รายงานวิจัยฉบับสมบูรณ์

โครงการ

ทฤษฎีจุดตรึงของการกระทำซ้ำเพื่อแก้ปัญหานัยทั่วไปของสภาพคงที่และปัญหาจุดตรึง

Fixed point theorem of iterative schemes for solving the generalized equilibrium problems and fixed point problems

โดย ผู้ช่วยศาสตราจารย์ ดร.อิสระ อินจันทร์

มิถุนายน 2555

สัญญาเลขที่ MRG5380081

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ผู้ช่วยศาสตราจารย์ ดร.อิสระ อินจันทร์ มหาวิทยาลัยราชภัฏอุตรดิตถ์
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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

กิตติกรรมประกาศ

งานวิจัยเรื่อง ทฤษฎีจุดตรึงของการกระทำซ้ำเพื่อแก้ปัญหานัยทั่วไปของสภาพคงที่และปัญหาจุดตรึง (MRG5380081) นี้ สำเร็จลุล่วงด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และ สำนักงานคณะกรรมการอุดมศึกษา (สกอ.) ประจำปี 2553-2555 และข้าพเจ้าขอขอบคุณ ศาสตราจารย์ ดร.สมยศ พลับเที่ยง ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร นักวิจัยที่ปรึกษา ที่ได้ให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา

Project Code: MRG5380081

(รหัสโครงการ)

Project Title: Fixed point theorem of iterative schemes for solving the generalized equilibrium problems and fixed point problems

(ชื่อโครงการ) ทฤษฎีจุดตรึงของการกระทำซ้ำเพื่อแก้ปัญหานัยทั่วไปของสภาพคงที่และปัญหาจุดตรึง

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Abstract

The aim of this project is to consider and study a new iterative scheme for solving the generalized equilibrium problems and fixed point problems. We plan to fine common solutions of fixed points of the generalized equilibrium problems and fixed point problems and also construct and discuss the convergence criterion for the iterative algorithm. Moreover, we will apply our results to generalized equilibrium problems.

In the first year, we will study and discuss some important basic results and consider some new theorems about the general iterative scheme for generalized equilibrium problems in the Hilbert spaces.

In the second year, we will focus our study to the heart of our project, that is, we will consider the general iterative scheme for generalized equilibrium problems in the Banach spaces.

In conclusion, we point out that the results of this project are the extension and improvements of the earlier and recent results in this field. Much work is needed to develop this interesting subject.

Keywords: Iterative approximation method/ Variational inequality problem/ Equilibrium problem/ generalized equilibrium problems/ fixed point problems/ Nonexpansive mapping / Optimization problem

บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้ คือ การศึกษากระบวนการทำซ้ำเพื่อหาผลเฉลยของปัญหาดุลยภาพทั่วไป และปัญหาจุดตรึง โดยจะหาผลเฉลยร่วมกันของเซตดุลยภาพทั่วไปเซตของจุดตรึงโดยใช้เทคนิดการลู่เข้าของกระบวนการทำซ้ำ และจะได้เสนอการประยุกต์ใช้ปัญหาดังกล่าว

โดยในปีแรกจะได้ศึกษาพื้นฐานของกระบวนการทำซ้ำและปัญหาดุลยภาพทั่วไปในปริภูมิฮิลเบิร์ต ในปีที่สองจะได้ศึกษาปัญหาดังกล่าวข้างต้นในปริภูมิบานาค

ผลที่ได้รับจากการศึกษานี้คือการขยายและปรับปรุงผลงานวิจัยของนักวิจัยหลาย ๆ ท่านเพื่อพัฒนาองค์ความรู้ให้ดีขึ้น

คำสำคัญ : วิธีการประมาณค่าแบบทำซ้ำ / ปัญหาอสมการเชิงแปรผัน/ปัญหาเชิงดุลยภาพ/ ปัญหาดุลยภาพทั่วไป/ ปัญหาจุดตรึง/ การส่งแบบไม่ขยาย / ปัญหาค่าเหมาะสมที่สุด

Executive Summary

1. บทคัดย่อเป็นภาษาอังกฤษ

Abstract

Title: Fixed point theorem of iterative schemes for solving the generalized equilibrium problems and fixed point problems

The aim of this project is to consider and study a new iterative scheme for solving the generalized equilibrium problems and fixed point problems. We plan to fine common solutions of fixed points of the generalized equilibrium problems and fixed point problems and also construct and discuss the convergence criterion for the iterative algorithm. Moreover, we will apply our results to generalized equilibrium problems.

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In conclusion, we point out that the results of this project are the extension and improvements of the earlier and recent results in this field. Much work is needed to develop this interesting subject.

2. ความสำคัญและที่มาของปัญหาที่ทำการวิจัย

ปัญหาสภาพคงที่ (equilibrium problems (EP)) นับได้ว่าเป็นเรื่องที่มีการศึกษากันอย่างต่อเนื่อง โดยได้นำไปประยุกต์ใช้ในสาขาวิทยาศาสตร์ประยุกต์หลายแขนง เช่น ฟิสิกส์ เคมี ชีววิทยา และในสาขาวิชาเศรษฐศาสตร์ โดยในสาขาวิชาเศรษฐศาสตร์นี้ได้มีการศึกษากันอย่างต่อเนื่อง เช่น การศึกษาสมดุลทางการตลาด (market equilibrium) เป็นต้น

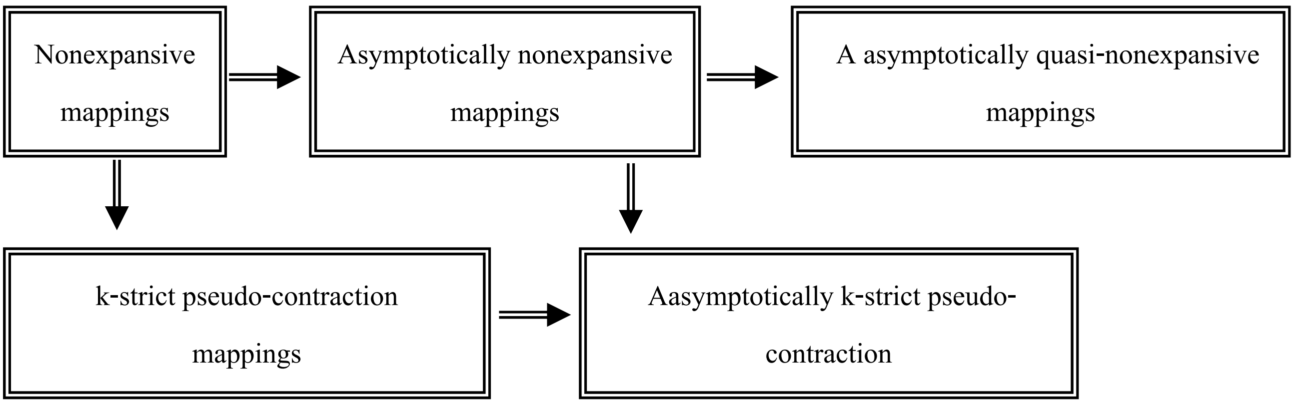
ปัญหาสภาพคงที่ (EP) ได้เริ่มมีการศึกษาบนปริภูมิฮิลเบิร์ต (Hilbert space) ในปี ค.ศ. 1994 และหลังจากนั้นเป็นต้นมาการศึกษาและพัฒนาเกี่ยวกับหัวข้อดังกล่าวจึงได้มีมาอย่างต่อเนื่อง เนื่องจากความรู้ที่ได้จากการศึกษาปัญหาสภาพคงที่เป็นพื้นฐานที่สามารถนำไปประยุกต์ใช้ให้เป็นประโยชน์ได้ในศาสตร์หลาย ๆแขนงวิชาทั้งวิทยาศาสตร์บริสุทธิ์และวิทยาศาสตร์ประยุกต์ เช่น physics (especially, mechanics), chemistry (chemical equilibrium), economics (market equilibrium) เป็นต้น

ซึ่งต่อมานักคณิตศาสตร์ได้สนใจศึกษาปัญหาสภาพคงที่ (EP) กันอย่างต่อเนื่อง โดยได้ศึกษาร่วมกับปัญหาจุดตรึง (fixed point problems) ซึ่งนับได้ว่าเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน (functional analysis) ในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการ และการพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ ๆ ที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่าง ๆนั้นแล้ว

บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ ทางวิทยาศาสตร์พื้นฐาน (basic science) อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ทฤษฎีจุดตรึงนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับ **การมีคำตอบของสมการต่าง ๆ** (existence of solution) และ **การมีเพียงคำตอบเดียว** ของสมการ (uniqueness of solution) ตลอดจนการคิดค้นหาวิธีในการประมาณหาคำตอบของสมการต่างๆ ดังนั้นการศึกษาทฤษฎีต่างๆ ที่เกี่ยวข้องกับการมีจุดตรึงของการส่งแบบไม่เป็นเชิงเส้นต่างๆ และการหาระเบียบวิธีต่างๆ ที่ใช้ในการประมาณค่าคำตอบนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์จำนวนมากให้ความสนใจศึกษาค้นคว้าวิจัย ซึ่งจากการศึกษาที่ผ่านมาพบว่าการศึกษาทฤษฎีจุดตรึงมีความสัมพันธ์ใกล้ชิดต่อการศึกษาปัญหาสภาพคงที่ (EP) โดยได้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้น ระเบียบวิธีการทำซ้ำของจุดตรึง(Fixed-point Iterations) ต่างๆที่ใช้ในการหาคำตอบ และ ประมาณคำตอบ เช่น การทำซ้ำแบบฮาลเพิล (Halpern iterative) วิธีการกระทำซ้ำไฮบริด (hybrid method iterative) ระเบียบวิธีของการกระทำซ้ำแบบมาน หรือ การกระทำซ้ำ 1 ขั้นตอน (Mann iteration or one-step iterations) การกระทำซ้ำแบบอิชิคาวา หรือการกระทำซ้ำ 2 ขั้นตอน (Ishikawa iteration or two-step iterations) เพื่อนำไปประยุกต์ใช้เกี่ยวข้องกับการแก้ปัญหาขีดต่ำสุด (minimization problem (MP)) ปัญหาอสมการแปรผัน (variational inequalities problem (VIP)) ปัญหาความเป็นไปได้แบบคอนเวกซ์ (convex feasibility problem (CFP)) และปัญหาสภาพคงที่ (equilibrium problem (EP))

การศึกษาเกี่ยวกับการกระทำซ้ำของจุดตรึงนั้นมีองค์ประกอบหลักที่สำคัญยิ่งคือ องค์ประกอบในเรื่องของฟังก์ชันหรือบางครั้งเรียกว่า การส่ง (mapping) เพื่อใช้ในกระบวนการพิจารณา การลู่เข้าของการกระทำซ้ำของจุดตรึงไปยังจุดตรึงจุดหนึ่งของการส่ง ซึ่งถูกเรียกว่า จุดตรึง(Fixed point) และตัวอย่างของการส่งแบบต่างที่สำคัญเช่น nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings, k-strict pseudo-contraction mappings, asymptotically k-strict pseudo-contraction mappings เป็นต้น ซึ่งสามารถแสดงความสัมพันธ์ได้ดังนี้



จากตัวอย่างข้างต้นและมีผลงานวิจัยอีกมากมาย ทำให้เกิดความตื่นตัวกันมากในวงการของการวิเคราะห์เชิงฟังก์ชัน โดยเฉพาะในเรื่องของการกระทำซ้ำของจุดตรึง ทำให้นักคณิตศาสตร์ได้ศึกษาทฤษฎีบทที่เกี่ยวกับการกระทำซ้ำของจุดตรึงมากขึ้น โดยได้มีการพัฒนาคิดค้นและวิจัยเกี่ยวกับด้านนี้กันอย่างต่อเนื่อง

จากความสำคัญที่กล่าวมาข้างต้นจะเห็นว่าทั้งการศึกษาความเกี่ยวข้องระหว่างปัญหาสภาพคงที่

(EP) และการศึกษาเกี่ยวกับการหาจุดตรึงของการส่งแบบไม่ขยายถือว่าเป็นการศึกษาที่เป็นประโยชน์อย่างยิ่ง ดังนั้นในโครงการวิจัยนี้ผู้ดำเนินการวิจัยจึงมีความสนใจที่จะศึกษาการทำซ้ำเพื่อให้ได้องค์ความรู้ใหม่ที่น่าสนใจเพื่อใช้ในการแก้ปัญหาสภาพคงที่ (EP) และปัญหาจุดตรึง ตามที่กล่าวไว้ข้างต้น

3. วัตถุประสงค์ของโครงการ

5.1 คิดค้นทฤษฎีและองค์ความรู้ใหม่ๆ เพื่อการค้นหากระบวนวิธีทำซ้ำและพิสูจน์ทฤษฎีการลู่เข้าอย่างอ่อนและอย่างเข้มสำหรับการส่งแบบไม่เป็นเชิงเส้น

5.2 นำผลไปใช้ประยุกต์เกี่ยวกับปัญหาสภาพคงที่ (EP), (VIP) และ (GEP)

4. ระเบียบวิธีวิจัย

8.1 ศึกษาความรู้ต่างๆ เกี่ยวกับคุณสมบัติของปริภูมิบานาคและปริภูมิฮิลเบิร์ตจากเอกสารที่เกี่ยวข้อง

8.2 ค้นคว้าหาเอกสาร ตำรา วารสาร และ เอกสารสิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยที่กำลังดำเนินการวิจัยอยู่จากแหล่งข้อมูลต่างๆ

8.3 โดยการอาศัยความรู้พื้นฐานที่ได้จากการศึกษาตามระเบียบวิธีตามข้อ 8.1 – 8.2 และประสบการณ์ที่ได้จากการแลกเปลี่ยนความคิดเห็นและปรึกษากับนักวิจัยที่มีความเชี่ยวชาญหาแนวทางในการคิดค้นทฤษฎีใหม่ๆ ตามวัตถุประสงค์ที่กำหนดไว้ในหัวข้อ 5.1 และ 5.2

5. แผนการดำเนินงานตลอดโครงการวิจัยและผล (output) ที่จะได้

กิจกรรมและขั้นตอนดำเนินงาน	2553 (6 เดือนแรก)					
	1	2	3	4	5	6
1. ค้นคว้าหาเอกสารที่เกี่ยวข้อง	■	■				
2. ศึกษาพื้นฐานเกี่ยวกับคุณสมบัติการลู่เข้าของวิธีการทำซ้ำ จากเอกสาร ที่เกี่ยวข้องพร้อมทั้งปรึกษานักวิจัยที่ปรึกษา			■	■		
3. คิดค้น และวิจัยเพื่อหาองค์ความรู้ใหม่เกี่ยวกับการลู่เข้าของลำดับเพื่อให้ได้ตามวัตถุประสงค์ 5.1 และ 5.2					■	■
4. รายงานความก้าวหน้าของโครงการใน 6 เดือนแรก						■

กิจกรรมและขั้นตอนดำเนินงาน	2553 (6 เดือนหลัง)					
	7	8	9	10	11	12
1. เดินทางไปหาเอกสารที่เกี่ยวข้อง						
2. คิดค้น และวิจัยเพื่อหาองค์ความรู้ใหม่เกี่ยวกับคุณสมบัติการลู่เข้าของวิธีการทำซ้ำอย่างต่อเนื่องจาก 6 เดือนแรก พร้อมทั้งปรึกษานักวิจัยที่ปรึกษา						
3.เขียน และ พิมพ์ผลงานวิจัยเกี่ยวกับคุณสมบัติการลู่เข้าของวิธีการทำซ้ำที่คิดค้นพร้อมทั้งส่งผลงานเพื่อลงตีพิมพ์ในวารสารนานาชาติ						
4. รายงานความก้าวหน้าของโครงการใน 6 เดือนที่สอง						

ผลงานที่คาดว่าจะได้รับการตีพิมพ์ และ วารสารนานาชาติ ในปี 2551 จำนวน 1 เรื่อง คือ

1. ชื่อเรื่อง "Strong convergence of iterative schemes for generalized equilibrium problems and fixed point problems"

วารสาร " Applied Mathematics and Computation" เป็นวารสารนานาชาติ

มี **Impact Factor : 0.821**

กิจกรรมและขั้นตอนดำเนินงาน	2554 (6 เดือนแรก)					
	1	2	3	4	5	6
1. เดินทางไปหาเอกสาร วารสาร เพิ่มเติม						
2. คิดค้น และวิจัยเพื่อหาองค์ความรู้ใหม่เกี่ยวกับคุณสมบัติการลู่เข้าของวิธีการทำซ้ำเพื่อให้ได้ตามวัตถุประสงค์ 5.1 และ 5.2						
3. เดินทางไปหาเอกสาร วารสาร เพิ่มเติมพร้อมทั้งปรึกษานักวิจัยที่ปรึกษา						
4. รายงานความก้าวหน้าของโครงการใน 6 เดือนแรก						

กิจกรรมและขั้นตอนดำเนินงาน	2554 (6 เดือนที่สอง)					
	7	8	9	10	11	12
1. ศึกษาบทความวิจัยที่เกี่ยวกับคุณสมบัติการลู่เข้าของวิธีการทำซ้ำเพิ่มเติม	■	■				
2. คิดค้น และวิจัยเพื่อหาองค์ความรู้ใหม่เกี่ยวกับคุณสมบัติการลู่เข้าของวิธีการทำซ้ำเพื่อให้ได้ตามวัตถุประสงค์ 5.1 และ 5.2 เพิ่มเติม		■	■	■		
3. เขียนและส่งผลงานเพื่อลงตีพิมพ์ในวารสารนานาชาติ					■	
4. เขียน และ พิมพ์ รายงานฉบับสมบูรณ์ของโครงการ					■	■
5. ส่งรายงานฉบับสมบูรณ์ต่อ สกว.						■

ผลงานที่คาดว่าจะได้รับการตีพิมพ์ และ วารสารนานาชาติ ในปี 2552 จำนวน 2 เรื่องคือ

1. ชื่อเรื่อง "Iterative schemes for generalized equilibrium problems and fixed point problems of asymptotically strict pseudo-contraction mappings"

วารสาร "Computer and Mathematics with Applications" เป็นวารสารนานาชาติ

มี **Impact factor** : **0.720**

2. ชื่อเรื่อง "Convergence theorems of iterative scheme for solving the generalized equilibrium problems in Hilbert spaces"

วารสาร "Applied Mathematical Modelling" เป็นวารสารนานาชาติ

มี **Impact factor** : **0.931**

6. ประโยชน์ที่คาดว่าจะได้รับ

1. ได้ความรู้และทฤษฎีใหม่ๆที่เกี่ยวกับทฤษฎีจุดตรึงทั้งในปริภูมิบานาค และปริภูมิฮิลเบิร์ต
2. การนำไปประยุกต์ใช้สำหรับปัญหาสภาพคงที่ (EP)
3. มีผลงานตีพิมพ์ในระดับนานาชาติเพื่อเป็นการเผยแพร่ผลงานและชื่อเสียงของนักคณิตศาสตร์ ไทย
4. เกิดความร่วมมือและแลกเปลี่ยนทางวิชาการระหว่างนักวิจัยรุ่นใหม่และผู้เชี่ยวชาญที่เป็นนักวิจัยอาวุโสทั้งในประเทศและต่างประเทศเพื่อนำไปสู่การพัฒนาความเป็นเลิศทางวิชาการของวงการคณิตศาสตร์ไทยและการพัฒนาประเทศชาติต่อไปในที่สุด

ผลงานที่คาดว่าจะได้รับการตีพิมพ์ ใน วารสารนานาชาติ ประมาณ 3 เรื่อง คือ

1. ชื่อเรื่อง "Strong convergence of iterative schemes for generalized equilibrium problems and fixed point problems"

วารสาร "Applied Mathematics and Computation" เป็นวารสารนานาชาติ มี **Impact Factor : 0.821**

2. ชื่อเรื่อง “Iterative schemes for generalized equilibrium problems and fixed point problems of Asymptotically strict pseudo-contraction mappings”

วารสาร “Computer and Mathematics with Applications” เป็นวารสารนานาชาติ **มี Impact factor** : **0.720**

3. ชื่อเรื่อง “Convergence theorems of iterative scheme for solving the generalized equilibrium problems in Hilbert spaces”

วารสาร “Applied Mathematical Modelling” เป็นวารสารนานาชาติ มี **Impact factor** : **0.931**

13. งบประมาณ

รายการ	**ปีที่ 1**	**ปีที่ 2**	**รวม**
1. หมวดค่าตอบแทน			
- ค่าตอบแทนหัวหน้าโครงการ(นายอิสระ อินจันทร์)	120,000	120,000	240,000
2. หมวดค่าวัสดุ			
- ค่าวัสดุสำนักงาน เช่น กระดาษ A4 (ปีละ 10 รีม) แฟ้ม ลวดเสียบ ฯ	5,000	5,000	10,000
- ค่าวัสดุคอมพิวเตอร์ เช่น หมึกพิมพ์สำหรับเครื่องพิมพ์เลเซอร์รุ่น รุ่น Lexmark E120 ปีละ 2 ตลับ, แผ่นCD เป็นต้น	15,000	15,000	30,000
3. หมวดค่าใช้สอย			
- ค่าพาหนะเดินทางค้นคว้าเอกสารและปรึกษางานวิจัย อุตรดิตถ์ -กรุงเทพฯ (ค้นคว้าวารสาร ณ หอสมุด จุฬาลงกรมหาวิทยาลัย) และ อุตรดิตถ์ –เชียงใหม่ (ค้นคว้าเอกสารและปรึกษานักวิจัยที่ปรึกษา) และค่าโรงแรมที่พัก อุตรดิตถ์ – พิษณุโลก (ปรึกษานักวิจัยที่ปรึกษา)	30,000	30,000	60,000
- ค่าเดินทางและที่พักเพื่อเข้าร่วมสัมมนาวิชาการคณิตศาสตร์ ปีละ 2 ครั้ง	10,000	10,000	20,000
- ค่าค่าไปรษณีย์ทั้งใน และ ต่างประเทศ	5,000	5,000	10,000
- ค่าสำเนาเอกสารและวารสารงานวิจัยที่เกี่ยวข้อง	15,000	15,000	30,000
4. หมวดค่าจ้าง			
- ค่าตอบแทนผู้พิมพ์งานวิจัย	5,000	5,000	10,000
- ค่าเข้ารูปเล่ม	-	3,000	3,000
รวมงบประมาณโครงการ	205,000	208,000	**413,000**

บทที่ 1

Introduction

การศึกษาเกี่ยวกับทฤษฎีการประมาณจุดตรึงและการประยุกต์ ได้มีผลงานเกี่ยวข้องอย่างมากมาย นับตั้งแต่ ปีค.ศ. 1953 Mann ได้นิยามการหาสูตรของเมตริกโดยวิธีการกระทำซ้ำ ซึ่งวิธีการกระทำซ้ำแบบมานน์ได้มีการศึกษาเพิ่มเติม ในปี ค.ศ. 1970 โดย Dotson และ Senter และ ในปี ค.ศ. 1974 Dotson นอกจากการประมาณจุดตรึงของ การส่งแบบไม่ขยาย (nonexpansive) แล้ววิธีการกระทำซ้ำแบบมานน์ยังมีประโยชน์ในการประมาณจุดตรึงของการส่งแบบไม่เป็นเชิงเส้นอื่นๆ เช่น การส่งแบบการหดเทียมอย่างเข้ม (strongly pseudo-contractive) ต่อมาจะพบว่าลำดับที่เกิดจากการกระทำซ้ำแบบมานน์จะลู่เข้าสู่จุดตรึงในกรณีที่ T เป็นการส่งลิฟชิทท์และการหดเทียมอย่างเข้ม อย่างไรก็ตามถ้า T เป็นการส่งแบบการหดเทียมแล้วลำดับที่เกิดจากการกระทำซ้ำของมานน์อาจจะไม่ลู่เข้าสู่จุดตรึงของ T ดังนั้นจึงเป็นไปได้ที่จะประมาณจุดตรึงของ T ด้วยลำดับที่เกิดจากการกระทำซ้ำแบบอื่นๆ

วิธีการกระทำซ้ำแบบ อิชิคาวา จึงได้ถูกนำเสนอ ในปี ค.ศ. 1974 โดย Ishikawa เพื่อประมาณหาจุดตรึงสำหรับการส่งแบบลิฟชิทท์ การหดเทียม ทั้งนี้เพราะว่าในกรณีที่ T เป็นแค่การส่งหดเทียมการกระทำซ้ำแบบมานน์ไม่สามารถทำให้ลำดับที่เกิดขึ้นลู่เข้าไปยังจุดตรึงของ T ได้

ต่อมาในปี ค.ศ. 1991 Schu ได้ปรับการกระทำซ้ำของมานน์เพื่อประมาณจุดตรึงของการส่งแบบไม่ขยายเชิงเส้นกำกับ (asymptotically nonexpansive) และปรับปรุงการกระทำซ้ำของอิชิคาวาเพื่อประมาณจุดตรึงของการส่งแบบการหดเทียมเชิงเส้นกำกับ จากที่กล่าวมาข้างตอนจะพบว่าลำดับที่เกิดจากการกระทำซ้ำของมานน์จะลู่เข้าสู่จุดตรึงของการส่ง T อย่างอ่อน ในปริภูมิฮิลเบิร์ตด้วย

สำหรับการศึกษาปัญหาสภาพคงที่ (EP) ได้มีผู้ศึกษาอย่างมาก โดยเริ่มศึกษาในปริภูมิฮิลเบิร์ต ในปี ค.ศ. 1994 โดย Blum และ Oettli โดยปัญหาสภาพคงที่สำหรับการส่ง $F: C\times C \to R$ คือ สามารถหา $x\in C$ ซึ่งทำให้

$$F(x,y)\ge 0, \qquad \forall y\in C \tag{1.1}$$

โดยเซตของผลเฉลยในสมการ (1) นี้จะแทนด้วยสัญลักษณ์ EP(F)

ปัญหามากมายในทาง physics, optimization และ economics ที่ยังทำการศึกษาการหาผลเฉลยของสมการ (1) ซึ่งวิธีการหาผลเฉลยของปัญหาสภาพคงที่ได้ถูกศึกษาโดย Flam และ Antipin และต่อมา Combettes และ Hirstoaga ได้ศึกษาการกระทำซ้ำและทฤษฎีการลู่เข้าของการกระทำซ้ำ โดยมีเงื่อนไขว่า $EP(F)\ne\phi$ ต่อมา Tada และ Takahashi ได้ศึกษาทฤษฎีการลู่เข้าอย่างอ่อน (weak convergence) และทฤษฎีการลู่เข้าอย่างเข้ม (strong convergence) ไปยังจุดใน $F(S)\cap EP(F)$ ของการส่ง S แบบไม่ขยาย (nonexpansive) ในปริภูมิฮิลเบิร์ต และ S. Takahashi และ Takahashi ได้ศึกษาการกระทำซ้ำโดยวิธีการประมาณแบบหนืด (viscosity approximation method) สำหรับการส่ง $S: C\to C$ ที่เป็นการส่งแบบไม่ขยายในปริภูมิฮิลเบิร์ต ซึ่ง $F(S)\cap EP(F)\ne\phi$, f contraction โดยที่ $\alpha\in(0,1)$, $x_1\in H$ $\{\alpha_n\}\subset[0,1]$, $\{r_n\}\subset(0,\infty)$ และ

$$\begin{cases} F(u_n,u)+\frac{1}{r_n}\langle u-u_n,u_n-x_n\rangle \geq 0, & \forall u\in C \\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Su_n, & n\geq 1 \end{cases} \tag{1.2}$$

โดยได้พิสูจน์ว่าลำดับ $\{x_n\}$ ที่กำหนดใน (2) ลู่เข้าอย่างเข้มไปยังจุดใน $F(S)\cap EP(F)$ ภายใต้เงื่อนไข

$$C1)\lim_{n\to\infty}\alpha_n=0,\sum_{n=1}^{\infty}\alpha_n=\infty,\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,$$

$$C2)\liminf_{n\to\infty} r_n>0,\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$$

ต่อมา Shang, Su และ Qin [17] ได้ศึกษาการกระทำซ้ำ ดังนี้

$$\begin{cases} F(u_n,u)+\frac{1}{r_n}\langle u-u_n,u_n-x_n\rangle \geq 0, & \forall u\in C \\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_n A)Su_n, & n\geq 1 \end{cases} \tag{1.3}$$

โดยได้แสดงว่าลำดับ $\{x_n\}$ ที่กำหนดใน (3) ลู่เข้าอย่างเข้มไปยังจุด $q\in F(S)\cap EP(F)$

ในปี ค.ศ. 2008 S. Takahashi และ Takahashi ได้ศึกษานัยทั่วไปของปัญหาสภาพคงที่ โดยให้ C เป็นเซตย่อยปิดแบบนูน (closed convex) ของ H ให้ $A:C\to H$ เป็นการส่งไม่เป็นเชิงเส้น แล้วนัยทั่วไปของปัญหาสภาพคงที่ **(generalized equilibrium problems (GEP))** คือ สามารถหา $z\in C$ ซึ่งทำให้

$$F(z,y)+\langle Az,y-z\rangle\geq 0, \quad \forall y\in C, \tag{1.4}$$

เซตของ $z\in C$ จะให้สัญลักษณ์แทนด้วย EP นั่นคือ

$$EP=\{z\in C: F(z,y)+\langle Az,y-z\rangle\geq 0, \quad \forall y\in C\}$$

จากสมการ (4) จะเห็นว่าถ้าให้ $A\equiv 0$ แล้วจะได้ว่าเซต EP จะเป็นเซต $EP(F)$ ในสมการ (1) และถ้าให้ $F\equiv 0$ แล้ว EP จะเป็นเซต $VI(A,C)$ ซึ่ง $VI(A,C)$ คือเซตของ ***อสมการแปรผัน*** **(Variational inequalities (VIP))** นั่นคือหาสมาชิก $z\in C$ ซึ่งทำให้

$$\langle Au,v-z\rangle\geq 0 \quad \text{สำหรับทุกๆ } v\in C \tag{1.5}$$

และสมาชิก $z\in C$ จะเรียกว่า *คำตอบของอสมการแปรผัน* (5)

โดย S. Takahashi และ Takahashi ได้พิสูจน์ว่า สำหรับ $u\in C$, $x_1\in C$ ถ้าให้ S เป็นการส่งแบบไม่ขยาย และให้ $\{x_n\},\{u_n\}\subset C$ เป็นลำดับที่กำหนดโดย

$$\begin{cases} F(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, & \forall y\in C \\ x_{n+1}=\beta_n x_n+(1-\beta_n)S[\alpha_n u+(1-\alpha_n)u_n], & n\geq 1 \end{cases} \tag{1.6}$$

เมื่อ $\{\alpha_n\}\subset[0,1],\{\beta_n\}\subset[0,1]$ และ $\{r_n\}\subset[0,2\alpha]$ และสอดคล้องเงื่อนไข

$$0<c\leq\beta_n\leq d<1, \qquad 0<a\leq r_n\leq b<2\alpha$$

$$\lim_{n\to\infty}(r_n-r_{n+1})=0, \quad \lim_{n\to\infty}\alpha_n=0 \quad \text{และ} \quad \sum_{n=1}^{\infty}\alpha_n=\infty$$

แล้ว ลำดับ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มไปยัง $z=P_{F(S)\cap EP}u$ เมื่อ P เป็น metric projection

ต่อมาในปี 2009 Ceng, Homidan, Ansari และ Yao ได้ศึกษาการกระทำซ้ำสำหรับการส่งแบบ k-strict pseudo-contraction โดยที่ $0\leq k<1$ ของลำดับ $\{x_n\}$ และ $\{u_n\}$ ที่กำหนดโดย สำหรับ $x_1\in C$

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \quad \forall y\in C \\ x_{n+1}=\alpha_n u_n+(1-\alpha_n)Su_n, \quad \forall\, n\ge 1 \end{cases} \tag{1.7}$$

ภายใต้เงื่อนไข ดังนี้

(*i*) $\{\alpha_n\}\subseteq[\alpha,\beta]$ สำหรับบาง $\alpha,\beta\in(k,1)$

(*ii*) $\{r_n\}\subset(0,\infty)$ และ $\liminf_{n\to\infty} r_n>0$

แล้วลำดับ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มไปยังจุดใน $F(S)\cap EP(F)$

จากผลงานวิจัยที่ได้กล่าวมาข้างต้น บทประยุกต์ที่เป็นประเด็นที่น่าสนใจเป็นอย่างยิ่ง ในเรื่องของ Fixed point iterations คือ กระบวนการในการนำการกระทำซ้ำ ไปหาคำตอบ(solutions)ให้กับ **(EP), (VIP)** และ **(GEP)** ซึ่งการศึกษาการกระทำซ้ำแบบต่างๆ เพื่อแก้ปัญหานัยทั่วไปของสภาพคงที่ **(GEP)** ในสมการ (4) นั้นจะทำให้เราสามารถแก้ปัญหาของ **(VIP)** และปัญหา **(EP)** พร้อมกันได้ ดังนั้นผู้วิจัยจึงมีแนวคิดขยายและศึกษาปัญหานัยทั่วไปของสภาพคงที่ **(GEP)** โดยใช้การกระทำซ้ำเป็นเครื่องมือในการแก้ปัญหา ซึ่งจะทำให้ได้ผลงานวิจัยที่มีประโยชน์และนำไปประยุกต์ใช้แก้ปัญหาได้ จึงขอเสนอการกระทำซ้ำแบบใหม่ๆ และทำการพิสูจน์การลู่เข้าของการกระทำซ้ำ สำหรับการส่งแบบต่างๆ เช่น การส่ง k-strict pseudo-contraction, asymptotically k-strict pseudo-contraction เป็นต้น โดยจะขอยกตัวอย่างงานวิจัยที่จะได้ทำการศึกษา ดังนี้

<u>ปัญหาที่ 1</u> จะได้ศึกษาการกระทำซ้ำแบบอิชิคาวา สำหรับการส่งแบบ k-strict pseudo-contraction โดยที่ $0\le k<1$ ของลำดับ $\{x_n\}$ และ $\{u_n\}$ ที่กำหนดโดย สำหรับ $x_1\in C$

$$\begin{cases} F(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \quad \forall y\in C \\ y_n=\alpha_n f(x_n)+(1-\alpha_n)Su_n, \\ x_{n+1}=(1-\beta_n)y_n+\beta_n Sy_n, \quad \forall\, n\ge 1 \end{cases} \tag{1.8}$$

โดยจะพิสูจน์ว่า ภายใต้เงื่อนไข $\{\alpha_n\}\subset[0,1],\{\beta_n\}\subset[0,1]$ และ $\{r_n\}\subset[0,2\alpha]$ และสอดคล้องเงื่อนไข

$\{\alpha_n\}\subseteq[\alpha,\beta]$ สำหรับบาง $\alpha,\beta\in(k,1)$

$\lim_{n\to\infty}\alpha_n=0$, $\quad \sum_{n=1}^{\infty}\alpha_n=\infty$, $\quad \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$

$\liminf_{n\to\infty} r_n>0$, $\quad \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$

$\lim_{n\to\infty}\beta_n=0$, $\quad \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$

แล้วลำดับ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มไปยังจุด $q\in F(S)\cap EP(F)\cap VI(A,C)$

ซึ่งเราจะเห็นว่า ถ้า $A\equiv 0$ และ $\beta_n\equiv o,\forall n$ ในสมการ (8) แล้วจะสามารถลดรูปไปเป็นสมการ (2) นั่นจะทำให้เป็นผลงานที่ครอบคลุมผลงานของ S. Takahashi และ Takahashi

<u>ปัญหาที่ 2</u> จากการศึกษาเราทราบว่าถ้า S เป็นการส่งแบบ k-strict pseudo-contraction แล้ว S จะเป็นการส่งแบบ asymptotically k-strict pseudo-contraction ดังนั้นถ้าเราให้ S ในสมการ (7) เป็นการส่งแบบ asymptotically k-strict pseudo-contraction โดยที่ $0\le k<1$ ของลำดับ $\{x_n\}$ และ $\{u_n\}$ ที่กำหนดโดย สำหรับ $x_1\in C$

$$\begin{cases} F(u_n, y) + \langle Ax_n, y-u_n \rangle + \dfrac{1}{r_n}\langle y-u_n, u_n - x_n \rangle \geq 0, & \forall y \in C \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)S^n u_n, & \forall\, n \geq 1 \end{cases} \qquad (1.9)$$

โดยจะพิสูจน์ว่า ภายใต้เงื่อนไข $\{\alpha_n\} \subset [0,1]$ และ $\{r_n\} \subset [0,2\alpha]$ และสอดคล้องเงื่อนไข

$\{\alpha_n\} \subseteq [\alpha, \beta]$ สำหรับบาง $\alpha, \beta \in (k,1)$

$\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$

$\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$

แล้วลำดับ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มไปยังจุด $q \in F(S) \cap EP(F)$

ซึ่งเราจะเห็นว่า ถ้า $A \equiv 0$ ในสมการ (9) แล้วจะสามารถลดรูปไปเป็นสมการ (7) นั่นจะทำให้เป็นผลงานที่ครอบคลุมผลงานของ Ceng, Homidan, Ansari และ Yao

นอกจากปัญหาทั้งสองข้อที่ได้กล่าวมาข้างต้นนั้น ผู้วิจัยยังจะได้ศึกษาการกระทำซ้ำหลาย ๆแบบ และศึกษาภายใต้การส่งหลาย ๆแบบ เพื่อให้ได้กระบวนการทำซ้ำไปช่วยแก้ปัญหา **(EP), (VIP)** และ **(GEP)** ได้ดียิ่งขึ้น

จากข้างต้นจะเห็นได้ว่าการศึกษาเกี่ยวกับ**(EP), (VIP)** และ **(GEP)** จะทำให้ ได้ผลลัพธ์คลุมผลงานวิจัยหลาย ๆ ชิ้น และที่สำคัญคือการนำไปประยุกต์ทางด้าน mechanics, physics, optimization and control, nonlinear programming and applied sciences และเพื่อให้ได้งานวิจัยที่เป็นองค์ความรู้ใหม่ ๆ ต่อวงการคณิตศาสตร์ไทย

บทที่ 2

Preliminaries

In this chapter, we give some defnitions, notations, and some useful results that will be used in the later chapters.

