



รายงานวิจัยฉบับสมบูรณ์

ความเป็นอามีนาเบิลของพืชชนิดตัวดำเนินการ
บนกรุปกระชั้นเฉพาะที่

โดย สมลักษณ์ อุดูดี

กันยายน 2561

สัญญาเลขที่ MRG5380028

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ผู้วิจัย สมลักษณ์ อุดดี

ต้นสังกัด คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยเชียงใหม่

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. และต้นสังกัดไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

พีชคณิตตัวดำเนินการคือพีชคณิตย่อยของตัวดำเนินการเชิงเส้นที่มีขอบเขตบนปริภูมิฮิลเบิร์ตเชิงซ้อนซึ่งเป็นเซตปิดภายใต้ตัวดำเนินการผูกพัน มีคลาสของพีชคณิตตัวดำเนินการที่พิจารณาสองคลาสคือ พีชคณิตซีสตาร์และพีชคณิตฟอนนอยมันน์

ให้ G กลุ่มกระชับเฉพาะที่และ $L^1(G)$ แทนปริภูมิของฟังก์ชันที่สามารถอินทิเกรตได้แบบสัมบูรณ์เทียบกับเมเชอร์ฮาร์ พีชคณิตซีสตาร์กรุปของ $G, C^*(G)$, เป็นส่วนปิดคลุมของตัวแทนสากลของ $L^1(G)$ ให้ $L^2(G)$ เป็นปริภูมิฮิลเบิร์ตของคลาสของฟังก์ชันที่ยกกำลังสองแล้วสามารถอินทิเกรตได้เทียบกับเมเชอร์ฮาร์

ให้ $(\lambda, L^2(G))$ เป็นตัวแทนปรกติทางซ้ายของ $L^1(G)$ พีชคณิตซีสตาร์กรุปลดรูป เป็นส่วนปิดคลุมภายใต้ไตนอร์มของ $\lambda(L^1(G))$ ใน $B(L^2(G))$ และแทนด้วย $C_r^*(G)$ พีชคณิตฟอนนอยมันน์กรุปของ G เป็นส่วนปิดคลุมแบบอ่อนของ $\lambda(L^1(G))$ ใน $B(L^2(G))$ และแทนด้วย $VN(G)$

เงื่อนไขความเป็นอามีนาเบิลสมมูลกับการจำแนกในหลากหลายแง่มุมซึ่งสะท้อนถึงการประยุกต์อย่างกว้างขวางของความเป็นอามีนาเบิล ในช่วงหลายปีที่ผ่านมา นักคณิตศาสตร์หลายท่านได้พิจารณาความเป็นอามีนาเบิลของพีชคณิตบานาคที่สร้างเหนือกลุ่มกระชับเฉพาะที่

งานวิจัยหลายผลงานถูกตีพิมพ์ งานเหล่านี้ให้นิยามที่สมมูลกันของความเป็นอามีนาเบิลและแนะนำความเป็นอามีนาเบิลในหลายรูปแบบของพีชคณิต เช่น ความเป็นอามีนาเบิล ความเป็นอามีนาเบิลแบบอ่อน ความเป็นอามีนาเบิลแบบเข้ม ความเป็นอามีนาเบิลไอดีล ความเป็นอามีนาเบิลสูงยิ่ง ความเป็นอามีนาเบิลสมมาตร เป็นต้น

ในงานวิจัยนี้ เราจะสำรวจความเป็นอามีนาเบิลของพีชคณิตที่นิยามบนกลุ่มกระชับเฉพาะที่ จะพิสูจน์ทฤษฎีบทเกี่ยวกับสมบัติของพีชคณิตตัวดำเนินการอามีนาเบิลบนกลุ่มกระชับเฉพาะที่ จะหาเงื่อนไขความเป็นอามีนาเบิลใหม่ที่สมมูลสำหรับพีชคณิตตัวดำเนินการ และจะหาความสัมพันธ์ของความเป็นอามีนาเบิลชนิดต่าง ๆ ของพีชคณิต

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คำสำคัญ: Amenability, Operator algebras, Locally compact group

Abstract

By operator algebras we mean subalgebras of bounded linear operators on a complex Hilbert space which are closed under the adjoint operation. There are two main classes of operator algebras that will be considered : C^* -algebras and von Neumann algebras.

Let G denote a locally compact group and $L^1(G)$ be the space of absolutely integrable functions with respect to Haar measure. The **group C^* -algebra** of G , $C^*(G)$, is the closure of the universal representation of $L^1(G)$. Let $L^2(G)$ be the Hilbert space of (class of) square integrable functions with respect to the Haar measure.

Let $(\lambda, L^2(G))$ be the left regular representation of $L^1(G)$. The **reduced group C^* -algebra** is the norm closure of $\lambda(L^1(G))$

in $B(L^2(G))$ and is denoted by $C_r^*(G)$. The **group von Neumann algebra** of G , is the weak closure of $\lambda(L^1(G))$ in $B(L^2(G))$ is denoted by $VN(G)$.

The amenability condition has many equivalent characterizations which reflect the wide range of applications of amenability. Over recent years, many mathematicians have considered the amenability of Banach algebras constructed over locally compact groups.

A number of papers have been published which give equivalent definitions of amenability and introduced various kinds of amenability of algebras, for examples, amenability, weak amenability, strong amenability, ideal amenability, ultra-amenability, symmetric amenability, etc..

In this research, we will investigate amenability of the certain algebras define on locally compact groups, prove theorems about properties of amenable operator algebras on locally compact groups, find new equivalent conditions for an operator algebra to be amenable and find relationships of various kinds of amenability of algebras.

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Project Title: Amenability of Operator Algebras on Locally Compact Groups

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Keywords: Amenability, Operator algebras, Locally compact group

Introduction to the research problem and its significance:

By operator algebras we mean subalgebras of bounded operators on a complex Hilbert space H which are closed under the adjoint operation $A \mapsto A^*$. There are two important classes of operator algebras : C^* -algebras and von Neumann algebras.

- C^* -algebras are operator algebras closed with respect to the uniform topology (the topology) defined by the operator norm.
- Von Neumann algebras are operator algebras closed with respect to the weak operator topology.

Let G denote a locally compact group; i.e., G is a group and a locally compact Hausdorff space and the map $(s,t) \mapsto s^{-1}t$ is continuous from $G \times G$ to G . Then there exists a nonzero left invariant Radon measure μ_G on G ; i.e. $\mu_G(sE) = \mu_G(E)$ for each Borel set E in G and each s in G . We call μ_G the **left Haar measure** on G (unique up to scalar multiples).

Let $L^1(G)$ be the space of absolutely integrable functions with respect to Haar measure. The **group C^* -algebra** of G is the closure of the universal representation of $L^1(G)$. That is, take π_u be the direct sum of all irreducible representations (up to unitary equivalence) of G . Define a C^* -norm on $L^1(G)$ by

$$\|f\| = \sup \{ \|\pi(f)\| \mid \pi \text{ is a } * \text{-representation of } L^1(G) \}.$$

The group C^* -algebra $C^*(G)$ is the completion of $L^1(G)$ in this norm.

Let $L^2(G)$ be the Hilbert space of (class of) square integrable functions with respect to the Haar measure. For each μ in $M(G)$ and f in $L^2(G)$ we can define the convolution $\mu \times f$ in $L^2(G)$. The map $\lambda : M(G) \rightarrow B(L^2(G))$ given by $\lambda(\mu)f = \mu \times f$ is a representation of $M(G)$ called the **(left) regular representation**.

The **reduced group C^* -algebra** is the norm closure of $\lambda(L^1(G))$ in $B(L^2(G))$ and is denoted by $C_r^*(G)$.

The **group von Neumann algebra** is the weak closure of $\lambda(L^1(G))$ in $B(L^2(G))$ is denoted by $VN(G)$. Thus $VN(G) = C_r^*(G)''$. Then $VN(G)$ is also the von Neumann algebra generated by $\lambda(M(G))$ or by $\lambda(G)$.

We denote by $A(G)$ the subspace of $B(G)$ consisting of functions whose associated functionals belong to $VN(G)_*$.

The set of functions f in $L^1(G)$ such that $f \geq 0$ and $\|f\|_1 = 1$ will play an important role in the following. We denote it by $S(G)$ and note that it can be identified with the set the normal states on $L^\infty(G)$ and that it is semigroup under convolution.

There are many alternative formulations of the notion of amenability of groups and algebras.

Amenability of a locally compact group

The amenability condition has many equivalent characterizations which reflect the wide range of applications of amenability.

Let G be a locally compact group. The following are equivalent.

- (a) G is amenable.
- (b) any $*$ -representation of $C^*(G)$ is weakly contained in its left regular representation of $C^*(G)$ is the unique extension of the left regular representation of $L^1(G)$;
- (c) the left regular representation of $C^*(G)$ is faithful;
- (d) $C^*(G) = C_r^*(G)$.
- (e) There is a nonzero left invariant functional (continuous or discontinuous) on $L^\infty(G)$.
- (f) For every probability measure in $M(G)$ the norm of the operator $f \mapsto u \times f, L^2(G) \rightarrow L^2(G)$, is equal to 1.
- (g) It has the fixed point property, i.e., every continuous and affine action of G

on a compact, convex subset C of a locally convex topological linear space has a fixed point.

(h) It has a certain Hahn-Banach type extension property.

(i) $\{f \in L^1(G) \mid \int_G f d\mu = 0\}$ has a left bounded approximate identity.

(j) There is a net $\{g_i\}$ in $S(G)$ such that $\|h \times g_i - g_i\| \rightarrow 0$ weak* in $(L^\infty(G))^*$ for each h in $S(G)$.

(k) There is a net $\{g_i\}$ in $S(G)$ such that $\|h \times g_i - g_i\| \rightarrow 0$ for each h in $S(G)$;

(l) For each compact set C and $\varepsilon > 0$ there is a g in $S(G)$ such that $\|\lambda_s g - g\| < \varepsilon$ for every s in C .

The class of groups containing abelian and compact groups and closed under extensions, dense homomorphic images, subgroups and direct limit is the class of elementary amenable groups. Amenable groups which are not elementary were considered by Grigorchuk.

Amenability of a C^* -algebra

Let A be a Banach algebra, then A is **amenable** if $H^1(A, X^*) = \{0\}$ for each Banach A -module X ; this definition was introduced by Johnson.

The Banach algebra A is **weakly amenable** if $H^1(A, A^*) = \{0\}$; this definition generalizes that introduced by Bade, Curtis and Dales.

Let A be a Banach algebra and let I be a closed two-sided ideal in A , A is **I-weakly amenable** if $H^1(A, I^*) = \{0\}$.

Let A be a Banach algebra and let I be a closed two-sided ideal in A , A is **n-I-weakly amenable** if $H^1(A, I^{(n)}) = \{0\}$.

Let A be a Banach algebra. A is **ideally amenable** if A is I -weakly amenable for every closed two-sided ideal I in A . This definition was introduced by Eshghi Gordji and Yazdanpanah.

A Banach algebra is called **symmetrically amenable** if it has a symmetric approximate diagonal. The group algebra $L^1(G)$ of a locally compact group is symmetrically amenable if and only if it is amenable whereas the Cuntz algebras $\mathcal{O}_n$ for $n \in \mathbb{N}, n \geq 2$ are amenable but not symmetrically amenable.

A Banach algebra U is said to be **ultra-amenable** if U_n is amenable for every ultrafilter U . Ultra-amenability implies amenability but is much stronger : a C^* -algebra is ultra-amenable if only if it is subhomogeneous and $L^1(G)$, for a discrete group G is ultra-amenable if and only if G is finite.

A C^* -algebra A is said to be **strongly amenable** if, whenever X is a Banach A -module and D is a derivation of A into X^* , there is a $f \in \overline{\{D(u)u^* \mid u \in (A_e)\}}$ with $D = \delta(f)$, where A_e is the C^* -algebra obtained by adjoining the identity e to A , X is made into a unital A_e -module by defining $xe = ex = x$ for all $x \in X$, D is extended to A_e by defining $D(e) = 0$, $U(A_e)$ is the unitary group of A_e , and $\overline{S}$ denotes the weak*-closed convex hull of a set S contained in X^* .

A C^* -algebra A is **strongly amenable** if and only if A_e is strongly amenable, and a C^* -algebra with identity is **strongly amenable** if and only if the definition is satisfied for all unital A -modules X with A_e replaced throughout by A .

The class of strongly amenable C^* -algebras includes all C^* -algebras which are **GCR**, uniformly hyperfinite, or the C^* -group algebra of a locally compact amenable group.

It is not known if there exist amenable C^* -algebras which are not strongly amenable.

For A a C^* -algebra, let $\hat{A} \otimes A$ be the completion of the algebraic tensor product $A \otimes A$ in the greatest cross-norm.

Let A be a C^* -algebra with identity e . Then the following seven statements are equivalent:

- (a) A is strongly amenable.
- (b) For all unital Banach A -modules X and $f \in X^*$, there exists $g \in \overline{\{ufu^* : u \in U(A)\}}$ such that $ag = ga$ for all $a \in A$.
- (c) For any $f \in (A \hat{\otimes} A)^*$ there exists $g \in \overline{\{ufu^* : u \in U(A)\}}$ such that $ag = ga$ for all $a \in A$.
- (d) There is a linear map T of $(A \hat{\otimes} A)^*$ into $C = \{g \in (A \hat{\otimes} A)^* : ag = ga \text{ for all } a \in A\}$ such that $T(a^\circ f) = a^\circ T(f)$, $T(f^\circ a) = T(f)^\circ a$, and $T(f) \in \overline{\{ufu^* : u \in U(A)\}}$ for all $a \in A, f \in (A \hat{\otimes} A)^*$.
- (e) Let X be a Banach A -module, S a w^* -closed convex subset of X^* such that $usu^* \in S$ for all $s \in S, u \in U(A)$. Then there exists an element $s \in S$ such that $usu^* = s$ for all $u \in U(A)$.
- (f) Let Y be a Banach A -module and X a subspace of Y such that $uxu^* \in X$ for all $x \in X, u \in U(A)$. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends f , there is an $h \in \overline{\{ugu^* : u \in U(A)\}}$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y, u \in U(A)$.
- (g) Let Y be a Banach A -module and X a two-sided A -submodule of Y . Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends f , there is an $h \in \overline{\{ugu^* : u \in U(A)\}}$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y, u \in U(A)$.

Let A be a C^* -algebra with unit e . Then the following three statements are equivalent:

- (a) A is amenable.

(b) There is a bounded linear map T of $(A \hat{\otimes} A)^*$ into $C = \{f \in (A \hat{\otimes} A)^* : af = fa \text{ for all } a \in A\}$ such that T restricted to C is the identity on C and $T(a^\circ f) = a^\circ T(f)$, $T(f^\circ a) = T(f)^\circ a$ for all $a \in A, f \in (A \hat{\otimes} A)^*$.

(c) Let Y be a Banach A -module and X a two-sided A -submodule of Y . Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then there is a $h \in Y^*$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y, u \in U(A)$.

Amenability of a von Neumann algebra

Connes (1976) and others proved that the following conditions on a von Neumann algebra M on a separable Hilbert space H are all equivalent:

(a) M is **hyperfinite or AFD or approximately finite dimensional or approximately finite**: this means the algebra contains an ascending sequence of finite dimensional subalgebras with dense union. (Warning: some authors use hyperfinite to mean AFD and finite.)

(b) M is **amenable**: this means that the derivations of M with values in a normal dual Banach bimodule are all inner.

(c) M has **Schwartz's property P**: for any bounded operator T on H the weak operator closed convex hull of the elements uTu^* contains an element commuting with M .

(d) M is **semidiscrete**: this means the identity map from M to M is a weak pointwise limit of completely positive maps of finite rank.

(e) M has **property E or the Hakeda-Tomiyama extension property**: this means that there is a projection of norm 1 from bounded operators on H to M' .

(f) M is **injective**: any completely positive linear map from any self adjoint closed subspace containing 1 of any unital C^* -algebra A to M can be extended to a completely positive map from A to M .

There is no generally accepted term for the class of algebras above; Connes has suggested that amenable should be the standard term.

Various authors have considered the amenability of Banach algebras constructed over locally compact groups. Locally compact groups are deeply related to their group C^* -algebras and group von Neumann algebras. For example, a locally compact group is amenable if and only if the group algebra is amenable as a Banach algebra and it is known that a discrete group G is amenable if and only if its reduced group C^* -algebra $C_r^*(G)$ is a nuclear C^* -algebra. This is also equivalent to that the group von Neumann algebra $VN(G)$ is injective (or hyperfinite).

Amenability is very important for many aspects of group theory and in particular for the study of representations. For examples, if G is amenable, then each bounded representation of G on a Hilbert space is equivalent to a unitary representation. Also, G is amenable if and only if each of its irreducible, unitary-representations is weakly contained in the regular representation on $L^2(G)$. Also, the generalized Stone-Weierstrass Theorem for separable C^* -algebras is true when the C^* -subalgebra is strongly amenable.

In this research we shall study the interactions between amenability properties of groups and group actions and amenability properties of the corresponding operator algebras and we will continue these investigations and prove the new results relating amenability and the representation theory of the object concerned.

Literature review:

In 1972, B.E.Johnson [11] initiated the theory of amenable Banach algebras. A locally compact group G is amenable in the usual sense if and only if its group algebras $L^1(G)$ is an amenable Banach algebra. Ever since the paper of Johnson was published, there have been ongoing to determine, for particular classes of Banach algebras, which algebras in them are the amenable ones. One specular result in this direction is the characterization of the amenable C^* -algebras : A C^* -algebra is amenable if and only if it is nuclear (this result, mostly credited to A. Connes and U.Haagerup, is the culmination of the efforts of many mathematicians.)

In the same year, J.Bunce [5] gave some conditions on a C^* -algebras that are equivalent to amenability or strong amenability and are analogous to some of the known equivalent definitions of amenable group.

In 1996, B.E.Johnson [12] proved that the group algebra $L^1(G)$ of a locally compact group is systematically amenable if and only if it is amenable. A. T-M Lau, R.J.Loy and G.A. Willis [14] gave several results about the certain algebras defined by locally compact groups. The algebras include the C^* -algebras and von Neumann algebras determined by the representation theory of the group, the Fourier algebra $A(G)$ and various subalgebras of these.

In 1996, F.Ghahramani, R.J.Roy and G.A. Willis [9] studied amenability and weak amenability of second conjugate Banach algebras. A.T.-M.Lau and R.J.Loy studied amenability of convolution algebras.

In 2002, C. Anantharaman-Delaroche [1] studied the relations between amenability (resp. amenability at infinity) of C^* -dynamical systems and equality or nuclearity (resp. exactness) of the corresponding crossed products.

In 2003, E.M.Gordji, F.Habibian, B.Hayati [10] studied the ideal amenability of Banach algebras and proved that A is ideally amenable for every closed two-sided ideal I in A . They showed that for ideal amenability, the homomorphism property for suitable direct summands is true similar to weak amenability and they applied this result for ideal amenability of Banach algebras on locally compact groups.

In 2006, J.Brodzki and J.B.Niblo [4] provided an illustration of an interesting and nontrivial interaction between analytic and geometric properties of groups and provided a short survey of approximation properties of operator algebras associated with discrete group. Moreover, they demonstrated directly that groups that satisfy the property RD with respect to a conditionally negative length function have the metric approximation property.

In 2008, L.Bartholdi [2] studied amenability of algebras and modules (based on the notion of

almost-invariant finite-dimensional subspace), and apply it to algebras associated with finitely generated groups. He showed that a group G is amenable if and only if its group ring $\mathbb{K}G$ is amenable for some (and therefore for any) field $\mathbb{K}$.

In the same year, M.Daws and V.Runde [8] proved that ultra-amenability implies amenability, but is much stronger : a C^* -algebra is ultra-amenable if and only if it is subhomogeneous and $l^1(G)$ for a discrete group G , is ultra-amenable if and only if G is finite.

In 2009, D.Kucerovsky, P-W NG [13] explored the connections between extension theory (i.e., KK-theory), the properties of operator algebras, and the properties of associated unitary groups.

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วัตถุประสงค์:

- เพื่อให้ได้ทฤษฎีบทเกี่ยวกับพีชคณิตตัวดำเนินการบนกรุป ซึ่งอาจจะทำให้ได้นิยามของความเป็นอามีนาเบิลใหม่ หรือได้นิยามของความเป็นอามีนาเบิลรูปแบบใหม่ที่สมมูลกับนิยามเดิม
- เพื่อหาเงื่อนไขของกรุปกระชับเฉพาะที่ทำให้พีชคณิตตัวดำเนินการบนกรุปนั้นเป็นพีชคณิตอามีนาเบิล
- เพื่อหาความสัมพันธ์ระหว่างสมบัติความเป็นอามีนาเบิลชนิดต่าง ๆ ของพีชคณิตตัวดำเนินการบนกรุปกระชับเฉพาะที่

ระเบียบวิธีวิจัย:

- ศึกษาความเป็นอามีนาเบิลของกรุปกระชับเฉพาะที่

- ศึกษาความเป็นอามีนาเบิลชนิดต่าง ๆ ของพีชคณิตตัวดำเนินการอามีนาเบิล
- ศึกษาสมบัติของพีชคณิตตัวดำเนินการอามีนาเบิลและพิสูจน์ทฤษฎีบทใหม่เกี่ยวกับสมบัติของพีชคณิตตัวดำเนินการอามีนาเบิล
- หาความสัมพันธ์ระหว่างความเป็นอามีนาเบิลชนิดต่าง ๆ ของพีชคณิตตัวดำเนินการ
- หาความเชื่อมโยงระหว่างสมบัติของกรุปกระทำเฉพาะที่และความเป็นอามีนาเบิลชนิดต่าง ๆ ของพีชคณิตตัวดำเนินการ

ขอบเขตของการวิจัย:

- พีชคณิตตัวดำเนินการอามีนาเบิลชนิดต่าง ๆ และทฤษฎีบทที่เกี่ยวข้องกับความเป็นอามีนาเบิลของพีชคณิตตัวดำเนินการ ซึ่งจะนำไปสู่นิยามของความเป็นอามีนาเบิลในรูปแบบใหม่ ทั้งนี้จะเน้นพีชคณิตซีสตาร์และพีชคณิตฟอนนอยมันน์
- ศึกษาและหาความสัมพันธ์ระหว่างความเป็นอามีนาเบิลชนิดต่าง ๆ ของพีชคณิตตัวดำเนินการ เพื่อให้ทราบถึงความเชื่อมโยงในภาพรวม
- หาเงื่อนไขของกรุปกระทำเฉพาะที่ที่ทำให้พีชคณิตตัวดำเนินการบนกรุปนั้นเป็นพีชคณิตตัวดำเนินการอามีนาเบิลชนิดต่าง ๆ

ผลลัพธ์ที่ได้:

- S. Utudee, Various kinds of amenability of operator algebras on locally compact groups. (ตีพิมพ์ในวารสารประชุม สกว. ปี 2555)
- W. Tapanyo, and S. Utudee, On Module Amenability of Ideals in Banach Algebras. (การประชุมวิชาการทางคณิตศาสตร์ ครั้งที่ 18 ประจำปี พ.ศ. 2556, Corresponding author)
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VARIOUS KINDS OF AMENABILITY OF OPERATOR ALGEBRAS ON LOCALLY COMPACT GROUPS



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ABSTRACT

Let G be a locally compact group. We can construct group algebra $L^1(G)$, measure algebra $M(G)$, full group C^* -algebra $C^*(G)$, reduced group C^* -algebra $C_r^*(G)$ and Fourier algebra $A(G)$ of G . In this research, we study various kinds of amenability of Banach algebras, of groups, of actions, and of operator algebras on G , for example, amenability, weak amenability, ideal amenability, super-amenability, ultra-amenability, module amenability, etc.

