รายงานวิจัยฉบับสมบูรณ์

โครงการ การคำนวณเชิงรบกวนของควอซีนอร์มอลโมดหลุมดำ

Perturbative Calculation of Quasi-normal Modes of Black Holes

โดย ดร.สุพจน์ มุศิริ

ตุลาคม พ.ศ. 2554

สัญญาเลขที่ MRG5280230

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คณะผู้วิจัย สังกัด
1. ดร.สุพจน์ มุศิริ ภาควิชาฟิสิกส์ คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินทรวิโรฒ

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Investigator : Suphot Musiri

Department of Physics, Srinakharinwiroth University, suphot@swu.ac.th

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บทคัดย่อ

งานวิจัยนี้เป็นการคำนวณหาควอซีนอร์มอลโมด ของระบบหลุมดำทอพอโลยี รีสเนอร์-นอร์สตรอม ที่มีค่าความโค้งส่วนย่อยเป็นศูนย์ ใน 4 มิติ โดยทำการรบกวนระบบด้วย สนามเสกลาร์ที่มีทั้งมวลและประจุ แล้วทำการเปรียบเทียบกับผลการคำนวณเชิงตัวเลข

งานวิจัยนี้ยังได้ทำการคำนวณหาควอซีนอร์มอลโมด ในกรณีที่การรบกวนด้วยสนามเสกลาร์แล้วมีผลทำให้สนามแม่เหล็กไฟฟ้าพื้นหลังเปลี่ยนแปลงไป และทำการคำนวณหาสภาพนำไฟฟ้าในระบบที่ศูนย์สัมบูรณ์ ค่าสภาพนำไฟฟ้าที่ได้มีค่าลู่ออก เมื่อกระแสอยู่ในสถานะของควอซีนอร์มอลโมด

Abstract :

The aim of this research is to calculate the quasinormal modes of the 4 dimensional topological Reissener-Nordstrom black holes with the zero sectional curvature. The system is perturbed by a massive and charged scalar field. We compare our result with the numerical work

We also calculate the quasinormal modes, when the scalar field perturbation alters the electromagnetic field. We calculate the conductivity of the system at the absolute temperature. We fond the conductivity diverged when the current is in the states of quasinormal modes.

หน้าสรุปโครงการ(Executive Summary)
ทุนพัฒนาศักยภาพในการทำงานวิจัยอาจารย์รุ่นใหม่

ชื่อโครงการ การคำนวณเชิงรบกวนของควอซีนอร์มอลโมดหลุมดำ
Perturbative Calculation of Quasi-normal Modes of Black Holes

ชื่อหัวหน้าโครงการ หน่วยงานที่สังกัด ที่อยู่ หมายเลขโทรศัพท์ โทรสาร และ e-mail
ดร.สุพจน์ มุศิริ
ภาควิชาฟิสิกส์ คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินวิโรฒ ประสานมิตร
อ.พระโขนง กรุงเทพฯ 10110
โทรศัพท์ 02-664-1000 ต่อ 8563 หรือ 8163 โทรสาร 02-644-1000 ต่อ 8163
e-mail: suphot@swu.ac.th

1. ความสำคัญและที่มาของปัญหา

ฟิสิกส์เป็นการศึกษาหาความรู้พื้นฐานของธรรมชาติ ทฤษฎีสำคัญในทางฟิสิกส์ที่ประสบความสำเร็จอย่างสูงในการอธิบายปรากฏการต่างๆ คือ ทฤษฎีสัมพัทธภาพ และ ทฤษฎีควอนตัม แต่ทฤษฎีทั้งสองเหมาะสมในการใช้อธิบายและแก้ปัญหาในสภาพเงื่อนไขที่ต่างกัน นั่นคือ ทฤษฎีสัมพัทธภาพเหมาะกับระบบมหัพภาคหรือเหมาะกับระบบที่มีอนุภาคที่มีความเร็วสูง และ ทฤษฎีควอนตัมเหมาะกับระบบจุลภาค การรวมทั้งสองทฤษฎีหรือการค้นหาทฤษฎีใหม่ที่ครอบคลุมทั้งสองทฤษฎีที่เรียกว่าควอนตัมกราวิตี(Quantum Gravity) จึงเป็นเรื่องสำคัญและท้าทายนักฟิสิกส์ทั่วโลก ปัจจุบันทฤษฎีสำคัญที่ถูกพัฒนาขึ้นมาคือ ทฤษฎีสตริง(String Theory)และทฤษฎีลูปควอนตัมกราวิตี(Loop Quantum Gravity Theory)

หลุมดำเป็นบริเวณที่มีความโน้มถ่วงสูงมากจนไม่มีสิ่งใดหลุดรอดไปได้ ความโน้มถ่วงที่สูงนี้ทำให้อนุภาคไม่เสถียรสลายตัวลงไปเป็นอนุภาคระดับมูลฐาน และยังทำให้อนุภาคมีความเร็วสูงขึ้นเข้าใกล้ความเร็วแสง ดังนั้นการศึกษาฟิสิกส์ของหลุมดำจึงจะเป็นประโยชน์ ในการศึกษาและทดสอบทฤษฎีใหม่ๆทางควอนตัมกราวีตี เช่น ทฤษฎีสตริง และทฤษฎีลูปควอนตัมกราวิตี

ควอนซีนอร์มอลโมลของหลุมดำ เป็นปรากฏการที่หลุมดำดูดกลืนและแผ่รังสีออกมาไม่ได้ทุกช่วงความถี่ หรือมีลักษณะคล้ายกับแบบจำลองอะตอมของโบร์(Bohr's Atom Model) หรือ อีกนัยหนึ่ง หลุมดำเกิดมีการควอนไทซ์ขึ้น ผลที่ได้จากการศึกษาควอนซีนอร์มอลโมลของหลุมดำสามารถนำไปเปรียบเทียบและทดสอบกับทฤษฎีใหม่ๆทางควอนตัมกราวีตีได้

วิธีการศึกษานคือการหาผลเฉลยของสมการคลื่นของหลุมดำในแบบจำลองที่สนใจ แล้วจัดผลเฉลยให้สมนัยการค่าขอบเขตของหลุมดำ โดยจะมีเพียงบางผลเฉลย(Quasinormal Modes)และความถี่

(Quasinormal Frequency) ที่สมนัยกับเงื่อนไขนี้ (รายละเอียดเกี่ยวกับวิธีดำเนินการวิจัย ดูเพิ่มเติมได้จาก (S. Musiri and G. Siopsis, Asymptotic form of Quasi-normal Modes of Large AdS Black Holes, hep-th/0308196 และ hep-th/0511113)

ในปี พ.ศ. 2547 (ค.ศ.2004) มาร์ตินซ์ ตรอนโคโส และ ซาเนลลี (C.Martinez, R.Tronoso and J. Zanelli, *Phys.Rev.*, **D70** (2004) 084035, hep-th/0406111.) ได้ค้นพบผลเฉลยของหลุมดำในปริภูมิเวลาแอนไทดิซิเตอร์ ที่ความโค้งเป็นลบ(Negative curvature) โดยให้มีอันตรกิริยาของสนามสเกลาร์(Scalar field) กระทำกับตัวเองกับหลุมดำ เรียกหลุมดำชนิดนี้ว่า หลุมดำเอ็มทีแซ็ด (MTZ) แต่เมื่อ ไม่มีสนามสเกลาร์แล้วผลเฉลยจะกลายเป็นหลุมดำทอพอโลยี่ (Topology black holes, TBH) และจากการศึกษาเทอร์โมไดนามิกส์ของหลุมดำที่มีมวลน้อยๆ ทั้งสองชนิด พบหลักฐาน การเปลี่ยนเฟสจากหลุมดำเอ็มทีแซ็ดไปเป็น หลุมดำทอพอโลยี่ได้ ที่อุณหภูมิวิกฤต

ในปี พ.ศ. 2549 (ค.ศ.2006) คูทโสมบัส สุพจน์ มุศิริ พาแพทโทโนพอลอส และ ซิอฟซีส (G.Koutsoumbas, S.Musiri, E.Papantonopoulos and G.Siopsis, *JHEP* **0610** (2006) 006, hep-th/0606096) ได้คำนวณหาความถี่ของควอซีนอร์มอลแบบเพอร์เทอร์เบชันลำดับหนึ่ง และ ศึกษาเทอร์โมไดนามิกส์ของหลุมดำเอ็มทีแซ็ดและหลุมดำทอพอโลยี่ โดยเพิ่มให้หลุมดำมีประจุไฟฟ้าด้วย พบมีการเปลี่ยนเฟสระหว่างหลุมดำทั้งสอง

งานวิจัยกำลังดำเนินการอยู่ เป็นการคำนวณหา ควอซีนอร์มอลโมดและความถี่ควอซีนอร์มอล ของหลุดดำทอพอโลยีชวาร์ซายด์ในปริภูมิ-เวลาแอนไทดิซิเตอร์ (Topological AdS Schwarzschild Black Holes) ที่มีค่าความโค้งส่วนย่อย k = -1 ใน 4 และ 5 มิติ

2. วัตถุประสงค์ของโครงการ

2.1 การคำนวณหาควอซีนอร์มอลโมลของหลุมดำเพื่อเป็นองค์ความรู้พื้นฐานใหม่

2.2 เพื่อใช้เปรียบเทียบและทดสอบทฤษฎีใหม่ๆ

2.3 ทำให้ผู้วิจัยมีโอกาสพัฒนาศักย์ภาพการทำวิจัยให้อยู่ในระดับนานาชาติ

2.4 เพื่อเป็นงานวิจัยต่อเนื่องในหลักสูตรปริญญาเอกและปริญญาโท ในภาควิชาฟิสิกส์ มศว.ประสานมิตร

2.5 เพื่อเป็นวิทยานิพนธ์ของนิสิตระดับบัณฑิตศึกษาทั้งในและนอกภาควิชา

2.6 เผยแพร่ ผลงานวิจัยที่ได้ใน สัมมนาวิชาต่างๆ และเผยแพร่สู่สาธารณะชน

3. ระเบียบวิธีวิจัย

3.1 สืบค้น ทบทวนเอกสารอ้างอิงทางวิชาการ ข้อมูล ตำราที่จำเป็น และที่เกี่ยวข้อง

3.2 ศึกษาและทำความเข้าใจกับ ขบวนการ วิธีการทางฟิสิกส์และคณิตศาสตร์ที่ใช้ในข้อ 3.1

3.3 วิเคราะห์ปัญหาของงานวิจัยที่ศึกษา และหาสมมุติฐานและหาปัญหาต่อยอดที่จะทำงานวิจัยจากข้อ 3.1

3.4 คิดค้นวิธีใหม่ๆ หรือประยุกต์ขบวนการและวิธีการทำวิจัยจากข้อ 3.1 ตัวอย่างเช่นการสร้างวิธีแบบใหม่เพื่อให้ได้คำตอบที่ลู่เข้าได้เร็วขึ้น โดยใช้ความรู้ทางฟิสิกส์เป็นตัวปรับเปลี่ยนตัวแปรต่างๆ ให้เหมาะสมกับความหมายทางกายภาพ ซึ่งจะเป็นประโยชน์ในการคำนวณแบบไม่ประมาณต่อไป

3.5 นำผลงานวิจัยที่ได้ไปเปรียบเทียบกับผลงานวิจัยอื่นๆที่เกี่ยวข้อง และวิเคราะห์ผลงานวิจัยเชิงคุณภาพทางฟิสิกส์หรือหาความหมายทางกายภาพ เช่น เปรียบเทียบกับผลการคำนวณแบบวิเคราะห์เชิงตัวเลข และ/หรือ งานวิจัยอื่นๆ

3.6 วิจารณ์ สรุปรายงานผล จัดทำรายงานฉบับสมบูรณ์

4. แผนการดำเนินงานตลอดโครงการ

ระยะเวลาของโครงการวิจัย 24 เดือน

ขั้นตอนของกิจกรรมวิจัย	เดือนที่								
	1-3	4-6	7-9	10-12	13-15	16-18	19-22	23	24
1. ศึกษาเอกสารและงานวิจัยที่เกี่ยวข้อง	←	─	→						
2. รายงานความก้าวหน้า		เดือนที่ 6							
3. ศึกษาหาปัญหาต่อยอด				↔					
4. รายงานความก้าวหน้า				เดือนที่ 12					
5. คิดค้นหาวิธีทำการวิจัย และทำวิจัย					←	─	→		
6. รายงานความก้าวหน้า						เดือนที่ 18			
7. เปรียบเทียบและวิเคราะห์								↔	
8. วิจารณ์ สรุปรายงานผล รายงานฉบับสมบูรณ์									↔

5. ผลงาน/หัวข้อเรื่องที่คาดว่าจะตีพิมพ์ในวารสารวิชาการระดับนานชาติในแต่ละปี

ปีที่ 1: ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : Perturbative Calculation of Quasi-normal modes of Topological Black Holes with Vector and Tensor Potentials ชื่อวารสารที่คาดว่าจะตีพิมพ์ : Physics Letters B, impact factor 4.189

ปีที่ 2 ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : Perturbative Calculation of Quasi-normal modes of Topological Black Holes with Scalar Potentials ชื่อวารสารที่คาดว่าจะตีพิมพ์ : Physics Letters B, impact factor 4.189

6. งบประมาณโครงการ

	ปีที่ 1	ปีที่ 2	รวม
1.หมวดค่าตอบแทน			
- ค่าตอบแทนหัวหน้าโครงการ 24 เดือน x 10,000.-	120,000.-	120,000.-	240,000.-
2. หมวดค่าวัสดุ			
-ค่ากระดาษที่ใช้ในการพิมพ์และคำนวณ	25,000.-	25,000.-	50,000.-
-วัสดุสำนักงาน	35,000.-	35,000.-	70,000.-
3. หมวดค่าใช้สอย			
-ค่าติดต่อ(ไปรษณีย์,โทรศัพท์)	15,000.-	15,000.-	30,000.-
-ค่าเดินทางภายในประเทศ	20,000.-	20,000.-	40,000.-
-ค่าจ้างจัดพิมพ์ เข้าเล่มรายงาน 4 ครั้ง	5,000.-	5,000.-	10,000.-
-ค่าใช้จ่ายอื่นๆ	20,000.-	20,000.-	40,000.-
4. หมวดค่าจ้าง			
ไม่มี	--	--	--
รวมงบประมาณโครงการ	240,000.-	240,000.-	480,000.-

(อาจมีการปรับเปลี่ยนความเหมาะสม และ/หรือ เบิกจ่ายตามความเป็นจริง)

1. Introduction (บทนำ)

Physics is the study of the physical nature and its principals. The two main important Physics theories, successfully explaining many physical phenomena, are the theory of relativity and quantum theory. However both theories describe completely different situations. Relativity is suitable for the macroscopic systems or is used in the systems with the speed of the particles within the system considerable to the speed of light. Meanwhile quantum theory is used to describe the microscopic system, the atom-sized systems. There are attempts to search for new theories that can describe very kinds of situations. Call such kind of these theories 'Quantum-Gravity Theory'. The main leading theories currently are String Theory and Loop Quantum Gravity Theory

Black Holes are the regions that the gravitation is extremely high. It is so high that even light can not escape when enters through the horizon. This excruciating gravitation can crash down any macroscopic particles into elementary particles and can accelerate the speed of these microscopic particles closer to the speed of light. To under the physical nature of the black holes we need a new theory that embraces both relativity and quantum. In other word the study of black hole physics can be used verify the candidate quantum-gravity theories, e.g. String theory and Quantum Loop Gravity.

One of the important puzzles of the black holes is its thermodynamics. According to classical relativity theory, black hole entropy is very large (near infinity), but their temperature is zero because they consume everything near them without giving anything back.

This classical black hole entropy contradicts with the first law of thermodynamics

$$dE = TdS$$

And the third law of thermodynamics, larger than zero

$$T > 0$$

In 1972 Bekenstein [1] pointed out the similarity between the non-decreasing area theorem and the second law of thermodynamics. He proposed that the area of the black hole horizon should be proportional to its entropy. This idea contradicted the traditional idea that entropy should be proportional to the volume of the system.

In 1973, Bardeen, Carter and Hawking [4] provided a rigorous proof of the first law of black holes,

$$\delta M = \frac{\kappa}{8\pi}\delta A$$

the second law, the non-decreasing of the horizon area of a black hole, is

$$\delta A > 0$$

and the third law T > 0 , where *M* is the mass of the black hole, *A*, the surface area of the black holes, is

$$S = \frac{A}{4}$$

and κ is the surface gravity. The temperature is, Hawking temperature

$$T = \frac{\kappa}{2\pi}$$

Their work is in agreement with Bekenstein's idea. This new black hole entropy and temperature are the consequence of the quantum effect of the system.

The above research has become the foundation of quantum gravity and can be used to verify the new quantum-gravity theory candidates, e.g. Loop Quantum Gravity Theory and (Super)String Theory

In loop quantum gravity theory, the space-times is not continuous and have been quantized, where the smallest length is the Plank's length. In 1996, Rovelli [4] calculated a black hole surface by counting the quantization of the area. He found that the area is in agreement with the entropy as Bekenstein has predicted. In 1998, Hod [5] noticed that the area of Schwarzschild black hole horizon equals to the frequency of the quasi-normal modes of the black holes.

String theory is currently the most successful in the calculation of the entropy and scattering cross section of the black holes, where both in macroscopic and microscopic pictures obtain the same result [6]. This calculation agreement has led to the conjecture of the correspondence between Anti de Sitter space-time gravity and Conformal Field Theory (AdS/CFT) [7]

AdS/CFT correspondence has generated the extensive research on the connection between the low effective energy string theory and conformal field theory. One of the important connections is that the poles of the propagator, which is the Green's functions in quantum field theory, turn out to be the quasi-normal frequencies of the AdS black holes.

Quasi-normal mode of black holes is the phenomena that the black holes absorb and radiate particles in certain ranges of frequency, similar to the Bohr's atom model. In other word, the black hole radiation is quantized. We expect that the quasi-normal mode research can be used to verify or/and test new coming quantum-gravity theory candidates.

The steps of the calculation is : a) finding the solutions to the interesting black hole wave equation, b) fitting the boundary conditions to the solutions, only ingoing wave at the horizon and outgoing wave at the infinity. c) taking the limit of the far away-zone solution and near but outside horizon-zone to the intermediate zone limit. d) only few solutions and frequencies satisfy the boundary conditions, called them quasi-normal modes and frequencies. Example of fitting the boundary condition is in [10].

There are many models of black holes, e.g. de sitter(dS), AdS, Schwarzschild, Reissner-Nordstr∅m, Kerr, etc. The research in this filed is both numerical and analytical. Our work is analytical one.

We will describe some related research as the following. In 2000, Horowitz and Hubeny [8] numerically calculated the lowest lying quasi-normal frequencies in the many-dimensional Schwarzschild AdS space-times. In 2002, Starinets [9] numerically calculated the higher overtones of the frequencies in the five-dimensional Schwarzschild AdS space-times. The higher frequency overtones, ω_n, are related to the lowest lying quasi-normal frequencies as

$$\omega_n = \omega_0 \pm 2n(1 \mp i)$$

In 2001, Cardoso and Lemos [11] calculated the quasi-normal modes in three-dimensional BTZ space-times. To make the solutions satisfy the boundary conditions they set the argument of the gamma function in the solutions to be zero.

