$, we will be interested in the set $var(L(t)) \setminus N$ of (labeled) variables occurring in t whose addresses do not contain any occurrences of $k \in N$.

We make the key observation that if t satisfies (3.1.2.1), then $var(\hat{\sigma}_t[t]) \subseteq \{x_{i_1}, \ldots, x_{i_j}\}$; if t satisfies (3.1.2.2), then $var(\hat{\sigma}_t^w[t]) \cap \{x_{i_{j+1}}, \ldots, x_{j_k}\} \neq \emptyset$ for all $w \in \mathbb{N}$. Now, we have the following.

Theorem 2.5. *Let $t = f(t_1, \ldots, t_n)$ be an n-ary term of type (n) with $op(t) > 1$ and $var(t) = \{x_{i_1}, \ldots, x_{i_j}\} \subset X_n$. If there is $j' < j$ such that $t_{i_1}, \ldots, t_{i_{j'}} \in var(t)$, $t_{i_{j'+1}}, \ldots, t_{i_j} \notin var(t)$, then either: the order of σ_t is equal to 1 plus the order of some transformation $\alpha \in T_{n_0}$ for some $n_0 < n$; or the order of σ_t is infinite.*

Proof. Assume that there is $j' < j$ such that $t_{i_1}, \ldots, t_{i_{j'}} \in var(t)$, $t_{i_{j'+1}}, \ldots, t_{i_j} \notin var(t)$. We separate to three subcases:

(3.1) $t_{i_1}, \ldots, t_{i_{j'}} \in \{x_{i_1}, \ldots, x_{i_{j'}}\}$,

(3.2) $t_{i_1}, \ldots, t_{i_{j'}} \in \{x_{i_{j'+1}}, \ldots, x_{i_j}\}$, i.e. $t_{i_1}, \ldots, t_{i_{j'}} \notin \{x_{i_1}, \ldots, x_{i_{j'}}\}$,

(3.3) $t_{i_1}, \ldots, t_{i_{j''}} \in \{x_{i_1}, \ldots, x_{i_{j'}}\}$ and $t_{i_{j''+1}}, \ldots, t_{i_{j'}} \in \{x_{i_{j'+1}}, \ldots, x_{i_j}\}$ for some $j'' \in \{1, \ldots, j'-1\}$.

Case (3.1). We separate to the following cases:

(3.1.1) $var(t_{i_{j'+1}}), \ldots, var(t_{i_j}) \subseteq \{t_{i_1}, \ldots, t_{i_{j'}}\}$,

(3.1.2) there is $l \in \{i_{j'+1}, \ldots, i_j\}$ such that $var(t_l) \cap \{x_{i_{j'+1}}, \ldots, x_{i_j}\} \neq \emptyset$.

(3.1.2.1) every t_l satisfying (3.1.2) have the property that every $x_{i_u} \in var(t_l) \cap \{x_{i_{j'+1}}, \ldots, x_{i_j}\}$ is not labeled with address containing the number i_v for every i_v with $x_{i_v} \in var(t_l) \cap \{x_{i_{j'+1}}, \ldots, x_{i_j}\}$,

(3.1.2.2) there is t_l satisfying (3.1.2) have the property that there are $x_{i_u}, x_{i_v} \in var(t_l) \cap \{x_{i_{j'+1}}, \ldots, x_{i_j}\}$ such that x_{i_u} is labeled by address containing i_v,

Case (3.1.1). Without of generality we may assume that $t_{i_1} = x_{r_{i_1}}, \ldots, t_{i_{j'}} = x_{r_{i_{j'}}}$ with $r_{i_1}, \ldots, r_{i_{j'}} \in \{i_1, \ldots, i_{j'}\}$. Define a mapping $\alpha \in T_n$ by, for $u \in \{1, \ldots, n\}$,

$$\alpha(u) := \begin{cases} r_{i_u} & \text{if } u \in \{i_1, \ldots, i_{j'}\}, \\ u & \text{otherwise.} \end{cases}$$

