รายงานวิจัยฉบับสมบูรณ์

คำตอบของจุดตรึงของระบบทั่วไปของอสมการแปรผัน ด้วยการประยุกต์เพื่อการหาค่าที่เหมาะสม

Fixed point solutions of a general system of variational inequalities with applications to optimization

ผศ.ดร.ภูมิ คำเอม

มิถุนายน 2555
สัญญาเลขที่ MRG5380044

รายงานวิจัยฉบับสมบูรณ์

คำตอบของจุดตรึงของระบบทั่วไปของอสมการแปรผัน
ด้วยการประยุกต์เพื่อการหาค่าที่เหมาะสม

Fixed point solutions of a general system of variational inequalities with applications to optimization

ผศ.ดร.ภูมิ คำเอม มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี

ศ.ดร.สมยศ พลับเที่ยง มหาวิทยาลัยนเรศวร (นักวิจัยที่ปรึกษา)

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานฉบับนี้เป็นของผู้วิจัย สกว. และสกอ. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยเรื่อง คำตอบของจุดตรึงของระบบทั่วไปของอสมการแปรผัน ด้วยการประยุกต์เพื่อการหาค่าที่เหมาะสม (MRG5380044) นี้ สำเร็จลุล่วงด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และ สำนักงานคณะกรรมการอุดมศึกษา (สกอ.) ประจำปี 2553-2555 และขอขอบคุณ ศ.ดร.สมยศ พลับเที่ยง ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร นักวิจัยที่ปรึกษา ที่ได้ให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา และสุดท้ายขอขอบคุณ มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี สถาบันต้นสังกัด

Project Code: MRG5380044
(รหัสโครงการ)
Project Title: Fixed point solutions of a general system of variational inequalities with applications to optimization
(ชื่อโครงการ) คำตอบของจุดตรึงของระบบทั่วไปของอสมการแปรผันด้วยการประยุกต์เพื่อการหาค่าที่เหมาะสม

Investigator: Asst. Prof. Dr. Poom Kumam
King Mongkut's University of Technology Thonburi
(ชื่อนักวิจัย) ผศ.ดร.ภูมิ คำเอม มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี
E-mail Address: poom.kum@kmutt.ac.th

Mentor: Professor Dr. Somyot Plubtieng Nareasuan University
(นักวิจัยที่ปรึกษา) ศ.ดร.สมยศ พลับเที่ยง มหาวิทยาลัยนเรศวร
E-mail Address: somyotp@nu.ac.th

Project Period: June 15, 2010 - June 14, 2012
(ระยะเวลาโครงการ) 15 มิถุนายน 2553 ถึงวันที่ 14 มิถุนายน 2555

ABSTRACT

The aim of this project is to consider and study the general systems of the generalized variational inequality problems and the system of general mixed equilibrium problems by using the fixed point methods. We prove strong convergence theorem for finding the common solutions of the general systems of variational inclusion problems and general system mixed equilibrium problems with related optimization problems in Hilbert and Banach spaces.

Keywords: Hybrid projection method/ Variational inequality problem/ General mixed equilibrium problems/ Nonlinear mappings/ Optimization problem

บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้เราพิจรณาและศึกษา ระบบทั่วไปของปัญหาอสมการเชิงแปรผันทั่วไป และระบบของปัญหาเชิงดุลยภาพผสมทั่วไป โดยใช้วิธีจุดตรึง เราพิศูจน์ทฤษฎีบทการลู่เข้าแบบเข้ม เพื่อหาคำตอบร่วมของระบบทั่วไปของปัญหาอสมการเชิงแปรผันทั่วไป และระบบของปัญหาเชิงดุลยภาพผสมทั่วไป ซึ่งสัมพันธ์กับปัญหาค่าเหมาะสมที่สุด ในปริภูมิฮิลเบร์ติและปริภูมิบานาค

คำสำคัญ : วิธีฉายไฮบริด / ปัญหาอสมการเชิงแปรผัน/ ปัญหาเชิงดุลยภาพผสมทั่วไป/ การส่งแบบไม่เชิงเส้น / ปัญหาค่าเหมาะสมที่สุด

สารบัญ

บทที่ 1

Introduction

1.1 Background

Variational inequality theory, which was introduced in 1960's by Stampacchia [324], has had a great impact and influence in the development of several branches of pure and applied sciences. The ideas and techniques of this theory are being used in a variety of diverse fields and proved to be productive and innovative, see [1-25] and the references therein. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. As a result of the interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various algorithms for solving variational inequalities and related optimization problems. Using the projection technique, one can establish the equivalence between the variational inequalities and fixed point problems. This alternative equivalent formulation has played an important role in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. It is now well-known that the variational inequalities are equivalent to the fixed-point problems, the origin of which can be traced back to Lions and Stampacchia [127]. This alternative formulation has been used to suggest and analyze projection iterative methods for solving the variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous. These conditions are very strict and rule out its application in several important problems. To overcome this drawback, Korpelevich [125] suggested and analyzed the extragradient method by using the technique of updating the solution. It has been shown that if the underlying operator is only monotone and Lipschitz continuous, then the approximate solution converges to the exact solution. Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the current interest in functional analysis. It is natural to consider a unified approach to these different problems, see, for example, [126, 137, 307, 152].

Equilibrium problems which were introduced by Blum and Oettli [108] and Noor and Oettli [106] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. It has been shown [108, 106] that equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. Hence collectively, equilibrium problems cover a vast range of applications. Due to the nature of the equilibrium problems, it is not possible to extend the projection and its variant forms for solving equilibrium problems. To overcome this drawback, one usually uses the auxiliary principle technique. The main and basic idea in this technique is to consider an auxiliary equilibrium problem related to the original problem and then show that the solution of the auxiliary problems is a solution of the original problem. This technique has been used to suggest and analyze a number of iterative methods for solving various classes of equilibrium problems and variational inequalities, see [111, 101, 102, 103, 104, 105] and the references therein.

1.2 Iterative Approximation of Fixed-Points

Let X be a nonempty set and $T : X \longrightarrow X$ a self map. We say that $p \in X$ is a *fixed point* of T if $p = Tp$ and denote by $F(T)$ the set of all fixed points of T. Having in view that many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator, on the other hand, the metrical fixed point theory has developed significantly in the second part of the 20th century.

As the constructive methods used in metrical fixed point theory are prevailingly iterative procedures, that is, approximate methods, it is also of crucial importance to have a priori or/and a posteriori error estimates or rate of convergence for such method. For example, the Banach fixed point theorem concerns certain contractions mappings of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point and it also given a constructive procedure for obtaining better and better approximations to the fixed point. By definition, this is a method such that we choose an arbitrary x_0 in a given set and calculate recursively a sequence $x_0, x_1, x_2, ...$ from a relation of the form

$$x_n = Tx_{n-1} = T^n x_0 \;\; n = 1, 2, 3, .. \tag{1.2.1}$$

That is, we choose an arbitrary x_0 and determine successively $x_1 = Tx_0, x_2 =$

$Tx_1, x_3 = Tx_2, \ldots$. It is also known as the Picard iteration starting at x_0.

Iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach fixed point theorem (or more difficult fixed point theorems). Many researchers are interested in obtaining (additional) condition on T and E as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed point of T. Moreover, if the Picard iteration converges to a fixed point of T, they will be interested in evaluating the error estimate (or alternatively, the rate of convergence) of the method, that is, in obtaining a stopping criterion for the sequence of successive approximations. However, the Picard iteration may not converge even in the weak topology.

Construction of fixed point iteration processes of nonlinear mappings is an important subject in the theory of nonlinear mappings, and finds application in a number of applied areas. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings, relatively nonexpansive mappings, hemirelatively nonexpansive mappings, generalized nonexpansive mappings and maximal monotone operators in various space have been studied by many mathematicians.

Let $(X, \|\cdot\|)$ be a real normed space and $C \subset X$ be a closed and convex. Three classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $T : C \longrightarrow C$.

Halpern's iteration

The first one is introduced by Halpern [16] which is defined as follows: $x_0 \in C$

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) T x_n, \;\; n \geq 0, \tag{1.2.2}$$

where $\{\alpha_n\}$ is a real sequence in [0,1].

Mann's iteration

The second iteration process is now known as Mann's iteration process [33] and is defined as follows: $x_0 \in C$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \;\; n \geq 0, \tag{1.2.3}$$

where $\{\alpha_n\}$ is a real sequence in [0,1].

Ishikawa's iteration

The third iteration process is referred to as Ishikawa's iteration [21] which is defined recursively by; $x_0 \in C$

$$x_{n+1} = \alpha_n T\big(\beta_n T x_n + (1-\beta_n)x_n\big) + (1-\alpha_n)x_n, \;\; n \geq 0, \tag{1.2.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1].

In general not much has been known regarding the convergence of the iteration processes (1.2.2)-(1.2.4) unless the underlying space has elegant properties which be briefly mention here.

Process (1.2.4) is indeed more general than process (1.2.3). But research has been concentrated on the latter due probably to the reasons that the formulation of process (1.2.3) is simpler than that of (1.2.4) and that a convergence theorem for process (1.2.3) may possibly lead to a convergence theorem for process (1.2.4) provided the sequence $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of process (1.2.4) has its own right. As a matter of fact, process (1.2.3) may fail to converge while process (1.2.4) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space. Both processes (1.2.3) and (1.2.4) have only weak convergence, in general. For example, Reich [42] proved that if X is a uniformly convex Banach space with a Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the Mann's iteration converges weakly to a fixed point of T. However, we note that Mann's iteration have only weak convergence even in a Hilbert space.

Normal Hybrid Method (or CQ method)

Attempts to modify the Mann's iteration method (1.2.2) so that strong convergence is guaranteed have recently been made. In 2003, Nakajo and Takahashi [39] proposed the following modification of the Mann's iteration method (1.2.2) by using the hybrid method in mathematical programming, for a single nonexpansive mapping T in a Hilbert space as follows: $x_0 = x \in C$

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)T x_n, \\ C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{1.2.5}$$

and they proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.2.5) converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

The iteration process (1.2.2) has been proved to be strong convergent in both Hilbert space and uniformly smooth Banach spaces unless the sequence satisfies conditions:

(i): $\lim_{n\longrightarrow\infty}\alpha_n = 0$ (ii): $\sum_{n=0}^{\infty}\alpha_n = \infty$ (iii): $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$

Due to the restriction of condition (ii), process (1.2.2) is widely believed to have slow convergence though the rate of convergence has not be determined. Moreover, Halpern [16] prove that conditions (i) and (ii) are indeed necessary in the sense that if process (1.2.2) is strongly convergent for all closed convex subsets C of a Hilbert space H and all nonexpansive mappings T on C, then the sequence $\{\alpha_n\}$ must satisfy conditions (i) and (ii). (However, it is unknown whether these two conditions are also sufficient). In 2006, Maritinez and Xu [35] develop the normal hybrid method for process (1.2.2) and proved the strong convergence of the method under condition (i) only. Moreover they extend Nakajo and Takahashi' s iteration process (1.2.5) to the Ishikawa iteration process. In 2005, Matsushita and Takahashi [37] extend the results of Nakajo and Takahashi [39] to a Banach space for a relatively nonexpansive mapping.

Note that the hybrid method iteration method presented by Massushita and Takahashi [37] can be used for relatively nonexpansive mapping, but it cannot be used for hemirelatively nonexpansive mapping.

Shrinking Projection Method

In 2008, Takahashi et. al. [49] introduced another hybrid method called the *shrinking projection method* for nonexpansive mapping T in a Hilbert space H as follows: $x_0 = x \in C$

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x \end{cases} \tag{1.2.6}$$

and they proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (6.4.2) converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$ and they proposed the following modification the iteration method (1.2.5) and (6.4.2) for a countable family of nonexpansive mappings satisfying NST-condition (see [49]) in a Hilbert space.

Monotone Hybrid Method

In 2008, Qin and Su [41] modified the iteration method (1.2.5), so call the *monotone hybrid method* for nonexpansive mapping T in a Hilbert space as follows: $x_0 = x \in C$

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{1.2.7}$$

By using this method, they proved strong convergence theorem under a control condition on the sequence $\{\alpha_n\}$ but the technic they used in this paper is different from Nakajo and Takahashi [21]. More precisely, they can show that the sequence generated by (6.4.3) is a Cauchy sequence, without the use of demiclosedness principle, Opial's condition and the Kadec-Klee property. Moreover, they extended the results to a Banach space for a relative nonexpansive mapping by using same method.

1.3 The Variational Inequality and the Equilibrium Problem

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself and let B be a β-inverse-strongly monotone of C into H. The variational inequality problem is to find $x \in C$ such that:

$$\langle Bx, y - x \rangle \geq 0 \quad for\ all\ y \in C. \tag{1.3.1}$$

The set of solutions of the variational inequality is denoted by $VI(C, B)$.

Let E be a real Banach space and let E^* be the dual space of E. Let A be a maximal monotone operator from E to E^*. It is well-known that many problems in nonlinear analysis and optimization can be formulated as follows: find a point $u \in E$ satisfying

$$0 \in Au. \tag{1.3.2}$$

We denote by $A^{-1}0$ the set of all points $u \in C$ such that $0 \in Au$. Such a problem contains numerous problems in economics, optimization and physics, and is connected with a variational inequality problem.

A well-known method to solve the problem (1.3.2) is call the *proximal point algorithm*: $x_0 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \;\; n = 0, 1, 2, 3, ..., \tag{1.3.3}$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resolvent of A. Many researchers have studies this algorithm in a Hilbert space and in a Banach space.

Let E be a real Banach space, let E^* be the dual space of E and let C be a closed subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find

$$\hat{x} \in C \text{ such that } F(\hat{x}, y) \geq 0, \; \forall y \in C.$$

The set of such solutions $\hat{x}$ is denoted by $EP(F)$.

Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [65], Combettes and Hirstoaga [69]. On the other hand, Ibaraki and Takahashi [18] introduced a new resovent of a maximal monotone operator in a Banach space and the concept of a generalized nonexpansive mapping in a Banach space. Kohsaka and Takahashi [31], Ibaraki and Takahashi [18] also studied some properties for generalized nonexpansive retractions in Banach spaces. Recently, Takahashi and Zembayashi [50] consider the following equilibrium problem with a bifunction defined on the dual space of a Banach space. Moreover, they proved a strong convergence theorem for finding a solution of the equilibrium problem which generalized the result of Combettes and Hirstoaga [69].

The aim of this project is to consider and study general systems of the generalized variational inequality problems for the single-valued and multi-valued nonlinear mappings. We plan to fine common solutions of fixed points and the solution of the variational inequality problems and also construct and discuss the convergence criterion for the iterative algorithm to approximate the solutions of the problems above, specially, we mainly focus to the generalized systems of resolvent equations and generalized systems of the variational inclusion problems for nonlinear mappings. Moreover, we will apply our results to (system) mixed equilibrium problems and optimization problems. However, it is worth mentioning that the class of variational inclusions inequality problems for nonlinear mappings have had a great impact and influence in the development of several branches of pure, applied and engineering sciences. In the first year, we will study and discuss some important basic results and consider some new theorems about the general systems of (generalized) variational (inclusion) inequality problems and fixed point problems for nonlinear mappings in the Hilbert spaces. In the second year, we will focus our study to the heart of our project, that is, we will consider the general systems of variational inclusion problems and general (system) mixed equilibrium problems with related optimization problems in the Banach spaces. In conclusion, we point out that the results of this project are the extension and improvements of the earlier and recent results in this field, and moreover, the study of this area is a fruitful and growing field of intellectual endeavor. Much work is needed to develop this interesting subject.

This research is divided into 7 chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with some preliminaries and give some useful results that will be used in later chapters. Chapter 3 we prove strong convergence theorems for finding a common element of the fixed point set. Chapter 4 we prove strong convergence theorems for finding a common element of the systems of generalized (mixed) equilibrium problems in Hilbert and Banach spaces. Chapter 5 we prove strong convergence theorems for finding a common element of the systems of variational inequality problems and the set of common fixed points. Chapter 6 we introduced and prove strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points which are application to optimization problems, Furthermore, we also give some applications and numerical example in the end of this section. The conclusion output of research is in Chapter 7.

บทที่ 2

Preliminaries

2.1 Linear Spaces and Metric Spaces

Definition 2.1.1. *Let* X *be a nonempty set, and assume that each pair of elements* x *and* y *in* X *can be combined by a process called* addition *to yield an element* z *in* X *denoted by* $x+y$. *Assume also that this operation of addition satisfies the following condition (1)–(4):*

(1) $(x+y)+z = x+(y+z)$;

(2) $x+y = y+x$;

(3) there exists a unique element in X, *denoted by* 0 *and called the zero element, or the origin, such that* $x+0 = x$ *for all* $x \in X$;

(4) each $x \in X$ *there corresponds a unique element in* X, *denoted by* $-x$ *and called the negative of* x, *such that* $x+(-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ *and each element* x *in* X *can be combined by a process called* scalar multiplication *to yield an element* y *in* X *denoted by* $y = \alpha x$ *satisfying (5)–(8):*

(5) $\alpha(\beta x) = (\alpha\beta)x$;

(6) $1 \cdot x = x$;

(7) $(\alpha+\beta)x = \alpha x + \beta x$;

(8) $\alpha(x+y) = \alpha x + \alpha y$.

The system $(X, \cdot, +)$ is called a *linear space* over $\mathbb{R}$ if it satisfies the conditions (1)–(8). A linear space is often called a *vector space*, and its elements are spoken as vectors.

Definition 2.1.2. *Let* X *be a nonempty set. A mapping* $d : X \times X \longrightarrow \mathbb{R}$, *satisfying the following conditions for all* x, y *and* z *in* X:

(A1) $d(x,y) = 0 \Longleftrightarrow x = y$;

(A2) $d(x,y) = d(y,x)$;

(A2) $d(x,y) \leq d(x,z) + d(z,y)$*. The conditions (A1)-(A3) are usually called the* metric axioms*.*

The function d assigns to each pair (x,y) of element of X a nonnegative real number $d(x,y)$, which does not on the order of the elements; $d(x,y)$ is called the *distance* between x and y. The set X together with a metric, denoted by (X,d), is called a *metric space*.

2.2 Normed Spaces and Banach Spaces

Definition 2.2.1. *Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be a norm on X if it satisfies the following conditions:*

(1) $\|x\| \geq 0, \forall x \in X$*;*

(2) $\|x\| = 0 \Leftrightarrow x = 0$*;*

(3) $\|x+y\| \leq \|x\| + \|y\|, \forall x,y \in X$*;*

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ *and* $\forall \alpha \in \mathbb{K}$*.*

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x,y) = \|x-y\|, \quad (x,y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space*.

Definition 2.2.2. *A normed space $(X, \|\cdot\|)$ is called strictly convex if for all $x,y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0,1)$.*

Definition 2.2.3. *Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to converge strongly in X if there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.*

Definition 2.2.4. *A sequence $\{x_n\}$ in a normed spaces is said to converge weakly to some vector x if $\lim_{n \longrightarrow \infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converges weakly to x.*

Definition 2.2.5. *Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m,n \geq N$. That is, $\{x_n\}$ is* a Cauchy sequence *in X if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m,n \longrightarrow \infty$.*

Theorem 2.2.6. [173] *Let* $\{x_n\}$ *be a sequence of a normed space* $(X, \|\cdot\|)$*,* $x \in X$ *and let* $x_n \to x$ *if and only if, for any subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$*, there exist a subsequence* $\{x_{n_{i_j}}\}$ *of* $\{x_{n_i}\}$ *converging to* x*.*

Definition 2.2.7. *A normed space* X *is called complete if every Cauchy sequence in* X *converges to an element in* X*.*

Definition 2.2.8. *A complete normed linear space over field* $\mathbb{K}$ *is called a Banach space over* $\mathbb{K}$*.*

Lemma 2.2.9. [174] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.

Definition 2.2.10. *Let* F *and* X *be linear spaces over the field* $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow X$ *is called a linear operator if* $T(x+y) = Tx + Ty$ *and* $T(\alpha x) = \alpha Tx, \forall x, y \in F,$ *and* $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ *is called a linear functional on* F *if* T *is a linear operator.*

Definition 2.2.11. *Let* F *and* X *be normed spaces over the field* $\mathbb{K}$ *and* $T : X \longrightarrow F$ *a linear operator.* T *is said to be bounded on* X *if there exists a real number* $M > 0$ *such that* $\|T(x)\| \leq M\|x\|, \forall x \in X$*.*

Definition 2.2.12. *Sequence* $\{x_n\}_{n=1}^{\infty}$ *in a normed linear space* X *is said to be a bounded sequence if there exists* $M > 0$ *such that* $\|x_n\| \leq M, \forall n \in \mathbb{N}$*.*

Definition 2.2.13. *A subset* C *of a normed linear space* X *is said to be convex subset in* X *if* $\lambda x + (1-\lambda)y \in C$ *for each* $x, y \in C$ *and for each scalar* $\lambda \in [0,1]$*.*

2.3 Inner Product Spaces and Hilber Spaces

Definition 2.3.1. *The real-value function of two variables* $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ *is called inner product on a real vector space* X *if for any* $x, y, z \in X$ *and* $\alpha, \beta \in \mathbb{R}$ *the following conditions are satisfied:*

(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$*;*

(2) $\langle x, y \rangle = \langle y, x \rangle$*;*

(3) $\langle x, x \rangle \geq 0$ *for each* $x \in X$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.3.2. *A* Hilbert spaces *is an inner product space which is complete under the norm induced by its inner product.*

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.3.3. [173]*(The Schwarz inequality) If x and y are any two vector in an inner product space X, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Remark 2.3.4. *In a Hilbert space H, weak convergence is defined by $\lim_{n\longrightarrow\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.*

Remark 2.3.5. *If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.*

Definition 2.3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called* lower semicontinuous *if for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.*

Lemma 2.3.7. [173] *Let X be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:*

$$\|x\| \leq \lim_{n\longrightarrow\infty} \inf \|x_n\|.$$

2.4 Basic Concepts in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle ., .\rangle$ and $\|.\|$, respectively. We have the following are hold:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{2.4.1}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.4.2}$$

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle, \tag{2.4.3}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.4.4}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Lemma 2.4.1. [176] *Let* $(E, \langle ., . \rangle)$ *be an inner product space. Then for all* $x, y, z \in E$ *and* $\alpha, \beta, \gamma \in [0, 1]$ *with* $\alpha + \beta + \gamma = 1,$ *we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.4.2. [177] *A Hilbert space* H *satisfies the* ***Opial condition*** *that is, for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$, *the inequality* $\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$, *holds for every* $y \in H$ *with* $y \neq x$.

Lemma 2.4.3. [178],[179] *A Hilbert space* H *satisfies the* ***Kadec-Klee property*** *that is, for any sequence* $\{x_n\}$ *with* $x_n \longrightarrow x$ *and* $\|x_n\| \longrightarrow \|x\|$ *together imply* $\|x_n - x\| \longrightarrow 0$.

2.5 Basic Concepts in Banach Spaces

Let E be a real Banach space and E^* be the dual space of E with norm $\|\cdot\|$ and duality pairing between E and E^* $\langle \cdot, \cdot \rangle$.

Definition 2.5.1. *We set* $E^{**} = (E^*)^*$. *If* E *be a Banach space, then there is a natural assignment of each* $x \in E$ *to a continuous linear functional* x^{**} *on* E^* *given by* $\langle x^{**}, f \rangle = \langle x, f \rangle$ *for all* $x \in E^*$. *Here* $\|x^{**}\| = \|x\|$. *We set* $b(x) = x^{**}$. *It* $b : E \to E^{**}$ *is surjective, then* E *is called* reflexive.

Definition 2.5.2. *Let* $U = \{x \in E : \|x\| = 1\}$. *A Banach space* E *is said to be* strictly convex *if* $\|\frac{x+y}{2}\| < 1$ *for all* $x, y \in U$ *with* $x \neq y$.

Definition 2.5.3. *Let* $U = \{x \in E : \|x\| = 1\}$. *A Banach space* E *is said to* uniformly convex *if, for any* $\epsilon \in (0, 2]$, *there exists* $\delta > 0$ *such that, for any* $x, y \in U$, $\|x - y\| \geq \epsilon$ *implies* $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Remark 2.5.4. *A uniformly convex Banach space is reflexive and strictly convex.*

Definition 2.5.5. *A Banach space* E *is said to be* smooth *if the limit* $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ *exists for all* $x, y \in U$.

Definition 2.5.6. *The* modulus of smoothness *of* E *is defined by*

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0, \infty) \to [0, \infty)$ *is a function.*

Definition 2.5.7. E *be an* uniformly smooth *if* $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$.

Definition 2.5.8. *Let* q *be a fixed real number with* $1 < q \leq 2$. *A Banach space* E *is said to be* q-uniformly smooth *if there exists a constant* $c > 0$ *such that* $\rho(\tau) \leq c\tau^q$ *for all* $\tau > 0$.

2.6 Some Nonlinear Mappings in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $T : C \to C$ a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Definition 2.6.1. *A mapping $S : C \to C$ is called L-*Lipschitz-continuous *if there exists a positive real number L such that*

$$\|Su - Sv\| \leq L\|u - v\|, \quad \forall u, v \in C. \tag{2.6.1}$$

Definition 2.6.2. *A mapping $f : C \to C$ is called a* contraction *on C if there exists a constant $\alpha \in (0,1)$ and $x, y \in C$ such that*

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|. \tag{2.6.2}$$

Definition 2.6.3. *A mapping T is called $nonexpansive$ if*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.6.3}$$

Theorem 2.6.4. [173] *(Banach's Contraction Mapping Principle) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, i.e. there exists a unique $x^* \in X$ such that $Tx^* = x^*$.*

Lemma 2.6.5. [181] *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Definition 2.6.6. *The* metric (nearest point) projection *P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property*

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H; \tag{2.6.4}$$

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \forall z \in C; \tag{2.6.5}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C; \tag{2.6.6}$$

and

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.6.7}$$

Definition 2.6.7. *A mapping A of C into H is called* monotone *if*

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u, v \in C. \tag{2.6.8}$$

Definition 2.6.8. *A is called* α-inverse-strongly monotone *if there exists a positive real number α such that*

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in C. \tag{2.6.9}$$

Lemma 2.6.9. *Let $A : H \to H$ be a α-inverse-strongly monotone mapping. If $\lambda \leq 2\alpha$, for any $\lambda > 0$ and $\alpha > 0$ then $I - \lambda A$ is a nonexpansive mapping from H into itself.*
Proof. *Let $u, v \in H$ and $\lambda > 0$,*

$$\begin{aligned}
\|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\
&= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\
&\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.
\end{aligned}$$

□

Remark 2.6.10. *It is easy to see that if A is an α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$—Lipschitz continuous.*

Definition 2.6.11. *The mapping $S : C \longrightarrow C$ is called a κ-strict pseudo-contraction mapping if there exists a constant $0 \leq \kappa < 1$ such that*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.6.10}$$

2.7 Some Geometric Properties of Banach Spaces

In this section, we discuss geometric properties of Banach spaces. When we deal with the nonlinear problems in Banach spaces, it is to hard to discuss them without geometric properties of Banach spaces.

Definition 2.7.1. ([24]) Let E and F be vector space. A *linear operator* from E into F is a function $f : E \longrightarrow F$ such that the following two conditions are satisfied whenever $x, y \in E$ and $\alpha \in \mathbb{F}$:

(1) $f(x + y) = f(x) + f(y)$;

(2) $f(\alpha x) = \alpha f(x)$

Definition 2.7.2. ([24]) A *linear functional* f is a linear operator with domain in a vector space E and range in the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.7.3. ([24]) Let E be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A linear functional $f : E \longrightarrow \mathbb{F}$ is said to be *bounded*, if there exist $k > 0$ such that $|f(x)| \leq k\|x\|$, for all $x \in E$.

Definition 2.7.4. ([24]) Let E be a normed space. Then the set of all bounded linear functionals on E constitutes a normed space with norm defined by

$$\|f\| = \sup_{x \neq 0 \in X} \frac{|f(x)|}{\|x\|}$$

which ia called the *dual space* of E and is denoted by E^*.

Theorem 2.7.5. ([47]) The dual space E^* of a normed space E is a Banach space.

Definition 2.7.6. ([47]) Let E be a Banach space and let E^* be it dual. With each $x \in E$, we associate the set $J(x) = \{f \in E^* |\ f(x) = \|x\|^2 = \|f\|^2\}$. The multivalued operator $J : E \longrightarrow E^*$ is called the *duality mapping* of E.

Theorem 2.7.7. ([47]) Let E be a Banach space and let J be duality mapping of E. Then:

(1) For $x \in E$, $J(x)$ is nonempty, bounded, closed and convex,

(2) $J(0) = \{0\}$,

(3) for $x \in E$ and a real α, $J(\alpha x) = \alpha J(x)$,

(4) for $x, y \in E$, $f \in J(x)$ and $g \in J(y)$, $\langle x - y, f - g \rangle \geq 0$,

(5) for $x, y \in E$ and $f \in J(y)$, $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, f \rangle$.

Definition 2.7.8. ([24]) Let E be normed space, for each $x \in E$ there corresponds a unique bounded linear functional $g_x \in E^{**}$ given by $g_x(f) = f(x)$, $f \in E^*$. A mapping $C : E \longrightarrow E^{**}$ defined by $x \mapsto g_x$, is called the *canonical mapping*.

Definition 2.7.9. ([47]) Let E be a Banach space and let $U = \{x \in E \mid \|x\| = 1\}$. Then a Banach space is said to be *smooth* provided the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.7.1}$$

exists for each $x, y \in U$. In this case, the norm of E is said to be $G\hat{a}teaux$ differentiable. The space E is said to have a uniformly $G\hat{a}teaux$ differentiable norm if for each $y \in U$, the limit (2.7.1) is attained uniformly for $x \in U$. The norm of E is said to be $Fr\acute{e}chet$ differentiable norm if for each $x \in U$, the limit (2.7.1) is attained uniformly for $y \in U$. The norm of E is said to be uniformly $Fr\acute{e}chet$ differentiable (and E is said to be *uniformly smooth*) if the limit (2.7.1) is attained uniformly for $(x, y) \in U \times U$.

Remark We know the following: see [47] more details;

(i) If E is smooth, then J is single-valued;

(ii) If E^* is is strictly convex, then J is single-valued;

(iii) If E is reflexive, then J is onto;

(iv) If E is strictly convex, then J is one-to-one;

(v) If E is strictly convex, then J is strictly monotone;

(vi) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

2.8 Basic Concept of Convex Analysis

Definition 2.8.1. ([48]) Let H be a Hilbert space and let C be nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\}$$

is closed. f is also called *convex* on if for any $x, y \in C$ and $t \in [0, 1]$, then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Theorem 2.8.2. ([48])(Minimization theorem)
Let C be a nonempty bounded closed convex subset of a Hilbert space H and let f be a proper lower semicontinuous convex function of C into $(-\infty, \infty]$. Then there exists $x_0 \in D(f)$ such that

$$f(x_0) = \min_{x \in C} f(x).$$

Definition 2.8.3. ([48]) Let H be a Hilbert space and let $f : H \longrightarrow (-\infty, \infty]$ be a proper convex function. Then, we define the subdifferential ∂f of f by

$$\partial f(x) = \{x \in H : f(y) \geq \langle y - x, z \rangle + f(x), \ \forall y \in H\}$$

for all $x \in H$. If $f(x) = \infty$, then $\partial f(x) = \emptyset$.

Lemma 2.8.4. ([48]) Let H be a Hilbert space and let $f : H \longrightarrow (-\infty, \infty]$ be a proper convex function. Let $z \in H$. Then

$$0 \in \partial f(z) \Leftrightarrow f(z) = \min_{x \in H} f(x).$$

Lemma 2.8.5. ([48]) Let E be a Banach space and let $f : E \longrightarrow (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E : f(y) \geq \langle y - x, x^* \rangle + f(x), \ \forall y \in E\}$$

for each $x \in E$. Then, ∂f is a maximal monotone operator.

Lemma 2.8.6. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Define the *indicator function* i_C of C by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & otherwise. \end{cases}$$

Then, i_C is proper, convex and semicontinuous and ∂i_C is a maximal monotone operator.

Definition 2.8.7. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H and $x \in C$. Then we define the set $N_C(x)$ of H by

$$N_C(x) = \{z \in H : \langle u - x, z \rangle \leq 0, \forall u \in C\}.$$

Such a set $N_C(x)$ is called the *normal cone* of C.

Remark The set $N_C(x)$ is a closed convex cone of H.

Definition 2.8.8. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let B be an operator of C into H. Consider the following problem: Find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0$$

for all $y \in C$. Such an $x \in C$ is called a *solution of the variational inequality* of B. We denote $VI(C, B)$ the set of all solutions of the variational inequality.

Definition 2.8.9. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let B be an operator of C into H. Then B is called *hemicotinuous* if for any $u, v \in C$ and $w \in H$, the function

$$t \mapsto \langle w, B(tu + (1 - t)v) \rangle$$

of [0,1] into $\mathbb{R}$ is contonuous.

Theorem 2.8.10. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $B : C \longrightarrow H$ be monotone and hemicontinuous and let $N_C(x)$ denote the normal cone of C at $x \in C$. Define

$$Tx = \begin{cases} Bx + N_C x, & \text{if} \quad v \in C, \\ \emptyset, & \text{if} \quad v \notin C. \end{cases}$$

Then $T : H \longrightarrow 2^H$ is a maximal monotone and $0 \in Tx$ iff $x \in VI(C, B)$.

Definition 2.8.11. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let B be an operator of C into H. Then B is called an *inverse strongly monotone operator* if there exists $\beta > 0$ such that

$$\langle x - y, Bx - By \rangle \geq \beta \|Bx - By\|^2$$

for all $x, y \in C$. Such a B is called *β-inverse strongly monotone*.

Remake. If B is a β-inverse strongly monotone operator of C to H, then it is obvious that B is $\frac{1}{\beta}$-Lipschitz continuous.

Lemma 2.8.12. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\beta > 0$ and let $B : C \longrightarrow H$ be β-inverse strongly monotone. If $0 < \lambda \leq 2\beta$, then $I - \lambda B$ is a nonexpansive mapping of C into H.

Lemma 2.8.13. ([48]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let B be an operator of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u \in VI(C, B) \Leftrightarrow u = P_C(I - \lambda B)u.$$

where P_C is the metric projection of H onto C.

Theorem 2.8.14. ([48]) Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\beta > 0$ and let $B : C \longrightarrow H$ be β-inverse strongly monotone. Then $VI(C, B) \neq \emptyset$.

Definition 2.8.15. ([48]) Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping B of C into H is called *monotone* if $\langle Bx - By, x - y \rangle \geq 0$ for all $x, y \in C$.

Definition 2.8.16. ([48]) Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. A mapping $T : C \longrightarrow C$ is called *strictly pseudocontractive* if there exists k with $0 \leq k < 1$ such that:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \textit{for all } x, y \in C.$$

Remark. If $k = 0$, then T is nonexpansive. Put $B = I - T$, where $T : C \longrightarrow C$ is a strictly pseudocontractive mapping with k. Then B is $\frac{1-k}{2}$-inverse-strongly monotone.

บทที่ 3

Fixed Point Problems

3.1 Strong Convergence Theorems

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 3.1.1 ([327]). *Assume $\{\alpha_n\}$ is a sequences of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \ \ n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\longrightarrow\infty} \delta_n/\gamma_n \geq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} \alpha_n$ exists.

Lemma 3.1.2 ([61], Lemma 3.2). *Let C be a nonempty closed subset of a Banach space and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{T_ny\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\longrightarrow\infty} T_ny \qquad \text{for all } y \in C.$$

Then $\lim_{n\longrightarrow\infty} \sup\{\|T_nz - Tz\| : z \in C\} = 0$.

3.1.1 A countable family of nonexpansive mappings

In this section, we prove some strong convergence theorems for monotone mappings and a countable family of nonexpansive mappings.

Theorem 3.1.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C,A) \neq \emptyset$, and f be a contraction of C into itself. Suppose $x_1 = x \in C$ and let $\{x_n\}$ be the iterative sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n P_C(x_n - \lambda_n A x_n),$$

for all $n = 0, 1, 2, ...,$ where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\longrightarrow\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. Then $\{x_n\}$ converges strongly to $z \in \mathrm{F}(S) \cap VI(C,A)$, where $z = P_{\mathrm{F}(S)\cap VI(C,A)} f(z)$.

Proof. Let $Q = P_{\mathrm{F}(S)\cap VI(C,A)}$. Then Qf is a contraction of H into C. In fact, there exists $k \in [0,1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \le \|f(x) - f(y)\| \le k\|x - y\|$$

for all $x, y \in H$. This implies that Qf is a contraction on H into C. Since H is complete, there exists a unique element of $z \in H$, such that $z = Qf(z)$. Such a $z \in H$ is an element of C.

Put $y_n = P_C(x_n - \lambda_n A x_n)$, for every $n \in \mathbb{N} \cup \{0\}$. Let $u \in F(S) \cap VI(C,A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n A u)$, we have

$$\begin{aligned}
\|y_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\
&\le \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \\
&\le \|(I - \lambda_n A)x_n - (I - \lambda_n A)u\| \\
&\le \|x_n - u\|,
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. We note that

$$\begin{aligned}
\|x_{n+1} - u\| &= \|\alpha_n(f(x_n) - u) + (1-\alpha_n)(S_n y_n - u)\| \\
&\le \alpha_n\|f(x_n) - u\| + (1-\alpha_n)\|S_n y_n - u\| \\
&\le \alpha_n(\|f(x_n) - f(u)\| + \|f(u) - u\|) + (1-\alpha_n)\|x_n - u\| \\
&\le \alpha_n k\|x_n - u\| + \alpha_n\|f(u) - u\| + (1-\alpha_n)\|x_n - u\| \\
&= (1 - (1-k)\alpha_n)\|x_n - u\| + (1-k)\alpha_n(\frac{1}{1-a}\|f(u) - u\|) \\
&\le \max\{\|x_n - u\|, \frac{1}{1-k}\|f(u) - u\|\}.
\end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we get

$$\|x_{n+1} - u\| \leq \max\{\|x_1 - u\|, \frac{1}{1-k}\|f(u) - u\|\}, \ n \geq 1. \tag{3.1.1}$$

Therefore $\{x_n\}$ is bounded. Hence, we also obtain that $\{y_n\}, \{S_n y_n\}$ and $\{f(x_n)\}$ are bounded.

Since $I - \lambda_n A$ is nonexpansive, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|P_C(x_{n+1} - \lambda_n A x_{n+1}) - P_C(x_n - \lambda_n A x_n)\| \\ &\leq \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_n A)x_n\| \\ &\leq \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| \\ &\leq \|(x_{n+1} - x_n)\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\|, \end{aligned} \tag{3.1.2}$$

for all $n \in \mathbb{N}$. So, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|[\alpha_n f(x_n) + (1-\alpha_n)S_n y_n] - [\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})S_{n-1}y_{n-1}]\| \\ &= \|\alpha_n[f(x_n) - f(x_{n-1})] + (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\ &\quad + (1-\alpha_n)(S_n y_n - S_{n-1}y_{n-1}) + (\alpha_{n-1} - \alpha_n)S_{n-1}y_{n-1}\| \\ &\leq \alpha_n k\|x_n - x_{n-1}\| + |(\alpha_n - \alpha_{n-1}|(\|f(x_{n-1}\| + \|S_{n-1}y_{n-1}\|) \\ &\quad + (1-\alpha_n)(\|S_n y_n - S_n y_{n-1}\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\|) \\ &\leq \alpha_n k\|x_n - x_{n-1}\| + |(\alpha_n - \alpha_{n-1}|(\|f(x_{n-1}\| + \|S_{n-1}y_n\|) \\ &\quad + (1-\alpha_n)\|y_n - y_{n-1}\| + (1-\alpha_n)\sup\{\|S_n z - S_{n-1}z\| : z \in \{y_{n-1}\}\} \\ &\leq \alpha_n k\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|fx_{n-1}\| + \|S_{n-1}y_{n-1}\|) \\ &\quad + (1-\alpha_n)(\|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|Ax_{n-1}\| \quad (\text{ by } 3.1.2) \\ &\quad + (1-\alpha_n)\sup\{\|S_n z - S_{n-1}z\| : z \in \{y_n\}\} \\ &\leq (1 - (1-k)\alpha_n)\|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n|M + |\alpha_n - \alpha_{n-1}|L \\ &\quad + (1-\alpha_n)\sup\{\|S_n z - S_{n-1}z\| : z \in \{y_n\}\}, \end{aligned}$$

for every $n \in \mathbb{N}$, where $L := \sup_{n\geq 1}\{\|fx_{n-1}\| + \|S_{n-1}y_{n-1}\|\}$ and $M := \sup_{n\geq 1}\{\|Ax_n\|\}$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=1}^{\infty}\{\|S_n z - S_{n+1}z\| : z \in \{y_n\}\} < \infty$, it follows by Lemma 3.1.1 that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0. \tag{3.1.3}$$

Then we also obtain $\lim_{n\longrightarrow\infty} \|y_{n+1} - y_n\| = 0$. Moreover, we note that

$$\begin{aligned} \|x_n - S_n y_n\| &\leq \|x_n - S_{n-1}y_{n-1}\| + \|S_{n-1}y_{n-1} - S_n y_{n-1}\| + \|S_n y_{n-1} - S_n y_n\| \\ &\leq \alpha_{n-1}\|f(x_{n-1}) - S_{n-1}y_{n-1}\| + \sup\{\|S_{n-1}z - S_n z\| : z \in \{y_{n-1}\}\} \\ &\quad + \|y_{n-1} - y_n\|, \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Thus, we have

$$\lim_{n\longrightarrow\infty}\|x_n - S_n y_n\| = 0. \tag{3.1.4}$$

From above, we obtain

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\le \alpha_n\|f(x_n)-u\|^2 + (1-\alpha_n)\|S_n y_n - u\|^2\\
&\le \alpha_n\|f(x_n)-u\|^2 + (1-\alpha_n)\|y_n - u\|^2\\
&= \alpha_n\|f(x_n)-u\|^2 + (1-\alpha_n)\|P_C(x_n-\lambda_n Ax_n) - u\|^2\\
&\le \alpha_n\|f(x_n)-u\|^2 + (1-\alpha_n)\|(I-\lambda_n A)x_n - (I-\lambda_n A)u\|^2\\
&\le \alpha_n\|f(x_n)-u\|^2 + (1-\alpha_n)(\|x_n-u\|^2 + \lambda_n(\lambda_n-2\alpha)\|Ax_n - Au\|^2)\\
&\le \alpha_n\|f(x_n)-u\|^2 + \|x_n-u\|^2 + (1-\alpha_n)a(b-2\alpha)\|Ax_n - Au\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
&-(1-\alpha_n)a(b-2\alpha)\|Ax_n-Au\|^2\\
\le\ &\alpha_n\|f(x_n)-u\|^2 + \|x_n-u\|^2 - \|x_{n+1}-u\|^2\\
\le\ &\alpha_n\|f(x_n)-u\|^2 + (\|x_n-u\| + \|x_{n+1}-u\|)\|x_{n+1}-x_n\|.
\end{aligned}$$

Since $\alpha_n \longrightarrow 0$ and $\|x_{n+1}-x_n\| \longrightarrow 0$, it follows that $\|Ax_n - Au\| \longrightarrow 0$. Further, we obtain

$$\begin{aligned}
\|y_n-u\|^2 &= \|P_C(x_n-\lambda_n Ax_n) - P_C(u-\lambda_n Au)\|^2\\
&\le \langle (x_n-\lambda_n Ax_n) - (u-\lambda_n Au), y_n - u\rangle\\
&= (1/2)\{\|(x_n-\lambda_n Ax_n) - (u-\lambda_n Au)\|^2 + \|y_n-u\|^2\\
&\quad -\|[(x_n-\lambda_n Ax_n) - (u-\lambda_n Au)] - (y_n-u)\|^2\}\\
&\le (1/2)\{\|x_n-u\|^2 + \|y_n-u\|^2 - \|(x_n-y_n) - \lambda_n(Ax_n-Au)\|^2\}\\
&= (1/2)\{\|x_n-u\|^2 + \|y_n-u\|^2 - \|(x_n-y_n)\|^2\\
&\quad +2\lambda_n\langle x_n-y_n, Ax_n-Au\rangle - \lambda_n^2\|Ax_n-Au\|^2\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|y_n-u\|^2 \le\ &\|x_n-u\|^2 - \|x_n-y_n\|^2\\
&+2\lambda_n\langle x_n-y_n, Ax_n-Au\rangle - \lambda_n^2\|Ax_n-Au\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)S_n y_n - u\|^2\\
&\le \alpha_n\|f(x_n)-u\|^2 + (1-\alpha_n)\|y_n-u\|^2\\
&\le \alpha_n\|f(x_n)-u\|^2 + \|x_n-u\|^2 - \|x_n-y_n\|^2\\
&\quad +2\lambda_n\langle x_n-y_n, Ax_n-Au\rangle - \lambda_n^2\|Ax_n-Au\|^2.
\end{aligned}$$

Since $\alpha_n \longrightarrow 0, \|x_{n+1} - x_n\| \longrightarrow 0$ and $\|Ax_n - Au\| \longrightarrow 0$, we have

$$\|x_n - y_n\| \longrightarrow 0. \tag{3.1.5}$$

>From $\|S_n y_n - y_n\| \le \|S_n y_n - x_n\| + \|x_n - y_n\|$, we obtain

$$\|S_n y_n - y_n\| \longrightarrow 0. \tag{3.1.6}$$

Next we show that

$$\limsup_{n \longrightarrow \infty} \langle f(z) - z, S_n y_n - z \rangle \le 0,$$

where $z = P_{F(S) \cap VI(C,A)} f(z)$. To show it, choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n \longrightarrow \infty} \langle f(z) - z, S_n y_n - z \rangle = \lim_{i \longrightarrow \infty} \langle f(z) - z, S_n y_{n_i} - z \rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$. Since $\|S_n y_n - y_n\| \longrightarrow 0$, we obtain $S_{n_i} y_{n_i} \rightharpoonup w$. We now show that $w \in F(S) \cap VI(C, A)$.

First, it follows by the same argument as in the proof of [258, Theorem 3.1, pp. 346-347] that $z \in VI(C, A)$. Let us show that $w \in F(S)$. Assume $w \notin F(S)$. From Opial's condition, we have

$$\begin{aligned}
\liminf_{n \longrightarrow \infty} \|y_{n_i} - w\| &< \liminf_{i \longrightarrow \infty} \|y_{n_i} - Sw\| \\
&= \liminf_{i \longrightarrow \infty} \|y_{n_i} - S_{n_i} y_{n_i} + S_{n_i} y_{n_i} - S y_{n_i} + S y_{n_i} - Sw\| \\
&\le \liminf_{i \longrightarrow \infty} \|S y_{n_i} - Sw\| \\
&\le \liminf_{i \longrightarrow \infty} \|y_{n_i} - w\|
\end{aligned}$$

This is a contradiction. Thus, we obtain $w \in F(S)$. Therefore $w \in F(S) \cap VI(C, A)$. Since $z = P_{F(S) \cap VI(C,A)} f(z)$, we have

$$\begin{aligned}
\limsup_{n \longrightarrow \infty} \langle f(z) - z), S_n y_n - z \rangle &= \lim_{i \longrightarrow \infty} \langle f(z) - z, S_n y_{n_i} - z \rangle \\
&= \langle f(z) - z, w - z \rangle \le 0.
\end{aligned}$$

for all $n \geq m$. For all $n \geq m$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)S_n y_n - z\|^2 \\
&= \|\alpha_n (f(x_n) - z) + (1-\alpha_n)(S_n y_n - z)\|^2 \\
&\leq \alpha_n^2 \|(f(x_n) - z)\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - z, S_n y_n - z\rangle \\
&\quad +(1-\alpha_n)^2 \|S_n y_n - z\|^2 \\
&\leq \alpha_n^2 \|(f(x_n) - z)\|^2 + (1-\alpha_n)^2 \|x_n - z\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n) - f(z), S_n y_n - z\rangle \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle \\
&\leq \alpha_n^2 \|(f(x_n) - z)\|^2 + (1 - 2\alpha_n + \alpha_n^2)\|x_n - z\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)k\|x_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle \\
&= [1 - 2\alpha_n + \alpha_n^2 + 2k\alpha_n(1-\alpha_n)]\|x_n - z\|^2 + \alpha_n^2\|f(x_n) - z\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle \\
&= (1-\bar{\alpha}_n)\|x_n - z\|^2 + \bar{\alpha}_n\bar{\beta}_n,
\end{aligned}$$

where

$$\begin{aligned}
\bar{\alpha}_n &= 2\alpha_n + \alpha_n^2 + 2k\alpha_n(1-\alpha_n), \\
\bar{\beta}_n &= \frac{\alpha_n\|f(x_n) - z\|^2 + 2(1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle.}{2 + \alpha_n + 2k(1-\alpha_n)}
\end{aligned}$$

It is easily see that $\bar{\alpha}_n \longrightarrow 0, \sum_{n=1}^{\infty} \bar{\alpha}_n = \infty$ and $\limsup_{n\longrightarrow\infty} \bar{\beta}_n \leq 0$. Hence, by Lemma 3.1.1, we obtain $x_n \to z = P_{F(S)\cap VI(C,A)} f(z)$. This completes the proof. □

Putting $f(y) = x \in C$ for all $y \in H$ in Theorem 3.1.3, we have the following result.

Theorem 3.1.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_n P_C(x_n - \lambda_n A x_n),$$

for all $n = 0, 1, 2, ...,$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$ with $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C and S be a mapping of C into itself defined by $Sz = \lim_{n\longrightarrow\infty} S_n z$ for all

$z \in C$ *and suppose that* $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$*. Then* $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap VI(C, A)$*, where* $z = P_{\mathrm{F}(S) \cap VI(C,A)} x_1$.

Proof. It follows by Theorem 3.1.3 that $x_n \longrightarrow z$, where $z = P_{F(S) \cap VI(C,A)} x_1$. □

Setting $S_n \equiv S$ in Theorem 3.1.3 and 3.1.4, we have the following results.

Corollary 3.1.5. (Chen, Zhang and Fan [?]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\alpha > 0$ *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H*. Let* S *be a sequence of nonexpansive mappings from* C *into itself such that* $\mathrm{F}(S) \cap VI(C, A) \neq \emptyset$*. Let* f *be a contraction of* C *into itself. Suppose* $x_1 = x \in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n),$$

for all $n = 0, 1, 2, ...,$ *where* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *with* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

$$\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap VI(C, A)$*, where* $z = P_{\mathrm{F}(S) \cap VI(C,A)} f(z)$.

By using the same argument in the proof of Theorem 3.1.3, we have the following theorem.

Theorem 3.1.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\{S_n\}$ *be a sequence of nonexpansive mappings from* C *into itself such that* $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ *and* f *be a contraction of* C *into itself. Suppose* $x_1 = x \in C$ *and let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n x_n$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$. *Suppose that* $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C*. Let* S *be a mapping defined by* $Sz = \lim_{n \longrightarrow \infty} S_n z$ *for all* $z \in C$ *and suppose that* $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$*. Then* $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S)$*, where* $z = P_{\mathrm{F}(S)} f(z)$.

Proof. Putting $Q = P_{F(S)}$ and $y_n = x_n$ in the proof of Theorem 3.1.3. Then, by using the same argument as in the proof of Theorem 3.1.3, we can show that $\{x_n\}$ converges strongly to a point $z \in F(S)$, where $z = P_{F(S)} f(z)$. □

Lemma 3.1.7. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4)*. Let* $\{r_n\}$ *be a sequence of positive integers and* T_{r_n} *be the mapping. Let* $\{r_n\}$ *be a sequence in* $(0, \infty)$ *such that* $inf\{r_n : n \in \mathbb{N}\} > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *then the following hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}}z - T_{r_n}z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C*,*

(i) $F(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_{r_n})$ *where* T *is a mapping defined by* $Tx = \lim_{n\longrightarrow\infty} T_{r_n}x$ *for all* $x \in C$*.*

Using Theorem 3.1.3 and Lemma 3.1.7, we have the following theorem.

Theorem 3.1.8. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4)*. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *such that* $VI(C, A) \cap \mathrm{EP}(F) \neq \varnothing$*. Let* f *be a contraction of* H *into itself. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \end{cases}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *and* $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ *satisfy* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *with* $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *with* $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*. Then* $\{x_n\}$ *converges strongly to* $w \in VI(C, A) \cap \mathrm{EP}(F)$*, moreover* $w = P_{\mathrm{EP}(F)\cap VI(C,A)} f(w)$.

Using Theorem 3.1.6 and Lemma 3.1.7, we have the following theorem.

Theorem 3.1.9. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *with* $EP(F) \neq \emptyset$ *and let* f *be a contraction of* C *into itself. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 = x \in C$ *and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \end{cases}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *with* $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*. Then* $\{x_n\}$ *converges strongly to* $w \in \mathrm{EP}(F)$*, moreover* $w = P_{\mathrm{EP}(F)} f(w)$.

3.1.2 Accretive operators

In this section, we consider the problem of finding a zero of an accretive operator. Let E be a real Banach space. Let p be a fixed real number with $p \geq 2$. A

Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. Observe that every p-uniform convex is uniformly convex. One should note that no a Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is *2-uniformly convex*, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by $J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$ for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. An operator $A \subset E \times E$ is said to be accretive if for each (x_1, y_1) and $(x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. An accretive operator A is said to satisfy the range condition of $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $D(A)$ is the domain of A, $R(I + \lambda A)$ is the range of $I + \lambda A$, and $\overline{D(A)}$ is the closure of $D(A)$. If A is an accretive operator which satisfies the range condition, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda A) \longrightarrow D(A)$ by $J_\lambda = (I - \lambda A)^{-1}$, which is called the resolvent of A. We know that J_λ is nonexpansive and $F(J_\lambda) = A^{-1}(0)$ for all $\lambda > 0$. An accretive operator A is said to be *m-accretive* if $R(I + \lambda A) = E$ for all $\lambda > 0$ (see also [61])

Lemma 3.1.10. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T \subset H \times H$ be an accretive operator such that $T^{-1}(0) \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$, and $\{r_n\}$ be a sequence in $(0, \infty)$. If $inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then the followings hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}}z - J_{r_n}z\| : z \in B\} < \infty$ for any bounded subset B of C,

(ii) $F(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n})$, where S is a mapping defined by $Sx = \lim_{n\longrightarrow\infty} J_{r_n}x$ for all $x \in C$.

Using Theorem 3.1.3 and Lemma 3.1.10, we have the following theorem.

Theorem 3.1.11. *Let $T \subset H \times H$ be an m-accretive operator with $T^{-1}(0) \neq \emptyset$ and let $C := \overline{D(T)}$. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H and let f be a contraction of C into itself. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} P_C(x_n - \lambda_n A x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{r_n\}$ is a sequence in $(0, \infty)$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a mapping defined by $Sx = \lim_{n\longrightarrow\infty} J_{r_n}x$ for all $x \in C$. If $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to $z \in T^{-1}(0) \cap VI(C, A)$, where $z = P_{T^{-1}(0) \cap VI(C,A)} f(z)$.

Proof. Since H is Hilbert space $C = \overline{D(T)}$ is closed and convex. By Lemma 3.1.10, we have the following

$$\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n}) = T^{-1}(0) \neq \emptyset.$$

Therefore, by Theorem 3.1.3, we obtain $\{x_n\}$ converges strongly to $z = P_{\mathrm{F}(S)\cap T^{-1}(0)} f(z)$. □

Using Theorem 3.1.6 and Lemma 3.1.10, we have the following theorem.

Theorem 3.1.12. *Let $T \subset H \times H$ be an m-accretive operator with $T^{-1}(0) \neq \emptyset$ and let $C := \overline{D(T)}$. Let f be a contraction of C into itself. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{r_n\}$ is a sequence in $(0, \infty)$. If $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to $z \in T^{-1}(0)$, where $z = P_{T^{-1}(0)} f(z)$.

3.1.3 Strictly pseudocontractive mappings

A mapping $T : C \longrightarrow C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \longrightarrow C$ is a strictly pseudocontractive mapping with k. We know that, A is $\frac{1-k}{2}-$ inverse strongly monotone and $A^{-1}(0) = F(T)$ (see [258]).

Now, using Theorem 3.1.3 we state a strong convergence theorem for a pair of a nonexpansive mapping and strictly pseudocontractive mapping as follows.

Theorem 3.1.13. *Let C be a closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself. Let T be a strictly pseudocontractive mapping with constant k of C into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n P_C((1 - \lambda_n)x_n + \lambda_n T x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{r_n\}$ is a sequence in $(0, \infty)$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a *mapping defined by* $Sz = \lim_{n\longrightarrow\infty} S_n z$ *for all* $z \in C$. *If* $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $inf\{r_n : n \in \mathbb{N}\} > 0$, *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *then* $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap F(T)$, *where* $z = P_{\mathrm{F}(S)\cap F(T)} f(z)$.

Proof. Put $A = I - T$. Then A is $\frac{1-k}{2}-$inverse-strongly monotone. We have that $F(T)$ is the solution set of $VI(A, C)$ i.e., $F(T) = VI(A, C)$ and

$$P_C(x_n - \lambda_n A x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n.$$

Therefore, by Theorem 3.1.3, the conclusion follows. □

Setting $f(y) = x$ for all $y \in C$ in Theorem 3.1.13, we have the following corollary.

Corollary 3.1.14. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $\{S_n\}$ *be a sequence of nonexpansive mappings of* C *into itself and let* T *be a strictly pseudocontractive mapping with constant* k *of* C *into itself such that* $\cap_{n=1}^{\infty} F(S_n) \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S_n P_C((1 - \lambda_n)x_n - \lambda_n T x_n)$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$, $\{r_n\}$ *is a sequence in* $(0, \infty)$ *and* $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ *satisfy* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. *Suppose that* S *is a mapping defined by* $Sz = \lim_{n\longrightarrow\infty} S_n z$ *for all* $z \in C$. *If* $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $inf\{r_n : n \in \mathbb{N}\} > 0$, *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *then* $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap F(T)$, *where* $z = P_{\mathrm{F}(S)\cap F(T)} x_1$.

3.2 Convergence Theorems by the Hybrid Projection Method

Let C be a closed and convex subset of E, a mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the fixed point set of T; that is, $F(T) = \{x \in C : Tx = x\}$. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{3.2.1}$$

where J is the normalized duality mapping.

If C is a nonempty, closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [319] introduced a generalized projection Π_C from E into C by

$$\Pi_C(x) = \mathrm{argmin}_{y\in C}\phi(y, x). \tag{3.2.2}$$

It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{3.2.3}$$

If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ and Π_C becomes the metric projection of E onto C. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y,x). \tag{3.2.4}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J (see, for example, [55, 319, 65, 69, 92]). In 2006, Wu and Huang [325] introduced a new generalized f-projection operator in Banach space. They extended the definition of the generalized projection operators introduced by Abler [318] and proved properties of the generalized f-projection operator. Next, we recall the concept of the generalized f-projection operator. Let $G : C \times E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a functional defined by

$$G(\xi, \varpi) = \|\xi\|^2 - 2\langle \xi, \varpi \rangle + \|\varpi\|^2 + 2\rho f(\xi), \tag{3.2.5}$$

where $\xi \in C$, $\varpi \in E^*$, ρ is positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From the definition of G, it is easy to see the following properties.

(1) $G(\xi, \varpi)$ is convex and continuous with respect to ϖ when ξ is fixed;

(2) $G(\xi, \varpi)$ is convex and lower semicontinuous with respect to ξ when ϖ is fixed.

Definition 3.2.1. *Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is a* generalized f-projection operator *if*

$$\pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{\xi \in C} G(\xi, \varpi), \ \forall \varpi \in E^*\}.$$

Definition 3.2.2. *Let C be a nonempty subset of E and let $\{T_n\}_{n=1}^{\infty}$ be a countable family of mappings from C into E. A point p in C is called an* asymptotic fixed point *of $\{T_n\}_{n=1}^{\infty}$ [82] if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. The asymptotic fixed point set of $\{T_n\}_{n=1}^{\infty}$ will be denoted by $\widehat{F}(\{T_n\}_{n=1}^{\infty})$. A mapping T_n from C into itself is called* countable family of relatively nonexpansive mappings *(see [89]) if*

$(R1)$ *$F(\{T_n\}_{n=1}^{\infty})$ is nonempty;*

$(R2)$ $\phi(p, T_n x) \leq \phi(p, x)$ *for all* $x \in C$ *and* $p \in F(\{T_n\}_{n=1}^{\infty})$*;*

$(R3)$ $\widehat{F}(\{T_n\}_{n=1}^{\infty}) = F(\{T_n\}_{n=1}^{\infty})$*.*

A sequence $\{T_n\}_{n=1}^{\infty}$ is called *countable family of relatively quasi-nonexpansive mappings* (or *countable family of quasi-ϕ-nonexpansive mappings*) if conditions $(R1)$ and $(R2)$ hold. It is obvious that a countable family of relatively nonexpansive mappings is a countable family of relatively quasi-nonexpansive mappings but the converse is not true. In order to explain this better, we give the following example.

Example 3.2.3. *Let* $E = \mathbb{R}$ *with the usual norm. We define a mapping* $T_n : E \longrightarrow E$ *by*

$$T_n(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{n}; \\ \frac{1}{n}, & \text{if } x > \frac{1}{n}, \end{cases}$$

for all $n \geq 0$ *and for each* $x \in \mathbb{R}$.
Then $\bigcap_{n=1}^{\infty} F(T_n) = F(T_n) = \{0\}$ *and*

$$\phi(0, T_n x) = \|0 - T_n x\| \leq \|0 - x\| = \phi(0, x), \ \forall x \in \mathbb{R}.$$

Hence, T *is a relatively quasi-nonexpansive mapping but not a relatively nonexpansive mapping.*

Definition 3.2.4. *A point* p *in* C *is called an* asymptotic fixed point *of* T *[82] if* C *contains a sequence* $\{x_n\}$ *which converges weakly to* p *such that* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$*. The asymptotic fixed point set of* T *will be denoted by* $\widehat{F}(T)$*. A mapping* T *from* C *into itself is called* relatively nonexpansive *(see [77, 86, 96]) if*

$(R1)'$ $F(T)$ *is nonempty;*

$(R2)'$ $\phi(p, Tx) \leq \phi(p, x)$ *for all* $x \in C$ *and* $p \in F(T)$*;*

$(R3)'$ $\widehat{F}(T) = F(T)$*.*

A mapping T is called *relatively quasi-nonexpansive* (or *quasi-ϕ-nonexpansive*) if conditions $(R1)'$ and $(R2)'$ hold. Obviously, relatively nonexpansive mappings implies relatively quasi-nonexpansive mappings but the converse is not true. Moreover, Definition 3.2.4 is a special case of Definition 3.2.2 when $T_n \equiv T$, for all $n \geq 0$. Relatively quasi-nonexpansive mappings are sometimes called hemirelatively nonexpansive mappings. The asymptotic behavior of a relatively nonexpansive mapping was studied in [62, 63, 64]. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings (see [62, 63, 64, 75, 84]) which requires the strong restriction: $F(T) = \widehat{F}(T)$. Furthermore, Su et al. [87, 88] gave an example of relatively quasi-nonexpansive mappings which is not relatively nonexpansive mapping.

Example 3.2.5. *(cf. [87, 95]) Let E be any smooth Banach space and let $x_0 \neq 0$ be any element of E. We define a mapping $T: E \longrightarrow E$ by*

$$T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^n})x_0, & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0; \\ -x, & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0. \end{cases}$$

Then T is a relatively quasi-nonexpansive mapping but not a relatively nonexpansive mapping. Actually, T above fails to have the condition $(R3)'$.

For other examples of relatively quasi-nonexpansive mappings such as the generalized projections others see [79, Examples 2.3 and 2.4].

There are many methods for approximating fixed points of a nonexpansive mapping. In 1953, Mann [74] introduced the following iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{3.2.6}$$

where the initial guess element $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is sequence in [0, 1]. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Bauschke and Combettes [58]. In an infinite-dimensional Hilbert space, Mann iteration can conclude *only weak convergence* (see [59, 68]). Attempts to modify the Mann iteration method (3.2.6) so that strong convergence is guaranteed have recently been made. Bauschke and Combettes [58] proposed the following modification of Mann iteration method

$$\begin{cases} x_1 = x \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n = 1, 2, 3, \ldots \quad . \end{cases} \tag{3.2.7}$$

They proved that if the sequence $\{\alpha_n\}$ bounded above from one, then $\{x_n\}$ defined by (3.2.7) converges strongly to $P_{F(T)}x$. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfy the following condition: if for each bounded subset B of C

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty. \tag{3.2.8}$$

Assume that if the mapping $T : C \to C$ defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, then $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$. Aoyama et al. [61, Lemma 3.1] proved that the sequence $\{T_n\}$ converges strongly to a point in C for all $x \in C$.

Very recently, Takahashi et al. [91] studied the strong convergence theorem by the new hybrid method for a family of nonexpansive mappings in Hilbert spaces: $x_0 \in H, C_1 = C$ and $x_1 = P_{C_1}x_0$ and let

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.9}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. They prove that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converge strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)}x_0$.

The ideas to generalize the process (3.2.6) from Hilbert spaces have recently been made. Matsushita and Takahashi [75] proposed the following hybrid iteration method (CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{3.2.10}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$. Many authors studied methods for approximating fixed points of countable family of (relatively quasi-) nonexpansive mappings (see [60, 61, 70, 77, 282, 83, 85, 93]). Plubtieng and Ungchittrakool [282] introduced a method for finding common fixed point of countable family of relatively nonexpansive mappings in a Banach space. Let $\hat{C}$ and C be two nonempty, closed and convex subsets of a uniformly smooth and uniformly convex Banach space E such that $C \subset \hat{C}$ and let $\{T_n\}$ be a sequence of relatively nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. Define $\{x_n\}$ in the following ways:

$$\begin{cases} x_0 \in \hat{C}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = J^{-1}(\alpha_n x_n + (1-\alpha_n)T_n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0 \end{cases} \tag{3.2.11}$$

and

$$\begin{cases} x_0 \in \hat{C}, \\ C_1 = C, \\ y_n = J^{-1}(\alpha_n x_n + (1-\alpha_n)T_n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{3.2.12}$$

where $\alpha_n \subset [0,1]$ satisfies some appropriate conditions. They proved that the processes (3.2.11) and (3.2.12) converge strongly to a common fixed point of a countable family of relatively nonexpansive mappings $\{T_n\}$ provided that $\{T_n\}$ satisfies some appropriate conditions.

