**รายงานวิจัยฉบับสมบูรณ์**

# การหาคำตอบของปัญหาเชิงดุลยภาพและปัญหาอสมการการแปรผัน

# Finding solutions of equilibrium problems and variational inequality problems

**โดย ดร.กมลรัตน์ แนมมณี**

**มิถุนายน 2555**

สัญญาเลขที่ MRG5380202

# รายงานวิจัยฉบับสมบูรณ์

# การหาคำตอบของปัญหาเชิงดุลยภาพและปัญหาอสมการการแปรผัน
# Finding solutions of equilibrium problems and variational inequality problems

ดร.กมลรัตน์ แนมมณี มหาวิทยาลัยพะเยา
ศาสตราจารย์ ดร.สุเทพ สวนใต้ มหาวิทยาลัยเชียงใหม่ (นักวิจัยที่ปรึกษา)

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยพะเยา

# กิตติกรรมประกาศ

งานวิจัยเรื่องการหาคำตอบของปัญหาเชิงดุลยภาพและปัญหาอสมการการแปรผัน (MRG5380202) นี้ ประสบความสำเร็จลุล่วงได้ด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจาก สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) สำนักงานคณะกรรมการอุดมศึกษา (สกอ.) และมหาวิทยาลัยพะเยา ประจำปี 2553 - 2555 และขอขอบคุณ ศาสตราจารย์ ดร.สุเทพ สวนใต้ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ นักวิจัยที่ปรึกษา สำหรับการให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา

บทคัดย่อ

---

รหัสโครงการ: **MRG5380202**

ชื่อโครงการ: **การหาคำตอบของปัญหาเชิงดุลยภาพและปัญหาอสมการการแปรผัน**

ชื่อนักวิจัย และสถาบัน: **ดร.กมลรัตน์ แนมมณี  มหาวิทยาลัยพะเยา**

อีเมล์: **kamonrat.na@up.ac.th**

ระยะเวลาโครงการ: **15 มิถุนายน 2553 ถึงวันที่ 14 มิถุนายน 2555**

บทคัดย่อ: **จุดประสงค์ของงานวิจัยนี้ คือ การสร้างระเบียบวิธีการทำซ้ำชนิดใหม่ต่าง ๆ เพื่อใช้ในการประมาณค่าจุดตรึงของการส่งแบบไม่เชิงเส้นและใช้แก้ปัญหาทางคณิตศาสตร์ เช่น ปัญหาอสมการการแปรผันและปัญหาเชิงดุลยภาพ ทั้งในปริภูมิฮิลเบิร์ตและปริภูมิบานาค**

**คำสำคัญ:** ระเบียบวิธีการทำซ้ำ / ปัญหาอสมการการแปรผัน / ปัญหาเชิงดุลยภาพ / การส่งแบบหดเทียมโดยแท้เชิงกำกับในแนวอินเตอร์มีเดียท / การส่งแบบไม่ขยายเชิงกึ่งกรุป

Abstract

---

Project Code: **MRG5380202**

Project Title: **Finding solutions of equilibrium problems and variational inequality problems**

Investigator: **Dr. Kamonrat Nammanee University of Phayao**

E-mail Address: **kamonrat.na@up.ac.th**

Project Period: **June 15, 2010 – June 14, 2012**

**Abstract:** The purposes of this research are to introduce several new iterative approximation methods for approximating the fixed points of nonlinear mappings and solving many mathematical problems such as variational inequality problems and equilibrium problems in both Hilbert spaces and Banach spaces

Keywords: **Iterative method / Variational inequality problems / Equilibrium problems / Asymptotically *k*-strict pseudo-contractive mapping in the intermediate Sense / Nonexpansive semigroup**

# บทที่ 1
# บทนำ (Introduction)

ทฤษฎีจุดตรึง (fixed point theory) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน (functional analysis) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการและ การพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ๆ ที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆ นั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน (basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ทฤษฎีจุดตรึงนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับ การมีคำตอบ (existence of solution) และ การมีเพียงคำตอบเดียว ของสมการ (uniqueness of solution) ตลอดจนการคิดค้นหาวิธีในการประมาณหาคำตอบของสมการต่างๆ ดังนั้นการศึกษาทฤษฎีต่างๆ ที่เกี่ยวข้องกับการมีจุดตรึงของการส่งต่างๆ และ การหาระเบียบวิธีต่างๆที่ใช้ในการประมาณค่าคำตอบนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์กลุ่มหนึ่งจำนวนมากให้ความสนใจศึกษา เมื่อศึกษาการมีคำตอบของสมการต่างๆแล้ว ปัญหาที่น่าสนใจต่อไปก็คือ เราจะหาคำตอบของสมการต่างๆ นั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้นระเบียบวิธีการกระทำซ้ำของจุดตรึง(fixed-point iterations) ต่างๆ ที่ใช้ในการหาคำตอบ และ ประมาณคำตอบ เพื่อนำไปประยุกต์ใช้เกี่ยวข้องกับการแก้ปัญหาในเรื่องของสมการตัวดำเนินการไม่เชิงเส้น (nonlinear operator equations) ในเรื่องของแก้ปัญหาอสมการแปรผัน (variational inequality problem (VIP)) และแก้สมการหาคำตอบของปัญหาดุลภาพ(equilibrium problems (EP)) ปัญหาที่ดีที่สุด(optimizations problems) ปัญหาน้อยที่สุด (minimizations problems) ทั้งในปริภูมิฮิลเบิตร์และปริภูมิบานาค ซึ่งปัญหาดังกล่าวเป็นปัญหาที่สำคัญที่มีประโยชน์มากมายในสาขาวิชาต่างๆ เช่น สาขาวิชาฟิสิกส์ คณิตศาสตร์ประยุกต์ วิศวกรรม และสาขาทางเศรษฐศาสตร์

จากความสำคัญข้างต้นเป็นผลให้นักคณิตศาสตร์จึงได้ศึกษาและวิจัยในแขนงดังกล่าว กันอย่างต่อเนื่อง ซึ่งการวิจัยเกี่ยวกับการกระทำซ้ำของจุดตรึงและการประมาณค่าจุดตรึงที่สำคัญนั้นสามารถนำมาแก้สมการหาคำตอบของปัญหาดุลภาพ เช่น ใน ปี 1997 Combettes และ Hirstoaga [25] ได้เริ่มต้นศึกษาและใช้วิธีการทำซ้ำในการหาการประมาณค่าที่ดีที่สุดเพื่อแก้ปัญหาดุลภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้ม (strong convergence theorems) และมีนักคณิตศาสตร์อีกมากมาย นำทฤษฎีบทการทำซ้ำดังกล่าวมาประยุกต์ใช้ในการแก้อสมการแปรผัน ปัญหาค่าน้อยสุด และปัญหาอื่นๆ ทางคณิตศาสตร์

ดังความสำคัญที่ได้กล่าวมาแล้วข้างต้น ผู้วิจัยจะกล่าวถึงที่มาของการทำวิจัยนี้ โดยเริ่มต้นจากในปี ค.ศ. 1994 Stampacchaia [82] ได้เป็นผู้คิดค้นใช้วิธีการประมาณค่าแบบซ้ำเพื่อแก้ไขปัญหาอสมการการแปรผันภายใต้ตัวดำเนินการทางเดียวอย่างเข้ม และต่อเนื่องแบบลิฟชิทพ์ ต่อมา Korpelevich [43]

เห็นว่า วิธีการประมาณค่าแบบซ้ำดังกล่าวมีข้อจำกัดมากมายเพื่อให้ได้มาซึ่งคำตอบของอสมการการแปรผัน จึงได้คิดค้นวิธีการประมาณค่าแบบซ้ำขึ้นมาใหม่ซึ่งเรียกว่า วิธีเอ็กซ์ตร้ากราเดียน (Extragradient method) และพบว่าการแก้ปัญหาอสมการการแปรผันดังกล่าวนั้น ตัวดำเนินการไม่จำเป็นต้องเป็น **ตัวดำเนินการทางเดียวอย่างเข้ม** และต่อเนื่องแบบลิฟชิทพ์ แต่ขอให้เป็น **ตัวดำเนินการทางเดียว** และต่อเนื่องแบบลิฟชิทพ์ ก็เพียงพอแล้ว นอกจากวิธีการประมาณค่าแบบซ้ำจะสามารถแก้ไขปัญหาอสมการการแปรผันแล้ว ยังสามารถประยุกต์ใช้ในการค้นหาจุดตรึงของการส่งแบบไม่ขยายดังรายละเอียดดังนี้

กำหนดให้ $H$ เป็นปริภูมิฮิลเบิร์ตบนเซตของจำนวนจริง และ $C$ เป็นเซตย่อยปิด (closed) นูน (convex) ของ $H$ กำหนดการส่ง $A:C\to H$ จะเรียกการส่ง $A$ ว่า การส่งทางเดียว (monotone mapping) ถ้า

$$\langle Au, Av\rangle \ge 0,\ \ \forall u,v\in C.$$

ปัญหาอสมการการแปรผัน (variational inequality problem(VIP)) คือการหา $u_0\in C$ ซึ่งทำให้อสมการต่อไปนี้เป็นจริง

$$\langle Au_0, u-u_0\rangle \ge 0,\quad \forall u\in C \qquad (1)$$

เซตคำตอบของปัญหาอสมการการแปรผันจะถูกเขียนแทนด้วย $VI(C,A)$

นั่นคือ $VI(C,A)=\{u\in C:\langle Au, v-u\rangle\ge 0\}$ ปัญหาอสมการการแปรผันนั่นได้ถูกศึกษากันอย่างกว้างขวางโดยดูได้จากเอกสารอ้างอิง [19, 26, 33, 64, 97, 99, 102, 103] สำหรับการส่ง $A:C\to H$ จะเรียกว่าเป็นการส่งแบบ $\alpha$*-ทางเดียวอย่างผกผัน* ($\alpha$-inverse-strongly monotone mapping) ถ้ามีจำนวนจริง $\alpha>0$ ซึ่งทำให้

$$\langle Au-Av, u-v\rangle \ge \alpha\|Au-Av\|^2,$$

สำหรับทุกๆ $u,v\in C$ และเรียก $T:C\to C$ ว่า*การส่งแบบไม่ขยาย* (nonexpansive mapping) ถ้า

$$\|Tx-Ty\|\le\|x-y\|$$

สำหรับทุกๆ $x,y\in C$ และกำหนดให้ $F(T)$ แทนเซตของจุดตรึงทั้งหมดของการส่ง $T$ นั่นคือ $F(T)=\{x\in C:Tx=x\}$ จากนิยามดังกล่าวจะเห็นได้ว่า $u$ เป็นคำตอบของอสมการการแปรผันในสมการที่ (1) ก็ต่อเมื่อ $u=P_C(u-\lambda Au)$ เมื่อ $\lambda>0$ และ $P_C$ เป็นภาพฉายระยะทาง (metric projection) นั่นแสดงให้เห็นว่าปัญหาอสมการการแปรผันเป็นความสัมพันธ์สมมูลกับปัญหาจุดตรึง (Fixed point problems) จึงเป็นผลให้ในปี ค.ศ. 2003 Takahashi และ Toyoda [91] ได้คิดค้นวิธีการทำซ้ำต่อไปนี้เพื่อค้นหาคำตอบร่วมระหว่างจุดตรึงของการส่งแบบไม่ขยายและผลเฉลยของอสมการการแปรผันนั่นคือเพื่อค้นหาคำตอบของ $F(S)\cap V(C,A)$ โดยกำหนดให้ $x_0\in C$ และนิยาม ลำดับ $\{x_n\}$ โดย

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)SP_C(x_n-\lambda_n Ax_n) \qquad (2)$$

สำหรับทุกๆ $n=0,1,2,...,$ เมื่อ $\{\alpha_n\}\subset(0,1)$, $\{\lambda_n\}\subset(0,2\alpha)$ และ $S:C\to C$ เป็นการส่งแบบไม่ขยายบน $C$ และ $P_C:H\to C$ เป็นภาพฉายระยะทางและ $A:C\to H$ เป็น $\alpha$-ทางเดียวอย่างผกผัน จากนั้น Takahashi และ Toyoda [91] ก็ได้พิสูจน์ว่าลำดับ $\{x_n\}$ ซึ่งนิยามโดยสมการ (2) ลู่เข้าอย่างอ่อนสู่สมาชิกร่วมของ $F(T)\cap VI(C,A)$ ในปริภูมิฮิลเบิร์ต ต่อมา Iiduka and Takahashi [49] ต้องการ

สร้างทฤษฎีบทการลู่เข้าอย่างเข้มสู่สมาชิกร่วมของ $F(T)\cap VI(C,A)$ จึงได้กำหนดวิธีการทำซ้ำแบบใหม่ ดังนี้ กำหนดให้ $x_0 = u \in C$ และนิยาม ลำดับ $\{x_n\}$ โดย

$$x_{n+1} = \alpha_n u + (1-\alpha_n) S P_C (x_n - \lambda_n A x_n) \qquad (3)$$

สำหรับทุกๆ $n = 0,1,2,...$ เมื่อ $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,2\alpha)$ และ $S : C \to C$ เป็นการส่งแบบไม่ขยายบน $C$ และ $P_C : H \to C$ เป็นภาพฉายระยะทางและ $A : C \to H$ เป็น $\alpha$-ทางเดียวอย่างผกผัน นอกจากนั้นแล้วโดยใช้วิธีการเอ็กซ์ตร้ากราเดียน (Extra) ซึ่งทั้ง Nadezhkina และ Takahashi [61] ได้ใช้การประมาณค่าแบบซ้ำชนิดใหม่เพื่อค้นหาผลเฉลยร่วมของ $F(T)\cap VI(C,A)$ ต่อมา Y. Yao และ J.C. Yao [109] ได้แนะนำวิธีการประมาณค่าแบบซ้ำเพื่อหาสมาชิกของ $F(T)\cap VI(C,A)$ ดังนี้

กำหนดให้ $A : C \to H$ เป็น $\alpha$-ทางเดียวอย่างผกผัน และ $S : C \to C$ เป็นการส่งแบบไม่ขยาย ซึ่ง $F(T)\cap VI(A,C) \neq \varnothing$ กำหนดให้ $x_0 = u \in C$ และนิยาม ลำดับ $\{x_n\}$ และ $\{y_n\}$ โดย

$$\begin{cases} y_n = P_C(x_n - \lambda_n x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C (y_n - \lambda_n A y_n) \end{cases} \qquad (4)$$

เมื่อ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0,1]$ และ $\{\lambda_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0,2\alpha]$ Y. Yao และ J.C. Yao [109] ได้พิสูจน์ว่าถ้าลำดับ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ สอดคล้องเงื่อนไขบางอย่าง แล้วลำดับ $\{x_n\}$ และ $\{y_n\}$ นิยามโดย (4) ลู่เข้าอย่างเข้มสู่จุดตรึงร่วมของเซตคำตอบของจุดตรึงของการส่งแบบไม่ขยายและเซตของผลเฉลยของอสมการการแปรผัน นั่นคือ $\{x_n\}$ และ $\{y_n\}$ ลู่เข้าอย่างเข้มสู่สมาชิกของ $F(T)\cap VI(A,C)$

อีกหนึ่งปัญหาทางคณิตศาสตร์ที่เป็นที่สนใจของนักคณิตศาสตร์หลายคนในปัจจุบันคือ **ปัญหาเชิงดุลยภาพ** (equilibrium problems (EP)) ซึ่งหมายถึง การหาค่าของ $x \in C$ ซึ่งสอดคล้องกับอสมการต่อไปนี้

$$F(x,y) \geq 0, \quad \forall y \in C \qquad (5)$$

เมื่อกำหนดฟังก์ชันโดเมนเชิงคู่ $F : C \times C \to \mathbb{R}$ และเซตคำตอบของปัญหาเชิงดุลยภาพ (5) ข้างบนนี้เราจะเขียนแทนด้วย $EP(F)$ ถ้ากำหนดการส่ง $T : C \to H$ ให้ $F(x,y) = \langle Tx, y-x \rangle$ สำหรับทุกๆ $x, y \in C$ แล้วจะได้ว่า $z \in EP(F)$ ก็ต่อเมื่อ $\langle Tz, y-z \rangle \geq 0, \forall y \in C$ นั่นแสดงให้เห็นว่าคำตอบของปัญหาเชิงดุลภาพสามารถแก้ไขบางปัญหาอสมการการแปรผัน นอกจากแล้วจะเห็นว่าปัญหาต่างๆ ในทางฟิสิกส์ หรือในทางเศรษฐศาสตร์ บางอย่างสามารถแปลงเป็นสมการหรืออสมการ ให้อยู่ในรูปอสมการ (5) ดังนั้นการหาผลเฉลยหรือการประมาณค่าของปัญหาเชิงดุลยภาพ (5) ถือเป็นการแก้ไขปัญหาในทางฟิสิกส์หรือในทางเศรษฐศาสตร์ได้อีกทางหนึ่ง ซึ่งสามารถดูได้จากเอกสารอ้างอิง [5], [26] และ [59] โดยในปี 1997 Combettes และ Hirstoaga [25] ได้เริ่มต้นศึกษาและใช้วิธีการประมาณค่าแบบซ้ำในการประมาณค่าที่ดีที่สุดเพื่อไปหาคำตอบ (solutions) ให้กับปัญหาเชิงดุลยภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้ม และเมื่อเร็วๆ นี้ เพื่อค้นหาคำตอบร่วมของ $EP(F)\cap F(T)$ S. Takahashi และ W. Takahashi [89] จึงได้ศึกษาวิธีการประมาณค่าแบบหนืด (viscosity approximation method) ในปริภูมิฮิลเบิร์ต โดยกำหนดให้ $S : C \to C$ เป็นการส่งแบบไม่ขยาย และให้ $x_1 \in C$ และนิยามลำดับ $\{x_n\}$ และ $\{u_n\}$ โดย

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n \end{cases}$$

สำหรับทุกๆ $n \in \mathbb{N}$ เมื่อ $\{\alpha_n\} \subset [0,1]$ และ $\{r_n\} \subset (0,\infty)$ ภายใต้เงื่อนไขที่เหมาะสมบางอย่างสำหรับลำดับ $\{\alpha_n\},\{\gamma_n\}$ S. Takahashi และ W. Takahashi [89] ได้พิสูจน์ว่า $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มสู่จุดตรึงร่วม $z \in F(T) \cap EP(F)$ เมื่อ $z = P_{F(T)\cap EP(F)} f(z)$ โดยใช้แนวคิดดังกล่าว ต่อมาในปี ค.ศ. 2007 Su, Shang และ Qin [83] ได้สร้างวิธีการประมาณค่าแบบซ้ำแบบใหม่เพื่อหาคำตอบร่วมระหว่าง เซตของจุดตรึงของการส่งแบบไม่ขยาย $F(S)$ เซตคำตอบของอสมการการแปรฝัน $VIP(A,C)$ (เมื่อ $A$ เป็นการส่งแบบ $\alpha-$inverse-strongly monotone) และ เซตคำตอบของปัญหาเชิงดุลยภาพ $EP(F)$ ในปริภูมิฮิลเบิร์ต โดยกำหนดให้ $x_1 \in C$ และนิยามลำดับโดย

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_c(u_n - \lambda_n Au_n), \ n \geq 1 \end{cases}$$

และ Su, Shang และ Qin [83] ได้พิสูจน์ $\{x_n\}$ และ $\{u_n\}$ ลู่เข้าอย่างเข้มสู่จุดตรึงร่วม $z \in F(T) \cap EP(F) \cap VI(A,C)$ เมื่อ $z = P_{F(T)\cap EP(F)\cap VI(A,C)} f(z)$ และเมื่อเร็วๆ นี้ Plubtieng และ Punpaeng [64] ได้นำเสนอวิธีการแก้ไขปัญหาเชิงดุลยภาพโดยใช้วิธีการเอ็กซ์ตร้ากราเดียนแบบใหม่ซึ่งแตกต่างจาก Su, Shang และ Qin [83] โดยผสมแนวคิดของ S. Takahashi และ W. Takahashi [89] และ Y. Yao และ J.C. Yao [109] ดังนี้ กำหนด $x_1 \in C$ และนิยามลำดับ $\{x_n\},\{y_n\}$ และ $\{u_n\}$ ดังนี้

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C \\ y_n = P_C(u_n - \lambda_n u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n) \end{cases} \qquad (6)$$

เมื่อ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0,1]$ และ $\{\lambda_n\}$ เป็นลำดับของจำนวนจริงในช่วงปิด $[0,2\alpha]$ ซึ่ง Plubtieng และ Punpaeng [63] ได้พิสูจน์ว่าถ้าลำดับ $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ สอดคล้องเงื่อนไขบางอย่าง แล้วลำดับ $\{x_n\}$ และ $\{y_n\}$ นิยามโดย (6) ลู่เข้าอย่างเข้มสู่จุดตรึงร่วมของเซตคำตอบของจุดตรึงของการส่งแบบไม่ขยาย เซตของผลเฉลยของอสมการการแปรผัน และ เซตคำตอบของปัญหาเชิงดุลยภาพ

จากผลงานวิจัยต่างๆ ที่กล่าวมาข้างต้นจะเห็นว่าพัฒนาการในเรื่องวิธีการประมาณค่านั้นได้ถูกคิดค้นอยู่เสมอๆ ในปริภูมิที่แตกต่างกันไป จึงเป็นเหตุผลที่ทำให้ผู้วิจัยต้องการที่จะค้นหาหรือนำเสนอวิธีการประมาณค่าแบบใหม่ๆ เพื่อให้สามารถประยุกต์ใช้กับปัญหาทางคณิตศาสตร์ในรูปแบบต่างๆ หรือบางปัญหาในทาง ฟิสิกส์ และ ทางเศรษฐศาสตร์ ได้มากขึ้น พร้อมทั้งยังเป็นการก่อให้เกิดองค์ความรู้ หรือทฤษฎีใหม่ๆ ในทางการวิเคราะห์เชิงฟังก์ชันหรือสาขาอื่นๆ ที่เกี่ยวข้อง

ดังนั้น การคิดค้นเพื่อให้เกิดวิธีการประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ๆ และทฤษฎีการลู่เข้าสู่จุดตรึงจึงเป็นองค์ความรู้ใหม่ที่คาดว่าจะได้รับ นอกจากนั้นแล้วยังสามารถใช้วิธีการประมาณค่าดังกล่าวเพื่อประยุกต์ใช้หาคำตอบของปัญหาเชิงดุลยภาพ และ ปัญหาอสมการการแปรผัน ปัญหาค่าน้อยสุด และ

ปัญหาอื่นๆ ทางคณิตศาสตร์ ซึ่งองค์ความรู้ใหม่ที่ได้นั้นจะเป็นพื้นฐานที่สำคัญในการพัฒนาสาขาวิชาการวิเคราะห์เชิงฟังก์ชันและสาขาวิชาอื่นๆ ที่เกี่ยวข้อง ดังที่ได้กล่าวมาแล้วข้างต้นอันจะเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

# CHAPTER 2

# PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapter.

Throughout this research, we let $\mathbb{R}$ stand for the set of all real numbers and $\mathbb{N}$ the set of all natural numbers.

## 2.1 Basic results

**Definition 2.1.1.** Let $E$ be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : E \longrightarrow \mathbb{R}$ is said to be *a norm on E* if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in E$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E$ and $\forall \alpha \in \mathbb{K}$.

