รายงานวิจัยฉบับสมบูรณ์

โครงการ "การระบายสีเส้นเชื่อมแบบเข้มในกราฟ"

Strong-edge colorings in graphs

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

โดย

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พฤษภาคม 2551 – เมษายน 2553

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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บทคัดย่อ: ใน [Y. Zhang and H.P. Yap, Equitable colorings of planar graphs, *J. Combin. Math. Conbin. Comput.* 27(1998), 97–105.] Zhang และ Yap ได้พิสูจน์ว่ากราฟระนาบที่มีระดับขั้นสูงสุด Δ ไม่น้อยกว่า 13 สามารถให้สีแบบเท่าเทียมได้ด้วยสี Δ สี ในรายงานวิจัยฉบับนี้เราได้พิสูจน์ว่ากราฟระนาบในคลาสต่าง ๆ โดยเฉพาะกราฟระนาบที่มีระดับขั้นสูงสุดคือ $9, 10, 11$ และ 12 สามารถให้สีแบบเท่าเทียมกันได้ด้วยสี Δ สีได้ จึงได้ผลตามมาว่า กราฟระนาบที่มีระดับขั้นสูงสุด Δ ไม่น้อยกว่า 9 สามารถให้สีแบบเท่าเทียมได้ด้วยสี Δ สี

คำหลัก: การให้สีแบบเท่าเทียม, กราฟระนาบ

Project Code:	MRG5180194
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Project Period:	May 2008–April 2010

Abstract: In [Y. Zhang and H.P. Yap, Equitable colorings of planar graphs, *J. Combin. Math. Conbin. Comput.* 27(1998), 97–105.], Zhang and Yap essentially proved that each planar graph with maximum degree Δ at least 13 has an equitable Δ-coloring. In this report, we proved that each planar graph in various classes has an equitable Δ-coloring, especially planar graphs with maximum degree $9, 10, 11$, and 12. Consequently, each planar graph with maximum degree Δ at least 9 has an equitable Δ-coloring.

Keywords: equitable colorings, planar graphs

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ในการทำงานวิจัยค้นคว้าครั้งนี้

ศาสตราจารย์ ดร. ณรงค์ ปั้นนิ่ม ที่ตอบรับเป็นนักวิจัยที่ปรึกษาให้กับโครงการนี้ และยังให้คำแนะนำที่ดีต่าง ๆ มาโดยตลอด

คณะผู้ประเมิน (referee) ของวารสาร Journal of Discrete Mathematics ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสาร

คณาจารย์ และเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ มหาวิทยาลัยขอนแก่น ที่ได้ช่วยเหลือโครงการวิจัยในครั้งนี้

Chapter 1

Executive Summary

An *equitable coloring* of a graph is a proper vertex coloring such that the sizes of every two color classes differ by at most 1. We say that G is *equitably k-colorable* if G has an equitable k-coloring.

It is known that determining if a planar graph with maximum vertex degree 4 is 3-colorable is NP-complete. For a given n-vertex planar graph G with maximum vertex degree 4, let G' be obtained from G by adding $2n$ isolated vertices. Then G is 3-colorable if and only if G' is equitably 3-colorable. Thus finding the minimum number of colors need to color a graph equitably even for planar graphs G is an NP-complete problem.

Hajnal and Szemerédi settled a conjecture of Erdős by proving that every graph G with maximum degree at most Δ has an equitable k-coloring for every $k \geq 1 + \Delta$. In its 'complementary' form this result concerns decompositions of a sufficiently dense graph into cliques of equal size. This result is now known as Hajnal and Szemerédi Theorem. Later, Kierstead and Kostochka gave a simpler proof of Hajnal and Szemerédi Theorem in the direct form of equitable coloring. The bound of the Hajnal-Szemerédi theorem is sharp, but it can be improved for some important classes of graphs. In fact, Chen, Lih, and Wu put forth the following conjecture.

Conjecture 1 *Every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd* *cycle or* Δ *is odd and* $G = K_{\Delta,\Delta}$.