INTRODUCTION

Amenable Groups

In 1929, class of amenable groups was introduced by von Neumann :

A group G is *amenable* if and only if there exists a left invariant mean on G .

Amenable Banach Algebras

In 1972, amenable Banach algebras were introduced and studied by B. E. Johnson :

The Banach algebra A is called *amenable* if and only if every derivation from A into any dual Banach A -module is inner.

For C^* -algebras

Amenability $\leftrightarrow$ Nuclearity

For von Neumann algebras

Injectivity $\leftrightarrow$ Amenability $\leftrightarrow$ Property P
Semidiscreteness $\leftrightarrow$ Amenability $\leftrightarrow$ Approximate finite-Dimensionality

NOTATIONS

$A(G)$	Fourier algebra of a group G
$C^*(G)$	Full group C^* -algebra of a group G
$C_r^*(G)$	Reduced group C^* -algebra of a group G
$B(\mathcal{H})$	Space of bounded linear operators on a Hilbert space $\mathcal{H}$
$L^p(G)$	Space of functions, f such that $ f ^p$ integrable with respect to (left) Haar measure on a group G where $p \geq 1$
$L^1(G, w)$	Beurling algebra of a group G
$M(G)$	Measure algebra of a group G
$VN(G)$	Von Neumann algebra of a group G

RESULTS

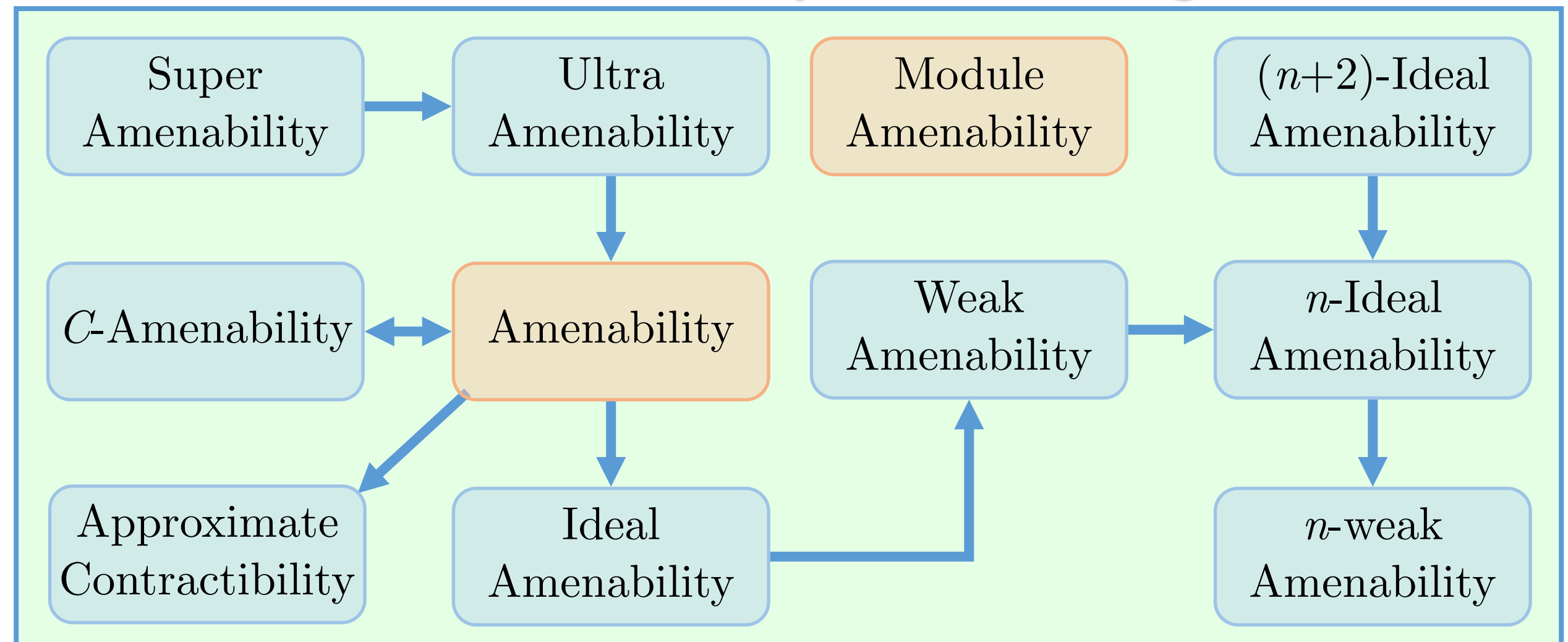
Table 1 : Relations between properties of group and amenability of its Banach algebras

Properties of Group G	Banach Algebras
G is amenable	$L^1(G)$ is amenable
G is finite	$L^1(G)^{**}$ is amenable
G is discrete and amenable	$M(G)$ is amenable
G is almost abelian locally compact	$A(G)$ is amenable
G is amenable and w is diagonally bounded on G	$L^1(G, w)$ is amenable
G is almost abelian	$VN(G)$ is amenable
Irreducible representation of G has bounded degree	$VN(G)$ is amenable
G is abelian	$L^2(G)$ is weak amenable
G is finite and locally compact	$L^1(G)$ is super-amenable
G is finite	$L^1(G, w)$ is super-amenable

In the case of a Hilbert space $\mathcal{H}$

Properties of $\mathcal{H}$	Banach Algebras
$\text{Dim } \mathcal{H} < \infty$	$B(\mathcal{H})$ is amenable

Various kinds of amenability of Banach algebras



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On Module Amenability of Ideals in Banach Algebras

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Abstract : Let $\mathfrak{A}$ and A be Banach algebras, and A be an $\mathfrak{A}$ -module amenable. In this research paper, we show that a closed ideal I of A containing a bounded approximate identity is also $\mathfrak{A}$ -module amenable.

Keywords : Banach algebras; Amenability; Module amenability.

2000 Mathematics Subject Classification : 46H10; 46H25

1 Introduction

The concept of amenability for Banach algebra was introduced by Johnson in 1972 [4]. He showed that amenability of a locally compact group G can be characterized by the vanishing of certain cohomology group of $L^1(G)$ that is a starting of amenability for Banach algebra. Every closed ideal in an amenable Banach algebra is amenable if and only if it has a bounded approximate identity (equivalently, it is weakly complemented in such Banach algebra (see e.g. [5] Theorem 2.3.7).

In 2004, [1], Amini introduced the concept of module amenability for a class of Banach algebras which is considered as a generalization of amenability. He extended some theorems of amenability to module amenability.

In [3], Jabbari considered in the case of commutative amenable Banach algebras. He showed that a closed ideal in a commutative module amenable Banach algebra is module amenable if and only if it has a bounded approximate identity (equivalently, it is weakly complemented in such Banach algebra.

An outline of the present paper is as follows. Definition and basic properties of module amenable Banach algebras are given in Section 2. The results proper state in Section 3 with a study of module amenability of Banach algebra having a bounded approximate identity. We prove that every closed ideal in a module amenable Banach algebra (not necessary commutative) containing a bounded approximate identity is also module amenable.

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2 Preliminaries

Let A and $\mathfrak{A}$ be Banach algebras such that A is a Banach $\mathfrak{A}$ -bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (2.1)$$

for every $a, b \in A$, $\alpha \in \mathfrak{A}$. Let X be a Banach A -bimodule and a Banach $\mathfrak{A}$ -bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot \alpha) \cdot x = a \cdot (\alpha \cdot x), \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \quad (2.2)$$

for every $a \in A$, $\alpha \in \mathfrak{A}$, $x \in X$ and the same for the right or two-side actions, we say that X is a *Banach A - $\mathfrak{A}$ -module*. In this case, X is said to be *commutative* if for every $\alpha \in \mathfrak{A}$, $x \in X$

$$\alpha \cdot x = x \cdot \alpha.$$

Let X be a Banach A -bimodule. X is *A -pseudo-unital* (pseudo-unital if not confused) if $AXA = X$. We say that X is *A -essential* (essential if not confused) if a linear combination of AXA is dense in X .

It is obvious that every pseudo-unital Banach A -bimodule is essential. If A has a bounded approximate identity, then the Cohen's factorization theorem, Theorem 11.10 of [2], implies that an essential Banach A -bimodule is pseudo-unital. Therefore, in the case A has a bounded approximate identity, an essential and pseudo-unital Banach A -bimodule are equivalent.

For a Banach space X , We denote its topological dual space by X^* and the action of $f \in X^*$ on an element $x \in X$ given by $\langle x, f \rangle$. If X is a (commutative) Banach A - $\mathfrak{A}$ -module, then so is X^* , where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$\langle x, \alpha \cdot f \rangle = \langle x \cdot \alpha, f \rangle, \quad \langle x, a \cdot f \rangle = \langle x \cdot a, f \rangle,$$

where $\alpha \in \mathfrak{A}$, $a \in A$, $f \in X^*$, $x \in X$, and the right actions is similarity defined.

Let A and B be Banach algebras which are Banach $\mathfrak{A}$ -bimodules with compatible actions (2.1). An $\mathfrak{A}$ -module map is a mapping $\varphi : A \rightarrow B$ with the following properties

- (i) $\varphi(a \pm b) = \varphi(a) \pm \varphi(b)$;
- (ii) $\varphi(\alpha \cdot a) = \alpha \cdot \varphi(a)$;
- (iii) $\varphi(a \cdot \alpha) = \varphi(a) \cdot \alpha$,

for every $a, b \in A$, $\alpha \in \mathfrak{A}$. An $\mathfrak{A}$ -module map φ is *bounded* if there exists $M > 0$ such that $\|\varphi(a)\| \leq M\|a\|$ for every $a \in A$.

Definition 2.1. Let X be a Banach A - $\mathfrak{A}$ -module, a bounded $\mathfrak{A}$ -module map $D : A \rightarrow X$ is called a *module derivation* if

$$D(ab) = a \cdot (Db) + (Da) \cdot b$$

for every $a, b \in A$. If X is commutative and $x \in X$, a module derivation $D_x : A \rightarrow X$ defined by

$$D_x(a) = a \cdot x - x \cdot a$$

where $a \in A$, is called an *inner module derivation*.

Note that, the module derivation is not necessarily linear but its continuity is accepted by boundedness. We write $Z_{\mathfrak{A}}^1(A, X)$ for the space of all module derivations from A into X , $\mathcal{B}_{\mathfrak{A}}^1(A, X)$ for the space of all inner module derivations, and the quotient space $\mathcal{H}_{\mathfrak{A}}^1(A, X) = Z_{\mathfrak{A}}^1(A, X)/\mathcal{B}_{\mathfrak{A}}^1(A, X)$ is called the *first relative (to $\mathfrak{A}$) Hochschild cohomology group* of A with coefficients in X .

Definition 2.2. Let A and $\mathfrak{A}$ be Banach algebras such that A is a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1). A is said to be *module amenable* (as an $\mathfrak{A}$ -module) if for any commutative Banach A - $\mathfrak{A}$ -module X , each bounded module derivation from A to a topological dual space X^* is inner, or equivalently, $\mathcal{H}_{\mathfrak{A}}^1(A, X^*) = \{0\}$.

In [1], Amini considered a discrete inverse semigroup S with E , the set of all idempotents of S . He let $\ell^1(E)$ act on $\ell^1(S)$ on the left trivially (zero actions) and on the right by, that is

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \delta_e = \delta_{se} = \delta_s * \delta_e,$$

for every $s \in S$, $e \in E$. By these actions, he showed in Theorem 3.1 that an inverse semigroup S is amenable if and only if $\ell^1(S)$ is $\ell^1(E)$ -module amenable.

The module amenability has some properties like the amenability such as [1] Proposition 2.2. This proposition stated that for a commutative Banach $\mathfrak{A}$ -bimodule A , if A is module amenable, then A has a bounded approximate identity.

Another one about the relation of two algebras, [1] Proposition 2.5. This proposition stated that for Banach algebras A and B and Banach $\mathfrak{A}$ -bimodules with compatible actions (2.1), if there is a continuous Banach algebra homomorphism and module morphism ϕ from A onto a dense subset of B , then module amenability of A implies module amenability of B .

In this case, we can apply to A and $B = A/I$ where I is a closed ideal and a closed submodule of A .

3 Main Results

[1] Lemma 2.1, Amini studied the Module Amenability of Banach Algebras with using an A -essential commutative Banach A - $\mathfrak{A}$ -module. It implies that, module amenability of Banach algebras can be obtained by considering only an A -essential commutative Banach A - $\mathfrak{A}$ -module that we show in the following proposition.

Proposition 3.1. Let A and $\mathfrak{A}$ be Banach algebras. Suppose that A is a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1). If A has a bounded approximate identity, then the following are equivalent.

- (i) A is module amenable.
- (ii) $H_{\mathfrak{A}}^1(A, X^*) = \{0\}$ for every A -pseudo-unital commutative Banach A - $\mathfrak{A}$ -module X .

Proof. The part (i) $\Rightarrow$ (ii) is trivial. Now, let X be any commutative Banach A - $\mathfrak{A}$ -module. By Lemma 2.1 of [1], we have $H_{\mathfrak{A}}^1(A, X^*) = H_{\mathfrak{A}}^1(A, (AXA)^*)$. Since A has a bounded approximate identity, we have AXA becomes A -pseudo-unital. Hence, $H_{\mathfrak{A}}^1(A, X^*) = \{0\}$. This shows that A is module amenable $\square$

Let A be a Banach algebra contained as a closed ideal in a Banach algebra B . Then the *strict topology* on B with respect to A is generated by the family of seminorms $(p_a)_{a \in A}$, where

$$p_a(b) := \|ba\| + \|ab\| \quad (b \in B),$$

an open ball at center b and radius ε with respect to p_a will be denote by $B_a(b; \varepsilon)$.

The strict topology on a Banach algebra is useful to extend a derivation on some algebra to a larger algebra, [5]. Now, we use it again for a module derivation.

Lemma 3.2. *Let A , B , and $\mathfrak{A}$ be Banach algebras such that A and B are Banach $\mathfrak{A}$ -bimodules with compatible actions (2.1). Suppose that A is contained as a closed ideal in B preserving module actions. Then for every $\alpha \in \mathfrak{A}$ the maps*

$$b \mapsto \alpha \cdot b \quad \text{and} \quad b \mapsto b \cdot \alpha$$

where $b \in B$, are continuous on the strict topology on B with respect to A .

Proof. Let $\alpha \in \mathfrak{A}$ and $(b_\beta)_\beta$ be a net in B converging to $b \in B$ respect to the strict topology. Given $a \in A$, so we have $p_a(b_\beta - b)$ converge to 0. Since A be a Banach $\mathfrak{A}$ -bimodule, there is $k > 0$ such that for each $c \in A$

$$\|\alpha \cdot c\| \leq k \|\alpha\| \|c\|.$$

Now, consider

$$p_a(\alpha \cdot b_\beta - \alpha \cdot b) = \|(\alpha \cdot b_\beta - \alpha \cdot b)a\| + \|a(\alpha \cdot b_\beta - \alpha \cdot b)\| \leq 2k \|a\| \|\alpha\| \|b_\beta - b\| \rightarrow 0.$$

Thus $\alpha \cdot b_\beta \rightarrow \alpha \cdot b$. Hence, the mapping $b \mapsto \alpha \cdot b$ is continuous. The continuity of the mapping $b \mapsto b \cdot \alpha$ is obtained similarly. $\square$

If A has a bounded approximate identity and can be contained as a closed ideal in a Banach algebra B , Proposition 2.1.6 of [5] show that any pseudo-unital Banach A -bimodule X is extend to be a Banach B -bimodule. Next lemma, we show that the B -module actions on X is continuous with respect to the strict topology on B .

Lemma 3.3. *Let B be a Banach algebra together with the strict topology respect to a closed ideal A and let X be an A -essential Banach A -bimodule. If A has a bounded approximate identity, then X is a Banach B -bimodule with module actions given by*

$$b \cdot x = ba \cdot y \quad \text{and} \quad x \cdot b = y' \cdot a' b$$

for every $b \in B$ and $x \in X$ where $x = a \cdot y = y' \cdot a'$ for some $a, a' \in A$, and some $y, y' \in X$. Moreover, for every $x \in X$ the maps

$$b \mapsto x \cdot b \quad \text{and} \quad b \mapsto b \cdot x$$

where $b \in B$, are continuous on the strict topology B with respect to A .

Proof. Let A have a bounded approximate identity, so the actions are well-defined from proposition 2.1.6 in [5]. So we will prove the continuity of the actions only. Let $x \in X$ and $(b_\beta)_\beta$ be a net in B converging to $b \in B$ respect to the strict topology. Given $a \in A$, so we have $p_a(b_\beta - b)$ converge to 0. Since X is A -essential and A has a bounded approximate identity, X is A -pseudo-unital. So we have $x = y \cdot a'$ for some $a' \in A$ and some $y \in X$. Since X is a Banach A -bimodule, there is $k > 0$ such that

$$\|y \cdot a'(b_\beta - b)\| \leq k\|y\|\|a'(b_\beta - b)\|.$$

Now, consider

$$\|x \cdot b_\beta - x \cdot b\| = \|x \cdot (b_\beta - b)\| = \|y \cdot a'(b_\beta - b)\| \leq k\|y\|\|a'(b_\beta - b)\| \rightarrow 0.$$

Hence, $b \mapsto x \cdot b$ is continuous. The continuity of the map $b \mapsto b \cdot x$ can be proved similarly. $\square$

Lemma 3.4. *Let A and $\mathfrak{A}$ be Banach algebras and I be a closed ideal of A . Suppose that A is a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1). If I has a bounded approximate identity, then I becomes a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1).*

Proof. Let $a \in I$, $\alpha \in \mathfrak{A}$ and $(e_\beta)_\beta$ be a bounded approximate identity for I . Since $\alpha \cdot ae_\beta = (\alpha \cdot a)e_\beta \in I$ for every β , it follow by the norm close ideal of I that

$$\alpha \cdot a = \lim_{\beta} \alpha \cdot (ae_\beta) \in I.$$

The right action is obtained similarly. Hence I is a Banach $\mathfrak{A}$ -bimodule. Compatibility of $\mathfrak{A}$ -module action on I is trivial. $\square$

Now we ready to extend a module derivation on some algebra to a larger algebra. We will show in the next proposition.

Proposition 3.5. *Let B and $\mathfrak{A}$ be Banach algebras such that B is a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1). Suppose that A is another Banach algebra contained as a closed ideal in B containing a bounded approximate identity. If X is an A -essential commutative Banach A - $\mathfrak{A}$ -module, then the following hold ;*

- (i) X is a commutative Banach B - $\mathfrak{A}$ -module with compatible actions (2.2),
- (ii) for every $D \in Z_{\mathfrak{A}}^1(A, X^*)$, there is $\tilde{D} \in Z_{\mathfrak{A}}^1(B, X^*)$ which is an extension of D and continuous with respect to the strict topology on B and the weak* topology on X^* .

Proof. Let X be an A -essential commutative Banach A - $\mathfrak{A}$ -module and let $(e_\beta)_\beta$ be a bounded approximate identity for A . Then X is A -pseudo-unital. By Lemma 3.3, X becomes a Banach B -bimodule. We only show that the B - $\mathfrak{A}$ -module actions on X are compatible with (2.2). Let $b \in B$, $\alpha \in \mathfrak{A}$ and $x \in X$. In the strict topology, we have $be_\beta \rightarrow b$. Then continuity of the action implies that

$$\alpha \cdot (b \cdot x) = \lim_{\beta} \alpha \cdot (be_\beta \cdot x) = \lim_{\beta} (\alpha \cdot be_\beta) \cdot x = (\alpha \cdot b) \cdot x.$$

Similarly,

$$b \cdot (\alpha \cdot x) = (b \cdot \alpha) \cdot x \text{ and } \alpha \cdot (x \cdot b) = (\alpha \cdot x) \cdot b.$$

For the right side, we obtain similarly. Therefore X is a Banach B - $\mathfrak{A}$ -module.

From Lemma 3.4, we have A is a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1). To prove (ii), let $D : A \rightarrow X^*$ be a module derivation. As in Theorem 2.2 of [3], for any $b \in B$, we define an operator D^b of A into X^* by

$$a \mapsto D(ba) - b \cdot Da.$$

To verify that $D^b a \in X^*$. Given $x, y \in X$ and k be a scalar. So we have

$$\begin{aligned} \langle kx + y, D^b a \rangle &= \langle kx + y, D(ba) - b \cdot (Da) \rangle \\ &= \langle kx + y, D(ba) \rangle - \langle kx + y, b \cdot (Da) \rangle \\ &= k \langle x, D(ba) - b \cdot (Da) \rangle + \langle y, D(ba) - b \cdot (Da) \rangle \\ &= k \langle x, D^b a \rangle + \langle y, D^b a \rangle, \end{aligned}$$

and

$$\|\langle x, D^b a \rangle\| = \|\langle x, D(ba) - b \cdot Da \rangle\| \leq \|x\| \|D(ba) - b \cdot Da\|.$$

Hence $D^b a \in X^*$. Since $(e_\beta)_\beta$ is a bounded net in A , it is obvious that $(D^b e_\beta)_\beta$ is a bounded net in X^* . Let $x \in X$. Since X is A -pseudo-unital, we can write $x = a' \cdot y \cdot a$ for some $a, a' \in A$ and some $y \in X$. So

$$\begin{aligned} \langle x, D^b e_\beta \rangle &= \langle a' \cdot y \cdot a, D(b e_\beta) - b \cdot D e_\beta \rangle \\ &= \langle a' \cdot y, a \cdot D(b e_\beta) - a b \cdot D e_\beta \rangle \\ &= \langle a' \cdot y, D(a b e_\beta) - (Da) \cdot b e_\beta - D(a b e_\beta) + D(a b) \cdot e_\beta \rangle \\ &= \langle a' \cdot y, D(a b) \cdot e_\beta - (Da) \cdot b e_\beta \rangle \\ &= \langle e_\beta a' \cdot y, D(a b) - (Da) \cdot b \rangle. \end{aligned}$$

Since $e_\beta a' \cdot y \rightarrow a' \cdot y$ and $D(a b) - (Da) \cdot b \in X^*$, we have

$$\langle x, D^b e_\beta \rangle = \langle e_\beta a' \cdot y, D(a b) - (Da) \cdot b \rangle \rightarrow \langle a' \cdot y, D(a b) - (Da) \cdot b \rangle.$$

Now we have $(D^b e_\beta)_\beta$ is a bounded net in a Banach space X^* such that $(\langle x, D^b e_\beta \rangle)_\beta$ is convergent for every $x \in X$. Then the net $(D^b e_\beta)_\beta$ weak* converge to the weak* limit $\lambda \in X^*$ which is defined by

$$\langle x, \lambda \rangle = \langle a' \cdot y, D(a b) - (Da) \cdot b \rangle.$$

We know that $(\langle x, D^b e_\beta \rangle)_\beta$ is a net in a Hausdorff topological space. Then its limit is unique and independent from a choice of $y \in X$ and $a, a' \in A$ such that $x = a' \cdot y \cdot a$. This implies that λ is well-defined.