2.1 Basic results.

Definition 2.1.1. Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ denoted for either $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a *norm* on X if it satisfies the following conditions:

(i) $\|x\| \geq 0$, $\forall x \in X$;

(ii) $\|x\| = 0 \Leftrightarrow x = 0$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$;

(iv) $\|\alpha x\| = |\alpha|\|x\|$, $\forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.1.2. Let X be a linear space over the field $\mathbb{K}$. A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ that assigns each ordered pair (x, y) of vectors in X to a scalar $\langle x, y \rangle$ is said to be an *inner product* on X if it satisfies the following conditions:

(i) $\langle x, x \rangle \geq 0, \quad \forall\ x \in X$ and $< x, x >= 0 \Leftrightarrow x = 0$;

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}, \quad \forall\ x, y \in X$;

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \quad \forall\ x, y \in X$ and $\forall \alpha \in \mathbb{K}$;

(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \quad \forall\ x, y, z \in X$.

Definition 2.1.3. A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or convergent in norm) if there exists $x \in X$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0 \text{ denoted by } x_n \to x.$$

Definition 2.1.4. A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there exists an element $x \in X$ such that

$$\lim_{n \to \infty} f(x_n) = f(x),$$

for all $f \in X^*$ where X^* is the dual space. Denoted by $x_n \rightharpoonup x$ or $\omega - \lim_{n \to \infty} x_n = x$.
It is clear that strong convergence implies weak convergence. And in a finite dimension normed space, weak convergence implies strong convergence.

Definition 2.1.5. A norm space X is said to be a *complete norm space* if every Cauchy sequence in X is a convergent sequence in X.

Definition 2.1.6. A complete norm linear space over the field $\mathbb{K}$ is called a *Banach space* over $\mathbb{K}$.

Definition 2.1.7. A subset C of a linear space X over the field $\mathbb{K}$ is *convex* if for any $x, y \in C$ implies

$$M = \{z \in X : z = \alpha x + (1-\alpha)y, 0 < \alpha < 1\} \subset C.$$

(M is called *closed segment with boundary point* x, y) or a subset C of X is *convex* if every $x, y \in C$ the segment joining x and y is contained in C.

Definition 2.1.8. A subset M of X is said to be *weakly compact* if every sequence $\{x_n\}$ in M contains a subsequence converging weakly to some point in M.

Theorem 2.1.9. Let $\{x_n\}$ be a sequence in extended real numbers and let $b = \limsup_{n\to\infty} x_n$. Then

$$(1) \quad r > b \Rightarrow x_n < r \quad ultimately;$$

$$(2) \quad r < b \Rightarrow x_n > r \quad frequently.$$

Ultimately means from some index onward ; *frequently* means for infinitely many indices.

Theorem 2.1.10. Let $\{x_n\}$ be a sequence in extended real numbers and let $c = \liminf_{n\to\infty} x_n$. Then

$$(1) \quad r < c \Rightarrow x_n > r \quad ultimately;$$

$$(2) \quad r > c \Rightarrow x_n < r \quad frequently.$$

Definition 2.1.11. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \text{ for all } x, y \in C.$$

Definition 2.1.12. Let X be a Banach space and let C be a nonempty subset of X. A mapping $T : C \to C$. A mapping T is called an *asymptotically nonexpansive mapping in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

Definition 2.1.13. Let C be a nonempty subset of a real normed space X. Let $P : X \to C$ be a *nonexpansive retraction* of X onto K i.e.,

$$\|Px - Py\| \le \|x - y\|$$

for all $x, y \in X$ and $Px = x$ for all $x \in C$, then C is said to be nonexpansive retract.

Definition 2.1.14. A mapping $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in (0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{2.1.1}$$

Definition 2.1.15. A mapping $T : C \to C$ is an asymptotically k-strict pseudo-contractive mapping if there exists a constant $0 \leq k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{2.1.2}$$

for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\gamma_n \geq 0$ for all n such that $\lim_{n\to\infty} k_n = 1$.

Definition 2.1.16. Let X be a Banach space. An element $x \in X$ is said to be a $fixed\ point$ of a mapping $T : X \to X$ if $Tx = x$.

Definition 2.1.17. A mapping $f : C \to C$ is *demiclosed* at y if for each $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $f(x_n) \to y$, then $f(x) = y$.

Definition 2.1.18. Let M be the set a mapping $f : M \to \mathbb{R}$ is $weak\ \ lower\ \ semi-continuous$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \text{ whenever } x_n \rightharpoonup x \text{ in } M.$$

Recall also that a one-parameter family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) $Lipschitian\ semigroup$ on C (see, e. g., [?]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$,

(ii) $T(t + s)x = T(t)T(s)x,\ t, s \geq 0, x \in C$,

(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,

(iv) there exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|,\ x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called $nonexpansive$ (or $a\ contraction\ semigroup$) if $L_t = 1$ for all $t > 0$, and $asymptotically\ nonexpansive$ if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

2.2 Useful lemmas.

The following lemmas will be useful for proving the convergence result of this research.

Lemma 2.2.1. Let H be a real Hilbert space. Then for any $x, y \in H$ we have

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$

(ii) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$

(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iv) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2,\ \forall t \in [0, 1]$.

Lemma 2.2.2. [13, 14] Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,

$$a_{n+1} \leq (1 - \gamma_n)a_n + b_n,\ n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$, and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that:

i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following condition:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y \in C$,

$$\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.2.3. [1] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{ for all } y \in C.$$

The following lemma was also given in [5].

Lemma 2.2.4. [1, 5, 11] Assume that $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4), and let $r > 0$ and $x \in H$. Define a mapping $T_r : H \to C$ as follows"

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. T_r is single- valued;
2. T_r is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$, for any $x, y \in H$;
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex;
5. $\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$, for all $s, t > 0$ and $x \in H$.

Lemma 2.2.5. Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows.

$$T_r(x) = \left\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:

(1) For each $x \in H$, $T_r \neq \emptyset$;

(2) T_r is single-valued;

(3) T_r firmly nonexpansive, i.e, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) $\text{Fix}(T_r) = MEP(F, \varphi)$ is closed and convex.

We also need the following lemmas.

Lemma 2.2.6. (see [24]) Let $\{x_n\}$ and $\{w_n\}$ be bounded sequence in a Banach space, let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ for all $n = 1, 2, \ldots$ Suppose that $x_{n+1} = (1 - \beta_n)w_n + \beta_n x_n$ for all $n = 1, 2, \ldots$ and $\limsup_{n\to\infty}(\|w_{n+1} - w_n\| + \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|w_n - x_n\| = 0$.

Lemma 2.2.7. Let C be a nonempty closed convex subset of real Hilbert space H and $T : C \to C$ be a k-strictly pseudocontractive mapping. Define a mapping $S : C \to C$ by $Sx = \delta x + (1 - \delta)Tx$ for all $x \in C$ and $\delta \in [k, 1)$. Then S is nonexpansive mapping such that $F(S) = F(T)$.

Definition 2.2.8. [20] Let C be nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudocontractive mappings of C into itself. For each $i = 1, 2, \ldots, N$, we define a mapping $S_i = \delta_i I + (1 - \delta_i)T_i$ where $\delta_i \in [k_i, 1)$ consider mapping K_n defined by

$$\begin{aligned}
U_{n,n+1} &= I \\
U_{n,n} &= \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I, \\
U_{n,n-1} &= \gamma_{n-1} S_{n-1} U_{n,n} + (1 - \gamma_{n-1})I, \\
&\vdots \\
U_{n,k} &= \gamma_k S_k U_{n,k+1} + (1 - \gamma_k)I, \\
U_{n,k-1} &= \gamma_{k-1} S_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \\
&\vdots \\
U_{n,2} &= \gamma_2 S_2 U_{n,3} + (1 - \gamma_2)I, \\
K_n = U_{n,1} &= \gamma_1 S_1 U_{n,2} + (1 - \gamma_1)I,
\end{aligned}$$

where $\gamma_1, \gamma_2, ...$ are real number such that $0 \leq \gamma_n \leq 1$.

As regards K_n, we have the following lemmas which are important for prove our main results.

Lemma 2.2.9. [20] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_i be nonexpansive mapping of C into itself such that $\cap_{i=1}^{\infty} F(S_i \neq \emptyset$ and $\gamma_1, \gamma_2, ...$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, ...$ Then for any $x \in C$ and $k \in \mathbb{N}$, the *limit* $\lim_{n\to\infty} U_{n,k}$ exists.

Using Lemma 2.2.9, one can define the mapping K of C into itself as follows:

$$Kx := \lim_{n\to\infty} K_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C.$$

Such a mapping K is called the modified K-mapping generated by $T_1, T_2, ..., \gamma_1, \gamma_2, ...$ and $\delta_1, \delta_2, ...$

Lemma 2.2.10. [20] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_i be nonexpansive mapping of C into itself such that $\cap_{i=1}^{\infty} F(S_i \neq \emptyset$ and $\gamma_1, \gamma_2, ...$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, ...$ Then $F(K) = \bigcap_{i=1}^{\infty} F(S_i)$.

Combining Lemma 2.2.7-2.2.10, one can get that $F(K) = \bigcap_{i=1}^{\infty} F(S_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Lemma 2.2.11. [20] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_i be nonexpansive mapping of C into itself such that $\cap_{i=1}^{\infty} F(S_i \neq \emptyset$ and $\gamma_1, \gamma_2, ...$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, ...$ If K is any bounded subset of C, then

$$\lim_{n\to\infty} \sup_{x\in K} \|K_n x - Kx\| = 0.$$

Lemma 2.2.12. [21] Let C be a nonempty closed convex subset of real Hilbert space H and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. For $r > 0$ anf $x \in C$, define a mapping $S_r : C \to C$ as follows:

$$S_r(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{2}\langle y - z, z - x\rangle \geq 0, \forall y \in C\} \tag{2.2.1}$$

for all $x \in C$. Assume that the condition $(H1) - (H5)$ hold. Then one has the following results:

(1) for each $x \in C, S_r(x) \neq \emptyset$ and S_r is single valued;

(2) S_r is firmly nonexpansive, that is, for any $x, y \in C$,

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle; \tag{2.2.2}$$

(3) $F(S_r) = EP$;

(4) EP is closed and convex.

Lemma 2.2.13. [36] Let $R : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a Lipschitz-continuous mapping. Then the mapping $(R + B) : H \to 2^H$ is maximal monotone.

Lemma 2.2.14. [37] Let $a_n, b_n,$ and c_n be three nonnegative real sequences satisfying

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $t_n \in [0, 1)$ with $\Sigma_{n=1}^{\infty} t_n = +\infty, b_n = o(t_n)$ and $\Sigma_{n=1}^{\infty} c_n < +\infty,$ then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2.15. [39] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2.16. [56] Let C be a nonempty subset of a Hilbert space H and let $T : C \to C$ a uniformly continuous asymptotically k-strict pseudo-contractive in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Lemma 2.2.17. [56] Let C be a nonempty closed convex subset of Hilbert space H and $T : C \longrightarrow C$ a continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that $\{x_n\}$ is sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{n\to\infty} \|x_n - T^n x_n\| = 0$, then $(I - T)x = 0$.

Remark 2.2.18. We note that if A is a α-inverse-strongly monotone, for all $u, v \in C$ and $\lambda_n > 0$,

$$\begin{aligned} \|(I - \lambda_n A)u - (I - \lambda_n A)v\|^2 &= \|(u - v) - \lambda_n(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda_n\langle u - v, Au - Av\rangle \\ &\quad +\lambda_n^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{2.2.3}$$

So, if $\lambda_n \leq 2\alpha$, then $I - \lambda_n A$ is a nonexpansive mapping from C to H.

Lemma 2.2.19. [67] Let $T : K \to H$ be a k-strictly pseudo-contraction. Defined $D : K \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in K$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.

บทที่ 3

Main Results

3.1 Hybrid extragradient method for general equilibrium problems and fixed point problems in Hilbert space

In this section, we show a strong convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem in a Hilbert space by using the hybrid extragradient method.

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{3.1.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{3.1.2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{3.1.3}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \lambda > 0. \tag{3.1.4}$$

A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, map of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v := \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$.

Theorem 3.1.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively.

Let S be a nonexpansive mappings from C into itself such that $F(S) \cap VI(C, A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $C_1 = C \subset H$, $x_1 = P_C x_0$;

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S(\beta_n x_n + (1 - \beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \ n \in \mathbb{N}, \end{cases} \tag{3.1.5}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:

(i) $0 < a_1 \leq \alpha_n, \beta_n \leq a_2 < 1$,

(ii) $0 \leq b \leq \lambda_n \leq c < \alpha < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$, for some $a, b, c, d, e \in \mathbb{R}$,

(iii) $\liminf_{n\to\infty} \lambda_n > 0$ and $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)\cap EP} x_0$.

Proof. We first show that $F(S) \cap EP \cap VI(A, C) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$, we can prove by induction. It is obvious that $F(S) \cap EP \cap VI(A, C) \subset C_1$. Let $p \in F(S) \cap VI(C, A) \cap EP$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2.4. Since $p \in VI(A, C)$, then $p = P_C(p - \lambda_n A p) = T_{r_n} p$ and $u_n = T_{r_n}(x_n - r_n B x_n)$. From (i), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n B x_n) - T_{r_n}(p - r_n B p)\|^2 \\ &\leq \|(x_n - r_n B x_n) - (p - r_n B p)\|^2 \\ &\leq \|(x_n - p) + r_n(Bp - Bx_n)\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2 \qquad (3.1.6) \\ &\leq \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bp - Bx_n\|^2 \\ &\leq \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bp\|^2 \\ &\leq \|x_n - p\|^2. \qquad (3.1.7) \end{aligned}$$

Put $v_n = P_C(u_n - \lambda_n Ay_n)$. From (3.1.3) and the monotonicity of A, we have

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|u_n - \lambda_n Ay_n - p\|^2 - \|u_n - \lambda_n Ay_n - v_n\|^2 \\
&= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle Ay_n, p - v_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - v_n\|^2 \\
&\quad +2\lambda_n(\langle Ay_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle + \langle Ay_n, y_n - v_n\rangle) \\
&\leq \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\
&\quad +2\lambda_n\langle Ay_n, y_n - v_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +2\langle u_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle.
\end{aligned}$$

Moreover, from $y_n = P_C(u_n - \lambda_n Au_n)$ and (3.1.2), we have

$$\langle u_n - \lambda_n Au_n - y_n, v_n - y_n\rangle \leq 0. \tag{3.1.8}$$

Since A is α-inverse strongly then A is also k-Lipschitz-continuous ($k = \frac{1}{\alpha}$), from (ii) we see that $\lambda_n < \frac{1}{k}$, it follows that

$$\begin{aligned}
\langle u_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, v_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n\rangle \\
&\leq \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n\rangle \\
&\leq \lambda_n k\|u_n - y_n\|\|v_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\
&\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad +\lambda_n^2 k^2\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\
&= \|u_n - p\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.1.9) \\
&\leq \|u_n - p\|^2, \qquad (3.1.10)
\end{aligned}$$

and hence

$$\|v_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.1.11}$$

Setting $w_n = \beta_n x_n + (1 - \beta_n)v_n$. Thus, from (3.1.11) we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|\beta_n x_n + (1 - \beta_n)v_n - p\|^2 \\
&= \|\beta_n(x_n - p) + (1 - \beta_n)(v_n - p)\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - v_n\|^2 \qquad (3.1.12) \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 \qquad (3.1.13) \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|x_n - p\|^2 \\
&= \|x_n - p\|^2. \qquad (3.1.14)
\end{aligned}$$

It follows that,

$$\begin{aligned}\|z_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sw_n - p\|^2 \\ &= \|\alpha_n(x_n - p) + (1-\alpha_n)(Sw_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sw_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sw_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sw_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \qquad (3.1.15) \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2. \qquad (3.1.16)\end{aligned}$$

So, we have $p \in C_{n+1}$ and hence

$$F(S) \cap VI(C,A) \cap EP \subset C_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \qquad (3.1.17)$$

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $c_j \in C_{m+1} \subset C_m$ with $c_j \to z$. Since C_m is closed, $z \in C_m$ and $\|z_m - c_j\| \leq \|c_j - x_m\|$. Then

$$\begin{aligned}\|z_m - z\| &= \|z_m - c_j + c_j - z\| \\ &\leq \|z_m - c_j\| + \|c_j - z\|\end{aligned} \qquad (3.1.18)$$

Taking $j \to \infty$,

$$\|z_m - z\| \leq \|z - x_m\|.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|z_m - x\| \leq \|x - x_m\|, \|z_m - y\| \leq \|y - x_m\|$, we have

$$\begin{aligned}\|z_m - z\|^2 &= \|z_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|\alpha(z_m - x) + (1-\alpha)(z_m - y)\|^2 \\ &= \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|(z_m - x) - (z_m - y)\|^2 \\ &\leq \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|y - x\|^2 \\ &\leq \alpha\|x_m - x\|^2 + (1-\alpha)\|x_m - y\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\ &= \|x_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|x_m - z\|^2. \qquad (3.1.19)\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{u_n\}$ are well-defined.

Since $F(S) \cap VI(C,A) \cap EP$ is a nonempty closed convex subset of H, there exists a unique $u \in F(S) \cap VI(C,A) \cap EP$ such that

$$u = P_{F(S)\cap VI(C,A)\cap EP}\, x_0.$$

From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0,$$

for all $y \in C_n$. Since $F(S) \cap VI(C,A) \cap EP \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0, \tag{3.1.20}$$

for all $u \in F(S) \cap VI(C,A) \cap EP$ and $n \in \mathbb{N}$. So, for $u \in F(S) \cap VI(C,A) \cap EP$, we have

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - u\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle, \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|. \end{aligned} \tag{3.1.21}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F(S) \cap VI(C,A) \cap EP \text{ and } n \in \mathbb{N}. \tag{3.1.22}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0, \tag{3.1.23}$$

for all $n \in \mathbb{N}$. So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - x_{n+1}\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle, \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned} \tag{3.1.24}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.1.25}$$

From (3.1.22) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From (3.1.11) and (3.1.14), $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are also bounded. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.1.23) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned} \tag{3.1.26}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.1.27}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{3.1.28}$$

Hence,

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0. \tag{3.1.29}$$

From $x_{n+1} = P_{C_{n+1}}x_0$, we obtain

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for all $z \in C_{n+1}$ and all $n \in \mathbb{N}$. Since $u \in F(S) \cap VI(C, A) \cap EP \subset C_{n+1}$ we have

$$\|x_{n+1} - x_0\| \leq \|u - x_0\| \tag{3.1.30}$$

all $n \in \mathbb{N} \cup \{0\}$. Since $x_{n+1} \in C_n$, we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (3.1.27), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - z_n\| = 0. \tag{3.1.31}$$

Since

$$\|x_n - z_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Sw_n\| = (1-\alpha_n)\|x_n - Sw_n\|,$$

it follows by (3.1.31) that

$$\lim_{n\longrightarrow\infty}\|x_n - Sw_n\| = 0. \tag{3.1.32}$$

From (3.1.12), (3.1.10), (3.1.6), we have

$$\begin{aligned}
\|w_n - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2] \\
&= \|x_n - p\|^2 - (1-\beta_n)2r_n\langle x_n - p, Bx_n - Bp\rangle + (1-\beta_n)r_n^2\|Bp - Bx_n\|^2 \\
&\leq \|x_n - p\|^2 - (1-\beta_n)2r_n\beta\|Bx_n - Bp\| + (1-\beta_n)r_n^2\|Bp - Bx_n\|^2 \\
&= \|x_n - p\|^2 - (1-\beta_n)r_n(2\beta - r_n)\|Bp - Bx_n\|^2, \qquad (3.1.33)
\end{aligned}$$

and from (3.1.15), (3.1.33),we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|w_n - p\|^2 \qquad (3.1.34) \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - (1-\beta_n)r_n(2\beta - r_n)\|Bp - Bx_n\|^2] \\
&= \|x_n - p\|^2 - (1-\alpha_n)(1-\beta_n)r_n(2\beta - r_n)\|Bp - Bx_n\|^2. \qquad (3.1.35)
\end{aligned}$$

and hence

$$\begin{aligned}\|Bp - Bx_n\|^2 &\leq \frac{1}{(1-\alpha_n)(1-\beta_n)r_n(2\beta - r_n)}\|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \frac{1}{(1-\alpha_n)(1-\beta_n)r_n(2\beta - r_n)}\|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \quad (3.1.36)\end{aligned}$$

From (3.1.31), $\liminf_{n\longrightarrow\infty} r_n > 0$, (i) and (ii), we obtain

$$\lim_{n\to\infty}\|Bp - Bx_n\| = 0. \quad (3.1.37)$$

We note from the proof of Theorem 3.1 [9], that $I - r_nB$ is nonexpansive, for all $n \in \mathbb{N}$. Since T_{r_n} is firmly nonexpansive and using Lemma 2.2.4 (2) and (3.1.6), we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\|^2 \\ &\leq \langle (x_n - r_nBx_n) - (p - r_nBp), u_n - p\rangle \\ &= \frac{1}{2}(\|(x_n - r_nBx_n) - (p - r_nBp)\|^2 + \|u_n - p\|^2 \\ &\quad -\|(x_n - r_nBx_n) - (p - r_nBp) - (u_n - p)\|^2) \\ &\leq \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Bx_n - Bp)\|^2) \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2).\end{aligned}$$

Thus, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2. \quad (3.1.38)$$

From (3.1.15), (3.1.13), (3.1.11) and (3.1.38) we can calculate

$$\begin{aligned}\|z_n - p\|^2 &= \|\alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2] \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 \\ &\quad +(1-\alpha_n)(1-\beta_n)[\|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2] \quad (3.1.39) \\ &\leq \|x_n - p\|^2 - (1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2 \\ &\quad +(1-\alpha_n)(1-\beta_n)2r_n\|x_n - u_n\|\|Bx_n - Bp\|, \quad (3.1.40)\end{aligned}$$

by (i), it follows that

$$\begin{aligned}(1-a_2)(1-a_2)\|x_n - u_n\|^2 &\leq (1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2 \\ &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\quad +(1-\alpha_n)(1-\beta_n)2r_n\|x_n - u_n\|\|Bx_n - Bp\| \\ &\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) \\ &\quad +(1-\alpha_n)(1-\beta_n)2r_n\|x_n - u_n\|\|Bx_n - Bp\|.\end{aligned}$$

From (3.1.31), (3.1.37), (i) and bounded of $\{x_n\}$ and $\{u_n\}$, we have

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0. \tag{3.1.41}$$

Since $\liminf_{n\longrightarrow\infty} r_n>0$, we obtain

$$\lim_{n\longrightarrow\infty}\|\frac{x_n-u_n}{r_n}\|=\lim_{n\longrightarrow\infty}\frac{1}{r_n}\|x_n-u_n\|=0. \tag{3.1.42}$$

Next, we show that $\lim_{n\to\infty}\|u_n-y_n\|=0$. Consider,

$$\begin{aligned}
\|y_n-p\|^2 &= \|P_C(u_n-\lambda_nAu_n)-P_C(p-\lambda_nAp)\|^2\\
&\leq \|(u_n-\lambda_nAu_n)-(p-\lambda_nAp)\|^2\\
&= \|(u_n-p)-\lambda_n(Au_n-Ap)\|^2\\
&= \|u_n-p\|^2-\lambda_n\langle u_n-p,Au_n-Ap\rangle+\lambda_n^2\|Au_n-Ap\|^2\\
&\leq \|x_n-p\|^2-2\lambda_n\beta\|Au_n-Ap\|^2+\lambda_n^2\|Au_n-Ap\|^2\\
&= \|x_n-p\|^2+\lambda_n(\lambda_n-2\alpha)\|Au_n-Ap\|^2.
\end{aligned}$$

From (3.1.15), (3.1.12), (3.1.11) we have

$$\begin{aligned}
\|z_n-p\|^2 &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2\\
&\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\beta_n\|x_n-p\|^2+(1-\beta_n)\|v_n-p\|^2-\beta_n(1-\beta_n)\|x_n-v_n\|^2]\\
&\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-v_n\|^2]\\
&\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-v_n\|^2]\\
&= \|x_n-p\|^2-(1-\alpha_n)\beta_n(1-\beta_n)\|x_n-v_n\|^2,
\end{aligned} \tag{3.1.43}$$

from (i), it follows that

$$\begin{aligned}
(1-a_2)a_1(1-a_2)\|x_n-v_n\|^2 &\leq (1-\alpha_n)\beta_n(1-\beta_n)\|x_n-v_n\|^2\\
&\leq \|x_n-p\|^2-\|z_n-p\|^2\\
&\leq \|x_n-z_n\|(\|x_n-p\|+\|z_n-p\|).
\end{aligned}$$

From (3.1.31) and (ii), we have

$$\lim_{n\longrightarrow\infty}\|x_n-v_n\|=0. \tag{3.1.44}$$

For $p \in F(S) \cap VI(C,A) \cap EP$, from (3.1.15), (3.1.13), (3.1.11), (3.1.9) we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2] \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)\|v_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)[\|u_n - p\|^2 \\
&\quad +(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2] \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)[\|x_n - p\|^2] \\
&\quad +(1-\alpha_n)(1-\beta_n)(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)[\|x_n - p\|^2] \\
&\quad +(1-\alpha_n)(1-\beta_n)(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|u_n - y_n\| &\leq \frac{1}{(1-\alpha_n)(1-\beta_n)(1-\lambda_n^2k^2)}(\|x_n - p\|^2 - \|z_n - p\|^2) \\
&= \frac{1}{(1-\alpha_n)(1-\beta_n)(1-\lambda_n^2k^2)}(\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|) \\
&\leq \frac{1}{(1-\alpha_n)(1-\beta_n)(1-\lambda_n^2k^2)}\|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

So, by (3.1.31) we obtain

$$\lim_{n\longrightarrow\infty} \|u_n - y_n\| = 0. \tag{3.1.45}$$

Since $\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\|$, from (3.1.41) and (3.1.45) we also have

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{3.1.46}$$

We note that $\|y_n - v_n\| \leq \|y_n - x_n\| + \|x_n - v_n\|$. From (3.1.44) and (3.1.46), we obtain

$$\lim_{n\longrightarrow\infty} \|v_n - y_n\| = 0. \tag{3.1.47}$$

Since

$$\begin{aligned}
\|w_n - x_n\| &= \|\beta_n x_n + (1-\beta_n)v_n - x_n\| \\
&= (1-\beta_n)\|v_n - x_n\|.
\end{aligned}$$

From (3.1.44), we have

$$\lim_{n\longrightarrow\infty} \|w_n - x_n\| = 0. \tag{3.1.48}$$

Note that

$$\begin{aligned}
\|Su_n - u_n\| &\leq \|Su_n - Sw_n\| + \|Sw_n - x_n\| + \|x_n - u_n\| \\
&\leq \|u_n - w_n\| + \|Sw_n - x_n\| + \|x_n - u_n\| \\
&\leq \|u_n - x_n\| + \|x_n - w_n\| + \|Sw_n - x_n\| + \|x_n - u_n\| \\
&= 2\|x_n - u_n\| + \|x_n - w_n\| + \|Sw_n - x_n\|.
\end{aligned}$$

From (3.1.32), (3.1.41) and (3.1.48), we obtain

$$\lim_{n\longrightarrow\infty}\|Su_n - u_n\| = 0. \tag{3.1.49}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup w$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. Since C is closed and convex, $w \in C$. We first show that $w \in EP$. It follows by (3.1.53) and (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}) \tag{3.1.50}$$

Put $y_t = ty + (1-t)p$ for all $t \in (0, 1]$ and $y \in C$. Thus, we have $y_t \in C$. So, from (3.1.41), we have

$$\langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, By_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \tfrac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y, u_{n_i})$$

and hence

$$\begin{aligned}\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \tfrac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y, u_{n_i})\\ &= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \langle y - u_{n_i}, \tfrac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, it follows that $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from monotonicity of B, we get

$$\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0.$$

So, from (A4), we have

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{3.1.51}$$

as $i \to \infty$. From (A1), (A4) and (3.1.51), we have

$$\begin{aligned}0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle\\ &\leq tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle\end{aligned}$$

and hence $0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle$. Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Bw\rangle$. This implies that $w \in EP$. Next, we show that $w \in F(S)$. Assume that $w \neq Sw$. From Opial's condition and $\|Su_{n_i} - u_{n_i}\| \to 0$, we have

$$\begin{aligned}\liminf_{i\to\infty}\|u_{n_i} - w\| &< \liminf_{i\to\infty}\|u_{n_i} - Sw\| = \liminf_{i\to\infty}\|(u_{n_i} - Su_{n_i}) + (Su_{n_i} - Sw)\|\\ &= \liminf_{i\to\infty}\|Su_{n_i} - Sw\| \leq \liminf_{i\to\infty}\|u_{n_i} - w\|.\end{aligned}$$

This is a contradiction. So, we have $w \in F(S)$. Therefore $w \in F(S) \cap EP$. Finally, we can show that $w \in VI(C, A)$. Define,

$$Tv = \begin{cases} Av + N_C v, & v \in C,\\ \emptyset, & v \notin C.\end{cases}$$

Then, T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, u - Av\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n Ay_n)$, we have $\langle v - v_n, v_n - (u_n -$

$\lambda_n Ay_n)\rangle \geq 0$, and hence, $\left\langle v - v_n, \frac{v_n - u_n}{\lambda_n} + Ay_n \right\rangle \geq 0$. Therefore, we have

$$\begin{aligned}
\langle v - v_{n_i}, u \rangle &\geq \langle v - v_{n_i}, Av \rangle \\
&\geq \langle v - v_{n_i}, Av \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\
&= \left\langle v - v_{n_i}, Av - Ay_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i} \rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle,
\end{aligned}$$

Since $\lim_{n \longrightarrow \infty} \|v_n - u_n\| = \lim_{n \longrightarrow \infty} \|v_n - y_n\| = 0$, $u_{n_i} \rightharpoonup p$ and A is Lipschitz continuous, we obtain that $\lim_{n \longrightarrow \infty} \|Av_n - Ay_n\| = 0$ and $v_{n_i} \rightharpoonup p$. From $\liminf_{n \longrightarrow \infty} \lambda_n > 0$ and $\lim_{n \longrightarrow \infty} \|v_n - u_n\| = 0$, we obtain

$$\langle v - z, u \rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Hence, we have $z \in F(S) \cap VI(C, A) \cap EP(F)$. Finally, we show that $x_n \longrightarrow z$, where $z = P_{F(S) \cap VI(C,A) EP(F)} x_0$. Since $x_n = P_{C_n} x_0$ and $z \in F(S) \cap VI(C, A) \cap EP(F) \subset C_n$, we have $\|x_n - x_0\| \leq \|z - x_0\|$. It follows from $z' = P_{F(S) \cap VI(C,A) \cap EP(F)} x_0$ and the lower semicontinuity of the norm that

$$\|z' - x_0\| \leq \|z - x_0\| \leq \text{liminf}_{i \longrightarrow \infty} \|x_{n_i} - x_0\| \leq \text{limsup}_{i \longrightarrow \infty} \|x_{n_i} - x_0\| \leq \|z' - x_0\|.$$

Thus, we obtain that $\lim_{k \longrightarrow \infty} \|x_{n_i} - x_0\| = \|z - x_0\| = \|z' - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\text{lim}_{i \longrightarrow \infty} x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_{F(T) \cap VI(C,A) \cap EP(F)} x_0$. □

Theorem 3.1.2. [2] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone of C into H and let S be a nonexpansive mappings from C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$, and $\{y_n\}$ be sequences generated by

$$\begin{cases}
x_1, u \in C \text{ chosen arbitrary,} \\
y_n = P_C(x_n - \lambda_n A x_n), \\
x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n x_n + (1 - \alpha_n) P_C(x_n - \lambda_n A y_n)), \forall n \geq 1,
\end{cases} \tag{3.1.52}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 1)$ satisfy the following condition:

(C1) $\lim_{n \to \infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

(C3) $\{\frac{\lambda_n}{\alpha}\} \subset (\tau, 1 - \delta)$ for some $\tau, \delta \in (0, 1)$ and $\lim_{n \to \infty} \lambda_n = 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(C,A)} u$.

Theorem 3.1.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse-strongly monotone mapping of C into H and let T be a strictly k-pseudocontractive mapping of C itself. Let S be a nonexpansive mappings from C into itself such that $F(S) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $C_1 = C \subset H$, $x_1 = P_C x_0$;

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle (I-T)x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S(\beta_n x_n + (1-\beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_n+1} x_0, \ \ n \in \mathbb{N}, \end{cases} \tag{3.1.53}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1], \{\lambda_n\} \subset (0, 2\alpha)$ and satisfy the following conditions:

(i) $0 < a \leq \alpha_n, \beta_n \leq b < 1$,

(ii) $0 \leq c \leq \lambda_n \leq d < \alpha$, for some $a, b, c, d \in \mathbb{R}$,

(iii) $\liminf_{n\to\infty} \lambda_n > 0$ and $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap EP} x_0$.

