รายงานวิจัยฉบับสมบูรณ์

โครงการ ทฤษฎีบทภาวะคู่กันสำหรับปริภูมิลำดับบนปริภูมิบานาค

โดย ดร. จิตติ รักบุตร และคณะ

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ผู้วิจัย	**สังกัด**
1. ดร. จิตติ รักบุตร	ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร
2. ศ. ดร. สุเทพ สวนใต้	ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

กิตติกรรมประกาศ

การวิจัยนี้ได้รับสนับสนุนด้านงบประมาณจากสำนักงานกองทุนสนับสนุนการวิจัย และสำนักงานคณะกรรมการการอุดมศึกษา ภายใต้ทุนพัฒนาศักยภาพการทำวิจัยของอาจารย์รุ่นใหม่ประจำปี 2551 ผู้วิจัยจึงขอขอบคุณผู้ให้ทุนมา ณ ที่นี้ และ ขอขอบคุณ ศ. ดร. สุเทพ สวนใต้ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้ให้คำปรึกษาและแนะนำผู้วิจัยตลอดมา ตลอดจนขอขอบคุณวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร จ. นครปฐม ต้นสังกัดของผู้วิจัย ที่อำนวยความสะดวกในการทำวิจัยนี้

จิตติ รักบุตร

15 พฤษภาคม 2553

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ชื่อนักวิจัย : นายจิตติ รักบุตร ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
มหาวิทยาลัยศิปากร จังหวัดนครปฐม 73000

E-mail Address : jitrakbud@yahoo.com

ระยะเวลาโครงการ : 2 ปี

บทคัดย่อ ในงานวิจัยนี้ เราให้ทฤษฎีบทโดยทั่วไปสำหรับภาวะคู่กันและการสะท้อนกลับสำหรับคลาส ๆ หนึ่งของปริภูมิลำดับค่าบานาค เราแสดงด้วยว่าผลที่ทราบกันดีของภาวะคู่กันและการสะท้อนกลับของปริภูมิลำดับ $l_p(X)$ และ $l_p[X]$ เป็นผลที่ได้โดยตรงจากทฤษฎีบทของเรา

คำสำคัญ: ปริภูมิลำดับค่าบานาค ภาวะคู่กัน การสะท้อนกลับ

Project Code : MRG5180358

Project Title : Duality Theorem for Sequence Spaces over Banach Spaces

Investigator : Mr. Jitti Rakbud, Deaprtment of Mathematics, Faculty of Science
Silpakorn University, Nakorn Pathom, Thailand, 73000

E-mail Address : jitrakbud@yahoo.com

Project Period : 2 Years

Abstract: In this research, we provide some general theorems on duality and reflexivity for a class of Banach-valued sequence spaces. We also show that the known results on the duality and reflexivity of the classical sequence space $l_p(X)$ as well as the sequence space $l_p[X]$ can be obtained from our results.

Keywords : Banach-valued sequence space, Duality, Reflexivity

สารบัญ

หน้า

บทที่ 1
บทนำ

1. 1 บทนำ

เป็นที่ทราบกันดีว่า สำหรับ $1 \le p < \infty$, l_p^* สมสัณฐานอย่างสมมาตรกับ l_q เมื่อ $\frac{1}{p} + \frac{1}{q} = 1$ และ c_0^* สมสัณฐานอย่างสมมาตรกับ l_∞ ผลที่เกี่ยวกับภาวะคู่กัน (duality) และภาวะคู่กันก่อน (preduality) สำหรับปริภูมิ l_p สำหรับ $1 \le p \le \infty$ นี้ได้ถูกวางนัยทั่วไปโดยนักคณิตศาสตร์หลาย ๆ ท่าน ซึ่งส่วนใหญ่จะศึกษาภาวะคู่ หรือ ปริภูมิคู่กันโคธี (Köthe dual) ของปริภูมิลำดับเฉพาะเจาะจงที่เป็นการวางนัยทั่วไปของปริภูมิ l_p เช่น ปริภูมิ

$$l_p(X) = \left\{ \{x_k\}_{k=1}^{\infty} \subset X : \sum_{k=1}^{\infty} \|x_k\|^p < \infty \right\}$$

และ

$$l_p(X) = \left\{ \{x_k\}_{k=1}^{\infty} \subset X : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \ \forall\, f \in X^* \right\}$$

ซึ่งนิยามบนปริภูมิบานาค X (ดู [1], [2], [3], [4], [5], [6], [7], [8], [10], [11] สำหรับเอกสารอ้างอิง)

ในงานวิจัยนี้ เราจะให้แนวทางที่สมเหตุสมผลบางประการในการศึกษาผลทางภาวะคู่กัน และภาวะคู่กันก่อน ของปริภูมิ l_p สำหรับ $1 \le p \le \infty$ ที่เก่าแก่นี้โดยทั่วไป อันเกิดจากข้อสังเกตเกี่ยวกับผลดังกล่าวดังต่อไปนี้

ข้อสังเกต ภาวะคู่กัน และภาวะคู่กันก่อนของปริภูมิ l_p สำหรับ $1 \le p \le \infty$ สามารถถูกมองในรูปความ สัมพันธ์เชิงภาวะคู่กันของปริภูมลำดับสี่ปริภูมิ $\left(A, B, C, l_p\right)$ ในลักษณะที่ A^* สมสัณฐานอย่างสมมาตรกับ B และ C^* สมสัณฐานอย่างสมมาตรกับ l_p โดยที่ A และ C เป็นโคลเชอร์ของเซตของลำดับของจำนวนเชิงซ้อนที่เทอมที่ไม่เป็นศูนย์เป็นจำนวนจำกัด ใน l_p และใน B ตามลำดับ สำหรับกรณีที่ $1 < p < \infty$ เราได้ว่า $A = l_p$ และ $B = C = l_q$ เมื่อ $\frac{1}{p} + \frac{1}{q} = 1$ สำหรับ $p = \infty$ เราได้ว่า $A = c_0$ และ $B = C = l_1$ ซึ่งทั้งสองกรณีนี้ เราได้การเคลื่อนที่ไปข้างหน้าของภาวะคู่กันของปริภูมลำดับสามปริภูมิ $\left(A, B, l_p\right)$ ในลักษณะที่ A^* สมสัณฐานอย่างสมมาตรกับ B และ B^* สมสัณฐานอย่างสมมาตรกับ l_p และสำหรับ $p = 1$ เราได้ว่า $A = l_1$, $B = l_\infty$ และ $C = c_0$

1.2 ที่มาและความสำคัญของปัญหา

ปริภูมิลำดับค่าบานาค (Banch-valued sequence space) ได้ถูกศึกษากันอย่างแพร่หลายโดยนักคณิตศาสตร์หลาย ๆ ท่าน งานวิจัยส่วนหนึ่งเกี่ยวกับปริภูมิคู่กัน หรือปริภูมิคู่กันโคธีของปริภูมิลำดับค่าบานาคเฉพาะเจาะจงบางปริภูมิ ช่น ปริภูมิ $l_p(X)$ และ $l_p[X]$ จากการศึกษางานวิจัยที่เกี่ยวข้องทำให้เราเห็นแนวทางทั่วไปแนวทางหนึ่งในการศึกษาปริภูมิคู่กันของปริภูมิลำดับค่าบานาค

1.3 วัตถุประสงค์

1. สร้างทฤษฎีบทโดยทั่วไปบนภาวะคู่กันสำหรับคลาส ๆ หนึ่งของปริภูมิลำดับค่าบานาค
2. นำเสนอการประยุกต์ทฤษฎีบทภาวะคู่กันในวัตถุประสงค์ข้อ1
3. ศึกษาการสะท้อนกลับของปริภูมิลำดับค่าบานาคที่ถูกศึกษาภาวะคู่ในวัตถุประสงค์ข้อ1

1.4 วิธีการวิจัย

การวิจัยจะถูกแบ่งเป็น 4 ขั้น ดังนี้

<u>ขั้นที่</u> 1 ศึกษาความรู้พื้นฐานและงานวิจัยที่เกี่ยวข้อง

<u>ขั้นที่</u> 2 ดำเนินการวิจัยเพื่อบรรลุวัตถุประสงค์ข้อ1

<u>ขั้นที่</u> 3 ดำเนินการวิจัยเพื่อบรรลุวัตถุประสงค์ข้อ2

<u>ขั้นที่</u> 4 ดำเนินการวิจัยเพื่อบรรลุวัตถุประสงค์ข้อ3

1.5 เอกสารอ้างอิง

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บทที่ 2
ผลการวิจัย

2.1 Class of Banach-valued function spaces with some certain properties

Let S be a non-empty set, let X be a Banach space, and let $\sum(S,X)$ be the set of all functions from S into X. For any $f\in\sum(S,X)$ and $A\subseteq S$, let $f_{[A]}:S\to X$ be defined by $f_{[A]}(x)=f(x)$ for all $x\in A$ and $f_{[A]}(x)=0$ otherwise. For any $t\in S$ and $x\in X$, let $e(t;x):S\to X$ be defined by $e(t;x)(t)=x$ and $e(t;x)(s)=0$ otherwise. For any $f\in\sum(S,X)$ and any finite subset A of S, we have $f_{[A]}=\sum_{t\in A}e(t;f(t))$. Let F be the family of all finite subsets of S. Then F is directed by the order $\succ$ defined by $A\succ B$ if and only if $B\subseteq A$. Next, suppose that $\|\|\cdot\|\|:\sum(S,X)\to[0,\infty]$ satisfies the following properties:

- (N1) For any $f\in\sum(S,X)$, $\|\|f\|\|=\sup\left\{\|\|f_{[A]}\|\|:A\in F\right\}$;
- (N2) There is a positive real number M such that for any $t\in S$ and $x\in X$, $\|\|e(t;x)\|\|\le M\|x\|$;
- (N3) There is a positive real number K such that for any $f\in\sum(S,X)$ and $t\in S$, $\|f(t)\|\le K\|\|f\|\|$;
- (N4) $\|\|f+g\|\|\le\|\|f\|\|+\|\|g\|\|$ for all $f,g\in\sum(S,X)$;
- (N5) $\|\|\alpha f\|\|=|\alpha|\|\|f\|\|$ for all $f\in\sum(S,X)$ and $\alpha\in\mathbb{C}$, under the convention that $0\cdot\infty=0$.

From (N3), the following properties is obtained:

- (N6) If $\|\|f\|\|=0$, then $f=0$.