Now define $\tilde{D} : B \rightarrow X^*$ by $\tilde{D}b = \lambda$. Let $c \in A$, $x = a' \cdot y \cdot a \in X$. Then

$$\langle x, \tilde{D}c \rangle = \langle a' \cdot y, D(ac) - (Da) \cdot c \rangle = \langle a' \cdot y, a \cdot Dc \rangle = \langle a' \cdot y \cdot a, Dc \rangle = \langle x, Dc \rangle,$$

so that $\tilde{D}$ is an extension of D .

Let $(b_\gamma)_\gamma$ be a net in B converging to $b \in B$ in the strict topology. We have $\|(b_\gamma - b)c\| + \|c(b_\gamma - b)\| = p_c(b_\gamma - b) \rightarrow 0$, i.e., $b_\gamma c \rightarrow bc$ and $cb_\gamma \rightarrow cb$ for each $c \in A$. Now we put $x = a' \cdot y \cdot a$ where $a, a' \in A, x \in X$ and consider

$$\begin{aligned} \langle x, \tilde{D}b_\gamma \rangle &= \langle a' \cdot y \cdot a, D(ab_\gamma) - (Da) \cdot b_\gamma \rangle \\ &= \langle y \cdot a, D(ab_\gamma) \cdot a' - (Da) \cdot b_\gamma a' \rangle \\ &\rightarrow \langle y \cdot a, D(ab) \cdot a' - (Da) \cdot ba' \rangle \\ &= \langle a' \cdot y \cdot a, D(ab) - (Da) \cdot b \rangle \\ &= \langle x, \tilde{D}b \rangle. \end{aligned}$$

This implies that $\tilde{D}$ is continuous with respect to the strict topology on B and the weak* topology on X^* . Next, we show that $\tilde{D}$ is an $\mathfrak{A}$ -module map. Let $b, c \in B, \alpha \in \mathfrak{A}, x = a' \cdot y \cdot a \in X$. We have

$$\begin{aligned} \langle x, \tilde{D}(b \pm c) \rangle &= \langle a' \cdot y, D(a(b \pm c)) - (Da) \cdot (b \pm c) \rangle \\ &= \langle a' \cdot y, (D(ab) - (Da) \cdot b) \rangle \pm \langle a' \cdot y, (D(ac) - (Da) \cdot c) \rangle \\ &= \langle x, \tilde{D}b \rangle \pm \langle x, \tilde{D}c \rangle \\ &= \langle x, \tilde{D}(b \pm c) \rangle, \end{aligned}$$

$$\begin{aligned} \langle x, \tilde{D}(\alpha \cdot b) \rangle &= \langle y, D(a(\alpha \cdot b)) \cdot a' - (Da) \cdot (\alpha \cdot ba') \rangle \\ &= \langle y, D(a(\alpha \cdot b)a') - a(\alpha \cdot b) \cdot Da' - D(a(\alpha \cdot b)a') + a \cdot D(\alpha \cdot ba') \rangle \\ &= \langle y, a \cdot D(\alpha \cdot ba') - a(\alpha \cdot b) \cdot Da' \rangle \\ &= \langle y \cdot a, \alpha \cdot D(ba') - (\alpha \cdot b) \cdot Da' \rangle \\ &= \langle y, a \cdot D(ba' \cdot \alpha) - ab \cdot D(a' \cdot \alpha) \rangle \\ &= \langle y, D(aba' \cdot \alpha) - (Da) \cdot (ba' \cdot \alpha) - D(aba' \cdot \alpha) + (D(ab)) \cdot (a' \cdot \alpha) \rangle \\ &= \langle y, (D(ab)) \cdot (a' \cdot \alpha) - (Da) \cdot (ba' \cdot \alpha) \rangle \\ &= \langle ((a' \cdot \alpha) \cdot y) \cdot a, \tilde{D}b \rangle \\ &= \langle a' \cdot (y \cdot a) \cdot \alpha, \tilde{D}b \rangle \\ &= \langle x, \alpha \cdot \tilde{D}b \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle x, \tilde{D}(b \cdot \alpha) \rangle &= \langle a' \cdot y, D(a(b \cdot \alpha)) - (Da) \cdot (b \cdot \alpha) \rangle \\ &= \langle a' \cdot y, D(ab) \cdot \alpha - ((Da) \cdot b) \cdot \alpha \rangle \\ &= \langle (\alpha \cdot (a' \cdot y)) \cdot a, \tilde{D}(b) \rangle \\ &= \langle x, \tilde{D}(b) \cdot \alpha \rangle, \end{aligned}$$

Then $\tilde{D}$ is a $\mathfrak{A}$ -module map. For a module derivation, we consider

$$\begin{aligned}
\langle x, \tilde{D}(bc) \rangle &= \langle a' \cdot y, D(abc) - (Da) \cdot (bc) \rangle \\
&= \langle a' \cdot y, D(abc) - (D(ab)) \cdot c + (D(ab)) \cdot c - (Da) \cdot (bc) \rangle \\
&= \langle a' \cdot y \cdot ab, \tilde{D}c \rangle + \langle ca' \cdot y, D(ab) - (Da) \cdot b \rangle \\
&= \langle x \cdot b, \tilde{D}c \rangle + \langle c \cdot x, \tilde{D}b \rangle \\
&= \langle x, b \cdot \tilde{D}c + (\tilde{D}b) \cdot c \rangle.
\end{aligned}$$

Then $\tilde{D}(bc) = b \cdot \tilde{D}c + (\tilde{D}b) \cdot c$, so that $\tilde{D}$ is a module derivation. Next, we show that $\tilde{D}$ is bounded. Turn back to the bounded net $(D^b e_\beta)_\beta$ and its weak* limit, $\tilde{D}b$, for every $x \in X$ we have

$$\|\langle x, D^b e_\beta \rangle\| = \|\langle x, D(be_\beta) - b \cdot De_\beta \rangle\| \leq \|x\| \|D(be_\beta) - b \cdot De_\beta\|.$$

This implies that $\|\langle x, \tilde{D}b \rangle\| \leq \|x\| \|D(be_\beta) - b \cdot De_\beta\|$. Take $\|x\| = 1$, we have

$$\|\tilde{D}b\| \leq \|D(be_\beta) - b \cdot De_\beta\|.$$

Since X^* is a Banach B -bimodule, there is $k > 0$ such that $\|b \cdot De_\beta\| \leq k \|b\| \|De_\beta\|$. Then

$$\|\tilde{D}b\| \leq \|D(be_\beta) - b \cdot De_\beta\| \leq \|b\| \sup_\beta \|e_\beta\| \|D\| (1 + k).$$

Hence $\tilde{D}$ is bounded, i.e., $\tilde{D} \in \mathcal{Z}_{\mathfrak{A}}^1(B, X^*)$ □

Theorem 3.6. *Let A and $\mathfrak{A}$ be Banach algebras and I be a closed ideal of A . Suppose that A is module amenable as a Banach $\mathfrak{A}$ -bimodule. If I has a bounded approximate identity, then I is module amenable. The converse is true if I is a commutative $\mathfrak{A}$ -module.*

Proof. By Lemma 3.4, I is a Banach $\mathfrak{A}$ -bimodule with compatible actions (2.1). Let X be a commutative Banach I - $\mathfrak{A}$ -module. By Proposition 3.1, we may assume that X is I -essential. Given $D \in \mathcal{Z}_{\mathfrak{A}}^1(I, X^*)$, by proposition 3.5, we can extend D to $\tilde{D} \in \mathcal{Z}_{\mathfrak{A}}^1(A, X^*)$. Since A is module amenable, $\tilde{D} \in B_{\mathfrak{A}}^1(A, X^*)$. Thus $D = \tilde{D}|_I \in B_{\mathfrak{A}}^1(I, X^*)$. Therefore, I is module amenable. □

The converse is obtained by proposition 2.2 of [1]. □

Corollary 3.7. *Let A and $\mathfrak{A}$ be Banach algebras and let I be a closed ideal of A . Suppose that A is a Banach $\mathfrak{A}$ -bimodule with compatible action (2.1) and $\mathfrak{A}$ has a bounded approximate identity for A . If I has a bounded approximate identity, then amenability of A implies module amenability of I .*

Proof. Since $\mathfrak{A}$ has a bounded approximate identity for A . Then by Proposition 2.1 of [1], A is module amenable. Since I has bounded approximate identity, Theorem 3.6 implies that I is module amenable. □

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RESEARCH

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Wavelet multilevel augmentation method for linear boundary value problems

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Abstract

This work presents a new approach to numerically solve the general linear two-point boundary value problems with Dirichlet boundary conditions. Multilevel bases from the anti-derivatives of the Daubechies wavelets are constructed in conjunction with the augmentation method. The accuracy of numerical solutions can be improved by increasing the number of basis levels, but the computational cost also increases drastically. The multilevel augmentation method can be applied to reduce the computational time by splitting the coefficient matrix into smaller submatrices. Then the unknown coefficients in the higher level can be solved separately. The convergent rate of this method is 2^s , where $1 \leq s \leq p + 1$, when the anti-derivatives of the Daubechies wavelets order p are applied. Some numerical examples are also presented to confirm our theoretical results.

MSC: 65J10; 65L10

Keywords: wavelets; multilevel augmentation method; boundary value problems; Dirichlet boundary conditions

1 Introduction

Boundary value problems can be viewed as mathematical models in science and engineering. For real world applications, exact solutions are not available. Numerical methods are required to solve numerically the models. The efficient methods provide approximate solutions by choosing the appropriate subspaces of solution spaces and their suitable bases. By applying suitable formulations, the linear model equation can be discretized to a linear system. A more accurate approximation can be obtained by increasing the number of basis functions. However, this leads to a larger discretized linear system. To save the computational cost, one can use the multilevel augmentation method. The resulting coefficient matrix corresponding to the finer level of approximate spaces is obtained by augmenting a matrix corresponding to a coarser level. Instead of solving the linear system at the finer level, the coefficient matrix can be separated so that the smaller system at the coarser level can be taken. Thus, the additional computational cost is proportional to the dimension of the different space between the spaces of the finer level and the coarser level, not the dimension of the whole finer level. This method allows us to develop faster and accurate algorithms for solving differential equations (see, e.g., [1–3], and [4]). These previous researches considered the second kind of equations and constructed piecewise polynomials as bases for the subspaces of the Sobolev spaces $H_0^m(0, 1)$ consisting of elements satisfying

the homogeneous boundary conditions

$$u^{(j)}(0) = u^{(j)}(1) = 0, \quad \text{for } j = 0, 1, \dots, m-1.$$

On the other hand, wavelets can be applied to discretize differential equations (see, *e.g.*, [5, 6]). Related numerical methods with the applications of Haar and Legendre wavelets for solving boundary value problem are proposed by Siraj-ul-Islam *et al.* [7, 8]. The advantage of wavelet basis is its capability to approximate solutions of differential equations. The wavelet Galerkin method is one of the most powerful methods that can be used to solve ordinary and partial differential equations (see, *e.g.*, [9–11], and [12]). In addition, the accuracy of the approximate solutions can easily be improved by merely increasing the numbers of wavelet basis functions and the orders of wavelets. However, the wavelet basis is not straightforwardly adjusted to satisfy general boundary conditions. In 1992, Xu and Shann introduced a different approach to handle the boundary conditions by using the anti-derivatives of Daubechies wavelets [13]. These anti-derivatives form bases for the finite-dimensional subspaces of Sobolev space H^1 and are used to construct an algorithm for approximating solutions.

In this work, we propose the method that combines the main advantages of wavelet bases and multilevel augmentation together. That is, we apply the multilevel augmentation of operators in conjunction with the anti-derivatives of Daubechies wavelets to approximate linear differential equations in the case of Dirichlet boundary conditions. The originality of this work is that we introduce the anti-derivatives of Daubechies wavelets for solving linear boundary value problems (see [13]) and apply this basis type with the augmentation method proposed by Chen (see, *e.g.*, [1–3] and [4]). By this concept, we obtain a new approach to reduce the computational time for solving the linear system resulting from discretizing a linear differential equation.

Given the interval $\Omega := (0, R)$, we use the notation $L^2(\Omega)$ to denote the space of square integrable functions on Ω with standard inner product $(\cdot, \cdot)$ defined by

$$(u, v) = \int_0^R u(x)v(x) dx$$

and the associated norm $\|\cdot\|$.

Let $H^s(\Omega)$ denote the standard Sobolev space with the norm $\|\cdot\|_s$ given by

$$\|v\|_s^2 = \sum_{i=0}^s \int_0^R |v^{(i)}(x)|^2 dx.$$

According to the boundary condition, we work on the solution space

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v(0) = v(R) = 0\},$$

equipped with the inner product

$$[u, v] = \int_0^R u'(x)v'(x) dx, \quad \text{for } u, v \in H_0^1(\Omega),$$

and its associated norm $|\cdot|$. It is well known that the norm $|\cdot|$ is equivalent to the standard norm $\|\cdot\|_1$ in this space.

Let $p \in \mathbb{N}$. We will apply the multilevel augmentation method and anti-derivatives of wavelets of order p to find numerical solutions of two-point boundary value problems with Dirichlet boundary conditions.

Assume that there exists a unique weak solution $u \in H_0^1(\Omega)$. To find numerical solutions, we propose the following steps:

1. The solution space $H_0^1(\Omega)$ is decomposed into orthogonal direct sum of subspaces. The anti-derivatives of the Daubechies wavelets are used to construct finite-dimensional subspaces.
2. For $n \in \mathbb{N}$, the multilevel method is applied to obtain the n th level solution by solving a linear system with matrix coefficients related to the anti-derivatives of the Daubechies wavelets.
3. To obtain a solution at a higher level, namely $(n+i)$ th level, the multilevel augmentation method is applied. By the algorithm to be presented, the computational time for solving the linear system is reduced since the dimension of the matrix coefficient is smaller.

Finally, this work is organized as follows. Section 2 gives an introduction to the anti-derivatives of the Daubechies wavelets and the finite-dimensional subspaces of the solution space $H_0^1(\Omega)$. In Section 3, we describe the algorithm to find approximate solutions using the multilevel augmentation method. The optimal error estimates for the approximate solutions are proven in Section 4, while some numerical examples are demonstrated in Section 5. Conclusions and future work are discussed in Section 6.

2 Bases for subspaces of $H_0^1(\Omega)$

In this section, we will introduce the wavelets of order p and their anti-derivatives. These functions form orthonormal bases for the finite-dimensional subspaces S_n of the solution space $H_0^1(\Omega)$. More details can be found in [5] and [13].

To define the Daubechies wavelets, we consider two functions: the *scaling function* $\phi(x)$ and the *wavelet function* $\psi(x)$. The scaling function is obtained from the dilation equation. The wavelet function is defined from the scaling function. Details are described as follows.

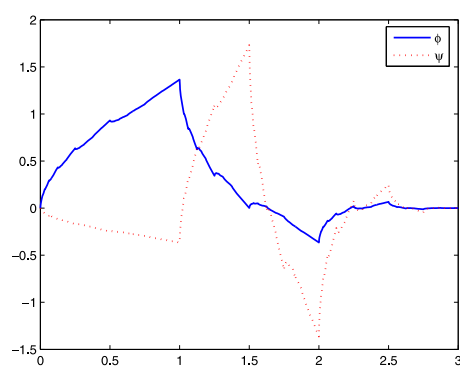
Given a positive integer p , consider a sequence $\{c_k\}_{k \in \mathbb{Z}}$ satisfying

$$\begin{aligned} c_k &= 0, \quad \text{for } k \notin \{0, 1, \dots, 2p-1\}, \\ \sum_k c_k &= 2, \\ \sum_k (-1)^k k^m c_k &= 2, \quad \text{for } 0 \leq m \leq p-1, \\ \sum_k c_k c_{k-2m} &= 2\delta_{0m}, \quad \text{for } 1-p \leq m \leq p-1. \end{aligned}$$

The scaling function $\phi(x)$ is the unique solution of the dilation equation

$$\phi(x) = \sum_{k=0}^{2p-1} c_k \phi(2x-k),$$

where c_k satisfy the above four properties.

Figure 1 ϕ and ψ for $p = 2$.


Let

$$\tilde{\psi}(x) = \sum_{k=0}^{2p-1} (-1)^k c_{2p-1-k} \phi(2x - k).$$

The wavelet function $\psi(x)$ is defined by

$$\psi(x) = \tilde{\psi}(x - p + 1).$$

The graphs of ϕ and ψ for $p = 2$ are shown in Figure 1.

Define

$$\psi_{jk}(x) = \begin{cases} \phi(x - k), & j = -1, \\ \sqrt{2^j} \psi(2^j x - k), & j \geq 0. \end{cases}$$

The functions ψ_{jk} with $j \geq -1$ and $(j, k) \in \mathbb{Z} \times \mathbb{Z}$ are called *wavelets of order p* . It is well known that the set of wavelets forms an orthonormal basis for $L^2(\mathbb{R})$.

For $j \geq -1$, define the index set I_j such that

$$k \in I_j \iff \begin{cases} 2 - 2p \leq k \leq 2p - 2, & j = -1, \\ 2 - 2p \leq k \leq 2^j(2p - 1) - 1, & j \geq 0. \end{cases}$$

$\{\psi_{jk}|_{\Omega} \mid j \geq -1, k \in I_j\}$ is a *frame* of $L^2(\Omega)$. That is, the $\text{span}\{\psi_{jk}|_{\Omega} \mid j \geq -1, k \in I_j\}$ consisting of all linear expansions is equal to $L^2(\Omega)$.

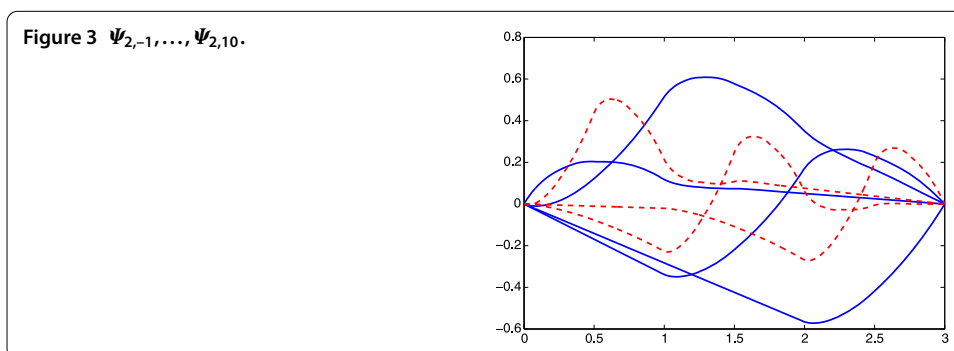
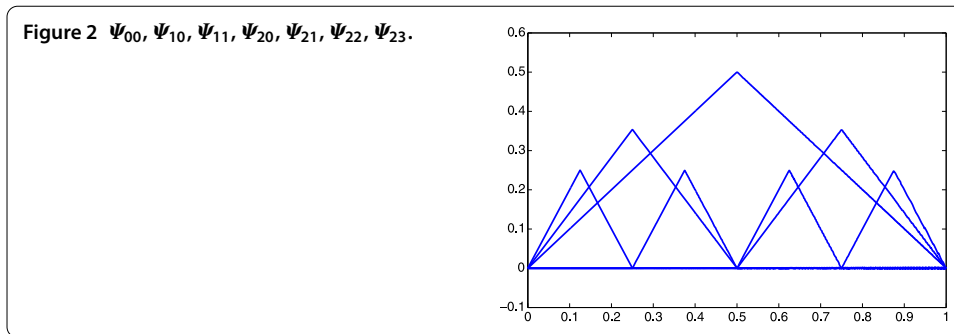
Next, we define the anti-derivatives of wavelets satisfying the Dirichlet boundary condition, namely,

$$\Psi_{jk}(x) = \int_0^x \psi_{jk} ds - \frac{x}{R} \int_0^R \psi_{jk} ds, \quad \text{for } 0 \leq x \leq R.$$

For $n \in \mathbb{N}_0$, we define a subspace S_n by

$$S_n = \text{span}\{\Psi_{jk} \mid -1 \leq j < n, k \in I_j\}.$$

It is obvious that these subspaces are finite-dimensional subspaces of $H_0^1(\Omega)$ and they are nested, i.e., $S_n \subset S_{n+1}$. Since $\{\Psi_{jk} \mid -1 \leq j < n, k \in I_j\}$ is a frame for S_n , it spans S_n but it



need not be linearly independent. Xu and Shann [13], introduced an index set D_j such that $\{\Psi_{jk} \mid -1 \leq j < n, k \in D_j\}$ is a basis for S_n . The index set D_j is defined by

$$k \in D_j \iff \begin{cases} 2 - 2p < k \leq 2p - 2, & j = -1, \\ 1 - p \leq k \leq 2^j(2p - 1) - p, & j \geq 0. \end{cases}$$

For $p = 1$, the basis functions $\{\Psi_{jk} \mid 0 \leq j < n, k \in D_j\}$ for S_n are the piecewise linear hierarchical basis functions with uniform mesh size, $h = 2^{-n}$. The graphs of $\Psi_{00}, \Psi_{10}, \dots, \Psi_{22}$, and Ψ_{23} are shown in Figure 2, while the graphs of $\Psi_{2,-1}, \dots, \Psi_{2,10}$ are shown in Figure 3.

It should be noted that the set

$$\{\Psi_{jk} \mid 0 \leq j < n, k \in D_j\}$$

is an orthonormal basis for S_n with the inner product $[\cdot, \cdot]$ of the Sobolev space $H_0^1(\Omega)$ but it is not orthonormal in $L^2(\Omega)$ when equipped with the standard inner product $(\cdot, \cdot)$. Since we require orthonormality in $L^2(\Omega)$ for our augmentation method, we apply the Gram-Schmidt process to obtain an orthonormal basis for S_n with the standard inner product $(\cdot, \cdot)$ in $L^2(\Omega)$. Thus, the set

$$\{\bar{\Psi}_{jk} \mid -1 \leq j < n, k \in D_j\}$$

is defined as orthonormal basis for S_n in the present method.