**Definition 2.1.2.** Let $(E, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset E$ is said to *converge strongly* in $X$ if there exists $x \in E$ such that $\lim_{n\longrightarrow\infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer $N$ such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n\longrightarrow\infty} x_n = x$ or $x_n \longrightarrow x$ to mean that $x$ is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset E$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer $N$ such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in $E$ if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

**Definition 2.1.3.** A normed space $E$ is called *complete* if every Cauchy sequence in $E$ converges to an element in $E$.

**Definition 2.1.4.** A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$.

**Definition 2.1.5.** Let $C$ be a nonempty subset of normed space $E$. A mapping $T : C \longrightarrow C$ is said to be *lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in C$

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.1.1}$$

The smallest number $k$ for which 2.1.1 holds is called the *Lipschitz constant* of $T$.

**Definition 2.1.6.** A lipschitzian mapping $T : C \longrightarrow C$ with Lipschitz constant $k < 1$ is said to be a *contraction mapping.*

**Definition 2.1.7.** An element $x \in C$ is said to be a *fixed point* of a mapping $T : C \longrightarrow C$ iff $Tx = x$.

**Definition 2.1.8.** [**Banach's contraction mapping principle**] Let $(M, d)$ be a complete metric spaces and let $T : M \longrightarrow M$ be a contraction. Then $T$ has a unique fixed point $x_0$.

**Definition 2.1.9.** Let $F$ and $E$ be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow E$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$ and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on F* if $T$ a is linear operator.

**Definition 2.1.10.** A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector $x$ if $\lim_{n\longrightarrow\infty} f(x_n) = f(x)$ holds for every continuous linear functional $f$. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converge weakly to $x$.

**Definition 2.1.11.** Let $F$ and $E$ be normed spaces over the field $\mathbb{K}$ and $T : F \longrightarrow E$ a linear operator. $T$ is said to be *bounded* on $F$ if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in F$.

**Definition 2.1.12.** Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space $X$ is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

**Definition 2.1.13.** Let $F$ and $E$ be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow E$ an operator and $c \in F$. We say that $T$ is *continuous at c* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If $T$ is continuous at each $x \in F$, then $T$ is said to be *continuous on F*.

**Definition 2.1.14.** Let $E$ and $F$ be normed spaces. The mapping $T : E \longrightarrow F$ is said to be *completely continuous* if and only if $T(C)$ is a compact subset of $F$ for every bounded subset $C$ of $E$.

**Definition 2.1.15.** A mapping $T : C \longrightarrow C$ is said to be *semicompact* if, for any sequence $\{x_n\}$ in $C$ such that $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x \in C$.

**Definition 2.1.16.** A subset $C$ of a normed linear space $E$ is said to be *convex set in X* if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

**Definition 2.1.17.** Let $E$ be a real normed space and $C$ a nonempty subset of $E$. A mapping $T : C \longrightarrow C$ is said to be

(a) *nonexpansive* whenever $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$;

(b) *asymptotically nonexpansive* on $C$ if there exists a sequence $\{k_n\}$ in $[1, \infty)$, with $\lim_{n\longrightarrow\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \tag{2.1.2}$$

for all $x, y \in C$ and each $n \geq 1$;

(c) *strict pseudo-contractive mapping* if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{2.1.3}$$

for all $x, y \in C$. (If (2.1.3) holds, we also say that $T$ is a $k$-strict pseudo-contraction.)

It is know that if $T$ is 0-strict pseudo-contractive mapping, $T$ is nonexpansive mapping.

(d) *asymptotically $k$-strict pseudo-contractive* if there exists a constant $0 \leq k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{2.1.4}$$

for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\gamma_n \geq 0$ for all $n$ such that $\lim_{n\to\infty} \gamma_n = 0$.

(e) asymptotically nonexpansive in the intermediate sense [6] provided $T$ is uniformly continuous and

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

(f) asymptotically $k$-strict pseudo-contractive mapping [39] with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, 1)$ with $\lim_{n\longrightarrow\infty} \gamma_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 \tag{2.1.5}$$

for all $x, y \in C$ and $n \in N$.

**Definition 2.1.18.** [74] Let $C$ be a nonempty subset of a Hilbert space $H$. A mapping $T : C \longrightarrow C$ will be called an asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, 1)$ with $\lim_{n\longrightarrow\infty} \gamma_n = 0$ such that

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x - y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \leq 0. \tag{2.1.6}$$

Throughout this paper we assume that

$$c_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - T^n x - (y - T^n y)\|^2)\}.$$

Then $c_n \geq 0$ for all $n \in N$, $c_n \to 0$ as $n \to \infty$ and (2.1.6) reduces to the relation

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n \tag{2.1.7}$$

for all $x, y \in C$ and $n \in N$

**Definition 2.1.19.** Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let $C$ be a closed convex subset of $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$.

**Definition 2.1.20.** Let $E$ be a real Banach space and $E^*$ the dual space of $E$. Let $K$ be a nonempty, closed and convex subset of $E$. A (one-parameter) nonexpansive semigroup is a family $\mathfrak{F} = \{T(t) : \ t \geq 0\}$ of self-mappings of $K$ such that

(i) $T(0)x = x$ for all $x \in K$;

(ii) $T(t+s)x = T(t)T(s)x$ for all $t, s \geq 0$ and $x \in K$;

(iii) for each $x \in K$, the mapping $T(\cdot)x$ is continuous;

(iv) for each $t \geq 0$, $T(t)$ is nonexpansive, that is,

$$\|T(t)x - T(t)y\| \leq \|x - y\|, \quad \forall x, y \in K.$$

We denote $F$ by the common fixed points set of $\mathfrak{F}$, that is, $F := \bigcap_{t \geq 0} F(T(t))$.

**Definition 2.1.21.** Let $C$ be a nonempty subset of a real Banach space $E$ and $F : C \times C \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, be a bifunction. The equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.1.8}$$

The solutions set of (2.1.8) is denoted by $EP(F)$.

For solving the equilibrium problem, we assume that:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, *i.e.* $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x, y)$;

(A4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semi-continuous.

**Definition 2.1.22.** A Banach space $E$ is called *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space $E$ is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The *modulus of convexity* of $E$ is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \ \|x\|, \|y\| \leq 1, \ \|x - y\| \geq \epsilon \right\},$$

for all $\epsilon \in [0, 2]$. $E$ is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of $E$ is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case $E$ is called *smooth*. The norm of $E$ is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. The norm of $E$ is called *uniformly Fréchet differentiable*, if the limit is attained uniformly for $x, y \in S(E)$. It is well known that (uniformly) *Fréchet differentiability* of the norm of $E$ implies (uniformly) *Gâteaux differentiability* of the norm of $E$.

**Definition 2.1.23.** The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{\, x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$.

It is also known that if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$ (see [88] for more details).

**Definition 2.1.24.** Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$.

From the definition of $\phi$, we see that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x, y) \leq \big(\|x\| + \|y\|\big)^2$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$

for all $x, y, z \in E$.

**Definition 2.1.25.** Let $C$ be a closed and convex subset of a smooth Banach space $E$ and let $T$ be a mapping from $C$ into itself. A point $p$ in $C$ is said to be an *asymptotic fixed point* of $T$ [8] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

The set of asymptotic fixed points of $T$ will be denoted by $\widehat{F}(T)$. A mapping $T$ is said to be *relatively nonexpansive* [8, 9] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. A point $p$ in $C$ is said to be a *strong asymptotic fixed point* of $T$ if $C$ contains a sequence $\{x_n\}$ which converges strongly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

The set of strong asymptotic fixed points of $T$ will be denoted by $\widetilde{F}(T)$. A mapping $T$ is said to be *weak relatively nonexpansive* [110] if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$.

It is obvious by definition that the class of weak relatively nonexpansive mappings contains the class of relatively nonexpansive mappings. Indeed, for any mapping $T : C \to C$, we see that $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if $T$ is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

**Definition 2.1.26.** Let $E$ be a reflexive, strictly convex and smooth Banach space and let $C$ be a nonempty, closed and convex subset of $E$. The *generalized projection mapping*, introduced by Alber [3], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$, that is, $\Pi_C(x) = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}.$$

In a Hilbert space, $\Pi_C$ is coincident with the metric projection denoted by $P_C$.

**Definition 2.1.27.** Let $E$ be a reflexive, strictly convex and smooth Banach space. It is known that $A : E \to 2^{E^*}$ is maximal monotone if and only if $R(J + \lambda A) = E^*$ for all $\lambda > 0$, where $R(B)$ stands for the range of $B$.

Define the *resolvent* of $A$ by $J_{\lambda A} = (J + \lambda A)^{-1} J$ for all $\lambda > 0$. It is known that $J_{\lambda A}$ is a single-valued mapping from $E$ to $D(A)$ and $A^{-1}(0^*) = F(J_{\lambda A})$ for all $\lambda > 0$. For each $\lambda > 0$, the *Yosida approximation* of $A$ is defined by

$$A_\lambda(x) = \frac{1}{\lambda}\big(J(x) - JJ_{\lambda A}(x)\big).$$

for all $x \in E$. We know that $A_\lambda(x) \in A\big(J_{\lambda A}(x)\big)$ for all $\lambda > 0$ and $x \in E$.

**Definition 2.1.28.** A continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ is said to be gauge function if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

**Definition 2.1.29.** Let $E$ be a normed space and $\varphi$ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \big\{f^* \in E^* : \ \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \ \|f^*\| = \varphi(\|x\|)\big\}, \quad x \in E$$

is called the duality mapping with gauge function $\varphi$.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the normalized duality mapping.

In the case $\varphi(t) = t^{q-1}, \ q > 1$, the duality mapping $J_\varphi = J_q$ is called the generalized duality mapping. It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ and $J_q(x) = \|x\|^{q-2} J(x), \ q > 1$.

*Remark* 2.1.30. *For the gauge function $\varphi$, the function $\Phi : [0, \infty) \to [0, \infty)$ defined by*

$$\Phi(t) = \int_0^t \varphi(s)ds \tag{2.1.9}$$

*is a continuous convex and strictly increasing function on $[0, \infty)$. Therefore, $\Phi$ has a continuous inverse function $\Phi^{-1}$.*

It is noted that if $0 \leq k \leq 1$, then $\varphi(kx) \leq \varphi(x)$. Further

$$\Phi(kt) = \int_0^{kt} \varphi(s)ds = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t).$$

*Remark* 2.1.31. *For each $x$ in a Banach space $E$, $J_\varphi(x) = \partial\Phi(\|x\|)$, where $\partial$ denotes the sub-differential.*

We also know the following facts:

(i) $J_\varphi$ is a nonempty, closed and convex set in $E^*$ for each $x \in E$.

(ii) $J_\varphi$ is a function when $E^*$ is strictly convex.

(iii) If $J_\varphi$ is single-valued, then

$$J_\varphi(\lambda x) = \frac{sign(\lambda)\varphi(\|\lambda x\|)}{\varphi(\|x\|)} J_\varphi(x), \quad \forall x \in E,\ \lambda \in \mathbb{R}$$

and

$$\langle x - y, J_\varphi(x) - J_\varphi(y)\rangle \geq \big(\varphi(\|x\|) - \varphi(\|y\|)\big)\big(\|x\| - \|y\|\big), \quad \forall x, y \in E.$$

Following Browder [7], we say that a Banach space $E$ has a weakly continuous duality mapping if there exists a gauge $\varphi$ for which the duality mapping $J_\varphi$ is single-valued and continuous from the weak topology to the weak$^*$ topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly$^*$ to $J_\varphi(x)$. It is known that the space $\ell^p$ has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Moreover, $\varphi$ is invariant on $[0, 1]$.

# CHAPTER 3

# MAIN RESULTS

## 3.1 Strong convergence theorem by hybrid method for non-Lipschitzian mapping

In this section, We introduce the hybrid method of modified Mann's iteration for an asymptotically $k$-strict pseudo-contractive mapping $T$ in the intermediate sense which is necessarily lipschitzian. We establish strong convergence theorem for such method.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [6, 32, 39, 90]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [39, 90].

Iteration method for finding a fixed point of an asymptotically $k$-strict pseudo-contractive mapping $T$ is the modified Mann's iteration method studied in [50, 75, 77, 94] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad n \geq 0, \tag{3.1.1}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ line in the interval $(0, 1)$.

In 2007, Takahashi, Takeuchi and Kubota [90] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of $C$ as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.1.2}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then they prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$. In 2008, Kumam [46], introduce an iterative scheme by a new hybrid method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for $\alpha$-inverse-strongly monotone mappings in a real Hilbert space.

In 2008, Inchan [30], introduce the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let $C$ be a nonempty closed bounded convex subset of a Hilbert space $H$, $T$ be an asymptotically nonexpansive mapping of $C$ into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.1.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then him prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, Inchan and Nammanee [31], introduce the modified Mann iteration processes for an asymptotically $k$-strict pseudo-contractive mapping. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $T$ be an asymptotically $k$-strict pseudo-contractive mapping of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.1.4}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0$ as $n \to \infty$ and $\limsup_{n \longrightarrow \infty} \alpha_n < 1-k$. Then they prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these fact, we introduce the modified Mann iteration processes for an asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense by idear in (3.1.4). Let $C$ be a closed convex subset of a Hilbert space $H$, $T : C \longrightarrow C$ be an asymptotically $k$-strictly pseudo-contractive mapping in the intermediate sense and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n \\ \qquad\qquad\qquad\qquad +c_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.1.5}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0, \ (n \to \infty)$.

We shall prove that the iteration generated by (3.1.5) converges strongly to $z_0 = P_{F(T)}x_0$.

In the sequel, the following lemmas are needed to prove our main results.

**Lemma 3.1.1.** *[56] There holds the identity in a Hilbert space $H$:*

*(i)* $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H.$

*(ii)* $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0,1]$.

**Lemma 3.1.2.** *[63] Let $C$ be a closed convex subset of a real Hilbert space $H$. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

**Lemma 3.1.3.** *[74] Let $C$ be a nonempty subset of a Hilbert space $H$ and let $T : C \to C$ a uniformly continuous asymptotically $k$-strict pseudo-contractive in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in $C$ such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

**Lemma 3.1.4.** *[74] Let $C$ be a nonempty closed convex subset of Hilbert space $H$ and $T : C \longrightarrow C$ a continuous asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that $\{x_n\}$ is sequence in $C$ such that $x_n \rightharpoonup x \in C$ and $\limsup_{n\to\infty} \|x_n - T^n x_n\| = 0$, then $(I - T)x = 0$.*

Now, we prove strong convergence theorem by hybrid method for asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense in Hilbert spaces.

**Theorem 3.1.5.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex bounded subset of $H$. Let $T$ be a uniformly continuous asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (3.1.5) converges strongly to $z_0 = P_{F(T)} x_0$.*

**Proof.** We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. For $u \in F(T) \subset C_m$. By lemma 3.1.1, we have,

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1 - \alpha_m) T^m x_m - u\|^2 \\
&= \|\alpha_m (x_m - u) + (1 - \alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m \|x_m - u\|^2 + (1 - \alpha_m)\|T^m x_m - u\|^2 - \alpha_m (1 - \alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m \|x_m - u\|^2 + (1 - \alpha_m)[(1 + \gamma_m)\|x_m - u\|^2 + k\|x_m - T^m x_m\|^2 + c_m] \\
&\quad - \alpha_m (1 - \alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1 + (1 - \alpha_m)\gamma_m)\|x_m - u\|^2 + (k - \alpha_m)(1 - \alpha_m)\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + (1 - \alpha_m)\gamma_m \|x_m - u\|^2 + [k - \alpha_m (1 - \alpha_m)]\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + [k - \alpha_m (1 - \alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \qquad (3.1.6)
\end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that $C_n$ is closed and convex for all $n \in \mathbb{N}$. It follows obvious that

$C_1 = C$ is closed and convex. Suppose that $C_m$ is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since $C_m$ is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \le \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$. Then

$$\begin{aligned}\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\ &= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\ &\le \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\ &\quad + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \le \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since $C_m$ is convex, $z \in C_m$ and $\|y_m - x\|^2 \le \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, $\|y_m - y\|^2 \le \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, we have

$$\begin{aligned}\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\ &= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\ &= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\ &\le \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\ &\quad + (1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\ &\quad - \alpha(1-\alpha)\|y - x\|^2 \\ &= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\ &\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\ &= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\ &\quad + \theta_m + c_m \\ &= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.\end{aligned}$$

Then $z \in C_{m+1}$, it follows that $C_{m+1}$ is closed and convex. Hence $C_n$ is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_0$. By Lemma 3.1.2, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0, \text{ for all } y \in C_n.$$

Moreover, by the same proof of Theorem 3.1 of [31], we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.1.7}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n, \quad (3.1.8)$$

By the definition of $y_n$, we have

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\ &= (1-\alpha_n)\|T^n x_n - x_n\|.\end{aligned}$$

From (3.1.8), we have

$$\begin{aligned}(1-\alpha_n)^2\|T^n x_n - x_n\|^2 &= \|y_n - x_n\|^2 \\ &= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \\ &\leq \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n \\ &\quad + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| \\ &\quad + \|y_n - x_{n+1}\|) + \theta_n + c_n.\end{aligned}$$

It follows that

$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n$.

Hence

$$(1-k-\alpha_n)\|T^n x_n - x_n\| \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n. \quad (3.1.9)$$

From $\limsup_{n\to\infty} \alpha_n < 1-k$, we can chosen $\epsilon > 0$ such that $\alpha_n \leq 1 - k - \epsilon$ for large enough $n$. From (3.1.7) and (3.1.9), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \quad (3.1.10)$$

From (3.1.7), (3.1.10) and Lemma 3.1.3, we have

$$\lim_{n\to\infty} \|T x_n - x_n\| = 0. \quad (3.1.11)$$

Since $H$ is reflexive and $\{x_n\}$ is bounded we get that $\emptyset \neq \omega_w(x_n)$. From Lemma 3.1.4, we have $\omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| &\leq \|x_0 - w\| \leq \liminf_{n\to\infty} \|x_0 - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\begin{aligned} \|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &\le 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This complete the proof. □

## 3.2 A General Iterative Method for a Nonexpansive Semigroup in Banach Spaces with Gauge Functions

In this section, we study strong convergence of the sequence generated by implicit and explicit general iterative methods for a one-parameter nonexpansive semigroup in a reflexive Banach space which admits the duality mapping $J_\varphi$, where $\varphi$ is a gauge function on $[0, \infty)$.

In 1967, Halpern [29] introduced the following classical iteration for a nonexpansive mapping $T : K \to K$ in a real Hilbert space:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{3.2.1}$$

where $\{\alpha_n\} \subset (0, 1)$ and $u \in K$.

In 1977, Lions [52] obtained a strong convergence provide the real sequence $\{\alpha_n\}$ satisfies the following conditions:

C1: $\lim_{n\to\infty} \alpha_n = 0$; C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$; C3: $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0$.

Reich [69] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lion's conditions imposed on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n + 1)$.

In 1992, Wittmann [101] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$ if $\{\alpha_n\}$ satisfies the following conditions:

C1: $\lim_{n\to\infty} \alpha_n = 0$; C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$; C3: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Shioji-Takahshi [80] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of Halpern iterative scheme has been widely used to approximate the fixed points for nonexpansive mappings (see, e.g., [4, 15, 17, 37, 68, 102, 103] and the reference cited therein).

Let $f : K \to K$ be a contraction. In 2000, Moudafi [60] introduced the explicit viscosity approximation method for a nonexpansive mapping $T$ as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{3.2.2}$$

where $\alpha_n \in (0,1)$. Xu [105] also studied the iteration process (3.2.2) in uniformly smooth Banach spaces.

Let $A$ be a strongly positive bounded linear operator on a real Hilbert space $H$: that is, there is a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A typical problem is to minimize a quadratic function over the fixed points set of a nonexpansive mapping on a Hilbert space $H$:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where $C$ is the fixed points set of a nonexpansive mapping $T$ on $H$ and $b$ is a given point in $H$.

In 2006, Marino-Xu [56] introduced the following general iterative method for a nonexpansive mapping $T$ in a Hilbert space $H$:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1, \tag{3.2.3}$$

where $\{\alpha_n\} \subset (0,1)$, $f$ is a contraction on $H$ and $A$ is a strongly positive bounded linear operator on $H$. They proved that the sequence $\{x_n\}$ generated by (3.2.3) converges strongly to a fixed point $x^* \in F(T)$ which also solves the variational inequality

$$\big\langle (A - \gamma f)x^*, x - x^*\big\rangle \geq 0, \ \forall x \in F(T)$$

which is the optimality condition for the minimization problem: $\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where $h$ is a potential function for $\gamma f$ (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

Suzuki [86] first introduced the following implicit viscosity method for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ in a Hilbert space:

$$x_n \;=\; \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{3.2.4}$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$. He proved strong convergence of iteration (3.2.4) under suitable conditions. Subsequently, Xu [106] extended Suzuki [86]'s result from a Hilbert space to a uniformly convex Banach space which admits a weakly sequentially continuous normalized duality mapping.

Motivated by Chen-Song [21], in 2007, Chen-He [11] investigated the implicit and explicit viscosity methods for a nonexpansive semigroup without integral in a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping:

$$x_n \;=\; \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{3.2.5}$$

$$x_{n+1} \;=\; \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{3.2.6}$$

where $\{\alpha_n\} \subset (0,1)$.

In 2008, Song-Xu [81] also studied the iterations (3.2.5) and (3.2.6) in a reflexive and strictly convex Banach space with a Gâteaux differentiable norm. Subsequently, Cholamjiak-Suantai [18] extended Song-Xu's results to a Banach space which admits duality mapping with a gauge function. Wangkeeree-Kamraksa [95] and Wangkeeree et al. [96] obtained the convergence results concerning the duality mapping with a gauge function in Banach spaces. The convergence of iterations for a nonexpansive semigroup has been studied by many authors (see, for instance, [33, 47, 48, 51, 64, 87]).

Let $E$ be a real reflexive Banach space which admits the duality mapping $J_\varphi$ with a gauge $\varphi$. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on $E$. Recall that an operator $A$ is said to be *strongly positive* if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|)$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} |\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|,$$

where $\alpha \in [0,1]$ and $\beta \in [-1,1]$.

Motivated by Chen-Song [21], Chen-He [11], Marino-Xu [56], Colao et al. [24] and Wangkeeree et al. [96], we study strong convergence of the following general iterative methods:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad n \geq 1, \tag{3.2.7}$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad n \geq 1, \tag{3.2.8}$$

where $\{\alpha_n\} \subset (0,1)$, $f$ is a contraction on $E$ and $A$ is a positive bounded linear operator on $E$.

In the sequel, the following lemmas are needed to prove our main results.

**Lemma 3.2.1.** [50] *Assume that a Banach space $E$ has a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi\big(\|x+y\|\big) \leq \Phi\big(\|x\|\big) + \big\langle y, J_\varphi(x+y)\big\rangle.$$

*In particular, for all $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in $E$ converges weakly to a point $x \in E$. Then the following holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x - y\|)$$

*for all $x, y \in E$.*

**Lemma 3.2.2.** [96] *Assume that a Banach space $E$ has a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$. Let $A$ be a strongly positive bounded linear operator on $E$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\bar{\gamma})$.*

**Lemma 3.2.3.** [103] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 1,$$

*where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

$$\text{(a) } \sum_{n=1}^{\infty} \gamma_n = \infty; \text{ (b) } \limsup_{n\to\infty} \delta_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty.$$

*Then $\lim_{n\to\infty} a_n = 0$.*

### 3.2.1 Implicit iteration scheme

In this section, we prove a strong convergence theorem of an implicit iterative method (3.2.7).