Chen, Lih, and Wu proved the conjecture for graphs with $\Delta(G) \leq 3$ of $\Delta(G) \geq |G|/2$. Lih and Wu proved the conjecture for bipartite graphs. Meyer proved that every forest with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$ colors. This result implies conjecture holds for forests. The bound of Meyer is attained at the complete bipartite $K_{1,m}$: in every proper coloring of $K_{1,m}$, the center vertex forms a color class, and hence the remaining vertices need at least $m/2$ colors. Yap and Zhang proved that the conjecture holds for outerplanar graphs. Later Kostochka extended the result for outerplanar graphs by proving that every outerplanar graph with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$. Again this bound is sharp.

Zhang and Yap essentially proved the conjecture holds for planar graphs with maximum degree $\Delta \geq 13$.

We are mainly interested in proving the conjecture in various classes of planar graphs. In this report, we proved that conjecture holds for planar graphs in various classes, especially a class of planar graphs with maximum degree $9, 10, 11$, and 12. Consequently, the conjecture holds for planar graphs with maximum degree at least 9.

Chapter 2

Main Results

Chen, Lih, and Wu put the following conjecture which is a main topic on the study of equitable coloring.

Conjecture 2 *Every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd cycle or Δ is odd and $G = K_{\Delta,\Delta}$.*

We published one paper in this topic (see Appendix **A1**). We proved that conjecture holds for planar graphs in various classes, especially a class of planar graphs with maximum degree $9, 10, 11$, and 12. Consequently, the conjecture holds for planar graphs with maximum degree at least 9.

2.1 Equitable Colorings of Planar Graphs with Maximum Degree at least Nine

Throughout this report, all graphs are finite, undirected and simple. We use $V(G)$, $|G|, E(G), e(G), \Delta(G)$, and $\delta(G)$, respectively, to denote vertex set, order, edge set, size, maximum degree, and minimum degree of a graph G. We write $xy \in E(G)$ if x and y are adjacent. The graph obtained by deleting an edge xy from G is denoted by $G\setminus\{xy\}$. For any vertex v in $V(G)$, let $N_G(v)$ be the set of all neighbors of v in G. *The degree of* v, denoted by $d_G(v)$, is equal to $|N_G(v)|$. We use $d(v)$ instead of $d_G(v)$ if no confusion arises. For disjoint subsets U and W of $V(G)$, the number of edges with one end in U and other in W is denoted by $e(U, W)$. we use $G[U]$ to denote the subgraph of G induced by U.

2.1.1 Preliminaries

Many proofs in this report involve edge-minimal planar graph that is not equitably m-colorable. In this section, we describe some properties of such graph that appear recurrently in later arguments. The following fact about planar graphs in general is well-known and can be found in standard texts about graph theory.

Lemma 1 *For any planar graph G of order $n, e(G) \leq 3n - 6$ and $\delta(G) \leq 5$.*

Let G be an edge-minimal planar graph with $|G| = mt$, where t is an integer, such that G is not equitably m-colorable. As G is planar, G has an edge xy where $d(x) = \delta \leq 5$. By edge-mimimality of G, the graph $G \setminus \{xy\}$ has an equitable m-coloring ϕ having color classes $V_1', V_2', \ldots, V_m'$. It suffices to consider only the case that $x, y \in V_1'$. Let $1 \leq \delta \leq 5$ be the minimum degree of non-isolated vertices. Choose x with degree δ and order $V_1', V_2', \ldots \cup V_\delta'$ in a way that $N(x) \subseteq V_1' \cup V_2' \cup \cdots \cup V_\delta'$. Define $V_1 = V_1' \setminus \{x\}$ and $V_i = V_i'$ for each $i = 2, 3, \ldots, m$.

We define $\mathcal{R}$ recursively. Let $V_1 \in \mathcal{R}$ and $V_j \in \mathcal{R}$ if there exists a vertex in V_j which has no neighbors in V_i for some $V_i \in \mathcal{R}$. Let $r = |\mathcal{R}|$. Let A and B denote $\bigcup_{V_i \in \mathcal{R}} V_i$ and $V(G) \setminus A$, respectively. Furthermore, we let A' denote $A \cup \{x\}$ and B' denote $B \setminus \{x\}$. From definition of $\mathcal{R}$ and $B, e(V_i, \{u\}) \geq 1$ for each $V_i \in \mathcal{R}$ and $u \in B$. Consequently $e(A, B) \geq r[(m - r)t + 1]$ and $e(A', B') \geq r(m - r)t$ if $r = |\mathcal{R}|$.