3 Multilevel augmentation method algorithm

In this section, we describe the multilevel augmentation method for solving boundary value problems.

Let $u \in H_0^1(\Omega)$ be the weak solution of a given differential equation. Suppose that the variational form of the differential equation is

$$a(u, v) = (f, v), \quad \text{for all } v \in H_0^1(\Omega). \quad (1)$$

Let the approximate solution u_n of the given equation be

$$u_n = \sum_{j=-1}^n \sum_{k \in D_j} \alpha_{jk} \bar{\Psi}_{jk}.$$

The main idea is to determine the coefficients α_{jk} in such a way that u_n behaves as if it were a weak solution on S_n , that is, u_n satisfies the linear system of equations,

$$\begin{aligned} a(u_n, \bar{\Psi}_{lm}) &= (f, \bar{\Psi}_{lm}), \quad \text{for all } -1 \leq l < n, m \in D_j, \\ \sum_{-1 \leq j < n} \sum_{k \in D_j} \alpha_{jk} a(\bar{\Psi}_{jk}, \bar{\Psi}_{lm}) &= (f, \bar{\Psi}_{lm}), \quad \text{for all } -1 \leq l < n, m \in D_j. \end{aligned}$$

Let $\iota : (j, k) \mapsto i$ be the lexicographically enumerating function. That is,

$$\iota(j, k) \leq \iota(l, m) \quad \text{if } j \leq l \text{ or } j = l \text{ and } k \leq m.$$

We then obtain a linear system of the form $\mathbf{A}_n \mathbf{u}_n = \mathbf{f}_n$, where $\mathbf{A}_n$ is the coefficient matrix, $\mathbf{u}_n$ is the unknown column vector, and $\mathbf{f}_n$ is the column vector defined by

$$\begin{aligned} \mathbf{A}_n &= [a(\bar{\Psi}_i, \bar{\Psi}_j) : i, j = 1, \dots, \dim S_n], \\ \mathbf{u}_n &= [\alpha_j : j = 1, \dots, \dim S_n], \\ \mathbf{f}_n &= [(f, \bar{\Psi}_j) : j = 1, \dots, \dim S_n]. \end{aligned}$$

The approximate solution u_n obtained in this way is called the *n*th multilevel solution of (1).

Next, we apply the augmentation method to approximate the next level solution, u_{n+1} . Suppose that u_n is already solved. That is, the matrix representation $\mathbf{u}_n$ of u_n satisfies the equation $\mathbf{A}_n \mathbf{u}_n = \mathbf{f}_n$. We augment the matrix $\mathbf{A}_n$ with submatrices $\mathbf{B}_n$, $\mathbf{C}_n$, and $\mathbf{D}_n$ where

$$\begin{aligned} \mathbf{B}_n &= [a(\bar{\Psi}_i, \bar{\Psi}_j) : i = 1, \dots, \dim S_n, j = \dim S_n + 1, \dots, \dim S_{n+1}], \\ \mathbf{C}_n &= [a(\bar{\Psi}_i, \bar{\Psi}_j) : i = \dim S_n + 1, \dots, \dim S_{n+1}, j = 1, \dots, \dim S_n], \\ \mathbf{D}_n &= [a(\bar{\Psi}_i, \bar{\Psi}_j) : i, j = \dim S_n + 1, \dots, \dim S_{n+1}]. \end{aligned}$$

The coefficient matrix $\mathbf{A}_{n+1}$ corresponding to the $(n+1)$ st level is identified as

$$\mathbf{A}_{n+1} = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix}.$$

Instead of finding the full $(n+1)$ th level solution, u_{n+1} , from $\mathbf{A}_{n+1} \mathbf{u}_{n+1} = \mathbf{f}_{n+1}$ of size $\dim S_{n+1}$, we will approximate u_{n+1} by decomposition the coefficient matrix $\mathbf{A}_{n+1}$ into the sum of an

upper triangular matrix and a lower triangular matrix:

$$\mathbf{A}_{n+1} = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ 0 & \mathbf{D}_n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{C}_n & 0 \end{bmatrix}.$$

If the matrices $\mathbf{A}_n$ and $\mathbf{D}_n$ are nonsingular, there exists a (unique) vector $\mathbf{u}_{n,1}$ satisfying

$$\begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ 0 & \mathbf{D}_n \end{bmatrix} \mathbf{u}_{n,1} + \begin{bmatrix} 0 & 0 \\ \mathbf{C}_n & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_n \\ 0 \end{bmatrix} = \mathbf{f}_{n+1}. \quad (2)$$

Let $u_{n,1} \in S_{n+1}$ be the corresponding element of $\mathbf{u}_{n,1}$. We call $u_{n,1}$ the $(n+1)$ st *multilevel augmentation solution* of (1). The index n refers to the *initial level* n , and 1 refer to a *one step* method to compute u_{n+1} approximately. The linear system of $\dim S_{n+1}$ can be solved by considering two linear systems. One is the size of $\dim S_n$, and another is the size of $\dim S_{n+1} - \dim S_n = 2^n(2p-1)$.

In general, an approximation $u_{n,i+1}$ for $(n+i+1)$ st multilevel solution is defined by setting

$$u_{n,0} = u_n.$$

Suppose that $u_{n,i}$ is already solved. We augment the matrix $\mathbf{A}_{n+i}$ with submatrices $\mathbf{B}_{n+i}$, $\mathbf{C}_{n+i}$, and $\mathbf{D}_{n+i}$ where

$$\mathbf{B}_{n+i} = [a(\overline{\Psi}_i, \overline{\Psi}_j) : i = 1, \dots, \dim S_{n+i}, j = (\dim S_{n+i}) + 1, \dots, \dim S_{n+i+1}],$$

$$\mathbf{C}_{n+i} = [a(\overline{\Psi}_i, \overline{\Psi}_j) : i = (\dim S_{n+i}) + 1, \dots, \dim S_{n+i+1}, j = 1, \dots, \dim S_{n+i}],$$

$$\mathbf{D}_{n+i} = [a(\overline{\Psi}_i, \overline{\Psi}_j) : i, j = (\dim S_{n+i}) + 1, \dots, \dim S_{n+i+1}].$$

The coefficient matrix $\mathbf{A}_{n+i+1}$ corresponding to the $(n+i+1)$ st level is identified as

$$\mathbf{A}_{n+i+1} = \begin{bmatrix} \mathbf{A}_{n+i} & \mathbf{B}_{n+i} \\ \mathbf{C}_{n+i} & \mathbf{D}_{n+i} \end{bmatrix}.$$

We split the coefficient matrix $\mathbf{A}_{n+i+1}$ and solve for $u_{n,i+1} \in S_{n+i+1}$ from the equation

$$\begin{bmatrix} \mathbf{A}_{n+i} & \mathbf{B}_{n+i} \\ 0 & \mathbf{D}_{n+i} \end{bmatrix} \mathbf{u}_{n,i+1} + \begin{bmatrix} 0 & 0 \\ \mathbf{C}_{n+i} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{n,i} \\ 0 \end{bmatrix} = \mathbf{f}_{n+i+1}. \quad (3)$$

This completes the multilevel augmentation algorithm.

4 Error analysis

In this section, we analyze the convergence of the multilevel augmentation method conjunction with the anti-derivatives of the Daubechies wavelets. First, we consider the distance between the weak solution u of (1) and the n th multilevel solution $u_n \in S_n$ obtained by the wavelets of order p . Theorem 4.1 in [13] stated that, for $u \in H_0^1(\Omega) \cap H^s(\Omega)$, there

exists a positive constant C such that

$$\|u - u_n\| + 2^{-n}\|u - u_n\|_1 \leq C(2^{-n})^s \|u\|_s, \quad 1 \leq s \leq p+1.$$

In particular, if we consider separately the distance between u and u_n with the standard L^2 norm $\|\cdot\|$, and the Sobolev norm $\|\cdot\|_1$, we obtain

$$\begin{aligned} \|u - u_n\| &\leq C(2^{-n})^s \|u\|_s, \quad 1 \leq s \leq p+1, \\ \|u - u_n\|_1 &\leq C(2^{-n})^{s-1} \|u\|_s, \quad 1 \leq s \leq p+1. \end{aligned}$$

The above estimations suggest that if we apply the wavelet of order p and $u \in H_0^1(\Omega) \cap H^s(\Omega)$, then the errors measured by the standard norm, and the Sobolev norm in L^2 , decrease by the factors of 2^{p+1} , and 2^p , respectively, from n to $n+1$ level.

Next, we consider the distance between the solution u and the $(n+i)$ th multilevel augmentation solution, $u_{n,i}$, of (1). In the remaining section, we denote by $\mathcal{A}$ the operator corresponding to the matrix $\mathbf{A}$, and we denote by $\mathbf{u}$ the column matrix representing element u .

For $n \in \mathbb{N}$, if the inverse operators $\mathcal{A}_n^{-1}$ and $\mathcal{D}_n^{-1}$ exist, then the $(n+1)$ th multilevel augmentation solution $u_{n,i}$ exists. If there also exist an $N \in \mathbb{N}$, $\alpha > 0$, and $\delta > 0$ such that

$$\|\mathcal{A}_n^{-1}\| \leq \alpha, \quad \|\mathcal{D}_n^{-1}\| \leq \delta, \quad \text{for } n \geq N,$$

and

$$\lim_{n \rightarrow \infty} \|\mathcal{B}_n\| = \lim_{n \rightarrow \infty} \|\mathcal{C}_n\| = 0,$$

the error for our method can be estimated as in the following theorem.

Theorem 1 (Error for multilevel augmentation method) *Let $u \in H_0^1(\Omega)$ be the solution of (1). Suppose that there exist an $N \in \mathbb{N}$, and positive constants α and δ such that for $n \geq N$ the inverse operators $\mathcal{A}_n^{-1}$, $\mathcal{D}_n^{-1}$ exist and*

$$\|\mathcal{A}_n^{-1}\| \leq \alpha, \quad \|\mathcal{D}_n^{-1}\| \leq \delta,$$

and

$$\lim_{n \rightarrow \infty} \|\mathcal{B}_n\| = \lim_{n \rightarrow \infty} \|\mathcal{C}_n\| = 0.$$

If $u \in H_0^1(\Omega) \cap H^s(\Omega)$, then there exist an $M \in \mathbb{N}$ and positive constant c_0 such that, for $n \geq M$ and $i \in \mathbb{N} \cup \{0\}$, we have the estimate

$$\|u - u_{n,i}\| \leq c_0 2^{-2(n+i)s} \|u\|_s, \quad 1 \leq s \leq p+1.$$

Proof By the hypotheses on $\mathcal{A}_n^{-1}$ and $\mathcal{D}_n^{-1}$, we see that the $(n+i)$ th multilevel augmentation solution u_{n+i} and the $(n+i)$ th multilevel augmentation solution $u_{n,i}$ exist for all $n \geq N$ and

$i \in \mathbb{N} \cup \{0\}$. By Theorem 4.1 in [13], there exists a positive constant C such that

$$\|u - u_{n+i}\| \leq C(2^{-(n+i)s})\|u\|_s, \quad 1 \leq s \leq p+1. \quad (4)$$

For $n \geq N$ and $i \in \mathbb{N}$, let W_{n+i-1} be the orthogonal complement of S_{n+i-1} in S_{n+i} . The element $u_{n,i} \in S_{n+i}$ can be uniquely written in the form

$$u_{n,i} = u_{n,i-1}^1 + v_{n,i-1}^1,$$

where $u_{n,i-1}^1 \in S_{n+i-1}$, and $v_{n,i-1}^1 \in W_{n+i-1}$. Since the column matrix $\mathbf{u}_{n,i}$ of $u_{n,i}$ satisfies the equation

$$\begin{bmatrix} \mathbf{A}_{n+i-1} & \mathbf{B}_{n+i-1} \\ 0 & \mathbf{D}_{n+i-1} \end{bmatrix} \mathbf{u}_{n,i} + \begin{bmatrix} 0 & 0 \\ \mathbf{C}_{n+i-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{n,i-1} \\ 0 \end{bmatrix} = \mathbf{f}_{n+i},$$

it also satisfies

$$\mathbf{A}_{n+i} \mathbf{u}_{n,i} = \mathbf{f}_{n+i} + \begin{bmatrix} 0 & 0 \\ \mathbf{C}_{n+i-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{n,i-1}^1 - \mathbf{u}_{n,i-1} \\ 0 \end{bmatrix}. \quad (5)$$

The $(n+i)$ th multilevel solution $u_{n+i} \in S_{n+i}$ satisfies the equation

$$\mathbf{A}_{n+i} \mathbf{u}_{n+i} = \mathbf{f}_{n+i}. \quad (6)$$

Subtracting (6) from (5), we have

$$\mathbf{A}_{n+i}(\mathbf{u}_{n,i} - \mathbf{u}_{n+i}) = \begin{bmatrix} 0 & 0 \\ \mathbf{C}_{n+i-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{n,i-1}^1 - \mathbf{u}_{n,i-1} \\ 0 \end{bmatrix}.$$

Since $\mathbf{A}_{n+i}$ is nonsingular, we have the equation

$$\mathbf{u}_{n,i} - \mathbf{u}_{n+i} = \mathbf{A}_{n+i}^{-1} \begin{bmatrix} 0 & 0 \\ \mathbf{C}_{n+i-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{n,i-1}^1 - \mathbf{u}_{n,i-1} \\ 0 \end{bmatrix}.$$

Since $\lim_{n \rightarrow \infty} \|\mathcal{B}_n\| = \lim_{n \rightarrow \infty} \|\mathcal{C}_n\| = 0$, there exist an $M \in \mathbb{N}$ and $c > 0$ such that for $n \geq M$ and $i \in \mathbb{N}$,

$$\|\mathcal{C}_{n+i-1}\| \leq \frac{2^{-s}}{3\alpha}$$

and

$$\|\mathbf{u}_{n,i-1}^1 - \mathbf{u}_{n,i-1}\| \leq c(2^{-(n+i-1)s})\|u\|_s, \quad 1 \leq s \leq p+1.$$

We have, for $n \geq M$, $i \in \mathbb{N}$, and $1 \leq s \leq p+1$,

$$\|u_{n,i} - u_{n+i}\| \leq \|\mathcal{A}_{n+i}^{-1}\| \|\mathcal{C}_{n+i-1}\| \|\mathbf{u}_{n,i-1}^1 - \mathbf{u}_{n,i-1}\| \leq \frac{c}{3} (2^{-(n+i)s})\|u\|_s. \quad (7)$$

From (4) and (7), we conclude that

$$\begin{aligned}\|u - u_{n,i}\| &\leq \|u - u_{n+i}\| + \|u_{n,i} - u_{n+i}\| \\ &\leq C(2^{-(n+i)s})\|u\|_s + \frac{c}{3}(2^{-(n+i)s})\|u\|_s \\ &\leq \left(C + \frac{c}{3}\right)(2^{-(n+i)s})\|u\|_s.\end{aligned}\quad \square$$

The above theorem suggests that, if the solution $u \in H_0^1(\Omega) \cap H^s(\Omega)$, and we apply the multilevel augmentation method from level $n + i - 1$ to $n + i$ by using the anti-derivatives wavelets of order p , the errors measured in $\|\cdot\|$ decrease by a factor of 2^{p+1} . Thus the behaviors of the decreasing error obtained by the multilevel, and the multilevel augmentation methods, are in the same order.

5 Examples

In this section, we illustrate the efficiency of the multilevel augmentation method in conjunction with the anti-derivatives of Daubechies wavelets of order p to find the numerical solutions of the following boundary value problem:

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (0, R), \quad (8)$$

with the Dirichlet conditions

$$u(0) = u(R) = 0,$$

where $R = 2p - 1$. It should be noted that the general interval (α, β) can be changed to $(0, R)$ by the method of changing variables.

We assume that $f \in L^2(\Omega)$, the coefficients q and r are smooth in the closed interval $[0, R]$ with $q > 0$ and $r \geq 0$.

The variational form of (8) is

$$a(u, v) = (f, v), \quad \text{for all } v \in H_0^1(\Omega), \quad (9)$$

where $a(\cdot, \cdot)$ is the bilinear form defined by

$$a(u, v) = \int_0^R qu'v' + ruv \, dx. \quad (10)$$

Since $A(\cdot, \cdot)$ is continuous and coercive on $H_0^1(\Omega)$, by the Lax-Milgram lemma, there exists a unique weak solution $u \in H_0^1(\Omega)$ for (9).

Note that we can also apply the multilevel augmentation method in conjunction with the anti-derivatives of the Daubechies wavelets order p for the cases with nonzero boundary conditions. For example, we consider the boundary value problem

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (\alpha, \beta), \quad (11)$$

with the boundary conditions

$$u(\alpha) = c, \quad u(\beta) = d.$$

We notice that the linear function

$$y(x) = \frac{\beta c - \alpha d}{\beta - \alpha} + \frac{d - c}{\beta - \alpha} x$$

satisfies the boundary conditions, that is, $y(\alpha) = c$ and $y(\beta) = d$. Let $u = y + w$ be the weak solution of (11). Since $u(\alpha) = c$ and $u(\beta) = d$, $w(\alpha) = w(\beta) = 0$. The variational form of the boundary value problem in this case is

$$a(w, v) = \int_{\alpha}^{\beta} f v \, dx - a(y, v), \quad \text{for all } v \in H_0^1(\alpha, \beta), \quad (12)$$

where $a(\cdot, \cdot)$ is the bilinear form

$$a(w, v) = \int_{\alpha}^{\beta} q w' v' + r w v \, dx. \quad (13)$$

Since $a(\cdot, \cdot)$ is continuous and coercive on $H_0^1(\alpha, \beta)$, by the Lax-Milgram lemma, there exists a unique weak solution $w \in H_0^1(\alpha, \beta)$ for (12).

Example 1 For the first example, we consider the two-point boundary value problem

$$-(xu')' + x^2 u = x^4 - x^3 - 4x + 1, \quad \text{for } x \in (0, 1), \quad (14)$$

with boundary conditions $u(0) = u(1) = 0$.

The exact solution is $u(x) = x^2 - x$. Here, we apply the Daubechies wavelets of order $p = 1$ to solve the problem. Numerical results for each level (n) are shown in Table 1. The column of $\|u - u_n\|$ presents the numerical results obtained by the standard multilevel method. When increasing the level of approximations, the norm of the L^2 error decreases by the factor of 2^s where $1 \leq s \leq 2$. The numerical results by the multilevel augmentation method starting from the second and the third levels are shown in the $\|u - u_{1,n-1}\|$ and $\|u - u_{2,n-2}\|$ columns, respectively. At the same level n , the L^2 errors are in the same order as those of the standard multilevel method, except that its values are slightly greater. These additional errors come from the augmentation part which can be seen from the proof of Theorem 1. It can be seen further that the error from the augmentation method is getting closer to the error from the standard multilevel method as n becomes large.

Example 2 Consider the boundary value problem

$$-(e^{-x} u')' = \pi e^{-x} (\cos(\pi x) + \pi \sin(\pi x)), \quad \text{for } x \in (0, 1), \quad (15)$$

with boundary conditions, $u(0) = u(1) = 0$.

The exact solution is $u = \sin(\pi x)$. Numerical results for $p = 1$ are shown in Table 2. The column of $\|u - u_n\|$ presents the numerical results obtained by the multilevel method.

Table 1 Example 1: Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{1,n-1}\ $	$\ u - u_{2,n-2}\ $
2	3	8.0368e-001		
3	7	5.6412e-001	5.7409e-001	
4	15	3.7341e-001	3.9796e-001	4.0020e-001
5	31	2.2508e-001	2.7097e-001	2.6985e-001
6	63	1.4568e-001	1.7275e-001	1.7293e-001
7	127	8.8603e-002	1.1148e-001	1.1143e-001

Table 2 Example 2: Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{1,n-1}\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $
2	3	1.6427			
3	7	0.6045	0.6458		
4	15	0.2424	0.2488	0.2494	
5	31	0.1124	0.1162	0.1162	0.1170
6	63	0.0589	0.0607	0.0607	0.0606
7	127	0.0309	0.0325	0.0325	0.0325
8	255	0.0154	0.0161	0.0161	0.0161
9	511	0.0085	0.0087	0.0087	0.0087

Table 3 Example 2: Numerical results for $p = 2$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{1,n-1}\ $
0	4	1.6268	
1	7	0.2097	
2	13	0.0250	0.0341
3	25	0.0030	0.0041
4	49	0.0004	0.0005

The L^2 error decreases by the factor of 2^s where $1 \leq s \leq 2$ as expected, and this agrees well with the theoretical results. The L^2 errors obtained by the multilevel augmentation method starting from levels 2, 3, and 4 are shown in the $\|u - u_{1,n-1}\|$, $\|u - u_{2,n-2}\|$, and $\|u - u_{3,n-3}\|$ columns, respectively. Their values are in the same order as the multilevel method.

To apply the Daubechies wavelets of order $p = 2$, we change the interval $[0, 1]$ to $[0, 3]$. So we obtain the following boundary value problem:

$$-(e^{-x}u')' = \frac{\pi}{3}e^{-x}\left(\cos\left(\frac{\pi}{3}x\right) + \frac{\pi}{3}\sin\left(\frac{\pi}{3}x\right)\right), \quad \text{for } x \in (0, 3), \quad (16)$$

with conditions $u(0) = u(3) = 0$.

The exact solution of this problem is $u(x) = \sin(\frac{\pi}{3}x)$. Numerical results from the wavelet basis of order $p = 2$ are shown in Table 3. The rate of L^2 error convergence is faster than that of the case $p = 1$. Here, the L^2 error is four times smaller than that of the previous error level. This agrees with the theoretical result that the L^2 error should decrease by a factor of 2^s , where $1 \leq s \leq 3$.

Example 3 Consider the boundary value problem with nonzero boundary conditions

$$u'' - u' = x^2 - 4x + 1, \quad \text{for } x \in (0, 1), \quad (17)$$

with boundary conditions, $u(0) = -1$, $u(1) = 1$.

Table 4 Example 3: Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{1,n-1}\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $
2	3	0.4221			
3	7	0.1993	0.2123		
4	15	0.0994	0.1051	0.1049	
5	31	0.0502	0.0561	0.0555	0.0555
6	63	0.0278	0.0303	0.0303	0.0302

Table 5 Example 3: Numerical results for $p = 2$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{1,n-1}\ $
0	4	3.3770	
1	7	0.2888	
2	13	0.0351	0.0378
3	25	0.0044	0.0047
4	49	0.0007	0.0008

The exact solution is $u = -\frac{x^3}{3} + x^2 + x + \frac{e^{x-3e+2}}{3(e-1)}$. Numerical results for $p = 1$ are shown in Table 4. Column of $\|u - u_n\|$ presents the numerical results obtained by the multilevel method. The L^2 error is twice smaller than that of the previous error level, and agrees well with the theoretical results. The L^2 errors obtained by the multilevel augmentation method starting from levels 3, 4, 5, and 6 are shown in the $\|u - u_{1,n-1}\|$, $\|u - u_{2,n-2}\|$, and $\|u - u_{3,n-3}\|$ columns, respectively. Their values are in the same order as the multilevel method.