Recently, Li et al. [73] introduced the following hybrid iterative scheme for approximation fixed points of relatively nonexpansive mapping using the generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$ and

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_{n+1} = \{w \in C_n : G(w, J y_n) \leq G(w, J x_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0. \end{cases} \tag{3.2.13}$$

They obtained strong convergence theorem for finding an element in the fixed point set of T. The result of Li et al. [73] extended and improved the results of Matsushita and Takahashi [75].

On the other hand, Nakajo et al. [76] introduced the following condition. Let C be a nonempty, closed and convex subset of a Banach space E, let $\{T_n\}$ be a family of mappings of C into itself such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\omega_w(z_n)$ denotes the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C. The sequence $\{T_n\}$ satisfy the NST-condition if for every bounded sequence $\{z_n\}$ in C

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \quad \text{implies} \qquad \omega_w(z_n) \subset \mathcal{F}.$$

Recall that a mapping $T : C \longrightarrow C$ is closed if for each $\{x_n\}$ in C, if $x_n \longrightarrow x$ and $T x_n \longrightarrow y$, then $Tx = y$. Let $\{T_n\}$ be a family of mappings of C into itself with $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. The sequence $\{T_n\}$ satisfy the $(*)$-condition [60] if for each bounded sequence $\{z_n\}$ in C

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \;\; \textit{and} \; \lim_{n\to+\infty} z_n = z \text{ imply } z \in \mathcal{F}. \tag{3.2.14}$$

It follows directly from the definitions above that if $\{T_n\}$ satisfies NST-condition, then $\{T_n\}$ satisfies $(*)$-condition. Hence the $(*)$-condition weaker than the NST-condition. If $T_n \equiv T$ and T is closed, then $\{T_n\}$ satisfies $(*)$-condition (see [60, 76] for more details). Now we give an example of a countable family of relatively quasi-nonexpansive mappings which are satisfy the $(*)$-condition.

Example 3.2.6. *Let $E = \mathbb{R}$. A mapping $T_n : E \longrightarrow E$ defined by Example 3.2.3. Hence, we have $\bigcap_{n=1}^{\infty} F(T_n) = F(T_n) = \{0\}$. For each bounded sequences $z_n \in E$, we observe that $T_n z_n = \frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$ and hence $z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} T_n z_n = 0$ as $n \longrightarrow \infty$, this implies that $z = 0 \in F(T_n)$. Therefore, T_n is a relatively quasi-nonexpansive mapping and satisfy the $(*)$-condition.*

The following questions naturally arise in connection with the above results.

Question 1 Can the algorithms (3.2.11), (3.2.12) and (3.2.13) still valid for relatively quasi-nonexpansive mappings which more general than relatively nonexpansive mappings?

Question 2 Is it possible to construct an approximate fixed point sequence for finding common fixed points of an infinite family of relatively quasi-nonexpansive mappings in more general Banach spaces?

The purpose of this section is to answer the above questions. Motivated and inspired by the works mentioned above, we introduce a new hybrid projection algorithm of the generalized f-projection operator which modify the iterative method introduced by Li et al. [73] for a countable family of relatively quasi-nonexpansive mappings in a uniformly smooth and uniformly convex Banach space by using the $(*)$-condition. By improving the main result of Li et al. [72] and Plubtieng and Ungchittrakool [282], we propose the new sufficient and uncomplicated condition in our main result which is more general than the formerly result. Our condition is weaker than the Plubtieng and Ungchittrakool's condition [282] in the reason that just only one condition will be needed. As applications, we apply our results to obtain new results for finding zeroes of general B-monotone and maximal monotone operators in a Banach space. The results presented in this paper generalize and improve previous results.

For the generalized f-projection operator, Wu and Hung [325] proved the following basic properties.

Lemma 3.2.1. (*Wu and Hung* [325]). *Let E be a real reflexive Banach space with its dual E^* and C be a nonempty, closed and convex subset of E. The following statement hold:*

(1) *$\pi_C^f \varpi$ is a nonempty, closed and convex subset of C for all $\varpi \in E^*$;*

(2) *if E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if*

$$\langle x - y, \varpi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \ \ \forall y \in C;$$

(3) *if E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then $\pi_C^f \varpi$ is single valued mapping.*

Recently, Fan et al. [323] show that the condition, f is positive homogeneous, which appeared in [323, Lemma 2.1 (iii)] can be removed.

Lemma 3.2.2. (*Fan et al.* [323]). *Let E be a real reflexive Banach space with its dual E^* and C be a nonempty, closed and convex subset of E. If E is strictly convex, then $\pi_C^f \varpi$ is single valued.*

Recall that J is single value mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$ where $x \in E$. This substitution in (5.3.8) give

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi). \tag{3.2.15}$$

Now we consider the second generalized f projection operator in Banach space (see [73]).

Definition 3.2.7. *Let E be a real smooth Banach space and let C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is generalized f-projection operator if*

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx), \ \forall x \in E\}.$$

Lemma 3.2.3. (*Deimling* [66]). *Let E be a Banach space and let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha, \ \forall x \in E.$$

Lemma 3.2.4. (*Li et al.* [73]). *Let E be a reflexive smooth Banach space and let C be a nonempty, closed and convex subset of E. The following statements hold*

(1) *$\Pi_C^f x$ is nonempty, closed and convex subset of C for all $x \in E$;*

(2) *for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \ \forall y \in C;$$

(3) *if E is strictly convex, then Π_C^f is single valued mapping.*

Lemma 3.2.5. (*Li et al.* [73]). *Let E be a real reflexive smooth Banach space, let C be a nonempty, closed and convex subset of E, $x \in E$ and let $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \ \forall y \in C.$$

Remark 3.2.6. Let E be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$, then Lemma 3.2.5 reduces to the property of the generalized projection operator considered by Alber [319].

Lemma 3.2.7. (*Qin et al.*[79]). *Let E be a real uniformly smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a closed and relatively quasi-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

3.2.1 A countable family of relatively quasi-nonexpansive mappings

In this section, by using the $(*)$-condition, we prove the convergence theorem for finding a common fixed points of a countable family of relatively quasi-nonexpansive mappings, in a uniformly convex and uniformly smooth Banach space.

Theorem 3.2.8. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of relatively quasi-nonexpansive mappings of C into E that satisfy the $(*)$-condition and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi^f_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0 \end{cases} \tag{3.2.16}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\liminf_{n\longrightarrow\infty}(1-\alpha_n) > 0$ then $\{x_n\}$ converges strongly to $\Pi^f_{\mathcal{F}} x_0$.

***Proof*.** We split the proof into five steps.

Step 1. We first show that C_n is closed and convex for each $n \in \mathbb{N}$.

Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know that $G(z, Jy_n) \leq G(z, Jx_n)$ is equivalent to

$$2\langle z, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2.$$

Therefore, C_{n+1} is closed and convex. This implies that $\Pi^f_{C_{n+1}} x_0$ is well defined.

Step 2. We will show that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$.

Next, we will show by induction that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \in \mathbb{N}$. Let $q \in \mathcal{F}$ and since $\{T_n\}$ is relatively quasi-nonexpansive mappings, we have

$$\begin{aligned}
G(q, Jy_n) &= G(q, \alpha_n Jx_n + (1-\alpha_n)JT_nx_n) \\
&= \|q\|^2 - 2\langle q, \alpha_n Jx_n + (1-\alpha_n)JT_nx_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JT_nx_n\|^2 + 2\rho f(q) \\
&\leq \|q\|^2 - 2\alpha_n\langle q, Jx_n\rangle - 2(1-\alpha_n)\langle q, JT_nx_n\rangle \\
&\quad + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|JT_nx_n\|^2 + 2\rho f(q) \\
&= \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, JT_nx_n) \\
&\leq \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, Jx_n) \\
&= G(q, Jx_n).
\end{aligned} \tag{3.2.17}$$

This shows that $q \in C_{n+1}$ which implies that $\mathcal{F} \subset C_{n+1}$. Hence $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$ and the sequence $\{x_n\}$ is well defined.

Step 3. We will show that $\{x_n\}$ is a Cauchy sequence in C and $\lim_{n\to\infty} G(x_n, Jx_0)$ exist.

Since $f : E \to \mathbb{R}$ is convex and lower semicontinuous function, from Lemma 3.2.3, we known that there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, x^*\rangle + \alpha, \forall y \in E.$$

Since $x_n \in E$, it follows that

$$\begin{aligned}
G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\
&\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho\langle x_n, x^*\rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho x^*\rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geq \|x_n\|^2 - 2\|x_n\|\|Jx_0 - \rho x^*\| + \|x_0\|^2 + 2\rho\alpha \\
&= (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha.
\end{aligned} \tag{3.2.18}$$

For each $q \in \mathcal{F}$ and $x_n = \Pi^f_{C_n} x_0$, we have

$$G(q, Jx_0) \geq G(x_n, Jx_0) \geq (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha.$$

This implies that $\{x_n\}$ is bounded and so are $\{G(x_n, Jx_0)\}$ and $\{y_n\}$. From the fact that $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and $x_n = \Pi^f_{C_n} x_0$, it follows from Lemma 3.2.5

$$0 \leq (\|x_{n+1} - \|x_n\|)^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0). \tag{3.2.19}$$

This implies that $\{G(x_n, Jx_0)\}$ is nondecreasing. Hence, we obtain that $\lim_{n\to\infty} G(x_n, Jx_0)$ exist. For any $m > n$, $x_n = \Pi^f_{C_n} x_0$, $x_m = \Pi^f_{C_m} x_0 \in C_m \subset C_n$ and from (3.2.19), we have

$$\phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0).$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. It follows that $\|x_n - x_m\| \longrightarrow 0$. Hence $\{x_n\}$ is a Cauchy sequence and by the completeness of E and the closedness of C, we can assume that there exists $p \in C$ such that $x_n \to p \in C$.

Step 4. We will show that $x_n \to p \in \mathcal{F}$.

In particular, since $\lim_{n\to\infty} G(x_n, Jx_0)$ exist from (3.2.19), we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.2.20}$$

It follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.2.21}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.2.22}$$

Since $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1}$, we get

$$G(x_{n+1}, Jy_n) \leq G(x_{n+1}, Jx_n)$$

is equivalent to

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

Then, we get

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.2.23}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = 0. \tag{3.2.24}$$

Assume that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n) JT_n x_n\| \\ &= \|(1-\alpha_n)Jx_{n+1} - (1-\alpha_n)JT_n x_n + \alpha_n Jx_{n+1} - \alpha_n Jx_n\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JT_n x_n\| - \alpha_n \|Jx_n - Jx_{n+1}\|, \end{aligned} \tag{3.2.25}$$

and therefore

$$\|Jx_{n+1} - JT_n x_n\| \leq \frac{1}{(1-\alpha_n)}(\|Jx_{n+1} - Jy_n\| + \alpha_n \|Jx_n - Jx_{n+1}\|) \tag{3.2.26}$$

since $\liminf_{n\longrightarrow\infty}(1-\alpha_n) > 0$, (4.2.91) and (4.2.101), one has

$$\lim_{n\to\infty} \|Jx_{n+1} - JT_n x_n\| = 0. \tag{3.2.27}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = 0. \tag{3.2.28}$$

Using the triangle inequality, we have

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\|.$$

From (4.2.90) and (3.2.28) we obtain

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.2.29}$$

Since $x_n \to p$, it follows from the $(*)$-condition that $p \in \mathcal{F} = \bigcap_{n=0}^{\infty} F(T_n)$.

Step 5. We will show that $p = \Pi_{\mathcal{F}}^f x_0$.

Since $\mathcal{F}$ is closed and convex set from Lemma 3.2.4, we have $\Pi_{\mathcal{F}}^f x_0$ is single value, denote by v. By definition $x_n = \Pi_{C_n}^f x_0$ and $v \in \mathcal{F} \subset C_n$, we also have

$$G(x_n, Jx_0) \leq G(v, Jx_0), \forall n \geq 1.$$

By the definition of G and f, we know that, for each given x, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to ξ. So

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(v, Jx_0).$$

From the definition of $\Pi_{\mathcal{F}}^f x_0$ and since $p \in \mathcal{F}$, we conclude that $v = p = \Pi_{\mathcal{F}}^f x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

Corollary 3.2.9. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of relatively quasi-nonexpansive mappings of C into E that satisfy the NST-condition and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^f x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0 \end{cases} \tag{3.2.30}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\liminf_{n\longrightarrow\infty}(1-\alpha_n) > 0$ then $\{x_n\}$ converges strongly to $p \in \mathcal{F}$, where $p = \Pi_{\mathcal{F}}^f x_0$.

Remark 3.2.10. Theorem 5.3.11 extends and improves the results of Li et al. [73] and Plubtieng and Ungchittrakool [282] from relatively nonexpansive mappings to a more general class of a countable family of relatively quasi-nonexpansive mappings.

Setting $T_n \equiv T$ in Theorem 5.3.11, then we obtain the following result.

Corollary 3.2.11. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$, be a relatively quasi-nonexpansive mapping and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous* *function with* $C \subset int(D(f))$*. Assume that* $F(T) \neq \emptyset$*. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi^f_{C_1} x_0$ *and* $C_1 = C$*, we define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_{n+1} = \{z \in C_n : G(z, J y_n) \leq G(z, J x_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0 \end{cases} \tag{3.2.31}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$*. If* $\liminf_{n \longrightarrow \infty}(1-\alpha_n) > 0$ *then* $\{x_n\}$ *converges strongly to* $\Pi^f_{F(T)} x_0$*.*

Remark 3.2.12. Corollary 3.2.11 extends and improves the result of Li et al. [73] from relatively nonexpansive mappings to more general relatively quasi-nonexpansive mappings.

Taking $f(x) = 0$ for all $x \in E$, we have $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi^f_C x = \Pi_C x$. From Theorem 5.3.11 we obtain the following corollaries.

Corollary 3.2.13. *Let* C *be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space* E*. Let* $\{T_n\}_{n=1}^{\infty}$ *be a countable family of relatively quasi-nonexpansive mappings of* C *into* E *that satisfy the* $(*)$*-condition. Assume that* $\mathcal{F} := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$*. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$*, we define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{3.2.32}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$*. If* $\liminf_{n \longrightarrow \infty}(1-\alpha_n) > 0$ *then* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$*.*

Corollary 3.2.14. *Let* C *be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space* E*. Let* $\{T_n\}_{n=1}^{\infty}$ *be a countable family of relatively quasi-nonexpansive mappings of* C *into* E *that satisfy the NST-condition. Assume that* $\mathcal{F} := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$*. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$*, we define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{3.2.33}$$

where $\{\alpha_n\}$ *is sequences in* $[0,1]$*. If* $\liminf_{n \longrightarrow \infty}(1-\alpha_n) > 0$ *then* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$*.*

Remark 3.2.15. Corollary 3.2.13 and 3.2.14 extend and improve the results of Plubtieng and Ungchittrakool [282] from relatively nonexpansive mappings to a more general class of a countable family of relatively quasi-nonexpansive mappings.

Corollary 3.2.16. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of relatively nonexpansive mappings of C into E that satisfy the $(*)$-condition and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $\mathcal{F} = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^f x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_nx_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0 \end{cases} \tag{3.2.34}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\liminf_{n\longrightarrow\infty}(1-\alpha_n) > 0$ then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}^f x_0$.

Remark 3.2.17. Corollary 3.2.16 extends and improves the result of Li et al. [73] from a single relatively nonexpansive mapping to the class of an infinite family of relatively quasi-nonexpansive mappings.

3.2.2 Zeroes of B-monotone mappings.

Let B be a mapping from E to E^*. A mapping B is called

(1) *monotone* if $\langle Bx - By, x - y\rangle \geq 0$ for all $x, y \in E$;

(2) *strictly monotone* if B monotone and $\langle Bx - By, x - y\rangle = 0$ if and only if $x = y$;

(3) *β- Lipschitz continuous* if there exist a constant $\beta \geq 0$ such that $\|Bx - By\| \leq \beta\|x - y\|$ for all $x, y \in E$.

Let M be a set-valued mapping from E to E^* with domain $D(M) = \{z \in E : Mz \neq 0\}$ and range $R(M) = \bigcup\{Mz : z \in D(M)\}$. A set value mapping M is called

(i) *monotone* if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(M)$ and $y_i \in Mx_i, i = 1, 2$;

(ii) *r-strongly monotone* if $\langle x_1 - x_2, y_1 - y_2\rangle \geq r\|x_1 - x_2\|$ for each $x_i \in D(M)$ and $y_i \in Mx_i, i = 1, 2$;

(iii) *maximal monotone* if M is monotone and its graph $G(M) = \{(x, y) : y \in Mx\}$ is not properly contained in the graph of any other monotone mapping;

(iv) *general* B*-monotone* if M is monotone and $(B+\lambda M)E = E^*$ holds for every $\lambda > 0$, where B is a mapping from E to E^*.

We consider the problem of finding a point $x^* \in E$ satisfying $0 \in Mx^*$. We denote by $M^{-1}0$ the set of all points $x^* \in C$ such that $0 \in Mx^*$, where M is maximal monotone operator from E to E^*.

Lemma 3.2.18. (*Li et al.* [73]). *Let E be a Banach space with the dual space E^*, let $B : E \to E^*$ be a strictly monotone mapping, and let $M : E \to 2^{E^*}$ be a general B-monotone mapping. Then M is maximal monotone mapping.*

Remark 3.2.19. *(*Li et al. *[73]). Let E be a Banach space with the dual space E^*, let $B : E \to E^*$ be a strictly monotone mapping, and let $M : E \to 2^{E^*}$ be a general B-monotone mapping. Then M is a maximal monotone mapping. Therefore, $M^{-1}0 = \{z \in D(M) : 0 \in Mz\}$ is closed and convex.*

Lemma 3.2.20. (*Alber.* [319]). *Let E be a uniformly convex and uniformly smooth Banach space, $\delta_E(\epsilon)$ is the modulus of convexity of E and $\rho_E(t)$ is the modulus of smoothness of E, then the inequalities*

$$8d^2\delta_E(\|x-\xi\|/4d) \leq \phi(x,\xi) \leq 4d^2\rho_E(4\|x-\xi\|/d)$$

hold for all x and ξ in E, where $d = \sqrt{(\|x\|^2 + \|\xi\|^2)/2}$.

Lemma 3.2.21. (*Xia and Huang.* [97]). *Let E be a Banach space with the dual space E^*, let $B : E \to E^*$ be a strictly monotone mapping, and let $M : E \to 2^{E^*}$ be a general B-monotone mapping. Then*

(1) *$(B+\lambda M)^{-1}$ is single valued;*

(2) *if E is reflexive and $M : E \to 2^{E^*}$ is a r-strongly monotone, then $(B+\lambda M)^{-1}$ is Lipschitz continuous with constant $\frac{1}{\lambda r}$ $(r > 0)$.*

Let E be a Banach space with the dual space E^*, $B : E \to E^*$ a strictly monotone mapping, and $M : E \to 2^{E^*}$ a general B-monotone mapping, for every $\lambda > 0$ and $x^* \in E^*$. From Lemma 3.2.21 there exists a unique $x \in D(M)$ such that $x = (B+\lambda M)^{-1}x^*$. We define a single valued mapping $T_\lambda : E \to D(M)$ by $T_\lambda x = (B+\lambda M)^{-1}Bx$. It is easy to see that $M^{-1}0 = F(T_\lambda)$ for all $\lambda > 0$. Indeed, we have

$$\begin{aligned} z \in M^{-1}0 \;&\Leftrightarrow\; 0 \in Mz \\ &\Leftrightarrow\; 0 \in \lambda Mz \\ &\Leftrightarrow\; Bz \in (B+\lambda M)z \\ &\Leftrightarrow\; z = (B+\lambda M)^{-1}Bz = T_\lambda z \\ &\Leftrightarrow\; z \in F(T_\lambda). \end{aligned} \tag{3.2.35}$$

Motivated by Li et al. [73] we obtain the following result.

Theorem 3.2.22. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with $\delta_E(\epsilon) \geq k\epsilon^2$ and $\rho_E(t) \leq ct^2$ for some $c, k > 0$. Let $B : E \to E^*$ a strictly monotone and β-Lipschitz continuous mapping and let $M : E \to 2^{E^*}$ be a general B-monotone and r-strongly monotone mapping with $r > 0$. Let $\{T_{\lambda_n}\} = (B+\lambda_n M)^{-1}B$ and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$ and suppose that for each $n \geq 0$ there exists $\lambda_n > 0$ such that $64c\beta^2 \leq min\{\frac{1}{2}k\lambda_n^2 r^2\}$. Assume that $\mathcal{F} := M^{-1}0 \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi^f_{C_1}x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_{\lambda_n}x_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}}x_0 \end{cases} \tag{3.2.36}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\liminf_{n\longrightarrow\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi^f_{\mathcal{F}}x_0$.

***Proof*.** We will show that $\{T_{\lambda_n}\}$ is a family of relatively quasi-nonexpansive mappings with common fixed point $\bigcap_{n=1}^{\infty} F(T_{\lambda_n}) = M^{-1}0$. We only need to show that $\phi(p, T_{\lambda_n}q) \leq \phi(p, q)$ for each $q \in E, p \in F(T_{\lambda_n}), n \geq 1$. From Lemma 3.2.20, and since B is a β-Lipschitz continuous mapping, we have

$$\begin{aligned} \phi(p, T_{\lambda_n}q) &= \phi(T_{\lambda_n}p, T_{\lambda_n}q) \\ &\leq 4d^2\rho_E(\tfrac{4\|T_{\lambda_n}p - T_{\lambda_n}q\|}{d}) \\ &\leq 64c\|T_{\lambda_n}p - T_{\lambda_n}q\|^2 \\ &= 64c\|(B+\lambda_n M)^{-1}Bp - (B+\lambda_n M)^{-1}Bq\|^2 \\ &\leq \tfrac{64c}{\lambda_n^2 r^2}\|Bp - Bq\|^2 \\ &\leq \tfrac{64c\beta^2}{\lambda_n^2 r^2}\|p - q\|^2 \end{aligned} \tag{3.2.37}$$

and we also have

$$\phi(p, q) \geq 8d^2\delta_E(\tfrac{\|p-q\|}{4d}) \geq \tfrac{1}{2}k\|p-q\|^2. \tag{3.2.38}$$

Since

$$64c\beta^2 \leq \frac{1}{2}k\lambda_n^2 r^2$$

it follows from (3.2.37) and (3.2.38) that $\phi(p, T_{\lambda_n}q) \leq \phi(p, q)$ for all $q \in E, p \in F(T_{\lambda_n}), n \geq 1$. Therefore $\{T_{\lambda_n}\}$ is a family of relatively quasi-nonexpansive mapping. Hence the result follows from Theorem 5.3.11. □

3.2.3 Zeroes of maximal monotone operators

In this section, we apply our results to find zeros of maximal monotone operator. Such a problem contains numerous problems in optimization, economics, and physics. The following result is also well known.

Lemma 3.2.23. (*Rockafellar.* [80]). *Let E be a reflexive strictly convex and smooth Banach space and let M be a monotone operator from E to E^*. Then M is maximal if and only if $R(J+\lambda M) = E^*$ for all $\lambda > 0$.*

Let E be a reflexive strictly convex and smooth Banach space, $B = J$ and let M be a maximal monotone operator from E to E^*. Using Lemma 3.2.23 and the strict convexity of E, we obtain that for every $\lambda > 0$ and $x \in E$, there exists a unique x_λ such that $Jx \in (Jx_\lambda + \lambda M x_\lambda)$. Then recall the single valued mapping $J_\lambda : E \longrightarrow D(M)$ by $J_\lambda = (J+\lambda M)^{-1}\mathrm{J}$ and J_λ is the *resovent* of M. We known that $M^{-1}0 = F(J_\lambda)$ (see [90, 92]).

Theorem 3.2.24. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $M \subset E \times E^*$ be a maximal monotone mapping such that $D(M) \subset C \subset J^{-1}(\cap_{\lambda_n > 0} R(J+\lambda_n M)$. Let $\{J_{\lambda_n}\} = (J+\lambda_n M)^{-1}J$ where $\lambda_n > 0$ and let $f : E \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $\mathcal{F} := M^{-1}0 \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi^f_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J J_{\lambda_n} x_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0 \end{cases} \tag{3.2.39}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\liminf_{n \longrightarrow \infty}(1-\alpha_n) > 0$ then $\{x_n\}$ converges strongly to $p \in \mathcal{F}$, where $p = \Pi^f_{\mathcal{F}} x_0$.

***Proof*.** First, we have $\bigcap_{n=1}^{\infty} F(J_{\lambda_n}) = M^{-1}0 \neq \emptyset$. Secondly, from the monotonicity of M, let $p \in \bigcap_{n=1}^{\infty} F(J_{\lambda_n})$ and $q \in E$, we have

$$\begin{aligned} \phi(p, J_{\lambda_n} q) &= \|p\|^2 - 2\langle p, JJ_{\lambda_n} q\rangle + \|J_{\lambda_n} q\|^2 \\ &= \|p\|^2 + 2\langle p, Jq - JJ_{\lambda_n} q - Jq\rangle + \|J_{\lambda_n} q\|^2 \\ &= \|p\|^2 + 2\langle p, Jq - JJ_{\lambda_n} q\rangle - 2\langle p, Jq\rangle + \|J_{\lambda_n} q\|^2 \\ &= \|p\|^2 - 2\langle J_{\lambda_n} q - p - J_{\lambda_n} q, Jq - JJ_{\lambda_n} q\rangle - 2\langle p, Jq\rangle + \|J_{\lambda_n} q\|^2 \\ &= \|p\|^2 - 2\langle J_{\lambda_n} q - p, Jq - JJ_{\lambda_n} q\rangle + 2\langle J_{\lambda_n} q, Jq - JJ_{\lambda_n} q\rangle - 2\langle p, Jq\rangle + \|J_{\lambda_n} q\|^2 \\ &\leq \|p\|^2 + 2\langle J_{\lambda_n} q, Jq - JJ_{\lambda_n} q\rangle - 2\langle p, Jq\rangle + \|J_{\lambda_n} q\|^2 \\ &= \|p\|^2 - 2\langle p, Jq\rangle + \|q\|^2 - \|J_{\lambda_n} q\|^2 + 2\langle J_{\lambda_n} q, Jq\rangle - \|q\|^2 \\ &= \phi(p, q) - \phi(J_{\lambda_n} q, q) \\ &\leq \phi(p, q) \end{aligned}$$

for all $n \geq 1$. Therefor $\{J_{\lambda_n}\}$ is a family of relatively quasi-nonexpansive mapping, for all $\lambda_n > 0$ with common fixed point set $\bigcap_{n=1}^{\infty} F(J_{\lambda_n}) = M^{-1}0$. Hence the result follows from Theorem 5.3.11. □

บทที่ 4

Equilibrium Problems

Let $\{f_i\}_{i\in\Gamma} : C\times C \longrightarrow \mathbb{R}$ be a bifunction, $\{\varphi_i\}_{i\in\Gamma} : C \longrightarrow \mathbb{R}$ be a real-valued function, and $\{B_i\}_{i\in\Gamma} : C \longrightarrow E^*$ be a monotone mapping, where Γ is an arbitrary index set. The *system of generalized mixed equilibrium problems* is to find $x \in C$ such that

$$f_i(x,y) + \langle B_i x, y-x\rangle + \varphi_i(y) - \varphi_i(x) \geq 0, \quad i \in \Gamma,\ \forall y \in C. \tag{4.0.1}$$

If Γ is a singleton, then problem (4.0.1) reduces to the *generalized mixed equilibrium problem*, which is to find $x \in C$ such that

$$f(x,y) + \langle Bx, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{4.0.2}$$

The set of solutions to (5.2.1) is denoted by GMEP(f,B,φ), i.e.,

$$\text{GMEP}(f,B,\varphi) = \{x \in C : f(x,y) + \langle Bx, y-x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in C\}. \tag{4.0.3}$$

If $B \equiv 0$, the problem (5.2.1) reduces into the *mixed equilibrium problem for* f, denoted by MEP(f,φ), which is to find $x \in C$ such that

$$f(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{4.0.4}$$

If $f \equiv 0$, the problem (5.2.1) reduces into the *mixed variational inequality* of Browder type, denoted by VI(C,B,φ), which is to find $x \in C$ such that

$$\langle Bx, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{4.0.5}$$

If $B \equiv 0$ and $\varphi \equiv 0$ the problem (5.2.1) reduces into the *equilibrium problem for* f, denoted by EP(f), which is to find $x \in C$ such that

$$f(x,y) \geq 0, \quad \forall y \in C. \tag{4.0.6}$$

If $f \equiv 0$, the problem (5.2.3) reduces into the *minimize problem*, denoted by Argmin(φ), which is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{4.0.7}$$

The above formulation (4.0.5) was shown in [5] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP(f). In other words, the EP(f) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP(f); see, for example [5, 10] and references therein. Some solution methods have been proposed to solve the EP(f); see, for example, [5, 10, 302, 15, 79, 93] and references therein.

4.1 The System of Generalized Mixed Equilibrium Problems in Hilbert Spaces

In this section, we prove a strong convergence theorem of the new shrinking projection method for finding a common element of the set of fixed points of strictly pseudocontractive mappings, the set of common solutions of generalized mixed equilibrium problems and the set of common solutions of the variational inequalities with inverse-strongly monotone mappings in Hilbert spaces.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function A and the set E:

(A1) $F(x,x) = 0$ for all $x \in E$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in E$;

(A3) for each $x, y, z \in E$, $\lim_{t \longrightarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in E, y \mapsto F(x,y)$ is convex and lower semicontinuous;

(A5) for each $y \in E, x \mapsto F(x,y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq E$ and $y_x \in E$ such that for any $z \in E \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{4.1.1}$$

(B2) E is a bounded set.

By similar argument as in the proof of Lemma 4.1.1, we have the following lemma appearing.

Lemma 4.1.1. *Let E be a nonempty closed convex subset of H. Let $F : E \times E \to \mathbb{R}$ be a bifunction satisfies (A1)-(A5) and let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to E$ as follows:*

$$T_r^F(x) = \Big\{ z \in E : \ F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in E \Big\},$$

for all $z \in H$. Then, the following hold:

(1) For each $x \in H, T_r^F(x) \neq \emptyset$;

(2) T_r^F is single-valued;

(3) T_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) $F(T_r^F) = MEP(F, \varphi)$;

(5) $MEP(F, \varphi)$ is closed and convex.

4.1.1 The shrinking projection method for common solutions of generalized mixed equilibrium problems

Theorem 4.1.2. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunction from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let A_1, A_2, B, C be four ρ, ω, β, ξ-inverse-strongly monotone mappings of E into H, respectively. Let $S : E \longrightarrow E$ be a k-strictly pseudocontractive mapping with a fixed point. Define a mapping $S_k : E \longrightarrow E$ by $S_k x = kx + (1-k)Sx, \quad \forall x \in E$. Suppose that*

$$\Theta := F(S) \cap GMEP(F_1, \varphi, A_1) \cap GMEP(F_2, \varphi, A_2) \cap VI(E, B) \cap VI(E, C) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{cases} x_0 \in H, \ \ E_1 = E, \ \ x_1 = P_{E_1}x_0, \ \ u_n \in E, \ \ v_n \in E, \\ F_1(u_n, u) + \varphi(u) - \varphi(u_n) + \langle A_1x_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in E, \\ F_2(v_n, v) + \varphi(v) - \varphi(v_n) + \langle A_2x_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \ \ \geq 0, \qquad \forall v \in E, \\ y_n = P_E(x_n - \lambda_n Bx_n), \ \ z_n = P_E(x_n - \mu_n Cx_n), \\ t_n = \alpha_n^{(1)}S_kx_n + \alpha_n^{(2)}y_n + \alpha_n^{(3)}z_n + \alpha_n^{(4)}u_n + \alpha_n^{(5)}v_n, \\ E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{E_{n+1}}x_0, \ \ \forall n \geq 0, \end{cases} \tag{4.1.2}$$

where $\{\alpha_n^{(i)}\}$ are sequences in $(0, 1)$, where $i = 1, 2, 3, 4, 5$, $r_n \in (0, 2\rho)$, $s_n \in (0, 2\omega)$ and $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy the following restrictions:

(C1) $\sum_{i=1}^{5} \alpha_n^{(i)} = 1$,

(C2) $\lim_{n\longrightarrow\infty} \alpha_n^{(i)} = \alpha^{(i)} \in (0, 1)$, *where* $i = 1, 2, 3, 4, 5$,

(C3) $a \leq r_n \leq 2\rho$ *and* $b \leq s_n \leq 2\omega$, *where* a, b *are two positive constants,*

(C4) $c \leq \lambda_n \leq 2\beta$ *and* $d \leq \mu_n \leq 2\xi$, *where* c, d *are two positive constants,*

(C5) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\mu_{n+1} - \mu_n| = 0$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

พิสูจน์. Let $p \in \Theta$ and Lemma 4.1.1, we obtain

$$p = P_E(p - \lambda_n Bp) = P_E(p - \mu_n Cp) = T_{r_n}^{F_1}(I - r_nA_1)p = T_{s_n}^{F_2}(I - s_nA_2)p.$$

Note that $u_n = T_{r_n}^{F_1}(I - r_nA_1)x_n \in \text{dom } \varphi$ and $v_n = T_{s_n}^{F_2}(I - s_nA_2)x_n \in \text{dom } \varphi$, we have

$$\|u_n - p\| \ = \ \|T_{r_n}^{F_1}(I - r_nA_1)x_n - T_{r_n}^{F_1}(I - r_nA_1)p\| \leq \|x_n - p\| \tag{4.1.3}$$

and

$$\|v_n - p\| \ = \ \|T_{s_n}^{F_2}(I - s_nA_2)x_n - T_{s_n}^{F_2}(I - s_nA_2)p\| \leq \|x_n - p\|. \tag{4.1.4}$$

Next, we will divide the proof into six steps.

Step 1. We show that $\{x_n\}$ is well defined and E_n is closed and convex for any $n \geq 1$.

From the assumption, we see that $E_1 = E$ is closed and convex. Suppose that E_k is closed and convex for some $k \geq 1$. Next, we show that E_{k+1} is closed and convex for some k. For any $p \in E_k$, we obtain

$$\| t_k - p \| \leq \| x_k - p \|$$

is equivalent to

$$\| t_k - p \|^2 + 2\langle t_k - x_k, x_k - p \rangle \leq 0.$$

Thus E_{k+1} is closed and convex. Then, E_n is closed and convex for any $n \geq 1$. This implies that $\{x_n\}$ is well defined.

Step 2. We show that $\Theta \subset E_n$ for each $n \geq 1$. From the assumption, we see that $\Theta \subset E = E_1$. Suppose $\Theta \subset E_k$ for some $k \geq 1$. For any $p \in \Theta \subset E_k$. Since $y_n = P_E(x_n - \lambda_n Bx_n)$ and $z_n = P_E(x_n - \mu_n Cx_n)$. For each $\lambda_n \leq 2\beta$ and $\mu_n \leq 2\xi$ by Lemma **??**, we have $I - \lambda_n B$ and $I - \mu_n C$ are nonexpansive. Thus, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|P_E(x_n - \lambda_n Bx_n) - P_E(p - \lambda_n Bp)\| \\
&\leq \|(x_n - \lambda_n Bx_n) - (p - \lambda_n Bp)\| \\
&= \|(I - \lambda_n B)x_n - (I - \lambda_n B)p\| \\
&\leq \|x_n - p\|,
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\| &= \|P_E(x_n - \mu_n Cx_n) - P_E(p - \mu_n Cp)\| \\
&\leq \|(x_n - \mu_n Cx_n) - (p - \mu_n Cp)\| \\
&= \|(I - \mu_n C)x_n - (I - \mu_n C)p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

From previous Lemma, we have S_k is nonexpansive with $F(S_k) = F(S)$. It follows that

$$\begin{aligned}
\|t_n - p\| &= \|\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n - p\| \\
&\leq \alpha_n^{(1)}\|S_k x_n - p\| + \alpha_n^{(2)}\|y_n - p\| + \alpha_n^{(3)}\|z_n - p\| + \alpha_n^{(4)}\|u_n - p\| + \alpha_n^{(5)}\|v_n - p\| \\
&\leq \alpha_n^{(1)}\|x_n - p\| + \alpha_n^{(2)}\|x_n - p\| + \alpha_n^{(3)}\|x_n - p\| + \alpha_n^{(4)}\|x_n - p\| + \alpha_n^{(5)}\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

It follows that $p \in E_{k+1}$. This implies that $\Theta \subset E_n$ for each $n \geq 1$.

Step 3. We claim that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - t_n\| = 0$.

From $x_n = P_{E_n} x_0$, we get

$$\langle x_0 - x_n, x_n - y \rangle \geq 0$$

for each $y \in E_n$. Using $\Theta \subset E_n$, we have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0 \text{ for each } p \in \Theta \text{ and } n \in \mathbb{N}.$$

Hence, for $p \in \Theta$, we obtain

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|. \end{aligned}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - p\|, \text{ for all } p \in \Theta \text{ and } n \in \mathbb{N}.$$

From $x_n = P_{E_n}x_0$ and $x_{n+1} = P_{E_{n+1}}x_0 \in E_{n+1} \subset E_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \tag{4.1.5}$$

For $n \in \mathbb{N}$, we compute

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned}$$

and then

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|, \text{ for all } n \in \mathbb{N}.$$

Thus the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists, That is, there exists m such that

$$m = \lim_{n\longrightarrow\infty} \|x_n - x_0\|. \tag{4.1.6}$$

From (4.1.34), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

By (4.1.35), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - x_{n+1}\| = 0. \tag{4.1.7}$$

Since $x_{n+1} = P_{E_{n+1}}x_0 \in E_{n+1} \subset E_n$, we have

$$\|x_n - t_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (6.4.13), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - t_n\| = 0. \tag{4.1.8}$$

Step 4. We claim that the following statements hold:

(S1) $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$;

(S2) $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0$;

(S3) $\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0$;

(S4) $\lim_{n\longrightarrow\infty}\|x_n - v_n\| = 0$.

For $p \in \Theta$, we note that

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_E(x_n - \mu_n Cx_n) - P_E(p - \mu_n Cp)\|^2 \\
&\leq \|(x_n - \mu_n Cx_n) - (p - \mu_n Cp)\|^2 \\
&= \|(x_n - p) - \mu_n(Cx_n - Cp)\|^2 \\
&\leq \|x_n - p\|^2 - 2\mu_n\langle x_n - p, Cx_n - Cp\rangle + \mu_n^2\|Cx_n - Cp\|^2 \\
&\leq \|x_n - p\|^2 + \mu_n(\mu_n - 2\xi)\|Cx_n - Cp\|^2 \\
&= \|x_n - p\|^2 - \mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2.
\end{aligned} \tag{4.1.9}$$

Similarly, we also have

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2. \tag{4.1.10}$$

We note that

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I - r_nA_1)x_n - T_{r_n}^{F_1}(I - r_nA_1)p\|^2 \\
&\leq \|(I - r_nA_1)x_n - (I - r_nA_1)p\|^2 \\
&= \|(x_n - p) - r_n(A_1x_n - A_1p)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, A_1x_n - A_1p\rangle + r_n^2\|A_1x_n - A_1p\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\rho\|A_1x_n - A_1p\|^2 + r_n^2\|A_1x_n - A_1p\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\rho)\|A_1x_n - A_1p\|^2 \\
&= \|x_n - p\|^2 - r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2.
\end{aligned} \tag{4.1.11}$$

Similarly, we also have

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2. \tag{4.1.12}$$

Observing that

$$\begin{aligned} & \|t_n - p\|^2 \\ \leq\ & \alpha_n^{(1)}\|S_kx_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2 \\ \leq\ & \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2. \end{aligned}$$

Substituting (6.1.28), (6.1.29), (4.1.11) and (6.1.30) into (4.1.41), we obtain

$$\begin{aligned} & \|t_n - p\|^2 \\ \leq\ & \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\Big\{\|x_n - p\|^2 - \lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2\Big\} \\ & + \alpha_n^{(3)}\Big\{\|x_n - p\|^2 - \mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2\Big\} \\ & + \alpha_n^{(4)}\Big\{\|x_n - p\|^2 - r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2\Big\} \\ & + \alpha_n^{(5)}\Big\{\|x_n - p\|^2 - s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2\Big\} \\ =\ & \|x_n - p\|^2 - \alpha_n^{(2)}\lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2 - \alpha_n^{(3)}\mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2 \\ & - \alpha_n^{(4)}r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2 - \alpha_n^{(5)}s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2. \end{aligned} \tag{4.1.13}$$

It follows that

$$\begin{aligned} & \alpha_n^{(3)}\mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2 \\ \leq\ & \|x_n - p\|^2 - \|t_n - p\|^2 - \alpha_n^{(2)}\lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2 \\ & - \alpha_n^{(4)}r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2 - \alpha_n^{(5)}s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2 \\ \leq\ & (\|x_n - p\| + \|t_n - p\|)\|x_n - t_n\|. \end{aligned}$$

From (C2), (C4) and (4.1.39), we have

$$\lim_{n\longrightarrow\infty} \|Cx_n - Cp\| = 0. \tag{4.1.14}$$

Since $s_n \in (0, 2\omega)$, we also have

$$\begin{aligned} & \alpha_n^{(5)}s_n(2\omega - s_n)\|A_2x_n - A_2p\|^2 \\ \leq\ & \|x_n - p\|^2 - \|t_n - p\|^2 - \alpha_n^{(2)}\lambda_n(2\beta - \lambda_n)\|Bx_n - Bp\|^2 \\ & -\alpha_n^{(3)}\mu_n(2\xi - \mu_n)\|Cx_n - Cp\|^2 - \alpha_n^{(4)}r_n(2\rho - r_n)\|A_1x_n - A_1p\|^2 \\ \leq\ & (\|x_n - p\| + \|t_n - p\|)\|x_n - t_n\|. \end{aligned}$$

From (C2), (C3) and (4.1.39), we obtain

$$\lim_{n\longrightarrow\infty} \|A_2x_n - A_2p\| = 0. \tag{4.1.15}$$

Similarly, (4.1.14) and (4.1.15), we can prove that

$$\lim_{n\longrightarrow\infty}\|Bx_n - Bp\| = \lim_{n\longrightarrow\infty}\|A_1x_n - A_1p\| = 0. \tag{4.1.16}$$

On the other hand, let $p \in \Theta$ for each $n \geq 1$, we get $p = T_{r_n}^{F_1}(I - r_nA_1)p$. Since $T_{r_n}^{F_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I - r_nA_1)x_n - T_{r_n}^{F_1}(I - r_nA_1)p\|^2 \\
&\leq \Big\langle (I - r_nA_1)x_n - (I - r_nA_1)p, u_n - p\Big\rangle \\
&= \frac{1}{2}\Big\{\|(I - r_nA_1)x_n - (I - r_nA_1)p\|^2 + \|u_n - p\|^2 \\
&\quad -\|(I - r_nA_1)x_n - (I - r_nA_1)p - (u_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(A_1x_n - A_1p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + \\
&\quad 2r_n\|x_n - u_n\|\|A_1x_n - A_1p\| - {r_n}^2\|A_1x_n - A_1p\|^2\Big\}.
\end{aligned}$$

So, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|A_1x_n - A_1p\|. \tag{4.1.17}$$

Observe that

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_E(x_n - \lambda_nBx_n) - P_E(p - \lambda_nBp)\|^2 \\
&\leq \langle (I - \lambda_nB)x_n - (I - \lambda_nB)p, y_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(I - \lambda_nB)x_n - (I - \lambda_nB)p\|^2 + \|y_n - p\|^2 \\
&\quad -\|(I - \lambda_nB)x_n - (I - \lambda_nB)p - (y_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|(x_n - y_n) - \lambda_n(Bx_n - Bp)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 - \lambda_n^2\|Bx_n - Bp\|^2 \\
&\quad + 2\lambda_n\langle x_n - y_n, Bx_n - Bp\rangle\Big\},
\end{aligned}$$

and hence

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Bx_n - Bp\|. \tag{4.1.18}$$

By using the same argument in (4.1.51) and (4.1.18), we can get

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s_n\|x_n - v_n\|\|A_2x_n - A_2p\| \tag{4.1.19}$$

and

$$\|z_n - p\|^2 \;\; \leq \;\; \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\mu_n\|x_n - z_n\|\|Cx_n - Cp\|. \quad (4.1.20)$$

Substituting (4.1.51), (4.1.54), (4.1.54) and (4.1.54) into (4.1.41), we obtain

$$\begin{aligned}
\|t_n - p\|^2 \;\; &\leq \;\; \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2 \\
&\leq \;\; \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\Big\{\|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Bx_n - Bp\|\Big\} \\
&\quad + \alpha_n^{(3)}\Big\{\|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\mu_n\|x_n - z_n\|\|Cx_n - Cp\|\Big\} \\
&\quad + \alpha_n^{(4)}\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|A_1x_n - A_1p\|\Big\} \\
&\quad + \alpha_n^{(5)}\Big\{\|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s_n\|x_n - v_n\|\|A_2x_n - A_2p\|\Big\} \\
&= \;\; \|x_n - p\|^2 - \alpha_n^{(2)}\|x_n - y_n\|^2 + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\|\|Bx_n - Bp\| \\
&\quad - \alpha_n^{(3)}\|x_n - z_n\|^2 + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\|\|Cx_n - Cp\| \\
&\quad - \alpha_n^{(4)}\|x_n - u_n\|^2 + 2r_n\alpha_n^{(4)}\|x_n - u_n\|\|A_1x_n - A_1p\| \\
&\quad - \alpha_n^{(5)}\|x_n - v_n\|^2 + 2s_n\alpha_n^{(5)}\|x_n - v_n\|\|A_2x_n - A_2p\|. \qquad (4.1.21)
\end{aligned}$$

It follows that

$$\begin{aligned}
\alpha_n^{(4)}\|x_n - u_n\|^2 \;\; &\leq \;\; \|x_n - p\|^2 - \|t_n - p\|^2 - \alpha_n^{(2)}\|x_n - y_n\|^2 + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\|\|Bx_n - Bp\| \\
&\quad - \alpha_n^{(3)}\|x_n - z_n\|^2 + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\|\|Cx_n - Cp\| \\
&\quad + 2r_n\alpha_n^{(4)}\|x_n - u_n\|\|A_1x_n - A_1p\| - \alpha_n^{(5)}\|x_n - v_n\|^2 \\
&\quad + 2s_n\alpha_n^{(5)}\|x_n - v_n\|\|A_2x_n - A_2p\| \\
&\leq \;\; (\|x_n - p\| + \|t_n - p\|)\|x_n - t_n\| + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\|\|Bx_n - Bp\| \\
&\quad + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\|\|Cx_n - Cp\| + 2r_n\alpha_n^{(4)}\|x_n - u_n\|\|A_1x_n - A_1p\| \\
&\quad + 2s_n\alpha_n^{(5)}\|x_n - v_n\|\|A_2x_n - A_2p\|.
\end{aligned}$$

From (C2), (4.1.39), (4.1.14), (4.1.15) and (4.1.16), we have

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0. \qquad (4.1.22)$$

By using the same argument, we can prove that

$$\lim_{n \longrightarrow \infty} \|x_n - y_n\| = \lim_{n \longrightarrow \infty} \|x_n - z_n\| = \lim_{n \longrightarrow \infty} \|x_n - v_n\| = 0. \qquad (4.1.23)$$

Applying (4.1.39), (5.4.21) and (4.1.23), we can obtain

$$\lim_{n \longrightarrow \infty} \|t_n - u_n\| = \lim_{n \longrightarrow \infty} \|t_n - y_n\| = \lim_{n \longrightarrow \infty} \|t_n - z_n\| = \lim_{n \longrightarrow \infty} \|t_n - v_n\| = 0. \quad (4.1.24)$$

Step 5. We show that

$$z \in F(S) \cap GMEP(F_1, \varphi, A_1) \cap GMEP(F_2, \varphi, A_2) \cap VI(E, B) \cap VI(E, C).$$

Assume that $\lambda_n \longrightarrow \lambda \in [c, 2\beta]$ and $\mu_n \longrightarrow \mu \in [d, 2\xi]$.

Define a mapping $\mathcal{P} : E \longrightarrow E$ by

$$\mathcal{P}x = \alpha^{(1)} S_k x + \alpha^{(2)} P_E(1-\lambda B)x + \alpha^{(3)} P_E(1-\mu C)x + \alpha^{(4)} T_r^{F_1}(I - rA_1)x + \alpha^{(5)} T_s^{F_2}(I - sA_2)x,$$

$\forall x \in E$, where $\lim_{n\longrightarrow\infty} \alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, when $i = 1,2,3,4,5$. By (C1), then we have$\sum_{i=1}^{5} \alpha_n^{(i)} = 1$. Since Lemma ??, we have $\mathcal{P}$ is nonexpansive and

$$\begin{aligned} F(\mathcal{P}) &= F(S_k) \cap F(P_E(1-\lambda B)) \cap F(P_E(1-\mu C)) \cap F(T_r^{F_1}(I - rA_1)) \cap F(T_s^{F_2}(I - sA_2)) \\ &= F(S_k) \cap GMEP(F_1, \varphi, A_1) \cap GMEP(F_2, \varphi, A_2) \cap VI(E, B) \cap VI(E, C). \quad (4.1.25) \end{aligned}$$

We note that

$$\begin{aligned} & \|\mathcal{P}x_n - x_n\| \\ \leq\ & \|\mathcal{P}x_n - t_n\| + \|t_n - x_n\| \\ =\ & \Bigg\| \Big[\alpha^{(1)} S_k x_n + \alpha^{(2)} P_E(1-\lambda B)x_n + \alpha^{(3)} P_E(1-\mu C)x_n \\ & + \alpha^{(4)} T_r^{F_1}(I - rA_1)x_n + \alpha^{(5)} T_s^{F_2}(I - sA_2)x_n\Big] \\ & - \Big[\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} P_E(1-\lambda_n B)x_n + \alpha_n^{(3)} P_E(1-\mu_n C)x_n + \alpha_n^{(4)} T_r^{F_1}(I - rA_1)x_n \\ & + \alpha_n^{(5)} T_s^{F_2}(I - sA_2)x_n\Big] \Bigg\| + \|t_n - x_n\| \\ \leq\ & |\alpha^{(1)} - \alpha_n^{(1)}| \|S_k x_n\| \\ & + \alpha^{(2)} \|P_E(I - \lambda B)x_n - P_E(I - \lambda_n B)x_n\| + |\alpha^{(2)} - \alpha_n^{(2)}| \|P_E(I - \lambda_n B)x_n\| \\ & + \alpha^{(3)} \|P_E(I - \mu C)x_n - P_E(I - \mu_n C)x_n\| + |\alpha^{(3)} - \alpha_n^{(3)}| \|P_E(I - \mu_n C)x_n\| \\ & + |\alpha^{(4)} - \alpha_n^{(4)}| \|T_r^{F_1}(I - rA_1)x_n\| + |\alpha^{(5)} - \alpha_n^{(5)}| \|T_s^{F_2}(I - sA_2)x_n\| + \|t_n - x_n\| \\ \leq\ & |\alpha^{(1)} - \alpha_n^{(1)}| \|S_k x_n\| + \alpha^{(2)} |\lambda_n - \lambda| \|Bx_n\| + |\alpha^{(2)} - \alpha_n^{(2)}| \|P_E(I - \lambda_n B)x_n\| \\ & + \alpha^{(3)} |\mu_n - \mu| \|Cx_n\| + |\alpha^{(3)} - \alpha_n^{(3)}| \|P_E(I - \mu_n C)x_n\| \\ & + |\alpha^{(4)} - \alpha_n^{(4)}| \|T_r^{F_1}(I - rA_1)x_n\| + |\alpha^{(5)} - \alpha_n^{(5)}| \|T_s^{F_2}(I - sA_2)x_n\| + \|t_n - x_n\| \\ \leq\ & K_1 \left(\sum_{i=1}^{5} |\alpha^{(i)} - \alpha_n^{(i)}| + |\lambda_n - \lambda| + |\mu_n - \mu| \right) + \|t_n - x_n\|, \end{aligned}$$

where K_1 is an appropriate constant such that

$$\begin{aligned} K_1 = \max\Big\{ & \sup_{n\geq 1} \|T_r^{F_1}(I - rA_1)x_n\|,\ \sup_{n\geq 1} \|T_s^{F_2}(I - sA_2)x_n\|,\ \sup_{n\geq 1} \|P_E(I - \lambda_n B)x_n\|, \\ & \sup_{n\geq 1} \|P_E(I - \mu_n C)x_n\|,\ \sup_{n\geq 1} \|Bx_n\|,\ \sup_{n\geq 1} \|Cx_n\|,\ \sup_{n\geq 1} \|S_k x_n\| \Big\}. \end{aligned}$$

From (C2), (C5) and (4.1.39), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - \mathcal{P}x_n\| = 0. \quad (4.1.26)$$

Since $\{x_{n_i}\}$ is bounded, There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup z$. It follows from (4.1.26), that

$$\lim_{n \longrightarrow \infty} \|x_{n_i} - \mathcal{P}x_{n_i}\| = 0.$$

It follows that $z \in F(\mathcal{P})$. By (4.1.25), we have $z \in \Theta$.

Step 6. Finally, we show that $x_n \to z$, where $z = P_\Theta x_0$.

Since Θ is nonempty closed convex subset of H, there exists a unique $z' \in \Theta$ such that $z' = P_\Theta x_0$. Since $z' \in \Theta \subset E_n$ and $x_n = P_{E_n} x_0$, we have

$$\|x_0 - x_n\| = \|x_0 - P_{E_n} x_0\| \leq \|x_0 - z'\| \tag{4.1.27}$$

for all $n \geq 1$. From (6.4.15), $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$. By the weak lower semicontinuity of the norm, we have

$$\|x_0 - z\| \leq \liminf_{i \longrightarrow \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z'\|. \tag{4.1.28}$$

Since $z \in \omega_w(x_n) \subset \Theta$, we obtain

$$\|x_0 - z'\| = \|x_0 - P_\Theta x_0\| \leq \|x_0 - z\|.$$

Using (6.4.15) and (6.4.15), we obtain $z' = z$. Thus $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So, we have

$$\|x_0 - z'\| \leq \|x_0 - z\| \leq \liminf_{i \longrightarrow \infty} \|x_0 - x_n\| \leq \limsup_{i \longrightarrow \infty} \|x_0 - x_n\| \leq \|x_0 - z'\|. \tag{4.1.29}$$

Thus,

$$\|x_0 - z\| = \lim_{i \longrightarrow \infty} \|x_0 - x_n\| = \|x_0 - z'\|.$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using Lemma 6.4.19, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \to 0$$

as $n \to \infty$ and hence $x_n \to z$ in norm. This completes of the proof. □

If the mapping S is nonexpansive, then $S_k = S_0 = S$. We can obtain the following result from Theorem 4.1.2 immediately.

Corollary 4.1.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunction from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let A_1, A_2, B, C be four ρ, ω, β, ξ-inverse-strongly monotone* *mapping of E into H, respectively. Let $S : E \longrightarrow E$ be a nonexpansive mapping with a fixed point. Suppose that*

$$\Theta := F(S) \cap GMEP(F_1, \varphi, A_1) \cap GMEP(F_2, \varphi, A_2) \cap VI(E, B) \cap VI(E, C) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm (6.5.3), where $\{\alpha_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $r_n \in (0, 2\rho)$, $s_n \in (0, 2\omega)$ and $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy (C1)-(C5) in Theorem 4.1.2. Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

If $\varphi = 0$ and $A_1 = A_2 = 0$ in Theorem 4.1.2, then we can obtain the following result immediately.

Corollary 4.1.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunction from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let B, C be two β, ξ-inverse-strongly monotone mapping of E into H, respectively. Let $S : E \longrightarrow E$ be a nonexpansive mapping with a fixed point. Suppose that*

$$\Theta := F(S) \cap EP(F_1) \cap EP(F_2) \cap VI(E, B) \cap VI(E, C) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{cases} x_0 \in H, \ E_1 = E, \ x_1 = P_{E_1} x_0, \ u_n \in E, \ v_n \in E, \\ F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in E, \\ F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \ \geq 0, \qquad \forall v \in E, \\ z_n = P_E(x_n - \mu_n C x_n), \\ y_n = P_E(x_n - \lambda_n B x_n), \\ t_n = \alpha_n^{(1)} S x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n, \\ E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $r_n \in (0,\infty)$, $s_n \in (0,\infty)$ and $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy the condition (C1)-(C5) in Theorem 4.1.2. Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

If $B = 0, C = 0$ and $F_1(u_n, u) = F_1(v_n, v) = 0$ in Corollary 4.1.4, then $P_E = I$ and we get $u_n = y_n = x_n$ and $v_n = z_n = x_n$, hence we can obtain the following result immediately.

Corollary 4.1.5. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $S : E \longrightarrow E$ be a k-strictly pseudocontractive mapping with a fixed point. Define a mapping $S_k : E \longrightarrow E$ by $S_k x = kx + (1-k)Sx, \quad \forall x \in E$. Suppose that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in H, \ \ E_1 = E, \ \ x_1 = P_{E_1}x_0 \\ t_n = \alpha_n S_k x_n + (1-\alpha_n)x_n, \\ E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{E_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ are sequences in $(0,1)$. Assume that the control sequences satisfy the condition $\lim_{n\longrightarrow\infty}\alpha_n = \alpha \in (0,1)$ in Theorem 4.1.2 Then, $\{x_n\}$ converges strongly to a point $P_{F(S)}x_0$.

4.1.2 Convex Feasibility Problem

Finally, we consider the following *Convex Feasibility Problem* (**CFP**): finding an $x \in \bigcap_{j=1}^{M} C_j$, where $M \geq 1$ is an integer and each C_i is assumed to be the solutions of equilibrium problem with the bifunction $F_j, j = 1, 2, 3, \ldots, M$ and the solution set of the variational inequality problem. There is a considerable investigation on **CFP** in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [220], computer tomography and radiation therapy treatment planning.

The following result can obtain from Theorem 4.1.2. We, therefore, omit the proof.

Theorem 4.1.6. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $\{F_j\}_{j=1}^{M}$ be a family of bifunction from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $A_j : E \longrightarrow H$ be ρ_j-inverse-strongly monotone mapping for each $j \in \{1, 2, 3, \ldots, M\}$. Let $B_i : E \longrightarrow H$ be β_i-inverse-strongly monotone mapping for each $i \in \{1, 2, 3, \ldots, N\}$. Let $S : E \longrightarrow E$ be a k-strictly pseudocontractive mapping with a fixed point. Define a mapping $S_k : E \longrightarrow E$ by $S_k x = kx + (1-k)Sx, \quad \forall x \in E$. Suppose that*

$$\Theta := F(S_k) \cap (\cap_{j=1}^{M} GMEP(F_j, \varphi, A_j)) \cap (\cap_{i=1}^{N} VI(E, B_i)) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{cases} x_0 \in H, \;\; E_1 = E, \;\; x_1 = P_{E_1}x_0, \;\; v_1, v_2, \ldots, v_M \in E, \\ F_1(v_{n,1}, v_1) + \varphi(v_1) - \varphi(v_{n,1}) + \langle A_1 x_n, v_1 - v_{n,1}\rangle + \frac{1}{r_1}\langle v_1 - v_{n,1}, v_{n,1} - x_n\rangle \geq 0, \quad \forall v_1 \in E, \\ F_2(v_{n,2}, v_2) + \varphi(v_2) - \varphi(v_{n,2}) + \langle A_2 x_n, v_2 - v_{n,2}\rangle + \frac{1}{r_2}\langle v_2 - v_{n,2}, v_{n,2} - x_n\rangle \geq 0, \quad \forall v_2 \in E, \\ \vdots \\ F_M(v_{n,M}, v_M) + \varphi(v_M) - \varphi(v_{n,M}) + \langle A_M x_n, v_M - v_{n,M}\rangle + \frac{1}{r_M}\langle v_M - v_{n,M}, v_{n,M} - x_n\rangle \geq 0, \\ \forall v_M \in E, \\ y_{n,1} = P_E(x_n - \lambda_{n,1}B_1 x_n), \\ y_{n,2} = P_E(x_n - \lambda_{n,2}B_2 x_n), \\ \vdots \\ y_{n,N} = P_E(x_n - \lambda_{n,N}B_N x_n), \\ t_n = \alpha_{n,0}S_k x_n + \sum_{i=1}^{N}\alpha_{n,i}y_{n,i} + \sum_{j=1}^{M}\alpha'_{n,j}v_{n,j}, \\ E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{E_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $\alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,N}$ and $\alpha'_{n,1}, \alpha'_{n,2}, \; \ldots, \alpha'_{n,M} \in (0,1)$ such that $\sum_{i=0}^{N}\alpha_{n,i} + \sum_{j=1}^{M}\alpha'_{n,j} = 1$, $\{\lambda_{n,i}\}$ are positive sequences in $(0,1)$. Assume that the control sequences satisfy the following restrictions:

(C1) $\lim_{n\longrightarrow\infty}\alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, *for each* $0 \leq i \leq N$,

(C2) $\lim_{n\longrightarrow\infty}\alpha_n'^{(j)} = \alpha'^{(j)} \in (0,1)$, *for each* $1 \leq j \leq M$,

(C3) $a_j \leq r_j \leq 2\rho_j$ *, where a_j is some positive constants for each* $1 \leq j \leq M$,

(C4) $c_i \leq \lambda_{n,i} \leq 2\beta_i$ *, where c_i is some positive constants for each* $1 \leq i \leq N$,

(C5) $\lim_{n\longrightarrow\infty}|\lambda_{n+1,i} - \lambda_{n,i}| = 0$, *for each* $1 \leq i \leq N$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

If $A_j = 0$, for each $1 \leq j \leq M$ and $F_i(v_{n,i}, v_i) = 0$, for each $1 \leq i \leq N$ in Theorem 4.1.6, then $v_{n,i} = x_n$, hence we can obtain the following result immediately.

Theorem 4.1.7. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : E \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $B_i : E \longrightarrow H$ be β_i-inverse-strongly monotone mapping for each $i \in \{1, 2, 3, \ldots, N\}$. Let $S : E \longrightarrow E$ be a k-strictly pseudocontractive mapping* *with a fixed point. Define a mapping* $S_k : E \longrightarrow E$ *by* $S_kx = kx+(1-k)Sx, \quad \forall x \in E$*. Suppose that*

$$\Theta := F(S_k) \cap (\cap_{i=1}^{N} VI(E, B_i)) \neq \emptyset.$$

Let $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in H, \ \ E_1 = E, \ \ x_1 = P_{E_1}x_0, \\ y_{n,1} = P_E(x_n - \lambda_{n,1}B_1x_n), \\ y_{n,2} = P_E(x_n - \lambda_{n,2}B_2x_n), \\ \vdots \\ y_{n,N} = P_E(x_n - \lambda_{n,N}B_Nx_n), \\ t_n = \alpha_{n,0}S_kx_n + \sum_{i=1}^{N} \alpha_{n,i}y_{n,i}, \\ E_{n+1} = \{w \in E_n : \|t_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{E_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $\alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,N} \in (0,1)$ *such that* $\sum_{i=0}^{N} \alpha_{n,i} = 1$*,* $\{\lambda_{n,i}\}$ *are positive sequences in* $(0,1)$*. Assume that the control sequences satisfy the following restrictions:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$*, for each* $0 \leq i \leq N$*,*

(C2) $c_i \leq \lambda_{n,i} \leq 2\beta_i$ *, where* c_i *is some positive constants for each* $1 \leq i \leq N$*,*

(C3) $\lim_{n\longrightarrow\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0,$ *for each* $1 \leq i \leq N$*.*

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

If $B_i = 0$, for each $1 \leq i \leq N$ in Theorem 4.1.6, we get $y_{n,i} = x_n$. Hence we can obtain the following result immediately.

Theorem 4.1.8. *Let* E *be a nonempty closed convex subset of a real Hilbert space* H*. Let be a* $\{F_j\}_{j=1}^{M}$ *be a family of bifunction from* $E \times E$ *to* $\mathbb{R}$ *satisfying (A1)-(A5) and let* $\varphi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with either (B1) or (B2). Let* $A_j : E \longrightarrow H$ *be* ρ_j*-inverse-strongly monotone mapping for each* $j \in \{1, 2, 3, \ldots, M\}$*. Let* $S : E \longrightarrow E$ *be a* k*-strictly pseudocontractive mapping with a fixed point. Define a mapping* $S_k : E \longrightarrow E$ *by* $S_kx = kx + (1-k)Sx, \quad \forall x \in E$*. Suppose that*

$$\Theta := F(S_k) \cap (\cap_{j=1}^{M} GMEP(F_j, \varphi, A_j)) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{cases} x_0 \in H, \;\; E_1 = E, \;\; x_1 = P_{E_1} x_0, \;\; v_1, v_2, \dots, v_M \in E, \\ F_1(v_{n,1}, v_1) + \varphi(v_1) - \varphi(v_{n,1}) + \langle A_1 x_n, v_1 - v_{n,1} \rangle + \frac{1}{r_1} \langle v_1 - v_{n,1}, v_{n,1} - x_n \rangle \geq 0, \quad \forall v_1 \in E, \\ F_2(v_{n,2}, v_2) + \varphi(v_2) - \varphi(v_{n,2}) + \langle A_2 x_n, v_2 - v_{n,2} \rangle + \frac{1}{r_2} \langle v_2 - v_{n,2}, v_{n,2} - x_n \rangle \geq 0, \quad \forall v_2 \in E, \\ \vdots \\ F_M(v_{n,M}, v_M) + \varphi(v_M) - \varphi(v_{n,M}) + \langle A_M x_n, v_M - v_{n,M} \rangle + \frac{1}{r_M} \langle v_M - v_{n,M}, v_{n,M} - x_n \rangle \geq 0, \\ \forall v_M \in E, \\ t_n = \alpha_{n,0} S_k x_n + \sum_{j=1}^{M} \alpha'_{n,j} v_{n,j}, \\ E_{n+1} = \{ w \in E_n : \|t_n - w\| \leq \|x_n - w\| \}, \\ x_{n+1} = P_{E_{n+1}} x_0, \quad \forall n \geq 1, \end{cases}$$

where $\alpha_{n,0}$ and $\alpha'_{n,1}, \alpha'_{n,2}, \; \dots, \alpha'_{n,M} \in (0,1)$ such that $\alpha_{n,0} + \sum_{j=1}^{M} \alpha'_{n,j} = 1$. Assume that the control sequences satisfy the following restrictions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n^{(0)} = \alpha^{(0)} \in (0,1)$*,*

(C2) $\lim_{n \longrightarrow \infty} \alpha_n^{\prime(j)} = \alpha^{\prime(j)} \in (0,1)$*, for each* $1 \leq j \leq M$*,*

(C3) $a_j \leq r_j \leq 2\rho_j$ *, where a_j is some positive constants for each $1 \leq j \leq M$.*

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

4.1.3 Hybrid algorithms of generalized mixed equilibrium problems and the common variational inequality problems

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of a common of generalized mixed equilibrium problems, the common solutions of the variational inequality for inverse-strongly monotone mapping and the set of fixed points of infinite family of nonexpansive mappings in the set of Hilbert spaces.

