รายงานวิจัยฉบับสมบูรณ์

โครงการ การประมาณค่าจุดตรึงของการส่งไม่ขยายในปริภูมิ CAT(0)

โดย อาจารย์ ดร. บัญชา ปัญญานาค และคณะ

มีนาคม 2554

สัญญาเลขที่ MRG5280025

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คณะผู้วิจัย	**สังกัด**
1. อาจารย์ ดร. บัญชา ปัญญานาค	มหาวิทยาลัยเชียงใหม่
2. ศาสตราจารย์ ดร. สมพงษ์ ธรรมพงษา	มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

กิติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณสำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้การสนับสนุนทุนวิจัยมาอย่างต่อเนื่อง ขอขอบพระคุณ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้ให้การสนับสนุนการทำวิจัยอย่างเต็มที่

อาจารย์ ดร. บัญชา ปัญญานาค
หัวหน้าโครงการวิจัย
3 มีนาคม 2554

สารบัญ

หน้า

สัญญาเลขที่: MRG5280025

ชื่อโครงการ: การประมาณค่าจุดตรึงของการส่งไม่ขยายในปริภูมิ CAT(0)

หัวหน้าโครงการ: อาจารย์ ดร. บัญชา ปัญญานาค
ภาควิชาคณิตศาสตร์, คณะวิทยาศาสตร์
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บทคัดย่อ

ในงานวิจัยนี้เราสนใจในการหาเงื่อนไขที่เพียงพอสำหรับสร้างทฤษฎีการลู่เข้าแบบเดลต้าและแบบเข้มของลำดับ $\{x_n\}$ ที่นิยามโดย $x_1 \in K,$

$$x_{n+1} = P\left(\left(1-\alpha_n\right)x_n + \alpha_n TP\left[\left(1-\beta_n\right)x_n + \beta_n Tx_n\right]\right), \quad n \geq 1 \text{.........................(1)}$$

เมื่อ K เป็นเซตย่อยปิด นูน และมีขอบเขตของปริภูมิเมตริกบริบูรณ์ CAT(0) X และ $P : X \to K$ เป็นฟังก์ชันการฉายของจุดใกล้สุดและ $T : K \to X$ เป็นการส่งแบบไม่ขยายที่ $F(T) := \{x \in K : Tx = x\} \neq \emptyset$

นอกจากนี้เรายังได้เงื่อนไขที่เพียงพอในการสร้างทฤษฎีการลู่เข้าแบบเดลต้าและแบบเข้มของลำดับ $\{x_n\}$ ที่นิยามโดย $x_1 \in K,$

$$z_n = \gamma_n T^n x_n + \left(1-\gamma_n\right)x_n$$

$$y_n = \beta_n T^n z_n + \left(1-\beta_n\right)x_n$$

$$x_{n+1} = \alpha_n T^n y_n + \left(1-\alpha_n\right)x_n, \qquad n \geq 1 \text{.........................(2)}$$

เมื่อ K เป็นเซตย่อยปิด นูน และมีขอบเขตของปริภูมิเมตริกบริบูรณ์ CAT(0) X และ $T : K \to K$ เป็นการส่งแบบไม่ขยายเชิงเส้นกำกับ

Abstract

In this research, we are interested in finding sufficient conditions for constructing Δ and strong convergence theorems for the sequence $\{x_n\}$ defined by $x_1 \in K$,

$$x_{n+1} = P\left(\left(1-\alpha_n\right)x_n + \alpha_n TP\left[\left(1-\beta_n\right)x_n + \beta_n Tx_n\right]\right), \quad n \geq 1 \text{........................(1)}$$

where K is a closed convex bounded subset of a complete CAT(0) space X and $P: X \to K$ is the nearest point projection from X onto K and $T: K \to X$ is a nonexpansive mapping with $F(T) := \{x \in K : Tx = x\} \neq \varnothing$

Moreover, we also obtain some sufficient conditions for constructing Δ and strong convergence theorems for the sequence $\{x_n\}$ defined by $x_1 \in K$,

$$\begin{aligned} z_n &= \gamma_n T^n x_n + \left(1-\gamma_n\right)x_n \\ y_n &= \beta_n T^n z_n + \left(1-\beta_n\right)x_n \\ x_{n+1} &= \alpha_n T^n y_n + \left(1-\alpha_n\right)x_n, \qquad n \geq 1 \text{........................(2)} \end{aligned}$$

where K is a closed convex bounded subset of a complete CAT(0) space X and $T: K \to X$ is an asymptotically nonexpansive mapping

Keywords: Fixed point, nonexpansive mapping, CAT(0) space, Δ convergence, strong convergence

สรุปผลการดำเนินงาน

1. รายละเอียดผลการดำเนินงานของโครงการ

1.1 สรุปย่อ (summary) ประกอบด้วยวัตถุประสงค์และการดำเนินงานวิจัย รวมทั้งผลงานวิจัยที่ได้รับอย่างย่อ

1.1.1. วัตถุประสงค์

(1) เพื่อศึกษาและหาเงื่อนไขที่จำเป็นและเพียงพอในการสร้างทฤษฎีการลู่เข้าแบบเดลต้าของลำดับ $\{x_n\}$ ที่นิยามโดย $x_1 \in K,$

$$x_{n+1} = P\left(\left(1-\alpha_n\right)x_n + \alpha_n TP\left[\left(1-\beta_n\right)x_n + \beta_n Tx_n\right]\right), \quad n \geq 1 \text{.........................(*)}$$

เมื่อ K เป็นเซตย่อยปิด นูน และมีขอบเขตของปริภูมิเมตริกบริบูรณ์ CAT(0) X และ $P : X \to K$ เป็นฟังก์ชันการฉายของจุดใกล้สุด (the nearest point projection) และ $T : K \to X$ เป็นการส่งแบบไม่ขยายที่ $F(T) := \{x \in K : Tx = x\} \neq \varnothing$