Numerical results from the wavelet basis of order $p = 2$ are shown in Table 5. The rate of L^2 error convergence is faster than that of the case $p = 1$. Here, the L^2 errors is eight times smaller than that of the previous error level.

6 Conclusions

This present work is our attempt to apply the multilevel augmentation method using the anti-derivatives of Daubechies wavelets for approximating two-point boundary value problems with Dirichlet boundary conditions. This method is extended from the multilevel augmentation method that uses polynomial wavelet basis. An error analysis has also been presented. The rate of convergence is by a factor of 2^s , $1 \leq s \leq p + 1$, where p is the Daubechies wavelet order. At the same level, the L^2 error of the multilevel augmentation method is greater than that of the multilevel method, but they are in the same order.

The difficulty of this approach is that the anti-derivatives of Daubechies wavelet cannot be expressed in explicit form. One is required to solve the dilation equation to obtain a wavelet basis in an implicit formula. Here, we have done this using a numerical approximation to obtain the basis function point by point. Also, it is not easy to extend this approach to problems in higher dimension.

The advantage of this method is that we need not solve the full linear system. The unknown coefficients from the previous level can be used to approximate additional unknowns in the next level. Thus, this method can reduce computational time and memory for storing matrix coefficients. Furthermore, by applying the general anti-derivatives of Daubechies wavelets, this method can be used to solve the boundary value problems with Neumann and mixed boundary conditions. Numerical experiments on these problems are in progress and will be reported elsewhere further.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Multilevel anti-derivative wavelets with augmentation for nonlinear boundary value problems

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Abstract

The multilevel augmentation method with the anti-derivatives of the Daubechies wavelets is presented for solving nonlinear two-point boundary value problems. The anti-derivatives of the Daubechies wavelets are applied as the multilevel bases for the subspaces of approximate solutions. This process results in a full nonlinear system that can be solved by the multilevel augmentation method for reducing computational cost. The convergence rate of the present method is shown. It is the order of 2^s , $0 \leq s \leq p$ when p is the order of the Daubechies wavelets. Various examples of the Dirichlet boundary conditions are shown to confirm the theoretical results.

1 Introduction

Many problems in science and engineering can be modeled by nonlinear differential equations. Due to their complexities of both differential equation forms and boundary condition types, analytical solutions are available for only simple problems. Efficient and accurate numerical solutions are then usually required in general. One of the most effective numerical methods relies on variational formulations; see [1, 2], and [3].

The multilevel basis method can be applied with the variational formulation to obtain the approximate solutions of nonlinear problem. This formulation results in the discretization of nonlinear systems with unknown coefficients in the approximate subspace of each basis level. A nonlinear solver such as the Newton iterative method can be used to find approximate solutions for each level required. In order to obtain more accurate results, the number of applied basis levels must be increased, resulting in large nonlinear systems. The computational time increases exponentially with only a small increase in the basis levels. In order to reduce the computational time, we apply the advantage of multilevel bases by connecting the information among basis levels. This approach, the augmentation method, was first introduced by [1, 2, 4]. The fully nonlinear system is divided into two smaller systems and then solved separately.

In multi-scale decompositions, multi-scale piecewise polynomials can be applied in variational formulation (see [1, 4]). These basis types are easily presented and implemented as a numerical algorithm. They can be used for specific types of boundary conditions, and in this case they can represent Dirichlet conditions with zero boundary values while the modified approximation technique can be applied for non-zero Dirichlet con-

ditions. To extend to a more general class of multilevel basis for solving various types of boundary conditions, the anti-derivatives of Daubechies wavelets introduced by Xu and Shann [5] can be applied to solve many kinds of boundary conditions: Dirichlet and Neumann types. Reference [6] has presented the case of the linear boundary value problem and shown that the Daubechies wavelets can be applied in conjunction with the augmentation method to save computational time for Dirichlet boundary value problem.

This study extends the multi-scale decomposition to a nonlinear boundary value problem. We apply the anti-derivatives of the Daubechies wavelets to solving nonlinear boundary value problems. The discretization of the nonlinear differential problem is represented by a nonlinear system that can be solved iteratively by the Newton method. To save computational time, the augmentation method presented by Chen, Chen, Wu and Xu (see *e.g.* [2, 4, 7, 8]) will be applied to solving the nonlinear system under the Daubechies wavelets. Combining these two concepts, as presented here, results in a new numerical method. The rate of convergence is also proved. It is of the order 2^s , $0 \leq s \leq p$ when p is the order of the Daubechies wavelets applied.

Given the interval (a, b) , we use the notation $L^2(a, b)$ to denote the space of square integrable functions on (a, b) with standard inner product $(\cdot, \cdot)$ defined by

$$(u, v) = \int_a^b u(x)v(x) dx,$$

and associated norm $\|\cdot\|$.

Let $H^s(a, b)$ denote the standard Sobolev space with the norm $\|\cdot\|_s$ given by

$$\|v\|_s^2 = \sum_{i=0}^s \int_a^b |v^{(i)}(x)|^2 dx,$$

and the seminorm $|\cdot|_s$ given by

$$|v|_s^2 = \int_a^b |v^{(s)}(x)|^2 dx.$$

For Dirichlet boundary conditions, we work on the solution space

$$H_0^1(a, b) = \{v \in H^1(a, b) | v(a) = v(b) = 0\},$$

equipped with the inner product

$$[u, v] = \int_a^b u'(x)v'(x) dx, \quad \text{for } u, v \in H_0^1(a, b),$$

and associated norm $|\cdot|_1$. It is well known that the norm $|\cdot|_1$ is equivalent to the standard norm $\|\cdot\|_1$ in this space.

We consider the following nonlinear differential equation:

$$\frac{d^2 u}{dx^2} - \beta(x) \frac{du}{dx} = f(x, u), \quad \text{for } x \in [a, b], \quad (1)$$

where $\beta(x) \in L^\infty(a, b)$ and $\varphi(x, u) \in C([a, b] \times \mathbb{R})$, with Dirichlet boundary conditions:

$$u(a) = 0 \quad \text{and} \quad u(b) = 0.$$

To solve numerically the nonlinear problem, the formulation can be summarized as follows.

1. Formulate the variational form of the considering problem. We determine approximate solution, if the variational form has a unique solution. Because it is not our main consideration in this work, we assume that the nonlinear boundary value problem and its variational form have the same isolated solution $u^* \in H_0^1(a, b)$.
2. Choose a sequence $\{S_n\}$ of nested finite-dimensional subspaces of the solution space $H_0^1(a, b)$ such that $\overline{\bigcup_{n \in \mathbb{N}} S_n} = H_0^1(a, b)$. At this step, such finite-dimensional subspaces have anti-derivatives of Daubechies wavelets as their orthonormal bases.
3. Apply the multilevel augmentation method (MAM) to obtain the n th level approximation, which is composed of two smaller systems. One is a linear system. Another one is a nonlinear system. The nonlinear system will be solved iteratively using the Newton method.

This work is organized as follows. In Section 2, we introduce the anti-derivatives of the Daubechies wavelets and the finite-dimensional subspaces of the solution subspaces of $H^1(a, b)$. The concept of multilevel augmentation method is presented in Section 3. The estimations of the optimal error rate are shown in Section 4. Some numerical examples are demonstrated in Section 5. Conclusions are finally drawn in Section 6.

2 Bases for subspaces of $H_0^1(a, b)$

To apply our method for solving the nonlinear boundary value problem, we construct a sequence $\{S_n\}$ of nested finite-dimensional subspaces of the solution space such that $\bigcup_{n \in \mathbb{N}} S_n$ is dense in the solution space. In this section, we will give a brief introduction to the anti-derivatives of wavelets that are the orthonormal bases for the finite-dimensional subspaces of the solution space $H_0^1(a, b)$.

Assume that p is a positive integer. For $j \geq -1$ and $(j, k) \in \mathbb{Z} \times \mathbb{Z}$, let

$$\psi_{jk}(x) = \begin{cases} \phi(x - k), & j = -1, \\ \sqrt{2^j} \psi(2^j x - k), & j \geq 0, \end{cases}$$

be the *Daubechies wavelets of order p* (see e.g. [5, 9] for the details of the construction and additional properties of the wavelets). We shift the interval $[0, 2p - 1]$ to $[a, b]$ by the transformation

$$y = \frac{b - a}{2p - 1} x + a, \quad x \in [0, 2p - 1].$$

Note that the support of the wavelet $\psi_{jk}(y)$ is the interval

$$\left[a + \left(\frac{k}{2^j} \right) \left(\frac{b - a}{2p - 1} \right), a + \left(\frac{k + 2p - 1}{2^j} \right) \left(\frac{b - a}{2p - 1} \right) \right].$$

Set

$$I_{-1} = \{k \in \mathbb{Z} | 2 - 2p \leq k \leq 2p - 2\}q \quad \text{and}$$

$$I_j = \{k \in \mathbb{Z} | 2 - 2p \leq k \leq 2^j(2p - 1) - 1\}, \quad \text{for } j \geq 0.$$

The wavelets $\{\psi_{jk} | j \geq -1, k \in I_j\}$ form a frame for $L^2(a, b)$, that is, the set consisting of all linear expansions is equal to $L^2(a, b)$.

In [5], Xu and Shann introduced the anti-derivatives of the Daubechies wavelets that form orthonormal bases for the finite-dimensional subspaces of solution spaces.

For $j \geq -1$ and $k \in I_j$, the anti-derivatives of wavelets are defined by

$$\Psi_{jk}(y) = \int_a^y \psi_{jk} ds - \frac{y-a}{b-a} \int_a^b \psi_{jk} ds, \quad \text{for } a \leq y \leq b.$$

Note that $\Psi_{jk} \in H_0^1(a, b)$. For $n \in \mathbb{N}_0$, define the finite-dimensional subspace

$$S_n = \text{span}\{\Psi_{jk}(y) | -1 \leq j < n, k \in I_j\}.$$

Let

$$D_{-1} = \{k \in \mathbb{Z} | 2 - 2p < k \leq 2p - 2\} \quad \text{and}$$

$$D_j = \{k \in \mathbb{Z} | 1 - p \leq k \leq 2^j(2p - 1) - p\}, \quad \text{for } j \geq 0.$$

The set $\{\Psi_{jk} | -1 \leq j < n, k \in D_j\}$ is a basis for S_n . Applying the Gram-Schmidt process, the resulting set,

$$\{\bar{\Psi}_{jk} | -1 \leq j < n, k \in D_j\},$$

is an orthonormal basis for S_n with the inner product $[\cdot, \cdot]$ in $H_0^1(a, b)$. For the sake of simplicity when we consider the algebraic system, we will enumerate the double indices lexicographically. The resulting set $\{\bar{\Psi}_j | 1 \leq j \leq \dim S_n\}$ is an orthonormal basis for S_n .

3 Multilevel augmentation method

In this section, we summarize the main concepts of multilevel augmentation method for solving the nonlinear boundary value problems. Readers can refer to [2, 4, 7, 8] for details.

The variational form of the nonlinear differential equation (1) is: Find $u \in H_0^1(a, b)$

$$(u', v') - A(u, v) = 0, \quad \text{for all } v \in H_0^1(a, b), \quad (2)$$

where

$$\begin{aligned} A(u, v) &= -(\beta(x)u' + \varphi(x, u), v) \\ &= -\int_a^b (\beta(x)u' + \varphi(x, u))v dx \\ &= -\int_a^b (\beta(x)u' + \varphi(x, u))v dx. \end{aligned}$$

Suppose that $u^* \in H_0^1(a, b)$ is the common isolated solution of given differential equation and its variational form. We will solve the variational form (2). Let S_n be a nested sequence of finite-dimensional subspaces of such that $\bigcup_{n \in \mathbb{N}} S_n$ is dense in H . Let $\{\bar{\Psi}_j | j = 1, 2, \dots, \dim S_n\}$ be an orthonormal basis for S_n and the n th level approximate solution

$u_n \in S_n$ be of the form

$$u_n = \sum_{j=1}^{\dim S_n} \alpha_j \bar{\Psi}_j,$$

satisfying

$$(u'_n, \bar{\Psi}'_k) - A(u_n, \bar{\Psi}_k) = 0, \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

Note that by the orthonormal property of the basis, we have

$$\alpha_k = A(u_n, \bar{\Psi}_k), \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

Suppose that u_n is already solved. Instead of solving u_{n+1} directly from the nonlinear system of $\dim S_{n+1}$, we apply the multilevel augmentation method to find an approximate solution in the next level, $u_{n,1}$. There are two main steps here. Firstly, we solve for $u_{n+1}^H = \sum_{j=\dim S_n+1}^{\dim S_{n+1}} \alpha_j^1 \bar{\Psi}_j \in S_{n+1} \setminus S_n$ from the system

$$((u_n + u_{n+1}^H)', \bar{\Psi}'_k) = A(u_n, \bar{\Psi}_k), \quad \text{for all } k = \dim S_n + 1, \dots, \dim S_{n+1}.$$

We then obtain

$$\alpha_j^1 = A(u_n, \bar{\Psi}_j), \quad \text{for all } j = \dim S_n + 1, \dots, \dim S_{n+1}.$$

Secondly, we solve for $u_{n+1}^L = \sum_{j=1}^{\dim S_n} \alpha_j^1 \bar{\Psi}_j \in S_n$ from the nonlinear system

$$\begin{aligned} ((u_{n+1}^L + u_{n+1}^H)', \bar{\Psi}'_k) &= A(u_{n+1}^L + u_{n+1}^H, \bar{\Psi}_k), \quad \text{for all } k = 1, 2, \dots, \dim S_n, \\ \left(\left(\sum_{j=1}^{\dim S_{n+1}} \alpha_j^1 \bar{\Psi}_j \right)', \bar{\Psi}'_k \right) \\ &= A \left(\sum_{j=1}^{\dim S_n} \alpha_j^1 \bar{\Psi}_j + \sum_{j=\dim S_n+1}^{\dim S_{n+1}} \alpha_j^1 \bar{\Psi}_j, \bar{\Psi}_k \right), \quad \text{for all } k = 1, 2, \dots, \dim S_n. \end{aligned}$$

That is, we solve for α_j^1 where $j = 1, \dots, \dim S_n$ from

$$\alpha_k^1 = A \left(\sum_{j=1}^{\dim S_n} \alpha_j^1 \bar{\Psi}_j + \sum_{j=\dim S_n+1}^{\dim S_{n+1}} \alpha_j^1 \bar{\Psi}_j, \bar{\Psi}_k \right), \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

Finally, we obtain the approximate solution at this level by setting

$$u_{n,1} = u_{n+1}^L + u_{n+1}^H = \sum_{j=1}^{\dim S_{n+1}} \alpha_j^1 \bar{\Psi}_j.$$

For $i \in \mathbb{N}$, suppose that $u_{n,i}$ is already solved, say,

$$u_{n,i} = \sum_{j=1}^{\dim S_{n+i}} \alpha_j^i \bar{\Psi}_j.$$

The $(n + i + 1)$ th multilevel augmentation solution, $u_{n,i+1}$, can be solved inductively. We begin with solving for $u_{n+i+1}^H = \sum_{j=\dim S_{n+1}}^{\dim S_{n+i+1}} \alpha_j^{i+1} \bar{\Psi}_j \in S_{n+i+1} \setminus S_n$ from the system

$$\alpha_j^{i+1} = A(u_{n,i}, \bar{\Psi}_j), \quad \text{for all } j = \dim S_n + 1, \dots, \dim S_{n+i+1}.$$

Then we solve $u_{n+i+1}^L = \sum_{j=1}^{\dim S_n} \alpha_j^{i+1} \bar{\Psi}_j \in S_n$ from the nonlinear system

$$\left((u_{n+i+1}^L + u_{n+i+1}^H)', \bar{\Psi}_k' \right) = A(u_{n+i+1}^L + u_{n+i+1}^H, \bar{\Psi}_k), \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

That is, we obtain α_j^{i+1} where $j = 1, \dots, \dim S_n$ from

$$\alpha_k^{i+1} = A \left(\sum_{j=1}^{\dim S_n} \alpha_j^{i+1} \bar{\Psi}_j + \sum_{j=\dim S_{n+1}}^{\dim S_{n+i+1}} \alpha_j^{i+1} \bar{\Psi}_j, \bar{\Psi}_k \right), \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

Finally, we obtain the approximate solution at this level by setting

$$u_{n,i+1} = u_{n+i+1}^L + u_{n+i+1}^H = \sum_{j=1}^{\dim S_{n+i+1}} \alpha_j^{i+1} \bar{\Psi}_j.$$

It should be noted that the original full nonlinear system of $\dim S_{n+i+1}$ can be solved in the augmentation method by just solving the smaller nonlinear system of the fixed size $\dim S_n$. The increasing number of unknown coefficients when increasing the level approximation can be solved by the corresponding linear systems. Specially for our presented orthonormal basis, the linear system is easy to solve. The unknown coefficients in the higher level are obtained directly. Overall, the computational time can then be reduced greatly by this method.

Algorithm: The multilevel augmentation method based on the Galerkin method.

Let n, i be two fixed positive integers.

Step 1: Solve the nonlinear system

$$(u_n', \bar{\Psi}_k') - A(u_n, \bar{\Psi}_k) = 0, \quad \text{for all } k = 1, 2, \dots, \dim S_n$$

and obtain the solution u_n . Let $u_{n,0} := u_n$ and $m := 1$.

Step 2: Compute

$$\alpha_j^m = A(u_{n,0}, \bar{\Psi}_j), \quad \text{for all } j = \dim S_n + 1, \dots, \dim S_{n+m}.$$

Define $u_{n,m}^H := \sum_{j=\dim S_{n+1}}^{\dim S_{n+m}} \alpha_j^m \bar{\Psi}_j$.

Step 3: Solve the nonlinear system

$$\alpha_k^m = A \left(\sum_{j=1}^{\dim S_n} \alpha_j^m \bar{\Psi}_j + \sum_{j=\dim S_{n+1}}^{\dim S_{n+m}} \alpha_j^m \bar{\Psi}_j, \bar{\Psi}_k \right), \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

Define $u_{n,m}^L := \sum_{j=1}^{\dim S_n} \alpha_j^m \bar{\Psi}_j$ and let $u_{n,m} = u_{n,m}^L + u_{n,m}^H$.

Step 4: Set $m \leftarrow m + 1$ and go back to Step 2 until $m = i$.

The computational complexity which is measured by the number of multiplications and functional evaluations used in the computation of the above algorithm is of the order $\mathcal{O}(\dim S_{n+m})$. More details of complexity analysis can be found in [4] and [2].

4 Error analysis

In this section, we will show the convergent rate of the multilevel augmentation method in conjunction with the anti-derivatives of the Daubechies wavelets. Let u^* be the isolated solution of (2), $u_n \in S_n$ be the n th (standard) multilevel solution obtained from the wavelets of order p and $u_{n,i}$ be the $(n + i)$ th multilevel augmentation solution of (2).

Throughout this section, we assume that φ satisfies the following conditions:

- (i) $\varphi(x, u)$ is a real continuous function in $(x, u) \in [a, b] \times \mathbb{R}$, and satisfies the Lipschitz condition with respect to u for $|u| \leq R, R \geq 0$, that is,

$$|\varphi(x, v) - \varphi(x, w)| \leq M_1 |u - w|, \quad |v| \leq R, |w| \leq R,$$

for some positive constant M_1 .

- (ii) $\varphi(x, u)$ is continuously differentiable with respect to u for all $x \in [a, b]$, and all $v \in B(u^*, \rho) := \{v | |v - u^*| \leq \rho\}$, for some $\rho > 0$, and there exists a positive constant M_2 such that

$$|\varphi_u(x, v) - \varphi_u(x, w)| \leq M_2 |v - w|, \quad \text{for all } v, w \in B(u^*, \rho).$$

The following lemma was proved in Section 3 of [4] but for convenience of the reader, we have reproduced and included its proof for ready reference.

Lemma 4.1 *Suppose that $\varphi(x, u)$ satisfies the conditions (i) and (ii). Then there exists a continuous and compact operator $\mathcal{K} : H_0^1(a, b) \rightarrow H_0^1(a, b)$ such that*

$$(\beta(x)u' + \varphi(x, u), v) = [\mathcal{K}u, v], \quad \text{for all } v \in H_0^1(a, b),$$

and that $\mathcal{K}$ is Fréchet differentiable on the closed ball $B(u^, \rho)$ and the Fréchet derivative $\mathcal{K}'$ satisfies the Lipschitz condition, that is, there exists a positive constant C*

$$|\mathcal{K}'(v) - \mathcal{K}'(w)|_1 \leq C |v - w|_1, \quad v, w \in B(u^*, \rho).$$

Proof For a given $u \in H_0^1(a, b)$, the operators $f_u(\cdot) := (\beta(x)u', \cdot)$ and $g_u(\cdot) := (\varphi(x, u), \cdot)$ are bounded linear functionals on $H_0^1(a, b)$. By the Riesz representation theorem, there exist $\mathcal{K}_1 u, \mathcal{K}_2 u \in H_0^1(a, b)$ such that

$$(\beta(x)u', v) = [\mathcal{K}_1 u, v], \quad \text{for all } v \in H_0^1(a, b),$$

and

$$(\varphi(x, u), v) = [\mathcal{K}_2 u, v], \quad \text{for all } v \in H_0^1(a, b).$$

We define the linear operator $\mathcal{K}_1 : H_0^1(a, b) \rightarrow H_0^1(a, b)$ by

$$\mathcal{K}_1 : u \mapsto \mathcal{K}_1 u, \quad \text{for all } u \in H_0^1(a, b),$$

and define the nonlinear operator $\mathcal{K}_2 : H_0^1(a, b) \rightarrow H_0^1(a, b)$ by

$$\mathcal{K}_2 : u \mapsto \mathcal{K}_2 u, \quad \text{for all } u \in H_0^1(a, b).$$

The operator $\mathcal{K}_1$ is compact. By the linearity of $\mathcal{K}_1$, we see that $\mathcal{K}_1$ is continuous (or bounded) and its Fréchet derivative is itself. Then there exists a positive constant c_1 such that

$$|\mathcal{K}_1'(v) - \mathcal{K}_1'(w)|_1 = |\mathcal{K}_1(v) - \mathcal{K}_1(w)|_1 \leq c_1 |v - w|_1.$$

By Proposition 3.1 in [4], the nonlinear $\mathcal{K}_2$ is continuous, compact and Fréchet differentiable on the closed ball $B(u^*, \rho)$. Moreover, for any $v, w \in B(u^*, \rho)$, there exists a positive constant c_2 such that

$$|\mathcal{K}_2'(v) - \mathcal{K}_2'(w)|_1 \leq c_2 |v - w|_1.$$

Define $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$. Then $\mathcal{K}$ continuous, compact and Fréchet differentiable on the closed ball $B(u^*, \rho)$. Set $C := c_1 + c_2$, the Fréchet derivative $\mathcal{K}'$ therefore satisfies the Lipschitz condition. $\square$

Next, we consider the difference between the isolated solution u^* and the $(n + i)$ th multilevel augmentation solution, $u_{n,i}$, of (1).