**Theorem 3.2.4.** *Let $E$ be a reflexive which admits a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi$ is invariant on $[0, 1]$. Let $\mathfrak{F} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on $E$ such that $F \neq \emptyset$. Let $f$ be a contraction on $E$ with the coefficient $\alpha \in (0, 1)$ and $A$ a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Then $\{x_n\}$ defined by (3.2.7) converges strongly to $q \in F$ which solves the following variational inequality:*

$$\big\langle (A - \gamma f)(q), J_\varphi(q - w)\big\rangle \leq 0, \quad \forall w \in F. \tag{3.2.9}$$

*Proof.* First, we prove the uniqueness of the solution to the variational inequality (3.2.9) in $F$. Suppose $p, q \in F$ satisfy (3.2.9), so we have

$$\big\langle (A - \gamma f)(p), J_\varphi(p - q)\big\rangle \leq 0$$

and

$$\big\langle (A - \gamma f)(q), J_\varphi(q - p)\big\rangle \leq 0.$$

Adding the above inequalities, we get

$$\big\langle A(p) - A(q) - \gamma(f(p) - f(q)), J_\varphi(p - q)\big\rangle \leq 0.$$

This shows that

$$\big\langle A(p - q), J_\varphi(p - q)\big\rangle \leq \gamma\big\langle (f(p) - f(q), J_\varphi(p - q)\big\rangle,$$

which implies by the strong positivity of $A$

$$\bar{\gamma}\|p - q\|\varphi(\|p - q\|) \leq \big\langle A(p - q), J_\varphi(p - q)\big\rangle \leq \gamma\alpha\|p - q\|\varphi(\|p - q\|).$$

Since $\varphi$ is invariant on $[0,1]$,

$$\varphi(1)\bar{\gamma}\|p-q\|\varphi(\|p-q\|) \leq \gamma\alpha\|p-q\|\varphi(\|p-q\|).$$

It follows that

$$\big(\varphi(1)\bar{\gamma}-\gamma\alpha\big)\|p-q\|\varphi(\|p-q\|) \leq 0.$$

Therefore $p=q$ since $0<\gamma<\frac{\bar{\gamma}\varphi(1)}{\alpha}$.

We next prove that $\{x_n\}$ is bounded. For each $w \in F$, by Lemma 3.2.2, we have

$$\begin{aligned}
\|x_n - w\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n A)T(t_n)x_n - w\| \\
&= \|(I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)w + \alpha_n(\gamma f(x_n) - A(w))\| \\
&\leq \varphi(1)(1-\alpha_n\bar{\gamma})\|x_n - w\| + \alpha_n\big(\gamma\alpha\|x_n-w\| + \|\gamma f(w) - A(w)\|\big) \\
&\leq \|x_n - w\| - \alpha_n\varphi(1)\bar{\gamma}\|x_n-w\| + \alpha_n\gamma\alpha\|x_n-w\| + \alpha_n\|\gamma f(w)-A(w)\|,
\end{aligned}$$

which yields

$$\|x_n - w\| \leq \frac{1}{\varphi(1)\bar{\gamma}-\gamma\alpha}\|\gamma f(w) - A(w)\|.$$

Hence $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{AT(t_n)x_n\}$.

We next prove that $\{x_n\}$ is relatively sequentially compact. By the reflexivity of $E$ and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a point $p$ in $E$ such that $x_{n_j} \rightharpoonup p$ as $j \to \infty$. Now we show that $p \in F$. Put $x_j = x_{n_j}$, $\beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j \in \mathbb{N}$, fix $t>0$. We see that

$$\begin{aligned}
\|x_j - T(t)p\| &\leq \sum_{k=0}^{[t/s_j]-1}\|T((k+1)s_j)x_j - T(ks_j)x_{j+1}\| \\
&\quad + \|T([t/s_j]s_j)x_j - T([t/s_j]s_j)p\| + \|T([t/s_j]s_j)p - T(t)p\| \\
&\leq [t/s_j]\|T(s_j)x_j - x_j\| + \|x_j - p\| + \|T(t-[t/s_j]s_j)p - p\| \\
&= [t/s_j]\beta_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j-p\| + \|T(t-[t/s_j]s_j)p-p\| \\
&\leq t\beta_j/s_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| \\
&\quad + \max\{\|T(s)p-p\| : 0 \leq s \leq s_j\}.
\end{aligned}$$

So we have

$$\limsup_{j\to\infty}\Phi(\|x_j - T(t)p\|) \leq \limsup_{j\to\infty}\Phi(\|x_j - p\|). \tag{3.2.10}$$

On the other hand, by Lemma 3.2.1 (ii), we have

$$\limsup_{j\to\infty}\Phi(\|x_j - T(t)p\|) = \limsup_{j\to\infty}\Phi(\|x_j-p\|) + \Phi(\|T(t)p-p\|). \tag{3.2.11}$$

Combining (3.2.10) and (3.2.11), we have

$$\Phi(\|T(t)p - p\|) \leq 0.$$

This implies that $p \in F$. Further, we see that

$$
\begin{aligned}
&\|x_j - p\|\varphi(\|x_j - p\|) \\
&= \big\langle x_j - p, J_\varphi(x_j - p)\big\rangle \\
&= \big\langle (I - \beta_j A)T(s_j)x_j - (I - \beta_j A)p + \beta_j(\gamma f(x_j) - A(p)), J_\varphi(x_j - p)\big\rangle \\
&= \big\langle (I - \beta_j A)T(s_j)x_j - (I - \beta_j A)p, J_\varphi(x_j - p)\big\rangle \\
&\quad + \beta_j\big\langle \gamma f(x_j) - \gamma f(p), J_\varphi(x_j - p)\big\rangle + \beta_j\big\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\big\rangle \\
&\le \varphi(1)(1 - \beta_j\bar{\gamma})\|x_j - p\|\varphi(\|x_j - p\|) \\
&\quad + \beta_j\gamma\alpha\|x_j - p\|\varphi(\|x_j - p\|) + \beta_j\big\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\big\rangle.
\end{aligned}
$$

So we have

$$
\|x_j - p\|\varphi(\|x_j - p\|) \le \frac{1}{\varphi(1)\bar{\gamma} - \gamma\alpha}\big\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\big\rangle.
$$

By the definition of $\Phi$, it is easily seen that

$$
\Phi(\|x_j - p\|) \le \|x_j - p\|\varphi(\|x_j - p\|).
$$

Hence

$$
\Phi(\|x_j - p\|) \le \frac{1}{\varphi(1)\bar{\gamma} - \gamma\alpha}\big\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\big\rangle.
$$

Therefore $\Phi(\|x_j - p\|) \to 0$ as $j \to \infty$ since $J_\varphi$ is weakly continuous; consequently, $x_j \to p$ as $j \to \infty$ by the continuity of $\Phi$. Hence $\{x_n\}$ is relatively sequentially compact.

Finally, we prove that $p$ is a solution in $F$ to the variational inequality (3.2.9). For any $w \in F$, we see that

$$
\begin{aligned}
\big\langle (I - T(t_n))x_n - (I - T(t_n))w, J_\varphi(x_n - w)\big\rangle &= \big\langle x_n - w, J_\varphi(x_n - w)\big\rangle \\
&\quad - \big\langle T(t_n)x_n - T(t_n)w, J_\varphi(x_n - w)\big\rangle \\
&\ge \|x_n - w\|\varphi\|x_n - w\| \\
&\quad - \|T(t_n)x_n - T(t_n)w\|\|J_\varphi(x_n - w)\| \\
&\ge \|x_n - w\|\varphi\|x_n - w\| \\
&\quad - \|x_n - w\|\|J_\varphi(x_n - w)\| \\
&= 0.
\end{aligned}
$$

On the other hand, we have

$$
(A - \gamma f)(x_n) = -\frac{1}{\alpha_n}(I - \alpha_n A)(I - T(t_n))x_n,
$$

which implies

$$
\begin{aligned}
\big\langle (A - \gamma f)(x_n), J_\varphi(x_n - w)\big\rangle &= -\frac{1}{\alpha_n}\big\langle (I - T(t_n))x_n - (I - T(t_n))w, J_\varphi(x_n - w)\big\rangle \\
&\quad + \big\langle A(I - T(t_n))x_n, J_\varphi(x_n - w\big\rangle \\
&\le \big\langle A(I - T(t_n))x_n, J_\varphi(x_n - w\big\rangle. \qquad (3.2.12)
\end{aligned}
$$

Observe
$$\|x_j - T(s_j)x_j\| = \beta_j\|\gamma f(x_j) - AT(s_j)x_j\| \to 0,$$
as $j \to \infty$. Replacing $n$ by $n_j$ and letting $j \to \infty$ in (3.2.12), we obtain
$$\big\langle (A-\gamma f)(p), J_\varphi(p-w)\big\rangle \le 0, \quad \forall w \in F.$$

So $p \in F$ is a solution of variational inequality (3.2.9); and hence $p = q$ by the uniqueness. In a summary, we have proved that $\{x_n\}$ is relatively sequentially compact and each cluster point of $\{x_n\}$ (as $n \to \infty$) equals $q$. Therefore $x_n \to q$ as $n \to \infty$. This completes the proof. □

### 3.2.2 Explicit iteration scheme

In this section, utilizing the implicit version in Theorem 3.3.9, we consider the explicit one in a reflexive Banach space which admits the duality mapping $J_\varphi$.

**Theorem 3.2.5.** *Let $E$ be a reflexive Banach space which admits a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi$ is invariant on $[0,1]$. Let $\{T(t) : t \ge 0\}$ be a nonexpansive semigroup on $E$ such that $F \ne \emptyset$. Let $f$ be a contraction on $E$ with the coefficient $\alpha \in (0,1)$ and $A$ a strongly positive bounded linear operator with coefficient $\bar\gamma > 0$ and $0 < \gamma < \frac{\bar\gamma\varphi(1)}{\alpha}$. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying $0 < \alpha_n < 1$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty}\frac{\alpha_n}{t_n} = 0$. Then $\{x_n\}$ defined by (3.2.8) converges strongly to $q \in F$ which also solves the variational inequality (3.2.9).*

*Proof.* Since $\alpha_n \to 0$, we may assume that $\alpha_n < \varphi(1)\|A\|^{-1}$ and $1-\alpha_n(\varphi(1)\bar\gamma - \gamma\alpha) > 0$ for all $n$. First we prove that $\{x_n\}$ is bounded. For each $w \in F$, by Lemma 3.2.2, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|\alpha_n\gamma f(T(t_n)x_n) + (I-\alpha_n A)T(t_n)x_n - w\| \\
&= \|\alpha_n(\gamma f(T(t_n)x_n) - A(w)) + (I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)w\| \\
&\le \|I-\alpha_n A\|\|T(t_n)x_n - T(t_n)w\| + \alpha_n\|\gamma f(T(t_n)x_n) - A(w)\| \\
&\le \varphi(1)(1-\alpha_n\bar\gamma)\|x_n - w\| + \alpha_n\gamma\alpha\|x_n - w\| + \alpha_n\|\gamma f(w) - A(w)\| \\
&= \big(\varphi(1) - \alpha_n(\varphi(1)\bar\gamma - \gamma\alpha)\big)\|x_n - w\| + \alpha_n\|\gamma f(w) - A(w)\| \\
&\le \big(1 - \alpha_n(\varphi(1)\bar\gamma - \gamma\alpha)\big)\|x_n - w\| + \alpha_n(\varphi(1)\bar\gamma - \gamma\alpha)\frac{\|\gamma f(w) - A(w)\|}{\varphi(1)\bar\gamma - \gamma\alpha}.
\end{aligned}$$

It follows from induction that
$$\|x_{n+1} - w\| \le \max\left\{\|x_1 - w\|, \frac{\|\gamma f(w) - A(w)\|}{\varphi(1)\bar\gamma - \gamma\alpha}\right\}, \quad n \ge 1.$$

Thus $\{x_n\}$ is bounded, and hence so are $\{f(x_n)\}$ and $\{AT(t_n)x_n\}$. From Theorem 3.3.9, there is a unique solution $q \in F$ to the following variational inequality:
$$\big\langle (A-\gamma f)q, J_\varphi(q-w)\big\rangle \le 0, \quad \forall w \in F.$$

Next we prove that

$$\limsup_{n\longrightarrow\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n+1})\rangle \leq 0.$$

Indeed, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\big\langle (A-\gamma f)q, J_\varphi(q-x_n)\big\rangle = \limsup_{j\longrightarrow\infty}\big\langle (A-\gamma f)q, J_\varphi(q-x_{n_j})\big\rangle.$$

Further, we can assume that $x_{n_j}\rightharpoonup p\in E$ by the reflexivity of $E$ and the boundedness of $\{x_n\}$. Now we show that $p\in F$. Put $x_j = x_{n_j}, \beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j\in\mathbb{N}$, fix $t>0$. We obtain

$$\begin{aligned}
\|x_{j+1}-T(t)p\| &\leq \sum_{k=0}^{[t/s_j]-1}\|T((k+1)s_j)x_j - T(ks_j)x_{j+1}\| \\
&\quad + \|T([t/s_j]s_j)x_j - T([t/s_j]s_j)p\| + \|T([t/s_j]s_j)p - T(t)p\| \\
&\leq [t/s_j]\|T(s_j)x_j - x_{j+1}\| + \|x_j - p\| + \|T(t-[t/s_j]s_j)p - p\| \\
&= [t/s_j]\beta_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j-p\| + \|T(t-[t/s_j]s_j)p-p\| \\
&\leq t\beta_j/s_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j-p\| \\
&\quad + \max\{\|T(s)p-p\| : 0\leq s\leq s_j\}.
\end{aligned}$$

It follows that $\limsup_{n\longrightarrow\infty}\Phi(\|x_j - T(t)p\|) \leq \limsup_{n\longrightarrow\infty}\Phi(\|x_j-p\|)$. From Lemma 3.2.1 (ii) we have

$$\limsup_{n\longrightarrow\infty}\Phi(\|x_j - T(t)p\|) = \limsup_{n\longrightarrow\infty}\Phi(\|x_j-p\|) + \Phi(\|T(t)p-p\|).$$

So we have $\Phi(\|T(t)p-p\|)\leq 0$ and hence $p\in F$. Since the duality mapping $J_\varphi$ is weakly sequentially continuous,

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n+1})\rangle &= \limsup_{j\longrightarrow\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n_j+1})\rangle \\
&= \langle (A-\gamma f)q, J_\varphi(q-p)\rangle \leq 0.
\end{aligned}$$

Finally, we show that $x_n\to q$. From Lemma 3.2.1 (i), we have

$$\begin{aligned}
\Phi\big(\|x_{n+1}-q\|\big) &= \Phi\Big(\big\|(I-\alpha_nA)T(t_n)x_n - (I-\alpha_nA)q + \alpha_n\big(\gamma f(x_n)-\gamma f(q)\big) \\
&\quad + \alpha_n\big(\gamma f(q) - A(q)\big)\big\|\Big) \\
&\leq \Phi\Big(\big\|(I-\alpha_nA)(T(t_n)x_n - q) + \alpha_n\big(\gamma f(x_n)-\gamma f(q)\big)\big\|\Big) \\
&\quad + \alpha_n\big\langle\gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\big\rangle \\
&\leq \Phi\Big(\varphi(1)(1-\alpha_n\bar{\gamma})\|x_n-q\| + \alpha_n\gamma\alpha\|x_n-q\|\Big) \\
&\quad + \alpha_n\big\langle\gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\big\rangle \\
&= \Phi\Big(\big(\varphi(1) - \alpha_n(\varphi(1)\bar{\gamma}-\gamma\alpha)\big)\|x_n-q\|\Big) \\
&\quad + \alpha_n\big\langle\gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\big\rangle \\
&\leq \big(1-\alpha_n(\varphi(1)\bar{\gamma}-\gamma\alpha)\big)\Phi\big(\|x_n-q\|\big) \\
&\quad + \alpha_n\big\langle\gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\big\rangle.
\end{aligned}$$

Note that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n \longrightarrow \infty} \big\langle \gamma f(q) - A(q), J_\varphi(x_{n+1} - q) \big\rangle \le 0$. Using Lemma 3.2.3, we have $x_n \to q$ as $n \to \infty$ by the continuity of $\Phi$. This completes the proof. □

## 3.3 Convergence theorems for maximal monotone operators, weak relatively nonexpansive mappings and equilibrium problems

This section, we introduce hybrid iterative schemes for solving a system of the zero-finding problems of maximal monotone operators, the equilibrium problem and the fixed point problem of weak relatively nonexpansive mappings. We then prove, in a uniformly smooth and uniformly convex Banach space, strong convergence theorems by using a shrinking projection method. We finally apply the obtained results to a system of convex minimization problems.

The problem of finding a zero point of maximal monotone operators plays an important role in optimizations. This is because it can be reformulated to a convex minimization problem and a variational inequality problem. Many authors have studied the convergence of such problems in various spaces (see, for examples, [10, 17, 20, 28, 41, 56, 62, 67, 73, 85, 97, 98, 99, 100, 107, 108]). Initiated by Martinet [57], in a real Hilbert space $H$, Rockafellar [72] introduced the following iterative scheme: $x_1 \in H$ and

$$x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \ge 1, \tag{3.3.1}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_\lambda$ is the resolvent of $A$ defined by $J_\lambda := J_{\lambda A} = (I + \lambda A)^{-1}$ for all $\lambda > 0$ and $A$ is a maximal monotone operator on $H$. Such an algorithm is called the *proximal point algorithm*. It was proved that the sequence $\{x_n\}$ generated by (3.3.1) converges weakly to an element in $A^{-1}(0)$ provided $\liminf_{n \to \infty} \lambda_n > 0$. Recently, Kamimura-Takahashi [34] introduced the following iteration in a real Hilbert space: $x_1 \in H$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \ge 1,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$. The weak convergence theorems are also established in a real Hilbert space under suitable conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2004, Kamimura et al. [36] extended the above iteration process to a much more general setting. In fact, they proposed the following algorithm: $x_1 \in E$ and

$$x_{n+1} = J^{-1}\big(\alpha_n J(x_n) + (1 - \alpha_n) J(J_{\lambda_n} x_n)\big), \quad \forall n \ge 1,$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, \infty)$ and $J_\lambda := J_{\lambda A} = (J + \lambda A)^{-1} J$ for all $\lambda > 0$. They proved, in a uniformly smooth and uniformly convex Banach space, a weak convergence theorem.

Recently, Takahashi-Zembayashi [92] introduced the following iterative scheme for a relatively nonexpansive mapping $T : C \to C$ in a uniformly smooth and uniformly convex Banach space: $x_1 \in C$ and

$$\begin{cases} C_1 = C, \\ y_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T x_n\big), \\ u_n \in C \ \ \textit{such that} \ \ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$. Such an algorithm is called the *shrinking projection method* which was introduced by Takahashi et al. [90]. They proved that the sequence $\{x_n\}$ converges strongly to an element in $F(T) \cap EP(F)$ under appropriate conditions. The equilibrium problem has been intensively studied by many authors (see, for examples, [19, 21, 22, 23, 44, 45, 66, 78, 79]).

Motivated by the previous results, we introduce a hybrid iterative scheme for finding a zero point of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2, \cdots, N)$ which is also a common element in the solutions set of an equilibrium problem for $F$ and in the fixed points set of weak relatively nonexpansive mappings $T_i : C \to C$ $(i = 1, 2, \cdots)$. Using the projection technique, we also prove that the sequence generated by a constructed algorithm converges strongly to an element in $\big[\bigcap_{i=1}^{N} A_i^{-1}(0^*)\big] \cap \big[\bigcap_{i=1}^{\infty} F(T_i)\big] \cap EP(F)$ in a uniformly smooth and uniformly convex Banach space. Finally, we apply our results to a system of convex minimization problems.

Now, we give some useful preliminaries and lemmas which will be used in the sequel.

**Lemma 3.3.1.** [35] *Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences in $E$. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

**Lemma 3.3.2.** [3, 35] *Let $C$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $x \in E$ and let $z \in C$. Then $z = \Pi_C(x)$ if and only if $\langle y - z, Jx - Jz\rangle \leq 0$ for all $y \in C$.*

**Lemma 3.3.3.** [3, 35] *Let $C$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall x \in C \ \textit{and} \ y \in E.$$

**Lemma 3.3.4.** [58] *Let $E$ be a smooth and strictly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $T$ be a mapping from $C$ into itself such that $F(T)$ is nonempty and $\phi(u, Tx) \leq \phi(u, x)$ for all $(u, x) \in F(T) \times C$. Then $F(T)$ is closed and convex.*

**Lemma 3.3.5.** [41] *Let $E$ be a smooth, strictly convex and reflexive Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}(0^*) \neq \emptyset$, and let $J_{\lambda A} = (J + \lambda A)^{-1} J$ for each $\lambda > 0$. Then*

$$\phi\big(p, J_{\lambda A}(x)\big) + \phi\big(J_{\lambda A}(x), x\big) \leq \phi(p, x)$$

*for all* $\lambda > 0$, $p \in A^{-1}(0^*)$, *and* $x \in E$.

**Lemma 3.3.6.** [5] *Let* $C$ *be a closed and convex subset of a smooth, strictly convex and reflexive Banach space* $E$, *let* $F$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4), *and let* $r > 0$ *and* $x \in E$. *Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

**Lemma 3.3.7.** [93] *Let* $C$ *be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space* $E$, *and let* $F$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4). For all $r > 0$ and $x \in E$, define the mapping $T_r : E \to C$ as follows:

$$T_r(x) = \Big\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\Big\}.$$

Then, the following hold:

(1) $T_r$ is single-valued;

(2) $T_r$ is a firmly nonexpansive-type mapping [42], *i.e.*, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

**Lemma 3.3.8.** [93] *Let* $C$ *be a closed and convex subset of a smooth, strictly and reflexive Banach space* $E$, *let* $F$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(\mathrm{A1}) - (\mathrm{A4})$, *let* $r > 0$. *Then*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

*for all* $x \in E$ *and* $p \in F(T_r)$.

Finally, we are now ready to prove our mail results.

**Theorem 3.3.9.** *Let* $E$ *be a uniformly smooth and uniformly convex Banach space and let* $C$ *be a nonempty, closed and convex subset of* $E$. *Let* $A_i : E \to 2^{E^*}$ $(i = 1, 2, \cdots, N)$ *be maximal monotone operators, let* $F : C \times C \to \mathbb{R}$ *be a bifunction, and let* $T_i : C \to C$ $(i = 1, 2, \cdots)$ *be weak relatively nonexpansive mappings such that* $\mathcal{F} := \big[\bigcap_{i=1}^{N} A_i^{-1}(0^*)\big] \cap \big[\bigcap_{i=1}^{\infty} F(T_i)\big] \cap EP(F) \neq \emptyset$. *Let* $\{e_n\}_{n=1}^{\infty} \subset E$ *be the sequence such that* $\lim_{n\to\infty} e_n = 0$. *Define the sequence* $\{x_n\}_{n=1}^{\infty}$ *in* $C$ *as follows:*

$$\begin{cases} x_1 \in C_1 = C, \\ y_n = J_{\lambda_n^N A_N} \circ J_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_n^1 A_1}(x_n + e_n), \\ u_n = T_{r_n} y_n, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \phi(z, T_i u_n) \leq \phi(z, x_n + e_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases}$$

*If* $\liminf_{n\to\infty} \lambda_n^i > 0$ *for each* $i = 1, 2, \cdots, N$ *and* $\liminf_{n\to\infty} r_n > 0$, *then the sequence* $\{x_n\}$ *converges strongly to* $q = \Pi_{\mathcal{F}}(x_1)$.