In 2003 Musiri and Siopsis [10] analytically calculated the quasi-normal modes and frequencies in three and five-dimensional AdS space-times. In three dimension space-times, we found the exact solutions. In the five dimension space-times, we analytically calculated the lowest lying quasi-normal frequencies. We also perform the first-order perturbative calculation of the modes and frequencies. Our analytical results are in agreement with the numerical results by Horowitz and Hubeny [8] and Starinets [9].

The method of connecting the far away-zone solution and the near horizon-zone was the standard procedure of the calculation in this field, where the calculation is performed outside the black holes. Until in 2003 Motl [12] did not only extend the inside-black hole solution to the far away zone-solution, but he also extended the real variable radius, *r*, to be a complex variable. The extension to the complex variable allows Molt to rotate his solution inside the black holes to some new appropriate regions. However all the physical quantity, e.g. frequencies, should not change according to the rotation. He could be able to obtain a new constraint equation of the frequencies and solved it, called his method 'Monodromy'. His result, zero-order perturbative frequencies, is in agreement with numerical results by Hod [5] and Nollert [13]. This is for the first time in Schwarzschild space-time that analytical result agrees with the numerical work.

In 2003, Musiri and Siopsis [10] took Motl's method, Monodromy [12], extended the calculation to the firs-order perturbative frequencies of large Schwarzschild black holes. Our first-order perturbation result is in agreement with Nollert's numerical result [13].

In 2003, Musiri and Siopsis [15] used monodromy method to calculate the higher tones of frequencies of large AdS black holes. Again the result agrees with Starinet's numerical result [9].

In 2004, Siopsis [19] calculated the large massive zero-order and first-order quasi-normal modes and frequencies of five-dimensional AdS black holes, where all the analytical work before is only massless case. He did not used monodromy method. The Heun's equation is perturbed instead. The result again agrees with Starinet's numerical result [9].

No hair theorem e.g.[21], one of the most important assumption of black holes, states that the black hole characteristics depend only three initial parameters, i.e. mass, charge and angular momentum, and the black holes are insensitive to any other outside parameters. However in 2004, C.Martinez, R.Troncoso and J.Zanelli [21] found an exact black hole solution with minimally coupled scalar field in the topological AdS space-time. For small mass black holes, they found an evidence of the black hole transformation from one kind to another kind of black holes when the black hole is perturbed by the scalar field.

In 2008 S.S.Gubser [34] has studied behaviors of the simple two gauge field model near a charged black hole in with the negative cosmological constant in 4 dimensions, where one of the fields is Maxwell's field, ($F_{\mu\nu}$) and the other is a massive and charged scalar field, (Ψ). The Lagrangian of the interaction is

$$L = -\frac{1}{4}F_{\mu\nu}^2 - \left|\partial_\mu \Psi - iqA_\mu \psi\right|^2 - m^2 |\psi|^2$$

At certain conditions of the system, he found a spontaneous symmetry breaking of the gauge invariance near the horizon. This causes the black holes going on the second order phase transition.

There are two research plans in this project.

1. Perturbative calculation of quasi-normal modes and frequencies of the scalar field, where there is no back reaction of the scalar field.
2. Phase transition of the black holes, quasi-normal modes and frequencies and conductivity of the current near the horizon, where the back reaction of the scalar field is considered.

1. Quasi-normal modes of the charged scalar field in electromagnetic background

The metric in this system is

$$ds^2 = -fdt^2 + \frac{1}{f}dr^2 + r^2 h_{ij} dx^i dx^j$$

$$f = k - \frac{2M}{r^{d-3}} + \frac{Q^2}{4r^{2d-6}} + \frac{r^2}{L^2}$$

where *M* is the black hole mass, *k* is the sectional curvature, *Q* is the black hole charge and *L* is the AdS radius. The horizon, r_+, can be solved from the condition

$$f(r_+) = 0$$

In this work, we simplify the problem by setting. The Hawking temperature is of the from $T = f'(r_+)/4\pi$

$$T = \frac{r_+^{-2d-1}}{16\pi L^2}\left[4(d-1)r_+^{2d=2} - (d-3)L^2(Q^2 r_+^6 - 4kr_+^{2d})\right]$$

The field potential, A_μ, in this system can be written in the form of the potential, $\Phi(r)$ as

$$A = \Phi(r)dt$$

By varying ψ and Φ in eq(1), the wave equations of ψ and Φ can be obtained from eq(1). The simple solution of the potential, Φ, for $\psi = 0$ is

$$\Phi = \sqrt{\frac{d-2}{2(d-3)}}\left(\frac{Q}{r^{d-3}} - \frac{Q}{r_+^{d-3}}\right)$$

The wave equation of the scalar field ψ

$$\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) - m_{eff}^2\right]\Psi = 0$$

where $m_{eff}^2 = m^2 + g^{tt}q^2\Phi^2$. The wave equation is separable when its ansatz is

$$\Psi = e^{-i\omega t} r^{\frac{2-d}{2}} R(r) S(x_i)$$

where $S(x_i)$ is a harmonic function with eigenvalue $\lambda^2 = l(l+d-3)$, l=0,1,2,3… To understand the boundary conditions at the horizon and the infinity, the tortoise variable, $dr_* = dr / f(r)$, is introduced into the radial wave equation

$$\frac{d^2 R}{dr_*^2} + \left[\omega^2 - V(r)\right] R(r) = 0$$

where the effective potential *V(r)* is $\quad l = 0$

$$V(r) = \frac{(d-2)(d-4)}{4r^2} f^2 + \frac{\lambda^2}{r^2} f + m_{eff}^2 f + \frac{d-2}{2r} ff'$$

At the horizon, $V(r \to r_+) \Rightarrow 0$, only the ingoing wave is allowed in this region $e^{-i\omega r_*} \approx (z-1)^{\alpha_1}$, whereas in the far away region $V(r \to \infty) \Rightarrow \infty$, the wave must be decayed.

In our calculation, we set k=0 and d=4. To solve the wave equation in region, $r_+ \le r < \infty$, let define a new variable $z = \frac{r_+}{r}$. The wave equation (4) changes to

$$z^2 (z-1)(z-a_2)(z-a_3)(z-a_4) \frac{d}{dz} (z-1)(z-a_2)(z-a_3)(z-a_4) \frac{dR}{dz} + K(z) R = 0$$

where

$$\begin{aligned}
K(z) &= \left(2r_+ / Q\right)^2 (\omega r_+)^2 z^2 - \lambda^2 \left(2r_+ / Q\right)^2 z^2 (z-1)(z-a_2)(z-a_3)(z-a_4) \\
&- m^2 r_+^2 \left(2r_+ / Q\right)^2 (z-1)(z-a_2)(z-a_3)(z-a_4) + 4q^2 r_+^2 \left(2r_+ / Q\right)^2 z^2 (z-1)^2 \\
&- 2(z-1)^2 (z-a_2)^2 (z-a_3)^2 (z-a_4)^2 + z(z-1)(z-a_2)^2 (z-a_3)^2 (z-a_4)^2 \\
&+ z(z-1)^2 (z-a_2)(z-a_3)^2 (z-a_4)^2 + z(z-1)^2 (z-a_2)^2 (z-a_3)(z-a_4)^2 \\
&+ z(z-1)^2 (z-a_2)^2 (z-a_3)^2 (z-a_4)
\end{aligned}$$

$$a_2 = A + B - \frac{p}{3}, \quad a_3 = -\frac{1}{2}(A+B) + \frac{i\sqrt{3}}{2}(A-B) - \frac{p}{3}, \quad a_4 = -\frac{1}{2}(A+B) - \frac{i\sqrt{3}}{2}(A-B) - \frac{p}{3}$$

$$A,B=\left[-\frac{1}{2}(-\frac{2p^3}{3^3}-\frac{p^2}{3}+p)\pm\sqrt{\frac{1}{2^2}(-\frac{2p^3}{3^3}-\frac{p^2}{3}+p)^2+\frac{1}{3^3}(p-\frac{p^2}{3})^3}\right]^{1/3}$$

$$p=-\frac{4r_+^2}{Q^2L^2}$$

To simplify the equation, we let

$$R=z^{\alpha_0}(z-1)^{\alpha_1}(z-a_{21})^{\alpha_2}(z-a_3)^{\alpha_3}(z-a_4)^{\alpha_4}F(z)$$

where

$$\alpha_0=\frac{1}{2}\pm\frac{1}{2}\sqrt{9+4m^2L^2}$$

$$\alpha_1=-i\frac{\omega r_+(2r_+/Q)^2}{(1-a_2)(1-a_3)(1-a_4)}$$

$$\alpha_2=-i\left[\begin{array}{l}\dfrac{(\omega r_+)^2(2r_+/Q)^4}{(a_2-1)^2(a_2-a_3)^2(a_2-a_4)^2}\\+\dfrac{4(qr_+)^2(2r_+/Q)^2}{(a_2-a_3)^2(a_2-a_4)^2}\end{array}\right]^{1/2}$$

$$\alpha_3=-i\left[\begin{array}{l}\dfrac{(\omega r_+)^2(2r_+/Q)^4}{(a_3-1)^2(a_3-a_2)^2(a_3-a_4)^2}\\+\dfrac{4(qr_+)^2(2r_+/Q)^2}{(a_3-a_2)^2(a_3-a_4)^2}\end{array}\right]^{1/2}$$

The negative sign in α_1 is chosen to satisfy the boundary condition at the horizon, only ingoing wave. To simplify the equation, take the term

$$z^{\alpha_0+1}(z-1)^{\alpha_1+1}(z-a_{21})^{\alpha_2+1}(z-a_3)^{\alpha_3+1}(z-a_4)^{\alpha_4+1}$$

out from the equation.

$$\begin{aligned}&z(z-1)(z-a_1)(z-a_2)(z-a_3)(z-a_4)\frac{d^2F}{dz^2}+2\alpha_0(z-a_1)(z-a_2)(z-a_3)(z-a_4)\frac{dF}{dz}\\&+(1+2\alpha_1)z(z-a_2)(z-a_3)(z-a_4)\frac{dF}{dz}+(1+2\alpha_2)z(z-a_1)(z-a_3)(z-a_4)\frac{dF}{dz}\\&+(1+2\alpha_3)z(z-a_1)(z-a_2)(z-a_4)\frac{dF}{dz}+(1+2\alpha_4)z(z-a_1)(z-a_2)(z-a_3)\frac{dF}{dz}\\&+\tilde{J}F=0\end{aligned}$$

where

$$
\begin{aligned}
\tilde{J} &= (1+\alpha_0+2\alpha_0\alpha_1)(z-a_2)(z-a_3)(z-a_4)+(1+\alpha_0+2\alpha_0\alpha_2)(z-a_1)(z-a_3)(z-a_4)\\
&+(1+\alpha_0+2\alpha_0\alpha_3)(z-a_1)(z-a_2)(z-a_4)+(1+\alpha_0+2\alpha_0\alpha_4)(z-a_1)(z-a_2)(z-a_3)\\
&+(\alpha_1+\alpha_2+2\alpha_1\alpha_2)z(z-a_3)(z-a_4)+(\alpha_1+\alpha_3+2\alpha_1\alpha_3)z(z-a_2)(z-a_4)\\
&+(\alpha_1+\alpha_4+2\alpha_1\alpha_4)z(z-a_2)(z-a_3)+(\alpha_2+\alpha_3+2\alpha_2\alpha_3)z(z-a_1)(z-a_4)\\
&+(\alpha_2+\alpha_4+2\alpha_2\alpha_4)z(z-a_1)(z-a_3)+(\alpha_3+\alpha_4+2\alpha_3\alpha_4)z(z-a_1)(z-a_2)\\
&-\lambda^2\frac{4r_+^2}{Q^2}z+m^2L^2[z^3-(a_1+a_2+a_3+a_4)z^2]\\
&+\alpha_1^2\left\{\begin{array}{l}(z-a_1)^3+(4a_1-a_2-a_3-a_4)(z-a_1)^2+\begin{bmatrix}a_1(a_1-a_2)+a_1(a_1-a_3)+a_1(a_1-a_4)\\+(a_1-a_2)(a_1-a_3)+(a_1-a_2)(a_1-a_4)\\+(a_1-a_3)(a_1-a_4)\end{bmatrix}(z-a_1)\\+a_1(a_1-a_2)(a_1-a_3)+a_1(a_1-a_2)(a_1-a_4)+a_1(a_1-a_3)(a_1-a_4)+(a_1-a_2)(a_1-a_3)(a_1-a_4)\end{array}\right\}\\
&+\alpha_2^2\left\{\begin{array}{l}(z-a_2)^3+(4a_2-a_1-a_3-a_4)(z-a_2)^2+\begin{bmatrix}a_2(a_2-a_1)+a_2(a_2-a_3)+a_2(a_2-a_4)\\+(a_2-a_1)(a_2-a_3)+(a_2-a_1)(a_2-a_4)\\+(a_2-a_3)(a_2-a_4)\end{bmatrix}(z-a_2)\\+a_2(a_2-a_1)(a_2-a_3)+a_2(a_2-a_1)(a_2-a_4)+a_2(a_2-a_3)(a_2-a_4)+(a_2-a_1)(a_2-a_3)(a_2-a_4)\end{array}\right\}\\
&+\alpha_3^2\left\{\begin{array}{l}(z-a_3)^3+(4a_3-a_1-a_2-a_4)(z-a_3)^2+\begin{bmatrix}a_3(a_3-a_1)+a_3(a_3-a_2)+a_3(a_3-a_4)\\+(a_3-a_1)(a_3-a_2)+(a_3-a_1)(a_3-a_4)\\+(a_3-a_2)(a_3-a_4)\end{bmatrix}(z-a_3)\\+a_3(a_3-a_1)(a_3-a_2)+a_3(a_3-a_1)(a_3-a_4)+a_3(a_3-a_2)(a_3-a_4)+(a_3-a_1)(a_3-a_2)(a_3-a_4)\end{array}\right\}\\
&+\alpha_4^2\left\{\begin{array}{l}(z-a_4)^3+(4a_4-a_1-a_2-a_3)(z-a_4)^2+\begin{bmatrix}a_4(a_4-a_1)+a_4(a_4-a_2)+a_4(a_4-a_3)\\+(a_4-a_1)(a_4-a_2)+(a_4-a_1)(a_4-a_3)\\+(a_4-a_3)(a_4-a_4)\end{bmatrix}(z-a_4)\\+a_4(a_4-a_1)(a_4-a_2)+a_4(a_4-a_1)(a_4-a_3)+a_4(a_4-a_2)(a_4-a_3)+(a_4-a_1)(a_4-a_2)(a_4-a_3)\end{array}\right\}
\end{aligned}
$$

Let divide the wave equation with the term

$$(z-a_2)(z-a_3)(z-a_4)$$

We expand the solution near the horizon by defining the variable $y=1-z$

$$
\begin{aligned}
&y(1-y)\frac{d^2F}{dy^2}\\
&+\left\{1+2\alpha_1-(1+2\alpha_0+2\alpha_1)y-(1+2\alpha_2)\frac{y(1-y)}{1-a_2-y}-(1+2\alpha_3)\frac{y(1-y)}{1-a_3-y}-(1+2\alpha_4)\frac{y(1-y)}{1-a_4-y}\right\}\frac{dF}{dy}\\
&-\frac{1}{(1-a_2-y)(1-a_3-y)(1-a_4-y)}\tilde{J}F=0
\end{aligned}
$$

Near the horizon $y\approx 0$, we can approximate $\tilde{J}$ as

$$
\begin{aligned}
J &\equiv \frac{1}{(1-a_2)(1-a_3)(1-a_4)}\tilde{J}(y\approx 0) \\
&= 1+\alpha_0+2\alpha_0\alpha_1+\frac{\alpha_1+\alpha_2+2\alpha_1\alpha_2}{1-a_2}+\frac{\alpha_1+\alpha_3+2\alpha_1\alpha_3}{1-a_3}+\frac{\alpha_1+\alpha_4+2\alpha_1\alpha_4}{1-a_4} \\
&-\frac{\lambda^2 4r_+^2/Q^2}{(1-a_2)(1-a_3)(1-a_4)}+\frac{m^2L^2p}{(1-a_2)(1-a_3)(1-a_4)}+\frac{\alpha_1^2}{(1-a_2)(1-a_3)(1-a_4)}\left[9-5k+p\right] \\
&+\frac{\alpha_1^2}{(1-a_2)(1-a_3)(1-a_4)}\left[(1-a_2)^3+(5a_2-k)(1-a_2)^2+(10a_2^2-4ka_2)(1-a_2)+9a_2^3-5ka_2^2+\frac{p}{a_2}\right] \\
&+\frac{\alpha_3^2}{(1-a_2)(1-a_3)(1-a_4)}\left[(1-a_3)^3+(5a_3-k)(1-a_3)^2+(10a_3^2-4ka_3)(1-a_3)+9a_3^3-5ka_3^2+\frac{p}{a_3}\right] \\
&+\frac{\alpha_4^2}{(1-a_2)(1-a_3)(1-a_4)}\left[(1-a_4)^3+(5a_4-k)(1-a_4)^2+(10a_4^2-4ka_4)(1-a_4)+9a_4^3-5ka_4^2+\frac{p}{a_4}\right]
\end{aligned}
$$

where $k=\frac{8Mr}{Q^2}$. After approximated, the wave equation is reduced to

$$
y(1-y)\frac{d^2F}{dy^2}+\left\{1+2\alpha_1-(1+2\alpha_0+2\alpha_1+\frac{1+2\alpha_2}{1-a_2}+\frac{1+2\alpha_3}{1-a_3}+\frac{1+2\alpha_4}{1-a_4})y\right\}\frac{dF}{dy}-JF=0
$$

which is the Hypergeometric function.

$$
y(1-y)\frac{d^2F}{dy^2}+\{1+2\alpha_1-(1+a+b)y\}\frac{dF}{dy}-JF=0
$$

where

$$
a,b=\alpha_0+\alpha_1+\frac{1+2\alpha_2}{2(1-a_2)}+\frac{1+2\alpha_3}{2(1-a_3)}+\frac{1+2\alpha_4}{2(1-a_4)}\pm\sqrt{(\alpha_0+\alpha_1+\frac{1+2\alpha_2}{2(1-a_2)}+\frac{1+2\alpha_3}{2(1-a_3)}+\frac{1+2\alpha_4}{2(1-a_4)})^2-J}
$$

To check our result, we will make certain approximation and compare our approximated result to the numerical work [35].

$$
J\approx 1+\alpha_0+2\alpha_0\alpha_1-\frac{4(m^2r_+^2+\lambda^2)/Q^2}{(1-a_2)(1-a_3)(1-a_4)}+\frac{\alpha_1+\alpha_2}{1-a_2}+\frac{\alpha_1+\alpha_3}{1-a_3}+\frac{\alpha_1+\alpha_4}{1-a_4}
$$

We keep only the linear terms of α, the small frequency and small charge in *J*, as the zero-order perturbation. At the horizon, y=0, the wave is only ingoing into black hole. Then the solution is

$$R(y) = y^{\alpha_1}{}_2F_1(a,b;c;y)$$

From the property of the Hypergoemetric functions,

$$F(a,b;c;y) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-y)$$
$$+(1-y)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-a)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-y)$$

In this far away zone the solution behavior is in the form of

$$R(z) \approx (\text{constant}) z^{\alpha_{0+}} + (\text{constant}) z^{\alpha_{0-}}$$