Theorem 4.2 *Let u^* be an isolated solution of (1) and $u_{n,i}$ be the $(n + i)$ th multilevel augmentation solution. Let $\mathcal{K}'(u^*)$ be the Fréchet derivative of $\mathcal{K}$ at u^* . If 1 is not an eigenvalue of $\mathcal{K}'(u^*)$ and if $u^* \in H^{s+1}(a, b)$, then there exist a positive constant β and a positive integer N such that, for all $n \geq N$, and $i \in \mathbb{N}_0$,*

$$\|u^* - u_{n,i}\| \leq c 2^{-(n+i+1)s} \|u^*\|_{s+1}, \quad 0 \leq s \leq p.$$

Proof The variational form of (1) can be written in the form of $(\mathcal{I} - \mathcal{K})u = 0$. By Lemma 4.1, the operator $\mathcal{K}$ is completely continuous and Fréchet differentiable on the closed ball $B(u^*, \rho)$ and the Fréchet derivative $\mathcal{K}'$ satisfies the Lipschitz condition.

Let $E_n := \inf\{|u^* - v|_1 | v \in S_n\}$. By Theorem 3.1 in [5],

$$E_n \leq C 2^{-(n+1)s} |u^*|_{s+1}, \quad 0 \leq s \leq p,$$

where C is a constant depending on n and s . Fix $0 \leq s \leq p$. For $n \in \mathbb{N}$, set $\gamma_n = C 2^{-(n+1)s} |u^*|_{s+1}$. Then $\gamma_{n+1}/\gamma_n \geq \sigma := 2^{-s} > 0$. Referring to Lemma 2.2 in [4], there exist a positive constant ρ and a positive integer N such that, for all $n \geq N$, and $i \in \mathbb{N}_0$,

$$|u^* - u_{n,i}|_1 \leq (\rho + 1) \gamma_{n+i}.$$

Thus

$$|u^* - u_{n,i}|_1 \leq (\rho + 1)C2^{-(n+i+1)s} |u^*|_{s+1}.$$

Since the norm $|\cdot|_1$ is equivalent to the standard norm $\|\cdot\|_1$, $\|u\| \leq \|u\|_1$, and $|u|_{s+1} \leq \|u\|_{s+1}$, there exists a positive constant c such that

$$\|u^* - u_{n,i}\| \leq c2^{-(n+i+1)s} \|u^*\|_{s+1}. \quad \square$$

The above estimation suggests that, if we apply the wavelet of order p , the solution $u \in H_0^1(a, b) \cap H^{s+1}(a, b)$. If we apply the multilevel augmentation method from level $n + i - 1$ to $n + i$ by the anti-derivatives wavelets of order p , the errors measured in L^2 -norm decrease at most by a factor of 2^p . Consequently, the errors obtained by the standard multilevel and the multilevel augmentation methods decrease with the same order.

5 Numerical examples

In this section, we illustrate the accuracy of the multilevel augmentation method in conjunction with the anti-derivatives of the Daubechies wavelets of order p for solving non-linear boundary value problems with Dirichlet boundary conditions.

Example 5.1 Consider the boundary value problem [4]

$$u''(x) = e^{u(x)}, \quad \text{for } x \in (0, 1), \quad (3)$$

with boundary conditions

$$u(0) = u(1) = 0.$$

The isolated solution is $u^*(x) = -\ln 2 + 2 \ln[c \sec\{c(x - 1/2)/2\}]$, with $c \approx 1.3360557 \dots$

The variational form is

$$(u'(x), v') = -(e^{u(x)}, v), \quad \text{for all } v \in H_0^1(0, 1),$$

where the inner product $(f, g) = \int_0^1 fg \, dx$. Equivalently,

$$(u'(x), \overline{\Psi}_j') = -(e^{u(x)}, \overline{\Psi}_j), \quad \text{for all } \overline{\Psi}_j \in H_0^1(0, 1),$$

where $\{\overline{\Psi}_j\}$ is an orthonormal basis for $H_0^1(0, 1)$.

Find the first level of approximate solution: standard multilevel method. For $p = 1$, the set $\{\overline{\Psi}_{00}\}$ is a basis for the subspace S_1 . Suppose that the approximate solution $u_1 \in S_1$ is of the form

$$u_1 = a_{00} \overline{\Psi}_{00}.$$

We need to know a_{00} . It can be approximated from

$$a_{00}(\overline{\Psi}_{00}', \overline{\Psi}_{00}') = -(e^{a_{00} \overline{\Psi}_{00}}, \overline{\Psi}_{00}).$$

This becomes a nonlinear equation with one unknown, a_{00} . It can be solved by the Newton method. At this step, we obtain the first level approximate solution.

Find the second level of approximate solution: standard multilevel method. For $p = 1$, the set of $\{\bar{\Psi}_{00}, \bar{\Psi}_{10}, \bar{\Psi}_{11}\}$ is represented as the multilevel bases for the subspace S_2 . Suppose that the approximate solution $u_2 \in S_2$ is in the form of

$$u_2 = b_{00}\bar{\Psi}_{00} + b_{10}\bar{\Psi}_{10} + b_{11}\bar{\Psi}_{11}.$$

We will find approximate solution $u_2 \in S_2$ by solving for b_{00}, b_{10}, b_{11} from

$$(u_2'(x), \bar{\Psi}_{00}') = -(e^{u_2(x)}, \bar{\Psi}_{00}'),$$

$$(u_2'(x), \bar{\Psi}_{10}') = -(e^{u_2(x)}, \bar{\Psi}_{10}'),$$

$$(u_2'(x), \bar{\Psi}_{11}') = -(e^{u_2(x)}, \bar{\Psi}_{11}'),$$

or

$$b_{00} = -(e^{u_2(x)}, \bar{\Psi}_{00}'),$$

$$b_{10} = -(e^{u_2(x)}, \bar{\Psi}_{10}'),$$

$$b_{11} = -(e^{u_2(x)}, \bar{\Psi}_{11}').$$

We obtain the system of nonlinear equations with three unknown b_{00}, b_{10} , and b_{11} . We can solve by the Newton method. At this step, we obtain the second level of approximate solution by the standard multilevel method.

Find the third level of approximate solution: standard multilevel method. For $p = 1$, the set $\{\bar{\Psi}_{00}, \bar{\Psi}_{10}, \bar{\Psi}_{11}, \bar{\Psi}_{20}, \bar{\Psi}_{21}, \bar{\Psi}_{22}, \bar{\Psi}_{23}\}$ is represented as the multilevel basis for the subspace S_3 . Suppose that the approximate solution $u_3 \in S_3$ is of the form

$$u_3 = c_{00}\bar{\Psi}_{00} + c_{10}\bar{\Psi}_{10} + c_{11}\bar{\Psi}_{11} + c_{20}\bar{\Psi}_{20} + c_{21}\bar{\Psi}_{21} + c_{22}\bar{\Psi}_{22} + c_{23}\bar{\Psi}_{23}.$$

We will find approximate solution $u_3 \in S_3$ by solving for $c_{00}, c_{10}, c_{11}, c_{20}, c_{21}, c_{22}, c_{23}$ from

$$c_{00} = -(e^{u_3(x)}, \bar{\Psi}_{00}'),$$

$$c_{10} = -(e^{u_3(x)}, \bar{\Psi}_{10}'),$$

$$c_{11} = -(e^{u_3(x)}, \bar{\Psi}_{11}'),$$

$$c_{20} = -(e^{u_3(x)}, \bar{\Psi}_{20}'),$$

$$c_{21} = -(e^{u_3(x)}, \bar{\Psi}_{21}'),$$

$$c_{22} = -(e^{u_3(x)}, \bar{\Psi}_{22}'),$$

$$c_{23} = -(e^{u_3(t)}, \bar{\Psi}_{23}').$$

We obtain a system of nonlinear equations with seven unknowns that can be solved by the Newton method. At this step, we obtain the third level of approximation by the standard

multilevel method. The calculations can be extended to any number of levels, depending on the numerical accuracy required.

Next, we will show the calculation steps of the multilevel augmentation method. Assume that we have obtained $u_2 \in S_2$: $u_2 = b_{00}\overline{\Psi}_{00} + b_{10}\overline{\Psi}_{10} + b_{11}\overline{\Psi}_{11}$ from the second level of the standard multilevel method. The third level of approximation can be obtained by the multilevel augmentation method as follows.

Find the third level of approximate solution: multilevel augmentation method. Next, we will show how to find $u_{2,1}$, which is the approximation of u_3 in the third level. Assume that we have already obtained the second level of the approximate solution. The approximate solution $u_{2,1} \in S_3$ is of the form

$$u_{2,1} = u_{2,1}^L + u_{2,1}^H,$$

where $u_{2,1}^L \in S_2$ and $u_{2,1}^H \in S_3 \setminus S_2$. Suppose that

$$\begin{aligned} u_{2,1}^L &= \alpha_{00}^1 \overline{\Psi}_{00} + \alpha_{10}^1 \overline{\Psi}_{10} + \alpha_{11}^1 \overline{\Psi}_{11}, \\ u_{2,1}^H &= \alpha_{20}^1 \overline{\Psi}_{20} + \alpha_{21}^1 \overline{\Psi}_{21} + \alpha_{22}^1 \overline{\Psi}_{22} + \alpha_{23}^1 \overline{\Psi}_{23}. \end{aligned}$$

There are two sub-steps to find $u_{2,1}^L$ and $u_{2,1}^H$.

We first solve for $\alpha_{20}^1, \alpha_{21}^1, \alpha_{22}^1, \alpha_{23}^1$ from

$$\begin{aligned} \alpha_{20}^1 &= -(e^{u_2(x)}, \overline{\Psi}_{20}), \\ \alpha_{21}^1 &= -(e^{u_2(x)}, \overline{\Psi}_{21}), \\ \alpha_{22}^1 &= -(e^{u_2(x)}, \overline{\Psi}_{22}), \\ \alpha_{23}^1 &= -(e^{u_2(x)}, \overline{\Psi}_{23}). \end{aligned}$$

Next, we can solve for $\alpha_{00}^1, \alpha_{10}^1, \alpha_{11}^1$ from

$$\begin{aligned} \alpha_{00}^1 &= -(e^{u_{2,1}(x)}, \overline{\Psi}_{00}), \\ \alpha_{10}^1 &= -(e^{u_{2,1}(x)}, \overline{\Psi}_{10}), \\ \alpha_{11}^1 &= -(e^{u_{2,1}(x)}, \overline{\Psi}_{11}). \end{aligned}$$

At this step, we solved iteratively by the Newton method. This procedure shows that the present scheme can reduce the computational time when solving the nonlinear systems.

The approximate solution $u_{1,2} \in S_3$ is finally obtained,

$$\begin{aligned} u_{1,2} &= u_{1,2}^L + u_{1,2}^H \\ &= \alpha_{00}^1 \overline{\Psi}_{00} + \alpha_{10}^1 \overline{\Psi}_{10} + \alpha_{11}^1 \overline{\Psi}_{11} + \alpha_{20}^1 \overline{\Psi}_{20} + \alpha_{21}^1 \overline{\Psi}_{21} + \alpha_{22}^1 \overline{\Psi}_{22} + \alpha_{23}^1 \overline{\Psi}_{23}. \end{aligned}$$

Find the fourth level of approximate solution: multilevel augmentation method. Next, we will show how to find $u_{2,2}$, which is the approximation of u_4 in the third level. Assume that we have already obtained the second level of approximate solution. The approximate solution $u_{2,2} \in S_4$ is of the form

$$u_{2,2} = u_{2,2}^L + u_{2,2}^H,$$

where $u_{2,2}^L \in S_2$ and $u_{2,2}^H \in S_4 \setminus S_2$. Suppose that

$$u_{2,2}^L = \alpha_{00}^2 \bar{\Psi}_{00} + \alpha_{10}^2 \bar{\Psi}_{10} + \alpha_{11}^2 \bar{\Psi}_{11},$$

$$u_{2,2}^H = \sum_{j=0}^3 \alpha_{2j}^2 \bar{\Psi}_{2j} + \sum_{j=0}^7 \alpha_{3j}^2 \bar{\Psi}_{3j}.$$

There are two sub-steps to find $u_{2,2}^L$ and $u_{2,2}^H$.

We first solve for $\alpha_{20}^2, \dots, \alpha_{23}^2, \alpha_{30}^2, \alpha_{31}^2, \dots, \alpha_{37}^2$, from

$$\alpha_{2j}^2 = -(e^{u_{2,1}(t)}, \bar{\Psi}_{2j}) \quad \text{for } j = 0, 1, 2, 3,$$

$$\alpha_{3j}^2 = -(e^{u_{2,1}(t)}, \bar{\Psi}_{3j}) \quad \text{for } j = 0, 1, 2, \dots, 7.$$

Next, we can solve for $\alpha_{00}^2, \alpha_{10}^2, \alpha_{11}^2$ from

$$\alpha_{00}^2 = -(e^{u_{2,2}(t)}, \bar{\Psi}_{00}),$$

$$\alpha_{10}^2 = -(e^{u_{2,2}(t)}, \bar{\Psi}_{10}),$$

$$\alpha_{11}^2 = -(e^{u_{2,2}(t)}, \bar{\Psi}_{11}).$$

The approximate solution $u_{2,2} \in S_4$ is finally obtained,

$$\begin{aligned} u_{2,2} &= u_{2,2}^L + u_{2,2}^H \\ &= \alpha_{00}^2 \bar{\Psi}_{00} + \alpha_{10}^2 \bar{\Psi}_{10} + \alpha_{11}^2 \bar{\Psi}_{11} + \sum_{j=0}^3 \alpha_{2j}^2 \bar{\Psi}_{2j} + \sum_{j=0}^7 \alpha_{3j}^2 \bar{\Psi}_{3j}. \end{aligned}$$

The multilevel augmentation method for calculating higher levels can be performed using the same procedure. The accuracy of the numerical solution depends on the starting level of the augmentation method.

Since we have to calculate the inner product of functions and bases, we perform it numerically by the trapezoidal rule in all of the examples. The derivatives are approximated using the central difference formula.

In Example 5.1, we apply the Daubechies wavelets of order $p = 1$ and $p = 2$ to solve the problem. The numerical results are shown in Tables 1 and 2, respectively. The column of $\|u - u_n\|$ shows the error in the L^2 norm obtained from the standard multilevel method. The L^2 error decreases by the factor of 2^1 when the applied level increases. The next column, Time_n , is the computing time in seconds run on the machine processor 2.3 GHz, Intel Core i7, memory 4 GB, 1600 MHz. It takes exponential order when the number of level increases. The L^2 error when starting with the levels 2 and 3 are also shown. The decreasing in L^2 error agrees with the theoretical results. Moreover, when we augment one more level, the computing time of our algorithm increases slightly, which is consistent with the linear complexity estimate.

The L^2 errors by the augmentation method are of the same order as those of the standard multilevel method, when applied at the same level. When comparing the results between $p = 1$ and $p = 2$, the L^2 errors for $p = 2$ decrease faster than those for $p = 1$. The rate of convergence decreases by the factor of $C2^2$, for some constants C . The results from Tables 1 and 2 confirm the theoretical results of our main theorem.

Table 1 Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	Time_n	$\ u - u_{1,n-1}\ $	$\text{Time}_{1,n-1}$	$\ u - u_{2,n-2}\ $	$\text{Time}_{2,n-2}$	$\ u - u_{3,n-3}\ $	$\text{Time}_{3,n-3}$
1	3	2.3549e-1	3.5800e-2	2.3962e+0	8.3000e-3				
2	7	1.0022e-1	2.3140e-1	1.3037e+0	3.7700e-1	1.0073e+0	1.7500e-2		
3	15	4.8223e-2	1.7408e+0	4.5435e-1	6.2230e-1	4.3987e-1	4.4840e-1	4.3458e-1	5.6600e-2
4	31	2.4206e-2	1.6873e+1	2.1642e-1	7.3150e-1	2.1044e-1	7.3400e-1	2.0832e-1	4.9160e-1
5	63	1.3392e-2	1.3638e+2	1.1440e-1	8.4620e-1	1.1187e-1	8.3810e-1	1.1097e-1	8.2520e-1

Table 2 Numerical results for $p = 2$

n	$\dim S_n$	$\ u - u_n\ $	Time_n	$\ u - u_{1,n-1}\ $	$\text{Time}_{1,n-1}$	$\ u - u_{2,n-2}\ $	$\text{Time}_{2,n-2}$	$\ u - u_{3,n-3}\ $	$\text{Time}_{3,n-3}$
1	7	9.6680e-5	3.8360e-1	7.9852e-2	2.4100e-2				
2	13	1.2588e-5	1.9599e+0	5.1073e-3	4.5320e-1	5.1347e-3	4.5900e-2		
3	25	2.2134e-06	1.2856e+01	1.6628e-04	6.4740e-1	1.8461e-4	5.1630e-1	1.8573e-4	1.2630e-1

Table 3 Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	Time_n	$\ u - u_{1,n-1}\ $	$\text{Time}_{1,n-1}$	$\ u - u_{2,n-2}\ $	$\text{Time}_{2,n-2}$	$\ u - u_{3,n-3}\ $	$\text{Time}_{3,n-3}$
1	3	2.3184e+0	3.5800e-2	2.3962e+0	8.3000e-3				
2	7	9.8550e-1	2.3140e-1	1.3037e+0	3.7700e-1	1.0073e+0	1.7500e-2		
3	15	4.3358e-1	1.7408e+0	4.5435e-1	6.2230e-1	4.3987e-1	4.4840e-1	4.3458e-1	5.6600e-2
4	31	2.0971e-1	1.6873e+1	2.1642e-1	7.3150e-1	2.1044e-1	7.3400e-1	2.0832e-1	4.9160e-1
5	63	1.1300e-1	1.3638e+2	1.1440e-1	8.4620e-1	1.1187e-1	8.3810e-1	1.1097e-1	8.2520e-1

Example 5.2 Consider the boundary value problem

$$u''(x) + u^2(x) = \sin^2(\pi x) - \pi^2 \sin(\pi x), \quad \text{for } x \in (0, 1), \quad (4)$$

with Dirichlet boundary conditions

$$u(0) = u(1) = 0.$$

The isolated solution is $u^*(x) = \sin(\pi x)$.

Numerical results for $p = 1$ is shown in Table 3. The L^2 norm of the error decreases by a factor of 2. The L^2 errors of the multilevel augmentation method are slightly greater than those of the multilevel method at the same level and decrease in the same order as the standard multilevel method.

In our last example, we will test our present method to solve the nonlinear boundary value problem with non-zero Dirichlet boundary conditions.

Consider

$$u'' - \beta(x)u' = \varphi(x, u), \quad \text{for } x \in (a, b), \quad (5)$$

with the Dirichlet boundary condition

$$u(a) = c \quad \text{and} \quad u(b) = d.$$

Table 4 Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	Time_n	$\ u - u_{1,n-1}\ $	$\text{Time}_{1,n-1}$	$\ u - u_{2,n-2}\ $	$\text{Time}_{2,n-2}$	$\ u - u_{3,n-3}\ $	$\text{Time}_{3,n-3}$
1	3	3.6441e+0	5.5200e-2	4.0184e+0	1.5600e-2				
2	7	1.6750e+0	3.4570e-1	2.0257e+0	4.5810e-1	1.7817e+0	3.6700e-2		
3	15	7.4431e-1	2.7760e+0	9.7397e-1	7.0520e-1	8.1878e-1	4.3340e-1	7.6238e-1	1.5370e-1
4	31	3.6599e-1	2.1067e+1	4.8121e-1	7.5820e-1	3.9986e-1	6.5820e-1	3.7317e-1	6.2760e-1
5	63	1.9502e-1	1.6468e+2	2.4916e-1	9.5190e-1	2.0838e-1	8.5460e-1	1.9563e-1	1.1097e+0

We assume that $\beta \in L^\infty(a, b)$, $\varphi \in C([a, b], \mathbb{R})$ and u is the unknown to be determined. Assume the solution u as

$$u = \left[\frac{d-c}{b-a}x + \frac{bc-ad}{b-a} \right] + w.$$

The variational formulation of (5) is to find $w \in H_0^1(a, b)$ such that

$$\int_a^b \left[w'' - \beta(x) \left(w' + \frac{d-c}{b-a} \right) \right] v \, dx - \int_a^b \varphi \left(x, w + \frac{d-c}{b-a}x + \frac{bc-ad}{b-a} \right) v \, dx = 0,$$

for all $v \in H_0^1(a, b)$.

By applying this technique, we can solve the nonlinear problem with non-zero Dirichlet conditions.

Example 5.3 Consider the boundary value problem

$$u'' + u^3 = (\sin(\pi x) + x)^3 - \pi^2 \sin(\pi x), \quad (6)$$

with Dirichlet boundary conditions

$$u(0) = 0, \quad u(1) = 1.$$

The isolated solution is $u^*(x) = \sin(\pi x) + x$.

The variational form of this problem is

$$((w+x)'' + (w+x)^3, v) = ((\sin(\pi x) + x)^3 - \pi^2 \sin(\pi x), v), \quad \text{for all } v \in H_0^1(0, 1).$$

Equivalently,

$$(w'' + (w+x)^3, v) = ((\sin(\pi x) + x)^3 - \pi^2 \sin(\pi x), v), \quad \text{for all } v \in H_0^1(0, 1).$$

Numerical results for the wavelet basis of order $p = 1$ are shown in Table 4. At the same level, the L^2 errors of the multilevel augmentation method are slightly greater than those of the standard multilevel method. The rate of convergence in the L^2 error agrees well with the theoretical results. The computing time of the augmentation method is of the same order when the number of augmented levels increases.

6 Conclusions

This study extends the multi-scale decomposition to a nonlinear boundary value problem. We apply the anti-derivatives of the Daubechies wavelets of order p to solve nonlinear two-point boundary value problems. The augmentation method is employed in a variational formulation for multilevel constructions. The present method can reduce computational time when solving the discretization of the full nonlinear system. The nonlinear system from the standard multilevel method can be separated or augmented into two smaller systems. One is linear and the other is a nonlinear one that can be solved iteratively by the Newton method. The numerical accuracy can be improved by increasing the resolutions or the level of approximations. The rate of convergence was shown to be at most on the order of 2^p where p is the order of the wavelet basis. We illustrate numerically in our examples that the L^2 error decreases when the number of basis levels increases. The rate of convergence from our estimations has been confirmed by many examples. Due to its advantages, the anti-derivatives of the Daubechies wavelets can be used to solve various kinds of boundary conditions. We are extending this study to apply this basis type with the augmentation method for solving Neumann type and mixed boundary conditions, without any modifications in the assumed form of approximate solution; these results will be reported elsewhere.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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MULTIRESOLUTION WAVELET BASES WITH AUGMENTATION METHOD
FOR SOLVING SINGULARLY PERTURBED REACTION-DIFFUSION NEUMANN PROBLEM

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ABSTRACT

This paper developed the anti-derivative wavelet bases to handle the more general types of boundary conditions: Dirichlet, mixed, and Neumann boundary conditions. The boundary value problem can be formulated by the variational approach, resulting in a system involving unknown wavelet coefficients. The wavelet bases are constructed to solve the unknown solutions corresponding to the types of solution spaces. The augmentation method is presented to reduce the dimension of the original system, while the convergence rate is in the same order as the multiresolution method. Some numerical examples have been shown to confirm the rate of convergence. The examples of the singularly perturbed problem with Neumann boundary conditions are also demonstrated, including highly oscillating cases.