Theorem 4.1.9. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous and convex function. Let A, B, D, E be $\alpha, \beta, \delta, \eta$-inverse-strongly monotone mapping of C into H, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite nonexpansive mapping such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, \varphi_1, A) \cap GMEP(F_2, \varphi_2, B) \cap VI(C, D) \cap VI(C, E) \neq \emptyset$.*

Assume that either (B1) or (B2) holds. Let $\{x_n\}$ *be a sequence generated by* $x_0 \in C,\ C_{1,i} = C,\ C_1 = \cap_{i=1}^{\infty} C_{1,i},\ x_1 = P_{C_1}x_0$ *and*

$$\begin{cases} t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n), \\ u_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_nBx_n), \\ w_n = \xi_nP_C(u_n - \lambda_nDu_n) + (1-\xi_n)P_C(t_n - \mu_nEt_n), \\ y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})T_iw_n, \\ C_{n+1,i} = \big\{z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\big\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}}x_0. \end{cases} \tag{4.1.30}$$

for every $n \geq 0$*, where* $\{r_n\}, \{s_n\} \subset (0,\infty),\ \lambda_n \in (0, 2\delta)$ *and* $\mu_n \in (0, 2\eta)$ *satisfying the following conditions: (i).* $0 < a \leq r_n \leq b < 2\alpha$;
(ii). $0 < c \leq s_n \leq d < 2\beta$;
(iii).$\lim_{n\to\infty} \alpha_{n,i} = 0$;
(iv).$\lim_{n\to\infty} \xi_n = \xi \in (0,1)$;
(v). $0 < e \leq \lambda_n \leq f < 2\delta$;
(vi). $0 < g \leq \mu_n \leq j < 2\eta$.
Then, $\{x_n\}$ *converges strongly to* $P_{\Theta}x_0$*.*

***Proof*.** Let $p \in \Theta$ then, $p = T_{r_n}^{(F_1,\varphi_1)}(p - r_nAp),\ p = T_{s_n}^{(F_2,\varphi_2)}(p - s_nBp),\ p = P_C(p - \lambda_nDp)$ and $p = P_C(p - \mu_nEp)$. By nonexpansiveness of $P_C,\ T_{r_n}^{(F_1,\varphi_1)}$ and $T_{s_n}^{(F_2,\varphi_2)}$, we have

$$\begin{aligned} & \|w_n - p\|^2 \\ = & \|\xi_nP_C(u_n - \lambda_nDu_n) + (1-\xi_n)P_C(t_n - \mu_nEt_n) - \xi_nP_C(p - \lambda_nDp) \\ & -(1-\xi_n)P_C(p - \mu_nEp)\|^2 \\ = & \Big\|\xi_n\big\{P_C(u_n - \lambda_nDu_n) - P_C(p - \lambda_nDp)\big\} + (1-\xi_n)\big\{P_C(t_n - \mu_nEt_n) - P_C(p - \mu_nEp)\big\}\Big\|^2 \\ \leq & \xi_n\|(u_n - \lambda_nDu_n) - (p - \lambda_nDp)\|^2 + (1-\xi_n)\|(t_n - \mu_nEt_n) - (p - \mu_nEp)\|^2 \\ = & \xi_n\|(u_n - p) - \lambda_n(Du_n - Dp)\|^2 + (1-\xi_n)\|(t_n - p) - \mu_n(Et_n - Ep)\|^2 \\ = & \xi_n\Big\{\|u_n - p\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\Big\} \\ & +(1-\xi_n)\Big\{\|t_n - p\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\Big\} \\ \leq & \xi_n\Big\{\|T_{s_n}^{(F_2,\varphi_2)}(x_n - s_nBx_n) - T_{s_n}^{(F_2,\varphi_2)}(p - s_nBp)\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\Big\} \\ & +(1-\xi_n)\Big\{\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n) - T_{r_n}^{(F_1,\varphi_1)}(p - r_nAp)\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\Big\} \\ \leq & \xi_n\{\|(x_n - s_nBx_n) - (p - s_nBp)\|^2\} + (1-\xi_n)\{\|(x_n - r_nAx_n) - (p - r_nAp)\|^2\} \quad (4.1.31) \\ \leq & \xi_n\|x_n - p\|^2 + (1-\xi_n)\|x_n - p\|^2 \\ \leq & \|x_n - p\|^2. \end{aligned}$$

Since both $I - r_nA$ and $I - s_nB$ are nonexpansive for each $n \geq 1$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{s_n}^{(F_2,\varphi_2)}(I - s_nB)x_n - T_{s_n}^{(F_2,\varphi_2)}(I - s_nB)p\|^2 \\ &\leq \|(I - s_nB)x_n - (I - s_nB)p\|^2 \\ &\leq \|x_n - p\|^2 + s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 \qquad (4.1.32) \\ &\leq \|x_n - p\|^2 \end{aligned}$$

and

$$\begin{aligned} \|t_n - p\|^2 &= \|T_{r_n}^{(F_1,\varphi_1)}(I - r_nA)x_n - T_{r_n}^{(F_1,\varphi_1)}(I - r_nA)p\|^2 \\ &\leq \|(I - r_nA)x_n - (I - r_nA)p\|^2 \\ &\leq \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2. \qquad (4.1.33) \end{aligned}$$

Therefore we obtain, $\|u_n - p\| \leq \|x_n - p\|$ and $\|t_n - p\| \leq \|x_n - p\|$.
Next, we will divide the proof into four steps.
Step 1. We show that $\{x_n\}$ is well defined. Let $n = 1$, then $C_{1,i} = C$ is closed and convex for each $i \geq 1$. Suppose that $C_{n,i}$ is closed convex for some $n > 1$. Then, by definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for $n \geq 1$. Hence, $C_{n,i}$ is closed convex for $n \geq 1$ and for each $i \geq 1$. This implies that C_n is closed convex for $n \geq 1$. Moreover, we show that $\Theta \subset C_n$. For $n = 1$, $\Theta \subset C = C_{1,i}$. For $n \geq 2$, let $p \in \Theta$. Then,

$$\begin{aligned} \|y_{n,i} - p\|^2 &= \|\alpha_{n,i}(x_0 - p)^2 + (1 - \alpha_{n,i})(T_iw_n - p)\|^2 \\ &\leq \alpha_{n,i}\|x_0 - p\|^2 + (1 - \alpha_{n,i})\|w_n - p\|^2 \\ &= \|w_n - p\|^2 + \alpha_{n,i}(\|x_0 - p\|^2 - \|w_n - p\|^2) \\ &\leq \|x_n - p\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle), \end{aligned}$$

which shows that $p \in C_{n,i},\ \forall n \geq 2,\ \forall i \geq 1$. So, $\Theta \subset C_{n,i},\ \forall n \geq 1,\ \forall i \geq 1$. Therefore, it follows that $\emptyset \neq \Theta \subset C_n,\ \forall n \geq 1$. This implies that $\{x_n\}$ is well defined.
Step 2. We claim that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|y_{n,i} - x_n\| = 0$.
From $x_n = P_{C_n}x_0$, we get

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Since $\Theta \subset C_n$, we have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0 \text{ for each } p \in \Theta \text{ and } n \in \mathbb{N}.$$

Hence, for $p \in \Theta$, we obtain

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - p\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|. \end{aligned}$$

It follows that

$$\|x_0 - x_n\| \le \|x_0 - p\|, \text{ for all } \ p \in \Theta \ \text{ and } \ n \in \mathbb{N}.$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0. \tag{4.1.34}$$

For $n \in \mathbb{N}$, we compute

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\| \end{aligned}$$

and then

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|, \text{ for all } \ n \in \mathbb{N}.$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists. That is, there exists m such that

$$m = \lim_{n\longrightarrow\infty} \|x_n - x_0\|. \tag{4.1.35}$$

Hence, $\{x_n\}$ is bounded and so are $\{Ax_n\}$, $\{Bx_n\}$, $\{u_n\}$, $\{Du_n\}$, $\{t_n\}$, $\{Et_n\}$, $\{w_n\}$, $\{T_i w_n\}$ and $\{y_{n,i}\}$ for $i = 1, 2, ...,$ and $n \ge 1$. From (4.1.34), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

By (4.1.35), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - x_{n+1}\| = 0. \tag{4.1.36}$$

Since $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\|y_{n,i} - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle). \tag{4.1.37}$$

By (iii) and (6.4.13), we get

$$\lim_{n\longrightarrow\infty}\|y_{n,i} - x_{n+1}\| = 0. \tag{4.1.38}$$

It follows that

$$\|y_{n,i} - x_n\| \leq \|y_{n,i} - x_{n+1}\| + \|x_n - x_{n+1}\|.$$

By (6.4.13) and (4.1.38), we have

$$\lim_{n\longrightarrow\infty}\|y_{n,i} - x_n\| = 0,\ \ i = 1, 2, ... \tag{4.1.39}$$

Step 3. We claim that the following statements hold:

(S1) $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$;

(S2) $\lim_{n\longrightarrow\infty}\|x_n - t_n\| = 0$;

(S3) $\lim_{n\longrightarrow\infty}\|w_n - x_n\| = 0$.

For (4.1.31), we note that

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\leq \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|T_i w_n - p\|^2 \\
&= \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\
&\leq \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\big\{\xi_n\|(x_n - s_nBx_n) - (p - s_nBp)\|^2 \\
&\quad +(1-\xi_n)\|(x_n - r_nAx_n) - (p - r_nAp)\|^2\big\} \\
&\leq \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\big\{\xi_n\big(\|x_n - p\|^2 + s_n(s_n - 2\beta)\|Bx_n - Bp\|^2\big) \\
&\quad +(1-\xi_n)\big(\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2\big)\big\} \\
&= \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\big\{\|x_n - p\|^2 + \xi_n s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 \\
&\quad +(1-\xi_n)r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2\big\} \\
&= \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 + (1-\alpha_{n,i})\xi_n s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 \\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \qquad (4.1.40) \\
&= \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 + (1-\alpha_{n,i})\xi_n s_n(s_n - 2\beta)\|Bx_n - Bp\|^2.
\end{aligned}$$

Since $0 < c \leq s_n \leq d \leq 2\beta,\ 0 \leq k_i \leq \alpha_{n,i} \leq h_i < 1$, we have

$$\begin{aligned}(1-h_i)\xi c(2\beta-d)\|Bx_n-Bp\|^2 &\leq \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2\\ &\leq \alpha_{n,i}\|x_0-p\|^2+\|y_{n,i}-x_n\|(\|x_n-p\|+\|y_{n,i}-p\|).\end{aligned}$$

By condition (iii) and (4.1.39), then $\lim_{n\to\infty}\|Bx_n-Bp\|=0$. By using the same method with (6.1.29). Hence, from (6.1.28) since $0<a\leq r_n\leq b\leq 2\alpha,\ 0\leq k_i\leq \alpha_{n,i}\leq h_i<1$, we have

$$\begin{aligned}(1-h_i)(1-\xi)a(2\alpha-b)\|Ax_n-Ap\|^2 &\leq \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2\\ &\leq \alpha_{n,i}\|x_0-p\|^2+\|y_{n,i}-x_n\|(\|x_n-p\|+\|y_{n,i}-p\|).\end{aligned}$$

By condition (iii) and (4.1.39), then we have $\lim_{n\to\infty}\|Ax_n-Ap\|=0$. On the other hand, we compute

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{s_n}^{(F_2,\varphi_2)}(I-s_nB)x_n-T_{s_n}^{(F_2,\varphi_2)}(I-s_nB)p\|^2\\ &\leq \langle (x_n-s_nBx_n)-(p-s_nBp),u_n-p\rangle\\ &= \frac{1}{2}\Big\{\|(x_n-s_nBx_n)-(p-s_nBp)\|^2+\|u_n-p\|^2\\ &\quad -\|(x_n-s_nBx_n)-(p-s_nBp)-(u_n-p)\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\|x_n-p\|^2+\|u_n-p\|^2-\|(x_n-s_nBx_n)-(p-s_nBp)-(u_n-p)\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|x_n-p\|^2+\|u_n-p\|^2-\|u_n-x_n\|^2+2s_n\langle x_n-u_n,Bx_n-Bp\rangle\\ &\quad -s_n^2\|Bx_n-Bp\|^2\Big\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_n-p\|^2 &\leq \|x_n-p\|^2-\|u_n-x_n\|^2+2s_n\langle x_n-u_n,Bx_n-Bp\rangle\\ &\quad -s_n^2\|Bx_n-Bp\|^2\\ &\leq \|x_n-p\|^2-\|u_n-x_n\|^2+2s_n\|x_n-u_n\|\|Bx_n-Bp\|. \quad (4.1.41)\end{aligned}$$

By using the same method as (4.1.41), we also have

$$\begin{aligned}\|t_n-p\|^2 &\leq \|x_n-p\|^2-\|t_n-x_n\|^2+2r_n\langle x_n-t_n,Ax_n-Ap\rangle\\ &\quad -r_n^2\|Ax_n-Ap\|^2\\ &\leq \|x_n-p\|^2-\|t_n-x_n\|^2+2r_n\|x_n-t_n\|\|Ax_n-Ap\|. \quad (4.1.42)\end{aligned}$$

Furthermore, we observe that

$$
\begin{aligned}
& \|y_{n,i}-p\|^2 \qquad (4.1.43)\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|T_iw_n-p\|^2\\
=\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|w_n-p\|^2\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\{\|\xi_nP_C(u_n-\lambda_nDu_n)+(1-\xi_n)P_C(t_n-\mu_nEt_n)-p\|^2\}\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\{\xi_n(\|u_n-p\|^2-\lambda_n(2\delta-\lambda_n)\|Du_n-Dp\|^2)\\
& +(1-\xi_n)(\|t_n-p\|^2-\mu_n(2\eta-\mu_n)\|Et_n-Ep\|^2)\}\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\{\xi_n\|u_n-p\|^2+(1-\xi_n)\|t_n-p\|^2\}\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\Big\{\xi_n\big(\|x_n-p\|^2-\|u_n-x_n\|^2+2s_n\|x_n-u_n\|\|Bx_n-Bp\|\big)\\
& +(1-\xi_n)\big(\|x_n-p\|^2-\|t_n-x_n\|^2+2r_n\|x_n-t_n\|\|Ax_n-Ap\|\big)\Big\}\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-(1-\alpha_{n,i})\xi_n\|u_n-x_n\|^2\\
& +(1-\alpha_{n,i})\xi_n2s_n\|x_n-u_n\|\|Bx_n-Bp\|-(1-\alpha_{n,i})(1-\xi_n)\|t_n-x_n\|^2\\
& +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\|Ax_n-Ap\| \qquad (4.1.44)\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2\\
& -(1-\alpha_{n,i})\xi_n\|u_n-x_n\|^2+(1-\alpha_{n,i})\xi_n2s_n\|x_n-u_n\|\|Bx_n-Bp\|\\
& +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\|Ax_n-Ap\|.
\end{aligned}
$$

By condition (i)-(iv), (4.1.39), $\lim_{n\to\infty}\|Ax_n-Ap\|=0$ and $\lim_{n\to\infty}\|Bx_n-Bp\|=0$, then we get

$$
\begin{aligned}
& (1-\alpha_{n,i})\xi_n\|u_n-x_n\|^2 \qquad (4.1.45)\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2+(1-\alpha_{n,i})\xi_n2s_n\|x_n-u_n\|\|Bx_n-Bp\|\\
& +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\|Ax_n-Ap\|\\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-y_{n,i}\|(\|x_n-p\|+\|y_{n,i}-p\|)\\
& +(1-\alpha_{n,i})\xi_n2s_n\|x_n-u_n\|\|Bx_n-Bp\|\\
& +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\|Ax_n-Ap\|. \qquad (4.1.46)
\end{aligned}
$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \qquad (4.1.47)$$

Similary (6.4.31), from (4.1.43) by conditions (i)-(iv), (4.1.39), $\lim_{n\to\infty}\|Ax_n-Ap\|=$

0 and $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$, then we get

$$\begin{aligned}
& (1-\alpha_{n,i})(1-\xi_n)\|t_n - x_n\|^2 \\
\leq\ & \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 + (1-\alpha_{n,i})\xi_n 2s_n\|x_n - u_n\|\|Bx_n - Bp\| \\
& +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n - t_n\|\|Ax_n - Ap\| \\
\leq\ & \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|(\|x_n - p\| + \|y_{n,i} - p\|) \\
& +(1-\alpha_{n,i})\xi_n 2s_n\|x_n - u_n\|\|Bx_n - Bp\| \\
& +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n - t_n\|\|Ax_n - Ap\|. \qquad (4.1.48)
\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \qquad (4.1.49)$$

From (4.1.30), (4.1.32) and (4.1.33), we have

$$\begin{aligned}
\|w_n - p\|^2 =\ & \|\xi_n P_C(u_n - \lambda_n Du_n) + (1-\xi_n)P_C(t_n - \mu_n Et_n) - \xi_n P_C(p - \lambda_n Dp) \\
& -(1-\xi_n)P_C(p - \mu_n Ep)\|^2 \\
=\ & \xi_n\|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|^2 \\
& +(1-\xi_n)\|P_C(t_n - \mu_n Et_n) - P_C(p - \mu_n Ep)\|^2 \\
\leq\ & \xi_n\{\|u_n - p\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\} + (1-\xi_n)\{\|t_n - p\|^2 \\
& -\mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\} \\
\leq\ & \xi_n\{\|x_n - p\|^2 + s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\} \\
& +(1-\xi_n)\{\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\} \\
\leq\ & \|x_n - p\|^2 + \xi_n s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 - \xi_n\lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2 \\
& +(1-\xi_n)r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 - (1-\xi_n)\mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2.
\end{aligned}$$

Furthermore, we observe that

$$\begin{aligned}
\|y_{n,i}-p\|^2 &\leq \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|T_iw_n-p\|^2\\
&= \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|w_n-p\|^2\\
&\leq \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\{\|x_n-p\|^2+\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad -\xi_n\lambda_n(2\delta-\lambda_n)\|Du_n-Dp\|^2+(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\quad -(1-\xi_n)\mu_n(2\eta-\mu_n)\|Et_n-Ep\|^2\}\\
&\leq \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2\\
&\quad +(1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad -(1-\alpha_{n,i})\xi_n\lambda_n(2\delta-\lambda_n)\|Du_n-Dp\|^2\\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\quad -(1-\alpha_{n,i})(1-\xi_n)\mu_n(2\eta-\mu_n)\|Et_n-Ep\|^2\\
&\leq \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2+(1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad -(1-\alpha_{n,i})\xi_n\lambda_n(2\delta-\lambda_n)\|Du_n-Dp\|^2\\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2.
\end{aligned}$$

Since $0<e\leq\lambda_n\leq f<2\delta,\ 0\leq k_i\leq\alpha_{n,i}\leq h_i<1,$ we have

$$\begin{aligned}
(1-h_i)\xi e(2\delta-f)\|Du_n-Dp\|^2 &\leq \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2\\
&\quad +(1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\leq \alpha_{n,i}\|x_0-p\|^2+\|y_{n,i}-x_n\|(\|x_n-p\|-\|y_{n,i}-p\|)\\
&\quad +(1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2.
\end{aligned}$$

By conditions (i)-(v), (4.1.39), $\lim_{n\to\infty}\|Ax_n-Ap\|=0$ and $\lim_{n\to\infty}\|Bx_n-Bp\|=0$, then $\lim_{n\to\infty}\|Du_n-Dp\|=0$. By using the same method with (4.1.50). Hence, from (4.1.50) and since $0<g\leq\mu_n\leq j\leq 2\eta,\ 0\leq k_i\leq\alpha_{n,i}\leq h_i\leq 1,$ we have

$$\begin{aligned}
(1-h_i)(1-\xi)g(2\eta-j)\|Et_n-Ep\|^2 &\leq \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2\\
&\quad +(1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\leq \alpha_{n,i}\|x_0-p\|^2+\|y_{n,i}-x_n\|(\|x_n-p\|-\|y_{n,i}-p\|)\\
&\quad +(1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad +(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2.
\end{aligned}$$

By conditions (i)-(iv), (vi), (4.1.39), $\lim_{n\to\infty}\|Ax_n-Ap\|=0$ and $\lim_{n\to\infty}\|Bx_n-$

$Bp\| = 0$, then $\lim_{n\to\infty} \|Et_n - Ep\| = 0$. From (4.1.30), we have

$$\begin{aligned} & \|w_n - p\|^2 \\ \leq\ & \Big\|\xi_n\big\{P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\big\} + (1-\xi_n)\big\{P_C(t_n - \mu_n Et_n) - P_C(p - \mu_n Ep)\big\}\Big\|^2 \\ \leq\ & \xi_n\|u_n' - p\|^2 + (1-\xi_n)\|t_n' - p\|^2. \end{aligned} \tag{4.1.50}$$

Assume that $u_n' = P_C(u_n - \lambda_n Du_n)$ and $t_n' = P_C(t_n - \mu_n Et_n)$. By nonexpansiveness of $I - \lambda_n D$ and $I - \mu_n E$, we also have

$$\begin{aligned} \|u_n' - p\|^2 &\leq \|P_C(I-\lambda_n D)u_n - P_C(I-\lambda_n D)p\|^2 \\ &\leq \langle (u_n - \lambda_n Du_n) - (p - \lambda_n Dp), u_n' - p\rangle \\ &= \frac{1}{2}\Big\{\|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\|^2 + \|u_n' - p\|^2 \\ &\quad -\|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp) - (u_n' - p)\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\|u_n - p\|^2 + \|u_n' - p\|^2 - \|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp) - (u_n' - p)\|^2\Big\} \\ &= \frac{1}{2}\Big\{\big\|T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n Bx_n) - T_{s_n}^{(F_2,\varphi_2)}(p - s_n Bp)\big\|^2 + \|u_n' - p\|^2 - \|u_n - u_n'\|^2 \\ &\quad +2\lambda_n\langle u_n - u_n', Du_n - Dp\rangle - \lambda_n^2\|Du_n - Dp\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n' - p\|^2 - \|u_n - u_n'\|^2 + \lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2\Big\}. \end{aligned}$$

It follows that

$$\|u_n' - p\|^2 \leq \|x_n - p\|^2 - \|u_n - u_n'\|^2 + \lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2. \tag{4.1.51}$$

Similaly (4.1.51), we obtain

$$\|t_n' - p\|^2 \leq \|x_n - p\|^2 - \|t_n - t_n'\|^2 + \mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2. \tag{4.1.52}$$

Substituting (4.1.51), (4.1.52) into (4.1.50)

$$\begin{aligned} \|w_n - p\|^2 &\leq \xi_n\|u_n' - p\|^2 + (1-\xi_n)\|t_n' - p\|^2 \\ &\leq \xi_n\{\|x_n - p\|^2 - \|u_n - u_n'\|^2 + \lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2\} \\ &\quad +(1-\xi_n)\{\|x_n - p\|^2 - \|t_n - t_n'\|^2 + \mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2\} \\ &\leq \|x_n - p\|^2 - \xi_n\|u_n - u_n'\|^2 + \xi_n\lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2 \\ &\quad -(1-\xi_n)\|t_n - t_n'\|^2 + (1-\xi_n)\mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2. \end{aligned} \tag{4.1.53}$$

By (4.1.53), we have

$$\begin{aligned}
& \|y_{n,i}-p\|^2 \\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|T_iw_n-p\|^2 \\
=\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|w_n-p\|^2 \\
=\ & \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\Big\{\|x_n-p\|^2-\xi_n\|u_n-u_n'\|^2 \\
& +\xi_n\lambda_n(\lambda_n-2\delta)\|Du_n-Dp\|^2-(1-\xi_n)\|t_n-t_n'\|^2+(1-\xi_n)\mu_n(\mu_n-2\eta)\|Et_n-Ep\|^2\Big\} \\
=\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-(1-\alpha_{n,i})\xi_n\|u_n-u_n'\|^2 \\
& +(1-\alpha_{n,i})\xi_n\lambda_n(\lambda_n-2\delta)\|Du_n-Dp\|^2-(1-\alpha_{n,i})(1-\xi_n)\|t_n-t_n'\|^2 \\
& +(1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n-2\eta)\|Et_n-Ep\|^2 \\
=\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-(1-\alpha_{n,i})\xi_n\|u_n-u_n'\|^2 \\
& +(1-\alpha_{n,i})\xi_n\lambda_n(\lambda_n-2\delta)\|Du_n-Dp\|^2+(1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n-2\eta)\|Et_n-Ep\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
& (1-\alpha_{n,i})\xi_n\|u_n-u_n'\|^2 \\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2 \\
& +(1-\alpha_{n,i})\xi_n\lambda_n(\lambda_n-2\delta)\|Du_n-Dp\|^2+(1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n-2\eta)\|Et_n-Ep\|^2 \\
\leq\ & \alpha_{n,i}\|x_0-p\|^2+\|x_n-y_{n,i}\|(\|x_n-p\|+\|y_{n,i}-p\|) \\
& +(1-\alpha_{n,i})\xi_n\lambda_n(\lambda_n-2\delta)\|Du_n-Dp\|^2+(1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n-2\eta)\|Et_n-Ep\|^2.
\end{aligned}$$

By conditions (iii)-(vi), (4.1.39), $\lim_{n\to\infty}\|Du_n-Dp\|=0$ and $\lim_{n\to\infty}\|Et_n-Ep\|=0$, then we get

$$\lim_{n\to\infty}\|u_n-u_n'\|=0. \tag{4.1.54}$$

By using the same argument (4.1.54), we can prove that

$$\lim_{n\to\infty}\|t_n-t_n'\|=0. \tag{4.1.55}$$

Applying (4.1.47) and (4.1.54), we also have

$$\lim_{n\to\infty}\|x_n-u_n'\|=0. \tag{4.1.56}$$

From (4.1.49) and (4.1.55), we obtain

$$\lim_{n\to\infty}\|x_n-t_n'\|=0. \tag{4.1.57}$$

Since $u_n'=P_C(u_n-\lambda_nDu_n)$ and $t_n'=P_C(t_n-\mu_nEt_n)$, we have

$$w_n-x_n=\xi_n(u_n'-x_n)+(1-\xi_n)(t_n'-x_n).$$

By (4.1.56) and (4.1.57), we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{4.1.58}$$

By condition (iii), we have $y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})T_i w_n$ which implies that

$$\|y_{n,i} - T_i w_n\| \;=\; \alpha_{n,i}\|x_0 - T_i w_n\| \to 0,\ n\to\infty,\ \forall i \geq 1.$$

From (4.1.39) and $\lim_{n\to\infty}\|y_{n,i} - T_i w_n\| = 0$, we have

$$\|x_n - T_i w_n\| \;\leq\; \|y_{n,i} - T_i w_n\| + \|y_{n,i} - x_n\| \to 0,\ n\to\infty,\ \forall i \geq 1. \tag{4.1.59}$$

Since

$$\|w_n - T_i w_n\| \;\leq\; \|w_n - x_n\| + \|x_n - T_i w_n\|.$$

By (4.1.58) and (4.1.59), hence $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0,\ \forall i = 1,2,....$

Step 4. We show that $z \in \Theta := (\cap_{i=1}^{\infty} F(T_i)) \cap GMEP(F_1,\varphi_1,A) \cap GMEP(F_2,\varphi_2,B) \cap VI(C,D) \cap VI(C,E)$.

First, we show that $z \in \cap_{i=1}^{\infty} F(T_i)$. Assume that $z \notin \cap_{i=1}^{\infty} F(T_i)$. Since $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z\| = 0$, we have that $\lim_{n\to\infty}\|w_n - z\| = 0$. By $\lim_{n\to\infty}\|w_n - z\| = 0$ and $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0$, $i = 1,2,...$, from Opial's condition, we have

$$\begin{aligned}
\liminf_{i\longrightarrow\infty}\|w_{n_i} - z\| &< \liminf_{i\longrightarrow\infty}\|w_{n_i} - T_i z\| \\
&\leq \liminf_{i\longrightarrow\infty}(\|w_{n_i} - T_i w_{n_i}\| + \|T_i w_{n_i} - T_i z\|) \\
&\leq \liminf_{i\longrightarrow\infty}\|w_{n_i} - z\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $z \in \cap_{i=1}^{\infty} F(T_i)$.

Next, we show that $z \in GMEP(F_1,\varphi,A)$. Since $t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n),\ n \geq 1$, we have for any $y \in C$ that

$$F_1(t_n,y) + \varphi_1(y) - \varphi_1(t_n) + \langle Ax_n, y - t_n\rangle + \frac{1}{r_n}\langle y - t_n, t_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), we also have

$$\varphi_1(y) - \varphi_1(t_n) + \langle Ax_n, y - t_n\rangle + \frac{1}{r_n}\langle y - t_n, t_n - x_n\rangle \geq F_1(y,t_n), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$. Then, we have

$$\begin{aligned}
&\langle y_t - t_{n_i}, Ay_t\rangle \\
\geq\;& \langle y_t - t_{n_i}, Ay_t\rangle - \varphi_1(y_t) + \varphi_1(t_{n_i}) - \langle y_t - t_{n_i}, Ax_{n_i}\rangle - \langle y_t - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F_1(y_t,t_{n_i}) \\
=\;& \langle y_t - t_{n_i}, Ay_t - At_{n_i}\rangle + \langle y_t - t_{n_i}, At_{n_i} - Ax_{n_i}\rangle - \varphi_1(y_t) + \varphi_1(t_{n_i}) \\
&-\langle y_t - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F_1(y_t,t_{n_i}).
\end{aligned}$$

Since $\|t_{n_i} - x_{n_i}\| \to 0$, we have $\|At_{n_i} - Ax_{n_i}\| \to 0$. Further, from an inverse strongly monotonicity of A, we have $\langle y_t - t_{n_i}, Ay_t - At_{n_i}\rangle \geq 0$. So, from (A4) and the weak lower semi-continuity of $\varphi_1, \frac{t_{n_i}-x_{n_i}}{r_{n_i}} \to 0$ and $t_{n_i} \rightharpoonup z$, we have at the limit

$$\langle y_t - z, Ay_t\rangle \geq -\varphi_1(y_t) + \varphi_1(z) + F_1(y_t, z) \tag{4.1.60}$$

as $i \to \infty$. From (A1),(A4) and (4.1.60), we also get

$$\begin{aligned}
0 &= F_1(y_t, y_t) + \varphi_1(y_t) - \varphi_1(y_t) \\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, z) + t\varphi_1(y) - (1-t)\varphi_1(z) - \varphi(y_t) \\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)[F_1(y_t, z) + \varphi_1(z) - \varphi_1(y_t)] \\
&\leq t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)\langle y_t - z, Ay_t\rangle \\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)t\langle y - z, Ay_t\rangle, \\
0 &\leq F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t) + (1-t)\langle y - z, Ay_t\rangle.
\end{aligned}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F_1(z, y) + \varphi_1(y) - \varphi_1(z) + \langle y - z, Az\rangle \geq 0.$$

This implies that $z \in GMEP(F_1, \varphi_1, A)$. By following the same arguments, we can show that $z \in GMEP(F_2, \varphi_2, B)$.

Lastly, by the same proof of [258, Theorem 3.1, pp. 346-347], we can show that $z \in VI(C, D)$ and $z \in VI(C, E)$. Therefore, $z \in (\cap_{i=1}^{\infty} F(T_i)) \cap GMEP(F_1, \varphi_1, A) \cap GMEP(F_2, \varphi_2, B) \cap VI(C, D) \cap VI(C, E)$ that is $z \in \Theta$.
Noting that since $x_n = P_{C_n}x_0$. By (??), we have

$$\langle x_0 - x_n, y - x_n\rangle \leq 0, \ \forall y \in C_n.$$

Since $\Theta \subset C_n$ and by the continuity of inner product, we obtain from the above inequality that

$$\langle x_0 - z, y - z\rangle \leq 0, \ \forall y \in C.$$

We conclude that $z = P_{\Theta}x_0$. This completes the proof. □

4.1.4 Complementarity Problem

Let C be a nonempty closed and convex cone in H and E be an operator of C into H. We define the *polar* of C in H to be the set

$$K^* := \{y^* \in H : \langle x, y^*\rangle \geq 0, \ \forall x \in C\}.$$

Then the element $u \in C$ is called a solution of the *complementarity problem* if

$$Eu \in K^*, \ \langle u, Eu \rangle = 0.$$

The set of solution of the complementarity problem is denoted by $C'(C, D), \ C'(C, E)$. We shall assume that $D, \ E$ satisfies the following conditions:

(E1) $D, \ E$ are $\delta, \ \eta$-inverse-strongly monotone mapping;

(E2) $C'(C, D), \ C'(C, E) \neq \emptyset$.

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C \cap dom(\varphi)$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x \rangle < \varphi(z);$$

(B2) C is a bounded set.

Corollary 4.1.10. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi continuous and convex function. Let A, B, D, E be $\alpha, \beta, \delta, \eta$-inverse-strongly monotone mapping of C into H, respectively. Let $T_1, T_2, \ldots$ be infinite nonexpansive mapping such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, \varphi_1, A) \cap GMEP(F_2, \varphi_2, B) \cap C'(C, D) \cap C'(C, E) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C, \ C_{1,i} = C, \ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \ x_1 = P_{C_1} x_0$ and*

$$\begin{cases} t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\ u_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n), \\ w_n = \xi_n P_C(u_n - \lambda_n D u_n) + (1-\xi_n) P_C(t_n - \mu_n E t_n), \\ y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) T_i w_n, \\ C_{n+1,i} = \Big\{ z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle) \Big\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{4.1.61}$$

for every $n \geq 0$, where $\{r_n\}, \{s_n\} \subset (0, \infty), \ \lambda_n \in (0, 2\delta)$ and $\mu_n \in (0, 2\eta)$ satisfy the following conditions:
(i). $0 < a \leq r_n \leq b < 2\alpha$;
(ii). $0 < c \leq s_n \leq d < 2\beta$;
(iii).$\lim_{n\to\infty} \alpha_{n,i} = 0$;
(iv).$\lim_{n\to\infty} \xi_n = \xi \in (0,1)$;

(v). $0 < e \le \lambda_n \le f < 2\delta$;

(vi). $0 < g \le \mu_n \le j < 2\eta$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$*.*

***Proof*.** Using Lemma 7.1.1 of [239], we have that $VI(C,D) = C'(C,D)$ and $VI(C,E) = C'(C,E)$. Hence, by Corollary 4.1.10 we can conclude the desired conclusion easily. This completes the proof. □

4.2 A System of Generalized Mixed Equilibrium Problems in Banach Spaces

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \le 0$ for all $x,y \in C$;

(A3) for each $x,y,z \in C$,

$$\lim_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x,y);$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x,y) = \langle Ax, y - x\rangle, \quad \forall x,y \in C.$$

Then, f satisfies (A1)–(A4). The following result is in Blum and Oettli [5].

Motivated by Combettes and Hirstoaga [10] in a Hilbert space and Takahashi and Zembayashi [50] in a Banach space, we obtained the following lemma.

Lemma 4.2.1. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \to E^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R}$ be a semicontinuous and convex functional. For $r > 0$ and let $x \in E$. Then, there exists $z \in C$ such that*

$$Q(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C,$$

where $Q(z,y) = f(z,y) + \langle Bz, y - z\rangle + \varphi(y) - \varphi(z), x, y \in C$. Furthermore, define a mapping $T_r : E \to C$ as follows:

$$T_r x = \left\{ z \in C : Q(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \ \forall y \in C \right\}.$$

Then the following hold:

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, i.e., for all* $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;

(3) $F(T_r) = \widetilde{F(T_r)} =$ *GMEP*(f, B, φ);

(4) *GMEP*(f, B, φ) *is closed and convex;*

(5) $\phi(p, T_r z) + \phi(T_r z, z) \leq \phi(p, z)$, $\forall p \in F(T_r)$ *and* $z \in E$.

4.2.1 A new modified block iterative algorithm for a system of generalized mixed equilibrium problems

In this section, we prove the new convergence theorems for finding the set of solutions of system of generalized mixed equilibrium problems, the common fixed point set of a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings, and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a *2*-uniformly convex and uniformly smooth Banach space.

Theorem 4.2.2. *Let* C *be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space* E. *For each* $j = 1, 2, \ldots, m$ *let* f_j *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies conditions (A1)–(A4),* $B_j : C \longrightarrow E^*$ *be a continuous and monotone mapping and* $\varphi_j : C \to \mathbb{R}$ *be a lower semicontinuous and convex function. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* E^* *satisfying* $\|Ay\| \leq \|Ay - Au\|$, $\quad \forall y \in C$ *and* $u \in$ *VI*$(A, C) \neq \emptyset$. *Let* $\{S_i\}_{i=1}^{\infty} : C \to C$ *be an infinite family of closed uniformly* L_i*-Lipschitz continuous and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *such that* $F := (\cap_{i=1}^{\infty} F(S_i)) \cap (\cap_{j=1}^{m}$*GMEP*$(f_j, B_j, \varphi_j))(\cap$ *VI*$(A, C))$ *is a nonempty and bounded subset in* C. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n v_n), \\ y_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J z_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{4.2.1}$$

where $\theta_n = \sup_{q \in F}(k_n - 1)\phi(q, x_n)$, *for each* $i \geq 0$, $\{\alpha_{n,i}\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$, $\{r_{j,n}\} \subset [d, \infty)$ *for some* $d > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with*

$0 < a < b < c^2\alpha/2$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity constant of* E. *If* $\sum_{i=0}^{\infty}\alpha_{n,i} = 1$ *for all* $n \geq 0$, $\liminf_{n\longrightarrow\infty}(1-\beta_n) > 0$ *and* $\liminf_{n\longrightarrow\infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Proof. We first show that C_{n+1} is closed and convex for each $n \geq 0$. Clearly, $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know $\phi(z, u_n) \leq \phi(z, x_n) + \theta_n$ is equivalent to $2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2 + \theta_n$. So, C_{n+1} is closed and convex.

Next, we show that $F \subset C_n$ for all $n \geq 0$. Since $u_n = \Omega_n^m y_n$, when $\Omega_n^j = T_{r_{j,n}}^{Q_j} T_{r_{j-1,n}}^{Q_{j-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1}$, $j = 1, 2, 3, \ldots, m$, $\Omega_n^0 = I$, by the convexity of $\|\cdot\|^2$, property of ϕ, and by uniformly quasi-ϕ-asymptotically nonexpansive of S_n for each $q \in F \subset C_n$, we have

$$
\begin{aligned}
\phi(q, u_n) &= \phi(q, \Omega_n^m y_n) \\
&\leq \phi(q, y_n) \\
&= \phi(q, J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n) \\
&= \|q\|^2 - 2\langle q, \beta_n Jx_n + (1-\beta_n)Jz_n\rangle + \|\beta_n Jx_n + (1-\beta_n)Jz_n\|^2 \\
&\leq \|q\|^2 - 2\beta_n\langle q, Jx_n\rangle - 2(1-\beta_n)\langle q, Jz_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|z_n\|^2 \\
&= \beta_n\phi(q, x_n) + (1-\beta_n)\phi(q, z_n) \qquad (4.2.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(q, z_n) &= \phi(q, J^{-1}(\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n)) \\
&= \|q\|^2 - 2\langle q, \alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\rangle + \|\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|^2 \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n v_n\rangle + \|\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n v_n\rangle + \alpha_{n,0}\|Jx_n\|^2 + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n v_n\|^2 \\
&\quad -\alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle + \alpha_{n,0}\|Jx_n\|^2 - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n v_n\rangle \\
&\quad + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n v_n\|^2 - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\
&= \alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\phi(q, S_i^n v_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\
&\leq \alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, v_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\|. \qquad (4.2.3)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\phi(q, v_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(q, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(q, Jx_n - \lambda_n Ax_n) \\
&\leq V(q, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, \lambda_n Ax_n\rangle \\
&= V(q, Jx_n) - 2\lambda_n\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n\rangle \\
&= \phi(q, x_n) - 2\lambda_n\langle x_n - q, Ax_n\rangle + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle. \qquad (4.2.4)
\end{aligned}
$$

Since $q \in \mathrm{VI}(A, C)$ and A is an α-inverse-strongly monotone mapping, we have

$$\begin{aligned} -2\lambda_n\langle x_n - q, Ax_n\rangle &= -2\lambda_n\langle x_n - q, Ax_n - Aq\rangle - 2\lambda_n\langle x_n - q, Aq\rangle \\ &\leq -2\lambda_n\langle x_n - q, Ax_n - Aq\rangle \\ &\leq -2\alpha\lambda_n\|Ax_n - Aq\|^2. \end{aligned} \tag{4.2.5}$$

From $\|Ax_n\| \leq \|Ax_n - Aq\|, \ \forall q \in \mathrm{VI}(A, C)$, we also have

$$\begin{aligned} 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\ &\leq \tfrac{4}{c^2}\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\ &= \tfrac{4}{c^2}\|Jx_n - \lambda_n Ax_n - Jx_n\|\|\lambda_n Ax_n\| \\ &= \tfrac{4}{c^2}\|\lambda_n Ax_n\|^2 \\ &= \tfrac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\ &\leq \tfrac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2. \end{aligned} \tag{4.2.6}$$

Substituting (4.2.80) and (4.2.81) into (4.2.3), we obtain

$$\begin{aligned} \phi(q, v_n) &\leq \phi(q, x_n) - 2\alpha\lambda_n\|Ax_n - Aq\|^2 + \tfrac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2 \\ &= \phi(q, x_n) + 2\lambda_n(\tfrac{2}{c^2}\lambda_n - \alpha)\|Ax_n - Aq\|^2 \\ &\leq \phi(q, x_n). \end{aligned} \tag{4.2.7}$$

Substituting (4.2.82) into (4.2.3), we also have

$$\begin{aligned} \phi(q, z_n) &\leq \alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \alpha_{n,0}k_n\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &= k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \phi(q, x_n) + \sup_{q\in F}(k_n - 1)\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &= \phi(q, x_n) + \theta_n - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \phi(q, x_n) + \theta_n. \end{aligned} \tag{4.2.8}$$

and substituting (4.2.83) into (4.2.79), we also have

$$\phi(q, u_n) \leq \phi(q, x_n) + \theta_n. \tag{4.2.9}$$

This shows that $q \in C_{n+1}$ implies that $F \subset C_{n+1}$ and hence, $F \subset C_n$ for all $n \geq 0$. This implies that the sequence $\{x_n\}$ is well defined. From definition of C_{n+1} that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0, \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{4.2.10}$$

Hence, we get

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n}x_0, x_0) \\ &\leq \phi(q, x_0) - \phi(q, x_n) \\ &\leq \phi(q, x_0), \quad \forall q \in F. \end{aligned} \tag{4.2.11}$$

From (4.2.85) and (4.2.86), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. In particular, by (3.2.3), the sequence $\{(\|x_n\|-\|x_0\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. Denote

$$M = \sup_{n\geq 0}\{\|x_n\|\} < \infty. \tag{4.2.12}$$

Moreover, by the definition of θ_n and (5.1.21), it follows that

$$\theta_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.2.13}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$, for $m > n$, we have

$$\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\
&\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty} \phi(x_n, x_0)$ exists and we take $m, n \to \infty$, we get $\phi(x_m, x_n) \to 0$.Then, we have $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Thus, $\{x_n\}$ is a Cauchy sequence, and by the completeness of E, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we claim that $\|Ju_n - Jx_n\| \to 0$, as $n \to \infty$. By definition of $x_n = \Pi_{C_n} x_0$, we have

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).
\end{aligned}$$

Since $\lim_{n\to\infty} \phi(x_n, x_0)$ exists, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.2.14}$$

Again we have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.2.15}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{4.2.16}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n.$$

By (4.2.13) and (4.2.89) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{4.2.17}$$

Again, we get

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{4.2.18}$$

Since

$$\begin{aligned} \|u_n - x_n\| &= \|u_n - x_{n+1} + x_{n+1} - x_n\| \\ &\leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \end{aligned}$$

It follows from (4.2.90) and (4.2.93) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.2.19}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{4.2.20}$$

Next, we will show that $p \in F := \cap_{j=1}^{m} \text{GMEP}(f_j, B_j, \varphi_j) \cap (\cap_{i=1}^{\infty} F(S_i)) \cap \text{VI}(A, C)$.

(a) We show that $p \in \cap_{i=1}^{\infty} F(S_i)$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, it follow from (4.2.83), we have

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + \theta_n,$$

by (4.2.13) and (4.2.89), we get

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0 \tag{4.2.21}$$

it follows that

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{4.2.22}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = 0. \tag{4.2.23}$$

From (4.2.78), we note that

$$\begin{aligned} \|Jx_{n+1} - Jz_n\| &= \|Jx_{n+1} - (\alpha_{n,0} Jx_n + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n v_n)\| \\ &= \|\alpha_{n,0} Jx_{n+1} - \alpha_{n,0} Jx_n + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i} Jx_{n+1} - \textstyle\sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n v_n\| \\ &= \|\alpha_{n,0}(Jx_{n+1} - Jx_n) + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n v_n)\| \\ &= \|\textstyle\sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n v_n) - \alpha_{n,0}(Jx_n - Jx_{n+1})\| \\ &\geq \textstyle\sum_{i=1}^{\infty} \alpha_{n,i} \|Jx_{n+1} - JS_i^n v_n\| - \alpha_{n,0} \|Jx_n - Jx_{n+1}\|, \end{aligned}$$

and hence

$$\|Jx_{n+1} - JS_i^n v_n\| \leq \tfrac{1}{\sum_{i=1}^{\infty} \alpha_{n,i}} (\|Jx_{n+1} - Jz_n\| + \alpha_{n,0} \|Jx_n - Jx_{n+1}\|). \tag{4.2.24}$$

From (4.2.91), (4.2.101) and $\liminf_{n\to\infty} \sum_{i=1}^{\infty} \alpha_{n,i} > 0$, we obtain that

$$\lim_{n\to\infty} \|Jx_{n+1} - JS_i^n v_n\| = 0. \tag{4.2.25}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - S_i^n v_n\| = 0. \tag{4.2.26}$$

Using the triangle inequality that

$$\begin{aligned} \|x_n - S_i^n v_n\| &= \|x_n - x_{n+1} + x_{n+1} - S_i^n v_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_i^n v_n\|. \end{aligned}$$

From (4.2.90) and (4.2.104), we have

$$\lim_{n\to\infty} \|x_n - S_i^n v_n\| = 0. \tag{4.2.27}$$

On the other hand, we note that

$$\begin{aligned} \phi(q, x_n) - \phi(q, u_n) + \theta_n &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle + \theta_n \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|q\|\|Jx_n - Ju_n\| + \theta_n. \end{aligned}$$

It follows from $\theta_n \longrightarrow 0$, $\|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$, that

$$\phi(q, x_n) - \phi(q, u_n) + \theta_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.2.28}$$

From (4.2.79), (4.2.3) and (4.2.82) that

$$\begin{aligned} \phi(q, u_n) &\leq \phi(q, y_n) \\ &\leq \beta_n\phi(q, x_n) + (1-\beta_n)\phi(q, z_n) \\ &\leq \beta_n\phi(q, x_n) + (1-\beta_n)[\alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, v_n) \\ &\quad -\alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\|] \\ &= \beta_n\phi(q, x_n) + (1-\beta_n)\alpha_{n,0}\phi(q, x_n) + (1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, v_n) \\ &\quad -(1-\beta_n)\alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \beta_n\phi(q, x_n) + (1-\beta_n)\alpha_{n,0}\phi(q, x_n) + (1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, v_n) \\ &\leq \beta_n\phi(q, x_n) + (1-\beta_n)\alpha_{n,0}\phi(q, x_n) \\ &\quad +(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n[\phi(q, x_n) - 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2] \\ &\leq \beta_n\phi(q, x_n) + (1-\beta_n)\alpha_{n,0}k_n\phi(q, x_n) \\ &\quad +(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, x_n) \\ &\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\ &\leq \beta_n k_n\phi(q, x_n) + (1-\beta_n)k_n\phi(q, x_n) \\ &\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\ &= k_n\phi(q, x_n) - (1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2] \\ &\leq \phi(q, x_n) + \sup_{q\in F}(k_n - 1)\phi(q, x_n) \\ &\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\ &= \phi(q, x_n) + \theta_n - (1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2, \end{aligned}$$

and hence

$$\begin{aligned} 2a(\alpha - \tfrac{2b}{c^2})\|Ax_n - Aq\|^2 &\leq 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\ &\leq \tfrac{1}{(1-\beta_n)\sum_{i=1}^{\infty}\alpha_{n,i}k_n}(\phi(q, x_n) - \phi(q, u_n) + \theta_n). \end{aligned} \tag{4.2.29}$$

From (4.2.28), $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $\liminf_{n\longrightarrow\infty}(1 - \beta_n) > 0$ and $\liminf_{n\longrightarrow\infty} \alpha_{n,0}\alpha_{n,i} > 0$, for $i \geq 0$ and $k_n \longrightarrow 1$ as $n \longrightarrow \infty$, we obtain that

$$\lim_{n\to\infty} \|Ax_n - Aq\| = 0. \tag{4.2.30}$$

From (4.2.81), we compute

$$\begin{aligned} \phi(x_n, v_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &\leq \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &= V(x_n, Jx_n - \lambda_n Ax_n) \\ &\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\ &= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle \\ &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle \\ &\leq \tfrac{4\lambda_n^2}{c^2}\|Ax_n - Aq\|^2 \\ &\leq \tfrac{4b^2}{c^2}\|Ax_n - Aq\|^2. \end{aligned}$$

From (4.2.108) that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0 \tag{4.2.31}$$

and we also obtain

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = 0. \tag{4.2.32}$$

Since S_i^n is continuous, for any $i \geq 1$

$$\lim_{n\to\infty} \|S_i^n x_n - S_i^n v_n\| = 0. \tag{4.2.33}$$

Again by the triangle inequality, we get

$$\|x_n - S_i^n x_n\| \leq \|x_n - S_i^n v_n\| + \|S_i^n v_n - S_i^n x_n\|.$$

From (4.2.105) and (4.2.111), we have

$$\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0, \quad \forall i \geq 1. \tag{4.2.34}$$

By using triangle inequality, we get

$$\|S_i^n x_n - p\| \leq \|S_i^n x_n - x_n\| + \|x_n - p\|, \quad \forall i \geq 1.$$

We know that $x_n \to p$ as $n \to \infty$ and from (4.2.112)

$$S_i^n x_n \longrightarrow p \quad \text{for each } i \geq 1.$$

Moreover, by the assumption that $\forall i \geq 1$, S_i is uniformly L_i-Lipschitz continuous, and hence we have.

$$\begin{aligned} \|S_i^{n+1}x_n - S_i^n x_n\| &\leq \|S_i^{n+1}x_n - S_i^{n+1}x_{n+1}\| + \|S_i^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - S_i^n x_n\| \\ &\leq (L_i + 1)\|x_{n+1} - x_n\| + \|S_i^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - S_i^n x_n\|. \end{aligned} \tag{4.2.35}$$

By (4.2.90) and (4.2.112), it yields that $\|S_i^{n+1}x_n - S_i^n x_n\| \to 0$. From $S_i^n x_n \longrightarrow p$, we have $S_i^{n+1}x_n \to p$, that is $S_i S_i^n x_n \to p$. In view of closeness of S_i, we have $S_i p = p$, for all $i \geq 1$. This implies that $p \in \cap_{i=1}^{\infty} F(S_i)$.

(b) We show that $p \in \cap_{j=1}^{m}$GMEP(f_j, B_j, φ_j).

Let $u_n = \Omega_n^m y_n$, when $\Omega_n^j = T_{r_{j,n}}^{Q_j} T_{r_{j-1,n}}^{Q_{j-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1}$, $j = 1, 2, 3, \ldots, m$ and $\Omega_n^0 = I$, we obtain

$$\begin{aligned} \phi(q, u_n) &= \phi(q, \Omega_n^m y_n) \\ &\leq \phi(q, \Omega_n^{m-1} y_n) \\ &\leq \phi(q, \Omega_n^{m-2} y_n) \\ &\vdots \\ &\leq \phi(q, \Omega_n^j y_n). \end{aligned} \tag{4.2.36}$$

By Lemma (4.2.1)(5), we have for $j = 1, 2, 3, \ldots, m$

$$\begin{aligned} \phi(\Omega_n^j y_n, y_n) + \theta_n &\leq \phi(q, y_n) - \phi(q, \Omega_n^j y_n) + \theta_n \\ &\leq \phi(q, x_n) - \phi(q, \Omega_n^j y_n) + \theta_n \\ &\leq \phi(q, x_n) - \phi(q, u_n) + \theta_n. \end{aligned} \tag{4.2.37}$$

From (4.2.13) and (4.2.28), we get $\phi(\Omega_n^j y_n, y_n) \to 0$ as $n \to \infty$, for $j = 1, 2, 3, \ldots, m$ and implies that

$$\lim_{n\to\infty} \|\Omega_n^j y_n - y_n\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{4.2.38}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, it follows from (4.2.79) and (4.2.83) that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \theta_n.$$

By (4.2.13) and (4.2.89), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

Applying previous Lemma that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{4.2.39}$$

Using the triangle inequality, we obtain

$$\|x_n - y_n\| \ \leq \ \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

From (4.2.90) and (4.2.39), we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.2.40}$$

Since $x_n \to p$ and $\|x_n - y_n\| \to 0$, we have $y_n \to p \ \ as \ \ n \to \infty$.
Again by using the triangle inequality, we have for $j = 1, 2, 3, \ldots, m$

$$\|p - \Omega_n^j y_n\| \ \leq \ \|p - y_n\| + \|y_n - \Omega_n^j y_n\|.$$

From (4.2.38) and $y_n \to p \ \ as \ \ n \to \infty$, we get

$$\lim_{n\to\infty} \|p - \Omega_n^j y_n\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{4.2.41}$$

By using the triangle inequality, we obtain

$$\|\Omega_n^j y_n - \Omega_n^{j-1} y_n\| \ \leq \ \|\Omega_n^j y_n - p\| + \|p - \Omega_n^{j-1} y_n\|.$$

From (4.2.41), we have

$$\lim_{n\to\infty} \|\Omega_n^j y_n - \Omega_n^{j-1} y_n\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{4.2.42}$$

Since $\{r_{j,n}\} \subset [d, \infty)$ and J is uniformly continuous on any bounded subset of E,

$$\lim_{n\to\infty} \tfrac{\|J\Omega_n^j y_n - J\Omega_n^{j-1} y_n\|}{r_{j,n}} = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{4.2.43}$$

From Lemma 4.2.1, we get for $j = 1, 2, 3, \ldots, m$

$$Q_j(\Omega_n^j y_n, y) + \frac{1}{r_{j,n}} \langle y - \Omega_n^j y_n, J\Omega_n^j y_n - J\Omega_n^{j-1} y_n \rangle \geq 0, \quad \forall y \in C.$$

From (A2),

$$\frac{1}{r_{j,n}} \langle y - \Omega_n^j y_n, J\Omega_n^j y_n - J\Omega_n^{j-1} y_n \rangle \geq Q_j(y, \Omega_n^j y_n), \quad \forall y \in C, \ \forall j = 1, 2, 3, \ldots, m.$$

From (4.2.41) and (4.2.43), we have

$$0 \geq Q_j(y, p), \quad \forall y \in C, \ \forall j = 1, 2, 3, \ldots, m. \tag{4.2.44}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)p$. Then, we get that $y_t \in C$. From (4.2.44), and it follows that

$$Q_j(y_t, p) \leq 0, \quad \forall y \in C, \ \forall j = 1, 2, 3, \ldots, m. \tag{4.2.45}$$

By the conditions (A1) and (A4), we have for $j = 1, 2, 3, \ldots, m$

$$\begin{aligned} 0 &= Q_j(y_t, y_t) \\ &\leq tQ_j(y_t, y) + (1-t)Q_j(y_t, p) \\ &\leq tQ_j(y_t, y) \\ &= Q_j(y_t, y). \end{aligned} \tag{4.2.46}$$

From (A3) and letting $t \longrightarrow 0$, This implies that $p \in \text{GMEP}(f_j, B_j, \varphi_j)$, $\forall j = 1, 2, 3, \ldots, m$. Therefore $p \in \cap_{j=1}^{m} \text{GMEP}(f_j, B_j, \varphi_j)$.

(c) We show that $p \in \text{VI}(A, C)$. Indeed, define $U \subset E \times E^*$ by

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{4.2.47}$$

Since U is maximal monotone and $U^{-1}0 = \text{VI}(A, C)$. Let $(v, w) \in G(U)$. Since $w \in Uv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.
From $v_n \in C$, we have

$$\langle v - v_n, w - Av \rangle \geq 0. \tag{4.2.48}$$

On the other hand, since $v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, we have

$$\langle v - v_n, Jv_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

and thus

$$\left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{4.2.49}$$

It follows from (4.2.117), (4.2.118) and A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous that

$$\begin{aligned} \langle v - v_n, w \rangle &\geq \langle v - v_n, Av \rangle \\ &\geq \langle v - v_n, Av \rangle + \left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \right\rangle \\ &= \langle v - v_n, Av - Ax_n \rangle + \left\langle v - zv_n, \frac{Jx_n - Jv_n}{\lambda_n} \right\rangle \\ &= \langle v - v_n, Av - Av_n \rangle + \langle v - v_n, Av_n - Ax_n \rangle + \left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} \right\rangle \\ &\geq -\|v - v_n\| \frac{\|v_n - x_n\|}{\alpha} - \|v - v_n\| \frac{\|Jx_n - Jv_n\|}{a} \\ &\geq -H \left(\frac{\|v_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jv_n\|}{a} \right), \end{aligned}$$

where $H = \sup_{n \geq 1} \|v - v_n\|$. Take the limit as $n \to \infty$, (4.2.109) and (4.2.110), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of B we have $p \in B^{-1}0$, that is $p \in \text{VI}(A, C)$. Hence, from (a), (b) and (c), we obtain $p \in F$.

Finally, we show that $p = \Pi_F x_0$. From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0$, $\forall z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - y \rangle \geq 0, \quad \forall y \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_0 - Jp, p - y \rangle \geq 0, \quad \forall y \in F.$$

We can conclude that $p = \Pi_F x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

4.2.2 A modified hybrid projection method for solving generalized mixed equilibrium problems

In this section, we prove the new convergence theorem for solving the set of solutions of a generalized mixed equilibrium problems and the common fixed point set of a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space E with Kadec-Klee property.

Theorem 4.2.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with Kadec-Klee property. Let $B : C \longrightarrow E^*$ be a continuous and monotone mapping and let $\varphi : C \to \mathbb{R}$ be a convex and lower semi-continuous. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let B be a continuous monotone mapping of C into E^*. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := \cap_{i=1}^{\infty} F(S_i) \cap GMEP(f, B, \varphi)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) Jz_n), \\ u_n \in C \text{ such that } u_n = K_{r_n} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{4.2.50}$$

where $\zeta_n = \sup_{q \in F}(k_n - 1)\phi(q, x_n)$, $\{\alpha_{n,i}\}, \{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$ and $\liminf_{n \longrightarrow \infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

***Proof*.** We first show that C_{n+1} is closed and convex for each $n \geq 0$. Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we known

$$\phi(z, u_n) \leq \phi(z, x_n) + \zeta_n \Leftrightarrow 2\langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + \zeta_n.$$

So, C_{n+1} is closed and convex. Therefore $\Pi_F x_0$ and $\Pi_{C_n} x_0$ are well defined.

Next, we show that $F \subset C_n$ for all $n \geq 0$. Indeed, since $u_n = K_{r_n}y_n$ for all $n \geq 0$. It is clear that $F \subset C_1 = C$. Suppose $F \subset C_n$ for $n \in \mathbb{N}$, by the convexity of $\|\cdot\|^2$, property of ϕ, and uniformly quasi-ϕ-asymptotically nonexpansive of S_n for each $q \in F \subset C_n$, we observe that

$$\begin{aligned}
\phi(q, u_n) &= \phi(q, K_{r_n}y_n) \\
&\leq \phi(q, y_n) \\
&= \phi(q, J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n)) \\
&= \|q\|^2 - 2\langle q, \beta_n Jx_n + (1-\beta_n)Jz_n\rangle + \|\beta_n Jx_n + (1-\beta_n)Jz_n\|^2 \\
&\leq \|q\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle q, Jz_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|z_n\|^2 \\
&= \beta_n\phi(q, x_n) + (1-\beta_n)\phi(q, z_n) \qquad (4.2.51)
\end{aligned}$$

and

$$\begin{aligned}
\phi(q, z_n) &= \phi(q, J^{-1}(\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n)) \\
&= \|q\|^2 - 2\langle q, \alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\rangle + \|\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n x_n\rangle + \|\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n x_n\rangle + \alpha_{n,0}\|Jx_n\|^2 + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n x_n\|^2 \\
&\quad -\alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle + \alpha_{n,0}\|Jx_n\|^2 - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n x_n\rangle \\
&\quad + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&= \alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\phi(q, S_i^n x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&\leq \alpha_{n,0}k_n\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&\leq k_n\phi(q, x_n).
\end{aligned}$$

$$(4.2.52)$$

Substituting (4.2.52) into (4.2.79), we get

$$\begin{aligned}
\phi(q, u_n) &\leq \beta_n\phi(q, x_n) + (1-\beta_n)\phi(q, z_n) \\
&\leq \beta_n\phi(q, x_n) + (1-\beta_n)k_n\phi(q, x_n) \\
&\leq \beta_n\phi(q, x_n) + (1-\beta_n)[\phi(q, x_n) + \sup_{q\in F}(k_n - 1)\phi(q, x_n)] \\
&\leq \phi(q, x_n) + (1-\beta_n)\sup_{q\in F}(k_n - 1)\phi(q, x_n) \\
&\leq \phi(q, x_n) + \zeta_n. \qquad (4.2.53)
\end{aligned}$$

This show that $q \in C_{n+1}$ implies that $F \subset C_{n+1}$ and hence, $F \subset C_n$ for all $n \geq 0$. Since F is nonempty, C_n is a nonempty closed convex subset of E and hence Π_{C_n} exist for all $n \geq 0$. This implies that the sequence $\{x_n\}$ is well defined.

From definition of C_{n+1} that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \qquad (4.2.54)$$

We note that

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0)\\ &\leq \phi(p, x_0) - \phi(p, x_n)\\ &\leq \phi(p, x_0), \quad \forall p \in F \subset C_n, \ \forall n \geq 0.\end{aligned} \tag{4.2.55}$$

From (4.2.85) and (4.2.86), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. In particular, by (3.2.3), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. Denote

$$K = \sup_{n\geq 0}\{\|x_n\|\} < \infty. \tag{4.2.56}$$

Moreover, by the definition of $\{\zeta_n\}$ and (5.1.21), it follows that

$$\zeta_n \longrightarrow 0, \ n \longrightarrow \infty. \tag{4.2.57}$$

Since

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \phi(x_n, x_0) &= \liminf_{n\longrightarrow\infty}\{\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\}\\ &\geq \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 = \phi(p, x_0),\end{aligned}$$

it follows that

$$\phi(p, x_0) \leq \liminf_{n\longrightarrow\infty} \phi(x_n, x_0) \leq \limsup_{n\longrightarrow\infty} \phi(x_n, x_0) \leq \phi(p, x_0).$$

This implies that $\lim_{n\longrightarrow\infty} \phi(x_n, x_0) = \phi(p, x_0)$. Hence, we get $\|x_n\| \longrightarrow \|p\|$ as $n \longrightarrow \infty$. In view of the Kadec-Klee property of E, we obtain that

$$\lim_{n\longrightarrow\infty} x_n = p.$$

Now, we claim that $\|Ju_n - Jx_n\| \to 0$, as $n \to \infty$. By definition of $\Pi_{C_n} x_0$, one has

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0)\\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).\end{aligned}$$

From the $\lim_{n\to\infty} \phi(x_n, x_0)$ exists, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.2.58}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \zeta_n.$$

By (4.2.57) and (4.2.89), we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{4.2.59}$$

From (3.2.3), we see that

$$\|u_n\| \longrightarrow \|p\|, \text{ as } n \longrightarrow \infty. \tag{4.2.60}$$

It follows that

$$\|Ju_n\| \longrightarrow \|Jp\|, \text{ as } n \longrightarrow \infty. \tag{4.2.61}$$

This implies that $\{\|Ju_n\|\}$ is bounded in E^*. Note that E is reflexive and E^* is also reflexive, we can assume that $Ju_n \rightharpoonup x^* \in E^*$. In view of the reflexive of E, wee see that $J(E) = E^*$. Hence there exist $x \in E$ such that $Jx = x^*$. It follows that

$$\begin{aligned} \phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2. \end{aligned}$$

Taking $\liminf_{n\longrightarrow\infty}$ on the both sides of equation above and in view of the weak lower semicontinuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, x^*\rangle + \|x^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 \\ &= \phi(p, x). \end{aligned}$$

Thai is, $p = x$, which implies that $x^* = Jp$. It follows that $Ju_n \rightharpoonup Jq \in E^*$. Since (3.2.3) and the Kadec-Klee property of E that

$$\lim_{n\longrightarrow\infty} u_n = p. \tag{4.2.62}$$

Since $\|x_n - u_n\| \leq \|x_n - p\| + \|p - u_n\|$. It follows that

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \tag{4.2.63}$$

From J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{4.2.64}$$

Next, we will show that $p \in F := GMEP(f, B, \varphi) \cap (\cap_{i=1}^{\infty} F(S_i))$.