The solution must be finite in this area. However

$$\alpha_{0-} = \frac{1}{2} - \frac{1}{2}\sqrt{9+4m^2L^2} < 0, \qquad \text{for} \qquad m^2L^2 > -2$$

this case $z^{\alpha_{0-}} \to \infty$ as $z \to 0$. To eliminate the divergent term we set the constant in the front of the second term in eq(7) or eq(8) to be zero by letting the arguments of gamma function dominators be negative integer number, $n = 0, 1, 2, 3, \ldots$, i.e.

$$\Gamma(a) \to \infty, \quad \text{or} \quad \Gamma(b) \to \infty$$

From the property of Gamma functions, their arguments must be

$$a = -n, \quad b = \text{-}n$$

a and b can be expressed in the terms of α_0, α_1, α_2, α_3, α_4 as

$$a,b = \alpha_0 + \alpha_1 + \frac{1+2\alpha_2}{2(1-a_2)} + \frac{1+2\alpha_3}{2(1-a_3)} + \frac{1+2\alpha_4}{2(1-a_4)} \pm \sqrt{(\alpha_0 + \alpha_1 + \frac{1+2\alpha_2}{2(1-a_2)} + \frac{1+2\alpha_3}{2(1-a_3)} + \frac{1+2\alpha_4}{2(1-a_4)})^2 - J}$$
$$= -n$$

or

$$n^2 + n(2\alpha_0 + 2\alpha_1 + \frac{1+2\alpha_2}{(1-a_2)} + \frac{1+2\alpha_3}{(1-a_3)} + \frac{1+2\alpha_4}{(1-a_4)}) + J = 0$$

The approximated quasinormal frequencies are solved from the above equation and are compared with the numerical result with the specific parameters as the following

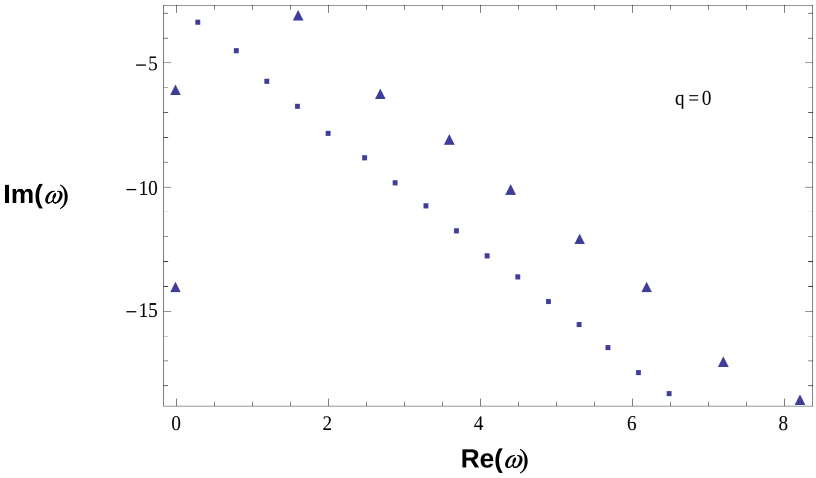


Figure 1 The frequency plots between the real part on the x-axis and the imaginary part on the y-axis where **q =0**, L=1.1, r_+ =Q=1 and m^2L^2=4. The markers ■ and ▲ represent our result and numerical result [35] respectively.

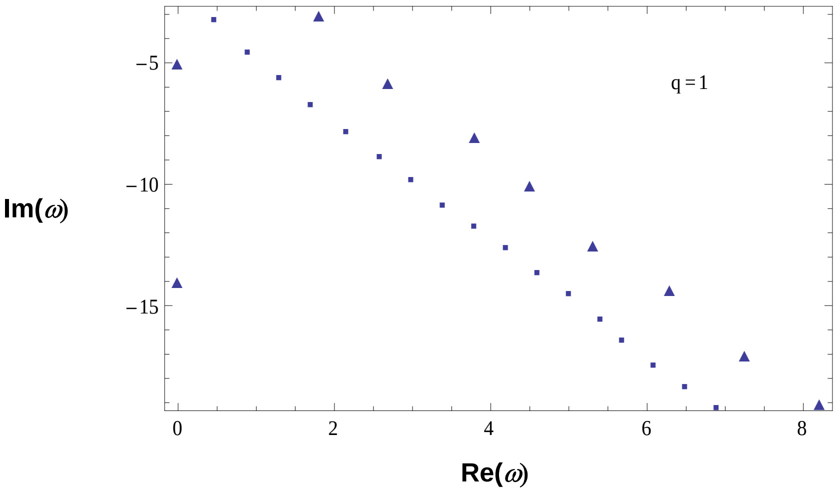


Figure 2 The frequency plots between the real part on the x-axis and the imaginary part on the y-axis where **q =1**, L=1.1, r_+ =Q=1 and m^2L^2=4. The markers ■ and ▲ represent our result and numerical result [35] respectively.

Our result is consistent with the numerical work. The slops of the both graphs in figure 1 and 2 are approximately -0.4. The space between the allowed frequencies is equally separated when the number n is large. The quasinormal frequencies are a discrete set of complex number with the equally spacing. This result is similar to those in many literatures, e.g. [17].

As the conclusion in this part, we make certain analytical approximation to the wave equation in the system, where there still some other numerical results that we have not approximated and compared with yet. Our further work is that we can take this approximated solution as the zero order perturbation and then perform the first order perturbation. The result can be compared with the available numerical work with various parameters.

2. Phase transition of the black holes, quasi-normal modes and frequencies and conductivity of the current near the horizon, where the back reaction of the scalar field is considered.

In the section we still study the Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}^{2} - \left|\partial_{\mu}\Psi - iqA_{\mu}\Psi\right|^{2} - m^{2}\left|\Psi\right|^{2}$$

In the first section, we take the Maxwell's scalar field to be $\Phi = \sqrt{\frac{d-2}{2(d-3)}}\left(\frac{Q}{r^{d-3}} - \frac{Q}{r_{+}^{d-3}}\right)$, when the perturbative scalar field, Ψ, is zero or small. However in this section, the black holes have no charge and the scalar field, Ψ, don't have to be small, then Maxwell's field does not have to be the same as in the previous section. The Maxwell's field and the scalar field will couple each other in the wave equations. We have to solve both wave equations together. Before doing that, let us explain some properties of this system. The metric in our case is

$$ds^{2} = -fdt^{2} + \frac{1}{f}dr^{2} + r^{2}h_{ij}dx^{i}dx^{j}$$

where $f = k - \frac{2M}{r^{d-3}} + \frac{r^{2}}{L^{2}}$ and the black holes has no charge and the sectional curvature $k = 0$

$$f(r) = \frac{2M}{r}\left(\frac{r^{3}}{2ML^{2}} - 1\right) = \frac{r^{2}}{L^{2}}\left(1 - \frac{2ML^{2}}{r^{3}}\right)$$

The singularities can be solve from setting *f(r)*=0. The outside singularity is the horizon, r_+

$$r_+ = (2ML^2)^{1/3}, \qquad r_2 = (2ML^2)^{1/3}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right), \qquad r_3 = (2ML^2)^{1/3}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

The wave equation of the Maxwell's equation, A_μ and the scalar field Ψ are obtained by varying the Lagrangian., $\frac{\delta L}{\delta A_\mu}$ and $\frac{\delta L}{\delta \Psi *}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

To simplify the problem we let the potential to be only scalar potential

$$A = \phi dt$$

and we let $|\Psi| = \psi$. Two wave equations are in the forms of

$$\frac{d^2\psi}{dr^2} + \left(\frac{df/dr}{f} + \frac{d-2}{r}\right)\frac{d\psi}{dr} + \frac{\phi^2}{f}\psi - \frac{m^2}{f}\psi = 0$$
$$\frac{d^2\phi}{dr^2} + \frac{d-2}{r}\frac{d\phi}{dr} - \frac{2\psi^2}{f}\phi = 0$$

In the scalar field ψ equation, there exists a non-linear term in ϕ. The same as in the scalar potential ϕ, there also exists the non-linear term in ψ.

We have to solve the equations in the far away region, then let $z = \frac{r_+}{r}$ and f(r) becomes

$$f = \frac{r_+^2}{L^2 z^2}(1-z^3)$$

Both wave equations change to

$$\frac{d^2\psi}{dz^2} + (1-z^3)\left[\frac{d}{dz}(1-z^3) - (d-2)\frac{(1-z^3)}{z}\right]\frac{d\psi}{dz} + \frac{\phi^2 L^4}{r_+^2(1-z^3)}\psi - \frac{m^2 L^2}{z^2(1-z^3)}\psi = 0$$

$$\frac{d^2\phi}{dz^2} - \frac{(d-4)}{z}\frac{d\phi}{dz} - \frac{2\psi^2 L^2}{z^2(1-z^3)}\phi = 0$$

In far away region the electric potential and the scalar field are of the forms

$$\phi \approx \mu - \frac{\rho}{r^{d-3}} + \ldots$$
$$\psi = \sum_n a_n z^{n+\Delta_\pm}$$

where μ and ρ are constant and

$$\Delta_\pm = \frac{d-1}{2} \pm \sqrt{\frac{(d-1)^2}{4} + m^2}$$

In [36] the scalar field ψ can be approximated as

$$\psi \approx \frac{1}{\sqrt{2}}(3-\Delta)(bz)^{3-\Delta} \tanh\left(\frac{bz}{\alpha}\right)^{2\Delta-3}$$

The simple Maxwell's field can be written in term of component as $A = (0,0,A_y,0)$. To simplify the problem we consider for the low bound of Δ as

$$\Delta = \frac{3}{2}$$

The potential V then can be written as

$$V = 2f\psi^2 = 2\frac{r_+^2}{L^2}\frac{(1-z^3)}{z^2}\psi^2$$
$$V \approx (3-\Delta)^2 b^2 r_+^2 (1-z^3)(bz)^{2(2-\Delta)} \tanh^2\left(\frac{bz}{\alpha}\right)^{2\Delta-3}$$

where the parameter b is approximated from [36] as $b^\Delta = \frac{\langle O_\Delta \rangle \alpha^{2\Delta-3}}{(3-\Delta)r_+^\Delta}$ or $V(z)$ in term of the variable z is

$$V \approx cz(1-z^3)$$

Where the constant $c = \frac{9}{4}L^4 r_+^2 b^3 \tanh^2(1)$. After put everything into the wave equation. It becomes

$$-(1-z^3)\frac{d}{dz}(1-z^3)\frac{dA}{dz} + VA = \omega^2 A$$

$$(z^3-1)^2\frac{d^2A}{dz^2} + 3z^2(z^3-1)\frac{dA}{dz} + cz(z^3-1)A + \frac{L^2\omega^2}{r_+^2}A = 0$$

From the identity

$$z^3 - 1 = (z-1)(z-a_2)(z-a_3)$$

and $$a_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad a_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The solution of the Maxwell's field, *A*, can be put in the form as

$$A = (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3}F(z)$$

where

$$\alpha_1 = -\frac{i\omega L}{3r_+}, \quad \alpha_2 = -\frac{i\omega L a_2}{3r_+}, \quad \alpha_3 = -\frac{i\omega L a_3}{3r_+},$$

The minus sign in the front of the frequency is chosen to represent the ingoing wave at the horizon and each singularity. After separate the singularities from the field *A* to be the field *F*, The wave equation becomes

$$\begin{aligned}
&(z-1)^{\alpha_1+2}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3+2}\frac{d^2F}{dz^2} + 2\alpha_1(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3+2}\frac{dF}{dz} \\
&+ 2\alpha_2(z-1)^{\alpha_1+2}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+2}\frac{dF}{dz} + 2\alpha_3(z-1)^{\alpha_1+2}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3+1}\frac{dF}{dz} \\
&+ \alpha_1(\alpha_1-1)(z-1)^{\alpha_1}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3+2}F + 2\alpha_1\alpha_2(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+2}F \\
&+ 2\alpha_1\alpha_3(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3+1}F + \alpha_2(\alpha_2-1)(z-1)^{\alpha_1+2}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3+2}F \\
&+ 2\alpha_2\alpha_3(z-1)^{\alpha_1+2}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+1}F + \alpha_3(\alpha_3-1)(z-1)^{\alpha_1+2}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3}F \\
&+3z^2(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+1}\frac{dF}{dz} + 3\alpha_1 z^2(z-1)^{\alpha_1}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+1}F \\
&+3\alpha_2 z^2(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3+1}F + 3\alpha_3 z^2(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3}F \\
&+cz(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+1}F + \frac{\omega^2L^2}{r_+^2}(z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3}F = 0
\end{aligned}$$

After being taken the term $(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+1}(z-a_3)^{\alpha_3+1}$ out from the equation, the wave equation reduces to.

$$\frac{d^2F}{dz^2} + \left[\frac{1+2\alpha_1}{z-1} + \frac{1+2\alpha_2}{z-a_2} + \frac{1+2\alpha_3}{z-a_3}\right]\frac{dF}{dz} + \frac{cz}{(z-1)(z-a_2)(z-1)(z-a_3)}F = 0$$

or $$(1-z^3)\frac{d^2F}{dz^2} + (-6\alpha_1 - 3z^2)\frac{dF}{dz} - czF = 0$$

To further reduce the equation we set

$$F = e^{6\alpha_1 z}H$$

In term of the field *H*, the wave equation is

$$(1-z^3)\frac{d^2H}{dz}-(6\alpha_1 z^3+3z^2)\frac{dH}{dz}+9\alpha_1^2(1-z^3)H-3\alpha_1(6\alpha_1+3z^2)H-czH=0$$

What we are interested is the physical properties of the system at the absolute temperature, where $z=\frac{r_+}{r}$. The temperature of the black holes is

$$T=\frac{3M^{1/3}}{4\pi L^{4/3}}=\frac{3r_+^{1/3}}{4\pi L^2}$$

Then as $T\to 0$, $r_+\to 0$. From $c=\frac{9}{4}L^4 b^3 r_+^2\tanh^2(1)=\frac{\langle O_\Delta\rangle^2 L^4}{r_+}\tanh^2(1)\sim\frac{1}{r_+}$, then

$$cz\to\frac{\langle O_\Delta\rangle^2 L^4\tanh^2(1)}{r}$$

At the absolute temperature, the equation is simplified to

$$\frac{d^2H}{dz^2}-\left(cz+\frac{9\alpha_1^2}{c}\right)H\approx 0$$

Let define the variable

$$y=c^{1/3}z-\frac{\omega^2L^2}{c^{2/3}}$$

then
$$\frac{d^2H}{dy^2}-yH=0$$

There are two solutions to this equation, the Airy functions

$$Ai(y)=\frac{\sqrt{y}}{\pi\sqrt{3}}K_{1/2}(\frac{2}{3}y^{3/2}),$$

$$Bi(y)=\frac{\sqrt{y}}{\pi\sqrt{3}}\left[I_{1/3}(\frac{2}{3}y^{2/3})+I_{-1/3}(\frac{2}{3}y^{2/3})\right]$$

At *y*=0 the Airy function *Bi(y)* diverges. However at the absolute temperature, the horizon shrink to zero, for which very value of z the solutions must be finite. The only allowed solution in this case is

$$H(z)=Ai(y)=\frac{\sqrt{y}}{\pi\sqrt{3}}K_{1/3}(\frac{2}{3}y^{3/2})$$

$$=\frac{1}{\pi\sqrt{3}}\left[c^{1/3}z-\frac{\omega^2L^2}{c^{2/3}}\right]^{1/2}K_{1/3}\left[\frac{2}{3}\left(c^{1/3}z-\frac{\omega^2L^2}{c^{2/3}}\right)^{3/2}\right]$$

$K_{1/3}(\frac{2}{3}y^{3/2})$ is a modified Bessel function, where

$$K_{1/3}(x)=\frac{\pi}{\sqrt{3}}\left[I_{-1/3}(x)-I_{1/3}(x)\right]$$

or

$$Ai(y)=\frac{\sqrt{y}}{3}\left[I_{-1/3}(\frac{2}{3}y^{2/3})-I_{1/3}(\frac{2}{3}y^{2/3})\right]$$

From our Maxwell's field which is written as $A=(z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3}e^{6\alpha_1}Ai(z)$, i.e.

$$\begin{aligned}A(z)&=(z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3}e^{6\alpha_1}Ai(y)\\&\approx(z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3}e^{6\alpha_1}\left[c^{1/3}z-\frac{\omega^2L^2}{c^{2/3}}\right]^{1/2}\\&\times\left[I_{-1/3}\left(\frac{2}{3}\left(c^{1/3}z-\omega^2L^2/c^{2/3}\right)^{3/2}\right)-I_{1/3}\left(\frac{2}{3}\left(c^{1/3}z-\omega^2L^2/c^{2/3}\right)^{3/2}\right)\right]\end{aligned}$$

the Maxwell's field can be expanded in the far region in term of $\frac{1}{r}$ or z as,

$$\begin{aligned}A(z)&=A^{(0)}+\frac{A^{(1)}}{r}+\ldots\\&=A(0)+\frac{r_+}{r}\frac{dA(0)}{dz}+\ldots\end{aligned}$$

Therefore from the expansion of the Airy function, Ai(y), the first term in the above equation is

$$A^{(0)}=a_2^{\alpha_2}a_3^{\alpha_3}\left(\frac{(-i\omega L)}{3c^{1/3}}\right)\left[I_{-1/3}\left(\frac{2(-i\omega L)^3}{3c}\right)-I_{1/3}\left(\frac{2(-i\omega L)^3}{3c}\right)\right]$$

where the modified Bessel function $I_\nu(x)=\left(\frac{x}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{k!\Gamma(\nu+k+1)}$, when its argument is a complex number

$$I_{\pm1/3}\left((-i)^3x\right)=\left(\frac{(-i)^3x}{2}\right)^{\pm1/3}\sum_{k=0}^{\infty}\frac{(-x/2)^{2k}}{k!\Gamma(\pm1/3+k+1)}=(-i)^{\pm1}J_{\pm1/3}(x)$$

Therefore the first term in the expansion is of the form

$$A^{(0)}=a_2^{\alpha_2}a_3^{\alpha_3}\frac{\omega L}{3c^{1/3}}\left[J_{-1/3}\left(\frac{2(\omega L)^3}{3c}\right)+J_{1/3}\left(\frac{2(\omega L)^3}{3c}\right)\right]$$

The second term in the far way region expansion can be obtained by differentiating the Maxwell's field respect to the variable *z,*

$$\begin{aligned}\frac{dA(z)}{dz} &= \alpha_1 (z-1)^{\alpha_1 - 1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai(y) + \alpha_2 (z-1)^{\alpha_1}(z-a_2)^{\alpha_2 - 1}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai(y) \\ &+ \alpha_3 (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3 - 1} e^{6\alpha_1 z} Ai(y) + 6\alpha_1 (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai(y) \\ &+ (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} \frac{d}{dz} Ai(y)\end{aligned}$$

where the derivative of the Airy function is

$$\begin{aligned}\frac{d}{dz} Ai(z) &= \frac{1}{3}\frac{d}{dz}\sqrt{y}\left[I_{-1/3}\left(2y^{3/2}/3\right) - I_{1/3}\left(2y^{3/2}/3\right)\right] \\ &= \left(\frac{3}{2}\right)^{1/3} \frac{d}{dz}(2y^{3/2}/3)^{1/3}\left[I_{-1/3}\left(2y^{3/2}/3\right) - I_{1/3}\left(2y^{3/2}/3\right)\right] \\ &= \left(\frac{3}{2}\right)^{1/3} \frac{d(2y^{3/2}/3)}{dz} \frac{d}{d\left(\frac{2y^{3/2}}{3}\right)} \left(\frac{2y^{3/2}}{3}\right)^{1/3}\left[I_{-1/3}\left(\frac{2y^{3/2}}{3}\right) - I_{1/3}\left(\frac{2y^{3/2}}{3}\right)\right]\end{aligned}$$