*Proof.* We split the proof into several steps as follows:

**Step 1**. $\mathcal{F} \subset C_n$ for all $n \geq 1$.

From Lemma 3.3.4, we know that $\bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex. From Lemma 3.3.7 (4), we also know that $EP(F)$ is closed and convex. On the other hand, since $A_i$ $(i = 1, 2, \cdots, N)$ are maximal monotone, $A_i^{-1}(0^*)$ are closed and convex for each $i = 1, 2, \cdots, N$; consequently, $\bigcap_{i=1}^{N} A_i^{-1}(0^*)$ is closed and convex. Hence $\mathcal{F}$ is a nonempty, closed and convex subset of $C$.

We next show that $C_n$ is closed and convex for all $n \geq 1$. Obviously, $C_1 = C$ is closed and convex. Now suppose that $C_k$ is closed and convex for some $k \in \mathbb{N}$. Then, for each $z \in C_k$ and $i \geq 1$, we see that $\phi(z, T_i u_k) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k\rangle - 2\langle z, JT_i u_k\rangle \leq \|x_k\|^2 - \|T_i u_k\|^2.$$

By the construction of the set $C_{k+1}$, we see that

$$\begin{aligned} C_{k+1} &= \big\{z \in C_k : \sup_{i\geq 1} \phi(z, T_i u_k) \leq \phi(z, x_k)\big\} \\ &= \bigcap_{i=1}^{\infty} \big\{z \in C_k : \phi(z, T_i u_k) \leq \phi(z, x_k)\big\}. \end{aligned}$$

Hence $C_{k+1}$ is closed and convex. This shows, by induction, that $C_n$ is closed and convex for all $n \geq 1$. It is obvious that $\mathcal{F} \subset C_1 = C$. Now, suppose that $\mathcal{F} \subset C_k$ for some $k \in \mathbb{N}$. For any $p \in \mathcal{F}$, by Lemma 3.3.5 and Lemma 3.3.8, we have

$$\begin{aligned} \phi(p, T_i u_k) \leq \phi(p, u_k) &= \phi(p, T_{r_k} y_k) \\ &\leq \phi(p, y_k) \\ &= \phi\big(p, J_{\lambda_k^N A_N} \circ J_{\lambda_k^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\leq \phi\big(p, J_{\lambda_k^{N-1} A_{N-1}} \circ J_{\lambda_k^{N-2} A_{N-2}} \circ \cdots \circ J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\cdots \\ &\leq \phi\big(p, J_{\lambda_k^2 A_2} \circ J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\leq \phi\big(p, J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\leq \phi(p, x_k + e_k). \end{aligned} \tag{3.3.2}$$

This shows that $\mathcal{F} \subset C_{k+1}$. By induction, we can conclude that $\mathcal{F} \subset C_n$ for all $n \geq 1$.

**Step 2.** $\lim_{n\to\infty} \phi(x_n, x_1)$ exists.

From $x_n = \Pi_{C_n}(x_1)$ and $x_{n+1} = \Pi_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \tag{3.3.3}$$

From Lemma 3.3.3, for any $p \in \mathcal{F} \subset C_n$, we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}(x_1), x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1). \tag{3.3.4}$$

Combining (3.3.3) and (3.3.4), we conclude that $\lim_{n\to\infty}\phi(x_n,x_1)$ exists.

**Step 3.** $\lim_{n\to\infty}\big\|J(T_iy_n)-J(x_n+e_n)\big\|=0$.

Since $x_m=\Pi_{C_m}(x_1)\in C_m\subset C_n$ for $m>n\geq 1$, by Lemma 3.3.3, it follows that

$$\begin{aligned}\phi(x_m,x_n)=\phi\big(x_m,\Pi_{C_n}(x_1)\big) &\leq \phi\big(x_m,x_1\big)-\phi\big(\Pi_{C_n}(x_1),x_1\big)\\ &= \phi(x_m,x_1)-\phi(x_n,x_1).\end{aligned}$$

Letting $m,n\to\infty$, we have $\phi(x_m,x_n)\to 0$. By Lemma 3.3.1, it follows that $\|x_m-x_n\|\to 0$ as $m,n\to\infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of the space $E$ and the closedness of $C$, we can assume that $x_n\to q\in C$ as $n\to\infty$. In particular, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Since $e_n\to 0$, we have

$$\lim_{n\to\infty}\|x_{n+1}-(x_n+e_n)\|=0. \tag{3.3.5}$$

Since $x_{n+1}=\Pi_{C_{n+1}}(x_1)\in C_{n+1}$, for each $i\geq 1$,

$$\begin{aligned}\phi(x_{n+1},T_iu_n) &\leq \phi(x_{n+1},x_n+e_n)\\ &= \big\langle x_{n+1},J(x_{n+1})-J(x_n+e_n)\big\rangle+\big\langle x_{n+1}-(x_n+e_n),J(x_{n+1})\big\rangle.\end{aligned}$$

Since $E$ is uniformly smooth, $J$ is uniformly norm-to-norm continuous on bounded sets. It follows from (3.3.5) and by the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty}\phi(x_{n+1},T_iu_n)=0$$

for all $i=1,2,\cdots$. So from Lemma 3.3.1, we have

$$\lim_{n\to\infty}\|x_{n+1}-T_iu_n\|=0$$

and

$$\lim_{n\to\infty}\|T_iu_n-x_n\|=0$$

and, since $e_n\to 0$, therefore

$$\lim_{n\to\infty}\|T_iu_n-(x_n+e_n)\|=0. \tag{3.3.6}$$

for all $i=1,2,\cdots$. Since $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$,

$$\lim_{n\to\infty}\big\|J(T_iu_n)-J(x_n+e_n)\big\|=0 \tag{3.3.7}$$

for all $i=1,2,\cdots$.

**Step 4.** $\lim_{n\to\infty}\|T_iu_n-u_n\|=0$ for all $i=1,2,\cdots$.

Denote $\Theta_n^i = J_{\lambda_n^i A_i} \circ J_{\lambda_n^{i-1} A_{i-1}} \circ \cdots \circ J_{\lambda_n^1 A_1}$ for each $i \in \{1, 2, \cdots, N\}$ and $\Theta_n^0 = I$ for each $n \geq 1$. We note that $y_n = \Theta_n^N(x_n + e_n)$ for each $n \geq 1$.

To this end, we will show that

$$\lim_{n\to\infty} \left\| J\big(\Theta_n^i(x_n + e_n)\big) - J\big(\Theta_n^{i-1}(x_n + e_n)\big) \right\| = 0$$

for all $i = 1, 2, \cdots, N$.

For any $p \in \mathcal{F}$, by (3.3.2), we see that

$$\begin{aligned} \phi\big(p, \Theta_n^{N-1}(x_n + e_n)\big) &\leq \phi\big(p, \Theta_n^{N-2}(x_n + e_n)\big) \\ &\leq \phi\big(p, \Theta_n^{N-3}(x_n + e_n)\big) \\ &\cdots \\ &\leq \phi\big(p, (x_n + e_n)\big). \end{aligned} \tag{3.3.8}$$

Since $p \in \mathcal{F}$, by Lemma 3.3.5 and (3.3.8), it follows that

$$\begin{aligned} & \phi\big(y_n, \Theta_n^{N-1}(x_n + e_n)\big) \\ \leq\ & \phi\big(p, \Theta_n^{N-1}(x_n + e_n)\big) - \phi(p, y_n) \\ \leq\ & \phi\big(p, (x_n + e_n)\big) - \phi(p, y_n) \\ \leq\ & \phi\big(p, (x_n + e_n)\big) - \phi(p, u_n) \\ \leq\ & \phi\big(p, (x_n + e_n)\big) - \phi(p, T_i u_n) \\ =\ & \|x_n + e_n\|^2 - \|T_i u_n\|^2 - 2\big\langle p, J(x_n + e_n) - J(T_i u_n)\big\rangle. \end{aligned}$$

From (3.3.6) and (3.3.7), we get that $\lim_{n\to\infty} \phi\big(y_n, \Theta_n^{N-1}(x_n + e_n)\big) = 0$. So we obtain

$$\lim_{n\to\infty} \big\|y_n - \Theta_n^{N-1}(x_n + e_n)\big\| = 0. \tag{3.3.9}$$

Again, since $p \in \mathcal{F}$,

$$\begin{aligned} & \phi\big(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)\big) \\ & \leq \phi\big(p, \Theta_n^{N-2}(x_n + e_n)\big) - \phi\big(p, \Theta_n^{N-1}(x_n + e_n)\big) \\ & \leq \phi\big(p, (x_n + e_n)\big) - \phi\big(p, \Theta_n^{N-1}(x_n + e_n)\big) \\ & \leq \phi\big(p, (x_n + e_n)\big) - \phi\big(p, T_i u_n\big). \end{aligned}$$

From (3.3.6) and (3.3.7), we get that

$$\lim_{n\to\infty} \phi\big(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)\big) = 0.$$

It also follows that

$$\lim_{n\to\infty} \big\|\Theta_n^{N-1}(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n)\big\| = 0.$$

Continuing in this process, we can show that

$$\lim_{n\to\infty} \big\|\Theta_n^{N-2}(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n)\big\| = \cdots = \lim_{n\to\infty} \big\|\Theta_n^{1}(x_n + e_n) - (x_n + e_n)\big\| = 0.$$

So, we now conclude that

$$\lim_{n\to\infty} \left\| \Theta_n^i(x_n + e_n) - \Theta_n^{i-1}(x_n + e_n) \right\| = 0 \tag{3.3.10}$$

for each $i = 1, 2, \cdots, N$. By the uniform norm-to-norm continuity of $J$, we also have

$$\lim_{n\to\infty} \left\| J\big(\Theta_n^i(x_n + e_n)\big) - J\big(\Theta_n^{i-1}(x_n + e_n)\big) \right\| = 0 \tag{3.3.11}$$

for each $i = 1, 2, \cdots, N$. Using (3.3.10), it is easily seen that

$$\lim_{n\to\infty} \|y_n - (x_n + e_n)\| = 0. \tag{3.3.12}$$

From $u_n = T_{r_n} y_n$, by Lemma 3.3.8, it follows that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\leq \phi(p, y_n) - \phi(p, T_{r_n} y_n) \\ &\leq \phi(p, x_n + e_n) - \phi(p, u_n) \\ &\leq \phi(p, x_n + e_n) - \phi(p, T_i u_n). \end{aligned}$$

This implies that $\lim_{n\to\infty} \phi(u_n, y_n) = 0$ and hence

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.3.13}$$

Combining (3.3.6), (3.3.12) and (3.3.13), we obtain

$$\lim_{n\to\infty} \|T_i u_n - u_n\| = 0 \tag{3.3.14}$$

for all $i \geq 1$.

**Step 5.** $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Since $x_n \to q$ and $e_n \to 0$, $x_n + e_n \to q$. So from (3.3.12) and (3.3.13), we have $u_n \to q$. Note that $T_i$ $(i = 1, 2, \cdots)$ are weak relatively nonexpansive. Using (3.3.14), we can conclude that $q \in \widetilde{F}(T_i) = F(T_i)$ for all $i \geq 1$. Hence $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

**Step 6.** $q \in \bigcap_{i=1}^{N} A_i^{-1}(0^*)$.

Noting that $\Theta_n^i(x_n + e_n) = J_{\lambda_n^i A_i} \Theta_n^{i-1}(x_n + e_n)$ for each $i = 1, 2, \cdots, N$, we obtain

$$\left\| A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n) \right\| = \frac{1}{\lambda_n^i} \left\| J\big(\Theta_n^{i-1}(x_n + e_n)\big) - J\big(\Theta_n^i(x_n + e_n)\big) \right\|.$$

From (3.3.11) and $\liminf_{n\to\infty} \lambda_n^i > 0$, we have

$$\lim_{n\to\infty} \left\| A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n) \right\| = 0. \tag{3.3.15}$$

We note that $\Big(\Theta_n^i(x_n+e_n), A_{\lambda_n^i}\Theta_n^{i-1}(x_n+e_n)\Big) \in G(A_i)$ for each $i = 1, 2, \cdots, N$. If $(w, w^*) \in G(A_i)$ for each $i = 1, 2, \cdots, N$, then it follows from the monotonicity of $A_i$ that

$$\Big\langle w^* - A_{\lambda_n^i}\Theta_n^{i-1}(x_n+e_n), w - \Theta_n^i(x_n+e_n)\Big\rangle \geq 0. \tag{3.3.16}$$

We see that $\Theta_n^i(x_n+e_n) \to q$ for each $i = 1, 2, \cdots, N$. Thus, from (3.3.15) and (3.3.16), we have

$$\langle w^*, w - q\rangle \geq 0.$$

By the maximality of $A_i$, it follows that $q \in A_i^{-1}(0^*)$ for each $i = 1, 2, \cdots, N$. Therefore $q \in \bigcap_{i=1}^N A_i^{-1}(0^*)$.

**Step 7.** $q \in EP(F)$.

From $u_n = T_{r_n}y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

By (A2), we have

$$\begin{aligned} \|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -F(u_n, y) \geq F(y, u_n), \quad \forall y \in C. \end{aligned}$$

Note that $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ since $\liminf_{n\to\infty} r_n > 0$. From (A4) and $u_n \to q$, we get $F(y, q) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)q$. Then $y_t \in C$, which implies that $F(y_t, q) \leq 0$. From (A1), we obtain that $0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, q) \leq tF(y_t, y)$. Thus $F(y_t, y) \geq 0$. From (A3), we have $F(q, y) \geq 0$ for all $y \in C$. Hence $q \in EP(F)$. From Step 5, Step 6 and Step 7, we now can conclude that $q \in \mathcal{F}$.

**Step 8.** $q = \Pi_{\mathcal{F}}(x_1)$.

From $x_n = \Pi_{C_n}(x_1)$, we have

$$\big\langle J(x_1) - J(x_n), x_n - z\big\rangle \geq 0, \quad \forall z \in C_n.$$

Since $\mathcal{F} \subset C_n$, we also have

$$\big\langle J(x_1) - J(x_n), x_n - z\big\rangle \geq 0, \quad \forall z \in \mathcal{F}. \tag{3.3.17}$$

Letting $n \to \infty$ in (3.3.17), we obtain

$$\big\langle J(x_1) - J(q), q - z\big\rangle \geq 0, \quad \forall z \in \mathcal{F}.$$

This shows that $q = \Pi_{\mathcal{F}}(x_1)$ by Lemma 3.3.2. We thus complete the proof. □

As a direct consequence of Theorem 3.3.9, we can also apply to a system of convex minimization problems.

**Theorem 3.3.10.** *Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $f_i : E \to (-\infty, \infty]$ $(i = 1, 2, \cdots, N)$ be proper lower semi-continuous convex functions, let $F : C \times C \to \mathbb{R}$ be a bifunction, and let $T_i : C \to C$ $(i = 1, 2, \cdots)$ be weak relatively nonexpansive mappings such that $\mathcal{F} := \big[\bigcap_{i=1}^{N}(\partial f_i^{-1})(0^*)\big] \cap \big[\bigcap_{i=1}^{\infty} F(T_i)\big] \cap EP(F) \neq \emptyset$. Let $\{e_n\}_{n=1}^{\infty} \subset E$ be the sequence such that $\lim_{n\to\infty} e_n = 0$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in $C$ as follows:*

$$\begin{cases} x_1 \in C_1 = C, \\ z_n^1 = \arg\ \min_{y\in E}\Big\{f_1(y) + \frac{1}{2\lambda_n^1}\|y\|^2 + \frac{1}{\lambda_n^1}\big\langle y, J(x_n + e_n)\big\rangle\Big\}, \\ \quad \cdots \\ z_n^{N-1} = \arg\ \min_{y\in E}\Big\{f_{N-1}(y) + \frac{1}{2\lambda_n^{N-1}}\|y\|^2 + \frac{1}{\lambda_n^{N-1}}\big\langle y, J(z_n^{N-2})\big\rangle\Big\}, \\ y_n = \arg\ \min_{y\in E}\Big\{f_N(y) + \frac{1}{2\lambda_n^N}\|y\|^2 + \frac{1}{\lambda_n^N}\big\langle y, J(z_n^{N-1})\big\rangle\Big\}, \\ u_n = T_{r_n} y_n, \\ C_{n+1} = \Big\{z \in C_n : \sup_{i\geq 1}\phi(z, T_i u_n) \leq \phi(z, x_n + e_n)\Big\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} \lambda_n^i > 0$ for each $i = 1, 2, \cdots, N$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges strongly to $q = \Pi_{\mathcal{F}}(x_1)$.*

*Proof.* By Rockafellar's theorem [70, 71], $\partial f_i$ are maximal monotone operators for each $i = 1, 2, \cdots, N$. Let $\lambda^i > 0$ for each $i = 1, 2, \cdots, N$. Then $z^i = J_{\lambda^i \partial f_i}(x)$ if and only if

$$\begin{aligned} 0 &\in \partial f_i(z^i) + \frac{1}{\lambda^i}\big(J(z^i) - J(x)\big) \\ &= \partial\Big(f_i + \frac{1}{\lambda^i}\big(\frac{\|\cdot\|^2}{2} - J(x)\big)\Big)(z^i), \end{aligned}$$

which is equivalent to

$$z^i = \arg\ \min_{y\in E}\Big\{f_i(y) + \frac{1}{\lambda^i}\big(\frac{\|y\|^2}{2} - \langle y, J(x)\rangle\big)\Big\}.$$

Using Theorem 3.3.9, we thus complete the proof. $\square$

# CHAPTER 4

# CONCLUSIONS

## 4.1 Outputs 3 papers (Supported by TRF : MRG5380202)

1. K. Nammanee and I. Inchan, Strong convergence theorem by hybrid method for non-Lipschitzian mapping, **Applied Mathematical Sciences**, Vol. 5, (2011), no. 52, 2581–2591.

2. Kamonrat Nammanee, Suthep Suantai and Prasit Cholamjiak, A General Iterative Method for a Nonexpansive Semigroup in Banach Spaces with Gauge Functions, **Journal of Applied Mathematics**, Volume 2012, Article ID 506976, 14 pages.

3. Kamonrat Nammanee, Suthep Suantai and Prasit Cholamjiak, Convergence theorems for maximal monotone operators, weak relatively nonexpansive mappings and equilibrium problems, **Journal of Applied Mathematics**, Volume 2012, Article Inpress.

## REFERENCES

# References


[1] R.P. Agarwal, Donal O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.* 8 (1) (2007) 61–79.

[2] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications. Springer, New York, USA (2009)

[3] Y.I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, pp. 15-50 (1996).

[4] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. *Nonlinear Anal.* 67, (2007) 2350-2360.

[5] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student* 63 (1994) 123-145.

[6] R. E. Bruck, T. Kuczumow and S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, *Colloq. Math.* 65 (1993) 169–179.

[7] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces. *Math. Z.* 100, (1967) 201-225.

[8] D. Butanriu, S. Reich, A.J. Zaslavski, Asymtotic behavior of relatively nonexpansive operators in Banach spaces, *J. Appl. Anal.* 7 (2001) 151-174.

[9] D. Butanriu, S. Reich, A.J. Zaslavski, Weakly convergence of orbits of nonlinear operators in reflexive Banach spaces, *Numer. Funct. Anal. Optim.* 24 (2003) 489-508.

[10] L.-C. Ceng, Y.-C. Liou, E. Naraghirad, Iterative approaches to find zeros of maximal monotone operators by hybrid approximate proximal point methods, *Fixed Point Theory Appl.* (2011), Art. ID 282171, 18 pages.

[11] R. Chen, H. He, Viscosity approximation of common fixed points of nonexpansive semigroups in Banach space. *Appl. Math. Lett.* 20, (2007) 751-757.

[12] R. Chen, Y. Song, Converence to common fixed point of nonexpansive semigroups. *J. Comput. Appl. Math.* 200, (2007) 566-575.

[13] C. Chidume, N. Shahzad, H. Zegeye, Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense, Numer. Funct. Anal. Optim. 25 (2004) 239–257.

[14] C.E. Chidume, Geometric properties of Banach spaces and nonlinear iterations. *Springer Verlag Series: Lecture Notes in Mathematics* Vol. 1965, XVII, 326p, ISBN 978-1-84882-189-7 (2009)

[15] C.E. Chidume, C.O. Chidume, Iterative approximation of fixed points of nonexpansive mappings. *J. Math. Anal. Appl.* 318, (2006) 288-295.

[16] Y.J. Cho, S.M. Kang, H. Zhou, Some control conditions on iterative methods. Comm. Appl. *Nonlinear Anal.* 12, (2005) 27-34.

[17] Y.J. Cho, S.M. Kang, H. Zhou, Approximate proximal point algorithms for finding zeroes of maximal monotone operators in Hilbert spaces, *J. Inequal. Appl.* (2008), Art. ID 598191, 10 pages.

[18] P. Cholamjiak, S. Suantai, Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions. *J. Glob. Optim.* Doi: 10.1007/s10898-011-9756-4.

[19] P. Cholamjiak, A hybrid iterative scheme for equilibrium problems, variational inequality problems and fixed point problems in Banach spaces, *Fixed Point Theory Appl.* (2009), Art. ID 719360, 18 pages.

[20] P. Cholamjiak, Y.J. Cho, S. Suantai, Composite iterative schemes for maximal monotone operators in reflexive Banach spaces, *Fixed Point Theory Appl.* 2011, 2011:7, 10 pages.

[21] P. Cholamjiak, S. Suantai, Convergence analysis for a system of equilibrium problems and a countable family of relatively quasi-nonexpansive mappings in Banach spaces, *Abstr. Appl. Anal.* (2010), Art. ID 141376, 17 pages.

[22] W. Cholamjiak, S. Suantai, A hybrid method for a countable family of multivalued maps, equilibrium problems, and variational inequality problems, *Discrete Dyn. Nat. Soc.* (2010), Art. ID 349158, 14 pages.

[23] W. Cholamjiak, S. Suantai, A new hybrid algorithm for variational inequality, generalized equilibrium problems and a countable family of quasi-nonexpansive mappings, *J. Nonlinear and Convex Anal.* 12 (2011) 381-398.

[24] V. Colao, G. Marino, H.K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems. *J. Math. Anal. Appl.* 344, (2008) 340-352.

[25] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming using proximal-like algorithms. *Math. Program* 78: 29-41 (1997).

[26] P. Daniele, F. Giannessi, A. Maugeri, Equilibrium Problems and Variational Models, Kluwer Academic Publishers, United Kingdom, 2003.

[27] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* 35 (1972), 171–174.

[28] O. Güler, On the convergence of the proximal point algorithm for convex minimization, *SIAM J. Control Optim.* 29 (1991) 403-419.