Clearly, the order of α is less than n^n. Assume that α has index a and period b. Then $\alpha^a = \alpha^{a+b}$. Since $var(t_{i_{j'+1}}), \ldots, var(t_{i_j}) \subseteq \{t_{i_1}, \ldots, t_{i_{j'}}\} \subset \{x_{i_1}, \ldots, x_{i_{j'}}\}$, we have

$$\hat{\sigma}_t[t] = \hat{\sigma}_t[f(t_1, \ldots, t_n)] = t(\hat{\sigma}_t[t_1], \ldots, \hat{\sigma}_t[t_n]) = t(x_{\alpha(1)}, \ldots, x_{\alpha(n)}).$$

It can be proved by induction that: for $k \in \mathbb{N}$, $(\hat{\sigma}_t)^k[t] = t(x_{\alpha^k(1)}, \ldots, x_{\alpha^k(n)})$. Therefore, $(\hat{\sigma}_t)^{a+1}[t] = (\hat{\sigma}_t)^{a+b+1}[t]$. This shows that the order of σ_t is equal to $a + b$. Hence, there is a transformation $\alpha_0 \in T_{n_0}$ for some $n_0 < n$ such that its index and period are the same as the index and period of α.

Case (3.1.2.1). Assume that t satisfies (3.1.2.1). Then $var(\hat{\sigma}_t[t]) \subseteq \{x_{i_1}, \ldots, x_{i_{j'}}\}$. Define a mapping $\alpha \in T_n$ by for $w \in \{1, \ldots, n\}$,

$$\alpha(w) := \begin{cases} i_w & \text{if } w \in \{1, \ldots, j\}, \\ w & \text{otherwise.} \end{cases}$$

We have that the order of α is less than n^n. Assume that α has index a and period b. Since $\hat{\sigma}_t[t] = t(\hat{\sigma}_t[t_1], \ldots, \hat{\sigma}_t[t_n])$ and $var(\hat{\sigma}_t[t]) \subseteq \{x_{i_1}, \ldots, x_{i_j}\}$ we have

$$\begin{aligned} \hat{\sigma}_t^2[t] &= \hat{\sigma}_t[t](\hat{\sigma}_t^2[t_1], \ldots, \hat{\sigma}_t^2[t_n]) \\ &= \hat{\sigma}_t[t](x_{\alpha(1)}, \ldots, x_{\alpha(n)}) \\ &= t(\hat{\sigma}_t[t_1](x_{\alpha(1)}, \ldots, x_{\alpha(n)}), \ldots, \hat{\sigma}_t[t_n](x_{\alpha(1)}, \ldots, x_{\alpha(n)})), \end{aligned}$$

and

$$\begin{aligned}\hat{\sigma}_t^3[t] &= \hat{\sigma}_t[t](\hat{\sigma}_t^2[t_1](x_{\alpha(1)},\ldots,x_{\alpha(n)}),\ldots,\hat{\sigma}_t^2[t_n](x_{\alpha(1)},\ldots,x_{\alpha(n)}))\\ &= \hat{\sigma}_t[t](x_{\alpha^2(1)},\ldots,x_{\alpha^2(n)}).\end{aligned}$$

It can be proved by induction that for $k \in \mathbb{N}$,

$$\hat{\sigma}_t^{k+1}[t] = \hat{\sigma}_t[t](x_{\alpha^k(1)},\ldots,x_{\alpha^k(n)})$$