Suppose there is $V_k \in \mathcal{R}$ for some $i \geq \delta + 1$. By definitions of $\mathcal{R}$, there exist $u_1 \in V_{i_1}, u_2 \in V_{i_2}, \ldots, u_s \in V_{i_s}, u_{i_{s+1}} \in V_{i_{s+1}} = V_k$ such that $e(V_1, \{u_1\}) = e(V_{i_1}, \{u_2\}) = \cdots = e(V_{i_s}, \{u_{s+1}\}) = 0$. Letting $W_1 = V_1 \cup \{u_1\}, W_{i_1} = V_{i_1} \cup$

$\{u_2\}\backslash\{u_1\}, \ldots, W_{i_s} = V_{i_s} \cup \{u_{s+1}\}\backslash\{u_s\}$, and $W_k = V_k \cup \{x\}\backslash\{u_{s+1}\}$, otherwise $W_i = V_i$, we get an equitable m-coloring of G. This contradicts to the fact that G is a counterexample.

Thus we assume $\mathcal{R} \subseteq \{V_1, V_2, \ldots, V_\delta\}$ where $\delta \leq 5$ is the minimum degree of non-isolated vertices.

We summarize our observation here.

Observation 1 *If G is an edge-minimal planar graph of order mt such that G is not equitably m-colorable, then we may assume*
(i) $\mathcal{R} \subseteq \{V_1, V_2, \ldots, V_\delta\}$ where $\delta \leq 5$ is the minimum degree of non-isolated vertices;
(ii) $e(u, V_i) \geq 1$ for each $u \in B$ and $V_i \in \mathcal{R}$;
(iii) $e(A, B) \geq r[(m-r)t+1]$ and $e(A', B') \geq r(m-r)t$.

2.1.2 Helpful Lemmas

Lemma 2 *Let $m \geq 1$ be a fixed integer. Suppose that any planar graph of order mt with maximum degree at most Δ is equitably m-colorable for any integer $t \geq 1$. Then any planar graph with maximum degree at most Δ is also equitably m-colorable.*

Lemma 3 *If G is a graph with maximum degree $\Delta \geq |G|/2$, then G is equitably Δ-colorable.*

By Lemmas 2 and 3, it suffices to consider only the planar graph of order mt where $t \geq 3$ is a positive integer.

Lemma 4 *Let H be a graph of order mt with chromatic number $\chi \leq m$. If $e(H) \leq (m-1)t$, then H is equitably m-colorable.*

Lemma 5 *If a planar graph G has an independent s-set V' and there exists $B \subseteq V(G) \setminus V'$ such that $|B| > (3s + e(V', B))/2$ and u has a neighbor in V' for all $u \in B$, then B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$.*

Proof. Let $B_1 \subseteq B$ be such that each $v \in B_1$ is adjacent to exactly one vertex of V'. Let $k = |B_1|$. Then $k + 2(|B| - r) \leq e(V', B)$, from which it follows that $k \geq 2|B| - e(V', B) > 3s$. Hence V' contains a vertex γ which is adjacent to at least four vertices of B_1. Since G is planar, G does not induce K_5. Hence B_1 contains two nonadjacent vertices α and β which are adjacent to γ.

Lemma 6 *If a graph G has an independent s-set V' and there exists $B \subseteq V(G) \setminus V'$ such that $e(u, V') \geq 1$ for all $u \in B$, and $e(G[B]) + e(V', B) < 2|B| - s$, then B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$.*

Proof. Let $B(v_i)$ consist of vertices in B whose only neighbor in V' is v_i. Note that $B(v_i)$ forms a clique in B, otherwise we get those desired vertices. Each of the remaining vertices in $B \setminus \bigcup_{i=1}^{s} B(v_i)$ has at least 2 neighbors in V'. So $e(G[B]) + e(V', B) \geq \sum_{i=1}^{s} \binom{|B(v_i)|}{2} + 2|B| - \sum_{i=1}^{s} |B(v_i)| \geq 2|B| - s$. We get a contradiction here.