(2) เพื่อศึกษาและหาเงื่อนไขที่จำเป็นและเพียงพอในการสร้างทฤษฎีการลู่เข้าแบบเข้มของลำดับ $\{x_n\}$ ที่นิยามโดย (*)

(3) เพื่อขยายวงความรู้ของการประมาณค่าจุดตรึงของการส่งไม่ขยายในปริภูมิ CAT(0)

1.1.2. การดำเนินงานวิจัย

ทีมวิจัยได้ดำเนินการตามแผนที่วางไว้จนสามารถส่งผลงานไปตีพิมพ์ที่วารสาร Fixed Point Theory and Applications , Volume 2010 Article ID 367274 ชื่อเรื่อง Approximating Fixed Points of Nonexpansive Nonself Mappings in CAT(0) Spaces (ดังเอกสารแนบหมายเลข 1) นอกจากนี้ทีมวิจัยยังได้ศึกษาวิธีการทำซ้ำนอร์ (Noor iteration) สำหรับการส่งไม่ขยายแบบเชิงเส้นกำกับในปริภูมิ CAT(0) อีกด้วย และได้ส่งผลงานดังกล่าวไปตีพิมพ์ที่วารสาร International of Mathematical Analysis Vol. 4, 2010, no. 13, 645 – 656 ชื่อเรื่อง Noor Iterations for Asymptotically Nonexpansive Mappings in CAT(0) Spaces (ดังเอกสารแนบหมายเลข 2)

จากการสนับสนุนการทำวิจัยครั้งนี้ ทีมวิจัยได้ผลิตนักศึกษาปริญญาโทจำนวน 1 คน ดังนี้

ชื่อนักศึกษา	ชื่อหัวข้อวิจัย/วิทยานิพนธ์	ปีที่สำเร็จการศึกษา
นางสาวเย็นฤดี นิวงษา (คณิตศาสตร์)	Noor Iterations for Asymptotically Nonexpansive Mappings in CAT(0) Spaces	2552

ทั้งนี้หัวหน้าโครงการวิจัยได้ตีพิมพ์ผลงานร่วมกับนักศึกษาท่านนี้อีกด้วย (ดังเอกสารแนบหมายเลข 2)

1.1.3. ผลงานวิจัยที่ได้รับ

ทีมวิจัยได้สร้างทฤษฎีใหม่ที่เกี่ยวกับการประมาณค่าจุดตรึงสำหรับการส่งไม่ขยาย ในปริภูมิ CAT(0) ดังนี้

ทฤษฎีบท 1 Let K be a nonempty closed convex subset of a complete CAT(0) space X and let T : K $\to$ X be a nonexpansive mapping with $x^* \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K,$ define the sequence $\{x_n\}$ by the recursion (1). Then $\lim_n d(x_n, x^*)$ exists.

ทฤษฎีบท 2 Let K be a nonempty closed convex subset of a complete CAT(0) space X and let T : K $\to$ X be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K,$ define the sequence $\{x_n\}$ by the recursion (1). Then $\lim_n d(x_n, Tx_n) = 0.$

ทฤษฎีบท 3 Let K be a nonempty closed convex subset of a complete CAT(0) space X and let T : K $\to$ X be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K,$ define the sequence $\{x_n\}$ by the recursion (1). Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

ทฤษฎีบท 4 Let K be a nonempty closed convex subset of a complete CAT(0) space X and let T : K $\to$ X be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K,$ define the sequence $\{x_n\}$ by the recursion (1). If T satisfies condition I, then $\{x_n\}$ converges strongly to a fixed point of T.

ทฤษฎีบท 5 Let K be a nonempty closed convex subset of a complete CAT(0) space X and let S,T : K $\to$ K be two nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K,$ define the sequence $\{x_n\}$ by the recursion

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S\left[(1-\beta_n)x_n + \beta_n Tx_n\right].$$

Then $\{x_n\}$ $\Delta-$converges to a common fixed point of S and T.

ทฤษฎีบท 6 Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let T : K $\to$ K be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$ satisfying

(i) 0 < lim inf $\alpha_n \leq$ lim sup α_n < 1 and

(ii) 0 < lim inf $\beta_n \leq$ lim sup β_n < 1.

For a given $x_1 \in C$, define

$$z_n = \gamma_n T^n x_n + (1-\gamma_n)x_n$$
$$y_n = \beta_n T^n z_n + (1-\beta_n)x_n$$
$$x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n.$$

Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

ทฤษฎีบท 7 Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let T : K $\rightarrow$ K be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$ satisfying

(i) 0 < lim inf $\alpha_n \leq$ lim sup α_n < 1 and

(ii) 0 < lim inf $\beta_n \leq$ lim sup β_n < 1.