Keywords: anti-derivative wavelets; multiresolution; Neumann boundary; mixed boundary; augmentation

1 Introduction

To solve numerically the differential equation with complex boundary conditions, one can find approximate solutions by choosing suitable bases and then discretizing the domain that results in a system involving unknown coefficients. Due to its advantage of the multiresolution wavelet basis, the full system for certain types of differential models needed not to be solved. Instead, it can be split into smaller sub-systems and solved separately. This is the concept of multiresolution augmentation method or multilevel augmentation method. This method allows one to develop faster and more accurate algorithms, see [3, 4, 5, 6] to solve boundary value problems.

The anti-derivatives of Daubechies wavelets can be used in the multiresolution augmentation method to solve linear and nonlinear two-point boundary value problems with Dirichlet boundary conditions (see [19, 20]). The accuracy of the numerical solution depends on the order p of the wavelet basis and the multiresolution levels. But the application of the augmentation method for solving Neumann and mixed boundary conditions is not straightforward. Recently, [13, 15], proposed a method by assuming that the second derivative of the solution can be expressed in terms of the Haar wavelet which is, in fact, the Daubechies wavelet of order $p = 1$, and then substitute it into the differential equation subjected to the types of boundary conditions. So, the wavelet order cannot be increased directly using this approach. It is required that a suitable wavelet basis is developed corresponding to the type of boundary condition that the wavelet orders can be increased.

In this work, we propose a new method to modify the anti-derivatives of Daubechies wavelets to be the subspace of solution spaces. The wavelet order can be increased when higher order approximation is required. Next, the variational approach is applied to the new approximation spaces and the full system is solved by the augmentation method to reduce computational time.

In view of fractional calculus, the fractional derivative of the Riemann Zeta function is recently presented in [11]. The fractional wavelet analysis based on the Gabor-Morlet mother wavelet is developed in [9, 10]. The wavelet spectrum and scalogram of the Weierstrass-Mandelbrot function (WMF) can be constructed. The spectral analysis of the WMF can describe the phenomena of various physical fields, including remote sensing, control theory, and quantum mechanics.

To begin with constructing the presented wavelet basis, the interval (a, b) is considered and the notation $L^2(a, b)$ used to denote the space of square integrable functions on (a, b) with the standard inner product $(\cdot, \cdot)$ defined by

$$(u, v) = \int_a^b u(x)v(x) dx,$$

and the associated norm $\|\cdot\|$.

Let $H^1(a, b)$ denote the Sobolev space with the norm $\|\cdot\|_1$ given by

$$\|v\|_1 = \sqrt{\int_a^b |v(x)|^2 dx + \int_a^b |v'(x)|^2 dx}.$$

In order to handle the various kinds of the boundary conditions, the following solution spaces are introduced:

$$\begin{aligned} H_\star^1(a, b) &= \{v \in H^1(a, b) \mid v(a) = 0\}, \\ H_\diamond^1(a, b) &= \{v \in H^1(a, b) \mid v(b) = 0\}, \\ H_0^1(a, b) &= \{v \in H^1(a, b) \mid v(a) = v(b) = 0\}. \end{aligned}$$

Details of the wavelet base construction on $H^1(a, b)$, $H_\star^1(a, b)$, $H_\diamond^1(a, b)$, and $H_0^1(a, b)$ are given in Section 2. The multiresolution augmentation method is presented in Section 3. The error analysis is shown in Section 4. Examples of the numerical results are demonstrated in Section 5. Finally, the conclusions have been drawn in Section 6.

2 Subspaces of solution spaces : $H_\star^1(a, b)$, $H_\diamond^1(a, b)$, $H^1(a, b)$, and $H_0^1(a, b)$

In this section, we introduce the anti-derivatives of wavelets that are orthonormal bases for finite-dimensional subspaces of the solution spaces $H_\star^1(a, b)$, $H_\diamond^1(a, b)$, $H^1(a, b)$ and $H_0^1(a, b)$.

Given a positive integer p , for $j \geq -1$ and $(j, k) \in \mathbb{Z} \times \mathbb{Z}$, let

$$\psi_{jk}(x) = \begin{cases} \phi(x - k), & j = -1, \\ \sqrt{2^j} \psi(2^j x - k), & j \geq 0, \end{cases}$$

be the *Daubechies wavelets of order p* (see, e.g., [8, 21] for the details of construction and additional properties of these wavelets). The supports $\text{supp } \phi = \text{supp } \psi = [0, 2p - 1]$. The interval $[0, 2p - 1]$ is shifted to $[a, b]$ by the transformation

$$y = \frac{b - a}{2p - 1} x + a, \quad x \in [0, 2p - 1].$$

In this paper, we simply denote $\psi_{jk}(y)$ by ψ_{jk} , whose support is

$$\left[a + \left(\frac{k}{2^j} \right) \left(\frac{b - a}{2p - 1} \right), a + \left(\frac{k + 2p - 1}{2^j} \right) \left(\frac{b - a}{2p - 1} \right) \right].$$

For the sake of simplicity, for each $j \geq -1$, the index sets I_j , D_j , and $\tilde{D}_j$ are defined by

$$\begin{aligned} k \in I_j &\iff \begin{cases} 2 - 2p \leq k \leq 2p - 2, & j = -1, \\ 2 - 2p \leq k \leq 2^j(2p - 1) - 1, & j \geq 0, \end{cases} \\ k \in D_j &\iff \begin{cases} 2 - 2p \leq k \leq 2p - 2, & j = -1, \\ 1 - p \leq k \leq 2^j(2p - 1) - p, & j \geq 0, \end{cases} \\ k \in \tilde{D}_j &\iff \begin{cases} 1 - 2p < k \leq 2p - 2, & j = -1, \\ 1 - p \leq k \leq 2^j(2p - 1) - p, & j \geq 0. \end{cases} \end{aligned}$$

The wavelets $\{\psi_{jk} \mid j \geq -1, k \in I_j\}$ form a frame for $L^2(a, b)$. To apply this study's proposed method for solving the boundary value problem, a sequence $\{S_n\}$ of nested closed subspaces of the solution space is constructed such that $\cup_{n \in \mathbb{N}} S_n$ is dense in the solution space. From [21] and [7], the anti-derivatives of scaling functions ϕ are used to construct the wavelet basis for these subspaces, but in the present method, the anti-derivatives of the wavelet functions ψ are used. This paper will prove for the solution spaces $H^1(a, b)$ and $H_0^1(a, b)$; the other subspaces can be proved in a similar way.

2.1 Subspaces of solution spaces $H^1(a, b)$

For $j \geq -1$, $k \in I_j$, define the anti-derivatives of wavelets, namely,

$$\widehat{\Psi}_{jk}(x) = \int_{-\infty}^x \psi_{jk} ds, \quad \text{for } a \leq x \leq b.$$

For $n \in \mathbb{N}$, define a subspace S_n by

$$S_n = \text{span}\{\widehat{\Psi}_{jk}(x) \mid -1 \leq j \leq n, k \in I_j\}.$$

It can be shown that the set $\{1, \widehat{\Psi}_{jk} \mid -1 \leq j \leq n, k \in D_j\}$ is a basis for S_n . The proof is given in Lemma 2.1 and Theorem 2.2.

Lemma 2.1. *For any $\alpha \in [-\infty, a)$, the set $\{\widehat{\Psi}_{jk} \mid j \geq -1, k \in I_j\}$ is a frame of $H^1(a, b)$.*

Proof. The set $\{\psi_{jk} \mid j \geq -1, k \in I_j\}$ is a frame of $L^2(a, b)$. Applying Lemma 3.1 in [21], $\{\widehat{\Psi}_{jk} \mid j \geq -1, k \in I_j\}$ is a frame of $H^1(a, b)$. $\square$

Theorem 2.2. *The set $\{1, \widehat{\Psi}_{jk} \mid -1 \leq j \leq n, k \in D_j\}$ is a basis for S_n .*

Proof. For any $i \geq -1$, $k \in I_j$ and $a \leq x \leq b$, we have

$$\widehat{\Psi}_{jk}(x) = \int_{-\infty}^x \psi_{jk} ds = \int_{-\infty}^a \psi_{jk} ds + \int_a^x \psi_{jk} ds.$$

Since $\int_{-\infty}^a \psi_{jk} ds$ is a constant and $\int_a^x \psi_{jk} ds = \Psi_{jk}(x) \in H^1(a, b)$, the set $\{1, \widehat{\Psi}_{jk} \mid -1 \leq j \leq n, k \in D_j\}$ spans S_n . Consequently, the set $\{1, \widehat{\Psi}_{jk} \mid -1 \leq j \leq n, k \in D_j\}$ is a basis for S_n . $\square$

2.2 Subspaces of solution space $H_0^1(a, b)$

We introduce the norm $|\cdot|_1$ of the space $H_0^1(a, b)$ by

$$|u|_1 = \sqrt{\int_a^b |u'(x)|^2 dx}, \quad \text{for } u \in H_0^1(a, b).$$

This norm is equivalent to the norm $\|\cdot\|_1$ defined previously in Section 1.

For $j \geq -1$ and $k \in I_j$, define the anti-derivatives of wavelets, namely,

$$\widetilde{\Psi}_{jk}(x) = \int_a^x \psi_{jk} ds - \frac{x-a}{b-a} \int_a^b \psi_{jk} dx, \quad \text{for } a \leq x \leq b.$$

For $n \in \mathbb{N}$, define

$$S_n = \text{span}\{\widetilde{\Psi}_{jk} \mid -1 \leq j \leq n, k \in I_j\}.$$

It can be shown that the set $\{\widetilde{\Psi}_{jk} \mid -1 \leq j \leq n, k \in \widetilde{D}_j\}$ is a basis for S_n . The proof can be seen in Lemma 2.3 and Theorem 2.4.

Lemma 2.3. *The set $\{\widetilde{\Psi}_{jk} \mid j \geq -1, k \in I_j\}$ is a frame of $H_0^1(a, b)$.*

Proof. Since $\widetilde{\Psi}'_{jk} = \psi_{jk} - \frac{1}{b-a} \int_a^b \psi_{jk} dx \in L^2(a, b)$ and $\widetilde{\Psi}_{jk}(a) = 0 = \widetilde{\Psi}_{jk}(b)$, $\widetilde{\Psi}_{jk} \in H_0^1(a, b)$. Let $u \in H_0^1(a, b)$. Since $u' \in L^2(a, b)$ and $\{\psi_{jk} \mid j \geq -1, k \in I_j\}$ forms a frame for $L^2(a, b)$, there are constants α_{jk} such that

$$\lim_{n \rightarrow \infty} \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right\| = 0.$$

Set $u_n = \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \widetilde{\Psi}_{jk}$. Then $u_n \in H_0^1(a, b)$ and $u'_n = \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} (\psi_{jk} -$

$\frac{1}{b-a} \int_a^b \psi_{jk} dx$). Applying triangle inequality, we have

$$\begin{aligned}
|u - u_n|_1 &= \sqrt{\int_a^b |u'(x) - u'_n(x)|^2 dx} \\
&= \|u' - u'_n\| \\
&= \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \left(\psi_{jk} - \frac{1}{b-a} \int_a^b \psi_{jk} dx \right) \right\| \\
&\leq \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right\| + \left\| \sum_{j=-1}^n \sum_{k \in I_j} \frac{\alpha_{jk}}{b-a} \int_a^b \psi_{jk} dx \right\| \\
&= \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right\| + \frac{1}{\sqrt{b-a}} \left| \int_a^b \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} dx \right|
\end{aligned}$$

Since $\int_a^b u' dx = u(b) - u(a) = 0$,

$$|u - u_n|_1 \leq \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right\| + \frac{1}{\sqrt{b-a}} \left| \int_a^b \left[u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right] dx \right|$$

Applying Hölder inequality, we have

$$|u - u_n|_1 \leq \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right\| + \left\| u' - \sum_{j=-1}^n \sum_{k \in I_j} \alpha_{jk} \psi_{jk} \right\|.$$

Then $\lim_{n \rightarrow \infty} |u - u_n|_1 = 0$. Hence $\text{span}\{\tilde{\Psi}_{jk} \mid j \geq -1, k \in I_j\}$ is dense in $H_0^1(a, b)$. That is, $\{\tilde{\Psi}_{jk} \mid j \geq -1, k \in I_j\}$ is a frame of $H_0^1(a, b)$. $\square$

Theorem 2.4. *The set $\{\tilde{\Psi}_{jk} \mid -1 \leq j \leq n, k \in \tilde{D}_j\}$ is a basis for S_n .*

Proof. By Proposition 4.2 in [21], $\{\psi_{jk} \mid 0 \leq j \leq n, k \in D_j\}$ is a basis for the subspace $\text{span}\{\psi_{jk} \mid 0 \leq j \leq n, k \in I_j\}$. Then $\text{span}\{\tilde{\Psi}_{jk} \mid 0 \leq j \leq n, k \in D_j\} = \text{span}\{\Psi_{jk} \mid 0 \leq j \leq n, k \in I_j\}$. Note that $\{\tilde{\Psi}_{-1,k} \mid k \in D_{-1}\}$ is linearly dependent. Indeed, from $\sum_{k \in D_{-1}} \psi_{-1,k} = 1$ we have

$$\begin{aligned}
\sum_{k \in D_{-1}} \tilde{\Psi}_{-1,k} &= \sum_{k \in D_{-1}} \left[\int_a^x \psi_{-1,k} ds - \frac{x-a}{b-a} \int_a^b \psi_{-1,k} dx \right] \\
&= \int_a^x \sum_{k \in D_{-1}} \psi_{-1,k} ds - \frac{x-a}{b-a} \int_a^b \sum_{k \in D_{-1}} \psi_{-1,k} dx \\
&= 0.
\end{aligned}$$

Next, assume that

$$\sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \tilde{\Psi}_{jk} = 0 \text{ on } [a, b].$$

Then for any $x \in [a, b]$

$$\int_a^x \sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} ds = \frac{x-a}{b-a} \int_a^b \sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} dx.$$

By the first fundamental theorem of calculus,

$$\sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} = \frac{1}{b-a} \int_a^b \sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} dx.$$

Set

$$\beta := \frac{1}{b-a} \int_a^b \sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} dx.$$

Rearranging,

$$\begin{aligned} 0 &= \sum_{j=-1}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} - \beta \\ &= \sum_{j=0}^n \sum_{k \in \tilde{D}_j} \alpha_{jk} \psi_{jk} + \sum_{k \in \tilde{D}_{-1}} \alpha_{-1,k} \psi_{-1,k} - \beta \end{aligned}$$

Since $\sum_{k \in D_{-1}} \psi_{-1,k} = 1$,

$$\begin{aligned} 0 &= \sum_{j=0}^n \sum_{k \in D_j} \alpha_{jk} \psi_{jk} + \sum_{k \in \tilde{D}_{-1}} \alpha_{-1,k} \psi_{-1,k} - \sum_{k \in D_{-1}} \beta \psi_{-1,k} \\ &= \sum_{j=0}^n \sum_{k \in D_j} \alpha_{jk} \psi_{jk} + \sum_{k \in \tilde{D}_{-1}} [\alpha_{-1,k} - \beta] \psi_{-1,k} - \beta \psi_{-1,2-2p} \end{aligned}$$

By linear independence of those ψ_{jk} , it can be concluded that $\beta = 0$ and $\alpha_{jk} = 0$ for $-1 \leq j \leq n$ and $k \in \tilde{D}_j$. Thus the set $\{\tilde{\Psi}_{jk} \mid -1 \leq j \leq n, k \in \tilde{D}_j\}$ is a basis for S_n . $\square$

2.3 Subspaces of solution space $H_\star^1(a, b)$

For $j \geq -1$ and $k \in I_j$, define the anti-derivatives of wavelets, namely,

$$\Psi_{jk}(x) = \int_a^x \psi_{jk} ds, \quad \text{for } a \leq x \leq b.$$

Graphs of Ψ_{jk} for $p = 1$ when $n = 1$, and 2 are shown in Figure 1, and graphs of Ψ_{jk} for $p = 2$ when $n = 1$ are shown in Figure 2. For $n \in \mathbb{N}$, define a subspace S_n by

$$S_n = \text{span}\{\Psi_{jk}(x) \mid -1 \leq j \leq n, k \in I_j\}.$$

Based on the proofs in Subsection 2.1 and Subsection 2.2, the set $\{\Psi_{jk} \mid -1 \leq j \leq n, k \in D_j\}$ is a basis for S_n .

2.4 Subspaces of solution space $H_{\diamond}^1(a, b)$

For $j \geq -1$ and $k \in D_j$, define the anti-derivatives of wavelets, namely,

$$\Psi_{jk}(x) = \int_x^b \psi_{jk} ds, \quad \text{for } a \leq x \leq b.$$

Graphs of Ψ_{jk} for $p = 1$ when $n = 1$, and 2 are shown in Figure 3, and graphs of Ψ_{jk} for $p = 2$ when $n = 1$ are shown in Figure 4.

Set

$$S_n = \text{span}\{\Psi_{jk} \mid -1 \leq j \leq n, k \in I_j\}.$$

Based on the proofs in Subsection 2.1 and Subsection 2.2, the set

$$\{\Psi_{jk} \mid -1 \leq j \leq n, k \in D_j\},$$

is a basis for S_n .

Note that, sometimes, the orthonormality of the basis for S_n may be required; then the Gram-Schmidt process can be applied with the standard inner product $(\cdot, \cdot)$ in $L^2(a, b)$.

3 Multiresolution augmentation method

3.1 Multiresolution method

Suppose that H is a solution subspace of $L^2(a, b)$ (that is, H can be one of the subspaces $H_{\star}^1(a, b)$, $H_{\diamond}^1(a, b)$, $H^1(a, b)$ and $H_0^1(a, b)$). Let $u \in H$ be the weak solution of a given differential equation. Suppose that the variational form of the differential equation is

$$A(u, v) = F(v), \quad \text{for all } v \in H. \quad (1)$$

Let S_n be a nested sequence of finite-dimensional subspaces such that $\cup_{n \in \mathbb{N}} S_n$ is dense in H . Let $\{\bar{\Psi}_j \mid j = 1, 2, \dots, \dim S_n\}$ be an orthonormal basis for S_n . Suppose that the n th level approximate solution $u_n \in S_n$ is of the form

$$u_n = \sum_{j=1}^{\dim S_n} \alpha_j \bar{\Psi}_j,$$

satisfying

$$\sum_{j=1}^{\dim S_n} \alpha_j A(\bar{\Psi}_j, \bar{\Psi}_k) = F(\bar{\Psi}_k), \quad \text{for all } k = 1, 2, \dots, \dim S_n.$$

Assume that A in (1) is a linear operator, the linear system that will be obtained is $\mathbf{A}_n \mathbf{u}_n = \mathbf{F}_n$, where

$$\begin{aligned} \mathbf{A}_n &= [A(\bar{\Psi}_j, \bar{\Psi}_k) : j, k = 1, \dots, \dim S_n], \\ \mathbf{u}_n &= [\alpha_j : j = 1, \dots, \dim S_n], \\ \mathbf{F}_n &= [F(\bar{\Psi}_k) : k = 1, \dots, \dim S_n]. \end{aligned}$$

The matrix system can be solved by any numerical methods, so $\mathbf{u}_n$ is obtained. This process is called the *multiresolution method (standard method)*. Next, the augmentation technique will be described when the approximate solution $\mathbf{u}_n$ is already obtained.

3.2 Multiresolution augmentation method

Suppose that $\mathbf{u}_n$ is already solved by the multiresolution method from the previous subsection. The augmentation method can be applied to find an approximate solution in the next level, $(n+1)$. Firstly, we augment the matrix $\mathbf{A}_n$ with submatrices $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{D}_n$ where

$$\begin{aligned}\mathbf{B}_n &= [A(\overline{\Psi}_j, \overline{\Psi}_k) : j = 1, \dots, \dim S_n, k = \dim S_n + 1, \dots, \dim S_{n+1}] , \\ \mathbf{C}_n &= [A(\overline{\Psi}_j, \overline{\Psi}_k) : j = \dim S_n + 1, \dots, \dim S_{n+1}, k = 1, \dots, \dim S_n] , \\ \mathbf{D}_n &= [A(\overline{\Psi}_j, \overline{\Psi}_k) : j, k = \dim S_n + 1, \dots, \dim S_{n+1}] .\end{aligned}$$

The augmented matrix $\mathbf{A}_{n+1}$ corresponding to the $(n+1)$ th level is identified as

$$\mathbf{A}_{n+1} = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} .$$

Secondly, we split the coefficient matrix $\mathbf{A}_{n+1}$ into the sum of upper and lower triangular matrices and find $\mathbf{u}_{n,1}$ satisfying,

$$\begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ 0 & \mathbf{D}_n \end{bmatrix} \mathbf{u}_{n,1} + \begin{bmatrix} 0 & 0 \\ \mathbf{C}_n & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_n \\ 0 \end{bmatrix} = \mathbf{F}_{n+1} .$$

We need to find the vector solution $\mathbf{u}_{n,1}$ which refers to an approximate solution from applying the augmentation method once. The augmentation process can be done inductively i times from $\mathbf{u}_n$ for finding $\mathbf{u}_{n,2}, \mathbf{u}_{n,3}, \dots, \mathbf{u}_{n,i}$ where $\mathbf{u}_{n,i}$ refers to the multiresolution augmentation solution at level $(n+i)$, and $\mathbf{u}_{n,i}$ are calculated from $\mathbf{u}_{n,i-1}$.

Note that the original system of the standard method at level $n+i$ has dimension $\dim S_{n+i}$ while that of the multiresolution augmentation method has dimensions $\dim S_{n+i-1}$ and $\dim S_{n+i} - \dim S_{n+i-1}$. It can be seen that the dimension of the original system can be reduced to smaller dimensions. This shows the advantage of the multiresolution augmentation method when the number of levels is very large. The multiresolution augmentation method for solving nonlinear operators with Dirichlet conditions can be found in [20]; but here we propose the augmentation method with the modified wavelet bases for solving the problems with Neumann and mixed boundary conditions.

