รายงานวิจัยฉบับสมบูรณ์

โครงการ
การจัดการรายได้เพื่อการแบ่งสรรเนื้อที่ในการขนส่งสินค้าทางอากาศ

โดย

กาญจ์นภา อมรัชกุล
วิชิต หล่อจีระชุณห์กุล

คณะสถิติประยุกต์ สถาบันบัณฑิตพัฒนบริหารศาสตร์

มีนาคม 2553

สัญญาเลขที่ MRG5280069

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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(ชื่อโครงการ) **การจัดการรายได้เพื่อการแบ่งสรรเนื้อที่ในการขนส่งสินค้าทางอากาศ**

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We consider a single air-cargo carrier, which wants to allocate cargo capacity to multiple forwarders before a booking horizon starts. A contribution that the carrier earns from each forwarder is based on the actual allotment usage at the end of the horizon. The airline's problem is to choose the allotments that maximize the expected total contribution. We derive a probability distribution of the actual usage by using a discrete Markov chain and solve the problem by using a dynamic programming method. Two heuristics for a large-scale allocation problem are also proposed, and their performance is tested via numerical experiments.

Keywords: Air Cargo; Capacity Management; Stochastic Model Applications

พิจารณาผู้ขนส่งสินค้าทางอากาศ (สายการบิน) หนึ่งราย ซึ่งต้องการจัดสรรเนื้อที่ขนส่งสินค้าทางอากาศให้กับผู้แทนขายหลายราย รายได้ที่สายการบินได้รับขึ้นอยู่กับปริมาณเนื้อที่ทั้งหมดซึ่งผู้แทนขายได้ใช้ไป สายการบินต้องการทราบว่า ควรจัดสรรเนื้อที่ให้ผู้แทนขาย แต่ละรายเป็นจำนวนเท่าใด เพื่อให้ค่าคาดหมายรายได้รวมสูงสุด การแจกแจงของเนื้อที่ทั้งหมดซึ่งใช้ไปหาได้จากลูกโซ่มาร์คอฟ และผลเฉลยของปัญหาข้างต้นหาได้จากโปรแกรมเชิงพลวัต นอกจากนี้ตัวแบบฮิวริสติกส์ได้ถูกสร้างขึ้นเพื่อแก้ปัญหาขนาดใหญ่ ผู้วิจัยทดสอบสมรรถนะของตัวแบบดังกล่าวโดยใช้การทดลองเชิงตัวเลข

(คำหลัก): สินค้าทางอากาศ การจัดการความจุ การประยุกต์ใช้ตัวแบบสโทแคสติก

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Air-Cargo Capacity Allocation for Multiple Freight Forwarders

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Abstract

We consider a single air-cargo carrier, which wants to allocate cargo capacity to multiple forwarders before a booking horizon starts. A contribution that the carrier earns from each forwarder is based on the actual allotment usage at the end of the horizon. The airline's problem is to choose the allotments that maximize the expected total contribution. We derive a probability distribution of the actual usage by using a discrete Markov chain and solve the problem by using a dynamic programming method. Two heuristics for a large-scale allocation problem are also proposed, and their performance is tested via numerical experiments.

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Introduction

Air-cargo operations generate significant revenue for passenger airlines, most of which carry cargo shipments in the belly of their planes. Air-cargo volumes are expected to grow 6 percent annually over the next two decades (Airbus 2007). The growth of air-cargo traffic partly results from global trade liberalization, and the emerging implementation of supply chain management strategies, which emphasize on short lead times.

Many carriers (airlines), especially those in Asia-Pacific, reserve large portions of cargo space on specific flights over a period of time for their key freight forwarders either as part of a binding contract or as part of a goodwill and gentleman's agreement (Billings et al. 2003). The allotted space is referred as an *allotment*. The forwarders re-sell their allotments to their own customers. They collect individual packages from shippers and transport the consolidated shipments to the airline. In addition, they provide value-adding services such as picking up from the shipper location and customs clearance for international cargo. Consequently, shippers typically prefer to use the forwarders unless a shipment is an emergency, or contains perishable or hazardous materials (Thuermer 2005).

An allotment contract is normally valid for the entire *season*, whose duration is specified by the International Air Transport Association (Slager and Kapteijns 2004). The airline awards the medium-term contracts to the forwarders before the season starts. At that time, the forwarders do not know the exact amount of their customer demand. After the forwarder's demand is materialized, the airline often allows the forwarder to return the unused portion of its allotment and charges only the actual usage. The airline's problem is to choose the allotments so that the expected total contribution is maximized. The *contribution* of the shipment is the contract rate minus the associated operational costs, e.g., the incremental fuel cost, and the handling costs.

In this article, we consider a single airline, which wants to allocate cargo capacity to multiple freight forwarders. During the booking horizon, the request from the forwarder arrives one by one. The forwarder's request is accepted, if its space requirement does not exceed the allotment the forwarder currently has. At the end of the booking horizon, the airline receives the contribution, proportional to the forwarder's actual allotment usage. We show that the sequence of the cumulative usages of the accepted shipments for each forwarder forms a discrete Markov chain. The expected actual allotment usage is neither concave nor differentiable. The airline's problem can be solved via dynamic programming, in which the number of computational steps quadratically increases in the capacity.

To develop some heuristic solutions, we assume that the shipment can be partially accepted. Under this assumption, the objective function is concave but still not differentiable. We present two heuristic approaches. In the first, the expected actual usage is approximated by a continuously differentiable function, and the heuristic solution is derived from the Karush-Kuhn-Tucker (KKT) conditions. The latter is based on Lagrangian relaxation, in which the capacity constraint is dualized. Through a series of numerical experiments, we show that the latter outperforms the first.

Since air-cargo capacity can be sold at different prices to heterogeneous customers but cannot be sold after the flight departure, it is a prime candidate for revenue management (RM) strategies. Books on RM theory and practice are, e.g., Ingold et al. (2000), Talluri and van Ryzin (2004), Yeoman and McMahon-Beattie (2004), and Phillips (2005). Descriptive papers on air-cargo RM can be found in e.g., Kasilingam (1996), Bazaraa et al. (2001), Billings et al. (2003), Slager and Kapteijns (2004) and Becker and Dill (2007). In general, there is a vast literature on capacity management and on supply chain contracts; see Van Mieghem (2003), Cachon (2003), and Lariviere (1999) for reviews. Unfortunately, "revenue managing the contractual terms under which inventory is sold, remains almost untouched in the RM literature in spite of the vast majority of business that is transacted under negotiated contracts (Boyd and Bilegan 2003)." Despite a large number of papers on a nonlinear resource allocation problem (cf. Patriksson 2008), there are few papers that present mathematical models of the air-cargo allocation problem. Below, we review some of them.

Hellermann (2006) proposes a capacity-option pricing contract between a single forwarder and a single airline. The capacity is sold in two stages. In the first stage, the capacity is sold up front through the medium-term contract. In the second stage, the airline sells on the spot market, in which price and demand are random. The interaction between the two parties is specified by a Stackelberg game. Under the terms of the contract, the forwarder pays a reservation fee to acquire the right (but not the obligation) to use capacity, and later pays an execution fee if it eventually uses the capacity. The airline, the Stackelberg leader, first announces the reservation and execution fees. Then, the forwarder decides on the amount of capacity to reserve. After the demand and the spot price materialize, it determines how many reservations to call on. Hellermann compares the performance of the option contract to that of a fixed-commitment contract. Hellermann consider the single-forwarder case, whereas we consider the multiple-forwarder case.