(a) First, we show that $p \in GMEP(f, B, \varphi)$. It follows from (4.2.79) and (4.2.52), that $\phi(p, y_n) \leq \phi(p, x_n) + \zeta_n$. By (4.2.57) and $u_n = K_{r_n} y_n$, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\ &\leq \phi(p, y_n) - \phi(p, K_{r_n} y_n) \\ &\leq \phi(p, x_n) - \phi(p, K_{r_n} y_n) + \zeta_n \\ &= \phi(p, x_n) - \phi(p, u_n) + \zeta_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{4.2.65}$$

From (3.2.3), we see that

$$\|u_n\| \longrightarrow \|y_n\|, \text{ as } n \longrightarrow \infty. \tag{4.2.66}$$

In view of $u_n \longrightarrow p$ as $n \longrightarrow \infty$, wee see that

$$\|y_n\| \longrightarrow \|p\|, \text{ as } n \longrightarrow \infty. \tag{4.2.67}$$

It follows that

$$\|Ju_n\| \longrightarrow \|Jp\|, \text{ as } n \longrightarrow \infty. \tag{4.2.68}$$

Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup z^* \in E^*$. In view of the reflexive of E, wee see that $J(E) = E^*$. Hence there exist $z \in E$ such that $Jz = z^*$. It follows that

$$\begin{aligned} \phi(u_n, y_n) &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|y_n\|^2 \\ &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|Jy_n\|^2. \end{aligned}$$

Taking $\liminf_{n\longrightarrow\infty}$ on the both sides of equality above yields that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, z^*\rangle + \|z^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jz\rangle + \|Jz\|^2 \\ &= \|p\|^2 - 2\langle p, Jz\rangle + \|z\|^2 \\ &= \phi(p, x). \end{aligned}$$

Thai is, $p = z$, which implies that $z^* = Jp$. It follows that $Jy_n \rightharpoonup Jq \in E^*$. Since (3.2.3) and the Kadec-Klee property of E that

$$Jy_n - Jp \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.2.69}$$

Since J^{-1} is norm-weak*-continuous. It follows that $y_n \rightharpoonup p$. Since (4.2.67) and E enjoys the KKadec-Klee property, we obtain that

$$y_n \longrightarrow p \text{ as } n \longrightarrow \infty. \tag{4.2.70}$$

It follows by (4.2.62) and (4.2.70), that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{4.2.71}$$

Since J is uniformly norm-to-norm continuous, we get

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{4.2.72}$$

From (A2), that

$$\varphi(y) - \varphi(u_n) + \langle By_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C,$$

and hence

$$\varphi(y)-\varphi(u_n)+\langle By_n, y-u_n\rangle+\langle y-u_n, \tfrac{Ju_n-Jy_n}{r_n}\rangle \geq f(y,u_n), \quad \forall y\in C. \quad (4.2.73)$$

For t with $0<t<1$ and $y\in C$, let $y_t=ty+(1-t)p$. Then $y_t\in C$ and hence

$$0\geq -\varphi(y_t)+\varphi(u_n)-\langle By_n, y_t-u_n\rangle-\langle y_t-u_n, \tfrac{Ju_n-Jy_n}{r_n}\rangle+f(y_t,u_n), \quad \forall y\in C.$$

It follows that

$$\begin{aligned}\langle By_t, y_t-u_n\rangle &\geq \langle By_t, y_t-u_n\rangle-\varphi(y_t)+\varphi(u_n)-\langle By_n, y_t-u_n\rangle-\langle y_t-u_n, \tfrac{Ju_n-Jy_n}{r_n}\rangle\\ &\quad +f(y_t,u_n), \quad \forall y_t\in C\\ &= \langle By_t, y_t-u_n\rangle-\varphi(y_t)+\varphi(u_n)-\langle Bu_n, y_t-u_n\rangle+\langle Bu_n, y_t-u_n\rangle\\ &\quad -\langle By_n, y_t-u_n\rangle-\langle y_t-u_n, \tfrac{Ju_n-Jy_n}{r_n}\rangle+f(y_t,u_n), \quad \forall y_t\in C\\ &= \langle By_t-Bu_n, y_t-u_n\rangle-\varphi(y_t)+\varphi(u_n)\\ &\quad +\langle Bu_n-By_n, y_t-u_n\rangle-\langle y_t-u_n, \tfrac{Ju_n-Jy_n}{r_n}\rangle+f(y_t,u_n), \quad \forall y_t\in C.\end{aligned}$$

By (4.2.97), we get $u_n\to p$ and $y_n\to p$ as $n\to\infty$. By the continuity of B, we obtain that $Bu_n-By_n\to 0$ as $n\to\infty$. From $r_n>0$ then $\frac{\|Ju_n-Jy_n\|}{r_n}\to 0$ as $n\to\infty$. Since B is monotone, we know that $\langle By_t-Bu_n, y_t-u_n\rangle\geq 0$. Thus, it follows from (A4) that

$$\begin{aligned}f(y_t,p)-\varphi(y_t)+\varphi(p) &\leq \liminf_{n\to\infty} f(y_t,u_n)-\varphi(y_t)+\varphi(u_n)\\ &\leq \lim_{n\to\infty}\langle By_t, y_t-u_n\rangle\\ &= \langle By_t, y_t-p\rangle.\end{aligned}$$

From the conditions (A1) and (A4), we obtain

$$\begin{aligned}0 &= f(y_t,y_t)+\varphi(y_t)-\varphi(y_t)\\ &\leq tf(y_t,y)+(1-t)f(y_t,p)+t\varphi(y)+(1-t)\varphi(p)-\varphi(y_t)\\ &= tf(y_t,y)+t\varphi(y)-t\varphi(y_t)+(1-t)f(y_t,p)+(1-t)\varphi(p)-(1-t)\varphi(y_t)\\ &\leq t(f(y_t,y)+\varphi(y)-\varphi(y_t))+(1-t)(\langle By_t, y_t-p\rangle\\ &\leq t(f(y_t,y)+\varphi(y)-\varphi(y_t))+(1-t)t\langle By_t, y-p\rangle\end{aligned}$$

dividing by t, we get

$$0\leq f(y_t,y)+\varphi(y)-\varphi(y_t)+(1-t)\langle By_t, y-p\rangle.$$

Letting $t\to 0$, we have

$$0\leq f(p,y)+\varphi(y)-\varphi(p)+\langle Bp, y-p\rangle, \quad \forall y\in C.$$

This implies that $p\in GMEP(f,B,\varphi)$.

(b) We show that $p\in\cap_{i=1}^{\infty}F(S_i)$. For any $j\geq 1$ and any $q\in F$, it follows from (4.2.79), (4.2.52) and (4.2.3) that

$$\alpha_{n,0}\alpha_{n,j}g(\|Jx_n-JS_j^n x_n\|)\leq \phi(q,x_n)-\phi(q,u_n)+\zeta_n\longrightarrow 0, \ n\longrightarrow\infty.$$

From the condition $\liminf_{n\longrightarrow\infty}\alpha_{n,0}\alpha_{n,i}>0$, we see that

$$g(\|Jx_n-JS_j^nx_n\|)\longrightarrow 0,\ n\longrightarrow\infty.$$

It follows from the property of g that

$$\lim_{n\to\infty}\|Jx_n-JS_i^nx_n\|=0,\quad \forall i\geq 1. \tag{4.2.74}$$

Since $x_n\longrightarrow p$ and J is uniformly continuous, it yields $Jx_n\longrightarrow Jp$. Thus from (4.2.113), we have

$$JS_i^nx_n\longrightarrow Jp,\quad \forall i\geq 1. \tag{4.2.75}$$

Since $J^{-1}:E^*\longrightarrow E$ is norm-weak*-continuous, we also have

$$S_i^nx_n\rightharpoonup p,\forall i\geq 1. \tag{4.2.76}$$

On the other hand, for each $i\geq 1$, we observe that

$$|\|S_i^nx_n\|-\|p\||=|\|J(S_i^nx_n)\|-\|Jp\||\leq ||J(S_i^nx_n)-Jp||.$$

In view of (4.2.114), we obtain $\|S_i^nx_n\|\longrightarrow\|p\|$ for each $i\geq 1$. Since E has the Kadee-Klee property, we get

$$S_i^nx_n\longrightarrow p\quad\text{for each}\quad i\geq 1.$$

By the assumption that for each $i\geq 1$, S_i is uniformly L_i-Lipschitz continuous, so we have

$$\begin{aligned}\|S_i^{n+1}x_n-S_i^nx_n\| &\leq \|S_i^{n+1}x_n-S_i^{n+1}x_{n+1}\|+\|S_i^{n+1}x_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\\ &\quad+\|x_n-S_i^nx_n\|\\ &\leq (L_i+1)\|x_{n+1}-x_n\|+\|S_i^{n+1}x_{n+1}-x_{n+1}\|+\|x_n-S_i^nx_n\|.\end{aligned} \tag{4.2.77}$$

By (4.2.62) and (4.2.115), it yields that $\|S_i^{n+1}x_n-S_i^nx_n\|\to 0$. From $S_i^nx_n\longrightarrow p$, we get $S_i^{n+1}x_n\to p$, that is $S_iS_i^nx_n\to p$. In view of closeness of S_i, we have $S_ip=p$, for all $i\geq 1$. This imply that $p\in\cap_{i=1}^{\infty}F(S_i)$.

Finally, we show that $x_n\longrightarrow p=\Pi_Fx_0$. Let $q=\Pi_Fx_0$. From $x_n=\Pi_{C_n}x_0$ and $q\in F\subset C_n$, we have

$$\phi(x_n,x_0)\leq\phi(q,x_0),\quad \forall n\geq 0.$$

This implies that

$$\phi(p,x_0)=\lim_{n\longrightarrow\infty}\phi(x_n,x_0)\leq\phi(q,x_0).$$

By definition of $p=\Pi_Fx_0$, we have $p=q$. Therefore $x_n\longrightarrow p=\Pi_Fx_0$. This completes the proof. □

4.2.3 Convergence theorems for mixed equilibrium problems and variational inequality problems

In this section, we prove the new convergence theorems for finding the set of solutions of a mixed equilibrium problem, the common fixed point set of a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings, and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a *2*-uniformly convex and uniformly smooth Banach space.

Theorem 4.2.4. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := \cap_{i=1}^{\infty} F(S_i) \cap MEP(f, \varphi) \cap VI(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n v_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n), \\ f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, z_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{4.2.78}$$

where $\theta_n = \sup_{q \in F}(k_n - 1)\phi(q, x_n)$, for each $i \geq 0$, $\{\alpha_{n,i}\}, \{\beta_n\}$ are sequences in $[0, 1]$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_{n\longrightarrow\infty}(1 - \beta_n) > 0$ and $\liminf_{n\longrightarrow\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

***Proof*.** We first show that C_{n+1} is closed and convex for each $n \geq 0$. Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we known $\phi(z, u_n) \leq \phi(z, x_n) + \theta_n$ is equivalent to $2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2 + \theta_n$. So, C_{n+1} is closed and convex.

Next, we show that $F \subset C_n$ for all $n \geq 0$. Indeed, put $u_n = T_{r_n} y_n$ for all $n \geq 0$. On the other hand, one has T_{r_n} is relatively quasi-nonexpansive mappings and $F \subset C_1 = C$. Suppose $F \subset C_n$ for $n \in \mathbb{N}$, by the convexity of $\|\cdot\|^2$, property of ϕ and by uniformly quasi-ϕ-asymptotically nonexpansive of S_n for each $q \in F \subset C_n$, we have

$$\begin{aligned}
\phi(q,u_n) &= \phi(q,T_{r_n}y_n)\\
&\leq \phi(q,y_n)\\
&= \phi(q,J^{-1}(\beta_n Jx_n+(1-\beta_n)Jz_n)\\
&= \|q\|^2-2\langle q,\beta_n Jx_n+(1-\beta_n)Jz_n\rangle+\|\beta_n Jx_n+(1-\beta_n)Jz_n\|^2\\
&\leq \|q\|^2-2\beta_n\langle q,Jx_n\rangle-2(1-\beta_n)\langle q,Jz_n\rangle+\beta_n\|x_n\|^2+(1-\beta_n)\|z_n\|^2\\
&= \beta_n\phi(q,x_n)+(1-\beta_n)\phi(q,z_n) \qquad (4.2.79)
\end{aligned}$$

and

$$\begin{aligned}
\phi(q,z_n) &= \phi(q,J^{-1}(\alpha_{n,0}Jx_n+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n))\\
&= \|q\|^2-2\langle q,\alpha_{n,0}Jx_n+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\rangle+\|\alpha_{n,0}Jx_n+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|^2\\
&= \|q\|^2-2\alpha_{n,0}\langle q,Jx_n\rangle-2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q,JS_i^n v_n\rangle+\|\alpha_{n,0}Jx_n+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|^2\\
&\leq \|q\|^2-2\alpha_{n,0}\langle q,Jx_n\rangle-2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q,JS_i^n v_n\rangle+\alpha_{n,0}\|Jx_n\|^2+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n v_n\|^2\\
&\quad -\alpha_{n,0}\alpha_{n,j}g\|Jv_n-JS_j^n v_n\|\\
&= \|q\|^2-2\alpha_{n,0}\langle q,Jx_n\rangle+\alpha_{n,0}\|Jx_n\|^2-2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q,JS_i^n v_n\rangle\\
&\quad +\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n v_n\|^2-\alpha_{n,0}\alpha_{n,j}g\|Jv_n-JS_j^n v_n\|\\
&= \alpha_{n,0}\phi(q,x_n)+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\phi(q,S_i^n v_n)-\alpha_{n,0}\alpha_{n,j}g\|Jv_n-JS_j^n v_n\|\\
&\leq \alpha_{n,0}\phi(q,x_n)+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,v_n)-\alpha_{n,0}\alpha_{n,j}g\|Jv_n-JS_j^n v_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(q,v_n) &= \phi(q,\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n))\\
&\leq \phi(q,J^{-1}(Jx_n-\lambda_n Ax_n))\\
&= V(q,Jx_n-\lambda_n Ax_n)\\
&\leq V(q,(Jx_n-\lambda_n Ax_n)+\lambda_n Ax_n)-2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-q,\lambda_n Ax_n\rangle\\
&= V(q,Jx_n)-2\lambda_n\langle J^{-1}(Jx_n-\lambda_n Ax_n)-q,Ax_n\rangle\\
&= \phi(q,x_n)-2\lambda_n\langle x_n-q,Ax_n\rangle+2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\rangle.
\end{aligned}$$

Since $q \in VI(A,C)$ and A is an α-inverse-strongly monotone mapping, we have

$$\begin{aligned}
-2\lambda_n\langle x_n-q,Ax_n\rangle &= -2\lambda_n\langle x_n-q,Ax_n-Aq\rangle-2\lambda_n\langle x_n-q,Aq\rangle\\
&\leq -2\lambda_n\langle x_n-q,Ax_n-Aq\rangle \qquad (4.2.80)\\
&\leq -2\alpha\lambda_n\|Ax_n-Aq\|^2.
\end{aligned}$$

From $\|Ax_n\| \leq \|Ax_n - Aq\|,\ \forall q \in VI(A, C)$, we also have

$$\begin{aligned} 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\ &\leq \tfrac{4}{c^2}\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\ &= \tfrac{4}{c^2}\|Jx_n - \lambda_n Ax_k - Jx_n\|\|\lambda_n Ax_n\| \\ &= \tfrac{4}{c^2}\|\lambda_n Ax_n\|^2 \\ &= \tfrac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\ &\leq \tfrac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2. \end{aligned} \tag{4.2.81}$$

Substituting (4.2.80) and (4.2.81) into (4.2.3), we obtain

$$\begin{aligned} \phi(q, v_n) &\leq \phi(q, x_n) - 2\alpha\lambda_n\|Ax_n - Aq\|^2 + \tfrac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2 \\ &= \phi(q, x_n) + 2\lambda_n(\tfrac{2}{c^2}\lambda_n - \alpha)\|Ax_n - Aq\|^2 \\ &\leq \phi(q, x_n). \end{aligned} \tag{4.2.82}$$

Substituting (4.2.82) into (4.2.3), we also have

$$\begin{aligned} \phi(q, z_n) &\leq \alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \alpha_{n,0}k_n\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &= k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \phi(q, x_n) + \sup_{q\in F}(k_n - 1)\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &= \phi(q, x_n) + \theta_n - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \phi(q, x_n) + \theta_n. \end{aligned} \tag{4.2.83}$$

and substituting (4.2.83) into (4.2.79), we also have

$$\phi(q, u_n) \leq \phi(q, x_n) + \theta_n. \tag{4.2.84}$$

This show that $q \in C_{n+1}$ implies that $F \subset C_{n+1}$ and hence, $F \subset C_n$ for all $n \geq 0$. This implies that the sequence $\{x_n\}$ is well defined. Since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \subset C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{4.2.85}$$

Then, we get

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n}x_0, x_0) \\ &\leq \phi(q, x_0) - \phi(q, x_n) \\ &\leq \phi(q, x_0), \quad \forall q \in F. \end{aligned} \tag{4.2.86}$$

From (4.2.85) and (4.2.86), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. In particular, by (3.2.3), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies $\{x_n\}, \{v_n\}, \{u_n\}, \{y_n\}$ and $\{z_n\}$ are also bounded. Denote

$$M = \sup_{n\geq 0}\{\|x_n\|\} < \infty. \tag{4.2.87}$$

Moreover, by the definition of $\{\theta_n\}$ and (5.1.21), it follows that

$$\theta_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.2.88}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$, for $m > n$, we have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned}$$

Since $\lim_{n\longrightarrow\infty} \phi(x_n, x_0)$ exists and we taking $m, n \to \infty$ then, we get $\phi(x_m, x_n) \to 0$. Then, we have $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Thus $\{x_n\}$ is a Cauchy sequence and by the completeness of E and there exist a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we claim that $\|Ju_n - Jx_n\| \to 0$, as $n \to \infty$. By definition of $\Pi_{C_n} x_0$, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \end{aligned}$$

Since $\lim\limits_{n\to\infty} \phi(x_n, x_0)$ exists, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.2.89}$$

It follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.2.90}$$

From J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{4.2.91}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n.$$

By (4.2.89), that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{4.2.92}$$

Then, we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{4.2.93}$$

Since

$$\begin{aligned} \|u_n - x_n\| &= \|u_n - x_{n+1} + x_{n+1} - x_n\| \\ &\leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.2.94}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{4.2.95}$$

Next, we will show that $p \in F := MEP(f,\varphi) \cap (\cap_{i=1}^{\infty} F(S_i)) \cap VI(A,C)$.

(a) First, we show that $p \in MEP(f,\varphi)$. From (4.2.79)-(4.2.83) and (4.2.88), we get $\phi(q,y_n) \leq \phi(q,x_n)$. Sine $u_n = T_{r_n} y_n$, we observe that

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\
&\leq \phi(q, y_n) - \phi(q, T_{r_n} y_n) \\
&\leq \phi(q, x_n) - \phi(q, T_{r_n} y_n) \\
&= \phi(q, x_n) - \phi(q, u_n) \\
&= \|q\|^2 - 2\langle q, Jx_n\rangle + \|x_n\|^2 - (\|q\|^2 - 2\langle q, Ju_n\rangle + \|u_n\|^2) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|q\|\|Jx_n - Ju_n\|.
\end{aligned} \tag{4.2.96}$$

From (4.2.94) and (4.2.95), we have

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{4.2.97}$$

Again since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{4.2.98}$$

From (A2), that

$$\varphi(y) - \varphi(u_n) + \tfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq f(y, u_n), \quad \forall y \in C,$$

$$\varphi(y) - \varphi(u_n) + \langle y - u_n, \tfrac{(Ju_n - Jy_n)}{r_n}\rangle \geq f(y, u_n), \quad \forall y \in C.$$

From $r_n > 0$ then $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ and $u_n \to p$ as $n \to \infty$, we obtain

$$f(y,p) + \varphi(p) - \varphi(y) \leq 0.$$

For t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Then $y_t \in C$ and hence $f(y_t,p) + \varphi(p) - \varphi(y_t) \leq 0$. By the conditions (A1), (A4) and convexity of φ, we have

$$\begin{aligned}
0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tf(y_t, y) + (1-t)f(y_t, p) + t\varphi(y) + (1-t)\varphi(p) - \varphi(y_t) \\
&\leq t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)(f(y_t, p) + \varphi(p) - \varphi(y_t)) \\
&\leq t[f(y_t, y) + \varphi(y) - \varphi(y_t)].
\end{aligned}$$

From (A3) and the weakly lower semicontinuity of φ, we also have $f(p,y)+\varphi(y)-\varphi(p)\geq 0, \quad \forall y\in C$. This implies $p\in MEP(f,\varphi)$.

(b) We show that $p\in\cap_{i=1}^{\infty}F(S_i)$. From definition of C_{n+1}, we have $\phi(z,z_n)\leq\phi(z,x_n)+\theta_n$. Since $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}$, we get $\phi(x_{n+1},z_n)\leq\phi(x_{n+1},x_n)+\theta_n$. It follows from (4.2.89), that

$$\lim_{n\to\infty}\phi(x_{n+1},z_n)=0 \tag{4.2.99}$$

it follows that

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \tag{4.2.100}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1}-Jz_n\|=0. \tag{4.2.101}$$

From (4.2.78), we note that

$$\begin{aligned}
\|Jx_{n+1}-Jz_n\| &= \|Jx_{n+1}-(\alpha_{n,0}Jx_n+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n)\|\\
&= \|\alpha_{n,0}Jx_{n+1}-\alpha_{n,0}Jx_n+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}Jx_{n+1}-\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|\\
&= \|\alpha_{n,0}(Jx_{n+1}-Jx_n)+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}(Jx_{n+1}-JS_i^n v_n)\|\\
&= \|\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}(Jx_{n+1}-JS_i^n v_n)-\alpha_{n,0}(Jx_n-Jx_{n+1})\|\\
&\geq \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|Jx_{n+1}-JS_i^n v_n\|-\alpha_{n,0}\|Jx_n-Jx_{n+1}\|,
\end{aligned}$$

and hence

$$\|Jx_{n+1}-JS_i^n v_n\|\leq\frac{1}{\sum_{i=1}^{\infty}\alpha_{n,i}}(\|Jx_{n+1}-Jz_n\|+\alpha_{n,0}\|Jx_n-Jx_{n+1}\|). \tag{4.2.102}$$

From (4.2.91), (4.2.101) and $\liminf_{n\to\infty}\sum_{i=1}^{\infty}\alpha_{n,i}>0$, we obtain that

$$\lim_{n\to\infty}\|Jx_{n+1}-JS_i^n v_n\|=0. \tag{4.2.103}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_{n+1}-S_i^n v_n\|=0. \tag{4.2.104}$$

Using the triangle inequality, that

$$\begin{aligned}
\|x_n-S_i^n v_n\| &= \|x_n-x_{n+1}+x_{n+1}-S_i^n v_n\|\\
&\leq \|x_n-x_{n+1}\|+\|x_{n+1}-S_i^n v_n\|.
\end{aligned}$$

From (4.2.90) and (4.2.104), we have

$$\lim_{n\to\infty}\|x_n-S_i^n v_n\|=0. \tag{4.2.105}$$

On the other hand, we note that

$$\phi(q,x_n)-\phi(q,u_n)+\theta_n = \|x_n\|^2-\|u_n\|^2-2\langle q,Jx_n-Ju_n\rangle+\theta_n.$$

It follows from $\theta_n \longrightarrow 0,\ \|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$, that

$$\phi(q, x_n) - \phi(q, u_n) + \theta_n \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \tag{4.2.106}$$

From (4.2.79), (4.2.3) and (4.2.82), that

$$\begin{aligned}
\phi(q, u_n) &\leq \phi(q, y_n) \\
&\leq \beta_n \phi(q, x_n) + (1-\beta_n)\phi(q, z_n) \\
&\leq \beta_n \phi(q, x_n) + (1-\beta_n)[\alpha_{n,0}\phi(q, x_n) + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n \phi(q, v_n) \\
&\quad -\alpha_{n,0}\alpha_{n,j} g\|Jv_n - JS_j^n v_n\|] \\
&= \beta_n \phi(q, x_n) + (1-\beta_n)\alpha_{n,0}\phi(q, x_n) + (1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n \phi(q, v_n) \\
&\quad -(1-\beta_n)\alpha_{n,0}\alpha_{n,j} g\|Jv_n - JS_j^n v_n\| \\
&\leq \beta_n \phi(q, x_n) + (1-\beta_n)\alpha_{n,0}\phi(q, x_n) + (1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n \phi(q, v_n) \\
&\leq \beta_n \phi(q, x_n) + (1-\beta_n)\alpha_{n,0}\phi(q, x_n) \\
&\quad +(1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n [\phi(q, x_n) - 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2] \\
&\leq \beta_n \phi(q, x_n) + (1-\beta_n)\alpha_{n,0} k_n \phi(q, x_n) + (1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n \phi(q, x_n) \\
&\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\
&= \beta_n k_n \phi(q, x_n) + (1-\beta_n) k_n \phi(q, x_n) \\
&\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\
&\leq k_n \phi(q, x_n) \\
&\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2] \\
&\leq \phi(q, x_n) + \sup_{q \in F}(k_n - 1)\phi(q, x_n) \\
&\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\
&\leq \phi(q, x_n) + \theta_n - (1-\beta_n)\textstyle\sum_{i=1}^{\infty} \alpha_{n,i} k_n 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
2a(\alpha - \tfrac{2b}{c^2})\|Ax_n - Aq\|^2 &\leq 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n)\|Ax_n - Aq\|^2 \\
&\leq \tfrac{1}{(1-\beta_n)\sum_{i=1}^{\infty} \alpha_{n,i} k_n}(\phi(q, x_n) - \phi(q, u_n) + \theta_n).
\end{aligned} \tag{4.2.107}$$

From (4.2.106), $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $\liminf_{n\longrightarrow\infty}(1 - \beta_n) > 0$ and $\liminf_{n\longrightarrow\infty} \alpha_{n,0}\alpha_{n,i} > 0$, for $i \geq 0$ and $k_n \longrightarrow 1$ as $n \longrightarrow \infty$, we obtain that

$$\lim_{n\to\infty} \|Ax_n - Aq\| = 0. \tag{4.2.108}$$

From (4.2.81), we compute

$$\begin{aligned}
\phi(x_n, v_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(x_n, Jx_n - \lambda_n Ax_n) \\
&\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\
&= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle \\
&= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle \\
&\leq \frac{4\lambda_n^2}{c^2}\|Ax_n - Aq\|^2 \\
&\leq \frac{4b^2}{c^2}\|Ax_n - Aq\|^2.
\end{aligned}$$

Applying by (4.2.108) that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0 \tag{4.2.109}$$

and we also obtain

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = 0 \tag{4.2.110}$$

From S_i^n is continuous, for any $i \geq 1$

$$\lim_{n\to\infty} \|S_i^n x_n - S_i^n v_n\| = 0. \tag{4.2.111}$$

Again by the triangle inequality, we get

$$\|x_n - S_i^n x_n\| \leq \|x_n - S_i^n v_n\| + \|S_i^n v_n - S_i^n x_n\|.$$

From (4.2.105) and (4.2.111), we have

$$\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0, \quad \forall i \geq 1. \tag{4.2.112}$$

Since J is uniformly continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|Jx_n - JS_i^n x_n\| = 0, \quad \forall i \geq 1. \tag{4.2.113}$$

Since $x_n \longrightarrow p$ and J is uniformly continuous, it yields $Jx_n \longrightarrow Jp$. Thus from (4.2.113), we get

$$JS_i^n x_n \longrightarrow Jp, \quad \forall i \geq 1. \tag{4.2.114}$$

Since $J^{-1} : E^* \longrightarrow E$ is norm-weake*-continuous, we have

$$S_i^n x_n \rightharpoonup p, \text{ for each } i \geq 1. \tag{4.2.115}$$

On the other hand, for each $i \geq 1$, we have

$$|\|S_i^n x_n\| - \|p\|| = |\|J(S_i^n x_n)\| - \|Jp\|| \leq \|J(S_i^n x_n) - Jp\|.$$

In view of (4.2.114), we obtain $\|S_i^n x_n\| \longrightarrow \|p\|$ for each $i \geq 1$. Since E is uniformly convex Banach spaces then E has the Kadec-Klee property, we get

$$S_i^n x_n \longrightarrow p \quad \text{for each} \quad i \geq 1.$$

Moreover, by the assumption that $\forall i \geq 1,\ S_i$ is uniformly L_i-Lipschitz continuous, hence we have.

$$\begin{aligned}
\|S_i^{n+1}x_n - S_i^n x_n\| &\leq \|S_i^{n+1}x_n - S_i^{n+1}x_{n+1}\| + \|S_i^{n+1}x_{n+1} - x_{n+1}\| \\
&\quad + \|x_{n+1} - x_n\| + \|x_n - S_i^n x_n\| \\
&\leq (L_i + 1)\|x_{n+1} - x_n\| + \|S_i^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - S_i^n x_n\|.
\end{aligned}$$

By (4.2.90) and (4.2.112), it yields that $\|S_i^{n+1}x_n - S_i^n x_n\| \to 0$. From $S_i^n x_n \longrightarrow p$, we have $S_i^{n+1}x_n \to p$, that is $S_i S_i^n x_n \to p$. In view of closeness of S_i, we have $S_i p = p$, for all $i \geq 1$. This imply that $p \in \cap_{i=1}^{\infty} F(S_i)$.

(c) We show that $p \in VI(A, C)$. Indeed, define $B \subset E \times E^*$ by

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{4.2.116}$$

Since B is maximal monotone and $B^{-1}0 = VI(A, C)$. Let $(v, w) \in G(B)$. Since $w \in Bv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.
From $v_n \in C$, we have

$$\langle v - v_n, w - Av \rangle \geq 0. \tag{4.2.117}$$

On the other hand, since $v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then by Lemma ??, we have

$$\langle v - v_n, Jv_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

and thus

$$\langle v - v_n, \tfrac{Jx_n - Jv_n}{\lambda_n} - Ax_n \rangle \leq 0. \tag{4.2.118}$$

It follows from (4.2.117), (4.2.118) and A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous, that

$$\begin{aligned}
\langle v - v_n, w \rangle &\geq \langle v - v_n, Av \rangle \\
&\geq \langle v - v_n, Av \rangle + \langle v - v_n, \tfrac{Jx_n - Jv_n}{\lambda_n} - Ax_n \rangle \\
&= \langle v - v_n, Av - Ax_n \rangle + \langle v - v_n, \tfrac{Jx_n - Jv_n}{\lambda_n} \rangle \\
&= \langle v - v_n, Av - Av_n \rangle + \langle v - v_n, Av_n - Ax_n \rangle + \langle v - v_n, \tfrac{Jx_n - Jv_n}{\lambda_n} \rangle \\
&\geq -\|v - v_n\| \tfrac{\|v_n - x_n\|}{\alpha} - \|v - v_n\| \tfrac{\|Jx_n - Jv_n\|}{a} \\
&\geq -G\left(\tfrac{\|v_n - x_n\|}{\alpha} + \tfrac{\|Jx_n - Jv_n\|}{a}\right),
\end{aligned}$$

where $G = \sup_{n \geq 1} \|v - v_n\|$. By (4.2.109), (4.2.110) and take the limit as $n \to \infty$, we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of B we have $p \in B^{-1}0$, that is $p \in VI(A, C)$.

Finally, we show that $p = \Pi_F x_0$. From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z\rangle \geq 0$, $\forall z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - y\rangle \geq 0, \quad \forall y \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_0 - Jp, p - y\rangle \geq 0, \quad \forall y \in F.$$

Then, we can conclude that $p = \Pi_F x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. $\square$

บทที่ 5

Variational Inequality Problems

5.1 Generalized Systems of Variational Inequalities for Inverse Strongly Monotone Operators

Consider the following problem of finding $(x^*, y^*) \in E \times E$ such that (see cf. Ceng et al. (2008) [299].)

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in E, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in E, \end{cases} \tag{5.1.1}$$

which is called *general system of variational inequalities* (GSVI) where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (5.1.1) reduces to finding $(x^*, y^*) \in E \times E$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in E, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in E, \end{cases} \tag{5.1.2}$$

which is defined by Verma (1999) [309] and Verma (2001) [310], and is called the new system of variational inequalities. Further, if $x^* = y^*$, then problem (5.1.2) reduces to the *classical variational inequality* $VI(A, E)$ i.e., find $x^* \in E$ such that $\langle Ax^*, x - x^* \rangle \geq 0$, $\forall x \in E$

We can characteristic problem, if $x^* \in F(S) \cap VI(A, E)$, then it follows that $x^* = Sx^* = P_E[x^* - \rho Ax^*]$, where $\rho > 0$ is a constant.

In 2008 Ceng et al [299], introduced a relaxed extragradient method for finding solutions of problem (5.1.1). Let the mappings $A, B : E \longrightarrow H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : E \longrightarrow E$ be a nonexpansive mapping. Suppose $x_1 = u \in E$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_E(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_E(y_n - \lambda_n Ay_n), \end{cases} \tag{5.1.3}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequence in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$. First, problem (5.1.1) is proven to be equivalent to a fixed point problem of nonexpansive mapping.

In this paper, motivation by above we consider generalized system of variational inequalities as follows:
Let E be a nonempty closed convex subset of a real Hilbert space H. Let $A, B, C : E \longrightarrow H$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in E \times E \times E$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in E, \\ \langle \mu B z^* + y^* - z^*, x - y^* \rangle \geq 0, & \forall x \in E, \\ \langle \tau C x^* + z^* - x^*, x - z^* \rangle \geq 0, & \forall x \in E, \end{cases} \tag{5.1.4}$$

which is called a *general system of variational inequalities* where $\lambda > 0$, $\mu > 0$ and $\tau > 0$ are three constants.

In particular, if $A = B = C$, then problem (5.1.4) reduces to finding $(x^*, y^*, z^*) \in E \times E \times E$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in E, \\ \langle \mu A z^* + y^* - z^*, x - y^* \rangle \geq 0, & \forall x \in E, \\ \langle \tau A x^* + z^* - x^*, x - z^* \rangle \geq 0, & \forall x \in E. \end{cases} \tag{5.1.5}$$

Next, we consider some special classes of the GSVI problem (5.1.4) reduce to the following GSVI:
(i) If $\tau = 0$, then the GSVI problems (5.1.4) reduce to GSVI problem (5.1.1).
(ii) If $\tau = \mu = 0$, then the GSVI problems (5.1.4) reduce to classical variational inequality VI(A,E) problem.
The above system enters a class of more general problems which originated mainly from the Nash equilibrium points and was treated from a theoretical viewpoint in [300, 301]. Observe at the same time that, to construct a mathematical model which is as close as possible to a real complex problem, we often have to use constraints which can be expressed as one several subproblems of a general problem. These constrains can be given for instance by variational inequalities, by fixed point problems or by problems of different types.

This section deals with a relaxed extragradient approximation method for solving a system of variational inequalities over the fixed-point sets of nonexpansive mapping. Under classical conditions, we prove a strong convergence theorem for method. Moreover, the proposed algorithm can be applied for instance to solving the classical variational inequality problems.

In this section, we introduce an iterative precess by the relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of the variational inequality problem for three inverse-strongly monotone mappings in a real Hilbert space. We prove that the iterative sequences converges strongly to a common element of the above two sets.

In order to prove our main result, the following lemmas are needed.

Lemma 5.1.1. *For given $x^*, y^*, z^* \in E \times E \times E$, (x^*, y^*, z^*) is a solution of problem* (5.1.4) *if and only if x^* is a fixed point of the mapping $G : E \longrightarrow E$ defined by*

$$G(x) = P_E\{P_E[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)]-\lambda AP_E[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)]\},$$

$\forall x \in E,$ *where $y^* = P_E(z^* - \mu Bz^*)$ and $z^* = P_E(x^* - \tau Cx^*)$.*

พิสูจน์.

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in E, \\ \langle \mu Bz^* + y^* - z^*, x - y^*\rangle \geq 0, & \forall x \in E, \\ \langle \tau Cx^* + z^* - x^*, x - z^*\rangle \geq 0, & \forall x \in E, \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \langle (-y^* + \lambda Ay^*) + x^*, x - x^*\rangle \geq 0, & \forall x \in E, \\ \langle (-z^* + \mu Bz^*) + y^*, x - y^*\rangle \geq 0, & \forall x \in E, \\ \langle (-x^* + \tau Cx^*) + z^*, x - z^*\rangle \geq 0, & \forall x \in E, \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \langle (y^* - \lambda Ay^*) - x^*, x^* - x\rangle \geq 0, & \forall x \in E, \\ \langle (z^* - \mu Bz^*) - y^*, y^* - x\rangle \geq 0, & \forall x \in E, \\ \langle (x^* - \tau Cx^*) - z^*, z^* - x\rangle \geq 0, & \forall x \in E, \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x^* = P_E(y^* - \lambda Ay^*) \\ y^* = P_E(z^* - \mu Bz^*) \\ z^* = P_E(x^* - \tau Cx^*), \end{cases}$$

$\Leftrightarrow$ $x^* = P_E[P_E(z^* - \mu Bz^*) - \lambda AP_E(z^* - \mu Bz^*)]$.

Thus

$$x^* = P_E\{P_E[P_E(x^*-\tau Cx^*)-\mu BP_E(x^*-\tau Cx^*)]-\lambda AP_E[P_E(x^*-\tau Cx^*)-\mu BP_E(x^*-\tau Cx^*)]\}.$$

□

Lemma 5.1.2. *The mapping G defined by Lemma 5.1.1 is nonexpansive mappings.*

พิสูจน์. For all $x, y \in E$

$$
\begin{aligned}
& \|G(x)-G(y)\| \\
= & \|P_E\{P_E[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)]-\lambda AP_E[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)]\} \\
& -P_E\{P_E[P_E(y-\tau Cy)-\mu BP_E(y-\tau Cy)]-\lambda AP_E[P_E(y-\tau Cy)-\mu BP_E(y-\tau Cy)]\} \\
\leq & \|\Big[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)\Big]-\lambda AP_E\Big[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)\Big] \\
& -\Big[P_E(y-\tau Cy)-\mu BP_E(y-\tau Cy)\Big]-\lambda AP_E\Big[P_E(y-\tau Cy)-\mu BP_E(y-\tau Cy)\Big]\| \\
= & \|(I-\lambda A)[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)]-(I-\lambda A)[P_E(y-\tau Cy)-\mu BP_E(y-\tau Cy)]\| \\
\leq & \|[P_E(x-\tau Cx)-\mu BP_E(x-\tau Cx)]-[P_E(y-\tau Cy)-\mu BP_E(y-\tau Cy)]\| \\
= & \|(I-\mu B)[P_E(x-\tau Cx)]-(I-\mu B)[P_E(y-\tau Cy)]\| \\
\leq & \|P_E(x-\tau Cx)-P_E(y-\tau Cy)\| \\
\leq & \|(x-\tau Cx)-(y-\tau Cy)\| \\
= & \|(I-\tau C)(x)-(I-\tau C)(y)\| \\
\leq & \|x-y\|.
\end{aligned}
$$

This shows that $G : E \longrightarrow E$ is a nonexpansive mapping. □

Throughout this paper, the set of fixed points of the mapping G is denoted by Γ. Now, we are ready to proof our main results in this paper.

Theorem 5.1.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A, B, C : E \longrightarrow H$ be α-inverse-strongly monotone, β-inverse-strongly monotone and γ-inverse-strongly monotone, respectively. Let S be a nonexpansive mapping of E into itself such that $F(S) \cap \Gamma \neq \emptyset$. Let f be a contraction of H into itself and given $x_1 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$
\begin{cases}
z_n = P_E(x_n - \tau Cx_n) \\
y_n = P_E(z_n - \mu Bz_n) \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SP_E(y_n - \lambda Ay_n), \;\; n \geq 0,
\end{cases}
\tag{5.1.6}
$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta), \tau \in (0, 2\gamma)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} \in F(S) \cap \Gamma$, where $\bar{x} = P_{F(S)\cap\Gamma} f(\bar{x})$ and $(\bar{x}, \bar{y}, \bar{z})$ is a solution of problem (5.1.4), where

$\bar{y} = P_E(\bar{z} - \mu B\bar{z})$ *and*

$\bar{z} = P_E(\bar{x} - \tau C\bar{x})$.

***Proof*.** Let $x^* \in F(S) \cap \Gamma$. Then $x^* = Sx^*$ and $x^* = Gx^*$, i.e. ,

$$x^* = P_E\{P_E[P_E(x^*-\tau Cx^*)-\mu BP_E(x^*-\tau Cx^*)]-\lambda AP_E[P_E(x^*-\tau Cx^*)-\mu BP_E(x^*-\tau Cx^*)]\}.$$

Put $x^* = P_E(y^* - \lambda Ay^*)$ and $t_n = P_E(y_n - \lambda Ay_n)$. Then $x^* = P_E[P_E(z^* - \mu Bz^*) - \lambda AP_E(z^* - \mu Bz^*)]$ implies that $y^* = P_E(z^* - \mu Bz^*)$, where $z^* = P_E(x^* - \tau Cx^*)$. Since $I - \lambda A$, $I - \mu B$ and $I - \tau C$ are nonexpansive mappings. We obtain that

$$\begin{aligned}
\|t_n - x^*\| &= \|P_E(y_n - \lambda Ay_n) - x^*\| \\
&= \|P_E(y_n - \lambda Ay_n) - P_E(y^* - \lambda Ay^*)\| \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| \\
&= \|(I - \lambda A)y_n - (I - \lambda A)y^*\| \\
&\leq \|y_n - y^*\| \qquad (5.1.7) \\
&= \|y_n - P_E(z^* - \mu Bz^*)\| \\
&= \|P_E(z_n - \mu Bz_n) - P_E(z^* - \mu Bz^*)\| \\
&\leq \|(I - \mu B)z_n - (I - \mu B)z^*\| \\
&\leq \|z_n - z^*\|, \qquad (5.1.8)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - z^*\| &= \|P_E(x_n - \tau Cx_n) - P_E(x^* - \tau Cx^*)\| \\
&\leq \|(x_n - \tau Cx_n) - (x^* - \tau Cx^*)\| \\
&= \|(I - \tau C)x_n - (I - \tau C)x^*\| \\
&\leq \|x_n - x^*\|. \qquad (5.1.9)
\end{aligned}$$

Substituting (5.1.9) into (5.1.8), we have

$$\|t_n - x^*\| \leq \|x_n - x^*\|, \qquad (5.1.10)$$

and by (5.1.7) also have

$$\|y_n - y^*\| \leq \|x_n - x^*\|. \qquad (5.1.11)$$

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n St_n$, we compute

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n St_n - x^*\| \\
&= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(St_n - x^*)\| \\
&\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|St_n - x^*\| \\
&\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|t_n - x^*\| \\
&\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
&= \alpha_n\|f(x_n) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&= \alpha_n\|f(x_n) - f(x^*) + f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\leq \alpha_n k\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&= (\alpha_n k + (1 - \alpha_n))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\
&= (1 - \alpha_n(1 - k))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\
&= (1 - \alpha_n(1 - k))\|x_n - x^*\| + \alpha_n(1 - k)\frac{\|f(x^*) - x^*\|}{(1 - k)}.
\end{aligned}
$$

By induction, we get

$$\|x_{n+1} - x^*\| \leq M,$$

where $M = \max\{\|x_0 - x^*\| + \frac{1}{(1-k)}\|f(x^*) - x^*\|\}, \quad n \geq 0$. Therefore, $\{x_n\}$ is bounded. Consequently, by (5.1.7),(5.1.8) and (5.1.9) the sequences $\{t_n\}, \{St_n\}, \{y_n\}, \{Ay_n\}$, $\{z_n\}, \{Bz_n\}, \{Cx_n\}$ and $\{f(x_n)\}$ are also bounded. Also, we observe that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \|P_E(x_{n+1} - \tau Cx_{n+1}) - P_E(x_n - \tau Cx_n)\| \\
&\leq \|(I - \tau C)x_{n+1} - (I - \tau C)x_n\| \\
&\leq \|x_{n+1} - x_n\|, \qquad (5.1.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\|t_{n+1} - t_n\| &= \|P_E(y_{n+1} - \lambda Ay_{n+1}) - P_E(y_n - \lambda Ay_n)\| \\
&\leq \|(y_{n+1} - \lambda Ay_{n+1}) - (y_n - \lambda Ay_n)\| \\
&= \|(I - \lambda A)y_{n+1} - (I - \lambda A)y_n\| \\
&\leq \|y_{n+1} - y_n\| \qquad (5.1.13) \\
&= \|P_E(z_{n+1} - \mu Bz_{n+1}) - P_E(z_n - \mu Bz_n)\| \\
&\leq \|z_{n+1} - z_n\| \\
&\leq \|x_{n+1} - x_n\|.
\end{aligned}
$$

Let $x_{n+1} = (1 - \beta_n)w_n + \beta_n x_n$. Thus, we get

$$w_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n f(x_n) + \gamma_n SP_C(y_n - \lambda_n Ay_n)}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n St_n}{1 - \beta_n}$$

it follows that

$$
\begin{aligned}
& w_{n+1} - w_n \\
= \; & \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} St_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n St_n}{1-\beta_n} \\
= \; & \frac{\alpha_{n+1} f(x_{n+1})}{1-\beta_{n+1}} + \frac{\gamma_{n+1} St_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1} f(x_n)}{1-\beta_{n+1}} + \frac{\alpha_{n+1} f(x_n)}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)}{1-\beta_n} - \frac{\gamma_n St_n}{1-\beta_n} \\
= \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) f(x_n) + \frac{\gamma_{n+1} St_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n St_n}{1-\beta_n} \\
= \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) f(x_n) \\
& + \frac{\gamma_{n+1} St_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_{n+1} St_n}{1-\beta_{n+1}} + \frac{\gamma_{n+1} St_n}{1-\beta_{n+1}} - \frac{\gamma_n St_n}{1-\beta_n} \\
= \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) f(x_n) \\
& + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1} - St_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) St_n \\
= \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) f(x_n) \qquad (5.1.14) \\
& + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) St_n + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1} - St_n) \\
= \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})(f(x_n) \qquad (5.1.15) \\
& + St_n) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1} - St_n).
\end{aligned}
$$

Combining (5.1.13) and (5.1.14), we obtain

$$
\begin{aligned}
& \|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| \\
\leq\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|f(x_{n+1})-f(x_n)\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)+St_n\| \\
& +|\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\|St_{n+1}-St_n\|-\|x_{n+1}-x_n\| \\
\leq\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|k\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)+St_n\| \\
& +|\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\| \\
\leq\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|k\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)+St_n\| \\
& +|\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\|x_{n+1}-x_n\|-\|x_{n+1}-x_n\| \\
=\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|k\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)+St_n\| \\
& +|\frac{\gamma_{n+1}-1+\beta_{n+1}}{1-\beta_{n+1}}|\|x_{n+1}-x_n\| \\
=\ & |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|k\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)+St_n\| \\
& +|\frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|x_{n+1}-x_n\|.
\end{aligned}
$$

This together with (i), (ii) and (iii) imply that

$$\limsup_{n\longrightarrow\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence, we have

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{5.1.16}$$

Consequently,

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=\lim_{n\longrightarrow\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{5.1.17}$$

From (5.1.12) and (5.1.13), we also have $\|z_{n+1}-z_n\|\longrightarrow 0$ $\|t_{n+1}-t_n\|\longrightarrow 0$ and $\|y_{n+1}-y_n\|\longrightarrow 0$ as $n\longrightarrow\infty$. Since

$$x_{n+1}-x_n=\alpha_n f(x_n)+\beta_n x_n+\gamma_n St_n-x_n=\alpha_n(f(x_n)-x_n)+\gamma_n(St_n-x_n),$$

it follows by (ii) and (5.1.17) that

$$\lim_{n\to\infty}\|x_n-St_n\|=0. \tag{5.1.18}$$

Since $x^* \in F(S) \cap \Gamma$ and from (5.1.11), we get

$$
\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= \;& \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\|^2 \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|S t_n - x^*\|^2 \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|P_E(y_n - \lambda A y_n) - P_E(y^* - \lambda A y^*)\|^2 \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\|^2 \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(y_n - y^*) - \lambda (A y_n - A y^*)\|^2 \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
& + \gamma_n \Big[\|y_n - y^*\|^2 - 2\lambda \langle y_n - y^*, A y_n - A y^* \rangle + \lambda^2 \|A y_n - A y^*\|^2 \Big] \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
& + \gamma_n \Big[\|y_n - y^*\|^2 - 2\lambda\alpha \|A y_n - A y^*\|^2 + \lambda^2 \|A y_n - A y^*\|^2 \Big] \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \Big[\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \Big] \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \Big[\|x_n - x^*\|^2 + \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \Big] \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + (\beta_n + \gamma_n) \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& -\gamma_n \lambda(\lambda - 2\alpha) \|A y_n - A y^*\|^2 \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
= \;& \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
\leq \;& \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_n - x_{n+1}\|.
\end{aligned}
\tag{5.1.19}
$$

From (ii), (iii) and $\|x_{n+1} - x_n\| \longrightarrow 0$, as $n \longrightarrow \infty$, we get $\|A y_n - A y^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Since $x^* \in F(S) \cap \Gamma$, from (5.1.7), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|y_n - y^*\|^2 \\
&= \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|P_E(z_n - \mu B z_n) - P_E(z^* - \mu B z^*)\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(z_n - \mu B z_n) - (z^* - \mu B z^*)\|^2 \\
&= \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(z_n - z^*) - (\mu B z_n - \mu B z^*)\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left[\|z_n - z^*\|^2 + \mu(\mu - 2\beta)\|B z_n - B z^*\|^2\right] \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n \mu(\mu - 2\beta)\|B z_n - B z^*\|^2.
\end{aligned}$$

Thus, we also have

$$\begin{aligned}
&-\gamma_n \mu(\mu - 2\beta)\|B z_n - B z^*\|^2 \\
\leq\ & \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
=\ & \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
\leq\ & \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|. \qquad (5.1.20)
\end{aligned}$$

By again (ii), (iii) and (5.1.17), we also get $\|B z_n - B z^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Let $x^* \in F(S) \cap \Gamma$, again from (5.1.8), (5.1.9), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|z_n - z^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(x_n - \tau C x_n) - (x^* - \tau C x^*)\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left[\|x_n - x^*\|^2 + \tau(\tau - 2\gamma)\|C x_n - C x^*\|^2\right] \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n \tau(\tau - 2\gamma)\|C x_n - C x^*\|^2.
\end{aligned}$$

Again, we have

$$\begin{aligned}
&-\gamma_n \tau(\tau - 2\gamma)\|C x_n - C x^*\|^2 \\
\leq\ & \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
=\ & \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
\leq\ & \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|. \qquad (5.1.21)
\end{aligned}$$

Similarly again by (ii), (iii) and $\|x_n - x_{n+1}\| \longrightarrow 0$ as $n \longrightarrow \infty$, from (5.1.21), we also that $\|C x_n - C x^*\| \longrightarrow 0$.

On the other hand, we compele that

$$
\begin{aligned}
& \|z_n - z^*\|^2 \\
= & \|P_E(x_n - \tau C x_n) - P_E(x^* - \tau C x^*)\|^2 \\
\leq & \langle (x_n - \tau C x_n) - (x^* - \tau C x^*), P_E(x_n - \tau C x_n) - P_E(x^* - \tau C x^*) \rangle \\
= & \langle (x_n - \tau C x_n) - (x^* - \tau C x^*), z_n - z^* \rangle \\
= & \frac{1}{2}\Big[\|(x_n - \tau C x_n) - (x^* - \tau C x^*)\|^2 + \|z_n - z^*\|^2 - \|(x_n - \tau C x_n) \\
& -(x^* - \tau C x^*) - (z_n - z^*)\|^2 \Big] \\
= & \frac{1}{2}\Big[\|(I - \tau C)x_n - (I - \tau C)x^*\|^2 + \|z_n - z^*\|^2 - \|(x_n - \tau C x_n) \\
& -(x^* - \tau C x^*) - (z_n - z^*)\|^2 \Big] \\
\leq & \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - z^*\|^2 - \|(x_n - z_n) - \tau(C x_n - C x^*) - (x^* - z^*)\|^2 \Big] \\
= & \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - z^*\|^2 - \| \big[(x_n - z_n) - (x^* - z^*)\big] - \tau(C x_n - C x^*)\|^2 \Big] \\
= & \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - z^*\|^2 - \|(x_n - z_n) - (x^* - z^*)\|^2 \\
& +2\tau \langle (x_n - z_n) - (x^* - z^*), C x_n - C x^* \rangle - \tau^2 \|C x_n - C x^*\|^2 \Big].
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
\|z_n - z^*\|^2 \leq & \|x_n - x^*\|^2 - \|(x_n - z_n) - (x^* - z^*)\|^2 \\
& +2\tau \langle (x_n - z_n) - (x^* - z^*), C x_n - C x^* \rangle - \tau^2 \|C x_n - C x^*\|^2.
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 = & \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\|^2 \\
\leq & \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|S t_n - x^*\|^2 \\
\leq & \alpha_n k \|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
\leq & \alpha_n k \|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|z_n - z^*\|^2 \\
\leq & \alpha_n k \|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 - \gamma_n \|(x_n - z_n) - (x^* - z^*)\|^2 \\
& +2\tau\gamma_n \langle (x_n - z_n) - (x^* - z^*), C x_n - C x^* \rangle - \tau^2 \gamma_n \|C x_n - C x^*\|^2 \\
= & \alpha_n k \|x_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \gamma_n \|(x_n - z_n) - (x^* - z^*)\|^2 \\
& +2\tau\gamma_n \langle (x_n - z_n) - (x^* - z^*), C x_n - C x^* \rangle - \tau^2 \gamma_n \|C x_n - C x^*\|^2 \\
\leq & \alpha_n k \|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n \|(x_n - z_n) - (x^* - z^*)\|^2 \\
& +2\tau\gamma_n \langle (x_n - z_n) - (x^* - z^*), C x_n - C x^* \rangle \\
\leq & \alpha_n k \|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n \|(x_n - z_n) - (x^* - z^*)\|^2 \\
& +2\tau\gamma_n \|(x_n - z_n) - (x^* - z^*)\| \|C x_n - C x^*\|,
\end{aligned}
$$

which implies that

$$\begin{aligned}
\gamma_n\|(x_n-z_n)-(x^*-z^*)\|^2 &\leq \alpha_n k\|x_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad +2\tau\gamma_n\|(x_n-z_n)-(x^*-z^*)\|\|Cx_n-Cx^*\| \\
&\leq \alpha_n k\|x_n-x^*\|^2+2\gamma_n\tau\|(x_n-z_n)-(x^*-z^*)\|\|Cx_n-Cx^*\| \\
&\quad +\|x_n-x_{n+1}\|\big(\|x_n-x^*\|+\|x_{n+1}-x^*\|\big). \qquad (5.1.22)
\end{aligned}$$

By (ii), (iii), $\|x_n-x_{n+1}\|\longrightarrow 0$ and $\|Cx_n-Cx^*\|\longrightarrow 0$ as $n\longrightarrow\infty$ from (5.1.22) we get
$\|(x_n-z_n)-(x^*-z^*)\|\longrightarrow 0$ as $n\longrightarrow\infty$. Now, observe that

$$\begin{aligned}
&\|(z_n-t_n)+(x^*-z^*)\|^2 \\
= \ &\|z_n-P_E(y_n-\lambda Ay_n)+P_E(y^*-\lambda Ay^*)-z^*\|^2 \\
= \ &\|z_n-P_E(y_n-\lambda Ay_n)+P_E(y^*-\lambda Ay^*)-z^*+\mu Bz_n-\mu Bz_n+\mu Bz^*-\mu Bz^*\|^2 \\
= \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)-[P_E(y_n-\lambda Ay_n)-P_E(y^*-\lambda Ay^*)]+\mu(Bz_n-Bz^*)\|^2 \\
\leq \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)-[P_E(y_n-\lambda Ay_n)-P_E(y^*-\lambda Ay^*)]\|^2 \\
&+2\mu\langle Bz_n-Bz^*, z_n-\mu Bz_n-(z^*-\mu Bz^*)-[P_E(y_n-\lambda Ay_n)-P_E(y^*-\lambda Ay^*)] \\
&+\mu(Bz_n-Bz^*)\rangle \\
= \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)-[P_E(y_n-\lambda Ay_n)-P_E(y^*-\lambda Ay^*)]\|^2 \\
&+2\mu\langle Bz_n-Bz^*, (z_n-t_n)+(x^*-z^*)\rangle \\
\leq \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)\|^2-\|P_E(y_n-\lambda_n Ay_n)-P_E(y^*-\lambda_n Ay^*)\|^2 \\
&+2\mu\|Bz_n-Bz^*\|\|(z_n-t_n)+(x^*-z^*)\| \\
\leq \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)\|^2-\|SP_E(y_n-\lambda_n Ay_n)-SP_E(y^*-\lambda_n Ay^*)\|^2 \\
&+2\mu\|Bz_n-Bz^*\|\|(z_n-t_n)+(x^*-z^*)\| \\
= \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)\|^2-\|St_n-Sx^*\|^2 \\
&+2\mu\|Bz_n-Bz^*\|\|(z_n-t_n)+(x^*-z^*)\| \\
\leq \ &\|z_n-\mu Bz_n-(z^*-\mu Bz^*)-(St_n-x^*)\| \\
&\times(\|z_n-\mu Bz_n-(z^*-\mu Bz^*)\|+\|St_n-x^*\|) \\
&+2\mu\|Bz_n-Bz^*\|\|(z_n-t_n)+(x^*-z^*)\|. \qquad (5.1.23)
\end{aligned}$$

Since $\|St_n-x_n\|\longrightarrow 0$, $\|(x_n-z_n)-(x^*-z^*)\|\longrightarrow 0$ and $\|Bz_n-Bz^*\|\longrightarrow 0,$ as $n\longrightarrow\infty$, it follows that

$$\|(z_n-t_n)+(x^*-z^*)\|\longrightarrow 0, \quad \text{as } n\longrightarrow\infty.$$

Since

$$\|St_n-t_n\| \ \leq \ \|St_n-x_n\|+\|(x_n-z_n)-(x^*-z^*)\|+\|(z_n-t_n)+(x^*-z^*)\|,$$

from above, we obtain

$$\lim_{n \longrightarrow \infty} \|St_n - t_n\| = 0. \tag{5.1.24}$$

Next, we show that

$$\limsup_{n \longrightarrow \infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x} \rangle \leq 0,$$

where $\bar{x} = P_{F(S) \cap \Gamma} f(\bar{x})$.
Indeed, since $\{t_n\}$ and $\{St_n\}$ are two bounded sequence in E, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that t_{n_i} of t_n such that $t_{n_i} \rightharpoonup z \in E$ and

$$\limsup_{n \longrightarrow \infty} \langle f(\bar{x}) - \bar{x}, St_n - \bar{x} \rangle = \lim_{i \longrightarrow \infty} \langle f(\bar{x}) - \bar{x}, St_{n_i} - \bar{x} \rangle.$$

Since $\lim_{n \longrightarrow \infty} \|St_n - t_n\| = 0$, we obtain $St_{n_i} \rightharpoonup z$ as $i \longrightarrow \infty$. Now we claim that $z \in F(S) \cap \Gamma$. It is easy to see that $z \in F(S)$.
Since $\|St_n - t_n\| \longrightarrow 0$, $\|St_n - x_n\| \longrightarrow 0$ and

$$\begin{aligned} \|t_n - x_n\| &= \|t_n - St_n + St_n - x_n\| \\ &\leq \|t_n - St_n\| + \|St_n - x_n\| \\ &= \|St_n - t_n\| + \|St_n - x_n\|, \end{aligned}$$

we conclude that $\|t_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Furthermore, by Lemma 5.1.2 that G is nonexpansive, then

$$\begin{aligned} \|t_n - G(t_n)\| &= \|G(x_n) - G(t_n)\| \\ &\leq \|x_n - t_n\|. \end{aligned}$$

Thus $\lim_{n \longrightarrow \infty} \|t_n - G(t_n)\| = 0$. Then, we obtain $z \in \Gamma$. Therefore there holds $z \in F(S) \cap \Gamma$.

On the other hand, it follows that

$$\begin{aligned} \limsup_{n \longrightarrow \infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x} \rangle &= \limsup_{n \longrightarrow \infty} \langle f(\bar{x}) - \bar{x}, St_n - \bar{x} \rangle \\ &= \lim_{n \longrightarrow \infty} \langle f(\bar{x}) - \bar{x}, St_{n_i} - \bar{x} \rangle \\ &= \langle f(\bar{x}) - \bar{x}, z - \bar{x} \rangle \\ &\leq 0. \end{aligned} \tag{5.1.25}$$

Finally, we show that $x_n \longrightarrow \bar{x}$, by (5.1.10) that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n St_n-\bar{x}\|^2\\
&\leq \|\beta_n(x_n-\bar{x})+\gamma_n(St_n-\bar{x})\|^2+2\alpha_n\langle f(x_n)-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&\leq \|\beta_n(x_n-\bar{x})+\gamma_n(St_n-\bar{x})\|^2+2\alpha_n\langle f(x_n)-f(\bar{x}), x_{n+1}-\bar{x}\rangle\\
&\quad +2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&\leq [\beta_n\|x_n-\bar{x}\|^2+\gamma_n\|St_n-\bar{x}\|^2]+2\alpha_n\|f(x_n)-f(\bar{x})\|\|x_{n+1}-\bar{x}\|\\
&\quad +2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&\leq [\beta_n\|x_n-\bar{x}\|^2+\gamma_n\|t_n-\bar{x}\|^2]+2\alpha_n\alpha\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\
&\quad +2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n)^2\|x_n-\bar{x}\|^2+\alpha_n\alpha(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2)\\
&\quad +2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n})\|x_n-\bar{x}\|^2+\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-\bar{x}\|^2\\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle\\
&:= (1-\sigma_n)\|x_n-\bar{x}\|^2+\delta_n,\ \ n\geq 0,
\end{aligned}$$

where $\sigma_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$ and $\delta_n=\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-\bar{x}\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle$. Therefore, by (5.1.25), we get that $\{x_n\}$ converges to $\bar{x}$, where $\bar{x}=P_{F(S)\cap\Gamma}f(\bar{x})$. This completes the proof.

Setting $A=B=C$ we obtain the following corollary:

Corollary 5.1.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A: E\longrightarrow H$ be α-inverse-strongly monotone. Let S be a nonexpansive mapping of E into itself such that $F(S)\cap\Gamma\neq\emptyset$. Let f be a contraction of H into itself and given $x_0\in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{cases} z_n=P_E(x_n-\tau Ax_n)\\ y_n=P_E(z_n-\mu Az_n)\\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n SP_E(y_n-\lambda Ay_n),\ \ n\geq 1,\end{cases} \tag{5.1.26}$$

where $\lambda,\mu,\tau\in(0,2\alpha)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n+\beta_n+\gamma_n=1$*,*

(ii) $\lim_{n\longrightarrow\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty}\beta_n \le \limsup_{n\longrightarrow\infty}\beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $\bar{x} \in F(S)\cap\Gamma$*, where* $\bar{x} = P_{F(S)\cap\Gamma}f(\bar{x})$ *and* $(\bar{x},\bar{y},\bar{z})$ *is a solution of problem (5.1.5), where*

$\bar{y} = P_E(\bar{z} - \mu A\bar{z})$ *and*

$\bar{z} = P_E(\bar{x} - \tau A\bar{x})$.

Setting $A \equiv B \equiv 0$ (the zero operators), we obtain the following corollary for solving the foxed points problem and the classical variational inequality problems.

Corollary 5.1.5. *Let* E *be a nonempty closed convex subset of a real Hilbert space* H*. Let the mapping* $A : E \longrightarrow H$ *be* α*-inverse-strongly monotone. Let* S *be a nonexpansive mapping of* E *into itself such that* $F(S)\cap VI(A,E) \neq \emptyset$*. Let* f *be a contraction of* H *into itself and given* $x_0 \in H$ *arbitrarily and* $\{x_n\}$ *is generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SP_E(x_n - \lambda Ax_n), \quad n \ge 1, \tag{5.1.27}$$

where $\lambda \in (0, 2\alpha)$ *and* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ *are three sequences in* $[0,1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty}\beta_n \le \limsup_{n\longrightarrow\infty}\beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $\bar{x} \in F(S)\cap VI(A,E)$*, where* $\bar{x} = P_{F(S)\cap VI(A,E)}f(\bar{x})$.

We recall that a mapping $T : E \longrightarrow E$ is called strictly pseudocontractive if there exists some k with $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x,y \in E.$$

For recent convergence result for strictly pseudocontractive mappings. Put $A = I - T$. Then we have

$$\|(I-A)x - (I-A)y\|^2 \le \|x-y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand,

$$\|(I-A)x - (I-A)y\|^2 \le \|x-y\|^2 + \|Ax-Ay\|^2 - 2\langle x-y, Ax-Ay\rangle.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \ge \frac{1-k}{2}\|Ax-Ay\|^2.$$

Consequently, if $T : E \longrightarrow E$ is a strictly pseudocontractive mapping with constant k, then the mapping $A = I - T$ is $(1-k)/2$-inverse-strongly monotone.

Setting $A = I - T$, $B = I - V$ and $C = I - W$ we obtain the following corollary:

Theorem 5.1.6. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let T, V, W be strictly pseudocontractive mappings with constant k of C into itself and let S be a nonexpansive mapping of E into itself such that $F(S) \cap \Gamma \neq \emptyset$. Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{cases} z_n = (I - \tau)x_n + \tau W x_n) \\ y_n = (I - \mu)z_n + \mu V z_n \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n), \ \ n \geq 1, \end{cases} \tag{5.1.28}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta), \tau \in (0, 2\gamma)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} \in F(S) \cap \Gamma$, where $\bar{x} = P_{F(S)\cap\Gamma} f(\bar{x})$ and $(\bar{x}, \bar{y}, \bar{z})$ is a solution of problem (5.1.4), where

$\bar{y} = P_E(\bar{z} - \mu B \bar{z})$ *and*

$\bar{z} = P_E(\bar{x} - \tau C \bar{x})$.