For a given $x_1 \in C$, define

$$z_n = \gamma_n T^n x_n + (1-\gamma_n)x_n$$
$$y_n = \beta_n T^n z_n + (1-\beta_n)x_n$$
$$x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

ผลงานวิจัยที่ตีพิมพ์ในวารสารวิชาการระดับนานาชาติ

(1) W. Laowang and B. Panyanak, Approximating Fixed Points of Nonexpansive Nonself Mappings in CAT(0) Spaces, Fixed Point Theory and Applications, Volume 2010, Article ID 367274, 11 pages (2009 Impact Factor 1.525)

(2) Yenruedee Niwongsa and Bancha Panyanak, Noor Iterations for Asymptotically Nonexpansive Mappings in CAT(0) Spaces International Journal of Mathematical Analysis, Volume 4, no. 13, 2010, 645 - 656

2. กิจกรรมอื่น ๆ ที่เกี่ยวข้อง

2.1 การไปเสนอผลงาน

1) เข้าร่วมและเสนอผลงานเรื่อง "Demiclosed Principle for Asymptotically Nonexpansive Mappings in CAT(0) Spaces" ในการประชุมวิชาการทฤษฎีจุดตรึงและการประยุกต์ครั้งที่ 3 (ปี 2552) ระหว่างวันที่ 4-6 กันยายน 2552 ณ มหาวิทยาลัยราชภัฏเลย จังหวัดเลย
2) A speaker in "The 9th International Conference on Fixed Point Theory and Its Applications" July 16-22,2009, National Changhua University of Education, Changhua, Taiwan
3) เข้าร่วมและเสนอผลงานเรื่อง "Fixed Point Theorems and Convergence Theorems for Multivalued Mappings in CAT(0) Spaces" ในการประชุมประจำปี "นักวิจัยรุ่นใหม่...พบ...เมธีวิจัยอาวุโส สกว." ครั้งที่ 9 ระหว่างวันที่ 15-17 ตุลาคม 2552 ณ โรงแรมฮอลิเดย์ อินน์ รีสอร์ท รีเจนท์ บีช ชะอำ จังหวัดเพชรบุรี
4) เข้าร่วมและเสนอผลงานเรื่อง "Demiclosed Principle for Asymptotically Nonexpansive Mappings in CAT(0) Spaces" ในงานวันวิชาการมหาวิทยาลัยเชียงใหม่ ครั้งที่ 5 ประจำปี 2552 "วิถีวิจัย : ทศวรรษที่ห้าสู่ความเป็นเลิศ" ระหว่างวันที่ 26-27 พฤศจิกายน 2552 ณ หอประชุมมหาวิทยาลัยเชียงใหม่

5) วิทยากรในโครงการประชุมเชิงปฏิบัติการเพื่อพัฒนานักวิจัยด้านทฤษฎีจุดตรึงและการประยุกต์ ระหว่างวันที่ 13-14 ตุลาคม 2552 ณ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏวไลอลงกรณ์ ในพระบรมราชูปถัมภ์
6) เข้าร่วมและเสนอผลงานเรื่อง "Approximating Fixed Points of Nonexpansive Mappings in CAT(0) Spaces" ในการประชุมประจำปี "นักวิจัยรุ่นใหม่...พบ...เมธีวิจัยอาวุโส สกว." ครั้งที่ 10 ระหว่างวันที่ 14-16 ตุลาคม 2553 ณ โรงแรมฮอลิเดย์ อินน์ รีสอร์ท รีเจนท์ บีช ชะอำ จังหวัดเพชรบุรี
7) เป็นผู้ทรงคุณวุฒิให้ตรวจพิจารณาบทความวิจัยให้แก่วารสารระดับนานาชาติ Fixed Point Theory and Applications จำนวน 3 บทความ
8) เป็นผู้ทรงคุณวุฒิให้ตรวจพิจารณาบทความวิจัยให้แก่วารสารระดับนานาชาติ Applied Mathematics Letters จำนวน 1 บทความ
9) เป็นผู้ทรงคุณวุฒิให้ตรวจพิจารณาบทความวิจัยให้แก่วารสารระดับนานาชาติ Topology and Its Applications จำนวน 1 บทความ
10) เป็นผู้ทรงคุณวุฒิให้ตรวจพิจารณาบทความวิจัยให้แก่วารสารระดับนานาชาติ Chiang Mai Journal of Sciences จำนวน 1 บทความ
11) เป็นผู้ทรงคุณวุฒิให้ตรวจพิจารณาบทความวิจัยให้แก่วารสารระดับนานาชาติ Applied Mathematics and Computations จำนวน 1 บทความ
12) เป็นประธาน/กรรมการสอบวิทยานิพนธ์นักศึกษาระดับบัณฑิตศึกษาภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ จำนวน 7 ครั้ง

2.2 การเชื่อมโยงทางวิชาการกับนักวิชาการอื่น ๆทั้งในและต่างประเทศ

1) ความเชื่อมโยงกับนักคณิตศาสตร์ในประเทศ

- ศ.ดร. สมพงษ์ ธรรมพงษา มหาวิทยาลัยเชียงใหม่
- ศ.ดร. สุเทพ สวนใต้ มหาวิทยาลัยเชียงใหม่
- ศ.ดร. สมยศ พลับเที่ยง มหาวิทยาลัยนเรศวร
- รศ.ดร. สาธิต แซ่จึง มหาวิยาลัยขอนแก่น
- ผศ.ดร. อรรถพล แก้วขาว มหาวิทยาลัยบูรพา
- ผศ.ดร. นรินทร์ เพชร์โรจน์ มหาวิทยาลัยนเรศวร
- ผศ.ดร. ระเบียน วังคีรี มหาวิทยาลัยนเรศวร
- ผศ.ดร. ภูมิ คำเอม มหาวิทยาลัยเทคโนโลยี พระจอมเกล้าธนบุรี
- ผศ.ดร. อัญชลีย์ แก้วเจริญ มหาวิทยาลัยนเรศวร

2) ความเชื่อมโยงกับนักคณิตศาสตร์ต่างประเทศ

- Professor Tomas Dominguez Benavides, Universidad de Sevilla, Spain.
- Professor William Art Kirk, University of Iowa, USA.
- Professor Brailey Sims, University of Newcastle, Australia

Output ที่ได้รับจากโครงการ

ผลงานวิจัยที่ได้รับการตีพิมพ์ในวารสารระดับนานาชาติ

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Research Article

Approximating Fixed Points of Nonexpansive Nonself Mappings in CAT(0) Spaces

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Suppose that K is a nonempty closed convex subset of a complete CAT(0) space X with the nearest point projection P from X onto K. Let $T : K \to X$ be a nonexpansive nonself mapping with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Suppose that $\{x_n\}$ is generated iteratively by $x_1 \in K$, $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$, $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Then $\{x_n\}$ Δ-converges to some point x^* in $F(T)$. This is an analog of a result in Banach spaces of Shahzad (2005) and extends a result of Dhompongsa and Panyanak (2008) to the case of nonself mappings.