Gupta (2008) studies flexible contracts between a single forwarder and a single airline, which receives two demand streams from the forwarder and the direct shippers. The paper identifies two contract schemes, which allow the carrier to achieve an efficient capacity allocation. In our article, the airline receives multiple demand streams from multiple forwarders. The functional form of the allotment usage in both Hellermann and Gupta resembles that in the newsvendor; specifically, the actual usage equals the minimum of the forwarder's demand and the allotment. We present a more detailed formulation; the sequence of the cumulative usages from one request to the next is modeled as a discrete Markov chain, and the expected actual usage is derived from the property of the Markov chain.

Kasilingam (1996) presents an allocation model in a network setting. A total contribution is maximized subject to a capacity constraint, which ensures that for each flight the total allotment from all associated routings and for all forwarders does not exceed the capacity, and a chance constraint, which states that for each forwarder the probability of the allotment exceeding the demand is within a specified threshold. The model assumes that the contribution for each route and for each forwarder is equal to the product of the amount of the allotment and the per-unit contribution. Putting it differently, Kasilingam (1996) assumes that the forwarder is charged for the whole allotment. Under this assumption, the total contribution is deterministic, and the objective function is linear in the allotment. However, we assume that the contribution is linear in the actual allotment usage, not the whole amount that the forwarder originally receives. Our assumption is consistent with the industry practice; see Bazaraa et al. (2001) and Hellermann (2006) Section 2.2.4.

The rest of the paper is organized as follows. In Section 1, we formulate the carrier's problem. The probabilistic analysis and the optimal solution of the carrier's problem are given in Section 2. We develop heuristic procedures in Section 3 and test their performance in Section 4. Finally, concluding remarks are provided in Section 5.

1 Problem Formulation

Throughout this article, we use $\mathbb{N}$ to denote the set of natural numbers, $\mathbb{Z}_+$ the set of nonnegative integers, and $\mathbb{R}_+$ the set of nonnegative real numbers.

The carrier has κ units of cargo capacity per flight, which would be allocated as allotments for m forwarders before the season starts. Let $x_i \geq 0$ denote the per-flight allotment for forwarder $i \in \{1, 2, \ldots, m\} := \mathcal{M}$. We assume that cargo capacity is one dimensional, because in practice most airlines manage their allotments only in terms of weight. (Nonetheless, modern short-haul planes are constrained by volume not weight due to recent advances in aircraft engine.) Alternatively, cargo capacity may be thought of as a number of standard containers in a cargo hold.

The carrier charges forwarder i based on its actual allotment usage $U_i(x_i)$ and receives a contribution of p_i per unit usage. Assume that the carrier is risk neutral, since it operates multiple repeat flights in one season. The carrier wants to maximize the expected total contribution subject to a capacity constraint:

$$\max \left\{ E\Big[\sum_{i=1}^{m} p_i U_i(x_i)\Big] : \sum_{i=1}^{m} x_i \leq \kappa, \quad x_i \geq 0 \text{ for each } i \in \mathcal{M} \right\} \tag{1}$$

The first constraint states that the carrier does not allocate more than its cargo capacity. In other words, it does not overbook their allotments. Theoretically, we could include an overbooking policy in (1): the objective function would include an expected oversale cost, and the right-hand-side of the capacity constraint would include an overbooking pad. In practice, the penalty cost from delaying forwarders' shipments is much higher than that from bumping passengers. Unlike passenger airlines which typically overbook their seats, air-cargo carriers rarely overbook their allotments.

In practice, the forwarders' usages are usually dependent, because they are affected by same factors, e.g., a gross domestic product (GDP), a weather condition, and a fuel price, etc. Even if the forwarders' usages $U_1(x_1), \ldots, U_m(x_m)$ are dependent, (1) remains valid. Recall that the expectation of a finite sum of random variables is the finite sum of their expected values, and that the expectation of a constant times a random variable is the constant times the expectation of the random variable. The objective function becomes

$$E\Big[\sum_{i=1}^{m} p_i U_i(x_i)\Big] = \sum_{i=1}^{m} p_i E\big[U_i(x_i)\big]$$

The expectation $E[U_i(x)]$ can be calculated with respect to the marginal distribution of random variable $U_i(x)$.

Assume that the request to book a shipment arrives to the carrier one by one throughout the booking horizon. When there is a medium-term contract between the forwarder and the carrier, the booking request from the forwarder is accepted, if its space requirement is within the contracted space (Bazaraa et al. 2001). In practice, the requests from the forwarder with the allotment are accepted on the first-come, first-served basis, but those from the forwarder without the allotment need not be. The latter is sometimes referred to as *ad-hoc/free sale* (Slager and Kapteijns 2004). The carrier might apply some rudimentary RM techniques in controlling the free sale space; e.g., it might reject the current request, if it anticipates the larger contribution from the future requests. We do not model the free sale, because many Asia Pacific airlines reserve large portions of their capacity as allotments (Hendricks and Elliott 2005). An omnibus model, which includes both allotment and free sale, is an interesting future research direction.

Consider the booking requests from forwarder i, which receives allotment x. Let N_i be its total number of requests, and let $W_{i,k}$ be the space requirement of its k-th request, where $k \in \{1, 2, \ldots, N_i\}$. Let $X_{i,k}(x)$ denote the cumulative usage after the carrier makes the k-th accept/reject decision [just before the arrival of the $(k+1)$-st booking request] for each $k \in \{1, 2, \ldots, N_i\}$. At the end of the booking horizon, the actual usage of forwarder i that receives allotment x is $U_i(x) = X_{i,N_i}(x)$. The finite sequence of the cumulative usages $\{X_{i,k}(x) : k = 1, 2, \ldots, N_i\}$ is determined by the following recurrence equation:

$$X_{i,k}(x) = \begin{cases} X_{i,k-1}(x) + W_{i,k} & \text{if } W_{i,k} \leq x - X_{i,k-1}(x) \\ X_{i,k-1}(x) & \text{otherwise} \end{cases} \tag{2}$$

for each $k = 1, 2, \ldots, N_i$, and $X_{i0}(x) = 0$. Equation (2) can be explained as follows. Prior to the arrival of the k-th request, the cumulative usage of the accepted shipments is $X_{i,k-1}(x)$, and the unused portion of the allotment is $x - X_{i,k-1}(x)$. If the unused portion is at least the space requirement of the k-th booking request, then the request is accepted, and the cumulative usage increases to $X_{i,k-1}(x) + W_{i,k}$. Otherwise, the request is rejected, and the cumulative usage stays the same.

Given that its total number of booking requests is $N_i = n \in \mathbb{N}$, we assume that the space requirements of the requests, $W_{i,1}, W_{i,2}, \ldots, W_{i,n}$, are independent and identically distributed (i.i.d.) nonnegative integer-valued random variables. The i.i.d. assumption is needed for mathematical tractability, and it is quite common among operations research papers with stochastic model applications. (Moreover, we perform a statistical test on some real data collected from a major airline in Thailand, to test the null hypothesis that the space requirements are i.i.d. The p-value is 0.4661 for the shipments on 19 June 2008 and 0.7373 on 24 April 2008. We conclude that there is insufficient evidence to reject the null hypothesis at 5% significant level. A similar conclusion can be made for many other dates.) Our assumption that the space requirements are integer-valued is not restrictive, because the units can be chosen arbitrarily. As mentioned earlier, most airlines manage their capacity in terms on weight. In practice, the unit of space requirement may be that of a weight measuring device. Furthermore, the freight charge is usually based on the nearest whole pound or kilogram.

