รายงานวิจัยฉบับสมบูรณ์

โครงการ : การมีอยู่จริงและขั้นตอนวิธีแก้ปัญหาสำหรับ
ระบบอสมการการแปรผัน ผนวกวิภัชนัยแบบสุ่ม

Existence and algorithms for the systems of

random fuzzy variational inclusion problems

โดย ผศ.ดร.นรินทร์ เพชร์โรจน์ และคณะ

15 มิถุนายน 2555

สัญญาเลขที่ MRG5380247

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย และมหาวิทยาลัยนเรศวร

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. และ สกอ. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

โครงการวิจัยนี้ได้รับทุนสนับสนุนตามโครงการความร่วมมือระหว่างสำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัย และมหาวิทยาลัยนเรศวร เพื่อเป็นการพัฒนาศักยภาพในการทำงานวิจัยอาจารย์รุ่นใหม่ ผู้วิจัยขอขอบพระคุณเจ้าของทุนเป็นอย่างสูงมา ณ โอกาสนี้

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นรินทร์ เพชร์โรจน์

Abstract

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Project Title: Existence and algorithms for the systems of random fuzzy variational inclusion problems

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In this project, firstly, we present some results on fixed point theorems, variational inequalities problems, system of variational inequalities problems, and their relationships. Then, by using those knowledge, we prove the results on random fuzzy variational inclusion problems, which are the main purposes of this project. It is worth to mention that, the results presented in this project are more general and are viewed as an extension, refinement, and improvement of the previously known results in the literature.

Keywords: Variational inequality problem**;** fixed point problem; random fuzzy mapping; weak distance; resolvent operator; regularization method, Hilbert spaces, Banach spaces

บทคัดย่อ

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ในโครงการนี้ ขั้นแรกเริ่มต้นด้วยการศึกษาทฤษฎีบทจุดตรึง ปัญหาอสมการการแปรผันและระบบของปัญหาอสมการการแปรผันและความสัมพันธ์ของปัญหาต่าง ๆเหล่านั้น และหลังจากนั้นโดยการใช้องค์ความรู้ที่ได้ศึกษาในขั้นแรกเราได้ศึกษาปัญหาอสมการการแปรผันผนวกวิภัชนัยแบบสุ่มซึ่งเป็นเป้าหมายหลักของโครงการนี้ ผลลัพธ์ที่ได้จากโครงการวิจัยนี้ทำให้ได้รับองค์ความรู้ที่ครอบคลุมและกว้างขวางขึ้น

คำหลัก: อสมการการแปรผัน; ปัญหาจุดตรึง; การส่งวิภัชนัยแบบสุ่ม; ระยะทางอย่างอ่อน; ตัวดำเนินการแก้ปัญหา; วิธีการทำให้เป็นระเบียบ; ปริภูมิฮิลเบิร์ต; ปริภูมิบานาค

บทนำ

การศึกษาเกี่ยวกับทฤษฎีบทอสมการการแปรผันได้เข้ามามีบทบาทสำคัญอย่างมากมากว่า 20 ปี เนื่องจากองค์ความรู้ที่ได้จากการศึกษาในเรื่องดังกล่าวเป็นเครื่องมือสำคัญซึ่งสามารถนำไปประยุกต์ใช้ในการตอบปัญหาของแบบจำลองพื้นฐานต่างๆที่เกิดจากในศาสตร์หลายๆ แขนงทั้งวิทยาศาสตร์บริสุทธิ์และวิทยาศาสตร์ประยุกต์ เช่น nonlinear programming, physics, economics, transportation equilibrium, regional และ engineering sciences เป็นต้น

ในปัจจุบันการศึกษาเกี่ยวกับทฤษฎีบทอสมการการแปรผันได้มีการพัฒนาขึ้นเป็นอย่างมาก โดยได้มีการขยายแนวคิดของปัญหาอสมการการแปรผันแบบที่ศึกษาโดย Hartman และ Stampacchia ในปี ค.ศ. 1964 ไปยังแนวคิดที่มีความเป็นนัยทั่วไปมากยิ่งขึ้น ซึ่งมีจุดประสงค์หลัก คือ การสร้างองค์ความรู้ใหม่ที่สามารถนำไปประยุกต์ใช้ได้อย่างแพร่หลายและกว้างขวางมากยิ่งขึ้น ในแนวทางดังกล่าว แนวคิดเกี่ยวกับอสมการการแปรผันผนวก (variational inclusion) ถือเป็นการวางนัยทั่วไปของปัญหาอสมการการแปรผันที่น่าสนใจซึ่งสามารถนำมาประยุกต์ใช้และมีประโยชน์อย่างมาก ซึ่งองค์ความรู้ที่ได้จากการศึกษาปัญหาอสมการการแปรผันผนวกนั้นได้มีส่วนสำคัญในการนำไปประยุกต์ใช้เกี่ยวกับการตอบปัญหา optimization theory, mathematical finance, decision sciences และ structural analysis เป็นต้น ดังนั้นสิ่งตามมาที่น่าสนใจคือปัญหาเกี่ยวกับกระบวนการหาคำตอบของวิธีการเชิงตัวเลขเพื่อให้ได้มาซึ่งคำตอบของปัญหาดังกล่าว

ในขั้นตอนของการพัฒนาการแนวคิดเกี่ยวกับการการประยุกต์ใช้องค์ความรู้ที่ได้รับจากศึกษาทฤษฎีบทอสมการการแปรผันนั้น ในปี 1985 Pang ได้แสดงว่า แบบจำลองของปัญหาดุลยภาพ (equilibrium model) เช่น the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium และ the general equilibrium programming problem สามารถนำมาศึกษาในรูปแบบที่เอกรูป(uniformly model) ได้ โดยใช้แนวคิดการศึกษาอสมการการแปรผันบนเซตผลคูณ (product set) ซึ่งในการศึกษาดังกล่าวนั้น Pang ได้ใช้วิธีการโดยแยกอสมการการแปรผันตั้งต้นสำหรับปัญหาบนเซตผลคูณดังกล่าวเป็นระบบของอสมการการแปรผัน (system of variational inequalities) และทำการศึกษาเกี่ยวกับกระบวนการหาคำตอบของระบบของอสมการการแปรผัน และได้แสดงให้เห็นว่าการศึกษาปัญหาอสมการการแปรผันบนเซตผลคูณและปัญหาระบบของอสมการการแปรผันเป็นการศึกษาที่สมมูลกัน ดังนั้นการศึกษาเกี่ยวกับกระบวนการหาคำตอบของระบบของอสมการการแปรผัน จึงนับว่าเป็นสิ่งที่น่าสนใจอย่างยิ่ง

ในอีกทางหนึ่งเป็นที่ทราบดีว่าการศึกษาเกี่ยวกับทฤษฎีเซตวิภัชนัย (fuzzy set theory) ซึ่งเริ่มมีขึ้นในปี ค. ศ. 1965 โดย Zadeh รวมถึงการศึกษาเกี่ยวกับสมการสุ่ม (random equations) ที่เกี่ยวพันกับตัวดำเนินการสุ่ม (random operator) นับว่าเป็นการศึกษาที่มีประโยชน์อย่างมากอีกเช่นกัน เนื่องจากมีการนำไปประยุกต์ใช้ในศาสตร์หลายๆแขนง อย่างเช่น physical, mathematical and engineering science, probabilistic model เป็นต้น ดังนั้นแนวคิดที่จะทำการศึกษาโดยการผสมผสานองค์ความรู้เกี่ยวกับ อสมการการแปรผันและระบบของอสมการการแปรผัน เซตวิภัชนัย และ สมการสุ่มที่เกี่ยวพันกับตัวดำเนินการสุ่ม เพื่อให้ได้องค์ความรู้ขึ้นมาใหม่นั้นจึงนับว่าเป็นเรื่องที่น่าสนใจเป็นอย่างมาก เนื่องจากองค์ความรู้ที่ได้รับจะครอบคลุมและสามารถนำไปประยุกต์ใช้ได้โดยรวมจากศาสตร์ทั้ง 3 แขนงที่กล่าวมาข้างต้นรวมถึงศาสตร์แขนงอื่นๆ ที่มีความเกี่ยวข้องได้อีกด้วย

จากที่กล่าวมาข้างต้นจะเห็นได้ว่าทั้งการศึกษาทฤษฎีบทเกี่ยวกับความสัมพันธ์ระหว่างระบบของอสมการการแปรผันที่มีความเกี่ยวข้องกับแนวคิดทฤษฎีเซตวิภัชนัยและสมการสุ่มนั้นถือว่าเป็นประโยชน์อย่างยิ่ง เนื่องจากปัญหาแต่ละชนิดต่างก็มีความสำคัญเด่นชัดในการนำไปประยุกต์ใช้แก้ปัญหาที่เกี่ยวข้อง ตัวอย่างเช่น เป็นที่ทราบดีว่าในการศึกษาหัวข้อเกี่ยวกับ mathematical programming จะพบว่า ปัญหาอสมการการแปรผันจะมีความเกี่ยวข้องกับปัญหาการประมาณ (optimization problem) ทำให้ผลที่ตามมาคือ การศึกษาเกี่ยวกับปัญหาอสมการการแปรผันที่มีความสัมพันธ์กับแนวคิดทฤษฎีเซตวิภัชนัยจะมีความเกี่ยวข้องกับปัญหาการประมาณเชิงวิภัชนัย (fuzzy optimization problem) ซึ่งเมื่อนำมาประกอบกับการพิจารณาในแนวคิดของสมการสุ่มซึ่งมีบทบาทสำคัญในการศึกษาเกี่ยวกับแบบจำลองในวิทยาศาสตร์ประยุกต์ (applied sciene) ร่วมด้วยจึงนับว่าเป็นที่น่าสนใจอย่างยิ่ง

ดังนั้นในโครงการวิจัยนี้ผู้ดำเนินการวิจัยจึงมีความสนใจที่จะศึกษาและแก้ปัญหาเพื่อให้ได้องค์ความรู้ใหม่ที่น่าสนใจ คือ ศึกษากระบวนการขั้นตอนวิธีในการหาคำตอบของระบบของอสมการการแปรผันผนวกรวมถึงสมการแปรผันผนวกวางนัยทั่วไปโดยมีความเกี่ยวข้องกับแนวคิดทฤษฎีเซตวิภัชนัยและสมการสุ่ม ซึ่งการศึกษาปัญหาดังกล่าวจะทำให้ทฤษฎีบทที่ค้นพบครอบคลุมการศึกษาที่มีอยู่เดิม ซึ่งมีผลต่อเนื่องทำให้ได้องค์ความรู้ใหม่ที่ได้สามารถนำไปใช้ได้อย่างกว้างขวางและมีศักยภาพมากยิ่งขึ้น ซึ่งจะเป็นพื้นฐานที่สำคัญในการพัฒนาวิชาการในสาขาวิชาที่เกี่ยวข้องอันจะเป็นพื้นฐานในการพัฒนาประเทศต่อไป

จุดประสงค์ของการวิจัย

วัตถุประสงค์ของโครงการวิจัยนี้ คือ คิดค้นทฤษฎีบทและองค์ความรู้ใหม่ๆ เกี่ยวกับ

1. กระบวนการขั้นตอนวิธีในการหาคำตอบของระบบของอสมการแปรผันผนวกสัมพันธ์กับแนวคิดทฤษฎีเซต วิภัชนัยและสมการสุ่มบนปริภูมิฮิลเบิร์ต
2. กระบวนการขั้นตอนวิธีในการหาคำตอบของระบบของอสมการแปรผันผนวกวางนัยทั่วไปสัมพันธ์กับ แนวคิดทฤษฎีเซตวิภัชนัยและสมการสุ่มปริภูมิฮิลเบิร์ต

ผลการวิจัย

1. Fixed point problems

Let (X, d) be a metric space and $2^X, CB(X)$, $Cl(X)$ denote the collections of nonempty subsets of X, nonempty closed bounded subsets of X and nonempty closed subsets of X, respectively. If $T : X \to 2^X$ is a mapping, then an element $x \in X$ is called a *fixed point* of T if $x \in T(x)$ and $Fix(T)$ denotes the set of fixed points of T, that is, $Fix(T) = \{x \in X : x \in T(x)\}$.

Recall that the function H on $CB(X)$ defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for all $A, B \in CB(X)$ is called the *Hausdorff metric*, where $d(x, B) = \inf_{b\in B}\{d(x, b)\}$.

1.1. **Yeol Je Cho, Soawapak Hirunworakit, Narin Petrot**, *Set-Valued Fixed Points Theorems for Generalized contractive mappings without the Hausdorff metric*, Applied Mathematics Letters 24 (2011) 1959—1967.

The concept of $\tau-$ distance on a metric space, which is a generalization of w- distance, introduced by T. Suzuki [T. Suzuki, Generalized Distance and Existence Theorems in Complete Metric Spaces, J. Math. Anal. Appl. 253 (2001), 440-458], as following. Let X be a metric space with metric d. Then a function p from $X \times X$ into $[0, \infty)$ is called $\tau-distance$ on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the followings are satisfied:

(τ1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$ and η is concave and continuous in its second variable;

(τ3) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_{n\to\infty} p(w, x_n)$ for all $w \in X$;

(τ4) $\lim_{n\to\infty} \sup\{p(x_n, y_m)) : m \geq n\} = 0$ and $\lim_{n\to\infty} \eta(x_n, t_n) = 0$ imply $\lim_{n\to\infty} \eta(y_n, t_n) = 0$;

(τ5) $\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

We define $D_p(x, A) = \inf\{p(x, y) | y \in A\}$. Then, in this part, we have the following results.

<u>**Theorem 1**</u> Let (X, d) be a metric space and $T : X \to Cl(X)$ is set-valued contractive mapping. If there exit a function $\varphi : [0, \infty) \to [0, 1)$ and a non-decreasing function $\theta : [0, \infty) \to [c, 1), c > 0$, such that

$$\varphi(t) < \theta(t)$$

for all $t \in [0, \infty)$ and

$$\limsup_{t\to r^+} \varphi(t) < \limsup_{t\to r^+} \theta(t)$$

for all $r \in [0, \infty)$, and there exists a τ-distance p on X such that for any $x \in X$ there exists $y \in T(x)$ satisfying

$$\theta(p(x, y))p(x, y) \leq D_p(x, T(x))$$

and

$$D_p(y, T(y)) \leq \varphi(p(x, y))p(x, y).$$

Then we have the following:

(a) For each $x_0 \in X$, there exists an orbit $\{x_n\} \in \mathcal{O}(T, x_0)$ such that $\{D_p(x_n, T(x_n))\}$ is decreasing to zero and the sequence $\{x_n\}$ is a Cauchy sequence.

(b) If $\{x_n\}$ converges to z and the function $f(x) := D_p(x, T(x))$ is T-orbitally lower semi-continuous at z with respect to x_0 then $z \in F(T)$. Moreover, if $T(z) = z$ then $p(z, z) = 0$.

<u>**Theorem 2**</u> Let (X, d) be a complete space. Suppose that $T : X \to Cl(X)$ be a set-valued mapping of X into itself. If there exists a function $\varphi : [0, \infty) \to [0, 1)$ such that

$$\limsup_{r\to t^+} \varphi(r) < 1$$

for any $t \in [0, \infty)$ and there exists a τ-distance p on X such that, for any $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq (2 - \varphi(p(x, y)))D_p(x, T(x))$$

and

$$D_p(y, T(y)) \leq \varphi(p(x, y))p(x, y).$$

Then we have the following:

(a) For any $x_0 \in X$, there exist an orbit $\{x_n\} \in \mathcal{O}(T, x_0)$ and $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

(b) If the function $f(x) := D_p(x, T(x))$ is T-orbitally lower semi-continuous at z with respect to x_0 then $z \in F(T)$. Moreover, if $T(z) = z$ then $p(z, z) = 0$.

Also, in the presented paper, some interesting remarks and examples are also discussed.

1.2. **Jittiporn Suwannawit and Narin Petrot**, *Common Fixed point theorems for hybrid generalized multivalued*, Thai Journal of Mathematics, 9(2) (2011), 417—427

Let X be a metric space. A subset $C \subset X$ is said to be approximative if the multivalued mapping

$$\mathcal{P}_C(x) = \{c \in C : d(x, c) = D(x, C)\}, \quad \forall x \in X$$

has nonempty values. The multivalued mapping $T : X \to 2^X$ is said to have approximative values if $T(x)$ is approximative for each $x \in X$.

Let $\propto \in (0, \infty], \mathcal{R}^+_\propto = [0, \propto)$. Let $\varphi : \mathcal{R}^+_\propto \to [0, \infty)$ satisfy

(i) $\varphi(t) < t$ for each $t \in (0, \propto)$;

(ii) φ is nondecreasing on $\mathcal{R}^+_\propto$

(iii) φ is upper-semicontinuous.

Define $\Phi[0,\propto) = \{\varphi : \varphi$ satisfies (i)-(iii) above$\}$.

From now on, for a metric space X, we let $\Gamma = \sup\{d(x,y) : x,y \in X\}$ and set $\propto = \Gamma$ if $\Gamma = \infty$, and $\propto > \Gamma$ if $\Gamma < \infty$.

Let J denotes an interval on $[0,\infty)$ containing 0, that is an interval of the form $[0,r],[0,r)$ or $[0,\infty)$, and we use the abbreviation φ^n for the nth iterate of a function φ. A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function on J if $\Sigma_{n=0}^{\infty}\varphi^n(t) < \infty$ for all $t \in J$.

Suppose that $S,T : E \to 2^E$ and $\varphi \in \Phi[0,\propto)$ satisfy

$$H(Sx,Ty) \leq \varphi(\rho(x,y)),$$

for each $x,y \in E$, where

$$\rho(x,y) = \max\{d(x,y), D(Sx,x), D(Ty,y), \frac{1}{2}\left[D(y,Sx) + D(x,Ty)\right]\}.$$

Then the pair S,T is called the hybrid generalized multivalued φ-weak contraction mapping.

Motivated and spirted by the research going on this field, in this work we prove that there is a common fixed point of hybrid generalized multivalued φ-weak contractions S,T on complete metric spaces X.

Theorem 1 Let (X,d) be a complete metric space. Let S,T be a pair of hybrid generalized multivalued φ-weak contractions on X. Assume that S,T have the approximative values and $\varphi_{|_J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_{\propto}$. If there is $x \in E$ such that either $D(x,Sx) \in J$ or $D(x,Tx) \in J$ then the mappings S and T have a common fixed point $u \in X$.

We also use the following concepts to present some further results.

Let $\propto \in (0,\infty], \mathcal{R}^+_{\propto} = [0,\propto)$. Let $f : [0,\infty) \to [0,\infty)$ satisfy

(i) $f(0) = 0$ and $f(t) > 0$ for each $t \in (0,\propto)$;

(ii) f is nondecreasing on $\mathcal{R}^+_{\propto}$;

(iii) f is continuous on $\mathcal{R}^+_{\propto}$;

(iv) $f(a+b) \leq f(a) + f(b)$ for all $a,b \in [0,\infty)$.

Define $\mathcal{F}[0,\propto) = \{f : f \text{ satisfies (i)-(iv) above}\}$.

Theorem 2 Let (X,d) be a complete metric space and $S,T : X \to 2^X$ be a pair of multivalued mappings. Suppose that $\varphi \in \Phi[0,\propto)$ and $f \in \mathcal{F}[0,\propto)$ satisfy

$$f(H(Sx,Ty)) \leq \varphi(f(\rho(x,y)))$$

for each $x,y \in X$. Assume that S,T have the approximative values and $\varphi_{|J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_\propto$. If there is $x \in X$ such that either $f(D(x,Sx)) \in J$ or $f(D(x,Tx)) \in J$ then the mappings S and T have a common fixed point $u \in X$.

2. Variational inequalities problems on Hilbert spaces

In this part we will use the following notations. Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

2.1. **Yeol Je Cho and Narin Petrot**, *Regularization and Iterative method for general variational inequality problem in Hilbert spaces*, Journal of Inequalities and Applications 2011, 2011:21.

In 1988, Noor [M. A. Noor, General variational inequalities, Appl. Math. Lett. 1 (1988) 119-121] introduced and studied a class of variational inequalities, which is known as general variational inequality, $GVI_C(A,g)$, as following: Find $u^* \in H, g(u^*) \in C$ such that

$$\langle A(u^*), g(v) - g(u^*) \rangle \geq 0, \quad \forall v \in H : g(v) \in C, \tag{1}$$

where C is a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, and $T, g : H \to H$ be mappings.

Motivated and inspired by the research going in this direction, in this paper, we present a method for finding a solution of the problem (1) which is related to the solution set of an inverse strongly monotone mapping as following: Find $u^* \in H, g(u^*) \in S(T)$ such that

$$\langle A(u^*), g(v) - g(u^*) \rangle \geq 0, \quad \forall v \in H : g(v) \in C, \tag{2}$$

when A is a generalized monotone mapping, $T : C \to H$ is an inverse strongly monotone mapping and $S(T) = \{x \in C : T(x) = 0\}$. We will denote by $GVI_C(A, g, T)$ for a set of solution to the problem (2).

Let $\alpha \in (0, 1)$ be a fixed positive real number. We now construct a regularization solution u_α for (2), by solving the following general variational inequality problem: find $u_\alpha \in H,\ g(u_\alpha) \in C$ such that

$$\langle A(u_\alpha) + \alpha^\mu (T \circ g)(u_\alpha) + \alpha g(u_\alpha), g(v) - g(u_\alpha)\rangle \geq 0 \quad \forall v \in H,\ g(v) \in C, \quad 0 < \mu < 1. \tag{3}$$

Theorem 1 (Regularization) Let C be a closed convex subset of a Hilbert space H and $g : H \to H$ be a mapping such that $C \subset g(H)$. Let $A : H \to H$ be a hemicontinuous on C and g-monotone mapping, $T : C \to H$ be λ-inverse strongly monotone mapping. If g is an expanding affine continuous mapping and $GVI_C(A, g, T) \neq \emptyset$, then the following conclusions are true:

(a) For each $\alpha \in (0, 1)$, the problem (3) has the unique solution u_α.

(b) If $\alpha \downarrow 0$ then $\{g(u_\alpha)\}$ converges. Moreover, $\lim\limits_{\alpha \to 0^+} g(u_\alpha) = g(u^*)$ for some $u^* \in GVI_C(A, g, T)$.

(c) There exists a positive constant M such that

$$\|g(u_\alpha) - g(u_\beta)\|^2 \leq \frac{M(\beta - \alpha)}{\alpha^2}, \tag{4}$$

when $0 < \alpha < \beta < 1$.

We also consider the regularization inertial proximal point algorithm

$$\langle c_n[A(z_{n+1}) + \alpha_n^\mu (T \circ g)(z_{n+1}) + \alpha_n g(z_{n+1})] + g(z_{n+1}) - g(z_n), g(v) - g(z_{n+1})\rangle \geq 0$$

$$\forall\ v \in H,\ g(v) \in K\ , z_1 \in H,\ g(z_1) \in K. \tag{5}$$

Theorem 2 (Iterative Method) Assume that all hypothesis of the Theorem 1 are satisfied. If the parameters c_n and α_n are chosen positive real numbers such that

(C1) $\lim\limits_{n \to \infty} \alpha_n = 0$, (C2) $\lim\limits_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0$,

(C3) $\liminf\limits_{n\to\infty}\ c_n\alpha_n > 0$,

then the sequence $\{g(z_n)\}$ defined by (5) converges strongly to the element $g(u^*)$ as $n \to +\infty$, where $u^* \in GVI_K(A, g, T)$.

2.2. **Suthep Suantai and Narin Petrot**, *Existence and stability of iterative algorithms for the system of nonlinear quasi mixed equilibrium problem*, Applied Mathematics Letters 24 (2011) 308–313.

Let $\Phi_1, \Phi_2 : H \times H \to H$ be given two bi-functions satisfying $\Phi_i(x, x) = 0$ for all $x \in H, i = 1, 2$. Let $T_i : H \times H \to H$ be a nonlinear mapping for each $i = 1, 2$. In this work, let $\mathcal{CC}(H)$ be denoted for the family of all nonempty subsets of H and let $C_i : H \to \mathcal{CC}(H)$ be a point-to-set mappings which associate a convex set $C_i(x)$ with any element x of H, for each $i = 1, 2$. We consider the problem of finding $(x^*, y^*) \in H \times H$ such that $x^* \in C_1(x^*), y^* \in C_2(y^*)$ and

$$\begin{cases} \Phi_1(x^*, z) + \langle T_1(x^*, y^*), z - x^*\rangle \geq 0, \quad \forall z \in C_1(x^*), \\ \Phi_2(y^*, z) + \langle T_2(x^*, y^*), z - y^*\rangle \geq 0, \quad \forall z \in C_2(y^*). \end{cases} \tag{6}$$

We have considered the following class of mappings in this part.