***Proof*.** Since $A = I - T$, $B = I - V$ and $C = I - W$, we have

$$\begin{aligned} P_E(x_n - \tau C x_n) &= (I - \tau)x_n + \tau W x_n \\ P_E(y_n - \lambda A y_n) &= (I - \lambda)y_n + \lambda T y_n \\ P_E(z_n - \mu B z_n) &= (I - \mu)z_n + \mu V z_n. \end{aligned}$$

Thus, the conclusion follows immediately from Theorem 5.1.3.

If $f(x) = x_0, \ \forall x \in E$ and $T = V = W$ in Theorem 5.1.6, we obtain the following corollary.

Corollary 5.1.7. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let T be strictly pseudocontractive mappings with constant k of C into itself and let S be a nonexpansive mapping of E into itself such that $F(S) \cap \Gamma \neq \emptyset$. Given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{cases} z_n = (I - \tau)x_n + \tau T x_n) \\ y_n = (I - \mu)z_n + \mu T z_n \\ x_{n+1} = \alpha_n x_0 + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n), \ \ n \geq 1, \end{cases} \tag{5.1.29}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta), \tau \in (0, 2\gamma)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$*,*

(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $\bar{x} \in F(S) \cap \Gamma$*, where* $\bar{x} = P_{F(S)\cap\Gamma}\bar{x}$ *and* $(\bar{x}, \bar{y}, \bar{z})$ *is a solution of problem (5.1.5), where*

$$\bar{y} = P_E(\bar{z} - \mu A\bar{z}) \text{ and}$$

$$\bar{z} = P_E(\bar{x} - \tau A\bar{x}).$$

5.2 General System of Variational Inequalities for Inverse Strongly Accretive Operators

Let $S : C \rightarrow C$ a nonlinear mapping. Let A be a monotone operator of C into H. The *variational inequality problem*, denote by $VI(C, A)$, is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0,$$

for all $x \in C$. Recall that an operator A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$

for all $x, y \in C$. An operator $A : C \rightarrow E$ is said to be β*-strongly accretive* if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta \|x - y\|^2 \quad \forall x, y \in C.$$

An operator A of C into E is said to be β*-inverse strongly accretive* if, for any $\beta > 0$,

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta \|Ax - Ay\|^2$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator.

Recently, Aoyama et al. first considered the following generalized variational inequality problem in a smooth Banach space. Let A be an accretive operator of C into E. Find a point $x \in C$ such that

$$\langle Ax, j(y - x) \rangle \geq 0, \tag{5.2.1}$$

for all $y \in C$. In order to find a solution of the variational inequality (5.2.1), the authors proved the following theorem in the framework of Banach spaces.

Theorem AIT. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$, and A be an α-inverse strongly accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \{x^* \in C : \langle Ax^*, j(x - x^*)\rangle \geq 0, \quad x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$, for some $a > 0$ and $\alpha_n \in [b, c]$, for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners: $x_1 - x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n),$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E and Q_C is a sunny nonexpansive retraction.

Let $A : C \to E$ be an β-inverse strongly accretive mapping. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, j(x - y^*)\rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{5.2.2}$$

Let *C* be nonempty closed convex subset of a real Banach space *E*. For given two operators $A, B : C \to E$, we consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, j(x - y^*)\rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{5.2.3}$$

where λ and μ are two positive real numbers. This system is called the *system of general variational inequalities* in a real Banach spaces. If we add up the requirement that $A = B$, then the problem (5.2.3) is reduced to the system (5.2.2).

An interesting problem to extend the above results to find a solution of a general system of variational inequalities.

In this section we introduce viscosity iterative scheme for finding solutions of a general system of variational inequalities (5.2.3) for two inverse-strongly accretive operators with a viscosity of modified extragradient methods and solutions of fixed point problems involving the nonexpansive mapping in Banach spaces. Then, we prove that the sequence $\{x_n\}$ defined by (5.2.6) below converge strongly to $\bar{x} = Q_{F(G)\cap F(S)} f(\bar{x})$ which $(\bar{x}, \bar{y})$ is a solution of the system of general variational inequalities (5.2.3), where $\bar{y} = Q_C(\bar{x} - \mu B\bar{x})$.

In this section, we always assume that E is a Banach space. Let D be a subset of C and $Q : C \to D$. Then Q is said to *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 5.2.1. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let Q : E → C be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$*;*
(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.

Proposition 5.2.2. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with* $F(T) \neq \emptyset$*. Then the set F(T) is a sunny nonexpansive retract of C.*

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [109, 144]. More precisely, take $t \in (0, 1)$ and define a contraction $S_t : C \longrightarrow C$ by

$$S_t x = tu + (1 - t)Sx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that S_t has a unique fixed point x_t in C. that is

$$x_t = tu + (1 - t)Sx_t.$$

We need the following lemmas for proving our main results.

Lemma 5.2.3. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_1 and S_2 be two nonexpansive mappings from C into itself with a common fixed point. Define a mapping $S : C \longrightarrow C$ by*

$$Sx = \delta S_1 x + (1 - \delta)S_2 x, \quad \forall x \in C,$$

where δ is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(S_1) \cap F(S_2)$.

Lemma 5.2.4. *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \qquad \forall x, y \in E.$$

Lemma 5.2.5. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 5.2.6. ([311]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n, \ n \ge 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 5.2.7. ([306]) *Let $(E, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 5.2.8. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be β-inverse-strongly accretive. Then, we have*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^2 \le \|x-y\|^2 + 2\lambda(\lambda K^2 - \beta)\|Ax - Ay\|^2.$$

If $\beta \ge \lambda K^2$, then $I - \lambda A$ is nonexpansive.

***Proof*.** For any $x, y \in C$, from Lemma 5.2.4, we have

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|(x-y) - \lambda(Ax - Ay)\|^2 \\
&\le \|x-y\|^2 - 2\lambda\langle Ax - Ay, j(x-y)\rangle + 2\lambda^2 K^2\|Ax - Ay\|^2 \\
&\le \|x-y\|^2 - 2\lambda\beta\|Ax - Ay\|^2 + 2\lambda^2 K^2\|Ax - Ay\|^2 \\
&= \|x-y\|^2 + 2\lambda(\lambda K^2 - \beta)\|Ax - Ay\|^2.
\end{aligned}$$

If $\beta \ge \lambda K^2$, then $I - \lambda A$ is nonexpansive. □

Lemma 5.2.9. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let the mapping $A, B : C \to E$ be β-inverse-strongly accretive and γ-inverse-strongly accretive, respectively. Let $G : C \to C$ be a mapping defined by*

$$G(x) = Q_C(Q_C(x - \mu Bx) - \lambda A Q_C(x - \mu Bx)) \quad \forall x \in C.$$

If $\beta \ge \lambda K^2$ and $\gamma \ge \mu K^2$, then G is nonexpansive.

Lemma 5.2.10. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be two possibly nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (5.2.3) if and only if $x^* = Q_C(y^* - \lambda Ay^*)$ where $y^* = Q_C(x^* - \mu Bx^*)$.*

***Proof*.** From (5.2.3), we rewrite as

$$\begin{cases} \langle x^* - (y^* - \lambda Ay^*), j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle y^* - (x^* - \mu Bx^*), j(x - y^*)\rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{5.2.4}$$

From Proposition 5.2.1 (iii), the system (5.2.4) equivalent to

$$\begin{cases} x^* = Q_C(y^* - \lambda Ay^*), \\ y^* = Q_C(x^* - \mu Bx^*). \end{cases} \tag{5.2.5}$$

□

Remark 5.2.11. From Lemma 5.2.10, we note that

$$x^* = Q_C(Q_C(x^* - \mu Bx^*) - \lambda AQ_C(x^* - \mu Bx^*)),$$

which implies that x^* is a fixed point of the mapping G.
Throughout this paper, the set of fixed points of the mapping G is denoted by $F(G)$.

In this section, we prove a strong convergence theorem.

Theorem 5.2.12. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonexpansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\mathcal{F} \neq \emptyset$ where G defined by Lemma 5.2.9 and let λ, μ are positive real numbers. The following conditions are satisfied:*
(C1). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ generated by

$$\begin{cases} y_n = Q_C(x_n - \mu Bx_n), \\ v_n = Q_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta Sx_n + (1 - \delta)v_n], \end{cases} \tag{5.2.6}$$

then $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of the problem (5.2.3), where $\bar{y} = Q_C(\bar{x} - \mu B\bar{x})$ and $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of C onto $\mathcal{F}$.

***Proof*.** First, we prove that $\{x_n\}$ bounded. Let $x^* \in \mathcal{F}$, from Lemma 5.2.10, we see that

$$x^* = Q_C(Q_C(x^* - \mu Bx^*) - \lambda AQ_C(x^* - \mu Bx^*)).$$

Put $y^* = Q_C(x^* - \mu Bx^*)$ and $v_n = Q_C(y_n - \lambda Ay_n)$. Then $x^* = Q_C(y^* - \lambda Ay^*)$. From Lemma 5.2.8, we have

$$\begin{aligned}
\|v_n - x^*\| &= \|Q_C(y_n - \lambda Ay_n) - Q_C(y^* - \lambda Ay^*)\| \\
&\leq \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| \\
&= \|(I - \lambda A)y_n - (I - \lambda A)y^*\| \\
&\leq \|y_n - y^*\| \\
&= \|Q_C(x_n - \mu Bx_n) - Q_C(x^* - \mu Bx^*)\| \\
&\leq \|(x_n - \mu Bx_n) - (x^* - \mu Bx^*)\| \\
&= \|(I - \mu B)x_n - (I - \mu B)x^*\| \\
&\leq \|x_n - x^*\| \qquad (5.2.7)
\end{aligned}$$

and put $e_n = \delta Sx_n + (1 - \delta)v_n$. From (5.2.7), we obtain

$$\begin{aligned}
\|e_n - x^*\| &= \|\delta Sx_n + (1 - \delta)v_n - x^*\| \\
&\leq \delta\|Sx_n - x^*\| + (1 - \delta)\|v_n - x^*\| \\
&\leq \delta\|x_n - x^*\| + (1 - \delta)\|x_n - x^*\| \\
&= \|x_n - x^*\|. \qquad (5.2.8)
\end{aligned}$$

We observe that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n e_n - x^*\| \\
&\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|e_n - x^*\| \\
&\leq \alpha\alpha_n\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
&= (1 - \alpha_n + \alpha\alpha_n)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - x^*\| + \alpha_n(1 - \alpha)\frac{\|f(x^*) - x^*\|}{1 - \alpha} \\
&\leq \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\}.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$, $\{v_n\}$, $\{e_n\}$, $\{Ay_n\}$ and $\{Bx_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Notice that

$$\begin{aligned}
\|v_{n+1}-v_n\| &= \|Q_C(y_{n+1}-\lambda Ay_{n+1})-Q_C(y_n-\lambda Ay_n)\| \\
&\le \|(y_{n+1}-\lambda Ay_{n+1})-(y_n-\lambda Ay_n)\| \\
&= \|(I-\lambda A)y_{n+1}-(I-\lambda A)y_n\| \\
&\le \|y_{n+1}-y_n\| \\
&= \|Q_C(x_{n+1}-\mu Bx_{n+1})-Q_C(x_n-\mu Bx_n)\| \\
&\le \|(x_{n+1}-\mu Bx_{n+1})-(x_n-\mu Bx_n)\| \\
&= \|(I-\mu B)x_{n+1}-(I-\mu B)x_n\| \\
&\le \|x_{n+1}-x_n\| \qquad (5.2.9)
\end{aligned}$$

it follows that

$$\begin{aligned}
\|e_{n+1}-e_n\| &= \|[\delta Sx_{n+1}+(1-\delta)v_{n+1}]-[\delta Sx_n+(1-\delta)v_n]\| \\
&\le \delta\|Sx_{n+1}-Sx_n\|+(1-\delta)\|v_{n+1}-v_n\| \\
&\le \delta\|x_{n+1}-x_n\|+(1-\delta)\|x_{n+1}-x_n\| \\
&= \|x_{n+1}-x_n\|. \qquad (5.2.10)
\end{aligned}$$

Setting $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$ for all $n\ge 0$, we see that $z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, then

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
=\ & \|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\| \\
=\ & \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}e_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n e_n}{1-\beta_n}\| \\
=\ & \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}e_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}+\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}-\frac{\gamma_{n+1}e_n}{1-\beta_{n+1}}+\frac{\gamma_{n+1}e_n}{1-\beta_{n+1}} \\
&-\frac{\alpha_n f(x_n)+\gamma_n e_n}{1-\beta_n}\| \\
=\ & \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(e_{n+1}-e_n)+(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})f(x_n) \\
&+(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})e_n\| \\
\le\ & \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|e_{n+1}-e_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\
&+|\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}-\frac{1-\beta_n-\alpha_n}{1-\beta_n}|\|e_n\| \\
\le\ & \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\
&+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|e_n\|) \\
\le\ & \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|e_n\|)+\|x_{n+1}-x_n\|.
\end{aligned}$$

Therefore

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|f(x_n)\| + \|e_n\|).$$

It follows from the condition (C1) and (C2), that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Applying Lemma 5.2.5, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and we also have

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|z_n - x_n\| \longrightarrow 0, \quad n \longrightarrow \infty.$$

Hence

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{5.2.11}$$

Next, we show that $x^* \in \mathcal{F}$. Define a mapping $T : C \to C$ by

$$Tx = \delta Sx + (1-\delta)Q_C(I-\lambda A)Q_C(I-\mu B), \quad \forall x \in C.$$

From Lemma 5.2.3 and Lemma 5.2.8, we see that T is a nonexpansive mapping with

$$\begin{aligned} F(T) &= F(S) \cap F(Q_C(I-\lambda A)Q_C(I-\mu B)) \\ &= F(S) \cap F(G). \end{aligned}$$

Therefore, we have $x^* \in \mathcal{F}$.

Next, we show that $\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_n - \bar{x})\rangle \le 0$, where $\bar{x} = Q_{\mathcal{F}}f(\bar{x})$. Since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup x^*$ such that

$$\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_n - \bar{x})\rangle = \lim_{i\to\infty}\langle (f-I)\bar{x}, J(x_{n_i} - \bar{x})\rangle. \tag{5.2.12}$$

Now, from (5.2.12), Proposition 5.2.1 (iii) and since J is strong to weak* uniformly continuous on bounded subset of E, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_n - \bar{x})\rangle &= \lim_{i\to\infty}\langle (f-I)\bar{x}, J(x_{n_i} - \bar{x})\rangle \\ &= \langle (f-I)\bar{x}, J(x^* - \bar{x})\rangle \le 0. \end{aligned} \tag{5.2.13}$$

From (5.2.11), it follows that

$$\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_{n+1} - \bar{x})\rangle \le 0. \tag{5.2.14}$$

Finally, we show that $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$. Observe that

$$\begin{aligned}
& \|x_{n+1} - \bar{x}\|^2 \\
= & \langle x_{n+1} - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
= & \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n e_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
= & \langle \alpha_n (f(x_n) - \bar{x}) + \beta_n (x_n - \bar{x}) + \gamma_n (e_n - \bar{x}), J(x_{n+1} - \bar{x})\rangle \\
= & \alpha_n \langle f(x_n) - f(\bar{x}), J(x_{n+1} - \bar{x})\rangle + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
& + \beta_n \langle x_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
& + \gamma_n \langle e_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
\leq & \alpha\alpha_n \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \beta_n \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
& + \gamma_n \|e_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
\leq & \alpha\alpha_n \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \beta_n \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
& + \gamma_n \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
= & \frac{\alpha\alpha_n + \beta_n + \gamma_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
= & \frac{\alpha\alpha_n + 1 - \alpha_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
= & \frac{1 - \alpha_n(1-\alpha)}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
\leq & \frac{1 - \alpha_n(1-\alpha)}{2}\|x_n - \bar{x}\|^2 + \frac{1}{2}\|x_{n+1} - \bar{x}\|^2 + \alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle
\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - \alpha_n(1-\alpha))\|x_n - \bar{x}\|^2 + 2\alpha_n \langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \tag{5.2.15}$$

Now, from (C1), (5.2.14) and applying Lemma 5.2.6 to (5.2.15), we get $\|x_n - \bar{x}\| \to 0$ as $n \to \infty$, where $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$. This completes the proof. □

5.3 Existence and Algorithm for the System of Mixed Variational Inequalities

We first introduce and consider the *system of mixed variational inequalities* (SMVI): is to find $\hat{x}$, $\hat{y}$, $\hat{z} \in C$ such that

$$\begin{cases}
\langle \delta_1 T_1 \hat{z} + J\hat{x} - J\hat{z}, y - \hat{x}\rangle + f_1(y) - f_1(\hat{x}) \geq 0, \ \forall y \in C, \\
\langle \delta_2 T_2 \hat{x} + J\hat{y} - J\hat{x}, y - \hat{y}\rangle + f_2(y) - f_2(\hat{y}) \geq 0, \ \forall y \in C, \\
\langle \delta_3 T_3 \hat{y} + J\hat{z} - J\hat{y}, y - \hat{z}\rangle + f_3(y) - f_3(\hat{z}) \geq 0, \ \forall y \in C,
\end{cases} \tag{5.3.1}$$

where $\delta_j > 0$, $T_j : C \to E^*$, $f_j : C \to \mathbb{R} \cup \{+\infty\}$ for $j = 1, 2, 3$ are mappings and J is the normalized duality mapping from E to E^*.

As special case of the problem (5.3.1), we have the following.

If $f_j(x) = 0$ for $j = 1, 2, 3$, $\forall x \in C$, (5.3.1) is equivalent to find $\hat{x}$, $\hat{y}$ and $\hat{z} \in C$ such that

$$\begin{cases} \langle \delta T_1 \hat{z} + J\hat{x} - J\hat{z}, y - \hat{x} \rangle \geq 0, \ \forall y \in C, \\ \langle \delta_2 T_2 \hat{x} + J\hat{y} - J\hat{x}, y - \hat{y} \rangle \geq 0, \ \forall y \in C, \\ \langle \delta_3 T_3 \hat{y} + J\hat{z} - J\hat{y}, y - \hat{z} \rangle \geq 0, \ \forall y \in C. \end{cases} \tag{5.3.2}$$

The problem (5.3.2) is call the *system of variational inequalities*. We denote by (SVI).

If $T_2 = T_3$, $f_2(x) = f_3(x)$, $\forall x \in C$ and $\hat{y} = \hat{z}$, then (5.3.1) is reduced to find $\hat{x}, \hat{y} \in C$ such that

$$\begin{cases} \langle \delta_1 T_1 \hat{y} + J\hat{x} - J\hat{y}, y - \hat{x} \rangle + f_1(y) - f_1(\hat{x}) \geq 0, \ \forall y \in C, \\ \langle \delta_2 T_2 \hat{x} + J\hat{y} - J\hat{x}, y - \hat{y} \rangle + f_2(y) - f_2(\hat{y}) \geq 0, \ \forall y \in C, \end{cases} \tag{5.3.3}$$

which is studied by Zhang et al. [338].

If $T = T_1 = T_2 = T_3$, $f_1(x) = f_2(x) = f_3(x)$, $\forall x \in C$ and $\hat{x} = \hat{y} = \hat{z}$, (5.3.1) is reduced to find $\hat{x}$ such that

$$\langle T\hat{x}, y - \hat{x} \rangle + f_1(y) - f_1(\hat{x}) \geq 0, \ \forall y \in C. \tag{5.3.4}$$

This iterative method is studied by Wu and Huang [326].

If $f_1(x) = 0, \forall x \in C$, (5.3.4) is reduced to find $\hat{x}$ such that

$$\langle T\hat{x}, y - \hat{x} \rangle \geq 0, \ \forall y \in C. \tag{5.3.5}$$

which is studied by Alber [319, 320], Li [72] and Fan [322]. If $E = H$ is a Hilbert space, (5.3.5) which is known as the *classical variational inequality* introduced and studied by Stampacchia [324].

If $E = H$ is a Hilbert space, then (5.3.1) is reduced to find $\hat{x}, \hat{y}, \hat{z} \in C$ such that

$$\begin{cases} \langle \delta_1 T_1 \hat{z} + \hat{x} - \hat{z}, y - \hat{x} \rangle + f_1(y) - f_1(\hat{x}) \geq 0, \ \forall y \in C, \\ \langle \delta_2 T_2 \hat{x} + \hat{y} - \hat{x}, y - \hat{y} \rangle + f_2(y) - f_2(\hat{y}) \geq 0, \ \forall y \in C, \\ \langle \delta_3 T_3 \hat{y} + \hat{z} - \hat{y}, y - \hat{z} \rangle + f_3(y) - f_3(\hat{z}) \geq 0, \ \forall y \in C. \end{cases} \tag{5.3.6}$$

If $f_j(x) = 0$ for $j = 1, 2, 3$, $\forall x \in C$, (5.3.6) reduces to the following (SVI):

$$\begin{cases} \langle \delta_1 T_1 \hat{z} + \hat{x} - \hat{z}, y - \hat{x} \rangle \geq 0, \ \forall y \in C, \\ \langle \delta_2 T_2 \hat{x} + \hat{y} - \hat{x}, y - \hat{y} \rangle \geq 0, \ \forall y \in C, \\ \langle \delta_3 T_3 \hat{y} + \hat{z} - \hat{y}, y - \hat{z} \rangle \geq 0, \ \forall y \in C. \end{cases} \tag{5.3.7}$$

The purpose of this paper is to study the existence and convergence analysis of solutions of the system of mixed variational inequalities in Banach spaces by using the generalized f-projection operator. The results presented in this paper improve and extend important recent results in the literature.

We also need the following lemmas for the proof of our main results.

Lemma 5.3.1. (*Xu*[327]) *Let $q > 1$ and $r > 0$ be two fixed real numbers. Let E be a q-uniformly convex Banach space if and only if there exists a continuous strictly increasing and convex function $g : [0, +\infty) \to [0, +\infty)$, $g(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^q + (1-\lambda)\|y\|^q - \varsigma_q(\lambda)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0, 1]$, where $\varsigma_q(\lambda) = \lambda(1-\lambda)^q + \lambda^q(1-\lambda)$.

For case $q = 2$, we have

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|).$$

Lemma 5.3.2. (*Change*[321]) *Let E be a uniformly convex and uniformly smooth Banach spaces. We have the following holds:*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J^*(\phi + \Phi)\rangle, \ \forall \phi, \Phi \in E^*.$$

Next we recall the concept of the generalized f-projection operator. Let $G : E^* \times C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, x) = \|\xi\|^2 - 2\langle \xi, x\rangle + \|x\|^2 + 2\rho f(x), \tag{5.3.8}$$

where $\xi \in E^*$, ρ is positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From definitions of G and f, it is easy to see the following properties:

(1) $(\|\xi\| - \|x\|)^2 + 2\rho f(x) \leq G(\xi, x) \leq (\|\xi\| + \|x\|)^2 + 2\rho f(x)$;

(2) $G(\xi, x)$ is convex and continuous with respect to x when ξ is fixed;

(3) $G(\xi, x)$ is convex and lower semicontinuous with respect to ξ when x is fixed.

Definition 5.3.1. *Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is* generalized f-projection operator *if*

$$\Pi_C^f \xi = \{u \in C : G(\xi, u) = \inf_{y \in C} G(\xi, y)\}, \ \forall \xi \in E^*.$$

In this paper, we fixed $\rho = 1$, we have

$$G(\xi, x) = \|\xi\|^2 - 2\langle \xi, x\rangle + \|x\|^2 + 2f(x).$$

For the generalized f-projection operator, Wu and Hung [326] proved the following basic properties.

Lemma 5.3.3. (*Wu and Hung* [325]) *Let E be a reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. The following statement holds:*

(1) $\Pi_C^f \xi$ *is nonempty closed convex subset of C for all $\xi \in E^*$;*

(2) *if E is smooth, then for all $\xi \in E^*$, $x \in \Pi_C^f \xi$ if and only if*

$$\langle \xi - Jx, x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \ \forall y \in C;$$

(3) *if E is smooth, then for any $\xi \in E^*$, $\Pi_C^f \xi = (J + \rho \partial f)^{-1} \xi$, where ∂f is the subdifferential of the proper convex and lower semi-continuous functional f.*

Lemma 5.3.4. (*Wu and Hung* [325]) *If $f(x) \geq 0$ for all $x \in C$, then for any $\rho > 0$,*

$$G(Jx, y) \leq G(\xi, y) + 2\rho f(y), \ \forall \xi \in E^*, \, y \in C, \, x \in \Pi_C^f \xi.$$

Lemma 5.3.5. (*Fan et al.* [323]) *Let E be a reflexive strictly convex Banach space with its dual E^* and C is a nonempty closed convex subset of E. If $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous, then*

(1) $\Pi_C^f : E^* \to C$ *is single valued and norm to weak continuous;*

(2) *if E has the property (h), that is, for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, implies $x_n \to x$, then $\Pi_C^f : E^* \to C$ is continuous.*

Defined the functional $G_2 : E \times C \to \mathbb{R} \cup \{+\infty\}$ by

$$G_2(x, y) = G(Jx, y), \ \forall \, x \in E, \ y \in C.$$

5.3.1 Generalized Projection Algorithms

Proposition 5.3.6. *Let C be a nonempty closed and convex subset of a reflexive strictly convex and smooth Banach space E. If $f_j : C \to \mathbb{R} \cup \{+\infty\}$ for $j = 1, 2, 3$ is proper, convex and lower semi-continuous, then $(\hat{x}, \hat{y}, \hat{z})$ is a solution of (SMVI) is equivalent to finding $\hat{x}, \hat{y}, \hat{z}$ such that*

$$\begin{cases} \hat{x} = \Pi_C^{f_1}(J\hat{z} - \delta_1 T_1 \hat{z}), \\ \hat{y} = \Pi_C^{f_2}(J\hat{x} - \delta_2 T_1 \hat{x}), \\ \hat{z} = \Pi_C^{f_3}(J\hat{y} - \delta_3 T_1 \hat{y}). \end{cases} \tag{5.3.9}$$

***Proof*.** From Lemma 5.3.3 (2) and E is a reflexive strictly convex and smooth Banach space, we known that J is single valued and $\Pi_C^{f_j}$ for $j = 1, 2, 3$ is well defined and single valued. So, we can conclude that Proposition 5.3.9 holds. □

For solving the system of mixed variational inequality (5.3.1), we defined some projection algorithms as follow:

Algorithm 5.3.7. *For an initial point $x_0, z_0 \in C$, we define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as follows:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} = \Pi_C^{f_2}(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} = \Pi_C^{f_3}(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{cases} \tag{5.3.10}$$

where $0 < a \leq \alpha_n \leq b < 1$.

If $f_j(x) = 0$, $j = 1, 2, 3$, for all $x \in C$ then Algorithm 5.3.7 reduces to the following iterative method for solving the system of variational inequalities (5.3.2).

Algorithm 5.3.8. *For an initial point $x_0, z_0 \in C$, we define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as follows:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Pi_C(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} = \Pi_C(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} = \Pi_C(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{cases} \tag{5.3.11}$$

where $0 < a \leq \alpha_n \leq b < 1$.

For solving the problem (5.3.6), we defined the algorithm as follows:

If $E = H$ is a Hilbert space, then Algorithm 5.3.7 reduces to the following.

Algorithm 5.3.9. *For an initial point $x_0, z_0 \in C$, we define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as follows:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} = \Pi_C^{f_2}(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} = \Pi_C^{f_3}(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{cases} \tag{5.3.12}$$

where $0 < a \leq \alpha_n \leq b < 1$.

If $f_j(x) = 0$, $j = 1, 2, 3$, for all $x \in C$, then Algorithm 5.3.9 reduces to the following iterative method for solving the problem (5.3.7) as follow:

Algorithm 5.3.10. *For an initial point $x_0, z_0 \in C$, we define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as follows:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} = P_C(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} = P_C(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{cases} \tag{5.3.13}$$

where $0 < a \leq \alpha_n \leq b < 1$.

5.3.2 Existence and Convergence Analysis

Theorem 5.3.11. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with dual space E^*. If the mapping $T_j : C \to E^*$ and $f_j : C \to \mathbb{R} \cup \{+\infty\}$ which is convex lower semi-continuous mappings for $j = 1, 2, 3$ satisfy the following conditions:*

(i) $\langle T_j x, J^*(Jx - \delta_j T_j x)\rangle \geq 0, \ \forall x \in C$ *for* $j = 1, 2, 3$*;*

(ii) $(J - \delta_j T_j)$ *are compact for* $j = 1, 2, 3$*;*

(iii) $f_j(0) = 0$ *and* $f_j(x) \geq 0, \ \forall x \in C$ *and* $j = 1, 2, 3$*.*

Then the system of mixed variational inequality (5.3.1) *have a solution* $(\hat{x}, \hat{y}, \hat{z})$ *and sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *defined by Algorithm 5.3.7 have convergent subsequences* $\{x_{n_i}\}, \{y_{n_i}\}$ *and* $\{z_{n_i}\}$*such that*

$$x_{n_i} \to \hat{x}, \ i \to \infty,$$

$$y_{n_i} \to \hat{y}, \ i \to \infty,$$

$$z_{n_i} \to \hat{z}, \ i \to \infty.$$

***Proof*.** Since E is uniformly convex and uniform smooth Banach spaces, we known that J is bijection from E to E^* and uniformly continuous on any bounded subsets of E. Hence $\Pi_C^{f_j}$ for $j = 1, 2, 3$ is well defined and single value implies that, $\{x_n\}, \{y_n\}$ and $\{z_n\}$ is well defined. Let $G_2(x, y) = G(Jx, y)$, for any $x \in C$ and $y = 0$, we have

$$\begin{aligned} G_2(x, 0) &= G(Jx, 0) \\ &= \|Jx\|^2 - 2\langle Jx, 0\rangle + 2f(0) \\ &= \|Jx\|^2 \\ &= \|x\|^2. \end{aligned} \tag{5.3.14}$$

By (5.3.14) and Lemma 5.3.4, we have

$$\begin{aligned} G_2(\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n), 0) &= G(J(\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)), 0) \\ &\leq G(Jz_n - \delta_1 T_1 z_n, 0) \\ &= \|Jz_n - \delta_1 T_1 z_n\|^2. \end{aligned} \tag{5.3.15}$$

From Lemma 5.3.2, and for all $x \in C$, $\langle T_1 x, J^*(Jx - \delta_1 T_1 x)\rangle \geq 0$, so for $z_n \in C$, we obtain

$$\begin{aligned} \|Jz_n - \delta_1 T_1 z_n\|^2 &\leq \|Jz_n\|^2 - 2\langle \delta_1 T_1 z_n, J^*(Jz_n - \delta_1 T_1 z_n)\rangle \\ &\leq \|Jz_n\|^2 \\ &\leq \|z_n\|^2. \end{aligned} \tag{5.3.16}$$

Again by Lemma 5.3.2, for all $x \in C$, $\langle T_2 x, J^*(Jx - \delta_2 T_2 x)\rangle \geq 0$, and for $x_{n+1} \in C$, we have

$$\begin{aligned}
\|y_{n+1}\|^2 &= G_2(y_{n+1}, 0) \\
&= G(Jy_{n+1}, 0) \\
&= G(J\Pi_C^{f_2}(Jx_{n+1} - \delta_2 T_2 x_{n+1}), 0) \\
&\leq G(Jx_{n+1} - \delta_2 T_2 x_{n+1}, 0) \\
&\leq \|Jx_{n+1} - \delta_2 T_2 x_{n+1}\|^2 \\
&\leq \|Jx_{n+1}\|^2 - 2\langle \delta_2 T_2 x_{n+1}, J^*(Jx_{n+1} - \delta_2 T_2 x_{n+1})\rangle \\
&\leq \|Jx_{n+1}\|^2 \\
&\leq \|x_{n+1}\|^2.
\end{aligned} \tag{5.3.17}$$

In similar way, for all $x \in C$, $\langle T_3 x, J^*(Jx - \delta_3 T_3 x)\rangle \geq 0$, and $z_{n+1} \in C$, we also have

$$\begin{aligned}
\|z_{n+1}\|^2 &= G(Jz_{n+1}, 0) \\
&\leq G(Jy_{n+1} - \delta_3 T_3 y_{n+1}, 0) \\
&= \|Jy_{n+1} - \delta_3 T_3 y_{n+1})\|^2 \\
&\leq \|Jy_{n+1}\|^2 - 2\langle \delta_3 T_3 y_{n+1}, J^*(Jy_{n+1} - \delta_3 T_3 y_{n+1})\rangle \\
&\leq \|y_{n+1}\|^2.
\end{aligned} \tag{5.3.18}$$

It follows from (5.3.17) and (5.3.18) that

$$\|z_{n+1}\|^2 \leq \|x_{n+1}\|^2, \ \forall n \in \mathbb{N}. \tag{5.3.19}$$

From (5.3.17) and (5.3.18), we compute

$$\begin{aligned}
\|x_{n+1}\|^2 &\leq (1 - \alpha_n)\|x_n\| + \alpha_n \|\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| \\
&\leq (1 - \alpha_n)\|x_n\| + \alpha_n \|z_n\| \\
&\leq (1 - \alpha_n)\|x_n\| + \alpha_n \|y_n\| \\
&\leq (1 - \alpha_n)\|x_n\| + \alpha_n \|x_n\| \\
&= \|x_n\|.
\end{aligned} \tag{5.3.20}$$

This implies that the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\}$ are bounded. For a positive number r such that $\{x_n\}, \{y_n\}, \{z_n\}, \{\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\} \in B_r$, by Lemma 5.3.1, for $q = 2$ there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for $\alpha_n \in [0, 1]$, we have

$$\begin{aligned}
\|x_{n+1}\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\|^2 \\
&\leq (1 - \alpha_n)\|x_n\|^2 + \alpha_n \|\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\|^2 \\
&\quad -\alpha_n(1 - \alpha_n) g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| \\
&= (1 - \alpha_n)\|x_n\|^2 + \alpha_n G_2(\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n, 0)) \\
&\quad -\alpha_n(1 - \alpha_n) g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\|.
\end{aligned} \tag{5.3.21}$$

Applying (5.3.15), (5.3.16) and (5.3.19), we have

$$\begin{aligned} & \alpha_n(1-\alpha_n)g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| \\ \le\ & (1-\alpha_n)\|x_n\|^2 - \|x_{n+1}\|^2 + \alpha_n G_2(\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n), 0) \\ \le\ & (1-\alpha_n)\|x_n\|^2 - \|x_{n+1}\|^2 + \alpha_n\|x_n\|^2 \\ =\ & \|x_n\|^2 - \|x_{n+1}\|^2. \end{aligned} \tag{5.3.22}$$

Summing (5.3.22), for $n = 0, 1, 2, 3, ..., k$, we have

$$\sum_{n=0}^{k} \alpha_n(1-\alpha_n)g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| \le \|x_0\|^2 - \|x_{k+1}\|^2 \le \|x_0\|^2,$$

taking $k \to \infty$, we get

$$\textstyle\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| \le \|x_0\|^2. \tag{5.3.23}$$

This show that series (5.3.23) is converge, we obtain that

$$\lim_{n\to\infty} \alpha_n(1-\alpha_n)g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| = 0. \tag{5.3.24}$$

From $0 < a \le \alpha_n \le b < 1$ for all n, thus $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) > 0$ and (5.3.24), we have

$$\lim_{n\to\infty} g\|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| = 0. \tag{5.3.25}$$

By property of functional g, we have

$$\lim_{n\to\infty} \|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| = 0. \tag{5.3.26}$$

Since $\{z_n\}$ is bounded sequence and $(J - \delta_1 T_1)$ is compact on C, then sequence $\{Jz_n - \delta_1 T_1 z_n\}$ have a convergence subsequence such that

$$\{Jz_{n_i} - \delta_1 T_1 z_{n_i}\} \to w_0 \in E^* \text{ as } i \to \infty. \tag{5.3.27}$$

By the continuity of the $\Pi_C^{f_1}$, we have

$$\lim_{i\to\infty} \Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) = \Pi_C^{f_1}(w_0). \tag{5.3.28}$$

Again since $\{x_n\}$, $\{y_n\}$ are bounded and $(J - \delta_2 T_2)$, $(J - \delta_3 T_3)$ are compact on C, then sequences $\{Jx_n - \delta_2 T_2 x_n\}$ and $\{Jy_n - \delta_3 T_3 y_n\}$ have convergence subsequences such that

$$\{Jx_{n_i} - \delta_2 T_2 x_{n_i}\} \to u_0 \in E^* \text{ as } i \to \infty, \tag{5.3.29}$$

and

$$\{Jy_{n_i} - \delta_3 T_3 y_{n_i}\} \to v_0 \in E^* \text{ as } i \to \infty. \tag{5.3.30}$$

By the continuity of $\Pi_C^{f_2}$ and $\Pi_C^{f_3}$, we have

$$\lim_{i\to\infty} \Pi_C^{f_2}(Jx_{n_i} - \delta_2 T_2 x_{n_i}) = \Pi_C^{f_2}(u_0), \tag{5.3.31}$$

and

$$\lim_{i\to\infty} \Pi_C^{f_3}(Jy_{n_i} - \delta_3 T_3 y_{n_i}) = \Pi_C^{f_3}(v_0). \tag{5.3.32}$$

Let

$$\Pi_C^{f_1}(w_0) = \hat{x}, \tag{5.3.33}$$

$$\Pi_C^{f_2}(u_0) = \hat{y}, \tag{5.3.34}$$

$$\Pi_C^{f_3}(v_0) = \hat{z}. \tag{5.3.35}$$

By using the triangle inequality, we have

$$\|x_{n_i} - \hat{x}\| \leq \|x_{n_i} - \Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i})\| + \|\Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) - \hat{x}\|.$$

From (5.3.26) and (5.3.28), we have

$$\lim_{i\to\infty} x_{n_i} = \hat{x}. \tag{5.3.36}$$

By definition of z_{n_i}, we get

$$\|z_{n_i} - \hat{z}\| \leq \|\Pi_C^{f_3}(Jy_{n_i} - \delta_3 T_3 y_{n_i}) - \hat{z}\|.$$

It follows by (5.3.32) and (5.3.35), we obtain

$$\lim_{i\to\infty} z_{n_i} = \hat{z}. \tag{5.3.37}$$

In the same way, we also have

$$\lim_{i\to\infty} y_{n_i} = \hat{y}. \tag{5.3.38}$$

By the continuity properties of $(J-\delta_1 T_1)$, $(J-\delta_2 T_2)$, $(J-\delta_3 T_3)$ and $\Pi_C^{f_j}$ for $j = 1, 2, 3$. We conclude that

$$\hat{x} = \Pi_C^{f_1}(J\hat{z} - \delta_1 T_1 \hat{z})$$
$$\hat{y} = \Pi_C^{f_2}(J\hat{x} - \delta T_2 \hat{x})$$
$$\hat{z} = \Pi_C^{f_3}(J\hat{y} - \delta_3 T_3 \hat{y}).$$

This complete of proof. □

Theorem 5.3.12. *Let C be a nonempty compact and convex subset of a uniformly convex and uniformly smooth Banach space E with dual space E^*. If the mapping $T_j : C \to E^*$ and $f_j : C \to \mathbb{R} \cup \{+\infty\}$ which is convex lower semi-continuous mappings for $j = 1, 2, 3$ satisfy the following conditions:*

(i) $\langle T_j x, J^*(Jx - \delta_j T_j x)\rangle \geq 0,\ \forall x \in C$ *for* $j = 1, 2, 3$*;*

(ii) $f_j(0) = 0$ *and* $f_j(x) \geq 0,\ \forall x \in C$ *for* $j = 1, 2, 3$*.*

Then the system of mixed variational inequality (5.3.1) *has a solution* $(\hat{x}, \hat{y}, \hat{z})$ *and sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *defined by Algorithm 5.3.7 have a convergent subsequences* $\{x_{n_i}\}, \{y_{n_i}\}$ *and* $\{z_{n_i}\}$ *such that*

$$x_{n_i} \to \hat{x},\ i \to \infty,$$

$$y_{n_i} \to \hat{y},\ i \to \infty,$$

$$z_{n_i} \to \hat{z},\ i \to \infty.$$

***Proof*.** In the same way to the proof in Theorem 5.3.11, we have

$$\lim_{n\to\infty} \|x_n - \Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\| = 0. \tag{5.3.39}$$

Hence there exist subsequences $\{x_{n_i}\} \subset \{x_n\}$ and $\{z_{n_i}\} \subset \{z_n\}$ such that

$$\lim_{i\to\infty} \|x_{n_i} - \Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i})\| = 0. \tag{5.3.40}$$

From the compactness of C, we have that

$$\{x_{n_i}\} \to \hat{x} \text{ as } i \to \infty$$

and

$$\{z_{n_i}\} \to \hat{z} \text{ as } i \to \infty,$$

where $\hat{x}, \hat{z}$ are points in C. Also for a sequence $\{y_n\} \supset \{y_{n_i}\} \to \hat{y}$ as $i \to \infty$, where $\hat{y}$ is a points in C. By the continuity properties of J, T_2, T_3 $\Pi_C^{f_2}$ and $\Pi_C^{f_3}$, we obtain that

$$\hat{y} = \Pi_C^{f_2}(J\hat{x} - \delta_2 T_2 \hat{x})$$

and

$$\hat{z} = \Pi_C^{f_3}(J\hat{y} - \delta_3 T_3 \hat{y}).$$

From definition of x_{n+1}, we get

$$\begin{aligned}
& \|\Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) - \hat{x}\| \\
= \ & \|\Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) - \hat{x} + x_{n_i+1} - (1-\alpha_n)x_{n_i} - \alpha_n \Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i})\| \\
= \ & \|x_{n_i+1} - \hat{x} + (1-\alpha_n)(\Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) - x_{n_i}\| \\
\leq \ & \|x_{n_i+1} - \hat{x}\| + (1-\alpha_n)\|x_{n_i} - \Pi_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i})\|.
\end{aligned}$$

By (5.3.36) and (5.3.39), we have

$$\hat{x} = \Pi_C^{f_1}(J\hat{z} - \delta_1 T_1 \hat{z}).$$

This complete of proof. □

Corollary 5.3.13. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with dual space E^*. If the mapping $T_j : C \to E^*$ for $j = 1, 2, 3$ satisfy the following conditions:*

(i) $\langle T_j x, J^*(Jx - \delta_j T_j x)\rangle \geq 0, \ \forall x \in C$ *for* $j = 1, 2, 3$*;*

(ii) $(J - \delta_j T_j)$ *are compact for* $j = 1, 2, 3$.

Then the system of mixed variational inequality (5.3.2) *have a solution* $(\hat{x}, \hat{y}, \hat{z})$ *and sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *defined by Algorithm 5.3.8 have convergent subsequences* $\{x_{n_i}\}, \{y_{n_i}\}$ *and* $\{z_{n_i}\}$ *such that* $x_{n_i} \to \hat{x},\ i \to \infty,\ y_{n_i} \to \hat{y},\ i \to \infty$ *and* $z_{n_i} \to \hat{z},\ i \to \infty$.

If $E = H$ is a Hilbert space, then $H^* = H,\ J^* = J = I$, so we obtain the following corollary.

Corollary 5.3.14. *Let C be a nonempty closed and convex subset of a Hilbert space H. If the mapping $T_j : C \to H$ and $f_j : C \to \mathbb{R} \cup \{+\infty\}$ which is convex lower semi-continuous mappings for $j = 1, 2, 3$ satisfy the following conditions:*

(i) $\langle T_j x, x - \delta_j T_j x\rangle \geq 0$ *for* $j = 1, 2, 3$*;*

(ii) $f_j(0) = 0$ *and* $f_j(x) \geq 0$ *for all* $x \in C$ *for* $j = 1, 2, 3$.

Then the system of mixed variational inequality (5.3.6) *have a solution* $(\hat{x}, \hat{y}, \hat{z})$ *and sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *defined by Algorithm 5.3.9 have a convergent subsequences* $\{x_{n_i}\}, \{y_{n_i}\}$ *and* $\{z_{n_i}\}$ *such that* $x_{n_i} \to \hat{x},\ i \to \infty,\ y_{n_i} \to \hat{y},\ i \to \infty$ *and* $z_{n_i} \to \hat{z},\ i \to \infty$.

Corollary 5.3.15. *Let C be a nonempty closed and convex subset of a Hilbert space H. If the mapping $T_j : C \to H$ for $j = 1, 2, 3$ satisfy the conditions: $\langle T_j x, x - \delta_j T_j x\rangle \geq 0$ for $j = 1, 2, 3$. Then the system of mixed variational inequality* (5.3.7) *have a solution $(\hat{x}, \hat{y}, \hat{z})$ and sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by Algorithm 5.3.10 have a convergent subsequences $\{x_{n_i}\}, \{y_{n_i}\}$ and $\{z_{n_i}\}$ such that $x_{n_i} \to \hat{x},\ i \to \infty,\ y_{n_i} \to \hat{y},\ i \to \infty$ and $z_{n_i} \to \hat{z},\ i \to \infty$.*

Remark 5.3.16. Theorem 5.3.11, 5.3.12 and Corollary 5.3.13 extend and improve the results of Zhang et al. [338] and Wu and Huang [326].

5.4 Variational Inequality Inclusion and Nonexpansive Semigroups

Let $B : H \longrightarrow H$ be a single-valued nonlinear mapping and $M : H \longrightarrow 2^H$ be a set-valued mapping. The *variational inclusion problem* is to find $\hat{x} \in H$ such that

$$\theta \in B(\hat{x}) + M(\hat{x}), \tag{5.4.1}$$

where θ is the zero vector in H. The set of solutions of problem (5.4.1) is denoted by $I(B, M)$. A set-valued mapping $M : H \longrightarrow 2^H$ is called *monotone* if for all $x, y \in H,\ f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H,\ \langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Definition5.4.1. A family $\mathcal{S} = \{S(s) : 0 \leq s \leq \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(1) $S(0)x = x$ for all $x \in C$;
(2) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$;
(3) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(4) for all $x \in C,\ s \mapsto S(s)x$ is continuous.

We denoted by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S} = \{S(s) : s \geq 0\}$, i.e., $F(\mathcal{S}) = \cap_{s\geq 0}F(S(s))$. It is know that $F(\mathcal{S})$ is closed and convex.

Definition5.4.2. Let $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|,\ \forall x, y \in C.$$

Let $\mathcal{K} : C \to \mathcal{R}$ be a differentiable functional on a convex set C, which is called:

(1) η*-convex* if

$$\mathcal{K}(y) - \mathcal{K}(x) \geq \Big\langle \mathcal{K}'(x), \eta(y, x)\Big\rangle,\ \forall x, y \in C,$$

where $\mathcal{K}'(x)$ is the Fréchet derivative of $\mathcal{K}$ at x;

(2) η*-strongly convex* if there exists a constant $\sigma > 0$ such that

$$\mathcal{K}(y) - \mathcal{K}(x) - \Big\langle \mathcal{K}'(x), \eta(y, x)\Big\rangle \geq \frac{\sigma}{2}\|x - y\|^2,\ \forall x, y \in C.$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in C$, then $\mathcal{K}$ is said to be *strongly convex*.

Definition5.4.3. Let $M : H \longrightarrow 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \longrightarrow H$ defined by

$$J_{M,\lambda}(\hat{x}) = (I + \lambda M)^{-1}(\hat{x}), \quad \hat{x} \in H \tag{5.4.2}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. The following characterizes the resolvent operator.

(R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \ \forall x, y \in H \ \text{ and } \ \forall \lambda > 0.$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; see([252]), that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \ \forall x, y \in H.$$

(R3) The solution of problem (5.4.1) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$, that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \ \forall \lambda > 0.$$

(R4) If $0 < \lambda \leq 2\beta$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \longrightarrow H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

Lemma 5.4.4. [252] *Let $M : H \longrightarrow 2^H$ be a maximal monotone mapping and let $B : H \longrightarrow H$ be a Lipshitz continuous mapping. Then the mapping $L = M + B : H \longrightarrow 2^H$ is a maximal monotone mapping.*

Lemma 5.4.5. *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(1). The weak ω-limit set $\omega_w(x_n) \subset C$,

(2). For each $z \in C$, $\lim_{n\longrightarrow\infty} \|x_n - z\|$ exists.

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 5.4.6. [232]. *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$, hold for each $y \in H$ with $y \neq x$.*

Lemma 5.4.7. [239] *Each Hilbert space H, satisfies the Kadec-Klee property, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ together imply $\|x_n - x\| \longrightarrow 0$.*

For solving the system of mixed equilibrium problem, let us assume that function $F_k : C \times C \longrightarrow \mathcal{R}, \ k = 1, 2, ..., N$ satisfies the following conditions:

(H1) F_k is monotone, i.e., $F_k(x,y)+F_k(y,x)\leq 0, \quad \forall x,y\in C$;

(H2) for each fixed $y\in C$, $x\mapsto F_k(x,y)$ is convex and upper semicontinuous;

(H3) for each fixed $x\in C, y\mapsto F_k(x,y)$ is convex.

Lemma 5.4.8. [223] *Let C be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from C to $\mathcal{R}$. Let F be a bifunction from $C\times C$ to $\mathcal{R}$ satisfying (H1)-(H3). Assume that*

(i) $\eta : C\times C\to H$ is k Lipschitz continuous with constant $k>0$ such that;

(a) $\eta(x,y)+\eta(y,x)=0, \quad \forall x,y\in C,$

(b) $\eta(\cdot,\cdot)$ is affine in the first variable,

(c) for each fixed $x\in C$, $y\mapsto\eta(x,y)$ is sequentially continuous from the weak topology to the weak topology,

(ii) $\mathcal{K}:C\to\mathcal{R}$ is η-strongly convex with constant $\sigma>0$ and its derivative $\mathcal{K}'$ is sequentially continuous from the weak topology to the strong topology;

(iii) for each $x\in C$, there exist a bounded subset $D_x\subset C$ and $z_x\in C$ such that for any $y\in C\backslash D_x$,

$$F(y,z_x)+\varphi(z_x)-\varphi(y)+\frac{1}{r}\Big\langle\mathcal{K}'(y)-\mathcal{K}'(x),\eta(z_x,y)\Big\rangle<0.$$

For given $r>0$, Let $K_r^F : C\to C$ be the mapping defined by:

$$K_r^F(x)=\Big\{y\in C : F(y,z)+\varphi(z)-\varphi(y)+\frac{1}{r}\Big\langle\mathcal{K}'(y)-\mathcal{K}'(x),\eta(z,y)\Big\rangle\geq 0, \quad \forall z\in C\Big\} \tag{5.4.3}$$

for all $x\in C$. Then the following hold

(1) K_r^F is single-valued;

(2) K_r^F is nonexpansive if $\mathcal{K}'$ is Lipschitz continuous with constant $\nu>0$ such that $\sigma\geq k\nu$;

(3) $F(K_r^F)=MEP(F,\varphi)$;

(4) $MEP(F,\varphi)$ is closed and convex.

Lemma 5.4.9. [184] *Let $V : C\to H$ be a ξ-strict pseudo-contraction, then*
(1) the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is

well defined;

(2) define a mapping $T : C \to H$ by

$$Tx = tx + (1-t)Vx, \forall x \in C \tag{5.4.4}$$

If $t \in [\xi, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly ξ-strict pseudo-contractions*, if there exists a constant $\xi \in [0, 1)$ such that

$$\|V_i x - V_i y\|^2 \leq \|x - y\|^2 + \xi\|(I - V_i)x - (I - V_i)y\|^2, \ \forall x, y \in C, \forall i \geq 1.$$

Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions. Let
$\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (5.4.4), i.e.,

$$T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [\xi, 1) \tag{5.4.5}$$

Let $\{T_i\}$ be a sequence of nonexpansive mappings of C into itself defined by (5.4.5) and let $\{\mu_i\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\
&\vdots \\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\
&\vdots \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I.
\end{aligned} \tag{5.4.6}$$

Such a mapping W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$. For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (6.1.4). Then we can have the following crucial conclusions concerning W_n.

Lemma 5.4.10. [238]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_i \leq b < 1$ for every $i \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using this Lemma, one can define a mapping $U_{\infty,k}$ and $W : C \to C$ as follows $U_{\infty,k}x = \lim_{n\to\infty} U_{n,k}x$ and

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \forall x \in C \tag{5.4.7}$$

Such a mapping W is called the W-mapping. Since W_n is nonexpansive and $F(W) = \cap_{i=1}^{\infty} F(T_i)$, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$ such that

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \le \|x - y\|.$$

Lemma 5.4.11. [238] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \le b < 1, \forall i \ge 1$. Then $F(W) = \cap_{i=1}^{\infty} F(T_i)$.*

Lemma 5.4.12. *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \le b < 1, \forall i \ge 1$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

Lemma 5.4.13. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \ge 0$,*

$$\lim_{t\longrightarrow\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)xds - T(h)\Big(\frac{1}{t}\int_0^t T(s)xds\Big) \right\| = 0.$$

Lemma 5.4.14. *Let C be a nonempty bounded closed convex subset of H, $\{x_n\}$ be a sequence in C and $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C. If the following conditions are satisfied:*

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{s\longrightarrow\infty} \limsup_{n\longrightarrow\infty} \|S(s)x_n - x_n\| = 0$, *then* $z \in F(\mathcal{S})$.

Theorem 5.4.15. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R}, \quad k = 1, 2, \ldots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \ge 1, t \in [\xi, 1)$, W_n be the W-mapping defined by*

(6.1.4) *and* W *be a mapping defined by* (6.1.5) *with* $F(W) \neq \emptyset$. *Let* $A, B : C \to H$ *be* γ, β*-inverse-strongly monotone mappings and* $M_1, M_2 : H \longrightarrow 2^H$ *be maximal monotone mappings such that*

$$\Theta := F(\mathcal{S}) \cap F(W) \cap \left(\cap_{k=1}^{N} SMEP(F_k)\right) \cap I(A, M_1) \cap I(B, M_2) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \ldots, N$, *which are constants. Let* $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ *and*

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ u_n = K^{F_N}_{r_{N,n}} K^{F_{N-1}}_{r_{N-1,n}} K^{F_{N-2}}_{r_{N-2,n}} \ldots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}} x_n, \\ y_n = J_{M_2,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M_1,\lambda_n}(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1-\alpha_n) \dfrac{1}{t_n} \displaystyle\int_0^{t_n} S(s) W_n v_n ds, \\ C_{n+1} = \left\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n) \left\| v_n - \dfrac{1}{t_n} \displaystyle\int_0^{t_n} S(s) W_n v_n ds \right\|^2 \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{5.4.8}$$

where $K^{F_k}_{r_k} : C \to C, \quad k = 1, 2, \ldots, N$ *is the mapping defined by* (5.4.3) *and* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *for all* $n \in \mathbb{N}$. *Assume the following conditions are satisfied:*

(C1) $\eta_k : C \times C \to H$ *is* L_k*-Lipschitz continuous with constant* $k = 1, 2, \ldots, N$ *such that*

- *(a)* $\eta_k(x, y) + \eta_k(y, x) = 0, \quad \forall x, y \in C,$
- *(b)* $x \mapsto \eta_k(x, y)$ *is affine,*
- *(c) for each fixed* $y \in C$, $y \mapsto \eta_k(x, y)$ *is sequentially continuous from the weak topology to the weak topology;*

(C2) $\mathcal{K}_k : C \to \mathcal{R}$ *is* η_k*-strongly convex with constant* $\sigma_k > 0$ *and its derivative* $\mathcal{K}'_k$ *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant* $\nu_k > 0$ *such that* $\sigma_k > L_k \nu_k$;

(C3) For each $k \in \{1, 2, \ldots, N\}$ *and for all* $x \in C$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \in C \backslash D_x$,

$$F_k(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_k} \Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y) \Big\rangle < 0;$$

(C4) $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (\xi, 1)$;

(C5) $\{\lambda_n\} \subset [a_1, b_1]$ *for some* $a_1, b_1 \in (0, 2\gamma]$;

(C6) $\{\delta_n\} \subset [a_2, b_2]$ *for some* $a_2, b_2 \in (0, 2\beta]$;

(C7) $\liminf_{n\to\infty} r_{k,n} > 0$ *for each* $k \in 1, 2, 3, ..., N$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z = P_\Theta x_0$.

***Proof*.** Pick any $p \in \Theta$. Taking $\Im_n^k = K^{F_k}_{r_{k,n}} K^{F_{k-1}}_{r_{k-1,n}} K^{F_{k-2}}_{r_{k-2,n}} \ldots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}}$ for $k \in \{1, 2, 3, \ldots, N\}$ and $\Im_n^0 = I$ for all $n \in \mathbb{N}$. From the definition of $K^{F_k}_{r_{k,n}}$ is nonexpansive for each $k = 1, 2, 3, \ldots, N$, then $\Im_n^k$ also and $p = \Im^{F_k}_{r_{k,n}} p$, we note that $u_n = \Im_n^N x_n$. If follows that

$$\|u_n - p\| = \|\Im_n^N x_n - \Im_n^N p\| \leq \|x_n - p\|.$$

Next, we will divide the proof into eight steps.

Step 1. We first show by induction that $\Theta \subset C_n$ for each $n \geq 1$.

Taking $p \in \Theta$, we get that $p = J_{M_1,\lambda_k}(p - \lambda_k Ap) = J_{M_2,\delta_k}(p - \delta_k Bp)$. Since J_{M_1,λ_k}, J_{M_2,δ_k} are nonexpansive. From the assumption, we see that $\Theta \subset C = C_1$. Suppose $\Theta \subset C_k$ for some $k \geq 1$. For any $p \in \Theta = C_k$, we have

$$\begin{aligned} \|v_k - p\| &= \|J_{M_1,\lambda_k}(y_k - \lambda_k Ay_k) - J_{M_1,\lambda_k}(p - \lambda_k Ap)\| \\ &\leq \|(y_k - \lambda_k Ay_k) - (p - \lambda_k Ap)\| \\ &\leq \|(I - \lambda_k A)y_k - (I - \lambda_k A)p\| \\ &\leq \|y_k - p\|, \end{aligned} \tag{5.4.9}$$

and

$$\begin{aligned} \|y_k - p\| &= \|J_{M_2,\delta_k}(u_k - \delta_k Bu_k) - J_{M_2,\delta_k}(p - \delta_k Bp)\| \\ &\leq \|(u_k - \delta_k Bu_k) - (p - \delta_k Bp)\| \\ &\leq \|u_k - p\| \\ &\leq \|x_k - p\|. \end{aligned} \tag{5.4.10}$$

Which yield that

$$\begin{aligned} \|z_k - p\|^2 &= \Big\|\alpha_k(v_k - p) + (1 - \alpha_k)\Big(\frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds - p\Big)\Big\|^2 \\ &\leq \alpha_k\|v_k - p\|^2 + (1 - \alpha_k)\Big\|\frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds - p\Big\|^2 \\ &\quad -\alpha_k(1 - \alpha_k)\Big\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\Big\|^2 \\ &\leq \alpha_k\|v_k - p\|^2 + (1 - \alpha_k)\|v_k - p\|^2 - \alpha_k(1 - \alpha_k)\Big\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\Big\|^2 \\ &\leq \|v_k - p\|^2 - \alpha_k(1 - \alpha_k)\Big\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\Big\|^2. \end{aligned} \tag{5.4.11}$$

Applying (5.4.9) and (5.4.10), we get

$$\|z_k - p\|^2 \leq \|x_k - p\|^2 - \alpha_k(1-\alpha_k)\Big\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\Big\|^2. \quad (5.4.12)$$

Hence $p \in C_{k+1}$. This implies that $\Theta \subset C_n$ for each $n \geq 1$.

Step 2. Next, we show that $\{x_n\}$ is well defined and C_n is closed and convex for any $n \in \mathbb{N}$.

It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. Now, we show that C_{k+1} is closed and convex for some k. For any $p \in C_k$, we obtain

$$\|z_k - p\|^2 \leq \|x_k - p\|^2$$

is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - p\rangle \leq 0. \quad (5.4.13)$$

Thus C_{k+1} is closed and convex. Then, C_n is closed and convex for any $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined.

Step 3. Next, we show that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Using $\Theta \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0, \quad \forall p \in \Theta \quad \text{and} \quad n \in \mathbb{N}.$$

So, for $p \in \Theta$. We observe that

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - p\|, \quad \forall p \in \Theta \quad \text{and} \quad n \in \mathbb{N}.$$

Hence, we get $\{x_n\}$ is bounded. It follows by (5.4.9)-(6.4.13), that $\{v_n\}, \{y_n\}$ and $\{W_n v_n\}$ are also bounded. From $x_n = P_{C_n}x_0$, and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \quad (5.4.14)$$

It follows that, we have for each $n \in \mathbb{N}$

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Thus, since the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n \longrightarrow \infty} \|x_n - x_0\|$ exists, that is

$$m = \lim_{n \longrightarrow \infty} \|x_n - x_0\|. \tag{5.4.15}$$

Step 4. Next, we show that $\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \longrightarrow \infty} \|x_n - z_n\| = 0$.

Applying (5.4.14), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0 \rangle + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Thus, by (5.4.15), we obtain

$$\lim_{n \longrightarrow \infty} \|x_n - x_{n+1}\| = 0. \tag{5.4.16}$$

On the other hand, from $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, which implies that

$$\|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\|. \tag{5.4.17}$$

It follows by (5.4.17), we also have

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (5.4.16), we obtain

$$\lim_{n \longrightarrow \infty} \|x_n - z_n\| = 0. \tag{5.4.18}$$

Step 5. Next, we show that

$$\lim_{n \longrightarrow \infty} \|\Im_n^k x_n - \Im_n^{k-1} x_n\| = 0 \tag{5.4.19}$$

for every $k \in \{1, 2, 3, \ldots, N\}$. Indeed, for $p \in \Theta$, note that $K^{F_k}_{r_{k,n}}$ is the firmly nonexpansie, so we have

$$\begin{aligned} \|\Im^k_n x_n - \Im^k_n p\|^2 &= \|K^{F_k}_{r_{k,n}} \Im^{k-1}_n x_n - K^{F_k}_{r_{k,n}} p\|^2 \\ &\leq \langle \Im^k_n x_n - p, \Im^{k-1}_n x_n - p\rangle \\ &= \frac{1}{2}\Big\{\|\Im^k_n x_n - p\|^2 + \|\Im^{k-1}_n x_n - p\|^2 - \|\Im^k_n x_n - \Im^{k-1}_n x_n\|^2\Big\}. \end{aligned}$$

Thus, we get

$$\|\Im^k_n x_n - \Im^k_n p\|^2 \leq \|\Im^{k-1}_n x_n - p\|^2 - \|\Im^k_n x_n - \Im^{k-1}_n x_n\|^2.$$

It follows that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|\Im^k_n x_n - \Im^k_n p\|^2 \\ &\leq \|\Im^{k-1}_n x_n - p\|^2 - \|\Im^k_n x_n - \Im^{k-1}_n x_n\|^2 \\ &\leq \|x_n - p\|^2 - \|\Im^k_n x_n - \Im^{k-1}_n x_n\|^2. \end{aligned} \quad (5.4.20)$$

By (5.4.9), (5.4.10), (6.4.13) and (5.4.20), we have for each $k \in \{1, 2, 3, ..., N\}$

$$\begin{aligned} \|z_n - p\|^2 &\leq \|v_n - p\|^2 \\ &\leq \|u_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \|\Im^k_n x_n - \Im^{k-1}_n x_n\|^2. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \|\Im^k_n x_n - \Im^{k-1}_n x_n\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \end{aligned}$$

Since (5.4.18) implies that for every $k \in \{1, 2, 3, ..., N\}$

$$\lim_{n\longrightarrow\infty} \|\Im^k_n x_n - \Im^{k-1}_n x_n\| = 0. \quad (5.4.21)$$

Step 6. Next, we show that $\lim_{n\longrightarrow\infty} \|y_n - v_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|\mathcal{K}_n W_n v_n - v_n\| = 0$, where $\mathcal{K}_n = \frac{1}{t_n}\int_0^{t_n} S(s)ds$

For any given $p \in \Theta$, $\lambda_n \in (0, 2\gamma]$, $\delta_n \in (0, 2\beta]$ and $p = J_{M_1,\lambda_n}(p - \lambda_n Ap) = J_{M_2,\delta_n}(p - \delta_n Bp)$. Since $I - \lambda_n A$ and $I - \delta_n B$ are nonexpansive, we have

$$\begin{aligned} \|v_n - p\|^2 &= \|J_{M_1,\lambda_n}(y_n - \lambda_n Ay_n) - J_{M_1,\lambda_n}(p - \lambda_n Ap)\|^2 \\ &\leq \|(y_n - \lambda_n Ay_n) - (p - \lambda_n Ap)\|^2 \\ &= \|(y_n - p) - \lambda_n(Ay_n - Ap)\|^2 \\ &\leq \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Ap\rangle + \lambda_n^2\|Ay_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2\lambda_n\gamma\|Ay_n - Ap\|^2 + \lambda_n^2\|Ay_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\gamma)\|Ay_n - Ap\|^2. \end{aligned} \quad (5.4.22)$$

Similarly, we can show that

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + \delta_n(\delta_n - 2\beta)\|Bu_n - Bp\|^2. \tag{5.4.23}$$

Observe that

$$\begin{aligned}
\|z_n - p\|^2 &= \Big\|\alpha_n(v_n - p) + (1-\alpha_n)\Big(\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - p\Big)\Big\|^2 \\
&\leq \alpha_n\|v_n - p\|^2 + (1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - p\Big\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\Big\|v_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds\Big\|^2 \\
&\leq \alpha_n\|v_n - p\|^2 + (1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - p\Big\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2.
\end{aligned} \tag{5.4.24}$$

Substituting (6.1.28) into (6.1.33) and using conditions (C4) and (C5), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\{\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\gamma)\|Ay_n - Ap\|^2\} \\
&= \|x_n - p\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\gamma)\|Ay_n - Ap\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-d)a_1(2\gamma - b_1)\|Ay_n - Ap\|^2 &\leq (1-\alpha_n)\lambda_n(2\gamma - \lambda_n)\|Ay_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

By (5.4.18), we obtain

$$\lim_{n\longrightarrow\infty}\|Ay_n - Ap\| = 0. \tag{5.4.25}$$

Since the resolvent operator J_{M_1,λ_n} is 1-inverse-strongly monotone, we obtain

$$\begin{aligned}
\|v_n - p\|^2 &= \|J_{M_1,\lambda_n}(y_n - \lambda_n Ay_n) - J_{M_1,\lambda_n}(p - \lambda_n Ap)\|^2 \\
&= \|J_{M_1,\lambda_n}(I - \lambda_n A)y_n - J_{M_1,\lambda_n}(I - \lambda_n A)p\|^2 \\
&\leq \Big\langle (I - \lambda_n A)y_n - (I - \lambda_n A)p, v_n - p\Big\rangle \\
&= \frac{1}{2}\Big\{\|(I - \lambda_n A)y_n - (I - \lambda_n A)p\|^2 + \|v_n - p\|^2 \\
&\quad - \|(I - \lambda_n A)y_n - (I - \lambda_n A)p - (v_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|y_n - p\|^2 + \|v_n - p\|^2 - \|(y_n - v_n) - \lambda_n(Ay_n - Ap)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|v_n - p\|^2 - \|y_n - v_n\|^2 \\
&\quad - \lambda_n^2\|Ay_n - Ap\|^2 + 2\lambda_n\langle y_n - v_n, Ay_n - Ap\rangle\Big\},
\end{aligned}$$

which yields that

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda_n \|y_n - v_n\| \|Ay_n - Ap\|. \tag{5.4.26}$$

Similarly, we can obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta_n \|u_n - y_n\| \|Bu_n - Bp\|. \tag{5.4.27}$$

Substituting (6.1.36) into (6.1.33), and using condition (C4) and (C5), we have

$$\begin{aligned} \|z_n - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda_n \|y_n - v_n\| \|Ay_n - Ap\|\Big\} \\ &= \|x_n - p\|^2 - (1-\alpha_n)\|y_n - v_n\|^2 + 2(1-\alpha_n)\lambda_n \|y_n - v_n\| \|Ay_n - Ap\|. \end{aligned}$$

It follows that

$$\begin{aligned} (1-\alpha_n)\|y_n - v_n\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)\lambda_n \|y_n - v_n\| \|Ay_n - Ap\| \\ &\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)\lambda_n \|y_n - v_n\| \|Ay_n - Ap\|. \end{aligned}$$

By (5.4.18) and (6.1.35), we get

$$\lim_{n \longrightarrow \infty} \|y_n - v_n\| = 0. \tag{5.4.28}$$

From (5.4.12) and (C4), we also have

$$\begin{aligned} \alpha_n(1-\alpha_n)\Big\|v_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds\Big\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \end{aligned}$$

Since $\mathcal{K}_n = \frac{1}{t_n}\int_0^{t_n} S(s)ds$, we obtain (5.4.18), we have

$$\lim_{n \longrightarrow \infty} \|\mathcal{K}_n W_n v_n - v_n\| = 0. \tag{5.4.29}$$

Since $\{W_n v_n\}$ is a bounded sequence in C, from Lemma 5.4.13 for all $h \geq 0$, we have

$$\lim_{n \to \infty} \|\mathcal{K}_n W_n v_n - S(h)\mathcal{K}_n W_n v_n\| = \lim_{n \to \infty} \left\| \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - S(h)\left(\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds\right)\right\| = 0. \tag{5.4.30}$$

It follows from (5.4.29) and (5.4.30), we get

$$\begin{aligned} \|v_n - S(s)v_n\| &\leq \|v_n - \mathcal{K}_n W_n v_n\| + \|\mathcal{K}_n W_n v_n - S(s)\mathcal{K}_n W_n v_n\| + \|S(s)\mathcal{K}_n W_n v_n - S(s)v_n\| \\ &\leq 2\|v_n - \mathcal{K}_n W_n v_n\| + \|\mathcal{K}_n W_n v_n - S(s)\mathcal{K}_n W_n v_n\|. \end{aligned}$$

So, we have

$$\lim_{n \to \infty} \|v_n - S(s)v_n\| = 0. \tag{5.4.31}$$

Step 7. Next, we show that $q \in \Theta := F(\mathcal{S}) \cap F(W) \cap \left(\cap_{k=1}^{N} SMEP(F_k)\right) \cap I(A, M_1) \cap I(B, M_2) \neq \emptyset$.