Notation. *Let $q_{m,\Delta}$ denote the maximum number not exceeding $3mt - 6$ such that each planar graph of order mt is equitably m-colorable if it has maximum degree at most Δ and size at most $q_{m,\Delta}$.*

Lemma 7 *Let G be an edge-minimal planar graph of order mt with maximum degree at most Δ that is not equitably m-colorable. If $e(G) \leq (r+1)(m-r)t - t + 2 + q_{r,\Delta}$, then B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1'$.*

Proof. If $e(G[A']) \leq q_{r,\Delta}$, then $G[A']$ is equitably r-colorable. Consequently, G is equitably m-colorable. So we suppose $e(G[A']) \geq q_{r,\Delta} + 1$. By Observation 1, $e(A' \setminus V_1', B') \geq (r-1)(m-r)t$. So $e(G[B']) + e(V_1', B') = e(G) - e(G[A']) - e(A' \setminus V_1', B') \leq 2mt - 2rt - t + 1 = 2|B'| - |V_1'|$. But $e(G[B]) = e(G[B'])$, $e(V_1', B) = e(V_1', B') + 1$, and $|B| = |B'| + 1$. So we have $e(G[B]) + e(V_1', B) + 1 \leq 2|B| - |V_1'|$. By Lemma 6, B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1'$.

Lemma 8 *If G is an edge-minimal planar graph of order mt with maximum degree at most Δ that is not equitably m-colorable, then $e(G) \geq r(m-r)t + q_{r,\Delta} + 1$.*

Proof. Suppose $e(G) \leq r(m-r)t + q_{r,\Delta}$. By Observation 1, $e(A', B') \geq r(m-r)t$. So $e(G[A']) \leq q_{r,\Delta}$, which implies $G[A']$ is equitably r-colorable. Thus G is equitably m-colorable. This contradiction completes the proof.

Lemma 9 *Let G be an edge-minimal planar graph of order mt with maximum degree at most Δ that is not equitably m-colorable. If B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1$, then $e(G) \geq r(m-r)t + q_{r,\Delta} + q_{m-r,\Delta} - \Delta + 4$.*

Proof. Suppose $e(G) \leq r(m-r)t + q_{r,\Delta} + q_{m-r,\Delta} - \Delta + 3$. If $e(G[A']) \leq q_{r,\Delta}$, then $G[A']$ is equitably r-colorable. Consequently, G is equitably m-colorable. So we suppose $e(G[A']) \geq q_{r,\Delta} + 1$. This with Observation 1 implies $e(G[A']) + e(A, B') \geq q_{r,\Delta} + 1 + r(m-r)t$. Note that $e(G[A']) + e(A, B') = e(G[A]) + e(A, B)$. Let $A_1 = A \setminus \{\gamma\} \cup \{\alpha, \beta\}$ and $B_1 = B \cup \{\gamma\} \setminus \{\alpha, \beta\}$. Then $e(G[A_1]) + e(A_1, B_1) \geq e(G[A]) + e(A, B) - \Delta + 2 \geq q_{r,\Delta} + 1 + r(m-r)t - \Delta + 2$. So $e(G[B_1]) = e(G) - e(G[A_1]) + e(A_1, B_1) \leq q_{m-r,\Delta}$ which implies $G[B_1]$ is equitably $(m-r)$-colorable. Combining with $V_1 \setminus \{\gamma\} \cup \{\alpha, \beta\}, V_2, \ldots, V_r$, we have G equitably m-colorable which is a contradiction.

Corollary 1 *Let G be an edge-minimal planar graph of order mt with maximum degree at most Δ that is not equitably m-colorable. Then $e(G) \geq r(m-r)t + q_{r,\Delta} + q_{m-r,\Delta} - \Delta + 4$ if one of the following conditions are satisfied;*
(i) $(m-r)t + 1 > (t-1)(3+\Delta)/2$;
(ii) $e(G) \leq (r+1)(m-r)t - t + 2 + q_{r,\Delta}$, *then* $e(G) \geq r(m-r)t + q_{r,\Delta} + q_{m-r,\Delta} - \Delta + 4$.

Proof. This is a direct consequence of Lemmas 5, 7, and 9.

2.1.3 Results on Planar Graphs

Lemma 10 *(i)* $q_{1,\Delta} = 0$. *(ii)* $q_{2,\Delta} \geq 2$. *(iii)* $q_{3,\Delta} \geq 3$. *(iv)* $q_{4,\Delta} \geq 3t$.