Therefore, $(\hat{\sigma}_t)^{a+1}[t] = (\hat{\sigma}_t)^{a+b+1}[t]$. This shows that the order of σ_t is equal to $a+b$. Since $\{x_{i_1},\ldots,x_{i_{j'}}\} \subset X_n$, there is a transformation $\alpha_0 \in T_{n_0}$ for some $n_0 < n$ such that its index and period are the same as the index and period of α.
Case (3.1.2.2). As mentioned, we have $var(\hat{\sigma}_t^w[t]) \cap \{x_{i_{j+1}},\ldots,x_{j_k}\} \neq \emptyset$ for all $w \in \mathbb{N}$. Then, for $k \in \mathbb{N}$, we have

$$\begin{aligned}op((\sigma_t)^{k+2}(f)) &= op((\hat{\sigma}_t)^k[f(t_1(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]),\ldots,t_n(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]))])\\ &= op((\sigma_t)^k(f)[(\hat{\sigma}_t)^k[t_1(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n])],\ldots,(\hat{\sigma}_t)^k[t_n(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n])]])\\ &> op((\sigma_t)^k(f))\end{aligned}$$

This shows that the order of σ_t is infinite.
Case (3.2). We have

$$\hat{\sigma}_t[t] = t(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]) = f(t_1(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]),\ldots,t_n(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n])).$$

Since $t_{i_{j+1}},\ldots,t_{i_k} \notin var(t)$ and $t_{i_1},\ldots,t_{i_j} \notin \{x_{i_1},\ldots,x_{i_j}\}$, it follows that

$$t_1(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]),\ldots,t_n(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]) \notin X_n.$$

Let $k \in \mathbb{N}$. We have $var((\sigma_t)^k(f)) \subseteq var(t)$. Now

$$\begin{aligned}op((\sigma_t)^{k+2}(f)) &= op((\hat{\sigma}_t)^k[f(t_1(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]),\ldots,t_n(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n]))])\\ &= op((\sigma_t)^k(f)[(\hat{\sigma}_t)^k[t_1(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n])],\ldots,(\hat{\sigma}_t)^k[t_n(\hat{\sigma}_t[t_1],\ldots,\hat{\sigma}_t[t_n])]])\\ &> op((\sigma_t)^k(f))\end{aligned}$$

This shows that the order of σ_t is infinite.
Case (3.3): there is $j'' \in \{1,\ldots,j'-1\}$ such that $t_{i_1},\ldots,t_{i_{j''}} \in \{x_{i_1},\ldots,x_{i_{j'}}\}$ and $t_{i_{j''+1}},\ldots,t_{i_{j'}} \in \{x_{i_{j''+1}},\ldots,x_{i_{j'}}\}$.
We separate to the following cases:

(3.3.1) $t_{i_1},\ldots,t_{i_{j''}} \in \{x_{i_1},\ldots,x_{i_{j''}}\}$,
(3.3.2) $t_{i_1},\ldots,t_{i_{j''}} \in \{x_{i_{j'+1}},\ldots,x_{i_{j'}}\}$,
(3.3.3) there is $j''' \in \{1,\ldots,j''\}$ such that $t_{i_1},\ldots,t_{i_{j'''}} \in \{x_{i_1},\ldots,x_{i_{j''}}\}$ and $t_{i_{j'''+1}},\ldots,t_{i_{j''}} \in \{x_{i_{j'+1}},\ldots,x_{i_{j''}}\}$.

Cases (3.3.1), (3.3.2) and (3.3.3) can be proved similarly as cases (3.1.1), (3.1.2) and (3.1.3), respectively.
Continue in this way, we can have only the following two cases left:

(I) $t_{i_1},\ldots,t_{i_{j^*}} \in \{x_{i_1},\ldots,x_{i_{j^*}}\}$,
(II) $t_{i_1},\ldots,t_{i_{j^*}} \in \{x_{i_{j^*+1}},\ldots,x_{i_{j^*}}\}$.

Both of the cases can be proved similarly as cases (3.1.1) and (3.1.2), respectively. This completes the proof. □

Using the main theorem, we have the following immediately.

Corollary 2.6. *The order of hypersubstitutions of type* (n) *for* $n \in \{1,2,3,4\}$ *is* $1,2,3,\ldots,n$ *or infinite.*

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