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [1]), Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [3]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [1]. The work by Burago et al. [4] contains a somewhat more elementary treatment, and by Gromov [5] a deeper study.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [6, 7]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and much papers have appeared (see, e.g., [8–19]).

In 2008, Kirk and Panyanak [20] used the concept of Δ-convergence introduced by Lim [21] to prove the CAT(0) space analogs of some Banach space results which involve weak convergence, and Dhompongsa and Panyanak [22] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting.

The purpose of this paper is to study the iterative scheme defined as follows. Let K is a nonempty closed convex subset of a complete CAT(0) space X with the nearest point projection P from X onto K. If $T : K \to X$ is a nonexpansive mapping with nonempty fixed point set, and if $\{x_n\}$ is generated iteratively by

$$x_1 \in K, \quad x_{n+1} = P\big((1-\alpha_n)x_n \oplus \alpha_n TP\big[(1-\beta_n)x_n \oplus \beta_n T x_n\big]\big), \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, we show that the sequence $\{x_n\}$ defined by (1.1) Δ-converges to a fixed point of T. This is an analog of a result in Banach spaces of Shahzad [23] and also extends a result of Dhompongsa and Panyanak [22] to the case of nonself mappings. It is worth mentioning that our result immediately applies to any CAT(κ) space with $\kappa \le 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \ge \kappa$ (see [1, page 165]).

2. Preliminaries and Lemmas

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [24]. In fact (cf. [1, page 163]), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

We now collect some elementary facts about CAT(0) spaces which will be used frequently in the proofs of our main results.

Lemma 2.1. *Let (X, d) be a CAT(0) space.*

(i) *[1, Proposition 2.4] Let K be a convex subset of X which is complete in the induced metric. Then, for every $x \in X$, there exists a unique point $P(x) \in K$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in K\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retract from X onto K.*

(ii) *[22, Lemma 2.1(iv)] For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = t d(x, y), \qquad d(y, z) = (1 - t) d(x, y). \tag{2.2}$$

one uses the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2).

(iii) *[22, Lemma 2.4] For $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d((1 - t)x \oplus ty, z) \leq (1 - t) d(x, z) + t d(y, z). \tag{2.3}$$

(iv) *[22, Lemma 2.5] For $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t) d(x, z)^2 + t d(y, z)^2 - t(1 - t) d(x, y)^2. \tag{2.4}$$

Let K be a nonempty subset of a CAT(0) space X and let $T : K \to X$ be a mapping. T is called *nonexpansive* if for each $x, y \in K$,

$$d(Tx, Ty) \leq d(x, y). \tag{2.5}$$

A point $x \in K$ is called a fixed point of T if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of T. The existence of fixed points for nonexpansive nonself mappings in a CAT(0) space was proved by Kirk [6] as follows.

Theorem 2.2. *Let K be a bounded closed convex subset of a complete CAT(0) space X. Suppose that $T : K \to X$ is a nonexpansive mapping for which*

$$\inf\{d(x, T(x)) : x \in K\} = 0. \tag{2.6}$$

Then T has a fixed point in K.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \tag{2.7}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{2.8}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{2.9}$$

It is known (see, e.g., [12, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition of Δ-convergence.

Definition 2.3 (see [20, 21]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one writes $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

The following lemma was proved by Dhompongsa and Panyanak (see [22, Lemma 2.10]).

Lemma 2.4. *Let K be a closed convex subset of a complete CAT(0) space X, and let $T : K \to X$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in K such that $\lim_n d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

We now turn to a wider class of spaces, namely, the class of hyperbolic spaces, which contains the class of CAT(0) spaces (see Lemma 2.8).

Definition 2.5 (see [16]). A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is such that

(W1) $d(z, W(x, y, \alpha)) \le (1-\alpha)d(z, x) + \alpha d(z, y)$;

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;

(W3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \le (1-\alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X, \alpha, \beta \in [0,1]$.

It follows from (W1) that for each $x, y \in X$ and $\alpha \in [0,1]$,

$$d(x, W(x, y, \alpha)) \le \alpha d(x, y), \qquad d(y, W(x, y, \alpha)) \le (1-\alpha)d(x, y). \tag{2.10}$$

In fact, we have

$$d(x, W(x, y, \alpha)) = \alpha d(x, y), \qquad d(y, W(x, y, \alpha)) = (1-\alpha)d(x, y), \tag{2.11}$$

since if

$$d(x, W(x, y, \alpha)) < \alpha d(x, y) \quad \text{or} \quad d(y, W(x, y, \alpha)) < (1-\alpha)d(x, y), \tag{2.12}$$

we get

$$\begin{aligned} d(x,y) &\le d(x,W(x,y,\alpha)) + d(W(x,y,\alpha),y) \\ &< \alpha d(x,y) + (1-\alpha)d(x,y) \\ &= d(x,y), \end{aligned} \tag{2.13}$$

which is a contradiction. By comparing between (2.2) and (2.11), we can also use the notation $(1-\alpha)x \oplus \alpha y$ for $W(x,y,\alpha)$ in a hyperbolic space (X,d,W).