A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be *ν-strongly monotone* if there exists a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \nu\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

And it is said to be *(τ, σ)-Lipschitz* if there exist constants $\tau, \sigma > 0$ such that

$$\|T(x_1, y_1) - T(x_2, y_2)\| \leq \tau\|x_1 - x_2\| + \sigma\|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{H}.$$

Theorem 1 (Existence theorem) For each $i = 1, 2$, let $\Phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a monotone function and $C_i : \mathcal{H} \to \mathcal{CC}(\mathcal{H})$. Let $T_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a ν_1-strongly monotone with respect to the first argument and (τ_1, σ_1)-Lipschitz mapping and $T_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a ν_2-strongly monotone with respect to the second argument and (τ_2, σ_2)-Lipschitz mapping. Suppose that there are positive real numbers ρ_1, ρ_2 which satisfy the following condition:

$$\begin{cases} (1 - 2\rho_1\nu_1 + \rho_1^2\tau_1^2)^{\frac{1}{2}} + \rho_2\tau_2 < 1 - \eta_1, \\ (1 - 2\rho_2\nu_2 + \rho_2^2\tau_2^2)^{\frac{1}{2}} + \rho_1\sigma_1 < 1 - \eta_2. \end{cases}$$

Then the set of solution of the problem (6) is a singleton.

Theorem 1 not only gives the conditions for the existence solution of the problems (6) but also provide the algorithm to find such solution for any initial vector $(x_0, y_0) \in \mathcal{H} \times \mathcal{H}$. In fact, by proceeding along the same lines as in Theorem 1, one can also show that the sequences $\{(x_n, y_n)\}$, defined by following Mann type perturbed iterative algorithm (MTA),

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J^{\rho_1}_{\Phi_1, C_1(x_n)}[x_n - \rho_1 T_1(x_n, y_n)], \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n J^{\rho_2}_{\Phi_2, C_2(y_n)}[y_n - \rho_2 T_2(x_n, y_n)], \end{cases} \tag{7}$$

converges strongly to the unique solution of the problem (6), when $\{\alpha_n\}$ is a sequence of real numbers such that $\alpha_n \in (0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. The stability analysis for (7) is also discussed. Firstly, we have observed the following facts. Let $(x, y) \in \mathcal{H} \times \mathcal{H}$. Then (x, y) is a solution of the problem (6) if and only if there exist positive real numbers ρ_1, ρ_2 such that (x, y) is a fixed point of the map $G_{\rho_1,\rho_2} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ defined by

$$G_{\rho_1,\rho_2}(x, y) = (A_{\rho_1}(x, y), B_{\rho_2}(x, y)), \quad \forall (x, y) \in \mathcal{H} \times \mathcal{H}, \tag{8}$$

where $A_{\rho_1}, B_{\rho_2} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ are defined by

$$A_{\rho_1}(x, y) = (1 - \lambda)x + \lambda J^{\rho_1}_{\Phi_1, C_1(x)}[x - \rho_1 T_1(x, y)]$$

$$B_{\rho_2}(x, y) = (1 - \lambda)y + \lambda J^{\rho_2}_{\Phi_2, C_2(y)}[y - \rho_2 T_2(x, y)],$$

where $\lambda \in (0, 1)$ is a fixed constant.

Now we give a definition, for stability analysis. Let $\mathcal{H}$ be a Hilbert space and let $A, B : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be nonlinear mappings. Let $G : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ be defined as $G(x, y) = (A(x, y), B(x, y))$ for any $(x, y) \in \mathcal{H} \times \mathcal{H}$, and let $(x_0, y_0) \in \mathcal{H} \times \mathcal{H}$. Assume that $(x_{n+1}, y_{n+1}) = f(G, x_n, y_n)$ defines an iteration procedure which yields a sequence of $\{(x_n, y_n)\}$ in $\mathcal{H} \times \mathcal{H}$. Suppose that $F(G) = \{(x, y) \in \mathcal{H} \times \mathcal{H} : G(x, y) = (x, y)\} \neq \emptyset$ and $\{(x_n, y_n)\}$ converges to some $(x^*, y^*) \in F(G)$. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $\mathcal{H} \times \mathcal{H}$ and $\varepsilon_n = \|(u_n, v_n) - f(G, x_n, y_n)\|$, for all $n \geq 0$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} (u_n, v_n) = (x^*, y^*)$, then the iterative procedure $\{(x_n, y_n)\}$ is said to be *G-stable* or *stable with respect to G*.

Theorem 2 (Stability analysis) Assume that all conditions of the Theorem 1 hold. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $\mathcal{H} \times \mathcal{H}$ and define $\{\delta_n\} \subset [0, \infty)$ by

$$\delta_n = \|(u_{n+1}, v_{n+1}) - (C_n, D_n)\|^+, \tag{9}$$

where

$$\begin{cases} C_n = (1 - \alpha_n)x_n + \alpha_n J^{\rho_1}_{\Phi_1, C_1(x_n)}[x_n - \rho_1 T_1(x_n, y_n)], \\ D_n = (1 - \alpha_n)y_n + \alpha_n J^{\rho_2}_{\Phi_2, C_2(y_n)}[y_n - \rho_2 T_2(x_n, y_n)], \end{cases} \tag{10}$$

where (x_n, y_n) is defined in (7), for each $n \in \mathbb{N}$. If G_{ρ_1, ρ_2} defined as in (8) then the iterative procedure (7) is G_{ρ_1, ρ_2}-stable.

2.3. **Ioannis K. Argyros, Yeol Je Cho and Narin Petrot**, *Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems*, Computers and Mathematics with Applications 60 (2010) 2292—2301.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $Q : C \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in \mathcal{H} \times C \times C$. We consider the following *generalized equilibrium problem*:

$$\begin{cases} \text{Find } x^* \in C \text{ such that} \\ \Phi(Qx^*, x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \end{cases} \tag{1.1}$$

We denote the set of solutions of the generalized equilibrium problem (1.1) by $GEP(C, Q, \Phi, \varphi)$.

On the other hand, for two nonlinear mappings $A, B : C \to \mathcal{H}$, we consider the following *system of nonlinear variational inequalities problems*:

$$\begin{cases} \text{Find } (x^*, y^*) \in C \times C \text{ such that} \\ \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \rho B x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

where λ and ρ are positive numbers.

Recall that a mapping $S : C \to C$ is said to be *Lipschitz continuous* if there exists a positive constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In the case $L = 1$, the mapping S is known as a *nonexpansive mapping*. If $S : C \to C$ is a mapping, we denote the set of fixed points of S by $F(S)$, that is, $F(S) = \{x \in C : Sx = x\}$.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $Q : C \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function. Let r be a positive number. For any $x \in C$, we consider the following problem:

$$\begin{cases} \text{Find } y \in C \text{ such that} \\ \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y \rangle \geq 0, \quad \forall z \in C, \end{cases} \tag{1.5}$$

which is known as the *auxiliary generalized equilibrium problem*.

Let $T^{(r)} : C \to C$ be the mapping such that, for each $x \in C$, $T^{(r)}(x)$ is the solution set of the auxiliary problem (1.5), i.e.,

$$T^{(r)}(x) = \{y \in C : \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y \rangle \geq 0,\ \forall z \in C\}, \quad \forall x \in C.$$

In this part, we have assumed the following **Condition (Δ):**

(a) $T^{(r)}$ is single-valued;

(b) $T^{(r)}$ is nonexpansive;

(c) $F(T^{(r)}) = GEP(C, Q, \Phi, \varphi)$.

Now, assuming that the Condition (Δ) is satisfied, then we can introduce the following algorithm:

Algorithm (I). Let ρ and λ be two positive numbers. Let $A, B : C \to \mathcal{H}$ and $S : C \to C$ be mappings. For any $u, x_1 \in C$, there exist sequences $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ in C such that

$$\begin{cases} \Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = P_C(x_n - \rho B x_n), \\ z_n = P_C(y_n - \lambda A y_n), \\ x_{n+1} = a_n u + b_n x_n + c_n \left[\gamma_1 S x_n + \gamma_2 u_n + \gamma_3 z_n\right], \quad \forall n \geq 1, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0, 1]$ and $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = 1$.

Theorem 1 Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : C \to \mathcal{H}$ be two nonlinear mappings and $S : C \to C$ be a nonexpansive mapping. Assume that the Condition (Δ) is satisfied and

$$\Omega = GEP(C, Q, \Phi, \varphi) \cap F(S) \cap F(D) \neq \emptyset,$$

where the mapping D is defined by

$$D(x) = P_C[P_C(x - \rho Bx) - \lambda A P_C(x - \rho Bx)], \quad \forall x \in C.$$

Let $u \in C$ be fixed and $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{x_n\}$ be four sequences in C generated by Algorithm (I). If the following conditions are satisfied:

(i) $(I - \lambda A)$ and $(I - \rho B)$ are nonexpansive mappings, where ρ and λ are positive constants;

(ii) $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,

then the sequence $\{x_n\}$ defined by the Algorithm (I) converges strongly to a point $\widetilde{x} = P_\Omega u$. Moreover, if $\widetilde{y} = P_C(\widetilde{x} - \rho B\widetilde{x})$, then $(\widetilde{x}, \widetilde{y})$ is a solution to the problem (1.3).

Also, some applications of this main results are also presented (please kindly see appendix).

2.4. Narin Petrot, *Some existence theorems for nonconvex variational inequalities problems*,Abstract and Applied Analysis, Volume 2010, Article ID 472760, 9 pages.

In this part, by using nonsmooth analysis knowledge, we provide the conditions for existence solutions of the variational inequalities problems in nonconvex setting. We also show that the strongly monotonic assumption of the mapping may not need for existence of solutions. In fact, we have considered the following problem: find $x^*, y^* \in C$ such that

$$\begin{cases} y^* - x^* - \rho T y^* \in N_C^P(x^*), \\ x^* - y^* - \eta T x^* \in N_C^P(y^*), \end{cases} \tag{11}$$

where ρ and η are fixed positive real numbers, C is a closed subset of H and $T : C \to H$ is a mapping.

We are deal with the following concepts. For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly prox-regular with respect to r if for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(\overline{x})$, one has

$$\left\langle \frac{z}{\|z\|}, x - \overline{x} \right\rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \quad \forall x \in C.$$

We make the convention $\frac{1}{r} = 0$ for $r = +\infty$.

It is well-known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, for the case of $r = \infty$, the uniform r-prox-regularity K is equivalent to the convexity of K. Moreover, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets.

Theorem 1 Let C be an uniformly r-prox-regular closed subset of a Hilbert space H and $T : C \to H$ be a nonlinear mapping. Let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping,

T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and the following conditions are satisfied

(a) $M^{\rho,\eta}\delta_{T(C)} < \xi$, where $\delta_{T(C)} = \sup\{\|u-v\|; u, v \in T(C)\}$;

(b) there exists $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ such that

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \rho, \eta < \min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s \mu_2}\right\}, \tag{12}$$

where $M^{\rho,\eta} = \max\{\rho, \eta\}, t_s = \frac{r}{r-s}$ and $\zeta = \frac{\sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$.

Then the problem (11) has a solution.

Notice that, in the presented paper, an iterative method for finding the solution of problem (11) is also showed and some special cases are also discussed.

3. Variational inequalities problems on Banach spaces

Let E be a real Banach space with its topological dual E^*, and $\langle \cdot, \cdot \rangle$ be the generalized duality pairing between E and E^*. Let $CB(E^*)$ be the family of all nonempty bounded and closed subsets of E^*. The Hausdorff metric, $H(\cdot, \cdot)$, on $CB(E^*)$ is defined by

$$H(C, D) = \max\left\{\sup_{x \in C} d(x, D), \sup_{y \in D} d(C, y)\right\}, \quad \forall C, D \in CB(E^*).$$

3.1. **Poom Kumama, Narin Petrot and Rabian Wangkeeree,** *Existence and iterative approximation of solutions of generalized mixed quasi-variational-like inequality problem in Banach spaces*, Applied Mathematics and Computation 217 (2011) 7496–7503.

Let K be a nonempty convex subset of E, in this paper, we devote our study to a class of generalized mixed quasi-variational-like inequality problem, which is stated as follows:

Let $T, A : K \to CB(E^*)$ be two set-valued mappings. $N : E^* \times E^* \to E^*$ and $\eta : K \times K \to E$ be two single-valued mappings. Let $\varphi : E \times E \to$

$(-\infty, +\infty]$ be a real bi-function. For a given $w^* \in E^*$, we shall study the following problem :

$$GMQVLIP(T, A, N, \eta, \varphi) \begin{cases} \text{find } \ u \in K, x, y \in E^* \ \text{ such that } x \in T(u), y \in A(u) \\ \langle N(x, y) - w^*, \eta(v, u)\rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \ \ \forall v \in K. \end{cases} \quad (13)$$

In case of (13), we will denote by $(u, x, y) \in GMQVLIP(T, A, N, \eta, \varphi)$.

We have considered the following classes of mappings. Let $T, A : K \to CB(E^*)$ be two set-valued mappings. Let $N : E^* \times E^* \to E^*, \eta : K \times K \to K$ be mappings. Then

(i) T is said to be η-cocoercive with respect to the first argument of $N(\cdot, \cdot)$, if there exists a constant $\tau > 0$, such that

$$\langle N(x, \cdot) - N(x', \cdot), \eta(u, v)\rangle \geq \tau\|N(x, \cdot) - N(x', \cdot)\|^2, \ \ \forall u, v \in K, x \in T(u), x' \in T(v);$$

(ii) $N(\cdot, \cdot)$ is Lipschitz continuous in the second argument with respect to the set-valued mapping A, if there exists a constant $\alpha > 0$ such that

$$\|N(\cdot, y) - N(\cdot, y')\| \leq \alpha\|u - v\|, \ \ \forall u, v \in K, y \in A(u), y' \in A(v);$$

(iii) $N(\cdot, \cdot)$ is η-strongly monotone in the first argument with respect to the set-valued mapping T if there exists a constant $\xi > 0$ such that

$$\langle N(x, \cdot) - N(x', \cdot), \eta(u, v)\rangle \geq \xi\|u - v\|^2, \ \ \forall u, v \in K, x \in T(u), x' \in T(v).$$

Similarly, η-strongly monotone of $N(\cdot, \cdot)$ in the second argument with respect to the set-valued mapping A can be defined;

(iv) T is said to be H-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$H((T(u), T(v)) \leq \gamma\|u - v\|, \ \ \forall u, v \in K;$$

(v) η is Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|,$$

for any $u, v \in K$.

In this work, we have assume that $N : E^* \times E^* \to E^*, \eta : K \times K \to E$ be two mappings satisfying the following conditions $(\mathcal{A})$:

(a) $\eta(u, v) = \eta(u, z) + \eta(z, v)$ for each $u, v, z \in K$;

(b) for each fixed $(u, x, y) \in K \times E^* \times E^*, v \mapsto \langle N(x, y), \eta(u, v)\rangle$ is a concave function.

(c) for each fixed $v \in K$, the functional $(u, x, y) \mapsto \langle N(x, y), \eta(u, v)\rangle$ is weakly lower semi-continuous function from $K \times E^* \times E^*$ to $\mathbb{R}$, i. e.,

$$u_n \rightharpoonup u, x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y \text{ imply } \langle N(x, y), \eta(u, v)\rangle \leq \liminf_{n\to\infty} \langle N(x_n, y_n), \eta(u_n, v)\rangle.$$

Theorem 1(Existence theorem) Let E be a real reflexive Banach space with the dual space E^*, and K be a nonempty convex subset of E. Let $T, A : K \to CB(E^*)$ be two set-valued mappings. Let $N : E^* \times E^* \to E^*$, and $\eta : K \times K \to E$. Let $\varphi : E \times E \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous such that $\text{int}\{u \in K : \varphi(u, u) < \infty\} \neq \emptyset$ and $\varphi(u, \cdot)$ is proper convex, for each $u \in E$. Suppose that:

(i) T is η-cocoercive with respect to the first argument of $N(\cdot, \cdot)$ with constant τ;

(ii) η is Lipschitz continuous with constant $\delta > 0$;

(iii) $N(\cdot, \cdot)$ is Lipschitz continuous and η-strongly monotone in the second argument with respect to A with constant $\alpha > 0$ and $\beta > 0$, respectively.

If condition $(\mathcal{A})$ is satisfied, then $GMQVLIP(T, A, N, \eta, \varphi) \neq \emptyset$.

Also, in this paper, we have constructed an iterative method for finding the solution of considered problem.

4. Random fuzzy variational inequalities problems on Banach spaces

Throughout this part, let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and X be a separable real Banach space endowed with dual space X^*, the norm $\|.\|$ and the dual pair $\langle .,. \rangle$ between X and X^*. We denote by $\mathcal{B}(X)$, $CB(X)$ and $\hat{H}(.,.)$ the class of Borel σ-fileds in X, the family of all nonempty closed bounded subsets of X and the Hausdorff metric

$$\hat{H}(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\}$$

on $CB(X)$, respectively.

4.1. **Narin Petrot and Javad Balooee**, *A New Class of General Nonlinear Random Set-valued Variational Inclusion Problems Involving A-maximal m-relaxed η-accretive Mappings and Random Fuzzy Mappings in Banach Spaces*, Journal of Inequalities and Applications 2012, 2012:98.

In what follows, we denote the collection of all fuzzy sets on X by $\mathfrak{F}(X) = \{A | A : X \to [0,1]\}$. For any set K, a mapping $\mathcal{S}$ from K into $\mathfrak{F}(X)$ is called a *fuzzy mapping*. If $\mathcal{S} : K \to \mathfrak{F}(X)$ is a fuzzy mapping, then $\mathcal{S}(x)$, for any $x \in K$, is a fuzzy set on $\mathfrak{F}(X)$ (in the sequel, we denote $\mathcal{S}(x)$ by $\mathcal{S}_x$) and $\mathcal{S}_x(y)$, for any $y \in X$, is the degree of membership of y in $\mathcal{S}_x$. For any $A \in \mathfrak{F}(X)$ and $\alpha \in [0,1]$, the set

$$(A)_\alpha = \{x \in X : A(x) \geq \alpha\}$$

is called a *α-cut* set of A.

We have considered the following classes of mappings. A fuzzy mapping $\mathcal{S} : \Omega \to \mathfrak{F}(X)$ is called *measurable* if, for any $\alpha \in (0,1]$, $(\mathcal{S}(.))_\alpha : \Omega \multimap X$ is a measurable set-valued mapping. A fuzzy mapping $\mathcal{S} : \Omega \times X \to \mathfrak{F}(X)$ is called a *random fuzzy mapping* if, for any $x \in X$, $\mathcal{S}(.,x) : \Omega \to \mathfrak{F}(X)$ is a measurable fuzzy mapping.

Now, let us introduce our main considered problem.

Suppose that $\mathcal{S}, \mathcal{T}, \mathcal{P}, \mathcal{Q}, \mathcal{G} : \Omega \times X \to \mathfrak{F}(X)$ are random fuzzy mappings, $A, p : \Omega \times X \to X$ and $\eta : \Omega \times X \times X \to X$, $N : \Omega \times X \times X \times X \to X$ are random single-valued mappings. Further, let $a, b, c, d, e : X \to [0,1]$ be any mappings and $M : \Omega \times X \times X \multimap X$ be a random set-valued mapping such that, for each fixed $t \in \Omega$ and $z(t) \in X$, $M(t,.,z(t)) : X \multimap X$ be an A-maximal m-relaxed η-accretive mapping with $Im(p) \cap dom M(t,.,z(t)) \neq \emptyset$. Now, we consider the following problem:

For any element $h : \Omega \to X$ and any measurable function $\lambda : \Omega \to (0, +\infty)$, find measurable mappings $x, \nu, u, v, \vartheta, w : \Omega \to X$ such that for each $t \in \Omega$, $x(t) \in X$, $\mathcal{S}_{t,x(t)}(\nu(t)) \geq a(x(t))$, $\mathcal{T}_{t,x(t)}(u(t)) \geq b(x(t))$, $\mathcal{P}_{t,x(t)}(v(t)) \geq c(x(t))$, $\mathcal{Q}_{t,x(t)}(\vartheta(t)) \geq d(x(t))$, $\mathcal{G}_{t,x(t)}(w(t)) \geq e(x(t))$ and

$$h(t) \in N_t(\nu, u, v) + \lambda(t) M_t(p_t(x) - \vartheta, w), \quad \forall t \in \Omega. \tag{14}$$

The problem (14) is called *the general nonlinear random A-maximal m-relaxed η-accretive equation with random relaxed cocoercive mappings and random fuzzy mappings in Banach spaces.*

The *generalized duality mapping* $J_q : X \multimap X^*$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is single-valued. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

The *modulus of smoothness* of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called *uniformly smooth* if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

Further, a Banach space X is called *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad q > 1.$$

It is well-known that Hilbert spaces, L_p(or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth.

Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces,we have the following result. Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Theorem 1 Let X be a q-uniformly smooth Banach space, A, p, η, M, N, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, h, λ be the same as in the problem (14) and $S, T, P, Q, G : \Omega \times X \to CB(X)$ be five random set-valued mappings induced by $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, respectively. Further, suppose that

(a) p is (γ, ϖ)-relaxed cocoercive and π-Lipschitz continuous;

(b) A is r-strongly η-accretive and σ-Lipschitz continuous;

(c) η is τ-Lipschitz continuous;

(d) S, T, P, Q and G are ξ-$\hat{H}$-Lipschitz continuous, ζ-$\hat{H}$-Lipschitz continuous, ς-$\hat{H}$-Lipschitz continuous, ϱ-$\hat{H}$-Lipschitz continuous and ι-$\hat{H}$-Lipschitz continuous, respectively;

(e) N is ϵ-Lipschitz continuous in the second argument, δ-Lipschitz continuous in the third argument and κ-Lipschitz continuous in the fourth argument;

(f) There exist measurable functions $\mu : \Omega \to (0, +\infty)$ and $\rho : \Omega \to (0, +\infty)$ with $\rho(t) \in (0, \frac{r(t)}{\lambda(t)m(t)})$, for all $t \in \Omega$, such that

$$\|J^{\eta_t, M_t(.,x)}_{\rho(t)\lambda(t), A_t}(z(t)) - J^{\eta_t, M_t(.,y)}_{\rho(t)\lambda(t), A_t}(z(t))\| \leq \mu(t)\|x(t) - y(t)\|, \qquad \forall t \in \Omega, x(t), y(t), z(t) \in X \tag{15}$$

and

$$\begin{cases} \varphi(t) = \varrho(t) + \mu(t)\iota(t) + \sqrt[q]{1 - q\varpi(t) + (q\gamma(t) + c_q)\pi^q(t)} < 1, \\ \sigma(t)(\pi(t) + \varrho(t)) + \rho(t)(\epsilon(t)\xi(t) + \delta(t)\zeta(t) + \kappa(t)\varsigma(t)) \\ < \tau^{1-q}(t)(1 - \varphi(t))(r(t) - \rho(t)\lambda(t)m(t)). \end{cases} \tag{16}$$

Then there exists a set of measurable mappings $x^*, \nu^*, u^*, \upsilon^*, \vartheta^*, w^* : \Omega \to X$ which is a random solution of the problem (14).

Also, in this paper, we have constructed an iterative method for finding the solution of considered problem.

Out Put จากโครงการที่ได้รับจาก สกว.