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to $q \in C$. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup q$.

(1) First, we prove that $q \in F(\mathcal{S})$. Indeed, from Lemma 5.4.14 and (5.4.31), we get $q \in F(\mathcal{S})$, i.e., $q = S(s)q, \forall s \geq 0$.

(2) We show that $q \in F(W) = \cap_{n=1}^{\infty} F(W_n)$, where $F(W_n) = \cap_{i=1}^{\infty} F(T_i), \forall n \geq 1$ and $F(W_{n+1}) \subset F(W_n)$. Assume that $q \notin F(W)$, then there exists a positive integer m such that $q \notin F(T_m)$ and so $q \notin \cap_{i=1}^{m} F(T_i)$. Hence for any $n \geq m$, $q \notin \cap_{i=1}^{n} F(T_i) = F(W_n)$, i.e., $q \neq W_n q$. This together with $q = S(s)q$, $\forall s \geq 0$ shows $q = S(s)q \neq S(s)W_n q$, $\forall s \geq 0$, therefore we have $q \neq \mathcal{K}_n W_n q$, $\forall n \geq m$. It follows from the Opial's condition and (5.4.29) that

$$\begin{aligned}
\liminf_{i\to\infty} \|v_{n_i} - q\| &< \liminf_{i\to\infty} \|v_{n_i} - \mathcal{K}_{n_i} W_{n_i} q\| \\
&\leq \liminf_{i\to\infty} (\|v_{n_i} - \mathcal{K}_{n_i} W_{n_i} v_{n_i}\| + \|\mathcal{K}_{n_i} W_{n_i} v_{n_i} - \mathcal{K}_{n_i} W_{n_i} q\|) \\
&\leq \liminf_{i\to\infty} \|v_{n_i} - q\|,
\end{aligned}$$

which is a contradiction. Thus, we get $q \in F(W)$.

(3) We prove that $q \in \cap_{k=1}^{N} SMEP(F_k, \varphi)$. Since $\Im_n^k = \mathcal{K}_{r_k}^{F_k}, k = 1, 2, \ldots, N$ and $u_n^k = \Im_n^k x_n$, we have

$$F_k(\Im_n^k x_n, x) + \varphi(x) - \varphi(\Im_n^k x_n) + \frac{1}{r_k}\left\langle \mathcal{K}'(\Im_n^k x_n) - \mathcal{K}'(\Im_n^{k-1} x_n), \eta(x, \Im_n^k x_n)\right\rangle \geq 0, \quad \forall x \in C.$$

It follows that

$$\frac{1}{r_k}\left\langle \mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i}), \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle \geq -F_k(\Im_{n_i}^k x_{n_i}, x) - \varphi(x) + \varphi(\Im_{n_i}^k x_{n_i}) \tag{5.4.32}$$

for all $x \in C$. From (5.4.21) and by conditions (C1)(c) and (C2), we get

$$\lim_{n_i\to\infty} \frac{1}{r_k}\left\langle \mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i}), \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle = 0.$$

By the assumption and by the condition (H1), we know that the function φ and the mapping $x \longmapsto (-F_k(x, y))$ both are convex and lower semicontinuous, hence they are weakly lower semicontinuous.

These together with $\frac{\mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i})}{r_k} \to 0$ and $\Im_{n_i}^k x_{n_i} \rightharpoonup q$, we have

$$\liminf_{n_i\to\infty} \left\langle \frac{\mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i})}{r_k}, \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle \geq \liminf_{n_i\to\infty} \{-F_k(\Im_{n_i}^k x_{n_i}, x) - \varphi(x) + \varphi(\Im_{n_i}^k x_{n_i})\}.$$

Then, we obtain

$$F_k(q, x) + \varphi(x) - \varphi(q) \geq 0, \quad \forall x \in C, \quad \forall k =, 1, 2, \ldots, N. \tag{5.4.33}$$

Therefore $q \in \cap_{k=1}^{N} SMEP(F_k, \varphi)$.

(4) Lastly, we prove that $q \in I(A, M_1) \cap I(B, M_2)$.

We observe that A is an $1/\gamma$-Lipschitz monotone mapping and $D(A) = H$. From Lemma 5.4.4, we know that $M_1 + A$ is maximal monotone. Let $(v, g) \in G(M_1 + A)$ that is, $g - Av \in M_1(v)$. Since $v_{n_i} = J_{M_1, \lambda_{n_i}}(y_{n_i} - \lambda_{n_i} A y_{n_i})$, we have

$$y_{n_i} - \lambda_{n_i} A y_{n_i} \in (I + \lambda_{n_i} M_1)(v_{n_i}),$$

that is,

$$\frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} A y_{n_i}) \in M_1(v_{n_i}). \tag{5.4.34}$$

By virtue of the maximal monotonicity of $M_1 + A$, we have

$$\left\langle v - v_{n_i}, g - Av - \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} A y_{n_i}) \right\rangle \geq 0, \tag{5.4.35}$$

and so

$$\begin{aligned} \left\langle v - v_{n_i}, g \right\rangle &\geq \left\langle v - v_{n_i}, Av + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} A y_{n_i}) \right\rangle \\ &= \left\langle v - v_{n_i}, Av - Av_{n_i} + Av_{n_i} - Ay_{n_i} + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle \\ &\geq 0 + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle + \left\langle v - v_{n_i}, \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle. \end{aligned} \tag{5.4.36}$$

By (5.4.28), $v_{n_i} \rightharpoonup q$ and A is inverse-strongly monotone, we obtain that $\lim_{n\to\infty} \|Ay_n - Av_n\| = 0$ and it follows that

$$\lim_{n_i \longrightarrow \infty} \langle v - v_{n_i}, g \rangle = \langle v - q, g \rangle \geq 0. \tag{5.4.37}$$

It follows from the maximal monotonicity of $M_1 + A$ that $\theta \in (M_1 + A)(q)$, that is, $q \in I(A, M_1)$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $q \in C$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup q$. In similar way, we can obtain $q \in I(B, M_2)$, hence $q \in I(A, M_1) \cap I(B, M_2)$

Step 8. Finally, we show that $x_n \longrightarrow z$ and $u_n \longrightarrow z$, where $z = P_\Theta x_0$.

Since Θ is nonempty closed convex subset of H, there exists a unique $z' \in \Theta$ such that $z' = P_\Theta x_0$. Since $z' \in \Theta \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_0 - x_n\| \leq \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z'\| \tag{5.4.38}$$

for all $n \in \mathbb{N}$. From (5.4.38) and $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$.
By the weakly lower semicontinuous of the norm, we have

$$\|x_0 - z\| \leq \liminf_{n_i \longrightarrow \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z'\|. \tag{5.4.39}$$

However, since $z \in \omega_w(x_n) \subset \Theta$, we have

$$\|x_0 - z'\| \leq \|x_0 - P_{C_n}x_0\| \leq \|x_0 - z\|.$$

Using (5.4.38) and (5.4.39), we obtain $z' = z$. Thus $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So, we have

$$\|x_0 - z'\| \leq \|x_0 - z\| \leq \liminf_{n\longrightarrow\infty} \|x_0 - x_n\| \leq \limsup_{n\longrightarrow\infty} \|x_0 - x_n\| \leq \|x_0 - z'\|.$$

Thus, we obtain that

$$\|x_0 - z\| = \lim_{n\longrightarrow\infty} \|x_0 - x_n\| = \|x_0 - z'\|.$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using the Kadec-Klee property, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

and hence $x_n \longrightarrow z$ in norm. Finally, noticing $\|u_n - z\| = \|\Im_n^N x_n - \Im_n^N z\| \leq \|x_n - z\|$. We also conclude that $u_n \longrightarrow z$ in norm. This completes the proof.

บทที่ 6

Optimization Problems

6.1 Optimization Problem

Definition 6.1.1. *Let A be a strongly positive bounded linear operator on H if there exists a constant $\bar{\gamma} > 0$ with the property*

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \qquad \forall x \in H. \tag{6.1.1}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{6.1.2}$$

where A is a nonexpansive mapping and b is a given point in H.

Optimization problem (for short, OP) as the following

$$\textbf{OP} : \min_{x\in F} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{6.1.3}$$

where $F = \cap_{n=1}^{\infty} C_n$, $C_1, C_2, \cdots$ are infinitely closed convex subsets of H such that $\cap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Lemma 6.1.2. [180] *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbb{R}\cup\{\infty\}$ be a proper lower-semicontinuous differentiable convex function. If z is a solution to the minimization problem*

$$g(z) = \inf_{x\in C} g(x),$$

then

$$\Big\langle g'(x), x - z\Big\rangle \geq 0, \quad x \in C.$$

In particular, if z solves problem OP, then

$$\Big\langle u + \big[\gamma f - (I+\mu A)\big]z, x - z\Big\rangle \leq 0.$$

Lemma 6.1.3. [231]. *Let E be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\bar{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \alpha\gamma$.

Lemma 6.1.4. [231]. *Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

For solving the mixed equilibrium problem for an equilibrium bifunction $\Theta : E \times E \longrightarrow \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(H1) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0, \quad \forall x, y \in E$;

(H2) for each fixed $y \in E$, $x \mapsto \Theta(x, y)$ is convex and upper semicontinuous;

(H3) for each $x \in E, y \mapsto \Theta(x, y)$ is convex.

Let $\eta : E \times E \longrightarrow H$, which is called Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda\|x - y\|, \quad \forall x, y \in E.$$

Let $K : E \longrightarrow \mathbb{R}$ be a differentiable functional on a convex set E, which is called:

(K1) η-convex [223] if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x) \rangle, \ \forall x, y \in E,$$

where $K'(x)$ is the Fréchet derivative at x;

(K2) η-strongly convex [243] if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x) \rangle \geq \frac{\sigma}{2}\|x - y\|^2, \ \forall x, y \in E.$$

Let E be a nonempty closed convex subset of a real Hilbert space H, let $\varphi : E \longrightarrow \mathbb{R}$ be a real-valued function and $\Theta : E \times E \longrightarrow \mathbb{R}$ be an equilibrium bifunction. Let r be a positive parameter. For a given point $x \in E$, the auxiliary problem for MEP consists of finding $y \in E$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y) \rangle \geq 0, \quad \forall z \in E.$$

Let $S_r : E \longrightarrow E$ be the mapping such that for each $x \in E$, $S_r(x)$ is the solution set of the auxiliary problem MEP, that is,

$$S_r(x) = \{y \in E : \Theta(y,z)+\varphi(z)-\varphi(y)+\frac{1}{r}\langle K'(y)-K'(x), \eta(z,y)\rangle \geq 0, \quad \forall z \in E\}, \quad \forall x \in E.$$

Definition 6.1.5. *Let $\{T_n\}$ be a sequence of nonexpansive mappings of E into itself and let $\{\mu_n\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of E into itself as follows:*

$$\begin{array}{rcl} U_{n,n+1} &=& I, \\ U_{n,n} &=& \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\ U_{n,n-1} &=& \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\ &\vdots& \\ U_{n,k} &=& \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\ U_{n,k-1} &=& \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\ &\vdots& \\ U_{n,2} &=& \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\ W_n = U_{n,1} &=& \mu_1 T_1 U_{n,2} + (1-\mu_1)I. \end{array} \tag{6.1.4}$$

Such a mapping W_n is nonexpansive from E to E and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (6.1.4). Then we can have the following crucial conclusions concerning W_n. You can find them in [238]. Now we only need the following similar version in Hilbert spaces.

Lemma 6.1.6. [238]. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of E into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in E$ and $k \in \mathbb{N}$, the limit $\lim_{n \longrightarrow \infty} U_{n,k}x$ exists.*

Using Lemma 6.1.6, one can define a mapping W of E into itself as follows:

$$Wx = \lim_{n \longrightarrow \infty} W_n x = \lim_{n \longrightarrow \infty} U_{n,1}x, \tag{6.1.5}$$

for every $x \in E$. Such a W is called the W-mapping generated by $T_1, T_2, ...$ and $\mu_1, \mu_2, ...$ Throughout this paper, we will assume that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, we have the following results.

Lemma 6.1.7. [238]. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 6.1.8. [314]. *If $\{x_n\}$ is a bounded sequence in E, then $\lim_{n\longrightarrow\infty} \|Wx_n - W_n x_n\| = 0$.*

Lemma 6.1.9. [307]. *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \le \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)v_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\longrightarrow\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\longrightarrow\infty} \|v_n - x_n\| = 0$.*

Lemma 6.1.10. *Let H be a real Hilbert space. Then the following inequalities hold:*

(1) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|x + y\|^2 \ge \|x\|^2 + 2\langle y, x\rangle$;

for all $x, y \in H$.

Lemma 6.1.11. [311]. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - l_n)a_n + \sigma_n, \ \forall n \ge 0,$$

where $\{l_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$

(2) $\limsup_{n\longrightarrow\infty} \frac{\sigma_n}{l_n} \le 0$ or $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Next, we prove a strong convergence theorem of a general iterative method (6.1.6) to compute the approximate solutions of the mixed equilibrium problems and optimization problems in Hilbert spaces.

Theorem 6.1.12. *Let E be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from E to $\mathbb{R}$. Let Θ be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying (H1)-(H3), let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself and let B be a ξ-inverse-strongly monotone mapping of C into H such that*

$$\Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP \cap VI(E, B) \neq \emptyset.$$

Let $\mu > 0$, $\gamma > 0$ and $r > 0$ be three constants. Let f be a contraction of E into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{\alpha}$. For given $x_1 \in H$ arbitrarily and fixed $u \in H$, suppose the $\{x_n\}$, $\{k_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated iteratively by

$$\begin{cases} \Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \quad \forall x \in C, \\ y_n = P_E(z_n - \delta_n B z_n), \\ k_n = \alpha_n x_n + (1-\alpha_n) W_n P_E(y_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n(u + \gamma f(W_n x_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A)) W_n P_E(k_n - \tau_n B k_n), \end{cases} \tag{6.1.6}$$

for all $n \in \mathbb{N}$, where W_n be the W-mapping defined by (6.1.4) and $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are three sequences in $(0,1)$. Assume the following conditions are satisfied:

(C1) $\eta : E \times E \longrightarrow H$ *is Lipschitz continuous with constant $\lambda > 0$ such that;*

- *(a)* $\eta(x,y) + \eta(y,x) = 0, \quad \forall x, y \in E$
- *(b)* $\eta(\cdot,\cdot)$ *is affine in the first variable,*
- *(c) for each fixed $y \in E$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(C2) $K : E \longrightarrow \mathbb{R}$ *is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\upsilon > 0$ such that $\sigma > \lambda\upsilon$;*

(C3) for each $x \in E$, there exist a bounded subset $D_x \subset E$ and $z_x \in E$ such that for any $y \in E \backslash D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0;$$

(C4) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(C5) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;

(C6) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\delta_{n+1} - \delta_n| = \lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$;

(C7) $\{\tau_n\}$, $\{\lambda_n\}$, $\{\delta_n\} \subset [a,b]$ *for some $a, b \in (0, 2\xi)$.*

Then, $\{x_n\}$ and $\{z_n\}$ converge strongly to $z \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP \cap VI(E,B)$ provided S_r is firmly nonexpansive, which solves the following optimization problem:

$$OP: \min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{6.1.7}$$

***Proof*.** Since $\epsilon_n \longrightarrow 0$ by the condition (C4) and (C5), we may assume, without loss of generality, that $\epsilon_n \leq (1-\beta_n)(1+\mu\|A\|)^{-1}$ for all $n \in \mathbb{N}$. First, we show that $I-\tau_n B$ is nonexpansive. Indeed, from the ξ-inverse-strongly monotone mapping definition on B and condition (C7), we observe that

$$\begin{aligned}\|(I-\tau_n B)x-(I-\tau_n B)y\|^2 &= \|(x-y)-\tau_n(Bx-By)\|^2\\ &= \|x-y\|^2-2\tau_n\langle x-y,Bx-By\rangle+\tau_n^2\|Bx-By\|^2\\ &\leq \|x-y\|^2-2\tau_n\xi\|Bx-By\|+\tau_n^2\|Bx-By\|^2\\ &= \|x-y\|^2+\tau_n(\tau_n-2\xi)\|Bx-By\|^2 \qquad (6.1.8)\\ &\leq \|x-y\|^2,\end{aligned}$$

if $\tau_n \leq 2\xi$ then the mapping $I-\tau_n B$ is nonexpansive, and so are $I-\lambda_n B$ and $I-\delta_n B$, if provided $\lambda_n, \delta_n \leq (o, 2\xi)$. On the other hand, since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\epsilon_n(I+\mu A))x,x\rangle &= 1-\beta_n-\epsilon_n-\epsilon_n\mu\langle Ax,x\rangle\\ &\geq 1-\beta_n-\epsilon_n-\epsilon_n\mu\|A\|\\ &\geq 0,\end{aligned}$$

this shows that $(1-\beta_n)I-\epsilon_n(I+\mu A)$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n(I+\mu A)\| &= \sup\{|\langle((1-\beta_n)I-\epsilon_n(I+\mu A))x,x\rangle| : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\epsilon_n-\epsilon_n\mu\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\leq 1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma}.\end{aligned}$$

We shall divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded. Indeed, pick any $p \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP \cap VI(E,B)$. From the definition of S_r, we note that $z_n = S_r x_n$. If follows that

$$\|z_n - p\| = \|S_r x_n - S_r p\| \leq \|x_n - p\|.$$

Since $I-\lambda_n B$, $I-\delta_n B$, P_E and W_n are nonexpansive and $p = W_n P_E(p-\lambda_n Bp) = W_n P_E(p-\delta_n Bp)$, we have

$$\begin{aligned}\|y_n-p\| &= \|W_nP_E(z_n-\delta_nBz_n)-W_nP_E(p-\delta_nBp)\|\\ &\leq \|P_E(z_n-\delta_nBz_n)-P_E(p-\delta_nBp)\|\\ &\leq \|(z_n-\delta_nBz_n)-(p-\delta_nBp)\|\\ &= \|(I-\delta_nB)z_n-(I-\delta_nB)p)\| \leq \|z_n-p\| \leq \|x_n-p\|.\end{aligned}$$

It follows that

$$
\begin{aligned}
\|k_n - p\| &= \|\alpha_n(x_n - p) + (1-\alpha_n)(W_nP_E(y_n - \lambda_n By_n) - p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|P_E(y_n - \lambda_n By_n) - P_E(p - \lambda_n Bp)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|(y_n - \lambda_n By_n) - (p - \lambda_n Bp)\| \\
&= \alpha_n\|x_n - p\| + (1-\alpha_n)\|(I - \lambda_n B)y_n - (I - \lambda_n B)p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\epsilon_n u + \epsilon_n(\gamma f(W_nx_n) - (I+\mu A)p) + \beta_n(x_n - p) \\
&\quad + ((1-\beta_n)I - \epsilon_n(I+\mu A))(W_nP_E(k_n - \tau_n Bk_n) - p)\| \\
&\leq (1 - \beta_n - \epsilon_n(1+\mu)\bar{\gamma})\|P_E(I - \tau_n B)k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|u\| \\
&\quad + \epsilon_n\|\gamma f(W_nx_n) - (I+\mu A)p\| \\
&\leq (1 - \beta_n - \epsilon_n(1+\mu)\bar{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|u\| \\
&\quad + \epsilon_n\|\gamma f(W_nx_n) - (I+\mu A)p\| \\
&\leq (1 - \beta_n - \epsilon_n(1+\mu)\bar{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|u\| \\
&\quad + \epsilon_n\gamma\|f(W_nx_n) - f(p)\| + \epsilon_n\|\gamma f(p) - (I+\mu A)p\| \\
&\leq (1 - \beta_n - \epsilon_n(1+\mu)\bar{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|u\| \\
&\quad + \epsilon_n\gamma\alpha\|x_n - p\| + \epsilon_n\|\gamma f(p) - (I+\mu A)p\| \\
&\leq (1 - \epsilon_n(1+\mu)\bar{\gamma} + \epsilon_n\gamma\alpha)\|x_n - p\| + \epsilon_n(\|\gamma f(p) - (I+\mu A)p\| + \|u\|) \\
&= (1 - ((1+\mu)\bar{\gamma} - \gamma\alpha)\epsilon_n)\|x_n - p\| + \epsilon_n(\|\gamma f(p) - (I+\mu A)p\| + \|u\|) \\
&= (1 - ((1+\mu)\bar{\gamma} - \gamma\alpha)\epsilon_n)\|x_n - p\| \\
&\quad + ((1+\mu)\bar{\gamma} - \gamma\alpha)\epsilon_n\frac{\|f(p) - (I+\mu A)p\| + \|u\|}{(1+\mu)\bar{\gamma} - \gamma\alpha}. \qquad (6.1.9)
\end{aligned}
$$

It follows that (6.1.9) and induction that

$$
\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - (I+\mu A)p\| + \|u\|}{(1+\mu)\bar{\gamma} - \gamma\alpha}\right\}, \quad n \geq 1. \qquad (6.1.10)
$$

Hence, $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{k_n\}$, $\{y_n\}$, $\{f(W_nx_n)\}$, $\{Bz_n\}$, $\{Bk_n\}$, $\{By_n\}$, $\{W_nk_n\}$ and $\{W_ny_n\}$.

Step 2. We claim that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|W_n\theta_n - x_n\| = 0$. Observing that $z_n = S_rx_n$ and $z_{n+1} = S_rx_{n+1}$, from the nonexpansive of S_r, we get

$$
\|z_{n+1} - z_n\| = \|S_rx_{n+1} - S_rx_n\| \leq \|x_{n+1} - x_n\|. \qquad (6.1.11)
$$

Put $\theta_n = P_E(k_n - \tau_n Bk_n)$ and $\phi_n = P_E(y_n - \lambda_n By_n)$. Since $I - \tau_n B$, $I - \lambda_n B$ and $I - \delta_n B$ are nonexpansive, then we have the following estimates:

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|P_E(z_{n+1} - \delta_{n+1}Bz_{n+1}) - P_E(z_n - \delta_n Bz_n)\| \\ &\leq \|(z_{n+1} - \delta_{n+1}Bz_{n+1}) - (z_n - \delta_n Bz_n)\| \\ &= \|(z_{n+1} - \delta_{n+1}Bz_{n+1}) - (z_n - \delta_{n+1}Bz_n) + (\delta_n - \delta_{n+1})Bz_n\| \\ &\leq \|(z_{n+1} - \delta_{n+1}Bz_{n+1}) - (z_n - \delta_{n+1}Bz_n)\| + |\delta_n - \delta_{n+1}|\|Bz_n\| \\ &= \|(I - \delta_{n+1}B)z_{n+1} - (I - \delta_{n+1}B)z_n\| + |\delta_n - \delta_{n+1}|\|Bz_n\| \\ &\leq \|z_{n+1} - z_n\| + |\delta_n - \delta_{n+1}|\|Bz_n\| \\ &\leq \|x_{n+1} - x_n\| + |\delta_n - \delta_{n+1}|\|Bz_n\|, \end{aligned} \tag{6.1.12}$$

$$\begin{aligned} \|\phi_{n+1} - \phi_n\| &\leq \|P_E(y_{n+1} - \lambda_{n+1}By_{n+1}) - P_E(y_n - \lambda_n By_n)\| \\ &\leq \|(y_{n+1} - \lambda_{n+1}By_{n+1}) - (y_n - \lambda_n By_n)\| \\ &\leq \|(y_{n+1} - \lambda_{n+1}By_{n+1}) - (y_n - \lambda_{n+1}By_n)\| + |\lambda_n - \lambda_{n+1}|\|By_n\| \\ &= \|(I - \lambda_{n+1}B)y_{n+1} - (I - \lambda_{n+1}B)y_n\| + |\lambda_n - \lambda_{n+1}|\|By_n\| \\ &\leq \|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|By_n\| \end{aligned} \tag{6.1.13}$$

and

$$\begin{aligned} \|\theta_{n+1} - \theta_n\| &\leq \|P_E(k_{n+1} - \tau_{n+1}Bk_{n+1}) - P_E(k_n - \tau_n Bk_n)\| \\ &\leq \|(k_{n+1} - \tau_{n+1}Bk_{n+1}) - (k_n - \tau_n Bk_n)\| \\ &\leq \|(k_{n+1} - \tau_{n+1}Bk_{n+1}) - (k_n - \tau_{n+1}Bk_n)\| + |\tau_n - \tau_{n+1}|\|Bk_n\| \\ &= \|(I - \tau_{n+1}B)k_{n+1} - (I - \tau_{n+1}B)k_n\| + |\tau_n - \tau_{n+1}|\|Bk_n\| \\ &\leq \|k_{n+1} - k_n\| + |\tau_n - \tau_{n+1}|\|Bk_n\|. \end{aligned} \tag{6.1.14}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned} \|W_{n+1}\phi_n - W_n\phi_n\| &= \|\mu_1 T_1 U_{n+1,2}\phi_n - \mu_1 T_1 U_{n,2}\phi_n\| \\ &\leq \mu_1\|U_{n+1,2}\phi_n - U_{n,2}\phi_n\| \\ &= \mu_1\|\mu_2 T_2 U_{n+1,3}\phi_n - \mu_2 T_2 U_{n,3}\phi_n\| \\ &\leq \mu_1\mu_2\|U_{n+1,3}\phi_n - U_{n,3}\phi_n\| \\ &\vdots \\ &\leq \mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}\phi_n - U_{n,n+1}\phi_n\| \\ &\leq M_2\prod_{i=1}^{n}\mu_i, \end{aligned} \tag{6.1.15}$$

where $M_2 \geq 0$ is a constant such that $\|U_{n+1,n+1}\phi_n - U_{n,n+1}\phi_n\| \leq M_2$ for all $n \geq 0$. Similarly, we can obtain that, there exist nonnegative numbers M_3 such that

$$\|U_{n+1,n+1}\theta_n - U_{n,n+1}\theta_n\| \leq M_3,$$

and so is

$$\|W_{n+1}\theta_n - W_n\theta_n\| \leq M_3 \prod_{i=1}^{n} \mu_i. \tag{6.1.16}$$

Observing that

$$\begin{cases} k_n = \alpha_n x_n + (1-\alpha_n)W_n\phi_n \\ k_{n+1} = \alpha_{n+1}x_{n+1} + (1-\alpha_{n+1})W_n\phi_{n+1}, \end{cases}$$

we obtain

$$k_n - k_{n+1} = \alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(W_n\phi_n - W_{n+1}\phi_{n+1}) + (W_{n+1}\phi_{n+1} - x_{n+1})(\alpha_{n+1} - \alpha_n),$$

which yields that

$$\begin{aligned} \|k_n - k_{n+1}\| &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|W_n\phi_n - W_{n+1}\phi_{n+1}\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\| \\ &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\{\|W_{n+1}\phi_{n+1} - W_{n+1}\phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\|\} \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\| \\ &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|\phi_{n+1} - \phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\|. \end{aligned} \tag{6.1.17}$$

Substitution of (6.1.13) and (6.1.15) into (6.1.17) yields that

$$\begin{aligned} \|k_n - k_{n+1}\| &= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\{\|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|By_n\|\} \\ &\quad + M_2\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\| \\ &= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|y_{n+1} - y_n\| + (1-\alpha_n)|\lambda_n - \lambda_{n+1}|\|By_n\| \\ &\quad + M_2\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\| \\ &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|y_{n+1} - y_n\| + M_2\prod_{i=1}^{n}\mu_i \\ &\quad + M_4(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|), \end{aligned} \tag{6.1.18}$$

where M_4 is an appropriate constant such that $M_4 = \max\{\sup_{n\geq 1}\|By_n\|, \sup_{n\geq 1}\|W_n\phi_n - x_n\|\}$.

Substitution of (6.1.12) into (6.1.18), we obtain

$$\begin{aligned}\|k_n - k_{n+1}\| &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|y_{n+1} - y_n\| + M_2\prod_{i=1}^{n}\mu_i \\ &\quad + M_4(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\ &\leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\{\|x_{n+1} - x_n\| + |\delta_n - \delta_{n+1}|\|Bz_n\|\} + M_2\prod_{i=1}^{n}\mu_i \\ &\quad + M_4(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\ &= \|x_n - x_{n+1}\| + (1-\alpha_n)\|x_{n+1} - x_n\| + (1-\alpha_n)|\delta_n - \delta_{n+1}|\|Bz_n\| + M_2\prod_{i=1}^{n}\mu_i \\ &\quad + M_4(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\ &\leq \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_5(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\delta_n - \delta_{n+1}|), \end{aligned} \tag{6.1.19}$$

where M_5 is an appropriate constant such that $M_5 = \max\{\sup_{n\geq 1}\|Bz_n\|, M_4\}$. Substituting (6.1.19) into (6.1.14), we obtain

$$\begin{aligned}\|\theta_{n+1} - \theta_n\| &\leq \|k_{n+1} - k_n\| + |\tau_n - \tau_{n+1}|\|Bk_n\| \\ &\leq \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_5(||\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\delta_n - \delta_{n+1}|) \\ &\quad + |\tau_n - \tau_{n+1}|\|Bk_n\| \\ &\leq \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i \\ &\quad + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\delta_n - \delta_{n+1}| + |\tau_n - \tau_{n+1}|), \end{aligned} \tag{6.1.20}$$

where M_6 is an appropriate constant such that $M_6 = \max\{\sup_{n\geq 1}\|Bk_n\|, M_5\}$.
Let $x_{n+1} = (1-\beta_n)v_n + \beta_n x_n,\ n \geq 1$. Where

$$v_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\epsilon_n(u + \gamma f(W_n x_n)) + ((1-\beta_n)I - \epsilon_n(I + \mu A))W_n\theta_n}{1-\beta_n}.$$

Then we have

$$\begin{aligned} v_{n+1} - v_n &= \frac{\epsilon_{n+1}(u + \gamma f(W_{n+1}x_{n+1})) + ((1-\beta_{n+1})I - \epsilon_{n+1}(I + \mu A))W_{n+1}\theta_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\epsilon_n(u + \gamma f(W_n x_n)) + ((1-\beta_n)I - \epsilon_n(I + \mu A))W_n\theta_n}{1-\beta_n} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(u + \gamma f(W_{n+1}x_{n+1})) - \frac{\epsilon_n}{1-\beta_n}(u + \gamma f(W_n x_n)) + W_{n+1}\theta_{n+1} - W_n\theta_n \\ &\quad + \frac{\epsilon_n}{1-\beta_n}(I + \mu A)W_n\theta_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(I + \mu A)W_{n+1}\theta_{n+1} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}((u + \gamma f(W_{n+1}x_{n+1})) - (I + \mu A)W_{n+1}\theta_{n+1}) \\ &\quad + \frac{\epsilon_n}{1-\beta_n}((I + \mu A)W_n\theta_n - u - \gamma f(W_n x_n)) \\ &\quad + W_{n+1}\theta_{n+1} - W_{n+1}\theta_n + W_{n+1}\theta_n - W_n\theta_n. \end{aligned} \tag{6.1.21}$$

It follows from (6.1.16), (6.1.20) and (6.1.21) that

$$\begin{aligned}
& \|v_{n+1}-v_n\|-\|x_{n+1}-x_n\| \qquad (6.1.22)\\
\leq\ & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|)\\
& +\frac{\epsilon_n}{1-\beta_n}(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|)\\
& +\|W_{n+1}\theta_{n+1}-W_{n+1}\theta_n\|+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\|\\
\leq\ & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|)\\
& +\frac{\epsilon_n}{1-\beta_n}(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|)+\|\theta_{n+1}-\theta_n\|\\
& +\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\|\\
\leq\ & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|)\\
& +\frac{\epsilon_n}{1-\beta_n}(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|)+M_3\prod_{i=1}^{n}\mu_i\\
& +M_2\prod_{i=1}^{n}\mu_i+M_6(|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\delta_n-\delta_{n+1}|+|\tau_n-\tau_{n+1}|)\\
\leq\ & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(W_{n+1}x_{n+1})\|+\|(I+\mu A)W_{n+1}\theta_{n+1}\|)\\
& +\frac{\epsilon_n}{1-\beta_n}(\|(I+\mu A)W_n\theta_n\|+\|u\|+\|\gamma f(W_nx_n)\|)+2L\prod_{i=1}^{n}\mu_i\\
& +M_6(|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\delta_n-\delta_{n+1}|+|\tau_n-\tau_{n+1}|), \qquad (6.1.23)
\end{aligned}$$

where L is an appropriate constant such that $L=\max\{M_2,M_3\}$.
It follows from condition (C4), (C5), (C6) and $0<\mu_i\leq b<1, \forall i\geq 1)$

$$\limsup_{n\longrightarrow\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence, by Lemma 6.1.9, we obtain

$$\lim_{n\longrightarrow\infty}\|v_n-x_n\|=0.$$

It follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=\lim_{n\longrightarrow\infty}(1-\beta_n)\|v_n-x_n\|=0. \qquad (6.1.24)$$

Applying (6.1.24) and condition in Theorem 5.4.15 to (6.1.11), (6.1.12), (6.1.14) and (6.1.20), we obtain that

$$\lim_{n\longrightarrow\infty}\|z_{n+1}-z_n\|=\lim_{n\longrightarrow\infty}\|y_{n+1}-y_n\|=\lim_{n\longrightarrow\infty}\|k_{n+1}-k_n\|=\lim_{n\longrightarrow\infty}\|\theta_{n+1}-\theta_n\|=0. \qquad (6.1.25)$$

By (6.1.25), (6.1.13), (C6) and $0 < \mu_i \leq b < 1, \forall i \geq 1)$, we also have

$$\lim_{n\longrightarrow\infty} \|\phi_{n+1} - \phi_n\| = 0. \tag{6.1.26}$$

Since $x_{n+1} = \epsilon_n(u + \gamma f(W_n x_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A))W_n\theta_n$, we have

$$\begin{aligned}
& \|x_n - W_n\theta_n\| \\
\leq\ & \|x_n - x_{n+1}\| + \|x_{n+1} - W_n\theta_n\| \\
=\ & \|x_n - x_{n+1}\| + \|\epsilon_n(u + \gamma f(W_n x_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I + \mu A))W_n\theta_n - W_n\theta_n\| \\
=\ & \|x_n - x_{n+1}\| + \|\epsilon_n((u + \gamma f(W_n x_n)) - (I + \mu A)W_n\theta_n) + \beta_n(x_n - W_n\theta_n)\| \\
\leq\ & \|x_n - x_{n+1}\| + \epsilon_n(\|u\| + \|\gamma f(W_n x_n)\| + \|(I + \mu A)W_n\theta_n\|) + \beta_n\|x_n - W_n\theta_n\|,
\end{aligned}$$

that is

$$\|x_n - W_n\theta_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}(\|u\| + \|\gamma f(W_n x_n)\| + \|(I + \mu A)W_n\theta_n\|).$$

By (C4), (C5) and (6.1.24) it follows that

$$\lim_{n\longrightarrow\infty} \|W_n\theta_n - x_n\| = 0. \tag{6.1.27}$$

Step 3. We claim that the following statements hold:

(1) $\lim_{n\longrightarrow\infty} \|x_n - \theta_n\| = 0$;

(2) $\lim_{n\longrightarrow\infty} \|W_n\theta_n - \theta_n\| = 0$.

Since B is a ξ-inverse-strongly monotone, by the assumptions imposed on $\{\tau_n\}$ for any $p \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP \cap VI(E, B)$, we have

$$\begin{aligned}
\|W_n\theta_n - p\|^2 &\leq \|P_E(k_n - \tau_n Bk_n) - P_E(p - \tau_n Bp)\|^2 \\
&\leq \|(k_n - \tau_n Bk_n) - (p - \tau_n Bp)\|^2 \\
&= \|(k_n - p) - \tau_n(Bk_n - Bp)\|^2 \\
&\leq \|k_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 + \tau_n(\tau_n - 2\xi)\|Bk_n - Bp\|^2. \qquad (6.1.28)
\end{aligned}$$

Similarly, we have

$$\|W_n\phi_n - p\|^2 \leq \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2. \tag{6.1.29}$$

Observe that

$$\begin{aligned}
& \|x_{n+1}-p\|^2 \\
= & \|((1-\beta_n)I-\epsilon_n(I+\mu A))(W_n\theta_n-p)+\beta_n(x_n-p)+\epsilon_n(u+\gamma f(W_nx_n)-(I+\mu A)p)\|^2 \\
= & \|((1-\beta_n)I-\epsilon_n(I+\mu A))(W_n\theta_n-p)+\beta_n(x_n-p)\|^2+\epsilon_n^2\|u+\gamma f(W_nx_n)-(I+\mu A)p\|^2 \\
& +2\beta_n\epsilon_n\langle x_n-p,u+\gamma f(W_nx_n)-(I+\mu A)p\rangle \\
& +2\epsilon_n\langle((1-\beta_n)I-\epsilon_n(I+\mu A))(W_n\theta_n-p),u+\gamma f(W_nx_n)-(I+\mu A)p\rangle \\
\leq & [((1-\beta_n)I-\epsilon_n(I+\mu A))\|W_n\theta_n-p\|+\beta_n\|x_n-p\|]^2+\epsilon_n^2\|u+\gamma f(W_nx_n)-(I+\mu A)p\|^2 \\
& +2\beta_n\epsilon_n\langle x_n-p,u+\gamma f(W_nx_n)-(I+\mu A)p\rangle \\
& +2\epsilon_n\langle((1-\beta_n)I-\epsilon_n(I+\mu A))(W_n\theta_n-p),u+\gamma f(W_nx_n)-(I+\mu A)p\rangle \\
\leq & [(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|W_n\theta_n-p\|+\beta_n\|x_n-p\|]^2+c_n \\
= & (1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|W_n\theta_n-p\|^2+\beta_n^2\|x_n-p\|^2 \\
& +2(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|W_n\theta_n-p\|x_n-p\||+c_n \\
\leq & (1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|W_n\theta_n-p\|^2+\beta_n^2\|x_n-p\|^2 \\
& +(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n(\|W_n\theta_n-p\|^2+\|x_n-p\|^2)+c_n \\
= & [(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2-2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n+\beta_n^2]\|W_n\theta_n-p\|^2+\beta_n^2\|x_n-p\|^2 \\
& +((1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n-\beta_n^2)(\|W_n\theta_n-p\|^2+\|x_n-p\|^2)+c_n \\
= & [(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n]\|W_n\theta_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
= & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|W_n\theta_n-p\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n, \qquad (6.1.30)
\end{aligned}$$

where

$$\begin{aligned}
c_n = & \epsilon_n^2\|u+\gamma f(x_n)-(I+\mu A)p\|^2+2\beta_n\epsilon_n\langle x_n-p,u+\gamma f(W_nx_n)-(I+\mu A)p\rangle \\
& +2\epsilon_n\langle((1-\beta_n)I-\epsilon_n(I+\mu A))(W_n\theta_n-p),u+\gamma f(W_nx_n)-(I+\mu A)p\rangle.
\end{aligned}$$

It follows from condition (C4) that

$$\lim_{n\longrightarrow\infty} c_n=0. \qquad (6.1.31)$$

Substituting (6.1.28) into (6.1.30), and using condition (C7), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|x_n-p\|^2+\tau_n(\tau_n-2\xi)\|Bk_n-Bp\|^2\} \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
= & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\tau_n(\tau_n-2\xi)\|Bk_n-Bp\|^2+c_n \\
\leq & \|x_n-p\|^2+\tau_n(\tau_n-2\xi)\|Bk_n-Bp\|^2+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned} a(2\xi-b)\|Bk_n-Bp\|^2 &\le \tau_n(2\xi-\tau_n)\|Bk_n-Bp\|^2 \\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n \\ &= (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n \\ &\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n. \end{aligned}$$

Since $c_n \longrightarrow 0$ as $n \longrightarrow \infty$ and (6.1.24), we obtain

$$\lim_{n\longrightarrow\infty}\|Bk_n-Bp\|=0. \tag{6.1.32}$$

Note that

$$\begin{aligned} \|k_n-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(W_n\phi_n-p)\|^2 \\ &\le \alpha_n\|x_n-p\|+(1-\alpha_n)\|W_n\phi_n-p\|^2 \\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\{\|x_n-p\|^2+\lambda_n(\lambda_n-2\xi)\|By_n-Bp\|^2\} \\ &= \|x_n-p\|^2+(1-\alpha_n)\lambda_n(\lambda_n-2\xi)\|By_n-Bp\|^2. \end{aligned} \tag{6.1.33}$$

Using (6.1.30) again, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|W_n\theta_n-p\|^2 \\ &\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\ &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-p\|^2 \\ &\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\ &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-p\|^2 \\ &\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n. \end{aligned} \tag{6.1.34}$$

Substituting (6.1.33) into (6.1.34), and using condition (C4) and (C7), we have

$$\begin{aligned} &\|x_{n+1}-p\|^2 \\ \le\ &(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|x_n-p\|^2+(1-\alpha_n)\lambda_n(\lambda_n-2\xi)\|By_n-Bp\|^2\} \\ &+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\ =\ &(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\lambda_n(\lambda_n-2\xi)\|By_n-Bp\|^2 \\ &+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2+c_n \\ \le\ &\|x_n-p\|^2+(1-\alpha_n)\lambda_n(\lambda_n-2\xi)\|By_n-Bp\|^2+c_n. \end{aligned}$$

It follows that

$$\begin{aligned} (1-\alpha_n)a(2\xi-b)\|By_n-Bp\|^2 &\le (1-\alpha_n)(\lambda_n(2\xi-\lambda_n)\|By_n-Bp\|^2 \\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n \\ &\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n. \end{aligned}$$

Since $c_n \longrightarrow 0$ as $n \longrightarrow \infty$ and (6.1.24), we obtain

$$\lim_{n\longrightarrow\infty} \|By_n - Bp\| = 0. \tag{6.1.35}$$

By (6.4.20), we also have

$$\begin{aligned}
\|\theta_n - p\|^2 &= \|P_E(k_n - \tau_n Bk_n) - P_E(p - \tau_n Bp)\|^2 \\
&= \|P_E(I - \tau_n B)k_n - P_E(I - \tau_n B)p\|^2 \\
&\leq \langle (I - \tau_n B)k_n - (I - \tau_n B)p, \theta_n - p\rangle \\
&= \frac{1}{2}\{\|(I - \tau_n B)k_n - (I - \tau_n B)p\|^2 + \|\theta_n - p\|^2 \\
&\quad - \|(I - \tau_n B)k_n - (I - \tau_n B)p - (\theta_n - p)\|^2\} \\
&\leq \frac{1}{2}\{\|k_n - p\|^2 + \|\theta_n - p\|^2 - \|(k_n - \theta_n) - \tau_n(Bk_n - Bp)\|^2\} \\
&\leq \frac{1}{2}\{\|x_n - p\|^2 + \|\theta_n - p\|^2 - \|k_n - \theta_n\|^2 \\
&\quad - \tau_n^2\|Bk_n - Bp\|^2 + 2\tau_n\langle k_n - \theta_n, Bk_n - Bp\rangle\},
\end{aligned}$$

which yields that

$$\|\theta_n - p\|^2 \leq \|x_n - p\|^2 - \|k_n - \theta_n\|^2 + 2\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\|. \tag{6.1.36}$$

Substituting (6.1.36) into (6.1.30), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
\leq\ & (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|W_n\theta_n - p\|^2 \\
&+ (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
\leq\ & (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|\theta_n - p\|^2 \\
&+ (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
\leq\ & (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\{\|x_n - p\|^2 - \|k_n - \theta_n\|^2 \\
&+ 2\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\|\} + (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
=\ & (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})^2\|x_n - p\|^2 - (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|k_n - \theta_n\|^2 \\
&+ 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n \\
\leq\ & \|x_n - p\|^2 - (1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|k_n - \theta_n\|^2 \\
&+ 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|k_n - \theta_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma}) \\
&(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&+ 2(1 - \epsilon_n - \epsilon_n\mu\bar{\gamma})(1 - \beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n.
\end{aligned}$$

Applying $\|x_{n+1}-x_n\| \longrightarrow 0$, $\|Bk_n - Bp\| \longrightarrow 0$ and $c_n \longrightarrow \infty$ as $n \longrightarrow \infty$ to the last inequality, we have

$$\lim_{n\longrightarrow\infty} \|k_n - \theta_n\| = 0. \tag{6.1.37}$$

On the other hand, we have

$$\begin{aligned}
\|W_n\theta_n - p\|^2 &\leq \|P_E(k_n - \tau_n Bk_n) - P_E(p - \tau_n Bp)\|^2 \\
&= \|P_E(I-\tau_n B)k_n - P_E(I-\tau_n B)p\|^2 \\
&\leq \langle (I-\tau_n B)k_n - (I-\tau_n B)p, W_n\theta_n - p\rangle \\
&= \frac{1}{2}\{\|(I-\tau_n B)k_n - (I-\tau_n B)p\|^2 + \|W_n\theta_n - p\|^2 \\
&\quad - \|(I-\tau_n B)k_n - (I-\tau_n B)p - (W_n\theta_n - p)\|^2\} \\
&\leq \frac{1}{2}\{\|k_n - p\|^2 + \|W_n\theta_n - p\|^2 - \|(k_n - W_n\theta_n) - \tau_n(Bk_n - Bp)\|^2\} \\
&\leq \frac{1}{2}\{\|x_n - p\|^2 + \|W_n\theta_n - p\|^2 - \|k_n - W_n\theta_n\|^2 \\
&\quad - \tau_n^2\|Bk_n - Bp\|^2 + 2\tau_n\langle k_n - W_n\theta_n, Bk_n - Bp\rangle\},
\end{aligned}$$

which yields that

$$\|W_n\theta_n - p\|^2 \leq \|x_n - p\|^2 - \|k_n - W_n\theta_n\|^2 + 2\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\|. \tag{6.1.38}$$

Similarly, we can prove

$$\|W_n\phi_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - W_n\phi_n\|^2 + 2\lambda_n\|y_n - W_n\phi_n\|\|By_n - Bp\|. \tag{6.1.39}$$

Substituting (6.1.38) into (6.1.30), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|W_n\theta_n - p\|^2 \\
&\quad + (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\{\|x_n - p\|^2 - \|k_n - W_n\theta_n\|^2 \\
&\quad + 2\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\|\} + (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})^2\|x_n - p\|^2 - (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad + 2(1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n \\
&\leq \|x_n - p\|^2 - (1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad + 2(1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n,
\end{aligned}$$

which yields that

$$\begin{aligned}
&(1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\|k_n - W_n\theta_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&+ 2(1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&+ 2(1-\epsilon_n - \epsilon_n\mu\bar{\gamma})(1-\beta_n - \epsilon_n - \epsilon_n\mu\bar{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n.
\end{aligned}$$

Applying (6.1.24) and (6.1.32) to the last inequality, we have

$$\lim_{n\longrightarrow\infty}\|k_n - W_n\theta_n\| = 0. \tag{6.1.40}$$

Using (6.1.34) again, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
\leq\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-p\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\leq\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|\alpha_n(x_n-p)+(1-\alpha_n)(W_n\phi_n-p)\|^2\}\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\leq\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|W_n\phi_n-p\|^2\}\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|W_n\phi_n-p\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\leq\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\{\|x_n-p\|^2-\|y_n-W_n\phi_n\|^2\\
&+2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|\}+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|x_n-p\|^2\\
&-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-p\|^2\\
&-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\;& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|+c_n\\
\leq\;& \|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|+c_n
\end{aligned}$$

which implies that

$$
\begin{aligned}
& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
\leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|+c_n \\
\leq\ & \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\alpha_n)\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|+c_n.
\end{aligned}
$$

From (6.1.24) and (6.1.35), we obtain

$$\lim_{n\longrightarrow\infty}\|y_n-W_n\phi_n\|=0. \tag{6.1.41}$$

Note that

$$k_n-W_n\phi_n=\alpha_n(x_n-W_n\phi_n)$$

Since $\alpha_n\longrightarrow\infty$ as $n\longrightarrow\infty$, we also have

$$\lim_{n\longrightarrow\infty}\|k_n-W_n\phi_n\|=0. \tag{6.1.42}$$

From (6.1.41) and (6.1.42), we have

$$\lim_{n\longrightarrow\infty}\|y_n-k_n\|=0. \tag{6.1.43}$$

On the other hand, we have

$$
\begin{aligned}
\|y_n-p\|^2 &\leq \|P_E(z_n-\delta_nBz_n)-P_E(p-\delta_nBp)\|^2 \\
&\leq \|(z_n-\delta_nBz_n)-(p-\delta_nBp)\|^2 \\
&= \|(z_n-p)-\delta_n(Bz_n-Bp)\|^2 \\
&\leq \|z_n-p\|^2-2\delta_n\langle z_n-p,Bz_n-Bp\rangle+\delta_n^2\|Bz_n-Bp\|^2 \\
&\leq \|x_n-p\|^2-2\delta_n\langle z_n-p,Bz_n-Bp\rangle+\delta_n^2\|Bz_n-Bp\|^2 \\
&\leq \|x_n-p\|^2+\delta_n(\delta_n-2\xi)\|Bz_n-Bp\|^2.
\end{aligned} \tag{6.1.44}
$$

Using (6.1.34) again, we obtain that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-p\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|(k_n-y_n)+(y_n-p)\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|k_n-y_n\|^2+\|y_n-p\|^2 \\
&\quad +2\|k_n-y_n\|\|y_n-p\|\}+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&= (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|y_n-p\|^2 \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\| \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \qquad (6.1.45) \\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|x_n-p\|^2+\delta_n(\delta_n-2\xi)\|Bz_n-Bp\|^2\} \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\| \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-p\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n(\delta_n-2\xi)\|Bz_n-Bp\|^2 \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\| \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&\le (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2 \\
&\quad +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n(\delta_n-2\xi)\|Bz_n-Bp\|^2 \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n \\
&\le \|k_n-y_n\|^2+\|x_n-p\|^2+\delta_n(\delta_n-2\xi)\|Bz_n-Bp\|^2 \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
a(2\xi-b)\|Bz_n-Bp\|^2 &\le \delta_n(2\xi-\delta_n)\|Bz_n-Bp\|^2 \\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\|k_n-y_n\|^2 \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n \\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\|k_n-y_n\|^2 \\
&\quad +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n
\end{aligned}$$

Since $c_n \longrightarrow 0$ as $n \longrightarrow \infty$, (6.1.24) and (6.1.43) ,we obtain

$$\lim_{n\longrightarrow\infty} \|Bz_n - Bp\| = 0. \tag{6.1.46}$$

We note that

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_E(z_n - \delta_n Bz_n) - P_E(p - \delta_n Bp)\|^2 \\
&= \|P_E(I - \delta_n B)z_n - P_E(I - \delta_n B)p\|^2 \\
&\leq \langle (I - \delta_n B)z_n - (I - \delta_n B)p, y_n - p\rangle \\
&= \frac{1}{2}\{\|(I - \delta_n B)z_n - (I - \delta_n B)p\|^2 + \|y_n - p\|^2 \\
&\quad - \|(I - \delta_n B)z_n - (I - \delta_n B)p - (y_n - p)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - p\|^2 + \|y_n - p\|^2 - \|(z_n - y_n) - \delta_n(Bz_n - Bp)\|^2 \\
&\leq \frac{1}{2}\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|z_n - y_n\|^2 \\
&\quad - \delta_n^2\|Bz_n - Bp\|^2 + 2\delta_n\langle z_n - y_n, Bz_n - Bp\rangle\}.
\end{aligned}$$

Then we derive

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - y_n\|^2 + 2\delta_n\|z_n - y_n\|\|Bz_n - Bp\|. \tag{6.1.47}$$

Using (6.1.45) again, we obtain that

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|y_n-p\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\| \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
\le\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|x_n-p\|^2-\|z_n-y_n\|^2 \\
& +2\delta_n\|z_n-y_n\|\|Bz_n-Bp\|\} \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\| \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
\le\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-p\|^2 \\
& -(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|z_n-y_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n\|z_n-y_n\|\|Bz_n-Bp\| \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\| \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
=\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2 \\
& -(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|z_n-y_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n\|z_n-y_n\|\|Bz_n-Bp\| \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
\le\ & \|k_n-y_n\|^2+\|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|z_n-y_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n\|z_n-y_n\|\|Bz_n-Bp\| \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|z_n-y_n\|^2 \\
\le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\|k_n-y_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n\|z_n-y_n\|\|Bz_n-Bp\| \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n \\
\le\ & \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\|k_n-y_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\delta_n\|z_n-y_n\|\|Bz_n-Bp\| \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|k_n-y_n\|\|y_n-p\|+c_n.
\end{aligned}$$

Applying $\|x_{n+1}-x_n\| \longrightarrow 0$, $\|y_n-k_n\| \longrightarrow 0$, $\|Bz_n-Bp\| \longrightarrow 0$ and $c_n \longrightarrow \infty$ as $n \longrightarrow \infty$ to the last inequality, we have

$$\lim_{n\longrightarrow\infty} \|z_n-y_n\| = 0. \tag{6.1.48}$$

On the other hand, we observe that

$$\|z_n-\theta_n\| \leq \|z_n-y_n\| + \|y_n-k_n\| + \|k_n-\theta_n\|.$$

Applying (6.1.37), (6.1.43) and (6.1.48), we have

$$\lim_{n\longrightarrow\infty} \|z_n-\theta_n\| = 0. \tag{6.1.49}$$

Let $p \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP \cap VI(E,B)$. Since $z_n = S_r x_n$ and S_r is firmly nonexpansive (Remark ??) , then we obtain

$$\begin{aligned}
\|z_n-p\|^2 &= \|S_r x_n - S_r p\|^2 \\
&\leq \langle S_r x_n - S_r p, x_n - p\rangle \\
&= \langle z_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|z_n-p\|^2 + \|x_n-p\|^2 - \|x_n-z_n\|^2).
\end{aligned}$$

So, we have

$$\|z_n-p\|^2 \leq \|x_n-p\|^2 - \|x_n-z_n\|^2.$$

Therefore, we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
\leq\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-p\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|(\theta_n-z_n)\\
&+(z_n-p)\|^2+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\leq\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|\theta_n-z_n\|^2+\|z_n-p\|^2+2\langle\theta_n-z_n,z_n-p\rangle\}\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\leq\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|z_n-p\|^2\\
&+2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\leq\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\{\|x_n-p\|^2-\|x_n-z_n\|^2\}\\
&+2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-p\|^2\\
&-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&+2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&+2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|+c_n\\
=\ & (1-2\epsilon_n(1+\mu)\bar{\gamma}+\epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&+2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|+c_n\\
\leq\ & \|x_n-p\|^2+\epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2-(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2\\
&+(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2\\
&+2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|+c_n.
\end{aligned}
$$

It follows that

$$\begin{aligned}
& (1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|x_n-z_n\|^2 \\
\leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|+c_n \\
\leq\ & \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-p\|^2 \\
& +(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|^2 \\
& +2(1-\epsilon_n-\epsilon_n\mu\bar{\gamma})(1-\beta_n-\epsilon_n-\epsilon_n\mu\bar{\gamma})\|\theta_n-z_n\|\|z_n-p\|+c_n.
\end{aligned}$$

Using $\epsilon_n \longrightarrow 0$, $c_n \longrightarrow 0$ as $n \longrightarrow \infty$, (6.1.24) and (6.1.49), we obtain

$$\lim_{n\longrightarrow\infty}\|z_n-x_n\|=0. \tag{6.1.50}$$

Note that

$$\|x_n-\theta_n\| \ \leq\ \|x_n-z_n\|+\|z_n-\theta_n\|,$$

thus from (6.1.49) and (6.1.50), we have

$$\lim_{n\longrightarrow\infty}\|x_n-\theta_n\|=0. \tag{6.1.51}$$

Observe that

$$\|W_n\theta_n-\theta_n\| \ \leq\ \|W_n\theta_n-x_n\|+\|x_n-\theta_n\|.$$

Applying (6.1.27) and (6.1.51), we obtain

$$\lim_{n\longrightarrow\infty}\|W_n\theta_n-\theta_n\|=0. \tag{6.1.52}$$

Let W be the mapping defined by (6.1.5). Since $\{\theta_n\}$ is bounded, applying Lemma 6.1.8 and (6.1.52), we have

$$\|W\theta_n-\theta_n\|\leq\|W\theta_n-W_n\theta_n\|+\|W_n\theta_n-\theta_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{6.1.53}$$

Step 4. We claim that

$$\limsup_{n\longrightarrow\infty}\langle u+[\gamma f-(I+\mu A)]z, x_n-z\rangle\leq 0,$$

where z is a solution of the optimization problem:

$$\text{OP}:\min_{x\in\Gamma}\frac{\mu}{2}\langle Ax,x\rangle+\frac{1}{2}\|x-u\|^2-h(x).$$

To show this inequality, we can choose a subsequence $\{\theta_{n_i}\}$ of $\{\theta_n\}$ such that

$$\lim_{i\longrightarrow\infty}\langle\langle u+[\gamma f-(I+\mu A)]z,\theta_{n_i}-z\rangle = \limsup_{n\longrightarrow\infty}\langle u+[\gamma f-(I+\mu A)]z,\theta_n-z\rangle. \quad (6.1.54)$$

Since $\{\theta_{n_i}\}$ is bounded, there exists a subsequence $\{\theta_{n_{i_j}}\}$ of $\{\theta_{n_i}\}$ which converges weakly to $w\in E$. Without loss of generality, we can assume that $\theta_{n_i}\rightharpoonup w$. From $\|W\theta_n-\theta_n\|\longrightarrow 0$, we obtain $W\theta_{n_i}\rightharpoonup w$. Next, we show that $w\in\Gamma$, where $\Gamma := \cap_{n=1}^{\infty}F(T_n)\cap MEP\cap VI(E,B)$. First, we prove $w\in MEP$. Since $z_n = S_r x_n$, we have

$$\Theta(z_n,x)+\varphi(x)-\varphi(z_n)+\frac{1}{r}\langle K'(z_n)-K'(x_n),\eta(x,z_n)\rangle\geq 0,\quad \forall x\in C.$$

From (H1), we also have

$$\frac{1}{r}\langle K'(z_n)-K'(x_n),\eta(x,z_n)\rangle+\varphi(x)-\varphi(z_n)\geq -\Theta(z_n,x)\geq\Theta(x,z_n)$$

and hence

$$\langle\frac{K'(z_{n_i})-K'(x_{n_i})}{r},\eta(x,z_{n_i})\rangle+\varphi(x)-\varphi(z_{n_i})\geq\Theta(x,z_{n_i}).$$

Since $\frac{K'(z_{n_i})-K'(x_{n_i})}{r}\longrightarrow 0$ and $z_{n_i}\rightharpoonup w$, from the weak lower semicontinuity of φ and $\Theta(x,y)$ in the second variable y, we also have $\Theta(x,w)+\varphi(w)-\varphi(x)\leq 0$ for all $x\in C$. For t with $0<t\leq 1$ and $x\in E$, let $x_t = tx+(1-t)w$. Since $x\in E$ and $w\in E$, we have $x_t\in E$ and hence $\Theta(x_t,w)+\varphi(w)-\varphi(x_t)\leq 0$. So, from the convexity of equilibrium bifunction $\Theta(x,y)$ in the second variable y, we have

$$\begin{aligned}0 &= \Theta(x_t,x_t)+\varphi(x_t)-\varphi(x_t)\\ &\leq t\Theta(x_t,x)+(1-t)\Theta(x_t,w)+t\varphi(x)+(1-t)\varphi(w)-\varphi(x_t)\\ &\leq t[\Theta(x_t,x)+\varphi(x)-\varphi(x_t)],\end{aligned}$$

and hence $\Theta(x_t,x)+\varphi(x)-\varphi(x_t)\geq 0$. Then, we have $\Theta(w,x)+\varphi(x)-\varphi(w)\geq 0$ for all $x\in E$ and hence $w\in MEP$.

Next, we show that $w\in\cap_{n=1}^{\infty}F(T_n)$. By Lemma 6.1.7, we have $F(W)=\cap_{n=1}^{\infty}F(T_n)$. Assume $w\notin F(W)$. Since $\|x_n-\theta_n\|\longrightarrow 0$ we know that $\theta_{n_i}\rightharpoonup w$ $(i\longrightarrow\infty)$ and $w\neq Ww$, it follows by the Opial's condition (Lemma 6.1.3) that

$$\begin{aligned}\liminf_{i\longrightarrow\infty}\|\theta_{n_i}-w\| &< \liminf_{i\longrightarrow\infty}\|\theta_{n_i}-Ww\|\\ &\leq \liminf_{i\longrightarrow\infty}(\|\theta_{n_i}-W\theta_{n_i}\|+\|W\theta_{n_i}-Ww\|)\\ &< \liminf_{i\longrightarrow\infty}\|\theta_{n_i}-w\|,\end{aligned}$$

which is a contradiction. Thus, we get $w\in F(W)=\cap_{n=1}^{\infty}F(T_n)$.