Definition 2.6 (see [16]). The hyperbolic space (X,d,W) is called uniformly convex if for any $r > 0$, and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$\left.\begin{aligned} d(x,a) &\le r \\ d(y,a) &\le r \\ d(x,y) &\ge \varepsilon r \end{aligned}\right\} \Longrightarrow d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \le (1-\delta)r. \tag{2.14}$$

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such a $\delta := \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called a modulus of uniform convexity.

Lemma 2.7 (see [16, Lemma 7]). *Let (X,d,W) be a uniformly convex hyperbolic with modulus of uniform convexity η. For any $r > 0$, $\varepsilon \in (0,2]$, $\lambda \in [0,1]$ and $a,x,y \in X$,*

$$\left.\begin{aligned} d(x,a) &\le r \\ d(y,a) &\le r \\ d(x,y) &\ge \varepsilon r \end{aligned}\right\} \Longrightarrow d((1-\lambda)x \oplus \lambda y, a) \le (1 - 2\lambda(1-\lambda)\eta(r,\varepsilon))r. \tag{2.15}$$

Lemma 2.8 (see [16, Proposition 8]). *Assume that X is a CAT(0) space. Then X is uniformly convex, and*

$$\eta(r,\varepsilon) = \frac{\varepsilon^2}{8} \tag{2.16}$$

is a modulus of uniform convexity.

The following result is a characterization of uniformly convex hyperbolic spaces which is an analog of Lemma 1.3 of Schu [25]. It can be applied to a CAT(0) space as well.

Lemma 2.9. *Let (X,d,W) be a uniformly convex hyperbolic space with modulus of convexity η, and let $x \in X$. Suppose that η increases with r (for a fixed ε) and suppose that $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_n d(x_n,x) \le r$, $\limsup_n d(y_n,x) \le r$, and $\lim_n d((1-t_n)x_n \oplus t_n y_n, x) = r$ for some $r \ge 0$. Then*

$$\lim_{n\to\infty} d(x_n,y_n) = 0. \tag{2.17}$$

Proof. The case $r = 0$ is trivial. Now suppose $r > 0$. If it is not the case that $d(x_n, y_n) \to 0$ as $n \to \infty$, then there are subsequences, denoted by $\{x_n\}$ and $\{y_n\}$, such that

$$\inf_n d(x_n, y_n) > 0. \tag{2.18}$$

Choose $\varepsilon \in (0,1]$ such that

$$d(x_n, y_n) \geq \varepsilon(r+1) > 0 \quad \forall n \in \mathbb{N}. \tag{2.19}$$

Since $0 < b(1-c) \leq 1/2$ and $0 < \eta(r,\varepsilon) \leq 1$, $0 < 2b(1-c)\eta(r,\varepsilon) \leq 1$. This implies $0 \leq 1 - 2b(1-c)\eta(r,\varepsilon) < 1$. Choose $R \in (r, r+1)$ such that

$$(1 - 2b(1-c)\eta(r,\varepsilon))R < r. \tag{2.20}$$

Since

$$\limsup_n d(x_n, x) \leq r, \quad \limsup_n d(y_n, x) \leq r, \quad r < R, \tag{2.21}$$

there are further subsequences again denoted by $\{x_n\}$ and $\{y_n\}$, such that

$$d(x_n, x) \leq R, \quad d(y_n, x) \leq R, \quad d(x_n, y_n) \geq \varepsilon R \quad \forall n \in \mathbb{N}. \tag{2.22}$$

Then by Lemma 2.7 and (2.20),

$$\begin{aligned} d((1-t_n)x_n \oplus t_n y_n, x) &\leq (1 - 2t_n(1-t_n)\eta(R,\varepsilon))R \\ &\leq (1 - 2b(1-c)\eta(r,\varepsilon))R < r \end{aligned} \tag{2.23}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) < r, \tag{2.24}$$

which contradicts to the hypothesis. □

3. Main Results

In this section, we prove our main theorems.

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T: K \to X$ be a nonexpansive mapping with $x^* \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1). *Then $\lim_{n\to\infty} d(x_n, x^*)$ exists.*

Proof. By Lemma 2.1(i) the nearest point projection $P : X \to K$ is nonexpansive. Then

$$\begin{aligned} d(x_{n+1}, x^*) &= d\big(P\big((1-\alpha_n)x_n \oplus \alpha_n TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big]\big), Px^*\big) \\ &\le d\big((1-\alpha_n)x_n \oplus \alpha_n TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], x^*\big) \\ &\le (1-\alpha_n)d(x_n, x^*) + \alpha_n d\big(TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], Tx^*\big) \\ &\le (1-\alpha_n)d(x_n, x^*) + \alpha_n d\big(P\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], x^*\big) \\ &\le (1-\alpha_n)d(x_n, x^*) + \alpha_n\big[(1-\beta_n)d(x_n, x^*) + \beta_n d(Tx_n, Tx^*)\big] \\ &\le (1-\alpha_n)d(x_n, x^*) + \alpha_n\big[(1-\beta_n)d(x_n, x^*) + \beta_n d(x_n, x^*)\big] \\ &= d(x_n, x^*). \end{aligned} \tag{3.1}$$

Consequently, we have

$$d(x_n, x^*) \le d(x_1, x^*) \quad \forall n \ge 1. \tag{3.2}$$

This implies that $\{d(x_n, x^*)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_n d(x_n, x^*)$ exists. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1)*. Then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.3}$$