2 Optimal Solution

We will derive a closed-form expression for $E[U_i(x)]$, the expected actual allotment usage of forwarder i that receives allotment $x \in \mathbb{N}$. It follows from the assumptions in Section 1 and the recursive equation (2) that the stochastic process $\{X_{i,k}(x) : k \in \mathbb{Z}_+\}$ is a discrete Markov chain with the state space $\{0, 1, \ldots, x\} := \mathcal{X}(x)$. Its initial distribution is $P(X_{i,0}(x) = 0) = 1$. Denote the (one-step) transition probability as $p_i(b, c|x) = P(X_{i,k}(x) = c|X_{i,k-1}(x) = b)$ for each $c, b \in \mathcal{X}(x)$.

Let h_i (resp., H_i) denote the probability mass (resp., cumulative distribution) function of a generic random variable $W_{i,1}$. We use a bar atop a distribution function to denote its complement; e.g., $\bar{H}_i(t) = 1 - H_i(t)$.

Lemma 1. *The transition probability of the Markov chain $\{X_{i,k}(x) : k \in \mathbb{Z}_+\}$ is given* *as follows:*

$$p_i(b,c|x) = \begin{cases} h_i(c-b) & \textit{if } b < c \leq x \\ \bar{H}_i(x-b) + h_i(0) & \textit{if } c = b \\ 0 & \textit{if } 0 \leq c < b \end{cases} \tag{3}$$

for each $b < x$, *and*

$$p_i(x,c|x) = \begin{cases} 1 & \textit{if } c = x \\ 0 & \textit{if } c \neq x \end{cases} \tag{4}$$

Proof. Suppose that the cumulative usage prior to the k-th arrival of the booking request is b. Then, the unused portion of the allotment is $(x-b)$. Let c denote the total usage prior to the $(k+1)$-st arrival of the booking request. Note that the sequence of total usages are nondecreasing: $X_{i,k-1}(x) \leq X_{i,k}(x)$ with probability one. Therefore, $p_i(b,c|x) = 0$ if $c < b$, which is the third case in (3). For the first case, the total usage strictly increases from b to c, if and only if the k-th booking request with the space requirement $(c-b)$ is accepted; i.e., $c - b \leq x - b$, or equivalently $c \leq x$. The event that the arriving request has space requirement $(c-b)$ occurs with probability $h_i(c-b)$. For the second case, the total usage remains the same, if and only if the k-th booking request is rejected, or its space requirement is zero. The first occurs with probability $\bar{H}_i(x-b)$, whereas the latter occurs with probability $h_i(0)$. Finally, in equation (4), the cumulative usage prior to the k-th arrival is equal to the allotment. The forwarder has no allotment left, and all of future requests are rejected; thus, the cumulative usage must remain the same. $\square$

Let $\mathbf{P}_i(x)$ be a (one-step) transition matrix, whose element in the b-th row and c-th column is $p_i(b,c|x)$ for each $b, c \in \mathcal{X}(x)$. From the Chapman-Kolmogorov equation, the k-step transition matrix, denoted by $\mathbf{P}_i^{(k)}(x)$, can be obtained by multiplying the one-step transition matrix with itself k times: $\mathbf{P}_i^{(k)}(x) = (\mathbf{P}_i(x))^k$ for each $k \in \mathbb{N}$. The distribution of the Markov chain $\{X_{i,k}(x) : k \in \mathbb{Z}_+\}$ is completely specified by the one-step transition probability $\mathbf{P}_i(x)$ and the initial distribution $P(X_{i,0}(x) = 0) = 1$.

Proposition 1. *The expected actual usage, if forwarder i receives allotment $a \in \mathbb{R}_+$, is given by*

$$E[U_i(a)] = \sum_{n=1}^{\infty} P(N_i = n) \sum_{t=1}^{x} t p_i^{(n)}(0, t|x)$$

where $x = \lfloor a \rfloor$, and $E[U_i(0)] = 0$.

Proof. Since N_i and $W_{i,k}$ are nonnegative integer-valued random variables, the expected actual usage $E[U_i(a)] = E[U_i(x)]$, where $x = \lfloor a \rfloor$. Recall that $U_i(x) = X_{i,N_i}(x)$. We calculate the expected total usage by conditioning on the number of booking requests:

$$\begin{aligned} E[X_{i,N_i}(x)] &= \sum_{n=0}^{\infty} P(N_i = n) E[X_{i,N_i}(x) | N_i = n] \\ &= 0P(N_i = 0) + \sum_{n=1}^{\infty} P(N_i = n) E[X_{i,n}(x)] && (5) \\ &= \sum_{n=1}^{\infty} P(N_i = n) \sum_{t=0}^{x} t P\big(X_{i,n}(x) = t | X_{i,0}(x) = 0\big) && (6) \\ &= \sum_{n=1}^{\infty} P(N_i = n) \sum_{t=0}^{x} t p_i^{(n)}(0, t|x) && (7) \end{aligned}$$

In equation (5), the total usage is zero, if the forwarder makes no booking request; i.e., $E[X_{i,N_i}(x)|N_i = 0] = 0$. In equation (6), we calculate $E[X_{i,n}(x)]$ by conditioning on the initial state $X_{i,0}(x)$. [Note that the initial distribution is $P(X_{i,0}(x) = 0) = 1$.] Equation (7) follows from the definition of the n-step transition probability $p_i^{(n)}(0, \cdot|x)$ given that the initial state is 0. Recall that $p_i^{(n)}(b, c|x)$ is the element in the b-th row and c-th column of the n-step transition matrix $\mathbf{P}_i^{(n)}(x) = (\mathbf{P}_i(x))^n$. The one-step transition matrix $\mathbf{P}_i(x)$ is given in Lemma 1. We can ignore the first term in the second summation in (7), since it equals zero. This completes the proof. □

Note that the expected actual allotment usage $E[U_i(x)]$ is nondecreasing but might not be concave on $\mathbb{Z}_+$. Non-concavity partly results from the assumption that the shipment of size $W_{i,k}$ is accepted on an all-or-none basis. For instance, suppose that the sequence of space requirements is $(1, 3, 9, 5, 2, 4)$. Then, the actual allotment usages, if the allotment are $(x : x = 4, 5, \ldots, 11)$ are $(u(x) : x = 4, 5, \ldots, 11) = (4, 4, 6, 6, 6, 9, 9, 11)$. The differences between two consecutive usages are $(u(x) - u(x-1) : x = 5, 6, \ldots, 11) = (0, 2, 0, 0, 3, 0, 2)$; clearly, the actual usage is not concave on $\mathbb{Z}_+$. [Recall that a function $f : \mathbb{Z}_+ \to \mathbb{R}$ is concave on $\mathbb{Z}_+$, if $f(a) - f(a-1)$ is nonincreasing in $a \in \mathbb{N}$.]