Proof. (i), (ii), (iii) are obvious. (iv) is the result of Lemma 4.

Lemma 11 $q_{5,\Delta} \geq \min\{5t+4, 7t-\Delta+3\}$ *for* $\Delta \geq 5$.

Proof. Let G be an edge-minimal graph counterexample of order $5t$ and $e(G) \leq \min\{5t+4, 7t-\Delta+3\}$. Let δ be the minimum degree of non-isolated vertices. For $\delta = 5$, color non-isolated vertices with 5 colors. If one color class has at least $t+1$ non-isolated vertices, then G has at least $5t+5$ edges which is a contradiction. Thus every color class has at most t non-isolated vertices. Next, we can add isolated vertices to each color class to have size t. This result is an equitable 5-coloring of G. This is also a contradiction.

So we suppose $r \leq \delta \leq 4$. If $r = 2, 3$, or 4, then $e(G) \geq \min\{6t+3, 7t+1\}$ by Lemmas 8 and 10. If $r = 1$, then $e(G) \geq 7t - \Delta + 4$ by Corollary 1 and Lemma 10. Since we obtain contradiction for all cases, the counterexample is impossible.

Lemma 12 $q_{6,\Delta} \geq \min\{11t-\Delta+5, 9t+2, 5t+q_{5,\Delta}-\Delta+3\}$ *for* $\Delta \geq 6$.

Proof. Let $\Delta \geq 6$. Suppose G' is a planar graph with maximum degree at most Δ and $e(G') \leq \min\{11t-\Delta+5, 9t+2, 5t+q_{5,\Delta}-\Delta+3\}$ but G' is not equitably 6-colorable. Let $G \subseteq G'$ be an edge-minimal graph that is not equitably 6-colorable. From Table 2.1.1, $e(G) > e(G')$. This contradiction completes the proof.

Lemma 13 *(i)* $q_{7,\Delta} \geq \min\{15t+6-\Delta, 14t+4, 10t+q_{5,\Delta}+5-\Delta, 6t+q_{6,\Delta}+3-\Delta\}$ *for* $7 \leq \Delta \leq 9$.
(ii) $q_{7,\Delta} \geq \min\{15t+6-\Delta, 10t+q_{5,\Delta}+5-\Delta, 11t+2, 6t+q_{6,\Delta}+3-\Delta\}$ *for* $\Delta \geq 10$.

Proof. Use Table 2.1.2 for an argument similar to the proof of Lemma 12.

r	lower bounds on size	Reason
5	$5t + q_{5,\Delta} + 1$	Lemma 8
4	$11t + 1$	Lemmas 8, 10
3	$9t + 4$	Lemmas 8, 10
2	$11t + 5$ or $11t - \Delta + 6$	Corollary 1(ii), Lemma 10
1	$9t + 3$ or $5t + q_{5,\Delta} - \Delta + 4$	Corollary 1(ii), Lemma 10

Table 2.1.1: Lower bounds on size of G in the proof of Lemma 12

r	lower bounds on size	Reason
5	$10t + q_{5,\Delta} + 1$	Lemma 8
4	$15t + 1$	Lemmas 8, 10
3	$15t + 6$ or $15t + 7 - \Delta$	Corollary 1(i), Lemma 10.
2	$14t + 5$ or $10t + q_{5,\Delta} + 6 - \Delta$	Corollary 1(i), Lemma 10
1	$6t + q_{6,\Delta} + 4 - \Delta$ for $7 \le \Delta \le 9$	Corollary 1(i), Lemma 10
1	$11t + 3$ or $6t + q_{6,\Delta} + 4 - \Delta$ for $\Delta \ge 10$	Corollary 1(ii), Lemma 10

Table 2.1.2: Lower bounds on size of G in the proof of Lemma 13

Lemma 14 *(i)* $q_{8,\Delta} \ge \min\{19t, 15t + q_{5,\Delta} + 6 - \Delta, 12t + q_{6,\Delta} + 5 - \Delta, 7t + q_{7,\Delta} + 3 - \Delta\}$ *for* $\Delta = 8$ *or* 9.
(ii) $q_{8,\Delta} \ge \min\{19t, 15t+q_{5,\Delta}+6-\Delta, 17t+4, 12t+q_{6,\Delta}+5-\Delta, 7t+q_{7,\Delta}+3-\Delta\}$ *for* $\Delta = 10$ *or* 11.
(iii) $q_{8,\Delta} \ge \min\{15t+q_{5,\Delta}+6-\Delta, 12t+q_{6,\Delta}+5-\Delta, 13t+2, 7t+q_{7,\Delta}+3-\Delta\}$ *for* $\Delta \ge 12$.