Proof. Let $x^* \in F(T)$. Then, by Theorem 3.1, $\lim_n d(x_n, x^*)$ exists. Let

$$\lim_{n\to\infty} d(x_n, x^*) = r. \tag{3.4}$$

If $r = 0$, then by the nonexpansiveness of T the conclusion follows. If $r > 0$, we let $y_n = P[(1-\beta_n)x_n \oplus \beta_n Tx_n]$. By Lemma 2.1(iv) we have

$$\begin{aligned} d(y_n, x^*)^2 &= d\big(P\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], Px^*\big)^2 \\ &\le d\big((1-\beta_n)x_n \oplus \beta_n Tx_n, x^*\big)^2 \\ &\le (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &\le (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(x_n, x^*)^2 \\ &= d(x_n, x^*)^2. \end{aligned} \tag{3.5}$$

Therefore

$$d(y_n, x^*) \le d\big((1-\beta_n)x_n \oplus \beta_n Tx_n, x^*\big) \le d(x_n, x^*). \tag{3.6}$$

It follows from (3.6) and Lemma 2.1(iv) that

$$\begin{aligned} d(x_{n+1},x^*)^2 &= d\left(P\left[(1-\alpha_n)x_n \oplus \alpha_n T y_n\right], Px^*\right)^2 \\ &\le d\left((1-\alpha_n)x_n \oplus \alpha_n T y_n, x^*\right)^2 \\ &\le (1-\alpha_n)d(x_n,x^*)^2 + \alpha_n d\left(Ty_n,x^*\right)^2 - \alpha_n(1-\alpha_n)d\left(x_n,Ty_n\right)^2 \\ &\le (1-\alpha_n)d(x_n,x^*)^2 + \alpha_n d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)d\left(x_n,Ty_n\right)^2 \\ &= d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)d\left(x_n,Ty_n\right)^2. \end{aligned} \tag{3.7}$$

Therefore

$$d(x_{n+1},x^*)^2 \le d(x_n,x^*)^2 - W(\alpha_n)d\left(x_n,Ty_n\right)^2, \tag{3.8}$$

where $W(\alpha)=\alpha(1-\alpha)$. Since $\alpha_n \in [\varepsilon, 1-\varepsilon], W(\alpha_n) \ge \varepsilon^2$.

By (3.8), we have

$$\varepsilon^2 \sum_{n=1}^{\infty} d\left(x_n,Ty_n\right)^2 \le d(x_1,x^*)^2 < \infty. \tag{3.9}$$

This implies $\lim_{n\to\infty} d(x_n,Ty_n)=0$.

Since T is nonexpansive, we get that $d(x_n,x^*) \le d(x_n,Ty_n)+d(y_n,x^*)$, and hence

$$r \le \liminf_{n\to\infty} d\left(y_n,x^*\right). \tag{3.10}$$

On the other hand, we can get from (3.6) that

$$\limsup_{n\to\infty} d\left(y_n,x^*\right) \le r. \tag{3.11}$$

Thus $\lim_n d(y_n,x^*)=r$. This fact and (3.6) imply

$$\lim_{n\to\infty} d\left((1-\beta_n)x_n \oplus \beta_n T x_n, x^*\right) = r. \tag{3.12}$$

Since T is nonexpansive,

$$\limsup_{n\to\infty} d(Tx_n,x^*) \le r. \tag{3.13}$$

It follows from (3.4), (3.12), (3.13), and Lemma 2.9 that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.14}$$

This completes the proof. □

The following theorem is an analog of [23, Theorem 3.5] and extends [22, Theorem 3.3] to nonself mappings.

Theorem 3.3. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T: K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1). *Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. By Theorem 3.2, $\lim_n d(x_n, Tx_n) = 0$. It follows from (3.2) that $\{d(x_n, v)\}$ is bounded and decreasing for each $v \in F(T)$, and so it is convergent. By Lemma 2.4, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that the sequence $\{x_n\}$ Δ-converges to an element of $F(T)$. □

We now state two strong convergence theorems. Recall that a mapping $T: K \to X$ is said to satisfy *Condition I* ([26]) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \forall x \in K. \tag{3.15}$$

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T: K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1). *Suppose that T satisfies condition I. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 3.5. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $T: K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1). *Then $\{x_n\}$ converges strongly to a fixed point of T.*

Another result in [23] is that the author obtains a common fixed point theorem of two nonexpansive self-mappings. The proof is metric in nature and carries over to the present setting. Therefore, we can state the following result.

Theorem 3.6. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S, T: K \to K$ be two nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n S\left[(1-\beta_n)x_n \oplus \beta_n T x_n\right]. \tag{3.16}$$

Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

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Noor Iterations for Asymptotically Nonexpansive Mappings in CAT(0) Spaces

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Abstract

In this paper, Δ and strong convergence theorems are established for the Noor iterations of asymptotically nonexpansive mappings in CAT(0) spaces. Our results extend and improve the recent ones announced by Dhompongsa and Panyanak [10], Nanjaras and Panyanak [25], Xu and Noor [29] and many others.

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Keywords: asymptotically nonexpansive mappings, fixed points, Noor iteration, $\Delta-$convergence, strong convergence, CAT(0) spaces

1 Introduction

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}-$trees (see [1]), Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [1]. Burago, et al. [4]

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contains a somewhat more elementary treatment, and Gromov [14] a deeper study.

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [17] and [18]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and much papers have appeared (see e. g., [19, 7, 5, 9, 12, 16, 23, 8, 20, 10, 28, 6, 11, 15, 21, 22, 27, 26]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see [1], p. 165).

In 2008, Kirk and Panyanak [20] used the concept of $\Delta-$convergence introduced by Lim [24] to prove the CAT(0) space analogs of some Banach space results which involve weak convergence; for instant, the demiclosedness principle for nonexpansive mappings, the Opial property and the Kadec-Klee property. In the same year, Dhompongsa and Panyanak [10] obtained $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iterations for nonexpansive mappings under some appropriate conditions.

Recently, Nanjaras and Panyanak [25] proved a $\Delta-$convergence theorem of the Krasnosel'skii-Mann iterations for asymptotically nonexpansive mappings in CAT(0) spaces.