4 Error analysis

In this section, the convergence rate of the multiresolution augmentation method in conjunction with anti-derivatives of Daubechies wavelets is presented. If u is the weak solution of (1) and $u_n \in S_n$ is the n th (standard) multiresolution solution obtained by using wavelet bases of order

p . Theorem 4.1 in [21] shows that, if $u \in H \cap H^s(a, b)$, there exists a positive constant C_1 such that

$$\|u - u_n\| + 2^{-n}\|u - u_n\|_1 \leq C_1(2^{-(n+1)})^s \|u\|_s, \quad 1 \leq s \leq p+1,$$

where $H^s(a, b)$ is the standard Sobolev space with the norm $\|\cdot\|_s$. In particular,

$$\|u - u_n\| \leq C_1(2^{-(n+1)})^s \|u\|_s, \quad 1 \leq s \leq p+1. \quad (2)$$

The estimation in (2) suggests that if we apply the wavelets of order p and $u \in H \cap H^s(a, b)$, then the errors from the n th level to the $(n+1)$ th level measured by the standard norms in L^2 decrease by the factors 2^{p+1} .

Next, we consider the difference between the weak solution u and the $(n+i)$ th multiresolution augmentation solution, $u_{n,i}$, of (1). Denote the operator corresponding to the matrix $\mathbf{A}$ by $\mathcal{A}$.

Suppose that there exists an $N \in \mathbb{N}$ such that for $n \geq N$, the inverse operators $\mathcal{A}_n^{-1}$ and $\mathcal{D}_n^{-1}$ exist. Under these assumptions, we have that the $(n+i)$ th multiresolution solution u_{n+i} and the $(n+i)$ th multiresolution augmentation solution $u_{n,i}$ exist for all $n \geq N$. Moreover, if there exist positive constants α and β such that

$$\|\mathcal{A}_n^{-1}\| \leq \alpha, \quad \|\mathcal{D}_n^{-1}\| \leq \beta,$$

and

$$\lim_{n \rightarrow \infty} \|\mathcal{B}_n\| = \lim_{n \rightarrow \infty} \|\mathcal{C}_n\| = 0,$$

by Theorem 3.5 and Corollary 3.6 in [3], there exist an $M \in \mathbb{N}$ and $C_2 > 0$ such that for $n \geq M$ and $i \in \mathbb{N}$,

$$\|u_{n,i} - u_{n+i}\| \leq C_2(2^{-(n+i+1)})^s \|u\|_s, \quad 1 \leq s \leq p+1. \quad (3)$$

From the estimations (2) and (3) and by applying the triangle inequality we can conclude that

$$\|u - u_{n,i}\| \leq (C_1 + C_2)2^{-(n+i+1)s} \|u\|_s, \quad 1 \leq s \leq p+1. \quad (4)$$

The error estimation in (4) suggests that, if the weak solution $u \in H \cap H^s(a, b)$ and we apply the multiresolution augmentation method from level $n+i-1$ to $n+i$ using the anti-derivatives wavelets of order p , then the errors measured in the L^2 -norm decrease at most by the factor of 2^{p+1} . Consequently, from (2), the errors obtained by the standard and the multiresolution augmentation methods decrease with the same order.

5 Numerical results and discussions

In this section, the efficiency of the multiresolution augmentation method in conjunction with the anti-derivatives of Daubechies wavelets of order p is illustrated to find the numerical solutions of two-point boundary value problems with various kinds of boundary conditions.

We consider the differential equation in the form

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (a, b),$$

where $f \in L^2(a, b)$ and the coefficients q and r are smooth functions in the closed interval $[a, b]$ with $q(x) > 0$ and $r(x) \geq 0$ for all $x \in [a, b]$. We give numerical examples of this differential equation with the mixed boundary conditions in examples 5.1-5.2 and with the Neumann boundary condition given in example 5.3. Singularly perturbed problems are presented in examples 5.4-5.5.

5.1 Mixed boundary conditions (Type I)

Consider

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (a, b), \quad (5)$$

with the following mixed boundary conditions :

$$u(a) = c, \text{ and } u'(b) = d.$$

Set $u = (x + c - a) + w$ is the weak solution of (5). Since $u(a) = c$ and $u'(b) = d$, $w(a) = 0$ and $w'(b) = d - 1$. The variational formulation of the boundary value problem (5) involves the finding of $w \in H_\star^1(a, b)$ such that

$$A(w, v) = \int_a^b f v \, dx + (d - 1)q(b)v(b) - A(x + c - a, v), \quad \text{for all } v \in H_\star^1(a, b),$$

where $A(\cdot, \cdot)$ is the bilinear form defined by

$$A(w, v) = \int_a^b q w' v' + r w v \, dx.$$

Since $A(\cdot, \cdot)$ is continuous and coercive on $H_\star^1(a, b)$, by the Lax-Milgram lemma, there exists a unique weak solution $u \in H_\star^1(a, b)$ for (5).

Example 5.1. Consider the boundary value problem

$$-u'' = \sin \pi x, \quad \text{for } x \in (0, 1), \quad (6)$$

with the following mixed boundary conditions:

$$u(0) = 0, \text{ and } u'(1) = 1.$$

The exact solution is $u(x) = x + \frac{1}{\pi^2}(\sin \pi x + \pi x)$.

Table 1
Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $	$\ u - u_{4,n-4}\ $
1	4	0.3616			
2	8	0.0924			
3	16	0.0244	0.0249		
4	32	0.0076	0.0076	0.0076	
5	64	0.0036	0.0036	0.0036	0.0036

Table 2
Numerical results for $p = 2$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $	$\ u - u_{4,n-4}\ $
1	5	$3.0498e - 01$			
2	8	$4.1191e - 02$			
3	14	$4.9862e - 03$	$7.0874e - 03$		
4	26	$6.1128e - 04$	$8.2725e - 04$	$8.1907e - 04$	
5	50	$7.6755e - 05$	$1.2692e - 04$	$9.1532e - 05$	$9.1205e - 05$

Numerical errors in terms of the L^2 norm for $p = 1$, and $p = 2$ are shown in Tables 1-2. Errors obtained from the standard method are shown in column 3, while errors obtained from the multiresolution augmentation method are shown in columns 4-6. The augmentation scheme starts at levels 2, 3, and 4. For $p = 1$, numerical errors for the first four levels decrease approximately 3-4 times from the previous level, while they decrease approximately 7-8 times for $p = 2$. These numerical results confirm the theoretical results that the error should decrease by the factor 2^{-s} , where $1 \leq s \leq p + 1$. The errors from the multiresolution augmentation method are in the same order, as expected of the standard method when the augmented level is large enough. For $p = 1$, the L^2 error in level 5 does not decrease by the same rate as expected. This is the result of unexpected additional errors from basis approximations which are composed of numerical integration and differentiation.

5.2 Mixed boundary conditions (Type II)

Consider

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (a, b), \quad (7)$$

with the following mixed boundary conditions :

$$u'(a) = c, \text{ and } u(b) = d.$$

Set $u = (x + d - b) + w$ be the weak solution of (7). Since $u(b) = d$, so $w(b) = 0$. The variational formulation involves the finding of $w \in H_\diamond^1(a, b)$ such that

$$A(w, v) = \int_a^b f v \, dx - (c - 1)q(a)v(a) - A(x + d - b, v), \quad \text{for all } v \in H_\diamond^1(a, b),$$

where $A(\cdot, \cdot)$ is the bilinear form defined by

$$A(w, v) = \int_a^b q w' v' + r w v \, dx.$$

Since $A(\cdot, \cdot)$ is continuous and coercive on $H_\diamond^1(a, b)$, by the Lax-Milgram lemma, there exists a unique weak solution $u \in H_\diamond^1(a, b)$ for (7).

Example 5.2. Consider the boundary value problem

$$u'' = \cos \pi x, \quad \text{for } x \in (0, 1), \quad (8)$$

with the following mixed boundary conditions :

$$u'(0) = 0, \text{ and } u(1) = 1.$$

The exact solution is $u(x) = 1 + \frac{1}{\pi^2}(\cos \pi - \cos \pi x)$.

Table 3
Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $	$\ u - u_{4,n-4}\ $
1	4	$5.1079e - 01$			
2	8	$1.2975e - 01$			
3	16	$3.3479e - 02$	$3.6188e - 02$		
4	32	$1.0197e - 02$	$1.0607e - 02$	$1.0424e - 02$	
5	64	$5.7858e - 03$	$5.8028e - 03$	$5.7952e - 03$	$5.7899e - 03$

Table 4
Numerical results for $p = 2$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $
1	5	$3.7558e - 01$		
2	8	$4.1778e - 02$		
3	14	$6.4214e - 03$	$6.6884e - 03$	
4	26	$4.1543e - 03$	$4.1715e - 03$	$4.1550e - 03$

The L^2 errors of example 5.2 are shown in Tables 3-4 for $p = 1$, and $p = 2$, respectively. The rate of convergence remains 2^{-s} where $1 \leq s \leq p + 1$. The numerical errors of the multiresolution augmentation method are in the same order as the standard method. This shows that the anti-derivative wavelet in conjunction with the matrix augmentation can be applied to solve the mixed boundary conditions and preserve the convergent rate.

5.3 Neumann boundary conditions

Consider

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (a, b), \quad (9)$$

with the following Neumann boundary conditions :

$$u'(a) = c, \text{ and } u'(b) = d,$$

The variational formulation of (9) involves the finding of $w \in H^1(a, b)$ such that

$$A(w, v) = \int_a^b f v \, dx + dq(b)v(b) - cq(a)v(a), \text{ for all } v \in H^1(a, b),$$

where $A(\cdot, \cdot)$ is the bilinear form defined by

$$A(w, v) = \int_a^b qw'v' + r w v \, dx.$$

Since $A(\cdot, \cdot)$ is continuous and coercive on $H^1(a, b)$, by the Lax-Milgram lemma, there exists a unique weak solution $u \in H^1(a, b)$ for (9).

Example 5.3. Consider the boundary value problem

$$-u'' + u = (1 + \pi^2) \sin \pi x + 1, \quad \text{for } x \in (0, 1), \quad (10)$$

with the following Neumann boundary conditions :

$$u'(0) = \pi, \text{ and } u'(1) = -\pi.$$

The exact solution is $u(x) = 1 + \sin \pi x$.

Table 5
Numerical results for $p = 1$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $	$\ u - u_{4,n-4}\ $
1	5	2.6941e + 00			
2	9	7.0689e - 01			
3	17	1.7981e - 01	6.2093e - 01		
4	33	4.6622e - 02	2.9533e - 01	1.5601e - 01	
5	65	1.3594e - 02	1.0759e - 01	7.3848e - 02	3.9654e - 02

Table 6
Numerical results for $p = 2$

n	$\dim S_n$	$\ u - u_n\ $	$\ u - u_{2,n-2}\ $	$\ u - u_{3,n-3}\ $
1	6	2.9812e + 00		
2	9	4.0325e - 01		
3	15	4.8843e - 02	6.2520e - 02	
4	27	5.9290e - 03	8.4895e - 03	6.8148e - 03
5	51	7.8698e - 04	1.4897e - 03	9.2025e - 04

The L^2 errors of example 5.3 are shown in Tables 5-6 for $p = 1$, and $p = 2$, respectively. This shows that the multiresolution augmentation method can be used to solve the problem with Neumann boundary conditions.

Example 5.4. Consider the following singularly perturbed one-dimensional reaction-diffusion problem

$$-\varepsilon^2 u'' + u = -\cos^2(\pi x) - 2\varepsilon^2 \pi^2 \cos(2\pi x), \quad \text{for } x \in (0, 1), \quad (11)$$

where ε is a small constant with the following Neumann boundary conditions :

$$u'(0) = 1, \quad u'(1) = 1.$$

From [18], the exact solution is $u(x) = -\cos^2(\pi x) + \varepsilon \frac{\left[\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon) \right]}{1 + \exp(-1/\varepsilon)}$.

Table 7
Numerical results for $p = 1$ and $\varepsilon = 0.001$

n	$\dim S_n$	$\ u - u_n\ $	RMS	$\ u - u_{2,n-2}\ $	RMS	$\ u - u_{3,n-3}\ $	RMS	$\ u - u_{4,n-4}\ $	RMS
1	5	5.4447e+00	4.2500e-02						
2	9	1.1189e+00	8.7000e-03						
3	17	2.6580e-01	2.1000e-03	4.2740e-01	3.3000e-03				
4	33	6.6200e-02	5.1727e-04	3.0110e-01	8.9700e-02	2.4000e-03	7.0044e-04		
5	65	1.7600e-02	1.3752e-04	2.1890e-01	5.6500e-02	2.1600e-02	1.7000e-03	4.4153e-04	1.6903e-04

Table 8
Numerical results for $p = 1$ and $\varepsilon = 0.0001$

n	$\dim S_n$	$\ u - u_n\ $	RMS	$\ u - u_{2,n-2}\ $	RMS	$\ u - u_{3,n-3}\ $	RMS	$\ u - u_{4,n-4}\ $	RMS
1	5	5.4440e+00	4.2500e-02						
2	9	1.1181e+00	8.7000e-03						
3	17	2.6580e-01	2.1000e-03	4.2590e-01	3.3000e-03				
4	33	6.5400e-02	5.1078e-04	2.9930e-01	8.7800e-02	2.3000e-03	6.8555e-04		
5	65	1.6400e-02	1.2776e-04	2.1896e-01	5.6541e-02	1.9400e-02	1.7000e-03	4.2297e-04	1.5135e-04

The L^2 and the root mean square (RMS) errors of example 5.4 for $p = 1$ are shown in Tables 7-8. In this example, we perform numerical experiments for two cases of small ε , 0.001 and 0.0001. The rate of convergence agrees well with the theoretical results. The multiresolution augmentation method can solve the singularly perturbed problem with the small parameter ε .

5.4 Dirichlet boundary conditions

Consider

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (a, b), \quad (12)$$

with the following Dirichlet boundary conditions :

$$u(a) = c, \text{ and } u(b) = d,$$

Set $u = \frac{bc - ad}{b - a} + \frac{d - c}{b - a}x + w$ is the weak solution of (12). Since $u(a) = c$ and $u(b) = d$, $w(a) = 0$ and $w(b) = 0$. The variational formulation of (12) involves the finding of $w \in H_0^1(a, b)$ such that

$$A(w, v) = \int_a^b f v \, dx - \left(\frac{bc - ad}{b - a} + \frac{d - c}{b - a}x, v \right), \text{ for all } v \in H_0^1(a, b),$$

where $A(\cdot, \cdot)$ is the bilinear form defined by

$$A(w, v) = \int_a^b qw'v' + r w v \, dx .$$

Since $A(\cdot, \cdot)$ is continuous and coercive on $H_0^1(a, b)$, by the Lax-Milgram lemma, there exists a unique weak solution $u \in H_0^1(a, b)$ for (12).

Example 5.5. Consider the boundary value problem ([22])

$$\varepsilon u'' + \frac{1}{(x+1)^2} u = 0, \quad \text{for } x \in (0, 1), \quad (13)$$

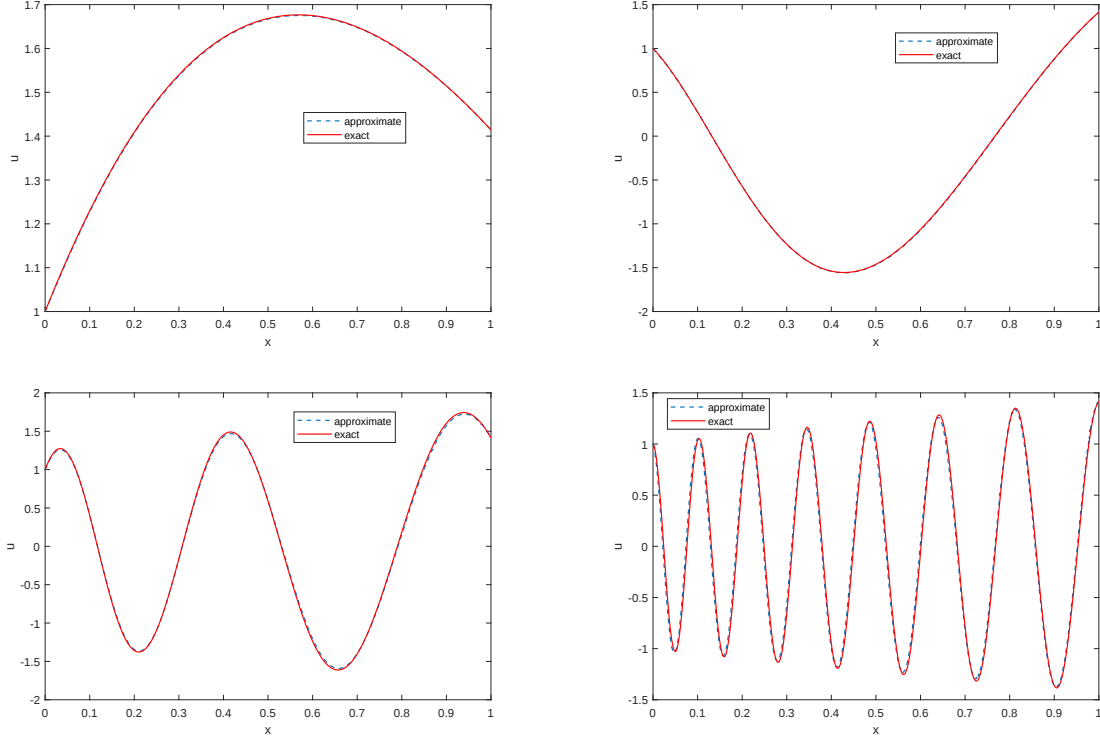
with the following Dirichlet boundary conditions :

$$u(0) = 1, \text{ and } u(1) = \sqrt{2} \quad (0 < \varepsilon < 4).$$

The exact solution is $u(x) = \sqrt{x+1} \cdot \frac{\cos \left[r \ln \left(\frac{x+1}{\sqrt{2}} \right) \right]}{\cos[r \ln(\sqrt{2})]}$, where $r := \sqrt{\frac{1}{\varepsilon} - \frac{1}{4}}$ and $\varepsilon \neq$

$$\frac{1}{\frac{1}{4} + \left[\frac{(2k+1)\pi}{\ln 2} \right]^2} .$$

Figure 5: The numerical results for $\varepsilon = 0.02, 0.002, 0.00025, 0.000025$



The numerical solutions for various values of ε are shown in Figure 5. We apply the anti-derivative wavelet at level 9 with $p = 1$ to solve the singularly perturbed problem. When ε becomes small, high oscillations are produced as $x \rightarrow 0$. This example shows the ability of the present method and the developed bases to solve high oscillation problems.

6 Conclusions

This work presents the multiresolution augmentation method using the anti-derivatives of Daubechies wavelets for numerically solving two-point boundary value problems subject to mixed and Neumann boundary conditions. Application of the wavelet basis with augmentation method for solving these types of boundary conditions is not straightforward. It requires suitable approximation spaces to estimate the unknown wavelet coefficients. So, we developed wavelet bases on the Sobolev space which can be easily applied by variational formulation. Due to the advantages of orthonormality and multiresolution of the wavelet basis constructed in the present study, the augmentation method can be applied to reduce the dimension of the original system. The unknown wavelet coefficients from the previous wavelet level can be used to approximate the increment of the unknown coefficients in the next level. The rate of convergence is still preserved to be of the same order as the multiresolution method, which is 2^s , $1 \leq s \leq p + 1$, where p is the Daubechies wavelet order. Numerical experiments were carried out to demonstrate the convergence rate. Additionally, the present method can be applied to solve singularly perturbed problems with highly oscillating behavior.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors have read and improved the final manuscript.

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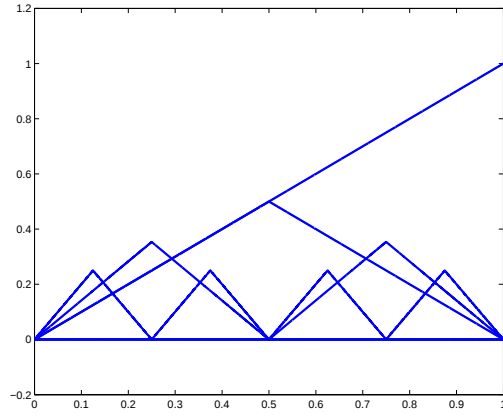


Figure 1: Ψ_{jk} for $p = 1$ when $n = 1$, and 2 of solution space $H_{\diamond}^1(a, b)$.

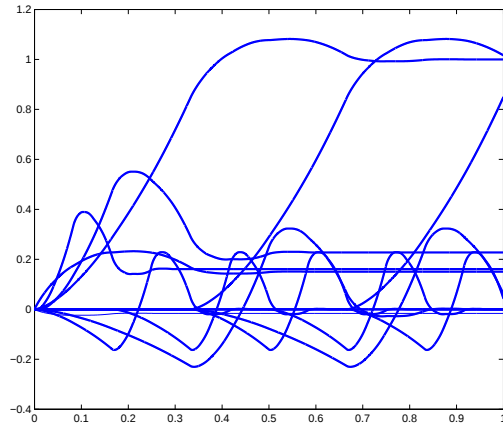


Figure 2: Ψ_{jk} for $p = 2$ when $n = 1$ of solution space $H_{\diamond}^1(a, b)$.

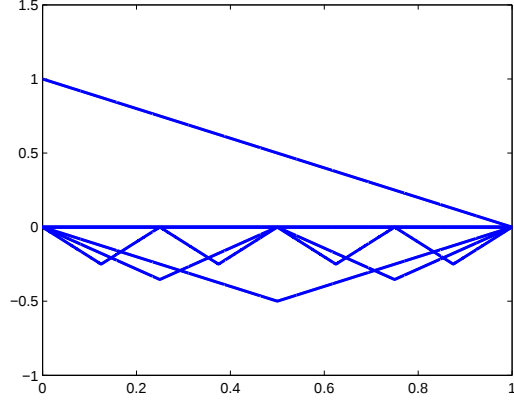


Figure 3: Ψ_{jk} for $p = 1$ when $n = 1$, and 2 of solution space $H_{\star}^1(a, b)$.

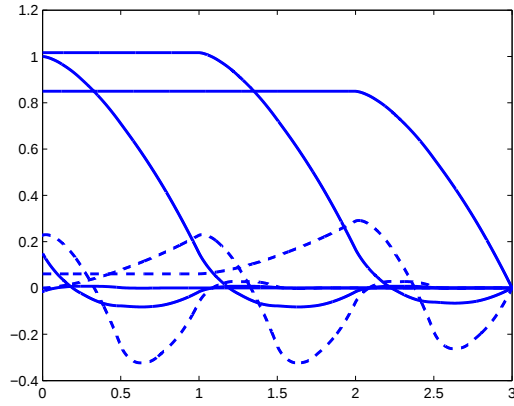


Figure 4: Ψ_{jk} for $p = 2$ when $n = 1$ of solution space $H_{\star}^1(a, b)$.