For short-hand notation, denote $\pi_i(a) = p_i E[U_i(a)]$ for each $a \in \mathbb{R}_+$. The carrier's problem can be expressed equivalently as

$$\zeta = \max \left\{ \sum_{i=1}^{m} \pi_i(a_i) : \sum_{i=1}^{m} a_i \leq \kappa, \quad a_i \geq 0 \text{ for each } i \in \mathcal{M} \right\} \tag{8}$$

Note that (8) is a special case of the nonlinear resource allocation problem. For each $i \in \mathcal{M}$, $\pi_i(a)$ is neither concave nor differentiable on $\mathbb{R}_+$. Problem (8) can be solved via dynamic programming. The number of stages equals m, the number of the forwarders. Define the state at the beginning of stage i as the amount of cargo capacity to be allocated to forwarders $i, i+1, \ldots, m$. At the beginning of stage i, the carrier observes the state y and determines the allotment for forwarder i, denoted by a_i. Let $u_i(y)$ be the value function at stage i; i.e., $u_i(y)$ is the maximum expected total contribution that can be earned from forwarders i through m, if the current state is y. The value function can be computed recursively via the Bellman optimality equation

$$u_i(y) = \max_{a_i = 0, 1, \ldots, y} \{\pi_i(a_i) + u_{i+1}(y - a_i)\}, \qquad i \in \mathcal{M} \tag{9}$$

and the boundary condition is $u_{m+1}(y) = 0$ for all y. Let $a_i^*(y) = \operatorname{argmax}_{a=0,1,\ldots,y} \{\pi_i(a) + u_{i+1}(y - a)\}$.

Proposition 2. *The optimal objective function in (8) is $\zeta = u_1(\kappa)$. The optimal allot-*

ment for forwarder $i \in \mathcal{M}$ is given as follows: $x_1^* = a_1^*(\kappa)$ and $x_i^* = a_i^*(\kappa - \sum_{k=1}^{i-1} x_k^*)$ for each $i \in \{2, 3, \ldots, m\}$.

Proof. Since the carrier has κ units of cargo capacity to allocate to forwarders 1 through m, the optimal objective function is $u_1(\kappa)$. From the definition, $a_i^*(y)$ represents the optimal allotment for forwarder i, if the carrier has y units to allocate to forwarders i through m. Thus, the optimal allotment for forwarder 1 is $x_1^* = a_1^*(\kappa)$. The carrier reserves x_1^* to forwarder 1, leaving $\kappa - x_1^*$ to allocate to forwarders 2 through m. Then, $x_2^* = a_2^*(\kappa - x_1^*)$. The carrier reserves x_2^* to forwarder 2, leaving $\kappa - (x_1^* + x_2^*)$ to allocate to forwarders 3 through m. Then, $x_3^* = a_3^*\big(\kappa - (x_1^* + x_2^*)\big)$. The carrier reserves x_3^* for forwarder 3, leaving $\kappa - (x_1^* + x_2^* + x_3^*)$ to allocate to forwarders 4 through m, and so on. $\square$

The number of iterations needed to solve (9) is of order $m\kappa^2$. This may create some computational burden, if the carrier is endowed with large cargo capacity. In the next section, we develop some heuristic solutions, implementable for an industry-sized problem.

3 Heuristic Solutions

The fact that the expected actual usage $E[U_i(a)]$ of forwarder i that receives allotment a is not concave in $a \in \mathbb{Z}_+$ results from the all-or-none acceptance rule. To develop some heuristics, we suppose that the shipment can be partially accepted. Upon receiving the k-th booking request with space requirement $W_{i,k}$ from forwarder i, the airline accepts $\min\{W_{i,k}, Z\}$ where Z is the unused portion of the allotment. We will show that the expected total usage is concave under the assumption that the request is partially accepted.

Suppose that forwarder i receives allotment x. Let $Y_{i,k}(x)$ denote its cumulative usage after the carrier's accept/reject decision of the k-th booking request. The actual allotment usage under the partial acceptance assumption is $V_i(x) = Y_{i,N_i}(x)$. The finite sequence of the cumulative usages $\{Y_{i,k}(x) : k = 1, 2, \ldots, N_i\}$ is determined by the following recurrence equation:

$$Y_{i,k}(x) = Y_{i,k-1}(x) + \min\Big\{\big(x - Y_{i,k-1}(x)\big), W_{i,k}\Big\} \tag{10}$$

for each $k = 1, 2, \ldots, N_i$, and $Y_{i,0}(x) = 0$. In (10), the carrier books the shipment up to the unused portion of the allotment $\big(x - Y_{i,k-1}(x)\big)$. We assume that the forwarder is willing to break the consolidated shipment and deliver $\{(x - Y_{i,k-1}(x)), W_{i,k}\}$.

From our construction, $Y_{i,k}(x) \geq X_{i,k}(x)$ for all k, with probability 1. Consequently, the expected actual allotment usage under the assumption that the request is partially accepted is at least that under the all-or-none assumption; i.e.,

$$E[V_i(x)] \geq E[U_i(x)] \tag{11}$$

Furthermore, if $W_{i,k}$ is a Bernoulli random variable, which takes on values $\{0, 1\}$, then with probability 1, $Y_{i,k}(x) = X_{i,k}(x)$ for all k, so $U_i(x) = V_i(x)$.

For each $i \in \mathcal{M}$, let D_i denote the sum of all space requirements of N_i booking requests from forwarder i; i.e., $D_i = \sum_{k=1}^{N_i} W_{i,k}$ is the total space requirement of forwarder i.

Proposition 3. *Suppose that forwarder i receives allotment $a \in \mathbb{R}_+$. Under the assumption that the request is partially accepted, $Y_{i,k}(a) = \min\{\sum_{j=1}^{k} W_{i,j}, a\}$ for each $k = 1, 2, \ldots, N_i$.*

Proof. The proof is done using mathematical induction. Clearly, $Y_{i,1}(x) = \min\{W_{i,1}, x\}$. Next, assume that $Y_{i,k-1}(x) = \min\{\sum_{j=1}^{k-1} W_{i,j}, x\}$. Substituting this expression in (10), we get

$$Y_{i,k}(x) = \min\{\sum_{j=1}^{k-1} W_{i,j}, x\} + \min\Big\{\big(x - \min\{\sum_{j=1}^{k-1} W_{i,j}, x\}\big), W_{i,k}\Big\}$$

If $\sum_{j=1}^{k-1} W_{i,j} < x$, then $Y_{i,k}(x) = \sum_{j=1}^{k-1} W_{i,j} + \min\{x - \sum_{j=1}^{k-1} W_{i,j}, W_{i,k}\} = \min\{x, \sum_{j=1}^{k} W_{i,j}\}$;

otherwise, $Y_{i,k}(x) = x$. Hence, $Y_{i,k}(x) = \min\{x, \sum_{j=1}^{k} W_{i,j}\}$. $\square$

For short-hand notation, denote $\rho_i(x) = p_i E[V_i(x)]$. It follows from (11) that the optimal solution to the following mathematical program

$$\xi = \max\left\{ \sum_{i=1}^{m} \rho_i(a_i) : \sum_{i=1}^{m} a_i \leq \kappa, \quad a_i \geq 0 \text{ for each } i \in \mathcal{M} \right\} \tag{12}$$

yields an upper bound on the carrier's problem (8); i.e., $\xi \geq \zeta$. Unlike $\pi_i(x)$ in the carrier's problem, the function $\rho_i(x)$ is concave on $\mathbb{Z}_+$. Similar to $\pi_i(a)$, the function $\rho_i(a)$ is not differentiable on $\mathbb{R}_+$, since $W_{i,k}$ and N_i are assumed to be nonnegative integer-valued random variables. To solve (12), we present two heuristic approaches.

1. Continuous approximation, in which the total space requirement is modeled as a nonnegative real-valued random variable, denoted by $\hat{D}_i$. Since the cumulative distribution function of $\hat{D}_i$ is continuous, the expected actual allotment usage of forwarder i that receives allotment x, $E[\min(\hat{D}_i, x)]$, is differentiable on $\mathbb{R}_+$. This allows us to apply a standard nonlinear programming technique (e.g., KKT conditions).