Proof A strictly k-pseudocontractive mapping is $\frac{1-k}{2}$-inverse strongly monotone. So, from Theorem 3.1.53, we obtain the desired result. □

3.2 Extragradient Method for Generalized Mixed Equilibrium Problems and Fixed Point Problems of Finite Family of Nonexpansive Mapping

In this section, we derive a strong convergence of an iterative algorithm of extragradient viscosity method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings and the set of the variational inequality for a monotone, k-Lipschitz continuous mapping in a Hilbert space.

Theorem 3.2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifucntion satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a paper lower semicontinuous and convex function. Let F be a β-inverse strongly monotone and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{T_1, T_2, \ldots\}$ be a family of infinitely k_i-strictly pseudocontractive mapping of C into itself, such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, ...$ and $\Omega = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \cap GMEP(F, \varphi) \neq \emptyset$. Let $\{S_n\}$ be the S-mapping generated by $\{T_1, T_2, \ldots, T_N\}$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Assume that either $(B1)$ or $(B2)$ holds, let $\{x_n\}$, $\{u_n\}$ and

$\{y_n\}$ be sequences generated by;

$$\begin{cases} x_1 = x \in C \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \gamma_n Au_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)S_n P_C(u_n - \gamma_n Ay_n) \end{cases} \tag{3.2.1}$$

where f is contraction of C into itself with $\alpha \in (0, 1)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $\{r_n\} \subseteq (0, \infty)$ are satisfy the following condition:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\lim_{n\to\infty} \gamma_n = 0$;

(v) $0 < r \leq 2\beta$;

(vi) $|\alpha_n^{(n+1)j} - \alpha_1^{nj}| \to 0$ and $|\alpha_3^{(n+1)j} - \alpha_3^{nj}| \to 0 \quad as \quad n \to \infty$, for all $j \in \{1, 2, \ldots, N\}$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ generated by (3.2.1) converges strongly to $z_0 \in \Omega$, where $z_0 = P_\Omega f(z_0)$.

Proof. Put $t_n = P_C(u_n - \gamma_n Ay_n)$ for every $n \in N$. First, we prove that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ax_n\}$, $\{Au_n\}$, $\{f(x_n)\}$ and $\{Ay_n\}$ are bounded. Let $x^* \in \Omega$ and let $\{T_{r_n}\}$ be a sequences of mapping defined as in Lemma 2.2.5 the $x^* = T_r(x^* - rFx^*)$. From $u_n = T_r(x_n - rFx_n)$, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_r(x_n - rFx_n) - T_r(x^* - rFx^*)\|^2 \\ &\leq \|(x_n - rFx_n) - (x^* - rFx^*)\|^2 \\ &= \|(x_n - x^*) - r(Fx_n - Fx^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2r\langle Fx_n - Fx^*, x_n - x^*\rangle + r^2\|Fx_n - Fx^*\|^2 \\ &\leq \|x_n - x^*\|^2 - 2r\beta\|Fx_n - Fx^*\|^2 + r^2\|Fx_n - Fx^*\|^2 \\ &\leq \|x_n - x^*\|^2 + r(r - 2\beta)\|Fx_n - Fx^*\|^2 \qquad (3.2.2) \\ &\leq \|x_n - x^*\|^2 . (since\ r < 2\beta) \qquad (3.2.3) \end{aligned}$$

From (3.1.3) the monotonicity of A and $x^* \in VI(C, A)$, we have

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - \gamma_n Ay_n - x^*\|^2 - \|u_n - \gamma_n Ay_n - t_n\|^2 \\
&= \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\gamma_n \langle Ay_n, x^* - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\gamma_n(\langle Ay_n - Ax^*, x^* - y_n\rangle \\
&\quad + \langle Ax^*, x^* - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\leq \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\gamma_n \langle Ay_n, y_n - t_n\rangle \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\quad + 2\gamma_n \langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \gamma_n Ay_n - y_n, t_n - y_n\rangle
\end{aligned}$$

Further, from $y_n = P_C(u_n - \gamma_n Au_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned}
\langle u_n - \gamma_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \gamma_n Au_n - y_n, t_n - y_n\rangle + \langle \gamma_n Au_n - \gamma_n Ay_n, t_n y_n\rangle \\
&\leq \langle \gamma_n Au_n - \gamma_n Ay_n, t_n - y_n\rangle \\
&\leq \gamma_n k \|u_n - y_n\| \|t_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\gamma_n k\|u_n - y_n\|\|t_n - y_n\| \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \gamma_n^2 k^2 \|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 + (\gamma_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\leq \|u_n - x^*\|^2, \quad (\gamma_n^2 k^2 - 1 \leq 0). \qquad (3.2.4)
\end{aligned}$$

From (3.2.1), (3.2.3) and (3.2.4), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n t_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + (1 - \alpha_n - \beta_n)\|S_n t_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + (1 - \alpha_n - \beta_n)\|t_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - x^*\| \\
&\leq \alpha_n \alpha \|x_n - x^*\| + \alpha_n \|f(x_n) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - x^*\| + \alpha_n(1 - \alpha_n)\frac{\|f(x^*) - x^*\|}{(1 - \alpha)} \\
&\leq \max\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\} \\
&\quad \vdots \\
&\leq \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\}.
\end{aligned}$$

Hence $\{x_n\}$ is bounded, we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded. From $y_n = P_C(u_n - \gamma_n Au_n)$ and the monotonicity and the Lipschitz continuous of A, we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P_C(u_n - \gamma_n Au_n) - P_C(x^* - \gamma_n Ax^*)\|^2 \\ &\leq \|u_n - \gamma_n Au_n - (x^* - \gamma_n Ax^*)\|^2 \\ &= \|u_n - x^*\|^2 - 2\gamma_n\langle Au_n - Ax^*, u_n - x^*\rangle + \gamma_n^2\|Au_n - Ax^*\|^2 \\ &\leq \|u_n - x^*\|^2 + \gamma_n^2 k\|u_n - x^*\|^2 \\ &= (1 + \gamma_n^2 k)\|u_n - x^*\|^2 \qquad (3.2.5)\end{aligned}$$

Hence, we obtain that $\{y_n\}$ is bounded, it follows that $\{Ax_n\}$, $\{Au_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$ are bounded. Now we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Consider,

$$\begin{aligned}\|t_{n+1} - t_n\| &= \|P_C(u_{n+1} - \gamma_{n+1}Ay_{n+1}) - P_C(u_n - \gamma_n Ay_n)\| \\ &\leq \|(u_{n+1} - \gamma_{n+1}Ay_{n+1}) - (u_n - \gamma_n Ay_n)\| \\ &\leq \|(u_{n+1} - u_n) - \gamma_{n+1}Ay_{n+1} - \gamma_{n+1}Ay_n + \gamma_{n+1}Ay_n + \gamma_n Ay_n\| \\ &\leq \|u_{n+1} - u_n\| + \gamma_{n+1}\|Au_{n+1} - Au_n\| + \gamma_{n+1}\|Au_n\| + \gamma_n\|Au_n\| \\ &\leq \|u_{n+1} - u_n\| + \gamma_{n+1}\|u_{n+1} - u_n\| + \gamma_{n+1}\|Au_n\| + \gamma_n\|Au_n\| \\ &\leq \|u_{n+1} - u_n\| + (2\gamma_{n+1} + \gamma_n)M_1, \qquad (3.2.6)\end{aligned}$$

when $M_1 \geq \sup\{k\|u_{n+1} - u_n\| + \|Au_n\|\}$. Since F is β-inverse-strongly monotone and $r < 2\beta$, we have for all $x, y \in C$

$$\begin{aligned}\|(I - rF)x - (I - rF)y\|^2 &= \|(x - y) - r(Fx - Fy)\|^2 \\ &= \|x - y\|^2 - 2r\langle Fx - Fy, x - y\rangle + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r\beta\|Ax - Ay\|^2 + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + r(r - 2\beta)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2,\end{aligned}$$

then $I - rF$ is nonexpansive. From $u_n = T_r(x_n - rFx_n)$, we get

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|T_r(x_{n+1} - rFx_{n+1}) - T_r(x_n - rFx_n)\| \\ &\leq |(I - rF)x_{n+1} - (I - rF)x_n\| \\ &= \|x_{n+1} - x_n\|. \qquad (3.2.7)\end{aligned}$$

From (3.2.6) and (3.2.7), we obtain that

$$\begin{aligned}\|t_{n+1} - t_n\| &\leq \|x_{n+1} - x_n\| + (2\gamma_{n+1} + \gamma_n)M_1 \\ &= \|x_{n+1} - x_n\| + c_n, \qquad (3.2.8)\end{aligned}$$

where $c_n := (2\gamma_{n+1} + \gamma_n)M_1$. Since $\lim_{n\to\infty}\gamma_n = 0$, we have $\lim_{n\to\infty} c_n = 0$. Next, we show

$$\lim_{n\to\infty}\|S_{n+1}x_n - S_n x_n\| = 0.$$

For $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}
\|U_{n+1,k}x_n - U_{n,k}x_n\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k}x_n + \alpha_2^{n+1,k}U_{n+1,k-1}x_n + \alpha_3^{n+1,k}x_n \\
&\quad -\alpha_1^{n,k}T_kU_{n,k-1}x_n - \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_3^{n,k}x_n\| \\
&= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}x_n - T_kU_{n,k-1}x_n) \\
&\quad +(\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}x_n \\
&\quad +(\alpha_3^{n+1,k} - \alpha_3^{n,k})x_n + \alpha_2^{n+1,k}(U_{n+1,k-1}x_n - U_{n,k-1}x_n) \\
&\quad +(\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}x_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}x_n\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| \\
&= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| \\
&\quad +|\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k}x_n\| \\
&\leq \|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |(\alpha_1^{n,k} - \alpha_1^{n+1,k} + \alpha_3^{n,k} - \alpha_3^{n+1,k})|\|U_{n,k-1}x_n\| \\
&\leq \|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |\alpha_1^{n,k} - \alpha_1^{n+1,k}|\|U_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n,k} - \alpha_3^{n+1,k}|\|U_{n,k-1}x_n\| \\
&= \|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_kU_{n,k-1}x_n\| + \|U_{n,k-1}x_n\|) \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|(\|x_n\| + \|U_{n,k-1}x_n\|) \qquad (3.2.9)
\end{aligned}$$

By (3.2.9), we obtain that for each $n \in \mathbb{N}$

$$\begin{aligned}
\|S_{n+1}x_n - S_nx_n\| &= \|U_{n+1,N}x_n - U_{n,N}x_n\| \\
&\leq \|U_{n+1,1}x_n - U_{n,1}x_n\| + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x_n\| \\
&\quad +\|U_{n,j-1}x_n\|) + \sum_{j=2}^{N}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|x_n\| + \|U_{n,j-1}x_n\|) \\
&= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1x_n - x_n\| + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x_n\| \\
&\quad +\|U_{n,j-1}x_n\|) + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|x_n\| + \|U_{n,j-1}x_n\|)
\end{aligned}$$

From condition (iv) $[|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ and $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0]$, we obtain

$$\lim_{n\to\infty}\|S_{n+1}x_n - S_nx_n\| = 0. \qquad (3.2.10)$$

Similarly

$$\lim_{n\to\infty} \|S_{n+1}t_n - S_n t_n\| = 0. \tag{3.2.11}$$

Let $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$, then we have

$$\begin{aligned} z_n &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)S_n t_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n)S_n t_n}{1-\beta_n}. \end{aligned}$$

We consider,

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})S_{n+1}t_{n+1}}{1-b_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n)S_n t_n}{1-\beta_n} \\ &= (\frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n} f(x_n)) + S_{n+1}t_{n+1} - S_n t_n \\ &\quad + \frac{\alpha_n}{1-\beta_n} S_n t_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}} S_{n+1}t_{n+1} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - S_{n+1}t_{n+1}) + \frac{\alpha_n}{1-\beta_n}(S_n t_n - \alpha_n f(x_n) + S_{n+1}t_{n+1} - S_n t_n). \end{aligned}$$

Then from (3.2.8), we have

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - S_{n+1}t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|S_n t_n - f(x_n)\| + \|S_{n+1}t_{n+1} - S_n t_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - S_{n+1}t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|S_n t_n - f(x_n)\| \\ &\quad + \|S_{n+1}t_{n+1} - S_{n+1}t_n\| + \|S_{n+1}t_n - S_n t_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - S_{n+1}t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - S_n t_n\| \\ &\quad + \|t_{n+1} - t_n\| + \|S_{n+1}t_n - s_n t_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - S_{n+1}t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - S_n t_n\| \\ &\quad + \|S_{n+1}t_n - s_n t_n\| + \|x_{n+1} - x_n\| + c_n. \end{aligned}$$

It follow that

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - S_{n+1}t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - S_n t_n\| \\ &\quad + \|S_{n+1}t_n - s_n t_n\| + c_n \end{aligned}$$

From (i), (3.2.11) and $\lim_{n\to\infty} c_n = 0$, we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.2.6, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Note that

$$\|x_{n+1} - x_n\| = \|\beta_n x_n + (1-\beta_n)z_n - x_n\| = (1-\beta_n)\|z_n - x_n\|$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0. \tag{3.2.12}$$

Consider,

$$\begin{aligned}\|S_nt_n-x_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-S_nt_n\| \\ &\le \|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-x_n\|+\beta_n\|S_nt_n-x_n\|,\end{aligned}$$

it follows that

$$(1-\beta_n)\|S_nt_n-x_n\| \le \|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-x_n\|$$

and hence

$$\|S_nt_n-x_n\| \le \frac{1}{1-\beta_n}(\|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-x_n\|)$$

From $(i),(ii)$ and (3.2.12), we obtain

$$\lim_{n\to\infty}\|S_nt_n-x_n\| = 0. \tag{3.2.13}$$

Next, we prove that

$$\lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}\|u_n-t_n\| = \lim_{n\to\infty}\|u_n-y_n\| = 0.$$

For $x^*\in\Omega$ we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_nf(x_n)+\beta_nx_n+(1-\alpha_n-\beta_n)S_nt_n-x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|S_nt_n-x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|t_n-x^*\| \\ &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)[\|u_n-x^*\|^2 \\ &\quad +(\gamma_n^2k^2-1)\|u_n-y_n\|^2] \\ &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|x_n-x^*\|^2 \\ &\quad +(1-\alpha_n-\beta_n)(\gamma_n^2k^2-1)\|u_n-y_n\|^2,\end{aligned} \tag{3.2.14}$$

it follows that

$$\begin{aligned}\|u_n-y_n\|^2 &\le \frac{\alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)} \\ &= \frac{\alpha_n(\|f(x_n)-x^*\|^2-\|x_n-x^*\|^2)}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)}+\frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)} \\ &\le \frac{\alpha_n(\|f(x_n)-x^*\|^2-\|x_n-x^*\|^2)}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)}+\frac{\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)}.\end{aligned}$$

From (i) and (3.2.12), we have

$$\lim_{n\to\infty}\|u_n-y_n\| = 0. \tag{3.2.15}$$

By the same argument as in (3.2.3), we also have

$$\begin{aligned}\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\gamma_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + \gamma_n^2 k^2\|t_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\gamma_n^2 k^2 - 1) - \|y_n - t_n\|^2. \qquad (3.2.16)\end{aligned}$$

From (3.2.14) and (3.2.16), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)S_n t_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|S_n t_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|t_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)[\|u_n - x^*\|^2 \\ &\quad +(\gamma_n^2 k^2 - 1)\|y_n - t_n\|^2] \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 \\ &\quad +(1 - \alpha_n - \beta_n)(\gamma_n^2 k^2 - 1)\|y_n - t_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|y_n - t_n\|^2 &\leq \frac{\alpha_n(\|f(x_n) - x^*\|^2 - \|x_n - x^*\|^2)}{(1 - \gamma_n^2 k^2)} + \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{(1 - \gamma_n^2 k^2)} \\ &\leq \frac{\alpha_n(\|f(x_n) - x^*\|^2 - \|x_n - x^*\|^2)}{(1 - \gamma_n^2 k^2)} + \frac{\|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|)}{(1 - \gamma_n^2 k^2)}.\end{aligned}$$

From (i) and (3.2.12), we obtain

$$\lim_{n\to\infty} \|y_n - t_n\| = 0. \qquad (3.2.17)$$

Note that

$$\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\| \to 0 \quad as \quad n \to \infty. \qquad (3.2.18)$$

From A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| = 0$. From (??) and (3.3.32), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + (1 - \alpha_n - \beta_n)(S_n t_n - x^*)\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|S_n t_n - x^*\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|t_n - x^*\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|u_n - x^*\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)[\|x_n - x^*\|^2 \\ &\quad +r(r - 2\beta)\|Fx_n - Fx^*\|^2] \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)r(r - 2\beta)\|Fx_n - Fx^*\|^2,\end{aligned}$$

it follows that

$$(1-\alpha_n-\beta_n)r(r-2\beta)\|Fx_n-Fx^*\|^2 \leq \alpha_n\|(f(x_n)-x^*)\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2$$
$$\leq \alpha_n\|(f(x_n)-x^*)\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|).$$

From (i) and (3.2.12), we get

$$\lim_{n\to\infty}\|Fx_n-Fx^*\|=0. \tag{3.2.19}$$

Since T_r is a firmly nonexpansive for $x^*\in\Omega$, we have

$$\begin{aligned}
\|u_n-x^*\|^2 &= \|S_r(x_n-rFx_n)-S_r(x^*-rFx^*)\|^2\\
&\leq \langle x_n-rFx_n-(x^*-rFx^*),u_n-x^*\rangle\\
&= \frac{1}{2}(\|x_n-rFx_n-(x^*-rFx^*)\|^2+\|u_n-x^*\|^2\\
&\quad -\|x_n-rFx_n-(x^*-rFx^*)-(u_n-x^*)\|^2)\\
&\leq \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n-r(Fx_n-Fx^*)\|^2)\\
&\leq \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n\|^2+2r\langle(Fx_n-Fx^*),x_n-u_n\rangle\\
&\quad -r^2\|Fx_n-Fx^*\|^2).
\end{aligned}$$

It follows that

$$\|u_n-x^*\|^2\leq\|x_n-x^*\|^2-\|x_n-u_n\|^2+2r\|Fx_n-Fx^*\|\|x_n-u_n\|. \tag{3.2.20}$$

Note that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(f(x_n)-x^*)+\beta_n(x_n-x^*)+(1-\alpha_n-\beta_n)(S_nt_n-x^*)\|^2\\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|S_nt_n-x^*\|^2\\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|\|t_n-x^*\|^2\\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|\|u_n-x^*\|^2\\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)[\|x_n-x^*\|^2\\
&\quad -\|x_n-u_n\|^2+2r\|Fx_n-Fx^*\|\|x_n-u_n\|]\\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+(1-\alpha_n)\|x_n-x^*\|^2-(1-\alpha_n-\beta_n)\|x_n-u_n\|^2\\
&\quad +2r\|Fx_n-Fx^*\|\|x_n-u_n\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-\alpha_n-\beta_n)\|x_n-u_n\|^2 &\leq \alpha_n\|(f(x_n)-x^*)\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad +2r\|Fx_n-Fx^*\|\|x_n-u_n\|\\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2-\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\
&\quad +2r\|Fx_n-Fx^*\|\|x_n-u_n\|.
\end{aligned}$$

From (i) (3.2.12) and (3.2.22) , we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.2.21}$$

Since

$$\begin{aligned}\|S_n u_n - u_n\| &\leq \|S_n u_n - S_n t_n\| + \|S_n t_n - x_n\| + \|x_n - u_n\| \\ &\leq \|u_n - t_n\| + \|S_n t_n - x_n\| + \|x_n - u_n\|.\end{aligned}$$

By (3.2.11), (3.2.16) and (3.2.21), we have

$$\lim_{n\to\infty} \|S_n u_n - u_n\| = 0. \tag{3.2.22}$$

Consider,

$$\|Su_n - u_n\| \leq \|Su_n - S_n u_n\| + \|S_n u_n - u_n\|.$$

By Lemma (2.2.11) and (3.2.22), we get

$$\lim_{n\to\infty} \|Su_n - u_n\| = 0. \tag{3.2.23}$$

Next we show that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle \leq 0,$$

where $z_0 = P_\Omega f(z_0)$. To show this inequality, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle f(z_0) - z_0, x_{n_i} - x^*\rangle = \limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_k}} \rightharpoonup w$. Without loss of generality, assume that $x_{n_i} \rightharpoonup w$. Consider, for all $x, y \in H$,

$$\begin{aligned}\|P_F(I-A)x - P_F(I-A)y\| &\leq \|(I-A)x - (I-A)y\| \\ &\leq \|I-A\|\|x-y\| \\ &\leq (1-\mu)\|x-y\|.\end{aligned}$$

Hence $P_\Omega(I-A)$ is contraction and has a unique fixed point, say $x^* \in \Omega$. That is, $x^* = P_\Omega(I-A)(x^*)$. We next prove that $w \in GMEP$. By $u_n = S_r(x_n - rFx_n)$, we know that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \ \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq \Theta(y, u_n), \ \ \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle \geq \Theta(y, u_{n_i}), \ \ \forall y \in C. \tag{3.2.24}$$

For $t \in (0,1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (3.2.24) we have

$$\begin{aligned}\langle y_t - u_{n_i}, Fy_t\rangle &\geq \langle y_t - u_{n_i}, Fy_t\rangle - \varphi(y_t) + \varphi_{(u_{n_i})} - \langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle + \Theta(y, u_{n_i}) \\ &= \langle y_t - u_{n_i}, Fy_t - Fu_{n_i}, Fu_{n_i} - Fx_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad -\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + \Theta(y, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Fu_{n_i} - Fx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of φ, $\frac{u_{n_i} - x_{n_i}}{r} \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Fy_t \rangle \geq -\varphi(y_t) + \varphi(w) + \Theta(y_t, w), \ \forall y \in C. \tag{3.2.25}$$

From (A1), (A4), and (3.2.25), we also have

$$\begin{aligned}
0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\
&= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[\Theta(y_t, w) + \varphi(w) - \varphi(y_t)] \\
&\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t, w, Fy_t \rangle \\
&\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y, w, Fy_t \rangle
\end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Fy_t \rangle.$$

Letting $t \to 0$, we have, for each $y \in C$

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw \rangle \geq 0.$$

for all $y \in C$ and hence $w \in GMEP(F, \varphi)$.

(b) Now we show that $w \in VI(C, A)$. For this purpose, we define a set-valued mapping $T : H \to 2^H$ by

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1 & \text{if } w_1 \in C, \\ \emptyset & \text{if } w_1 \notin C. \end{cases}$$

where $N_C w_1$ is the normal cone to C at $w_1 \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tw_1$ if and only if $w_1 \in VI(C, A)$. Let $(w_1, g) \in G(T)$. Then $Tw_1 = Aw_1 + N_C w_1$ and hence $g - Aw_1 \in N_C w_1$. So, we have $\langle w_1 - t, g - Aw_1 \rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \gamma_n A y_n)$ and $w_1 \in C$ we have

$$\langle u_n - \gamma_n A y_n - t_n, t_n - w_1 \rangle \geq 0$$

and hence

$$\langle w_1 - t_n, \frac{t_n - u_n}{\gamma_n} + A y_n \rangle \geq 0$$

Therefor, we have

$$\begin{aligned}
\langle w_1 - t_{n_i}, g\rangle &\leq \langle w_1 - t_{n_i}, Aw_1\rangle \\
&\leq \langle w_1 - t_{n_i}, Aw_1\rangle - \langle w_1 - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}} + Ay_{n_i}\rangle \\
&= \langle w_1 - t_{n_i}, Aw_1 - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle \\
&= \langle w_1 - t_{n_i}, Aw_1 - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle \\
&= \langle w_1 - t_{n_i}, Aw_1 - At_{n_i}\rangle + \langle w_1 - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle \\
&\leq \langle w_1 - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle w_1 - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle.
\end{aligned}$$

Hence we obtain $\langle w_1 - w, g\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$.

(C) We next show that $w \in F(S) = \bigcap_{i=1}^{N} F(T_i)$. Suppose the contrary, $w \notin F(S)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Sw$, from the Opial's condition we have

$$\begin{aligned}
\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\| \\
&= \liminf_{i\to\infty} \|u_{n_i} - Su_{n_i} + Su_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \|Su_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \|u_{n_i} - w\|
\end{aligned}$$

which is a contradiction. So, we get $w \in F(S) = \bigcap_{i=1}^{N} F(T_i)$. This implies $w \in \Omega$. Therefor, we have

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{n\to\infty} \langle f(z_0) - z_0, x_{n_i} - z_0\rangle = \langle f(z_0) - z_0, w - z_0\rangle \leq 0. \quad (3.2.26)$$

Consider,

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n + \beta_n) S_n t_n - z_0\|^2 \\
&= \|\alpha_n (f(x_n) - z_0) + \beta_n (x_n - z_0) + (1 - \alpha_n + \beta_n)(S_n t_n - z_0)\|^2 \\
&\leq \|\beta_n (x_n - z_0) + (1 - \alpha_n + \beta_n)(S_n t_n - z_0)\|^2 + 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\beta_n \|x_n - z_0\| + (1 - \alpha_n + \beta_n)\|S_n t_n - z_0\|)^2 + 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\beta_n \|x_n - z_0\| + (1 - \alpha_n + \beta_n)\|t_n - z_0\|)^2 + 2\alpha_n \alpha \|x_n - z_0\| \|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\beta_n \|x_n - z_0\| + (1 - \alpha_n + \beta_n)\|x_n - z_0\|)^2 + 2\alpha_n \alpha \|x_n - z_0\| \|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - z_0\|)^2 + \alpha_n \alpha (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle
\end{aligned}$$

then we obtain

$$(1-\alpha_n\alpha)\|x_{n+1}-z_0\|^2 \leq ((1-\alpha_n)^2+\alpha_n\alpha)\|x_n-z_0\|^2+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle,$$

it follows that

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &\leq \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z_0\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle \\
&= \frac{1-2\alpha_n\alpha_n^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z_0\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle \\
&= \frac{1-2\alpha_n+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z_0\|^2+\frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-z_0\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle \\
&= 1-\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\|x_n-z_0\|^2+\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\{\frac{\alpha_n M}{2(1-\alpha)} \\
&\quad +\frac{1}{(1-\alpha)}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\} \\
&\leq (1-\delta_n)\|x_n-z_0\|^2+\delta_n b_n,
\end{aligned}$$

when $M=\sup\|x_n-z_o\|^2$, $\delta_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}$ and $b_n=\{\frac{\alpha_n M}{2(1-\alpha)}+\frac{1}{(1-\alpha)}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\}$ From $\lim_{n\to\infty}\alpha_n=0$ and $\limsup_{n\to\infty}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\leq 0$, we have $\limsup_{n\to\infty}b_n\leq 0$, and from

$$\sum_{n=1}^{\infty}\alpha_n=\infty,\ \text{implies}\ \sum_{n=1}^{\infty}\delta_n=\infty$$

By Lemma 2.2.2 we have $\lim_{n\to\infty}\|x_n-z_0\|=0$. This complete the proof. □

3.3 Convergence Theorem of a New Iterative Method for Mixed Equilibrium Problems and Variational Inclusions: Approach to Variational Inequalities

In this section, we derive a strong convergence of an iterative algorithm which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points nonexpansive mapping of C into itself and the set of the variational inclusion in Hilbert space.

Theorem 3.3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T:C\to C$ be a nonexpansive mapping such that $F:=F(T)\cap EP\cap I(B,R)\neq\emptyset$. Let $\Theta:C\times C\to\mathbb{R}$ be a bifunction satisfying $(H1)-(H4)$, let F,B be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let A be a strongly positive bounded linear operator with coefficient $0<\mu<1$ and $R:H\to 2^H$ be a maximal monotone mapping. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $x_0=C$;

$$\begin{cases}\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-(x_n-rFx_n)\rangle\geq 0,\ \forall y\in C\\ y_n=P_C[(I-\alpha_nA)J_{R,\lambda}(I-\lambda B)u_n],\\ x_{n+1}=\beta_nu+(1-\beta_n)Ty_n,\ n\geq 0\end{cases}\tag{3.3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ are satisfying: $\{r_n\} \subset (0,\infty)$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \beta_n = 0$, $\Sigma_{n=1}^{\infty}\beta_n = \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(iii) $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$,

(iv) $0 < r \le 2\alpha, 0 < \lambda \le 2\beta$,

Then $\{x_n\}$ converge strongly to $z_0 = P_F u$ which solves the following variational inequality

$$\langle(Ax, y - x\rangle \ge 0, \quad \forall y \in F. \tag{3.3.2}$$

Proof. Since F is α-inverse strongly monotone and B is β-inverse strongly monotone, we have

$$\|(I - rF)x - (I - rF)y\|^2 \le \|x - y\|^2 + r(r - 2\alpha)\|Fx - Fy\|^2, \tag{3.3.3}$$

$$\|(I - \lambda B)x - (I - \lambda B)y\|^2 \le \|x - y\|^2 + \lambda(\lambda - 2\beta)\|Bx - By\|^2. \tag{3.3.4}$$

It is clear that if $0 < r \le 2\alpha$ and $0 < \lambda \le 2\beta$, then $(I - rF)$ and $(I - \lambda B)$ are all nonexpansive. Set $w_n = J_{R,\lambda}(u_n - \lambda Bu_n), n \ge 0$. It follows that

$$\|w_n - x^*\| = \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\| \le \|(u_n - \lambda Bu_n) - (x^* - \lambda Bx^*)\| \le \|u_n - x^*\|. \tag{3.3.5}$$

By Lemma 2.2.1, we have $u_n = S_r(x_n - rFx_n)$ for all $n \ge 0$. Then, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|S_r(x_n - rFx_n) - S_r(x^* - rFx^*)\|^2 \\ &\le \|(x_n - rFx_n) - (x^* - rFx^*)\|^2 \\ &\le \|(x_n - x^*) - r_n(Bx_n - Bx^*)\|^2 \\ &\le \|x_n - x^*\|^2 - 2r\langle Fx_n - Fx^*, x_n - x^*\rangle + r\|Fx^* - Fx_n\|^2 \qquad (3.3.6) \\ &\le \|x_n - x^*\|^2 - 2r\alpha\|Fx_n - Fx^*\|^2 + r^2\|Fx^* - Fx_n\|^2 \\ &\le \|x_n - x^*\|^2. \qquad (3.3.7) \end{aligned}$$

Hence,

$$\|w_n - x^*\| \le \|u_n - x^*\| \le \|x_n - x^*\|. \tag{3.3.8}$$

Since A is linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}.$$

Observe that

$$\langle(I - \alpha_n A)u, u\rangle = 1 - \alpha_n\langle Au, u\rangle \ge 1 - \alpha_n\|A\| \ge 0,$$

that is to say $I - \alpha_n A$ is positive operator. It follows that

$$\begin{aligned} \|(I - \alpha_n A)\| &= \sup\{|\langle(I - \alpha_n A)u, u\rangle| : u \in H, \|u\| = 1\} \\ &= \sup\{\langle(I - \alpha_n A)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1 - \alpha_n\mu. \end{aligned}$$

From (??), we deduce that

$$\begin{aligned}\|y_n - x^*\| &= \|P_C[(I-\alpha_n A)w_n] - x^*\| \\ &\leq \|(I-\alpha_n A)w_n - x^*\| \\ &= \|(I-\alpha_n A)(w_n - x^*) - \alpha_n Ax^*\| \\ &\leq \|I-\alpha_n A\|\|w_n - x^*\| + \alpha_n\|Ax^*\| \\ &\leq (1-\alpha_n\mu)\|w_n - x^*\| + \alpha_n\|Ax^*\| \\ &= (1-\alpha_n\mu)\|w_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu} \\ &\leq (1-\alpha_n\mu)\|x_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu},\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\beta_n u + (1-\beta_n)Ty_n - x^*\| \\ &= \|\beta_n(u - x^*) + (1-\beta_n)(Ty_n - x^*)\| \\ &\leq \beta_n\|u - x^*\| + (1-\beta_n)\|Ty_n - x^*\| \\ &\leq \beta_n\|u - x^*\| + (1-\beta_n)\|y_n - x^*\| \\ &\leq \beta_n\|u - x^*\| + (1-\beta_n)[(1-\alpha_n\mu)\|x_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu}] \\ &\leq \beta_n \max\{\|u - x^*\|, \|x_0 - x^*\|, \frac{\|Ax^*\|}{\mu}\} \\ &\quad +(1-\beta_n)\max\{\|u - x^*\|, \|x_0 - x^*\|, \frac{\|Ax^*\|}{\mu}\} \\ &= \max\{\|u - x^*\|, \|x_0 - x^*\|, \frac{\|\|Ax^*\|}{\mu}\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{u_n\}, \{y_n\}, \{Ty_n\}$ and $\{Ay_n\}$ are all bounded.