Proof. Use Table 2.1.3 for an argument similar to the proof of Lemma 12.

Lemma 15 *(i)* $q_{9,\Delta} \ge \min\{20t + q_{5,\Delta}, 23t, 18t + q_{6,\Delta} + 6 - \Delta, 14t + q_{7,\Delta} + 5 - \Delta, 8t + q_{8,\Delta} + 3 - \Delta\}$ *for* $9 \le \Delta \le 11$.
(ii) $q_{9,\Delta} \ge \min\{20t+q_{5,\Delta}, 23t, 18t+q_{6,\Delta}+6-\Delta, 20t+4, 14t+q_{7,\Delta}+5-\Delta, 8t+q_{8,\Delta} + 3 - \Delta\}$ *if* $\Delta = 12$.

Proof. Use Table 2.1.4 for an argument similar to the proof of Lemma 12.

r	lower bounds on size	Reason
5	$15t + q_{5,\Delta} + 1$	Lemma 8
4	$19t + 1$	Lemmas 8 and 10
3	$19t + 6$ or $15t + q_{5,\Delta} + 7 - \Delta$	Corollary 1(i) and Lemma 10
2	$17t + 5$ or $12t + q_{6,\Delta} + 6 - \Delta$ for $\Delta \geq 10$	Corollary 1(i) and Lemma 10
2	$12t + q_{6,\Delta} + 6 - \Delta$for $8 \leq \Delta \leq 9$	Corollary 1(ii) and Lemma 10
1	$13t + 2$ or $7t + q_{7,\Delta} + 4 - \Delta$ for $\Delta \geq 12$	Corollary 1(i) and Lemma 10
1	$7t + q_{7,\Delta} + 4 - \Delta$ for $8 \leq \Delta \leq 11$	Corollary 1(ii) and Lemma 10

Table 2.1.3: Lower bounds on size of G in the proof of Lemma 14
endtable

r	lower bounds on size	Reason
5	$20t + q_{5,\Delta} + 1$	Lemma 8
4	$23t + 1$	Lemmas 8 and 10
3	$23t + 6$ or $18t + q_{6,\Delta} + 7 - \Delta$	Corollary 1(i) and Lemma 10.
2	$20t + 5$ or $14t + q_{7,\Delta} + 6 - \Delta$ for $\Delta = 12$	Corollary 1(i) and Lemma 10
2	$14t + q_{7,\Delta} + 6 - \Delta$ for $9 \leq \Delta \leq 11$	Corollary 1(ii) and Lemma 10
1	$8t + q_{8,\Delta} + 4 - \Delta$	Corollary 1(ii) and Lemma 10

Table 2.1.4: Lower bounds on size of G in the proof of Lemma 15

Lemma 16 $q_{10,\Delta} \geq \min\{25t + q_{5,\Delta}, 27t, 21t + q_{7,\Delta} + 6 - \Delta, 16t + q_{8,\Delta} + 5 - \Delta, 9t + q_{9,\Delta} + 3 - \Delta\}$ *for* $\Delta = 10, 11,$ *or* 12.

Proof. Use Table 2.1.5 for an argument similar to the proof of Lemma 12.

Lemma 17 $q_{11,\Delta} \geq \min\{30t + q_{5,\Delta}, 31t, 24t + q_{8,\Delta} + 6 - \Delta, 18t + q_{9,\Delta} + 5 - \Delta, 10t + q_{10,\Delta} + 3 - \Delta\}$ *for* $\Delta = 11$ *or* 12.

Proof. Use Table 2.1.6 for an argument similar to the proof of Lemma 12.