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติจำนวน 8 ผลงาน โดยอยู่ในฐานข้อมูล ISI จำนวน 7 ผลงาน และอยู่ในฐานข้อมูล Scopus จำนวน 1 ผลงาน ได้แก่

1.1 Yeol Je Cho, Soawapak Hirunworakit, Narin Petrot, Set-Valued Fixed Points Theorems for Generalized contractive mappings without the Hausdorff metric, Applied Mathematics Letters 24 (2011) 1959–1967. (ISI)

1.2 Jittiporn Suwannawit and Narin Petrot, Common Fixed point theorems for hybrid generalized multivalued, Thai Journal of Mathematics, 9(2) (2011), 417–427. (Scopus)

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1.6 Narin Petrot, Some existence theorems for nonconvex variational inequalities problems, Abstract and Applied Analysis, Volume 2010, Article ID 472760, 9 pages. (ISI)

1.7 Poom Kumama, Narin Petrot and Rabian Wangkeeree, Existence and iterative approximation of solutions of generalized mixed quasi-variational-like inequality problem in Banach spaces, Applied Mathematics and Computation 217 (2011) 7496–7503. (ISI)

1.8 Narin Petrot and Javad Balooee, A New Class of General Nonlinear Random Set-valued Variational Inclusion Problems Involving A-maximal m-relaxed η-accretive Mappings and Random Fuzzy Mappings in Banach Spaces, Journal of Inequalities and Applications 2012, 2012:98. (ISI)

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการ และเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอนและมีเครือข่ายความร่วมมือสร้างกระแสความสนใจในวงกว้าง

3. อื่น ๆ: การเสนอผลงานในที่ประชุมวิชาการ

3.1 วันที่ 1 ก. พ.- 3 ก.พ. 2555

หัวข้อ: Variational inequality problems for fuzzy mappings on a class of nonconvex sets

ชื่อการประชุม: the Franco-Thai Symposium, Bangkok, Thailand,

3.2 วันที่ 2 ส.ค- 5 ส.ค. 2554

หัวข้อ: Fixed point theorems for contractive multi-valued mappings induced by generalized distances in metric spaces

ชื่อการประชุม : The seventh international conference on Nonlinear Analysis and Convex Analysis (NACA2011), Pukyong National University, Busan, Korea.

3.2 วันที่ 9 ก.ย- 12 ก.ย. 2553

หัวข้อ: Sufficient conditions for a solution of system of nonconvex variational inequalities involving non-monotone mappings

ชื่อการประชุม: The Second Asian Conference on Nonlinear Analysis and Optimization, at Royal Paradise Hotel & Spa Patong beach, Phuket, Thailand

ภาคผนวก

ภาคผนวก 1

Set-Valued Fixed Points Theorems for Generalized Contractive Mappings without The Hausdorff Metric

Yeol Je Cho, Soawapak Hirunworakit and Narin Petrot

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Set-valued fixed-point theorems for generalized contractive mappings without the Hausdorff metric

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ABSTRACT

In this paper, the concept of a set-valued contractive mapping is considered by using the idea of a generalized distance, such as the τ-distance, in metric spaces without using the concept of the Hausdorff metric. Furthermore, under some mild conditions, we provide the existence theorems for fixed-point problems of the considered mapping. Hence, our results can be viewed as a generalization and improvement of many recent results.

1. Introduction and preliminaries

Let (X, d) be a metric space and let 2^X, $CB(X)$, and $Cl(X)$ denote the collections of nonempty subsets of X, nonempty closed bounded subsets of X, and nonempty closed subsets of X, respectively. If $T : X \to 2^X$ is a mapping, then an element $x \in X$ is called a *fixed point* of T if $x \in T(x)$. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in X : x \in T(x)\}$.

Recall that the function H on $CB(X)$ defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for all $A, B \in CB(X)$ is called the *Hausdorff metric*, where $d(x, B) = \inf_{b\in B}\{d(x, b)\}$. By using the concept of the Hausdorff metric, Nadler [1] established the following result for fixed-point problems for a multi-valued contractive mapping in a complete metric space, which in turn is a generalization of the well-known Banach contraction principle [2].

Theorem 1.1 ([1]). *Let (X, d) be a complete space and let T be a mapping from X into $CB(X)$. Assume that there exists $\kappa \in (0, 1)$ such that*

$$H(T(x), T(y)) \le \kappa d(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in T(z)$.

Nadler's fixed-point theorem for multi-valued contractive mappings has been generalized in many directions and applied in nonlinear analysis (see [3–13,1,14–18]).

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In 1996, the concept of a w-distance on a metric space was introduced by Kada et al. [7] as follows.

Definition 1.2 ([7]). Let (X, d) be a metric space. A function $\omega : X \times X \to [0, \infty)$ is called a *w-distance* on X if the following conditions are satisfied:

(w_1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$ for all $x, y, z \in X$;
(w_2) a mapping $\omega(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous for each fixed $x \in X$;
(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply that $d(x, y) \leq \varepsilon$ for all $x, y, z \in X$.

They also gave some examples of the w-distance and, by using the concept of such a w-distance, they generalized Caristi's fixed-point theorem [3], Ekeland's variational principle [5], and Takahashi's nonconvex minimization theorem [17]. In particular, if (X, d) is a metric space, then the metric d is a w-distance on (X, d), which makes this class of great importance.

In 2009, Latif and Abdou [10] proved the following fixed-point theorem.

Theorem 1.3 ([10]). *Let (X, d) be a complete metric space with a w-distance ω. Let $T : X \to Cl(X)$ be a set-valued mapping satisfying the following conditions:*

(i) *there exists a function $\varphi : [0, \infty) \to [0, 1)$ and a function $\theta : [0, \infty) \to [c, 1)$, with $c > 0$ and θ nondecreasing, such that*

$$\varphi(t) < \theta(t), \limsup_{r \to t^+} \varphi(r) < \limsup_{r \to t^+} \theta(r)$$

for all $t \in [0, \infty)$;

(ii) *for any $x \in X$, there exists $y \in T(x)$ such that*

$$\theta(\omega(x, y))\omega(x, y) \leq W(x, T(x))$$

and

$$W(y, T(y)) \leq \varphi(\omega(x, y))\omega(x, y);$$

(iii) *the real-valued function f on X defined by $f(x) = W(x, T(x))$ is lower semi-continuous, where $W(u, K) = \inf_{y \in K} \omega(u, y)$.*

Then there exists $z \in X$ such that $f(z) = 0$. Further, if $\omega(z, z) = 0$, then $z \in F(T)$.

Note that, if we take $\varphi =: h < \kappa, h \in (0, 1)$, then we obtain the result presented by Latif and Abdou [9]. Moreover, if $\omega = d$, then Theorem 1.3 reduces to a fixed-point theorem presented by Ćirić [4], Klim and Wardowski [8], Latif and Albar [11], and Feng and Liu [6].

Evidently, Theorem 1.3 generalizes and improves a number of well-known fixed-point results given by many authors. Thus, in this paper, we are interested in providing some fixed-point theorems related to Theorem 1.3.

To do so, let us recall the concept of a τ-distance on a metric space, which is a generalization of the w-distance, introduced by Suzuki [14], as follows.

Definition 1.4 ([14]). Let X be a metric space with metric d. Then a function p from $X \times X$ into $[0, \infty)$ is called the *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the followings are satisfied:

$(\tau 1)$ $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
$(\tau 2)$ $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable;
$(\tau 3)$ $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply that $p(w, x) \leq \liminf_{n\to\infty} p(w, x_n)$ for all $w \in X$;
$(\tau 4)$ $\lim_{n\to\infty} \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_{n\to\infty} \eta(x_n, t_n) = 0$ imply that $\lim_{n\to\infty} \eta(y_n, t_n) = 0$;
$(\tau 5)$ $\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

In this paper, we will develop some fixed-point theorems by using the concept of the τ-distance. In order to obtain fixed-point theorems by using the τ-distance, the following concepts and lemmas (see [15]) are crucial.

Definition 1.5. Let (X, d) be a metric space and let p be a τ-distance on X. Then a sequence $\{x_n\}$ in X is called p-Cauchy if there exists a function $\eta : X \times [0, \infty) \to [0, \infty)$ satisfying $(\tau 2)-(\tau 5)$ and a sequence $\{z_n\}$ in X such that $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$.

Lemma 1.6. *Let (X, d) be a metric space and let p be a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a p-Cauchy sequence.*

Lemma 1.7. *Let (X, d) be a metric space and let p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $\lim_{n\to\infty} \sup\{p(x_n, y_m) : m > n\} = 0$, then $\{y_n\}$ is a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

Let (X, d) be a metric space. For any fixed $x_0 \in X$, a sequence $\{x_n\} = \{x_0, x_1, x_2, \ldots\} \subset X$ such that $x_{n+1} \in T(x_n)$ is called an *orbit* of x_0 with respect to mapping $T : X \to 2^X$. We will denote by $\mathcal{O}(T, x_0)$ the set of all orbital sequences of x_0 with respect to mapping T.

Definition 1.8. Let (X, d) be a metric space and let $x_0, z \in X$. A mapping $f : X \to \mathbb{R}$ is said to be *T-orbitally lower semi-continuous* at z with respect to x_0 if $\{x_n\} \in \mathcal{O}(T, x_0)$ and $x_n \to z$ imply that $f(z) \le \liminf_{n\to\infty} f(x_n)$.

2. Main results

Let (X, d) be a metric space and let A be a subset of X. If p is a τ-distance on X and $x \in X$, from now on, we define $D_p(x, A) = \inf\{p(x, y) | y \in A\}$.

In this section, inspired by Latif and Abdou [10], we now give some results which generalize Theorem 1.3.

Theorem 2.1. *Let (X, d) be a metric space and let $T : X \to Cl(X)$ be a set-valued mapping. If there exists a function $\varphi : [0, \infty) \to [0, 1)$ and a nondecreasing function $\theta : [0, \infty) \to [c, 1)$, $c > 0$, such that*

$$\varphi(t) < \theta(t) \tag{2.1}$$

for all $t \in [0, \infty)$ and

$$\limsup_{t\to r^+} \varphi(t) < \limsup_{t\to r^+} \theta(t) \tag{2.2}$$

for all $r \in [0, \infty)$, and there exists a τ-distance p on X such that, for any $x \in X$, there exists $y \in T(x)$ satisfying

$$\theta(p(x, y))p(x, y) \le D_p(x, T(x)) \tag{2.3}$$

and

$$D_p(y, T(y)) \le \varphi(p(x, y))p(x, y), \tag{2.4}$$

then we have the following.

(a) *For each $x_0 \in X$, there exists an orbit $\{x_n\} \in \mathcal{O}(T, x_0)$ such that $\{D_p(x_n, T(x_n))\}$ is decreasing to zero and the sequence $\{x_n\}$ is a Cauchy sequence.*
(b) *If $\{x_n\}$ converges to z and the function $f(x) := D_p(x, T(x))$ is T-orbitally lower semi-continuous at z with respect to x_0, then $z \in F(T)$. Moreover, if $T(z) = z$, then $p(z, z) = 0$.*

Proof. To prove (a), let $x_0 \in X$ be given. First, we show that there exists a sequence $\{x_0, x_1, x_2, \ldots\}$ in (X, d) such that $x_{n+1} \in T(x_n)$ and $\{D_p(x_n, T(x_n))\}$ is a decreasing sequence that converges to zero. Indeed, by (2.3) and (2.4), we can choose $x_1 \in T(x_0)$ such that

$$\theta(p(x_0, x_1))p(x_0, x_1) \le D_p(x_0, T(x_0)) \tag{2.5}$$

and

$$D_p(x_1, T(x_1)) \le \varphi(p(x_0, x_1))p(x_0, x_1). \tag{2.6}$$

By using (2.5) and (2.6), we get

$$D_p(x_1, T(x_1)) \le \frac{\varphi(p(x_0, x_1))}{\theta(p(x_0, x_1))} D_p(x_0, T(x_0)). \tag{2.7}$$

Now, define a function $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{\varphi(t)}{\theta(t)}$$

for all $t \in [0, \infty)$. Notice that, from (2.1) and (2.2), it follows that

$$\psi(t) < 1 \tag{2.8}$$

for all $t \in [0, \infty)$, and

$$\limsup_{t\to r^+} \psi(t) < 1 \tag{2.9}$$

for all $r \in [0, \infty)$. Moreover, by (2.7), we also have

$$D_p(x_1, T(x_1)) \le \psi(p(x_0, x_1))D_p(x_0, T(x_0)).$$

Again, by using (2.3) and (2.4), we can choose $x_2 \in T(x_1)$ such that

$$\theta(p(x_1, x_2))p(x_1, x_2) \le D_p(x_1, T(x_1))$$

and

$$D_p(x_2, T(x_2)) \le \varphi(p(x_1, x_2))p(x_1, x_2).$$

Moreover, by the definition of ψ, we have

$$D_p(x_2, T(x_2)) \le \psi(p(x_1, x_2))D_p(x_1, T(x_1)).$$

Continuing this process and denoting $p_n = p(x_n, x_{n+1})$ and $D_n = D_p(x_n, T(x_n))$, we can obtain an iterative sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} \in T(x_n)$,

$$\theta(p_n)p_n \le D_n, \tag{2.10}$$

and

$$D_{n+1} \le \varphi(p_n)p_n \tag{2.11}$$

for all $n \ge 0$, and so, from (2.10) and (2.11),

$$D_{n+1} \le \psi(p_n)D_n. \tag{2.12}$$

Thus, it follows from (2.12) and (2.8) that

$$D_{n+1} < D_n$$

for all $n \ge 0$; that is, we have that $\{D_n\}$ is a strictly monotone decreasing sequence. Moreover, since θ is a nondecreasing function, we know that $\{p_n\}$ is also a strictly monotone decreasing sequence. Consequently, there exist $\delta \ge 0$ and $\beta \ge 0$ such that

$$\lim_{n\to\infty} D_n = \delta \quad \text{and} \quad \lim_{n\to\infty} p_n = \beta.$$

Furthermore, it follows from (2.12) that

$$\delta \le (\limsup_{n\to\infty} \psi(p_n))\delta = (\limsup_{p_n\to\beta} \psi(p_n))\delta.$$

Since $\limsup_{p_n\to\beta} \psi(p_n) < 1$, we conclude that $\delta = 0$.

Next, we show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Let us consider a behavior of the sequence $\{p_n\}_{n=0}^{\infty}$. Since $0 < c \le \theta(t)$ for all $t \in [0, \infty)$, it follows from (2.10) that $cp_n \le \theta(p_n)p_n \le D_n$, and hence

$$p_n \le \frac{1}{c}D_n. \tag{2.13}$$

Now, put $\alpha = \limsup_{p_n\to 0^+} \psi(p_n)$. Then, by (2.9), we can choose a real number q such that $q \in (\alpha, 1)$, and so there exists a positive integer n_1 such that $\psi(p_n) < q$ for all $n \ge n_1$. Thus, from (2.12), we have $D_n \le qD_{n-1}$ for all $n \ge n_1$. This implies that

$$D_m \le q^{m-n}D_n \tag{2.14}$$

for all $m > n \ge n_1 + 1$. Moreover, from (2.13) and (2.14), we get

$$p_m \le \frac{1}{c}q^{m-n}D_n \tag{2.15}$$

for all $m > n > n_1 + 1$. This implies that

$$\sum_{k=n}^{m} p_k \le \frac{1}{c}\sum_{k=n}^{m} q^{k-n}D_n \le \frac{1}{c}\left(\frac{1}{1-q}\right)D_n$$

for all $m > n \ge n_1 + 1$. Thus, using this together with $\lim_{n\to\infty} D_n = 0$ and Lemma 1.6, we know that $\{x_n\}$ is a p-Cauchy sequence. Consequently, from Lemma 1.7, we see that (a) is followed.

To prove (b), assume that $\lim_{n\to\infty} x_n = z$ and that the function $f(x) := D_p(x, T(x))$ is T-orbitally lower semi-continuous at z with respect to x_0. Thus it follows that

$$0 \le D_p(z, T(z)) = f(z) \le \liminf_{n\to\infty} f(x_n) \le \lim_{n\to\infty} D_n = 0.$$

Thus $f(z) = 0$. Consequently, there exists a sequence $\{z_n\} \subset T(z)$ such that $\lim_{n\to\infty} p(z, z_n) = 0$. Therefore,

$$0 \le \lim_n \sup\{p(x_n, z_m) : m > n\} \le \lim_n \sup\{p(x_n, z) + p(z, z_n) : m > n\} = 0.$$

This implies, by Lemma 1.7 and the closedness of $T(z)$, that $z \in T(z)$.

If $T(z) = z$ then, by using (2.4), we see that $0 \leq p(z,z) = D_p(z,Tz) \leq \varphi(p(z,z))p(z,z)$. Since $\varphi([0,\infty)) \subset [0,1)$, we must have $p(z,z) = 0$. This completes the proof. □

Immediately, from Theorem 2.1, we can obtain the following result.

Corollary 2.2. *Let (X,d) be a complete metric space. Suppose that $T : X \to Cl(X)$ satisfies all the conditions of Theorem 2.1. If a real-valued function $f(x) = D_p(x, T(x))$ is lower semi-continuous on (X,d), then there exists $z \in X$ such that $z \in F(T)$.*

Remark 2.3. Since the class of τ-mappings is wider than the class of w-mappings, Corollary 2.2 can be viewed as a generalization of Theorem 1.3. Moreover, we do not need the assumption $\tau(z,z) = 0$, which has been proposed in Theorem 1.3.

Next, we provide another generalization of Theorem 1.3.

Theorem 2.4. *Let (X,d) be a complete metric space. Suppose that $T : X \to Cl(X)$ is a set-valued mapping of X into itself. If there exists a function $\varphi : [0,\infty) \to [0,1)$ such that*

$$\limsup_{r \to t^+} \varphi(r) < 1 \tag{2.16}$$

for any $t \in [0,\infty)$, and there exists a τ-distance p on X such that, for any $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x,y) \leq (2 - \varphi(p(x,y)))D_p(x,T(x)) \tag{2.17}$$

and

$$D_p(y,T(y)) \leq \varphi(p(x,y))p(x,y), \tag{2.18}$$

then we have the following.

(a) *For any $x_0 \in X$, there exist an orbit $\{x_n\} \in \mathcal{O}(T,x_0)$ and $z \in X$ such that $\lim_{n\to\infty} x_n = z$.*
(b) *If the function $f(x) := D_p(x,T(x))$ is T-orbitally lower semi-continuous at z with respect to x_0, then $z \in F(T)$. Moreover, if $T(z) = z$, then $p(z,z) = 0$.*

Proof. (a) First, since $\varphi(p(x,y)) < 1$ for all $x, y \in X$, it follows that $2 - \varphi(p(x,y)) > 1$ for all $x, y \in X$. Let $x_0 \in X$ be any initial point. Then, by (2.17) and (2.18), there exists $x_1 \in T(x_0)$ such that

$$p(x_0,x_1) \leq (2 - \varphi(p(x_0,x_1)))D_p(x_0,T(x_0)) \tag{2.19}$$

and

$$D_p(x_1,T(x_1)) \leq \varphi(p(x_0,x_1))p(x_0,x_1). \tag{2.20}$$

Thus, it follows from (2.19) and (2.20) that

$$D_p(x_1,T(x_1)) \leq \varphi(p(x_0,x_1))(2 - \varphi(p(x_0,x_1)))D_p(x_0,T(x_0)). \tag{2.21}$$

Now, define a function $\psi : [0,\infty) \to [0,+\infty)$ by

$$\psi(t) = \varphi(t)(2 - \varphi(t)) \tag{2.22}$$

for any $t \in [0,\infty)$. Notice that $\varphi(t) < 1$ and $\limsup_{r\to t^+} \varphi(r) < 1$ for any $t \in [0,\infty)$. This gives

$$\psi(t) = \varphi(t)(2 - \varphi(t)) = 1 - (1 - \varphi(t))^2 < 1 \tag{2.23}$$

and

$$\limsup_{r \to t^+} \psi(r) < 1 \tag{2.24}$$

for any $t \in [0,\infty)$. Moreover, by (2.21) and (2.22), we can write

$$D_p(x_1,T(x_1)) \leq \psi(p(x_0,x_1))D_p(x_0,T(x_0)). \tag{2.25}$$

Next, again by using (2.17), (2.18) and (2.22), we can find $x_2 \in T(x_1)$ such that

$$p(x_1,x_2) \leq (2 - \varphi(p(x_1,x_2)))D_p(x_1,T(x_1))$$

and

$$D_p(x_2,T(x_2)) \leq \psi(p(x_1,x_2))D_p(x_1,T(x_1)).$$

Continuing this process, we can choose an iterative sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} \in T(x_n)$,

$$p(x_n, x_{n+1}) \leq (2 - \varphi(p(x_n, x_{n+1})))D_p(x_n, T(x_n)), \tag{2.26}$$

and

$$D_p(x_{n+1}, T(x_{n+1})) \leq \psi(p(x_n, x_{n+1}))D_p(x_n, T(x_n)) \tag{2.27}$$

for all $n \geq 0$.

From now on, put $p_n = p(x_n, x_{n+1})$ and $D_n = D_p(x_n, T(x_n))$ for all $n \geq 0$. Then, from (2.26) and (2.27), and $\varphi(t) < 1$, for all $t \geq 0$, we get

$$D_{n+1} \leq \psi(p_n)D_n \tag{2.28}$$

and

$$D_n \leq p_n \leq 2D_n. \tag{2.29}$$

Furthermore, by (2.23) and (2.28), we know that $\{D_n\}_{n=0}^{\infty}$ is a strictly decreasing sequence of nonnegative real numbers. Therefore, there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} D_n = \delta. \tag{2.30}$$

Thus, by (2.29), we see that the sequence $\{p_n\}_{n=0}^{\infty}$ is also bounded, and so there exists $\beta \geq \delta$ such that

$$\liminf_{n\to\infty} p_n = \beta. \tag{2.31}$$

Now, we claim that $\delta = 0$. Consider the following possible two cases.

Case 1: If $\beta > \delta$, then, from (2.30) and (2.31), we can find a positive integer n_0 such that

$$\delta \leq D_n \leq \delta + \frac{\beta - \delta}{4} \tag{2.32}$$

and

$$\beta - \frac{\beta - \delta}{4} < p_n \tag{2.33}$$

for all $n \geq n_0$. Thus, by using (2.32), (2.33) and (2.26), we have

$$\delta + 3\left(\frac{\beta - \delta}{4}\right) = \beta - \frac{\beta - \delta}{4} < p_n \leq (2 - \varphi(p_n))D_n \leq (2 - \varphi(p_n))\left(\delta + \frac{\beta - \delta}{4}\right)$$

for all $n \geq n_0$. This gives

$$1 + \frac{2(\beta - \delta)}{3\delta + \beta} < 1 + (1 - \varphi(p_n))$$

for all $n > n_0$, which implies that

$$-(1 - \varphi(p_n))^2 < -\left[\frac{2(\beta - \delta)}{3\delta + \beta}\right]^2$$

for all $n > n_0$. Thus we have

$$\psi(p_n) = 1 - (1 - \varphi(p_n))^2 < 1 - \left[\frac{2(\beta - \delta)}{3\delta + \beta}\right]^2 =: h$$

for all $n \geq n_0$. Thus, it follows from (2.28) that

$$D_{n+1} \leq hD_n \tag{2.34}$$

for all $n \geq n_0$. Consequently, from (2.32) and (2.34), we obtain

$$\delta \leq D_{n_0+k} \leq hD_{n_0+k-1} \leq h^2 D_{n_0+k-2} \leq \cdots \leq h^k D_{n_0} \leq h^k\left(\delta + \frac{\beta - \delta}{4}\right) \tag{2.35}$$

for all $k \geq 1$. Since $h \in (0, 1)$, we have $\lim_{k\to\infty} h^k = 0$. Using this and (2.35), we have $\delta = 0$.

Case 2: If $\beta = \delta$, then, from (2.31), we can find a subsequence $\{p_{n_k}\}_{k=0}^{\infty}$ of $\{p_n\}$ such that

$$\lim_{k\to\infty} p_{n_k} = \delta.$$

Thus, by (2.24), it follows that

$$\limsup_{p_{n_k}\to\delta^+} \psi(p_{n_k}) < 1. \tag{2.36}$$

Also, from (2.28), we have

$$D_{n_k+1} \le \psi(p_{n_k})D_{n_k}.$$

Thus, it follows from (2.30) that

$$\delta = \lim_{k\to\infty} D_{n_k+1} \le \limsup_{k\to+\infty}(\psi(p_{n_k})D_{n_k}) = (\limsup_{p_{n_k}\to\delta^+}\psi(p_{n_k}))\delta.$$

Since $\limsup_{p_{n_k}\to\delta^+}\psi(p_{n_k}) < 1$, this inequality implies that $\delta = 0$. Therefore, from Cases 1 and 2, we conclude that

$$\lim_{n\to\infty} D_n = 0, \tag{2.37}$$

and so our claim is proved.

Now, using (2.24), (2.28), (2.29) and (2.37), as in the proof of Theorem 2.1, we know that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

(b) The proof is similar to that of Theorem 2.1. □

Remark 2.5. Theorem 2.4 recovers a result presented by Latif and Abdou [10].

As a special case of Theorem 2.4, we can obtain the result presented by Ćirić [4] as follows.