Step 2 We must show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (3.3.1), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\beta_n u + (1-\beta_n)Ty_n - (\beta_{n-1}u + (1-\beta_{n-1})Ty_{n-1})\| \\ &= \|(1-\beta_n)(Ty_n - Ty_{n-1}) + (1-\beta_n)Ty_{n-1} + (\beta_n - \beta_{n-1})u \\ &\quad +(1-\beta_{n-1})Ty_{n-1})\| \\ &= \|(1-\beta_n)(Ty_n - Ty_{n-1}) + (\beta_n - \beta_{n-1})(u - Ty_{n-1})\| \\ &\leq (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|. \qquad (3.3.9)\end{aligned}$$

Note that,

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|P_C(I-\alpha_n A)w_n - P_C(I-\alpha_{n-1}A)w_{n-1}\| \\ &\leq \|(I-\alpha_n A)w_n - (I-\alpha_{n-1}A)w_{n-1}\| \\ &= \|(I-\alpha_n A)(w_n - w_{n-1}) + (I-\alpha_n A)w_{n-1} - (I-\alpha_{n-1}A)w_{n-1}\| \\ &\leq (1-\alpha_n\mu)\|w_n - w_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\|, \qquad (3.3.10)\end{aligned}$$

and from $(I-\lambda B)$ and $(I-rF)$ are nonexpansive, we have

$$\begin{aligned}
\|w_n - w_{n-1}\| &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(u_{n-1} - \lambda Bu_{n-1})\| \\
&\leq \|(I-\lambda B)u_n - (I-\lambda B)u_{n-1}\| \\
&= \|u_n - u_{n-1}\| \\
&= \|S_r(x_n - rFx_n) - S_r(x_{n-1} - rFx_{n-1})\| \\
&\leq \|(I-rF)x_n - (I-rF)x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\|. \qquad (3.3.11)
\end{aligned}$$

Substituting (3.3.11) in (3.3.10), we get

$$\|y_n - y_{n-1}\| \leq (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\|, \qquad (3.3.12)$$

and substituting (3.3.12) into (3.3.9), we get

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\| + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|, \quad (3.3.13)$$

and we have

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n\mu)\|x_n - x_{n-1}\| + \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|\mu\frac{\|Aw_{n-1}\|}{\mu} + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|. \quad (3.3.14)$$

Put $t_n := \alpha_n\mu$, $b_n := \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|\mu\frac{\|Aw_{n-1}\|}{\mu}$ and $c_n := |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|$ from $(i), (ii), (iii)$ and bounded of $\{\|u - Ty_{n-1}\|\}$ and by Lemma 2.2.14, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3 Prove that $\|Fx_n - F_{x^*}\| \to 0$ and $\|Bu_n - B_{x^*}\| \to 0$ as $n \to \infty$. Consider

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\leq \|(I-\lambda B)u_n - (I-\lambda B)x^*\|^2 \\
&= \|u_n - x^*\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bx^*\|^2 \\
&= \|x_n - x^*\|^2 + r(r-2\alpha)\|Fx_n - Fx^*\|^2 \\
&\quad +\lambda(\lambda - 2\beta)\|Bu_n - Bx^*\|^2, \qquad (3.3.15) \\
&\leq \|x_n - x^*\|^2, (\text{ since } r < 2\alpha \text{ and } \lambda < 2\beta). \qquad (3.3.16)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(I-\alpha_n A)w_n - x^*\|^2 \\
&\leq \|(I-\alpha_n A)w_n - x^*\|^2 \\
&= \|w_n - x^* - \alpha_n Aw_n\|^2 \\
&= \|w_n - x^*\|^2 - 2\alpha_n\langle w_n - x^*, Aw_n\rangle + \alpha_n\|Aw_n\|^2 \qquad (3.3.17) \\
&= \|w_n - x^*\|^2 + \alpha_n(2\|w_n - x^*\|\|Aw_n\| + \|Aw_n\|^2) \\
&= \|w_n - x^*\| + d_n, \qquad (3.3.18)
\end{aligned}$$

where $d_n = \alpha_n(2\|w_n - x^*\|\|Aw_n\| + \|Aw_n\|^2)$. From $\lim_{n\to\infty}\alpha_n = 0$ and boundedness, we have $\lim_{n\to\infty} d_n = 0$, there exists $N \in \mathbb{N}$ such that

$$\|y_n - x^*\|^2 \leq \|w_n - x^*\|^2 \leq \|x_n - x^*\|^2, \forall n \geq N. \qquad (3.3.19)$$

Note that from (3.3.15) and (3.3.18), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\beta_n u+(1-\beta_n)Ty_n-x^*\|^2\\
&= \|\beta_n(u-x^*)+(1-\beta_n)(Ty_n-x^*)\|^2\\
&= \beta_n\|u-x^*\|^2+(1-\beta_n)\|Ty_n-x^*\|^2\\
&\leq \beta_n\|u-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2 \qquad (3.3.20)\\
&\leq \beta_n\|u-x^*\|^2+\|w_n-x^*\|^2+\alpha_n(2\|w_n-x^*\|\|Aw_n\|+\|Aw_n\|^2)\\
&\leq \|x_n-x^*\|^2+r(r-2\alpha)\|Fx_n-Fx^*\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bx^*\|^2\\
&\quad +\beta_n\|u-x^*\|^2+d_n. \qquad (3.3.21)
\end{aligned}$$

It follows that

$$\begin{aligned}
r(2\alpha-r)\|Fx_n-Fx^*\|^2 &+ \lambda(2\beta-\lambda)\|Bu_n-Bx^*\|^2\\
&\leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad +d_n+\beta_n\|u-x^*\|^2\\
&\leq \|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_n-x^*\|)\\
&\quad +d_n+\beta_n\|u-x^*\|^2.
\end{aligned}$$

From $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\beta_n=0$, we have

$$\lim_{n\to\infty}\|Fx_n-Fx^*\|=0=\lim_{n\to\infty}\|Bu_n-Bx^*\|.$$

Next, we show that $\lim_{n\to\infty}\|u_n-x_n\|=0=\lim_{n\to\infty}\|u_n-w_n\|$. Since S_r is a firmly nonexpansive, we have

$$\begin{aligned}
\|u_n-x^*\|^2 &= \|S_r(x_n-rFx_n)-S_r(x^*-rFx^*)\|^2\\
&\leq \langle x_n-rFx_n-(x^*-rFx^*),u_n-x^*\rangle\\
&= \frac{1}{2}(\|x_n-rFx_n-(x^*-rFx^*)\|^2+\|u_n-x^*\|^2\\
&\quad -\|x_n-rFx_n-(x^*-rFx^*)-(u_n-x^*)\|^2)\\
&\leq \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n-r(Fx_n-Fx^*)\|^2)\\
&\leq \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n\|^2\\
&\quad +2r\langle(Fx_n-Fx^*),x_n-u_n\rangle-r^2\|Fx_n-Fx^*\|^2)
\end{aligned}$$

It follows that

$$\|u_n-x^*\|^2\leq\|x_n-x^*\|^2-\|x_n-u_n\|^2+2r\|Fx_n-Fx^*\|\|x_n-u_n\|. \qquad (3.3.22)$$

Since $J_{R,\lambda}$ is $1-$inverse strongly monotone, we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\le \langle (u_n - \lambda Bu_n) - (x^* - \lambda Bx^*), w_n - x^* \rangle \\
&= \frac{1}{2}(\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*)\|^2 + \|w_n - x^*\|^2 \\
&\quad -\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*) - (w_n - x^*)\|^2) \\
&\le \frac{1}{2}(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n - \lambda(Bu_n - Bx^*)\|^2) \\
&\le \frac{1}{2}(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2 \\
&\quad +2\lambda\langle (Bu_n - Bx^*), u_n - w_n \rangle - \lambda^2\|Bu_n - Bx^*\|^2).
\end{aligned}$$

Which implies that

$$\|w_n - x^*\|^2 \le \|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2 + 2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\|. \tag{3.3.23}$$

By (3.3.22) and (3.3.23), we have

$$\begin{aligned}
\|w_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\
&\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\|.
\end{aligned} \tag{3.3.24}$$

Substituting (3.3.24) into (3.3.17), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\
&\quad +2\lambda\|Bu_n - Bx^*)\|\|u_n - w_n\| + d_n \\
&\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\
&\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n.
\end{aligned} \tag{3.3.25}$$

Substituting (3.3.25) into (3.3.20), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\
&\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - u_n\|^2 + \|u_n - w_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\
&\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n + \beta_n\|u - x^*\|^2 \\
&\le \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad +2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\
&\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n + \beta_n\|u - x^*\|^2.
\end{aligned}$$

From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 = \lim_{n\to\infty}\|Fx_n - Fx^*\| = 0 = \lim_{n\to\infty}\|Bu_n - Bx^*\|$, $\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n$ and bounded of sequences, we get

$$\lim_{n\to\infty}\|u_n - x_n\| = 0 = \lim_{n\to\infty}\|u_n - w_n\|.$$

Step 4 Prove that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_F u$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{i\to\infty}\langle u - z_0, x_{n_i} - z_0\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_k}} \rightharpoonup w$. Without loss of generality, assume that $x_{n_i} \rightharpoonup w$. Consider, for all $x, y \in H$,

$$\begin{aligned} \|P_F(I-A)x - P_F(I-A)y\| &\le \|(I-A)x - (I-A)y\| \\ &\le \|I-A\|\|x-y\| \\ &\le (1-\mu)\|x-y\|. \end{aligned}$$

Hence $P_F(I-A)$ is contraction and has a unique fixed point, say $x^* \in F$. That is, $x^* = P_F(I-A)(x^*)$. We next prove that $w \in EP$. By $u_n = S_r(x_n - rFx_n)$, we know that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge 0,\ \ \forall y \in C.$$

It follows from (H2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge \Theta(y, u_n),\ \ \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle \ge \Theta(y, u_{n_i}),\ \ \forall y \in C. \tag{3.3.26}$$

For $t \in (0,1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (3.2.24) we have

$$\begin{aligned} \langle y_t - u_{n_i}, Fy_t\rangle &\ge \langle y_t - u_{n_i}, Fy_t\rangle - \varphi(y_t) + \varphi_{(u_{n_i})} \\ &\quad -\langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle + \Theta(y, u_{n_i}) \\ &= \langle y_t - u_{n_i}, Fy_t - Fu_{n_i}, Fu_{n_i} - Fx_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad -\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + \Theta(y, u_{n_i}). \end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Fu_{n_i} - Fx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of φ, $\frac{u_{n_i} - x_{n_i}}{r} \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Fy_t\rangle \ge -\varphi(y_t) + \varphi(w) + \Theta(y_t, w),\ \ \forall y \in C. \tag{3.3.27}$$

From (H1), (H4), and (3.3.27), we also have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\le t\Theta(y_t, y) + (1-t)\Theta(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[\Theta(y_t, w) + \varphi(w) - \varphi(y_t)] \\ &\le t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t, w, Fy_t\rangle \\ &\le t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y, w, Fy_t\rangle \end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Fy_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw\rangle \geq 0.$$

This implies that $w \in EP$.

Next, we show that $w \in I(B, R)$. In fact, since B is β-inverse strongly monotone, B is Lipschitz continuous monotone mapping. It follows from Lemma 2.2.13 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$ that is, $g - Bv \in R(v)$. Again since $w_{n_i} = J_{R,\lambda}(u_{n_i} - \lambda Bu_{n_i})$, we have $u_{n_i} - \lambda u_{n_i} \in (I + \lambda R)(w_{n_i})$, that is, $(1/\lambda)(u_{n_i} - w_{n_i} - \lambda Bu_{n_i}) \in R(w_{n_i})$. By virtue of the maximal monotonicity of $R + B$, we have

$$\langle v - w_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda Bu_{n_i})\rangle \geq 0,$$

and so

$$\begin{aligned}
\langle v - w_{n_i}, g\rangle &\geq \langle v - w_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda Bu_{n_i}\rangle \\
&= \langle v - w_{n_i}, Bv - Bw_{n_i} + Bw_{n_i} - Bu_{n_i} + \frac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle \\
&\geq \langle v - w_{n_i}, Bw_{n_i} - Bu_{n_i}\rangle + \langle v - w_{n_i}, \frac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle.
\end{aligned}$$

It follows from $\|u_n - w_n\| \to 0, \|Bu_n - Bw_n\| \to 0$ and $w_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \to \infty} \langle v - w_{n_i}, g\rangle = \langle v - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $B + R$ that $\theta \in (R + B)(w)$, that is $w \in I(B, R)$. Next, we can show that $w \in F(T)$. Consider,

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(I - \alpha_n A)w_n - P_C(I - \alpha_n A)x^*\|^2 \\
&\leq \langle (I - \alpha_n A)w_n - (I - \alpha_n A)x^*, y_n - x^*\rangle \\
&= \frac{1}{2}(\|(I - \alpha_n A)w_n - (I - \alpha_n A)x^*\|^2 + \|y_n - x^*\|^2 \\
&\quad - \|(I - \alpha_n A)w_n - (I - \alpha_n A)x^* - (y_n - x^*)\|^2) \\
&\leq \frac{1}{2}(\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2 \\
&\quad + 2\alpha_n\langle w_n - y_n, Aw_n - Ay_n\rangle - \alpha_n^2\|Aw_n - Ay_n\|^2) \\
&\leq \frac{1}{2}(\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2 \\
&\quad + 2\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|),
\end{aligned}$$

it follows that

$$\|w_n - y_n\|^2 \leq \|w_n - x^*\|^2 - \|y_n - x^*\|^2 + \alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|. \tag{3.3.28}$$

From (3.3.20), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|u - x^*\|^2 + (1 - \beta_n)\|y_n - x^*\|^2 \\
&\leq \beta_n\|u - x^*\|^2 + \|y_n - x^*\|^2
\end{aligned}$$

Then, we get

$$- \|y_n - x^*\|^2 \leq \beta_n\|u - x^*\|^2 - \|x_{n+1} - x^*\|^2. \qquad (3.3.29)$$

Replace (3.3.8) and (3.3.29) into (3.3.28), we have

$$\begin{aligned} \|w_n - y_n\|^2 &\leq \|x_n - x^*\|^2 + (\beta_n\|u - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\ &\quad +\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\| \\ &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n\|u - x^*\|^2 \\ &\quad +\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\| \\ &\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\ &\quad +\beta_n\|u - x^*\|^2 + \alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|. \end{aligned}$$

From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0$ and boundedness, we obtain

$$\lim_{n\to\infty} \|w_n - y_n\| = 0.$$

Note that

$$\begin{aligned} \|Tu_n - u_n\| &\leq \|Tu_n - Tw_n\| + \|Tw_n - Ty_n\| + \|Ty_n - x_{n+1}\| \\ &\quad +\|x_{n+1} - x_n\| + \|x_n - u_n\| \\ &\leq \|u_n - w_n\| + \|w_n - y_n\| + \beta_n\|Ty_n - u\| + \|x_{n+1} - x_n\| \\ &\quad +\|x_n - u_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

From $u_{n_i} \rightharpoonup w$ and H satisfying Opial's condition, it is easy to prove that $w \in F(T)$. Therefore, $w \in F$. It follows that

$$\begin{aligned} \limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle &= \lim_{i\to\infty} \langle u - z_0, x_{n_i} - z_0\rangle \\ &= \langle u - z_0, w - z_0\rangle \leq 0. \end{aligned} \qquad (3.3.30)$$

From (??), we have for any $n \geq \mathbb{N}$

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \|\beta_n u + (1 - \beta_n)Ty_n - z_0\|^2 \\ &= \|(1 - \beta_n)(Ty_n - z_0) + \beta_n(u - z_0)\|^2 \\ &\leq (1 - \beta_n)\|Ty_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\leq (1 - \beta_n)\|y_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\leq (1 - \beta_n)\|x_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \end{aligned} \qquad (3.3.31)$$

Since $\Sigma_{n=1}^{\infty}\beta_n = \infty$, (3.3.30) and Lemma 2.2.14, we have $\lim_{n\to\infty} \|x_n - z_0\| = 0$, we get that $\{x_n\}$ converges strongly to $z_0 = P_F u$. □

Corollary 3.3.2. [38] Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\Omega := \cap EP \cap I(B, R) \neq \emptyset$. Let $\Theta : H \times C \to \mathbb{R}$ be a bifunction satisfying $(H1) - (H4)$, let F, B be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $r > 0$ and $\lambda > 0$

be two constants such that $r < 2\alpha$ and $\lambda < 2\beta$. Let A be a strongly positive bounded linear operator with coefficient $0 < \mu < 1$ and $R : H \to 2^H$ be a maximal monotone mapping. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $x_0 = C$;

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n], \end{cases} \tag{3.3.32}$$

where $\{\alpha_n\}[0, 1]$ are satisfying: $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$,

Then $\{x_n\}$ converge strongly to $x^* \in \Omega$ which solves the following variational inequality

$$\langle (Ax, y - x\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.3.33}$$

Proof. In Theorem 3.3.1, let $T = I$ and $\beta_n = 0$ for all $n \in \mathbb{N}$, then, we can obtain Corollary. This completes the proof. □

3.4 Strong Convergence Theorem by Hybrid Method for Non-Lipschitzian Mapping

this section, we prove strong convergence theorem by hybrid method for asymptotically k-strict pseudo-contractive mapping in the intermediate sense in Hilbert spaces.

Theorem 3.4.1. Let H be a Hilbert space and let C be a nonempty closed convex bounded subset of H. Let T be a uniformly continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C, C_1 = C, x_1 = P_{C_1}(x_0) \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1 - \alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.4.1}$$

where $\theta_n = (diamC)^2(1 - \alpha_n)\gamma_n \to 0$, $(n \to \infty)$, converges strongly to $z_0 = P_{F(T)}x_0$.

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. For $u \in F(T) \subset C_m$.

By lemma 2.2.1, we have,

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\
&= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 \\
&\quad + k\|x_m - T^m x_m\|^2 + c_m] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + (k - \alpha_m)(1-\alpha_m)\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m
\end{aligned} \tag{3.4.2}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$. Then

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\
&= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\
&\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&\quad + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.
\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, we have

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\
&= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\
&\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad + (1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad - \alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\
&\quad + \theta_m + c_m \\
&= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$. By Lemma 2.2.15, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \text{ for all } y \in C_n.$$

Moreover, by the same proof of Theorem 3.1 of [48], we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.4.3}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n, \tag{3.4.4}$$

By the definition of y_n, we have

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\ &= (1-\alpha_n)\|T^n x_n - x_n\|.\end{aligned}$$

From (3.4.4), we have

$$\begin{aligned}(1-\alpha_n)^2\|T^n x_n - x_n\|^2 &= \|y_n - x_n\|^2 \\ &= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \\ &\leq \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 \\ &\quad +2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 \\ &\quad +\theta_n + c_n + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 \\ &\quad +2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) \\ &\quad +\theta_n + c_n.\end{aligned}$$

It follows that

$$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n.$$

Hence

$$(1 - k - \alpha_n)\|T^n x_n - x_n\| \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n. \tag{3.4.5}$$

From $\limsup_{n\to\infty} \alpha_n < 1 - k$, we can chosen $\epsilon > 0$ such that $\alpha_n \leq 1 - k - \epsilon$ for large enough n. From (3.4.3) and (3.4.5), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.4.6}$$

From (3.4.3), (3.4.6) and Lemma 2.2.16, we have

$$\lim_{n\to\infty} \|T x_n - x_n\| = 0. \tag{3.4.7}$$

Since H is reflexive and $\{x_n\}$ is bounded we get that $\emptyset \neq \omega_w(x_n)$. From Lemma 2.2.17, we have $\omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| \leq \|x_0 - w\| &\leq \liminf_{n\to\infty} \|x_0 - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\begin{aligned}\|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.\end{aligned}$$

This complete the proof. □

3.5 Strong Convergence Theorem by Hybrid Iterative Scheme for Generalized Equilibrium Problems and Fixed Point Problems of Strictly Pseudo-Contraction Mappings

In this section, we prove a strong convergence theorem of the hybrid method for strictly pseudo-contractive mappings in a real Hilbert space.

Theorem 3.5.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \longrightarrow \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive for some $0 \leq k < 1$. Defined a mapping $S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume that $F := F(S) \cap VI(C, A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by

$$\begin{cases} x_1 = C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ w_n = \beta_n S_k y_n + (1-\beta_n) y_n, \\ z_n = \alpha_n x_n + (1-\alpha_n) S_k w_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1 \end{cases} \tag{3.5.1}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:

(i) $k \leq \alpha_n, \beta_n \leq a < 1$,

(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$, for some $a, b, c, d, e \in \mathbb{R}$.

Then $\{x_n\}$ converge strongly to z, where $z = P_F x_1$.

Proof. Let $p \in F$ since $0 \leq r_n < 2\beta$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(p - r_n Bp)\|^2 \\ &\leq \|(x_n - r_n Bx_n) - (p - r_n Bp)\|^2 \\ &\leq \|(x_n - p) - r_n(Bx_n - Bp)\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bp - Bx_n\|^2 \qquad (3.5.2) \\ &\leq \|x_n - p\|^2. \qquad (3.5.3) \end{aligned}$$

First we show that $F \subset C_n$ for all $n \in \mathbb{N}$, we can prove by induction. It is obvious that $F \subset C_1$. Let $p \in F$, we known that $I - \lambda_n A$ is nonexpansive, for all $n \in \mathbb{N}$ and from $p \in VI(C, A)$ we get $p = P_C(p - \lambda_n Ap)$. It follows that

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2 \\ &\leq \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^2 \\ &\leq \|u_n - p\|^2. \qquad (3.5.4) \end{aligned}$$

Consider,

$$\begin{aligned} \|w_n - p\| &= \|\beta_n(S_k y_n - p) + (1 - \beta_n)(y_n - p)\| \\ &\leq \beta_n\|S_k y_n - p\| + (1 - \beta_n)\|y_n - p\| \\ &\leq \beta_n\|y_n - p\| + (1 - \beta_n)\|y_n - p\| \\ &= \|y_n - p\| \\ &= \|u_n - p\| \\ &= \|x_n - p\|. \qquad (3.5.5) \end{aligned}$$

Thus, we have

$$\begin{aligned} \|z_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n)S_k w_n - p\| \\ &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_k w_n - p)\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|S_k w_n - p\| \qquad (3.5.6) \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|w_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= \|x_n - p\|. \qquad (3.5.7) \end{aligned}$$

So, we have $p \in C_{n+1}$ and hence $F \subset C_n$, for all $n \in \mathbb{N}$.

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $c_j \in C_{m+1} \subset C_m$

with $c_j \to z$. Since C_m is closed, $z \in C_m$ and $\|z_m - c_j\| \leq \|c_j - x_m\|$. Then

$$\begin{aligned} \|z_m - z\| &= \|z_m - c_j + c_j - z\| \\ &\leq \|z_m - c_j\| + \|c_j - z\| \end{aligned} \tag{3.5.8}$$

Taking $j \to \infty$,

$$\|z_m - z\| \leq \|z - x_m\|.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|z_m - x\| \leq \|x - x_m\|, \|z_m - y\| \leq \|y - x_m\|$, we have

$$\begin{aligned} \|z_m - z\|^2 &= \|z_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|\alpha(z_m - x) + (1-\alpha)(z_m - y)\|^2 \\ &= \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|(z_m - x) - (z_m - y)\|^2 \\ &\leq \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|y - x\|^2 \\ &\leq \alpha\|x_m - x\|^2 + (1-\alpha)\|x_m - y\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\ &= \|x_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|x_m - z\|^2. \end{aligned} \tag{3.5.9}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_1$, we have

$$\langle x_1 - x_n, x_n - y \rangle \geq 0, \text{ for all } y \in C_n.$$

Since $F \subset C_n$, we obtain

$$\langle x_1 - x_n, x_n - u \rangle \geq 0 \text{ for all } u \in F \text{ and } n \in \mathbb{N}. \tag{3.5.10}$$

So, for $u \in F$, we get

$$\begin{aligned} 0 \leq \langle x_1 - x_n, x_n - u \rangle &= \langle x_1 - x_n, x_n - x_1 + x_1 - u \rangle \\ &= -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - u \rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_1 - x_n\|\|x_1 - u\|. \end{aligned}$$

This implies that

$$\|x_1 - x_n\|^2 \leq \|x_1 - x_n\|\|x_1 - u\|,$$

hence

$$\|x_1 - x_n\| \leq \|x_1 - u\| \quad \text{for all } u \in F \text{ and } n \in \mathbb{N}. \tag{3.5.11}$$

From $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\langle x_1 - x_n, x_n - x_{n+1} \rangle \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.5.12}$$

So, for $x_{n+1} \in C_n$, we also have, for $n \in \mathbb{N}$

$$0 \leq \langle x_1 - x_n, x_n - x_{n+1}\rangle = \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle$$
$$= -\langle x_n - x_1, x_n - x_1\rangle + \langle x_1 - x_n, x_1 - x_{n+1}\rangle$$
$$\leq -\|x_n - x_1\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.$$

This implies that

$$\|x_1 - x_n\|^2 \leq \|x_1 - x_n\|\|x_1 - x_{n+1}\|$$

and we get

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.5.13}$$

From (3.5.11), we have $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.5.12), we note that

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_1) + (x_1 - x_{n+1})\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 - 2\langle x_1 - x_n, x_1 - x_n\rangle - 2\langle x_1 - x_n, x_n - x_{n+1}\rangle \\
&\quad +\|x_1 - x_{n+1}\|^2 \\
&\leq \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2 \\
&= -\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.5.14}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ imply that

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{3.5.15}$$

Further, we get

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

From (3.5.14) and (3.5.15), we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.5.16}$$

Next, we show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. For $p \in \Theta$. From (3.5.5), (3.5.2) and by (ii), we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(S_k w_n - p)\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_k w_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - S_k w_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - S_k w_n\|^2 \qquad (3.5.17) \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 \\
&\quad +r_n^2\|Bp - Bx_n\|^2] \qquad (3.5.18) \\
&= \|x_n - p\|^2 + d(e - 2\beta)\|Bx_n - Bp\|^2,
\end{aligned}$$

and hence

$$\begin{aligned} d(2\beta - e)\|Bx_n - Bp\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &= \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \end{aligned}$$

From (3.5.16), we have $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$. From remark 2.2.18 that for $\lambda_n \le 2\beta$ then $I - r_n B$ is nonexpansive, for all $n \in \mathbb{N}$, T_{r_n} is firmly nonexpansive and by using Lemma 2.2.3, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\|^2 \\ &\le \langle (x_n - r_nBx_n) - (p - r_nBp), u_n - p\rangle \\ &= \frac{1}{2}(\|(x_n - r_nBx_n) - (p - r_nBp)\|^2 + \|u_n - p\|^2 \\ &\quad -\|(x_n - r_nBx_n) - (p - r_nBp) - (u_n - p)\|^2) \\ &\le \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Bx_n - Bp)\|^2) \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2). \end{aligned}$$

Thus, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2. \tag{3.5.19}$$

From (3.5.19), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_kw_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_kw_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_kw_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 \qquad (3.5.20) \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2] \qquad (3.5.21) \\ &\le \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bp\|, \end{aligned}$$

it follows that

$$\begin{aligned} (1 - a)\|x_n - u_n\|^2 &\le (1 - \alpha_n)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bp\| \\ &\le \|x_n - z_n\|(\|x_n - p\| - \|z_n - p\|) \\ &\quad +2r_n\|x_n - u_n\|\|Bx_n - Bp\|. \end{aligned}$$

Using (3.5.16) and $\|Bx_n - Bp\| \to 0$, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.5.22}$$

Consider,

$$\begin{aligned}
\|w_n - p\|^2 &= \|\beta_n(S_k y_n - p) + (1-\beta_n)(y_n - p)\|^2 \\
&= \beta_n\|S_k y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&\leq \beta_n\|y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&= \|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2, \qquad (3.5.23)
\end{aligned}$$

From (3.5.20) and (3.5.23) we also have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2] \qquad (3.5.24) \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&\leq \|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-a)k(1-a)\|y_n - S_k y_n\|^2 &\leq (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

From (3.5.16), we have

$$\lim_{n\to\infty} \|S_k y_n - y_n\| = 0. \qquad (3.5.25)$$

Next, we show that $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Consider

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(p - \lambda_n A p)\|^2 \\
&\leq \|(u_n - \lambda_n A u_n) - (p - \lambda_n A p)\|^2 \\
&= \|(u_n - p) - \lambda_n(A u_n - A p)\|^2 \\
&= \|u_n - p\|^2 - \lambda_n\langle u_n - p, A u_n - A p\rangle + \lambda_n^2\|A u_n - A p\|^2 \\
&\leq \|x_n - p\|^2 - 2\lambda_n\alpha\|A u_n - A p\|^2 + \lambda_n^2\|A u_n - A p\|^2 \\
&= \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A u_n - A p\|^2 \\
&\leq \|x_n - p\|^2 + b(c - 2\alpha)\|A u_n - A p\|^2.
\end{aligned}$$

From (3.5.24) and (ii), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + b(c - 2\alpha)\|A u_n - A p\|^2] \\
&\leq \|x_n - p\|^2 + (1-\alpha_n)b(c - 2\alpha)\|A u_n - A p\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-a)b(2\alpha - c)\|A u_n - A p\|^2 &\leq (1-\alpha_n)b(2\alpha - c)\|A u_n - A p\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

From (??), that

$$\lim_{n\to\infty} \|Au_n - Ap\| = 0. \tag{3.5.26}$$

From (3.1.1), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\leq \langle (u_n - \lambda_n Au_n) - (p - \lambda_n Ap), y_n - p\rangle \\
&= \frac{1}{2}\{\|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\|^2 + \|y_n - p\|^2 \\
&\quad -\|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap) - (y_n - p)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) - \lambda_n(Au_n - Ap)\|^2\} \\
&= \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle \\
&\quad -\lambda_n^2\|Au_n - Ap\|^2\},
\end{aligned}$$

so, we obtain

$$\|y_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2. \tag{3.5.27}$$

From (3.5.4), (3.5.24), (3.5.27) and (i) we have

$$\begin{aligned}
\|z_n - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|u_n - p\|^2 - \|u_n - y_n\|^2 \\
&\quad +2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2] \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\|u_n - y_n\|^2 \\
&\quad +(1-\alpha_n)2\lambda_n\|u_n - p\|\|Au_n - Ap\| \\
&\leq \|x_n - p\|^2 - (1-\alpha_n)\|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-a)\|u_n - y_n\|^2 &\leq (1-\alpha_n)\|u_n - y_n\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\| \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2\lambda_n\|u_n - p\|\|Au_n - Ap\|,
\end{aligned}$$

From (i), (??) and (3.5.26), we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.5.28}$$

Next, we show that $\lim_{n\to\infty} \|S_k u_n - u_n\| = 0$, consider

$$\begin{aligned}
\|S_k u_n - u_n\| &\leq \|S_k u_n - S_k y_n\| + \|S_k y_n - y_n\| + \|y_n - u_n\| \\
&\leq 2\|y_n - u_n\| + \|S_k y_n - y_n\|.
\end{aligned}$$

From (3.5.28) and (3.5.25) we obtain that

$$\lim_{n\to\infty} \|S_k u_n - u_n\| = 0. \tag{3.5.29}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup w$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. Since C is closed and convex, $w \in C$. Next, we show that $w \in F$. First, we show that $w \in VI(C, A)$. Define,

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, u - Av\rangle \geq 0$. On the other hand, from $y_n = P_C(u_n - \lambda_n A u_n)$, we have $\langle v - y_n, y_n - (u_n - \lambda_n A u_n)\rangle \geq 0$, that is,

$$\left\langle v - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, u\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle,
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty} \|y_n - u_n\| = 0$ and A is Lipschitz continuous, we obtain

$$\langle v - w, u\rangle \geq 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. Next, we show that $w \in EP$. It follows by (3.1.3) and (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}) \tag{3.5.30}$$

Put $y_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. Since $y \in C$ and $w \in C$, we have $y_t \in C$. So, from (3.5.22), we have

$$\langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, By_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \tfrac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$$

and hence

$$\begin{aligned}
\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \tfrac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \langle y - u_{n_i}, \tfrac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, it follows that $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from monotonicity of B, we get

$$\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0.$$

So, from (A4), we have

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{3.5.31}$$

as $i \to \infty$. From (A1), (A4) and (3.5.31), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle$$
$$\leq tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle$$

and hence $0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle$. Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Bw\rangle$. This implies that $w \in EP$. Next, we show that $w \in F(S)$. From Lemma 2.2.19, we have $F(S_k) = F(S)$, we may assume that $w \neq S_k w$, by Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - S_k w\| = \liminf_{i\to\infty} \|(u_{n_i} - S_k u_{n_i}) + (S_k u_{n_i} - S_k w)\| \\
&= \liminf_{i\to\infty} \|S_k u_{n_i} - S_k w\| \leq \liminf_{i\to\infty} \|u_{n_i} - w\|.
\end{aligned}$$

This is a contradiction. So, we have $w \in F(S_k) = F(S)$. Therefore $w \in F$.

Finally, we show that $x_n \longrightarrow z$, where $z = P_F x_1$. Since $x_n = P_{C_n} x_1$ and $z \in F \subset C_n$, we have

$$\|x_n - x_1\| \leq \|z - x_1\|.$$

It follows from $z' = P_F x_1$ and the lower semicontinuity of the norm that

$$\|z' - x_1\| \leq \|z - x_1\| \leq \liminf_{i\longrightarrow\infty} \|x_{n_i} - x_1\| \leq \limsup_{i\longrightarrow\infty} \|x_{n_i} - x_1\| \leq \|z' - x_1\|.$$

Thus, we obtain that $\lim_{k\longrightarrow\infty} \|x_{n_i} - x_1\| = \|z - x_1\| = \|z' - x_1\|$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_F x_1$. □

Theorem 3.5.2. [61] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \longrightarrow \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive self mapping for some $0 \leq k < 1$. Defined a mapping $S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume that $F := F(S) \cap VI(C, A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $C_1 = C \subset H$, $x_1 = P_C x_0$;

$$\begin{cases}
x_1 \in C, \\
C_1 = C, \\
F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\
y_n = P_C(u_n - \lambda_n A u_n) \\
z_n = \alpha_n x_n + (1-\alpha_n) S_k y_n, \\
C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0
\end{cases} \tag{3.5.32}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:

(i) $k \leq \alpha_n \leq a < 1$,

(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$, for some $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ converge strongly to z, where $z = P_F x_0$.

Proof. If $\beta_n = 0$ for all $n \in \mathbf{N}$, by theorem 3.5.1, we obtain the desired result. □

บทที่ 4

CONCLUSIONS

4.1 Outputs 5 papers (Supported by TRF: MRG5380081)

1. Hybrid extragradient method for general equilibrium problems and fixed point problems in Hilbert space. Nonlinear Analysis: Hybrid Systems, 5 (2011) 467—478.

2. Extragradient Method for Generalized Mixed Equilibrium Problems and Fixed Point Problems of Finite Family of Nonexpansive Mapping. Applied Mathematical Sciences, Vol. 5, 2011, no. 72, 3585 - 3606.

3. Convergence Theorem of a New Iterative Method for Mixed Equilibrium Problems and Variational Inclusions: Approach to Variational Inequalities. Applied Mathematical Sciences, Vol. 6, 2012, no. 16, 747 - 763.

4. Strong Convergence Theorem by Hybrid Method for Non-Lipschitzian Mapping. Applied Mathematical Sciences, Vol. 5, 2011, no. 52, 2581 - 2591.

5. Strong Convergence Theorem by Hybrid Iterative Scheme for Generalized Equilibrium Problems and Fixed Point Problems of Strictly Pseudo-Contraction Mappings. International Mathematical Forum, 5, 2010, no. 60, 2953 - 2969.

หนังสืออ้างอิง

[1] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student. 63 (1994) 123-145.

[2] A. Bnouhachem, M. Aslam Noor and Z. Hao, Some new extragradient iterative methods for variational inequalities, Nonlinear Analysis (2008), doi:10.1016/j.na.2008.02.014.

[3] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197-228.

[4] L.C. Ceng, S. Al-Homidan, Q.H. Ansari and J.C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Comput. Appl. Math. (2008), doi:10.1016/j.cam.2008.03.032.

[5] P. L. Combettes, S. A. Hirstoaga, Equilibrium programing using proximal-like algorithms, Math. Program. 78 (1997) 29-41.

[6] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Matecon 12 (1976) 747-756.

[7] Z. Opial, Weak convergence of the sequence of successive approxmation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591-597.

[8] S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Com. 197 (2008) 548-558.

[9] W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417-428.

[10] W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276-286.

[11] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mappings in a Hilbert spaces, Nonlinear Anal. 69 (2008) 1025-1033.

[12] S. Takahashi, W, Takahashi, Viscosity approxmation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331(1) (2007) 506-515.

[13] H. K. Xu, An iterative approach to guadratic optimazation, J. Optim. Theory Appl. 116 (2003) 659-678.

[14] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math Anal. Appl. 298 (2004) 279-291.

[15] Y. Yao, Y. C. Liou, R. Chen, Convergence theorem for for fixed point problems and variational inequality problems, Nonlinear Anal. (2008), doi:10.1016/j.na.2008.02.094.

[16] J.-C. Yao, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, in: J.P. Crouzeix, N. Haddjissas, S. Schaible (Eds.), Handbook of Generalized Convexity and Monotonicity, 2005 501-558.

[17] L.C. Zeng, S. Schaible, J.C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, J. Optim. Theory Appl. 124 (2005) 725-738.

[18] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems. Taiwanese J. Math. 10 (2006)1293-1303.

[19] S. Atsushiba, W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, Indian J. Math. 41 (1999) 435-453.

[20] A. Kangtunyakarn, S. Suantai, Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis: Hybrid Systems 3 (2009) 296-309.

[21] J.W. Peng, J.C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems nd variational inequality problems*, Taiwanese J. Math. 12 (6) (2008) 1401—1432.

[22] R.T. Rockafellar, *On the maximality of suma of nonlinear monotone operators*, Trans. Amer. Math. Soc. 149 (1970) 75——88.

[23] H. K. Xu, *Iterative algorithims for nonlinear operators*, J. London. Math. Soc 66 (1) (2002) 240—256.

[24] T. Suzuki, *Strong convergence theorems for an infinite family of nonexpansive mappings in general Banach space*, Fixed Point Theory Appl. 1 (2005) 103—123.

[25] L.-C. Ceng and J.-C. Yao, “A hybrid iterative scheme for mixed equilibrium problems and fixed point problems,” Journal of Computational and Applied Mathematics, vol. 214, no. 1, pp. 186-201, 2008.