Let

$$\Lambda\left(S,X,\|\|\cdot\|\|\right)=\left\{f\in\sum(S,X):\|\|f\|\|<\infty\right\};$$

and

$$\Lambda_0\left(S,X,\|\|\cdot\|\|\right)=\left\{f\in\Lambda\left(S,X,\|\|\cdot\|\|\right):\text{the net }\left\{\|\|f_{[A]}-f\|\|\right\}_{A\in F}\text{ converges to }0\right\}.$$

It is obvious that $f_{[A]}$ belongs to $\Lambda_0\left(S,X,\|\|\cdot\|\|\right)$ for all $f\in\sum(S,X)$ and $A\in F$. From the properties (N4)-(N6), we have the function $\|\|\cdot\|\|$ is indeed a norm on $\Lambda\left(S,X,\|\|\cdot\|\|\right)$. From now on, we will assume for convenience that the constants M and K appearing on (N2) and (N3) are equal to 1.

Theorem 2.1.1. *Both* $\Lambda\left(S, X, |||\cdot|||\right)$ *and* $\Lambda_0\left(S, X, |||\cdot|||\right)$ *equipped with the norm* $|||\cdot|||$ *are Banach spaces.*

Proof Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\Lambda\left(S, X, |||\cdot|||\right)$. Then by (N3), we have for each $t \in S$ that $\|f_n(t) - f_m(t)\| \le |||f_n - f_m|||$ for all n, m. This implies that $\{f_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence in X for all $t \in S$. Thus, by the completeness of X, there is, for each $t \in S$, an element $f(t) \in X$ such that $f_n(t) \to f(t)$. Let $f : S \to X$ be defined by $t \mapsto f(t)$. We will show that $f \in \Lambda\left(S, X, |||\cdot|||\right)$ and $f_n \to f$. For each $A \in F$, we have

$$\left|\left|\left|(f_n)_{[A]} - f_{[A]}\right|\right|\right| = \left|\left|\left|(f_n - f)_{[A]}\right|\right|\right| = \left|\left|\left|\sum_{t \in A} e\left(t; (f_n - f)_{[A]}(t)\right)\right|\right|\right|$$
$$\le \sum_{t \in A} \left|\left|\left|e\left(t; (f_n - f)_{[A]}(t)\right)\right|\right|\right|$$
$$= \sum_{t \in A} \|f_n(t) - f(t)\| \to 0 \text{ as } n \to \infty.$$

Then $(f_n)_{[A]} \to f_{[A]}$ as $n \to \infty$ for all $A \in F$. Let $\varepsilon > 0$. Then there is a positive integer N such that,

$$|||f_n - f_m||| < \frac{\varepsilon}{2} \text{ for all } n, m \ge N.$$

Thus, by (N1), we have for each $A \in F$ that

$$\left|\left|\left|(f_n - f_m)_{[A]}\right|\right|\right| \le |||f_n - f_m||| < \frac{\varepsilon}{2} \text{ for all } n, m \ge N.$$

Hence, by taking the limit as $m \to \infty$, we have for each $A \in F$ that

$$\left|\left|\left|(f_n - f)_{[A]}\right|\right|\right| \le \frac{\varepsilon}{2} \text{ for all } n \ge N.$$

Thus, by (N1) again, we obtain

$$|||f_n - f||| < \varepsilon \text{ for all } n \ge N.$$

This implies that $f \in \sum(S, X)$ and $f_n \to f$.

To see that $\Lambda_0\left(S, X, |||\cdot|||\right)$ is a Banach space, suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\Lambda_0\left(S, X, |||\cdot|||\right)$ converging to an element f in $\Lambda\left(S, X, |||\cdot|||\right)$. Let $\varepsilon > 0$. Then there is a positive integer N such that $|||f_N - f||| < \frac{\varepsilon}{3}$. Since f_N belongs to $\Lambda_0\left(S, X, |||\cdot|||\right)$, there is an $A_0 \in F$ such that $\left|\left|\left|(f_N)_{[A]} - f_N\right|\right|\right| < \frac{\varepsilon}{3}$ for all $A \succ A_0$.

Consequently, we have by (N1) that

$$\left|\left|\left|f_{[A]} - f\right|\right|\right| \le |||f_N - f||| + \left|\left|\left|(f_N)_{[A]} - f_N\right|\right|\right| + \left|\left|\left|(f_N)_{[A]} - f_{[A]}\right|\right|\right|$$

$$= \left\| f_N - f \right\| + \left\| (f_N)_{[A]} - f_N \right\| + \left\| (f_N - f)_{[A]} \right\|$$

$$\le \left\| f_N - f \right\| + \left\| (f_N)_{[A]} - f_N \right\| + \left\| f_N - f \right\|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for all} \quad A \succ A_0 .$$

It follows that $f \in \Lambda_0\left(S, X, \|\cdot\|\right)$. □

The Banach space $\Lambda\left(S, X, \|\cdot\|\right)$ was first considered in [9] by O. Wootijirutikal , S.-C. Ong, P. Chaisuruya, and J. Rakbud.

If the function $\|\cdot\|$ has the following additional property :

(N7) for any function $\lambda : S \to \mathbb{C}$ with $|\lambda(t)| = 1$ for all $t \in S$, $\|\lambda f\| \le |\lambda| \|f\|$ for all $f \in \sum(S, X)$, where $\lambda f : S \to X$ defined by $\lambda f(t) = \lambda(t) f(t)$ for all $t \in S$,

we call the Banach space $\Lambda\left(S, X, \|\cdot\|\right)$ the X *-valued function space defined by* $\|\cdot\|$, or simply, an X -valued function space . For convenience, we may sometimes denote the function space $\Lambda\left(S, X, \|\cdot\|\right)$ by just $\Lambda\left(S, X\right)$. When S is the set $\mathbb{N}$ of all positive integers, we call $\Lambda\left(\mathbb{N}, X, \|\cdot\|\right)$ specifically the X *-valued sequence space defined by* $\|\cdot\|$, or simply, an X -valued sequence space .

2.2 Dual of $\Lambda_0\left(S, X, \|\cdot\|\right)$

Let $\Lambda\left(S, X, \|\cdot\|\right)$ be an X -valued function space. For any $\varphi \in \sum\left(S, X^*\right)$, we define

$$\|\varphi\|^* = \sup\left\{ \left| \sum_{t \in S} \varphi(t)(f(t)) \right| : f \in \Lambda\left(S, X, \|\cdot\|\right), \|f\| \le 1 \right\}$$

If the supremum is finite, and $\|\varphi\|^* = \infty$ otherwise. For any $z \in \mathbb{C}$, let $\operatorname{dir}(z) = \frac{\overline{z}}{z}$ if $z \ne 0$ and $\operatorname{dir}(z) = 1$ if $z = 0$.

Theorem 2.2.1. *The set*

$$\Delta = \left\{ \varphi \in \sum\left(S, X^*\right) : \sum_{t \in S} |\varphi(t)(f(t))| \text{ converges for all } f \in \Lambda\left(S, X, \|\cdot\|\right) \right\}$$

is the X^* *-valued function space defined by* $\|\cdot\|^*$*, or symbolically,* $\Delta = \Lambda\left(S, X^*, \|\cdot\|^*\right)$.

Proof We must show first that for any $\varphi \in \sum\left(S, X^*\right)$, $\left\|\left|\varphi\right\|\right|^* < \infty$ if and only if $\varphi \in \Delta$. Suppose that $\varphi \in \Delta$. Then the linear functional T on $\Lambda(S, X)$ defined by $T(f) = \sum_{t \in S} \varphi(t)(f(t))$ for all $f \in \Lambda(S, X)$ is well defined. For each $A \in F$, we have by (N3) that the linear functional T_A on $\Lambda(S, X)$ defined by $T_A(f) = \sum_{t \in A} \varphi(t)(f(t))$ for all $f \in \Lambda(S, X)$ is bounded. For each $f \in \Lambda(S, X)$, we have $\left|T_A(f)\right| \le \sum_{t \in A} \left|\varphi(t)(f(t))\right|$ for all $A \in F$. It follows by the uniform boundedness principle that $\sup\left\{\left\|T_A\right\| : A \in F\right\} < \infty$. Since $T_A(f) \to T(f)$, we obtain

$$\sup\left\{\left|\sum_{t \in S} \varphi(t)(f(t))\right| : f \in \Lambda\left(S, X, \left\|\left|\cdot\right\|\right|\right), \left\|\left|f\right\|\right| \le 1\right\} = \left\|T\right\| \le \sup\left\{\left\|T_A\right\| : A \in F\right\} < \infty .$$

Conversely, suppose that $\left\|\left|\varphi\right\|\right|^* < \infty$.Then $\sum_{t \in S} \varphi(t)(f(t))$ converges for all $f \in \Lambda(S, X)$. Let $f \in \Lambda(S, X)$ with $\left\|\left|f\right\|\right| \le 1$, and let $\lambda : S \to \mathbb{C}$ be defined by $\lambda(t) = \mathrm{dir}(\varphi(t)(f(t)))$ for all $t \in S$. Then $\left|\lambda(t)\right| = 1$ for all t. Thus, by (N7), we have $\left\|\left|\lambda f\right\|\right| \le \left\|\left|f\right\|\right| \le 1$, and hence

$$\begin{aligned}
\sum_{t \in S} \left|\varphi(t)(f(t))\right| &= \sum_{t \in S} \mathrm{dir}(\varphi(t)(f(t)))\varphi(t)(f(t)) \\
&= \sum_{t \in S} \varphi(t)(\mathrm{dir}(\varphi(t)(f(t)))f(t)) \\
&= \sum_{t \in S} \varphi(t)(\lambda(t)f(t))
\end{aligned}$$

converges. For any $f \in \Lambda(S, X)$, we have $\left\|\left|\frac{1}{\left\|\left|f\right\|\right|} f\right\|\right| = 1$. Thus

$$\sum_{t \in S} \left|\varphi(t)(f(t))\right| = \left\|\left|f\right\|\right| \sum_{t \in S} \left|\varphi(t)\left(\frac{1}{\left\|\left|f\right\|\right|} f(t)\right)\right|$$

Converges for $f \in \Lambda(S, X)$.

The rest of the proof is to show that the function $\left\|\left|\cdot\right\|\right|^*$ satisfies the properties (N1)-(N7).