2. Lagrangian relaxation, in which we dualize the capacity constraint, $\sum_{i=1}^{m} x_i \leq \kappa$. The objective function of the relaxed problem is concave, so the relaxed problem for a fixed value of the Lagrange multiplier is easy to solve. We then use subgradient optimization to update the Lagrange multiplier in such a way that the capacity constraint is likely to be tighter on the subsequent iteration.

We use boldface type to denote an m-dimensional vector; e.g., $\mathbf{x} = (x_1, x_2, \ldots, x_m)$.

3.1 Continuous Approximation

Recall that the total space requirement of forwarder i, $D_i = \sum_{k=1}^{N_i} W_{i,k}$, is a $\mathbb{Z}_+$-valued random variable. Using continuous approximation, we model it as an $\mathbb{R}_+$-valued random variable $\hat{D}_i$, whose mean μ_i and variance σ_i^2 are chosen such that

$$\begin{aligned} \mu_i &= E[D_i] = E[N_i]E[W_{i,1}], \\ \sigma_i^2 &= \operatorname{var}(D_i) = \operatorname{var}(W_{i,1})E[N_i] + (E[W_{i,1}])^2 \operatorname{var}(N_i), \end{aligned} \tag{13}$$

Problem (12) becomes

$$\hat{\xi} = \max\Big\{ \sum_{i=1}^{m} p_i E[\min(\hat{D}_i, x_i)] : \sum_{i=1}^{m} x_i \leq \kappa, \quad x_i \geq 0 \text{ for each } i \in \mathcal{M} \Big\} \tag{14}$$

Let F_i denote the cumulative distribution function of $\hat{D}_i$. Index the forwarders such that $p_i \geq p_{i+1}$ for all $i \in \mathcal{M}$, where $p_{m+1} = 0$. Since $\hat{D}_i$ is an $\mathbb{R}_+$-valued random variable, F_i is continuous and strictly increasing, and the quantile F_i^{-1} is a well-defined function.

Proposition 4. *The necessary and sufficient conditions for $\hat{\mathbf{x}}^*$ to be an optimal solution to (14) are as follows: There exists $\ell^* \in \mathcal{M}$ and a positive $\lambda^* \in [p_{\ell^*+1}, p_{\ell^*})$ such that*

$$\hat{x}_i^* = \begin{cases} F_i^{-1}(1 - \lambda^*/p_i) & \textit{for each } i = 1, 2, \ldots, \ell^* \\ 0 & \textit{for each } i = \ell^* + 1, \ell^* + 2, \ldots, m \end{cases}$$

and $\sum_{i=1}^{\ell^*} \hat{x}_i^* = \kappa$.

Proof. Since F_i is continuously differentiable for each $i \in \mathcal{M}$, the objective function $\sum_{i=1}^{m} p_i E[\min(\hat{D}_i, x_i)]$ is continuously differentiable on $\mathbb{R}_+^m$. Moreover, it possesses continuous second partial derivatives, so the KKT conditions are necessary for a point to be an optimal solution. The KKT conditions are also sufficient, since the objective function is concave on $\mathbb{R}_+^m$. (All constraints are linear, so the constraint qualifications/regularity conditions are satisfied.)

We associate a vector of multipliers $\boldsymbol{\mu} \geq \mathbf{0}$ with the nonnegativity constraints and $\lambda \geq$

0 with the capacity constraint to form the Lagragian function:

$$\mathcal{L}(\mathbf{x}|\boldsymbol{\mu}, \lambda) = \sum_{i=1}^{m} p_i E[\min(\hat{D}_i, x_i)] + \lambda(\kappa - \sum_{i=1}^{m} x_i) + \sum_{i=1}^{m} \mu_i x_i$$

For a feasible solution $\mathbf{x}$ to be an optimal solution to (14), the KKT conditions

$$\begin{aligned} p_i \bar{F}_i(x_i) - \lambda + \mu_i &= 0 && \text{for } i \in \mathcal{M} \\ \lambda(\kappa - \sum_{i=1}^{m} x_i) &= 0 && \\ \mu_i x_i &= 0 && \text{for } i \in \mathcal{M} \end{aligned}$$

are both necessary and sufficient. Since $\boldsymbol{\mu} \geq \mathbf{0}$, we can eliminate them; the above conditions can be written as

$$p_i \bar{F}_i(x_i) \leq \lambda \qquad \text{for } i \in \mathcal{M} \tag{15}$$

$$\lambda(\kappa - \sum_{i=1}^{m} x_i) = 0 \tag{16}$$

$$[p_i \bar{F}_i(x_i) - \lambda] x_i = 0 \qquad \text{for } i \in \mathcal{M} \tag{17}$$

Note that $\bar{F}_i(0) = 1$ and $\bar{F}_i(x)$ is strictly decreasing on $[0, \kappa]$. From (17), we conclude that $x_i > 0$ if and only if $p_i > \lambda$. From this result and the way the forwarders are indexed, there exists $\ell \in \mathcal{M}$ such that $x_i > 0$ for $i \leq \ell$ and $x_i = 0$ for $i > \ell$. For each $i \leq \ell$, $p_i \bar{F}_i(x_i) = \lambda$; the multiplier must be $\lambda < p_\ell$. For $i > \ell$, $x_i = 0$ and $p_{\ell+1} \leq \lambda$. Finally, suppose that $\lambda = 0$. Then, it follows from (15) that x_i is the largest possible value of $\hat{D}_i$, and the capacity constraint $\sum_{i=1}^{m} x_i \leq \kappa$ is violated. Hence, $\lambda > 0$, and (16) becomes $\sum_{i=1}^{m} x_i = \kappa$. □

Proposition 4 asserts that the first ℓ^* forwarders receive positive allotments, whereas the last $(m - \ell^*)$ forwarders receive zero allotments. For forwarders 1 through ℓ^*, its allotment is chosen such that the marginal revenue is equal to the Lagrange multiplier [i.e., $p_i\bar{F}_i(\hat{x}_i^*) = \lambda^*$], and that the sum of their allotments equals the capacity. That is,

$$\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda^*/p_i) = \kappa \text{ where } p_{\ell^*+1} \leq \lambda^* < p_{\ell^*} \tag{18}$$

The two-phase algorithm in Table 1 finds an optimal solution to (14). We search for the optimal number of forwarders that would receive positive allotments (denoted as ℓ^*) in Phase I and for the optimal Lagrange multiplier (denoted as λ^*) in Phase II. The values of ℓ^* and λ^* must satisfy (18).

The idea behind Phase I is as follows. We want to find ℓ^* such that $\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda/p_i) = \kappa$ for some $\lambda \in [p_{\ell^*+1}, p_{\ell^*})$. For a fixed value of $\ell \in \mathcal{M}$, the function $\sum_{i=1}^{\ell} F_i^{-1}(1-\lambda/p_i)$ is decreasing in $\lambda \in [p_{\ell+1}, p_\ell)$. Then,

$$\sum_{i=1}^{\ell} F_i^{-1}(1-p_\ell/p_i) < \sum_{i=1}^{\ell} F_i^{-1}(1-\lambda/p_i) \leq \sum_{i=1}^{\ell} F_i^{-1}(1-p_{\ell+1}/p_i)$$

Let LB(ℓ) and UB(ℓ) denote the quantities on the left- and right-hand sides, respectively. Note that $\sum_{i=1}^{\ell} F_i^{-1}(1-\lambda/p_i)$ is continuous in λ. The intermediate value theorem asserts that if LB(ℓ) $\leq \kappa \leq$ UB(ℓ), then there exists $\lambda(\ell) \in [p_{\ell+1}, p_\ell)$ such that $\sum_{i=1}^{\ell} F_i^{-1}(1-\lambda(\ell)/p_i) = \kappa$. Phase I determines the smallest integer ℓ^* such that LB(ℓ^*) $\leq \kappa \leq$ UB(ℓ^*).