[26] O. Chadli, S. Schaible, and J. C. Yao, “Regularized equilibrium problems with application to noncoercive hemivariational inequalities,” Journal of Optimization Theory and Applications, vol. 121, no. 3, pp. 571-596, 2004.

[27] O. Chadli, N. C. Wong, and J. C. Yao, “Equilibrium problems with applications to eigenvalue problems,” Journal of Optimization Theory and Applications, vol. 117, no. 2, pp. 245-266, 2003.

[28] P. L. Combettes and S. A. Hirstoaga, “Equilibrium programming in Hilbert spaces,” Journal of Nonlinear and Convex Analysis, vol. 6, no. 1, pp. 117-136, 2005.

[29] S. D. Flam and A. S. Antipin, “Equilibrium programming using proximal-like algorithms,” Mathematical Programming, vol. 78, no. 1, pp. 29-41, 1997.

[30] I. V. Konnov, S. Schaible, and J. C. Yao, “Combined relaxation method for mixed equilibrium problems,” Journal of Optimization Theory and Applications, vol. 126, no. 2, pp. 309-322, 2005.

[31] B. Lemaire, “Which fixed point does the iteration method select?” in Recent Advances in Optimization (Trier, 1996), vol. 452 of Lecture Notes in Economics and Mathematical Systems, pp. 154-167, Springer, Berlin, Germany, 1997.

[32] J.-W. Peng and J.-C. Yao, Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems, Mathematical and Computer Modelling 49 (2009) 1816-1828.

[33] W. Takahashi, K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, Math. Comput. Modelling 32 (2000) 1463-1471.

[34] Y. Yao, Y.-C. Liou, and J.-C. Yao, “Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings,” Fixed Point Theory and Applications, vol. 2007, Article ID 64363, 12 pages, 2007.

[35] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, “Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems,” Taiwanese Journal of Mathematics, vol. 10, no. 6, pp. 1497-1514, 2006.

[36] H. Brezis, Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematics Studies, no. 5, North-Holland, Amsterdam, The Netherlands, 1973.

[37] L. S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194, 114-125, (1995).

[38] Y. C. Liou, An Iterative Algorithm for Mixed Equilibrium Problems and Variational Inclusions Approach to Variational Inequalities, Fixed Point Theory and Applications Volume 2010, Article ID 564361, 15 pages doi:10.1155/2010/564361.

[39] S. Plubtieng and R. Punpaeng, “A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces,” Journal of Mathematical Analysis and Applications, vol. 336, no. 1, pp. 455- 469, 2007.

[40] S. M. Robinson, “Generalized equations and their solutions. I. Basic theory,” Mathematical Programming Study, no. 10, pp. 128-141, 1979.

[41] R. T. Rockafellar, “Monotone operators and the proximal point algorithm,” SIAM Journal on Control and Optimization, vol. 14, no. 5, pp. 877—898, 1976.

[42] R.P. Agarwal, Donal O’Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61—79

[43] R. E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, colloq. Math. 65 (1993) 169—179.

[44] C. Chidume, N. Shahzad, H. Zegeye, Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense, Numer. Funct. Anal. Optim. 25 (2004) 239—257.

[45] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **35** (1972) 171-174.

[46] S. Ishikawa, Fixed point theorems for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **44** (1974) 147-150.

[47] I. Inchan, Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, Int. Journal of Math. Analysis. **2**(23) (2008) 1135-1145.

[48] I. Inchan, K. Nammanee, Strong convergence theorems by hybrid methods for asymptotically k-strict pseudo-contractive mappings in Hilbert space, Nonlinear Anal. : Hybrid syst. **3**(4) (2009) 380-385.

[49] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, Journal of Applied Mathematics and Computing. **29** (2009) 263—280.

[50] G.E. Kim, T.H. Kim, Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces, Comp. Math. Appl. 42 (2001) 1565—1570.

[51] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear. Anal. **64** (2006) 1140—1152.

[52] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically k-strict pseudcontractions, Nonlinear. Anal. **68** (2008) 2828—2836.

[53] T.C. Lim, H.K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal.**22** (1994) 1345—1355.

[54] W. A. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. **4**(1953) 506—510.

[55] G. Marino, H.K. Xu, Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl.**329** (2007) 336—349.

[56] D. R. Sahu, H. K. Xu and J. C. Yao, Asymptotically strict pseudocontractive mapping in the intermediate sense, Nonlinear Anal. **70** (2009) 3502—3511

[57] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Amer. Math. Soc. **43** (1991) 153—159.

[58] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. **158** (1991) 407—413.

[59] J. Schu, Approximation of fixed points of asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **112** (1991) 143—151.

[60] K.K. Tan, H.K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc.**122**(1994) 733—739.

[61] Y. J. Cho, X. Qin and J. I. Kang, Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, Nonlinear Anal. doi:10.1016/j.na.2009.02.106.

[62] Y.J. Cho, et al., Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.02.106

[63] Y. Liu, A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, Nonlinear Anal., (in press) doi:10.1016/j.na.2009.03.060

[64] J. W. Peng, Y. C. Liou and J. C. Yao, An Iterative Algorithm Combining Viscosity Method with Parallel Method for a Generalized Equilibrium Problem and Strict Pseudocontractions, Fixed Point Theory and Applications (2009), Article ID 794178, 21 pages doi:10.1155/2009/794178

[65] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. **118** (2003) 417-428.

[66] S. Takahashi, W, Takahashi, Viscosity approxmation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. **331(1)**(2007) 506-515.

[67] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert space, Nonlinear Anal. **69 (2)**, (2008) 456—462.

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Hybrid extragradient method for general equilibrium problems and fixed point problems in Hilbert space☆

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ABSTRACT

In this paper, we introduce an iterative scheme by the hybrid methods for finding a common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem in a Hilbert space. Then, we prove the strongly convergent theorem by a hybrid extragradient method to the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem. Our results extend and improve the results of Bnouhachem et al. [A. Bnouhachem, M. Aslam Noor, Z. Hao, Some new extragradient iterative methods for variational inequalities, Nonlinear Analysis (2008) doi:10.1016/j.na.2008.02.014] and many others.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively, and C is a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ are real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (2.1) (see [1,2]). In 1997, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $\mathrm{VI}(C, A)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [3] and the references therein.

Let $B : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem: Find

$$z \in C \quad \text{such that } F(z, y) + \langle Bz, y - z \rangle \geq 0, \ \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by GEP, i.e.,

$\mathrm{GEP} = \{z \in C : F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C\}$. In the case of $B \equiv 0$, GEP is denoted by $\mathrm{EP}(F)$.

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In the case of $F \equiv 0$, GEP is also denoted by VI(C, A). A mapping A of C into H is called α-inverse-strongly monotone [4] if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$. Recently, Takahashi and Toyoda [5], Yao et al. [6] and Plubtieng and Punpaeng [7] introduced an iterative method for finding an element of VI$(C, A) \cap F(S)$, where $A : C \to H$ is an α-*inverse-strongly monotone* mapping.

We recall that, a mapping $A : C \longrightarrow H$ is said to be *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \text{for all } u, v \in C;$$

A is said to be *k-Lipschitz continuous* if there exists a positive real number k such that

$$\|Au - Av\| \leq k\|u - v\|, \quad \text{for all } u, v \in C;$$

Remark 1.1. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous.

It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then the variational inequality problem has a unique solution. How to actually find a solution of the variational inequality problem is one of the most important topics in the study of the variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g., [3,8] and the references therein.

In 1976, Korpelevich [9] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n) \end{cases} \tag{1.3}$$

for all $n \geq 0$, where $\lambda \in \left(0, \frac{1}{k}\right)$, C is a nonempty closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if VI(C, A) is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.3), converge to the same point $z \in$ VI(C, A).

In 2003, Takahashi and Toyoda [5], introduced the following iterative scheme:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap \text{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to some $z \in F(S) \cap \text{VI}(C, A)$. Recently, Zeng and Yao [10] proved the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions: (i) $\lambda_n k \subset (0, 1-\delta)$ for some $\delta \in (0, 1)$ and (ii) $\alpha_n \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$. They proved that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to the same point $P_{F(S)\cap \text{VI}(C,A)}x_0$ provided that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

In 2007, Takahashi et al. [11] introduced the modified Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then we prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2008, Bnouhachem et al. [12] introduced the following new extragradient iterative method for finding an element of $F(S) \cap \text{VI}(C, A)$. Let C be a closed convex subset of a real Hilbert space H, A be α-inverse strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \text{VI}(C, A) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ be given by

$$\begin{cases} x_1, u \in C \quad \text{chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n Ay_n)), \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common element of $F(S) \cap \text{VI}(C, A)$.

In this paper, motivated and inspired by the results of Bnouhachem et al. [12] and Takahashi et al. [11], we introduce a new iterative scheme by the hybrid extragradient method, as follows: $x_0 = x \in HC_1 = C$, $x_1 = P_C x_0$ and let

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S(\beta_n x_n + (1-\beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. We will prove that $\{x_n\}$ and $\{u_n\}$ in (1.8) strongly converge to a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for nonexpansive mappings in a Hilbert space. Our results extend and improved that the corresponding ones announced by Bnouhachem et al. [12].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.3}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in \mathrm{VI}(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{2.4}$$

A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v := \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$. It is also known that H satisfies *Opial's condition* [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$
(ii) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$
(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$
(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall t \in [0, 1]$.

Lemma 2.2 ([14,15]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,*

$$a_{n+1} \leq (1 - \gamma_n) a_n + b_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that:

(i) $\Sigma_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

To solve the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [16].

Lemma 2.3 *([16]). Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The following lemma was also given in [2].

Lemma 2.4 *([16,2,17]). Assume that $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)–(A4), and let $r > 0$ and $x \in H$. Define a mapping $T_r : H \to C$ as follows*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$, for any $x, y \in H$;*
3. *$F(T_r) = \mathrm{EP}(F)$;*
4. *$\mathrm{EP}(F)$ is closed and convex;*
5. *$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$, for all $s, t > 0$ and $x \in H$.*

3. Strong convergence theorems

In this section, we show a strong convergence theorem to find a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem in a Hilbert space by using the hybrid extragradient method.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4) and let A be an α-inverse-strongly monotone mapping of C into H and let B be a β-inverse-strongly monotone mapping of C into H, respectively. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP} \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $C_1 = C \subset H$, $x_1 = P_C x_0$;*

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S(\beta_n x_n + (1 - \beta_n) P_C(u_n - \lambda_n A y_n)), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:

(i) $0 < a_1 \leq \alpha_n, \beta_n \leq a_2 < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ *and* $0 < d \leq r_n \leq e < 2\beta$, *for some* $a, b, c, d, e \in \mathbb{R}$,
(iii) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{VI}(C,A)\cap \mathrm{GEP}} x_0$.

Proof. First we show that $F(S) \cap \mathrm{GEP} \cap \mathrm{VI}(C, A) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$, we can prove by induction. It is obvious that $F(S) \cap \mathrm{GEP} \cap \mathrm{VI}(C, A) \subset C_1$. Let $p \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP}$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.4.

Since $p \in \mathrm{VI}(C, A)$, then $p = P_C(p - \lambda_n Ap) = T_{r_n}(p - r_n Bp)$ and $u_n = T_{r_n}(x_n - r_n Bx_n)$. From (i), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(p - r_n Bp)\|^2 \\ &\le \|(x_n - r_n Bx_n) - (p - r_n Bp)\|^2 \\ &\le \|(x_n - p) + r_n(Bp - Bx_n)\|^2 \\ &\le \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2 \qquad (3.2) \\ &\le \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bp - Bx_n\|^2 \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2. \qquad (3.3) \end{aligned}$$

Put $v_n = P_C(u_n - \lambda_n Ay_n)$. From (2.3) and the monotonicity of A, we have

$$\begin{aligned} \|v_n - p\|^2 &\le \|u_n - \lambda_n Ay_n - p\|^2 - \|u_n - \lambda_n Ay_n - v_n\|^2 \\ &= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle Ay_n, p - v_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n(\langle Ay_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle + \langle Ay_n, y_n - v_n\rangle) \\ &\le \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle. \end{aligned}$$

Moreover, from $y_n = P_C(u_n - \lambda_n Au_n)$ and (2.2), we have

$$\langle u_n - \lambda_n Au_n - y_n, v_n - y_n\rangle \le 0. \qquad (3.4)$$

Since A is α-inverse strongly then A is also k-Lipschitz-continuous $\left(k = \frac{1}{\alpha}\right)$, from (ii) we see that $\lambda_n < \frac{1}{k}$, it follows that

$$\begin{aligned} \langle u_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, v_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n\rangle \\ &\le \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n\rangle \\ &\le \lambda_n k\|u_n - y_n\|\,\|v_n - y_n\|. \end{aligned}$$

So, we have

$$\begin{aligned} \|v_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|u_n - y_n\|\,\|v_n - y_n\| \\ &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2k^2\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\ &= \|u_n - p\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 \qquad (3.5) \\ &\le \|u_n - p\|^2, \qquad (3.6) \end{aligned}$$

and hence

$$\|v_n - p\| \le \|u_n - p\| \le \|x_n - p\|. \qquad (3.7)$$

Setting $w_n = \beta_n x_n + (1 - \beta_n)v_n$. Thus, from (3.7) we have

$$\begin{aligned} \|w_n - p\|^2 &= \|\beta_n x_n + (1 - \beta_n)v_n - p\|^2 \\ &= \|\beta_n(x_n - p) + (1 - \beta_n)(v_n - p)\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - v_n\|^2 \qquad (3.8) \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 \qquad (3.9) \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2. \qquad (3.10) \end{aligned}$$

It follows that,

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Sw_n - p\|^2 \\ &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(Sw_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|Sw_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Sw_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|Sw_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 \qquad (3.11) \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2. \qquad (3.12) \end{aligned}$$

So, we have $p \in C_{n+1}$ and hence

$$F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP} \subset C_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.13}$$

Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $c_j \in C_{m+1} \subset C_m$ with $c_j \to z$. Since C_m is closed, $z \in C_m$ and $\|z_m - c_j\| \le \|c_j - x_m\|$. Then

$$\begin{aligned} \|z_m - z\| &= \|z_m - c_j + c_j - z\| \\ &\le \|z_m - c_j\| + \|c_j - z\|. \end{aligned} \tag{3.14}$$

Taking $j \to \infty$,

$$\|z_m - z\| \le \|z - x_m\|.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0, 1]$. Since C_m is convex, $z \in C_m$ and $\|z_m - x\| \le \|x - x_m\|$, $\|z_m - y\| \le \|y - x_m\|$, we have

$$\begin{aligned} \|z_m - z\|^2 &= \|z_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|\alpha(z_m - x) + (1-\alpha)(z_m - y)\|^2 \\ &= \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|(z_m - x) - (z_m - y)\|^2 \\ &\le \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|y - x\|^2 \\ &\le \alpha\|x_m - x\|^2 + (1-\alpha)\|x_m - y\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\ &= \|x_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|x_m - z\|^2. \end{aligned} \tag{3.15}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{u_n\}$ are well-defined.

Since $F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP}$ is a nonempty closed convex subset of H, there exists a unique $u \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP}$ such that

$$u = P_{F(S)\cap \mathrm{VI}(C,A)\cap \mathrm{EP}} x_0.$$

From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0,$$

for all $y \in C_n$. Since $F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \ge 0, \tag{3.16}$$

for all $u \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP}$ and $n \in \mathbb{N}$. So, for $u \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP}$, we have

$$\begin{aligned} 0 \le \langle x_0 - x_n, x_n - u\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle, \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\,\|x_0 - u\|. \end{aligned} \tag{3.17}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\,\|x_0 - u\|,$$

hence

$$\|x_0 - x_n\| \le \|x_0 - u\| \quad \text{for all } u \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP} \text{ and } n \in \mathbb{N}. \tag{3.18}$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0, \tag{3.19}$$

for all $n \in \mathbb{N}$. So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned} 0 \le \langle x_0 - x_n, x_n - x_{n+1}\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle, \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\,\|x_0 - x_{n+1}\|. \end{aligned} \tag{3.20}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\,\|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.21}$$

From (3.18) we have $\{x_n\}$ is bounded, and $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From (3.7) and (3.10), $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are also bounded. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.19) we have

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned} \tag{3.22}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.23}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\|. \tag{3.24}$$

Hence,

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{3.25}$$

From $x_{n+1} = P_{C_{n+1}} x_0$, we obtain

$$\|x_{n+1} - x_0\| \le \|z - x_0\|$$

for all $z \in C_{n+1}$ and all $n \in \mathbb{N}$. Since $u \in F(S) \cap VI(C, A) \cap EP \subset C_{n+1}$ we have

$$\|x_{n+1} - x_0\| \le \|u - x_0\| \tag{3.26}$$

all $n \in \mathbb{N} \cup \{0\}$. Since $x_{n+1} \in C_n$, we have

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_n - x_{n+1}\|.$$

By (3.23), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - z_n\| = 0. \tag{3.27}$$

Since

$$\|x_n - z_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n) S w_n\| = (1 - \alpha_n)\|x_n - S w_n\|,$$

it follows by (3.27) that

$$\lim_{n\longrightarrow\infty} \|x_n - S w_n\| = 0. \tag{3.28}$$

From (3.8), (3.6), (3.2), we have

$$\begin{aligned}
\|w_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - v_n\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2] \\
&= \|x_n - p\|^2 - (1 - \beta_n)2r_n\langle x_n - p, Bx_n - Bp\rangle + (1 - \beta_n)r_n^2\|Bp - Bx_n\|^2 \\
&\le \|x_n - p\|^2 - (1 - \beta_n)2r_n\beta\|Bx_n - Bp\| + (1 - \beta_n)r_n^2\|Bp - Bx_n\|^2 \\
&= \|x_n - p\|^2 - (1 - \beta_n)r_n(2\beta - r_n)\|Bp - Bx_n\|^2,
\end{aligned} \tag{3.29}$$

and from (3.11) and (3.29), we obtain

$$\|z_n - p\|^2 \le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 \tag{3.30}$$

$$\begin{aligned}
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - (1 - \beta_n)r_n(2\beta - r_n)\|Bp - Bx_n\|^2] \\
&= \|x_n - p\|^2 - (1 - \alpha_n)(1 - \beta_n)r_n(2\beta - r_n)\|Bp - Bx_n\|^2.
\end{aligned} \tag{3.31}$$

And hence

$$\|Bp - Bx_n\|^2 \le \frac{1}{(1-\alpha_n)(1-\beta_n)r_n(2\beta - r_n)}\|x_n - p\|^2 - \|z_n - p\|^2$$
$$\le \frac{1}{(1-\alpha_n)(1-\beta_n)r_n(2\beta - r_n)}\|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \quad (3.32)$$

From (3.27), $\liminf_{n\longrightarrow\infty} r_n > 0$, (i) and (ii), we obtain

$$\lim_{n\to\infty} \|Bp - Bx_n\| = 0. \quad (3.33)$$

We note from the proof of Theorem 3.1 [18], that $I - r_n B$ is nonexpansive, for all $n \in \mathbb{N}$. Since T_{r_n} is firmly nonexpansive and using Lemma 2.4(2) and (3.2), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(p - r_n Bp)\|^2 \\
&\le \langle (x_n - r_n Bx_n) - (p - r_n Bp), u_n - p\rangle \\
&= \frac{1}{2}(\|(x_n - r_n Bx_n) - (p - r_n Bp)\|^2 + \|u_n - p\|^2 - \|(x_n - r_n Bx_n) - (p - r_n Bp) - (u_n - p)\|^2) \\
&\le \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Bx_n - Bp)\|^2) \\
&= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2).
\end{aligned}$$

Thus, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2. \quad (3.34)$$

From (3.11), (3.9), (3.7) and (3.34) we can calculate

$$\begin{aligned}
\|z_n - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2] \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)\|u_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)[\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2] \\
&\le \|x_n - p\|^2 - (1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2 + (1-\alpha_n)(1-\beta_n)2r_n\|x_n - u_n\|\,\|Bx_n - Bp\|, \quad (3.35)
\end{aligned}$$

by (i), it follows that

$$\begin{aligned}
(1-a_2)(1-a_2)\|x_n - u_n\|^2 &\le (1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2 \\
&\le \|x_n - p\|^2 - \|z_n - p\|^2 + (1-\alpha_n)(1-\beta_n)2r_n\|x_n - u_n\|\,\|Bx_n - Bp\| \\
&\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + (1-\alpha_n)(1-\beta_n)2r_n\|x_n - u_n\|\,\|Bx_n - Bp\|.
\end{aligned}$$

From (3.27), (3.33), (i) and the boundedness of $\{x_n\}$ and $\{u_n\}$, we have

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \quad (3.36)$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, we obtain

$$\lim_{n\longrightarrow\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\longrightarrow\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \quad (3.37)$$

Next, we show that $\lim_{n\to\infty}\|u_n - y_n\| = 0$. Consider,

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\le \|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\|^2 \\
&= \|(u_n - p) - \lambda_n(Au_n - Ap)\|^2 \\
&= \|u_n - p\|^2 - \lambda_n\langle u_n - p, Au_n - Ap\rangle + \lambda_n^2\|Au_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2\lambda_n\beta\|Au_n - Ap\|^2 + \lambda_n^2\|Au_n - Ap\|^2 \\
&= \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Ap\|^2.
\end{aligned}$$

From (3.11), (3.8), (3.7) we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2] \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2] \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - v_n\|^2] \\ &= \|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|x_n - v_n\|^2, \end{aligned} \tag{3.38}$$

from (i), it follows that

$$\begin{aligned}(1-a_2)a_1(1-a_2)\|x_n - v_n\|^2 &\le (1-\alpha_n)\beta_n(1-\beta_n)\|x_n - v_n\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

From (3.27) and (ii), we have

$$\lim_{n\longrightarrow\infty}\|x_n - v_n\| = 0. \tag{3.39}$$

For $p \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{EP}$, from (3.11), (3.9), (3.7), (3.5) we obtain

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2] \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)\|v_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)[\|u_n - p\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2] \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)[\|x_n - p\|^2] \\ &\quad + (1-\alpha_n)(1-\beta_n)(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 \\ &= \|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|u_n - y_n\| &\le \frac{1}{(1-\alpha_n)(1-\beta_n)(1-\lambda_n^2k^2)}(\|x_n - p\|^2 - \|z_n - p\|^2) \\ &= \frac{1}{(1-\alpha_n)(1-\beta_n)(1-\lambda_n^2k^2)}(\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|) \\ &\le \frac{1}{(1-\alpha_n)(1-\beta_n)(1-\lambda_n^2k^2)}\|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

So, by (3.27) we obtain

$$\lim_{n\longrightarrow\infty}\|u_n - y_n\| = 0. \tag{3.40}$$

Since $\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - y_n\|$, from (3.36) and (3.40) we also have

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0. \tag{3.41}$$

We note that $\|y_n - v_n\| \le \|y_n - x_n\| + \|x_n - v_n\|$. From (3.39) and (3.41), we obtain

$$\lim_{n\longrightarrow\infty}\|v_n - y_n\| = 0. \tag{3.42}$$

Since

$$\begin{aligned}\|w_n - x_n\| &= \|\beta_n x_n + (1-\beta_n)v_n - x_n\| \\ &= (1-\beta_n)\|v_n - x_n\|.\end{aligned}$$

From (3.39), we have

$$\lim_{n\longrightarrow\infty}\|w_n - x_n\| = 0. \tag{3.43}$$

Note that

$$\begin{aligned}\|Su_n - u_n\| &\le \|Su_n - Sw_n\| + \|Sw_n - x_n\| + \|x_n - u_n\| \\ &\le \|u_n - w_n\| + \|Sw_n - x_n\| + \|x_n - u_n\|\end{aligned}$$

$$\leq \|u_n - x_n\| + \|x_n - w_n\| + \|Sw_n - x_n\| + \|x_n - u_n\|$$
$$= 2\|x_n - u_n\| + \|x_n - w_n\| + \|Sw_n - x_n\|.$$

From (3.28), (3.36) and (3.43), we obtain

$$\lim_{n\longrightarrow\infty} \|Su_n - u_n\| = 0. \tag{3.44}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup w$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. Since C is closed and convex, $w \in C$. We first show that $w \in \mathrm{EP}$. It follows by (4.2) and (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}). \tag{3.45}$$

Put $y_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. Thus, we have $y_t \in C$. So, from (3.36), we have

$$\langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, By_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Bx_{n_i}\rangle - \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y, u_{n_i})$$

and hence

$$\begin{aligned}\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y, u_{n_i})\\ &= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, it follows that $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from monotonicity of B, we get

$$\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0.$$

So, from (A4), we have

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{3.46}$$

as $i \to \infty$. From (A1), (A4) and (3.46), we have

$$\begin{aligned}0 &= F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle\\ &\leq tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle\end{aligned}$$

and hence $0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle$. Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Bw\rangle$. This implies that $w \in \mathrm{GEP}$. Next, we show that $w \in F(S)$. Assume that $w \neq Sw$. From Opial's condition and $\|Su_{n_i} - u_{n_i}\| \to 0$, we have

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\| = \liminf_{i\to\infty} \|(u_{n_i} - Su_{n_i}) + (Su_{n_i} - Sw)\|\\ &= \liminf_{i\to\infty} \|Su_{n_i} - Sw\| \leq \liminf_{i\to\infty} \|u_{n_i} - w\|.\end{aligned}$$

This is a contradiction. So, we have $w \in F(S)$. Therefore $w \in F(S) \cap \mathrm{GEP}$.

Finally, we can show that $w \in \mathrm{VI}(C, A)$. Define,

$$Tv = \begin{cases} Av + N_C v, & v \in C,\\ \emptyset, & v \notin C.\end{cases}$$

Then, T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, u - Av\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n Ay_n)$, we have $\langle v - v_n, v_n - (u_n - \lambda_n Ay_n)\rangle \geq 0$, and hence, $\left\langle v - v_n, \frac{v_n - u_n}{\lambda_n} + Ay_n\right\rangle \geq 0$. Therefore, we have

$$\begin{aligned}\langle v - v_{n_i}, u\rangle &\geq \langle v - v_{n_i}, Av\rangle\\ &\geq \langle v - v_{n_i}, Av\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle\\ &= \left\langle v - v_{n_i}, Av - Ay_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle\end{aligned}$$

$$= \langle v - v_{n_i}, Av - Av_{n_i}\rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle$$
$$\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle.$$

Since $\lim_{n\longrightarrow\infty} \|v_n - u_n\| = \lim_{n\longrightarrow\infty} \|v_n - y_n\| = 0$, $u_{n_i} \rightharpoonup p$ and A is Lipschitz continuous, we obtain that $\lim_{n\longrightarrow\infty} \|Av_n - Ay_n\| = 0$ and $v_{n_i} \rightharpoonup p$. From $\liminf_{n\longrightarrow\infty} \lambda_n > 0$ and $\lim_{n\longrightarrow\infty} \|v_n - u_n\| = 0$, we obtain

$$\langle v - z, u\rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \mathrm{VI}(C, A)$. Hence, we have $z \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP}$. Finally, we show that $x_n \longrightarrow z$, where $z = P_{F(S)\cap\mathrm{VI}(C,A)\mathrm{EP}(F)}x_0$. Since $x_n = P_{C_n}x_0$ and $z \in F(S) \cap \mathrm{VI}(C, A) \cap \mathrm{GEP} \subset C_n$, we have $\|x_n - x_0\| \leq \|z - x_0\|$. It follows from $z' = P_{F(S)\cap\mathrm{VI}(C,A)\cap\mathrm{GEP}}x_0$ and the lower semicontinuity of the norm that

$$\|z' - x_0\| \leq \|z - x_0\| \leq \liminf_{i\longrightarrow\infty} \|x_{n_i} - x_0\| \leq \limsup_{i\longrightarrow\infty} \|x_{n_i} - x_0\| \leq \|z' - x_0\|.$$

Thus, we obtain that $\lim_{k\longrightarrow\infty} \|x_{n_i} - x_0\| = \|z - x_0\| = \|z' - x_0\|$. Using the Kadec–Klee property of H, we obtain that

$$\lim_{i\longrightarrow\infty} x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_{F(T)\cap\mathrm{VI}(C,A)\cap\mathrm{GEP}}x_0$. □

4. Application

Theorem 4.1 (*[12]*). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone of C into H and let S be a nonexpansive mapping from C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1, u \in C \quad \text{chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n x_n + (1 - \alpha_n)P_C(x_n - \lambda_n Ay_n)), \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 1)$ satisfy the following condition:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,
(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(C3) $\left\{\frac{\lambda_n}{\alpha}\right\} \subset (\tau, 1 - \delta)$ *for some* $\tau, \delta \in (0, 1)$ *and* $\lim_{n\to\infty} \lambda_n = 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap\mathrm{VI}(C,A)}u$.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4) and let A be an α-inverse-strongly monotone mapping of C into H and let T be a strictly k-pseudocontractive mapping of C itself. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap \mathrm{GEP} \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $C_1 = C \subset H$, $x_1 = P_C x_0$;*

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle (I - T)x_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \lambda_n Au_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n)S(\beta_n x_n + (1 - \beta_n)P_C(u_n - \lambda_n Ay_n)), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_n+1}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $u_n = T_{r_n}(x_n - r_n(I - T)x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, 2\alpha)$, and$\{r_n\} \subset (0, 1 - k)$ satisfy the following conditions:

(i) $0 < a \leq \alpha_n, \beta_n \leq b < 1$, *for some* $a, b \in \mathbb{R}$,
(ii) $0 \leq c \leq \lambda_n \leq d < \alpha$, *for some* $c, d \in \mathbb{R}$,
(iii) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap\mathrm{GEP}}x_0$.

Proof. A strictly k-pseudocontractive mapping is $\frac{1-k}{2}$-inverse strongly monotone. So, from Theorem 3.1, we obtain the desired result. □

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References

[1] L.C. Ceng, S. Al-Homidan, Q.H. Ansari, J.C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Comput. Appl. Math. (2008) doi:10.1016/j.cam.2008.03.032.
[2] P.L. Combettes, S.A. Hirstoaga, Equilibrium programing using proximal-like algorithms, Math. Program. 78 (1997) 29–41.
[3] J.-C. Yao, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, in: J.P. Crouzeix, N. Haddjissas, S. Schaible (Eds.), Handbook of Generalized Convexity and Monotonicity, 2005, pp. 501–558.
[4] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197–228.
[5] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417–428.
[6] Y. Yao, Y.C. Liou, R. Chen, Convergence theorem for fixed point problems and variational inequality problems, Nonlinear Anal. (2008) doi:10.1016/j.na.2008.02.094.
[7] S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. 197 (2008) 548–558.
[8] L.C. Zeng, S. Schaible, J.C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, J. Optim. Theory Appl. 124 (2005) 725–738.
[9] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Matecon 12 (1976) 747–756.
[10] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese J. Math. 10 (2006) 1293–1303.
[11] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276–286.
[12] A. Bnouhachem, M. Aslam Noor, Z. Hao, Some new extragradient iterative methods for variational inequalities, Nonlinear Anal. (2008) doi:10.1016/j.na.2008.02.014.
[13] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.
[14] H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003) 659–678.
[15] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.
[16] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.
[17] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mappings in a Hilbert spaces, Nonlinear Anal. 69 (2008) 1025–1033.
[18] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (1) (2007) 506–515.

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Extragradient Method for Generalized Mixed Equilibrium Problems and Fixed Point Problems of Finite Family of Nonexpansive Mapping

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Abstract

In this paper, we introduce the extragradient method for finding a common element of the set of solutions of generalized mixed equilibrium problem, the set of common fixed point of family of nonexpansive mappings the set of variational inequality for monotone, Lipschitz continuous mapping in a Hilbert space. Then we prove the strong convergence of iterative algorithm to a common element of this sets. The result extend and improve the result of Jian-Wen Peng and Jen-Chih Yao [Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed problems, Math. Comp. Modelling, 49(2009) 1816 - 1828.]

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1 Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T $(i.e. F(T) = \{x \in H : Tx = x\})$. Let $f : C \to C$ be a contraction mapping if $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for $\alpha \in (0, 1)$ and for all $x, y \in C$. Let $F : C \to H$ be a nonlinear mapping, let $\varphi : C \to \mathbb{R}$ be a function, and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$. Now we consider the following generalized mixed equilibrium problem: to find $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) + \langle Fu, y - u \rangle \geq 0, \ \forall y \in C. \tag{1}$$

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The set of solution of problem (1) is denoted by $GMEP(\Theta,\varphi)$.

If $F=0$, then the generalized mixed equilibrium problem (1) becomes the followsing mixed equilibrium: to find $u\in C$ such that

$$\Theta(u,y)+\varphi(y)-\varphi(u)\geq 0,\forall y\in C, \tag{2}$$

which was considered by Ceng and Yao [10]. If $\varphi=0$, then the generalized mixed equilibrium problem (1) becomes the following equilibrium: to find $u\in C$ such that

$$\Theta(u,y)+\langle Fu,y-u\rangle\geq 0,\ \forall y\in C, \tag{3}$$

which was studied by S. Takahashi and W. Takahashi [19]. If $\varphi=0$ and $F=0$, then the generalized mixed equilibrium problem (1) becomes the following equilibrium problem: to find $u\in C$ such that

$$\Theta(u,y)\geq 0,\ \forall y\in C. \tag{4}$$

If $\Theta(x,y)=0$ for all $x,y\in C$, the generalized mixed equilibrium problem (1) becomes the following variational inequality problem: to find $u\in C$ such that

$$\varphi(y)-\varphi(u)+\langle Fu,y-u\rangle\geq 0,\ \forall y\in C. \tag{5}$$

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [9, 14, 23, 12, 11, 15] Some methods have been proposed to solve the mixed equilibrium problem and the equilibrium problem. In 1997, Flaim and Antipen [14] introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, S. Takahashi and W. Takahashi [21] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem(2) and the set of fixed point points of a nonexpansive mapping. Furthermore, Yao et al. [22] introduced some new iterative schemes for finding a common element of the set of solutions of the equilibrium problem (2) and the set of common fixed points of finitely (infinitely) nonexpansive mappings. Very recently, Ceng and Yao [10] considered a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings. Peng and Yao [17] developed a CQ method. They obtained some strong convergence results for finding a common element of the set of solutions of the mixed equilibrium problem (1) and the set of the variational inequality and the set of fixed points of a nonexpansive mapping. Their results extend and improve the corresponding results in [10, 13, 16, 21].

In 1999, Atsushiba and Takahashi [1] defined the mapping W_n as follow:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1+(1-\lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1}+(1-\lambda_{n,2})I,\\
U_{n,3} &= \lambda_{n,3}T_3U_{n,2}+(1-\lambda_{n,3})I,\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}T_N-1U_{n,N-2}+(1-\lambda_{n,N-1})I,\\
W_n &= U_{n,N}=\lambda_{n,N}T_NU_{n,N-1}+(1-\lambda_{n,N})I,
\end{aligned} \tag{6}$$

when $\{\lambda_{n,i}\}_i^N \subseteq [0,1]$. This mapping is called the $W-mapping$ generated by $T_1, T_2, ..., T_N$ and $\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,N}$. In 2000, Takahashi and Shimoji [20] proved that if X is a strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,1} < 1 \quad i = 1, 2, ..., N$.