(N1) Let $\varphi \in \Delta$, and let T and T_A for each $A \in F$ be the functions defined in the preceding paragraph. It is clear that $\left\|T_A\right\| = \left\|\left|\varphi_{[A]}\right\|\right|^*$ for all $A \in F$, and therefore we have $\left\|\left|\varphi\right\|\right|^* = \left\|T\right\| \le \sup\left\{\left\|T_A\right\| : A \in F\right\} = \sup\left\{\left\|\left|\varphi_{[A]}\right\|\right|^* : A \in F\right\}$. Let $f \in \Lambda(S, X)$ with $\left\|\left|f\right\|\right| \le 1$, and let λ be the function defined in the preceding paragraph. Then by (N7), $\left\|\left|\lambda f\right\|\right| \le \left\|\left|f\right\|\right| \le 1$. This yields for each $A \in F$ that

$$\begin{aligned}|T_A(f)| &= \left|\sum_{t\in A}\varphi(t)(f(t))\right| \le \sum_{t\in A}|\varphi(t)(f(t))| \\ &= \sum_{t\in A}\mathrm{dir}(\varphi(t)(f(t)))\varphi(t)(f(t)) \\ &= \sum_{t\in A}\lambda(t)\varphi(t)(f(t)) \quad = \sum_{t\in A}\varphi(t)(\lambda(t)f(t)) \\ &= \left|T\left((\lambda f)_{[A]}\right)\right| \le \|T\| \left\|\left|(\lambda f)_{[A]}\right|\right\| \\ &\le \|\|\varphi\|\|^* \,\|\|\lambda f\|\| \le \|\|\varphi\|\|^*.\end{aligned}$$

It follows that $\|\|\varphi_{[A]}\|\|^* = \|T_A\| \le \|\|\varphi\|\|^*$ for all $A \in F$. Consequently, (N1) holds.

(N2) For any $t \in S$ and $y \in X^*$, we have by (N4) that

$$\begin{aligned}\|\|e(t;y)\|\|^* &= \sup\left\{\left|\sum_{t\in S}e(t,y)(s)(f(s))\right| : f \in \Lambda(S,X),\ \|\|f\|\| \le 1\right\} \\ &= \sup\left\{|e(t,y)(t)(f(t))| : f \in \Lambda(S,X),\ \|\|f\|\| \le 1\right\} \\ &= \sup\left\{|y(f(t))| : f \in \Lambda(S,X),\ \|\|f\|\| \le 1\right\} \\ &= \|y\| \sup\left\{\|\|f\|\| : f \in \Lambda(S,X),\ \|\|f\|\| \le 1\right\} \le \|y\|.\end{aligned}$$

Thus (N2) holds.

(N3) Let $\varphi \in \Delta$ and $t \in S$. For any $x \in X$ with $\|x\| \le 1$, we have by (N2) (of $\|\|\cdot\|\|$) that $\|\|e(t;x)\|\| \le \|x\| \le 1$. Thus

$$|\varphi(t)(x)| = |\varphi(t)(e(t;x)(t))| = \left|\sum_{s\in S}\varphi(s)(e(t;x)(s))\right| \le \|\|\varphi\|\|^*.$$

It follows that $\|\varphi(t)\| \le \|\|\varphi\|\|^*$, and hence (N3) holds.

The properties (N4) and (N5) follow directly from the definition of $\|\|\varphi\|\|^*$.

(N7) Let $\varphi \in \Sigma\left(S, X^*\right)$, and let $\lambda : S \to \mathbb{C}$ with $|\lambda(t)| = 1$ for all $t \in S$. Suppose that $\|\|\varphi\|\|^* < \infty$. Then for any $f \in \Lambda(S,X)$ with $\|\|f\|\| \le 1$, we have by the property (N7) of $\|\|\cdot\|\|$ that

$$\left|\sum_{t\in S}\lambda(t)\varphi(t)(f(t))\right| = \left|\sum_{t\in S}\varphi(t)(\lambda(t)f(t))\right| \le \|\|\varphi\|\|^*.$$

Consequently, $\|\|\lambda\varphi\|\|^* \le \|\|\varphi\|\|^*$. □

We call the X^*-valued function spaces $\Lambda\left(S, X^*, \|\|\cdot\|\|^*\right)$ defined by $\|\|\cdot\|\|^*$ the *Köthe dual* of $\Lambda\left(S, X, \|\|\cdot\|\|\right)$ and call $\|\|\cdot\|\|^*$ the dual norm of $\|\|\cdot\|\|$. The main goal of this section is to identify the dual $\Lambda_0\left(S,X,\|\|\cdot\|\|\right)^*$ of $\Lambda_0\left(S,X,\|\|\cdot\|\|\right)$ with the Köthe dual $\Lambda\left(S,X^*,\|\|\cdot\|\|^*\right)$ of $\Lambda\left(S,X,\|\|\cdot\|\|\right)$.

Let $\Psi\in\Lambda_0\left(S,X,\|\|\cdot\|\|\right)^*$. We then define, for each $t\in S$, the function $y_t:X\to\mathbb{C}$ by $y_t(x)=\Psi(e(t;x))$ for all $x\in X$. Clearly, $\|y_t\|\le\|\Psi\|$ for all $t\in S$, and hence $y_t\in X^*$. Let $\varphi^{(\Psi)}:S\to X^*$ be defined by $t\mapsto y_t$.

Theorem 2.2.2. $\Lambda_0\left(S,X,\|\|\cdot\|\|\right)^*$ *is isometrically isomorphic to* $\Lambda\left(S,X^*,\|\|\cdot\|\|^*\right)$ *by the isomorphism* $\Psi\mapsto\varphi^{(\Psi)}$.

Proof We will show first that $\varphi^{(\Psi)}\in\Lambda\left(S,X^*\right)$. To see this, let $f\in\Lambda(S,X)$, and let $\lambda:S\to\mathbb{C}$ be defined by $\lambda(t)=\operatorname{dir}(\Psi(e(t;f(t))))$ for all $t\in S$. Then we have for each $A\in F$ that

$$\begin{aligned}\sum_{t\in S}\left|\varphi^{(\Psi)}(t)(f(t))\right| &=\sum_{t\in S}\left|y_t(f(t))\right|=\sum_{t\in S}\left|\Psi(e(t;f(t)))\right|\\ &=\sum_{t\in S}\lambda(t)\Psi(e(t;f(t)))=\sum_{t\in S}\Psi(\lambda(t)e(t;f(t)))\\ &=\sum_{t\in S}\Psi(e(t;\lambda(t)f(t)))=\Psi\left(\sum_{t\in A}e(t;\lambda(t)f(t))\right)\\ &=\Psi\left((\lambda f)_{[A]}\right)\le\|\Psi\|\left\|\left|(\lambda f)_{[A]}\right|\right\|\le\|\Psi\|.\end{aligned}$$

This yields $\varphi^{(\Psi)}\in\Lambda(S,X)$ and $\left\|\left|\varphi^{(\Psi)}\right|\right\|^*\le\|\Psi\|$. Next, we will show that $\left\|\left|\varphi^{(\Psi)}\right|\right\|^*\ge\|\Psi\|$. To get this, let $f\in\Lambda_0(S,X)$ and let $\lambda:S\to\mathbb{C}$ be defined above. Then for each $A\in F$,

$$\begin{aligned}\left|\Psi\left(f_{[A]}\right)\right| &=\left|\Psi\left(\sum_{t\in A}e(t;f(t))\right)\right|\le\sum_{t\in A}\left|\Psi(e(t;f(t)))\right|\\ &=\sum_{t\in A}\lambda(t)\Psi(e(t;f(t)))=\sum_{t\in A}y_t(\lambda(t)f(t))\\ &=\sum_{t\in A}\varphi^{(\Psi)}(t)(\lambda(t)f(t))\le\left\|\left|\varphi^{(\Psi)}\right|\right\|^*.\end{aligned}$$

Thus, by the continuity of Ψ, we have $\left|\Psi(f)\right|\le\left\|\left|\varphi^{(\Psi)}\right|\right\|^*$ for all $f\in\Lambda_0(S,X)$ with $\|\|f\|\|\le 1$. Hence $\left\|\left|\varphi^{(\Psi)}\right|\right\|^*\ge\|\Psi\|$, and so $\left\|\left|\varphi^{(\Psi)}\right|\right\|^*=\|\Psi\|$. Finally, we will show that the linear map $\Psi\mapsto\varphi^{(\Psi)}$ is onto. Let $\varphi\in\Lambda\left(S,X^*\right)$. Then the linear functional $\Psi:\Lambda_0(S,X)\to\mathbb{C}$ defined by $\Psi(f)=\sum_{t\in S}\varphi(t)(f(t))$ for all $f\in\Lambda_0(S,X)$

is bounded and $\varphi^{(\Psi)} = \varphi$. Therefore, we have the linear map $\Psi \mapsto \varphi^{(\Psi)}$ is onto as required. The proof is complete. □

In the following theorem, the space $\Lambda\left(S, X^*\right)$ will be regarded as the space of all bounded linear functional Φ_φ on $\Lambda(S, X)$ defined for each $\varphi \in \Lambda\left(S, X^*\right)$ by $\Phi_\varphi(f) = \sum_{t \in S} \varphi(t)(f(t))$ for all $f \in \Lambda(S, X)$. It is clear that $\Lambda\left(S, X^*\right)$ is a closed subspace of $\Lambda(S, X)^*$.

Theorem 2.2.3. *If* $\Lambda_0(S, X) \neq \Lambda(S, X)$*, then the annihilator* $\Lambda_0(S, X)^\perp$ *of* $\Lambda_0(S, X)$ *is a nontrivial closed subspace of* $\Lambda(S, X)^*$ *and*

$$\Lambda(S, X)^* = \Lambda\left(S, X^*\right) \oplus \Lambda_0(S, X)^\perp$$

Proof Suppose that $\Lambda_0(S, X) \neq \Lambda(S, X)$. Then by the Hahn-Banach extension theorem, we have $\Lambda_0(S, X)^\perp$ is a nontrivial closed subspace of $\Lambda(S, X)^*$. For any $\Psi \in \Lambda(S, X)^*$, let $\Omega_\Psi = \Psi - \Phi_{\varphi^{(\Psi)}}$. Then $\Omega_\Psi \in \Lambda_0(S, X)^\perp$, and hence $\Lambda(S, X)^* = \Lambda\left(S, X^*\right) + \Lambda_0(S, X)^\perp$. For any $\varphi \in \Lambda\left(S, X^*\right)$, if $\Phi_\varphi \in \Lambda_0(S, X)^\perp$, then $\Phi_\varphi(f) = \lim_{A \in F} \Phi_\varphi\left(f_{[A]}\right) = 0$ for all $f \in \Lambda(S, X)$. It follows that $\Lambda\left(S, X^*\right) \cap \Lambda_0(S, X)^\perp = \{0\}$. Consequently, $\Lambda(S, X)^* = \Lambda\left(S, X^*\right) \oplus \Lambda_0(S, X)^\perp$. □

2.3 Predual of $\Lambda(S, X)$

The aim of this section is to complete the duality relation among the four function spaces $\left\{\Lambda_0(S, X), \Lambda\left(S, X^*\right), \Lambda_0\left(S, X^*\right), \Lambda(S, X)\right\}$. From the previous section we have done the the duallty relation between the first two spaces $\Lambda_0(S, X)$ and $\Lambda\left(S, X^*\right)$. The rest is to investigate the the duallty relation between $\Lambda_0\left(S, X^*\right)$ and $\Lambda(S, X)$. We expect to have $\Lambda_0\left(S, X^*\right)^*$ is isometrically isomorphic to $\Lambda(S, X)$.