Corollary 2 *(A1)* $q_{5,9}$ *is at least* $5t, 5t + 2,$ *and* $5t + 4$ *for* t *at least* $3, 4,$ *and* 5, *respectively.*

r	lower bounds on size	Reason
5	$25t + q_{5,\Delta} + 1$	Lemma 8
4	$27t + 1$	Lemmas 8 and 10
3	$21t + q_{7,\Delta} + 7 - \Delta$	Corollary 1(ii) and Lemma 10.
2	$16t + q_{8,\Delta} + 6 - \Delta$	Corollary 1(ii) and Lemma 10
1	$9t + q_{9,\Delta} + 4 - \Delta$	Corollary 1(ii) and Lemma 10

Table 2.1.5: Lower bounds on size of G in the proof of Lemma 16

r	lower bounds on size	Reason
5	$30t + q_{5,\Delta} + 1$	Lemma 8
4	$31t + 1$	Lemmas 8 and 10
3	$24t + q_{8,\Delta} + 7 - \Delta$	Corollary 1(ii) and Lemma 10.
2	$18t + q_{9,\Delta} + 6 - \Delta$	Corollary 1(ii) and Lemma 10
1	$10t + q_{10,\Delta} + 4 - \Delta$	Corollary 1(ii) and Lemma 10

Table 2.1.6: Lower bounds on size of G in the proof of Lemma 17

(A2) $q_{5,10}$ *is at least* $5t - 1, 5t + 1,$ *and* $5t + 4$ *for* t *at least* $3, 4,$ *and* 6, *respectively.*

(A3) $q_{5,11}$ *is at least* $5t - 2, 5t,$ *and* $5t + 4$ *for* t *at least* $3, 4,$ *and* 6, *respectively.*

(A4) $q_{5,12}$ *is at least* $5t - 3, 5t + 1,$ *and* $5t + 4$ *for* t *at least* $3, 5,$ *and* 7, *respectively.*

(B1) $q_{6,9}$ *is at least* $9t - 3, 9t,$ *and* $9t + 2$ *for* t *at least* $3, 4,$ *and* 5, *respectively.*

(B2) $q_{6,10}$ *is at least* $9t - 5, 9t - 2,$ *and* $9t + 2$ *for* t *is at least* $3, 4,$ *and* 6, *respectively.*

(B3) $q_{6,11}$ *is at least* $9t - 7, 9t - 4,$ *and* $9t + 2$ *for* t *at least* $3, 4,$ *and* 6, *respectively.*

(B4) $q_{6,12}$ *is at least* $9t - 9, 9t - 3,$ *and* $9t + 2$ *for* t *at least* $3, 5,$ *and* 7, *respectively.*

(C1) $q_{7,9}$ *is at least* $14t - 6, 14t - 2, 14 + 1,$ *and* $14t + 4$ *for* t *at least* $3, 4, 5,$ *and* 8, *respectively.*

(C2) $q_{7,10}$ *is at least* $11t$ *and* $11t + 2$ *for* t *at least* 3 *and* 4, *respectively.*

(C3) $q_{7,11}$ *is at least* $11t-3$ *and* $11t+2$ *for* t *at least* 3 *and* 4, *respectively.*

(C4) $q_{7,12}$ *is at least* $11t-6$ *and* $11t+2$ *for* t *at least* 3 *and* 5, *respectively.*

(D1) $q_{8,9}$ *is at least* $19t-6$ *and* $19t$ *for* t *at least* 3 *and* 4, *respectively.*

(D2) $q_{8,10}$ *is at least* $17t-4, 17t-1, 17t+1,$ *and* $17t+4$ *for* t *at least* $3, 4, 6,$ *and* 9, *respectively.*

(D3) $q_{8,11}$ *is at least* $17t-8, 17t-2, 17t,$ *and* $17t+4$ *for* t *at least* $3, 4, 6,$ *and* 10, *respectively.*

(D4) $q_{8,12}$ *is at least* $13t$ *and* $13t+2$ *for* t *at least* 3 *and* 4, *respectively.*

(E1) $q_{9,10}$ *is at least* $23t-5$ *and* $23t$ *for* t *at least* 3 *and* 4, *respectively.*

(E2) $q_{9,11}$ *is at least* $23t-10, 23t-2,$ *and* $23t$ *for* t *at least* $3, 4,$ *and* 5, *respectively.*