From the Bessel properties

$$\frac{d}{dx} x^{1/3} I_{-1/3}(x) = x^{1/3} I_{2/3}(x), \qquad \frac{d}{dx} x^{1/3} I_{1/3}(x) = x^{1/3} I_{-2/3}(x)$$

The derivative of the Airy function is reduced to

$$\frac{d}{dz} Ai(z) = c^{1/3} y \left[I_{2/3}\left(\frac{2y^{3/2}}{3}\right) - I_{-2/3}\left(\frac{2y^{3/2}}{3}\right)\right]$$

Again the derivative of the Maxwell's equation is

$$\begin{aligned}\frac{dA(z)}{dz} &= \alpha_1 (z-1)^{\alpha_1 - 1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai(y) + \alpha_2 (z-1)^{\alpha_1}(z-a_2)^{\alpha_2 - 1}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai(y) \\ &+ \alpha_3 (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3 - 1} e^{6\alpha_1 z} Ai(y) + 6\alpha_1 (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai(y) \\ &+ (z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3} e^{6\alpha_1 z} c^{1/3} y \left[I_{2/3}\left(\frac{2y^{3/2}}{3}\right) - I_{-2/3}\left(\frac{2y^{3/2}}{3}\right)\right]\end{aligned}$$

After the derivative, we substitute z=0

$$\frac{dA(0)}{dz} = \alpha_1(-1)^{\alpha_1-1}(-a_2)^{\alpha_2}(-a_3)^{\alpha_3} Ai\left((-i\omega L)^2 / c^{2/3}\right) + \alpha_2(-1)^{\alpha_1}(-a_2)^{\alpha_2-1}(-a_3)^{\alpha_3} e^{6\alpha_1 z} Ai\left((-i\omega L)^2 / c^{2/3}\right)$$
$$+ \alpha_3(-1)^{\alpha_1}(-a_2)^{\alpha_2}(-a_3)^{\alpha_3-1} Ai\left((-i\omega L)^2 / c^{2/3}\right) + 6\alpha_1(-1)^{\alpha_1}(-a_2)^{\alpha_2}(-a_3)^{\alpha_3} Ai\left((-i\omega L)^2 / c^{2/3}\right)$$
$$+ (-1)^{\alpha_1}(-a_2)^{\alpha_2}(-a_3)^{\alpha_3} c^{1/3} y\left[I_{2/3}\left(\frac{2(-i\omega L)^3}{3c}\right) - I_{-2/3}\left(\frac{2(-i\omega L)^3}{3c}\right)\right]$$

From the relation of the conductivity in this system

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = -\frac{\langle J_x \rangle}{\dot{A}_x} = -\frac{i\langle J_x \rangle}{\omega A_x} = -\frac{iA_x^{(i)}}{\omega A_x^{(0)}} = -\frac{ir_+}{\omega A(0)}\frac{dA(0)}{dz}$$

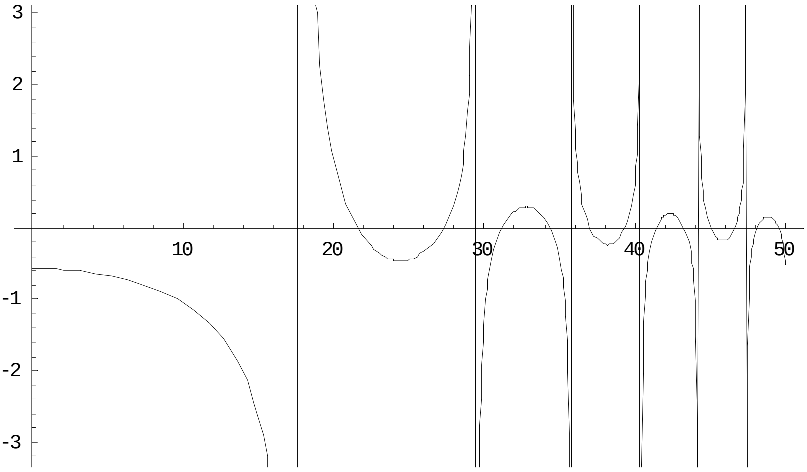
finally the conductivity at the absolute temperature can be written as

$$\sigma(\omega) = -1 - i\frac{\left[J_{2/3}\left(2\omega^3 L^3/3c\right) - J_{-2/3}\left(2\omega^3 L^3/3c\right)\right]}{\left[J_{-1/3}\left(2\omega^3 L^3/3c\right) + J_{1/3}\left(2\omega^3 L^3/3c\right)\right]}$$

At this absolute temperature the frequency ω is possibly be real number. From this equation the conductivity is complex number, the real part of $\sigma_R = -1$ and the imaginary part is

$$\sigma_I(\omega) = -i\frac{\left[J_{2/3}\left(2\omega^3 L^3/3c\right) - J_{-2/3}\left(2\omega^3 L^3/3c\right)\right]}{\left[J_{-1/3}\left(2\omega^3 L^3/3c\right) + J_{1/3}\left(2\omega^3 L^3/3c\right)\right]}$$

The real and imaginary parts of conductivity can be plotted against the frequency ω



The conductivity, $\sigma(\omega)$, diverges at some values of the frequencies ω which satisfy the constraint

$$J_{-1/3}\left(\frac{2\omega^3L^3}{3c}\right)+J_{1/3}\left(\frac{2\omega^3L^3}{3c}\right)=0$$

or the dominator in the conductivity equation is zero. The frequencies that satisfies this equation is actually the quasinormal frequencies and the solutions to this frequencies are quasinormal modes.

The divergences of the conductivity at these frequency values imply that there is a tendency of the discrete energy values of the current in the system. Or it means that there are discrete energy levels of the charge particles or the allowed states of the current.

As in the conclusion of this section, our work is to make the approximation to the wave equation, i.e. at the absolute temperature and the potential at the lower bound and solving for the solution, which in this case is the Airy function. Many physical properties and quantities can be studied and calculated from this approximated solution e.g. the conductivity. We can take this approximated solution as the zero order perturbation and calculate the first order where the temperature is not zero but small. We can also extend our work by studying the case that the potential is not at the lower bound and solve for a new approximated solution.

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Appendix

This appendix contains the three manuscripts in pdf and Microsoft-word formats. The first one has been posted on Las Alamo web site, http://www.arxiv.org, as the preprint numbers ‘arXiv:1011.2938. The second manuscripts is a full paper for an oral presentation in Thailand Sciences and Technologies Congress 2011. The last one is a full paper for an oral presentation of Siam Physics Congress 2011.

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Holographic superconductors near the Breitenlohner-Freedman bound

George Siopsis* and Jason Therrien†
Department of Physics and Astronomy, The University of Tennessee, Knoxville, TN 37996 - 1200, USA.

Suphot Musiri
Department of Physics, Srinakharinwirot University, Bangkok 10110, Thailand.‡
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We discuss holographic superconductors in an arbitrary dimension whose dual black holes have scalar hair of mass near the Breitenlohner-Freedman bound. We concentrate on low temperatures in the probe limit. We show analytically that when the bound is saturated, the condensate diverges at low temperatures as $|\ln T|^\delta$, where δ depends on the dimension. This mild divergence was missed in earlier numerical studies. We calculate the conductivity analytically and show that in the zero temperature limit all poles move to the real axis. We obtain an infinite tower of real poles which are determined by the zeroes of the Airy function in 2+1 dimensions and the poles of the digamma function in 3+1 dimensions. Our analytic results are in good agreement with numerical results whenever the latter are available.

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I. INTRODUCTION

Using the AdS/CFT correspondence [1] it has been shown that if an abelian symmetry has been broken outside of a black hole in AdS space, scalar hair can form creating a holographic superconductor in the dual CFT[2–5]. In the last few years it has been seen that these systems exhibit several characteristics seen in real world strongly coupled superconductors and seem very promising. The AdS/CFT correspondence has also been applied to other areas of condensed matter physics [6–9]; some reviews are [10–12].

Studying the conductivity of holographic superconductors, Horowitz and Roberts [4, 14] noted that at the Breitenlohner-Freedman (BF) bound of the scalar hair of the dual black hole, quasinormal modes appeared to move toward the real axis at low temperatures. Once the back reaction to the metric was included, they showed that the quasinormal modes never became normal. Their number was determined by the height of an effective potential associated with the wave equation of an electromagnetic perturbation. The study was done numerically which limited the ability to go to very low temperatures.

Using the analytic tools developed recently in [15], we explore the zero temperature limit of holographic superconductors near the BF bound. We find that in the probe limit, when the BF bound is saturated, the condensate diverges at low temperatures as $|\ln T|^\delta$, where δ is a constant that we compute and depends on the dimension of spacetime. This is a very mild divergence which explains why it was not detected in earlier numerical studies [4]. It signals the breakdown of the probe limit at very low temperatures. When back reaction to the metric is included, the effective potential associated with the electromagnetic perturbation that determines the conductivity has a finite height. This results in a finite number of quasinormal modes. As one approaches the probe limit, the height of the potential increases with an attendant increase in the number of modes. The latter approach the real axis as the temperature is lowered. In the limit of zero temperature at the BF bound, we obtain infinitely many modes whose frequencies we compute exactly. In 2+1 (3+1) dimensions they are given in terms of the zeroes of the Airy (poles of the digamma) function.

Our paper is organized as follows. In section II we discuss solutions to the field equations at the critical temperature and near zero temperature in the probe limit. In section III we calculate the conductivity at the BF bound at low temperatures. Finally, section IV contains our concluding remarks.

*Electronic address: siopsis@tennessee.edu
†Electronic address: jtherrie@utk.edu
‡Electronic address: suphot@swu.ac.th

II. FIELD EQUATIONS

We are interested in the dynamics of a scalar field of mass m and electric charge q coupled to a $U(1)$ vector potential in the backgound of a $d+1-$ dimensional AdS black hole. The action is

$$S = \int d^{d+1}\sqrt{-g}\left[\frac{R+d(d-1)/l^2}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\Psi|^2 - m^2|\Psi|^2\right] \tag{1}$$

where $F = dA$. We shall adopt units in which $l = 1$.

To find a solution of the field equations, consider the metric *ansatz*

$$ds^2 = \frac{1}{z^2}\left[-f(z)e^{-\chi(z)}dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right] \tag{2}$$

where $\vec{x} \in \mathbb{R}^{d-1}$, representing an AdS black hole of planar horizon. The AdS boundary is at $z = 0$. We shall choose units so that the horizon is at $z = 1$, therefore we require $f(1) = 0$. This is possible because of scaling symmetries of the system and can be done without loss of generality as long as one is careful to only consider physical quantities which are scale invariant.

The Hawking temperature is

$$T = -\frac{f'(1)}{4\pi}e^{-\chi(1)} \tag{3}$$

Assuming that the scalar field is a real function $\Psi(z)$ and the potential is an electrostatic scalar potential, $A = \Phi(z)dt$, the field equations are [3]

$$\begin{aligned}
\Psi'' + \left[\frac{f'}{f} - \frac{\chi'}{2} - \frac{d-1}{z}\right]\Psi' + \left[\frac{q^2\Phi^2 e^{\chi}}{f^2} - \frac{m^2}{z^2 f}\right]\Psi &= 0 \\
\Phi'' + \left[\frac{\chi'}{2} - \frac{d-3}{z}\right]\Phi' - \frac{2q^2\Psi^2}{z^2 f}\Phi &= 0 \\
-\frac{d-1}{2}\chi' + z\Psi'^2 + \frac{zq^2\Phi^2\Psi^2}{f^2}e^{\chi} &= 0 \\
\frac{f}{2}\Psi'^2 + \frac{z^2}{4}\Phi'^2 e^{\chi} - \frac{d-1}{2}\frac{f'}{z} + \frac{d(d-1)}{2}\frac{f-1}{z^2} + \frac{m^2\Psi^2}{2z^2} + \frac{q^2\Psi^2\Phi^2 e^{\chi}}{2f} &= 0
\end{aligned} \tag{4}$$

where prime denotes differentiation with respect to z, to be solved in the interval $(0,1)$, where $z = 1$ is the horizon and $z = 0$ is the boundary.

We are interested in solving the system of non-linear equations (4) in the limit of large q (probe limit). To this end, we shall expand the fields as series in $1/q$ as follows:

$$\begin{aligned}
\Psi &= \frac{1}{q}\left[\Psi_0 + \Psi_1\frac{1}{q^2} + \dots\right] \\
\Phi &= \frac{1}{q}\left[\Phi_0 + \Phi_1\frac{1}{q^2} + \dots\right] \\
f &= f_0 + f_1\frac{1}{q^2} + \dots \\
\chi &= \chi_0 + \chi_1\frac{1}{q^2} + \dots
\end{aligned} \tag{5}$$

and consider the zeroth order system ($q \to \infty$) first and then discuss the addition of first-order ($\mathcal{O}(1/q^2)$) corrections in order to obtain a physically sensible system.

Near the boundary ($z \to 0$), we have $f \to 1$, $\chi \to 0$ and so approximately

$$\Psi \approx \Psi^{(\pm)}z^{\Delta_\pm} \quad , \quad \Phi \approx \mu - \rho z^{d-2} \tag{6}$$

where

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2} \tag{7}$$

While a linear combination of asymptotics is allowed by the field equations, it turns out that any such combination is unstable [20]. However, if the horizon has negative curvature, such linear combinations lead to stable configurations in certain cases [21].

Thus, the system is labeled uniquely by the dimension $\Delta = \Delta_\pm$. The mass of the scalar field is bounded from below by the BF bound [18], $m^2 \geq -\frac{d^2}{4}$, and there appears to be a quantum phase transition at $m^2 = 0$. There is also a unitarity bound that requires $\Delta > \frac{d-2}{2}$.

Demanding at the horizon

$$\Phi(1) = 0 \ , \tag{8}$$

(gauge choice ensuring that $A = \Phi dt$ is regular at the horizon [2]), μ is interpreted as the chemical potential of the dual theory on the boundary. ρ is the charge density on the boundary and the leading coefficient in the expansion of the scalar yields vacuum expectation values of operators of dimension $\Delta_\pm$,

$$\langle \mathcal{O}_{\Delta_\pm} \rangle = \sqrt{2}\Psi^{(\pm)} \tag{9}$$

The field equations admit non-vanishing solutions for the scalar below a critical temperature T_c where these operators condense. In view of (6) and (9), it is convenient to define

$$\Psi(z) = \frac{1}{\sqrt{2}q} b^\Delta z^\Delta F(z) \quad , \quad b = \langle q\mathcal{O}_\Delta \rangle^{1/\Delta} \tag{10}$$

with $F(0) = 1$.

Above the critical temperature, $\Psi = 0$ and the field equations are solved by the AdS Reissner-Nördstrom black hole with flat horizon,

$$f(z) = 1 - \left(1 + \frac{(d-2)\rho^2}{4}\right) z^d + \frac{(d-2)\rho^2}{4} z^{2(d-1)} \quad , \quad \chi(z) = 0 \quad , \quad \Phi(z) = \rho\left(1 - z^{d-2}\right) \tag{11}$$

whose Hawking temperature (3) is

$$T = \frac{d}{4\pi}\left[1 - \frac{(d-2)^2\rho^2}{4d}\right] \tag{12}$$

The corresponding scale-invariant quantity (reduced critical temperature) is

$$\hat{T} = \frac{T}{(q\rho)^{1/(d-1)}} \tag{13}$$

Right at the critical temperature, Ψ obeys the scalar field equation (4) in the Reissner-Nördstrom background (11) with $\rho = \rho_c$. Thus F (eq. (10)) at $T = T_c$ obeys the field equation

$$F'' + \left[\frac{f'}{f} + \frac{2\Delta + 1 - d}{z}\right] F' + \left[\Delta \frac{(d-\Delta)(1-f) + zf'}{z^2 f} + q^2\rho_c^2 \frac{(1-z^{d-2})^2}{f^2}\right] F = 0 \tag{14}$$

For a given q, ρ_c is an eigenvalue which is determined by solving this equation for F subject to the boundary condition at the AdS boundary $F(0) = 1$. We also demand that at the horizon $F(1)$ be finite, and that there be no contribution of the other solution (behaving as $F(z) \sim z^{d-2\Delta}$ as $z \to 0$). The latter condition implies that F has a Taylor expansion around $z = 0$ with the properties $F(0) = 1$, $F'(0) = 0$ (as can be easily deduced from (14)).

To solve eq. (14) for large q (probe limit), use the expansion (5) to write

$$F = F_0 + F_1 \frac{1}{q^2} + \dots \quad , \quad \rho = \frac{1}{q}\left[\rho_0 + \rho_1 \frac{1}{q^2} + \dots\right] \tag{15}$$

Then at zeroth order ($q \to \infty$ limit), the background (11) turns into an AdS Schwarzschild black hole, so

$$f_0(z) = 1 - z^d \quad , \quad \chi_0 = 0 \tag{16}$$

and we obtain the field equation at the critical temperature

$$-F_0'' + \frac{1}{z}\left[\frac{d}{1-z^d} - 2\Delta - 1\right] F_0' + \Delta^2 \frac{z^{d-2}}{1-z^d} F_0 = \rho_{0c}^2 \frac{(1-z^{d-2})^2}{(1-z^d)^2} F_0 \tag{17}$$

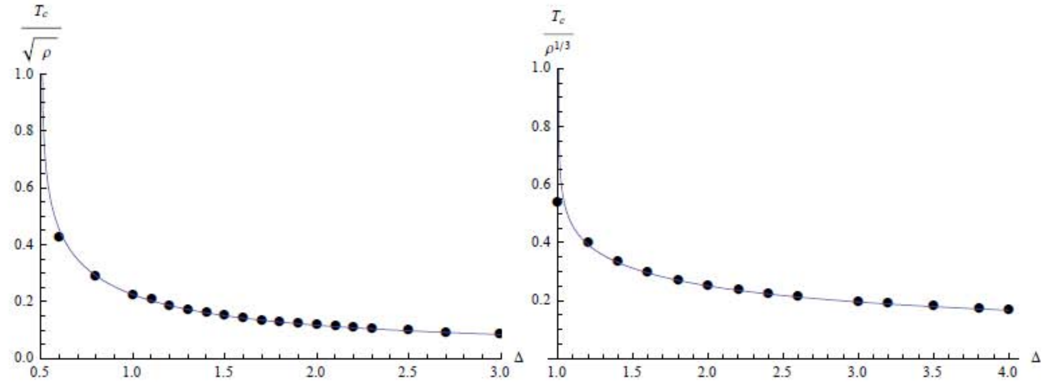

FIG. 1: The critical temperature T_c *vs* the scaling dimension Δ for $d = 3$ (left panel) and $d = 4$ (right panel). Data points represent exact values; solid line is obtained by minimizing (19) with the trial function (20).

which yields the estimate of the reduced (eq. (13)) critical temperature

$$\hat{T}_c = \frac{d}{4\pi}\rho_{0c}^{-\frac{1}{d-1}} \tag{18}$$

The eigenvalue ρ_{0c}^2 minimizes the expression

$$\rho_{0c}^2 = \frac{\int_0^1 dz\, z^{2\Delta-d+1}\{(1-z^d)[F_0'(z)]^2 + \Delta^2 z^{d-2}[F_0(z)]^2\}}{\int_0^1 dz\, z^{2\Delta-d+1}\frac{(1-z^{d-2})^2}{1-z^d}[F_0(z)]^2} \tag{19}$$

We can estimate the eigenvalue by substituting the trial function

$$F_0 = F_\alpha(z) \equiv 1 - \alpha z^{d-1} \tag{20}$$

which obeys the boundary conditions $F_\alpha(0) = 1$, $F_\alpha'(0) = 0$ and $F_\alpha(1)$ is finite.
For $\Delta = \frac{d}{2}$ and $d = 3, 4$, we obtain, respectively,

$$\rho_{0c}^2 \approx 6.3,\ 4.2 \quad , \quad T_c \approx 0.15\sqrt{q\rho},\ 0.2(q\rho)^{1/3} \tag{21}$$

in very good agreement with the exact $T_c = 0.15\sqrt{q\rho},\ 0.25(q\rho)^{1/3}$. In fig. 1 we extend the comparison to the entire range of the scaling dimension Δ for $d = 3, 4$ demonstrating the accuracy of the estimate (19) with the trial function (20) for the critical temperature (18).