For any $f \in \Lambda(S, X)$, we define a function $\tilde{\Theta}_f : \Lambda\left(S, X^*\right) \to \mathbb{C}$ by $\tilde{\Theta}_f(\varphi) = \sum_{t \in S} \varphi(t)(f(t))$ for all $\varphi \in \Lambda\left(S, X^*\right)$ and let Θ_f be the restriction of $\tilde{\Theta}_f$ to $\Lambda_0\left(S, X^*\right)$. It is clear that $\left\|\Theta_f\right\| \leq \left\|\tilde{\Theta}_f\right\| \leq \||f\||$ for all $f \in \Lambda(S, X)$.

Proposition 2.3.1. *For any* $f \in \Lambda(S, X)$, $\left\|\Theta_f\right\| = \||f\|| = \left\|\tilde{\Theta}_f\right\|$.

Proof Let $f \in \Lambda(S,X)$, let $\varepsilon > 0$, and let $A \in F$. Then we have by the Hahn-Banach extension theorem that there is $\Psi \in \Lambda(S,X)^*$ with $\|\Psi\| \le 1$ such that $\left\|\left| f_{[A]} \right\|\right| \le \left|\Psi\left(f_{[A]}\right)\right| + \varepsilon$. By Theorem 2.2.3, we have $\Psi\left(f_{[A]}\right) = \Phi_{\varphi^{(\Psi)}}\left(f_{[A]}\right)$. Thus, by Theorem 2.2.2 and the property (N1) of $\left\|\left|\cdot\right\|\right|^*$, we have

$$\begin{aligned}
\left\|\left| f_{[A]} \right\|\right| &\le \left|\Psi\left(f_{[A]}\right)\right| + \varepsilon = \left|\Phi_{\varphi^{(\Psi)}}\left(f_{[A]}\right)\right| + \varepsilon \\
&= \left|\tilde{\Theta}_{f_{[A]}}\left(\varphi^{(\Psi)}\right)\right| + \varepsilon = \left|\tilde{\Theta}_f\left(\left(\varphi^{(\Psi)}\right)_{[A]}\right)\right| + \varepsilon \\
&= \left|\Theta_f\left(\left(\varphi^{(\Psi)}\right)_{[A]}\right)\right| + \varepsilon \le \left\|\Theta_f\right\| \left\|\left|\left(\varphi^{(\Psi)}\right)_{[A]}\right\|\right|^* + \varepsilon \\
&\le \left\|\Theta_f\right\| \left\|\left|\varphi^{(\Psi)}\right\|\right|^* + \varepsilon = \left\|\Theta_f\right\| \left\|\Psi\right\| + \varepsilon \\
&\le \left\|\Theta_f\right\| + \varepsilon .
\end{aligned}$$

Since ε was given arbitrarily, we obtain $\left\|\left| f_{[A]} \right\|\right| \le \left\|\Theta_f\right\|$ for all $A \in F$. Thus, by (N1) of $\left\|\left|\cdot\right\|\right|$, we have $\left\|\left| f \right\|\right| \le \left\|\Theta_f\right\|$. It follows that $\left\|\Theta_f\right\| = \left\|\left| f \right\|\right| = \left\|\tilde{\Theta}_f\right\|$. □

We now have the function $f \mapsto \Theta_f$ is an isometric isomorphism from $\Lambda(S,X)$ into $\Lambda_0\left(S,X^*\right)^*$. In the following theorem, we provide a necessary and sufficient condition the function to be onto.

Theorem 2.3.2. *The isometric isomorphism* $f \mapsto \Theta_f$ *from from* $\Lambda(S,X)$ *into* $\Lambda_0\left(S,X^*\right)^*$ *is onto if and only if* X *is reflexive.*

Proof Suppose that X is reflexive. We want to show that the map $f \mapsto \Theta_f$ is onto. To see this, let $\Psi \in \Lambda_0\left(S,X^*\right)^*$, and for each $t \in S$, let $\phi_t : X^* \to \mathbb{C}$ be defined by $\phi_t(y) = \Psi(e(t;y))$ for all $y \in X^*$. Then $\phi_t \in X^{**}$ for all $t \in S$. Thus, by the reflexivity of X, there exists, for each $t \in S$, an $x_t \in X$ such that $\phi_t(y) = y(x_t)$ for all $y \in X^*$. Let $f : S \to X$ be defined by $f(t) = x_t$. We will prove that $f \in \Lambda(S,X)$ and then $\Psi = \Theta_f$. Let $A \in F$. Then we have for any $\varphi \in \Lambda\left(S,X^*\right)$ with $\left\|\left|\varphi\right\|\right|^* \le 1$ by (N1) of $\left\|\left|\cdot\right\|\right|^*$ that

$$\begin{aligned}
\left|\Theta_{f_{[A]}}(\varphi)\right| &= \left|\sum_{t\in A} \varphi(t)(f(t))\right| = \left|\sum_{t\in A} \varphi(t)(x_t)\right| \\
&= \left|\sum_{t\in A} \phi_t(\varphi(t))\right| = \left|\sum_{t\in A} \Psi(e(t;\varphi(t)))\right| \\
&= \left|\Psi\left(\varphi_{[A]}\right)\right| \le \left\|\Psi\right\| \left\|\left|\varphi_{[A]}\right\|\right|^* \le \left\|\Psi\right\|.
\end{aligned}$$

It follows that $\left\|\left| f_{[A]} \right|\right\| = \left\| \Theta_{\varphi_{[A]}} \right\| \le \|\Psi\|$ for all $A \in F$. Hence, by (N1) of $\left\|\left|\cdot\right|\right\|$, we obtain $\left\|\left| f \right|\right\| \le \|\Psi\|$, and therefore $f \in \Lambda(S, X)$. To see that $\Psi = \Theta_f$, let $\varphi \in \Lambda_0\left(S, X^*\right)$. Then we have for each $A \in F$ that

$$\Psi\left(\varphi_{[A]}\right) = \Psi\left(\sum_{t \in A} e(t; \varphi(t))\right) = \sum_{t \in A} \Psi(e(t; \varphi(t)))$$
$$= \sum_{t \in A} \phi_t(\varphi(t)) = \sum_{t \in A} \varphi(t)(f(t)) = \Theta_f\left(\varphi_{[A]}\right).$$

Thus, by the continuity of both Ψ and Θ_f, we have $\Psi = \Theta_f$ as required.

Conversely, suppose that the map $f \mapsto \Theta_f$ is onto. To show that X is reflexive, let $\varphi \in X^{**}$, and let $t_0 \in S$ be fixed. Then the linear functional Ψ on $\Lambda\left(S, X^*\right)$ defined by $\Psi(\varphi) = \phi(\varphi(t_0))$ for all $\varphi \in \Lambda\left(S, X^*\right)$ is bounded. Thus, by the assumption, there is an $f \in \Lambda(S, X)$ such that $\Theta_f = \Psi$. From this, we have for any $y \in X^*$ that

$$\phi(y) = \phi(e(t_0; y)(t_0)) = \Psi(e(t_0; y)) = \Theta_f(e(t_0; y)) = e(t_0; y)(t_0)(f(t_0)) = y(f(t_0)).$$

Therefore X is reflexive. □

2.4 Reflexivity

In this section, we establish a reflexivity theorem for our Banach-valued function spaces. We denote here the isomorphism $\Psi \mapsto \varphi^{(\Psi)}$ from $\Lambda_0(S, X)^*$ onto $\Lambda\left(S, X^*\right)$ by N, the isomorphism $f \mapsto \Theta_f$ from $\Lambda(S, X)$ onto $\Lambda_0\left(S, X^*\right)^*$ by M. Let P be the isometric isomorphism from the subspace $\left\{\Theta_f : f \in \Lambda(S, X)\right\}$ of $\Lambda_0\left(S, X^*\right)^*$ into $\Lambda\left(S, X^*\right)^*$ defined by $\Theta_f \mapsto \tilde{\Theta}_f$.

Lemma 2.4.1. *$N^* PM(f) = Q(f)$ for all $f \in \Lambda_0(S, X)$, where N^* is the adjoint of N and $Q : \Lambda_0(S, X) \to \Lambda_0(S, X)^{**}$ is the natural map.*

Proof Let $f \in \Lambda_0(S, X)$. Then for each $A \in F$,

$$N^* PM\left(f_{[A]}\right) = N^*\left(P\left(M\left(f_{[A]}\right)\right)\right) = N^* P\left(\Theta_{f_{[A]}}\right) = \tilde{\Theta}_{f_{[A]}} N.$$

Next, let $\Psi \in \Lambda(S, X)^*$, then we have for each $A \in F$ that

$$\tilde{\Theta}_{f_{[A]}} N(\Psi) = \tilde{\Theta}_{f_{[A]}}\left(\varphi^{(\Psi)}\right) = \sum_{t \in A} \Psi(e(t; f(t))) = \Psi\left(f_{[A]}\right) = Q\left(f_{[A]}\right)(\Psi).$$

Accordingly, $N^* PM\left(f_{[A]}\right) = Q\left(f_{[A]}\right)$ for each $A \in F$. It follows by the continuous of the maps $N^* PM$ and Q that $N^* PM(f) = Q(f)$. The proof is finished. □

An X-valued function space $\Lambda(S,X)$ is called *a GAK-space* (see [4] for the original definition) if $\Lambda_0(S,X)=\Lambda(S,X)$.