Finally, we show that $w \in VI(E,B)$. Define

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & w_1 \in E, \\ \emptyset, & w_1 \notin E. \end{cases}$$

Then, T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_E w_1$ and $\theta_n \in E$, we have $\langle w_1 - \theta_n, w_2 - Bw_1 \rangle \geq 0$. On the other hand, from $\theta_n = P_C(k_n - \tau_n B k_n)$, we have

$$\langle w_1 - \theta_n, \theta_n - (k_n - \tau_n B k_n) \rangle \geq 0,$$

and hence

$$\langle w_1 - \theta_n, \frac{\theta_n - k_n}{\tau_n} + Bk_n \rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} \langle w_1 - \theta_{n_i}, w \rangle &\geq \langle w_1 - \theta_{n_i}, Bw_1 \rangle \\ &\geq \langle w_1 - \theta_{n_i}, Bw_1 \rangle - \langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} + Bk_{n_i} \rangle \\ &= \langle w_1 - \theta_{n_i}, Bw_1 - Bk_{n_i} - \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} \rangle \\ &= \langle w_1 - \theta_{n_i}, Bv - B\theta_{n_i} \rangle + \langle w_1 - \theta_{n_i}, B\theta_{n_i} - Bk_{n_i} \rangle - \langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} \rangle \\ &\geq \langle w_1 - \theta_{n_i}, B\theta_{n_i} - Bk_{n_i} \rangle - \langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} \rangle. \end{aligned}$$

Noting that $\|\theta_{n_i} - k_{n_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$ and B is Lipschitz continuous implies that

$$\langle w_1 - w, w_2 \rangle \geq 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(E,B)$. That is $w \in \Gamma := \cap_{n=1}^{\infty} F(T_n) \cap MEP \cap VI(E,B)$. Therefore, from Lemma 6.1.2, $\|x_n - \theta_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ and (6.1.54), we have

$$\begin{aligned} \limsup_{n \longrightarrow \infty} \langle u + [\gamma f - (I + \mu A)]z, x_n - z \rangle &= \limsup_{n \longrightarrow \infty} \langle u + [\gamma f - (I + \mu A)]z, \theta_n - z \rangle \\ &= \lim_{i \longrightarrow \infty} \langle u + [\gamma f - (I + \mu A)]z, \theta_{n_i} - z \rangle \\ &= \langle u + [\gamma f - (I + \mu A)]z, w - z \rangle \leq 0. \end{aligned} \tag{6.1.55}$$

It follows from the last inequality, (6.1.27) and (6.1.51) that

$$\limsup_{n \longrightarrow \infty} \langle u + [\gamma f - (I + \mu A)]z, W_n \theta_n - z \rangle \leq 0. \tag{6.1.56}$$

Step 5. Finally, we prove $\{x_n\}$ and $\{z_n\}$ converges strongly to $z \in \Gamma$. From (6.1.6), we obtain

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \\
= \;& \|\epsilon_n(u+\gamma f(W_nx_n)) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n(I+\mu A))W_n\theta_n - z\|^2 \\
= \;& \|((1-\beta_n)I - \epsilon_n(I+\mu A))(W_n\theta_n - z) + \beta_n(x_n - z) + \epsilon_n(u+\gamma f(W_nx_n) - (I+\mu A)z)\|^2 \\
= \;& \|((1-\beta_n)I - \epsilon_n(I+\mu A))(W_n\theta_n - z) + \beta_n(x_n - z)\|^2 + \epsilon_n^2\|u+\gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
& + 2\beta_n\epsilon_n\langle x_n - z, u+\gamma f(W_nx_n) - (I+\mu A)z\rangle \\
& + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n(I+\mu A))(W_n\theta_n - z), u+\gamma f(W_nx_n) - (I+\mu A)z\rangle \\
\leq \;& [(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma})\|W_n\theta_n - z\| + \beta_n\|x_n - z\|]^2 + \epsilon_n^2\|u+\gamma f(W_nx_n) \\
& -(I+\mu A)z\|^2 + 2\beta_n\epsilon_n\gamma\langle x_n - z, f(W_nx_n) - f(z)\rangle + 2\beta_n\epsilon_n\langle x_n - z, u+\gamma f(z) \\
& -(I+\mu A)z\rangle + 2(1-\beta_n)\gamma\epsilon_n\langle W_n\theta_n - z, f(W_nx_n) - f(z)\rangle \\
& +2(1-\beta_n)\epsilon_n\langle W_n\theta_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& - 2\epsilon_n^2\langle(I+\mu A)(W_n\theta_n - z), u+\gamma f(z) - (I+\mu A)z\rangle \\
\leq \;& [(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma})\|W_n\theta_n - z\| + \beta_n\|x_n - z\|]^2 + \epsilon_n^2\|u+\gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
& + 2\beta_n\epsilon_n\gamma\|x_n - z\|\|f(W_nx_n) - f(z)\| + 2\beta_n\epsilon_n\langle x_n - z, u+\gamma f(z) \\
& -(I+\mu A)z\rangle + 2(1-\beta_n)\gamma\epsilon_n\|W_n\theta_n - z\|\|f(W_nx_n) - f(z)\| \\
& +2(1-\beta_n)\epsilon_n\langle W_n\theta_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& - 2\epsilon_n^2\langle(I+\mu A)(W_n\theta_n - z), u+\gamma f(z) - (I+\mu A)z\rangle \\
\leq \;& [(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma})\|\theta_n - z\| + \beta_n\|x_n - z\|]^2 + \epsilon_n^2\|u+\gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
& + 2\beta_n\epsilon_n\gamma\|x_n - z\|\|f(W_nx_n) - f(z)\| + 2\beta_n\epsilon_n\langle x_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& + 2(1-\beta_n)\gamma\epsilon_n\|\theta_n - z\|\|f(W_nx_n) - f(z)\| \\
& +2(1-\beta_n)\epsilon_n\langle W_n\theta_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& - 2\epsilon_n^2\langle(I+\mu A)(W_n\theta_n - z), u+\gamma f(z) - (I+\mu A)z\rangle \\
\leq \;& [(1-\beta_n-\epsilon_n(1+\mu)\bar{\gamma})\|x_n - z\| + \beta_n\|x_n - z\|]^2 + \epsilon_n^2\|u+\gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
& + 2\beta_n\epsilon_n\gamma\alpha\|x_n - z\|^2 + 2\beta_n\epsilon_n\langle x_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& + 2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n - z\|^2 + 2(1-\beta_n)\epsilon_n\langle W_n\theta_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& - 2\epsilon_n^2\langle(I+\mu A)(W_n\theta_n - z), u+\gamma f(z) - (I+\mu A)z\rangle \\
= \;& [(1-\epsilon_n(1+\mu)\bar{\gamma})^2 + 2\beta_n\epsilon_n\gamma\alpha \\
& +2(1-\beta_n)\gamma\epsilon_n\alpha]\|x_n - z\|^2 + \epsilon_n^2\|u+\gamma f(W_nx_n) - (I+\mu A)z\|^2 \\
& + 2\beta_n\epsilon_n\langle x_n - z, u+\gamma f(z) - (I+\mu A)z\rangle + 2(1-\beta_n)\epsilon_n\langle W_n\theta_n - z, u+\gamma f(z) - (I+\mu A)z\rangle \\
& - 2\epsilon_n^2\langle(I+\mu A)(W_n\theta_n - z), u+\gamma f(z) - (I+\mu A)z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq [1-2((1+\mu)\bar{\gamma}-\alpha\gamma)\epsilon_n]\|x_n-z\|^2 \\
&\quad +\epsilon_n^2(1+\mu)^2\bar{\gamma}^2\|x_n-z\|^2+\epsilon_n^2\|u+\gamma f(W_nx_n)-(I+\mu A)z\|^2 \\
&\quad +2\beta_n\epsilon_n\langle x_n-z,u+\gamma f(z)-(I+\mu A)z\rangle+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,u+\gamma f(z)-(I+\mu A)z\rangle \\
&\quad +2\epsilon_n^2\|(I+\mu A)(W_n\theta_n-z)\|\|u+\gamma f(z)-(I+\mu A)z\| \\
&= [1-2((1+\mu)\bar{\gamma}-\alpha\gamma)\epsilon_n]\|x_n-z\|^2+\epsilon_n\{\epsilon_n[(1+\mu)^2\bar{\gamma}^2\|x_n-z\|^2 \\
&\quad +\|u+\gamma f(W_nx_n)-(I+\mu A)z\|^2 \\
&\quad +2\|(I+\mu A)(W_n\theta_n-z)\|\|u+\gamma f(z)-(I+\mu A)z\|]+2\beta_n\langle x_n-z,u+\gamma f(z)-(I+\mu A)z\rangle \\
&\quad +2(1-\beta_n)\langle W_n\theta_n-z,u+\gamma f(z)-(I+\mu A)z\rangle\}.
\end{aligned}
$$

Since $\{x_n\}$, $\{f(W_nx_n)\}$ and $\{W_n\theta_n\}$ are bounded, we can take a constant $M>0$ such that $(1+\mu)^2\bar{\gamma}^2\|x_n-z\|^2+\|u+\gamma f(W_nx_n)-(I+\mu A)z\|^2+2\|(I+\mu A)(W_n\theta_n-z)\|\|u+\gamma f(z)-(I+\mu A)z\|\leq M$, for all $n\geq 0$. It follows that

$$
\|x_{n+1}-z\|^2\leq(1-l_n)\|x_n-z\|^2+\epsilon_n\sigma_n, \tag{6.1.57}
$$

where

$$
\begin{aligned}
l_n &= 2((1+\mu)\bar{\gamma}-\alpha\gamma)\epsilon_n, \\
\sigma_n &= \epsilon_nM+2\beta_n\langle x_n-z,u+\gamma f(z)-(I+\mu A)z\rangle \\
&\quad +2(1-\beta_n)\langle W_n\theta_n-z,u+\gamma f(z)-(I+\mu A)z\rangle.
\end{aligned}
$$

Using (C4), (6.1.55) and (6.1.56), we get $l_n\longrightarrow 0$, $\sum_{n=1}^{\infty}l_n=\infty$ and $\limsup_{n\longrightarrow\infty}\frac{\sigma_n}{l_n}\leq 0$. Applying Lemma 6.1.11 and (6.1.55) to (6.1.57), we conclude that $x_n\longrightarrow z$ in norm. Finally, noticing $\|z_n-z\|=\|S_rx_n-S_rz\|\leq\|x_n-z\|$. We also conclude that $z_n\longrightarrow z$ in norm. This completes the proof.

6.2 Multi-Objective Optimization problem

In this section, we study a kind of multi-objective optimization problem by using the result of this paper. We will give an iterative algorithm of solution for the following *optimization problem* with nonempty set of solutions

$$
\begin{cases}
\min \quad h_1(x) \\
\min \quad h_2(x) \\
x\in C,
\end{cases} \tag{6.2.1}
$$

where $h(x)$ is a convex and lower semi-continuous functional and define C is a closed convex subset of a real Hilbert space H. We denote the set of solutions of (6.2.1) by $M(h_1)$ and $M(h_2)$. Let $F_i : C \times C \to \mathbb{R}$ be a bifunction defined by $F_i(x,y) = h_i(y) - h_i(x)$. We consider the equilibrium problem, it is obvious that $EP(F_i) = M(h_i),\ i = 1,2$. Therefore, from Theorem 4.2.4, we obtained the following corollary.

Corollary 6.2.1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi continuous and convex function. Let A, B, D, E be $\alpha, \beta, \delta, \eta$-inverse-strongly monotone mapping of C into H, respectively. Let $T_1, T_2, \ldots$ be an infinite nonexpansive mapping such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap MEP(F_1, \varphi_1) \cap MEP(F_2, \varphi_2) \cap VI(C,D) \cap VI(C,E) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C,\ C_{1,i} = C,\ C_1 = \cap_{i=1}^{\infty} C_{1,i},\ x_1 = P_{C_1} x_0$ and*

$$\begin{cases} h_1(t) - h_1(t_n) + \frac{1}{r_n}\langle t - t_n, t_n - x_n\rangle \geq 0, & \forall t \in C, \\ h_2(u) - h_2(u_n) + \frac{1}{s_n}\langle u - u_n, u_n - t_n\rangle \geq 0, & \forall u \in C, \\ w_n = \xi_n P_C(u_n - \lambda_n D u_n) + (1-\xi_n) P_C(t_n - \mu_n E t_n), \\ y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) T_i w_n, \\ C_{n+1,i} = \Big\{ z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\Big\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{6.2.2}$$

for every $n \geq 0$, where $\{r_n\}, \{s_n\} \subset (0,\infty),\ \lambda_n \in (0, 2\delta)$ and $\mu_n \in (0, 2\eta)$ satisfy the following conditions:
(i).$\lim_{n\to\infty} \alpha_{n,i} = 0$;
(ii).$\lim_{n\to\infty} \xi_n = \xi \in (0,1)$;
(iii). $0 < e \leq \lambda_n \leq f < 2\delta$;
(iv). $0 < g \leq \mu_n \leq j < 2\eta$.
Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

***Proof*.** From Theorem 4.2.4 put $F_1(t_n, t) = h_1(t) - h_1(t_n),\ F_2(u_n, u) = h_2(u) - h_2(u_n)$ and $A,\ B,\ \varphi_1,\ \varphi_2 \equiv 0$. The conclusion of Corollary 6.2.1 can be obtained from Theorem 4.2.4 immediately. □

6.3 Minimizer of a continuously Frechet Differentiable Convex Functional

In this section, we study the problem for finding a minimizer of a continuously Frèchet differentiable convex functional in a Hilbert space.

First, we use the following lemma in our result:

Lemma 6.3.1. [252] *Let E be a Banach space, let f be a continuously Frèchet differentiable convex functional on E and let ∇f be the gradient of f. If ∇f is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is an α-inverse-strongly monotone.*

Let $f_1,\ f_2$ be functionals on H which satisfies the following conditions:

(C1) $f_1,\ f_2$ be a continuously Frèchet differentiable convex functional on H and $\nabla f_1,\ \nabla f_2$ be $\frac{1}{\delta},\ \frac{1}{\eta}$- Lipschitz continuous,

(C2) $(\nabla f_1)^{-1}0 = \{z_1 \in H : f_1(z_1) = \min_{y_1 \in H} f_1(y_1)\} \neq \emptyset$ and $(\nabla f_2)^{-1}0 = \{z_2 \in H : f_2(z_2) = \min_{y_2 \in H} f_2(y_2)\} \neq \emptyset$.

Corollary 6.3.2. *Let H be a real Hilbert Space. Let F_1, F_2 be a bifunction of $H \times H$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi continuous and convex function. Let A, B be α, β-inverse-strongly monotone mapping of H into H, respectively. Let $T_1, T_2, \ldots$ be infinite nonexpansive mappings. Let $f_1,\ f_2$ be functionals on H which satisfies the conditions (C1) and (C2). Suppose that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, \varphi_1) \cap GMEP(F_2, \varphi_2) \cap (\nabla f_1)^{-1}0 \cap (\nabla f_2)^{-1}0 \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C,\ C_{1,i} = C,\ C_1 = \cap_{i=1}^{\infty} C_{1,i},\ x_1 = P_{C_1} x_0$ and*

$$\begin{cases} t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\ u_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n), \\ w_n = \xi_n(u_n - \lambda_n \nabla f_1(u_n)) + (1-\xi_n)(t_n - \mu_n \nabla f_2(t_n)), \\ y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) T_i w_n, \\ C_{n+1,i} = \Big\{ z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle) \Big\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{6.3.1}$$

for every $n \geq 0$, where $\{r_n\}, \{s_n\} \subset (0, \infty),\ \lambda_n \in (0, 2\delta)$ and $\mu_n \in (0, 2\eta)$ satisfying the following conditions: (i). $0 < a \leq r_n \leq b < 2\alpha$;
(ii). $0 < c \leq s_n \leq d < 2\beta$;
(iii).$\lim_{n\to\infty} \alpha_{n,i} = 0$;
(iv).$\lim_{n\to\infty} \xi_n = \xi \in (0,1)$;
(v). $0 < e \leq \lambda_n \leq f < 2\delta$;
(vi). $0 < g \leq \mu_n \leq j < 2\eta$.
Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

Proof. We know form condition (C1) and Lemma 6.3.1 that $\nabla f_1,\ \nabla f_2$ are $\delta,\ \eta$-inverse-strongly monotone operators from H in to itself. The conclusion of Corollary 6.3.2 can be obtained from Theorem 4.2.4 immediately. □

6.4 Minimization Problem

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle x, y\rangle, \ \forall x \in F(S), \tag{6.4.1}$$

where A is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S and y is a given point in H [263].

In 2006, Marino and Xu [263] introduced a general iterative method for nonexpansive mapping. They defined the sequence $\{x_n\}$ generated by the algorithm $x_0 \in C$,

$$x_{n+1} \;=\; \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad n \geq 0 \tag{6.4.2}$$

where $\{\alpha_n\} \subset (0,1)$ and A is a strongly positive linear bounded operator. They proved that if $C = H$ and the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6.4.2) converge strongly to a fixed point of S (say $\bar{x} \in H$) which is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)\bar{x}, x - \bar{x}\rangle \geq 0, \ \forall x \in F(S), \tag{6.4.3}$$

which is the optimality condition for the *minimization problem*

$$\min_{x \in F(S) \cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{6.4.4}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of the variational inequalities. Let P_C be the projection of H onto C. In 2005, Iiduka and Takahashi [258] introduced following iterative process for $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \ \forall n \geq 0, \tag{6.4.5}$$

where $u \in C$, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\beta$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (6.4.5) converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping (say $\bar{x} \in C$) which solve some variational inequality

$$\langle \bar{x} - u, x - \bar{x}\rangle \geq 0, \ \forall x \in F(S) \cap VI(C, A). \tag{6.4.6}$$

In 2008, Su et al. [268] introduced the following iterative scheme by the viscosity approximation method in a real Hilbert space: $x_1, u_n \in H$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) SP_C(u_n - \lambda_n Au_n), \end{cases} \tag{6.4.7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate conditions. Furthermore, they proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $z \in F(S) \cap VI(C,A) \cap EP(F)$, where $z = P_{F(S)\cap VI(C,A)\cap EP(F)} f(z)$.

Let $\{T_i\}$ be an infinite family of nonexpansive mappings of H into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq 1$ for every $i \in N$. For $n \geq 1$, we defined a mapping W_n of H into itself as follows:

$$\begin{aligned} &U_{n,n+1} := I, \\ &U_{n,n} := \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ &\vdots \\ &U_{n,k} := \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ &\vdots \\ &U_{n,2} := \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ &W_n := U_{n,1} := \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{aligned} \tag{6.4.8}$$

In 2011, He et al. [274] introduced following iterative process for $\{T_n : C \to C\}$ which is a sequence of nonexpansive mappings. Let $\{z_n\}$ be the sequence defined by

$$z_{n+1} = \epsilon_n \gamma f(z_n) + (I - \epsilon_n) W_n K^1_{r_1,n} K^2_{r_2,n} \cdot \ldots \cdot K^K_{r_K,n} z_n, \ \forall n \in N \tag{6.4.9}$$

The sequence $\{z_n\}$ defined by (6.4.9) converges strongly to a common element of the set of fixed points of nonexpansive mappings, the set of solutions of the variational inequality and the generalized equilibrium problem. Recently, Jitpeera and Kumam [275] introduced the following a new general iterative method for finding a common element of the set of solutions of fixed point for nonexpansive mappings, the set of solution of generalized mixed equilibrium problems and the set of solutions of the variational inclusion for a β-inverse-strongly monotone mapping in a real Hilbert space.

In this section, we modify the iterative methods (6.4.2), (6.4.7) and (6.4.9) by purposing the new general viscosity iterative method. We show that under some control conditions the sequence $\{x_n\}$ converges strongly to a common element of the set of common fixed points of nonexpansive mappings, the solution of the system of mixed equilibrium problems and the set of solutions of the variational inclusion in a real Hilbert space. Moreover, we apply our results to the class of strictly pseudocontractive mappings. Finally, we give a numerical example which support our main theorem in the last part. Our results improve and extend the corresponding results of Marino and Xu (2006), Su et al. (2008), He et al. (2011) and some authors.

Lemma 6.4.1. *[327] Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \ \ \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathcal{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(ii) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 6.4.2. *[251] Let* C *be a closed convex subset of a real Hilbert space* H *and let* $T : C \longrightarrow C$ *be a nonexpansive mapping. Then* $I - T$ *is demiclosed at zero, that is,*

$$x_n \rightharpoonup x \ \mathit{and} \ x_n - Tx_n \to 0$$

implies $x = Tx$.

Lemma 6.4.3. *[274] Let* C *be a nonempty closed and convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i\in N}$ *be an infinite family of nonexpansive mappings of* C *into itself such that* $\cap_{i\in N} F(T_i) \neq \emptyset$ *and let* $\{\lambda_i\}$ *be an real sequence such that* $0 \leq \lambda_i \leq \mathrm{b} < 1$ *for every* $i \in N$. *Then* $\mathrm{F(W)} = \cap_{i\in N} F(T_i) \neq \emptyset$.

Lemma 6.4.4. *[274] Let* C *be a nonempty closed and convex subset of a strictly convex Banach space. Let* $\{T_i\}$ *be an infinite family of nonexpansive mappings of* C *into itself and let* $\{\lambda_i\}$ *be a real sequence such that* $0 \leq \lambda_i \leq \mathrm{b} < 1$ *for every* $i \in N$. *Then, for every* $x \in C$ *and* $k \in N$, *the limit* $\lim_{n\longrightarrow\infty} U_{n,k}$ *exist.*

In view of the previous lemma, we define

$$Wx := \lim_{n\longrightarrow\infty} U_{n,1}x = \lim_{n\longrightarrow\infty} W_n x.$$

Next we stat our main result, we show a strong convergence theorem which solves the problem of finding a common element of the common fixed points, the common solution of a system of mixed equilibrium problems and variational inclusion of inverse-strongly monotone mappings in a Hilbert space.

Theorem 6.4.5. *Let H be a real Hilbert space, C be a nonempty close and convex subset of H and B be a* β*-inverse-strongly monotone mapping. Let* φ *: C*$\longrightarrow$*R be a convex and lower semicontinuous function, f : C* $\longrightarrow$ *C be a contraction mapping with coefficient* α*(0 <* α *< 1), M : H* $\longrightarrow$ 2^H *be a maximal monotone mapping. Let A be a strongly positive linear bounded operator of H into itself with coefficient*

$\overline{\gamma} > 0$. *Assume that* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ *and* $\lambda \in (0, 2\beta)$. *Let* $\{T_n\}$ *be a family of nonexpansive mappings of H into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq \emptyset.$$

Suppose that $\{x_n\}$ *is a sequence generated by the following algorithm for* $x_0 \in C$ *arbitrarily and*

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \ \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n \gamma f(x_n) + (I - \epsilon_n A) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \tag{6.4.10}$$

for all n = 1,2,3,..., *where*

$$K_{r_{i,n}}^{F_i}(x) = \{u_n \in C : F_i(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_{i,n}}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C\}$$

for all $i = 1, 2, 3, \ldots, N$ *and the following conditions are satisfied*

(C1): $\{\epsilon_n\} \subset (0, 1)$, $\lim_{n \longrightarrow 0} \epsilon_n = 0$, $\sum_{n=1}^{\infty} \epsilon_n = \infty$, $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$;

(C2): $\{r_n\} \subset [c, d]$ *with* $c, d \in (0, 2\sigma)$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to* $q \in \theta$, *where* $q = P_\theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle (\gamma f - A)q, p - q\rangle \leq 0, \quad \forall p \in \theta, \tag{6.4.11}$$

which is the optimality condition for the minimization problem

$$\min_{q \in \theta} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{6.4.12}$$

where h is a potential function for γf *(i.e.,* $h'(q) = \gamma f(q)$ *for* $q \in H$*).*

Proof. Since condition (C1), we may assume without loss of generality, then $\epsilon_n \in (0, \| A \|^{-1})$ for all n. Then, we have $\|I - \epsilon_n A\| \leq 1 - \epsilon_n \bar{\gamma}$. Next, we will assume that $\|I - A\| \leq \|1 - \bar{\gamma}\|$.

Next, we will divide the proof into six steps.

Step 1. First, will show that $\{x_n\}$ and $\{u_n\}$ are bounded. Since B is β-inverse-strongly monotone mappings, we have

$$\begin{aligned}
\|(I-\lambda B)x-(I-\lambda B)y\|^2 &= \|Ix-\lambda Bx-Iy+\lambda By\|^2\\
&= \|x-y-\lambda Bx+\lambda By\|^2\\
&= \|(x-y)-\lambda(Bx+By)\|^2\\
&\leq \|x-y\|^2-2\lambda\langle x-y\rangle\langle Bx+By\rangle\\
&\quad +\lambda^2\|Bx-By\|^2\\
&\leq \|x-y\|^2-2\lambda\beta\|Bx+By\|^2\\
&\quad +\lambda^2\|Bx-By\|^2\\
&\leq \|x-y\|^2+\lambda(\lambda-2\beta\|Bx+By\|^2 \qquad (6.4.13)
\end{aligned}$$

if $0<\lambda<2\beta,\ then\ I-\lambda$ B is nonexpansive.
Put $y_n := J_{M,\lambda}(u_n-\lambda Bu_n),\ n\geq 0$. Since $J_{M,\lambda}$ and $I-\lambda B$ are nonexpansive mapping, it follows that

$$\begin{aligned}
\|y_n-q\| &= \|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(q-\lambda Bq)\|\\
&\leq \|(u_n-\lambda Bu_n)-(q-\lambda Bq)\|\\
&\leq \|u_n-q\|. \qquad (6.4.14)
\end{aligned}$$

By Lemma **??**, we have

$$\begin{aligned}
&u_n = K^{F_N}_{r_n,n}\cdot K^{F_{N-1}}_{r_{n-1},n}\cdot K^{F_{N-2}}_{r_{n-2},n}\cdot ...\cdot K^{F_2}_{r_2,n}\cdot K^{F_1}_{r_1,n}\cdot x_n,\ \ for\ n\geq 0\\
&\tau^k_n = K^{F_k}_{r_k,n}\cdot K^{F_{k-1}}_{r_{k-1},n}\cdot ...\cdot K^{F_2}_{r_2,n}\cdot K^{F_1}_{r_1,n},\quad for\ k\in\{0,1,2,...,N\}
\end{aligned}$$

and $\tau^0_n = I$ for all $n\in N,\quad q=\tau^{F_k}_{r_k,n}q,\quad u_n=\tau^N_{r_k,N}x_n$ Then, we have

$$\begin{aligned}
\|u_n-q\|^2 &= \|\tau^N_{r_k,n}x_n-\tau^{F_k}_{r_k,n}q\|^2\\
&= \|x_n-q\|^2. \qquad (6.4.15)
\end{aligned}$$

Hence, we get

$$\|y_n-q\|\leq\|x_n-q\|. \qquad (6.4.16)$$

From (6.4.10), we deduce that

$$\begin{aligned}
\|x_{n+1}-q\| &= \|P_C(\epsilon_n\gamma f(x_n)+(I-\epsilon_n A)W_n y_n)-P_C q\| \\
&\leq \|\epsilon_n(\gamma f(x_n)-Aq)+(I-\epsilon_n A)(W_n y_n-q)\| \\
&\leq \epsilon_n\|\gamma f(x_n)-Aq)\|+(1-\epsilon_n\bar{\gamma})\|(y_n))-q\| \\
&\leq \epsilon\gamma\epsilon_n\|x_n-q\|+\epsilon_n\|\gamma f(g)-Aq\| \\
&\quad +(1-\epsilon_n\bar{\gamma})\|x_n-q\| \qquad (6.4.17)\\
&= (1-(\bar{\gamma}-\gamma\epsilon)\epsilon_n)\|x_n-q\|-\epsilon_n\|\gamma f(q)-Aq\| \\
&= (1-(\bar{\gamma}-\gamma\epsilon)\epsilon_n)\|x_n-q\|+(\bar{\gamma}-\gamma\epsilon)\epsilon_n\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\epsilon} \\
&\quad \vdots \\
&\leq \max\Big\{\|x_n-q\|,\ \frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\epsilon}\Big\}.
\end{aligned}$$

It follows by induction that

$$\|x_n-q\| \leq \max\{\|x_0-q\|,\ \frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\epsilon}\}, n\geq 0. \qquad (6.4.18)$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Bu_n\}$, $\{f(x_n)\}$ and $\{AW_n y_n\}$.

Step 2. We claim that $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. From (6.4.10), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|P_C(\epsilon_n\gamma f(x_n)+(I-\epsilon_n A)W_n y_n)-P_C(\epsilon_{n-1}\gamma f(x_{n-1})+(I-\epsilon_{n-1}A)W_n y_{n-1}\| \\
&\leq \|(I-\epsilon_n A)(W_n y_n-W_n y_{n-1})-(\epsilon_n-\epsilon_{n-1})AW_n y_{n-1}+ \\
&\quad \gamma\epsilon_n(f(x_n)-f(x_{n-1}))+\gamma(\epsilon_n-\epsilon_{n-1})f(x_{n-1})\| \\
&\leq (1-\epsilon_n\bar{\gamma})\|y_n-y_{n-1}\|+|\epsilon_n-\epsilon_{n-1}|\|AW_n y_n\|+\gamma\epsilon\epsilon_n\|x_n-x_{n-1}\| \\
&\quad +\gamma|\epsilon_n-\epsilon_{n-1}|\|f(x_{n-1})\|. \qquad (6.4.19)
\end{aligned}$$

Since $J_{M,\lambda}$ *and* $I-\lambda B$ are nonexpansive, we also have

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(u_{n-1}-\lambda Bu_{n-1})\| \\
&\leq \|(u_n-\lambda Bu_n)-(u_{n-1}-\lambda Bu_{n-1})\| \qquad (6.4.20)\\
&\leq \|u_n-u_{n-1}\|.
\end{aligned}$$

On the other hand, from $u_{n-1}=\tau^N_{r_k,n-1}x_{n-1}$ and $u_n=\tau^N_{r_k,n}x_n$, it follows that

$$F(u_{n-1},y)+\varphi(y)-\varphi(u_{n-1})+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0, \quad \forall y\in C \quad (6.4.21)$$

and

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y\in C. \qquad (6.4.22)$$

Substituting $y = u_n$ into (6.4.21) and $y = u_{n-1}$ into (6.4.22), we get

$$F(u_{n-1}, u_n) + \varphi(u_n) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0 \quad (6.4.23)$$

and

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, un - x_n\rangle \geq 0. \quad (6.4.24)$$

From (A2), we obtain

$$\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\rangle \geq 0, \quad (6.4.25)$$

and

$$\langle u_n - u_{n-1}, u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0, \quad (6.4.26)$$

so,

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0. \quad (6.4.27)$$

It follows that

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0,$$

and

$$\langle u_n - u_{n-1}, u_{n-1} - u_n\rangle + \langle u_n - u_{n-1}, (1 - \frac{r_{n-1}}{r_n})(u_n - x_n)\rangle \geq 0. \quad (6.4.28)$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \Big\langle u_n - u_{n-1}, \Big(1 - \frac{r_{n-1}}{r_n}\Big)(u_n - x_n)\Big\rangle \\ &\leq \|u_n - u_{n-1}\|\Big\{|1 - \frac{r_{n-1}}{r_n}\Big|\|u_n - x_n\|\Big\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|, \quad (6.4.29) \end{aligned}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting (6.4.29) into (6.4.20), we have

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|. \quad (6.4.30)$$

Substituting (6.4.30) into (6.4.19), we get

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\epsilon_n\bar{\gamma})\Big(\|x_n-x_{n-1}\|+\frac{M_1}{c}|r_n-r_{n-1}|\Big)+|\epsilon_n-\epsilon_{n-1}|\|AW_ny_{n-1}\| \\ &\quad +\gamma\epsilon\epsilon_n\|x_n-x_{n-1}\|+\gamma|\epsilon_n-\epsilon_{n-1}|\|f(x_{n-1})\| \\ &= (1-\epsilon_n\bar{\gamma})\|x_n-x_{n-1}\|+(1-\epsilon_n\bar{\gamma})\frac{M_1}{c}|r_n-r_{n-1}|+|\epsilon_n-\epsilon_{n-1}|\|AW_ny_{n-1}\| \\ &\quad +\gamma\epsilon\epsilon_n\|x_n-x_{n-1}\|+\gamma|\epsilon_n-\epsilon_{n-1}|\|f(x_{n-1})\| \\ &\leq (1-(\bar{\gamma}-\gamma\epsilon)\epsilon_n)\|x_n-x_{n-1}\|+\frac{M_1}{c}|r_n-r_{n-1}|+|\epsilon_n-\epsilon_{n-1}|\|AW_ny_{n-1}\| \\ &\quad +\gamma|\epsilon_n-\epsilon_{n-1}|\|f(x_{n-1})\| \\ &\leq (1-(\bar{\gamma}-\gamma\epsilon)\epsilon_n)\|x_n-x_{n-1}\|+\frac{M_1}{c}|r_n-r_{n-1}|+M_2|\epsilon_n-\epsilon_{n-1}|,\end{aligned}$$

where $M_2=\sup\big\{\max\{\|AW_ny_{n-1}\|,\|f(x_{n-1})\|:n\in\mathbb{N}\}\big\}$. Since conditions (C1)-(C2) and by Lemma 6.4.1, we have $\|x_{n+1}-x_n\|\longrightarrow 0$ as $n\to\infty$. From (6.4.30), we also have $\|y_{n+1}-y_n\|\longrightarrow 0$ as $n\to\infty$.

Step 3. Next, we show that $\lim_{n\longrightarrow\infty}\|Bu_n-Bq\|=0$.

For $q\in\theta$ hence $q=J_{M,\lambda}(q-\lambda Bq)$. By (6.4.13) and (6.4.15), we get

$$\begin{aligned}\|y_n-q\|^2 &= \|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(q-\lambda Bq)\|^2 \\ &\leq \|(u_n-\lambda Bu_n)-(q-\lambda Bq)\|^2 \\ &\leq \|u_n-q\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2 \\ &\leq \|x_n-q\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2. \qquad (6.4.31)\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|P_C(\epsilon_n\gamma f(x_n)+(I-\epsilon_nA)W_ny_n)-P_C(q)\|^2 \\ &\leq \|\epsilon_n(\gamma f(x_n)-Aq)+(I-\epsilon_nA)(W_ny_n-q)\|^2 \\ &\leq \big(\epsilon_n\|\gamma f(x_n)-Aq\|+(1-\epsilon_n\bar{\gamma})\|y_n-q\|\big)^2 \\ &\leq \epsilon_n\|\gamma f(x_n)-Aq\|^2+(1-\epsilon_n\bar{\gamma})\|y_n-q\|^2 \\ &\quad +2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\| \qquad (6.4.32) \\ &\leq \epsilon_n\|\gamma f(x_n)-Aq\|^2+2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\| \\ &\quad +(1-\epsilon_n\bar{\gamma})\Big(\|x_n-q\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2\Big) \\ &\leq \epsilon_n\|\gamma f(x_n)-Aq\|^2+2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\| \\ &\quad +\|x_n-q\|^2+(1-\epsilon_n\bar{\gamma})\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2.\end{aligned}$$

So, we obtain

$$\begin{aligned}
& (1-\epsilon_n\bar{\gamma})\lambda(2\beta-\lambda)\|Bu_n-Bq\|^2 \\
\leq\ & \epsilon_n\|\gamma f(x_n)-Aq\|^2+\|x_n-x_{n+1}\|\big(\|x_n-q\|+\|x_{n+1}-q\|\big)+\xi_n, \quad (6.4.33)
\end{aligned}$$

where $\xi_n = 2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|$. By conditions (C1),(C3) and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, then, we obtain that $\|Bu_n-Bq\|\longrightarrow 0$ as $n\longrightarrow\infty$.

Step 4. We show the followings:

(i) $\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0$;

(ii) $\lim_{n\longrightarrow\infty}\|u_n-y_n\|=0$;

(iii) $\lim_{n\longrightarrow\infty}\|y_n-W_ny_n\|=0$.

Since $K_{r_n}(x)$ is firmly nonexpansive, we observe that

$$\begin{aligned}
\|u_n-q\|^2 &= \|\tau^N_{r_n,n}\ x_n-\tau^N_{r_n,n}\ q\|^2 \\
&\leq \langle x_n-q,u_n-q\rangle \\
&= \frac{1}{2}\Big(\|x_n-q\|^2+\|u_n-q\|^2-\|x_n-q-u_n-q\|^2\Big) \quad (6.4.34) \\
&\leq \frac{1}{2}\Big(\|x_n-q\|^2+\|u_n-q\|^2-\|x_n-u_n\|^2\Big)
\end{aligned}$$

it follows that

$$\|u_n-q\|^2 \leq \|x_n-q\|^2-\|x_n-u_n\|^2.$$

Since $J_{M,\lambda}$ is 1-inverse-strongly monotone, we compute

$$\begin{aligned}
\|y_n-q\|^2 &= \|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(q-\lambda Bq)\|^2 \\
&\leq \langle(u_n-\lambda Bu_n)-(q-\lambda Bq),y_n-q\rangle \\
&= \frac{1}{2}\Big(\|(u_n-\lambda Bu_n)-(q-\lambda Bq)\|^2+\|y_n-q\|^2 \\
&\quad -\|(u_n-\lambda Bu_n)-(q-\lambda Bq)-(y_n-q)\|^2\Big) \quad (6.4.35) \\
&\leq \frac{1}{2}\Big(\|u_n-q\|^2+\|y_n-q\|^2-\|(u_n-y_n)-\lambda(Bu_n-Bq)\|^2\Big) \\
&= \frac{1}{2}\Big(\|u_n-q\|^2+\|y_n-q\|^2-\|u_n-y_n\|^2 \\
&\quad +2\lambda\langle u_n-y_n,Bu_n-Bq\rangle-\lambda^2\|Bu_n-Bq\|^2\Big),
\end{aligned}$$

which implies that

$$\|y_n - q\|^2 \;\leq\; \|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Bu_n - Bq\|. \quad (6.4.36)$$

Substitute (6.4.36) into (6.4.32), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|y_n - q\|^2 + 2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\ &\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \Big(\|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Bu_n - Bq\|\Big) \\ &\quad +2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|. \qquad (6.4.37)\end{aligned}$$

Then, we derive

$$\begin{aligned}&\|x_n - u_n\|^2 + \|u_n - y_n\|^2 \\ \leq\;& \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &+2\lambda\|u_n - y_n\|\|Bu_n - Bq\| + 2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|. \\ =\;& \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - x_{n+1}\|(\|x_n - q\| + \|x_{n+1} - q\|) \qquad (6.4.38) \\ &+2\lambda\|u_n - y_n\|\|Bu_n - Bq\| + 2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|.\end{aligned}$$

By condition (C1), $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\|Bu_n - Bq\| = 0$. So, we have $\|x_n - u_n\| \to 0,\ \|u_n - y_n\| \to 0$ as $n \to \infty$. It follows that

$$\|x_n - y_n\| \;\leq\; \|x_n - u_n\| + \|u_n - y_n\| \to 0, \text{ as } n \to \infty. \qquad (6.4.39)$$

From (6.4.10), we have

$$\begin{aligned}\|x_n - W_n y_n\| &\leq \|x_n - W_n y_{n-1}\| + \|W_n y_{n-1} - W_n y_n\| \\ &\leq \Big\|P_C\big(\epsilon_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}A)W_n y_{n-1}\big) - P_C(W_n y_{n-1})\Big\| \\ &\quad +\|y_{n-1} - y_n\| \qquad (6.4.40) \\ &\leq \epsilon_{n-1}\|\gamma f x_{n-1} - AW_n y_{n-1}\| + \|y_{n-1} - y_n\|.\end{aligned}$$

By condition (C1) and $\lim_{n\to\infty}\|y_{n-1} - y_n\| = 0$, we obtain that $\|x_n - W_n y_n\| \to 0$ as $n \to \infty$.

Hence, we have

$$\begin{aligned}\|x_n - W_n x_n\| &\leq \|x_n - W_n y_n\| + \|W_n y_n - W_n x_n\| \\ &\leq \|x_n - W_n y_n\| + \|y_n - x_n\|. \qquad (6.4.41)\end{aligned}$$

By (6.4.39) and $\lim_{n\to\infty}\|x_n - W_n y_n\| = 0$, we obtain $\|x_n - W_n x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover, we also have

$$\|y_n - W_n y_n\| \leq \|y_n - x_n\| + \|x_n - W_n y_n\|.$$

By (6.4.39) and $\lim_{n\to\infty}\|x_n - W_n y_n\| = 0$, we obtain $\|y_n - W_n y_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Step 5. We show that $q \in \theta := \bigcap_{n=1}^{\infty} F(T_n) \cap (\bigcap_{k=1}^{N} SMEP(F_k)) \cap I(B, M)$ and $\limsup_{n\longrightarrow\infty}\langle(\gamma f - A)q, W_n y_n - q\rangle \leq 0$. It is easy to see that $P_\theta(\gamma f + (I - A))$ is a contraction of H into itself.
Indeed, since $0 < \gamma < \frac{\bar{\gamma}}{\epsilon}$, we have

$$\begin{aligned}\|P_\theta(\gamma f + (I-A))x - P_\theta(\gamma f + (I-A))y\| &\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ &\leq \gamma\epsilon\|x - y\| + (1 - \bar{\gamma})\|x - y\| \quad (6.4.42) \\ &\leq (1 - \bar{\gamma} + \gamma\epsilon)\|x - y\|.\end{aligned}$$

Since H is complete, then there exists a unique fixed point $q \in H$ such that $q = P_\theta(\gamma f + (I - A))(q)$. Hence, we obtain that $\langle(\gamma f - A)q, w - q\rangle \leq 0$ for all $w \in \theta$.

Next, we show that $\limsup_{n\longrightarrow\infty}\langle(\gamma f - A)q, W_n y_n - q\rangle \leq 0$, where $q = P_\theta(\gamma f + I - A)(q)$ is the unique solution of the variational inequality $\big\langle(\gamma f - A)q, w - q\big\rangle \geq 0,\ \forall w \in \theta$. We can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle(\gamma f - A)q, W_n y_n - q\rangle = \lim_{i\longrightarrow\infty}\langle(\gamma f - A)q, W_n y_{n_i} - q\rangle. \quad (6.4.43)$$

As $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$.
Next we claim that $w \in \theta$. Since $\|y_n - W_n y_n\| \longrightarrow 0, \|x_n - W_n x_n\| \longrightarrow 0$ and $\|x_n - y_n\| \longrightarrow 0$ and by Lemma 6.4.2, we have $w \in \bigcap_{n=1}^{\infty} F(T_n)$.

Next, we show that $w \in \bigcap_{k=1}^{\infty} SMEP(F_k)$. Since $u_n = \tau_{r_k,n}^{N}\, x_n$, for $k = 1, 2, 3, \ldots, N$, we know that

$$F_k(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \quad (6.4.44)$$

It follows by (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_k(y, u_n), \quad \forall y \in C. \quad (6.4.45)$$

Hence, for $k = 1, 2, 3, \ldots, N$, we get

$$\varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq F_k(y, u_{n_i}), \quad \forall y \in C. \quad (6.4.46)$$

For $t \in (0, 1]$ and $y \in H$, let $y_t = ty + (1 - t)w$. From (6.4.46), we have

$$0 \;\geq\; \varphi(y_t) + \varphi(u_{n_i}) - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_k(y_t, u_{n_i}) \quad (6.4.47)$$

Since $\|u_{n_i} - x_{n_i}\| \longrightarrow 0$, from (A4) and the weakly lower semicontinuity of φ, $\frac{(u_{n_i}-x_{n_i})}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$. From (A1), (A4) and we have

$$\begin{aligned} 0 &= F_k(y_t, y_t) - \varphi(y_t) + \varphi(y_t) \\ &\leq tF_k(y_t, y) + (1-t)F_k(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t[F_k(y_t, y) + \varphi(y) - \varphi(y_t)]. \end{aligned} \tag{6.4.48}$$

Deviding by t, we get

$$F_k(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

The weakly lower semicontinuity of φ for $k = 1, 2, 3, \ldots, N$, we get

$$F_k(w, y) + \varphi(y) \geq \varphi(w).$$

So, we have

$$F_k(w, y) + \varphi(y) - \varphi(w) \geq 0, \ \forall k = 1, 2, 3, \ldots, N.$$

This implies that $w \in \bigcap_{k=1}^{N} SMEP(F_k)$.

Lastly, we show that $w \in I(B, M)$. In fact, since B is β-inverse strongly monotone, hence B is a monotone and Lipschitz continuous mapping. It follows that $M + B$ is a maximal monotone. Let $(v, g) \in G(M + B)$, since $g - Bv \in M(v)$. Again since $y_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda Bu_{n_i})$, we have $u_{n_i} - \lambda Bu_{n_i} \in (I + \lambda M)(y_{n_i})$, that is, $\frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i}) \in M(y_{n_i})$. By virtue of the maximal monotonicity of $M + B$, we have

$$\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i})\rangle \geq 0,$$

and hence

$$\begin{aligned} \langle v - y_{n_i}, g\rangle &\geq \Big\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i})\Big\rangle \\ &= \langle v - y_{n_i}, Bv - By_{n_i}\rangle + \langle v - y_{n_i}, By_{n_i} - Bu_{n_i}\rangle \\ &\quad + \Big\langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i})\Big\rangle. \end{aligned} \tag{6.4.49}$$

It follows from $\lim_{n\to\infty} \|u_n - y_n\| = 0$, we have $\lim_{n\to\infty} \|Bu_n - By_n\| = 0$ and $y_{n_i} \rightharpoonup w$, it follows that

$$\limsup_{n\to\infty} \langle v - y_n, g\rangle = \langle v - w, g\rangle \geq 0. \tag{6.4.50}$$

It follows from the maximal monotonicity of $B + M$ that $\theta \in (M + B)(w)$, that is, $w \in I(B, M)$. Therefore, $w \in \theta$. We observe that

$$\limsup_{n\longrightarrow\infty} \langle(\gamma f - A)q, W_n y_n - q\rangle = \lim_{i\longrightarrow\infty} \langle(\gamma f - A)q, W_n y_{n_i} - q\rangle = \langle(\gamma f - A)q, w - q\rangle \leq 0.$$

Step 6. Finally, we prove $x_n \longrightarrow q$. By using (6.4.10) and together with Schwarz inequality, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P_C\big(\epsilon_n\gamma f(x_n)+(I-\epsilon_nA)W_ny_n\big)-P_C(q)\|^2\\
&\leq \|\epsilon_n(\gamma f(x_n)-Aq)+(I-\epsilon_nA)(W_ny_n-q)\|^2\\
&\leq (I-\epsilon_nA)^2\|(W_ny_n-q)\|^2+\epsilon_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\epsilon_n\langle(I-\epsilon_nA)(W_ny_n-q),\gamma f(x_n)-Aq\rangle\\
&\leq (1-\epsilon_n\bar{\gamma})^2\|y_n-q\|^2+\epsilon_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\epsilon_n\langle W_ny_n-q,\gamma f(x_n)-Aq\rangle-2\epsilon_n^2\langle A(W_ny_n-q),\gamma f(x_n)-Aq\rangle\\
&\leq (1-\epsilon_n\bar{\gamma})^2\|x_n-q\|^2+\epsilon_n^2\|\gamma f(x_n)-Aq\|^2+2\epsilon_n\langle W_ny_n-q,\gamma f(x_n)-\gamma f(q)\rangle\\
&\quad +2\epsilon_n\langle W_ny_n-q,\gamma f(q)-Aq\rangle-2\epsilon_n^2\langle A(W_ny_n-q),\gamma f(x_n)-Aq\rangle\\
&\leq (1-\epsilon_n\bar{\gamma})^2\|x_n-q\|^2+\epsilon_n^2\|\gamma f(x_n)-Aq\|^2+2\epsilon_n\|W_ny_n-q\|\|\gamma f(x_n)-\gamma f(q)\|\\
&\quad +2\epsilon_n\langle W_ny_n-q,\gamma f(q)-Aq\rangle-2\epsilon_n^2\langle A(W_ny_n-q),\gamma f(x_n)-Aq\rangle\\
&\leq (1-\epsilon_n\bar{\gamma})^2\|x_n-q\|^2+\epsilon_n^2\|\gamma f(x_n)-Aq\|^2+2\gamma\epsilon\epsilon_n\|y_n-q\|\|x_n-q\|\\
&\quad +2\epsilon_n\langle W_ny_n-q,\gamma f(q)-Aq\rangle-2\epsilon_n^2\langle A(W_ny_n-q),\gamma f(x_n)-Aq\rangle\\
&\leq (1-\epsilon_n\bar{\gamma})^2\|x_n-q\|^2+\epsilon_n^2\|\gamma f(x_n)-Aq\|^2+2\gamma\epsilon\epsilon_n\|x_n-q\|^2\\
&\quad +2\epsilon_n\langle W_ny_n-q,\gamma f(q)-Aq\rangle-2\epsilon_n^2\langle A(W_ny_n-q),\gamma f(x_n)-Aq\rangle\\
&\leq \big((1-\epsilon_n\bar{\gamma})^2+2\gamma\epsilon\epsilon_n\big)\|x_n-q\|^2+\epsilon_n\Big\{\epsilon_n\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\langle W_ny_n-q,\gamma f(q)-Aq\rangle-2\epsilon_n\|A(W_ny_n-q)\|\|\gamma f(x_n)-Aq\|\Big\}\\
&= \big(1-2(\bar{\gamma}-\gamma\epsilon)\epsilon_n\big)\|x_n-q\|^2+\epsilon_n\Big\{\epsilon_n\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\langle W_ny_n-q,\gamma f(q)-Aq\rangle-2\epsilon_n\|A(W_ny_n-q)\|\ |\gamma f(x_n)-Aq\|\\
&\quad +\epsilon_n\bar{\gamma}^2\|x_n-q\|^2\Big\}. \qquad (6.4.51)
\end{aligned}$$

Since $\{x_n\}$ is bounded, where $\eta \geq \|\gamma f(x_n)-Aq\|^2-2\|A(W_ny_n-q)\|\|\gamma f(x_n)-Aq\|+\bar{\gamma}^2\|x_n-q\|^2$ for all $n \geq 0$. It follows that

$$\|x_{n+1}-q\|^2 \leq \big(1-2(\bar{\gamma}-\gamma\epsilon)\epsilon_n\big)\|x_n-q\|^2+\epsilon_n\delta_n, \qquad (6.4.52)$$

where $\delta_n = 2\langle W_ny_n-q,\gamma f(q)-Aq\rangle+\eta\alpha_n$. Since $\limsup_{n\longrightarrow\infty}\langle(\gamma f-A)q, W_ny_n-q\rangle \leq 0$, we get $\limsup_{n\longrightarrow\infty}\delta_n \leq 0$. Applying Lemma 6.4.1, we can conclude that $x_n \longrightarrow q$. This completes the proof. □

Corollary 6.4.6. *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H. Let B be β-inverse-strongly monotone and $\varphi : C \to \mathcal{R}$ is convex and lower semicontinuous function. Let $f : C \longrightarrow C$ be a contraction with coefficient α*

$(0 < \alpha < 1)$, $M : H \longrightarrow 2^H$ *be a maximal monotone mapping and* $\{T_n\}$ *be a family of nonexpansive mappings of* H *into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq 0.$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm for* $x_0,\ u_n \in C$ *arbitrarily:*

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \;\; \forall n \in N \\ x_{n+1} = P_C[\epsilon_n f(x_n) + (I - \epsilon_n) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \tag{6.4.53}$$

for all $n = 0, 1, 2, \ldots$, *and the conditions (C1)-(C3) in Theorem 6.4.5 are satisfied.*

Then, the sequence $\{x_n\}$ *converges strongly to* $q \in \theta$, *where* $q = P_\theta(f + I)(q)$ *which solves the following variational inequality:*

$$\langle (f - I)q, p - q \rangle \leq 0, \;\; \forall p \in \theta.$$

Proof. Putting $A \equiv I$ and $\gamma \equiv 1$ in Theorem 6.4.5, we can obtain desired conclusion immediately. □

Corollary 6.4.7. *Let* H *be a real Hilbert space,* C *be a nonempty closed and convex subset of* H. *Let* B *be* β*-inverse-strongly monotone,* $\varphi : C \to \mathcal{R}$ *is convex and lower semicontinuous function and* $M : H \longrightarrow 2^H$ *be a maximal monotone mapping. Let* $\{T_n\}$ *be a family of nonexpansive mappings of* H *into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq 0.$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm for* $x_0, u \in C$ *and* $u_n \in C$*:*

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \;\; \forall n \in N \\ x_{n+1} = P_C[\epsilon_n u + (I - \epsilon_n) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \tag{6.4.54}$$

for all $n = 0, 1, 2, \ldots$, *and the conditions (C1)-(C3) in Theorem 6.4.5 are satisfied.*

Then, the sequence $\{x_n\}$ *converges strongly to* $q \in \theta$, *where* $q = P_\theta(q)$ *which solves the following variational inequality:*

$$\langle u - q, p - q \rangle \leq 0, \;\; \forall p \in \theta.$$

Proof. Putting $f(x) \equiv u$, $\forall x \in C$ in Corollary 6.4.6, we can obtain desired conclusion immediately. □

Corollary 6.4.8. *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H and B be β-inverse-strongly monotone mapping, A a strongly positive linear bounded operator of H into itself with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $f : C \longrightarrow C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and $\{T_n\}$ be a family of nonexpansive mappings of H into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0 \in C$ arbitrarily:

$$x_{n+1} = P_C\Big[\epsilon_n \gamma f(x_n) + (I - \epsilon_n A)W_n P_C(x_n - \lambda B x_n)\Big] \qquad (6.4.55)$$

for all $n = 0, 1, 2, ...$, and the conditions (C1)-(C3) in Theorem 6.4.5 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle (\gamma f - A)q, p - q\rangle \leq 0, \ \ \forall p \in \theta.$$

Proof. Taking $F \equiv 0$, $\varphi \equiv 0$, $u_n = x_n$ and $J_{M,\lambda} = P_C$ in Theorem 6.4.5, we can obtain desired conclusion immediately. □

Remark 6.4.9. Corollary 6.4.8 generalizes and improves the result of Klin-eam and Suantai [260].

6.5 Some Applications

In this section, we apply the iterative scheme for finding a common fixed point of nonexpansive mapping and strictly pseudocontractive mapping.

Definition6.5.1. A mapping $S : C \longrightarrow C$ is called *strictly pseudo-contraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

If $\kappa = 0$, then S is nonexpansive. In this case, we say that $S : C \longrightarrow C$ is a κ-strictly pseudo-contraction. Putting $B = I - S$. Then, we have

$$\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + \kappa\|Bx - By\|^2, \quad \forall x, y \in C.$$

Observe that

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By\rangle, \quad \forall x, y \in C.$$

Hence, we obtain

$$\langle x-y, Bx-By\rangle \geq \frac{1-\kappa}{2}\|Bx-By\|^2, \quad \forall x,y\in C.$$

Then, B is a $\frac{1-\kappa}{2}$-inverse-strongly monotone mapping.

Using Theorem 6.4.5, we first prove a strongly convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudo-contraction.

Theorem 6.5.2. *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H and B be an β-inverse-strongly monotone, $\varphi : C \to \mathcal{R}$ is convex and lower semicontinuous function, $f : C \longrightarrow C$ be a contraction with coefficient α $(0<\alpha<1)$ and A be a strongly positive linear bounded operator of H into itself with coefficient $\bar{\gamma}>0$. Assume that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself and let S be a κ-strictly pseudo-contraction of C into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n)\cap(\cap_{k=1}^{N} SMEP(F_k))\cap F(S)\neq 0.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0,u_n\in C$ arbitrarily:

$$\begin{cases} u_n = K^{F_N}_{r_n,n}\cdot K^{F_{N-1}}_{r_{n-1},n}\cdot K^{F_{N-2}}_{r_{n-2},n}\cdot \ldots \cdot K^{F_2}_{r_2,n}\cdot K^{F_1}_{r_1,n}\cdot x_n, \;\; \forall n\in N \\ x_{n+1} = P_C[\epsilon_n\gamma f(x_n)+(I-\epsilon_n A)W_n(1-\lambda)x_n+\lambda Sx_n] \end{cases} \tag{6.5.1}$$

for all $n=0,1,2,\ldots$, and the conditions (C1)-(C3) in Theorem 6.4.5 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q\in\theta$, where $q=P_\theta(\gamma f+I-A)(q)$ which solves the following variational inequality:

$$\langle(\gamma f-A)q, p-q\rangle\leq 0, \;\; \forall p\in\theta$$

which is the optimality condition for the minimization problem

$$\min_{q\in\theta}\frac{1}{2}\langle Aq,q\rangle - h(q), \tag{6.5.2}$$

where h is a potential function for γf (i.e., $h'(q)=\gamma f(q)$ for $q\in H$).

Proof. Put $B\equiv I-T$, then B is $\frac{1-\kappa}{2}$ inverse-strongly monotone and $F(S) = I(B,M)$ and $J_{M,\lambda}(x_n-\lambda Bx_n)=(1-\lambda)x_n+\lambda Tx_n$. So by Theorem 6.4.5, we obtain the desired result. □

Corollary 6.5.3. *Let H be a real Hilbert space, C be a closed convex subset of H and B be β-inverse-strongly monotone, $\varphi : C\to\mathcal{R}$ is convex and lower semicontinuous* *function. Let $f : C \longrightarrow C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and T_n be a nonexpansive mapping of H into itself and let S be a κ-strictly pseudo-contraction of C into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap F(S) \neq 0.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0 \in C$ arbitrarily:

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \ \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n f(x_n) + (I - \epsilon_n) W_n((1-\lambda)u_n + \lambda S u_n)] \end{cases} \tag{6.5.3}$$

for all $n = 0, 1, 2, \ldots$, and the conditions (C1)-(C3) in Theorem 6.4.5 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(f + I)(q)$, which solves the following variational inequality:

$$\langle (f - I)q, p - q \rangle \leq 0, \ \ \forall p \in \theta$$

which is the optimality condition for the minimization problem

$$\min_{q \in \theta} \frac{1}{2} \langle Aq, q \rangle - h(q), \tag{6.5.4}$$

where h is a potential function for γf (i.e., $h'(q) = \gamma f(q)$ for $q \in H$).

Proof. Put $A \equiv I$ and $\gamma \equiv 1$ in Theorem 6.5.2, we obtain the desired result. □

6.6 Numerical example

Now, we give a real numerical example in which the condition satisfy the ones of theorem 6.4.5 and some numerical experiment results to explain the main result theorem 6.4.5 as follows:

Example 6.6.1. Let $H = R, C = [-1, 1], T_n = I, \ \lambda_n = \beta \in (0, 1), \ n \in N, \ F_k(x, y) = 0, \forall x, y \in C, r_{n,n} = 1, \ k \in \{1, 2, 3, \ldots, N\}, \ \varphi(x) = 0, \ \forall x \in C, \ B = A = I, \ f(x) = \frac{1}{5}x, \ \forall x \in H, \ \lambda = \frac{1}{2}$ with contraction coefficient $\alpha = \frac{1}{10}, \ \epsilon_n = \frac{1}{n}$ for every n $\in N$ and $\gamma = 1$. Then $\{x_n\}$ is the sequence generated by

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n \tag{6.6.1}$$

and $x_n \longrightarrow 0$ as $n \longrightarrow \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x \in C} = \frac{2}{5}x^2 + q.$$

Proof. We prove the Example 6.6.1 by step 1, step 2, step 3. By step 4, we give two numerical experiment results which can directly explain the sequence $\{x_n\}$ strongly converges to 0.

Step 1. We show

$$K_{r_n,n}^{F_N}x = P_C x,\ \forall x \in H, F_N \in \{1,2,3,...,N\}, \tag{6.6.2}$$

where

$$P_C x = \begin{cases} \frac{x}{|x|}, & x \in H\backslash C \\ x, & x \in C. \end{cases} \tag{6.6.3}$$

Indeed, since $F_k(x,y) = 0,\ \forall x, y \in C,\ n \in \{1,2,3,...,N\}$, due to the definition of K_r(x), $\forall x \in H$, as lemma **??**, we have

$$K_r(x) = \Big\{u \in C : \langle y-u, u-x\rangle \geq 0, \forall y \in C\Big\}.$$

Also by the equivalent property of the nearest projection P_C from $H \longrightarrow C$, we obtain this conclusion, when we take $x \in C$, $K_{r_n,n}^{F_N}x = P_C x = Ix$. By (iii) in lemma **??**, we have

$$\bigcap_{k=1}^{N} SMEP(F_k) = C. \tag{6.6.4}$$

Step 2. We show

$$W_n = I. \tag{6.6.5}$$

Indeed. By (6.4.8), we have

$$W_1 = U_{11} = \lambda_1 T_1 U_{12} + (1-\lambda_1)I = \lambda_1 T_1 + (1-\lambda_1)I, \tag{6.6.6}$$

$$\begin{aligned} W_2 = U_{21} &= \lambda_1 T_1 U_{22} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{23} + (1-\lambda_2)I) + (1-\lambda_1)I \\ &= \lambda_1\lambda_2 T_1 T_2 + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I, \\ W_3 = U_{31} &= \lambda_1 T_1 U_{32} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{33} + (1-\lambda_2)I) + (1-\lambda_1)I \\ &= \lambda_1\lambda_2 T_1 T_2 U_{33} + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I, \\ &= \lambda_1\lambda_2 T_1 T_2(\lambda_3 T_3 U_{34} + (1-\lambda_3 I)) + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I, \\ &= \lambda_1\lambda_2\lambda_3 T_1 T_2 T_3 + \lambda_1\lambda_2(1-\lambda_3 T_1 T_2 + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I. \end{aligned}$$

Compute in this way by (6.4.8), we obtain

$$\begin{aligned} W_n = U_{n1} &= \lambda_1\lambda_2\cdots\lambda_n T_1T_2\cdots T_n + \lambda_1\lambda_2\cdots\lambda_{n-1}(1-\lambda_n)T_1T_2\cdots T_{n-1} \\ &\quad +\lambda_1\lambda_2\cdots\lambda_{n-2}(1-\lambda_{n-1})T_1T_2\cdots T_{n-2} + \cdots + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I. \end{aligned}$$

Since $T_n = I,\ \lambda_n = \beta,\ n \in N$, thus

$$W_n = [\beta^n + \beta^{n-1}(1-\beta) + \cdots + \beta(1-\beta) + (1-\beta)]I = I.$$

Step 3. We show

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n \ and \ x_{n+1} \longrightarrow 0,\ as\ n \longrightarrow \infty, \qquad (6.6.7)$$

where 0 is the unique solution of the minimization problem

$$\min_{x\in C} = \frac{2}{5}x^2 + q.$$

Indeed, we can see A = I is a strongly position bounded linear operator with coefficient $\bar{\gamma} = \frac{1}{2},\ \gamma$ is a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, so we can take $\gamma = 1$. Due to (6.6.1), (6.6.3) and (6.6.5), we can obtain an special sequence $\{x_n\}$ of (6.4.10) in theorem 6.4.5 as follows:

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n$$

Since $T_n = I,\ n \in N$, so,

$$\cap_{n=1}^{\infty} F(T_n) = H,$$

combining with (6.6.4), we have

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) = C = [-1, 1].$$

By Lemma 6.4.1, it is obviously that $z_n \longrightarrow 0$, 0 is the unique solution of the minimization problem

$$\min_{x\in C} = \frac{2}{5}x^2 + q,$$

where q is a constant number.

Step 4. We give the numerical experiment results using software Mathlab 7.0 and get the figure 1 to figure 4, which show that the iteration process of the sequence $\{x_n\}$ is a monotone decreasing sequence and converges to 0, but the more the iteration steps are, the more showily the sequence $\{x_n\}$ converges to 0.

Now we turn to realizing (6.4.10) for approximating a fixed point of T. We take the initial valued $x_1 = 1$ and x_1 = 1/2, respectively. All the numerical results are given in Tables 1 and 2. The corresponding graph appears in Figure 1 (i) and (ii).

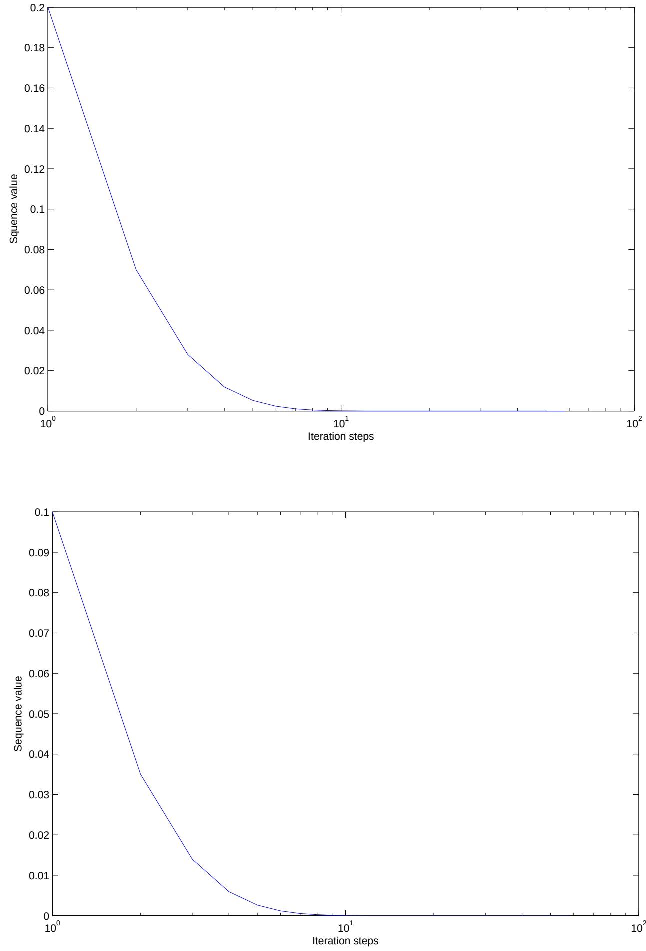
Table 1 This table shows the value of sequence $\{x_n\}$ on each iteration steps (initial value $x_1 = 1$)

n	x_n	n	x_n
1	1.000000000000000	31	0.000000000054337
2	0.200000000000000	32	0.000000000026643
3	0.070000000000000	33	0.000000000013072
4	0.028000000000000	34	0.000000000006417
$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	0.000000301580666	39	0.000000000000184
20	0.000000146028533	40	0.000000000000091
21	0.000000070823839	41	0.000000000000045
$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	0.000000000226469	47	0.000000000000001
30	0.000000000110892	48	0.000000000000000

Table 2 This table shows the value of sequence $\{x_n\}$ on each iteration steps (initial value $x_1 = \frac{1}{2}$)

n	x_n	n	x_n
1	0.500000000000000	31	0.000000000027168
2	0.100000000000000	32	0.000000000013321
3	0.035000000000000	33	0.000000000006536
4	0.014000000000000	34	0.000000000003208
$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	0.000000150790333	39	0.000000000000092
20	0.000000073014267	40	0.000000000000045
21	0.000000035411919	41	0.000000000000022
$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	0.000000000113235	46	0.000000000000001
30	0.000000000055446	47	0.000000000000000

The numerical results that support our main theorem as shown by calculating and plotting graphs using Matlab 7.11.0.