[29] B. Halpern, Fixed points of nonexpanding maps. *Bull. Am. Math. Soc.* 73, (1967) 957-961.

[30] I. Inchan, Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, *Int. Journal of Math. Analysis.* Vol. 2, 23 (2008) 1135–1145.

[31] I. Inchan, K. Nammanee, Strong convergence theorems by hybrid methods for asymptotically $k$-strict pseudo-contractive mappings in Hilbert space, *Nonlinear Anal. : Hybrid syst.* 3(4) (2009) 380-385.

[32] S. Ishikawa, Fixed point by a new iteration. *Proc. Amer. Math. Soc.* 44 (1974) 147–150.

[33] J. Kang, Y. Su, X Zhang, General iterative algorithm for nonexpansive semigroups and variational inequalities in Hilbert spaces. *J. Ineq. Appl.*, Art. ID 264052 (2010)

[34] S. Kamimura, W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, *J. Approx. Theor.* 106 (2000) 226-240.

[35] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.* 13 (2002) 938-945.

[36] S. Kamimura, F. Kohsaka and W. Takahashi, Weak and strong convergence of a maximal monotone operators in a Banach space, *Set-Valued Anal.* 12 (2004) 417-429.

[37] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations. *Nonlinear Anal.* 61, (2005) 51-60.

[38] G.E. Kim and T.H. Kim, Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces, *Comp. and Math. Appl.* 42 (2001) 1565–1570.

[39] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, *Nonlinear. Anal.* 64 (2006) 1140–1152.

[40] T. H. Kim and H. K. Xu, Convergence of the modified Mann's iterations method for asymptotically strict pseudo-contractions, *Nonlinear. Anal.* 68 (2008) 2828–2836.

[41] F. Kohsaka and W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, *Abstr. Appl. Anal.* 3 (2004) 239-249.

[42] F. Kohsaka, W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces, *SIAM J. Optim.* 19 (2008) 824-835.

[43] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon* 12 (1976) 747-756.

[44] P. Kumam, A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive mapping, *Nonlinear Anal.: Hybr. Syst.* 2 (2008) 1245-1255.

[45] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, *J. Appl. Math. Comput.* 29 (2009) 263-280.

[46] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, *Journal of Applied Mathematics and Computing.* 29 (2009) 263–280.

[47] X.N. Li, J.S. Gu, Strong convergence of modified Ishikawa iteration for a nonexpansive semigroup in Banach spaces. *Nonlinear Anal.* 73, (2010) 1085-1092.

[48] S. Li, L. Li, Y. Su, General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space. *Nonlinear Anal.* 70, (2009) 3065-3071.

[49] H. Iiduka, w. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.* 61, (2005) 341-350.

[50] T.C. Lim, H.K. Xu, Fixed point theorems for asymptotically nonexpansive mappings. *Nonlinear Anal.* 22, (1994) 1345-1355.

[51] Q. Lin, Viscosity approximation to common fixed points of a nonexpansive semigroup with a generalized contraction mappings. *Nonlinear Anal.* 71, (2009) 5451-5457.

[52] P.L. Lions, Approximation de points fixes de contractions. C.R. *Acad. Sci. Paris Ser.* A-B 284, (1977) A1357-A1359.

[53] W. R. Mann, Mean value methods in iteration. *Proc. Amer. Math. Soc.* 4 (1953) 506–510.

[54] G. Marino, H.K. Xu, Convergence of generalized proximal point algorithm, *Comm. Pure Appl. Anal.* 3 (2004) 791-808.

[55] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* 318, (2006) 43-52.

[56] G. Marino, H.K. Xu, Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.* 329 (2007) 336–349.

[57] B. Martinet, Régularisation d'inéquations variationelles par approximations successives. Rev. Francaise d'Informatique et de Recherche Opérationelle 4 (1970) 154-159.

[58] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theor.* 134 (2005) 257-266.

[59] A. Moudafi and M. Thera, Proximal and dynamical approaches to equilibrium problems, Lecture note in Economics and Mathematical Systems, Springer-Verlag, New York, 477, (1999) 187–201.

[60] A. Moudafi, Viscosity approximation methods for fixed point problems. *J. Math. Anal. Appl.* (2000) 241, 46-55.

[61] N. Nadezhkina, W. Takahashi, Weak convergence theorems by extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 128 (2006) 191-201.

[62] N. Onjai-uea, P. Kumam, A new iterative scheme for equilibrium problems, fixed point problems for nonexpansive mappings and zero points of maximal monotone operators, *Fixed Point Theory Appl.* (2012) doi:10.1186/1687-1812-2012-27.

[63] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math Anal. Appl.* 336 (2007) 455-469.

[64] S. Plubtieng, R. Punpaeng, Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces. *Math. Comput. Modelling.* 48, (2008) 279-286.

[65] S. Plubtieng and R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, *Applied Mathematics and Computation* 197, (2008) 548-558.

[66] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, *J. Comput. Appl. Math.* 225 (2009) 20-30.

[67] X. Qin, S.M. Kang, Y.J. Cho, Approximating zeros of monotone operators by proximal point algorithms, *J. Glob. Optim.* 46 (2010) 75-87.

[68] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces. *J. Math. Anal. Appl.* 75, (1980) 287-292.

[69] S. Reich, Approximating fixed points of nonexpansive mappings. *PanAmer. Math. J.* 4, (1994) 23-28.

[70] R.T. Rockafellar, Characterizations of the subdifferentials of convex functions, *Pacific J. Math.* 17 (1966) 497-510.

[71] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* 33 (1970) 209-216.

[72] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* 14 (1976) 877-898.

[73] S. Saewan, P. Kumam, A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems, *Abstr. Appl. Anal.* (2010), Art. ID 123027, 31 pages, 2010.

[74] D. R. Sahu, H. K. Xu and J. C. Yao, Asymptotically strict pseudocontractive mapping in the intermediate sense, *Nonlinear Anal.* **70** (2009) 3502–3511.

[75] J. Schu, Approximation of fixed points of asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **112** (1991) 143–151.

[76] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* 43 (1991) 153–159.

[77] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings. *J. Math. Anal. Appl.* 158 (1991) 407–413.

[78] Y. Shehu, Iterative approximation method for finite family of relatively quasi nonexpansive mappings and systems of equilibrium problems, *J. Glob. Optim.* DOI 10.1007/s10898-010-9619-4.

[79] Y. Shehu, A new hybrid iterative scheme for countable families of relatively quasi-nonexpansive mappings and system of equilibrium problems, *Int. J. Math. Math. Sci.* (2011), Art. ID 131890, 23 pages.

[80] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mapping in Banach spaces. *Proc. Am. Math. Soc.* 125, (1997) 3641-3645.

[81] Y. Song, S. Xu, Strong convergence theorems for nonexpansive semigroup in Banach spaces. *J. Math. Anal. Appl.* 338, (2008) 152-161.

[82] G. Stampacchia (1964). Formes bilineaires coercitivies sur les ensembles convexes. *C. R. Acad. Sci. Paris*, 258, 4413-4416.

[83] Y. Su, M. Shang and X. Qin, An iterative method of solution for equilibrium and optimization problems, *Nonlinear Anal.* 69(8) (2008) 2709-2719.

[84] Y. Su, H.-K. Xu, X. Zhang, Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications, *Nonlinear Anal.* 73 (2010) 3890-3906.

[85] P. Sunthrayuth, P. Kumam, A system of generalized mixed equilibrium problems, maximal monotone operators and fixed point problems with application to optimization problems, Abstr. Appl. Anal. (2012), Art. ID 316276, 39 pages.

[86] T. Suzuki, On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces. *Proc. Am. Math. Soc.* 131, (2002) 2133-2136.

[87] T. Suzuki, Strong convergence of Krasnoselskii and Manns type sequences for one parameter nonexpansive semigroups without Bochner integrals. *J. Math. Anal. Appl.* 305, (2005) 227-239.

[88] W. Takahashi, Nonlinear Function Analysis. Yokahama Publishers, Yokahama (2000)

[89] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* 311(1) (2007) 506-515.

[90] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* 341 (2008) 276-286.

[91] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.* 118, (2003) 417-428.

[92] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, *Fixed Point Theory Appl.* (2008), Art. ID 528476, 11 pages.

[93] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, *Nonlinear Anal.* 70 (2009) 45-57.

[94] K.K. Tan, H. K. Xu, Fixed point iteration processes for asymptotically nonexpansive mapping, *Proc. Amer. Math. Soc.* 122 (1994), 733–739.

[95] R. Wangkeeree, U. Kamraksa, Strong convergence theorems of viscosity iterative methods for a countable family of strict pseudo-contractions in Banach spaces. *Fixed Point Theory and Appl.* (2010) doi:10.1155/2010/579725.

[96] R. Wangkeeree, N. Petrot, R. Wangkeeree, The general iterative methods for nonexpansive mappings in Banach spaces. *J. Glob. Optim.* doi 10.1007/s10898-010-9617-6

[97] K. Wattanawitoon, P. Kumam, Hybrid proximal-point methods for zeros of maximal monotone operators, variational inequalities and mixed equilibrium problems, *Int. J. Math. Math. Sci.* (2011), Art. ID 174796, 31 pages.

[98] K. Wattanawitoon, P. Kumam, Generalized mixed equilibrium problems for maximal monotone operators and two relatively quasi-nonexpansive mappings, *Thai J. Math.* 9 (2011) 165-189.

[99] K. Wattanawitoon, P. Kumam, A new iterative scheme for generalized mixed equilibrium, variational inequality problems and a zero point of maximal monotone operators, *J. Appl. Math.* (2012), Art. ID 692829, 27 pages.

[100] U. Witthayarat, Y. J. Cho, P. Kumam, Convergence of an iterative algorithm for common solutions for zeros of maximal accretive operator with applications, *J. Appl. Math.* (2012), Art. ID 185104, 17 pages.

[101] R. Wittmann, Approximation of fixed points of nonexpansive mappings. *Arch. Math.* 58, (1992) 486-491.

[102] H.K. Xu, Another control condition in iterative method for nonexpansive mappings. *Bull. Astral. Math. Soc.* 65, (2002) 109-113.

[103] H.K. Xu, Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* 66, (2002) 240-256.

[104] H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.* 116, (2003) 659-678.

[105] H.K. Xu, Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.* 298, (2004) 279-291.

[106] H.K. Xu, A strong convergence theorem for contraction semigroups in Banach spaces. *Bull. Aust. Math. Soc.* 72, (2005) 371-379.

[107] Y. Yao, M.A. Noor, On convergence criteria of generalized proximal point algorithms, *J. Comput. Appl. Math.* 217 (2008) 46-55.

[108] Y. Yao, N. Shahzad, Strong convergence of a proximal point algorithm with general errors, *Optim. Lett.* 6(4) (2012) 621-628, doi: 10.1007/s11590-011-0286-2

[109] Y. Yao, J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings. *Appl. Math. Comput.* 186(2), (2007) 1551-1558.

[110] H. Zegeye, N. Shahzad, Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, *Nonlinear Anal.* 70 (2009) 2707-2716.

# APPENDIX

**Applied Mathematical Sciences, Vol. 5, 2011, no. 52, 2581 - 2591**

# Strong Convergence Theorem by Hybrid Method for Non-Lipschitzian Mapping

**K. Nammanee**[a1] **and I. Inchan**[b]

[a]Department of Mathematics, School of Science,
University of Phayao, 56000, Thailand

[b]Department of Mathematics and Computer, Faculty of Science and Technology
Uttaradit Rajabhat University, 53000, Thailand

**Abstract**

We introduce the hybrid method of modified Mann's iteration for an asymptotically $k$-strict pseudo-contractive mapping $T$ in the intermediate sense which is necessarily lipschitzian. We establish strong convergence theorem for such method. The result extend and improve the recent ones announced by Inchan and Nammanee, Inchan and concern result of Takahashi, Takeuchi and Kubota [Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 341 (2008), 276–286], and many others.

**Mathematics Subject Classification:** 46C05, 47D03, 47H09, 47H10, 47H20

**Keywords:** asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense; Mann's iteration method

## 1 Introduction

Let $C$ be a nonempty subset of a Hilbert space $H$ and $T : C \to C$ a mapping. Recall the following concepts.

(i) $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

[1]Corresponding author;
*Email addresses:* kamonrat.na@up.ac.th (K. Nammanee) ; peissara@uru.ac.th (I. Inchan)

ii) $T$ is asymptotically nonexpansive (cf. [4]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying $\lim_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all integers $n \ge 1$ and $x, y \in C$.

iii) $T$ is uniformly Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|$ for all integers $n \ge 1$ and all $x, y \in C$.

(iv) $T$ is asymptotically nonexpansive in the intermediate sense [2] provided $T$ is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by Agarwal, O'Regan and Sahu [1], Bruck, Kuczumow and Reich [2], Chidume, Shahzad and Zegeye [3], Kim and Kim [9] and many others.

In 2008, Kim and Xu [11] introduced the concept of asymptotically $k$-strict pseudo-contractive mappings in Hilbert space as below:

**Definition 1.1.** *Let $C$ be a nonempty subset of a Hilbert space $H$. A mapping $T : C \to C$ is said to be an asymptotically $k$-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, 1)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that*

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 \qquad (1)$$

*for all $x, y \in C$ and $n \in N$.*

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically $k$-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{\frac{k+\sqrt{1+(1-k)\gamma_n}}{1+k} : n \in N\}$.

Recently, Sahu et al.[16] introduced the concept of asymptotically $k$-strict pseudo-contractive mappings in the intermediate sense which are not necessarily Lipschitzian (see Lemma 2.6 [16]) as below:

**Definition 1.2.** *Let $C$ be a nonempty subset of a Hilbert space $H$. A mapping $T : C \to C$ will be called an asymptotically $k$-strict pseudo-contractive* *mapping in the intermediate sense with sequence* $\{\gamma_n\}$ *if there exist a constant* $k \in [0,1)$ *and a sequence* $\{\gamma_n\}$ *in* $[0,1)$ *with* $\lim_{n\to\infty} \gamma_n = 0$ *such that*

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \le 0. \tag{2}$$

*Throughout this paper we assume that*

$$c_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - T^n x - (y - T^n y)\|^2)\}.$$

*Then* $c_n \ge 0$ *for all* $n \in N$, $c_n \to 0$ *as* $n \to \infty$ *and (2) reduces to the relation*

$$\|T^n x - T^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n \tag{3}$$

*for all* $x, y \in C$ *and* $n \in N$

**Remark 1.3.** *If* $c_n = 0$ *for all* $n \in N$ *in (3) then* $T$ *is an asymptotically* $k$*-strict pseudocontractive mapping with sequence* $\{\gamma_n\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [5, 13, 17, 20]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [10, 20].

Iteration method for finding a fixed point of an asymptotically $k$-strict pseudo-contractive mapping $T$ is the modified Mann's iteration method studied in [12, 18, 19, 21] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \ge 0, \tag{4}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ line in the interval $(0,1)$.

In 2007, Takahashi, Takeuchi and Kubota [20] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of $C$ as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases} \tag{5}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then they prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$. In 2008, Kumam [8], introduce an iterative scheme by a new hybrid method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for $\alpha$-inverse-strongly monotone mappings in a real Hilbert space.

In 2008, Inchan [6], introduce the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let $C$ be a nonempty closed bounded convex subset of a Hilbert space $H$, $T$ be an asymptotically nonexpansive mapping of $C$ into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then him prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, Inchan and Nammanee [7], introduce the modified Mann iteration processes for an asymptotically $k$-strict pseudo-contractive mapping. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $T$ be an asymptotically $k$-strict pseudo-contractive mapping of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then they prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these fact, it is the purpose of this paper to introduce the modified Mann iteration processes for an asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense by idear in (7). Let $C$ be a closed convex subset of a Hilbert space $H$, $T : C \to C$ be an asymptotically $k$-strictly pseudo-contractive mapping in the intermediate sense and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0,\ (n \to \infty)$.

We shall prove that the iteration generated by (8) converges strongly to $z_0 = P_{F(T)}x_0$.

# 2 Preliminary

A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of *fixed points* of $T$; that is, $F(T) = \{x \in C : Tx = x\}$. Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let $C$ be a closed convex subset of $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$.

We collect some lemmas which will be used in the proof for the main result.

**Lemma 2.1.** *[14] There holds the identity in a Hilbert space $H$:*

*(i)* $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H.$

*(ii)* $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0, 1]$.

**Lemma 2.2.** *[15] Let $C$ be a closed convex subset of a real Hilbert space $H$. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

**Lemma 2.3.** *[16] Let $C$ be a nonempty subset of a Hilbert space $H$ and let $T : C \to C$ a uniformly continuous asymptotically $k$-strict pseudo-contractive in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in $C$ such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

**Lemma 2.4.** *[16] Let $C$ be a nonempty closed convex subset of Hilbert space $H$ and $T : C \to C$ a continuous asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that $\{x_n\}$ is sequence in $C$ such that $x_n \rightharpoonup x \in C$ and $\limsup_{n\to\infty} \|x_n - T^n x_n\| = 0$, then $(I - T)x = 0$.*

# 3 Main Results

In this section, we prove strong convergence theorem by hybrid method for asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense in Hilbert spaces.

**Theorem 3.1.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex bounded subset of $H$. Let $T$ be a uniformly continuous asymptotically $k$-strict pseudo-contractive mapping in the intermediate sense of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1-k$. Then $\{x_n\}$ generated by (8) converges strongly to $z_0 = P_{F(T)}x_0$.*

**Proof.** We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. For $u \in F(T) \subset C_m$. By lemma 2.1, we have,

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\
&= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 \\
&\quad + k\|x_m - T^m x_m\|^2 + c_m] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1+(1-\alpha_m)\gamma_m)\|x_m - u\|^2 + (k-\alpha_m)(1-\alpha_m)\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \qquad (1)
\end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that $C_n$ is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that $C_m$ is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since $C_m$ is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$. Then

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\
&= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\
&\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&\quad + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.
\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since $C_m$ is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, we have

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\
&= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\
&\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad +(1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad -\alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&\quad +[k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\
&\quad +\theta_m + c_m \\
&= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that $C_{m+1}$ is closed and convex. Hence $C_n$ is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_0$. By Lemma 2.2, we have

$$\langle x_0 - x_n, x_n - y \rangle \geq 0, \text{ for all } y \in C_n.$$

Moreover, by the same proof of Theorem 3.1 of [7], we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n, \tag{3}$$

By the definition of $y_n$, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\
&= (1-\alpha_n)\|T^n x_n - x_n\|.
\end{aligned}$$

From (3), we have

$$\begin{aligned}(1-\alpha_n)^2\|T^n x_n - x_n\|^2 &= \|y_n - x_n\|^2\\ &= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2\\ &\leq \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2\\ &\quad +2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\|\\ &\leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2\\ &\quad +\theta_n + c_n + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\|\\ &= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2\\ &\quad +2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|)\\ &\quad +\theta_n + c_n.\end{aligned}$$

It follows that

$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n.$

Hence

$$(1-k-\alpha_n)\|T^n x_n - x_n\| \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n. \tag{4}$$

From $\limsup_{n\to\infty} \alpha_n < 1-k$, we can chosen $\epsilon > 0$ such that $\alpha_n \leq 1-k-\epsilon$ for large enough $n$. From (2) and (4), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{5}$$

From (2), (5) and Lemma 2.3, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{6}$$

Since $H$ is reflexive and $\{x_n\}$ is bounded we get that $\emptyset \neq \omega_w(x_n)$. From Lemma 2.4, we have $\omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| &\leq \|x_0 - w\| \leq \liminf_{n\to\infty} \|x_0 - x_n\|\\ &\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\|x_n - z_0\|^2 = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2$$

$$\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0 \rangle) \to 0 \text{ as } n \to \infty.$$

This complete the proof. □

Using this Theorem 3.1, we have the following corollaries.

**Corollary 3.2.** *[7] Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T$ be an asymptotically $k$-strict pseudo-contractive mapping of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (7) converges strongly to $z_0 = P_{F(T)}x_0$.*

**Corollary 3.3.** *[6] Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T$ be an asymptotically nonexpansive mapping of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

*where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (7) converges strongly to $z_0 = P_{F(T)}x_0$.*

**Corollary 3.4.** *([20] Theorem 4.1) Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into $H$ such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of $C$ as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

*where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.*

**Acknowledgements.** The authors would like to thank the referee(s) and professor Suthep Suantai for his comments and suggestions on the manuscript. This work was supported by the Thailand Research Fund and the Commission on Higher Education under grant MRG5380202 and MRG5380081.

# References


[1] R.P. Agarwal, Donal O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61–79

[2] R. E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, colloq. Math. 65 (1993) 169–179.

[3] C. Chidume, N. Shahzad, H. Zegeye, Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense, Numer. Funct. Anal. Optim. 25 (2004) 239–257.

[4] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **35** (1972) 171-174.

[5] S. Ishikawa, Fixed point theorems for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **44** (1974) 147-150.

[6] I. Inchan, Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, Int. Journal of Math. Analysis. **2**(23) (2008) 1135-1145.

[7] I. Inchan, K. Nammanee, Strong convergence theorems by hybrid methods for asymptotically $k$-strict pseudo-contractive mappings in Hilbert space, Nonlinear Anal. : Hybrid syst. **3**(4) (2009) 380-385.

[8] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, Journal of Applied Mathematics and Computing. **29** (2009) 263–280.

[9] G.E. Kim, T.H. Kim, Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces, Comp. Math. Appl. 42 (2001) 1565–1570.

[10] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear. Anal. **64** (2006) 1140–1152.

[11] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically k-strict pseudcontractions, Nonlinear. Anal. **68** (2008) 2828–2836.

[12] T.C. Lim, H.K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal.**22** (1994) 1345–1355.

[13] W. A. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. **4**(1953) 506–510.

[14] G. Marino, H.K. Xu, Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl.**329** (2007) 336–349.

[15] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math Anal. Appl. **336** (2007) 455–469.

[16] D. R. Sahu, H. K. Xu and J. C. Yao, Asymptotically strict pseudocontractive mapping in the intermediate sense, Nonlinear Anal. **70** (2009) 3502–3511

[17] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Amer. Math. Soc. **43** (1991) 153–159.

[18] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. **158** (1991) 407–413.

[19] J. Schu, Approximation of fixed points of asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. **112** (1991) 143–151.

[20] W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. **341** (2008) 276–286.

[21] K.K. Tan, H.K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc.**122**(1994) 733–739.

**Received: March, 2011**

Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 506976, 14 pages
doi:10.1155/2012/506976

*Research Article*

# A General Iterative Method for a Nonexpansive Semigroup in Banach Spaces with Gauge Functions

**Kamonrat Nammanee,[1] Suthep Suantai,[2,3] and Prasit Cholamjiak[1,3]**

[1] *School of Science, University of Phayao, Phayao 56000, Thailand*
[2] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[3] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*

Correspondence should be addressed to Prasit Cholamjiak, prasitch2008@yahoo.com

Received 23 November 2011; Accepted 27 January 2012

Academic Editor: Giuseppe Marino



We study strong convergence of the sequence generated by implicit and explicit general iterative methods for a one-parameter nonexpansive semigroup in a reflexive Banach space which admits the duality mapping $J_\varphi$, where $\varphi$ is a gauge function on $[0,\infty)$. Our results improve and extend those announced by G. Marino and H.-K. Xu (2006) and many authors.