Theorem 2.4.2. (Reflexivity theorem for the Banach-valued function spaces) *Let $\Lambda(S,X)$ be an X-valued function space. Then the following are equivalent:*

(1) $\Lambda(S,X)$ *is reflexive;*

(2) $\Lambda_0(S,X)$ *is reflexive;*

(3) X *is reflexive and both* $\Lambda(S,X)$ *and its* Köthe *dual are GAK.*

Proof $(1)\Rightarrow(2)$. Obvious

$(2)\Rightarrow(3)$. Suppose that (3) holds. If $\Lambda(S,X)$ is not GAK, then we have by Lemma 2.4.1 that $\Lambda(S,X)$ is not reflexive. Next, suppose that X is not reflexive or $\Lambda\left(S,X^*\right)$ is not GAK. We will prove that each of these two conditions implies that

$$\left\{\tilde{\Theta}_f : f\in\Lambda(S,X)\right\}\neq\Lambda\left(S,X^*\right)^* \qquad (*)$$

If X is not reflexive , then (*) holds by Theorem 2.3.2. Suppose that $\Lambda_0\left(S,X^*\right)\neq\Lambda\left(S,X^*\right)$. Then by the Hahn-Banach extension theorem, there exists $\psi\in\Lambda\left(S,X^*\right)^*$ such that $\|\psi\|\neq 0$ and $\Lambda_0\left(S,X^*\right)\subseteq\ker\psi$. If $\psi=\tilde{\Theta}_f$ for some $f\in\Lambda(S,X)$, then $\Theta_f=0$. This yields $\|\psi\|=\|\tilde{\Theta}_f\|=\|\Theta_f\|=0$, which is a contradiction. Thus (*) holds. Since (*) holds, it follows immediately from Lemma 2.4.1 that $\Lambda_0(S,X)$ is not reflexive.

$(3)\Rightarrow(1)$. It follows directly from Theorem 2.3.2 and Lemma 2.4.1. □

2.5 Applications to the sequence spaces $l_p(X)$ and $l_p[X]$

In this section, we show that the well-known results on duality and reflexivity of the sequence spaces $l_p(X)$ and $l_p[X]$ can easily be deduced from our theorems.

It is clear that for each $1\le p\le\infty$, the sequence spaces $l_p(X)$ is the X-valued sequence space defined by $\|\cdot\|_p$ and it is GAK, except for the case where $p=\infty$. Thus, by the duality theorem, $l_p(X)^*$ is isometrically isomorphic to its Köthe dual $\Lambda\left(\mathbb{N}X^*,\|\cdot\|_p^*\right)$. It can be shown by the unifiom boundedness principle that for $1\le p<\infty$, $\Lambda\left(\mathbb{N}X^*,\|\cdot\|_p^*\right)=l_q(X)$, where $\frac{1}{p}+\frac{1}{q}=1$. Thus

$\Lambda\left(\mathbb{N}, X^*, \|\cdot\|_p^*\right)$ is GAK for all $1 < p < \infty$, whereas $\Lambda\left(\mathbb{N}, X^*, \|\cdot\|_1^*\right)$ is not GAK. Therefore, by the reflexivity theorem, $l_1(X)$ and $l_\infty(X)$ are not reflexive, while for $1 < p < \infty$, $l_p(X)$ is reflexive if and only if X is reflexive.

It is also clear that $l_p[X]$ is the X-valued sequence space defined by $|||\cdot|||_p$, where $|||\cdot|||_p$ is given by

$$|||\{x_k\}_{k=1}^{\infty}|||_p = \sup\left\{\left(\sum_{k=1}^{\infty}|f(x_k)|^p\right)^{1/p} : f \in X^*, \|f\| \le 1\right\}.$$

It was proved in [10] by C. K. Wu and Q. Y. Bu that the Köthe dual of $l_p[X]$ is GAK for all $1 < p < \infty$. Thus, by the reflexivity theorem, we have that the sequence space $l_p[X]$ is reflexive if and only if X is reflexive and $l_p[X]$ is GAK. This is exactly the same as that given in [1] by Q. Y. Bu.

บทที่ 3
บทสรุปและวิจารณ์ผลการวิจัย

จากการวิจัยนี้ ผู้วิจัยได้สร้างทฤษฎีบทภาวะคู่กันและทฤษฎีบทการสะท้อนกลับสำหรับคลาส ๆ หนึ่งของปริภูมิฟังก์ชันค่าบานาค ทฤษฎีบทดังกล่าวเป็นการวางนัยทั่วไปของผลที่เก่าแก่ที่เกี่ยวกับภาวะคู่กันและการสะท้อนกลับของปริภูมิ l_p ในมุมมองตามข้อสังเกตที่ถูกกล่าวถึงในบทนำ ผู้วิจัยคาดหวังว่าทฤษฎีบทภาวะคู่กันและทฤษฎีบทการสะท้อนกลับที่ถูกสร้างขึ้นมาในงานวิจัยนี้ จะเป็นเครื่องมือหนึ่งในการศึกษาภาวะคู่กันและการสะท้อนกลับของปริภูมิฟังก์ชันค่าบานาคเฉพาะเจาะจงที่อาจจะถูกนิยามขึ้นมาใหม่ในอนาคต หรือแม้แต่ปริภูมิฟังก์ชันค่าบานาคที่มีอยู่แล้วในปัจจุบัน

Output

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   <u>ผู้แต่ง</u> Jitti Rakbud, Sutep Suantai
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ภาคผนวก

DUALITY THEOREM FOR BANACH-VALUED FUNCTION SPACES

JITTI RAKBUD AND SUTEP SUANTAI

ABSTRACT. In this paper, we provide some general theorems on duality and reflexivity for a class of Banach-valued functions spaces. We also show that the known results on the duality and reflexivity of the classical sequence space $l_p(X)$ as well as the sequence space $l_p[X]$ can be obtained from our results.

1. Introduction and preliminaries

It is well known that $c_0^* \cong l_1$ and $l_p^* \cong l_q$ for $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. General studies of these classical results on the duality and preduality of the classical Banach sequence spaces l_p have been contributed by many people (see [1] [2] [3] [4] [5] [6] [7] [8] [11] [10] for references). Most of them deal with the Köthe dual of a fixed Banach-valued sequence space which is a generalization of the sequence space l_p. Well-known ones are the following spaces:

$$l_p(X) = \left\{ \{x_k\}_{k=1}^{\infty} \subset X : \sum_{k=1}^{\infty} \|x_k\|^p < \infty \right\};$$

and

$$l_p[X] = \left\{ \{x_k\}_{k=1}^{\infty} \subset X : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \ \forall f \in X^* \right\},$$

which are defined over any Banach space X. In this paper, we provide a general study of these classical results by a way analogous to the following observation.

Observation: The duality and preduality of the classical sequence spaces l_p can be viewed in the form of the duality relation among the four spaces $\{A, B, C, l_p\}$, in the sense that $A^* \cong B$ and $C^* \cong l_p$, where A and C are the closure of the set of scalar sequences with finitely many non-zero terms, in l_p and in B respectively. In the case where $1 < p < \infty$, A is equal to l_p itself, and $B = C = l_q$ when $\frac{1}{p} + \frac{1}{q} = 1$. For $p = \infty$, we have $A = c_0$, and $B = C = l_1$. Thus, in these two cases, the duality progression of the three spaces $\{A, B, l_p\}$, in the sense that $A^* \cong B$ and $B^* \cong l_p$, holds. For p=1, we have $A = l_1$, $B = l_\infty$, and $C = c_0$.

Let S be a non-empty set, let X be a Banach space, and let $\Sigma(S, X)$ be the set of all functions from S into X. For any $f \in \Sigma(S, X)$ and $A \subseteq S$, let $f_{[A]} : S \to X$ be defined

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by $f_{[A]} = f$ on A and $f_{[A]} = 0$ otherwise. For any $t \in S$ and $x \in X$, let $e(t;x) : S \to X$ be defined by $e(t;x)(t) = x$ and $e(t;x)(s) = 0$ otherwise. For any $f \in \Sigma(S,X)$ and any finite subset A of S, we have $f_{[A]} = \sum_{t \in A} e(t; f(t))$. Let $\mathcal{F}$ be the family of all finite subsets of S. Then $\mathcal{F}$ is directed by the order $\succeq$ defined by $A \succeq B$ if and only if $B \subseteq A$. Next, suppose that $|||\cdot||| : \Sigma(S,X) \to [0,\infty]$ satisfying the following properties.

(N1) For any $f \in \Sigma(S,X)$, $|||f||| = \sup_{A \in \mathcal{F}} \left|\left|\left| f_{[A]} \right|\right|\right|$.

(N2) There is a positive real number M such that for any $t \in S$ and $x \in X$, $|||e(t;x)||| \leq M \|x\|$.

(N3) There is a positive real number K such that for any $f \in \Sigma(S,X)$ and $t \in S$, $\|f(t)\| \leq K\, |||f|||$.

(N4) $|||f+g||| \leq |||f||| + |||g|||$ for all $f, g \in \Sigma(S,X)$.

(N5) $|||\alpha f||| = |\alpha|\, |||f|||$ for all $f \in \Sigma(S,X)$ and $\alpha \in \mathbb{C}$, under the convention that $0 \cdot \infty = 0$.

From (N3), the following property is obtained.

(N6) If $|||f||| = 0$, then $f = 0$.

Let

$$\Lambda(S,X,|||\cdot|||) = \{f \in \Sigma(S,X) : |||f||| < \infty\};$$

and

$$\Lambda_0(S,X,|||\cdot|||) = \left\{ f \in \Lambda(S,X,|||\cdot|||) : \text{ the net } \left\{ \left|\left|\left| f_{[A]} - f \right|\right|\right| \right\}_{A \in \mathcal{F}} \text{ converges to } 0 \right\}.$$

It is obvious that $f_{[A]}$ belongs to $\Lambda_0(S,X,|||\cdot|||)$ for all $f \in \Lambda(S,X,|||\cdot|||)$ and $A \in \mathcal{F}$. From the properties N(4), N(5) and (N6), we have that the function $|||\cdot|||$ is indeed a norm on $\Lambda(S,X,|||\cdot|||)$. From now on, we will assume for convenience that the constants M and K appearing in (N2) and (N3) are equal to 1.

Theorem 1.1. *Both $\Lambda(S,X,|||\cdot|||)$ and $\Lambda_0(S,X,|||\cdot|||)$ equipped the norm $|||\cdot|||$ are Banach spaces.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\Lambda(S,X,|||\cdot|||)$. Then by (N3), we have for each $t \in S$ that $\|f_n(t) - f_m(t)\| \leq |||f_n - f_m|||$ for all n, m. This implies that $\{f_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space X for all t. Let, for each $t \in S$, $f(t) = \lim_{n\to\infty} f_n(t)$, and let $f : S \to X$ be defined by $t \mapsto f(t)$. We will show that $f \in \Lambda(S,X,|||\cdot|||)$ and $f_n \to f$ as $n \to \infty$. For each $A \in \mathcal{F}$, we have

$$\left|\left|\left| (f_n)_{[A]} - f_{[A]} \right|\right|\right| = \left|\left|\left| (f_n - f)_{[A]} \right|\right|\right| = \left|\left|\left| \sum_{t \in A} e\left(t; (f_n - f)_{[A]}(t)\right) \right|\right|\right|$$

$$\begin{aligned}&\leq \sum_{t\in A} \left\|\left| e\left(t; (f_n - f)_{[A]}(t)\right)\right|\right\| \\ &= \sum_{t\in A} \|f_n(t) - f(t)\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $(f_n)_{[A]} \to f_{[A]}$ as $n \to \infty$ for all $A \in \mathcal{F}$. Let $\epsilon > 0$. Then there is a positive integer N such that for any $A \in \mathcal{F}$

$$\left|\left\|(f_n)_{[A]} - (f_m)_{[A]}\right\|\right| \leq \||f_n - f_m\|| < \frac{\epsilon}{2} \text{ for all } n, m \geq N.$$

Hence, by taking the limit as $m \to \infty$, we have for each $A \in \mathcal{F}$ that $\left|\left\|(f_n)_{[A]} - (f)_{[A]}\right\|\right| \leq \frac{\epsilon}{2}$ for all $n \geq N$. Thus, by (N1), $\||f_n - f\|| < \epsilon$ for all $n \geq N$. This implies that $f \in \Lambda(S, X)$ and $f_n \to f$ as $n \to \infty$ as required.