In 2009, J. W. Peng and J. C. Yao [18], introduce a new iterative scheme based on the extragradient method generated by;

$$\begin{cases} x_1 = x \in C \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n)\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \gamma_n A u_n) \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \gamma_n A y_n). \end{cases} \tag{7}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ the sequences $\{x_n\}$ and $\{u_n\}$ generated by (7) converges strongly to $z = P_{\cap_{i=1}^N F(T_i) \cap VI(C,A) \cap MEP(\Theta,\varphi)} f(z)$ for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping in Hilbert spaces.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in N$,and $j = 1, 2, ..., N$, let $\alpha_j^n = (\alpha_1^{nj}, \alpha_2^{nj}, \alpha_3^{nj})$ be such that $\alpha_1^{nj}, \alpha_2^{nj}, \alpha_3^{nj} \in [0,1]$ with $\alpha_1^{nj} + \alpha_2^{nj} + \alpha_3^{nj} = 1$. We define mapping $S_n : C \to C$ as follow:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I \\ &\vdots \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I \\ S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I, \end{aligned} \tag{8}$$

In this paper inspired and motivate J. W. Peng and J. C. Yao [18], we introduce an iterative extragradient method generated by

$$\begin{cases} x_1 = x \in C \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \gamma_n A u_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n P_C(u_n - \gamma_n A y_n). \end{cases} \tag{9}$$

Then we proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ the sequences $\{x_n\}$ and $\{u_n\}$ generated by (9) converges strongly to $z = P_{\cap_{i=1}^N F(T_i) \cap VI(C,A) \cap GMEP(\Theta,\varphi)} f(z)$.

2 Preliminaries

Let H be a real Hilbert space with inner prduct $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonexpansive closed convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the matrix projection of H onto C. We kwon that P_C is nonexpansive mapping from H onto C. It is also known that $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \tag{10}$$

for all $x \in H$ and $y \in C$. It is easy to see that (10) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{11}$$

for all $x \in H$ and $y \in C$. A mapping A of C into H is called monotone if

$$\langle x - y, Ax - Ay \rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into H is called β-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \beta \|Ax - Ay\|^2$$

for all $x, y \in C$. A mapping $A : C \to H$ is called k-Lipschitz continuous if there exists a positive real number α such that

$$\|Ax - Ay\| \leq k\|x - y\|$$

for all $x, y \in C$. It is easy to see that if A is β-inverse-strongly-monotone mappings, then A is monotone and Lipschitz continuous. The converse is not true in general. The class of β-inverse-strongly-monotone mappings dose not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A is monotone and Lipschitz continuous, but not α-inverse-strongly-monotone.

Let A be monotone mapping of c into H. In the context of the variational inequality problem the characterization of projection (10) implies the following:

$u \in VI(C, A) \Rightarrow u = P_C(u - \lambda Au), \quad \lambda > 0,$
and

$u = P_C(u - \lambda Au)$ for some $\lambda > 0 \Rightarrow u \in VI(C, A)$

It also known that H satisfies the Opial condition (see [4]), i.e. for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in T_x$ and $g \in T_y$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. A be a monotone, k-Lipschitz continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, i.e, $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \ \ \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & if \ v \in C, \\ \emptyset & if \ v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in T_v$ if and only if $v \in VI(C, A)$ (see [5]).

For solving the mixed equilibrium problem, let us give the give the following assumption for the bifunction F, φ and the set C :
(A1) $F(x, y) = 0$ for all $x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $y \in C, x \mapsto F(x, y)$ is weakly upper semicontinuous;
(A4) for each $y \in C, y \mapsto F(x, y)$ is convex;
(A5) for each $y \in C, y \mapsto F(x, y)$ is lower semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) C is a bounded set.
By similar argument as in the proof of lemma in (see [3]), we have the following result.

Lemma 2.1. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows.*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:
(1) For each $x \in H$, $T_r \neq \emptyset$;
(2) T_r is single-valued;
(3) T_r firmly nonexpansive, i.e, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) $Fix(T_r) = MEP(F, \varphi)$ is closed and convex.

We also need the following lemmas.

Lemma 2.2. *Let H be a real Hilbert space. Then for any $x, y \in H$, we have*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$

(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0,1].$

Lemma 2.3. *(see [6, 7]). Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n$$

where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4. *(see [8]) Let $\{x_n\}$ and $\{w_n\}$ be bounded sequence in a Banach space, let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ for all $n = 1, 2, \ldots$ Suppose that $x_{n+1} = (1-\beta_n)w_n + \beta_n x_n$ for all $n = 1, 2, \ldots$ and $\limsup_{n\to\infty}(\|w_{n+1} - w_n\| + \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|w_n - x_n\| = 0$.*

Lemma 2.5. *Let C be a nonempty closed convex subset of real Hilbert space H and $T : C \to C$ be a k-strictly pseudocontractive mapping. Define a mapping $S : C \to C$ by $Sx = \delta x + (1-\delta)Tx$ for all $x \in C$ and $\delta \in [k, 1)$. Then S is nonexpansive mapping such that $F(S) = F(T)$.*

Definition 2.6. *[2] Let C be nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of k_i-strictly pseudocontractive mappings of C into itself. For each $i = 1, 2, \ldots, N$, we define a mapping $S_i = \delta_i I + (1-\delta_i)T_i$ where $\delta_i \in [k_i, 1)$ consider mapping K_n defined by*

$$\begin{aligned}
U_{n,n+1} &= I \\
U_{n,n} &= \gamma_n S_n U_{n,n+1} + (1-\gamma_n)I, \\
U_{n,n-1} &= \gamma_{n-1} S_{n-1} U_{n,n} + (1-\gamma_{n-1})I, \\
&\vdots \\
U_{n,k} &= \gamma_k S_k U_{n,k+1} + (1-\gamma_k)I, \\
U_{n,k-1} &= \gamma_{k-1} S_{k-1} U_{n,k} + (1-\gamma_{k-1})I, \\
&\vdots \\
U_{n,2} &= \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I, \\
K_n = U_{n,1} &= \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned}$$

where $\gamma_1, \gamma_2, \ldots$ are real number such that $0 \leq \gamma_n \leq 1$.

As regards K_n, we have the following lemmas which are important for prove our main results.

Lemma 2.7. *[2] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_i be nonexpansive mapping of C into itself such that $\cap_{i=1}^{\infty} F(S_i \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, \ldots$. Then for any $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}$ exists.*

Using Lemma 2.7, one can define the mapping K of C into itself as follows:

$$Kx := \lim_{n\to\infty} K_n x = \lim_{n\to\infty} U_{n,1} x, \quad x \in C.$$

Such a mapping K is called the modified K-mapping generated by $T_1, T_2, ..., \gamma_1, \gamma_2, ...$ and $\delta_1, \delta_2, ...$

Lemma 2.8. *[2] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_i be nonexpansive mapping of C into itself such that $\cap_{i=1}^{\infty} F(S_i \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, \ldots$. Then $F(K) = \bigcap_{i=1}^{\infty} F(S_i)$.*

Combining Lemma 2.5-2.8, one can get that $F(K) = \bigcap_{i=1}^{\infty} F(S_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Lemma 2.9. *[2] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let S_i be nonexpansive mapping of C into itself such that $\cap_{i=1}^{\infty} F(S_i \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, \ldots$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|K_n x - Kx\| = 0.$$

3 Main Result

In this section, we derive a strong convergence of an iterative algorithm of extragradient viscosity method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings and the set of the variational inequality for a monotone, k-Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifucntion satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a paper lower semicontinuous and convex function. Let F be a β-inverse strongly monotone and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{T_1, T_2, \ldots\}$ be a family of infinitely k_i-strictly pseudocontractive mapping of C into itself, such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, \ldots$ and $\Omega = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \cap GMEP(F, \varphi) \neq \emptyset$. Let $\{S_n\}$ be the S-mapping generated by $\{T_1, T_2, \ldots, T_N\}$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Assume that either $(B1)$ or $(B2)$ holds, let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by;*

$$\begin{cases} x_1 = x \in C \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \gamma_n A u_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n P_C(u_n - \gamma_n A y_n) \end{cases} \tag{12}$$

where f is contraction of C into itself with $\alpha \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, $\{r_n\} \subseteq (0,\infty)$ are satisfy the following condition:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\lim_{n\to\infty} \gamma_n = 0$;

(v) $0 < r \leq 2\beta$;

(vi) $|\alpha_n^{(n+1)j} - \alpha_1^{nj}| \to 0$ *and* $|\alpha_3^{(n+1)j} - \alpha_3^{nj}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, \dots, N\}$.
Then the sequence $\{x_n\}$ and $\{u_n\}$ generated by (12) converges strongly to $z_0 \in \Omega$, where $z_0 = P_\Omega f(z_0)$.

Proof. Put $t_n = P_C(u_n - \gamma_n Ay_n)$ for every $n \in N$. First, we prove that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ax_n\}$, $\{Au_n\}$, $\{f(x_n)\}$ and $\{Ay_n\}$ are bounded. Let $x^* \in \Omega$ and let $\{T_{r_n}\}$ be a sequences of mapping defined as in Lemma 2.1 the $x^* = T_r(x^* - rFx^*)$. From $u_n = T_r(x_n - rFx_n)$, we have

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \|T_r(x_n - rFx_n) - T_r(x^* - rFx^*)\|^2 \\
&\leq \|(x_n - rFx_n) - (x^* - rFx^*)\|^2 \\
&= \|(x_n - x^*) - r(Fx_n - Fx^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r\langle Fx_n - Fx^*, x_n - x^*\rangle + r^2\|Fx_n - Fx^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2r\beta\|Fx_n - Fx^*\|^2 + r^2\|Fx_n - Fx^*\|^2 \\
&\leq \|x_n - x^*\|^2 + r(r - 2\beta)\|Fx_n - Fx^*\|^2 \qquad (13) \\
&\leq \|x_n - x^*\|^2 . (\textit{since } r < 2\beta) \qquad (14)
\end{aligned}
$$

From (11) the monotonicity of A and $x^* \in VI(C, A)$, we have

$$
\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - \gamma_n Ay_n - x^*\|^2 - \|u_n - \gamma_n Ay_n - t_n\|^2 \\
&= \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\gamma_n\langle Ay_n, x^* - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\gamma_n(\langle Ay_n - Ax^*, x^* - y_n\rangle \\
&\quad + \langle Ax^*, x^* - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\leq \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\gamma_n\langle Ay_n, y_n - t_n\rangle \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\quad + 2\gamma_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \gamma_n Ay_n - y_n, t_n - y_n\rangle
\end{aligned}
$$

Further, from $y_n = P_C(u_n - \gamma_n Au_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned}\langle u_n - \gamma_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \gamma_n Au_n - y_n, t_n - y_n\rangle + \langle \gamma_n Au_n - \gamma_n Ay_n, t_n y_n\rangle \\ &\leq \langle \gamma_n Au_n - \gamma_n Ay_n, t_n - y_n\rangle \\ &\leq \gamma_n k\|u_n - y_n\|\|t_n - y_n\|.\end{aligned}$$

So, we have

$$\begin{aligned}\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\gamma_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \gamma_n^2 k^2\|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\gamma_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &\leq \|u_n - x^*\|^2, \quad (\gamma_n^2 k^2 - 1 \leq 0). \qquad (15)\end{aligned}$$

From (12), (14) and (15), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)S_n t_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + (1 - \alpha_n - \beta_n)\|S_n t_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + (1 - \alpha_n - \beta_n)\|t_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| \\ &\quad +(1 - \alpha_n - \beta_n)\|x_n - x^*\| \\ &\leq \alpha_n\alpha\|x_n - x^*\| + \alpha_n\|f(x_n) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &= (1 - \alpha_n(1 - \alpha))\|x_n - x^*\| + \alpha_n(1 - \alpha_n)\frac{\|f(x^*) - x^*\|}{(1 - \alpha)} \\ &\leq \max\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\} \\ &\quad \vdots \\ &\leq \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\}.\end{aligned}$$

Hence $\{x_n\}$ is bounded, we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded. From $y_n = P_C(u_n - \gamma_n Au_n)$ and the monotonicity and the Lipschitz continuous of A, we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P_C(u_n - \gamma_n Au_n) - P_C(x^* - \gamma_n Ax^*)\|^2 \\ &\leq \|u_n - \gamma_n Au_n - (x^* - \gamma_n Ax^*)\|^2 \\ &= \|u_n - x^*\|^2 - 2\gamma_n\langle Au_n - Ax^*, u_n - x^*\rangle + \gamma_n^2\|Au_n - Ax^*\|^2 \\ &\leq \|u_n - x^*\|^2 + \gamma_n^2 k\|u_n - x^*\|^2 \\ &= (1 + \gamma_n^2 k)\|u_n - x^*\|^2 \qquad (16)\end{aligned}$$

Hence, we obtain that $\{y_n\}$ is bounded, it follows that $\{Ax_n\}$, $\{Au_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$ are bounded. Now we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Consider,

$$\begin{aligned}\|t_{n+1}-t_n\| &= \|P_C(u_{n+1}-\gamma_{n+1}Ay_{n+1})-P_C(u_n-\gamma_nAy_n)\|\\ &\le \|(u_{n+1}-\gamma_{n+1}Ay_{n+1})-(u_n-\gamma_nAy_n)\|\\ &\le \|(u_{n+1}-u_n)-\gamma_{n+1}Ay_{n+1}-\gamma_{n+1}Ay_n+\gamma_{n+1}Ay_n+\gamma_nAy_n\|\\ &\le \|u_{n+1}-u_n\|+\gamma_{n+1}\|Au_{n+1}-Au_n\|+\gamma_{n+1}\|Au_n\|+\gamma_n\|Au_n\|\\ &\le \|u_{n+1}-u_n\|+\gamma_{n+1}\|u_{n+1}-u_n\|+\gamma_{n+1}\|Au_n\|+\gamma_n\|Au_n\|\\ &\le \|u_{n+1}-u_n\|+(2\gamma_{n+1}+\gamma_n)M_1, \qquad (17)\end{aligned}$$

when $M_1 \ge \sup\{k\|u_{n+1}-u_n\|+\|Au_n\|\}$. Since F is β-inverse-strongly monotone and $r<2\beta$, we have for all $x,y\in C$

$$\begin{aligned}\|(I-rF)x-(I-rF)y\|^2 &= \|(x-y)-r(Fx-Fy)\|^2\\ &= \|x-y\|^2-2r\langle Fx-Fy,x-y\rangle+r^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2-2r\beta\|Ax-Ay\|^2+r^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2+r(r-2\beta)\|Ax-Ay\|^2\\ &\le \|x-y\|^2,\end{aligned}$$

then $I-rF$ is nonexpansive. From $u_n=T_r(x_n-rFx_n)$, we get

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|T_r(x_{n+1}-rFx_{n+1})-T_r(x_n-rFx_n)\|\\ &\le |(I-rF)x_{n+1}-(I-rF)x_n\|\\ &= \|x_{n+1}-x_n\|. \qquad (18)\end{aligned}$$

From (17) and (18), we obtain that

$$\begin{aligned}\|t_{n+1}-t_n\| &\le \|x_{n+1}-x_n\|+(2\gamma_{n+1}+\gamma_n)M_1\\ &= \|x_{n+1}-x_n\|+c_n, \qquad (19)\end{aligned}$$

where $c_n:=(2\gamma_{n+1}+\gamma_n)M_1$. Since $\lim_{n\to\infty}\gamma_n=0$, we have $\lim_{n\to\infty}c_n=0$. Next, we show

$$\lim_{n\to\infty}\|S_{n+1}x_n-S_nx_n\|=0.$$

For $k\in\{2,3,\ldots,N\}$, we have

$$\begin{aligned}\|U_{n+1,k}x_n-U_{n,k}x_n\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k}x_n+\alpha_2^{n+1,k}U_{n+1,k-1}x_n+\alpha_3^{n+1,k}x_n\\ &\quad -\alpha_1^{n,k}T_kU_{n,k-1}x_n-\alpha_2^{n,k}U_{n,k-1}x_n-\alpha_3^{n,k}x_n\|\\ &= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}x_n-T_kU_{n,k-1}x_n)\\ &\quad +(\alpha_1^{n+1,k}-\alpha_1^{n,k})T_kU_{n,k-1}x_n\\ &\quad +(\alpha_3^{n+1,k}-\alpha_3^{n,k})x_n+\alpha_2^{n+1,k}(U_{n+1,k-1}x_n-U_{n,k-1}x_n)\\ &\quad +(\alpha_2^{n+1,k}-\alpha_2^{n,k})U_{n,k-1}x_n\|\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}x_n\| \\
&\quad +\alpha_2^{n+1,k}\|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| \\
&= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| \\
&\quad +|\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k}x_n\| \\
&\leq \|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |(\alpha_1^{n,k} - \alpha_1^{n+1,k} + \alpha_3^{n,k} - \alpha_3^{n+1,k})|\|U_{n,k-1}x_n\| \\
&\leq \|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x_n\| + |\alpha_1^{n,k} - \alpha_1^{n+1,k}|\|U_{n,k-1}x_n\| \\
&\quad +|\alpha_3^{n,k} - \alpha_3^{n+1,k}|\|U_{n,k-1}x_n\| \\
&= \|U_{n+1,k-1}x_n - U_{n,k-1}x_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_k U_{n,k-1}x_n\| + \|U_{n,k-1}x_n\|) \\
&\quad +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|(\|x_n\| + \|U_{n,k-1}x_n\|) \qquad (20)
\end{aligned}
$$

By (20), we obtain that for each $n \in \mathbb{N}$

$$
\begin{aligned}
\|S_{n+1}x_n - S_n x_n\| &= \|U_{n+1,N}x_n - U_{n,N}x_n\| \\
&\leq \|U_{n+1,1}x_n - U_{n,1}x_n\| + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_j U_{n,j-1}x_n\| \\
&\quad +\|U_{n,j-1}x_n\|) + \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|x_n\| + \|U_{n,j-1}x_n\|) \\
&= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1x_n - x_n\| + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_j U_{n,j-1}x_n\| \\
&\quad +\|U_{n,j-1}x_n\|) + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|x_n\| + \|U_{n,j-1}x_n\|)
\end{aligned}
$$

From condition (iv) $[|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ and $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0]$, we obtain

$$\lim_{n\to\infty} \|S_{n+1}x_n - S_n x_n\| = 0. \qquad (21)$$

Similarly

$$\lim_{n\to\infty} \|S_{n+1}t_n - S_n t_n\| = 0. \qquad (22)$$

Let $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$, then we have

$$
\begin{aligned}
z_n &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)S_n t_n - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n)S_n t_n}{1-\beta_n}.
\end{aligned}
$$

We consider,

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1}-\beta_{n+1})S_{n+1}t_{n+1}}{1-b_{n+1}}-\frac{\alpha_n f(x_n)+(1-\alpha_n-\beta_n)S_nt_n}{1-\beta_n}\\
&= (\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}f(x_n))+S_{n+1}t_{n+1}-S_nt_n\\
&\quad +\frac{\alpha_n}{1-\beta_n}S_nt_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}S_{n+1}t_{n+1}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-S_{n+1}t_{n+1})+\frac{\alpha_n}{1-\beta_n}(S_nt_n-\alpha_nf(x_n)+S_{n+1}t_{n+1}-S_nt_n).
\end{aligned}
$$

Then from (19), we have

$$
\begin{aligned}
\|z_{n+1}-z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_{n+1}t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|S_nt_n-f(x_n)\|+\|S_{n+1}t_{n+1}-S_nt_n\|\\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_{n+1}t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|S_nt_n-f(x_n)\|\\
&\quad +\|S_{n+1}t_{n+1}-S_{n+1}t_n\|+\|S_{n+1}t_n-S_nt_n\|\\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_{n+1}t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_nt_n\|\\
&\quad +\|t_{n+1}-t_n\|+\|S_{n+1}t_n-s_nt_n\|\\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_{n+1}t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_nt_n\|\\
&\quad +\|S_{n+1}t_n-s_nt_n\|+\|x_{n+1}-x_n\|+c_n.
\end{aligned}
$$

It follow that

$$
\begin{aligned}
\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-S_{n+1}t_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-S_nt_n\|\\
&\quad +\|S_{n+1}t_n-s_nt_n\|+c_n
\end{aligned}
$$

From (i), (22) and $\lim_{n\to\infty} c_n=0$, we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

By Lemma 2.4, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0.$$

Note that

$$\|x_{n+1}-x_n\|=\|\beta_nx_n+(1-\beta_n)z_n-x_n\|=(1-\beta_n)\|z_n-x_n\|$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{23}$$

Consider,

$$\begin{aligned}\|S_nt_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - S_nt_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - x_n\| + \beta_n\|S_nt_n - x_n\|,\end{aligned}$$

it follows that

$$(1-\beta_n)\|S_nt_n - x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - x_n\|$$

and hence

$$\|S_nt_n - x_n\| \leq \frac{1}{1-\beta_n}(\|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - x_n\|)$$

From $(i), (ii)$ and (23), we obtain

$$\lim_{n\to\infty} \|S_nt_n - x_n\| = 0. \tag{24}$$

Next, we prove that

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|u_n - t_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0.$$

For $x^* \in \Omega$ we obtain that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)S_nt_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1-\alpha_n-\beta_n)\|S_nt_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1-\alpha_n-\beta_n)\|t_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1-\alpha_n-\beta_n)[\|u_n - x^*\|^2 \\ &\quad +(\gamma_n^2k^2 - 1)\|u_n - y_n\|^2] \\ &\leq \alpha_n\|f(x_n) - x^*\| + \beta_n\|x_n - x^*\|^2 + (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 \\ &\quad +(1-\alpha_n-\beta_n)(\gamma_n^2k^2 - 1)\|u_n - y_n\|^2,\end{aligned} \tag{25}$$

it follows that

$$\begin{aligned}\|u_n - y_n\|^2 &\leq \frac{\alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)} \\ &= \frac{\alpha_n(\|f(x_n) - x^*\|^2 - \|x_n - x^*\|^2)}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)} + \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)} \\ &\leq \frac{\alpha_n(\|f(x_n) - x^*\|^2 - \|x_n - x^*\|^2)}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)} + \frac{\|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|)}{(1-\alpha_n-\beta_n)(1-\gamma_n^2k^2)}.\end{aligned}$$

From (i) and (23), we have

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{26}$$

By the same argument as in (14), we also have

$$\begin{aligned}\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\gamma_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + \gamma_n^2k^2\|t_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\gamma_n^2k^2 - 1) - \|y_n - t_n\|^2.\end{aligned} \tag{27}$$

From (25) and (27), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)S_n t_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|S_n t_n-x^*\|^2 \\
&\leq \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|t_n-x^*\| \\
&\leq \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)[\|u_n-x^*\|^2 \\
&\quad +(\gamma_n^2k^2-1)\|y_n-t_n\|^2] \\
&\leq \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|x_n-x^*\|^2 \\
&\quad +(1-\alpha_n-\beta_n)(\gamma_n^2k^2-1)\|y_n-t_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|y_n-t_n\|^2 &\leq \frac{\alpha_n(\|f(x_n)-x^*\|^2-\|x_n-x^*\|^2)}{(1-\gamma_n^2k^2)}+\frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{(1-\gamma_n^2k^2)} \\
&\leq \frac{\alpha_n(\|f(x_n)-x^*\|^2-\|x_n-x^*\|^2)}{(1-\gamma_n^2k^2)}+\frac{\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)}{(1-\gamma_n^2k^2)}.
\end{aligned}$$

From (i) and (23), we obtain

$$\lim_{n\to\infty}\|y_n-t_n\|=0. \tag{28}$$

Note that

$$\|u_n-t_n\|\leq\|u_n-y_n\|+\|y_n-t_n\|\to 0 \quad as \quad n\to\infty. \tag{29}$$

From A is k-Lipschitz continuous, we have $\|Ay_n-At_n\|=0$. From (12) and (13), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(f(x_n)-x^*)+\beta_n(x_n-x^*)+(1-\alpha_n-\beta_n)(S_n t_n-x^*)\|^2 \\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|S_n t_n-x^*\|^2 \\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|t_n-x^*\|^2 \\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|u_n-x^*\|^2 \\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)[\|x_n-x^*\|^2 \\
&\quad +r(r-2\beta)\|Fx_n-Fx^*\|^2] \\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)r(r-2\beta)\|Fx_n-Fx^*\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-\alpha_n-\beta_n)r(r-2\beta)\|Fx_n-Fx^*\|^2 &\leq \alpha_n\|(f(x_n)-x^*)\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\leq \alpha_n\|(f(x_n)-x^*)\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|).
\end{aligned}$$

From (i) and (23), we get

$$\lim_{n\to\infty}\|Fx_n-Fx^*\|=0. \tag{30}$$

Since T_r is a firmly nonexpansive for $x^* \in \Omega$, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|S_r(x_n - rFx_n) - S_r(x^* - rFx^*)\|^2 \\ &\leq \langle x_n - rFx_n - (x^* - rFx^*), u_n - x^*\rangle \\ &= \frac{1}{2}(\|x_n - rFx_n - (x^* - rFx^*)\|^2 + \|u_n - x^*\|^2 \\ &\quad -\|x_n - rFx_n - (x^* - rFx^*) - (u_n - x^*)\|^2) \\ &\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n - r(Fx_n - Fx^*)\|^2) \\ &\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\langle(Fx_n - Fx^*), x_n - u_n\rangle \\ &\quad -r^2\|Fx_n - Fx^*\|^2). \end{aligned}$$

It follows that

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\|. \tag{31}$$

Note that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + (1 - \alpha_n - \beta_n)(S_n t_n - x^*)\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|S_n t_n - x^*\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|\|t_n - x^*\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|\|u_n - x^*\|^2 \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \alpha_n - \beta_n)[\|x_n - x^*\|^2 \\ &\quad -\|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\|] \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - (1 - \alpha_n - \beta_n)\|x_n - u_n\|^2 \\ &\quad +2r\|Fx_n - Fx^*\|\|x_n - u_n\|, \end{aligned}$$

it follows that

$$\begin{aligned} (1 - \alpha_n - \beta_n)\|x_n - u_n\|^2 &\leq \alpha_n\|(f(x_n) - x^*)\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad +2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\ &\leq \alpha_n\|(f(x_n) - x^*)\|^2 - \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\ &\quad +2r\|Fx_n - Fx^*\|\|x_n - u_n\|. \end{aligned}$$

From (i) (23) and (33) , we obtain that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{32}$$

Since

$$\begin{aligned} \|S_n u_n - u_n\| &\leq \|S_n u_n - S_n t_n\| + \|S_n t_n - x_n\| + \|x_n - u_n\| \\ &\leq \|u_n - t_n\| + \|S_n t_n - x_n\| + \|x_n - u_n\|. \end{aligned}$$

By (22), (27) and (32), we have

$$\lim_{n\to\infty} \|S_n u_n - u_n\| = 0. \tag{33}$$

Consider,

$$\|Su_n - u_n\| \le \|Su_n - S_n u_n\| + \|S_n u_n - u_n\|.$$

By Lemma (2.9) and (33), we get

$$\lim_{n\to\infty} \|Su_n - u_n\| = 0. \tag{34}$$

Next we show that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle \le 0,$$

where $z_0 = P_\Omega f(z_0)$. To show this inequality, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle f(z_0) - z_0, x_{n_i} - x^*\rangle = \limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_k}} \rightharpoonup w$. Without loss of generality, assume that $x_{n_i} \rightharpoonup w$. Consider, for all $x, y \in H$,

$$\begin{aligned}\|P_F(I-A)x - P_F(I-A)y\| &\le \|(I-A)x - (I-A)y\| \\ &\le \|I-A\|\|x-y\| \\ &\le (1-\mu)\|x-y\|.\end{aligned}$$

Hence $P_\Omega(I-A)$ is contraction and has a unique fixed point, say $x^* \in \Omega$. That is, $x^* = P_\Omega(I-A)(x^*)$. We next prove that $w \in GMEP$. By $u_n = S_r(x_n - rFx_n)$, we know that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge 0, \ \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge \Theta(y, u_n), \ \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle \ge \Theta(y, u_{n_i}), \ \forall y \in C. \tag{35}$$

For $t \in (0, 1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (35) we have

$$\begin{aligned}\langle y_t - u_{n_i}, Fy_t\rangle &\ge \langle y_t - u_{n_i}, Fy_t\rangle - \varphi(y_t) + \varphi_{(u_{n_i})} - \langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle + \Theta(y, u_{n_i}) \\ &= \langle y_t - u_{n_i}, Fy_t - Fu_{n_i}, Fu_{n_i} - Fx_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + \Theta(y, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Fu_{n_i} - Fx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of φ, $\frac{u_{n_i} - x_{n_i}}{r} \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Fy_t\rangle \ge -\varphi(y_t) + \varphi(w) + \Theta(y_t, w), \ \forall y \in C. \tag{36}$$

From (A1), (A4), and (36), we also have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[\Theta(y_t, w) + \varphi(w) - \varphi(y_t)] \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t, w, Fy_t\rangle \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y, w, Fy_t\rangle \end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Fy_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw\rangle \geq 0.$$

for all $y \in C$ and hence $w \in GMEP(F, \varphi)$.

(b) Now we show that $w \in VI(C, A)$. For this purpose, we define a set-valued mapping $T : H \to 2^H$ by

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1 & \text{if } w_1 \in C, \\ \emptyset & \text{if } w_1 \notin C. \end{cases}$$

where $N_C w_1$ is the normal cone to C at $w_1 \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tw_1$ if and only if $w_1 \in VI(C, A)$. Let $(w_1, g) \in G(T)$. Then $Tw_1 = Aw_1 + N_C w_1$ and hence $g - Aw_1 \in N_C w_1$. So, we have $\langle w_1 - t, g - Aw_1\rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \gamma_n Ay_n)$ and $w_1 \in C$ we have

$$\langle u_n - \gamma_n Ay_n - t_n, t_n - w_1\rangle \geq 0$$

and hence

$$\langle w_1 - t_n, \tfrac{t_n - u_n}{\gamma_n} + Ay_n\rangle \geq 0$$

Therefor, we have

$$\begin{aligned} \langle w_1 - t_{n_i}, g\rangle &\leq \langle w_1 - t_{n_i}, Aw_1\rangle \\ &\leq \langle w_1 - t_{n_i}, Aw_1\rangle - \langle w_1 - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}} + Ay_{n_i}\rangle \\ &= \langle w_1 - t_{n_i}, Aw_1 - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle \\ &= \langle w_1 - t_{n_i}, Aw_1 - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle \\ &= \langle w_1 - t_{n_i}, Aw_1 - At_{n_i}\rangle + \langle w_1 - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle \\ &\leq \langle w_1 - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle w_1 - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\gamma_{n_i}}\rangle. \end{aligned}$$

Hence we obtain $\langle w_1 - w, g\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$.

(C) We next show that $w \in F(S) = \bigcap_{i=1}^{N} F(T_i)$. Suppose the contrary, $w \notin F(S)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Sw$, from the Opial's condition we have

$$\begin{aligned}
\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\| \\
&= \liminf_{i\to\infty} \|u_{n_i} - Su_{n_i} + Su_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \|Su_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \|u_{n_i} - w\|
\end{aligned}$$

which is a contradiction. So, we get $w \in F(S) = \bigcap_{i=1}^{N} F(T_i)$. This implies $w \in \Omega$. Therefor, we have

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{n\to\infty}\langle f(z_0) - z_0, x_{n_i} - z_0\rangle = \langle f(z_0) - z_0, w - z_0\rangle \leq 0. \quad (37)$$

Consider,

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n + \beta_n)S_n t_n - z_0\|^2 \\
&= \|\alpha_n(f(x_n) - z_0) + \beta_n(x_n - z_0) + (1 - \alpha_n + \beta_n)(S_n t_n - z_0)\|^2 \\
&\leq \|\beta_n(x_n - z_0) + (1 - \alpha_n + \beta_n)(S_n t_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\beta_n\|x_n - z_0\| + (1 - \alpha_n + \beta_n)\|S_n t_n - z_0\|)^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\beta_n\|x_n - z_0\| + (1 - \alpha_n + \beta_n)\|t_n - z_0\|)^2 + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (\beta_n\|x_n - z_0\| + (1 - \alpha_n + \beta_n)\|x_n - z_0\|)^2 + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - z_0\|)^2 + \alpha_n\alpha(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle
\end{aligned}$$

then we obtain

$$(1 - \alpha_n\alpha)\|x_{n+1} - z_0\|^2 \leq ((1 - \alpha_n)^2 + \alpha_n\alpha)\|x_n - z_0\|^2 + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle,$$

it follows that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \frac{(1 - \alpha_n)^2 + \alpha_n\alpha}{1 - \alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1 - \alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \frac{1 - 2\alpha_n\alpha_n^2 + \alpha_n\alpha}{1 - \alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1 - \alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-2\alpha_n+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z_0\|^2+\frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-z_0\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle \\
&= 1-\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\|x_n-z_0\|^2+\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\{\frac{\alpha_n M}{2(1-\alpha)} \\
&\quad +\frac{1}{(1-\alpha)}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\} \\
&\le (1-\delta_n)\|x_n-z_0\|^2+\delta_n b_n,
\end{aligned}
$$

when $M=\sup\|x_n-z_o\|^2$, $\delta_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}$ and $b_n=\{\frac{\alpha_n M}{2(1-\alpha)}+\frac{1}{(1-\alpha)}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\}$
From $\lim_{n\to\infty}\alpha_n=0$ and $\limsup_{n\to\infty}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\le 0$, we have $\limsup_{n\to\infty} b_n\le 0$,
and from

$$\sum_{n=1}^{\infty}\alpha_n=\infty,\ \text{implies}\ \sum_{n=1}^{\infty}\delta_n=\infty$$

By Lemma 2.3 we have $\lim_{n\to\infty}\|x_n-z_0\|=0$. This complete the proof. □

4 Application

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A5) and $\varphi: C\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\Omega := \bigcap_{i=1}^n F(T_i)\cap VI(C,A)\cap MEP(\Theta,\varphi)\neq\emptyset$. Let W_n be the W-mappings generated by $T_1,T_2,\dots,T_N$ and $\lambda_{n1},\lambda_{n2},\dots,\lambda_{nN}$. Assume that either $(B1)$ or $(B2)$ holds and that v is an arbitrary point in C. Let $\{x_n\}\subseteq C$, $\{u_n\}\subseteq C$ and $\{y_n\}\subseteq C$ be sequences generated by;*

$$
\begin{cases}
x_1=x\in C\\
\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C\\
y_n=P_C(u_n-\gamma_n Au_n)\\
x_{n+1}=\alpha_n v+\beta_n x_n+(1-\alpha_n-\beta_n)W_nP_C(u_n-\gamma_n Ay_n)
\end{cases}
\tag{38}
$$

for every $n=1,2,...,$ where $\{\gamma_n\},\{r_n\},\{\alpha_n\},\{\lambda_{n1}\},\{\lambda_{n2}\},...,\{\lambda_{nN}\}$ and $\{\beta_n\}$ are sequences of numbers satisfying the condition:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$*;*

(iv) $\lim_{n\to\infty} \gamma_n = 0$*;*

(v) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, ..., N$.
Then $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converges strongly to* $w = P_\Omega(v)$.

Proof. Let $F \equiv 0$ and replace S_n by W_n and $f(x) = v$ for all $x \in C$, we have equation (12) reduce to (38). From Theorem 3.1, the sequence $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converges strongly to $w = P_\Omega(v)$. This complete the proof. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H. $VI(C, A) \cap GMEP(\Theta, \varphi) \neq \emptyset$. Assume that either $(B1)$ or $(B2)$ holds and that v is an arbitrary point in C. Let $\{x_n\} \subseteq C$, $\{u_n\} \subseteq C$ and $\{y_n\} \subseteq C$ be sequences generated by;*

$$\begin{cases} x_1 = x \in C \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n)\rangle \geq 0, \quad \forall y \in C \\ y_n = P_C(u_n - \gamma_n A u_n) \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) P_C(u_n - \gamma_n A y_n) \end{cases} \tag{39}$$

for every $n = 1, 2, ...,$ where $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences of numbers satisfying the condition in Theorem 3.1. Then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converges strongly to $w = P_{VI(A,C)\cap MEP(\Theta,\varphi)}(v)$.