Theorem 2.6 ([4]). *Let (X, d) be a complete space. Suppose that $T : X \to Cl(X)$ is a set-valued mapping of X into itself. If there exists a function $\varphi : [0,\infty) \to [0,1)$ such that*

$$\limsup_{r\to t^+}\varphi(r) < 1 \tag{2.38}$$

for any $t \in [0,\infty)$ and, for any $x \in X$, there exists $y \in T(x)$ satisfying

$$d(x,y) \le (2-\varphi(d(x,y)))D(x,T(x)) \tag{2.39}$$

and

$$D(y,T(y)) \le \varphi(d(x,y))d(x,y), \tag{2.40}$$

then T has a fixed point in X provided that $f(x) = D(x,T(x))$ is lower semi-continuous.

In [10], the authors give an example showing that Theorem 1.3 is a genuine generalization of the result of Theorem 2.6. Here, we provide another one.

Example 2.7. Let $X = [0,\infty)$ and let $d : X\times X \to [0,\infty)$ be a usual metric. Let $T : X \to Cl(X)$ be defined by

$$T(x) = \begin{cases} \left\{\frac{1}{2}x^2\right\}, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right]; \\ \left\{\frac{17}{96}, \frac{1}{4}\right\}, & \text{if } x = \frac{15}{32}; \\ \left[0, \frac{1}{4}\right] \cup \left\{\frac{2x-1}{2}\right\}, & \text{if } x \in (1,\infty). \end{cases}$$

Now, we show that the given mapping T does not satisfy the assumptions of Theorem 2.6. To do this, let us consider a point $x = \frac{3}{2}$. Then we have $T(x) = \left[0, \frac{1}{4}\right] \cup \{1\}$, and it follows that $D(x, T(x)) = \frac{1}{2}$. Now, let $\varphi : [0,\infty) \to [0,1)$ be any real-valued function. Notice that only the real number $y = 1 \in T(x)$ satisfies (2.39) and, consequently,

$$D(y, Ty) = d\left(1, \frac{1}{2}\right) = d\left(\frac{3}{2}, 1\right) = d(x,y).$$

Therefore, since $\varphi([0,\infty)) \subset [0,1)$, we see that (2.40) cannot be satisfied.

On the other hand, we show that T satisfies all hypotheses of our Theorem 2.4. Define now a function $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) = \begin{cases} \frac{8}{5}t, & \text{if } x \in \left[0, \frac{7}{24}\right) \cup \left(\frac{7}{24}, \frac{1}{2}\right); \\ \frac{5}{8}, & \text{if } t = \frac{7}{24}; \\ \frac{4}{5}, & \text{if } t \in \left[\frac{1}{2}, \infty\right). \end{cases}$$

Then, obviously, such a function φ satisfies (2.16) of Theorem 2.4.

Further, let us define a function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \begin{cases} d(x, y), & \text{if } \{x, y\} \subset [0, 1]; \\ 1, & \text{if } \{x, y\} \not\subset [0, 1]. \end{cases}$$

It follows that p is a w-distance on X, and hence, it is a τ-distance (see [16]).

We consider the following cases.

Case 1: For $x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right]$, we have $T(x) = \{\frac{1}{2}x^2\}$. Consequently, for $y = \frac{1}{2}x^2$, we get

$$\begin{aligned} D_p(y, T(y)) &= p\left(\frac{1}{2}x^2, \frac{1}{8}x^4\right) \\ &= \frac{1}{2}\left(x + \frac{1}{2}x^2\right)\left(x - \frac{1}{2}x^2\right) \\ &= \frac{1}{2}\left(x + \frac{1}{2}x^2\right)p(x, y) \\ &\le \frac{8}{5}\left(x - \frac{1}{2}x^2\right)p(x, y) \\ &= \varphi(p(x, y))p(x, y). \end{aligned}$$

Moreover, we have

$$p(x, y) = D_p(x, Tx) \le (2 - \varphi(p(x, y)))D_p(x, Tx).$$

Case 2: Let $x = \frac{15}{32}$. For $y = \frac{17}{96} \in T(x)$, we have

$$p(x, y) = \frac{7}{24} < \left(2 - \frac{5}{8}\right)\frac{7}{32} = (2 - \varphi(p(x, y)))D_p(x, Tx)$$

and

$$D_p(y, T(y)) = p\left(\frac{17}{96}, \frac{1}{2} \cdot \frac{(17)^2}{(96)^2}\right) < \frac{17}{96} < \frac{5}{8} \cdot \frac{7}{24} = \varphi(p(x, y))p(x, y).$$

Case 3: Let $x \in (1, \infty)$. Notice that $D_p(x, Tx) = 1$. If we now choose $y = \frac{1}{4} \in T(x)$, then

$$p(x, y) = 1 < \left(2 - \frac{4}{5}\right)(1) = (2 - \varphi(1))(1) = (2 - \varphi(p(x, y)))D_p(x, Tx)$$

and

$$D_p(y, T(y)) = p\left(\frac{1}{4}, \frac{1}{2} \cdot \frac{1}{16}\right) = p\left(\frac{1}{4}, \frac{1}{32}\right) = \frac{7}{32} < \left(\frac{4}{5}\right)(1) = \varphi(p(x, y))p(x, y).$$

Therefore, from above three cases, we see that (2.17) and (2.18) of Theorem 2.4 are satisfied.

Moreover, we have

$$f(x) = D_p(x, T(x)) = \begin{cases} x - \frac{1}{2}x^2, & x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right]; \\ \frac{7}{32}, & \text{if } x = \frac{15}{32}; \\ 1, & \text{if } x \in (1, \infty), \end{cases}$$

which is a lower semi-continuous function. Therefore, all assumptions of Theorem 2.4 are satisfied. In fact, we can check that $F(T) = \{0\}$.

Remark 2.8. We do not use the concept of the Hausdorff metric in the proofs of Theorems 2.1 and 2.4.

3. Conclusion

We note that the results presented by Latif and Abdou [10] are interesting and important. Therefore, in this paper, we have considered and improved their result, Theorem 1.3. In particular, we have been interested in considering and proving the main results by using concepts of the generalized distance, namely the τ-distance. Hence, the results presented in this paper are general and, consequently, they can be applied in various ways.

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Common Fixed Point Theorem for Hybrid Generalized Multivalued[1]

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Abstract : In this work, the common fixed point theorems for a pair of hybrid generalized multivalued φ-weak contraction are proven. Consequently, since the concept of hybrid generalized multivalued φ-weak contraction includes almost concepts of the generalizations of Banach contraction principle as special cases, our results can be viewed as a refinement and improvement of the previously known results for metric fixed-point theory.

Keywords : Common fixed points; Hybrid generalized multivalued φ-weak contractions; Bianchini-Grandolfi gauge function; Hausdorff pseudometric.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction and Preliminaries

Let E be a complete metric space with distance $d(\cdot,\cdot)$. Let 2^E denote the family consisting of all nonempty subsets of E. We define the Hausdorff pseudometric, $H : 2^E \times 2^E \to [0,\infty]$ by

$$H(A,B) = \max\{D(a,B), D(A,b)\},$$

where $D(a,B) = \inf_{b\in B} d(a,b)$, $\quad D(A,b) = \inf_{a\in A} d(a,b)$.

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Definition 1.1. Let E be a metric space. A subset $C \subset E$ is said to be *approximative* if the multivalued mapping

$$\mathcal{P}_C(x) = \{c \in C : d(x,c) = D(x,C)\}, \quad \forall x \in E$$

has nonempty values. The multivalued mapping $T : E \to 2^E$ is said to have *approximative values* if $T(x)$ is approximative for each $x \in E$.

Let $\propto \in (0,\infty], \mathcal{R}^+_\propto = [0,\propto)$. Let $\varphi : \mathcal{R}^+_\propto \to [0,\infty)$ satisfy

(i) $\varphi(t) < t$ for each $t \in (0,\propto)$;

(ii) φ is nondecreasing on $\mathcal{R}^+_\propto$;

(iii) φ is upper-semicontinuous.

Define $\Phi[0,\propto) = \{\varphi : \varphi \text{ satisfies (i)-(iii) above}\}$.

From now on, for a metric space E, we let $\Gamma = \sup\{d(x,y) : x,y \in E\}$ and set $\propto = \Gamma$ if $\Gamma = \infty$, and $\propto > \Gamma$ if $\Gamma < \infty$.

Definition 1.2. Let E be a matric space. Suppose that $S,T : E \to 2^E$ and $\varphi \in \Phi[0,\propto)$ satisfy

$$H(Sx,Ty) \le \varphi(\rho(x,y)),$$

for each $x,y \in E$, where

$$\rho(x,y) = \max\left\{d(x,y), D(Sx,x), D(Ty,y), \frac{1}{2}\left[D(y,Sx) + D(x,Ty)\right]\right\}.$$

Then the pair S,T is called the *hybrid generalized multivalued φ-weak contraction mapping*.

Remark 1.3. *Let E be a Banach algebra with the norm $\|\cdot\|$ and the metric $d(\cdot,\cdot)$ generated by it. In Definition 1.2, let $\rho(x,y) = d(x,y)$; so*

$$H(Tx,Ty) \le \varphi(d(x,y))$$

for all $x,y \in E$. Then the multivalued mapping T is called a nonlinear D-contraction with a contraction function φ (see [1, 2]). In addition, let $\varphi(t) = kt$ with $k > 0$ and $\rho(x,y) = d(x,y)$; then

$$H(Tx,Ty) \le \varphi(d(x,y))$$

for all $x,y \in E$. In this case the mapping T is nothing but the multivalued Lipschitz operator defined by [3]. Moreover, if $0 < k < 1$ then the mapping T is called a multivalued contraction on E which was first studied by Markin [4] and Nadler [5].

During the last few decades, since the pioneering works of Markin [4] and Nadler [5], an extensive literature has been developed, consisting in many theorems which deal with fixed points for multi-valued mappings (see [6, 7, 8, 9, 10]), or may be related to various classes of φ-contractions, which are obtained for different collection of properties of the function φ (see for example, [11, 12, 13]), especially the monograph of Rus [14, 15], for the good survey and several still open problems. Equally important is the concept of hybrid contractive mapping of the metric fixed-point theory which have been obtained by mathematical researcher, for example [16, 17, 18, 19, 20].

Motivated and spirted by the research going on this field, in this work we prove that there is a common fixed point of hybrid generalized multivalued φ-weak contractions S, T on complete metric spaces E. Since the concept of hybrid generalized multivalued φ-weak contraction includes almost concepts of the generalization of Banach contraction principle as special cases (both singlevalued and multivalued settings), results obtained in this paper continue to hold for those problems. Our results can be viewed as a refinement and improvement of the previously known results for metric fixed-point theory. To reach the goal, we also need the following concepts:

Let J denotes an interval on $[0,\infty)$ containing 0, that is an interval of the form $[0,r], [0,r)$ or $[0,\infty)$, and we use the abbreviation φ^n for the nth iterate of a function φ.

Definition 1.4. A nondecreasing function $\varphi : J \to J$ is said to be a *Bianchini-Grandolfi gauge function* [21] *on* J if $\Sigma_{n=0}^{\infty}\varphi^n(t) < \infty$ for all $t \in J$.

As for the investigations of the Bianchini-Grandolfi gauge function we also refer to [22]. The following lemma is quite important one.

Lemma 1.5 ([23])**.** *Let E be a metric space and B be a nonempty subset of E. Then $D(x,B) \le d(x,y) + D(y,B)$, for any $x, y \in E$.*

2 Common Fixed Point Theorems

Theorem 2.1. *Let (E,d) be a complete metric space. Let S,T be a pair of hybrid generalized multivalued φ-weak contractions on E. Assume that S,T have the approximative values and $\varphi_{|_J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}_{\propto}^{+}$. If there is $x \in E$ such that either $D(x,Sx) \in J$ or $D(x,Tx) \in J$ then the mappings S and T have a common fixed point $u \in E$.*

Proof. Without loss of generality, we will assume that there is $u_0 \in E$ such that $D(u_0, Su_0) \in J$. Take $u_0 \in E$, since Su_0 is approximative it follows that there exists $u_1 \in Su_0$ such that $d(u_0,u_1) = D(u_0,Su_0)$. Next, since Tu_1 is approximative, there exists $u_2 \in Tu_1$ such that $d(u_1,u_2) = D(u_1,Tu_1)$. Moreover,

$$d(u_1,u_2) = D(u_1,Tu_1) \le \sup_{x\in Su_0} D(x,Tu_1) \le H(Su_0,Tu_1).$$

It follows that

$$\begin{aligned}
d(u_1,u_2) &\le H(Su_0,Tu_1) \le \varphi(\rho(u_0,u_1)) \\
&= \varphi\left(\max\left\{d(u_0,u_1), D(u_1,Tu_1), D(u_0,Su_0), \frac{1}{2}\left[D(u_0,Tu_1)+D(u_1,Su_0)\right]\right\}\right) \\
&\le \varphi\left(\max\left\{d(u_0,u_1), d(u_1,u_2), d(u_0,u_1), \frac{1}{2}\left[d(u_0,u_1)+d(u_1,u_2)\right]\right\}\right) \\
&\le \varphi\left(\max\{d(u_0,u_1), d(u_1,u_2)\}\right). \qquad (2.1)
\end{aligned}$$

Write $\omega = \max\{d(u_0,u_1), d(u_1,u_2)\}$. Observe that, if $\omega = 0$ then $u_0 = u_1 = u_2$ and it follows that $u_0 = u_1 \in Su_0$ and $u_0 = u_2 \in Tu_1 = Tu_0$, i.e., u_0 is a common fixed point of mappings S and T, and then our proof is completed. On the other hand, if $0 < \omega = d(u_1,u_2)$ then using $\varphi(t) < t$ for $t \in (0,\propto)$, from (2.1) we have

$$d(u_1,u_2) \le \varphi(d(u_1,u_2)) < d(u_1,u_2)$$

which is a contradiction. Therefore, $\omega = d(u_0,u_1)$ and from (2.1) we obtain

$$d(u_1,u_2) \le \varphi(\rho(u_0,u_1)) \le \varphi(d(u_0,u_1)) < d(u_0,u_1). \qquad (2.2)$$

We continue the procedure of constructing u_n inductively, we can choose a sequence $\{u_n\}$ in E such that for all $n \ge 1$, $u_{2n} \in Tu_{2n-1}, u_{2n+1} \in Su_{2n}$ and

$$d(u_{2n},u_{2n+1}) = D(u_{2n},Su_{2n}),\ d(u_{2n+1},u_{2n+2}) = D(u_{2n+1},Tu_{2n+1}).$$

Moreover,

$$D(u_{2n},Su_{2n}) \le \sup_{x\in Tu_{2n-1}} D(x,Su_{2n}) \le H(Tu_{2n-1},Su_{2n}),$$

and

$$D(u_{2n+1},Tu_{2n+1}) \le \sup_{x\in Tu_{2n+1}} D(x,Su_{2n}) \le H(Su_{2n},Tu_{2n+1})$$

for all $n \ge 1$. Therefore, by using an argument similar to the above we get,

$$d(u_{2n},u_{2n+1}) \le \varphi(\rho(u_{2n-1},u_{2n})) < d(u_{2n-1},u_{2n}) \qquad (2.3)$$

and

$$d(u_{2n+1},u_{2n+2}) \le \varphi(\rho(u_{2n},u_{2n+1})) < d(u_{2n},u_{2n+1}) \qquad (2.4)$$

for all $n \ge 1$. Thus, from (2.3) and (2.4), we get

$$d(u_n,u_{n+1}) \le \varphi(\rho(u_{n-1},u_n)) < d(u_{n-1},u_n) \qquad (2.5)$$

for all $n \ge 1$. Using (2.2) and (2.5), we repeat the procedure to obtain

$$d(u_n,u_{n+1}) \le \varphi(\rho(u_{n-1},u_n)) \le \varphi^2(d(u_{n-2},u_{n-1})) \le \cdots \le \varphi^n(d(u_0,u_1))$$

for all $n \geq 1$. Therefore, for positive integers m, k, we get

$$\begin{aligned} d(u_k, u_{k+m}) &\leq d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2}) + \cdots + d(u_{k+m-1}, u_{k+m}) \\ &\leq \sum_{i=k}^{k+m-1} \varphi^i(d(u_0, u_1)) = \sum_{i=k}^{k+m-1} \varphi^i(D(u_0, Su_0)). \end{aligned}$$

Since $D(u_0, S(u_0)) \in J$ and $\varphi_{|_J}$ is a Bainchini-Grandolfi gauging function on J, the above inequality implies that $\{u_n\}$ is a Cauchy sequence in E. By virtue of the completeness of E, there exists $u \in E$ such that $u_n \to u$ for $n \to \infty$. Now, we prove that $u \in Tu$ and $u \in Su$, i.e., u is a common fixed point of S and T. To do this, we note that

$$\begin{aligned} D(u_{2n}, Su) &\leq H(Tu_{2n-1}, Su) \leq \varphi(\rho(u_{2n-1}, u)) \\ &= \varphi\Big(\max\Big\{d(u_{2n-1}, u), D(u_{2n-1}, Tu_{2n-1}), D(u, Su), \\ &\qquad \frac{1}{2}[D(u_{2n-1}, Su) + D(u, Tu_{2n-1})]\Big\}\Big) \\ &\leq \varphi\Big(\max\Big\{d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\qquad \frac{1}{2}[d(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})]\Big\}\Big). \end{aligned}$$

Denote by

$$\begin{aligned} \alpha(u_n, u) =: \max\Big\{&d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\frac{1}{2}[D(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})]\Big\} \end{aligned}$$

the right hand side of the above inequality. Then, $\alpha(u_n, u) \to D(u, Su)$ as $n \to \infty$. Therefore, in view of Lemma 1.5 and the upper semi-continuity of φ, we get

$$D(u, Su) = \lim_{n\to\infty} D(u_{2n}, Su) \leq \limsup_{n\to\infty} \varphi(\alpha(u_n, u)) \leq \varphi(D(u, Su)).$$

This implies $D(u, Su) = 0$. Since Su is approximative, there exists $y \in Su$ such that $d(u, y) = 0$, i.e., $u = y$. Hence $u \in Su$. As

$$\begin{aligned} D(u, Tu) &\leq H(Su, Tu) \\ &\leq \varphi\left(\max\left\{d(u, u), D(u, Tu), D(u, Su), \frac{1}{2}[D(u, Tu) + D(u, Su)]\right\}\right) \\ &= \varphi(D(u, Tu)), \end{aligned}$$

which gives $D(u, Tu) = 0$, and this reduces to $u \in Tu$. This completes the proof. □

Remark 2.2. *Under the hypothesis of Theorem 2.1, S and T have a unique common fixed point if the following condition is satisfied:*

$$d(x, y) \leq H(Sx, Ty), \quad \forall x, y \in E. \qquad \textbf{(C)}$$

Proof. Let u and v be common fixed points of S and T. Then, by the condition (**C**), we have

$$\begin{aligned} d(u,v) &\leq H(Su,Tv) \leq \varphi(\rho(u,v)) \\ &= \varphi\left(\max\left\{d(u,v), D(v,Tv), D(u,Tu), \frac{1}{2}[D(u,Tv)+D(v,Su)]\right\}\right) \\ &\leq \varphi\left(\max\left\{d(u,v), \frac{1}{2}[d(u,v)+D(v,Tv)+d(v,u)+D(u,Su)]\right\}\right) = \varphi(d(u,v)). \end{aligned}$$

Hence $u = v$. The proof is completed. □

By Theorem 2.1, we get the following results immediately.

Corollary 2.3. *Let (E,d) be a complete metric space. Let T be a hybrid generalized multivalued φ-weak contractions on E. Assume that T has the approximative values and $\varphi_{|J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_\propto$. If there is $x \in E$ such that $D(x,Tx) \in J$ then the mapping T has a fixed point $u \in E$.*

Corollary 2.4. *Let (E,d) be a complete metric space. Let S,T be a pair of hybrid generalized multivalued φ-weak contractions on E. If S,T have the approximative values and $\sum_{i=1}^{\infty}\varphi^i(t) < \infty$ for all $t \in (0,\propto)$, then the pair S,T has a common fixed point $u \in E$.*

3 Further Results

Let $\propto \in (0,\infty]$, $\mathcal{R}^+_\propto = [0,\propto)$. Let $f : [0,\infty) \to [0,\infty)$ satisfy

(i) $f(0) = 0$ and $f(t) > 0$ for each $t \in (0,\propto)$;

(ii) f is nondecreasing on $\mathcal{R}^+_\propto$;

(iii) f is continuous on $\mathcal{R}^+_\propto$;

(iv) $f(a+b) \leq f(a) + f(b)$ for all $a,b \in [0,\infty)$.

Define $\mathcal{F}[0,\propto) = \{f : f \text{ satisfies (i)-(iv) above}\}$.

Example 3.1. *The following examples were partially given in [24]:*

(i) Let ϕ is nonnegative, nondecreasing, Lebesgue integrable on $[0,\propto)$ and satisfies

$$\int_0^t \phi(s)ds > 0, \quad t \in (0,\propto).$$

Define $f(t) = \int_0^t \phi(s)ds$ then $f \in \mathcal{F}[0,\propto)$.

(ii) Let ψ be a nonnegative, Lebesgue integrable on $[0, \propto)$ and satisfies

$$\int_0^t \psi(s)ds > 0, \quad t \in (0, \propto)$$

and θ be a nonnegative, Lebesgue integrable on $[0, \int_0^{\propto} \psi(s)ds)$ and satisfies

$$\int_0^t \theta(s)ds > 0, \quad t \in [0, \int_0^{\propto} \psi(s)ds).$$

If ψ and θ are nondecreasing and we define $f(t) = \int_0^{\int_0^t \psi(s)ds} \theta(\tau)d\tau$, then $f \in \mathcal{F}[0, \propto)$.

Using above concepts, Theorem 2.1 could be further extended to more general results. In fact, the proof of next Theorem is similar to that of Theorem 2.1, however, for the sake of completeness we will present it.

Theorem 3.2. *Let (E, d) be a complete metric space and $S, T : E \to 2^E$ be a pair of multivalued mappings. Suppose that $\varphi \in \Phi[0, \propto)$ and $f \in \mathcal{F}[0, \propto)$ satisfy*

$$f(H(Sx, Ty)) \leq \varphi(f(\rho(x, y))) \tag{3.1}$$

for each $x, y \in E$. Assume that S, T have the approximative values and $\varphi_{|J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_{\propto}$. If there is $x \in E$ such that either $f(D(x, Sx)) \in J$ or $f(D(x, Tx)) \in J$ then the mappings S and T have a common fixed point $u \in E$.