In this paper, motivated by the above results, we prove Δ and strong convergence theorems of the Noor iterative schemes for asymptotically nonexpansive mappings in the CAT(0) space setting. Our results extend and improve the corresponding ones announced by Dhompongsa and Panyanak [10], Nanjaras and Panyanak [25], Xu and Noor [29] and many others.

2 Preliminary Notes

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [3]. In fact (cf. [1], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

Definition 2.1 Let C be a nonempty subset of a CAT(0) space X and $T : C \to X$ be a mapping. T is said to be *asymptotically nonexpansive* if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ and such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y), \text{ for all } n \geq 1 \text{ and } x, y \in C.$$

A point $x \in C$ is called a fixed point of T if $x = Tx$. We shall denote with $F(T)$ the set of fixed points of T. The existence of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces was proved by Kirk [18] as the following statement.

Theorem 2.2 *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X and $T : C \to C$ be asymptotically nonexpansive. Then T has a fixed point.*

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [9] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition of $\Delta-$convergence.

Definition 2.3 ([20, 24]) A sequence $\{x_n\}$ in a metric space X is said to $\Delta-$*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the $\Delta-$*limit* of $\{x_n\}$.

Recall that a subset K in a metric space X is said to be $\Delta-$*compact* ([24]) if every sequence in K has a $\Delta-$convergent subsequence. A mapping T from a metric space X to a metric space Y is said to be *completely continuous* if $T(K)$ is a compact subset of Y whenever K is a $\Delta-$compact subset of X.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.4 ([20]) *Every bounded sequence in a complete CAT(0) space always has a $\Delta-$convergent subsequence.*

Lemma 2.5 ([8]) *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.6 ([25]) *Let C be a closed and convex subset of a complete CAT(0) space X and $T : C \to X$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_n d(x_n, Tx_n) = 0$ and $\Delta - \lim_n x_n = x$. Then $x = Tx$.*

Lemma 2.7 ([10]) *Let (X, d) be a CAT(0) space.*

(i) For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \;\; and \;\; d(y, z) = (1 - t)d(x, y). \tag{1}$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (1).

(ii) For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

(iii) For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2.$$

3 $\Delta-$convergence theorems

Lemma 3.1 ([30]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+b_n)a_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 3.2 *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$. For a given $x_1 \in C$, consider the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by*

$$\begin{aligned} z_n &= \gamma_n T^n x_n \oplus (1-\gamma_n)x_n \\ y_n &= \beta_n T^n z_n \oplus (1-\beta_n)x_n \qquad n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n)x_n. \end{aligned}$$

Then $\lim_n d(x_n, x^*)$ exists for all $x^* \in F(T)$.

Proof. We first note that $F(T) \neq \emptyset$ by Theorem 2.2. For each $x^* \in F(T)$, we have

$$\begin{aligned} d(z_n, x^*) &= d(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, x^*) \\ &\leq \gamma_n d(T^n x_n, x^*) + (1-\gamma_n)d(x_n, x^*) \\ &\leq \gamma_n k_n d(x_n, x^*) + (1-\gamma_n)d(x_n, x^*) \\ &= (1+\gamma_n k_n - \gamma_n)d(x_n, x^*). \end{aligned} \tag{2}$$

Also

$$\begin{aligned} d(y_n, x^*) &= d(\beta_n T^n z_n \oplus (1-\beta_n)x_n, x^*) \\ &\leq \beta_n k_n d(z_n, x^*) + (1-\beta_n)d(x_n, x^*). \end{aligned} \tag{3}$$

By (2) and (3), we have

$$\begin{aligned} d(x_{n+1}, x^*) &= d(\alpha_n T^n y_n \oplus (1-\alpha_n)x_n, x^*) \\ &\leq \alpha_n k_n d(y_n, x^*) + (1-\alpha_n)d(x_n, x^*) \\ &\leq \alpha_n k_n \left[\beta_n k_n d(z_n, x^*) + (1-\beta_n)d(x_n, x^*)\right] + (1-\alpha_n)d(x_n, x^*) \\ &\leq \alpha_n k_n \left[\beta_n k_n (1+\gamma_n k_n - \gamma_n)d(x_n, x^*) + (1-\beta_n)d(x_n, x^*)\right] \\ &\quad + (1-\alpha_n)d(x_n, x^*) \\ &= (\alpha_n\beta_n\gamma_n k_n^2 + \alpha_n\beta_n k_n + \alpha_n)(k_n - 1)d(x_n, x^*) + d(x_n, x^*) \\ &\leq (k_n^2 + k_n + 1)(k_n - 1)d(x_n, x^*) + d(x_n, x^*) \\ &= \left[1 + (k_n^2 + k_n + 1)(k_n - 1)\right] d(x_n, x^*). \end{aligned}$$

Since $\{k_n\}$ is bounded, there exists $M > 0$ such that

$$d(x_{n+1}, x^*) \leq (1 + M(k_n - 1))d(x_n, x^*).$$

By Lemma 3.1 and the fact that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we get $\lim_{n\to\infty} d(x_n, x^*)$ exists. ■

Lemma 3.3 *Let* $C, X, T, \{k_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ *are as in Lemma 3.2.*
(i) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*, then*

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0.$$

(ii) If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n > 0$*, then*

$$\lim_{n\to\infty} d(T^n z_n, x_n) = 0.$$

Proof. By Lemma 2.7(iii) along with the proof of Lemma 2.2 in [29] with $p = 2$ and $w(\lambda) = \lambda(1 - \lambda)$ for $\lambda \in [0, 1]$, we can obtain the desired result. ■