Proof. Let $A \equiv 0$, we have equation (12) reduce to (38). From Theorem 3.1, the sequence $\{x_n\}$ and $\{u_n\}$ generated by (38) converges strongly to $x^* \in VI(A, C) \cap MEP(\Theta, \varphi)$. This complete the proof. □

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References

[1] S. Atsushiba, W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, Indian J. Math. 41 (1999) 435-453.

[2] A. Kangtunyakarn, S. Suantai, Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis: Hybrid Systems 3 (2009) 296-309.

[3] J.W. Peng, J.C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems nd variational inequality problems*, Taiwanese J. Math. 12 (6) (2008) 1401–1432.

[4] Z. Opial, *Weak convergence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967) 591-597.

[5] R.T. Rockafellar, *On the maximality of suma of nonlinear monotone operators*, Trans. Amer. Math. Soc. 149 (1970) 75-88.

[6] H. K. Xu, *Iterative algorithims for nonlinear operators*, J. London. Math. Soc 66 (1) (2002) 240–256.

[7] H. K. Xu, *An iterative approach to guadratic optimazation*, J. Optim. Theory Appl. (2003) 659–678.

[8] T. Suzuki, *Strong convergence theorems for an infinite family of nonexpansive mappings in general Banach space*, Fixed Point Theory Appl. 1 (2005) 103–123.

[9] E. Blum and W. Oettli, Fromoptimization and variational inequalities to equilibrium problems, The Mathematics Student, vol. 63, no. 14, pp. 123-145, 1994.

[10] L.-C. Ceng and J.-C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, Journal of Computational and Applied Mathematics, vol. 214, no. 1, pp. 186-201, 2008.

[11] O. Chadli, S. Schaible, and J. C. Yao, Regularized equilibrium problems with application to noncoercive hemivariational inequalities, Journal of Optimization Theory and Applications, vol. 121, no. 3, pp. 571-596, 2004.

[12] O. Chadli, N. C. Wong, and J. C. Yao, Equilibrium problems with applications to eigenvalue problems, Journal of Optimization Theory and Applications, vol. 117, no. 2, pp. 245-266, 2003.

[13] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, Journal of Nonlinear and Convex Analysis, vol. 6, no. 1, pp. 117-136, 2005.

[14] S. D. Flam and A. S. Antipin, Equilibrium programming using proximal-like algorithms, Mathematical Programming, vol. 78, no. 1, pp. 29-41, 1997.

[15] I. V. Konnov, S. Schaible, and J. C. Yao, Combined relaxation method for mixed equilibrium problems, Journal of Optimization Theory and Applications, vol. 126, no. 2, pp. 309-322, 2005.

[16] B. Lemaire, Which fixed point does the iteration method select? in Recent Advances in Optimization (Trier, 1996), vol. 452 of Lecture Notes in Economics and Mathematical Systems, pp. 154-167, Springer, Berlin, Germany, 1997.

[17] J.-W. Peng and J.-C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems, Taiwanese Journal of Mathematics, vol. 12, no. 6, pp. 1401-1432, 2008.

[18] J.-W. Peng and J.-C. Yao, Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems, Mathematical and Computer Modelling 49 (2009) 1816-1828.

[19] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis: Theory, Methods and Applications, vol. 69, no. 3, pp. 1025-1033, 2008.

[20] W. Takahashi, K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, Math. Comput. Modelling 32 (2000) 1463-1471.

[21] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol. 331, no. 1, pp. 506-515, 2007.

[22] Y. Yao, Y.-C. Liou, and J.-C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory and Applications, vol. 2007, Article ID 64363, 12 pages, 2007.

[23] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems, Taiwanese Journal of Mathematics, vol. 10, no. 6, pp. 1497-1514, 2006.

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Convergence Theorem of a New Iterative Method for Mixed Equilibrium Problems and Variational Inclusions: Approach to Variational Inequalities

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Abstract

The purpose of this work, we present a new iterative algorithm for finding a common of the set of solutions of a mixed equilibrium problem, the set of a variational inclusion and the set of fixed point of nonexpansive mapping in a real Hilbert space. Under suitable conditions, some strong convergence theorems for approximating a common element of the above three sets are obtained. The results presented in the paper improve some recent results of Y. C. Liou, [An Iterative Algorithm for Mixed Equilibrium Problems and Variational Inclusions Approach to Variational Inequalities, Fixed Point Theory and Applications, Volume 2010, Article ID 564361, 15 pages. doi:10.1155/2010/564361].

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Keywords: mixed equilibrium problem; variational inclusion; maximal monotone

1 Introduction

Let C be a closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a nonlinear mapping, let $\varphi : C \to R$ be a function, and let Θ be a bifunction

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of $C \times C$ into R. Now we consider the following mixed equilibrium problem: to find $u \in C$ such that

$$\Theta(u,y) + \varphi(y) - \varphi(u) + \langle Fu, y-u \rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

The set of solution of problem (1.1) is denoted by EP.

If $F = 0$, then the mixed equilibrium problem (1.1) becomes the followsing mixed equilibrium: to find $u \in C$ such that

$$\Theta(u,y) + \varphi(y) - \varphi(u) \geq 0, \forall y \in C, \tag{1.2}$$

which was considered by Ceng and Yao [3]. If $\varphi = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium: to find $u \in C$ such that

$$\Theta(u,y) + \langle Fu, y-u \rangle \geq 0, \ \forall y \in C, \tag{1.3}$$

which was studied by S. Takahashi and W. Takahashi [16]. If $\varphi = 0$ and $F = 0$, then the mixed equilibrium problem (1.1) becomes the following problem: to find $u \in C$ such that

$$\Theta(u,y) \geq 0, \ \forall y \in C. \tag{1.4}$$

If $\Theta(x,y) = 0$ for all $x, y \in C$, the mixed equilibrium problem (1.1) becomes the following variational inequality problem: to find $u \in C$ such that

$$\varphi(y) - \varphi(u) + \langle Fu, y-u \rangle \geq 0, \ \forall y \in C. \tag{1.5}$$

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [2, 7, 19, 5, 4, 8] Some methods have been proposed to solve the mixed equilibrium problem and the equilibrium problem. In 1997, Flaim and Antipen [7] introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, S. Takahashi and W. Takahashi [17] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem(1.2) and the set of fixed point points of a nonexpansive mapping. Furthermore, Yao et al. [18] introduced some new iterative schemes for finding a common element of the set of solutions of the equilibrium problem (1.2) and the set of common fixed points of finitely (infinitely) nonexpansive mappings. Very recently, Ceng and Yao [3] considered a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings. Peng and Yao [12] developed a CQ method.

They obtained some strong convergence results for finding a common element of the set of solutions of the mixed equilibrium problem (1.1) and the set of the variational inequality and the set of fixed points of a nonexpansive mapping. Their results extend and improve the corresponding results in [3, 6, 9, 17].

Recall that a mapping $B : C \to C$ is said to be β-inverse strongly monotone if there exists a constant $\beta > 0$ such that $\langle Bx - By, x - y\rangle \geq \beta\|Bx - By\|^2$, for all $x, y \in C$. A mapping A is strongly positive on H if there exists a constants $\mu > 0$ such that $\langle Ax, x\rangle \geq \mu\|x\|^2$ for all $x \in H$. Let $B : H \to H$ be a single-valued nonlinear mapping and let $R : H \to 2^H$ be a set-valued mapping. Now we concern the following variational inclusion, which is to find a point $x \in H$ such that

$$\theta \in B(x) + R(x), \tag{1.6}$$

where θ is the zero vector in H. The set of solution of problem (1.6) is denoted by $I(B, R)$. If $H = R^m$, then problem (1.6) becomes the generalized equation introduced by Robinson [14]. If $B = 0$, then problem (1.6) becomes the inclusion problem introduced by Rockafellar [15].

In 2010, Y. C. Liou [11], introduce iterative algorithm generated by for $x_0 \in C$ the sequence $\{x_n\}$ and $\{u_n\}$ as follows:

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n], \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Then $\{x_n\}$ strongly convergent to a common element of the set of solutions of a mixed equilibrium problem and the set of a variational inclusion in a real Hilbert space.

Inspired and motivate by the work in this paper, we present an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of variational inclusion and the set of fixed point of nonexpansive mappings in real Hilbert space.

2 Preliminary

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x - P_Cx\| \leq \|x - y\| \text{ for all } y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2,$$

for every $x, y \in H$. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Leftrightarrow \langle x - x^*, x^* - y\rangle \geq 0, \ \forall y \in C. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let the set-valued mapping $R : H \to 2^H$ be maximal monotone. We define the resolvent operator

$$J_{R,\lambda}(x) = (I + \lambda R)^{-1}(x), \ x \in H, \tag{2.3}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single valued, nonexpansive, and 1-inverse strong monotone and that a solution of problem (1.6) is a fixed point of the operator $J_{R,\lambda}(I - \lambda B)$ for all $\lambda > 0$, see, for instance, [9].

Throughout this paper, we assume that a bifunction $\Theta : C \times C \to \mathbb{R}$ and a convex function $\varphi : C \to \mathbb{R}$ satisfy the following condition:

$(H1)$ $\Theta(x, x) = 0$ for all $x \in C$;

$(H2)$ Θ is monotone, that is $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

$(H3)$ for each $y \in C, x \mapsto \Theta(x, y)$ is weakly upper semicontinuous;

$(H4)$ for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous;

$(H5)$ for each $x \in C$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{2}\langle y_x - z, z - x\rangle < 0. \tag{2.4}$$

Lemma 2.1. *[12] Let C be a nonempty closed convex subset of real Hilbert space H and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. For $r > 0$ anf $x \in C$, define a mapping $S_r : C \to C$ as follows:*

$$S_r(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{2}\langle y - z, z - x\rangle \geq 0, \forall y \in C\} \tag{2.5}$$

for all $x \in C$. Assume that the condition $(H1)-(H5)$ hold. Then one has the following results:

(1) *for each $x \in C, S_r(x) \neq \emptyset$ and S_r is single valued;*

(2) *S_r is firmly nonexpansive, that is, for any $x, y \in C$,*

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle; \tag{2.6}$$

(3) *$F(S_r) = EP$;*

(4) *EP is closed and convex.*

Lemma 2.2. *[1] Let $R : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a Lipschitz-continuous mapping. Then the mapping $(R + B) : H \to 2^H$ is maximal monotone.*

Lemma 2.3. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$

(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2,\ \forall t \in [0, 1].$

Lemma 2.4. *[10] Let $a_n, b_n,$ and c_n be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $t_n \in [0, 1)$ with $\Sigma_{n=1}^{\infty} t_n = +\infty, b_n = o(t_n)$ and $\Sigma_{n=1}^{\infty} c_n < +\infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[13] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

The following lemmas will be useful for proving the convergence result of this paper.

3 Main Results

In this section, we derive a strong convergence of an iterative algorithm which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points nonexpansive mapping of C into itself and the set of the variational inclusion in Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F := F(T) \cap EP \cap I(B,R) \neq \emptyset$. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying $(H1)-(H4)$, let F, B be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let A be a strongly positive bounded linear operator with coefficient $0 < \mu < 1$ and $R : H \to 2^H$ be a maximal monotone mapping. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $x_0 = C$;*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \forall y \in C \\ y_n = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n], \\ x_{n+1} = \beta_n u + (1 - \beta_n)Ty_n, \ n \geq 0 \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ are satisfying: $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:

(*i*) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(*ii*) $\lim_{n\to\infty} \beta_n = 0$, $\Sigma_{n=1}^{\infty}\beta_n = \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(*iii*) $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$,

(*iv*) $0 < r \leq 2\alpha, 0 < \lambda \leq 2\beta$,

Then $\{x_n\}$ converge strongly to $z_0 = P_F u$ which solves the following variational inequality

$$\langle (Ax, y - x\rangle \geq 0, \quad \forall y \in F. \tag{3.2}$$

Proof. Since F is α-inverse strongly monotone and B is β-inverse strongly monotone, we have

$$\|(I - rF)x - (I - rF)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Fx - Fy\|^2, \tag{3.3}$$

$$\|(I - \lambda B)x - (I - \lambda B)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\beta)\|Bx - By\|^2. \tag{3.4}$$

It is clear that if $0 < r \leq 2\alpha$ and $0 < \lambda \leq 2\beta$, then $(I - rF)$ and $(I - \lambda B)$ are all nonexpansive. Set $w_n = J_{R,\lambda}(u_n - \lambda Bu_n), n \geq 0$. It follows that

$$\|w_n - x^*\| = \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\| \leq \|(u_n - \lambda Bu_n) - (x^* - \lambda Bx^*)\| \leq \|u_n - x^*\|. \tag{3.5}$$

By Lemma 2.3, we have $u_n = S_r(x_n - rFx_n)$ for all $n \geq 0$. Then, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|S_r(x_n - rFx_n) - S_r(x^* - rFx^*)\|^2 \\
&\leq \|(x_n - rFx_n) - (x^* - rFx^*)\|^2 \\
&\leq \|(x_n - x^*) - r_n(Bx_n - Bx^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2r\langle Fx_n - Fx^*, x_n - x^*\rangle + r\|Fx^* - Fx_n\|^2 \qquad (3.6) \\
&\leq \|x_n - x^*\|^2 - 2r\alpha\|Fx_n - Fx^*\|^2 + r^2\|Fx^* - Fx_n\|^2 \\
&\leq \|x_n - x^*\|^2. \qquad (3.7)
\end{aligned}$$

Hence,

$$\|w_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\|. \tag{3.8}$$

Since A is linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}.$$

Observe that

$$\langle (I - \alpha_n A)u, u\rangle = 1 - \alpha_n\langle Au, u\rangle \geq 1 - \alpha_n\|A\| \geq 0,$$

that is to say $I - \alpha_n A$ is positive operator. It follows that

$$\begin{aligned}
\|(I - \alpha_n A)\| &= \sup\{|\langle (I - \alpha_n A)u, u\rangle| : u \in H, \|u\| = 1\} \\
&= \sup\{\langle (I - \alpha_n A)u, u\rangle : u \in H, \|u\| = 1\} \\
&= \sup\{1 - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\} \\
&\leq 1 - \alpha_n\mu.
\end{aligned}$$

From (3.1), we deduce that

$$\begin{aligned}
\|y_n - x^*\| &= \|P_C[(I - \alpha_n A)w_n] - x^*\| \\
&\leq \|(I - \alpha_n A)w_n - x^*\| \\
&= \|(I - \alpha_n A)(w_n - x^*) - \alpha_n Ax^*\| \\
&\leq \|I - \alpha_n A\|\|w_n - x^*\| + \alpha_n\|Ax^*\| \\
&\leq (1 - \alpha_n\mu)\|w_n - x^*\| + \alpha_n\|Ax^*\| \\
&= (1 - \alpha_n\mu)\|w_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu} \\
&\leq (1 - \alpha_n\mu)\|x_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu},
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\beta_n u+(1-\beta_n)Ty_n-x^*\| \\
&= \|\beta_n(u-x^*)+(1-\beta_n)(Ty_n-x^*)\| \\
&\leq \beta_n\|u-x^*\|+(1-\beta_n)\|Ty_n-x^*\| \\
&\leq \beta_n\|u-x^*\|+(1-\beta_n)\|y_n-x^*\| \\
&\leq \beta_n\|u-x^*\|+(1-\beta_n)[(1-\alpha_n\mu)\|x_n-x^*\|+\alpha_n\mu\frac{\|Ax^*\|}{\mu}] \\
&\leq \beta_n\max\{\|u-x^*\|,\|x_0-x^*\|,\frac{\|Ax^*\|}{\mu}\} \\
&\quad +(1-\beta_n)\max\{\|u-x^*\|,\|x_0-x^*\|,\frac{\|Ax^*\|}{\mu}\} \\
&= \max\{\|u-x^*\|,\|x_0-x^*\|,\frac{\|\|Ax^*\|}{\mu}\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{u_n\},\{y_n\},\{Ty_n\}$ and $\{Ay_n\}$ are all bounded.

Step 2 We must show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\beta_n u+(1-\beta_n)Ty_n-(\beta_{n-1}u+(1-\beta_{n-1})Ty_{n-1})\| \\
&= \|(1-\beta_n)(Ty_n-Ty_{n-1})+(1-\beta_n)Ty_{n-1}+(\beta_n-\beta_{n-1})u \\
&\quad +(1-\beta_{n-1})Ty_{n-1})\| \\
&= \|(1-\beta_n)(Ty_n-Ty_{n-1})+(\beta_n-\beta_{n-1})(u-Ty_{n-1})\| \\
&\leq (1-\beta_n)\|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\|u-Ty_{n-1}\| \\
&\leq \|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\|u-Ty_{n-1}\|. \qquad (3.9)
\end{aligned}$$

Note that,

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|P_C(I-\alpha_nA)w_n-P_C(I-\alpha_{n-1}A)w_{n-1}\| \\
&\leq \|(I-\alpha_nA)w_n-(I-\alpha_{n-1}A)w_{n-1}\| \\
&= \|(I-\alpha_nA)(w_n-w_{n-1})+(I-\alpha_nA)w_{n-1}-(I-\alpha_{n-1}A)w_{n-1}\| \\
&\leq (1-\alpha_n\mu)\|w_n-w_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Aw_{n-1}\|, \qquad (3.10)
\end{aligned}$$

and from $(I-\lambda B)$ and $(I-rF)$ are nonexpansive, we have

$$\begin{aligned}
\|w_n-w_{n-1}\| &= \|J_{R,\lambda}(u_n-\lambda Bu_n)-J_{R,\lambda}(u_{n-1}-\lambda Bu_{n-1})\| \\
&\leq \|(I-\lambda B)u_n-(I-\lambda B)u_{n-1}\| \\
&= \|u_n-u_{n-1}\| \\
&= \|S_r(x_n-rFx_n)-S_r(x_{n-1}-rFx_{n-1})\| \\
&\leq \|(I-rF)x_n-(I-rF)x_{n-1}\| \\
&\leq \|x_n-x_{n-1}\|. \qquad (3.11)
\end{aligned}$$

Substituting (3.11) in (3.10), we get

$$\|y_n - y_{n-1}\| \leq (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\|, \tag{3.12}$$

and substituting (3.12) into (3.9), we get

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\| + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|, \tag{3.13}$$

and we have

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n\mu)\|x_n - x_{n-1}\| + \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|\mu\frac{\|Aw_{n-1}\|}{\mu} + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|. \tag{3.14}$$

Put $t_n := \alpha_n\mu$, $b_n := \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|\mu\frac{\|Aw_{n-1}\|}{\mu}$ and $c_n := |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|$ from $(i), (ii), (iii)$ and bounded of $\{\|u - Ty_{n-1}\|\}$ and by Lemma 2.4, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3 Prove that $\|Fx_n - F_{x^*}\| \to 0$ and $\|Bu_n - B_{x^*}\| \to 0$ as $n \to \infty$. Consider

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\leq \|(I-\lambda B)u_n - (I-\lambda B)x^*\|^2 \\
&= \|u_n - x^*\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bx^*\|^2 \\
&= \|x_n - x^*\|^2 + r(r-2\alpha)\|Fx_n - Fx^*\|^2 \\
&\quad +\lambda(\lambda - 2\beta)\|Bu_n - Bx^*\|^2, && (3.15) \\
&\leq \|x_n - x^*\|^2, (\text{ since } r < 2\alpha \text{ and } \lambda < 2\beta). && (3.16)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(I-\alpha_n A)w_n - x^*\|^2 \\
&\leq \|(I-\alpha_n A)w_n - x^*\|^2 \\
&= \|w_n - x^* - \alpha_n Aw_n\|^2 \\
&= \|w_n - x^*\|^2 - 2\alpha_n\langle w_n - x^*, Aw_n\rangle + \alpha_n\|Aw_n\|^2 && (3.17) \\
&= \|w_n - x^*\|^2 + \alpha_n(2\|w_n - x^*\|\|Aw_n\| + \|Aw_n\|^2) \\
&= \|w_n - x^*\| + d_n, && (3.18)
\end{aligned}$$

where $d_n = \alpha_n(2\|w_n - x^*\|\|Aw_n\| + \|Aw_n\|^2)$. From $\lim_{n\to\infty}\alpha_n = 0$ and boundedness, we have $\lim_{n\to\infty} d_n = 0$, there exists $N \in \mathbb{N}$ such that

$$\|y_n - x^*\|^2 \leq \|w_n - x^*\|^2 \leq \|x_n - x^*\|^2, \forall n \geq N. \tag{3.19}$$

Note that from (3.15) and (3.18), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n u+(1-\beta_n)Ty_n-x^*\|^2\\ &= \|\beta_n(u-x^*)+(1-\beta_n)(Ty_n-x^*)\|^2\\ &= \beta_n\|u-x^*\|^2+(1-\beta_n)\|Ty_n-x^*\|^2\\ &\le \beta_n\|u-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2 \qquad (3.20)\\ &\le \beta_n\|u-x^*\|^2+\|w_n-x^*\|^2+\alpha_n(2\|w_n-x^*\|\|Aw_n\|+\|Aw_n\|^2)\\ &\le \|x_n-x^*\|^2+r(r-2\alpha)\|Fx_n-Fx^*\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bx^*\|^2\\ &\quad +\beta_n\|u-x^*\|^2+d_n. \qquad (3.21)\end{aligned}$$

It follows that

$$\begin{aligned}r(2\alpha-r)\|Fx_n-Fx^*\|^2 &+ \lambda(2\beta-\lambda)\|Bu_n-Bx^*\|^2\\ &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad +d_n+\beta_n\|u-x^*\|^2\\ &\le \|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_n-x^*\|)\\ &\quad +d_n+\beta_n\|u-x^*\|^2\end{aligned}$$

.

From $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\beta_n=0$, we have

$$\lim_{n\to\infty}\|Fx_n-Fx^*\|=0=\lim_{n\to\infty}\|Bu_n-Bx^*\|.$$

Next, we show that $\lim_{n\to\infty}\|u_n-x_n\|=0=\lim_{n\to\infty}\|u_n-w_n\|$. Since S_r is a firmly nonexpansive, we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|S_r(x_n-rFx_n)-S_r(x^*-rFx^*)\|^2\\ &\le \langle x_n-rFx_n-(x^*-rFx^*),u_n-x^*\rangle\\ &= \frac{1}{2}(\|x_n-rFx_n-(x^*-rFx^*)\|^2+\|u_n-x^*\|^2\\ &\quad -\|x_n-rFx_n-(x^*-rFx^*)-(u_n-x^*)\|^2)\\ &\le \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n-r(Fx_n-Fx^*)\|^2)\\ &\le \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n\|^2\\ &\quad +2r\langle(Fx_n-Fx^*),x_n-u_n\rangle-r^2\|Fx_n-Fx^*\|^2)\end{aligned}$$

It follows that

$$\|u_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-u_n\|^2+2r\|Fx_n-Fx^*\|\|x_n-u_n\|. \qquad (3.22)$$

Since $J_{R,\lambda}$ is $1-$inverse strongly monotone, we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\ &\leq \langle (u_n - \lambda Bu_n) - (x^* - \lambda Bx^*), w_n - x^*\rangle \\ &= \frac{1}{2}(\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*)\|^2 + \|w_n - x^*\|^2 \\ &\quad -\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*) - (w_n - x^*)\|^2) \\ &\leq \frac{1}{2}(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n - \lambda(Bu_n - Bx^*)\|^2) \\ &\leq \frac{1}{2}(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2 \\ &\quad +2\lambda\langle (Bu_n - Bx^*), u_n - w_n\rangle - \lambda^2\|Bu_n - Bx^*\|^2).\end{aligned}$$

Which implies that

$$\|w_n - x^*\|^2 \leq \|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2 + 2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\|. \tag{3.23}$$

By (3.22) and (3.23), we have

$$\begin{aligned}\|w_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\|. \end{aligned}\tag{3.24}$$

Substituting (3.24) into (3.17), we have

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\ &\quad +2\lambda\|Bu_n - Bx^*)\|\|u_n - w_n\| + d_n \\ &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n. \end{aligned}\tag{3.25}$$

Substituting (3.25) into (3.20), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2 \\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - u_n\|^2 + \|u_n - w_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n + \beta_n\|u - x^*\|^2 \\ &\leq \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\ &\quad +2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n + \beta_n\|u - x^*\|^2.\end{aligned}$$

From $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0 = \lim_{n\to\infty}\|Fx_n - Fx^*\| = 0 = \lim_{n\to\infty}\|Bu_n - Bx^*\|$, $\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n$ and bounded of sequences, we get

$$\lim_{n\to\infty}\|u_n - x_n\| = 0 = \lim_{n\to\infty}\|u_n - w_n\|.$$

Step 4 Prove that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_F u$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{i\to\infty}\langle u - z_0, x_{n_i} - z_0\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_k}} \rightharpoonup w$. Without loss of generality, assume that $x_{n_i} \rightharpoonup w$. Consider, for all $x, y \in H$,

$$\begin{aligned}\|P_F(I-A)x - P_F(I-A)y\| &\le \|(I-A)x - (I-A)y\| \\ &\le \|I-A\|\|x-y\| \\ &\le (1-\mu)\|x-y\|.\end{aligned}$$

Hence $P_F(I-A)$ is contraction and has a unique fixed point, say $x^* \in F$. That is, $x^* = P_F(I-A)(x^*)$. We next prove that $w \in EP$. By $u_n = S_r(x_n - rFx_n)$, we know that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge 0, \ \forall y \in C.$$

It follows from (H2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge \Theta(y, u_n), \ \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle \ge \Theta(y, u_{n_i}), \ \forall y \in C. \quad (3.26)$$

For $t \in (0,1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (3.26) we have

$$\begin{aligned}\langle y_t - u_{n_i}, Fy_t\rangle &\ge \langle y_t - u_{n_i}, Fy_t\rangle - \varphi(y_t) + \varphi_{(u_{n_i})} \\ &\quad -\langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\rangle + \Theta(y, u_{n_i}) \\ &= \langle y_t - u_{n_i}, Fy_t - Fu_{n_i}, Fu_{n_i} - Fx_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad -\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + \Theta(y, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Fu_{n_i} - Fx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of φ, $\frac{u_{n_i} - x_{n_i}}{r} \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Fy_t\rangle \ge -\varphi(y_t) + \varphi(w) + \Theta(y_t, w), \ \forall y \in C. \quad (3.27)$$

From (H1), (H4), and (3.27), we also have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[\Theta(y_t, w) + \varphi(w) - \varphi(y_t)] \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t, w, Fy_t\rangle \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y, w, Fy_t\rangle \end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Fy_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw\rangle \geq 0.$$

This implies that $w \in EP$.

Next, we show that $w \in I(B, R)$. In fact, since B is β-inverse strongly monotone, B is Lipschitz continuous monotone mapping. It follows from Lemma 2.2 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$ that is, $g - Bv \in R(v)$. Again since $w_{n_i} = J_{R,\lambda}(u_{n_i} - \lambda Bu_{n_i})$, we have $u_{n_i} - \lambda u_{n_i} \in (I + \lambda R)(w_{n_i})$, that is, $(1/\lambda)(u_{n_i} - w_{n_i} - \lambda Bu_{n_i}) \in R(w_{n_i})$. By virtue of the maximal monotonicity of $R + B$, we have

$$\langle v - w_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda Bu_{n_i})\rangle \geq 0,$$

and so

$$\begin{aligned} \langle v - w_{n_i}, g\rangle &\geq \langle v - w_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda Bu_{n_i}\rangle \\ &= \langle v - w_{n_i}, Bv - Bw_{n_i} + Bw_{n_i} - Bu_{n_i} + \frac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle \\ &\geq \langle v - w_{n_i}, Bw_{n_i} - Bu_{n_i}\rangle + \langle v - w_{n_i}, \frac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle. \end{aligned}$$

It follows from $\|u_n - w_n\| \to 0, \|Bu_n - Bw_n\| \to 0$ and $w_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \to \infty} \langle v - w_{n_i}, g\rangle = \langle v - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $B + R$ that $\theta \in (R + B)(w)$, that is $w \in I(B,R)$. Next, we can show that $w \in F(T)$. Consider,

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(I-\alpha_n A)w_n - P_C(I-\alpha_n A)x^*\|^2 \\
&\leq \langle (I-\alpha_n A)w_n - (I-\alpha_n A)x^*, y_n - x^* \rangle \\
&= \frac{1}{2}(\|(I-\alpha_n A)w_n - (I-\alpha_n A)x^*\|^2 + \|y_n - x^*\|^2 \\
&\quad -\|(I-\alpha_n A)w_n - (I-\alpha_n A)x^* - (y_n - x^*)\|^2) \\
&\leq \frac{1}{2}(\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2 \\
&\quad +2\alpha_n\langle w_n - y_n, Aw_n - Ay_n\rangle - \alpha_n^2\|Aw_n - Ay_n\|^2) \\
&\leq \frac{1}{2}(\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2 \\
&\quad +2\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|),
\end{aligned}
$$

it follows that

$$
\|w_n - y_n\|^2 \leq \|w_n - x^*\|^2 - \|y_n - x^*\|^2 + \alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|. \quad (3.28)
$$

From (3.20), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|u - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\
&\leq \beta_n\|u - x^*\|^2 + \|y_n - x^*\|^2
\end{aligned}
$$

Then, we get

$$
-\|y_n - x^*\|^2 \leq \beta_n\|u - x^*\|^2 - \|x_{n+1} - x^*\|^2. \quad (3.29)
$$

Replace (3.8) and (3.29) into (3.28), we have

$$
\begin{aligned}
\|w_n - y_n\|^2 &\leq \|x_n - x^*\|^2 + (\beta_n\|u - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\
&\quad +\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\| \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n\|u - x^*\|^2 \\
&\quad +\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\| \\
&\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad +\beta_n\|u - x^*\|^2 + \alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|.
\end{aligned}
$$

From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\beta_n = 0$ and boundedness, we obtain

$$
\lim_{n\to\infty}\|w_n - y_n\| = 0.
$$

Note that

$$
\begin{aligned}
\|Tu_n - u_n\| &\leq \|Tu_n - Tw_n\| + \|Tw_n - Ty_n\| + \|Ty_n - x_{n+1}\| \\
&\quad +\|x_{n+1} - x_n\| + \|x_n - u_n\| \\
&\leq \|u_n - w_n\| + \|w_n - y_n\| + \beta_n\|Ty_n - u\| + \|x_{n+1} - x_n\| \\
&\quad +\|x_n - u_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

From $u_{n_i} \rightharpoonup w$ and H satisfying Opial's condition, it is easy to prove that $w \in F(T)$. Therefore, $w \in F$. It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle &= \lim_{i\to\infty}\langle u - z_0, x_{n_i} - z_0\rangle \\ &= \langle u - z_0, w - z_0\rangle \leq 0. \end{aligned} \tag{3.30}$$

From (3.1), we have for any $n \geq \mathbb{N}$

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\beta_n u + (1-\beta_n)Ty_n - z_0\|^2 \\ &= \|(1-\beta_n)(Ty_n - z_0) + \beta_n(u - z_0)\|^2 \\ &\leq (1-\beta_n)\|Ty_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\leq (1-\beta_n)\|y_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\leq (1-\beta_n)\|x_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \end{aligned} \tag{3.31}$$

Since $\Sigma_{n=1}^{\infty}\beta_n = \infty$, (3.30) and Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - z_0\| = 0$, we get that $\{x_n\}$ converges strongly to $z_0 = P_F u$. □

Corollary 3.2. *[11] Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\Omega := \cap EP \cap I(B,R) \neq \emptyset$. Let $\Theta : H \times C \to \mathbb{R}$ be a bifunction satisfying $(H1)-(H4)$, let F, B be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $r > 0$ and $\lambda > 0$ be two constants such that $r < 2\alpha$ and $\lambda < 2\beta$. Let A be a strongly positive bounded linear operator with coefficient $0 < \mu < 1$ and $R : H \to 2^H$ be a maximal monotone mapping. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $x_0 = C$;*

$$\begin{cases}\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n],\end{cases} \tag{3.32}$$

where $\{\alpha_n\}[0,1]$ are satisfying: $\{r_n\} \subset (0,\infty)$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$,

Then $\{x_n\}$ converge strongly to $x^* \in \Omega$ which solves the following variational inequality

$$\langle (Ax, y - x\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.33}$$

Proof. In Theorem 3.1, let $T = I$ and $\beta_n = 0$ for all $n \in \mathbb{N}$, then, we can obtain Corollary. This completes the proof. □

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References

[1] H. Brezis, Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematics Studies, no. 5, North-Holland, Amsterdam, The Netherlands, 1973.

[2] E. Blum and W. Oettli, Fromoptimization and variational inequalities to equilibrium problems, The Mathematics Student, vol. 63, no. 14, pp. 123-145, 1994.

[3] L.-C. Ceng and J.-C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, Journal of Computational and Applied Mathematics, vol. 214, no. 1, pp. 186-201, 2008.

[4] O. Chadli, S. Schaible, and J. C. Yao, Regularized equilibrium problems with application to noncoercive hemivariational inequalities, Journal of Optimization Theory and Applications, vol. 121, no. 3, pp. 571-596, 2004.

[5] O. Chadli, N. C. Wong, and J. C. Yao, Equilibrium problems with applications to eigenvalue problems, Journal of Optimization Theory and Applications, vol. 117, no. 2, pp. 245-266, 2003.

[6] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, Journal of Nonlinear and Convex Analysis, vol. 6, no. 1, pp. 117-136, 2005.

[7] S. D. Flam and A. S. Antipin, Equilibrium programming using proximal-like algorithms, Mathematical Programming, vol. 78, no. 1, pp. 29-41, 1997.

[8] I. V. Konnov, S. Schaible, and J. C. Yao, Combined relaxation method for mixed equilibrium problems, Journal of Optimization Theory and Applications, vol. 126, no. 2, pp. 309-322, 2005.

[9] B. Lemaire, Which fixed point does the iteration method select? in Recent Advances in Optimization (Trier, 1996), vol. 452 of Lecture Notes in Economics and Mathematical Systems, pp. 154-167, Springer, Berlin, Germany, 1997.

[10] L. S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194, 114-125, (1995).

[11] Y. C. Liou, An Iterative Algorithm for Mixed Equilibrium Problems and Variational Inclusions Approach to Variational Inequalities, Fixed Point Theory and Applications Volume 2010, Article ID 564361, 15 pages doi:10.1155/2010/564361.

[12] J.-W. Peng and J.-C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems, Taiwanese Journal of Mathematics, vol. 12, no. 6, pp. 1401-1432, 2008.

[13] S. Plubtieng and R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol. 336, no. 1, pp. 455- 469, 2007.

[14] S. M. Robinson, Generalized equations and their solutions. I. Basic theory, Mathematical Programming Study, no. 10, pp. 128-141, 1979.

[15] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM Journal on Control and Optimization, vol. 14, no. 5, pp. 877898, 1976.

[16] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis: Theory, Methods and Applications, vol. 69, no. 3, pp. 1025-1033, 2008.

[17] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol. 331, no. 1, pp. 506-515, 2007.

[18] Y. Yao, Y.-C. Liou, and J.-C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory and Applications, vol. 2007, Article ID 64363, 12 pages, 2007.

[19] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems, Taiwanese Journal of Mathematics, vol. 10, no. 6, pp. 1497-1514, 2006.

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Strong Convergence Theorem by Hybrid Method for Non-Lipschitzian Mapping

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Abstract

We introduce the hybrid method of modified Mann's iteration for an asymptotically k-strict pseudo-contractive mapping T in the intermediate sense which is necessarily lipschitzian. We establish strong convergence theorem for such method. The result extend and improve the recent ones announced by Inchan and Nammanee, Inchan and concern result of Takahashi, Takeuchi and Kubota [Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 341 (2008), 276–286], and many others.

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1 Introduction

Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ a mapping. Recall the following concepts.