To see that $\Lambda_0(S, X, \||\cdot\||)$ is a Banach space, suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $\Lambda_0(S, X, \||\cdot\||)$ converging to a function f in $\Lambda(S, X, \||\cdot\||)$. Let $\epsilon > 0$. Then there is a positive integer N such that $\||f_N - f\|| < \frac{\epsilon}{3}$. Since $f_N \in \Lambda_0(S, X, \||\cdot\||)$, there is $A_0 \in \mathcal{F}$ such that $\left|\left\|(f_N)_{[A]} - f_N\right\|\right| < \frac{\epsilon}{3}$ for all $A \succeq A_0$. Consequently, by (N1), we obtain that

$$\begin{aligned}\left|\left\|f_{[A]} - f\right\|\right| &\leq \||f_N - f\|| + \left|\left\|(f_N)_{[A]} - f_N\right\|\right| + \left|\left\|(f_N)_{[A]} - f_{[A]}\right\|\right| \\ &= \||f_N - f\|| + \left|\left\|(f_N)_{[A]} - f_N\right\|\right| + \left|\left\|(f_N - f)_{[A]}\right\|\right| \\ &\leq \||f_N - f\|| + \left|\left\|(f_N)_{[A]} - f_N\right\|\right| + \||f_N - f\|| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon \text{ for all } A \succeq A_0.\end{aligned}$$

It follows that $f \in \Lambda_0(S, X, \||\cdot\||)$. □

The Banach space $\Lambda(S, X, \||\cdot\||)$ was first considered in [9] by O. Woottijirutikal, S.-C. Ong, P. Chaisuriya, and J. Rakbud.

If the function $\||\cdot\||$ has the following additional property:

(N7) for any function $\lambda : S \to \mathbb{C}$ with $|\lambda(t)| = 1$ for all $t \in S$ and $f \in \Sigma(S, X)$, $\||\lambda f\|| \leq \||f\||$, where $\lambda f(t) := \lambda(t)f(t)$ for all $t \in S$,

we call the Banach space $\Lambda(S, X, \||\cdot\||)$ the *X-valued function space defined by* $\||\cdot\||$, or simply, an X-valued function space. For convenience, we may sometimes denote $\Lambda(S, X, \||\cdot\||)$ by just $\Lambda(S, X)$. When S is the set $\mathbb{N}$ of all positive integers, we call $\Lambda(\mathbb{N}, X, \||\cdot\||)$ specifically the *X-valued sequence space* defined by $\||\cdot\||$, or shortly, an X-valued sequence space.

2. Dual of $\Lambda_0(S,X)$

Let $\Lambda(S,X,|||\cdot|||)$ be an X-valued function space. For any $\varphi \in \Sigma(S,X^*)$, we define

$$|||\varphi|||^* = \sup\left\{\left|\sum_{t\in S}\varphi(t)(f(t))\right| : f \in \Lambda(S,X,|||\cdot|||), |||f||| \leq 1\right\}$$

if the supremum is finite, and $|||\varphi|||^* = \infty$ otherwise. For any $z \in \mathbb{C}$, let $\mathrm{dir}(z) = \frac{\bar{z}}{|z|}$ if $z \neq 0$ and $\mathrm{dir}(z) = 1$ if $z = 0$.

Theorem 2.1. *The set*

$$\triangle := \left\{\varphi \in \Sigma(S,X^*) : \sum_{t\in S}|\varphi(t)(f(t))| \text{ converges for all } f \in \Lambda(S,X,|||\cdot|||)\right\}$$

is the X^*-valued function space defined by $|||\cdot|||^*$, or symbolically, $\triangle = \Lambda(S,X^*,|||\cdot|||^*)$.

Proof. We must show first that $|||\varphi|||^* < \infty$ if and only if $\varphi \in \triangle$. Suppose that $\varphi \in \triangle$. Then the linear functional T on $\Lambda(S,X)$ defined by $T(f) = \sum_{t\in S}\varphi(t)(f(t))$ for all $f \in \Lambda(S,X)$ is well defined. For each $A \in \mathcal{F}$, we have by (N3) that the linear functional T_A on $\Lambda(S,X)$ defined by $T_A(f) = \sum_{t\in A}\varphi(t)(f(t))$ for all $f \in \Lambda(S,X)$ is bounded. For each $f \in \Lambda(S,X)$, we have $|T_A(f)| \leq \sum_{t\in S}|\varphi(t)(f(t))|$ for all $A \in \mathcal{F}$. It follows by the uniform boundedness principle that $\sup_{A\in\mathcal{F}}\|T_A\| < \infty$. Since $T_A(f) \to T(f)$, we obtain

$$\sup\left\{\left|\sum_{t\in S}\varphi(t)(f(t))\right| : f \in \Lambda(S,X), |||f||| \leq 1\right\} = \|T\| \leq \sup_{A\in\mathcal{F}}\|T_A\| < \infty.$$

Conversely, suppose that $|||\varphi|||^* < \infty$. Then $\sum_{t\in S}\varphi(t)(f(t))$ converges for all $f \in \Lambda(S,X)$. Let $f \in \Lambda(S,X)$ with $|||f||| = 1$, and let $\lambda : S \to \mathbb{C}$ be defined by $\lambda(t) = \mathrm{dir}(\varphi(t)(f(t)))$ for all $t \in S$. Then $|\lambda(t)| = 1$ for all t. Thus, by (N7), we have $|||\lambda f||| = |||f||| \leq 1$, and hence

$$\begin{aligned}\sum_{t\in S}|\varphi(t)(f(t))| &= \sum_{t\in S}\mathrm{dir}(\varphi(t)(f(t)))\varphi(t)(f(t)) \\ &= \sum_{t\in S}\varphi(t)(\mathrm{dir}(\varphi(t)(f(t)))f(t)) \\ &= \sum_{t\in S}\varphi(t)(\lambda(t)f(t))\end{aligned}$$

converges. For any $f \in \Lambda(S,X)$, we have $\left|\left|\left|\frac{1}{|||f|||}f\right|\right|\right| = 1$. Thus

$$\sum_{t\in S}|\varphi(t)(f(t))| = |||f|||\sum_{t\in S}\left|\varphi(t)\left(\frac{1}{|||f|||}f(t)\right)\right|$$

converges for all $f \in \Lambda(S, X)$.

The rest of the proof is to show that $\|\!|\cdot|\!\|^*$ satisfies the properties (N1)-(N7).

(N1). Let $\varphi \in \triangle$, and let T and T_A for each $A \in \mathcal{F}$ be the linear functionals defined in the beginning of the preceding paragraph. It is clear that $\|T_A\| = \left\|\!\left|\varphi_{[A]}\right|\!\right\|^*$ for all $A \in \mathcal{F}$, and hence we have $\|\!|\varphi|\!\|^* = \|T\| \leq \sup_{A\in\mathcal{F}} \|T_A\| = \sup_{A\in\mathcal{F}} \left\|\!\left|\varphi_{[A]}\right|\!\right\|^*$. Let $f \in \Lambda(S, X)$ with $\|\!|f|\!\| \leq 1$, and let $\lambda : S \to \mathbb{C}$ be the function defined by $\lambda(t) = \operatorname{dir}(\varphi(t)(f(t)))$ for all $t \in S$. Then $|\lambda(t)| = 1$ for all $t \in S$ and by (N7), $\|\!|\lambda f|\!\| \leq \|\!|f|\!\| \leq 1$. This yields for any $A \in \mathcal{F}$ that

$$
\begin{aligned}
|T_A(f)| &= \left|\sum_{t\in A} \varphi(t)(f(t))\right| \leq \sum_{t\in A} |\varphi(t)(f(t))| \\
&= \sum_{t\in A} \operatorname{dir}(\varphi(t)(f(t)))\varphi(t)(f(t)) \\
&= \sum_{t\in A} \lambda(t)\varphi(t)(f(t)) = \sum_{t\in A} \varphi(t)(\lambda(t)f(t)) \\
&= |T((\lambda f)_{[A]})| \leq \|T\| \left\|\!\left|(\lambda f)_{[A]}\right|\!\right\| \\
&\leq \|\!|\varphi|\!\|^* \, \|\!|\lambda f|\!\| \leq \|\!|\varphi|\!\|^* .
\end{aligned}
$$

It follows that $\left\|\!\left|\varphi_{[A]}\right|\!\right\|^* = \|T_A\| \leq \|\!|\varphi|\!\|^*$ for all $A \in \mathcal{F}$. Accordingly, (N1) holds.

(N2). For any $t \in S$ and $y \in X^*$, we have by (N4) that

$$
\begin{aligned}
\|\!|e(t; y)|\!\|^* &= \sup\left\{\left|\sum_{s\in S} e(t; y)(s)(f(s))\right| : f \in \Lambda(S, X), \|\!|f|\!\| \leq 1\right\} \\
&= \sup\{|e(t; y)(t)(f(t))| : f \in \Lambda(S, X), \|\!|f|\!\| \leq 1\} \\
&= \sup\{|y(f(t))| : f \in \Lambda(S, X), \|\!|f|\!\| \leq 1\} \\
&\leq \sup\{\|y\| \, \|f(t)\| : f \in \Lambda(S, X), \|\!|f|\!\| \leq 1\} \\
&\leq \|y\| \left(\sup\{\|\!|f|\!\| : f \in \Lambda(S, X), \|\!|f|\!\| \leq 1\}\right) \leq \|y\| .
\end{aligned}
$$

Thus (N2) holds.