Proof. Without loss of generality, we will assume that there is $u_0 \in E$ such that $f(D(u_0, Su_0)) \in J$. Take $u_0 \in E$, since Su_0 is approximative it follows that there exists $u_1 \in Su_0$ such that $d(u_0, u_1) = D(u_0, Su_0)$. Next, since Tu_1 is approximative, there exists $u_2 \in Tu_1$ such that $d(u_1, u_2) = D(u_1, Tu_1)$. Moreover,

$$d(u_1, u_2) = D(u_1, Tu_1) \leq \sup_{x \in Su_0} D(x, Tu_1) \leq H(Su_0, Tu_1).$$

It follows that

$$\begin{aligned}
&f(d(u_1, u_2)) \leq f(H(Su_0, Tu_1)) \leq \varphi(f(\rho(u_0, u_1))) \\
&= \varphi\left(f\left(\max\left\{d(u_0, u_1), D(u_1, Tu_1), D(u_0, Su_0), \frac{1}{2}\left[D(u_0, Tu_1) + D(u_1, Su_0)\right]\right\}\right)\right) \\
&\leq \varphi\left(f\left(\max\left\{d(u_0, u_1), d(u_1, u_2), d(u_0, u_1), \frac{1}{2}\left[d(u_0, u_1) + d(u_1, u_2)\right]\right\}\right)\right) \\
&\leq \varphi(f\left(\max\{d(u_0, u_1), d(u_1, u_2)\}\right)).
\end{aligned} \tag{3.2}$$

Write $\omega = \max\{d(u_0, u_1), d(u_1, u_2)\}$. Observe that, if $\omega = 0$ then $u_0 = u_1 = u_2$ and it follows that $u_0 = u_1 \in Su_0$ and $u_0 = u_2 \in Tu_1 = Tu_0$, i.e., u_0 is a common fixed point of mappings S and T, and then our proof is completed. On the other hand, if $0 < \omega = d(u_1, u_2)$ then using $\varphi(t) < t$ for $t \in (0, \propto)$, from (3.2) we have

$$f(d(u_1, u_2)) \leq \varphi(f(d(u_1, u_2))) < f(d(u_1, u_2))$$

which is a contradiction. Therefore, $\omega = d(u_0, u_1)$ and from (3.2) we obtain

$$f(d(u_1, u_2)) \leq \varphi(f(\rho(u_0, u_1))) \leq \varphi(f(d(u_0, u_1))). \tag{3.3}$$

We continue the procedure of constructing u_n inductively, we can choose a sequence $\{u_n\}$ in E such that for all $n \geq 1$, $u_{2n} \in Tu_{2n-1}, u_{2n+1} \in Su_{2n}$ and

$$d(u_{2n}, u_{2n+1}) = D(u_{2n}, Su_{2n}),\ d(u_{2n+1}, u_{2n+2}) = D(u_{2n+1}, Tu_{2n+1}).$$

Moreover,

$$D(u_{2n}, Su_{2n}) \leq \sup_{x \in Tu_{2n-1}} D(x, Su_{2n}) \leq H(Tu_{2n-1}, Su_{2n}),$$

and

$$D(u_{2n+1}, Tu_{2n+1}) \leq \sup_{x \in Tu_{2n+1}} D(x, Su_{2n}) \leq H(Su_{2n}, Tu_{2n+1})$$

for all $n \geq 1$. Therefore, by using an argument similar to the above we get,

$$f(d(u_{2n}, u_{2n+1})) \leq \varphi(f(\rho(u_{2n-1}, u_{2n}))) < f(d(u_{2n-1}, u_{2n})), \tag{3.4}$$

and

$$f(d(u_{2n+1}, u_{2n+2})) \leq \varphi(f(\rho(u_{2n}, u_{2n+1}))) < f(d(u_{2n}, u_{2n+1})) \tag{3.5}$$

for all $n \geq 1$. Thus, from (3.4) and (3.5), we get

$$f(d(u_n, u_{n+1})) \leq \varphi(f(\rho(u_{n-1}, u_n))) < f(d(u_{n-1}, u_n)) \tag{3.6}$$

for all $n \geq 1$. Using (3.3) and (3.6), we repeat the procedure to obtain

$$f(d(u_n, u_{n+1})) \leq \varphi(f(\rho(u_{n-1}, u_n))) \leq \varphi^2(f(d(u_{n-2}, u_{n-1}))) \leq \cdots \leq \varphi^n(f(d(u_0, u_1)))$$

for all $n \geq 1$. Therefore, for positive integers m, k, we get

$$\begin{aligned} f(d(u_k, u_{k+m})) &\leq f(d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2}) + \cdots + d(u_{k+m-1}, u_{k+m})) \\ &\leq f(d(u_k, u_{k+1})) + f(d(u_{k+1}, u_{k+2})) + \cdots + f(d(u_{k+m-1}, u_{k+m})) \\ &\leq \sum_{i=k}^{k+m-1} \varphi^i(f(d(u_0, u_1))) = \sum_{i=k}^{k+m-1} \varphi^i(f(D(u_0, Su_0))). \end{aligned}$$

Since $f(D(u_0, S(u_0))) \in J$ and $\varphi_{|_J}$ is a Bainchini-Grandolfi gauging function on J, in light of the continuity of the function f, the above inequality implies that $\{u_n\}$ is a Cauchy sequence in E. By virtue of the completeness of E, there exists $u \in E$ such that $u_n \to u$ for $n \to \infty$. Now, we prove that $u \in Tu$ and $u \in Su$, i.e.,

u is a common fixed point of S and T. Now, since f is a nondecreasing function, we have

$$\begin{aligned} f(D(u_{2n}, Su)) &\le f(H(Tu_{2n-1}, Su)) \le \varphi(f(\rho(u_{2n-1}, u))) \\ &= \varphi\Big(f\Big(\max\Big\{d(u_{2n-1}, u), D(u_{2n-1}, Tu_{2n-1}), D(u, Su), \\ &\qquad \frac{1}{2}\left[D(u_{2n-1}, Su) + D(u, Tu_{2n-1})\right]\Big\}\Big)\Big) \\ &\le \varphi\Big(f\Big(\max\Big\{d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\qquad \frac{1}{2}\left[d(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})\right]\Big\}\Big)\Big). \end{aligned}$$

Denote by

$$\begin{aligned} \alpha(u_n, u) =: \max\Big\{&d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\frac{1}{2}\left[D(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})\right]\Big\} \end{aligned}$$

the right hand side of the above inequality. Then $\alpha(u_n, u) \to D(u, Su)$ as $n \to \infty$, and consequently, $f(\alpha(u_n, u)) \to f(D(u, Su))$ as $n \to \infty$. Therefore, in view of Lemma 1.5 and the upper semi-continuity of φ, we get

$$\begin{aligned} f(D(u, Su)) &= f\left(\lim_{n\to\infty} D(u_{2n}, Su)\right) = \lim_{n\to\infty} f\left(D(u_{2n}, Su)\right) \\ &\le \limsup_{n\to\infty} \varphi(f(\alpha(u_n, u))) \le \varphi(f(D(u, Su))). \end{aligned}$$

Thus $f(D(u, Su)) = 0$, which implies that $D(u, Su) = 0$. Since Su is approximative, there exists $y \in Su$ such that $d(u, y) = 0$, i.e., $u = y$. Hence $u \in Su$. As

$$\begin{aligned} f(D(u, Tu)) &\le f(H(Su, Tu)) \\ &\le \varphi\left(f\left(\max\left\{d(u, u), D(u, Tu), D(u, Su), \frac{1}{2}\left[D(u, Tu) + D(u, Su)\right]\right\}\right)\right) \\ &= \varphi(f(D(u, Tu))), \end{aligned}$$

which gives $f(D(u, Tu)) = 0$, and this reduces to $u \in Tu$. This completes the proof. □

Remark 3.3. *Theorem 3.2 is a genuine generalization of Lemma 3.1 of a paper by Hong et al. [25], which is the important result for such paper. However, it has been observed that a proof of such lemma contains an error. The proof of such lemma at line 14, p. 5, presented as:*

$$f(d(u_{m+1}, u_{n+1})) \le f(H(Tu_m, Tu_n)) \le (f(\rho(u_m, u_n, \delta))), \tag{3.7}$$

where $\delta \in (0,1]$. It is related to the procedure of constructing the sequence $\{u_n\}$, that we only have

$$d(u_{n-1}, u_n) \leq H(Tu_{n-2}, Tu_{n-1})), \quad for \quad n = 2, 3, ...$$

Hence, the first inequality is not assuredly hold and this is a point which may break down the conclusion of such lemma. Because, if the first inequality is not true, then the conclusion that $\{u_n\}$ is a Cauchy sequence would be failed, but this result is an important step in the proof of the lemma.

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ภาคผนวก 3

Regularization and Iterative method for general variational inequality problem in Hilbert spaces

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Regularization and iterative method for general variational inequality problem in hilbert spaces

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Abstract

Without the strong monotonicity assumption of the mapping, we provide a regularization method for general variational inequality problem, when its solution set is related to a solution set of an inverse strongly monotone mapping. Consequently, an iterative algorithm for finding such a solution is constructed, and convergent theorem of the such algorithm is proved. It is worth pointing out that, since we do not assume strong monotonicity of general variational inequality problem, our results improve and extend some well-known results in the literature.

Keywords: general variational inequality problem, regularization, inertial proximal point algorithm, monotone mapping, inverse strongly monotone mapping

1. Introduction

It is well known that the ideas and techniques of the variational inequalities are being applied in a variety of diverse fields of pure and applied sciences and proven to be productive and innovative. It has been shown that this theory provides the most natural, direct, simple, unified, and efficient framework for a general treatment of a wide class of linear and nonlinear problems. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In 1988, Noor [1] introduced and studied a class of variational inequalities, which is known as general variational inequality, $GVI_K(A, g)$, is as follows: Find $u^* \in H$, $g(u^*) \in K$ such that

$$\langle A(u^*), g(v) - g(u^*)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{1.1}$$

where K is a nonempty closed convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$, and T, g: $H \to H$ be mappings. It is known that a class of nonsymmetric and odd-order obstacle, unilateral, and moving boundary value problems arising in pure and applied can be studied in the unified framework of general variational inequalities (e.g., [2] and the references therein). Observe that to guarantee the existence and uniqueness of a solution of the problem (1.1), one has to impose conditions on the operator A and g, see [3] for example in a more general case. By the way, it is worth noting that, if A fails to be Lipschitz continuous or strongly monotone, then the solution set of the problem (1.1) may be an empty one.

Related to the variational inequalities, we have the problem of finding the fixed points of the nonlinear mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems (e.g., [3-8]). Motivated and inspired by the research going in this direction, in this article, we present a method for finding a solution of the problem (1.1), which is related to the solution set of an inverse strongly monotone mapping and is as follows: Find $u^* \in H$, $g(u^*) \in S(T)$ such that

$$\langle A(u^*), g(v) - g(u^*)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{1.2}$$

when A is a generalized monotone mapping, T: $K \to H$ is an inverse strongly monotone mapping, and $S(T) = \{x \in K: T(x) = 0\}$. We will denote by $GVI_K(A, g, T)$ for a set of solution to the problem (1.2). Observe that, if $T =: 0$, the zero operator, then the problem (1.2) reduces to (1.1). Moreover, we would also like to notice that although many authors have proven results for finding the solution of the variational inequality problem and the solution set of inverse strongly monotone mapping (e.g., [4,8,9]), it is clear that it cannot be directly applied to the problem $GV I_K(A, g, T)$ due to the presence of g.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot\rangle$ and $||\cdot||$, respectively. Let K be a nonempty closed convex subset of H. In this section, we will recall some well-known results and definitions.

Definition2.1. Let A: $H \to H$ be a mapping and $K \subset H$. Then, A is said to be semi-continuous at a point x in K if

$$\lim_{t\to 0}\langle A(x + th), y\rangle = \langle A(x), y\rangle, \; x + th \in K, \; y \in H.$$

Definition2.2. A mapping T: $K \to H$ is said to be λ-inverse strongly monotone, if there exists a $\lambda > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \lambda||T(x) - T(y)||^2, \quad \forall x, y \in K.$$

Recall that a mapping B: $K \to H$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$||Bx - By||^2 \leq ||x - y||^2 + k||(I - B)(x) - (I - B)(y)||^2, \quad \forall x, y \in K.$$

Let I be the identity operator on K. It is well known that if B: $K \to H$ is a k-strictly pseudocontrative mapping, then the mapping $T := I - B$ is a $\left(\frac{1-k}{2}\right)$-inverse strongly monotone, see [4]. Conversely, if T: $K \to H$ is a λ-inverse strongly monotone with $\lambda \in (0, \frac{1}{2}]$, then $B := I - T$ is $(1 - 2\lambda)$-strictly pseudocontrative mapping. Actually, for all $x, y \in K$, we have

$$\langle T(x) - T(y), x - y\rangle \geq \lambda||T(x) - T(y)||^2$$

On the other hand, since H is a real Hilbert space, we have

$$||(I-T)(x)-(I-T)(y)||^2 = ||x-y||^2 + ||T(x)-T(y)||^2 - 2\langle T(x)-T(y), x-y\rangle.$$

Hence,

$$||(I-T)(x)-(I-T)(y)||^2 = ||x-y||^2 + (1-2\lambda)||T(x)-T(y)||^2.$$

Moreover, we have the following result:

Lemma 2.3. [10]*Let K be a nonempty closed convex subset of a Hilbert space H and B: $K \to H$ a k-strictly pseudocontractive mapping. Then, I - B is demiclosed at zero, that is, whenever $\{x_n\}$ is a sequence in K such that $\{x_n\}$ converges weakly to $x \in K$ and $\{(I - B)(x_n)\}$ converges strongly to* 0, *we must have* $(I - B)(x) = 0$.

Definition2.4. Let A, g: $H \to H$. Then A is said to be g-monotone if

$$\langle A(x)-A(y), g(x)-g(y)\rangle \geq 0, \quad \forall x, y \in H$$

For $g = I$, the identity operator, Definition 2.4 reduces to the well-known definition of monotonicity. However, the converse is not true.

Now we show an example in proof of our main problem (1.2).

Example 2.5. Let a, b be strictly positive real numbers. Put $H = \{(x_1, x_2)|\ -a \leq x_1 \leq a, -b \leq x_2 \leq b\}$ with the usual inner product and norm. Let $K = \{(x_1, x_2) \in H: 0 \leq x_1 \leq x_2\}$ be a closed convex subset of H. Let T: $K \to H$ be a mapping defined by $T(x) = (I - P_\Delta)(x)$, where $\Delta = \{x := (x_1, x_2) \in H: x_1 = x_2\}$ is a closed convex subset of H, and P_Δ is a projection mapping from K onto Δ. Clearly, T is $\frac{1}{2}$-inverse strongly monotone, and $S(T) = \Delta \cap K$. Now, if $A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ is a considered matrix operator and $g = -I$, where I is the 2×2 identity matrix. Then, we can verify that A is a g-monotone operator. Indeed, for each $x := (x_1, x_2)$, $y := (y_1, y_2) \in H$, we have

$$\begin{aligned}\langle A(x)-A(y), g(x)-g(y)\rangle &= \left([x_1-y_1\ x_2-y_2] \times \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}\right) \times \begin{bmatrix} -(x_1-y_1) \\ -(x_2-y_2) \end{bmatrix} \\ &= (x_1-y_1)^2 - 2(x_1-y_1)(x_2-y_2) + (x_2-y_2)^2 \\ &= ((x_1-y_1)-(x_2-y_2))^2 \geq 0.\end{aligned}$$

Moreover, if $u^* := (u_1^*, u_2^*) \in GVI_K(A, g)$, then we must have $\langle A(u^*), g(y) - g(u^*)\rangle \geq 0$, for all $y = (y_1, y_2) \in H$, $g(y) \in K$. This equivalence becomes

$$\frac{2u_1^*-u_2^*}{u_1^*} \geq \frac{u_1^*-y_1}{u_2^*-y_2}, \tag{2.1}$$

for all $y = (y_1, y_2) \in H$, $g(y) \in K$. Notice that $g^{-1}(K) = \{(y_1, y_2) \in H|y_1 \geq y_2\}$. Thus, in view of (2.1), it follows that $\{x = (x_1, x_2) \in H|x_1 = x_2\} \subset GVI_K(A, g)$. Hence, $GVI_K(A, g, T) \neq \emptyset$.

Remark 2.6. In Example 2.5, the operator A is not a monotone mapping on H.

We need the following concepts to prove our results.

Let $\mathcal{R}$ stand for the set of real numbers. Let $F: K \times K \to \mathcal{R}$ be an equilibrium bifunction, that is, $F(u, u) = 0$ for every $u \in K$.

Definition2.7. The equilibrium bifunction $F: K \times K \to \mathcal{R}$ is said to be (*i*) monotone, if for all u, $v \in K$, then we have

$$F(u,v) + F(v,u) \leq 0, \tag{2.2}$$

(*ii*) strongly monotone with constant τ; if for all u, $v \in K$, then we have

$$F(u,v) + F(v,u) \leq -\tau||u-v||^2, \tag{2.3}$$

(*iii*) hemicontinuous in the first variable u; if for each fixed v, then we have

$$\lim_{t\to+0} F(u+t(z-u),v) = F(u,v), \quad \forall (u,z) \in K \times K. \tag{2.4}$$

Recall that the equilibrium problem for $F : K \times K \to \mathcal{R}$ is to find $u^* \in K$ such that

$$F(u^*,v) \geq 0, \quad \forall v \in K. \tag{2.5}$$

Concerning to the problem (2.5), the following facts are very useful.

Lemma 2.8. [11]*Let* $F : K \times K \to \mathcal{R}$*be such that* $F(u, v)$ *is convex and lower semicontinuous in the variable* v *for each fixed* $u \in K$. *Then,*

(1) *if* $F(u, v)$ *is hemicontinuous in the first variable and has the monotonic property, then* $U^* = V^*$, *where* U^* *is the solution set of* (2.5), *and* V^* *is the solution set of* $F(u, v^*) \leq 0$ *for all* $u \in K$. *Moreover, in this case, they are closed and convex;*
(2) *if* $F(u, v)$ *is hemicontinuous in the first variable for each* $v \in K$ *and* F *is strongly monotone, then* U^* *is a nonempty singleton. In addition, if* F *is a strongly monotone mapping, then* $U^* = V^*$ *is a singleton set.*

The following basic results are also needed.

Lemma 2.9. *Let* $\{x_n\}$ *be a sequence in* H. *If* $x_n \to x$ *wealky and* $||x_n|| \to ||x||$, *then* $x_n \to x$ *strongly.*

Lemma 2.10. [12]. *Let* a_n, b_n, c_n *be the sequences of positive real numbers satisfying the following conditions.*

(*i*) $a_{n+1} \leq (1 - b_n)a_n + c_n$, $b_n < 1$,
(*ii*) $\sum_{n=0}^{\infty} b_n = +\infty$, $\lim_{n\to+\infty}(\frac{c_n}{b_n}) = 0$.

Then, $\lim_{n\to+\infty} a_n = 0$.

3. Regularization

Let $\alpha \in (0, 1)$ be a fixed positive real number. We now construct a regularization solution u_α for (1.2), by solving the following general variational inequality problem: find $u_\alpha \in H$, $g(u_\alpha) \in K$ such that

$$\langle A(u_\alpha) + \alpha^\mu (T\circ g)(u_\alpha) + \alpha g(u_\alpha), g(v) - g(u_\alpha)\rangle \geq 0\, \forall v \in H, \quad g(v) \in K, \quad 0 < \mu < 1. \tag{3.1}$$

Theorem 3.1. *Let* K *be a closed convex subset of a Hilbert space* H *and* g: $H \to H$ *be a mapping such that* $K \subset g(H)$. *Let* A: $H \to H$ *be a hemicontinuous on* K *and* g-

monotone mapping, T: $K \to H$ be λ-inverse strongly monotone mapping. If g is an expanding affine continuous mapping and $GVI_K(A, g, T) \neq \emptyset$, then the following conclusions are true.

(a) *For each $\alpha \in (0, 1)$, the problem* (3.1) *has the unique solution u_α:*

(b) *If $\alpha \downarrow 0$, then $\{g(u_\alpha)\}$ converges. Moreover, $\lim_{\alpha \to 0^+} g(u_\alpha) = g(u^*)$ for some $u^* \in GVI_K(A, g, T)$.*

(c) *There exists a positive constant M such that*

$$\|g(u_\alpha) - g(u_\beta)\|^2 \leq \frac{M(\beta - \alpha)}{\alpha^2}, \tag{3.2}$$

when $0 < \alpha < \beta < 1$.

Proof. First, in view of the definition 2.2, we will always assume that $\lambda \in (0, \frac{1}{2}]$. Now, related to mappings A, T, and g, we define functions $F_A, F_T : g^{-1}(K) \times g^{-1}(K) \to \mathcal{R}$ by

$$F_A(u, v) = \langle A(u), g(v) - g(u)\rangle \text{ and } F_T(u, v) = \langle (T \circ g)(u), g(v) - g(u)\rangle,$$

for all $(u, v) \in g^{-1}(K) \times g^{-1}(K)$. Note that, F_A, F_T are equilibrium monotone bifunctions, and $g^{-1}(K)$ is a closed convex subset of H.

Now, let $\alpha \in (0, 1)$ be a given positive real number. We construct a function $F_\alpha : g^{-1}(K) \times g^{-1}(K) \to \mathcal{R}$ by

$$F_\alpha(u, v) = [F_A + \alpha^\mu F_T](u, v) + \alpha\langle g(u), g(v) - g(u)\rangle, \tag{3.3}$$

for all $(u, v) \in g^{-1}(K) \times g^{-1}(K)$.

(a) Observe that, the problem (3.1) is equivalent to the problem of finding $u_\alpha \in g^{-1}(K)$ such that

$$F_\alpha(u_\alpha, v) \geq 0, \quad \forall v \in g^{-1}(K). \tag{3.4}$$

Moreover, one can easily check that $F_\alpha(u, v)$ is a monotone hemicontinuous in the variable u for each fixed $v \in g^{-1}(K)$. Indeed, it is strongly monotone with constant $\alpha\xi > 0$, where g is an ξ-expansive. Thus, by Lemma 2.8(ii), the problem (3.4) has a unique solution $u_\alpha \in g^{-1}(K)$ for each $\alpha > 0$. This prove (a).

(b) Note that for each $y \in GVI_K(A, g, T)$ we have $[F_A + \alpha^\mu F_T](y, u_\alpha) \geq 0$. Consequently, by (3.4), we have

$$\begin{aligned}
0 &\geq -F_\alpha(u_\alpha, y) \\
&= -\left[F_A(u_\alpha, y) + \alpha^\mu F_T(u_\alpha, y) + \alpha\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle\right] \\
&\geq -\left[F_A(u_\alpha, y) + \alpha^\mu F_T(u_\alpha, y) + \alpha\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle\right] - [F_A(y, u_\alpha) + \alpha^\mu F_T(y, u_\alpha)] \\
&= -[F_A(u_\alpha, y) + F_A(y, u_\alpha)] - \alpha^\mu[F_T(u_\alpha, y) + F_T(y, u_\alpha)] - \alpha\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle \\
&\geq \alpha\langle g(u_\alpha), g(u_\alpha) - g(y)\rangle.
\end{aligned}$$

This means

$$\langle g(u_\alpha), g(y) - g(u_\alpha)\rangle \geq 0, \ \forall y \in GVI_K(A, g, T).$$

Consequently,

$$||g(u_\alpha)||\ ||g(y)|| \geq \langle g(u_\alpha), g(y)\rangle \geq \langle g(u_\alpha), g(u_\alpha)\rangle = ||g(u_\alpha)||^2, \tag{3.5}$$

that is, $||g(u_\alpha)|| \leq ||g(y)||$ for all $y \in GVI_K(A, g, T)$. Thus, $\{g(u_\alpha)\}$ is a bounded subset of K. Consequently, the set of weak limit points as $\alpha \to 0$ of the net $(g(u_\alpha))$ denoted by $\omega_w(g(u_\alpha))$ is nonempty. Pick $z \in \omega_w(g(u_\alpha))$ and a null sequence $\{\alpha_k\}$ in the interval (0, 1) such that $\{g(u_{\alpha_k})\}$ weakly converges to z as $k \to \infty$. Since K is closed and convex, we know that K is weakly closed, and it follows that $z \in K$. Now, since $K \subset g(H)$, we let $u^* \in H$ be such that $z = g(u^*)$ and claim that $u^* \in GVI_K(A, g, T)$.

To prove such a claim, we will first show that $g(u^*) \in S(T)$. To do so, let us pick a fixed $y \in GVI_K(A, g, T)$. By (3.3) and the monotonicity of F_A, we have

$$\alpha_k^\mu F_T(u_{\alpha_k}, y) + \alpha_k\langle g(u_{\alpha_k}), g(y) - g(u_{\alpha_k})\rangle \geq -F_A(u_{\alpha_k}, y) \geq F_A(y, u_{\alpha_k}) \geq 0,$$

equivalently,

$$F_T(u_{\alpha_k}, y) + \alpha_k^{1-\mu}\langle g(u_{\alpha_k}), g(y) - g(u_{\alpha_k})\rangle \geq 0,$$

for each $k \in \mathbb{N}$. Using the above together with the assumption that T is an λ-inverse strongly monotone mapping, we have

$$\begin{aligned}\lambda||T(g(u_{\alpha_k})) - T(g(y))||^2 &\leq \langle T(g(u_{\alpha_k})), g(u_{\alpha_k}) - g(y)\rangle \\ &= -F_T(u_{\alpha_k}, y) \\ &\leq \alpha_k^{1-\mu}\langle g(u_{\alpha_k}), g(y) - g(u_{\alpha_k})\rangle \\ &\leq \alpha_k^{1-\mu}\left[||g(u_{\alpha_k})||\ ||g(y)|| - ||g(u_{\alpha_k})||^2\right] \\ &\leq \alpha_k^{1-\mu}||g(y)||^2\end{aligned}$$

]

for each $k \in \mathbb{N}$. Letting $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} ||T(g(u_{\alpha_k})) - T(g(y))|| = \lim_{k\to+\infty} ||T(g(u_{\alpha_k}))|| = 0.$$

On the other hand, we know that the mapping $I - T$ is a strictly pseudocontractive, thus by lemma 2.3, we have T demiclosed at zero. Consequently, since $\{g(u_{\alpha_k})\}$ weakly converges to $g(u^*)$, we obtain $T(g(u^*)) = T(g(y)) = 0$. This proves $g(u^*) \in S(T)$, as required.