Phase II searches for $\lambda^* \in [p_{\ell^*+1}, p_{\ell^*})$ that solves $\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda^*/p_i) = \kappa$, where ℓ^* is found in Phase I. This can be done using a one-dimensional search procedure. In Table 1 Phase II, we present a bisection method. In iteration $t \geq 1$, we employ the midpoint rule (traditionally called the Bolzano search plan) for selecting the trial solution $\lambda_t = (\lambda_t' + \lambda_t'')/2$. If $\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda_t/p_i) > \kappa$, we need to increase the lower bound $\lambda_{t+1}' = \lambda_t$. If $\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda_t/p_i) < \kappa$, we need to decrease the upper bound $\lambda_{t+1}'' = \lambda_t$. [Again, note that $\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda/p_i)$ is decreasing in λ.] By construction, the length of the interval of uncertainty in iteration $(t+1)$ is halved that in iteration t. We stop when $\sum_{i=1}^{\ell^*} F_i^{-1}(1-\lambda_t/p_i)$ is close to κ.

- Phase I: Search for $\ell^* \in \mathcal{M}$.

  1. Initialization: Set $\ell = 1$.
  2. Iteration $\ell \geq 1$:
     (a) Set $\mathrm{UB}(\ell) = \sum_{i=1}^{\ell} F_i^{-1}(1 - p_{\ell+1}/p_i)$.
     (b) Set $\mathrm{LB}(\ell) = \sum_{i=1}^{\ell} F_i^{-1}(1 - p_{\ell}/p_i)$.
  3. Stopping:
     (a) If $\ell = m$, set $\ell^* = m$ and go to Phase II.
     (b) If $\mathrm{LB}(\ell) \leq \kappa \leq \mathrm{UB}(\ell)$, set $\ell^* = \ell$ and go to Phase II. Otherwise, set $\ell = \ell + 1$ and perform the next iteration.

- Phase II: Search for $\lambda^* \in [p_{\ell^*+1}, p_{\ell^*})$.

  1. Initialization: Select a small tolerance $\epsilon > 0$.
     (a) Set initial lower bound $\lambda_1' = p_{\ell^*+1}$.
     (b) Set initial upper bound $\lambda_1'' = p_{\ell^*}$.
  2. Iteration $t \geq 1$:
     (a) Compute new multiplier $\lambda_t = (\lambda_t' + \lambda_t'')/2$.
     (b) Computer allotment $x_{it} = F_i^{-1}(1 - \lambda_t/p_i)$ for $i = 1, 2, \ldots, \ell^*$.
  3. Stopping: If $|\sum_{i=1}^{\ell^*} x_{it} - \kappa| < \epsilon$, set $\lambda^* = \lambda_t$ and $\hat{x}_i^* = x_{it}$, and stop. Otherwise,
     (a) If $\sum_{i=1}^{\ell^*} x_{it} > \kappa$, set $\lambda_{t+1}' = \lambda_t$, and $\lambda_{t+1}'' = \lambda_t''$.
     (b) If $\sum_{i=1}^{\ell^*} x_{it} < \kappa$, set $\lambda_{t+1}'' = \lambda_t$, and $\lambda_{t+1}' = \lambda_t'$.
     (c) Set $t = t + 1$ and perform the next iteration.

Table 1: Continuous approximation heuristic procedure

3.2 Lagrangian relaxation

In Problem (12), we relax the capacity constraint by multiplying it by Lagrange multiplier $\nu \geq 0$ and bringing it into the objective function, which now becomes

$$\sum_{i=1}^{m} \rho_i(a_i) + \nu(\kappa - \sum_{i=1}^{m} a_i) = \sum_{i=1}^{m}[\rho_i(a_i) - \nu a_i] + \nu\kappa$$

For short-hand notation, denote $c_i(a|\nu) = \rho_i(a) - \nu a$. We want to solve

$$\min\{\xi(\nu) : \nu \geq 0\} \text{ where } \xi(\nu) = \max\{\sum_{i=1}^{m} c_i(a_i|\nu) : a_i \in \mathbb{Z}_+ \text{ for each } i \in \mathcal{M}\} \tag{19}$$

Recall that in Problem (12), $\rho_i(a) = p_i E[\min(D_i, a)]$, and D_i is a $\mathbb{Z}_+$-valued random variable, so we can restrict our attention to a nonnegative integer allotment. For a fixed value of the Lagrange multiplier, the maximization $\xi(\nu)$ is easy to solve, since the objective function is separable, and $c_i(a|\nu)$ is concave on $\mathbb{Z}_+$ for each $i \in \mathcal{M}$.

Let G_i denote the cumulative distribution function of D_i for each $i \in \mathcal{M}$.

Proposition 5. *For a fixed value of the Lagrange multiplier ν, an optimal solution of $\xi(\nu)$ is as follows. If $\nu > p_i$, then $a_i^*(\nu) = 0$; otherwise,*

$$a_i^*(\nu) = \text{argmax}\left\{a \in \mathbb{N} : \bar{G}_i(a-1) \geq \nu/p_i\right\} \tag{20}$$

Proof. Since the objective function is separable, we can individually maximize each term $c_i(a|\nu)$. Since D_i is a $\mathbb{Z}_+$-valued random variable, we have that for each $a \in \mathbb{Z}_+$

$$E[\min(D_i, a)] = \sum_{t=0}^{\infty} P(\min(D_i, a) > t) = \sum_{t=0}^{a-1} P(D_i > t) = \sum_{t=0}^{a-1} \bar{G}_i(t) \tag{21}$$

where the first equation follows from the result $E[Z] = \sum_{t=0}^{\infty} P(Z > t)$ for a $\mathbb{Z}_+$-valued random variable Z. Using (21), we obtain the expression

$$c_i(a|\nu) = p_i E[\min(D_i, a)] - \nu a = p_i \sum_{t=0}^{a-1} \bar{G}_i(t) - \nu a$$

and the first difference is

$$c_i(a|\nu) - c_i(a-1|\nu) = p_i \bar{G}_i(a-1) - \nu$$

If $\nu > p_i$, then the first difference is negative for all $a \geq 0$, so $c_i(a|\nu)$ is nonincreasing, and $a_i^*(\nu) = 0$. Otherwise, we obtain the expression (20). □

The solution $\mathbf{a}^*(\nu)$ found in Proposition 5 can be used to obtain an upper bound:

$$\sum_{i=1}^{m} c_i(a_i^*(\nu)|\nu) \geq \xi \geq \zeta$$

The first inequality follows from the fact that $\xi(\nu) \geq \xi$ for all $\nu \geq 0$, and the second inequality follows from (11). Also, it can be used to obtain a lower bound. Let

$$x_i^*(\nu) = \lfloor \tilde{x}_i^*(\nu) \rfloor \text{ where } \tilde{x}_i^*(\nu) = \left[a_i^*(\nu) - \frac{1}{m^*(\nu)} (\sum_{i=1}^{m} a_i^*(\nu) - \kappa)^+ \right]^+ \tag{22}$$

where $m^*(\nu)$ is the number of non-zero allotments, i.e., the size of the set $\{i \in \mathcal{M} : a_i^*(\nu) > 0\}$. Note that $\mathbf{x}^*(\nu)$ is a feasible solution and can be used to obtain a lower bound $\sum_{i=1}^{m} \rho_i(x_i^*(\nu))$ to the approximated problem (12) or $\sum_{i=1}^{m} \pi_i(x_i^*(\nu))$ to the original problem (8).