Next we consider the zero temperature limit. Because we are working in units in which the radius of the horizon is fixed ($z = 1$), in this limit, $\hat{T} \to 0$ (eq. (13)) whereas T is bounded. Therefore, $\rho \to \infty$. Also the condensate diverges in these units for the same reason, so $b \to \infty$ (eq. (10)). We are interested in calculating scale-invariant quantities, such as $\langle\mathcal{O}\rangle^{1/\Delta}/T_c \sim b/(q\rho)^{1/(d-1)}$.

We shall consider the probe limit ($q \to \infty$) and then discuss how first-order corrections (in a $1/q$ expansion) can be added to obtain a physically meaningful system.

In the probe limit, as we lower the temperature, $F_0(z)$ (eq. (15)) has a smooth limit as $T \to 0$ for $\Delta = \Delta_- \le \frac{d}{2}$. This is not the case for $\Delta = \Delta_+ > \frac{d}{2}$ and care should be exercised in taking the zero temperature limit in that case.
We need to solve the system of zeroth-order equations,

$$\begin{aligned} -F_0'' + \frac{1}{z}\left[\frac{d}{1-z^d} - 1 - 2\Delta\right]F_0' + \frac{\Delta^2 z^{d-2}}{1-z^d}F_0 - \frac{1}{(1-z^d)^2}\Phi_0^2 F_0 &= 0 \\ \Phi_0'' - \frac{d-3}{z}\Phi_0' - \frac{b^{2\Delta}z^{2(\Delta-1)}}{1-z^d}F_0^2\Phi_0 &= 0 \end{aligned} \tag{22}$$

Let us first discuss the $T=0$ limit. It is evident from the field equation for Φ_0 (and can be easily confirmed numerically for arbitrary regular functions $F_0(z)$) that $\Phi_0 \to 0$ as $b\to\infty$ for $z \gtrsim 1/b$. Then for $z \gtrsim 1/b$, we obtain

$$F_0(z) = \mathcal{A}\, F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; 1; 1-z^d\right) \tag{23}$$

which is regular at the horizon. At $T=0$, this is valid in the entire interval, because $1/b \to 0$. We deduce the $T=0$ function

$$F_0(z) = \frac{\Gamma^2(1-\frac{\Delta}{d})}{\Gamma(1-\frac{2\Delta}{d})} F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; 1; 1-z^d\right) \tag{24}$$

where we used $F_0(0)=1$ and standard hypergeometric identities.

Next, we wish to solve the equations close to but above $T=0$. To this end, we shall use iteration as follows,

$$\begin{aligned} -F_0^{(n+1)\prime\prime} + \frac{1}{z}\left[\frac{d}{1-z^d} - 1 - 2\Delta\right] F_0^{(n+1)\prime} + \frac{\Delta^2 z^{d-2}}{1-z^d} F_0^{(n+1)} &= \frac{\mu^2}{(1-z^d)^2}[\hat{\Phi}_0^{(n+1)}]^2 F_0^{(n)} \\ \hat{\Phi}_0^{(n+1)\prime\prime} - \frac{d-3}{z}\hat{\Phi}_0^{(n+1)\prime} - \frac{b^{2\Delta} z^{2(\Delta-1)}}{1-z^d}[F_0^{(n)}]^2 \hat{\Phi}_0^{(n+1)} &= 0 \end{aligned} \tag{25}$$

starting with

$$F_0^{(0)}(z) = 1 \ , \quad \hat{\Phi}_0^{(0)}(z) = 0 \tag{26}$$

We defined

$$\Phi_0(z) = \mu \hat{\Phi}_0(z) \ , \quad \hat{\Phi}_0(0) = 1 \tag{27}$$

where μ is the chemical potential.

At the nth step, we obtain for the scalar field

$$\begin{aligned} F_0^{(n+1)}(z) &= \mathcal{F}_1(z)\left[1 + \mu^2 \int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_2(z')[\hat{\Phi}_0^{(n+1)}(z')]^2 F_0^{(n)}(z')\right] \\ &\quad -\mathcal{F}_2(z)\mu^2 \int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_1(z')[\hat{\Phi}_0^{(n+1)}(z')]^2 F_0^{(n)}(z') \end{aligned} \tag{28}$$

where

$$\mathcal{F}_1(z) = F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; \frac{2\Delta}{d}; z^d\right) \ , \quad \mathcal{F}_2(z) = \frac{z^{d-2\Delta}}{d-2\Delta} F\left(1-\frac{\Delta}{d}, 1-\frac{\Delta}{d}; 2-\frac{2\Delta}{d}; z^d\right) \tag{29}$$

and we imposed the boundary condition $F_0^{(n+1)}(0)=1$. At the horizon, this function diverges. Demanding regularity at $z=1$ fixes the chemical potential μ.

For $n=0$, we obtain for the electrostatic potential

$$\hat{\Phi}_0^{(1)}(z) = \frac{2}{\Gamma(\nu)(2\Delta)^\nu}(bz)^{\frac{d-2}{2}}\left[K_\nu\left(\frac{(bz)^\Delta}{\Delta}\right) - \frac{K_\nu\left(\frac{b^\Delta}{\Delta}\right)}{I_\nu\left(\frac{b^\Delta}{\Delta}\right)} I_\nu\left(\frac{(bz)^\Delta}{\Delta}\right)\right] \ , \quad \nu = \frac{d-2}{2\Delta} \tag{30}$$

where we imposed the boundary condition (8). Notice that the second Bessel function has an exponentially small coefficient, $\mathcal{O}(\sim e^{-2b^\Delta/\Delta})$, and can be neglected at low temperatures.

The charge density is found by using (6) and (15) to be

$$\frac{\rho_0}{b^{d-2}} = -\frac{\mu}{(2\Delta)^{2\nu}} \tag{31}$$

For the scalar field we obtain

$$F_0^{(1)}(z) = \mathcal{F}_1(z)\left[1 + \mu^2\int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_2(z')[\hat{\Phi}_0^{(1)}(z')]^2\right] - \mathcal{F}_2(z)\mu^2\int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_1(z')[\hat{\Phi}_0^{(1)}(z')]^2 \tag{32}$$

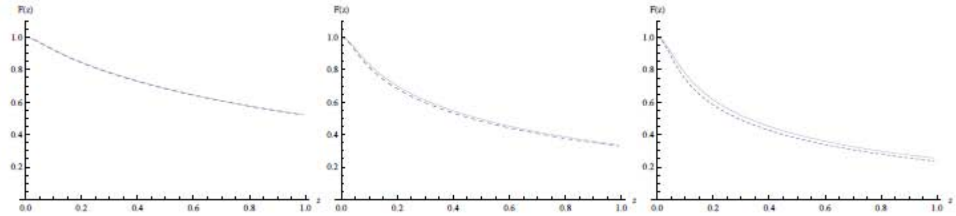

FIG. 2: The field F (eq. (10)) for $\Delta = 1.2$ (left panel), 1.4 (middle panel), 1.5 (right panel) and $d = 3$. Solid curves are first-order analytic expression (32), and dashed curves are exact numerical results (almost indistinguishable) at $T/T_c \approx 0.1$.

The logarithmic singularity at the horizon is found using

$$\mathcal{F}_1(z) \approx -\frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})}\ln(1-z) \quad , \quad \mathcal{F}_2(z) \approx -\frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})}\ln(1-z) \tag{33}$$

Near the horizon, we deduce

$$F_0^{(1)}(z) \approx -\left[\frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})}(1+\mu^2 a_2) - \frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})}\mu^2 a_1\right]\ln(1-z) \tag{34}$$

where

$$a_i = \int_0^1 \frac{dz}{1-z^d} z^{2\Delta+1-d}\mathcal{F}_i(z)\left[\hat{\Phi}_0^{(1)}(z)\right]^2 \quad , \quad i = 1,2 \tag{35}$$

Demanding regularity at the horizon, we need

$$\frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})}(1+\mu^2 a_2) - \frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})}\mu^2 a_1 = 0 \tag{36}$$

which fixes the chemical potential μ,

$$\frac{1}{\mu^2} = \frac{\Gamma(2-\frac{2\Delta}{d})\Gamma^2(\frac{\Delta}{d})}{(d-2\Delta)\Gamma(\frac{2\Delta}{d})\Gamma^2(1-\frac{\Delta}{d})}a_1 - a_2 \tag{37}$$

Explicitly,

$$a_1 = \frac{1}{b^{2\Delta+2-d}}\frac{(d-2)\Gamma(1-\nu)}{(2\Delta)^{2\nu}\Gamma(\nu)} + \dots \ , \quad a_2 = \frac{1}{b^2}\frac{\sqrt{\pi}\Delta^{\frac{2}{\Delta}-1}\Gamma(\frac{1}{\Delta})\Gamma(\frac{d-1}{\Delta})\Gamma(\frac{d}{2\Delta})}{(d-2\Delta)\Gamma^2(\nu)2^{2\nu}\Gamma(\frac{d+\Delta}{2\Delta})} + \dots \tag{38}$$

Evidently, for $\Delta < \frac{d}{2}$, $a_2/a_1 \to 0$ as $b \to \infty$, therefore

$$\mu^2 \approx \mathcal{C} b^{2\Delta+2-d} \ , \quad \mathcal{C} = \frac{(d-2\Delta)(2\Delta)^{2\nu}\Gamma(\nu)\Gamma(\frac{2\Delta}{d})\Gamma^2(1-\frac{\Delta}{d})}{(d-2)\Gamma(1-\nu)\Gamma(2-\frac{2\Delta}{d})\Gamma^2(\frac{\Delta}{d})} \tag{39}$$

It is easily seen (using standard hypergeometric identities) that the low temperature expression (32) reduces to the $T = 0$ one (24) as $b \to \infty$ in the entire interval $[0,1]$.

Before we consider the next iterative order, we note that at finite temperature, the first-order expression (32) is in excellent agreement with numerical results even at $T/T_c \sim 0.1$, which is the lowest temperature at which a numerical solution is available. This is shown in figs. 2 and 3 in which the corresponding curves are almost indistinguishable, implying that the next iterative order introduces negligible corrections to the first-order expression (32) for temperatures $T/T_c \lesssim 0.1$.

We can repeat the above steps for the next iterative order to calculate $F_0^{(2)}$ and $\hat{\Phi}_0^{(2)}$. The resulting functions are very close to the their first-order counterparts, showing that the iteration converges rather rapidly. In fact, the second order quantities are subleading in $1/b$ and vanish as $b \to \infty$ ($T \to 0$). This is the case for all values of the scaling

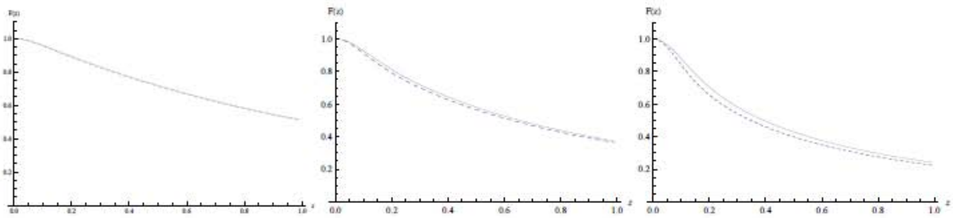

FIG. 3: The field F (eq. (10)) for $\Delta = 1.6$ (left panel), 1.8 (middle panel), 2 (right panel) and $d = 4$. Solid curves are first-order analytic expression (32), and dashed curves are exact numerical results (almost indistinguishable) at $T/T_c \approx 0.2$.

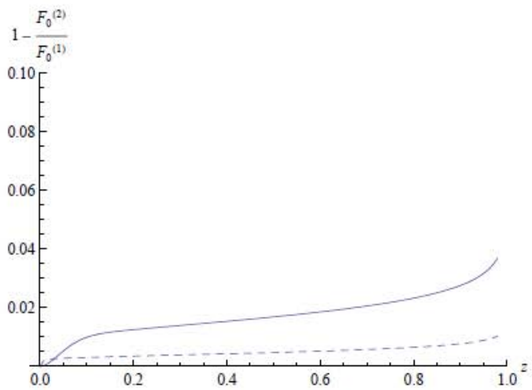

FIG. 4: The second order correction to the scalar field F for $d = 3$, $\Delta = 1.4$ at $b = 20$ ($T/T_c \sim 0.1$) (solid line) and $b = 200$ ($T/T_c \sim 0.01$) (dashed line).

dimension Δ. In fig. 4, we show the difference between second and first iterative order for $d = 3$, $\Delta = 1.4$ (all other values of Δ are similar). The error $(1 - \frac{F_0^{(2)}}{F_0^{(1)}})$ is less than 0.05 in the entire interval $[0, 1]$. As the temperature decreases from $T/T_c \sim 0.1$ to $T/T_c \sim 0.01$, the error decreases to less than 0.01. To demonstrate that the error is subleading in $1/b$, in fig. 5 we plot it at the mid-point ($z = \frac{1}{2}$) as well as the horizon ($z = 1$) (at $z = 0$ the error vanishes by design). Evidently, it goes to zero as $1/b$, showing that the second iterative order introduces subleading corrections at low temperature.

For the charge density we deduce from (31)

$$\rho_0 \sim b^{\frac{d}{2}+\Delta-1} \tag{40}$$

Using

$$\frac{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} \sim b\rho_0^{-\frac{1}{d-1}} \quad , \quad \frac{T}{T_c} \sim \rho_0^{-\frac{1}{d-1}} \tag{41}$$

we finally obtain

$$\frac{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} = \gamma \left(\frac{T}{T_c} \right)^{-\frac{d/2-\Delta}{d/2+\Delta-1}} \tag{42}$$

showing that the condensate diverges as $T \to 0$. The exponent depends on the dimensions of the operator and space-time. The expression for the exponent in (42) corrects an earlier analytic result [15]. The constant of proportionality γ can be found analytically. It is plotted in fig. 6 *vs* Δ. As Δ approaches the BF bound, $\gamma \to 0$, showing that the power law behavior changes, as we discuss next.

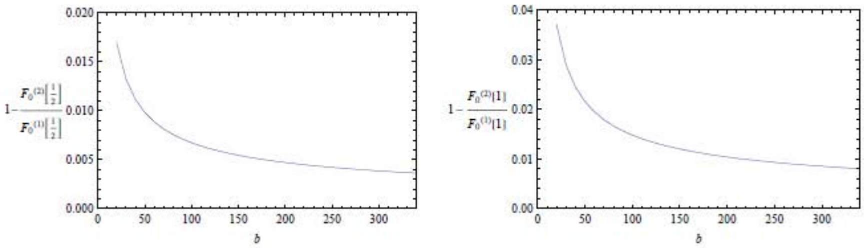

FIG. 5: The second order correction to the scalar field F for $d = 3$, $\Delta = 1.4$ as a function of temperature at the mid-point, $z = \frac{1}{2}$, (left panel) and the horizon, $z = 1$, (right panel). The horizontal axis corresponds to the temperature range $0.01 \lesssim \frac{T}{T_c} \lesssim 0.1$ with T decreasing to the right.

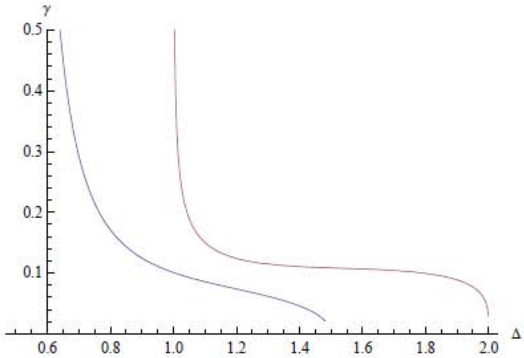

FIG. 6: The parameter γ in the low temperature expression (42) for the condensate *vs* Δ. Curve on left (right) is for $d = 3$ ($d = 4$).

Indeed, at the end point (BF bound), $\Delta = \frac{d}{2}$, we need to exercise care. Letting $\Delta = \frac{d}{2} - \epsilon$, we obtain from (37) and (38),

$$\frac{1}{\mu^2} = \frac{(d-2)\Gamma(\frac{2}{d})}{d^{2(1-\frac{2}{d})}\Gamma(1-\frac{2}{d})}\left[\frac{1}{2\epsilon b^{2-2\epsilon}} - \frac{1}{2\epsilon b^2} + \dots\right] \tag{43}$$

and taking the limit $\epsilon \to 0$, we deduce at the BF bound

$$\frac{\mu^2}{b^2} = \frac{d^{2(1-\frac{2}{d})}\Gamma(1-\frac{2}{d})}{(d-2)\Gamma(\frac{2}{d})[\ln b + \beta_d + o(b^0)]} \tag{44}$$

where β_d is a constant that depends on the dimension and is easily computed (e.g., for $d = 3$, $\beta_3 \approx 1.75$).

Higher-order corrections are computed as before by considering the next iterative order. This introduces corrections that are subleading and vanish as $T \to 0$. Already at $T/T_c \sim 0.1$, our first-order analytic results are almost indistinguishable from numerical results (see right panels of figs. 2 and 3), the discrepancy being $\sim 1\%$. Below that temperature, numerical results are not available. Nevertheless the error can be estimated by calculating the correction at the second iterative order, as before. One finds that the error tends to zero as $T \to 0$ in the entire interval $[0, 1]$.

For the charge density, we have

$$\rho_0 \sim b^{d-1}(\ln b)^{-1/2} \tag{45}$$

therefore,

$$\frac{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} \sim (\ln b)^{\frac{1}{2(d-1)}} \sim \left(\ln \frac{T_c}{T}\right)^{\frac{1}{2(d-1)}} \tag{46}$$

showing that the condensate diverges at the BF bound, albeit very mildly. This mild divergence was missed in earlier numerical studies [4].

The BF bound can also be approached from above. However, the calculation becomes considerably more invloved, because for $\Delta > d/2$, as $T \to 0$, we have $F_0 \approx 1$ near the boundary ($z = 0$), but asymptotically ($z \gtrsim 1/b$), $F_0 \sim z^{d-2\Delta}$, which does not have a smooth limit as $T \to 0$. Therefore, we cannot apply perturbation theory and a different approach is called for [15]. For example, one can approximate F_0 by

$$F_0(z) = \begin{cases} 1 & , \; z \le \alpha \\ \left(\frac{z}{\alpha}\right)^{d-2\Delta} & , \; z > \alpha \end{cases} \tag{47}$$

and find α by a variational method. We shall not dwell on this further here.