## 1. Introduction

Let $E$ be a real Banach space and $E^*$ the dual space of $E$. Let $K$ be a nonempty, closed, and convex subset of $E$. A (one-parameter) nonexpansive semigroup is a family $\mathfrak{F} = \{T(t) : t \geq 0\}$ of self-mappings of $K$ such that

(i) $T(0)x = x$ for all $x \in K$,

(ii) $T(t+s)x = T(t)T(s)x$ for all $t, s \geq 0$ and $x \in K$,

(iii) for each $x \in K$, the mapping $T(\cdot)x$ is continuous,

(iv) for each $t \geq 0$, $T(t)$ is nonexpansive, that is,

$$\|T(t)x - T(t)y\| \leq \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

We denote $F$ by the common fixed points set of $\mathfrak{F}$, that is, $F := \bigcap_{t\geq 0} F(T(t))$.

In 1967, Halpern [1] introduced the following classical iteration for a nonexpansive mapping $T : K \to K$ in a real Hilbert space:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$.

In 1977, Lions [2] obtained a strong convergence provide the real sequence $\{\alpha_n\}$ satisfies the following conditions:

C1: $\lim_{n\to\infty}\alpha_n = 0$; C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$; C3: $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1})/\alpha_n^2 = 0$.

Reich [3] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lion's conditions imposed on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$.

In 1992, Wittmann [4] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$ if $\{\alpha_n\}$ satisfies the following conditions:

C1: $\lim_{n\to\infty}\alpha_n = 0$; C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$; C3: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Shioji and Takahashi [5] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of the Halpern iterative scheme has been widely used to approximate the fixed points for nonexpansive mappings (see, e.g., [6–12] and the reference cited therein).

Let $f : K \to K$ be a contraction. In 2000, Moudafi [13] introduced the explicit viscosity approximation method for a nonexpansive mapping $T$ as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

where $\alpha_n \in (0,1)$. Xu [14] also studied the iteration process (1.3) in uniformly smooth Banach spaces.

Let $A$ be a strongly positive bounded linear operator on a real Hilbert space $H$, that is, there is a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.4}$$

A typical problem is to minimize a quadratic function over the fixed points set of a nonexpansive mapping on a Hilbert space $H$:

$$\min_{x\in C}\frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where $C$ is the fixed points set of a nonexpansive mapping $T$ on $H$ and $b$ is a given point in $H$.

In 2006, Marino and Xu [15] introduced the following general iterative method for a nonexpansive mapping $T$ in a Hilbert space $H$:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1, \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$, $f$ is a contraction on $H$, and $A$ is a strongly positive bounded linear operator on $H$. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a fixed point $x^* \in F(T)$ which also solves the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{1.7}$$

which is the optimality condition for the minimization problem: $\min_{x\in C}(1/2)\langle Ax, x\rangle - h(x)$, where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Suzuki [16] first introduced the following implicit viscosity method for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.8}$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$. He proved strong convergence of iteration (1.8) under suitable conditions. Subsequently, Xu [17] extended Suzuki's [16] result from a Hilbert space to a uniformly convex Banach space which admits a weakly sequentially continuous normalized duality mapping.

Motivated by Chen and Song [18], in 2007, Chen and He [19] investigated the implicit and explicit viscosity methods for a nonexpansive semigroup without integral in a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.9}$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.10}$$

where $\{\alpha_n\} \subset (0,1)$.

In 2008, Song and Xu [20] also studied the iterations (1.9) and (1.10) in a reflexive and strictly convex Banach space with a Gâteaux differentiable norm. Subsequently, Cholamjiak and Suantai [21] extended Song and Xu's results to a Banach space which admits duality mapping with a gauge function. Wangkeeree and Kamraksa [22] and Wangkeeree et al. [23] obtained the convergence results concerning the duality mapping with a gauge function in Banach spaces. The convergence of iterations for a nonexpansive semigroup and nonlinear mappings has been studied by many authors (see, e.g., [24–38]).

Let $E$ be a real reflexive Banach space which admits the duality mapping $J_\varphi$ with a gauge $\varphi$. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on $E$. Recall that an operator $A$ is said to be *strongly positive* if there exists a constant $\overline{\gamma} > 0$ such that

$$\begin{gathered}\langle Ax, J_\varphi(x)\rangle \geq \overline{\gamma}\|x\|\varphi(\|x\|), \\ \|\alpha I - \beta A\| = \sup_{\|x\|\leq 1}\left|\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle\right|,\end{gathered} \tag{1.11}$$

where $\alpha \in [0,1]$ and $\beta \in [-1,1]$.

Motivated by Chen and Song [18], Chen and He [19], Marino and Xu [15], Colao et al. [39], and Wangkeeree et al. [23], we study strong convergence of the following general iterative methods:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad n \geq 1, \tag{1.12}$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad n \geq 1, \tag{1.13}$$

where $\{\alpha_n\} \subset (0,1)$, $f$ is a contraction on $E$ and $A$ is a positive bounded linear operator on $E$.

## 2. Preliminaries

A Banach space $E$ is called *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space $E$ is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The *modulus of convexity* of $E$ is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x + y)\right\| : \|x\|, \|y\| \leq 1,\ \|x - y\| \geq \epsilon\right\}, \tag{2.1}$$

for all $\epsilon \in [0,2]$. $E$ is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of $E$ is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in S(E)$. In this case $E$ is called *smooth*. The norm of $E$ is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. The norm of $E$ is called *uniformly Fréchet differentiable*, if the limit is attained uniformly for $x, y \in S(E)$. It is well known that (uniformly) *Fréchet differentiability* of the norm of $E$ implies (uniformly) *Gâteaux differentiability* of the norm of $E$.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the *modulus of smoothness* of $E$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 :\ x \in S(E),\quad \|y\| \leq t\right\}. \tag{2.3}$$

A Banach space $E$ is called *uniformly smooth* if $\rho_E(t)/t \to 0$ as $t \to 0$. See [40–42] for more details.

We need the following definitions and results which can be found in [40, 41, 43].

*Definition 2.1.* A continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ is said to be gauge function if $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$.

*Definition 2.2.* Let $E$ be a normed space and $\varphi$ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|),\ \|f^*\| = \varphi(\|x\|)\}, \quad x \in E, \tag{2.4}$$

is called the duality mapping with gauge function $\varphi$.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the normalized duality mapping.

In the case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called the generalized duality mapping. It follows from the definition that $J_\varphi(x) = \varphi(\|x\|)/\|x\| J(x)$ and $J_q(x) = \|x\|^{q-2} J(x)$, $q > 1$.

*Remark 2.3.* For the gauge function $\varphi$, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds \tag{2.5}$$

is a continuous convex and strictly increasing function on $[0,\infty)$. Therefore, $\Phi$ has a continuous inverse function $\Phi^{-1}$.

It is noted that if $0 \le k \le 1$, then $\varphi(kx) \le \varphi(x)$. Further

$$\Phi(kt) = \int_0^{kt} \varphi(s)ds = k\int_0^t \varphi(kx)dx \le k\int_0^t \varphi(x)dx = k\Phi(t). \tag{2.6}$$

*Remark 2.4.* For each $x$ in a Banach space $E$, $J_\varphi(x) = \partial\Phi(\|x\|)$, where $\partial$ denotes the subdifferential.

We also know the following facts:

(i) $J_\varphi$ is a nonempty, closed, and convex set in $E^*$ for each $x \in E$,

(ii) $J_\varphi$ is a function when $E^*$ is strictly convex,

(iii) If $J_\varphi$ is single-valued, then

$$\begin{gathered} J_\varphi(\lambda x) = \frac{sign(\lambda)\varphi(\|\lambda x\|)}{\varphi(\|x\|)} J_\varphi(x), \qquad \forall x \in E, \lambda \in \mathbb{R}, \\ \langle x - y, J_\varphi(x) - J_\varphi(y) \rangle \ge (\varphi(\|x\|) - \varphi(\|y\|))(\|x\| - \|y\|), \qquad \forall x, y \in E. \end{gathered} \tag{2.7}$$

Following Browder [43], we say that a Banach space $E$ has a weakly continuous duality mapping if there exists a gauge $\varphi$ for which the duality mapping $J_\varphi$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that the space $\ell^p$ has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Moreover, $\varphi$ is invariant on $[0, 1]$.

**Lemma 2.5** (See [44]). *Assume that a Banach space $E$ has a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle. \tag{2.8}$$

*In particular, for all $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle. \tag{2.9}$$

(ii) *Assume that a sequence $\{x_n\}$ in $E$ converges weakly to a point $x \in E$. Then the following holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x - y\|) \tag{2.10}$$

*for all $x, y \in E$.*

**Lemma 2.6** (See [23]). *Assume that a Banach space $E$ has a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$. Let $A$ be a strongly positive bounded linear operator on $E$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\overline{\gamma})$.*

**Lemma 2.7** (See [12]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 1, \tag{2.11}$$

*where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*
*(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (b) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$.*
*Then $\lim_{n\to\infty} a_n = 0$.*

## 3. Implicit Iteration Scheme

In this section, we prove a strong convergence theorem of an implicit iterative method (1.12).

**Theorem 3.1.** *Let $E$ be a reflexive which admits a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi$ is invariant on $[0,1]$. Let $\mathfrak{F} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on $E$ such that $F \neq \emptyset$. Let $f$ be a contraction on $E$ with the coefficient $\alpha \in (0,1)$ and $A$ a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. Then $\{x_n\}$ defined by (1.12) converges strongly to $q \in F$ which solves the following variational inequality:*

$$\langle (A - \gamma f)(q), J_\varphi(q - w)\rangle \leq 0, \quad \forall w \in F. \tag{3.1}$$

*Proof.* First, we prove the uniqueness of the solution to the variational inequality (3.1) in $F$. Suppose that $p, q \in F$ satisfy (3.1), so we have

$$\langle (A-\gamma f)(p), J_\varphi(p-q)\rangle \le 0,$$
$$\langle (A-\gamma f)(q), J_\varphi(q-p)\rangle \le 0. \tag{3.2}$$

Adding the above inequalities, we get

$$\langle A(p)-A(q)-\gamma(f(p)-f(q)), J_\varphi(p-q)\rangle \le 0. \tag{3.3}$$

This shows that

$$\langle A(p-q), J_\varphi(p-q)\rangle \le \gamma\langle f(p)-f(q), J_\varphi(p-q)\rangle, \tag{3.4}$$

which implies by the strong positivity of $A$

$$\overline{\gamma}\|p-q\|\varphi(\|p-q\|) \le \langle A(p-q), J_\varphi(p-q)\rangle \le \gamma\alpha\|p-q\|\varphi(\|p-q\|). \tag{3.5}$$

Since $\varphi$ is invariant on $[0,1]$,

$$\varphi(1)\overline{\gamma}\|p-q\|\varphi(\|p-q\|) \le \gamma\alpha\|p-q\|\varphi(\|p-q\|). \tag{3.6}$$

It follows that

$$(\varphi(1)\overline{\gamma}-\gamma\alpha)\|p-q\|\varphi(\|p-q\|) \le 0. \tag{3.7}$$

Therefore $p = q$ since $0 < \gamma < (\overline{\gamma}\varphi(1))/\alpha$.

We next prove that $\{x_n\}$ is bounded. For each $w \in F$, by Lemma 2.6, we have

$$\begin{aligned}
\|x_n - w\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n A)T(t_n)x_n - w\| \\
&= \|(I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)w + \alpha_n(\gamma f(x_n) - A(w))\| \\
&\le \varphi(1)(1-\alpha_n\overline{\gamma})\|x_n - w\| + \alpha_n(\gamma\alpha\|x_n - w\| + \|\gamma f(w) - A(w)\|) \\
&\le \|x_n - w\| - \alpha_n\varphi(1)\overline{\gamma}\|x_n - w\| + \alpha_n\gamma\alpha\|x_n - w\| + \alpha_n\|\gamma f(w) - A(w)\|,
\end{aligned} \tag{3.8}$$

which yields

$$\|x_n - w\| \le \frac{1}{\varphi(1)\overline{\gamma}-\gamma\alpha}\|\gamma f(w) - A(w)\|. \tag{3.9}$$

Hence $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{AT(t_n)x_n\}$.

We next prove that $\{x_n\}$ is relatively sequentially compact. By the reflexivity of $E$ and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a point $p$ in $E$ such that $x_{n_j} \rightharpoonup p$ as $j \to \infty$. Now we show that $p \in F$. Put $x_j = x_{n_j}$, $\beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j \in \mathbb{N}$, fix $t > 0$. We see that

$$\begin{aligned}
\|x_j - T(t)p\| \le{}& \sum_{k=0}^{[t/s_j]-1} \|T((k+1)s_j)x_j - T(ks_j)x_{j+1}\| \\
&+ \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)x_j - T\left(\left[\frac{t}{s_j}\right]s_j\right)p\right\| + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)p - T(t)p\right\| \\
\le{}& \left[\frac{t}{s_j}\right]\|T(s_j)x_j - x_j\| + \|x_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\
={}& \left[\frac{t}{s_j}\right]\beta_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\
\le{}& \frac{t\beta_j}{s_j}\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| \\
&+ \max\{\|T(s)p - p\| : 0 \le s \le s_j\}.
\end{aligned} \tag{3.10}$$

So we have

$$\limsup_{j\to\infty}\Phi(\|x_j - T(t)p\|) \le \limsup_{j\to\infty}\Phi(\|x_j - p\|). \tag{3.11}$$

On the other hand, by Lemma 2.5 (ii), we have

$$\limsup_{j\to\infty}\Phi(\|x_j - T(t)p\|) = \limsup_{j\to\infty}\Phi(\|x_j - p\|) + \Phi(\|T(t)p - p\|). \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\Phi(\|T(t)p - p\|) \le 0. \tag{3.13}$$

This implies that $p \in F$. Further, we see that

$$\begin{aligned}
\|x_j - p\|\varphi(\|x_j - p\|) ={}& \langle x_j - p, J_\varphi(x_j - p)\rangle \\
={}& \langle (I - \beta_j A)T(s_j)x_j - (I - \beta_j A)p, J_\varphi(x_j - p)\rangle \\
&+ \beta_j\langle \gamma f(x_j) - \gamma f(p), J_\varphi(x_j - p)\rangle + \beta_j\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle \\
\le{}& \varphi(1)(1 - \beta_j\overline{\gamma})\|x_j - p\|\varphi(\|x_j - p\|) \\
&+ \beta_j\gamma\alpha\|x_j - p\|\varphi(\|x_j - p\|) + \beta_j\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle.
\end{aligned} \tag{3.14}$$

So we have

$$\|x_j - p\|\varphi(\|x_j - p\|) \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha}\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle. \tag{3.15}$$

By the definition of $\Phi$, it is easily seen that

$$\Phi(\|x_j - p\|) \leq \|x_j - p\|\varphi(\|x_j - p\|). \tag{3.16}$$

Hence

$$\Phi(\|x_j - p\|) \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha}\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle. \tag{3.17}$$

Therefore $\Phi(\|x_j - p\|) \to 0$ as $j \to \infty$ since $J_\varphi$ is weakly continuous; consequently, $x_j \to p$ as $j \to \infty$ by the continuity of $\Phi$. Hence $\{x_n\}$ is relatively sequentially compact.

Finally, we prove that $p$ is a solution in $F$ to the variational inequality (3.1). For any $w \in F$, we see that

$$\begin{aligned}
\langle (I - T(t_n))x_n - (I - T(t_n))w, J_\varphi(x_n - w)\rangle &= \langle x_n - w, J_\varphi(x_n - w)\rangle \\
&\quad - \langle T(t_n)x_n - T(t_n)w, J_\varphi(x_n - w)\rangle \\
&\geq \|x_n - w\|\varphi\|x_n - w\| \\
&\quad - \|T(t_n)x_n - T(t_n)w\|\|J_\varphi(x_n - w)\| \\
&\geq \|x_n - w\|\varphi\|x_n - w\| \\
&\quad - \|x_n - w\|\|J_\varphi(x_n - w)\| \\
&= 0.
\end{aligned} \tag{3.18}$$

On the other hand, we have

$$(A - \gamma f)(x_n) = -\frac{1}{\alpha_n}(I - \alpha_n A)(I - T(t_n))x_n, \tag{3.19}$$

which implies

$$\begin{aligned}
\langle (A - \gamma f)(x_n), J_\varphi(x_n - w)\rangle &= -\frac{1}{\alpha_n}\langle (I - T(t_n))x_n - (I - T(t_n))w, J_\varphi(x_n - w)\rangle \\
&\quad + \langle A(I - T(t_n))x_n, J_\varphi(x_n - w)\rangle \\
&\leq \langle A(I - T(t_n))x_n, J_\varphi(x_n - w)\rangle.
\end{aligned} \tag{3.20}$$

Observe

$$\|x_j - T(s_j)x_j\| = \beta_j\|\gamma f(x_j) - AT(s_j)x_j\| \to 0, \tag{3.21}$$

as $j \to \infty$. Replacing $n$ by $n_j$ and letting $j \to \infty$ in (3.20), we obtain

$$\langle (A-\gamma f)(p), J_\varphi(p-w)\rangle \le 0, \quad \forall w \in F. \tag{3.22}$$

So $p \in F$ is a solution of variational inequality (3.1); and hence $p = q$ by the uniqueness. In a summary, we have proved that $\{x_n\}$ is relatively sequentially compact and each cluster point of $\{x_n\}$ (as $n \to \infty$) equals $q$. Therefore $x_n \to q$ as $n \to \infty$. This completes the proof. □

## 4. Explicit Iteration Scheme

In this section, utilizing the implicit version in Theorem 3.1, we consider the explicit one in a reflexive Banach space which admits the duality mapping $J_\varphi$.

**Theorem 4.1.** *Let $E$ be a reflexive Banach space which admits a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi$ is invariant on [0,1]. Let $\{T(t) : t \ge 0\}$ be a nonexpansive semigroup on $E$ such that $F \neq \emptyset$. Let $f$ be a contraction on $E$ with the coefficient $\alpha \in (0,1)$ and $A$ a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\,\varphi(1)/\alpha$. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying $0 < \alpha_n < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. Then $\{x_n\}$ defined by* (1.13) *converges strongly to $q \in F$ which also solves the variational inequality* (3.1).

*Proof.* Since $\alpha_n \to 0$, we may assume that $\alpha_n < \varphi(1)\|A\|^{-1}$ and $1-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha) > 0$ for all $n$. First we prove that $\{x_n\}$ is bounded. For each $w \in F$, by Lemma 2.6, we have

$$\begin{aligned}
\|x_{n+1}-w\| &= \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n A)T(t_n)x_n - w\right\| \\
&= \left\|(I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)w + \alpha_n(\gamma f(x_n) - A(w))\right\| \\
&\le \varphi(1)(1-\alpha_n\overline{\gamma})\|x_n-w\| + \alpha_n\gamma\alpha\|x_n-w\| + \alpha_n\left\|\gamma f(w)-A(w)\right\| \\
&= (\varphi(1)-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha))\|x_n-w\| + \alpha_n\left\|\gamma f(w)-A(w)\right\| \\
&\le (1-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha))\|x_n-w\| + \alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha)\frac{\left\|\gamma f(w)-A(w)\right\|}{\varphi(1)\overline{\gamma}-\gamma\alpha}.
\end{aligned} \tag{4.1}$$

It follows from induction that

$$\|x_{n+1}-w\| \le \max\left\{\|x_1-w\|, \frac{\left\|\gamma f(w)-A(w)\right\|}{\varphi(1)\overline{\gamma}-\gamma\alpha}\right\}, \quad n \ge 1. \tag{4.2}$$

Thus $\{x_n\}$ is bounded, and hence so are $\{f(x_n)\}$ and $\{AT(t_n)x_n\}$. From Theorem 3.1, there is a unique solution $q \in F$ to the following variational inequality:

$$\langle (A-\gamma f)q, J_\varphi(q-w)\rangle \le 0, \quad \forall w \in F. \tag{4.3}$$

Next we prove that

$$\limsup_{n\to\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n+1})\rangle \le 0. \tag{4.4}$$

Indeed, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (A-\gamma f)q, J_\varphi(q-x_n)\rangle = \limsup_{j\to\infty}\left\langle (A-\gamma f)q, J_\varphi\left(q-x_{n_j}\right)\right\rangle. \tag{4.5}$$

Further, we can assume that $x_{n_j} \rightharpoonup p \in E$ by the reflexivity of $E$ and the boundedness of $\{x_n\}$. Now we show that $p \in F$. Put $x_j = x_{n_j}, \beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j \in \mathbb{N}$, fix $t > 0$. We obtain

$$\begin{aligned}
\|x_{j+1} - T(t)p\| &\le \sum_{k=0}^{[t/s_j]-1} \|T((k+1)s_j)x_j - T(ks_j)x_{j+1}\| \\
&\quad + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)x_j - T\left(\left[\frac{t}{s_j}\right]s_j\right)p\right\| + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)p - T(t)p\right\| \\
&\le \left[\frac{t}{s_j}\right]\|T(s_j)x_j - x_{j+1}\| + \|x_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\
&= \left[\frac{t}{s_j}\right]\beta_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\
&\le \frac{t\beta_j}{s_j}\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| \\
&\quad + \max\{\|T(s)p - p\| : 0 \le s \le s_j\}.
\end{aligned} \tag{4.6}$$

It follows that $\limsup_{n\to\infty}\Phi(\|x_j - T(t)p\|) \le \limsup_{n\to\infty}\Phi(\|x_j - p\|)$. From Lemma 2.5 (ii) we have

$$\limsup_{n\to\infty}\Phi(\|x_j - T(t)p\|) = \limsup_{n\to\infty}\Phi(\|x_j - p\|) + \Phi(\|T(t)p - p\|). \tag{4.7}$$

So we have $\Phi(\|T(t)p - p\|) \le 0$ and hence $p \in F$. Since the duality mapping $J_\varphi$ is weakly sequentially continuous,

$$\begin{aligned}
\limsup_{n\to\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n+1})\rangle &= \limsup_{j\to\infty}\left\langle (A-\gamma f)q, J_\varphi\left(q-x_{n_j+1}\right)\right\rangle \\
&= \langle (A-\gamma f)q, J_\varphi(q-p)\rangle \le 0.
\end{aligned} \tag{4.8}$$

Finally, we show that $x_n \to q$. From Lemma 2.5 (i), we have

$$\begin{aligned}
\Phi(\|x_{n+1}-q\|) &= \Phi(\|(I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)q + \alpha_n(\gamma f(x_n) - \gamma f(q)) \\
&\qquad + \alpha_n(\gamma f(q) - A(q))\|) \\
&\le \Phi(\|(I-\alpha_n A)(T(t_n)x_n - q) + \alpha_n(\gamma f(x_n) - \gamma f(q))\|) \\
&\quad + \alpha_n\langle \gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\rangle \\
&\le \Phi(\varphi(1)(1-\alpha_n\overline{\gamma})\|x_n - q\| + \alpha_n\gamma\alpha\|x_n - q\|) \\
&\quad + \alpha_n\langle \gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\rangle \\
&= \Phi((\varphi(1) - \alpha_n(\varphi(1)\overline{\gamma} - \gamma\alpha))\|x_n - q\|) \\
&\quad + \alpha_n\langle \gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n(\varphi(1)\overline{\gamma} - \gamma\alpha))\Phi(\|x_n - q\|) \\
&\quad + \alpha_n\langle \gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\rangle.
\end{aligned} \tag{4.9}$$

Note that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\langle \gamma f(q) - A(q), J_\varphi(x_{n+1}-q)\rangle \le 0$. Using Lemma 2.7, we have $x_n \to q$ as $n \to \infty$ by the continuity of $\Phi$. This completes the proof. □

*Remark 4.2.* Theorems 3.1 and 4.1 improve and extend the main results proved in [15] in the following senses:

(i) from a nonexpansive mapping to a nonexpansive semigroup,

(ii) from a real Hilbert space to a reflexive Banach space which admits a weakly continuous duality mapping with gauge functions.