We want to solve (19). The Lagragian relaxation algorithm is presented in Table 2. In iteration k, for a fixed value of the Lagrange multiplier ν_k, we solve $\xi(\nu_k)$: Its solution $\mathbf{a}^*(\nu_k)$ is given in Proposition 5, and it is used to construct a feasible solution $\mathbf{x}^*(\nu_k)$ as in (22). The first and latter are used to compute the upper and lower bounds of ξ in (12), respectively. The best upper and lower bounds that we have found during iterations 1 through k can be determined. With the bounds in hand, we obtain the updated Lagrange multiplier ν_{k+1} using a subgradient optimization method. The process continues until the best lower and upper bounds are close enough. Steps in Table 2 are similar to those in a generic Lagrangian relaxation algorithm; cf. Fisher (1985).

4 Numerical Examples

We illustrate the performance of our heuristic solutions via numerical examples. We also compare them with a proportional allocation scheme, whose variants may be used in practice because of their simplicity. In this scheme, the allotment for forwarder i is $a_i = \frac{\mu_i}{\sum_{i=1}^m \mu_i}\kappa$ for each $i \in \mathcal{M}$, where μ_i is defined as in (13). Note that the allotment that forwarder i receives is proportional to its mean. The larger the mean demand, the larger the allotment.

Below, we describe the setup in the numerical experiments. Suppose that the carrier manages its air-cargo capacity in terms of weight. Assume that there are $m = 3$ forwarders, whose per-kilogram contributions are 1.2, 1.0, and 0.8 respectively. We consider two sets of experiments. In the first (resp., second) set, small (resp., large) problem instances are considered; the capacity is discretized so that one unit equals 300 (resp., 50) kilograms, and the per-unit contribution (p_1, p_2, p_3) is $(360, 200, 240)$ [resp., $(60, 50, 40)$]. The small problem instances are solved to optimality. The number of shipment requests from forwarder i is a Poisson random variable with mean $E[N_i] = \eta_i = 12 - 0.03p_i$. Note that the mean number of arrivals (demand) is linearly decreasing in the per-unit contribution (margin). Assume that the random requirements of all forwarders are i.i.d. negative binomial random variables with parameters r and p. Then, $E[W_{i,k}] = rq/p$ and $\text{var}(W_{i,k}) = rq/p^2$ where $q = 1 - p$.

In the continuous approximation heuristic, we model $\hat{D}_i$ using the gamma distribution with shape and scale parameters a_i and b_i, respectively. Then, $E[\hat{D}_i] = a_i b_i$ and $\text{var}(\hat{D}_i) =$

1. Initialization:

   (a) Determine initial multiplier $\nu_1 = \frac{1}{m}\sum_{i=1}^{m} p_i$.

   (b) Select a small tolerance $\epsilon > 0$.

   (c) Set the best upper bound $\mathcal{U}_0 = \infty$, the best lower bound $\mathcal{L}_0 = -\infty$, and a constant $\alpha_1 = 2$.

2. Iteration $k \geq 1$:

   (a) Obtain $\mathbf{a}^*(\nu_k)$ in Proposition 5, and $\mathbf{x}^*(\nu)$ as in (22).

   (b) Find an upper bound $\mathcal{U}_k^* = \sum_{i=1}^{m} c_i(a_i^*(\nu_k)|\nu_k)$, and the best upper bound $\mathcal{U}_k = \min\{\mathcal{U}_k^*, \mathcal{U}_{k-1}\}$.

   (c) Find a lower bound $\mathcal{L}_k^* = \sum_{i=1}^{m} \rho_i(x_i^*(\nu_k))$, and the best lower bound $\mathcal{L}_k = \max\{\mathcal{L}_k^*, \mathcal{L}_{k-1}\}$.

   (d) Compute a stepsize
   $$t_k = \frac{\alpha_k(\mathcal{U}_k - \mathcal{L}_k)}{\left(\kappa - \sum_{i=1}^{m} a_i^*(\nu_k)\right)^2}$$

   (e) Update the Lagrange multiplier
   $$\nu_{k+1} = \max\left\{0, \nu_k - t_k\left(\kappa - \sum_{i=1}^{m} a_i^*(\nu_k)\right)\right\}$$

   (f) Modify the constant α_k. If the best upper bound $\mathcal{U}_k$ fails to go down for some consecutive number of iterations (e.g., 4 consecutive iterations in the numerical example), then the value of α_k is halved; i.e., $\alpha_{k+1} = \alpha_k/2$. Otherwise, it remains unchanged; i.e., $\alpha_{k+1} = \alpha_k$.

   (g) Stop if $\mathcal{U}_k - \mathcal{L}_k < \epsilon$. Otherwise, perform the next iteration $k = k + 1$.

Table 2: Lagrangian relaxation heuristic procedure

$\%\Delta$	Proportional	Continuous	Lagrangian
Minimum	3.05	1.78	0.00
Maximum	13.19	14.07	12.71
Average	6.10	5.83	2.48

Table 3: Percent differences between the optimal expected contribution and the expected contribution if the heuristic solution is used

$a_i b_i^2$. It follows from (13) that $a_i = \eta_i rq/(1+rq)$ and $b_i = (1+rq)/p$.

In the Lagragian relaxation algorithm, we compute the distribution of D_i by conditioning on N_i. If $N_i = n \in \mathbb{N}$, then the n-fold convolution $\sum_{k=1}^{n} W_{i,k}$ follows the negative binomial distribution with parameters nr and p, since $W_{i,1}, W_{i,2}, \ldots$ are i.i.d. negative binomial random variables with parameters r and p.

Example 1 (Small Problem Instances)**.** Let $(r,p) = (12, 0.79)$. Then, the mean of total capacity requirements of booking requests from all three forwarders is $E[\sum_{i=1}^{m} D_i] = 28.7$, or equivalently 8,610 kilograms. This is about two thirds of the cargo capacity of Airbus A330-300 based on the normal operating conditions and full passenger loads. Table 3 shows the maximum, minimum, and average, of the percent differences between the optimal expected contribution and the expected contribution if the heuristic solution is used, when the capacity is varied from 18 to 38. From Table 3, the average, minimum, and maximum from the Lagrangian relaxation algorithm are smaller than those from the other heuristics. The average and minimum from the continuous approximation algorithm are smaller than those from the proportional allocation scheme, whereas the maximum from the proportional allocation scheme is smaller than that from the continuous approximation algorithm. If the proportional allocation is replaced with the Lagrangian relaxation heuristic, then the expected incremental benefit is on average 3.62 percent.