Having understood the probe limit, we now turn to the first-order corections in a $1/q^2$ expansion. For $\Delta < d/2$, it is necessary to include these corrections in order to obtain a physical system at low temperatures, because in the $q \to \infty$ limit the condensate diverges as $T \to 0$ (eqs. (42) and (46)).

At first order, we obtain for the functions determining the metric,

$$\begin{aligned} zf_1' - df_1 &= \frac{(bz)^{2\Delta}}{4(d-1)} \left[\left(m^2 + \Delta^2 f_0 + \frac{z^2 \Phi_0^2}{(bz)^{2\Delta}} \right) F_0^2 + 2\Delta z f_0 F_0 F_0' + z^2 f_0 F_0'^2 + \frac{z^4}{(bz)^{2\Delta}} {\Phi_0'}^2 \right] \\ z\chi_1' &= \frac{(bz)^{2\Delta}}{d-1} \left[\left(\Delta^2 + \frac{z^2 \Phi_0^2}{f_0^2} \right) F_0^2 + 2\Delta z F_0 F_0' + z^2 {F_0'}^2 \right] \end{aligned} \tag{48}$$

They can be solved at low temperature using our zeroth-order results above. We obtain

$$f_1(z) = -\frac{\Delta}{4(d-1)} (bz)^{2\Delta} \left[2 - z^d - z^{d-2\Delta} \right] + \dots \; , \; \chi_1(z) = -\frac{\Delta}{2(d-1)} (bz)^{2\Delta} + \dots \tag{49}$$

For the temperature, we deduce the first-order expression

$$T = \frac{d}{4\pi} \left[1 + \frac{\Delta^2}{2d(d-1)} \frac{b^{2\Delta}}{q^2} + \dots \right] \tag{50}$$

showing that the temperature receives a positive correction away from the probe limit. Moreover, it is now clear when the probe limit fails. Indeed, for the expansion in $1/q^2$ to be valid, we ought to have

$$b \lesssim q^{1/\Delta} \tag{51}$$

For a given q, this places a lower bound on the temperature. While zero temperature is unattainable for finite q, the temperature can be made arbitrarily low by choosing a sufficiently large q. It follows that, even though the probe limit ($q \to \infty$) is not a physical system, its properties are a good approximation to corresponding properties of physical systems (of finite q). The approximation becomes better with increasing q and the $1/q^2$ expansion is valid.

III. CONDUCTIVITY

Next, we calculate the low temperature conductivity at the BF bound. For explicit analytic results, we concentrate on two cases, $d = 3$ and $d = 4$. We shall obtain the conductivity σ as a function of the rescaled frequency

$$\hat{\omega} = \frac{\omega}{b} = \frac{\omega}{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}} \tag{52}$$

The function $\sigma(\hat{\omega})$ has a well-defined limit as $q \to \infty$ (probe limit) down to zero temperature even though the condensate $\langle \mathcal{O}_\Delta \rangle$ diverges. Thus, the probe limit, which is not a physical state at low temperatures, can be arbitrarily well approximated by physical states of sufficiently large q. The conductivity of these states can be obtained as a $1/q^2$ expansion with the conductivity in the probe limit serving as the zeroth order term in the expansion.

A. $d = 3$

The conductivity on the AdS boundary is found by applying a sinusoidal electromagnetic perturbation in the bulk of frequency ω obeying the wave equation

$$-\frac{d^2 A}{dr_*^2} + VA = \omega^2 A \quad , \quad V = \frac{2q^2}{z^2} f \Psi^2 \tag{53}$$

where A is any component of the perturbing electromagnetic potential along the boundary. Eq. (53) is to be solved subject to ingoing boundary condition at the horizon

$$A \sim e^{-i\omega r_*} \sim (1-z)^{-i\omega/3} \tag{54}$$

as $z \to 1$ ($r_* \to -\infty$), where r_* is the tortoise coordinate

$$r_* = \int \frac{dz}{f(z)} = \frac{1}{6}\left[\ln\frac{(1-z)^3}{1-z^3} - 2\sqrt{3}\tan^{-1}\frac{\sqrt{3}z}{2+z}\right] \tag{55}$$

with the integration constant chosen so that the boundary is at $r_* = 0$. We with to solve this equation at low temperatures.

Using (10) with $d=3$, the wave equation reads

$$\frac{d}{dz}\left[(1-z^3)\frac{dA}{dz}\right] - \left[b^{2\Delta}z^{2\Delta-2}F^2(z) - \frac{\omega^2}{1-z^3}\right]A = 0 \tag{56}$$

To account for the boundary condition at the horizon, set

$$A = (1-z)^{-i\omega/3}e^{-i\omega z/3}\mathcal{A}(z) \tag{57}$$

where we included a factor $e^{-i\omega z/3}$ for convenience, so that only $\mathcal{A}(z)$ will contribute to the conductivity. The wave equation becomes

$$\begin{aligned}&-3(1-z^3)\mathcal{A}'' + z\left[9z - 2(1+z+z^2)i\omega\right]\mathcal{A}' \\ &+\left[3b^{2\Delta}z^{2\Delta-2}F^2(z) - (1+2z+3z^2)i\omega - \frac{(3+2z+z^2)(3+z+z^2+z^3)}{3(1+z+z^2)}\omega^2\right]\mathcal{A} = 0\end{aligned} \tag{58}$$

Regularity of the wavefunction $\mathcal{A}$ at the horizon ($z=1$) implies the boundary condition

$$(3-2i\omega)\mathcal{A}'(1) + \left(b^{2\Delta}F^2(1) - 2i\omega - \frac{4\omega^2}{3}\right)\mathcal{A}(1) = 0 \tag{59}$$

In the zero temperature limit, $b \to \infty$, it is convenient to rescale $z \to z/b$. The wave equation can be solved as a series expansion in $1/q^2$. The zeroth order term is given by replacing F by F_0 (eq. (15)). For $\Delta \le \frac{3}{2}$, $F_0(z)$ has a smooth zero-temperature limit, so after rescaling and letting $b \to \infty$, we obtain the zero-temperature wave equation

$$-\mathcal{A}'' + \left[z^{2\Delta-2} - \hat{\omega}^2\right]\mathcal{A} = 0 \tag{60}$$

where we used $F(z/b) \to F(0) = 1$, as $b \to \infty$. For $1 < \Delta \le \frac{3}{2}$, there are two linearly independent solutions, $\mathcal{A}_\pm$, distinguished by their asymptotic behavior,

$$\mathcal{A}_\pm \sim e^{\pm\frac{1}{\Delta}z^\Delta} \quad , \quad z \to \infty \tag{61}$$

The general solution can be written as a linear combination,

$$\mathcal{A} = c^+\mathcal{A}_+ + c^-\mathcal{A}_- \tag{62}$$

Applying the boundary condition (59), we deduce

$$\frac{c^+}{c^-} \sim e^{-\frac{2}{\Delta}b^\Delta} \tag{63}$$

so at zero temperature

$$c^+ = 0 \tag{64}$$

i.e., $\mathcal{A} \to 0$ as $z \to \infty$.

For $\Delta = \frac{3}{2}$, we obtain the exact explicit solution

$$\mathcal{A}(z) = \mathcal{A}_-(z) = \mathrm{Ai}(bz - \hat{\omega}^2) \tag{65}$$

whereas $\mathcal{A}_+(z) = \mathrm{Bi}(bz - \hat{\omega}^2)$, with arbitrary normalization, where we restored the scaling parameter b.

At zero temperature in the probe limit, the quasinormal frequencies have moved to the real axis yielding an infinite set of normal frequencies which are solutions of

$$\mathrm{Ai}(-\hat{\omega}^2) = 0 \tag{66}$$

Thus we obtain an infinite tower of real frequencies given by the zeroes of the Airy function.

The zero temperature conductivity in the probe limit is

$$\sigma(\hat{\omega}) = \frac{i}{\hat{\omega}} \frac{\mathrm{Ai}'(-\hat{\omega}^2)}{\mathrm{Ai}(-\hat{\omega}^2)} \tag{67}$$

The real frequencies that solve (66) are the poles of the conductivity. Notice that at zero temperature $\Re\sigma = 0$, except at the poles of $\Im\sigma$ where $\Re\sigma$ diverges as a δ-function.

At low temperatures, we can calculate the first-order correction analytically by considering the zero temperature wave equation (60) as the zeroth-order equation. Then for the first-order correction $\delta\mathcal{A}$ to the potential at low temperatures, we obtain from (58),

$$-\delta\mathcal{A}'' + [z - \hat{\omega}^2]\delta\mathcal{A} = -\frac{1}{3(1-z^3)}\mathcal{H}_1\mathcal{A} \tag{68}$$

where

$$\mathcal{H}_1 = z\left[9z - 2(1+z+z^2)i\omega\right]\frac{d}{dz} + 3b^3 z(2F_1(z) + z^3) - (1+2z+3z^2)i\omega + \frac{z^2(1-15z-12z^2-10z^3)}{3(1+z+z^2)}\omega^2 \tag{69}$$

The first-order potential leads to quasinormal modes which are zeroes of $\mathcal{A} + \delta\mathcal{A}$. Thus the zero temperature real frequencies (66) get shifted at finite (low) temperatures away from the real axis. We obtain $\hat{\omega} \to \hat{\omega} + \delta\hat{\omega}$, where

$$\delta\hat{\omega} = \frac{\pi \mathrm{Bi}(-\hat{\omega}^2)}{3\hat{\omega}\mathrm{Ai}'(-\hat{\omega}^2)}\int_0^1 \frac{dz}{1-z^3}\,\mathrm{Ai}(bz - \hat{\omega}^2)\mathcal{H}_1\mathrm{Ai}(bz - \hat{\omega}^2) \tag{70}$$

This first-order expression is valid for low frequencies. As we heat up the system, most modes disappear and we are left with a finite number of quasinormal modes. Their number decreases as we increase the temperature. Conversely, as we cool down the system, (70) becomes increasingly accurate for an increasing number of modes. These modes shift to the real axis ($\delta\hat{\omega} \to 0$ as $T \to 0$) and at zero temperature we obtain an infinite number of real frequencies given by (66).

This shifting of quasinormal modes can be seen in plots of the conductivity. As the mode frequency approaches the real axis, the corresponding spike in the plot of the imaginary part of the conductivity becomes more pronounced. To demonstrate this, we calculated the conductivity using the first-order approximation (32) to the scalar field. In figure 7, we show the imaginary part of the conductivity at temperature $T/T_c \approx .1$ and compare with the exact numerical solution. The agreement is very good even at such high temperature at which only one quasinormal mode is left. Unfortunately, this is the low temperature limit attained by numerical analysis as numerical instabilities prohibit one from lowering the temperature further. Using our analytical results, we see in figure 8 the emergence of an increasing number of poles as we lower the temperature to $T/T_c \approx .06$ and .04. Finally in figure 9 we compare the lower temperature ($T/T_c \approx .01$) result with the zero temperature analytic expression (67) demonstrating convergence.

For $\Delta > 3/2$, the potential is

$$V = b^{2\Delta} z^{2\Delta-2}(1-z^3)F(bz) \tag{71}$$

with F given approximately by (47). It attains a maximum of order $b^{2(2-\Delta)}$ for $\Delta < 2$. Therefore, at zero temperature it has infinite height. However, the width becomes infinitely narrow leading to a finite tower of poles for the conductivity (quasinormal modes). In the zero temperature limit, the number of modes increases as one approaches the BF bound and decreases away from it. For $\Delta \geq 2$, the height of the potential becomes finite at zero temperature. It turns out that the potential is too narrow to possess bound states, so no poles exist for $\Delta \geq 2$.

B. $d = 4$

The $d = 4$ case is similar. Working as in the $d = 3$ case, at zero temperature, the wave equation for $\Delta = 2$ (at the BF bound) in the probe limit reduces to

$$A'' - \frac{1}{z}A' - [b^4 z^2 - \omega^2]A = 0 \tag{72}$$

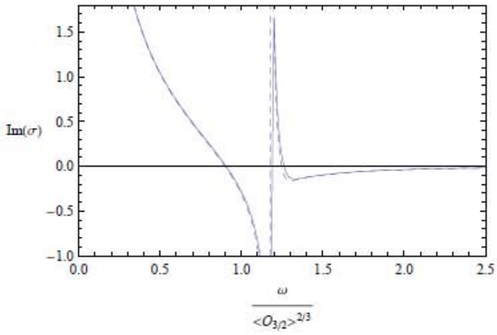

FIG. 7: The imaginary part of the conductivity in $d = 3$ using the expression (32) for the scalar field (dotted line) compared with the exact numerical solution (solid line) at $\frac{T}{T_c} \approx .1$

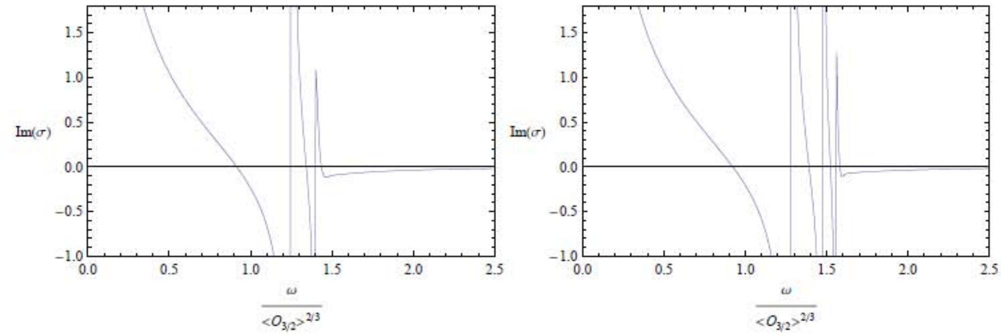

FIG. 8: The imaginary part of the conductivity *vs.* frequency in $d = 3$ using the expression (32) for F at $\frac{T}{T_c} \approx .05$ (left), .04 (right). As the temperature decreases, poles move on to the real axis.

whose acceptable solution can be written in terms of a Whittaker function,

$$A = W_{\frac{\hat{\omega}^2}{4},\frac{1}{2}}(b^2z^2) \tag{73}$$

(The other solution diverges as $z \to \infty$.) At the boundary ($z \to 0$), it has a logarithmic divergence which we need to subtract before we can calculate QNMs and the conductivity [4]. The conductivity is then given by

$$\sigma(\hat{\omega}) = \frac{2}{i\hat{\omega}}\frac{A_2}{A_0} + \frac{i\hat{\omega}}{2} \tag{74}$$

where

$$A(z) = A_0 + A_2 b^2 z^2 - A_0 \frac{\hat{\omega}^2}{2} b^2 z^2 \ln(b^2 z^2) + \dots \tag{75}$$

with an arbitrarily chosen cutoff.

Using the expansion for small arguments,

$$W_{\frac{\hat{\omega}^2}{4},\frac{1}{2}}(b^2z^2) = -\frac{2}{\hat{\omega}^2\Gamma(-\hat{\omega}^2/4)}\left\{1 - \left[1 + \hat{\omega}^2\left(2\gamma - 1 + \ln(b^2z^2) + \psi(1-\hat{\omega}^2/4)\right)\right]\frac{b^2z^2}{2}\right\} + \dots \tag{76}$$

we deduce the zero temperature conductivity in the probe limit

$$\sigma(\hat{\omega}) = \frac{1}{i\hat{\omega}} + i\hat{\omega}\left[2\gamma - \frac{1}{2} + \psi(1-\hat{\omega}^2/4)\right] \tag{77}$$

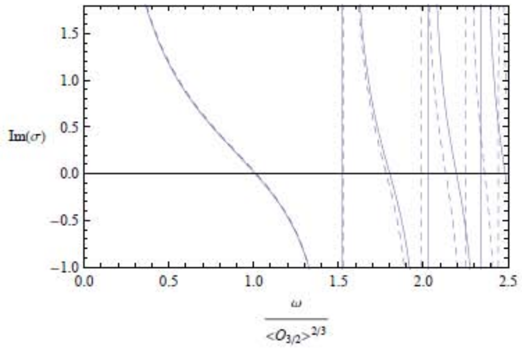

FIG. 9: Comparison of the imaginary part of the conductivity in $d = 3$ using the expression (32) for F at $\frac{T}{T_c} \approx .01$ (dotted line) and the zero temperature limit (67) (solid line).

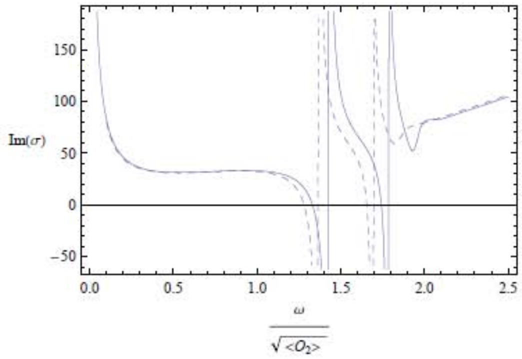

FIG. 10: The imaginary part of the conductivity in $d = 4$ using the expression (32) for the scalar field (dotted line) compared with the exact numerical solution (solid line) at $\frac{T}{T_c} \approx .17$

We have a pole at $\omega = 0$, as expected and an infinite tower of real poles determined by the poles of the digamma function. The poles have real frequencies

$$\hat{\omega} = \frac{\omega_n}{\langle \mathcal{O} \rangle^{1/2}} = 2\sqrt{n} \ , \quad n = 0, 1, 2, \ldots \tag{78}$$

As we increase the temperature, these poles move away from the real axis and turn into quasinormal modes. At any given temperature we have a finite number of such modes with the number increasing as we approach zero temperature. To demonstrate this, we have calculated the conductivity using the first-order approximation (32) to the scalar field. In figure 10 we compare with numerical results at temperature $T/T_c \approx .17$ and find good agreement. As we go to lower temperature, numerical instabilities arise and it is no longer possible to compare our analytical results with their numerical counterparts. We find convergence to the zero temperature limit (77) but much slower than in $d = 3$. In figure 11 we show the imaginary part of the conductivity at $T/T_c \approx 0.1$ and 0.04. As we lower the temperature, the number of poles increases and the poles shift to the right on the real axis approaching the limiting values (78) which correspond to the zero temperature limit of the conductivity shown in figure 12.

IV. CONCLUSION

We discussed holographic superconductors in the probe limit when the scalar hair of the dual black hole is near the BF bound. Using the analytic tools developed in [15], we analyzed the zero temperature limit trying to understand the

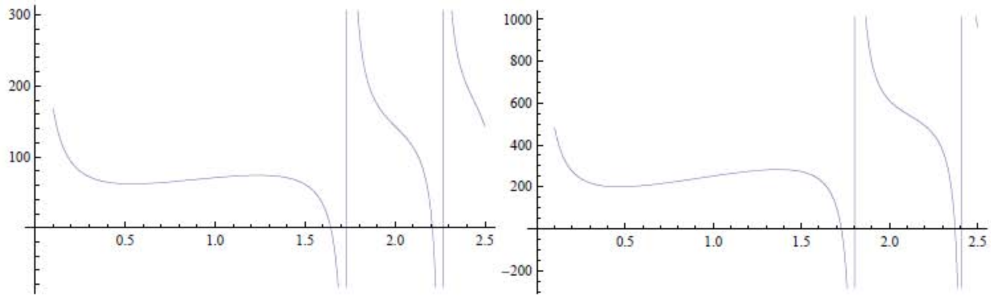

FIG. 11: The imaginary part of the conductivity *vs.* frequency in $d = 4$ using the expression (32) for F at $\frac{T}{T_c} \approx .1$ (left), .04 (right).