Lemma 3.4 *Let* C *be a nonempty closed, bounded and convex subset of a complete CAT(0) space* X *and let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $\{k_n\}$ *satisfying* $\{k_n\} \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be real sequences in [0,1] satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
For a given $x_1 \in C$*, define*

$$\begin{aligned} z_n &= \gamma_n T^n x_n \oplus (1 - \gamma_n)x_n \\ y_n &= \beta_n T^n z_n \oplus (1 - \beta_n)x_n \qquad n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n. \end{aligned}$$

Then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. From Lemma 3.3, we have

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0 \text{ and } \lim_{n\to\infty} d(T^n z_n, x_n) = 0.$$

Thus

$$\begin{aligned} d(T^n x_n, x_n) &\leq d(T^n x_n, T^n y_n) + d(T^n y_n, x_n) \\ &\leq k_n d(x_n, y_n) + d(T^n y_n, x_n) \end{aligned}$$

$$\leq k_n\beta_n d(T^n z_n, x_n) + d(T^n y_n, x_n) \to 0 \quad \text{as } n \to \infty, \tag{4}$$

so that

$$\begin{aligned} d(x_{n+1}, T^n x_{n+1}) &\leq d(x_{n+1}, x_n) + d(T^n x_{n+1}, T^n x_n) + d(T^n x_n, x_n) \\ &\leq d(x_{n+1}, x_n) + k_n d(x_{n+1}, x_n) + d(T^n x_n, x_n) \\ &= (1+k_n) d(\alpha_n T^n y_n \oplus (1-\alpha_n) x_n, x_n) + d(T^n x_n, x_n) \\ &\leq (1+k_n)\alpha_n d(T^n y_n, x_n) + d(T^n x_n, x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{5}$$

By (4) and (5), we have

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\leq d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, Tx_{n+1}) \\ &\leq d(x_{n+1}, T^{n+1}x_{n+1}) + k_1 d(T^n x_{n+1}, x_{n+1}) \to 0 \quad (\text{as } n \to \infty), \end{aligned}$$

which implies $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ as desired. ■

Now, we are ready to prove the $\Delta-$convergence theorem.

Theorem 3.5 *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in [0,1] satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n \oplus (1-\gamma_n) x_n \\ y_n &= \beta_n T^n z_n \oplus (1-\beta_n) x_n \qquad\qquad n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n) x_n. \end{aligned}$$

Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

Proof. It follows from Lemma 3.4 that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Now we let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.4 and 2.5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Since $\lim_n d(v_n, Tv_n) = 0$, then $v \in F(T)$ by Lemma 2.6. We claim that

$u = v$. Suppose not, since $v \in F(T)$, by Lemma 3.2 $\lim_n d(x_n, v)$ exists. By the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_n d(v_n, v) &< \limsup_n d(v_n, u)\\ &\leq \limsup_n d(u_n, u)\\ &< \limsup_n d(u_n, v)\\ &= \limsup_n d(x_n, v)\\ &= \limsup_n d(v_n, v)\end{aligned}$$

a contradiction, and hence $u = v \in F(T)$. To show that $\{x_n\}$ $\Delta-$converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.4 and 2.5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not, since $\lim_n d(x_n, v)$ exists, then by the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_n d(v_n, v) &< \limsup_n d(v_n, x)\\ &\leq \limsup_n d(x_n, x)\\ &< \limsup_n d(x_n, v)\\ &= \limsup_n d(v_n, v)\end{aligned}$$

a contradiction, and hence the conclusion follows. ∎

For $\gamma_n \equiv 0$ in Theorem 3.5, we can obtain Ishikawa-type convergence result as the following statement.

Theorem 3.6 *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in [0,1] satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(ii) $\limsup_{n\to\infty} \beta_n < 1$.
For a given $x_1 \in C$, define

$$y_n = \beta_n T^n x_n \oplus (1 - \beta_n)x_n$$

$$x_{n+1} = \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n, \qquad n \geq 1.$$

Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

Proof. By combining between the proofs of Theorem 3.5 and Theorem 2.2 of [29] we can get the desired result. ■

For $\beta_n \equiv 0$, Theorem 3.6 reduces to the following result which is a refinement of Theorem 5.7 in [25].

Theorem 3.7 *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in [0,1] satisfying*
$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *For a given $x_1 \in C$, define*

$$x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n)x_n, \qquad n \geq 1.$$

Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

4 Strong convergence theorems

By using the same ideas and techniques as in Section 3, we can also obtain strong convergence theorems for completely continuous asymptotically nonexpansive mappings. Therefore we can state the following results without proofs.

Theorem 4.1 *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in [0,1] satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_1 \in C$, define

$$z_n = \gamma_n T^n x_n \oplus (1 - \gamma_n)x_n$$

$$y_n = \beta_n T^n z_n \oplus (1 - \beta_n)x_n \qquad n \geq 1$$

$$x_{n+1} = \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 4.2 *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in [0,1] satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*

(ii) $\limsup_{n\to\infty} \beta_n < 1$.

For a given $x_1 \in C$, *define*

$$y_n = \beta_n T^n x_n \oplus (1-\beta_n)x_n$$

$$x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n)x_n. \quad n \geq 1$$

Then $\{x_n\}$ *converges strongly to a fixed point of* T.

Theorem 4.3 *Let* C *be a nonempty closed, bounded and convex subset of a complete CAT(0) space* X *and let* $T : C \to C$ *be a completely continuous asymptotically nonexpansive mapping with* $\{k_n\}$ *satisfying* $\{k_n\} \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{\alpha_n\}$ *be real sequence in [0,1] satisfying* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *For a given* $x_1 \in C$, *define*

$$x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n)x_n, \quad n \geq 1.$$

Then $\{x_n\}$ *converges strongly to a fixed point of* T.

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