## Acknowledgments

The authors wish to thank the editor and the referee for valuable suggestions. K. Nammanee was supported by the Thailand Research Fund, the Commission on Higher Education, and the University of Phayao under Grant MRG5380202. S. Suantai and P. Cholamjiak wish to thank the Thailand Research Fund and the Centre of Excellence in Mathematics, Thailand.

## References

[1] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[2] P.-L. Lions, "Approximation de points fixes de contractions," *Comptes Rendus de l'Académie des Sciences*, vol. 284, no. 21, pp. A1357–A1359, 1977.

[3] S. Reich, "Approximating fixed points of nonexpansive mappings," *Panamerican Mathematical Journal*, vol. 4, no. 2, pp. 23–28, 1994.

[4] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.

[5] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[6] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, “Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.
[7] C. E. Chidume and C. O. Chidume, “Iterative approximation of fixed points of nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 288–295, 2006.
[8] Y. J. Cho, S. M. Kang, and H. Zhou, “Some control conditions on iterative methods,” *Communications on Applied Nonlinear Analysis*, vol. 12, no. 2, pp. 27–34, 2005.
[9] T.-H. Kim and H.-K. Xu, “Strong convergence of modified Mann iterations,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 61, no. 1-2, pp. 51–60, 2005.
[10] S. Reich, “Strong convergence theorems for resolvents of accretive operators in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.
[11] H.-K. Xu, “Another control condition in an iterative method for nonexpansive mappings,” *Bulletin of the Australian Mathematical Society*, vol. 65, no. 1, pp. 109–113, 2002.
[12] H.-K. Xu, “Iterative algorithms for nonlinear operators,” *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.
[13] A. Moudafi, “Viscosity approximation methods for fixed-points problems,” *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.
[14] H.-K. Xu, “Viscosity approximation methods for nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
[15] G. Marino and H.-K. Xu, “A general iterative method for nonexpansive mappings in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.
[16] T. Suzuki, “On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces,” *Proceedings of the American Mathematical Society*, vol. 131, no. 7, pp. 2133–2136, 2003.
[17] H.-K. Xu, “A strong convergence theorem for contraction semigroups in Banach spaces,” *Bulletin of the Australian Mathematical Society*, vol. 72, no. 3, pp. 371–379, 2005.
[18] R. Chen and Y. Song, “Convergence to common fixed point of nonexpansive semigroups,” *Journal of Computational and Applied Mathematics*, vol. 200, no. 2, pp. 566–575, 2007.
[19] R. Chen and H. He, “Viscosity approximation of common fixed points of nonexpansive semigroups in Banach space,” *Applied Mathematics Letters*, vol. 20, no. 7, pp. 751–757, 2007.
[20] Y. Song and S. Xu, “Strong convergence theorems for nonexpansive semigroup in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 152–161, 2008.
[21] P. Cholamjiak and S. Suantai, “Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions,” *Journal of Global Optimization*. In press.
[22] R. Wangkeeree and U. Kamraksa, “Strong convergence theorems of viscosity iterative methods for a countable family of strict pseudo-contractions in Banach spaces,” *Fixed Point Theory and Applications*, vol. 2010, Article ID 579725, 21 pages, 2010.
[23] R. Wangkeeree, N. Petrot, and R. Wangkeeree, “The general iterative methods for nonexpansive mappings in Banach spaces,” *Journal of Global Optimization*, vol. 51, no. 1, pp. 27–46, 2011.
[24] I. K. Argyros, Y. J. Cho, and X. Qin, “On the implicit iterative process for strictly pseudo-contractive mappings in Banach spaces,” *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 208–216, 2009.
[25] S.-S. Chang, Y. J. Cho, H. W. J. Lee, and C. K. Chan, “Strong convergence theorems for Lipschitzian demicontraction semigroups in Banach spaces,” *Fixed Point Theory and Applications*, vol. 11, Article ID 583423, 10 pages, 2011.
[26] Y. J. Cho, L. Ćirić, and S.-H. Wang, “Convergence theorems for nonexpansive semigroups in CAT(0) spaces,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 17, pp. 6050–6059, 2011.
[27] Y. J. Cho, S. M. Kang, and X. Qin, “Some results on $k$-strictly pseudo-contractive mappings in Hilbert spaces,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 5, pp. 1956–1964, 2009.
[28] Y. J. Cho, S. M. Kang, and X. Qin, “Strong convergence of an implicit iterative process for an infinite family of strict pseudocontractions,” *Bulletin of the Korean Mathematical Society*, vol. 47, no. 6, pp. 1259–1268, 2010.
[29] Y. J. Cho, X. Qin, and S. M. Kang, “Strong convergence of the modified Halpern-type iterative algorithms in Banach spaces,” *Analele Stiintifice ale Universitatii Ovidius Constanta*, vol. 17, no. 1, pp. 51–67, 2009.
[30] W. Guo and Y. J. Cho, “On the strong convergence of the implicit iterative processes with errors for a finite family of asymptotically nonexpansive mappings,” *Applied Mathematics Letters*, vol. 21, no. 10, pp. 1046–1052, 2008.

[31] H. He, S. Liu, and Y. J. Cho, "An explicit method for systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings," *Journal of Computational and Applied Mathematics*, vol. 235, no. 14, pp. 4128–4139, 2011.

[32] J. Kang, Y. Su, and X. Zhang, "General iterative algorithm for nonexpansive semigroups and variational inequalities in Hilbert spaces," *journal of Inequalities and Applications*, vol. 2010, Article ID 264052, 10 pages, 2010.

[33] X. N. Li and J. S. Gu, "Strong convergence of modified Ishikawa iteration for a nonexpansive semigroup in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 4, pp. 1085–1092, 2010.

[34] S. Li, L. Li, and Y. Su, "General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 9, pp. 3065–3071, 2009.

[35] Q. Lin, "Viscosity approximation to common fixed points of a nonexpansive semigroup with a generalized contraction mapping," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5451–5457, 2009.

[36] S. Plubtieng and R. Punpaeng, "Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces," *Mathematical and Computer Modelling*, vol. 48, no. 1-2, pp. 279–286, 2008.

[37] X. Qin, Y. J. Cho, S. M. Kang, and H. Zhou, "Convergence of a modified Halpern-type iteration algorithm for quasi-$\phi$-nonexpansive mappings," *Applied Mathematics Letters*, vol. 22, no. 7, pp. 1051–1055, 2009.

[38] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[39] V. Colao, G. Marino, and H.-K. Xu, "An iterative method for finding common solutions of equilibrium and fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 340–352, 2008.

[40] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, vol. 6, Springer, New York, NY, USA, 2009.

[41] C. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, vol. 1965 of *Lecture Notes in Mathematics*, Springer, London, UK, 2009.

[42] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[43] F. E. Browder, "Convergence theorems for sequences of nonlinear operators in Banach spaces," *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.

[44] T.-C. Lim and H. K. Xu, "Fixed point theorems for asymptotically nonexpansive mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 22, no. 11, pp. 1345–1355, 1994.

# Convergence theorems for maximal monotone operators, weak relatively nonexpansive mappings and equilibrium problems

**Kamonrat Nammanee**[a], **Suthep Suantai**[b,c] **and Prasit Cholamjiak**[a,c] *

[a] *School of Science, University of Phayao, Phayao 56000, Thailand*

[b] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

[c] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand*

---

**Abstract**

In this work, we introduce hybrid iterative schemes for solving a system of the zero-finding problems of maximal monotone operators, the equilibrium problem and the fixed point problem of weak relatively nonexpansive mappings. We then prove, in a uniformly smooth and uniformly convex Banach space, strong convergence theorems by using a shrinking projection method. We finally apply the obtained results to a system of convex minimization problems.

**Keywords:** Maximal monotone operator; equilibrium problem; fixed point; weak relatively nonexpansive; strong convergence.

**AMS Subject Classification:** 47H09, 47H10.

## 1 Introduction

Let $E$ be a real Banach space and $C$ a nonempty subset of $E$. Let $E^*$ be the dual space of $E$. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x\rangle$. Let $T : C \to C$ be

*Corresponding author

*Email addresses:* kamonrat.na@up.ac.th (K. Nammanee), scmti005@chiangmai.ac.th (S. Suantai), prasitch2008@yahoo.com (P. Cholamjiak).

a nonlinear mapping. We denote by $F(T)$ the fixed points set of $T$, that is, $F(T) = \{x \in C : x = Tx\}$. Let $A : E \to 2^{E^*}$ be a set-valued mapping. We denote $D(A)$ by the *domain* of $A$, that is, $D(A) = \{x \in E : \ Ax \neq \emptyset\}$ and also denote $G(A)$ by the *graph* of $A$, that is, $G(A) = \{(x, x^*) \in E \times E^* : \ x^* \in Ax\}$. A set-valued mapping $A$ is said to be *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in G(A)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operators on $E$. It is known that if $A$ is maximal monotone, then the set $A^{-1}(0^*) = \{z \in E : \ 0^* \in Az\}$ is closed and convex.

The problem of finding a zero point of maximal monotone operators plays an important role in optimizations. This is because it can be reformulated to a convex minimization problem and a variational inequality problem. Many authors have studied the convergence of such problems in various spaces (see, for examples, [6, 10, 11, 12, 16, 22, 23, 25, 29, 30, 38, 39, 40, 41, 42, 43]). Initiated by Martinet [20], in a real Hilbert space $H$, Rockafellar [28] introduced the following iterative scheme: $x_1 \in H$ and

$$x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_\lambda$ is the resolvent of $A$ defined by $J_\lambda := J_{\lambda A} = (I + \lambda A)^{-1}$ for all $\lambda > 0$ and $A$ is a maximal monotone operator on $H$. Such an algorithm is called the *proximal point algorithm*. It was proved that the sequence $\{x_n\}$ generated by (1.1) converges weakly to an element in $A^{-1}(0)$ provided $\liminf_{n\to\infty} \lambda_n > 0$. Recently, Kamimura-Takahashi [13] introduced the following iteration in a real Hilbert space: $x_1 \in H$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$. The weak convergence theorems are also established in a real Hilbert space under suitable conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2004, Kamimura et al. [15] extended the above iteration process to a much more general setting. In fact, they proposed the following algorithm: $x_1 \in E$ and

$$x_{n+1} = J^{-1}\big(\alpha_n J(x_n) + (1 - \alpha_n) J(J_{\lambda_n} x_n)\big), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, \infty)$ and $J_\lambda := J_{\lambda A} = (J + \lambda A)^{-1} J$ for all $\lambda > 0$. They proved, in a uniformly smooth and uniformly convex Banach space, a weak convergence theorem.

Let $F : C \times C \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, be a bifunction. The equilibrium problem is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solutions set of (1.2) is denoted by $EP(F)$.

For solving the equilibrium problem, we assume that:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, *i.e.* $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x,y)$;

(A4) for all $x \in C$, $F(x,\cdot)$ is convex and lower semi-continuous.

Recently, Takahashi-Zembayashi [37] introduced the following iterative scheme for a relatively nonexpansive mapping $T : C \to C$ in a uniformly smooth and uniformly convex Banach space: $x_1 \in C$ and

$$\begin{cases} C_1 = C, \\ y_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T x_n\big), \\ u_n \in C \ \ \textit{such that} \ \ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$. Such an algorithm is called the *shrinking projection method* which was introduced by Takahashi et al. [35]. They proved that the sequence $\{x_n\}$ converges strongly to an element in $F(T) \cap EP(F)$ under appropriate conditions. The equilibrium problem has been intensively studied by many authors (see, for examples, [5, 7, 8, 9, 18, 19, 24, 32, 33]).

Motivated by the previous results, we introduce a hybrid iterative scheme for finding a zero point of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2, \cdots, N)$ which is also a common element in the solutions set of an equilibrium problem for $F$ and in the fixed points set of weak relatively nonexpansive mappings $T_i : C \to C$ $(i = 1, 2, \cdots)$. Using the projection technique, we also prove that the sequence generated by a constructed algorithm converges strongly to an element in $\big[\bigcap_{i=1}^N A_i^{-1}(0^*)\big] \cap \big[\bigcap_{i=1}^\infty F(T_i)\big] \cap EP(F)$ in a uniformly smooth and uniformly convex Banach space. Finally, we apply our results to a system of convex minimization problems.

## 2 Preliminaries and lemmas

In this section, we give some useful preliminaries and lemmas which will be used in the sequel.

Let $E$ be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of $E$. A Banach space $E$ is said to be *strictly convex* if for any $x, y \in U$,

$$x \neq y \ \text{ implies } \ \|x + y\| < 2.$$

A Banach space $E$ is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \ \text{ implies } \ \|x + y\| < 2(1 - \delta).$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. The function $\delta : [0, 2] \to [0, 1]$ which called the *modulus of convexity* of $E$ is defined as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \ x, y \in E, \ \|x\| = \|y\| = 1, \ \|x - y\| \geq \varepsilon \right\}.$$

Then $E$ is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space $E$ is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{ \ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}$$

for all $x \in E$. It is also known that if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$ (see [34] for more details).

Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. From the definition of $\phi$, we see that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x, y) \leq \big(\|x\| + \|y\|\big)^2$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$

for all $x, y, z \in E$.

Let $C$ be a closed and convex subset of $E$ and let $T$ be a mapping from $C$ into itself. A point $p$ in $C$ is said to be an *asymptotic fixed point* of $T$ [3] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\widehat{F}(T)$. A mapping $T$ is said to be *relatively nonexpansive* [3, 4] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. A point $p$ in $C$ is said to be a *strong asymptotic fixed point* of $T$ if $C$ contains a sequence $\{x_n\}$ which converges strongly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of $T$ will be denoted by $\widetilde{F}(T)$. A mapping $T$ is said to be *weak relatively nonexpansive* [44] if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. It is obvious by definition that the class of weak relatively nonexpansive mappings contains the class of relatively nonexpansive mappings. Indeed, for any mapping $T : C \to C$, we see that $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if $T$ is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

Non-trivial examples of weak relatively nonexpansive mappings which are not relatively nonexpansive can be found in [31].

Let $E$ be a reflexive, strictly convex and smooth Banach space and let $C$ be a nonempty, closed and convex subset of $E$. The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$, that is, $\Pi_C(x) = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}.$$

In a Hilbert space, $\Pi_C$ is coincident with the metric projection denoted by $P_C$.

**Lemma 2.1.** [14] *Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences in $E$. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

**Lemma 2.2.** [1, 14] *Let $C$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $x \in E$ and let $z \in C$. Then $z = \Pi_C(x)$ if and only if $\langle y - z, Jx - Jz\rangle \le 0$ for all $y \in C$.*

**Lemma 2.3.** [1, 14] *Let $C$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y) \quad \forall x \in C \text{ and } y \in E.$$

**Lemma 2.4.** [21] *Let $E$ be a smooth and strictly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $T$ be a mapping from $C$ into itself such that $F(T)$ is nonempty and $\phi(u,Tx) \leq \phi(u,x)$ for all $(u,x) \in F(T) \times C$. Then $F(T)$ is closed and convex.*

Let $E$ be a reflexive, strictly convex and smooth Banach space. It is known that $A : E \to 2^{E^*}$ is maximal monotone if and only if $R(J+\lambda A) = E^*$ for all $\lambda > 0$, where $R(B)$ stands for the range of $B$.

Define the *resolvent* of $A$ by $J_{\lambda A} = (J+\lambda A)^{-1}J$ for all $\lambda > 0$. It is known that $J_{\lambda A}$ is a single-valued mapping from $E$ to $D(A)$ and $A^{-1}(0^*) = F(J_{\lambda A})$ for all $\lambda > 0$. For each $\lambda > 0$, the *Yosida approximation* of $A$ is defined by

$$A_\lambda(x) = \frac{1}{\lambda}\big(J(x) - JJ_{\lambda A}(x)\big).$$

for all $x \in E$. We know that $A_\lambda(x) \in A\big(J_{\lambda A}(x)\big)$ for all $\lambda > 0$ and $x \in E$.

**Lemma 2.5.** [16] *Let $E$ be a smooth, strictly convex and reflexive Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}(0^*) \neq \emptyset$, and let $J_{\lambda A} = (J+\lambda A)^{-1}J$ for each $\lambda > 0$. Then*

$$\phi\big(p, J_{\lambda A}(x)\big) + \phi\big(J_{\lambda A}(x), x\big) \leq \phi(p,x)$$

*for all $\lambda > 0$, $p \in A^{-1}(0^*)$, and $x \in E$.*

**Lemma 2.6.** [2] *Let $C$ be a closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4)*, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

**Lemma 2.7.** [36] *Let $C$ be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$, and let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). For all $r > 0$ and $x \in E$, define the mapping $T_r : E \to C$ as follows:

$$T_r(x) = \Big\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \ \forall y \in C\Big\}.$$

Then, the following hold:

(1) $T_r$ is single-valued;

(2) $T_r$ is a firmly nonexpansive-type mapping [17], *i.e.*, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

**Lemma 2.8.** [36] *Let $C$ be a closed and convex subset of a smooth, strictly and reflexive Banach space $E$, let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) − (A4), *let $r > 0$. Then*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

*for all $x \in E$ and $p \in F(T_r)$.*

# 3 Strong convergence theorems

In this section, we are now ready to prove our main theorem.

**Theorem 3.1.** *Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $A_i : E \to 2^{E^*}$ $(i = 1, 2, \cdots, N)$ be maximal monotone operators, let $F : C \times C \to \mathbb{R}$ be a bifunction, and let $T_i : C \to C$ $(i = 1, 2, \cdots)$ be weak relatively nonexpansive mappings such that $\mathcal{F} := \big[\bigcap_{i=1}^N A_i^{-1}(0^*)\big] \cap \big[\bigcap_{i=1}^\infty F(T_i)\big] \cap EP(F) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset E$ be the sequence such that $\lim_{n\to\infty} e_n = 0$. Define the sequence $\{x_n\}_{n=1}^\infty$ in $C$ as follows:*

$$\begin{cases} x_1 \in C_1 = C, \\ y_n = J_{\lambda_n^N A_N} \circ J_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_n^1 A_1}(x_n + e_n), \\ u_n = T_{r_n} y_n, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \phi(z, T_i u_n) \leq \phi(z, x_n + e_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} \lambda_n^i > 0$ for each $i = 1, 2, \cdots, N$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges strongly to $q = \Pi_{\mathcal{F}}(x_1)$.*

*Proof.* We split the proof into several steps as follows:

**Step 1**. $\mathcal{F} \subset C_n$ for all $n \geq 1$.

From Lemma 2.4, we know that $\bigcap_{i=1}^\infty F(T_i)$ is closed and convex. From Lemma 2.7 (4), we also know that $EP(F)$ is closed and convex. On the other hand, since $A_i$ $(i = 1, 2, \cdots, N)$ are maximal monotone, $A_i^{-1}(0^*)$ are closed and convex for each $i = 1, 2, \cdots, N$; consequently, $\bigcap_{i=1}^N A_i^{-1}(0^*)$ is closed and convex. Hence $\mathcal{F}$ is a nonempty, closed and convex subset of $C$.

We next show that $C_n$ is closed and convex for all $n \geq 1$. Obviously, $C_1 = C$ is closed and convex. Now suppose that $C_k$ is closed and convex for some $k \in \mathbb{N}$. Then, for each $z \in C_k$ and $i \geq 1$, we see that $\phi(z, T_i u_k) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k\rangle - 2\langle z, JT_i u_k\rangle \leq \|x_k\|^2 - \|T_i u_k\|^2.$$

By the construction of the set $C_{k+1}$, we see that

$$\begin{aligned} C_{k+1} &= \big\{z \in C_k : \sup_{i\geq 1} \phi(z, T_i u_k) \leq \phi(z, x_k)\big\} \\ &= \bigcap_{i=1}^{\infty} \big\{z \in C_k : \phi(z, T_i u_k) \leq \phi(z, x_k)\big\}. \end{aligned}$$

Hence $C_{k+1}$ is closed and convex. This shows, by induction, that $C_n$ is closed and convex for all $n \geq 1$. It is obvious that $\mathcal{F} \subset C_1 = C$. Now, suppose that $\mathcal{F} \subset C_k$ for some $k \in \mathbb{N}$. For any $p \in \mathcal{F}$, by Lemma 2.5 and Lemma 2.8, we have

$$\begin{aligned} \phi(p, T_i u_k) \leq \phi(p, u_k) &= \phi(p, T_{r_k} y_k) \\ &\leq \phi(p, y_k) \\ &= \phi\big(p, J_{\lambda_k^N A_N} \circ J_{\lambda_k^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\leq \phi\big(p, J_{\lambda_k^{N-1} A_{N-1}} \circ J_{\lambda_k^{N-2} A_{N-2}} \circ \cdots \circ J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\cdots \\ &\leq \phi\big(p, J_{\lambda_k^2 A_2} \circ J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\leq \phi\big(p, J_{\lambda_k^1 A_1}(x_k + e_k)\big) \\ &\leq \phi(p, x_k + e_k). \end{aligned} \tag{3.1}$$

This shows that $\mathcal{F} \subset C_{k+1}$. By induction, we can conclude that $\mathcal{F} \subset C_n$ for all $n \geq 1$.

**Step 2.** $\lim_{n\to\infty} \phi(x_n, x_1)$ exists.

From $x_n = \Pi_{C_n}(x_1)$ and $x_{n+1} = \Pi_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \tag{3.2}$$

From Lemma 2.3, for any $p \in \mathcal{F} \subset C_n$, we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}(x_1), x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1). \tag{3.3}$$

Combining (3.2) and (3.3), we conclude that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists.

**Step 3.** $\lim_{n\to\infty} \big\|J(T_i y_n) - J(x_n + e_n)\big\| = 0.$

Since $x_m = \Pi_{C_m}(x_1) \in C_m \subset C_n$ for $m > n \geq 1$, by Lemma 2.3, it follows that

$$\begin{aligned}\phi(x_m, x_n) = \phi\big(x_m, \Pi_{C_n}(x_1)\big) &\leq \phi\big(x_m, x_1\big) - \phi\big(\Pi_{C_n}(x_1), x_1\big) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1).\end{aligned}$$

Letting $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. By Lemma 2.1, it follows that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of the space $E$ and the closedness of $C$, we can assume that $x_n \to q \in C$ as $n \to \infty$. In particular, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $e_n \to 0$, we have

$$\lim_{n\to\infty} \|x_{n+1} - (x_n + e_n)\| = 0. \tag{3.4}$$

Since $x_{n+1} = \Pi_{C_{n+1}}(x_1) \in C_{n+1}$, for each $i \geq 1$,

$$\begin{aligned}\phi(x_{n+1}, T_i u_n) &\leq \phi(x_{n+1}, x_n + e_n) \\ &= \big\langle x_{n+1}, J(x_{n+1}) - J(x_n + e_n)\big\rangle + \big\langle x_{n+1} - (x_n + e_n), J(x_{n+1})\big\rangle.\end{aligned}$$

Since $E$ is uniformly smooth, $J$ is uniformly norm-to-norm continuous on bounded sets. It follows from (3.4) and by the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, T_i u_n) = 0$$

for all $i = 1, 2, \cdots$. So from Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - T_i u_n\| = 0$$

and

$$\lim_{n\to\infty} \|T_i u_n - x_n\| = 0$$

and, since $e_n \to 0$, therefore

$$\lim_{n\to\infty} \|T_i u_n - (x_n + e_n)\| = 0. \tag{3.5}$$

for all $i = 1, 2, \cdots$. Since $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$,

$$\lim_{n\to\infty} \big\|J(T_i u_n) - J(x_n + e_n)\big\| = 0 \tag{3.6}$$

for all $i = 1, 2, \cdots$.