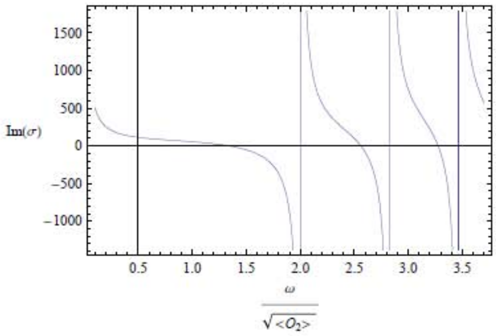

FIG. 12: The imaginary part of the conductivity at zero temperature in $d = 4$ (77).

ground state of the system. Undeterred by numerical instabilities, we found that at low temperatures the condensate diverges as $|\ln T|^\delta$ where δ depends on the dimension of spacetime (eq. (46)). This signals the breakdown of the probe limit at low temperatures even at the BF bound. The divergence is very mild which explains why it was missed in earlier numerical analyses. Even though the probe limit at the BF bound cannot be a physical state, it is still useful to analyze it because it is a limit of physical states. The latter are obtained by including back reaction to the bulk metric which sets the charge of the scalar hair q to a fixed (finite) value. The probe limit is then $q \to \infty$. Thus, any quantity (such as the conductivity) calculated in the probe limit can be approximated by the corresponding (physical) quantity at finite q. This can be done with increasing accuracy by increasing q.

We calculated the zero temperature conductivity at the BF found in the probe limit and found exact analytic expressions in $d = 3, 4$. Thus, we showed that the conductivity has an infinite tower of real poles determined by the zeroes of the Airy function in $d = 3$ (eq. (66)) and the poles of the digamma function in $d = 4$ (eq. (78)). As we heat up the system, only a finite number of poles remains and their positions move off of the real axis. Thus at low temperature, we obtain a finite tower of quasinormal modes. Their number increases as we lower the temperature of the system and diverges at zero temperature.

The probe limit we studied here can be used as a zeroth-order contribution to a perturbative expansion in $1/q^2$. Our results can be extended in a systematic way to include back reaction to the bulk metric in order to analyze physical states. This will greatly facilitate the probe of the zero temperature limit in which numerical methods fail due to numerical instabilities. Work in this direction is in progress.

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Quasinormal Modes of Flat AdS Reissener-Nordstrom Black Holes with the Maxwell Field Background in 3+1 Dimensions

Ortira Chabkaew[1,2] and Suphot Musiri[1*]

[1] *Department of Physics, Faculty of Sciences, Srinakharinwirot University, Bangkok, 10110, Thailand*
[2] *Section of Science, Bangrachanwittaya school, Singburi, 16130, Thailand*
**Corresponding author. E-mail: suphot@swu.ac.th*

Abstract

The quasinormal modes(QNM) of the 3+1 dimensional anti de Sitter(AdS) Reissener-Nordstrom(RN) black holes interacting with the scalar field are analytically calculated for the zero order perturbation. To simplify the problem we study for the case of the sectional curvature k=0. For the zero and small perturbative charge case our result is consistent with the numerical result.

Keywords: Quasinormal modes, Quasinormal frequencies, AdS black holes, Reissener-Nordrom black holes, Black hole phase transition

Introduction

Quasinormal modes of black holes are the solutions to the Einstein wave equation, where the boundary conditions are only the ingoing wave at the horizon of the black holes and only outgoing or decayed wave at the infinity [1]. These conditions allow only certain number of the wave frequencies and cause the frequencies becoming complex number.

Quasinormal modes are the wave that the black holes respond to the perturbation. The frequencies are inversely proportional to time that the system uses to recoil back to the equilibrium. The wave carries information of black hole property and space-time geometry, in some cases also including the perturbative particles.

The correspondence between anti de Sitter space-time and the conformal field theory (AdS/CFT) [2] proposes that the general relativity in d+1 dimensional AdS space-time is equivalent to the quantum field in d dimensions at the boundary of the system. This allows us to study a microscopic system with strong interactions through studying general relativity of a corresponding AdS space-time.

The AdS black holes have recently become more intense research topics, due to the arrival of the AdS/CFT correspondence. In many cases, for example, large-mass black holes in spherical-symmetric space-times, the perturbation is stable. However, in [3] the black hole solution in 3+1 dimensions with the sectional curvature k=-1, hyperbolic symmetry, and perturbed by a scalar field is found. When the scalar field is turned off, the black hole changes to an AdS black hole, called this phenomenon, black hole phase transition.

In this work, the quasinormal modes of charged AdS black holes, Reissner-Norstrom(RN) [4] perturbed by a small charge is analytically calculated for the zero-order perturbation. We consider for the case of the sectional curvature k=0 and in 3+1dimensions. Let Q be the black hole charge which is the source of electromagnetic field, $F_{\mu\nu}$ in this space-time and let ψ be the charge scalar filed that perturbs the system.

The Lagrangian of the fields and the interaction can be written as [5]

$$L = -\frac{1}{4}F_{\mu\nu}^2 - \left|\partial_\mu \psi - iqA_\mu \psi\right|^2 - m^2|\psi|^2 \qquad (1)$$

where A_μ the potentials of $F_{\mu\nu}$ and m is the mass of the charge q.

We arrange this article as following, the first section is the anti de Siitter Reissner-Nordstrom black holes, which describes some properties of these black holes. The second section is the quasinormal modes of AdS RN black hole perturbed by a charge scalar field with k=0 in 3+1 dimensions, which presents our calculation. The third section is the result and conclusion, which discusses the result, compares with the numerical work. The forth section is conclusion, where some further suggestion is given.

Anti de Sitter Reissner-Nordstrom Black Holes

The wave equations

The metric of d-dimensional AdS RN space-times is [6]

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 h_{ij}dx^i dx^j$$

where

$$f = k - \frac{2M}{r^{d-3}} + \frac{Q^2}{4r^{2d-6}} + \frac{r^2}{L^2}$$

M and Q are the black hole mass and charge respectively. L is the AdS radius. k is the sectional curvature number, 1 for spherical, 0 for flat and -1 for hyperbolic symmetry. The Hawking temperature can be calculated from the equation, $T = f'(r_+)/4\pi$, where r_+ is the horizon of the black hole.

$$T = \frac{r_+^{-2d-1}}{16\pi L^2}\left[4(d-1)r_+^{2d+2} - (d-3)L^2(Q^2 r_+^6 - 4kr_+^{2d})\right]$$

The field potential, A_μ, in this system can be written in the form of the potential, $\Phi(r)$ as

$$A = \Phi(r)dt$$

By varying ψ and Φ in eq(1), the wave equations of ψ and Φ can be obtained from eq(1). The simple solution of the potential, Φ, for $\psi = 0$ is

$$\Phi = \sqrt{\frac{d-2}{2(d-3)}}\left(\frac{Q}{r^{d-3}} - \frac{Q}{r_+^{d-3}}\right)$$

The wave equation of the scalar field ψ

$$\left[\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - m_{eff}^2\right]\psi = 0 \qquad (2)$$

where $m_{eff}^2 = m^2 + g^{tt}q^2\Phi^2$. The wave equation is separable when its ansatz is

$$\psi = e^{-i\omega t}r^{(2-d)/2}R(r)S(x_i) \qquad (3)$$

where $S(x_i)$ is a harmonic function with eigenvalue $\lambda^2 = l(l+d-3)$, l=0,1,2,3… To understand the boundary conditions at the horizon and the infinity, the tortoise variable, $dr_* = dr/f(r)$, is introduced into the radial wave equation

$$\frac{d^2R(r)}{dr_*^2} + \left[\omega^2 - V(r)\right]R(r) = 0 \qquad (4)$$

where the effective potential *V(r)* is

$$V(r) = \frac{(d-2)(d-4)}{4r^2}f^2 + \frac{\lambda^2}{r^2}f + m_{eff}^2 f + \frac{d-2}{2r}ff'$$

At the horizon, $V(r \to r_+) \Rightarrow 0$, only the ingoing wave is allowed in this region $e^{-i\omega r_*} \approx (z-1)^{\alpha_1}$, whereas in the far away region $V(r \to \infty) \Rightarrow \infty$, the wave must be decayed.

Quasinormal Modes of AdS RN Black Holes

In our calculation, we set k=0 and d=4. To solve the wave equation in region, $r_+ \le r < \infty$, let define a new variable $z = \frac{r_+}{r}$. The wave equation (4) changes to

$$\begin{aligned}&z^2(z-1)(z-a_2)(z-a_3)(z-a_4)\times\\&\frac{d}{dz}(z-1)(z-a_2)(z-a_3)(z-a_4)\frac{dR}{dz}\\&+K(z)R = 0\end{aligned} \qquad (5)$$

where

$$\begin{aligned}K &= (2r_+/Q)^2(\omega r_+)^2 z^2\\&-\lambda^2(2r_+/Q)^2 z^2(z-1)(z-a_2)(z-a_3)(z-a_4)\\&-m^2r_+^2(2r_+/Q)^2(z-1)(z-a_2)(z-a_3)(z-a_4)\\&+4q^2r_+^2(2r_+/Q)^2 z^2(z-1)^2\\&-2(z-1)^2(z-a_2)^2(z-a_3)^2(z-a_4)^2\\&+z(z-1)(z-a_2)^2(z-a_3)^2(z-a_4)^2\\&+z(z-1)^2(z-a_2)(z-a_3)^2(z-a_4)^2\\&+z(z-1)^2(z-a_2)^2(z-a_3)(z-a_4)^2\\&+z(z-1)^2(z-a_2)^2(z-a_3)^2(z-a_4)\end{aligned}$$

and the new parameters in the equation are

$$\begin{aligned}a_2 &= A+B-p/3\\a_3 &= -(A+B)/2 + i\sqrt{3}/2(A-B)\\a_4 &= -(A+B)/2 - i\sqrt{3}/2(A-B)\end{aligned}$$

$$A,B = \left[\begin{array}{l}-\frac{1}{2}(\frac{2p^3}{3^3} - \frac{p^2}{3} + p)\\ \pm\sqrt{\frac{1}{4}(\frac{2p^3}{3^3} - \frac{p^2}{3} + p)^2 + \frac{1}{3^3}(p - \frac{p^2}{3})^3}\end{array}\right]^{1/3}$$

$$p = -\frac{4r_+^2}{Q^2L^2}$$

To simplify the equation, we let

$$R = z^{\alpha_0}(z-1)^{\alpha_1}(z-a_2)^{\alpha_2}(z-a_3)^{\alpha_3}(z-a_4)^{\alpha_4}F(z)$$

where

$$\alpha_0 = \frac{1}{2} \pm \frac{1}{2}\sqrt{9 + 4m^2L^2}$$

$$\alpha_1 = i\frac{\omega r_+(2r_+/Q)^2}{(1-a_2)(1-a_3)(1-a_4)}$$

$$\alpha_2 = i\left[\begin{array}{l}\dfrac{(\omega r_+)^2(2r_+/Q)^4}{(a_2-1)^2(a_2-a_3)^2(a_2-a_4)^2}\\ +\dfrac{4(qr_+)^2(2r_+/Q)^2}{(a_2-a_3)^2(a_2-a_4)^2}\end{array}\right]^{1/2}$$

$$\alpha_3 = i\left[\begin{array}{l}\dfrac{(\omega r_+)^2(2r_+/Q)^4}{(a_3-1)^2(a_3-a_2)^2(a_3-a_4)^2} \\ +\dfrac{4(qr_+)^2(2r_+/Q)^2}{(a_3-a_2)^2(a_3-a_4)^2}\end{array}\right]^{1/2}$$

$$\alpha_4 = i\left[\begin{array}{l}\dfrac{(\omega r_+)^2(2r_+/Q)^4}{(a_4-1)^2(a_4-a_2)^2(a_4-a_3)^2} \\ +\dfrac{4(qr_+)^2(2r_+/Q)^2}{(a_4-a_2)^2(a_4-a_3)^2}\end{array}\right]^{1/2}$$

We take the factor

$z^{\alpha_0+1}(z-1)^{\alpha_1+1}(z-a_2)^{\alpha_2+2}(z-a_3)^{\alpha_3+2}(z-a_4)^{\alpha_4+2}$

out of the wave equation (5) and change variable to $y=1-z$, near the horizon. We are going to calculate the zero order solution, where the frequency and charge, q, are small. Then the wave equation is reduced into the form of

$$y(1-y)\frac{d^2F}{dy}+(c-(a+b+1)y)\frac{dF}{dy}-JF=0$$

which is a hypergeometric function where

$$a,b=\alpha_0+\alpha_1+\frac{1+2\alpha_2}{1-a_2}+\frac{1+2\alpha_3}{1-a_3}+\frac{1+2\alpha_4}{1-a_4}$$
$$\pm\sqrt{(\alpha_0+\alpha_1+\frac{1+2\alpha_2}{1-a_2}+\frac{1+2\alpha_3}{1-a_3}+\frac{1+2\alpha_4}{1-a_4})^2-J} \tag{6}$$

$$J\approx 1+\alpha_0(1+2\alpha_1)-\frac{4m^2r_+^4/Q^2}{(1-a_2)(1-a_3)(1-a_4)}$$
$$+\frac{\alpha_1+\alpha_2}{1-a_2}+\frac{\alpha_1+\alpha_3}{1-a_3}+\frac{\alpha_1+\alpha_4}{1-a_4}$$

We keep only the linear terms of α, the small frequency and small charge in J, as the zero-order perturbation. At the horizon, y=0, the wave is only ingoing into black hole. Then the solution is

$$R(y)=y^{\alpha_1}F(a,b;c;y)$$

From the property of the hypergoemetric functions,

$$F(a,b;c;y)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-y)$$
$$+(1-y)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b-c+1;1-y) \tag{7}$$

In this far away zone the solution behavior is in the form of

$$R(z)\approx(\text{constant})z^{\alpha_{0+}}+(\text{constant})z^{\alpha_{0-}} \tag{8}$$

The solution must be finite in this area. However $\alpha_{0-}=1/2-1/2\sqrt{9+4m^2L^2}<0$ for $m^2L^2>-2$ this causes $z^{\alpha_{0-}}\to\infty$ as $z\to 0$. To eliminate the divergent term we set the constant in the front of the second term in eq(7) or eq(8) to be zero by letting the arguments of gamma function dominators be negative integer number, $n=0,1,2,3\ldots$, i.e.,

$$a=-n,\quad b=-n$$

or in term of $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4$ from eq(6),

$$n^2+n(\alpha_0+\alpha_1+\frac{1+2\alpha_2}{1-a_2}+\frac{1+2\alpha_3}{1-a_3}+\frac{1+2\alpha_4}{1-a_4})$$
$$+J=0 \tag{9}$$

Results and Discussion

The approximated quasinormal frequencies could be solved from Eq(9).

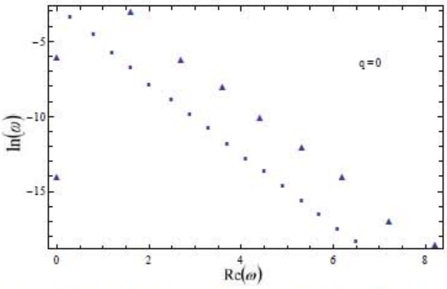

Figure 1 The frequency plot between the real part on the x-axis and the imaginary part on the y-axis where **q** =0, L=1.1, r₊ =Q=1 and m²L²=4. The markers ■ and ▲ represent our result and numerical result [6] respectively.

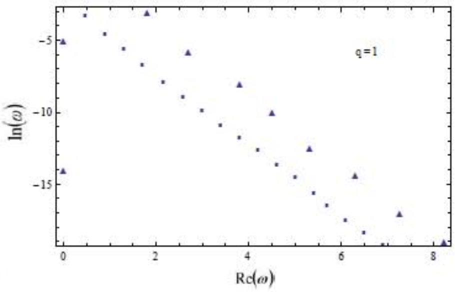

Figure 2 The frequency plot between the real part on the x-axis and the imaginary part on the y-axis where **q** =1, L=1.1, r_+ =Q=1 and m^2L^2=4. The markers ■ and ▲ represent our result and numerical result [6] respectively.

Our result is consistent with the numerical work. The slops of the both graphs in figure 1 and 2 are approximately -0.4. The space between the allowed frequencies is equally separated when the number n is large. The quasinormal frequencies are a discrete set of complex number with the equally spacing. This result is similar to those in many literatures, e.g. [7].

Conclusions

Our work aims to have better understanding analytically of the AdS black hole system, which is perturbed by a simply gauge field. We take k=0 and d=4 to simplify the problem. The zero-order perturbation gives the quasinormal modes and frequencies of the RN black holes interacting with the scalar field, where the frequencies and the charge q are taken to be small. Even thought our work is in agreement with the numerical result, there are things that can be enhanced to improve the result. For example, we could find a better approximated zero order solution for large number of q and other parameters, or continue to calculate the first order perturbation, etc.

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ANALYTICAL CALCULATION OF NON-ROTATING KALUZA-KLEIN BLACK HOLE QUASINORMAL MODES WITH SQUASHED HORIZONS

Supakchai Ponglertsakul,[1,*] Suphot Musiri,[2]

[1] Department of Physics, Faculty of Sciences, Chulalongkorn University Bangkok 10330, Thailand
[2] Department of Physics, Faculty of Sciences, Srinakharinwiroth University Bangkok 10110 Thailand
*e-mail: supakchai.p@gmail.com

Abstract: We study the Kaluza-Klein black holes, with squashed horizons. The system is perturbed by a massless scalar field. We analytically calculate the solution to the wave equation, which is the quasinormal modes. We approximate the non-rotating case, where the frequencies to solution are a set of discrete complex number. Our analytical result is very well in agreement with the numerical work. Both results imply the existence of the stable ground state and the few lower excited states.

Introduction: The extra dimensions have become mandatory to sustain or generalize the principles or theories in many physics areas of research. The Kaluza-Klein theory has combined the general relativity and electrodynamics into a 5-dimensional theory. Black holes contained in the general relativity can be viewed as singularities that dictate the spacetime curvature and the behaviors of particle and fields in the systems. The fields in the black hole systems must satisfy the boundary conditions that only ingoing wave at the horizons of the black holes and only outgoing wave or the vanishing wave at the infinity are allowed. The waves that obey these boundary conditions called quasinormal modes and their frequencies as quasinormal frequencies.[1] The gravitational wave signals from supermassive black holes are expected to be detected by Laser Interfermeter Space Antenna (LISA).[2] In this work, the non-rotating Kaluza-Klein black holes with the squashed horizon are studied and the quasinormal modes of these systems are analytically calculated, where we expect the frequencies to be a set of discrete complex number.