(i) T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

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ii) T is asymptotically nonexpansive (cf. [4]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying $\lim_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all integers $n \ge 1$ and $x, y \in C$.

iii) T is uniformly Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|$ for all integers $n \ge 1$ and all $x, y \in C$.

(iv) T is asymptotically nonexpansive in the intermediate sense [2] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by Agarwal, O'Regan and Sahu [1], Bruck, Kuczumow and Reich [2], Chidume, Shahzad and Zegeye [3], Kim and Kim [9] and many others.

In 2008, Kim and Xu [11] introduced the concept of asymptotically k-strict pseudo-contractive mappings in Hilbert space as below:

Definition 1.1. *Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, 1)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that*

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 \qquad (1)$$

for all $x, y \in C$ and $n \in N$.

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{\frac{k+\sqrt{1+(1-k)\gamma_n}}{1+k} : n \in N\}$.

Recently, Sahu et al.[16] introduced the concept of asymptotically k-strict pseudo-contractive mappings in the intermediate sense which are not necessarily Lipschitzian (see Lemma 2.6 [16]) as below:

Definition 1.2. *Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ will be called an asymptotically k-strict pseudo-contractive* *mapping in the intermediate sense with sequence* $\{\gamma_n\}$ *if there exist a constant* $k \in [0,1)$ *and a sequence* $\{\gamma_n\}$ *in* $[0,1)$ *with* $\lim_{n\to\infty} \gamma_n = 0$ *such that*

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \leq 0. \tag{2}$$

Throughout this paper we assume that

$$c_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - T^n x - (y - T^n y)\|^2)\}.$$

Then $c_n \geq 0$ *for all* $n \in N$, $c_n \to 0$ *as* $n \to \infty$ *and (2) reduces to the relation*

$$\|T^n x - T^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n \tag{3}$$

for all $x, y \in C$ *and* $n \in N$

Remark 1.3. *If* $c_n = 0$ *for all* $n \in N$ *in (3) then* T *is an asymptotically* k*-strict pseudocontractive mapping with sequence* $\{\gamma_n\}$*.*

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [5, 13, 17, 20]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [10, 20].

Iteration method for finding a fixed point of an asymptotically k-strict pseudo-contractive mapping T is the modified Mann's iteration method studied in [12, 18, 19, 21] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T^n x_n, \quad n \geq 0, \tag{4}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ line in the interval $(0, 1)$.

In 2007, Takahashi, Takeuchi and Kubota [20] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n) T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases} \tag{5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then they prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$. In 2008, Kumam [8], introduce an iterative scheme by a new hybrid method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a real Hilbert space.

In 2008, Inchan [6], introduce the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let C be a nonempty closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then him prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, Inchan and Nammanee [7], introduce the modified Mann iteration processes for an asymptotically k-strict pseudo-contractive mapping. Let C be a nonempty closed convex subset of a Hilbert space H, T be an asymptotically k-strict pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} \alpha_n < 1-k$. Then they prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these fact, it is the purpose of this paper to introduce the modified Mann iteration processes for an asymptotically k-strict pseudo-contractive mapping in the intermediate sense by idear in (7). Let C be a closed convex subset of a Hilbert space H, $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping in the intermediate sense and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0,\ (n \to \infty)$.

We shall prove that the iteration generated by (8) converges strongly to $z_0 = P_{F(T)}x_0$.

2 Preliminary

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

We collect some lemmas which will be used in the proof for the main result.

Lemma 2.1. *[14] There holds the identity in a Hilbert space H:*

(i) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H.$

(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0,1]$.

Lemma 2.2. *[15] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.3. *[16] Let C be a nonempty subset of a Hilbert space H and let $T : C \to C$ a uniformly continuous asymptotically k-strict pseudo-contractive in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.4. *[16] Let C be a nonempty closed convex subset of Hilbert space H and $T : C \to C$ a continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that $\{x_n\}$ is sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{n\to\infty} \|x_n - T^n x_n\| = 0$, then $(I - T)x = 0$.*

3 Main Results

In this section, we prove strong convergence theorem by hybrid method for asymptotically k-strict pseudo-contractive mapping in the intermediate sense in Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty closed convex bounded subset of H. Let T be a uniformly continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1-k$. Then $\{x_n\}$ generated by (8) converges strongly to $z_0 = P_{F(T)}x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. For $u \in F(T) \subset C_m$. By lemma 2.1, we have,

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\
&= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 \\
&\quad + k\|x_m - T^m x_m\|^2 + c_m] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + (k-\alpha_m)(1-\alpha_m)\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \qquad (1)
\end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$. Then

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\
&= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\
&\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&\quad + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.
\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \le \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, $\|y_m - y\|^2 \le \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, we have

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\
&= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\
&\le \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad +(1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad -\alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&\quad +[k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\
&\quad +\theta_m + c_m \\
&= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_0$. By Lemma 2.2, we have

$$\langle x_0 - x_n, x_n - y \rangle \ge 0, \text{ for all } y \in C_n.$$

Moreover, by the same proof of Theorem 3.1 of [7], we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n, \tag{3}$$

By the definition of y_n, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\
&= (1-\alpha_n)\|T^n x_n - x_n\|.
\end{aligned}$$

From (3), we have

$$\begin{aligned}(1-\alpha_n)^2\|T^nx_n-x_n\|^2 &= \|y_n-x_n\|^2\\ &= \|y_n-x_{n+1}+x_{n+1}-x_n\|^2\\ &\leq \|y_n-x_{n+1}\|^2+\|x_{n+1}-x_n\|^2\\ &\quad +2\|y_n-x_{n+1}\|\|x_{n+1}-x_n\|\\ &\leq \|x_n-x_{n+1}\|^2+[k-\alpha_n(1-\alpha_n)]\|x_n-T^nx_n\|^2\\ &\quad +\theta_n+c_n+\|x_{n+1}-x_n\|^2+2\|y_n-x_{n+1}\|\|x_{n+1}-x_n\|\\ &= [k-\alpha_n(1-\alpha_n)]\|x_n-T^nx_n\|^2\\ &\quad +2\|x_{n+1}-x_n\|(\|x_{n+1}-x_n\|+\|y_n-x_{n+1}\|)\\ &\quad +\theta_n+c_n.\end{aligned}$$

It follows that

$((1-\alpha_n)^2-(k-\alpha_n(1-\alpha_n)))\|x_n-T^nx_n\|^2 \leq 2\|x_{n+1}-x_n\|(\|x_{n+1}-x_n\|+\|y_n-x_{n+1}\|)+\theta_n+c_n.$

Hence

$$(1-k-\alpha_n)\|T^nx_n-x_n\| \leq 2\|x_{n+1}-x_n\|(\|x_{n+1}-x_n\|+\|y_n-x_{n+1}\|)+\theta_n+c_n. \tag{4}$$

From $\limsup_{n\to\infty}\alpha_n < 1-k$, we can chosen $\epsilon > 0$ such that $\alpha_n \leq 1-k-\epsilon$ for large enough n. From (2) and (4), we have

$$\lim_{n\to\infty}\|T^nx_n-x_n\| = 0. \tag{5}$$

From (2), (5) and Lemma 2.3, we have

$$\lim_{n\to\infty}\|Tx_n-x_n\| = 0. \tag{6}$$

Since H is reflexive and $\{x_n\}$ is bounded we get that $\emptyset \neq \omega_w(x_n)$. From Lemma 2.4, we have $\omega_w(x_n) \subset F(T)$. By the fact that $\|x_n-x_0\| \leq \|z_0-x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0-z_0\| &\leq \|x_0-w\| \leq \liminf_{n\to\infty}\|x_0-x_n\|\\ &\leq \limsup_{n\to\infty}\|x_0-x_n\| \leq \|x_0-z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\|x_n-z_0\|^2 = \|x_n-x_0\|^2+2\langle x_n-x_0, x_0-z_0\rangle+\|x_0-z_0\|^2$$

$$\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.$$

This complete the proof. □

Using this Theorem 3.1, we have the following corollaries.

Corollary 3.2. *[7] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically k-strict pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (7) converges strongly to $z_0 = P_{F(T)}x_0$.*

Corollary 3.3. *[6] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (7) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.4. *([20] Theorem 4.1) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

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References

[1] R.P. Agarwal, Donal O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61–79

[2] R. E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, colloq. Math. 65 (1993) 169–179.

[3] C. Chidume, N. Shahzad, H. Zegeye, Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense, Numer. Funct. Anal. Optim. 25 (2004) 239–257.

[4] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **35** (1972) 171-174.

[5] S. Ishikawa, Fixed point theorems for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **44** (1974) 147-150.

[6] I. Inchan, Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, Int. Journal of Math. Analysis. **2**(23) (2008) 1135-1145.

[7] I. Inchan, K. Nammanee, Strong convergence theorems by hybrid methods for asymptotically k-strict pseudo-contractive mappings in Hilbert space, Nonlinear Anal. : Hybrid syst. **3**(4) (2009) 380-385.

[8] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, Journal of Applied Mathematics and Computing. **29** (2009) 263–280.

[9] G.E. Kim, T.H. Kim, Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces, Comp. Math. Appl. 42 (2001) 1565–1570.

[10] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear. Anal. **64** (2006) 1140–1152.

[11] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically k-strict pseudcontractions, Nonlinear. Anal. **68** (2008) 2828–2836.

[12] T.C. Lim, H.K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal.**22** (1994) 1345–1355.

[13] W. A. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. **4**(1953) 506–510.

[14] G. Marino, H.K. Xu, Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl.**329** (2007) 336–349.

[15] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math Anal. Appl. **336** (2007) 455–469.

[16] D. R. Sahu, H. K. Xu and J. C. Yao, Asymptotically strict pseudocontractive mapping in the intermediate sense, Nonlinear Anal. **70** (2009) 3502–3511

[17] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Amer. Math. Soc. **43** (1991) 153–159.

[18] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. **158** (1991) 407–413.

[19] J. Schu, Approximation of fixed points of asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **112** (1991) 143–151.

[20] W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. **341** (2008) 276–286.

[21] K.K. Tan, H.K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc.**122**(1994) 733–739.

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Strong Convergence Theorem by Hybrid Iterative Scheme for Generalized Equilibrium Problems and Fixed Point Problems of Strictly Pseudo-Contraction Mappings

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Abstract

The purpose of this work is to introduce a hybrid iterative scheme for finding a common element of the set of a generalized equilibrium problem, the set of solutions to a variational inequality and the set of fixed points of a strict pseudo-contraction mappings in a real Hilbert space. The results obtained in this paper extend and improve the result of Cho, Qin and Kang [Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, Nonlinear Anal. doi:10.1016/j.na.2009.02.106], and many authors.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping S of C into itself is nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$. The set of fixed points of S is denoted by $F(S)$. Let F be

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a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \ \text{ for all } \ y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [6] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(C, A)$, is to find $x^* \in C$ such that $\langle Ax^*, v - x^* \rangle \geq 0$ for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [15] and the references therein. Let $B : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem(GEP): Find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C \tag{2}$$

The set of such $z \in C$ is denoted by EP, i.e.,

$EP = \{z \in C : F(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in C\}$.

In the case of $B \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is also denoted by $VI(C, A)$. A mapping A of C into H is called α-inverse-strongly monotone [2] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$. Recently, Takahashi and Toyoda [11] and Yao et al. [16] introduced an iterative method for finding an element of $VI(C, A) \cap F(S)$, where $A : C \to H$ is an α-inverse-strongly monotone mapping. Let A be a strongly positive bounded linear operator on H: that is, there exists a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2 \ \text{ for all } \ x \in H. \tag{3}$$

A mapping $S : C \to C$ is called a *k-strict pseudo-contraction mapping* if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \tag{4}$$

for all $x, y \in C$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty. It is well-known that S is nonexpansive if and only if S is 0-strictly pseudo-contractive. The mapping S is also said to be pseudo-contractive if $k = 1$ and S is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $S - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings.

In 1967, Browder and Petryshyn [2] established the first convergence result for k-strict pseudo-contraction in real Hilbert spaces. They proved weak and strong convergence theorem by using iteration with a constant control sequence $\{\alpha_n\} = \alpha$ for all n. Many authors have appeared in the literature on the existence of solution equilibrium, see also, for example [1, 5, 8, 12] and references therein. To find an element of $EP(F) \cap F(S)$, Takahashi and Takahashi [12] introduced the an iterative scheme for nonexpansive mappings by the hybrid method in a Hilbert space.

Recently, in 2008, Takahashi and Takahashi [10] introduced a hybrid iterative method for finding a common element of EP and $F(S)$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} u_n \in C, \quad \text{such that} \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(a_n u + (1 - a_n u_n)), \quad \forall n \in \mathbf{N}. \end{cases} \tag{5}$$

where B be an β-inverse strongly monotone mapping of C into H with positive real number α, and proved strong convergence theorems in the framework of a Hilbert space, under some suitable conditions on parameters $\{a_n\}, \{\beta_n\}$ and $\{\lambda_n\}$.

Very recently, Cho, et al. [4], Ceng et al. [5], Liu [7] and Peng et al. [9] established an iterative scheme for finding a common element of the set of solution of an equilibrium problem (1), generalized equilibrium problem (2) and the set of fixed point of a k-strict pseudo-contraction mapping in the setting of real Hilbert space. They also studied some weak and strong convergence theorem for k-strict pseudo-contraction mappings of the sequence generated by their algorithm.

In 2009, Cho, Qin and Kang [3] introduce the hybrid methods for finding a common element of $F(S) \cap VI(C, A) \cap EP$. Let S be a k-strict pseudo-contraction mapping and defined $_K x = kx(1 - k)Sx$ for all $x \in C$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n Au_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n)S_k y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ n \geq 1 \end{cases}$$

A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. They proved strong convergence theorems in the framework of a Hilbert space, under some suitable conditions on parameters $\{a_n\}, \{r_n\}$ and $\{\lambda_n\}$.

In this paper, we extend and improve the result of Cho, Qin and Kang [3]. Then, we obtain the strong convergence theorem for the sequences generated by these processes. Furthermore, using the theorem we also obtain strong convergence theorems for finding elements of fixed points, equilibrium problems and the set of solutions to a variational inequality, respectively.

2 Preliminary

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfied

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{6}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following propertied: $P_C x \in C$ and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{7}$$

for all $x \in H$, $y \in C$. The following is the property in Hilbert spaces: for any $x, y \in H$, we have

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$

(ii) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$

(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iv) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \ \forall t \in [0, 1]$.

Remark 2.1 *We note that if A is a α-inverse-strongly monotone, for all $u, v \in C$ and $\lambda_n > 0$,*

$$\begin{aligned} \|(I - \lambda_n A)u - (I - \lambda_n A)v\|^2 &= \|(u - v) - \lambda_n(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda_n\langle u - v, Au - Av\rangle \\ &\quad + \lambda_n^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{8}$$

So, if $\lambda_n \leq 2\alpha$, then $I - \lambda_n A$ is a nonexpansive mapping from C to H.

Lemma 2.2 *[17] Let $T : K \to H$ be a k-strictly pseudo-contraction. Defined $D : K \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in K$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y \in C$, $\lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.3 *[1] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$ for all $y \in C$.*

Lemma 2.4 [1, 6, 10] *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)-(A4), and let $r > 0$ and $x \in H$. Then,there exists unique $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$ for all $y \in C$. Moreover, let T_r be a mapping of H into C defined by $T_r(x) = z$ for all $x \in H$. Then, the following hold:*

1.T_r is single- valued;

2.T_r is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$, for any $x, y \in H$;

3.$F(T_r) = EP(F)$;

4.$EP(F)$ is closed and convex;

5.$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$, for all $s, t > 0$ and $x \in H$.

Lemma 2.5 *(see [13, 14]) Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,*

$$a_{n+1} \leq (1 - \gamma_n)a_n + b_n, \ n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence in $\mathbf{R}$ such that:

i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

In this section, we prove a strong convergence theorem of the hybrid method for strictly pseudo-contractive mappings in a real Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive for some $0 \leq k < 1$. Defined a mapping*

$S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume that $F := F(S) \cap VI(C,A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by

$$\begin{cases} x_1 = C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ w_n = \beta_n S_k y_n + (1-\beta_n) y_n, \\ z_n = \alpha_n x_n + (1-\alpha_n) S_k w_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1 \end{cases} \tag{9}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0,\infty)$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:
(i) $k \leq \alpha_n, \beta_n \leq a < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$,for some $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ converge strongly to z, where $z = P_F x_1$.

Proof. Let $p \in F$ since $0 \leq r_n < 2\beta$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n B x_n) - T_{r_n}(p - r_n B p)\|^2 \\ &\leq \|(x_n - r_n B x_n) - (p - r_n B p)\|^2 \\ &\leq \|(x_n - p) - r_n(Bx_n - Bp)\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bp - Bx_n\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 + r_n^2\|Bp - Bx_n\|^2 \qquad (10) \\ &\leq \|x_n - p\|^2. \qquad (11) \end{aligned}$$

First we show that $F \subset C_n$ for all $n \in \mathbf{N}$, we can prove by induction. It is obvious that $F \subset C_1$. Let $p \in F$, we known that $I - \lambda_n A$ is nonexpansive, for all $n \in \mathbf{N}$ and from $p \in VI(C,A)$ we get $p = P_C(p - \lambda_n A p)$. It follows that

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(p - \lambda_n A p)\|^2 \\ &\leq \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^2 \\ &\leq \|u_n - p\|^2. \end{aligned} \tag{12}$$

Consider,

$$\begin{aligned} \|w_n - p\| &= \|\beta_n(S_k y_n - p) + (1-\beta_n)(y_n - p)\| \\ &\leq \beta_n\|S_k y_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\leq \beta_n\|y_n - p\| + (1-\beta_n)\|y_n - p\| \\ &= \|y_n - p\| \\ &= \|u_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{13}$$

Thus, we have

$$\begin{aligned}\|z_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)S_k w_n - p\| \\ &= \|\alpha_n(x_n - p) + (1-\alpha_n)(S_k w_n - p)\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|S_k w_n - p\| \qquad (14)\\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|w_n - p\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\ &= \|x_n - p\|. \qquad (15)\end{aligned}$$

So, we have $p \in C_{n+1}$ and hence $F \subset C_n$, for all $n \in \mathbf{N}$.

Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbf{N}$. Let $c_j \in C_{m+1} \subset C_m$ with $c_j \to z$. Since C_m is closed, $z \in C_m$ and $\|z_m - c_j\| \le \|c_j - x_m\|$. Then

$$\begin{aligned}\|z_m - z\| &= \|z_m - c_j + c_j - z\| \\ &\le \|z_m - c_j\| + \|c_j - z\| \qquad (16)\end{aligned}$$

Taking $j \to \infty$, we have $\|z_m - z\| \le \|z - x_m\|$. Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|z_m - x\| \le \|x - x_m\|, \|z_m - y\| \le \|y - x_m\|$, we have

$$\begin{aligned}\|z_m - z\|^2 &= \|z_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|\alpha(z_m - x) + (1-\alpha)(z_m - y)\|^2 \\ &= \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|(z_m - x) - (z_m - y)\|^2 \\ &\le \alpha\|z_m - x\|^2 + (1-\alpha)\|z_m - y\|^2 - \alpha(1-\alpha)\|y - x\|^2 \\ &\le \alpha\|x_m - x\|^2 + (1-\alpha)\|x_m - y\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\ &= \|x_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|x_m - z\|^2. \qquad (17)\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_1$, we have $\langle x_1 - x_n, x_n - y\rangle \ge 0$, for all $y \in C_n$. Since $F \subset C_n$, we obtain

$$\langle x_1 - x_n, x_n - u\rangle \ge 0 \text{ for all } u \in F \text{ and } n \in \mathbf{N}. \qquad (18)$$

So, for $u \in F$, we get

$$\begin{aligned}0 \le \langle x_1 - x_n, x_n - u\rangle &= \langle x_1 - x_n, x_n - x_1 + x_1 - u\rangle \\ &= -\langle x_n - x_1, x_n - x_1\rangle + \langle x_1 - x_n, x_1 - u\rangle \\ &\le -\|x_n - x_1\|^2 + \|x_1 - x_n\|\|x_1 - u\|.\end{aligned}$$

This implies that $\|x_1 - x_n\|^2 \le \|x_1 - x_n\|\|x_1 - u\|$, hence

$$\|x_1 - x_n\| \le \|x_1 - u\| \text{ for all } u \in F \text{ and } n \in \mathbf{N}. \qquad (19)$$

From $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\langle x_1 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \text{for all } n \in \mathbf{N}. \tag{20}$$

So, for $x_{n+1} \in C_n$, we also have, for $n \in \mathbf{N}$

$$\begin{aligned} 0 \leq \langle x_1 - x_n, x_n - x_{n+1}\rangle &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &= -\langle x_n - x_1, x_n - x_1\rangle + \langle x_1 - x_n, x_1 - x_{n+1}\rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|. \end{aligned}$$

This implies that $\|x_1 - x_n\|^2 \leq \|x_1 - x_n\|\|x_1 - x_{n+1}\|$ and we get

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\| \quad \text{for all } n \in \mathbf{N}. \tag{21}$$

From (19), we have $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (20), we note that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_1) + (x_1 - x_{n+1})\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\langle x_1 - x_n, x_1 - x_n\rangle - 2\langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2 \\ &= -\|x_n - x_1\|^2 + \|x_1 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we obtain

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{22}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ imply that

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{23}$$

Further, we get $\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|$.
From (22) and (23), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{24}$$

Next, we show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. For $p \in \Theta$. From (13), (10) and by (ii), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_k w_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_k w_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_k w_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_k w_n\|^2 \quad (25) \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - 2r_n\beta\|Bx_n - Bp\|^2 \\ &\quad + r_n^2\|Bp - Bx_n\|^2] \quad (26) \\ &= \|x_n - p\|^2 + d(e - 2\beta)\|Bx_n - Bp\|^2, \end{aligned}$$

and hence

$$\begin{aligned} d(2\beta - e)\|Bx_n - Bp\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &= \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \end{aligned}$$

From (24), we have $\lim_{n\to\infty} \|Bx_n - Bp\| = 0$. From remark 2.1 that for $\lambda_n \leq 2\beta$ then $I - r_nB$ is nonexpansive, for all $n \in \mathbf{N}$, T_{r_n} is firmly nonexpansive and by using Lemma 2.4, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\|^2 \\ &\leq \langle (x_n - r_nBx_n) - (p - r_nBp), u_n - p\rangle \\ &= \frac{1}{2}(\|(x_n - r_nBx_n) - (p - r_nBp)\|^2 + \|u_n - p\|^2 \\ &\quad - \|(x_n - r_nBx_n) - (p - r_nBp) - (u_n - p)\|^2) \\ &\leq \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Bx_n - Bp)\|^2) \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2). \end{aligned}$$

Thus, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2. \tag{27}$$

From (27), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_kw_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_kw_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - S_kw_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 \qquad (28) \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Bx_n - Bp\rangle - r_n^2\|Bx_n - Bp\|^2] \qquad (29) \\ &\leq \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bp\|, \end{aligned}$$

it follows that

$$\begin{aligned} (1 - a)\|x_n - u_n\|^2 &\leq (1 - \alpha_n)\|x_n - u_n\|^2 \\ &\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bp\| \\ &\leq \|x_n - z_n\|(\|x_n - p\| - \|z_n - p\|) \\ &\quad + 2r_n\|x_n - u_n\|\|Bx_n - Bp\|. \end{aligned}$$

Using (24) and $\|Bx_n - Bp\| \to 0$, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{30}$$

Consider,

$$\begin{aligned}\|w_n - p\|^2 &= \|\beta_n(S_k y_n - p) + (1-\beta_n)(y_n - p)\|^2 \\ &= \beta_n\|S_k y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\ &\le \beta_n\|y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\ &= \|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2, \qquad (31)\end{aligned}$$

From (28) and (31) we also have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - S_k y_n\|^2] \qquad (32) \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\ &\le \|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}(1-a)k(1-a)\|y_n - S_k y_n\|^2 &\le (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

From (24), we have

$$\lim_{n\to\infty} \|S_k y_n - y_n\| = 0. \qquad (33)$$

Next, we show that $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Consider

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(p - \lambda_n A p)\|^2 \\ &\le \|(u_n - \lambda_n A u_n) - (p - \lambda_n A p)\|^2 \\ &= \|(u_n - p) - \lambda_n(A u_n - A p)\|^2 \\ &= \|u_n - p\|^2 - \lambda_n\langle u_n - p, A u_n - A p\rangle + \lambda_n^2\|A u_n - A p\|^2 \\ &\le \|x_n - p\|^2 - 2\lambda_n\alpha\|A u_n - A p\|^2 + \lambda_n^2\|A u_n - A p\|^2 \\ &= \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A u_n - A p\|^2 \\ &\le \|x_n - p\|^2 + b(c - 2\alpha)\|A u_n - A p\|^2.\end{aligned}$$

From (32) and (ii), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|y_n - S_k y_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + b(c - 2\alpha)\|A u_n - A p\|^2] \\ &\le \|x_n - p\|^2 + (1-\alpha_n)b(c - 2\alpha)\|A u_n - A p\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}(1-a)b(2\alpha - c)\|A u_n - A p\|^2 &\le (1-\alpha_n)b(2\alpha - c)\|A u_n - A p\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).\end{aligned}$$

From (24), that

$$\lim_{n\to\infty} \|Au_n - Ap\| = 0. \tag{34}$$

From (6), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\leq \langle (u_n - \lambda_n Au_n) - (p - \lambda_n Ap), y_n - p\rangle \\
&= \frac{1}{2}\{\|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\|^2 + \|y_n - p\|^2 \\
&\quad - \|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap) - (y_n - p)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) - \lambda_n(Au_n - Ap)\|^2\} \\
&= \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle \\
&\quad - \lambda_n^2\|Au_n - Ap\|^2\},
\end{aligned}$$

so, we obtain

$$\|y_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2. \tag{35}$$

From (12), (32), (35) and (i) we have

$$\begin{aligned}
\|z_n - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 - (1 - \alpha_n)\beta_n(1 - \beta_n)\|y_n - S_k y_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|u_n - p\|^2 - \|u_n - y_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2] \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\|u_n - y_n\|^2 \\
&\quad + (1 - \alpha_n)2\lambda_n\|u_n - p\|\|Au_n - Ap\| \\
&\leq \|x_n - p\|^2 - (1 - \alpha_n)\|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1 - a)\|u_n - y_n\|^2 &\leq (1 - \alpha_n)\|u_n - y_n\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\| \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2\lambda_n\|u_n - p\|\|Au_n - Ap\|,
\end{aligned}$$

From (i), (24) and (34), we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{36}$$

Next, we show that $\lim_{n\to\infty} \|S_k u_n - u_n\| = 0$, consider

$$\begin{aligned}
\|S_k u_n - u_n\| &\leq \|S_k u_n - S_k y_n\| + \|S_k y_n - y_n\| + \|y_n - u_n\| \\
&\leq 2\|y_n - u_n\| + \|S_k y_n - y_n\|.
\end{aligned}$$

From (36) and (33) we obtain that

$$\lim_{n\to\infty} \|S_k u_n - u_n\| = 0. \tag{37}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup w$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. Since C is closed and convex, $w \in C$. Next, we show that $w \in F$. First, we show that $w \in VI(C, A)$. Define,

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{38}$$

Then, T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, u - Av\rangle \geq 0$. On the other hand, from $y_n = P_C(u_n - \lambda_n A u_n)$, we have $\langle v - y_n, y_n - (u_n - \lambda_n A u_n)\rangle \geq 0$, that is,

$$\left\langle v - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, u\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle,
\end{aligned}$$

Since $\lim_{n\to\infty} \|y_n - u_n\| = 0$ and A is Lipschitz continuous, we obtain

$$\langle v - w, u\rangle \geq 0. \tag{39}$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. Next, we show that $w \in EP$. It follows by (9) and (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}) \tag{40}$$

Put $y_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Since $y \in C$ and $w \in C$, we have $y_t \in C$. So, from (30), we have

$\langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, By_t\rangle = 0 \geq -\langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$

and hence

$\langle y_t - u_{n_i}, By_t\rangle \geq \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$

$= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})$.

Since $\|u_{n_i} - x_{n_i}\| \to 0$, it follows that $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from monotonicity of B, we get $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4), we have

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{41}$$

as $i \to \infty$. From (A1), (A4) and (41), we have

$$\begin{aligned} 0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle \\ &\leq tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle \end{aligned}$$

and hence $0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle$. Letting $t \to 0$, we have for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Bw\rangle$. This implies that $w \in EP$. Next, we show that $w \in F(S)$. From Lemma 2.2, we have $F(S_k) = F(S)$, we may assume that $w \neq S_k w$, by Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - S_k w\| = \liminf_{i\to\infty} \|(u_{n_i} - S_k u_{n_i}) + (S_k u_{n_i} - S_k w)\| \\ &= \liminf_{i\to\infty} \|S_k u_{n_i} - S_k w\| \leq \liminf_{i\to\infty} \|u_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So, we have $w \in F(S_k) = F(S)$. Therefore $w \in F$.

Finally, we show that $x_n \to z$, where $z = P_F x_1$. Since $x_n = P_{C_n} x_1$ and $z \in F \subset C_n$, we have $\|x_n - x_1\| \leq \|z - x_1\|$. It follows from $z' = P_F x_1$ and the lower semicontinuity of the norm that

$$\|z' - x_1\| \leq \|z - x_1\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_1\| \leq \limsup_{i\to\infty} \|x_{n_i} - x_1\| \leq \|z' - x_1\|. \tag{42}$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_i} - x_1\| = \|z - x_1\| = \|z' - x_1\|$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_F x_1$. ⋄

Theorem 3.2 *[3] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive self mapping for some $0 \leq k < 1$. Defined a mapping $S_k : C \to C$ by $S_k x = kx + (1-k)Sx$ for all $x \in C$. Assume* *that* $F := F(S) \cap VI(C,A) \cap EP \neq \emptyset$. *Let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $C_1 = C \subset H$, $x_1 = P_C x_0$*;*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1-\alpha_n) S_k y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{43}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ *and* $\{r_n\} \subset (0,\infty)$. *Assume that the control sequences* $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0, 2\alpha)$ *and* $\{r_n\} \subset (0, 2\beta)$ *satisfy the following conditions: (i)* $k \leq \alpha_n \leq a < 1$, *(ii)* $0 \leq b \leq \lambda_n \leq c < 2\alpha$ *and* $0 \leq d \leq r_n \leq e < 2\beta$, *for some* $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ *converge strongly to* z, *where* $z = P_F x_0$.

Proof. If $\beta_n = 0$ for all $n \in \mathbf{N}$, by Thm 3.1, we obtain the desired result. $\diamond$

Corollary 3.3 [4, Theorem 3.1] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F : C \times C \to \mathbf{R}$ *be a bifunction satisfying (A1)-(A4). Let* B *be an* β*-inverse-strongly monotone mapping of* C *into* H. *Let* $T : C \to C$ *be a* k*-strictly pseudo-contractive self mapping for some* $0 \leq k < 1$ *such that* $\Theta := F(T) \cap VI(C,A) \cap EP \neq \emptyset$. *Let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $C_1 = C \subset H$, $x_1 = P_C x_0$*;*

$$\begin{cases} u_n \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ z_n = \alpha_n x_n + (1-\alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{44}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$, $S = kx + (1-k)T$ *for all* $x \in C$ *and* $\{r_n\} \subset (0,\infty)$. *Assume that the control sequences* $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, 2\beta)$ *satisfy the following conditions:*

(i) $0 \leq k < \alpha_n \leq a < 1$,

(ii) $0 \leq d \leq r_n \leq e < 2\beta$, *for some* $a, d, e \in \mathbf{R}$.

Then $\{x_n\}$ *converge strongly to* z, *where* $z = P_\Theta x_0$.

Proof. Put $A \equiv 0$, $\beta_n = 0$ for all $n \in \mathbf{N}$, and from Lemma 2.2, we have $F(S) = F(T)$ and Theorem 3.1, we obtain the desired result. $\diamond$

4 Applications

In this section, we obtain some strong convergence theorems by applying $F \equiv 0$ and $\beta_n \equiv 0$ for all $n \in \mathbf{N}$ in Theorem 3.1.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F := F(S) \cap VI(C, A) \cap EP \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} x_1 = C, \\ C_1 = C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n) S y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1 \end{cases} \tag{45}$$

where $u_n = T_{r_n}(x_n - r_n B x_n)$ and $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\} \subset [0, 1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:
(i) $k \leq \alpha_n \leq a < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ and $0 \leq d \leq r_n \leq e < 2\beta$, for some $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ converge strongly to z, where $z = P_F x_1$.

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let B be an β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a k-strictly pseudo-contractive for some $0 \leq k < 1$. Defined a mapping $S_k : C \to C$ by $S_k x = kx + (1 - k)Sx$ for all $x \in C$. Assume that $F := F(S) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $C_1 = C \subset H$, $x_1 = P_C x_0$ and $u_n \in C$;*

$$\begin{cases} \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n A u_n) \\ z_n = \alpha_n x_n + (1 - \alpha_n) S_k y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{46}$$

where $\{r_n\} \subset (0, \infty)$. Assume that the control sequences $\{\alpha_n\} \subset [0, 1], \{\lambda_n\} \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, 2\beta)$ satisfy the following conditions:

(i) $k \leq \alpha_n \leq a < 1$,
(ii) $0 \leq b \leq \lambda_n \leq c < 2\alpha$ *and* $0 \leq d \leq r_n \leq e < 2\beta$, *for some* $a, b, c, d, e \in \mathbf{R}$.
Then $\{x_n\}$ *converge strongly to* z, *where* $z = P_F x_0$.

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References

[1] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student. **63**(1994) 123-145.

[2] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. **20** (1967) 197-228.

[3] Y. J. Cho, X. Qin and J. I. Kang, Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, Nonlinear Anal. doi:10.1016/j.na.2009.02.106.

[4] Y.J. Cho, et al., Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.02.106

[5] L.C. Ceng, S. Al-Homidan, Q.H. Ansari and J.C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Comput. Appl. Math. (2008), doi:10.1016/j.cam.2008.03.032.

[6] P. L. Combettes, S. A. Hirstoaga, Equilibrium programing using proximal-like algorithms, Math. Program. **78** (1997), 29-41.

[7] Y. Liu, A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, Nonlinear Anal., (in press) doi:10.1016/j.na.2009.03.060

[8] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math Anal. Appl. **336** (2007), 455-469.

[9] J. W. Peng, Y. C. Liou and J. C. Yao, An Iterative Algorithm Combining Viscosity Method with Parallel Method for a Generalized Equilibrium Problem and Strict Pseudocontractions, Fixed Point Theory and Applications (2009), Article ID 794178, 21 pages doi:10.1155/2009/794178

[10] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mappings in a Hilbert spaces, Nonlinear Anal. **69** (2008) 1025-1033.

[11] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. **118** (2003) 417-428.

[12] S. Takahashi, W, Takahashi, Viscosity approxmation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. **331(1)**(2007) 506-515.

[13] H. K. Xu, An iterative approach to guadratic optimazation, J. Optim. Theory Appl. **116** (2003) 659-678.

[14] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math Anal. Appl. **298** (2004), 279-291.

[15] J. C. Yao, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, in: J.P. Crouzeix, N. Haddjissas, S. Schaible (Eds.),Handbook of Generalized Convexity and Monotonicity, Kluwer Academic, (2005) 501-558.

[16] Y. Yao, Y. C. Liou, R. Chen, Convergence theorem for fixed point problems and variational inequality problems, Nonlinear Anal. (2008), doi:10.1016/j.na.2008.02.094.

[17] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert space, Nonlinear Anal. **69 (2)**, (2008) 456–462.

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