**Step 4.** $\lim_{n\to\infty} \|T_i u_n - u_n\| = 0$ for all $i = 1, 2, \cdots$.

Denote $\Theta_n^i = J_{\lambda_n^i A_i} \circ J_{\lambda_n^{i-1} A_{i-1}} \circ \cdots \circ J_{\lambda_n^1 A_1}$ for each $i \in \{1, 2, \cdots, N\}$ and $\Theta_n^0 = I$ for each $n \geq 1$. We note that $y_n = \Theta_n^N(x_n + e_n)$ for each $n \geq 1$.

To this end, we will show that

$$\lim_{n\to\infty}\Big\|J\big(\Theta_n^i(x_n+e_n)\big)-J\big(\Theta_n^{i-1}(x_n+e_n)\big)\Big\|=0$$

for all $i=1,2,\cdots,N$.

For any $p\in\mathcal{F}$, by (3.1), we see that

$$\begin{aligned}\phi\big(p,\Theta_n^{N-1}(x_n+e_n)\big) &\le \phi\big(p,\Theta_n^{N-2}(x_n+e_n)\big)\\ &\le \phi\big(p,\Theta_n^{N-3}(x_n+e_n)\big)\\ &\cdots\\ &\le \phi\big(p,(x_n+e_n)\big).\end{aligned}\tag{3.7}$$

Since $p\in\mathcal{F}$, by Lemma 2.5 and (3.7), it follows that

$$\begin{aligned}&\phi\big(y_n,\Theta_n^{N-1}(x_n+e_n)\big)\\ \le\;& \phi\big(p,\Theta_n^{N-1}(x_n+e_n)\big)-\phi(p,y_n)\\ \le\;& \phi\big(p,(x_n+e_n)\big)-\phi(p,y_n)\\ \le\;& \phi\big(p,(x_n+e_n)\big)-\phi(p,u_n)\\ \le\;& \phi\big(p,(x_n+e_n)\big)-\phi(p,T_iu_n)\\ =\;& \|x_n+e_n\|^2-\|T_iu_n\|^2-2\big\langle p,J(x_n+e_n)-J(T_iu_n)\big\rangle.\end{aligned}$$

From (3.5) and (3.6), we get that $\lim_{n\to\infty}\phi\big(y_n,\Theta_n^{N-1}(x_n+e_n)\big)=0$. So we obtain

$$\lim_{n\to\infty}\big\|y_n-\Theta_n^{N-1}(x_n+e_n)\big\|=0.\tag{3.8}$$

Again, since $p\in\mathcal{F}$,

$$\begin{aligned}&\phi\big(\Theta_n^{N-1}(x_n+e_n),\Theta_n^{N-2}(x_n+e_n)\big)\\ &\le \phi\big(p,\Theta_n^{N-2}(x_n+e_n)\big)-\phi\big(p,\Theta_n^{N-1}(x_n+e_n)\big)\\ &\le \phi\big(p,(x_n+e_n)\big)-\phi\big(p,\Theta_n^{N-1}(x_n+e_n)\big)\\ &\le \phi\big(p,(x_n+e_n)\big)-\phi\big(p,T_iu_n\big).\end{aligned}$$

From (3.5) and (3.6), we get that

$$\lim_{n\to\infty}\phi\big(\Theta_n^{N-1}(x_n+e_n),\Theta_n^{N-2}(x_n+e_n)\big)=0.$$

It also follows that

$$\lim_{n\to\infty}\big\|\Theta_n^{N-1}(x_n+e_n)-\Theta_n^{N-2}(x_n+e_n)\big\|=0.$$

Continuing in this process, we can show that

$$\lim_{n\to\infty}\|\Theta_n^{N-2}(x_n+e_n)-\Theta_n^{N-3}(x_n+e_n)\|=\cdots=\lim_{n\to\infty}\|\Theta_n^{1}(x_n+e_n)-(x_n+e_n)\|=0.$$

So, we now conclude that

$$\lim_{n\to\infty}\left\|\Theta_n^{i}(x_n+e_n)-\Theta_n^{i-1}(x_n+e_n)\right\|=0 \tag{3.9}$$

for each $i=1,2,\cdots,N$. By the uniform norm-to-norm continuity of $J$, we also have

$$\lim_{n\to\infty}\left\|J\big(\Theta_n^{i}(x_n+e_n)\big)-J\big(\Theta_n^{i-1}(x_n+e_n)\big)\right\|=0 \tag{3.10}$$

for each $i=1,2,\cdots,N$. Using (3.9), it is easily seen that

$$\lim_{n\to\infty}\|y_n-(x_n+e_n)\|=0. \tag{3.11}$$

From $u_n=T_{r_n}y_n$, by Lemma 2.8, it follows that

$$\begin{aligned}\phi(u_n,y_n) &= \phi(T_{r_n}y_n,y_n)\\ &\le \phi(p,y_n)-\phi(p,T_{r_n}y_n)\\ &\le \phi(p,x_n+e_n)-\phi(p,u_n)\\ &\le \phi(p,x_n+e_n)-\phi(p,T_iu_n).\end{aligned}$$

This implies that $\lim_{n\to\infty}\phi(u_n,y_n)=0$ and hence

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.12}$$

Combining (3.5), (3.11) and (3.12), we obtain

$$\lim_{n\to\infty}\|T_iu_n-u_n\|=0 \tag{3.13}$$

for all $i\ge 1$.

**Step 5.** $q\in\bigcap_{i=1}^{\infty}F(T_i)$.

Since $x_n\to q$ and $e_n\to 0$, $x_n+e_n\to q$. So from (3.11) and (3.12), we have $u_n\to q$. Note that $T_i$ $(i=1,2,\cdots)$ are weak relatively nonexpansive. Using (3.13), we can conclude that $q\in\widetilde{F}(T_i)=F(T_i)$ for all $i\ge 1$. Hence $q\in\bigcap_{i=1}^{\infty}F(T_i)$.

**Step 6.** $q\in\bigcap_{i=1}^{N}A_i^{-1}(0^*)$.

Noting that $\Theta_n^i(x_n+e_n)=J_{\lambda_n^iA_i}\Theta_n^{i-1}(x_n+e_n)$ for each $i=1,2,\cdots,N$, we obtain

$$\left\|A_{\lambda_n^i}\Theta_n^{i-1}(x_n+e_n)\right\|=\frac{1}{\lambda_n^i}\left\|J\big(\Theta_n^{i-1}(x_n+e_n)\big)-J\big(\Theta_n^{i}(x_n+e_n)\big)\right\|.$$

From (3.10) and $\liminf_{n\to\infty} \lambda_n^i > 0$, we have

$$\lim_{n\to\infty} \left\| A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n) \right\| = 0. \tag{3.14}$$

We note that $\Big(\Theta_n^i(x_n+e_n), A_{\lambda_n^i}\Theta_n^{i-1}(x_n+e_n)\Big) \in G(A_i)$ for each $i = 1, 2, \cdots, N$. If $(w, w^*) \in G(A_i)$ for each $i = 1, 2, \cdots, N$, then it follows from the monotonicity of $A_i$ that

$$\Big\langle w^* - A_{\lambda_n^i}\Theta_n^{i-1}(x_n + e_n), w - \Theta_n^i(x_n + e_n)\Big\rangle \geq 0. \tag{3.15}$$

We see that $\Theta_n^i(x_n+e_n) \to q$ for each $i = 1, 2, \cdots, N$. Thus, from (3.14) and (3.15), we have

$$\langle w^*, w - q\rangle \geq 0.$$

By the maximality of $A_i$, it follows that $q \in A_i^{-1}(0^*)$ for each $i = 1, 2, \cdots, N$. Therefore $q \in \bigcap_{i=1}^N A_i^{-1}(0^*)$.

**Step 7.** $q \in EP(F)$.

From $u_n = T_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

By (A2), we have

$$\begin{aligned} \|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -F(u_n, y) \geq F(y, u_n), \quad \forall y \in C. \end{aligned}$$

Note that $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ since $\liminf_{n\to\infty} r_n > 0$. From (A4) and $u_n \to q$, we get $F(y, q) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)q$. Then $y_t \in C$, which implies that $F(y_t, q) \leq 0$. From (A1), we obtain that $0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, q) \leq tF(y_t, y)$. Thus $F(y_t, y) \geq 0$. From (A3), we have $F(q, y) \geq 0$ for all $y \in C$. Hence $q \in EP(F)$. From Step 5, Step 6 and Step 7, we now can conclude that $q \in \mathcal{F}$.

**Step 8.** $q = \Pi_{\mathcal{F}}(x_1)$.

From $x_n = \Pi_{C_n}(x_1)$, we have

$$\Big\langle J(x_1) - J(x_n), x_n - z\Big\rangle \geq 0, \quad \forall z \in C_n.$$

Since $\mathcal{F} \subset C_n$, we also have

$$\Big\langle J(x_1) - J(x_n), x_n - z\Big\rangle \geq 0, \quad \forall z \in \mathcal{F}. \tag{3.16}$$

Letting $n \to \infty$ in (3.16), we obtain

$$\big\langle J(x_1) - J(q), q - z\big\rangle \geq 0, \quad \forall z \in \mathcal{F}.$$

This shows that $q = \Pi_{\mathcal{F}}(x_1)$ by Lemma 2.2. We thus complete the proof. □

As a direct consequence of Theorem 3.1, we can also apply to a system of convex minimization problems.

**Theorem 3.2.** *Let $E$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $f_i : E \to (-\infty, \infty]$ $(i = 1, 2, \cdots, N)$ be proper lower semi-continuous convex functions, let $F : C \times C \to \mathbb{R}$ be a bifunction, and let $T_i : C \to C$ $(i = 1, 2, \cdots)$ be weak relatively nonexpansive mappings such that $\mathcal{F} := \big[\bigcap_{i=1}^{N} (\partial f_i^{-1})(0^*)\big] \cap \big[\bigcap_{i=1}^{\infty} F(T_i)\big] \cap EP(F) \neq \emptyset$. Let $\{e_n\}_{n=1}^{\infty} \subset E$ be the sequence such that $\lim_{n\to\infty} e_n = 0$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in $C$ as follows:*

$$\begin{cases} x_1 \in C_1 = C, \\ z_n^1 = \arg\ \min_{y\in E} \Big\{ f_1(y) + \frac{1}{2\lambda_n^1}\|y\|^2 + \frac{1}{\lambda_n^1}\big\langle y, J(x_n + e_n)\big\rangle \Big\}, \\ \quad \cdots \\ z_n^{N-1} = \arg\ \min_{y\in E} \Big\{ f_{N-1}(y) + \frac{1}{2\lambda_n^{N-1}}\|y\|^2 + \frac{1}{\lambda_n^{N-1}}\big\langle y, J(z_n^{N-2})\big\rangle \Big\}, \\ y_n = \arg\ \min_{y\in E} \Big\{ f_N(y) + \frac{1}{2\lambda_n^N}\|y\|^2 + \frac{1}{\lambda_n^N}\big\langle y, J(z_n^{N-1})\big\rangle \Big\}, \\ u_n = T_{r_n} y_n, \\ C_{n+1} = \Big\{ z \in C_n : \sup_{i\geq 1} \phi(z, T_i u_n) \leq \phi(z, x_n + e_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} \lambda_n^i > 0$ for each $i = 1, 2, \cdots, N$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges strongly to $q = \Pi_{\mathcal{F}}(x_1)$.*

*Proof.* By Rockafellar's theorem [26, 27], $\partial f_i$ are maximal monotone operators for each $i = 1, 2, \cdots, N$. Let $\lambda^i > 0$ for each $i = 1, 2, \cdots, N$. Then $z^i = J_{\lambda^i \partial f_i}(x)$ if and only if

$$\begin{aligned} 0 \ &\in \ \partial f_i(z^i) + \frac{1}{\lambda^i}\big(J(z^i) - J(x)\big) \\ &= \ \partial\Big(f_i + \frac{1}{\lambda^i}\big(\frac{\|\cdot\|^2}{2} - J(x)\big)\Big)(z^i), \end{aligned}$$

which is equivalent to

$$z^i \ = \ \arg\ \min_{y\in E} \Big\{ f_i(y) + \frac{1}{\lambda^i}\big(\frac{\|y\|^2}{2} - \langle y, J(x)\rangle\big) \Big\}.$$

Using Theorem 3.1, we thus complete the proof. □

If $E = H$ is a real Hilbert space, we then obtain the following results:

**Corollary 3.3.** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $A_i : H \to 2^H$ $(i = 1, 2, \cdots, N)$ be maximal monotone operators, let $F : C \times C \to \mathbb{R}$ be a bifunction, and let $T_i : C \to C$ $(i = 1, 2, \cdots)$ be weak relatively nonexpansive mappings such that $\mathcal{F} := \big[\bigcap_{i=1}^N A_i^{-1}(0)\big] \cap \big[\bigcap_{i=1}^\infty F(T_i)\big] \cap EP(F) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset H$ be the sequence such that $\lim_{n\to\infty} e_n = 0$. Define the sequence $\{x_n\}_{n=1}^\infty$ in $C$ as follows:*

$$\begin{cases} x_1 \in C_1 = C, \\ y_n = J_{\lambda_n^N A_N} \circ J_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_n^1 A_1}(x_n + e_n), \\ u_n = T_{r_n} y_n, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \|z - T_i u_n\| \leq \|z - (x_n + e_n)\|\big\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} \lambda_n^i > 0$ for each $i = 1, 2, \cdots, N$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}(x_1)$.*

**Corollary 3.4.** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $f_i : H \to (-\infty, \infty]$ $(i = 1, 2, \cdots, N)$ be proper lower semi-continuous convex functions, let $F : C \times C \to \mathbb{R}$ be a bifunction, and let $T_i : C \to C$ $(i = 1, 2, \cdots)$ be weak relatively nonexpansive mappings such that $\mathcal{F} := \big[\bigcap_{i=1}^N (\partial f_i^{-1})(0)\big] \cap \big[\bigcap_{i=1}^\infty F(T_i)\big] \cap EP(F) \neq \emptyset$. Let $\{e_n\}_{n=1}^\infty \subset H$ be the sequence such that $\lim_{n\to\infty} e_n = 0$. Define the sequence $\{x_n\}_{n=1}^\infty$ in $C$ as follows:*

$$\begin{cases} x_1 \in C_1 = C, \\ z_n^1 = \arg\ \min_{y\in H} \Big\{f_1(y) + \frac{1}{2\lambda_n^1}\|y\|^2 + \frac{1}{\lambda_n^1}\langle y, x_n + e_n\rangle\Big\}, \\ \quad \cdots \\ z_n^{N-1} = \arg\ \min_{y\in H} \Big\{f_{N-1}(y) + \frac{1}{2\lambda_n^{N-1}}\|y\|^2 + \frac{1}{\lambda_n^{N-1}}\langle y, z_n^{N-2}\rangle\Big\}, \\ y_n = \arg\ \min_{y\in H} \Big\{f_N(y) + \frac{1}{2\lambda_n^N}\|y\|^2 + \frac{1}{\lambda_n^N}\langle y, z_n^{N-1}\rangle\Big\}, \\ u_n = T_{r_n} y_n, \\ C_{n+1} = \Big\{z \in C_n : \sup_{i\geq 1} \|z - T_i u_n\| \leq \|z - (x_n + e_n)\|\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} \lambda_n^i > 0$ for each $i = 1, 2, \cdots, N$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}(x_1)$.*

**Remark 3.5.** Using the shrinking projection method, we can construct a hybrid proximal point algorithm for solving a system of the zero-finding problems, the equilibrium problems and the fixed point problems of weak relatively nonexpansive mappings.

**Remark 3.6.** Since every relatively nonexpansive mapping is weak relatively nonexpansive, our results also hold if $T_i : C \to C$ $(i = 1, 2, \cdots)$ are relatively nonexpansive mappings.

**Acknowledgement.** The authors thank the editor and the referee(s) for valuable suggestions. The first author was supported by the Thailand Research Fund, the Commission on Higher Education and the university of Phayao under grant MRG5380202. The second and the third authors wish to thank the Thailand Research Fund and the Centre of Excellence in Mathematics, Thailand.

## References

[1] Y.I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, pp. 15-50 (1996).

[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123-145.

[3] D. Butanriu, S. Reich, A.J. Zaslavski, Asymtotic behavior of relatively nonexpansive operators in Banach spaces, J. Appl. Anal. 7 (2001) 151-174.

[4] D. Butanriu, S. Reich, A.J. Zaslavski, Weakly convergence of orbits of nonlinear operators in reflexive Banach spaces, Numer. Funct. Anal. Optim. 24 (2003) 489-508.

[5] P. Cholamjiak, A hybrid iterative scheme for equilibrium problems, variational inequality problems and fixed point problems in Banach spaces, Fixed Point Theory Appl. (2009), Art. ID 719360, 18 pages, 2009.

[6] P. Cholamjiak, Y.J. Cho, S. Suantai, Composite iterative schemes for maximal monotone operators in reflexive Banach spaces, Fixed Point Theory Appl. 2011, 2011:7, 10 pages.

[7] P. Cholamjiak, S. Suantai, Convergence analysis for a system of equilibrium problems and a countable family of relatively quasi-nonexpansive mappings in Banach spaces, Abstr. Appl. Anal. (2010), Art. ID 141376, 17 pages, 2010.

[8] W. Cholamjiak, S. Suantai, A hybrid method for a countable family of multivalued maps, equilibrium problems, and variational inequality problems, Discrete Dyn. Nat. Soc. (2010), Art. ID 349158, 14 pages, 2010.

[9] W. Cholamjiak, S. Suantai, A new hybrid algorithm for variational inequality, generalized equilibrium problems and a countable family of quasi-nonexpansive mappings, J. Nonlinear and Convex Anal. 12 (2011) 381-398.

[10] Y.J. Cho, S.M. Kang, H. Zhou, Approximate proximal point algorithms for finding zeroes of maximal monotone operators in Hilbert spaces, J. Inequal. Appl. (2008), Art. ID 598191, 10 pages, 2008.

[11] L.-C. Ceng, Y.-C. Liou, E. Naraghirad, Iterative approaches to find zeros of maximal monotone operators by hybrid approximate proximal point methods, Fixed Point Theory Appl. (2011), Art. ID 282171, 18 pages, 2011.

[12] O. Güler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim. 29 (1991) 403-419.

[13] S. Kamimura, W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theor. 106 (2000) 226-240.

[14] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938-945.

[15] S. Kamimura, F. Kohsaka and W. Takahashi, Weak and strong convergence of a maximal monotone operators in a Banach space, Set-Valued Anal. 12 (2004) 417-429.

[16] F. Kohsaka and W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, Abstr. Appl. Anal. 3 (2004) 239-249.

[17] F. Kohsaka, W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces, SIAM J. Optim. 19 (2008) 824-835.

[18] P. Kumam, A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive mapping, Nonlinear Anal.: Hybr. Syst. 2 (2008) 1245-1255.

[19] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, J. Appl. Math. Comput. 29 (2009) 263-280.

[20] B. Martinet, Régularisation d'inéquations variationelles par approximations successives. Rev. Francaise d'Informatique et de Recherche Opérationelle 4 (1970) 154-159.

[21] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theor. 134 (2005) 257-266.

[22] G. Marino, H.-K. Xu, Convergence of generalized proximal point algorithm, Comm. Pure Appl. Anal. 3 (2004) 791-808.

[23] N. Onjai-uea, P. Kumam, A new iterative scheme for equilibrium problems, fixed point problems for nonexpansive mappings and maximal monotone operators, Fixed Point Theory Appl. (In press).

[24] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009) 20-30.

[25] X. Qin, S.M. Kang, Y.J. Cho, Approximating zeros of monotone operators by proximal point algorithms, J. Glob. Optim. 46 (2010) 75-87.

[26] R.T. Rockafellar, Characterizations of the subdifferentials of convex functions, Pacific J. Math. 17 (1966) 497-510.

[27] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, Pacific J. Math. 33 (1970) 209-216.

[28] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877-898.

[29] S. Saewan, P. Kumam, A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems, Abstr. Appl. Anal. (2010), Art. ID 123027, 31 pages, 2010.

[30] P. Sunthrayuth, P. Kumam, A system of generalized mixed equilibrium problems, maximal monotone operators and fixed point problems with application to optimization problems, Abstr. Appl. Anal. (2012), Art. ID 316276, 39 pages, 2012.

[31] Y. Su, H.-K. Xu, X. Zhang, Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications, Nonlinear Anal. 73 (2010) 3890-3906.

[32] Y. Shehu, Iterative approximation method for finite family of relatively quasi nonexpansive mappings and systems of equilibrium problems, J. Glob. Optim. (In press).

[33] Y. Shehu, A new hybrid iterative scheme for countable families of relatively quasi-nonexpansive mappings and system of equilibrium problems, Int. J. Math. Math. Sci. (2011), Art. ID 131890, 23 pages, 2011.

[34] W. Takahashi, Nonlinear Functional Analysis, Fixed Point Theory and Its Application, Yokohama-Publishers, Yokohama, Japan 2000.

[35] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276-286.

[36] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70 (2009) 45-57.

[37] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory Appl. (2008), Art. ID 528476, 11 pages, 2008.

[38] K. Wattanawitoon, P. Kumam, Hybrid proximal-point methods for zeros of maximal monotone operators, variational inequalities and mixed equilibrium problems, Int. J. Math. Math. Sci. (2011), Art. ID 174796, 31 pages, 2011.

[39] K. Wattanawitoon, P. Kumam, Generalized mixed equilibrium problems for maximal monotone operators and two relatively quasi-nonexpansive mappings, Thai J. Math. 9 (2011) 165-189.

[40] K. Wattanawitoon, P. Kumam, A new iterative scheme for generalized mixed equilibrium, variational inequality problems and a zero point of maximal monotone operators, J. Appl. Math. (2012), Art. ID 692829, 27 pages, 2012.

[41] U. Witthayarat, Y. J. Cho, P. Kumam, Convergence of an iterative algorithm for common solutions for zeros of maximal accretive operator with applications, J. Appl. Math. (In press).

[42] Y. Yao, M.A. Noor, On convergence criteria of generalized proximal point algorithms, J. Comput. Appl. Math. 217 (2008) 46-55.

[43] Y. Yao, N. Shahzad, Strong convergence of a proximal point algorithm with general errors, Optim. Lett. (In press).

[44] H. Zegeye, N. Shahzad, Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, Nonlinear Anal. 70 (2009) 2707-2716.