In this introduction section we describe the Kaluza-Klein black hole system and the squashed shape of the horizon. The wave equation in this curved spacetime is considered. The five-dimensional rotating Kaluza-Klein black holes with the squashed horizons have the metric in the form of [3]

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta}k^2 dr^2 + \frac{r^2+a^2}{4}[k(\sigma_1^2+\sigma_2^2)+\sigma_3^2] + \frac{\mu}{r^2+a^2}(dt - \frac{a}{2}\sigma_3)^2 \qquad (1)$$

where

$$\sigma_1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \qquad \sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi, \qquad \sigma_3 = d\psi + \cos\theta d\phi \qquad (2)$$

$$\Sigma(r) = r^2(r^2+a^2), \qquad (3)$$

$$\Delta(r) = (r^2+a^2)^2 - \mu r^2 \qquad (4)$$

$$k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}. \qquad (5)$$

Parameters μ and a correspond to black hole mass and angular momentum respectively. The variables have the range $0 \le r \le r_\infty$, $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$, where the radius has the upper limit r_∞ The sphere of the horizon is squashed by the term $k(r)$. The singularities of the metric are obtained by setting $\Delta(r_\pm)=0$. r_+ and r_- are outer and inner singularities respectively, which are depend on μ and a as

$$r_\pm=\sqrt{\frac{(\mu-2a^2)\pm\sqrt{\mu^2-4a^2\mu}}{2}} \tag{6}$$

Since the metric is apparently singular at $r=r_\infty$, the radial coordinate r is restricted within the range $0<r<r_\infty$. Note that, if we take $r_\infty\to\infty$ which causes $k(r)\to 1$ therefore our metric reduces to five-dimensional Kerr black hole. Moreover, the shape of the event horizon is also characterized by the function $k(r_+),k(r_-)$.[3]

The metric above describes the space-time geometry of the rotating squashed Kaluza-Klein black hole which looks like a five-dimensional squashed black hole near the horizons, and like the Kaluza-Klein geometry at $r\to r_\infty$. To see the asymptotic behavior of this metric, let us define a new radial coordinate [4]

$$\rho=\rho_0\frac{r^2}{r_\infty^2-r^2} \tag{7}$$

where

$$\rho_0=\sqrt{\frac{k_0(r_\infty^2+a^2)}{4}}, \text{ and } \qquad k_0=k(r=0)=\frac{(r_\infty^2+a^2)^2-\mu r_\infty^2}{r_\infty^4} \tag{8}$$

In a new radial coordinate ρ varies from 0 to ∞ while r varies from 0 to r_∞. By transforming (1) via a new coordinate (7) and take limit $\rho\to\infty$, the metric becomes

$$ds^2=-dt^2+d\rho^2+\rho^2(\sigma_1^2+\sigma_2^2)+\frac{r_\infty^2+a^2}{4}\sigma_3^2+\frac{\mu}{r^2+a^2}(dt-\frac{a}{2}\sigma_3)^2 \tag{9}$$

To remove the cross-term between dt and σ_3, let define new coordinates as

$$\tilde{\psi}=\psi-\frac{2\mu a}{(r_\infty^2+a^2)^2+\mu a^2}t$$
$$\tilde{t}=\sqrt{\frac{(r_\infty^2+a^2)^2-\mu a^2}{(r_\infty^2+a^2)^2+\mu a^2}}t \tag{10}$$

and define a new notation $\tilde{\sigma}_3=d\tilde{\psi}+\cos\theta d\phi$, and replace all these new coordinates and notation into (9). Then, the asymptotic structure of the rotating squashed Kaluza-Klein black hole is revealed

$$ds^2=-d\tilde{t}^2+d\rho^2+\rho^2(\sigma_1^2+\sigma_2^2)+\frac{(r_\infty^2+a^2)^2+\mu a^2}{4(r_\infty^2+a^2)}\tilde{\sigma}_3^2 \tag{11}$$

The first three terms on the RHS of (11) represent a four dimensional Minkowski spacetime while the rest is a twisted S^1 bundle. The size of the compacified dimension at infinity is also obtained [4]

$$r_\infty'^2=\frac{(r_\infty^2+a^2)^2+\mu a^2}{r_\infty^2+a^2} \tag{12}$$

The size of the extra dimension depends on three parameters r_∞,μ and a. Note that, for $a\to 0$ or $r_\infty^2>>a^2$, the radius of the compactified dimension (12) could be interpreted by r_∞.

Klein-Gordon equation in curved background

Our aim is to study a scalar field which evolves in the rotating squashed black hole spacetime. Hence, we need to construct equation of motion for a scalar field in curved background. An equation of motion for a real scalar field is so called Klein-Gordon equation. To derive a Klein-Gordon equation in a curved background, let consider an action for a single scalar field in curved spacetime

$$s=\int\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}(\nabla_{\mu}\Phi)(\nabla_{\nu}\Phi)-\frac{1}{2}m^{2}\Phi^{2}\right]d^{4}x \qquad (13)$$

Here m stands for mass of a scalar field, where g is determinant of the metric tensor. For a scalar field case, it is possible to replace the covariant derivative with an ordinary partial derivative. By varying this action with respect to the scalar field, we obtain an equation of motion for a scalar field in curved background.

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}\,g^{\mu\nu}\partial_{\nu}\Phi\right)-m^{2}\Phi=0 \qquad (15)$$

This equation will be used as an important part of our calculation in the next section. Its solution describes the behavior of the scalar field in the curved spacetime. For simplicity in our work, we consider only for a massless scalar field.

Methodology: In this section we study the wave equation of the perturbed scalar field. The equation is separable. We analytically solve the radius wave equation. We obtain and present the frequency-constrain equation which correspond to the each quasinormal mode.

Equation of motion for a real scalar field in a rotating squashed Kaluza-Klein black hole

We are going to calculate an equation of motion for a scalar particle in our particular metric (1). First, we have defined the proper time $dt=Bd\tau$ and $B\equiv\frac{(r_{\infty}^{2}+a^{2})^{2}}{2\rho_{0}r_{\infty}^{3}}$ is a constant.[5]

The former metric (1) becomes

$$ds^{2}=-B^{2}d\tau^{2}+\frac{\Sigma}{\Delta}k^{2}dr^{2}+\frac{r^{2}+a^{2}}{4}[k(\sigma_{1}^{2}+\sigma_{2}^{2})+\sigma_{3}^{2}]+\frac{\mu}{r^{2}+a^{2}}(Bd\tau-\frac{a}{2}\sigma_{3})^{2} \qquad (16)$$

Therefore, we can calculate components of metric tensor $g_{\mu\nu}$, and its inverse $g^{\mu\nu}$ as

$$g_{\mu\nu}=\begin{pmatrix}
-\left(1-\frac{\mu}{r^{2}+a^{2}}\right)B^{2} & 0 & 0 & -\frac{a\mu\cos\theta}{2(r^{2}+a^{2})}B & -\frac{a\mu}{2(r^{2}+a^{2})}B \\
0 & \frac{\Sigma k^{2}}{\Delta} & 0 & 0 & 0 \\
0 & 0 & \frac{k^{2}(r^{2}+a^{2})}{4} & 0 & 0 \\
-\frac{a\mu\cos\theta}{2(r^{2}+a^{2})}B & 0 & 0 & \frac{r^{2}+a^{2}}{4}(k\sin^{2}\theta+\cos^{2}\theta)+\frac{\mu a^{2}\cos^{2}\theta}{4(r^{2}+a^{2})} & \frac{\cos\theta}{4}\left(\frac{(r^{2}+a^{2})^{2}+\mu a^{2}}{r^{2}+a^{2}}\right) \\
-\frac{a\mu}{2(r^{2}+a^{2})}B & 0 & 0 & \frac{\cos\theta}{4}\left(\frac{(r^{2}+a^{2})^{2}+\mu a^{2}}{r^{2}+a^{2}}\right) & \frac{(r^{2}+a^{2})^{2}+a^{2}\mu}{4(r^{2}+a^{2})}
\end{pmatrix}$$

$$g^{\mu\nu}=\begin{pmatrix} -\left(\frac{(r^2+a^2)^2+a^2\mu}{(r^2+a^2)^2-r^2\mu}\right)\frac{1}{B^2} & 0 & 0 & 0 & -\left(\frac{2a\mu}{(r^2+a^2)^2-r^2\mu}\right)\frac{1}{B} \\ 0 & \frac{\Delta}{\Sigma k^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{k^2(r^2+a^2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{k(r^2+a^2)\sin^2\theta} & -\frac{4\cos\theta}{k(r^2+a^2)\sin^2\theta} \\ -\left(\frac{2a\mu}{(r^2+a^2)^2-r^2\mu}\right)\frac{1}{B} & 0 & 0 & -\frac{4\cos\theta}{k(r^2+a^2)\sin^2\theta} & \frac{4\cos^2\theta}{k(r^2+a^2)\sin^2\theta}+\frac{4(r^2+a^2-\mu)}{(r^2+a^2)^2-r^2\mu} \end{pmatrix}$$

where $\sqrt{-g}=\frac{k^2\sin\theta B}{8}\sqrt{(r^2+a^2)\Sigma}$, In our calculation, we denote spacetime indices by $(\tau,r,\theta,\phi,\psi)\to(0,1,2,3,4)$. It is convenient to use a new radial coordinate ρ which defined by (7). Then, take the ansatz for a scalar field $\Phi(\tau,\rho,\theta,\phi,\psi)=e^{-i\omega\tau}R(\rho)e^{im\phi+i\lambda\psi}S(\theta)$, where $S(\theta)$ is the spheroidal harmonics. After change to a new radial coordinate and put the ansatz into (15), we can separate the angular variable part from the radial and the time parts. For the angular part, it reads [5]

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}\right]S(\theta)-\left[\frac{(m-\lambda\cos\theta)^2}{\sin^2\theta}-E_{lm\lambda}\right]S(\theta)=0 \tag{17}$$

Here the eigenvalue of the angular equation is $E_{lm\lambda}=l(l+1)-\lambda^2$. The parameters l is the angular number of the variable θ and λ is integer number for the 5th dimension represented by the angle variable, ψ. However, the time variable can be removed in the final step of the calculation. So, the only remaining part is the radial component which takes the form

$$\Theta\frac{d^2R(\rho)}{d\rho^2}+\frac{d\Theta}{d\rho}\frac{dR(\rho)}{d\rho}+\left[\frac{\tilde{\mathrm{N}}^2}{\Theta}+\Lambda-l(l+1)+\lambda^2\right]R(\rho)=0 \tag{18}$$

and

$$\begin{aligned}
\Theta(\rho)&=\frac{(r_\infty^2+a^2)}{4r_\infty^4\rho_0^2}\left[(\rho r_\infty^2+a^2(\rho+\rho_0))^2-\mu\rho(\rho+\rho_0)r_\infty^2\right],\\
\tilde{\mathrm{N}}^2&=\frac{\mu r_\infty^2(\rho+\rho_0)^4}{N^4(r_\infty^2+a^2)^2}\left[\omega-\frac{\lambda a\mathrm{N}^2(r_\infty^2+a^2)}{\rho_0 r_\infty^3}\right]^2,\\
\Lambda&=\frac{4\rho_0^2r_\infty^6(\rho+\rho_0)^2\omega^2}{\mathrm{N}^2(r_\infty^2+a^2)^4}-\frac{4\lambda^2(\rho+\rho_0)^2}{r_\infty^2+a^2},\\
\mathrm{N}^2&=\frac{\rho+\rho_0}{\rho+\frac{a^2}{r_\infty^2+a^2}\rho_0}.\\
\rho_+&=\rho_0\frac{r_+^2}{r_\infty^2-r_+^2},\\
r_\infty^2&=\frac{(\mu-2a^2)\pm\sqrt{\mu^2-4a^2\mu}}{2}.
\end{aligned} \tag{19}$$

In order to obtain the quasinormal frequencies ω, we have to solve (18) under certain boundary conditions as mentioned before. In this work our limit ourselves to our the case of non-rotation $a=0$ and the equation of motion becomes

$$\rho(\rho-\rho_+)^2\frac{d^2R(\rho)}{d\rho^2}+(\rho-\rho_+)(2\rho-\rho_+)\frac{dR(\rho)}{d\rho}+\frac{\mu\omega^2\rho(\rho+\rho_0)^2}{r_\infty^2}R(\rho)+(\rho-\rho_+)\left[\frac{4\rho_0^2(\rho+\rho_0)\rho\omega^2}{r_\infty^2}-\frac{4\lambda^2(\rho+\rho_0)^2}{r_\infty^2}-E_{lm\lambda}\right]R(\rho)=0 \tag{20}$$

To simplify the solution let us separate the singularity at horizon by writing the $R(\rho)$ function as,

$$R(\rho)=(\rho-\rho_+)^\alpha F(\rho)$$

where
$$\alpha\equiv-\frac{i\omega r_\infty}{2\left(\frac{r_\infty^2}{\mu}-1\right)^{1/2}} \tag{21}$$

The minus sign of α represents the incoming wave at the horizon. After substituted $R(\rho)$ in equation (20) the wave equation changes to

$$\rho(\rho-\rho_+)\frac{d^2F(\rho)}{d\rho^2}+2\alpha\rho\frac{dF(\rho)}{d\rho}+(2\rho-\rho_+)\frac{dF(\rho)}{d\rho}+\left(\alpha(\alpha-1)+2\alpha\right)F(\rho)+$$
$$\frac{\mu\omega^2}{r_\infty^2}\left[(\rho-\rho_+)^2+2(\rho-\rho_+)(\rho_++\rho_0)+\rho_+(\rho-\rho_+)+2\rho_+(\rho_++\rho_0)+(\rho_++\rho_0)^2\right]F(\rho)+$$
$$\left[\frac{4\rho_0^2(\rho_++\rho_0)\rho\omega^2}{r_\infty^2}-\frac{4\lambda^2(\rho+\rho_0)^2}{r_\infty^2}-E_{lm\lambda}\right]F(\rho)=0 \tag{22}$$

Let's define a new variable $x=\dfrac{\rho}{\rho_+}$, where $x=1$ at the horizon. The radius wave equation is reduced to

$$x(1-x)\frac{d^2F(x)}{dx^2}+\left[1-(2\alpha+2)x\right]\frac{dF(x)}{dx}+\left[\alpha'+\beta'x+\gamma'x^2\right]F(x)=0 \tag{23}$$

$$\alpha'=\frac{4\lambda^2\rho_0^2}{r_\infty^2}+E_{lm\lambda}-\alpha$$
$$\beta'=-\frac{\mu\omega^2}{r_\infty^2}\rho_+(\rho_++\rho_0)-\rho_0\rho_+\omega^2+\frac{8\lambda^2\rho_+\rho_0}{r_\infty^2} \tag{24}$$
$$\gamma'=-\omega^2\rho_+^2+\frac{4\lambda^2\rho_+^2}{r_\infty^2}$$

To further simplify the solution let us define $F=e^{Cx}H(x)$, where we choose $C^2=-\gamma'$. Then the above equation, (23) change to

$$x(x-1)\frac{d^2H}{dx^2}+\left[2Cx(x-1)-1+2(1+\alpha)x\right]\frac{dH}{dx}+(\alpha'-C)H+\left[-C^2+2C+2C\alpha+\beta'\right]xH=0 \tag{25}$$

Change the variable from x to $v=x-1$, where at the horizon $v=0$. Next divide the equation with $-2Cx$ and the wave equation becomes

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$$(-2Cv)\frac{d^2H}{d(-2Cv)^2}+\left[1+2\alpha-(-2Cv)+1-\frac{1}{x}\right]\frac{dH}{d(-2Cv)}-\left[1+\alpha+\frac{\beta'+\gamma'}{2C}\right]H-\frac{\alpha'-C}{2Cx}H=0 \tag{26}$$

The wave equation, near the horizon, $x=1$, can be approximated as

$$(-2Cv)\frac{d^2H}{d(-2Cv)^2}+\left[1+2\alpha-(-2Cv)\right]\frac{dH}{d(-2Cv)}-\left[\frac{1}{2}+\alpha+\frac{\alpha'+\beta'+\gamma'}{2C}\right]H=0 \qquad (27)$$

The solutions to the above wave equation, (27) are the confluent hypergeometric function. However we need only the ingoing solution to the horizon in order to satisfy the boundary condition

$$R=e^{Cx}(\rho-\rho_+)^{\alpha}F\left(\hat{a};\hat{b};-2Cv\right) \qquad (28)$$

where $\hat{a}=\frac{1}{2}+\alpha+\frac{\alpha'+\beta'+\gamma'}{2C}$ and $\hat{b}=1+2\alpha$. In the region the value of x approaching infinity the solution must be vanishing into order to satisfy the other boundary condition where the potential in this region diverged. We can approximate the solution at the infinity by using the property of the confluent hypergeometric function

$$R(v\to\infty)\approx e^{Cx-i\pi\hat{a}}(-2C)^{-\hat{a}}v^{\alpha-\hat{a}}\frac{1}{\Gamma(\hat{b}-\hat{a})}+e^{-Cx}(-2C)^{\hat{a}-\hat{b}}v^{\alpha+\hat{a}-\hat{b}}\frac{1}{\Gamma(\hat{a})} \qquad (29)$$

As $v\to\infty$, the first term decays, while the second term diverges. To get rid of the divergence, set the argument of the Gamma function to be negative integer.

$$\hat{a}=\frac{1}{2}+\alpha+\frac{\alpha'+\beta'+\gamma'}{2C}=-n, \qquad n=0,1,2,3,... \qquad (30)$$

The above equation, (30), is a constrain equation for the frequencies, where it can be written down in term of a dimensionless $\omega\rho_+$ parameter as

$$\left(2n+1-\frac{2i\omega\rho_+r_\infty}{\sqrt{\mu}}\right)\sqrt{-(\omega\rho_+)^2+\frac{\lambda^2\mu}{r_\infty^2\left(r_\infty^2-\mu\right)}}-\frac{i\omega\rho_+r_\infty}{\sqrt{\mu}}+(\omega\rho_+)^2\left(1+\frac{r_\infty^2}{\mu}\right)-\frac{\lambda^2\mu}{r_\infty^2-\mu}-l(l+1)=0$$

(31)

The quasinormal frequencies can be solved from this equation. There are four roots to equation (31), but there is only one that gives positive real number and negative imaginary number, corresponding to the ingoing wave at the horizon and decaying wave at the far away region respectively. We will compare our result with numerical work in the next section

Results, Discussion and Conclusion: The quasinormal frequencies are obtained from equation (31). We put some specific parameters in order to compare with the numerical result as the following [6]

Table 1 The quasinorma frequencies from WKB method and our analytical result, where $l=10$, $\rho_0/\rho_+=3$ or $r_\infty/r_+=2$ and $n=2$

λ	WKB[6]	Analytical work
0	3.5149 - 0.15956i	3.4921 - 0.16667i
0.5	3.5336 - 0.15821i	3.4940 - 0.16643i
1	3.5898 - 0.15413i	3.4999 - 0.16572i
1.5	3.6842 - 0.14729i	3.5098 - 0.16455i
2	3.8178 - 0.13759i	3.5236 - 0.16291i
2.5	3.9924 - 0.12493i	3.5413 - 0.16083i
3	4.2103 - 0.10912i	3.5629 - 0.15832i
3.5	4.4747 - 0.08988i	3.5883 - 0.15539i

From both results the imaginary part is negative, causing the wave vanishing in the far away zone. As λ increases, the real part increases while the imaginary part decreases.

Our frequencies change more slowly than WKB result when λ increases. To improve our frequency result, we can take our approximated solution as the zero order perturbation. We can continue to perform the first order. Also we can add the rotation $a \neq 0$ to black hole to the problem in our future work.

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