Now, we will show that $u^* \in GVI_K(A, g, T)$. Notice that, from the monotonic property of F_α and (3.4), we have

$$F_A(v, u_{\alpha_k}) + \alpha_k^\mu F_T(v, u_{\alpha_k}) + \alpha_k\langle g(v), g(u_{\alpha_k}) - g(v)\rangle = F_\alpha(v, u_{\alpha_k}) \leq -F_\alpha(u_{\alpha_k}, v) \leq 0,$$

for all $v \in g^{-1}(K)$. That is,

$$F_A(v, u_{\alpha_k}) + \alpha_k^\mu F_T(v, u_{\alpha_k}) \leq \alpha_k\langle g(v), g(v) - g(u_{\alpha_k})\rangle, \tag{3.6}$$

for all $v \in g^{-1}(K)$. Since $\alpha_k \downarrow 0$ as $k \to \infty$, we see that (3.6) implies $F_A(v, u^*) \leq 0$ for any $v \in H$, $g(v) \in K$. Consequently, in view of Lemma 2.8(1), we obtain our claim immediately.

Next we observe that the sequence $\{g(u_{\alpha_k})\}$ actually converges to $g(u^*)$ strongly. In fact, by using a lower semi-continuous of norm and (3.5), we see that

$$||g(u^*)|| \le \liminf_{k\to\infty} ||g(u_{\alpha_k})|| \le \limsup_{k\to\infty} ||g(u_{\alpha_k})|| \le ||g(u^*)||,$$

since $u^* \in GVI_K(A, g, T)$. That is, $||g(u_{\alpha_k})|| \to ||g(u^*)||$ as $k \to \infty$. Now, it is straight-forward from Lemma 2.9, that the weak convergence to $g(u^*)$ of $\{g(u_{\alpha_k})\}$ implies strong convergence to $g(u^*)$ of $\{g(u_{\alpha_k})\}$. Further, in view of (3.5), we see that

$$||g(u^*)|| = \inf\{||g(y)|| : y \in GVI_K(A, g, T)\}. \tag{3.7}$$

Next, we let $\{g(u_{\alpha_j})\} \subset (g(u_\alpha))$, where $\{\alpha_j\}$ be any null sequence in the interval (0, 1). By following the lines of proof as above, and passing to a subsequence if necessary, we know that there is $\tilde{u} \in GVI_K(A, g, T)$ such that $g(u_{\alpha_j}) \to g(\tilde{u})$ as $j \to \infty$. Moreover, in view of (3.5) and (3.7), we have $||g(\tilde{u})|| = ||g(u^*)||$. Consequently, since the function $||g(\cdot)||$ is a lower semi-continuous function and $GVI_K(A, g, T)$ is a closed convex set, we see that (3.7) gives $u^* = \tilde{u}$. This has shown that $g(u^*)$ is the strong limit of the net $(g(u_\alpha))$ as $\alpha \downarrow 0$.

(c) Let $0 < \alpha < \beta < 1$ and u_α, u_β are solutions of the problem (3.1). Thus, since F_A and F_T are monotone mappings, by (3.4), we have

$$0 \le (\beta^\mu - \alpha^\mu)F_T(u_\beta, u_\alpha) + \beta\langle g(u_\beta), g(u_\alpha) - g(u_\beta)\rangle + \alpha\langle g(u_\alpha), g(u_\beta) - g(u_\alpha)\rangle,$$

that is,

$$\left\langle g(u_\alpha) - \frac{\beta}{\alpha}g(u_\beta), g(u_\alpha) - g(u_\beta)\right\rangle \le \left(\frac{\beta^\mu - \alpha^\mu}{\alpha}\right)F_T(u_\beta, u_\alpha). \tag{3.8}$$

Notice that,

$$\begin{aligned}\left\langle g(u_\alpha) - \frac{\beta}{\alpha}g(u_\beta), g(u_\alpha) - g(u_\beta)\right\rangle &= ||g(u_\alpha) - g(u_\beta)||^2 + \frac{\alpha-\beta}{\alpha}\langle g(u_\beta), g(u_\alpha))\rangle - \frac{\alpha-\beta}{\alpha}||g(u_\beta)||^2 \\ &\ge ||g(u_\alpha) - g(u_\beta)||^2 + \frac{\alpha-\beta}{\alpha}\langle g(u_\beta), g(u_\alpha))\rangle,\end{aligned}$$

since $0 < \alpha < \beta$. Using the above, by (3.8), we have

$$||g(u_\alpha) - g(u_\beta)||^2 \le \frac{\beta-\alpha}{\alpha}\theta^2 + \frac{\beta^\mu - \alpha^\mu}{\alpha}F_T(u_\beta, u_\alpha), \tag{3.9}$$

where $\theta = \sup\{||g(u_\alpha)||: \alpha \in (0, 1)\}$. Moreover, since F_T is a Lipschit continuous mapping (with Lipschitz constant $\frac{1}{\lambda}$), it follows that

$$||g(u_\alpha) - g(u_\beta)||^2 \le \frac{\beta-\alpha}{\alpha}\theta^2 + \frac{\beta^\mu - \alpha^\mu}{\alpha}M_1$$

for some $M_1 > 0$. Further, by applying the Lagranges mean-value theorem to a continuous function $h(t) = t^{-\mu}$ on $[1, +\infty)$, we know that

$$||g(u_\alpha) - g(u_\beta)||^2 \le \frac{M(\beta-\alpha)}{\alpha^2}, \tag{3.10}$$

for some $M > 0$. This completes the proof. □

Remark 3.2. If $g =: I$, the identity operator on H, then we see that Theorem 3.1 reduces to a result presented by Kim and Buong [9].

4. Iterative Method

Now, we consider the regularization inertial proximal point algorithm:

$$\begin{gathered}\langle c_n[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})+\alpha_n g(z_{n+1})]+g(z_{n+1})-g(z_n),g(v)-g(z_{n+1})\rangle\geq 0\\ \forall v\in H,\ g(v)\in K,\ z_1\in H,\ g(z_1)\in K.\end{gathered}\tag{4.1}$$

The well definedness of (4.1) is guaranteed by the following result.

Proposition 4.1. *Assume that all hypothesis of the Theorem 3.1 are satisfied. Let $z \in g^{-1}(K)$ be a fixed element. Define a bifunction $F_z : g^{-1}(K) \times g^{-1}(K) \to \mathbb{R}$ by*

$$F_z(u,v):=\langle c[A(u)+\alpha^{\mu}(T\circ g)(u)+\alpha g(u)]+g(u)-g(z),g(v)-g(u)\rangle,$$

where c, α are positive real numbers. Then, there exists the unique element $u^* \in g^{-1}(K)$ such that $F_z(u^*, v) \geq 0$ for all $v \in g^{-1}(K)$.

Proof. Assume that g is an ξ- expanding mapping. Then, for each $u, v \in g^{-1}(K)$, we see that

$$\begin{aligned}F_z(u,v)+F_z(v,u)&\leq(1+c\alpha)\langle g(u)-g(v),g(v)-g(u)\rangle\\&=-(1+c\alpha)||g(u)-g(v)||^2\\&\leq-\xi(1+c\alpha)||u-v||^2.\end{aligned}$$

This means F is $\xi(1 + c\alpha)$-strongly monotone. Consequently, by Lemma 2.8, the proof is completed. □

The result of the next theorem shows some sufficient conditions for the convergent of regularization inertial proximal point algorithm (4.1).

Theorem 4.2. *Assume that all the hypotheses of the Theorem 3.1 are satisfied. If the parameters c_n and α_n are chosen as positive real numbers such that*

(C1) $\lim\limits_{n\to\infty}\alpha_n=0$,
(C2) $\lim\limits_{n\to\infty}\frac{\alpha_n-\alpha_{n+1}}{\alpha_{n+1}^2}=0$,
(C3) $\liminf\limits_{n\to\infty}c_n\alpha_n>0$,

then the sequence $\{g(z_n)\}$ defined by (4.1) converges strongly to the element $g(u^*)$ as $n \to +\infty$, where $u^* \in GVI_K(A, g, T)$.

Proof. From (4.1) we have

$$\langle c_n[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})]+(1+c_n\alpha_n)g(z_{n+1})-g(z_n),g(v)-g(z_{n+1})\rangle\geq 0$$

that is

$$\langle c_n[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})]+(1+c_n\alpha_n)g(z_{n+1}),g(v)-g(z_{n+1})\rangle\geq\langle g(z_n),g(v)-g(z_{n+1})\rangle,$$

or equivalently,

$$(1+c_n\alpha_n)\left\langle\frac{c_n}{(1+c_n\alpha_n)}[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})]+g(z_{n+1}),g(v)-g(z_{n+1})\right\rangle\geq\\\langle g(z_n),g(v)-g(z_{n+1})\rangle,$$

so

$$\left\langle \frac{c_n}{(1+c_n\alpha_n)}[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})]+g(z_{n+1}),g(v)-g(z_{n+1})\right\rangle \geq \frac{1}{(1+c_n\alpha_n)}\langle g(z_n),g(v)-g(z_{n+1})\rangle.$$

Hence

$$\langle \kappa_n[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})]+g(z_{n+1}),g(v)-g(z_{n+1})\rangle \geq \beta_n\langle g(z_n),g(v)-g(z_{n+1})\rangle,$$

where

$$\beta_n=\frac{1}{(1+c_n\alpha_n)},\ \text{and}\ \kappa_n=c_n\beta_n. \tag{4.2}$$

On the other hand, by Theorem 3.1, there is $u_n \in g^{-1}(K)$ such that

$$\langle A(u_n)+\alpha^{\mu}(T\circ g)(u_n)+\alpha g(u_n),g(v)-g(u_n)\rangle \geq 0, \tag{4.3}$$

for all $n \in \mathbb{N}$. This implies

$$\langle c_n[A(u_n)+\alpha_n^{\mu}(T\circ g)(u_n)]+(1+c_n\alpha_n)g(u_n)-g(u_n),g(v)-g(u_n)\rangle \geq 0,$$

and so

$$\left\langle \frac{c_n}{(1+c_n\alpha_n)}[A(u_n)+\alpha_n^{\mu}(T\circ g)(u_n)]+g(u_n),g(v)-g(u_n)\right\rangle \geq \left\langle \frac{1}{(1+c_n\alpha_n)}\langle g(u_n),g(v)-g(u_n)\right\rangle.$$

Thus,

$$\langle \kappa_n[A(u_n)+\alpha_n^{\mu}(T\circ g)(u_n)]+g(u_n),g(v)-g(u_n)\rangle \geq \beta_n\langle g(u_n),g(v)-g(u_n)\rangle. \tag{4.4}$$

By setting $v = u_n$ in (4.2) we have

$$\langle \kappa_n[A(z_{n+1})+\alpha_n^{\mu}(T\circ g)(z_{n+1})]+g(z_{n+1}),g(u_n)-g(z_{n+1})\rangle \geq \beta_n\langle g(z_n),g(u_n)-g(z_{n+1})\rangle,$$

and $v = z_{n+1}$ in (4.4) we have

$$\langle \kappa_n[A(u_n)+\alpha_n^{\mu}(T\circ g)(u_n)]+g(u_n),g(z_{n+1})-g(u_n)\rangle \geq \beta_n\langle g(u_n),g(z_{n+1})-g(u_n)\rangle,$$

and adding one obtained result to the other, we get

$$\begin{aligned}\kappa_n\langle A(z_{n+1})-A(u_n)+\alpha_n^{\mu}((T\circ g)(z_{n+1})-(T\circ g)(u_n))),g(u_n)-g(z_{n+1})\rangle+\langle g(z_{n+1})-g(u_n),g(u_n)-g(z_{n+1})\rangle\\ \geq \beta_n\langle g(z_n)-g(u_n),g(u_n)-g(z_{n+1})\rangle.\end{aligned} \tag{4.5}$$

Notice that, since A is a g-monotone mapping, and T is a λ-inverse strongly monotone, we have

$$\langle A(z_{n+1})-A(u_n),g(u_n)-g(z_{n+1})\rangle \leq 0,$$

and

$$\langle (T\circ g)(z_{n+1}))-(T\circ g)(u_n)),g(u_n)-g(z_{n+1})\rangle \leq 0.$$

Thus, by (4.5), we obtain

$$\langle g(z_{n+1})-g(u_n),g(u_n)-g(z_{n+1})\rangle \geq \beta_n\langle g(z_n)-g(u_n),g(u_n)-g(z_{n+1})\rangle,$$

that is,

$$\langle g(z_{n+1}) - g(u_n), g(z_{n+1}) - g(u_n)\rangle \le \beta_n \langle g(z_n) - g(u_n), g(z_{n+1}) - g(u_n)\rangle.$$

Consequently,

$$||g(z_{n+1}) - g(u_n)||^2 \le \beta_n ||g(z_n) - g(u_n)||\ ||g(z_{n+1}) - g(u_n)||,$$

which implies that

$$||g(z_{n+1}) - g(u_n)|| \le \beta_n ||g(z_n) - g(u_n)||. \tag{4.6}$$

Using the above Equation 4.6 and (3.2), we know that

$$\begin{aligned} ||g(z_{n+1}) - g(u_{n+1})|| &\le ||g(z_{n+1}) - g(u_n)|| + ||g(u_n) - g(u_{n+1})|| \\ &\le \beta_n ||g(z_n) - g(u_n)|| + \sqrt{\frac{M(\alpha_n - \alpha_{n+1})}{\alpha_{n+1}^2}} \\ &\le (1 - b_n)||g(z_n) - g(u_n)|| + d_n \end{aligned}$$

where

$$b_n = \frac{c_n \alpha_n}{(1 + c_n \alpha_n)}, \quad d_n = \sqrt{\frac{M(\alpha_n - \alpha_{n+1})}{\alpha_{n+1}^2}}.$$

Consequently, by the condition (C3), we have $\sum_{n=1}^{\infty} b_n = \infty$. Meanwhile, the conditions (C2) and (C3) imply that $\lim_{n\to\infty} \frac{d_n}{b_n} = 0$. Thus, all the conditions of Lemma 2.10 are satisfied, then it follows that $||g(z_{n+1}) - g(u_{n+1})|| \to 0$ as $n \to \infty$. Moreover, by (C1) and Theorem 3.1, we know that there exists $u^* \in GVI_K(A, g, T)$ such that $g(u_n)$ converges strongly to $g(u^*)$. Consequently, we obtain that $g(z_n)$ converges strongly to $g(u^*)$ as $n \to +\infty$. This completes the proof. □

Remark 4.3. The sequences $\{\alpha_n\}$ and $\{c_n\}$ which are defined by

$$\alpha_n = \left(\frac{1}{n}\right)^p, \ 0 < p < 1, \quad \text{and} \quad c_n = \frac{1}{\alpha_n}$$

satisfy all the conditions in Theorem 4.2.

Remark 4.4. It is worth noting that, because of condition (C2) of Theorem 4.2, the important natural choice $\{1/n\}$ does not include in the class of parameters $\{\alpha_n\}$. This leads to a question: Can we find another regularization inertial proximal point algorithm for the problem (1.2) that includes a natural parameter choice $\{1/n\}$?

Remark 4.5. If F is a nonexpansive mapping, then $I - F$ is an inverse strongly monotone mapping, and the fixed points set of mapping F and the solution set $S(I - F)$ are equal. This means that our results contain the study of finding a common element of (general) variational inequalities problems and fixed points set of nonexpansive mapping, which were studied in [4-8] as special cases.

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Both authors contributed equally in this paper. They read and approved the final manuscript.

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ภาคผนวก 4

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Existence and stability of iterative algorithms for the system of nonlinear quasi-mixed equilibrium problems☆

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ABSTRACT

In this paper, we consider the system of nonlinear quasi-mixed equilibrium problems. The existence theorems of solutions of such problems are provided by considering the limit point of an iterative algorithm. This means, we not only give the conditions for the existence theorems of the presented problems but also provide the algorithm to find such solutions. Moreover, the stability of such an algorithm is also discussed. The results presented in this paper are more general, and may be viewed as an extension, refinement and improvement of the previously known results in the literature.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\Phi_1, \Phi_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be given two bi-functions satisfying $\Phi_i(x, x) = 0$ for all $x \in \mathcal{H}, i = 1, 2$. Let $T_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a nonlinear mapping for each $i = 1, 2$. In this work, let $\mathcal{CC}(\mathcal{H})$ be the family of all nonempty closed convex subsets of $\mathcal{H}$ and $C_i : \mathcal{H} \to \mathcal{CC}(\mathcal{H})$ be a point-to-set mapping which associate a nonempty closed convex set $C_i(x)$ with any element x of $\mathcal{H}$, for each $i = 1, 2$. We consider the problem of finding $(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$ such that $x^* \in C_1(x^*)$, $y^* \in C_2(y^*)$ and

$$\begin{cases} \Phi_1(x^*, z) + \langle T_1(x^*, y^*), z - x^* \rangle \geq 0, & \forall z \in C_1(x^*), \\ \Phi_2(y^*, z) + \langle T_2(x^*, y^*), z - y^* \rangle \geq 0, & \forall z \in C_2(y^*). \end{cases} \tag{1.1}$$

Since in many important problems the closed convex set C also depends upon the solutions explicitly or implicitly, it is worth mentioning that the problem of type (1.1) is of interest to study; see [1] for more details. Consequently, problem (1.1) is called *the system of nonlinear quasi-mixed equilibrium problems*.

For each $i = 1, 2$ if the convex set $C(u)$ is of the form

$$C_i(u) = m_i(u) + C_i, \tag{1.2}$$

where C_i is a fixed closed convex set and m_i is a point-to-point mapping, then problem (1.1) is equivalent to finding $(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$ such that $x^* - m_1(x^*) \in C_1$ and $y^* - m_2(x^*) \in C_2$ and

$$\begin{cases} \Phi_1(x^*, z) + \langle T_1(x^*, y^*), z - x^* \rangle \geq 0, & \forall z \in C_1(x^*), \\ \Phi_2(y^*, z) + \langle T_2(x^*, y^*), z - y^* \rangle \geq 0, & \forall z \in C_2(y^*). \end{cases} \tag{1.3}$$

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A problem related to (1.3) was studied by Ding et al. [2]. Moreover, if we set $m_1 = m_2 \equiv 0$ then problem (1.3) is reduced to finding $x^*, y^* \in C_1 \times C_2$ such that

$$\begin{cases} \Phi_1(x^*, z) + \langle T_1(x^*, y^*), z - x^* \rangle \geq 0, & \forall z \in C_1, \\ \Phi_2(y^*, z) + \langle T_2(x^*, y^*), z - y^* \rangle \geq 0, & \forall z \in C_2, \end{cases} \tag{1.4}$$

which is due to Cho and Petrot [3], when $C_1 = C_2$.

If for each $i = 1, 2$, let $S_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the nonlinear mapping and ζ, ϑ are fixed positive real numbers. Let $T_1(x, y) = \zeta S_1(y, x) + x - y$, $T_2(x, y) = \vartheta S_2(x, y) + y - x$ for all $x, y \in \mathcal{H}$ and $\Phi_i(x, z) = \psi_i(z) - \psi_i(x)$ for all $x, z \in \mathcal{H}$, where $\psi_i : \mathcal{H} \to \mathbb{R}$ is a real valued function, for each $i = 1, 2$. Then problem (1.4) reduces to finding $x^*, y^* \in \mathcal{H}$ such that

$$\begin{cases} \langle \zeta S_1(y^*, x^*) + x^* - y^*, z - x^* \rangle + \psi_1(z) - \psi_1(x^*) \geq 0, & \forall z \in C, \\ \langle \vartheta S_2(x^*, y^*) + y^* - x^*, z - y^* \rangle + \psi_2(z) - \psi_2(y^*) \geq 0, & \forall z \in C, \end{cases} \tag{1.5}$$

which is called the *system of nonlinear mixed variational inequalities problems*. A special case of problem (1.5), has been studied by many authors; see [4–10] for examples. Evidently, the examples described above shown that a number of classes of variational inequalities and related optimization problems can be obtained as special cases of the system of mixed equilibrium problems (1.1).

Motivated and inspired by these works, in this paper, we provide the existence theorem for problem (1.1) and the uniqueness of solution. The stability of the iterative algorithm and some important remarks are also discussed. To do so, we need the following basic concepts and lemmas.

Definition 1.1 (*Blum and Oettli [11]*). A real valued bi-function $\Phi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(i) monotone if

$$\Phi(x, y) + \Phi(y, x) \leq 0, \quad \forall x, y \in \mathcal{H};$$

(ii) strictly monotone if

$$\Phi(x, y) + \Phi(y, x) < 0, \quad \forall x, y \in \mathcal{H} \text{ with } x \neq y;$$

(iii) upper hemicontinuous if

$$\limsup_{t \to 0^+} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y), \quad \forall x, y, z \in \mathcal{H}.$$

Definition 1.2. A function $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semi-continuous at x_0 if for all $\alpha < f(x_0)$, there exists a constant $\beta > 0$ such that

$$\alpha \leq f(x), \quad \forall x \in B(x_0, \beta),$$

where $B(x_0, \beta)$ denotes the ball with the center x_0 and the radius β, i.e.,

$$B(x_0, \beta) = \{y : \|y - x_0\| \leq \beta\}.$$

f is said to be lower semi-continuous if it is lower semi-continuous at every point of E.

Lemma 1.3 (*Combettes and Hirstoaga [12]*). *Let C be a nonempty closed convex subset of $\mathcal{H}$ and Φ be a bi-function of $\mathcal{H} \times \mathcal{H}$ into $\mathbb{R}$ satisfying the following conditions:*

(C1) *Φ is monotone and upper hemicontinuous;*
(C2) *$\Phi(x, \cdot)$ is convex and lower semi-continuous for each $x \in C$.*

Let $\rho > 0$ be fixed. Define a mapping $J^\rho_{\Phi,C} : \mathcal{H} \to C$ as follows:

$$J^\rho_{\Phi,C}(x) = \{w \in C : \rho\Phi(w, z) + \langle w - x, z - w \rangle \geq 0, \ \forall z \in C\},$$

for all $x \in \mathcal{H}$. Then $J^\rho_{\Phi,C}$ is a single valued mapping.

Definition 1.4. Let $M \subset \mathcal{H} \times \mathcal{H}$ be a set-valued mapping. Then M is called *monotone* if for any $(x_1, y_1), (x_2, y_2) \in M$,

$$\langle y_1 - y_2, x_1, x_2 \rangle \geq 0.$$

Lemma 1.5 (*[3]*). *Let C be a nonempty closed convex subset of $\mathcal{H}$. If $\Phi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a monotone function, then the operator $J^\rho_{\Phi,C}$ is a non-expansive mapping, that is,*

$$\|J^\rho_{\Phi,C}(x) - J^\rho_{\Phi,C}(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

In this paper, we are interested in the following classes of nonlinear mappings.

Definition 1.6. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be ν-*strongly monotone* if there exists a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \nu\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

Definition 1.7. A mapping $T : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be (τ, σ)-*Lipschitz* if there exist constants $\tau, \sigma > 0$ such that

$$\|T(x_1, y_1) - T(x_2, y_2)\| \le \tau\|x_1 - x_2\| + \sigma\|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{H}.$$

2. The existence theorems

In this section we will provide the existence theorem for the solution of problem (1.1). We begin with an important lemma,

Lemma 2.1. *$(x^*, y^*) \in \mathcal{H} \times \mathcal{H}$ is a solution of problem (1.1) if and only if*

$$x^* = J^{\rho}_{\Phi,C_1(x^*)}[x^* - \rho_1 T_1(x^*, y^*)],$$
$$y^* = J^{\rho}_{\Phi,C_2(y^*)}[y^* - \rho_2 T_2(x^*, y^*)].$$

Proof. The proof directly follows from the definitions of $J^{\rho}_{\Phi,C_1(x^*)}$ and $J^{\rho}_{\Phi,C_2(y^*)}$. □

From Lemma 2.1, we see that the system of nonlinear quasi-mixed implicit equilibrium problems (1.1) is equivalent to the fixed point problems:

$$\begin{cases} x^* = (1-\lambda)x^* + \lambda J^{\rho_1}_{\Phi_1,C_1(x^*)}[x^* - \rho_1 T_1(x^*, y^*)] \\ y^* = (1-\lambda)y^* + \lambda J^{\rho_2}_{\Phi_2,C_2(y^*)}[y^* - \rho_2 T_2(x^*, y^*)] \end{cases} \tag{2.1}$$

where $\lambda \in (0, 1)$ is a parameter. The fixed point formulation (2.1) enables us to suggest the following iterative scheme.