(N3). Let $\varphi \in \triangle$ and $t \in S$. For any $x \in X$ with $\|x\| \leq 1$, we have by (N2) (of $\|\!|\cdot|\!\|$) that $\|\!|e(t; x)|\!\| \leq \|x\| \leq 1$. Thus

$$
|\varphi(t)(x)| = |\varphi(t)(e(t; x)(t))| = \left|\sum_{s\in S} \varphi(s)(e(t; x)(s))\right| \leq \|\!|\varphi|\!\|^* .
$$

It follows that $\|\varphi(t)\| \leq \|\|\varphi\|\|^*$.

The properties (N4) and (N5) follow directly from the definition of $\|\|\cdot\|\|^*$.

(N7). Let $\varphi \in \Sigma(S, X^*)$, and let $\lambda : S \to \mathbb{C}$ with $\lambda(t) = 1$ for all $t \in S$. Suppose that $\|\|\varphi\|\|^* < \infty$. Then for any $f \in \Lambda(S, X)$ with $\|\|f\|\| \leq 1$, we have by the property (N7) of $\|\|\cdot\|\|$ that

$$\left|\sum_{t\in S}\lambda(t)\varphi(t)(f(t))\right| = \left|\sum_{t\in S}\varphi(t)(\lambda(t)f(t))\right| \leq \|\|\varphi\|\|^* .$$

It follows that $\|\|\lambda\varphi\|\|^* \leq \|\|\varphi\|\|^*$. □

We call the X^*-valued function space $\Lambda(S, X^*, \|\|\cdot\|\|^*)$ the *Köthe dual* of $\Lambda(S, X, \|\|\cdot\|\|)$ and call $\|\|\cdot\|\|^*$ the *dual norm* of $\|\|\cdot\|\|$. The main goal of this section is to identify the dual $\Lambda_0(S, X, \|\|\cdot\|\|)^*$ of $\Lambda_0(S, X, \|\|\cdot\|\|)$ with the Köthe dual $\Lambda(S, X^*, \|\|\cdot\|\|^*)$ of $\Lambda(S, X, \|\|\cdot\|\|)$.

Let $\Psi \in \Lambda_0(S, X, \|\|\cdot\|\|)^*$. We then define, for each $t \in S$, the function $y_t : X \to \mathbb{C}$ by $y_t(x) = \Psi(e(t; x))$ for all $x \in X$. Clearly, $\|y_t\| \leq \|\Psi\|$ for all $t \in S$ and hence $y_t \in X^*$. Let $\varphi^{(\Psi)} : S \to X^*$ be defined by $t \mapsto y_t$.

Theorem 2.2. $\Lambda_0(S, X, \|\|\cdot\|\|)^*$ *is isometrically isomorphic to* $\Lambda(S, X^*, \|\|\cdot\|\|^*)$ *by the isomorphism* $\Psi \mapsto \varphi^{(\Psi)}$.

Proof. We will show first that $\varphi^{(\Psi)} \in \Lambda(S, X^*)$. To see this, let $f \in \Lambda(S, X)$, and let $\lambda : S \to \mathbb{C}$ be defined by $\lambda(t) = \mathrm{dir}(\Psi(e(t; f(t))))$ for all $t \in S$. Then for each $A \in \mathcal{F}$,

$$\begin{aligned}
\sum_{t\in A}\left|\varphi^{(\Psi)}(t)(f(t))\right| &= \sum_{t\in A}|y_t(f(t))| = \sum_{t\in A}|\Psi(e(t; f(t)))| \\
&= \sum_{t\in A}\lambda(t)\Psi(e(t; f(t))) = \sum_{t\in A}\Psi(\lambda(t)e(t; f(t))) \\
&= \sum_{t\in A}\Psi(e(t; \lambda(t)f(t))) = \Psi\left(\sum_{t\in A}e(t; \lambda(t)f(t))\right) \\
&= \Psi\left((\lambda f)_{[A]}\right) \leq \|\Psi\| \left\|\left\|(\lambda f)_{[A]}\right\|\right\| \leq \|\Psi\| .
\end{aligned}$$

Consequently, $\varphi^{(\Psi)} \in \Lambda(S, X^*)$ and $\left\|\left\|\varphi^{(\Psi)}\right\|\right\|^* \leq \|\Psi\|$. Next, we will show that $\|\Psi\| \leq \left\|\left\|\varphi^{(\Psi)}\right\|\right\|^*$. Let $f \in \Lambda_0(S, X)$ with $\|\|f\|\| \leq 1$, and let $\lambda : S \to \mathbb{C}$ be the function defined above. Then for each $A \in \mathcal{F}$,

$$\left|\Psi\left(f_{[A]}\right)\right| = \left|\Psi\left(\sum_{t\in A}e\left(t; f(t)\right)\right)\right| \leq \sum_{t\in A}|\Psi(e(t; f(t)))|$$

$$= \sum_{t\in A}\lambda(t)y_t(f(t)) = \sum_{t\in A}y_t(\lambda(t)f(t))$$

$$= \sum_{t\in A}\varphi^{(\Psi)}(t)(\lambda(t)f(t)) \le \left|\!\left|\!\left|\varphi^{(\Psi)}\right|\!\right|\!\right|^*.$$

Thus, by the continuity of Ψ, $|\Psi(f)| \le \left|\!\left|\!\left|\varphi^{(\Psi)}\right|\!\right|\!\right|^*$ for all $f \in \Lambda_0(S,X)$ with $|\!|\!|f|\!|\!| \le 1$. Hence $\|\Psi\| = \left|\!\left|\!\left|\varphi^{(\Psi)}\right|\!\right|\!\right|^*$. Finally, we will show that the function $\Psi \mapsto \varphi^{(\Psi)}$ is onto. Let $\varphi \in \Lambda(S,X^*)$. Then the linear functional $\Psi : \Lambda_0(S,X) \to \mathbb{C}$ defined by $\Psi(f) = \sum_{t\in S}\varphi(t)(f(t))$ for all $f \in \Lambda_0(S,X)$ is bounded and $\varphi^{(\Psi)} = \varphi$. □

In the following theorem, the space $\Lambda(S,X^*)$ of functions will be considered as the space of bounded linear functionals Φ_φ on $\Lambda(S,X)$ defined for each $\varphi \in \Lambda(S,X^*)$ by $\Phi_\varphi(f) = \sum_{t\in S}\varphi(t)(f(t))$ for all $f \in \Lambda(S,X)$. It is clear that $\Lambda(S,X^*)$ is a closed subspace of $\Lambda(S,X)^*$.

Theorem 2.3. *If $\Lambda_0(S,X) \subsetneq \Lambda(S,X)$, then the annihilator $\Lambda_0(S,X)^\perp$ of $\Lambda_0(S,X)$ is a non-trivial closed subspace of $\Lambda(S,X)^*$, and $\Lambda(S,X)^* = \Lambda(S,X^*) \oplus \Lambda_0(S,X)^\perp$.*

Proof. Suppose that $\Lambda_0(S,X) \subsetneq \Lambda(S,X)$. Then by the Hahn-Banach extension theorem, we have $\Lambda_0(S,X)^\perp$ is a non-trivial closed subspace of $\Lambda(S,X)^*$. For any $\Psi \in \Lambda(S,X)^*$, let $\Omega_\Psi = \Psi - \Phi_{\varphi^{(\Psi)}}$. Then $\Omega_\Psi \in \Lambda_0(S,X)^\perp$, and hence $\Lambda(S,X)^* = \Lambda(S,X^*) + \Lambda_0(S,X)^\perp$. For any $\varphi \in \Lambda(S,X^*)$, if $\Phi_\varphi \in \Lambda_0(S,X)^\perp$, then $\Phi_\varphi(f) = \lim_{A\in\mathcal{F}}\Phi_\varphi\left(f_{[A]}\right) = 0$ for all $f \in \Lambda(S,X)$. Thus $\Lambda(S,X^*) \cap \Lambda_0(S,X)^\perp = \{0\}$. It follows that $\Lambda(S,X)^* = \Lambda(S,X^*) \oplus \Lambda_0(S,X)^\perp$. The proof is complete. □

3. Predual of $\Lambda(S,X)$

The aim of this section is to complete the duality relation among the four spaces: $\Lambda_0(S,X)$, $\Lambda(S,X^*)$, $\Lambda_0(S,X^*)$, and $\Lambda(S,X)$. From the previous section, we have obtained Theorem 2.2 which shows the duality relation between the first two spaces $\Lambda_0(S,X)$ and $\Lambda(S,X^*)$. The rest is to investigate the preduality of the space $\Lambda(S,X)$. We expect to have $\Lambda_0(S,X^*)^* \cong \Lambda(S,X)$.

For any $f \in \Lambda(S,X)$, we define a function $\widetilde{\Theta}_f : \Lambda(S,X^*) \to \mathbb{C}$ by $\widetilde{\Theta}_f(\varphi) = \sum_{t\in S}\varphi(t)(f(t))$ for all $\varphi \in \Lambda(S,X^*)$ and let Θ_f be the restriction of $\widetilde{\Theta}_f$ to $\Lambda_0(S,X^*)$. It is clear that $\|\Theta_f\| \le \left\|\widetilde{\Theta}_f\right\| \le |\!|\!|f|\!|\!|$ for all $f \in \Lambda(S,X)$.

Proposition 3.1. *For any $f \in \Lambda(S,X)$, $\left\|\widetilde{\Theta}_f\right\| = |\!|\!|f|\!|\!| = \|\Theta_f\|$.*

Proof. Let $f \in \Lambda(S,X)$, and let $\epsilon > 0$, and let $A \in \mathcal{F}$. Then we have by the Hahn-Banach extension theorem that there is $\Psi \in \Lambda(S,X)^*$ with $\|\Psi\| \le 1$ such that $\left\|\left| f_{[A]} \right|\right\| < \left|\Psi\left(f_{[A]}\right)\right| + \epsilon$. By Theorem 2.3, we have $\Psi\left(f_{[A]}\right) = \Phi_{\varphi^{(\Psi)}}\left(f_{[A]}\right)$. Thus by Theorem 2.2 and the property (N1) of $\||\cdot|\|^*$, we have

$$\begin{aligned}
\left\|\left| f_{[A]} \right|\right\| &< \left|\Psi\left(f_{[A]}\right)\right| + \epsilon = \left|\Phi_{\varphi^{(\Psi)}}\left(f_{[A]}\right)\right| + \epsilon \\
&= \left|\widetilde{\Theta}_{f_{[A]}}\left(\varphi^{(\Psi)}\right)\right| + \epsilon = \left|\widetilde{\Theta}_f\left(\left(\varphi^{(\Psi)}\right)_{[A]}\right)\right| + \epsilon \\
&= \left|\Theta_f\left(\left(\varphi^{(\Psi)}\right)_{[A]}\right)\right| + \epsilon \le \|\Theta_f\| \left\|\left|(\varphi^{(\Psi)})_{[A]}\right|\right\|^* + \epsilon \\
&\le \|\Theta_f\| \left\|\left|\varphi^{(\Psi)}\right|\right\|^* + \epsilon = \|\Theta_f\| \|\Psi\| + \epsilon \\
&\le \|\Theta_f\| + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $\left\|\left| f_{[A]} \right|\right\| \le \|\Theta_f\|$ for all $A \in \mathcal{F}$. Thus, by (N1) of $\||\cdot|\|$, we obtain $\||f|\| \le \|\Theta_f\|$. It follows that $\left\|\widetilde{\Theta}_f\right\| = \||f|\| = \|\Theta_f\|$ as asserted. □

We now have the function $f \mapsto \Theta_f$ is an isometric isomorphism from $\Lambda(S,X)$ into $\Lambda_0(S,X^*)^*$. In the following theorem, we provide a necessary and sufficient condition for the function to be onto.