(E3) $q_{9,12}$ *is at least* $20t-6, 20t-2, 20t,$ *and* $20t+4$ *for* t *at least* $3, 5, 7,$ *and* 13, *respectively.*

(F1) $q_{10,11}$ *is at least* $27t-3$ *and* $27t$ *for* t *at least* 3 *and* 4, *respectively.*

(F2) $q_{10,12}$ *is at least* $27t-9, 27t-1,$ *and* $27t$ *for* t *at least* $3, 5,$ *and* 6, *respectively.*

(G1) $q_{11,12}$ *is at least* $31t$ *for* t *at least* 3.

Proof. The result can be calculated directly from Lemmas 11 to 17.

Theorem 1 *Each planar graph with maximum degree at most* $\Delta \geq 9$ *has an equitable* Δ*-coloring.*

Proof. Since Zhang and Yap proved the case of $\Delta \geq 13$, it suffices to show only the case $\Delta = 9, 10, 11$, or 12. By Lemmas 2 and 3, we consider only the case $|G| = \Delta t$ where $t \geq 3$ is a positive integer. Let G be an edge-minimal planar graph with maximum degree at most Δ but is not equitably Δ-colorable.

For $r = 5$, we have $e(G) \geq 5(\Delta-5)t + q_{5,\Delta} + q_{\Delta-5,\Delta} - \Delta + 4$ by Corollary 1. But $5(\Delta-5)t + q_{5,\Delta} + q_{\Delta-5,\Delta} - \Delta + 4 > 3\Delta t - 6$ by Corollary 2.

For $r = 4$, we have $e(G) \geq 4(\Delta-4)t + q_{4,\Delta} + q_{\Delta-4,\Delta} - \Delta + 4$ by Corollary 1. But $5(\Delta-4)t + q_{4,\Delta} + q_{\Delta-4,\Delta} - \Delta + 4 > 3\Delta t - 6$ by Lemma 10 and Corollary 2.

Consider the case $r = 3$. We have $e(B', V_1) \geq (\Delta-3)t$, by Observation 1. But y has at most $\Delta - 1$ neighbors in B' because $xy \in E(G)$, so

$(t-1)\Delta-1 \geq e(B', V_1)$. Consequently, $(t-1)\Delta-1 \geq (\Delta-3)t$. That is $t \geq 4$ when $\Delta = 9, 10$, and 11, and $t \geq 5$ when $\Delta = 12$. By Corollary 1, $e(G) \geq 3(\Delta-3)t+q_{3,\Delta}+q_{\Delta-3,\Delta}-\Delta+4$. But $3(\Delta-3)t+q_{3,\Delta}+q_{\Delta-3,\Delta}-\Delta+3 > 3\Delta t-6$ by Lemma 10 and Corollary 2.

Consider the case $r = 2$. Similar to the above case, we have $(t-1)\Delta-1 \geq (\Delta-2)t$. That is $t \geq 5$ when $\Delta = 9$, $t \geq 6$ when $\Delta = 10$ and 11, and $t \geq 7$ when $\Delta = 12$. By Corollary 1, $e(G) \geq 2(\Delta-2)t + q_{2,\Delta} + q_{\Delta-2,\Delta} - \Delta + 4$. But $2(\Delta-2)t + q_{2,\Delta} + q_{\Delta-2,\Delta} - \Delta + 3 > 3\Delta t - 6$ by Lemma 10 and Corollary 2.

Consider the case $r = 1$. Similar to the above case, we have $(t-1)\Delta-1 \geq (\Delta-1)t$. That is $t \geq \Delta+1$. By Corollary 1, $e(G) \geq 1(\Delta-1)t+q_{1,\Delta}+q_{\Delta-1,\Delta}-\Delta+4$. But $1(\Delta-1)t + q_{1,\Delta} + q_{\Delta-1,\Delta} - \Delta + 3 > 3\Delta t - 6$ by Lemma 10 and Corollary 2.

Since we obtain contradiction for all cases, the counterexample is impossible.

Appendix

A1 Kittikorn Nakprasit, *Equitable Colorings of Planar Graphs with Maximum Degree at least Nine*, Discrete Math., to be appeared