Algorithm (I). Let ρ_1, ρ_2 be fixed positive constants. For given $(x_0, y_0) \in \mathcal{H} \times \mathcal{H}$. Define $\{(x_n, y_n)\} \subset \mathcal{H} \times \mathcal{H}$ by

$$\begin{cases} x_{n+1} = (1-\lambda)x_n + \lambda J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_n - \rho_1 T_1(x_n, y_n)], \\ y_{n+1} = (1-\lambda)y_n + \lambda J^{\rho_2}_{\Phi_2,C_2(y_n)}[y_n - \rho_2 T_2(x_n, y_n)], \end{cases} \tag{2.2}$$

where $\lambda \in (0, 1)$ is a fixed parameter.

Of course, we will use Algorithm (I) as a tool for obtaining our main result, that is, the existence theorem solutions to problem (1.1). To do this, from now on, we will assume the following condition:

Condition (Δ). For each $i = 1, 2$ there exists $\eta_i > 0$ such that

$$\|J^{\rho}_{\Phi,C_i(u)}z - J^{\rho}_{\Phi,C_i(v)}z\| \le \eta_i\|u - v\|, \quad \forall u, v, z \in \mathcal{H}.$$

Remark 2.2. Let C be a closed convex subset of $\mathcal{H}$. It is clear that Condition (Δ) is satisfied for the case $C(u) = C$ for all $u \in H$, with $\eta = 0$. We also remark that Condition (Δ) is true for the case $C(u) = m(u) + C$, as defined by (1.2) when m is a μ-Lipschitz continuous and the function Φ satisfies $\Phi(x - y, z) = \Phi(x, z - y)$ for all $x, y, z \in C$. Indeed, for each $u, z \in \mathcal{H}$ we observe that

$$J^{\rho}_{\Phi,C(u)}z = J^{\rho}_{\Phi,m(u)+C}z = m(u) + J^{\rho}_{\Phi,C}[z - m(u)]. \tag{2.3}$$

It follows that

$$\begin{aligned} \|J^{\rho}_{\Phi,C(u)}z - J^{\rho}_{\Phi,C(v)}z\| &= \|m(u) + J^{\rho}_{\Phi,C}[z - m(u)] - m(v) - J^{\rho}_{\Phi,C}[z - m(v)]\| \\ &\le \|m(u) - m(v)\| + \|J^{\rho}_{\Phi,C}[z - m(u)] - J^{\rho}_{\Phi,C}[z - m(v)]\| \\ &\le 2\|m(u) - m(v)\| \le 2\mu\|u - v\|, \end{aligned}$$

this shows that Condition (Δ) holds for $\eta = 2\mu$.

Theorem 2.3. *For each $i = 1, 2$, let $\Phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a monotone function and $C_i : \mathcal{H} \to \mathcal{CC}(\mathcal{H})$. Let $T_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a ν_1-strongly monotone with respect to the first argument and (τ_1, σ_1)-Lipschitz mapping and $T_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a ν_2-strongly monotone with respect to the second argument and (τ_2, σ_2)-Lipschitz mapping. Suppose that there are positive real numbers ρ_1, ρ_2 which satisfy the following condition:*

$$\begin{cases} (1 - 2\rho_1\nu_1 + \rho_1^2\tau_1^2)^{\frac{1}{2}} + \rho_2\tau_2 < 1 - \eta_1, \\ (1 - 2\rho_2\nu_2 + \rho_2^2\tau_2^2)^{\frac{1}{2}} + \rho_1\sigma_1 < 1 - \eta_2. \end{cases} \tag{2.4}$$

Then the set of solution of problem (1.1) is a singleton.

Proof. Since $J^{\rho_1}_{\Phi_1,C_1}$ and $J^{\rho_2}_{\Phi_2,C_2}$ are non-expansive mappings, we have the following estimate:

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\lambda)\|x_n-x_{n-1}\| + \lambda\|J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_n-\rho_1T_1(x_n,y_n)] - J^{\rho_1}_{\Phi_1,C_1(x_{n-1})}[x_{n-1}-\rho_1T_1(x_{n-1},y_{n-1})]\| \\ &\le (1-\lambda)\|x_n-x_{n-1}\| + \lambda\|J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_n-\rho_1T_1(x_n,y_n)] - J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_{n-1}-\rho_1T_1(x_{n-1},y_{n-1})]\| \\ &\quad + \lambda\|J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_{n-1}-\rho_1T_1(x_{n-1},y_{n-1})] - J^{\rho_1}_{\Phi_1,C_1(x_{n-1})}[x_{n-1}-\rho_1T_1(x_{n-1},y_{n-1})]\| \\ &\le (1-\lambda(1-\eta_1))\|x_n-x_{n-1}\| + \lambda\|x_n-x_{n-1}-\rho_1[T_1(x_n,y_n)-T_1(x_{n-1},y_n)]\| \\ &\quad + \lambda\rho_1\|T(x_{n-1},y_n)-T(x_{n-1},y_{n-1})\|. \end{aligned} \tag{2.5}$$

Since for each $w\in\mathcal{H}$ the mapping $T_1(\cdot,w):\mathcal{H}\to\mathcal{H}$ is a ν_1- strongly monotone, and the mapping $T_1(w,\cdot):\mathcal{H}\to\mathcal{H}$ is a τ_1-Lipschitz, we obtain

$$\begin{aligned}\|x_n-x_{n-1}-\rho_1[T_1(x_n,y_n)-T_1(x_{n-1},y_n)]\|^2 &= \|x_n-x_{n-1}\|^2 - 2\rho_1\langle x_n-x_{n-1}, T_1(x_n,y_n)-T_1(x_{n-1},y_n)\rangle \\ &\quad + \rho_1^2\|T_1(x_n,y_n)-T_1(x_{n-1},y_n)\|^2 \\ &\le \|x_n-x_{n-1}\|^2 - 2\rho_1\nu_1\|x_n-x_{n-1}\| + \rho_1^2\tau_1^2\|x_n-x_{n-1}\|^2 \\ &= (1-2\rho_1\nu_1+\rho_1^2\tau_1^2)\|x_n-x_{n-1}\|^2, \end{aligned} \tag{2.6}$$

and

$$\|T(x_{n-1},y_n)-T(x_{n-1},y_{n-1})\| \le \sigma_1\|y_n-y_{n-1}\|. \tag{2.7}$$

Consequently, from (2.5)–(2.7), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\lambda(1-\eta_1))\|x_n-x_{n-1}\| + \lambda(1-2\rho_1\nu_1+\rho_1^2\tau_1^2)^{\frac12}\|x_n-x_{n-1}\| + \lambda\rho_1\sigma_1\|y_n-y_{n-1}\| \\ &= (1-\lambda(1-(\eta_1+\theta_1)))\|x_n-x_{n-1}\| + \lambda\rho_1\sigma_1\|y_n-y_{n-1}\|, \end{aligned} \tag{2.8}$$

where $\theta_1 = (1-2\rho_1\nu_1+\rho_1^2\tau_1^2)^{\frac12}$.

Similarly, we have the following inequality

$$\|y_{n+1}-y_n\| \le (1-\lambda(1-(\eta_2+\theta_2)))\|y_n-y_{n-1}\| + \lambda\rho_2\tau_2\|x_n-x_{n-1}\|, \tag{2.9}$$

where $\theta_2 = (1-2\rho_2\nu_2+\rho_2^2\sigma_2^2)^{\frac12}$.

Consequently, from (2.8) and (2.9), we have

$$\|x_{n+1}-x_n\| + \|y_{n+1}-y_n\| \le \max\{\kappa_1,\kappa_2\}(\|x_n-x_{n-1}\| + \|y_n-y_{n-1}\|), \tag{2.10}$$

where

$$\kappa_1 = 1-\lambda[1-(\eta_1+\theta_1+\rho_2\tau_2)], \qquad \kappa_2 = 1-\lambda[1-(\eta_2+\theta_2+\rho_1\sigma_1)]. \tag{2.11}$$

Now, define the norm $\|\cdot\|^+$ on $\mathcal{H}\times\mathcal{H}$ by

$$\|(x,y)\|^+ = \|x\| + \|y\|, \quad \forall (x,y)\in\mathcal{H}\times\mathcal{H}.$$

Notice that $(\mathcal{H}\times\mathcal{H}, \|\cdot\|^+)$ is a Banach space and

$$\|(x_{n+1},y_{n+1})-(x_n,y_n)\|^+ \le \max\{\kappa_1,\kappa_2\}\|(x_n,y_n)-(x_{n-1},y_{n-1})\|^+. \tag{2.12}$$

By condition (2.4), we see that $\kappa := \max\{\kappa_1,\kappa_2\} < 1$. Write $a_n := (x_n,y_n)$. From (2.12) we have

$$\|a_{n+1}-a_n\|^+ \le \kappa^n\,\|a_1-a_0\|^+, \tag{2.13}$$

for all $n\ge 1$. Hence, for any $m\ge n>1$, it follows that

$$\|a_m-a_n\|^+ \le \sum_{i=n}^{m-1}\|a_{i+1}-a_i\|^+ \le \sum_{i=n}^{m-1}\kappa^i\|a_1-a_0\|^+. \tag{2.14}$$

Since $\kappa<1$, it follows from (2.14) that $\|a_m-a_n\|^+\to 0$ as $n\to\infty$ and hence $\{a_n\}$ is a Cauchy sequence in $(\mathcal{H}\times\mathcal{H}, \|\cdot\|^+)$. Consequently, there exists $(x^*,y^*)\in\mathcal{H}\times\mathcal{H}$ such that $(x_n,y_n)\to(x^*,y^*)$ as $n\to\infty$. Now we show that (x^*,y^*) is a solution of problem (1.1). In fact, by Condition (Δ), we note that

$$\begin{aligned}&\|J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_n-\rho_1T_1(x_n,y_n)] - J^{\rho_1}_{\Phi_1,C_1(x^*)}[x^*-\rho_1T_1(x^*,y^*)]\| \\ &\quad\le \|J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_n-\rho_1T_1(x_n,y_n)] - J^{\rho_1}_{\Phi_1,C_1(x_n)}[x^*-\rho_1T_1(x^*,y^*)]\| \\ &\qquad + \|J^{\rho_1}_{\Phi_1,C_1(x_n)}[x^*-\rho_1T_1(x^*,y^*)] - J^{\rho_1}_{\Phi_1,C_1(x^*)}[x^*-\rho_1T_1(x^*,y^*)]\| \\ &\quad\le \|x_n-x^*-\rho_1(T_1(x_n,y_n)-T_1(x^*,y^*))\| + \eta_1\|x_n-x^*\| \\ &\quad\le [(2+\rho_1\tau_1+\eta_1)\|x_n-x^*\| + \rho_1\sigma_1\|y_n-y^*\|] \longrightarrow 0 \quad \text{as } n\to\infty. \end{aligned} \tag{2.15}$$

And, similarly,

$$\|J^{\rho_1}_{\Phi_2,C_2(y_n)}[y_n - \rho_2 T_2(x_n, y_n)] - J^{\rho_2}_{\Phi_2,C_2(y^*)}[y^* - \rho_2 T_2(x^*, y^*)]\| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{2.16}$$

Using (2.15) and (2.16), from the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$x^* = J^{\rho_1}_{\Phi_1,C_1(x^*)}[x^* - \rho_1 T_1(x^*, y^*)] \in C_1(x^*),$$
$$y^* = J^{\rho_2}_{\Phi_2,C_2(y^*)}[y^* - \rho_2 T_2(x^*, y^*)] \in C_2(y^*).$$

Thus, by Lemma 2.1, we conclude that (x^*, y^*) is a solution for problem (1.1).

Next, assume that there also exists $(u^*, v^*) \in \mathcal{H} \times \mathcal{H}$ such that $u^* \in C_1(u^*)$, $v^* \in C_2(v^*)$ and

$$u^* = J^{\rho_1}_{\Phi_1,C_1(u^*)}[u^* - \rho_1 T_1(u^*, v^*)],$$
$$v^* = J^{\rho_2}_{\Phi_2,C_2(v^*)}[v^* - \rho_2 T_2(u^*, v^*)].$$

Using the same lines as obtaining (2.12), we know that

$$\|(x^* - u^*, y^* - v^*)\|^+ \le \kappa \|(x^* - u^*, y^* - v^*)\|^+. \tag{2.17}$$

Since, $\kappa < 1$, we must have $x^* = u^*$ and $y^* = v^*$. Hence, the set of solution of problem (1.1) is a singleton. This completes the proof. □

Remark 2.4. Theorem 2.3 not only gives the conditions for the existence solution of problem (1.1) but also provide the algorithm to find such a solution for any initial vector $(x_0, y_0) \in \mathcal{H} \times \mathcal{H}$. In fact, by proceeding along the same lines as in Theorem 2.3, one can also show that the sequences $\{(x_n, y_n)\}$, defined by following Mann type perturbed iterative algorithm (MTA),

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^{\rho_1}_{\Phi_1,C_1(x_n)}[x_n - \rho_1 T_1(x_n, y_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n J^{\rho_2}_{\Phi_2,C_2(y_n)}[y_n - \rho_2 T_2(x_n, y_n)], \end{cases} \tag{2.18}$$

converges strongly to the unique solution of problem (1.1), when $\{\alpha_n\}$ is a sequence of real numbers such that $\alpha_n \in (0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let C be a fixed closed convex subset of a Hilbert space $\mathcal{H}$. If $C_1(u) = C_2(u) = C$ for all $u \in \mathcal{H}$, we have the following result.

Theorem 2.5 ([3]). *For each $i = 1, 2$, let $\Phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a monotone function and $C_i : \mathcal{H} \to \mathcal{CC}(\mathcal{H})$. Let $T_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a ν_1-strongly monotone with respect to the first argument and (τ_1, σ_1)-Lipschitz mapping and $T_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a ν_2-strongly monotone with respect to the second argument and (τ_2, σ_2)-Lipschitz mapping. Suppose that there are positive real numbers ρ_1, ρ_2 which satisfy the following condition:*

$$\begin{cases} (1 - 2\rho_1\nu_1 + \rho_1^2\tau_1^2)^{\frac{1}{2}} + \rho_2\tau_2 < 1, \\ (1 - 2\rho_2\nu_2 + \rho_2^2\tau_2^2)^{\frac{1}{2}} + \rho_1\sigma_1 < 1. \end{cases} \tag{2.19}$$

Then the set of solution of problem (1.1) is a singleton.

Proof. The result is followed immediately from Remark 2.2 and Theorem 2.3. □

3. Stability analysis

In this section, we will study stability of the Mann type perturbed iterative algorithm (2.18). Firstly, in view of fixed point formulation (2.1), the following remark is clear.

Remark 3.1. Let $(x, y) \in \mathcal{H} \times \mathcal{H}$. Then (x, y) is a solution of problem (1.1) if and only if there exist positive real numbers ρ_1, ρ_2 such that (x, y) is a fixed point of the map $G_{\rho_1,\rho_2} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ defined by

$$G_{\rho_1,\rho_2}(x, y) = (A_{\rho_1}(x, y), B_{\rho_2}(x, y)), \quad \forall (x, y) \in \mathcal{H} \times \mathcal{H}, \tag{3.1}$$

where $A_{\rho_1}, B_{\rho_2} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ are defined by

$$A_{\rho_1}(x, y) = (1-\lambda)x + \lambda J^{\rho_1}_{\Phi_1,C_1(x)}[x - \rho_1 T_1(x, y)]$$
$$B_{\rho_2}(x, y) = (1-\lambda)y + \lambda J^{\rho_2}_{\Phi_2,C_2(y)}[y - \rho_2 T_2(x, y)],$$

where $\lambda \in (0, 1)$ is a fixed constant.

Using the idea as in Theorem 2.3, we have another version for the existence solution of problem (1.1).

Theorem 3.2. *Assume that all assumptions of Theorem 2.3 hold. Then the mapping G_{ρ_1,ρ_2}, which is defined as in (3.1), has a unique fixed point.*

Now we give a definition, which can be viewed as an extension of the concept of stability of iteration procedure given by Harder and Hick [13].

Definition 3.3 ([14]). Let $\mathcal{H}$ be a Hilbert space and let $A, B : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be nonlinear mappings. Let $G : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ be defined as $G(x, y) = (A(x, y), B(x, y))$ for any $(x, y) \in \mathcal{H} \times \mathcal{H}$, and let $(x_0, y_0) \in \mathcal{H} \times \mathcal{H}$. Assume that $(x_{n+1}, y_{n+1}) = f(G, x_n, y_n)$ defines an iteration procedure which yields a sequence of $\{(x_n, y_n)\}$ in $\mathcal{H} \times \mathcal{H}$. Suppose that $F(G) = \{(x, y) \in \mathcal{H} \times \mathcal{H} : G(x, y) = (x, y)\} \neq \emptyset$ and $\{(x_n, y_n)\}$ converges to some $(x^*, y^*) \in F(G)$. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $\mathcal{H} \times \mathcal{H}$ and $\varepsilon_n = \|(u_n, v_n) - f(G, x_n, y_n)\|$, for all $n \geq 0$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty}(u_n, v_n) = (x^*, y^*)$, then the iterative procedure $\{(x_n, y_n)\}$ is said to be *G-stable* or *stable with respect to G*.

Theorem 3.4. *Assume that all conditions of Theorem 3.2 hold. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $\mathcal{H} \times \mathcal{H}$ and define $\{\delta_n\} \subset [0, \infty)$ by*

$$\delta_n = \|(u_{n+1}, v_{n+1}) - (C_n, D_n)\|^+, \tag{3.2}$$

where

$$\begin{cases} C_n = (1 - \alpha_n)x_n + \alpha_n J^{\rho_1}_{\Phi_1, C_1(x_n)}[x_n - \rho_1 T_1(x_n, y_n)], \\ D_n = (1 - \alpha_n)y_n + \alpha_n J^{\rho_2}_{\Phi_2, C_2(y_n)}[y_n - \rho_2 T_2(x_n, y_n)], \end{cases} \tag{3.3}$$

where (x_n, y_n) is defined in (2.18), for each $n \in \mathbb{N}$. If G_{ρ_1,ρ_2} is defined as in (3.1) then the iterative procedure (2.18) is G_{ρ_1,ρ_2}-stable.

Proof. Assume that $\lim_{n\to\infty} \delta_n = 0$. Let (x^*, y^*) be the unique fixed point of the mapping G_{ρ_1,ρ_2}, this means,

$$x^* = J^{\rho_1}_{\Phi_1, C_1(x^*)}[x^* - \rho_1 T_1(x^*, y^*)]$$
$$y^* = J^{\rho_1}_{\Phi_2, C_2(y^*)}[y^* - \rho_2 T_2(x^*, y^*)].$$

Now from (3.2) and (3.3), we have

$$\|(u_{n+1}, v_{n+1}) - (x^*, y^*)\|^+ \leq \delta_n + \|C_n - x^*\| + \|D_n - y^*\|. \tag{3.4}$$

Notice that $(C_n, D_n) = \{(x_{n+1}, y_{n+1})\}$ for each $n \in \mathbb{N}$, which implies that $\lim_{n\to\infty} C_n = x^*$ and $\lim_{n\to\infty} D_n = y^*$. Using this one and the assumption $\lim_{n\to\infty} \delta_n = 0$, in view of (3.4), we have $\lim_{n\to\infty}(u_{n+1}, v_{n+1}) = (x^*, y^*)$. This completes the proof. □

Remark 3.5. It is worth noting that for a suitable and appropriate choice of the operators T_1, T_2, Φ_1, Φ_2 and point-to-set mappings C_1, C_2, one can obtain a large number of various classes of variational inequalities. This means that problem (1.1) is quite general and unifying.

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ภาคผนวก 5

Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems

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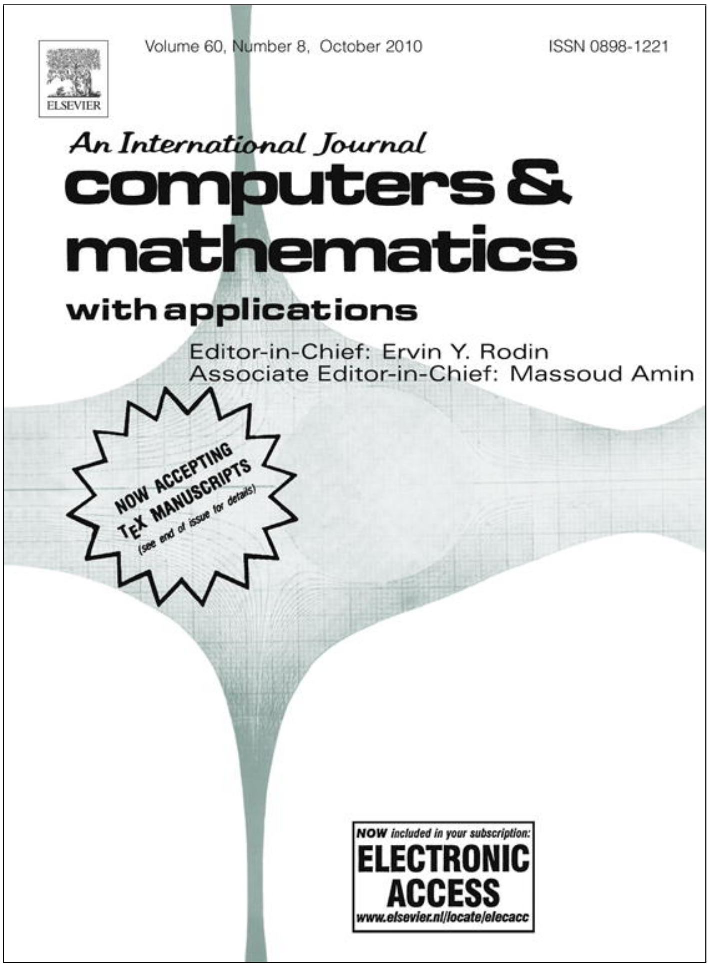
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Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems

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ABSTRACT

In this paper, we introduce an iterative method for finding a common element of the set of solutions of the generalized equilibrium problems, the set of solutions for the systems of nonlinear variational inequalities problems and the set of fixed points of nonexpansive mappings in Hilbert spaces. Furthermore, we apply our main result to the set of fixed points of an infinite family of strict pseudo-contraction mappings. The results obtained in this paper are viewed as a refinement and improvement of the previously known results.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $Q : C \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H}\times C\times C \to \mathbb{R}$ be an equilibrium-like function, that is, $\Phi(w,u,v)+\Phi(w,v,u)=0$ for all $(w,u,v)\in\mathcal{H}\times C\times C$. We consider the following *generalized equilibrium problem*:

$$\begin{cases}\text{Find } x^*\in C \text{ such that}\\ \Phi(Qx^*,x^*,y)+\varphi(y)-\varphi(x^*)\geq 0, \quad \forall y\in C.\end{cases} \tag{1.1}$$

We denote the set of solutions of the generalized equilibrium problem (1.1) by $GEP(C,Q,\Phi,\varphi)$.

Special cases of the problem (1.1) are as follows:

(I) Let $\Phi(w,u,v)=F(u,v)$, where $F : C\times C\to\mathbb{R}$. Then the problem (1.1) reduces to the following equilibrium problem:

$$\text{Find } x^*\in C \text{ such that } F(x^*,v)+\varphi(v)-\varphi(x^*)\geq 0, \quad \forall v\in C.$$

This problem was studied by Flores-Bazan [1].

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(II) If $\varphi = 0$ and $\Phi(w, u, v) = F(u, v)$, where $F : C \times C \to \mathbb{R}$, then the problem (1.1) becomes the following equilibrium problem:

$$\text{Find } x^* \in C \text{ such that } F(x^*, v) \geq 0, \quad \forall v \in C. \tag{1.2}$$

(III) If $\Phi(w, u, v) = \langle w, v - u\rangle$ for all $(w, u, v) \in \mathcal{H} \times C \times C$, then the problem (1.1) reduces to the following problem:

$$\text{Find } x^* \in C \text{ such that } \langle Qx^*, v - x^*\rangle + \varphi(v) - \varphi(x^*) \geq 0, \quad \forall v \in C.$$

This problem was studied by Dien [2] and Noor [3].

(IV) If $\varphi = 0$ and $\Phi(w, u, v) = \langle w, v - u\rangle$ for all $(w, u, v) \in \mathcal{H} \times C \times C$, then the problem (1.1) reduces to the following classical variational inequality: problem:

$$\text{Find } x^* \in C \text{ such that } \langle Qx^*, v - x^*\rangle \geq 0, \quad \forall v \in C.$$

In brief, for an appropriate choice of the mapping Q, the functions Φ, φ and the convex set C, one can obtain a number of the various classes of equilibrium problems as special cases.

In particular, the equilibrium problems (1.2) which were introduced by Blum-Oettli [4] and Noor-Oettli [5] in 1994 have had a great impact and influence on the development of several branches of pure and applied sciences. In [4,5], it has been shown that equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. This means that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. Hence collectively, equilibrium problems cover a vast range of applications.