(i) (ii)

Figure 1. The iteration comparison chart of different initial values. (i) $x_1 = 1$ and (ii) $x_1 = \frac{1}{2}$.

บทที่ 7

Conclusion

ผลงานวิจัยตีพิมพ์ระดับนานาชาติ 23 บทความ ดังนี้ (MRG5380044)

(1) C. Jaiboon and **P. Kumam**, A general iterative method for addressing mixed equilibrium problems and optimization problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 73 (2010) pp. 1180-1202. **(2009 Impact Factor=1.487)**

(2) W. Chantarangsi, C. Jaiboon, and **P. Kumam**, A viscosity hybrid steepest descent method for generalized mixed equilibrium problems and variational inequalities for relaxed cocoercive mapping in Hilbert spaces, Abstract and Applied Analysis, Volume 2010 (2010), Article ID 390972, 39 pages **(2009 Impact Factor: 2.221)**

(3) P. Kumam and **C. Jaiboon**, A system of generalized mixed equilibrium problems and fixed point problems for pseudo-contraction mappings in Hilbert spaces, Fixed Point Theory and Applications, Volume 2010, Article ID 361512, 33 pages. **(20010 Impact Factor 1.936)**

(4) S. Saewan and **P. Kumam**, "Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-ϕ-asymptotically nonexpansive mappings " Abstract and Applied Analysis, Volume 2010, Article ID 357120, 22 pages **(2009 Impact Factor: 2.221)**

(5) S. Saewan and **P. Kumam,** "A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems," Abstract and Applied Analysis, Volume 2010, Article ID 123027, 31 pages **(2009 Impact Factor: 2.221)**

(6) T. Jitpeera, P. Katchang, and **P. Kumam**, A viscosity of Cesàro mean approximation methods for mixed equilibrium, variational inequalities and fixed point problems, Fixed Point Theory and Applications, Volume 2011, Article ID 945051, 24 pages doi:10.1155/2011/945051 **(2010 Impact Factor 1.936)**

(7) T. Jitpeera and **P. Kumam**, "A New Hybrid Algorithm for a System of Mixed Equilibrium Problems, Fixed Point Problems for Nonexpansive Semigroup, and Variational Inclusion Problem," Fixed Point Theory and Applications, vol. 2011, Article ID 217407, 27 pages, 2011. doi:10.1155/2011/217407. **(2010 Impact Factor 1.936)**

(8) T. Jitpeera and **P. Kumam**, The shrinking projection method for a system of generalized mixed equilibrium problems and fixed point problems for pseudocontractive mappings, Journal of Inequalities and Applications, Volume 2011, Article ID 840319, 25 pages. **(2010 Impact Factor 0.88)**

(9) P. Kumam and S. Plubtieng, 'Viscosity approximation methods for monotone mappings and a countable family of nonexpansive mappings', Mathematica Slovaca, Math. Slovaca, 61 (2) (2011), 257-274. **(2010 Impact Factor 0.316)**

(10) T. Jitpeera, U. Witthayarat and **P. Kumam** , "Hybrid algorithms of common solutions of generalized mixed equilibrium problems and the common variational inequality problems with applications," Fixed Point Theory and Applications, Volume 2011, Article ID 971479, 28 pages. **(2010 Impact Factor 1.936)**

(11) W. Kumam, P. Junlouchai and **P. Kumam**, Generalized Systems of Variational Inequalities and Projection Methods for Inverse-Strongly Monotone Mappings, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 976505, 24 pages doi:10.1155/2011/976505 **(2010 Impact Factor 0.967)**

(12) **P. Kumam** and S. Plubtieng, "Convergence Theorems by hybrid methods for Monotone Mappings and a Countable Family of Nonexpansive Mappings and its Applications, International Journal of Pure and Applied Mathematics, Volume 70 No. 1 2011, 81-107. (No Impact Factor)

(13) P. Katchang and **P. Kumam**, An iterative algorithm for finding a common solution of fixed points and a general system of variational inequalities for two inverse strongly accretive operators, Positivity, Volume 15, Number 2, (2011) 281-295. **(2010 Impact Factor 0.578)**

(14) S. Saewan and **P. Kumam**, "The shrinking projection method for solving generalized equilibrium problem and common fixed points for asymptotically quasi-

ϕ-nonexpansive mappings," Fixed Point Theory and Applications 2011, 2011:9 http://dx.doi.org/10.1186/1687-1812-2011-9 **(2010 Impact Factor 1.936)**

(15) S. Saewan and **P.Kumam**, "Strong convergence theorems for countable families of uniformly quasi-phi-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems," Abstract and Applied Analysis, Volume 2011, Article ID 701675, 27 pages. **(2010 Impact Factor 1.442)**

(16) S. Saewan and P. Kumam, "A modified hybrid projection method for solving generalized mixed equilibrium problems and fixed point problems in Banach spaces", Computers and Mathematics with Applications 62 (2011) 1723-1735. **(2010 Impact Factor 1.472)**

(17) T. Jitpeera and **P. Kumam**, " Hybrid algorithms for minimization problems over the solutions of generalized mixed equilibrium and variational inclusion problems," Mathematical Problems in Engineering, Volume 2011, Article ID 648617, 26 pages **(2010 Impact Factor 0.689)**

(18) S. Saewan and **P. Kumam**, A new modified block iterative algorithm for uniformly quasi-ϕ-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems, Fixed Point Theory and Applications 2011, 2011:35 doi:10.1186/1687-1812-2011-35 **(2010 Impact Factor 1.936)**

(19) S. Saewan and **P. Kumam**, Convergence theorems for mixed equilibrium problems, variational inequality problem and uniformly quasi-ϕ-asymptotically nonexpansive mappings, Applied Mathematics and Computation, 218 (2011) 3522-3538. **(2010 Impact Factor 1.534)**

(20) T. Chamnarnpan and **P. Kumam**, "Iterative Algorithms for Solving the System of Mixed Equilibrium Problems, Fixed-Point Problems, and Variational Inclusions with Application to Minimization Problem" Volume 2012, Article ID 538912, 29 pages doi:10.1155/2012/538912 **(2010 Impact Factor 0.630)**

(21) P. Katchang and **P. Kumam**, "Hybrid-extragradient type methods for a generalized equilibrium problem and variational inequality problems of nonexpansive semigroups" Fixed Point Theory, 13 (2012), 107-120. **(2010 Impact Factor 1.03)**

(22) S. Saewan and **P. Kumam**, Existence and Algorithm for solving the system of mixed variational inequalities in Banach spaces, Journal of Applied Mathematics, Volume 2012, Article ID 413468, 16 pages **(2010 Impact Factor 0.630)**

(23) S. Saewan and **P. Kumam**, "A strong convergence theorem concerning a hybrid projection method for finding common fixed points of a countable family of relatively quasi-nonexpansive mappings" Journal of Nonlinear and Convex Analysis, Volume 13, Number 2, 2012, **(2010 Impact Factor 0.738)**

บรรณานุกรม

[1] Y. I. Alber, *Metric and generalized projection operators in Banach space: properties and applications,* in: A. G. Katrosatos (ED), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15–50.

[2] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Student, **63** (1994) 123–145.

[3] F.E. Browder, *Fixed point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. **43** (1965) 1272 - 1276.

[4] F.E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Natl. Acad. Sci. USA **54** (1965) 1041-1044

[5] F.E. Browder, *Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces* Arch. Rational Mech. Anal. **24** (1967) 82-90.

[6] F.E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces,* Bull. Amer. Math. Soc. **74** (1968) 660–665.

[7] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Student, **63** (1994) 123–145.

[8] D. Butnariu, S. Reich, A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces,* J. Appl. Anal., **7** (2001) 151–174.

[9] L. C. Ceng, C. Lee, J. C. Yao, *Strong weak convergence theorems of implicit hybrid steepest-descent methods for variational inequalities,* Taiwanese J. Math., **12** (2008) 227–244.

[10] L. C. Ceng, Q. H. Ansari, J. C. Yao, *Viscosity approximation methods for generalized equilibrium problems and fixed point problems,* J. Global Optim., **43** (2009) 487–502.

[11] J. Chen, L. Zhang and T. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mapping,* J. Math. Appl. **334** (2007) 1450—1461.

[12] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces,* J. Nonlinear Convex Anal., **6** (2005) 117—136.

[13] A. Goebel, W.A. Kirk, " Topics in Metric Fixed Point Theory" Cambridge Stud. Adv. Math., vol. 28, Cambridge Univ. Prees, Cambridge, 1990.

[14] K. Goebel, M.A. Pineda, *On a type os generalized nonexpansiveness,* Proc. of the 8th International Conference on Fixed Point Theory and its Application **74** (2007) 660—665.

[15] D. Gohde, *Zum prinzip der kontraktiven abbildung,* Math. Nach. **30** (1965) 251—258.

[16] B. Halpern *Fixed points of nonexpanding maps,* Bull. Amer. Math. Soc., **73** (1967) 957—961.

[17] T. Ibaraki, W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces,* J. Approx. Theory, **149** (2007) 1—14.

[18] T. Ibaraki, W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces,* to appear

[19] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings,* Nonlinear Anal. **61** (2005) 341—350.

[20] G. Inoue, W. Takahashi, K. Zembayashi, *Strong convergence theorems by hybrid methods for maximal monotone operator and relatively nonexpansive mappings in Banach spaces,* J. Convex Anal., to appear.

[21] S. Ishikawa, *Fixed points by a new iteration method,* Proc. Amer. Math. Soc., **44** (1974) 147—150.

[22] S. Kamimura, W. Takahashi, *Strong convergence of proximal-type algorithm in a Banach space,* SIAM J. Optim., **13** (2002) 938—945.

[23] S. Kamimura, F. Kohsaka, W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space,* Set-valued Anal., **12** (2004) 417—429.

[24] E. Kreyszig, “Introductory functional analysis with applications”, John Willey and Sons, New York, (1978).

[25] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances,* Amer. Math. Monthly **72** (1965) 1004—1006.

[26] C. Klin-eam, S. Suantai, *Strong Convergence of Monotone Hybrid Method for Maximal Monotone Operators and Hemirelatively Nonexpansive Mappings*, Fixed Point Theory and Applications, Vol. 2009, Article ID 261932, (2009) 14 pages.

[27] C. Klin-eam, S. Suantai, *A New Approximation Method for Solving Variational Inequalities and Fixed Points of Monexpansive Mappings*, Journal of Inequalities and Applications, Vol.2009, Article ID 520301,(2009) 16 pages.

[28] C. Klin-eam, S. Suantai, W. Takahashi, *Strong Convergence of Generalized Projection Algorithms for Nonlinear Operators*, Abstract and Applied Analysis, Vol. 2009, Article ID 649831,(2009) 18 pages.

[29] F. Kohsaka, W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space,* Abstr. Appl. Anal., **2004** (2004) 239—249.

[30] F. Kohsaka, W. Takahashi, *Block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces,* Fixed Point Theory Appl., vol. 2007, Article ID 21972, (2007) 18 pages.

[31] F. Kohsaka, W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces,* J. Nonlinear Convex Anal., **8** (2007) 197—209.

[32] F. Kohsaka, W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach space,* SIAM J. Optim., **19** (2008) 824—835.

[33] W. R. Mann, *Mean Vauled methods in iteration,* Proc. Amer. Math. Soc., **4** (1953) 506—510.

[34] G. Marino, H.K. Xu, *A general iterative method for nonexpansive mapping in Hilbert space,*J. Math. Anal. Appl., **318** (2006)43—52.

[35] C. Martinez-Yanes, H. K. Xu, *Strong convergence of the CQ method for fixed point iteration processes,* Nonlinear Anal., **64** (2006) 2400—2411.

[36] S. Matsushita, W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach space,* Fixed Point Theory Appl., Vol.2004 (2004) 37—47.

[37] S. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space,* J. Approx. Theory, **134** (2005) 257—266.

[38] A. Moudafi, *Viscosity approximation methods for fixed points problems,* J. Math. Anal. Appl. **241** (2000)46—55.

[39] K. Nakajo, W, Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups,* J. Math. Anal. Appl. **279** (2003) 372—379.

[40] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc. **73** (1967)591—597.

[41] X. Qin, Y. Su, *Strong convergence theorems for relatively nonexpansive mappings in a Banach space,* Nonlinear Anal., **67** (2007) 1958—1965.

[42] S. Reich, *Weak convergence theorems for nonexpansive mapping in a Banach space,* J. Math. Anal. Appl. **67** (1979) 247—276.

[43] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators,* Trans. Amer. Math. Soc., **194** (1970) 75—88.

[44] S. Schaible, J. C. Yao, L. C. Zeng *A proximal method for pseudomonotone type variational-like inequalities,* Taiwanese J. Math., **10** (2006) 497—513.

[45] M. V. Solodov, B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space,* Math. Program., **87** (2000) 189—202.

[46] Y. Su, D. Wang, M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings,* Fixed Point Theory Applications, vol. 2008, Article ID 284613,(2008) 8 pages

[47] W. Takahashi, "Nonlinear Functional Analysis - Fixed Point Theory and its Applications", Yokohama Publishers inc, Yokohama, 2000 (in English).

[48] W. Takahashi, " Introduction to Nonlinear and Convex Analysis", Yokohama Publishers inc, Yokohama, 2009 (Japanese).

[49] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl., **341** (2008) 276—286.

[50] W. Takahashi, K. Zembayashi, *A strong convergence theorem for the equilibrium problem with a bifunction defined on the dual space of a Banach space,* Proceeding of the 8th International Conference on Fixed Point Theory and Its Applications, (2007) 197—209.

[51] R. Wittmann, *Approximation of fixed points of nonexpansive mappings,* Arch. Math. (Basel) **58** (1992) 491—498.

[52] H.K. Xu, *Iterative algorithm for nonlinear operators,* J. London. Math. Soc.**2** (2002) 240—256.

[53] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings,* J. Math. Anal. Appl. **298** (2004) 279-291.

[54] C. Zalinescu, *On uniformly convex functions,* J. Math. Anal. Appl. **95** (1983) 344—374.

[55] Ya. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer. Math. J. **4** (1994), 39—54.

[56] Ya. I. Alber, *Generalized projection operators in Banach spaces: Properties and applications*, in: Proceedings of the Israel Seminar, Ariel, Israel, Funct. Differential Equation 1 (1994) 1—21.

[57] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, pp. 15—50.

[58] H. H. Bauschke and P. L. Combettes, *A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces*, Math. Oper. Res. **26** (2001), 248—264.

[59] H. H. Bauschke, E. Matoušková and S. Reich, *Projection and proximal point methods: convergence results and counterexamples*, Nonlinear Anal. **56** (2004), 715—738.

[60] D. Boonchari and S. Saejung, *Approximation of common fixed points of a countable family of relatively nonexpansive mappings*, Fixed Point Theory and Appl. **2010** (2010), Article ID 407651, 26 pp.

[61] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350—2360.

[62] D. Butnariu, S. Reich and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. **7** (2001), 151—174.

[63] D. Butnariu, S. Reich and A.J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **24** (2003), 489—508.

[64] Y. Censor and S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization **37** (1996), 323—339.

[65] I. Cioranescu, *Geometry of Banach spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[66] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[67] J. H. Fan, X. Liu and J. L. Li, *Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces*, Nonlinear Anal. **70** (2009), 3997—4007.

[68] A. Genel and J. Lindenstrauss, *An example concerning fixed points*, Israel J. of Math. **22** (1975), 81—86.

[69] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938—945.

[70] P. Kumam and S. Plubtieng, *Viscosity approximation methods for monotone mappings and a countable family of nonexpansive mappings*, Math. Slovaca, **61** (2011), 257—274.

[71] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

[72] J. L. Li, *The generalized projection operator on reflexive Banach spaces and its applications*, J. Math. Anal. Appl. **306** (2005), 55—71.

[73] X. Li, N. Huang and D. O'Regan, *Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications*, Comput. Math. Appl. 60 (2010), 1322—1331.

[74] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506—510,

[75] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, **134** (2005), 257—266.

[76] K. Nakajo, K. Shimoji and W. Takahashi, *Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces*, Taiwanese J. Math. **10** (2006), 339—360.

[77] W. Nilsrakoo and S. Saejung, *Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings*, Fixed Point Theory and Appl. **2008** (2008), Article ID 312454, 19 pp.

[78] S. Plubtieng and K. Ungchittrakool, *Approximation of common fixed points for a countable family of relatively nonexpansive mappings in a Banach space and applications*, Nonlinear Anal. **72** (2010), 2896—2908

[79] X. Qin, Y. J. Cho and S. M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces*, J. Comput. Appl. Math. **225** (2009), 20—30.

[80] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75—88.

[81] S. Reich, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Bull. Amer. Math. Soc. **26** (1992), 367—370.

[82] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, pp 313—318 (1996).

[83] S. Plubtieng and P. Kumam, *Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings*, J. Comput. Appl. Math. **224** (2009), 614—621.

[84] S. Saewan, P. Kumam and K. Wattanawitoon, *Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces*, Abstr. Appl. Anal. **2010** (2010), Article ID 734126, 26 pp.

[85] S. Saewan and P. Kumam *A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings*

for generalized mixed equilibrium and variational inequality problems, Abstr. Appl. Anal. **2010** (2010), Article ID 123027, 31 pp.

[86] Y. Su, D. Wang and M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings*, Fixed Point Theory and Appl. **2008** (2008), Article ID 284613, 8 pp.

[87] Y. Su, Z. Wang and H. K. Xu, *Strong convergence theorems for a common fixed point of two hemi-relatively nonexpansive mappings*, Nonlinear Anal. **71** (2009), 5616—5628.

[88] Y. Su, M. Li and H. Zhang, *New monotone hybrid algorithm for hemi-relatively nonexpansive mappings and maximal monotone operators*, Appl. Math. Comput. **217** (2011), 5458—5465

[89] Y. Su, H.-K. Xu, and X. Zhang, *Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications*, Nonlinear Anal. **73** (2010), 3890—3906.

[90] W. Takahashi, *Convex Analysis and Approximation Fixed Points*, Yokohama-Publishers, 2009.

[91] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276—286.

[92] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, 2000.

[93] K. Wattanawitoon and P. Kumam, *Strong convergence to common fixed points for countable families of asymptotically nonexpansive mappings and semigroups*, Fixed Point Theory and Appl. **2010** (2010), Article ID 301868, 16 pp.

[94] K. Q. Wu and N. J. Huang, *The generalized f-projection operator with an application*, Bull. Aust. Math. Soc. **73** (2006), 307—317.

[95] Z. Wang, Y. Su, D. Wang and Y. Dong, *A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexpansive mappings and systems of equilibrium problems in Banach spaces*, J. Comput. Appl. Math. **235** (2011), 2364—2371.

[96] H. Zegeye and N. Shahzad, *Strong convergence for monotone mappings and relatively weak nonexpansive mappings*, Nonlinear Anal. **70** (2009), 2707—2716.

[97] F. Q. Xia and N. J. Huang, *Variational inclusions with a general H monotone operator in Banach spaces*, Comput. Math. Appl. **54** (2007), 24—30.

[98]] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal., 67(2007) 2350-2360.

[99]] H.H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1996) 150-159.

[100]] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (1996) 367-426.

[101]] M. Aslam Noor, Generalized mixed quasi-equilibrium problems with trifunction, Appl. Math. Lett. 18 (2005) 695—700.

[102]] M. Aslam Noor, On a class of nonconvex equilibrium problems, Appl. Math. Comput. 157 (2004) 653—666.

[103]] M. Aslam Noor, Multivalued general equilibrium problems, J. Math. Anal. Appl. 283 (2003) 140—149.

[104]] M. Aslam Noor, Themistocles M. Rassias, On nonconvex equilibrium problems, J. Math. Anal. Appl. 312 (2005) 289—299.

[105]] M. Aslam Noor, Themistocles M. Rassias, On general hemiequilibrium problems, J. Math. Anal. Appl. 324 (2006) 1417—1428.

[106]] M. Aslam Noor, W. Oettli, On general nonlinear complementarity problems and quasi equilibria, Matematiche (Catania) 49 (1994) 313-331.

[107]] G. Bigi, M. Castellani, G. Kassay, A dual view of equilibrium problems. J. Math. Anal. Appl. 342, 17—26 (2008)

[108]] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123-145.

[109]] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proceedings of the National Academy of Sciences of the United States of America, 53 (1965), 1272-1276.

[110]] A. Cabot, Proximal point algorithm controlled by a slowly vanishing term: applications to hierarchical minimization. SIAM J. Optim. 15(2), 555—572 (2005)

[111]] L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214 (2008) 186-201.

[112]] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117—136.

[113]] P.L. Combettes, The foundations of set theoretic estimation, Proc. IEEE 81 (1993) 182-208.

[114]] A. Chinchuluun, P. Pardalos, A. Migdalas, L. Pitsoulis, Pareto Optimality, Game Theory and Equilibria, Edward Elgar Publishing, (2008).

[115]] X.P. Ding, Auxiliary principle and algorithm for mixed equilibrium problems and bilevel mixed equilibrium problems in Banach spaces. J. Optim. Theory Appl., 146(2), 347—357 (2010)

[116]] X.P. Ding, Existence and Algorithm of Solutions for Mixed Equilibrium Problems and Bilevel Mixed Equilibrium Problems in Banach Spaces, Acta Mathematica Sinica, English Series, DOI: 10.1007/s10114-011-9730-6.

[117]] X.P. Ding, Existence and iterative algorithm of solutions for a class of bilevel generalized mixed equilibrium problems in Banach spaces, J Glob Optim DOI 10.1007/s10898-011-9724-z.

[118]] X.P. Ding, Iterative algorithm of solutions for generalized mixed implicit equilibrium-like problems. Appl. Math. Comput. 162(2), 799—809 (2005)

[119]] X. P. Ding, Y. C. Liou, J. C. Yao, Existence and algorithms for bilevel generalized mixed equilibrium problems in Banach spaces, J Glob Optim DOI 10.1007/s10898-011-9712-3.

[120]] X.P. Ding, T.C. Lai, S.J. Yu, Systems of generalized vector quasi-variational inclusion problems and application to mathematical programs. Taiwanese J. Math. 13(5), 1515—1536 (2009)

[121]] X.P. Ding, Y.C. Lin, J.C. Yao, Predictor-corrector algorithms for solving generalized mixed implicit quasi-equilibrium problems. Appl. Math. Mech. 27(9), 1157—1164 (2006)

[122]] N.J. Huang, H.Y. Lan, Y.J. Cho, Sensitivity analysis for nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings. J. Comput. Appl. Math. 196, 608—618 (2006)

[123]] K.R. Kazmi, F.A. Khan, Existence and iterative approximation of solutions of generalized mixed equilibrium problems. Comput. Math. Appl. 56, 1314—1321 (2008)

[124]] I.V.Konnov, Application of the proximal method to nonmonotone equilibrium problems. J. Optim. Theory Appl. 119, 317—333 (2003)

[125]] G. M. Korpelevich, The extragradient method for finding saddle points and other problems. Matecon 12 (1976), 747-756.

[126]] H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansine mappings and inverse-strongly monotone mappigs, Nonlinear Anal. 61 (2005), 341-350.

[127]] J.L. Lions, G. Stampacchia, Variational inequalities. Comm. Pure Apl. Math. 20 (1967), 493-512.

[128]] A.N. Iusem, A.R. De Pierro, On the convergence of Han's method for convex programming with quadratic objective, Math. Program. Ser. B 52 (1991) 265—284.

[129]] Z.-Q. Luo, J.-S. Pang, D. Ralph, Mathematical Programs With Equilibrium Constraints. Cambridge University Press, Cambridge (1996)

[130]] Z.-Q. Luo, J.-S. Pang, D. Ralph, Mathematical Programs With Equilibrium Constraints. Cambridge University Press, Cambridge (1996).

[131]] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. (2006)

[132]] G. Mastroeni, On auxiliary principle for equilibrium problems. Publicatione del Departimento di Mathematica Dell'Universita di Pisa 3, 1244—1258 (2000)

[133] A. Moudafi, Mixed equilibrium problems: sensitivity analysis and algorithmic aspects. Comput. Math. Appl. 44, 1099—1108 (2002)

[134]] A. Moudafi, Proximal point algorithm extended for equilibrium problems. J. Nat. Geom. 15, 91—100 (1999)

[135]] A. Moudafi, Proximal methods for a class of bilevel monotone equilibrium problems. J. Glob. Optim. (2010), Volume 47 Issue 2, 287-292, DOI: 10.1007/s10898-009-9476-1.

[136]] A. Moudafi, Proximal methods for a class of bilevel monotone equilibrium problems. J. Glob. Optim. Math., 47(2), 287—29 (2010)

[137]] N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive and monotone mappings. J. Optim. Theory Appl. 128 (2006), 191-201.

[138]] P.M. Pardalos, T.M. Rassias, A.A. Khan, Nonlinear Analysis and Variational Problems, Springer, (2010).

[139]] J.W. Peng, Iterative algorithms for mixed equilibrium problems, strict pseudocontractions and monotone mappings. J. Optim. Theory Appl. (2009). doi:10.1007/s10957-009-9585-5

[140]] J.W. Peng, Iterative algorithms for mixed equilibrium problems, strict pseudocontractions and monotone mappings. J. Optim. Theory Appl., 144(1), 107—119 (2010)

[141]] J.W. Peng, J.C. Yao, Some new iterative algorithms for generalized mixed equilibrium problems with strict pseudo-contractions and monotone mappings. Taiwan. J. Math. 13, 1537—1582 (2009)

[142]] J.W. Peng, J.C. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings. Nonlinear Anal., Ser. A, Theory Methods Appl. 71, 6001—6010 (2009)

[143]] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 336 (2007) 455—469.

[144]] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, Journal of Mathematical Analysis and Applications, 75 (1980), 287-292.

[145]] M. Shang, Y. Su, X. Qin, Strong convergence theorems for a finite family of nonexpansive mappings, Fixed Point Theory Appl. 2007 (2007) Art. ID 76971, 9 pages.

[146]] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese J. Math. 5 (2001) 387-404.

[147]] M. Solodov, An explicit descentmethod for bilevel convex optimization, J. Convex Anal. 14(2), 227—237 (2007) (to appear)

[148]] G. Stampacchia, Formes bilineaires coercitivies sur les ensembles convexes, C. R. Acad. Sciences, Paris, 258(1964), 4413-4416.

[149]] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for oneparameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005), 227-239.

[150]] A. Tada, W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, in: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, 2007, pp. 609—617.

[151]] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506—515.

[152]] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mapping, J. Optim. Theory Appl. 118 (2003), 417-428

[153]] D.Q. Tran, L.D. Muu , V.H. Nguyen, Extragradient algorithms extended to solving equilibrium problems. Optimization 57(6), 749—776 (2008)

[154]] D.T. Tuc, N.X. Tan, : Existence conditions in variational inclusions with constraints. Optimization 53(5—6), 505—515 (2004)

[155]] N.T.T. Van, J.J. Strodiot , V.H. Nguyen, A bundle method for solving equilibrium problems. Math. Program. 116(1—2), Ser. B, 529—552 (2009)

[156]] I. Yamada, N. Ogura, Hybrid steepest descentmethod for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings. Num. Funct. Anal. Optim. 25(7—8), 619—655 (2004)

[157]] Y. Yao, Y.C. Liou, J.C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory Appl. 2007 (2007), Article ID 64363, 12 pp.

[158]] R. Wangkeeree and U. Kamraksa, A General Iterative Method for Solving the variational Inequality Problem and Fixed Point problem of an Infinite family of nonexpansive mappings in Hilbert spaces, Fixed Point Theory and Applications, Vol. 2009, Article ID 369215, 23 pages doi:10.1155/2009/369215.

[159]] R. Wangkeeree, N. Petrot, and R. Wangkeeree, The general iterative methods for nonexpansive mappings in Banach spaces, Journal of Global Optimization, DOI 10.1007/s10898-010-9617-6.

[160]] R. Wangkeeree, An Extragradient Approximation Method for Equilibrium Problems and Fixed Point Problems of a Countable Family of Nonexpansive Mappings, Fixed Point Theory and Applications, Volume 2008 (2008), Article ID 134148, 17 pages, doi:10.1155/2008/134148.

[161]] R. Wangkeeree, and U. Kamraksa, An iterative approximation method for solving a general system of variational inequality problems and mixed equilibrium problems, Nonlinear Analysis: Hybrid Systems 3 (2009), 615-630.

[162]] F.Q. Xia, X.P. Ding, Predictor-corrector algorithms for solving generalized mixed implicit quasiequilibrium problems. Appl. Math. Comput. 188(1), 173—179 (2007)

[163]] H.K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, Journal of Mathematical Analysis and Applications, 314 (2006), 631-643.

[164]] D.C. Youla, Mathematical theory of image restoration by the method of convex projections, in: H. Stark (Ed.), Image Recovery: Theory and Applications, Academic Press, Florida, 1987, pp. 29-77.

[165] Mann, W.R., 1953, “Mean value methods in iterations", **Proceedings of the American Mathematical Society**, Vol. 4, pp. 506—510.

[166] Halpern, B., 1967, “Fixed points of nonexpansive maps", **Bulletin of the American Mathematical Society**, Vol. 73, pp. 957—961.

[167] Ishikawa, S., 1974, “Fixed point by a new iterations methods", **Proceedings of the American Mathematical Society**, Vol. 44, pp. 147—150.

[168] Noor, M.A., 2000, “New approximation schemes for general variational inequalities", **Journal of Mathematical Analysis and Applications**, Vol. 251, pp. 217—229.

[169] Noor, M.A., 2001, “Three-step iterative algorithms for multivalued quasi variational inclusions", **Journal of Mathematical Analysis and Applications**, Vol. 255, pp. 589—604.

[170] Korpelevich, G.M., 1976, "The extragradient method for finding saddle points and other problems", **Journal Matecon**, Vol. 12, pp. 747—756.

[171] Takahashi, W., Takeuchi, Y. and Kubota, R., 2008, "Strong Convergence Theorems by Hybrid Methods for Families of Nonexpansive Mappings in Hilbert Spaces", **Journal of Mathematical Analysis and Applications**, Vol. 341, pp. 276—286

[172] Nakajo, K. and Takahashi, W., 2003, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups", **Journal of Mathematical Analysis and Applications**, Vol. 279, pp. 372—379.

[173] Takahashi, W., 2000, **Introduction to Nonlinear and Convex Analysis**, Yokohama—Publishers, Yokohama, Japan.

[174] Suzuki, T., 2005, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals", **Journal of Mathematical Analysis and Applications**, Vol. 305, pp. 227—239.

[175] Xu, H.K., 2004, "Viscosity approximation methods for nonexpansive mappings", **Journal of Mathematical Analysis and Applications**, Vol. 298, pp. 279—291.

[176] Osilike, M.O. and Igbokwe, D.I., 2000, "Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations", **Computers & Mathematics with Applications**, Vol. 40, pp. 559-567.

[177] Opial, Z., 1967, "Weak convergence of successive approximations for nonexpansive mappings", **Bulletin of the American Mathematical Society**, Vol. 73, pp. 591——597.

[178] Goebel, K. and Kirk, W.A., 1990, **Topics in metric fixed point theory**, Cambridge University Press, Cambridge.

[179] Takahashi, W., 2000, **Nonlinear Functional Analysis**, Yokohama Publishers, Yokohama.

[180] Yao, Y., Noor, M.A., Zainab S. and Liouc, Y.C., 2009, "Mixed Equilibrium Problems and Optimization Problems", **Journal of Mathematical Analysis and Applications**, Vol. 354, pp. 319—329.

[181] Marino, G. and Xu, H.-K.A, 2006, "General iterative method for nonexpansive mappings in Hilbert spaces", **Journal of Mathematical Analysis and Applications**, Vol. 318, pp. 43—52.

[182] Hanson, M.A., 1981, “On sufficiency of the Kuhn—Tucker conditions", **Journal of Mathematical Analysis and Applications**, Vol. 80, 545——550.

[183] Ansari, Q.H. and Yao, J.C., 2001, “Iterative schems for solving mixed variational-like inequalities", **Journal Optimization Theory & Applications**, Vol. 108, pp. 527——541.

[184] Zhou, H., 2008, “Convergence theorems of fixed Points for k-strict pseudo-contractions in Hilbert spaces", **Nonlinear Analysis**, Vol. 69, pp. 456——462.

[185] Shimoji, K. and Takahashi, W., 2001, “Strong convergence to common fixed points of infinite nonexpansive mappings and applications", **Taiwanese Journal of Mathematics**, Vol. 5, pp. 387—404.

[186] Chang, S.S., 2007, **Variational Inequalities and Related Problems**, Chongqing Publishing House.

[187] Shimizu, T. and Takahashi, W., 1997, “Strong convergence to common fixed points of families of nonexpansive mappings", **Journal of Mathematical Analysis and Applications**, Vol. 211, pp. 71—83.

[188] Tan, K.K. and Xu, H.K., 1992, “The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces", **Proceedings of the American Mathematical Society**, Vol. 114, pp. 399——404.

[189] Plubtieng, S. and Thammathiwat, T., 2008, “A Viscosity approximation method for finding a common fixed point of nonexpansive and firmly nonexpansive mappings in Hilbert spaces", **Thai journal of Mathematics**, Vol. 6, pp. 377—390.

[190] Atsushiba, S. and Takahashi, W., 1999, “Strong convergence theorems for a finite family of nonexpansive mappings and applications", **Indian Journal Mathematics**, Vol. 41, pp. 435—453.

[191] Colao, V., Marino G. and Xu, H.-K., 2008, “An iterative method for finding common solutions of equilibrium and fixed point problems", **Journal of Mathematical Analysis and Applications**, Vol. 344, pp. 340—352.

[192] Cho, Y.J. and Qin, X.L., 2009, “Convergence of a general iterative method for nonexpansive mappings in Hilbert spaces", **Journal of Computational and Applied Mathematics**, Vol. 228. pp. 458—465.

[193] Bruck, R.E., 1973, “Nonexpansive projections on subsets of Banach spaces", **Pacific Journal of Mathematics**, Vol. 47, pp. 341—355.

[194] Reich, S., 1973, “Asymptotic bebavior of contractions in Banach space", **Journal of Mathematical Analysis and Applications**, Vol. 44, pp. 57—70.

[195] Kirk, W.A. and Sims, B., 2001, **Handbook of metric fixed point theory**, Kluwer Academic Publishers.

[196] Kitahara, S. and Takahashi, W., 1993, “Image recovery by convex combinations of sunny nonexpansive retractions", **Topological Methods in Nonlinear Analysis**, Vol. 2, pp. 333—342.

[197] Cai, G. and Hu, C.S., 2010, “Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces", **Computers and Mathematics with Applications**, Vol. 59, pp. 149—160.

[198] Xu, H.K., 1991, “Inequalities in Banach spaces with applications", **Nonlinear Anal.**, Vol. 16, pp. 1127—1138.

[199] Qin, X., Cho, Y.J., Kang J.I. and Kang, S.M., 2009, “Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces", **Journal of Computational and Applied Mathematics**, Vol. 230, pp. 121—127.

[200] Xu, H.K., 2006, “Strong convergence of an iterative method for nonexpansive and accretive operators", **Journal of Mathematical Analysis and Applications**, Vol. 314, pp. 631—643.

[201] Wittmann, R., 1992, “Approximation of fixed points of nonexpansive mappings", **Archiv der Mathematik**, Vol. 58, pp. 486—491.

[202] Bruck, R.E., 1973, “Properties of fixed point sets of nonexpansive mappings in Banach spaces", **Transactions of the American Mathematical Society**, Vol. 179, pp. 251—262.

[203] Browder, F.E., 1968, “Semicontractive and semiaccretive nonlinear mappings in Banach spaces", **Bulletin of the American Mathematical Society**, Vol. 74, pp. 660-665.

[204] Verma, R.U., 1999, “On a new system of nonlinear variational inequalities and associated iterative algorithms", **Mathematical Sciences Research**, Vol. 3, pp. 65—68.

[205] Verma, R.U., 2001, “Iterative algorithms and a new system of nonlinear quasi-variational inequalities", **Advances in Nonlinear Variational Inequalities**, Vol. 4, pp. 117—124.

[206] Stampacchia, G., 1964, “Formes bilineaires coercitives sur les ensembles convexes", **Comptes rendus Academy of Sciences**, Vol. 258, pp. 4413——4416.

[207] Brézis, H., 1973, “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert", **North-Holland Mathematics Studies**, Notas de Mathemática, North-Holland, Amsterdam, The Netherlands.

[208] Zhang, S.-S., Lee, J.H.W. and Chan, C.K., 2008, “Algorithms of common solutions to quasi variational inclusion and fixed point problems", **Applied Mathematics and Mechanics**, Vol 29, pp. 571-—581.

[209] Rockafellar, R.T., 1976, “Monotone operators and proximal point algorithm", **SIAM Journal on Control and Optimization**, Vol. 14, pp. 877—898.

[210] Rockafellar, R.T., 1970, “On the maximality of sums of nonlinear monotone operators", **Transactions of the American Mathematical Society**, Vol. 149. pp. 75—88.

[211] Aoyama, K., Iiduka H. and Takahashi, W., 2006, “Weak convergence of an iterative sequence for accretive operators in Banach spaces", **Fixed Point Theory and Applications**, Vol. 2006, 13 pages.

[212] Ceng, L.C. and Yao, J.C., 2008, “A hybrid iterative scheme for mixed equilibrium problems and fixed point problems", **Journal of Computational and Applied Mathematics**, Vol. 214, pp. 186—201.

[213] Blum, E.and Oettli, W., 1994, “From optimization and variational inequalities to equilibrium problems", **Math. Student.**, Vol. 63, pp. 123—145.

[214] Combettes, P.L. and Hirstoaga, S.A., 2005, “Equilibrium programming in Hilbert spaces", **Journal of Nonlinear and Convex Analysis**, Vol. 6, pp. 117—136.

[215] Peng, J.-W. and Yao, J.-C., 2009, “Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems", **Mathematical and Computer Modelling**, Vol. 49, pp. 1816—1828.

[216] Yao, Y., Liou, Y.-C. and Yao, J.-C., 2007, “Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings", **Fixed Point Theory and Applications**, Vol. 2007, 12 pages.

[217] M. Aslam Noor and W. Ottli (1994). On general nonlinear complementarity problems and quasi equilibria. *Le Mathematics (Catania).* 49:313—331.

[218] E. Blum, W. Oettli (1994). From optimization and variational inequalities to equilibrium problems. *Math. Student.* 63:123—145.

[219] F.E. Browder and W.V. Petryshyn (1967). Construction of fixed points of nonlinear mappings in Hilbert space. *J. Math. Anal. Appl.* 20:197—228.

[220] P.L. Combettes (1997). Hilbertian convex feasibility problem: Convergence of projection methods. *Appl. Math. Optim.* 35:311-330.

[221] P.L. Combettes and S.A. Hirstoaga (1997). Equilibrium programming using proximal-like algorithms. *Math. Program.* 78:29—41.

[222] L. C. Ceng and J. C. Yao (2005). Iterative algorithm for grneralized set-valued strong nonlinear mixed variational-like inequalities. *J. optim. Theory Appl.* 124:725—738.

[223] L.C. Ceng and J.C. Yao (2008). A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. Appl. Math.* 214:186-201.

[224] S.-s. Changa, H.W.J. Lee and C.K. Chan (2008). A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. *Nonlinear Analysis.* doi:10.1016/j.na.2008.04.035

[225] F. Deutsch and I. Yamada (1998). Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings. *Numer. Funct. Anal. Optim.* 19:33—56.

[226] D. Gabay (1983). Applications of the Method of Multipliers to Variational Inequalities, Augmented Lagrangian Methods, Edited by M. Fortin and R. Glowinski, North-Holland, Amsterdam, Holland. 299——331.

[227] H. Iiduka and W. Takahashi (2005). Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings. *Nonlinear Analysis.* 61:341—350.

[228] C. Jaiboon, P. Poom. and U.W. Humphries (2009) Convergence Theorems by the Viscosity Approximation Method for Equilibrium Problems and Variational Inequality Problems. a accepted for publication and to appear in *J. Comput. Math. Optim.* SAS International Publications.

[229] I. V. Konnov, S. Schaible and J.C. Yao (2005). Combined relaxation method for mixed equilibrium problems. *J. Optim. Theory Appl.* 126:309-322.

[230] F. Liu and M.Z. Nashed (1998). Regularization of nonlinear Ill-posed variational inequalities and convergence rates. *Set-Valued Anal.* 6:313—344.

[231] G. Marino and H.K. Xu (2006). A general iterative method for nonexpansive mapping in Hilbert spaces. *J. Math. Anal. Appl.* 318:43—52.

[232] Z. Opial (1967). Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.* 73:595—597.

[233] X. Qin, M. Shang and Y. Su (2008). A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces. *Nonlinear Analysis.* 69(8):3897—3909.

[234] X. Qin, M. Shang and Y. su (2008). Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems. *Math. Comput. Model.* 48:1033—1046.

[235] R.T. Rockafellar (1970). On the maximality of sums of nonlinear monotone operators. Trans. *Amer. Math. Soc.* 149:75—88.

[236] T. Suzuki (2005). Strong convergence of krasnoselskii and mann's type sequences for one-parameter nonexpansive semigroups without bochner integrals. *J. Math. Anal. Appl.* 305:227—239.

[237] G. Stampacchia (1964). Formes bilineaires coercivites sur les ensembles convexes. *Comptes rendus Acad. Sci. Paris.* 258:4413——4416.

[238] K. Shimoji and W. Takahashi (2001) Strong convergence to common fixed points of infinite nonexpansive mappings and applications. *Taiwanese J. Math.* 5:387—404.

[239] W. Takahashi (2000). Nonlinear functional analysis. *Yokohama Publishers.* Yokohama.

[240] RU. Verma (2005). General convergence analysis for two—step projection methods and application to variational problems. *J. Optim. Theory Appl.* 18(11):1286—1292.

[241] H.K. Xu (2003). An iterative approach to quadratic optimization. *J. Optim. Theory Appl.* 116:659—678.

[242] H.K. Xu (2004). Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.* 298:279—291.

[243] Y. Yao (2007). A general iterative method for a finite family of nonexpansive mappings. *Nonlinear Analysis*. 66(12):2676—2687.

[244] J. C. Yao and O. Chadli (2005). Pseudomonotone complementarity problems and variational in- equalities, in: J.P. Crouzeix, N. Haddjissas, S. schaible(Eds), Handbook of Generalized Convexity and Monotonicity. *Kluwer Academic*. 501—558.

[245] Y. Yao, Y.C. Liou and J.C. Yao (2008). Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings. *Fixed Point Theory and Applications*.

[246] Y. H. Yao, Y. C. Liou and J. C. Yao (2008). A New Hybrid Iterative Algorithm for Fixed-Point Problems, Variational Inequality Problems, and Mixed Equilibrium Problems. *Fixed Point Theory and Applications*. Article ID 417089, 15 pages.

[247] Y. Yao, M. A. Noor and Y.C. Liou (2009). On iterative methods for equilibrium problems. *Nonlinear Analysis*. 70(1):479—509.

[248] Y. Yao, M. A. Noor, S. Zainab and Y.C. Liouc (2009). Mixed E-quilibrium Problems and Optimization Problems. *J. Math. Anal. Appl* .doi:10.1016/j.jmaa.2008.12.005.

[249] I. Yamada, N. Ogura, Y. Yamashita and K. Sakaniwa (1998). Quadratic optimization of fixed point of nonexpansive mapping in Hilbert space. *Numer. Funct. Anal. Optim*. 19(1,2):165—190.

[250] F.E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Natl. Acad. Sci. USA*, 56 (1966) 1080—1086.

[251] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Proc. Symp. Pure. Math,* 18 (1976) 78—81.

[252] H. Brézis, Opérateur maximaux monotones, *in Mathematics Studies*, vol. 5, North-Holland, Amsterdam, The Netherlands, 1973.

[253] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student.*, 63 (1994) 123—145.

[254] P.L. Combettes and S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, 6 (2005) 117—136.

[255] S.D. Flam and A.S. Antipin, Equilibrium progamming using proximal-link algolithms, *Math. Program.*, 78 (1997) 29—41.

[256] Y. Hao, Some results of variational inclusion problems and fixed point problems with applications, *App. Math. Mech.*, 30(12) (2009) 1589—1596.

[257] P. Hartman and G. Stampacchia, On some nonlinear elliptic differential functional equations, *Acta Math.*, 115 (1966) 271—310.

[258] H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mapping and inverse-strong monotone mappings, *Nonlinear Anal.*, 61 (2005) 341—350.

[259] W.A. Kirk, Fixed point theorem for mappings which do not increase distance, *Amer. Math. Monthly.*, 72 (1965) 1004—1006.

[260] C. Klin-eam and S. Suantai, A new approximation method for solving variational inequalities and fixed points of nonexpansive mappings, *J. Inequalities Appl.* Vol. 2009, Article ID 520301, 16 pages.

[261] B. Lemaire, Which fixed point does the iteration method select, in Recent Advances in Optimization (Trier, 1996), vol. 452 of Lecture Note in Economics and Mathematical Systems *Springer*, Berlin, Germany, 1997.

[262] M. Liu, S. S. Chang and P. Zuo, An algorithm for finding a common solution for a system of mixed equilibrium problem, quasivariational inclusion problems of nonexpansive semigroup, *J. Inequal. Appl.*, vol 2010, Article ID 895907, 23 pages, 2010.

[263] G. Marino and H.K. Xu, A general iterative method for nonexpansive mapping in Hilbert space, *J. Math. Anal. Appl.*, 318 (2006) 43—52.

[264] A. Moudafi, Viscosity approximation methods for fixed points problems, *J. Math. Anal. Appl.*, 241 (2000) 46—55.

[265] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, 73 (1967) 591——597.

[266] J.-W. Peng, Y.-C. Liou and J.-C. Yao, An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions, *Fixed Point Theory Appl.*, vol 2009, Article ID 794178, 21 pages, 2009.

[267] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.*, 149 (2000) 46—55.

[268] Y. Su, M. Shang and X. Qin, An iterative method of solution for equilibrium and optimization problems, *Nonlinear Anal.* 69 (2008) 2709–2719.

[269] Y. Shehu, Iterative methods for family of strictly psedocontractive mappings and system of generalized mixed equilibrium problems and variational inequality problems, *Fixed Point Theory and Appl.*, vol 2011, Article ID 852789, 22 pages.

[270] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, 331 (2007) 506–515.

[271] H.K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, 66 (2002) 240–256.

[272] J.C. Yao and O. Chadli, Pseudomonotone complementarity problems and variational inequalities, in: J.P. Crouzeix, N. Haddjissas, S. Schaible (Eds.), *Handbook of Generalized Convexity and Monotonicity*, (2005) 501–558.

[273] S.S. Zhang, H.W. Joseph Lee and C.K. Chan, Algorithms of common solutions to quasi variational inclusion and fixed point problems, *App. Math. Mech.*, 29(5) (2008).

[274] H. He, S. Liu and Y. J. Cho, An explicit method for systems of equilibrium problems and fixed points of infinite family of nonexpansive mapping, *Journal of Computational and Applied Mathematics*, Vol. 235, (2011), Pages 4128-4139

[275] T. Jitpeera and P. Kumam, A General Iterative Algorithm for Generalized Mixed Equilibrium Problems and Variational Inclusions Approach to Variational Inequalities, *International Journal of Mathematics and Mathematical Science*, vol 2011, Article ID 619813, 25 pages.

[276] T. Jitpeera, U. Witthayarat and P. Kumam , "Hybrid algorithms of common solutions of generalized mixed equilibrium problems and the common variational inequality problems with applications," Fixed Point Theory and Applications, Volume 2011, Article ID 971479, 28 pages.

[277] T. Jitpeera and P. Kumam, "An extragradient type method for a system of equilibrium problems, variational inequality problems and fixed points of finitely many nonexpansive mappings" Journal of Nonlinear Analysis and Optimization: Theory & Applications, Vol. 1 No. 1 (2010), 71-91.

[278] T. Jitpeera and P. Kumam, A new Hybrid Algorithm for A system of equilibrium problems and variational inclusion, Ann Univ Ferrara (2011) 57:89-108

[279] T. Jitpeera and P. Kumam, " Hybrid algorithms for minimization problems over the solutions of generalized mixed equilibrium and variational inclusion problems," Mathematical Problems in Engineering, Volume 2011, Article ID 648617, 26 pages

[280] T. Jitpeera and P. Kumam, A New Hybrid Algorithm for a System of Mixed Equilibruim Problems, Fixed Point Problems for Nonexpansive Semigroup, and Variational Inclusion Problem, *Fixed Point Theory Appl.*, vol 2011, Article ID 217407, 27 pages.

[281] P. Kumam, U. Hamphries and P. Katchang, Common solutions of generalized mixed equilibrium problems, variational inclusions and common fixed points for nonexpansive semigroups and strictly pseudo-contractive mappings," Journal of Applied Mathematics, Volume 2011, Article ID 953903, 27 pages

[282] P. Sunthrayuth and P. Kumam "A new general iterative method for solution of a new general system of variational inclusions for nonexpansive semigroups in Banach spaces," Journal of Applied Mathematics, vol. 2011, Article ID 187052, 29 pages, 2011.

[283] P. Katchang and P. Kumam, Convergence of iterative algorithm for finding common solution of fixed points and general system of variational inequalities for two accretive operators, Thai Iournal of mathematics. volume 9 (2011) number 2 : 319-335.

[284] W. Kumam, P. Junlouchai, and P. Kumam , Generalized Systems of Variational Inequalities and Projection Methods for Inverse-Strongly Monotone Mappings, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 976505, 24 pages

[285] P. Kumam, A relaxed extragradient approximation method of two inverse-strongly monotone mappings for a general system of variational inequalities, fixed point and equilibrium problems, Bulletin of the Iranian Mathematical Society, Vol. 36 No. 1 April 2010, 227-252.

[286] P. Kumam and C. Jaiboon "Approximation of common solutions to system of mixed equilibrium problems, variational inequality problems and strict pseudo-contractive mappings," Fixed Point Theory and Applications, Volume 2011, Article ID 347204, 30 pages.

[287] C. Jaiboon and P. Kumam, A general iterative method for addressing mixed equilibrium problems and optimization problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 73 (2010) pp. 1180-1202.

[288] Y. J. Cho, I. K. Argyros, N. Petrot, Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems, *Computers and Mathematics with Applications.*, 60 (2010) 2292—2301.

[289] Y. J. Cho, N. Petrot and Suthep Suantai, Fixed point theorems for nonexpansive mappings with applications to generalized equilibrium and system of nonlinear variational inequalities problems, *Journal of Nonlinear Analysis and Optimization.*, Vol. 1 No. 1 (2010), 45-53.

[290] Y. J. Cho, N. Petrot, On the System of Nonlinear Mixed Implicit Equilibrium Problems in Hilbert Spaces, *Journal of Inequalities and Applications.*, (2010), Article ID 437976, doi:10.1155/2010/437976.

[291] Y. Yao, Y. J. Cho and Y.-C. Liou, Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems, European Journal of Operational Research, 212 (2011), 242-250.

[292] Y. Yao and N. Shahzad, New methods with perturbations for non-expansive mappings in Hilbert spaces, Fixed Point Theory and Applications, in press.

[293] Y. Yao and N. Shahzad, Strong convergence of a proximal point algorithm with general errors, Optimization Letters, doi: 10.1007/s11590-011-0286-2.

[294] Yonghong Yao, Yeong-Cheng Liou and Chia-Ping Chen, Algorithms construction for nonexpansive mappings and inverse-strongly monotone mappings, Taiwanese Journal of Mathematics, 15(2011), 1979-1998.

[295] Y. Yao, R. Chen and Y.-C. Liou, A unified implicit algorithm for solving the triple-hierarchical constrained optimization problem, Mathematical and Computer Modelling, doi:10.1016/j.mcm.2011.06.038.

[296] Y. Yao, Y.-C. Liou, S. M Kang and Y. Yu, Algorithms with strong convergence for a system of nonlinear variational inequalities in Banach spaces, Nonlinear Analysis, 74(2011), 6024-6034.

[297] Y. Yao, Y.-C. Liou, and S. M. Kang, Two-step projection methods for a system of variational inequality problems in Banach spaces, Journal of Global Optimization, doi: 10.1007/s10898-011-9804-0.

[298] Goebel, K. and Kirk, W. A.: Topics on Metric Fixed-Point Theory, Cambridge University Press, Cambridge. England, 1990.

[299] Ceng, L.-C., Wang, C.-Y., and Yao, J.-C.: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, *Math. Meth. Oper. Res.*, 67:375—390, 2008.

[300] Kassay, G., Kolumban, J., Pales, Zs.: Factorization of Minty and Stampacchia variational inequality systems, *Euro. J. Oper. Res.*, 143 (2002), 377-389.

[301] Kassay, G., Kolumban, J.: System of multi-valued variational inequalities, *Publ. Math.*, 56 (2000), 185-195.

[302] P. Kumam, A relaxed extragradient approximation method of two inverse-strongly monotone mappings for a general system of variational inequalities, fixed point and equilibrium problems, Bulletin of the Iranian Mathematical Society, Vol. 36 No. 1 April 2010, 227-252.

[303] W. Kumam and P. Kumam, Hybrid iterative scheme by relaxed extragradient method for solutions of equilibrium problems and a general system of variational inequalities with application to optimization, Nonlinear Analysis: Hybrid Systems 3 (2009), 640-656.

[304] Liu, F. Nashed, M.Z. and Takahashi, W.: Regularization of nonlinear ill-posed variational inequalities and convergence rates, *Set-Value Analy* 6, 313-344 (1998).

[305] Nadezhkina, N. and Takahashi, W.: Weak convergence theorem by an extra-gradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 128, 191-201 (2006).

[306] Osilike, M.O. and Igbokwe, D. I.: Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, *Comp. Math. Appl.* 40, 559-567 (2000).

[307] Suzuki, T.: Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. and Appl.* 305, 227—239 (2005).

[308] Takahashi, W. and Toyoda, M.: Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 118, 417-428 (2003).

[309] Verma RU.: On a new system of nonlinear variational inequalities and associated iterative algorithms, Math. Sci. Res., Hot-Line 3(8): 65—68.

[310] Verma RU.: Iterative algorithms and a new system of nonlinear variational inequalities. Adv. Nonlinear Var. Inequal. 3(8):117—124.

[311] Xu, H. K.: Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298, 279-291 (2004).

[312] Yao, J.-C. and Chadli, O.: Pseudomonotone complementarity problems and variational inequalities, In: Handbook of Generalized Convexity and Monotonicity (Eds.: J.P. Crouzeix, N. Haddjissas and S. Schaible) 501-558, Springer Netherlands (2005).

[313] Yao, Y. and Yao, J.-C.: On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* 186, 1551-1558 (2007).

[314] Yao, Y., Liou Y. C. and Yao, J.-C.: An Extragradient Method for Fixed Point Problems and Variational Inequality Problems, *Journal of Inequalities and Applications* Volume 2007, Article ID 38752, 12 pages doi:10.1155/2007/38752 (2007).

[315] Zeng, L.C. and Yao, J.-C.: Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.* 10, 1293-1303 (2006).

[316] Zeng, L.C., Wong, N.C. and Yao, J.-C.: Strong convergence theorems for strictly pseudocontractive mapping of Browder-Petryshyn type, *Taiwanese J. Math.* 10(4), 837-849 (2006).

[317] Zhang, S., Lee, J. and Chan, C.: Algorithms of common solutions to quasi variational inclusion and fixed point problems, *Appl. Math. Mech.-Engl. Ed.,*29(5), 571-581 (2008).

[318] Ya. Alber, Generalized projection operators in Banach spaces: Properties and applications, in: Proceedings of the Israel Seminar, Ariel, Israel, Funct. Differential Equation 1 (1994) 1—21.

[319] Ya. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, pp. 15—50.

[320] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: Proceedings of the Israel Seminar on Functional Differential Equations, Ariel, Israel, vol. 1, 1994, pp. 1-21.

[321] S.S. Chang, *On Chidumes open questions and approximate solutions of multi-valued strongly accretive mapping in Banach spaces*, J. Math. Anal. Appl. 216 (1997) 94——111.

[322] J.H. Fan, *A Mann type iterative scheme for variational inequalities in noncompact subsets of Banach spaces*, J. Math. Anal. Appl. 337 (2008) 1041——1047.

[323] J.H. Fan, X. Liu, J.L. Li, *Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces*, Nonlinear Anal. 70(11) (2009) 3997—4007.

[324] G. Stampacchia, *Formes bilinearies coercivities sur les ensembles convexes*, C. R. Acad. Sci., Paris 258 (1964) 4413-—4416.

[325] K. Q. Wu, N. J. Huang, *The generalised f-projection operator with an application*, Bull. Aust. Math. Soc. 73(2006) 307-317.

[326] K. Q. Wu, N. J. Huang, *Properties of the generalised f-projection operator and its applications in Banach spaces*, Comput. Math. Appl. 54(2007) 399—406.

[327] H.K. Xu, *Inequlities in Banch spaces with applications*, Nonlinear Anal. 16 (1991) 1127——1138.

[328] R. P. Agarwal, Y. J. C. and N. Petrot, Systems of general nonlinear set-valued mixed variational inequalities problems in Hilbert spaces, Fixed Point Theory and Applications 2011, 2011:31.

[329] Y. J. Cho and N. Petrot, Regularization and Iterative method for general variational inequality problem in Hilbert spaces, Journal of Inequalities and Applications 2011, 2011:21 doi:10.1186/1029-242X-2011-21

[330] . P. Kumam, N. Petrot, R. Wangkeeree, Existence and iterative approximation of solutions of generalized mixed quasi-variational-like inequality problem in Banach spaces, Applied Mathematics and Computation 217 (2011) 7496—7503.

[331] S. Suantai, N. Petrot, Existence and stability of iterative algorithms for the system of nonlinear quasi-mixed equilibrium problems, Applied Mathematics Letters 24(3) (2011) 308—313. (IF= 1.155)

[332] I. Inchan and N. Petrot, System of general variational inequalities involving different nonlinear operators related to fixed point problems and its applications, Fixed Point Theory and Applications, Volume 2011, Article ID 689478, 17 pages, doi:10.1155/2011/689478.

[333] Y. J. Cho, N. Petrot and Suthep Suantai, Fixed point theorems for nonexpansive mappings with applications to generalized equilibrium and system of nonlinear variational inequalities problems, Journal of Nonlinear Analysis and Optimization, Vol. 1 No. 1 (2010), 45-53.

[334] N. Onjai-uea and P. Kumam, "Algorithms of common solutions to generalized mixed equilibrium problems and a system of quasivariational inclusions for two difference nonlinear operators in Banach spaces," Fixed Point Theory and Applications, Volume 2011, Article ID 601910, 23 pages

[335] N. Onjai-uea and P. Kumam, Existence and Convergence Theorems for the new system of generalized mixed variational inequalities in Banach spaces, Journal of Inequalities and Applications 2012, 2012:9 doi:10.1186/1029-242X-2012-9

[336] N. Petrot, Resolvent operator technique for approximate solvability of generalized System mixed variational inequalities and fixed point problems, Applied Mathematics Letters, 23(4) (2010), 440-445.

[337] Y. J Cho and N. Petrot, On the System of Nonlinear Mixed Implicit Equilibrium Problems in Hilbert Spaces, Journal of Inequalities and Applications, (2010), Article ID 437976, doi:10.1155/2010/437976,

[338] Q-b Zhang, R. Denga, L. Liu *Projection algorithms for the system of mixed variational inequalities in Banach spaces*, Mathematical and Computer Modelling 53 (2011) 1692-1699.

ภาคผนวก
Outputs

International Journal

(1) C. Jaiboon and **P. Kumam**, A general iterative method for addressing mixed equilibrium problems and optimization problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 73 (2010) pp. 1180-1202. **(2009 Impact Factor=1.487)**

(2) W. Chantarangsi, C. Jaiboon, and **P. Kumam**, A viscosity hybrid steepest descent method for generalized mixed equilibrium problems and variational inequalities for relaxed cocoercive mapping in Hilbert spaces, Abstract and Applied Analysis, Volume 2010 (2010), Article ID 390972, 39 pages **(2009 Impact Factor: 2.221)**

(3) P. Kumam and **C. Jaiboon**, A system of generalized mixed equilibrium problems and fixed point problems for pseudo-contraction mappings in Hilbert spaces, Fixed Point Theory and Applications, Volume 2010, Article ID 361512, 33 pages. **(20010 Impact Factor 1.936)**

(4) S. Saewan and **P. Kumam**, "Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-ϕ-asymptotically nonexpansive mappings " Abstract and Applied Analysis, Volume 2010, Article ID 357120, 22 pages **(2009 Impact Factor: 2.221)**

(5) S. Saewan and **P. Kumam,** "A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems," Abstract and Applied Analysis, Volume 2010, Article ID 123027, 31 pages **(2009 Impact Factor: 2.221)**

(6) T. Jitpeera, P. Katchang, and **P. Kumam**, A viscosity of Cesàro mean approximation methods for mixed equilibrium, variational inequalities and fixed point problems, Fixed Point Theory and Applications, Volume 2011, Article ID 945051, 24 pages doi:10.1155/2011/945051 **(2010 Impact Factor 1.936)**

(7) T. Jitpeera and **P. Kumam**, "A New Hybrid Algorithm for a System of Mixed Equilibrium Problems, Fixed Point Problems for Nonexpansive Semigroup, and Variational Inclusion Problem," Fixed Point Theory and Applications, vol. 2011, Article ID 217407, 27 pages, 2011. doi:10.1155/2011/217407. **(2010 Impact Factor 1.936)**

(8) T. Jitpeera and **P. Kumam**, The shrinking projection method for a system of generalized mixed equilibrium problems and fixed point problems for pseudocontractive mappings, Journal of Inequalities and Applications, Volume 2011, Article ID 840319, 25 pages. **(2010 Impact Factor 0.88)**

(9) P. Kumam and S. Plubtieng, 'Viscosity approximation methods for monotone mappings and a countable family of nonexpansive mappings', Mathematica Slovaca, Math. Slovaca, 61 (2) (2011), 257-274. **(2010 Impact Factor 0.316)**

(10) T. Jitpeera, U. Witthayarat and **P. Kumam** , "Hybrid algorithms of common solutions of generalized mixed equilibrium problems and the common variational inequality problems with applications," Fixed Point Theory and Applications, Volume 2011, Article ID 971479, 28 pages. **(2010 Impact Factor 1.936)**

(11) W. Kumam, P. Junlouchai and **P. Kumam**, Generalized Systems of Variational Inequalities and Projection Methods for Inverse-Strongly Monotone Mappings, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 976505, 24 pages doi:10.1155/2011/976505 **(2010 Impact Factor 0.967)**

(12) **P. Kumam** and S. Plubtieng, "Convergence Theorems by hybrid methods for Monotone Mappings and a Countable Family of Nonexpansive Mappings and its Applications, International Journal of Pure and Applied Mathematics, Volume 70 No. 1 2011, 81-107. (No Impact Factor)

(13) P. Katchang and **P. Kumam**, An iterative algorithm for finding a common solution of fixed points and a general system of variational inequalities for two inverse strongly accretive operators, Positivity, Volume 15, Number 2, (2011) 281-295. **(2010 Impact Factor 0.578)**

(14) S. Saewan and **P. Kumam**, "The shrinking projection method for solving generalized equilibrium problem and common fixed points for asymptotically quasi-ϕ-nonexpansive mappings," Fixed Point Theory and Applications 2011, 2011:9 http://dx.doi.org/10.1186/1687-1812-2011-9 **(2010 Impact Factor 1.936)**

(15) S. Saewan and **P.Kumam**, "Strong convergence theorems for countable families of uniformly quasi-phi-asymptotically nonexpansive mappings and a system of

generalized mixed equilibrium problems," Abstract and Applied Analysis, Volume 2011, Article ID 701675, 27 pages. **(2010 Impact Factor 1.442)**

(16) S. Saewan and P. Kumam, "A modified hybrid projection method for solving generalized mixed equilibrium problems and fixed point problems in Banach spaces", Computers and Mathematics with Applications 62 (2011) 1723-1735. **(2010 Impact Factor 1.472)**

(17) T. Jitpeera and **P. Kumam**, " Hybrid algorithms for minimization problems over the solutions of generalized mixed equilibrium and variational inclusion problems," Mathematical Problems in Engineering, Volume 2011, Article ID 648617, 26 pages **(2010 Impact Factor 0.689)**

(18) S. Saewan and **P. Kumam**, A new modified block iterative algorithm for uniformly quasi-ϕ-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems, Fixed Point Theory and Applications 2011, 2011:35 doi:10.1186/1687-1812-2011-35 **(2010 Impact Factor 1.936)**

(19) S. Saewan and **P. Kumam**, Convergence theorems for mixed equilibrium problems, variational inequality problem and uniformly quasi-ϕ-asymptotically nonexpansive mappings, Applied Mathematics and Computation, 218 (2011) 3522-3538. **(2010 Impact Factor 1.534)**

(20) T. Chamnarnpan and **P. Kumam**, "Iterative Algorithms for Solving the System of Mixed Equilibrium Problems, Fixed-Point Problems, and Variational Inclusions with Application to Minimization Problem" Volume 2012, Article ID 538912, 29 pages doi:10.1155/2012/538912 **(2010 Impact Factor 0.630)**

(21) P. Katchang and **P. Kumam**, "Hybrid-extragradient type methods for a generalized equilibrium problem and variational inequality problems of nonexpansive semigroups" Fixed Point Theory, 13 (2012), 107-120. **(2010 Impact Factor 1.03)**

(22) S. Saewan and **P. Kumam**, Existence and Algorithm for solving the system of mixed variational inequalities in Banach spaces, Journal of Applied Mathematics, Volume 2012, Article ID 413468, 16 pages **(2010 Impact Factor 0.630)**

(23) S. Saewan and **P. Kumam**, "A strong convergence theorem concerning a hybrid projection method for finding common fixed points of a countable family of relatively quasi-nonexpansive mappings" Journal of Nonlinear and Convex Analysis, Volume 13, Number 2, 2012, **(2010 Impact Factor 0.738)**