Example 2 (Large Problem Instances)**.** Let $(r,p) = (36, 0.79)$. Then, the mean of total capacity requirements of booking requests from all three forwarders is $E[\sum_{i=1}^{m} D_i] = 301$, or equivalently 15,050 kilograms. This is approximately the cargo capacity of Airbus A330-300. Figure 1 shows the expected total contributions from different schemes and

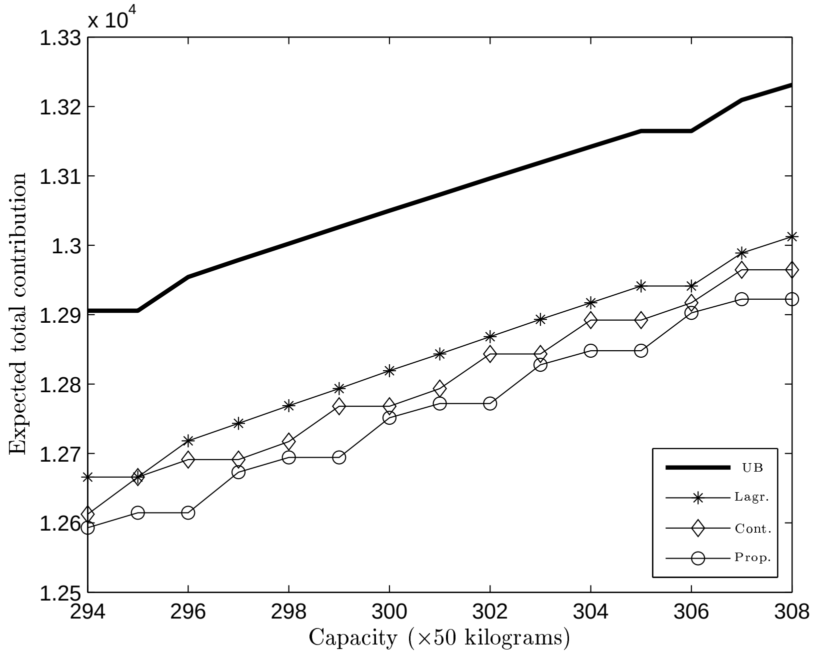

Figure 1: Expected total contributions from different schemes

the upper bound. The cargo capacity κ ranges from 294 to 308. As before, the expected contribution from the Lagrangian relaxation heuristic is at least that from the other two heuristics in all problem instances considered. Also, the expected contribution from the continuous approximation heuristic is at least that from the proportional allocation scheme in every case.

5 Conclusion

The air-cargo business has recently become a significant source of revenue for passenger airlines, some of which sell a large portion of their space to forwarders as allotments. Despite its importance, the number of journal articles on air-cargo revenue management is much less than that on passenger revenue management. Few papers study the allotment decisions, and most of them restrict their attention to a case of single forwarder. The major contribution of the paper to the literature is to extend the study of allotment management to the case of multiple forwarders. The multiple-forwarder case is observed more frequently in practice compared to the single-forwarder case.

We contribute to the literature by developing a mathematical model to evaluate the expected profit for a given allotment combination and formulating a dynamic programming problem to find optimal allotments for multiple forwarders. Upon recognizing that the computation requirement associated with dynamic programming may be intensive especially when the cargo capacity is large, we develop two heuristic solutions applicable to the industry-sized problem. Through a series of numerical experiments, we evaluate the performance of our proposed heuristics and the proportional allocation scheme, of which variant might be implemented in practice. In one set of the experiments, we find that our heuristic based on the Lagrangian relaxation approach outperforms the other two schemes. In the other set, if it replaces the proportional allocation scheme, the expected contribution gain is about 3.62 percent. This additional revenue becomes crucial for the multi-billion dollar airline industry, which is often operating on slim margins.

The problem that the resource can be sold in advance to intermediaries can be found in other settings, e.g., hotel, cruise, and tour operations. In tour operations, large portions of vacation packages are reserved for travel agencies. Our study may apply to these settings as well.

Lastly, we discuss some possible future research directions. After the forwarders return unwanted space, the carrier sells it to a direct shipper on an ad-hoc/free sale basis. What should be an optimal mix between the allotment and the free-sale? This requires building an omnibus model that captures both the short- and medium-term booking processes. Also, one could study an allocation problem in a network environment, in which each shipment needs to be routed so that it reaches its destination on time. The sequence of flights, upon which each shipment is sent, must be determined. What should be an optimal allotment on each leg and flight? Finally, how should the carrier strike a medium-term contract, when it does not know the forwarders' demands? Since the carrier does not directly deal with the forwarders' customers, their demand distributions might not be known to the carrier and become the forwarders' private information. The carrier needs to design contracts, which create enough incentive for the forwarders to share information. We hope to pursue these and other related questions in the future.

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References

Airbus. The Airbus Market Outlook for 2006–2025, 2007. Retrieved October 16, 2007 from `http://www.airbus.com/en/corporate/gmf/index.html`.

M. Bazaraa, J. D. Hurley, E. L. Johnson, G. L. Nemhauser, J. S. Sokol, E.-P. Chew, C.-H. Huang, I. Mok, K.-C. Tan, and C.-P. Teo. The Asia Pacific air cargo system. 2001. Research Paper No: TLI-AP/00/01. The Logistics Institute-Asia Pacific. National University of Singapore and Georgia Institute of Technology. Retrieved January 06, 2007 from `http://www.tliap.nus.edu.sg/Library/default.aspx`.

B. Becker and N. Dill. Managing the complexity of air cargo revenue management. *Journal of Revenue and Pricing Management*, 6(3):175–187, 2007.

J. Billings, A. Diener, and B. Yuen. Cargo revenue optimisation. *Journal of Revenue and Pricing Management*, 2(1):69–79, 2003.

E. A. Boyd and I. C. Bilegan. Revenue management and e-commerce. *Management Science*, 49:1363–1386, 2003.

G. P. Cachon. Supply chain coordination with contracts. *Handbooks in Operations Research and Management Science: Supply Chain Management*, 2003. edited by A.G. de Kok and S.C. Graves, Elsevier, Amsterdam.

M. L. Fisher. An applications oriented guide to Lagrangian relaxation. *Interfaces*, 15(2): 10–21, 1985.

D. Gupta. Flexible carrier-forwarder contracts for air cargo business. 2008. To appear in *Journal of Pricing and Revenue Management*.

R. Hellermann. *Capacity Options for Revenue Management: Theory and Applications in the Air Cargo Industry*. Springer, 2006.

G. L. Hendricks and T. Elliott. Implementing revenue management techniques in an air cargo environment. *UNISYS: Transportation solutions, Insights, and White Paper*, 2005. Retrieved December 06, 2005 from `http://www.unisys.com/transportation/insights/insights_compendium`.

A. Ingold, U. McMahon-Beattie, and I. Yeoman. *Yield Management: Strategies for the Service Industries*. South-Western Cengage Learning, Bedford Row, London, 2000.

R. G. Kasilingam. Air cargo revenue management: Characteristics and complexities. *European Journal of Operational Research*, 96(1):36–44, 1996.

M. A. Lariviere. Supply chain contracting and coordination with stochastic demand. In S. Tayur, R. Ganeshan, and M. Magazine, editors, *Quantitative Models for Supply Chain Management*. Kluwer Academic Publishers, Boston, 1999.

M. Patriksson. A survey on the continuous nonlinear resource allocation problem. *European Journal of Operational Research*, 185(1):1–46, 2008.

R. Phillips. *Pricing and Revenue Optimization*. Stanford University Press, Stanford, CA, 2005.

B. Slager and L. Kapteijns. Implementation of cargo revenue management at KLM. *Journal of Revenue and Pricing Management*, 3(1):80–90, 2004.

K. Talluri and G. J. van Ryzin. *The Theory and Practice of Revenue Management*. Kluwer Academic Publishers, Boston, Massachusetts, 2004.

K. E. Thuermer. Air cargo pricing: No simple matter. *Logistics Management*, 44(11): 36–42, 2005.

J. A. Van Mieghem. Capacity management, investment, and hedging: Review and recent developments. *Manufacturing & Service Operations Management*, 5(4):269–302, 2003.

I. Yeoman and U. McMahon-Beattie. *Revenue Management and Pricing: Case Studies and Applications*. Thomson Learning, Bedford Row, London, 2004.