Theorem 3.2. *The isomorphism $f \mapsto \Theta_f$ from $\Lambda(S,X)$ into $\Lambda_0(S,X^*)^*$ is onto if and only if X is reflexive.*

Proof. Suppose that X is reflexive. We want to show that the isomorphism $f \mapsto \Theta_f$ is onto. To see this, let $\Psi \in \Lambda_0(S,X^*)^*$, and for each $t \in S$, let $\phi_t : X^* \to \mathbb{C}$ be defined by $\phi_t(y) = \Psi(e(t;y))$ for all $y \in X^*$. Then $\phi_t \in X^{**}$ for all $t \in S$. Thus, by the reflexivity of X, there exists, for each $t \in S$, an x_t in X such that $\phi_t(y) = y(x_t)$ for all $y \in X^*$. Let $f(t) = x_t$ for all $t \in S$. We will show that $f \in \Lambda(S,X)$ and $\Psi = \Theta_f$. For each $A \in \mathcal{F}$, we have for any $\varphi \in \Lambda(S,X^*)$ with $\||\varphi|\|^* \le 1$ by (N1) of $\||\cdot|\|^*$ that

$$\begin{aligned}
\left|\Theta_{f_{[A]}}(\varphi)\right| &= \left|\sum_{t\in A}\varphi(t)(f(t))\right| = \left|\sum_{t\in A}\varphi(t)(x_t)\right| \\
&= \left|\sum_{t\in A}\phi_t(\varphi(t))\right| = \left|\sum_{t\in A}\Psi(e(t;\varphi(t)))\right| \\
&= \left|\Psi\left(\varphi_{[A]}\right)\right| \le \|\Psi\|.
\end{aligned}$$

It follows that $\left\|\left| f_{[A]} \right|\right\| = \left\|\Theta_{\varphi_{[A]}}\right\| \le \|\Psi\|$ for all $A \in \mathcal{F}$. Consequently, by (N1) of $\||\cdot|\|$, we have $\||f|\| \le \|\Psi\|$, and hence $f \in \Lambda(S,X)$. To see that $\Psi = \Theta_f$, let $\varphi \in \Lambda_0(S,X^*)$.

Then we have for each $A \in \mathcal{F}$ that

$$\Psi\left(\varphi_{[A]}\right) = \Psi\left(\sum_{t\in A} e(t;\varphi(t))\right) = \sum_{t\in A} \Psi(e(t;\varphi(t)))$$
$$= \sum_{t\in A} \phi_t(\varphi(t)) = \sum_{t\in A} \varphi(t)(f(t)) = \Theta_f\left(\varphi_{[A]}\right).$$

Thus, by the continuity of both Ψ and Θ_f, we have $\Psi = \Theta_f$ as required. Conversely, suppose that the isomorphism $f \mapsto \Theta_f$ is onto. Let $\phi \in X^{**}$, and let $t_0 \in S$ be fixed. Then the linear functional Ψ on $\Lambda(S, X^*)$ defined by $\Psi(\varphi) = \phi(\varphi(t_0))$ for all $\varphi \in \Lambda(S, X^*)$ is bounded. Thus, by the assumption, there is an $f \in \Lambda(S, X)$ such that $\Theta_f = \Psi$. From this, we have $\phi(y) = \phi(e(t_0; y)(t_0)) = \Psi(e(t_0; y)) = \Theta_f(e(t_0; y)) = e(t_0; y)(t_0)(f(t_0)) = y(f(t_0))$ for all $y \in X^*$. Therefore, the reflexivity of X is obtained. □

4. Reflexivity

In this section, we establish a reflexivity theorem for our Banach-valued function spaces. We denote here the isomorphism $\Psi \mapsto \varphi^{(\Psi)}$ from $\Lambda_0(S, X)^*$ onto $\Lambda(S, X^*)$ by N, the isomorphism $f \mapsto \Theta_f$ from $\Lambda(S, X)$ into $\Lambda_0(S, X^*)^*$ by M. Let P be the isometric isomorphism from the space $\{\Theta_f : f \in \Lambda(S, X)\}$ into $\Lambda(S, X^*)^*$ defined by $\Theta_f \mapsto \widetilde{\Theta}_f$.

Lemma 4.1. *$N^*PM(f) = Q(f)$ for all $f \in \Lambda_0(S, X)$, where N^* is the adjoint of N and $Q : \Lambda_0(S, X) \to \Lambda_0(S, X)^{**}$ is the natural map.*

Proof. Let $f \in \Lambda_0(S, X)$. Then for every A in $\mathcal{F}$,

$$N^*PM\left(f_{[A]}\right) = N^*\left(P\left(M\left(f_{[A]}\right)\right)\right) = N^*\left(P\left(\Theta_{f_{[A]}}\right)\right) = \widetilde{\Theta}_{f_{[A]}}N.$$

Let $\Psi \in \Lambda(S, X)^*$. Then we have for each A in $\mathcal{F}$ that

$$\widetilde{\Theta}_{f_{[A]}}N(\Psi) = \widetilde{\Theta}_{f_{[A]}}\left(\varphi^{(\Psi)}\right) = \sum_{t\in A} \Psi(e(t; f(t))) = \Psi\left(f_{[A]}\right) = Q\left(f_{[A]}\right)(\Psi).$$

It follows that $N^*PM\left(f_{[A]}\right) = Q\left(f_{[A]}\right)$ for all A in $\mathcal{F}$. Therefore, $N^*PM(f) = Q(f)$. □

An X-valued function space $\Lambda(S, X)$ is called a *GAK-space* (see [4]) if $\Lambda_0(S, X) = \Lambda(S, X)$.

Theorem 4.2. (Reflexivity theorem for Banach-valued function spaces) *Let $\Lambda(S, X)$ be an X-valued function space. Then the following are equivalent:*

1. *$\Lambda(S, X)$ is reflexive;*
2. *$\Lambda_0(S, X)$ is reflexive;*
3. *X is reflexive, and both $\Lambda(S, X)$ and its Köthe dual $\Lambda(S, X^*)$ are GAK.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Suppose that (3) doesn't hold. If $\Lambda(S,X)$ is not GAK, then we have by Lemma 4.1 that $\Lambda(S,X)$ is not reflexive. Next, suppose that X is not reflexive or $\Lambda(S,X^*)$ is not GAK. We will show that each of these two conditions implies that

$$\left\{\widetilde{\Theta}_f : f \in \Lambda(S,X)\right\} \neq \Lambda(S,X^*)^*. \qquad (*)$$

If X is not reflexive, then $(*)$ holds by Theorem 3.2. Suppose that $\Lambda_0(S,X^*) \neq \Lambda(S,X^*)$. Then by the Hahn-Banach extension theorem, there exists $\psi \in \Lambda(S,X^*)^*$ such that $\|\psi\| \neq 0$ and $\Lambda_0(S,X^*) \subseteq \ker\psi$. If $\psi = \widetilde{\Theta}_f$ for some $f \in \Lambda(S,X)$, then $\Theta_f = 0$. This implies that $\|\psi\| = 0$, which is a contradiction. Accordingly, $(*)$ holds. Hence, by Lemma 4.1, $\Lambda_0(S,X)$ is not reflexive.

(3) $\Rightarrow$ (1). It follows immediately from Theorem 3.2 and Lemma 4.1. The proof is finished □

5. Applications to the sequence spaces $l_p(X)$ and $l_p[X]$

In this section, we show that the well-known results on the duality and reflexivity of the sequence spaces $l_p(X)$ and $l_p[X]$ can be deduced from our reflexivity theorem.

It is clear that for any $1 \leq p \leq \infty$, $l_p(X)$ is the X-valued sequence space defined by $\|\cdot\|_p$ and it is GAK, except for the case where $p = \infty$. Thus, by the duality theorem, $l_p(X)^*$ is isometrically isomorphic to its the Köthe dual $\Lambda\left(\mathbb{N}, X^*, \|\cdot\|_p^*\right)$. It is well-known for each $1 \leq p < \infty$ that $\Lambda\left(\mathbb{N}, X^*, \|\cdot\|_p^*\right) = l_q(X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus $\Lambda\left(\mathbb{N}, X^*, \|\cdot\|_p^*\right)$ is GAK for all $1 < p < \infty$, whereas $\Lambda\left(\mathbb{N}, X^*, \|\cdot\|_1^*\right)$ is not GAK. It follows immediately from the reflexivity theorem that $l_1(X)$ and $l_\infty(X)$ are not reflexive, while for $1 < p < \infty$, l_p is reflexive if and only if X is reflexive.

It is also clear that $l_p[X]$ is the X-valued sequence space defined by $|\!|\!|\cdot|\!|\!|_p$, where $|\!|\!|\cdot|\!|\!|_p$ is given by

$$|\!|\!|\{x_k\}_{k=1}^\infty|\!|\!|_p = \sup_{f \in X^*,\ \|f\| \leq 1} \left(\sum_{k=1}^{\infty} |f(x_k)|^p\right)^{1/p}.$$

It was proved in [10] by C. X. Wu and Q. Y. Bu that the Köthe dual of $l_p[X]$ is GAK for all $1 < p < \infty$. Thus, by the reflexivity theorem, we have that the sequence space $l_p[X]$ for $1 < p < \infty$ is reflexive if and only if X is reflexive and $l_p[X]$ is GAK. This is exactly the same as that given in [1] by Q. Y. Bu.

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Department of Mathematics, Faculty of Science, Silpakorn University, Nakorn Pathom, 73000, Thailand

E-mail address: jitti@su.ac.th

Department of Mathematics, Faculty of Science, Chiangmai University, Chiangmai, 50200, Thailand

E-mail address: suthep@math.science.cmu.ac.th