Related to the equilibrium problems, we also have the problems of finding the fixed points of the nonlinear mappings, which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of nonlinear mappings (for examples, see [6–12] and the references therein).

On the other hand, for two nonlinear mappings $A, B : C \to \mathcal{H}$, we consider the following *system of nonlinear variational inequalities problems*:

$$\begin{cases} \text{Find } (x^*, y^*) \in C \times C \text{ such that} \\ \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \langle \rho Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

where λ and ρ are positive numbers.

In particular, if $A = B$, then the problem (1.3) was studied by Verma [13–16]. Recently, Ceng-Wang-Yao [17] considered an iterative method for the system of variational inequalities (1.3) and obtained a strong convergence theorem for the problem (1.3) and a fixed point problem for a single nonexpansive mapping (see [17] for more details).

Motivated by the recent research work going on in this fascinating field, in this paper we introduce a general iterative method for finding a common element of the set of solutions for the problem (1.1), the set of solutions for the problem (1.3) and the set of fixed points of a nonexpansive mapping. Consequently, we apply our main result to the set of fixed points of an infinite family of nonexpansive mappings and also the set of fixed points of an infinite family of strict pseudo-contraction mappings. The results obtained in this paper can be viewed as an important extension of the previously known results.

We now recall some well-known concepts and results.

Definition 1.1. A mapping $S : C \to C$ is said to be *Lipschitz continuous* if there exists a positive constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In the case $L = 1$, the mapping S is known as a *nonexpansive mapping*. If $S : C \to C$ is a mapping, we denote the set of fixed points of S by $F(S)$, that is, $F(S) = \{x \in C : Sx = x\}$.

Let C be a nonempty closed convex subset of $\mathcal{H}$. It is well known that, for any $z \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C z$, such that

$$\|z - P_C z\| \leq \|z - y\|, \quad \forall y \in C.$$

Such a mapping P_C is called the *metric projection* of $\mathcal{H}$ on to C. We know that P_C is nonexpansive. Furthermore, for any $z \in \mathcal{H}$ and $u \in C$,

$$u = P_C z \Longleftrightarrow \langle u - z, w - u\rangle \geq 0, \quad \forall w \in C. \tag{1.4}$$

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $Q : C \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function. Let r be a positive number. For any $x \in C$, we consider the following problem:

$$\begin{cases} \text{Find } y \in C \text{ such that} \\ \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \dfrac{1}{r}\langle y - x, z - y\rangle \geq 0, \quad \forall z \in C, \end{cases} \tag{1.5}$$

which is known as the *auxiliary generalized equilibrium problem*.

Let $T^{(r)} : C \to C$ be the mapping such that, for each $x \in C$, $T^{(r)}(x)$ is the solution set of the auxiliary problem (1.5), i.e.,

$$T^{(r)}(x) = \left\{ y \in C : \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \quad \forall z \in C \right\}, \forall x \in C.$$

From now on, we will assume the following *Condition* (Δ):

(a) $T^{(r)}$ is single-valued;
(b) $T^{(r)}$ is nonexpansive;
(c) $F(T^{(r)}) = GEP(C, Q, \Phi, \varphi)$.

The following example shows the sufficient conditions for the existence of the Condition (Δ).

Example 1.2 (*[7]*). Let C be a nonempty bounded closed convex subset of a real Hilbert space $\mathcal{H}$ and $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex functional. Let $Q : C \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function that satisfies the following conditions:

(Φ1) for any fixed $y \in C$, $(w, x) \mapsto \Phi(w, x, y)$ is an upper semi-continuous function from $\mathcal{H} \times C$ to $\mathbb{R}$, that is, whenever $w_n \to w$ and $x_n \to x$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \Phi(w_n, x_n, y) \leq \Phi(w, x, y);$$

(Φ2) for any fixed $(w, y) \in \mathcal{H} \times C$, $x \mapsto \Phi(w, x, y)$ is a concave function;
(Φ3) for any fixed $(w, x) \in \mathcal{H} \times C$, $y \mapsto \Phi(w, x, y)$ is a convex function.

Then (a) and (c) of the Condition (Δ) hold true. If, in addition, the mapping $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ satisfies the following:

$$\Phi(Qx_1, T^{(r)}(x_1), T^{(r)}(x_2)) + \Phi(Qx_2, T^{(r)}(x_2), T^{(r)}(x_1)) \leq 0, \quad \forall (x_1, x_2) \in C \times C,$$

then the mapping $T^{(r)}$ is firmly nonexpansive, that is,

$$\|T^{(r)}u - T^{(r)}v\|^2 \leq \langle T^{(r)}u - T^{(r)}v, u - v\rangle, \quad \forall u, v \in C.$$

Remark 1.3. The boundedness of the convex set C in the Example 1.2 can be replaced by the following weaker condition: For any $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that

$$\Phi(Qx, y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle y - x, z_x - y\rangle < 0, \quad \forall y \in C \setminus D_x.$$

Now, assuming that the Condition (Δ) is satisfied, then we can introduce the following algorithm:

Algorithm (I). Let ρ and λ be two positive numbers. Let $A, B : C \to \mathcal{H}$ and $S : C \to C$ be mappings. For any $u, x_1 \in C$, there exist sequences $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ in C such that

$$\begin{cases} \Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = P_C(x_n - \rho Bx_n), \\ z_n = P_C(y_n - \lambda Ay_n), \\ x_{n+1} = a_n u + b_n x_n + c_n [\gamma_1 Sx_n + \gamma_2 u_n + \gamma_3 z_n], \quad \forall n \geq 1, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0, 1]$ and $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = 1$.

Of course, we will use the Algorithm (I) to obtain our main results in this paper. To do this, we also need the following lemmas:

Lemma 1.4 (*[18]*). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. If, for each $n \geq 1$, $T_n : C \to C$ is a nonexpansive mapping, then there exists a nonexpansive mapping $T : C \to C$ such that*

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

In particular, if $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, then the mapping $T = \sum_{n=1}^{\infty} \mu_n T_n$ satisfies the above requirement, where $\{\mu_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \mu_n = 1$.

Lemma 1.5 (*[17]*). *Let ρ and λ be positive numbers. For any $x^*, y^* \in C$ with $y^* = P_C(x^* - \rho Bx^*)$, (x^*, y^*) is a solution of the problem (1.3) if and only if x^* is a fixed point of the mapping $D : C \to C$ defined by*

$$D(x) = P_C[P_C(x - \rho Bx) - \lambda AP_C(x - \rho Bx)], \quad \forall x \in C.$$

Lemma 1.6 ([19]). *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero, i.e., if $\{x_n\}$ converges weakly to a point $x \in C$ and $\{x_n - Sx_n\}$ converges to zero, then $x = Sx$.*

Lemma 1.7 ([20]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space E and b_n be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1.$$

Suppose that $x_{n+1} = (1 - b_n)l_n + b_n x_n$ for all $n \ge 1$ and

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|l_n - x_n\| = 0$.

Lemma 1.8 ([21]). *Assume that $\{\theta_n\}$ is a sequence of nonnegative real numbers such that*

$$\theta_{n+1} \le (1 - a_n)\theta_n + \delta_n, \quad \forall n \ge 1,$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} a_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{a_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \theta_n = 0$.

2. Main results

Now, we are in a position to state and prove our main results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : C \to \mathcal{H}$ be two nonlinear mappings and $S : C \to C$ be a nonexpansive mapping. Assume that the Condition (Δ) is satisfied and*

$$\Omega = GEP(C, Q, \Phi, \varphi) \cap F(S) \cap F(D) \ne \emptyset,$$

where the mapping D is defined by Lemma 1.5. Let $u \in C$ be fixed and $\{u_n\}, \{y_n\}, \{z_n\}, \{x_n\}$ be four sequences in C generated by Algorithm (I). If the following conditions are satisfied:

(i) *$(I - \lambda A)$ and $(I - \rho B)$ are nonexpansive mappings, where ρ and λ are positive constants;*
(ii) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*
(iii) *$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$,*

then the sequence $\{x_n\}$ defined by the Algorithm (I) converges strongly to a point $\widetilde{x} = P_{\Omega}u$. Moreover, if $\widetilde{y} = P_C(\widetilde{x} - \rho B\widetilde{x})$, then $(\widetilde{x}, \widetilde{y})$ is a solution to the problem (1.3).

Proof. Note that the second part follows directly from the first part and Lemma 1.5. Now, the proof of the first part is divided into the six steps as follows:

Step 1: P_{Ω} is well defined.
In fact, firstly, since $T^{(r)}$ is a nonexpansive mapping, $\Omega \ne \emptyset$ and

$$F(T^{(r)}) = GEP(C, Q, \Phi, \varphi),$$

we have $GEP(C, Q, \Phi, \varphi)$ is a nonempty closed convex set.

Next, by the definition of the mapping D, we observe that

$$D = P_C[P_C(I - \rho B) - \lambda AP_C(I - \rho B)] = P_C(I - \lambda A)P_C(I - \rho B).$$

Consequently, since $I - \lambda A$ and $I - \rho B$ are nonexpansive mappings, we know that D is a nonexpansive mapping and hence $F(D)$ is a closed convex set.

On the other hand, since the mapping S is nonexpansive, we have the set $F(S)$ is a closed convex subset of $\mathcal{H}$. Therefore, it follows that $\Omega = GEP(C, Q, \Phi, \varphi) \cap F(D) \cap F(S)$ is a nonempty closed convex subset of $\mathcal{H}$. Thus the mapping P_{Ω} is well defined.

Step 2: The sequence $\{x_n\}$ is bounded.
In fact, let $x^* \in \Omega$. Since $x^* = Dx^*$, we have

$$x^* = P_C\left[P_C(x^* - \rho Bx^*) - \lambda AP_C(x^* - \rho Bx^*)\right].$$

Putting $y^* = P_C(x^* - \rho Bx^*)$, we have

$$x^* = P_C(y^* - \lambda Ay^*).$$

Let $e_n = \gamma_1 Sx_n + \gamma_2 u_n + \gamma_3 z_n$ for all $n \geq 1$ and consider the following computation:

$$\begin{aligned}\|e_n - x^*\| &= \|\gamma_1 Sx_n + \gamma_2 u_n + \gamma_3 z_n - x^*\| \\ &\leq \gamma_1\|Sx_n - x^*\| + \gamma_2\|u_n - x^*\| + \gamma_3\|z_n - x^*\| \\ &\leq \gamma_1\|x_n - x^*\| + \gamma_2\|T^{(r)}x_n - T^{(r)}x^*\| + \gamma_3\|P_C(I-\lambda A)y_n - P_C(y^* - \lambda Ay^*)\| \\ &\leq \gamma_1\|x_n - x^*\| + \gamma_2\|x_n - x^*\| + \gamma_3\|y_n - y^*\| \\ &= \gamma_1\|x_n - x^*\| + \gamma_2\|x_n - x^*\| + \gamma_3\|P_C(I-\rho B)x_n - P_C(I-\rho B)x^*\| \\ &\leq \gamma_1\|x_n - x^*\| + \gamma_2\|x_n - x^*\| + \gamma_3\|x_n - x^*\| \\ &= \|x_n - x^*\|, \quad \forall n \geq 1,\end{aligned}$$

and

$$\begin{aligned}\|x_2 - x^*\|^2 &= \|a_1 u + b_1 x_1 + c_1 e_1 - x^*\|^2 \\ &\leq a_1\|u - x^*\| + b_1\|x_1 - x^*\| + c_1\|e_1 - x^*\| \\ &\leq a_1\|u - x^*\| + b_1\|x_1 - x^*\| + c_1\|x_1 - x^*\| \\ &\leq a_1\|u - x^*\| + (1-a_1)\|x_1 - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.\end{aligned} \tag{2.1}$$

From (2.1) and induction, we know that the sequence $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$.

Step 3: $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

To do this, in view of condition (iii), without loss of generality we may assume that $b_n \in (0, 1)$ for all $n \in \mathbb{N}$. Consequently, this allows us to put

$$l_n = \frac{x_{n+1} - b_n x_n}{1 - b_n}, \quad \forall n \geq 1, \tag{2.2}$$

which implies that

$$x_{n+1} - x_n = (1-b_n)(l_n - x_n), \quad \forall n \geq 1. \tag{2.3}$$

Now, by (2.2), (2.3), Lemma 1.7 and condition (iii), we show that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.4}$$

First, we compute $l_{n+1} - l_n$. Consider the following computation:

$$\begin{aligned}l_{n+1} - l_n &= \frac{a_{n+1}u + c_{n+1}e_{n+1}}{1 - b_{n+1}} - \frac{a_n u + c_n e_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1-b_{n+1}}u + \frac{1 - b_{n+1} - a_{n+1}}{1 - b_{n+1}}e_{n+1} - \frac{a_n}{1-b_n}u - \frac{1-b_n-a_n}{1-b_n}e_n \\ &= \frac{a_{n+1}}{1-b_{n+1}}(u - e_{n+1}) + \frac{a_n}{1-b_n}(e_n - u) + e_{n+1} - e_n, \quad \forall n \geq 1,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\|e_{n+1} - e_n\| &= \|\gamma_1 Sx_{n+1} + \gamma_2 u_{n+1} + \gamma_3 z_{n+1} - (\gamma_1 Sx_n + \gamma_2 u_n + \gamma_3 z_n)\| \\ &\leq \gamma_1\|Sx_{n+1} - Sx_n\| + \gamma_2\|u_{n+1} - u_n\| + \gamma_3\|z_{n+1} - z_n\| \\ &= \gamma_1\|Sx_{n+1} - Sx_n\| + \gamma_2\|T^{(r)}x_{n+1} - T^{(r)}x_n\| + \gamma_3\|z_{n+1} - z_n\| \\ &\leq \gamma_1\|x_{n+1} - x_n\| + \gamma_2\|x_{n+1} - x_n\| + \gamma_3\|z_{n+1} - z_n\|, \quad \forall n \geq 1,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|P_C(y_{n+1} - \lambda Ay_{n+1}) - P_C(y_n - \lambda Ay_n)\| \\ &\leq \|(I-\lambda A)y_{n+1} - (I-\lambda A)y_n\| \\ &\leq \|y_{n+1} - y_n\| \\ &= \|P_C(x_{n+1} - \rho Bx_{n+1}) - P_C(x_n - \rho Bx_n)\| \\ &\leq \|(I-\rho B)x_{n+1} - (I-\rho B)x_n\| \\ &\leq \|x_{n+1} - x_n\|, \quad \forall n \geq 1.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) yields that

$$\|e_{n+1}-e_n\| \le \|x_{n+1}-x_n\|, \quad \forall n \ge 1. \tag{2.8}$$

Using (2.5) and (2.8), we have

$$\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\| \le \frac{a_{n+1}}{1-b_{n+1}}\|u-e_{n+1}\| + \frac{a_n}{1-b_n}\|e_n-u\|, \quad \forall n \ge 1. \tag{2.9}$$

thus it follows from conditions (ii) and (iii) that

$$\limsup_{n\to\infty} (\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|) \le 0,$$

that is, (2.4) is satisfied.

Step 4: $x_n - e_n \to 0$ as $n \to \infty$.
From Algorithm (I), we have

$$c_n(e_n-x_n) = x_{n+1}-x_n + a_n(x_n-u),$$

which implies that

$$c_n\|e_n-x_n\| \le \|x_{n+1}-x_n\| + a_n\|(x_n-u)\|$$

and so, from conditions (ii) and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, it follows that

$$\lim_{n\to\infty}\|e_n-x_n\| = 0. \tag{2.10}$$

Step 5: $\limsup_{n\to\infty}\langle u-\widetilde{x}, x_n-\widetilde{x}\rangle \le 0$, where $\widetilde{x} = P_\Omega u$.

Since $\{x_n\}$ is a bounded sequence, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in C$ such that $\{x_{n_j}\}$ converges weakly to a point p as $j \to \infty$ and

$$\limsup_{n\to\infty}\langle u-\widetilde{x}, x_n-\widetilde{x}\rangle = \limsup_{j\to\infty}\langle u-\widetilde{x}, x_{n_j}-\widetilde{x}\rangle. \tag{2.11}$$

Now, we show that $p \in \Omega = GEP(C, Q, \Phi, \varphi) \cap F(D) \cap F(S)$. To show this, define a mapping $G : C \to C$ by

$$Gx = \gamma_1 Sx + \gamma_2 T^{(r)}x + \gamma_3 Dx, \quad \forall x \in C.$$

From Lemma 1.4, it follows that G is a nonexpansive mapping such that

$$F(G) = F(S) \cap F(T^{(r)}) \cap F(D).$$

On the other hand, from (2.10), we obtain

$$\lim_{j\to\infty}\|Gx_{n_j}-x_{n_j}\| = 0.$$

Thus, by Lemma 1.6, we have $p \in F(G) = \Omega$. Consequently, from (1.4) and (2.11), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-\widetilde{x}, x_n-\widetilde{x}\rangle &= \limsup_{j\to\infty}\langle u-\widetilde{x}, x_{n_j}-\widetilde{x}\rangle\\ &= \langle u-\widetilde{x}, p-\widetilde{x}\rangle\\ &\le 0.\end{aligned} \tag{2.12}$$

Step 6: $x_n \to \widetilde{x}$ as $n \to \infty$.
Notice that

$$\begin{aligned}\|x_{n+1}-\widetilde{x}\|^2 &= \|a_n u + b_n x_n + c_n e_n - \widetilde{x}\|^2\\ &= \langle a_n(u-\widetilde{x}) + b_n(x_n-\widetilde{x}) + c_n(e_n-\widetilde{x}), x_{n+1}-\widetilde{x}\rangle\\ &\le a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + b_n\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\| + c_n\|e_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\|\\ &\le a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + b_n\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\| + c_n\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\|\\ &= a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + (1-a_n)\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\|\\ &\le a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + \frac{(1-a_n)}{2}\left(\|x_n-\widetilde{x}\|^2 + \|x_{n+1}-\widetilde{x}\|^2\right).\end{aligned} \tag{2.13}$$

This implies that

$$\|x_{n+1}-\widetilde{x}\|^2 \le (1-a_n)\|x_n-\widetilde{x}\| + 2a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle. \tag{2.14}$$

Therefore, using (2.12) together with the conditions (ii) and (iii), (2.14) and Lemma 1.8, it follows that $x_n \to \widetilde{x}$ as $n \to \infty$. This completes the proof. □

Now, we give an example for the nonlinear mappings $A, B : C \to \mathcal{H}$ given in Theorem 2.1.
Recall that a nonlinear mapping $A : C \to \mathcal{H}$ is said to be:
(1) α-*cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax-Ay, x-y\rangle \ge \alpha\|Ax-Ay\|^2, \quad \forall x, y \in C;$$

(2) β-*strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Ax-Ay, x-y\rangle \ge \beta\|x-y\|^2, \quad \forall x, y \in C;$$

(3) *relaxed* (ζ, β)-*cocoercive* if there exist constants $\zeta, \beta > 0$ such that

$$\langle Ax-Ay, x-y\rangle \ge (-\zeta)\|Ax-Ay\|^2 + \beta\|x-y\|^2, \quad \forall x, y \in C.$$

Example 2.2. Let $A : C \to \mathcal{H}$ be a nonlinear mapping and λ be a positive constant. Assume that
(A1) A is α-cocoercive mapping and $\lambda \in (0, 2\alpha]$;
(A2) A is β-strongly monotone and L-Lipschitz continuous mapping and $\lambda \in \left(0, \frac{2\beta}{L}\right]$;
(A3) A is relaxed (ζ, β)-cocoercive and L-Lipschitz continuous mapping with $\beta - L\zeta > 0$ and $\lambda \in \left(0, \frac{2(\beta-L\zeta)}{L}\right]$.

If, either (A1), (A2) or (A3) is satisfied, then $I - \lambda A$ is a nonexpansive mapping. Indeed, if (A1) is satisfied, then we have

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y-\lambda(Ax-Ay)\|^2\\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2 - 2\alpha\lambda\|Ax-Ay\|^2 + \lambda^2\|Ax-Ay\|^2\\ &= \|x-y\|^2 - \lambda(2\alpha-\lambda)\|Ax-Ay\|^2\\ &\le \|x-y\|^2, \quad \forall x, y \in C.\end{aligned}$$

Similarly, by using (A2) or (A3), we can show that $I - \lambda A$ is a nonexpansive mapping.
Using the technique as in Theorem 2.1, one can prove the following results.

Corollary 2.3. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : C \to \mathcal{H}$ be two nonlinear mappings. Assume that the Condition (Δ) is satisfied and*

$$\Omega = GEP(C, Q, \Phi, \varphi) \cap F(D) \ne \emptyset,$$

where the mapping D is defined by Lemma 1.5. Let $u \in C$ be fixed and $\{u_n\}, \{y_n\}, \{z_n\}, \{x_n\}$ be four sequences in C generated by

$$\begin{cases}\Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \dfrac{1}{r}\langle u_n - x_n, v - u_n\rangle \ge 0, \quad \forall v \in C,\\ y_n = P_C(x_n - \rho Bx_n),\\ z_n = P_C(y_n - \lambda Ay_n),\\ x_{n+1} = a_n u + b_n x_n + c_n[\gamma u_n + (1-\gamma)z_n], \quad \forall n \ge 1,\end{cases} \tag{2.15}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0, 1]$ and $\gamma \in (0, 1)$ such that $a_n + b_n + c_n = 1$ for all $n \ge 1$. If the following conditions are satisfied:

(i) *$(I - \lambda A)$ and $(I - \rho B)$ are nonexpansive mappings, where ρ and λ are positive constants;*
(ii) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*
(iii) *$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$,*

then the sequence $\{x_n\}$ defined by the iterative algorithm (2.15) converges strongly to a point $\widetilde{x} = P_\Omega u$. Moreover, if $\widetilde{y} = P_C(\widetilde{x} - \rho B\widetilde{x})$, then $(\widetilde{x}, \widetilde{y})$ is a solution to the problem (1.3).

Corollary 2.4. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : C \to \mathcal{H}$ be two nonlinear mappings and $S : C \to C$ be a nonexpansive mapping. Assume that*

$$\Omega = F(D) \cap F(S) \ne \emptyset,$$

where the mapping D is defined by Lemma 1.5. Let $u \in C$ be fixed and $\{y_n\}, \{z_n\}, \{x_n\}$ be three sequences in C generated by

$$\begin{cases} y_n = P_C(x_n - \rho B x_n), \\ z_n = P_C(y_n - \lambda A y_n), \\ x_{n+1} = a_n u + b_n x_n + c_n[\gamma S x_n + (1-\gamma) z_n], \quad \forall n \geq 1, \end{cases} \tag{2.16}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0, 1]$ and $\gamma \in (0, 1)$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$. If the following conditions are satisfied:

(i) *$(I - \lambda A)$ and $(I - \rho B)$ are nonexpansive mappings, where ρ and λ are positive constants;*
(ii) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*
(iii) *$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*

then the sequence $\{x_n\}$ defined by the iterative algorithm (2.16) converges strongly to a point $\widetilde{x} = P_\Omega u$. Moreover, if $\widetilde{y} = P_C(\widetilde{x} - \rho B \widetilde{x})$, then $(\widetilde{x}, \widetilde{y})$ is a solution to the problem (1.3).

Corollary 2.5. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $S : C \to C$ be a nonexpansive mappings. Assume that the Condition (Δ) is satisfied and*

$$\Omega = GEP(C, Q, \Phi, \varphi) \cap F(S) \neq \emptyset.$$

Let $u \in C$ be fixed and $\{u_n\}, \{x_n\}$ be two sequences in C generated by

$$\begin{cases} \Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \dfrac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = a_n u + b_n x_n + c_n[\gamma S x_n + (1-\gamma) u_n], \quad \forall n \geq 1, \end{cases} \tag{2.17}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0, 1]$ and $\gamma \in (0, 1)$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$. If the following conditions are satisfied:

(i) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*
(ii) *$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*

then the sequence $\{x_n\}$ defined by the iterative algorithm (2.17) converges strongly to a point $\widetilde{x} = P_\Omega u$.

Corollary 2.6. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : C \to \mathcal{H}$ be two nonlinear mappings. Assume that $F(D) \neq \emptyset$, where the mapping D is defined by Lemma 1.5. Let $u \in C$ be fixed and $\{y_n\}, \{z_n\}, \{x_n\}$ be three sequences in C generated by*

$$\begin{cases} y_n = P_C(x_n - \rho B x_n), \\ z_n = P_C(y_n - \lambda A y_n), \\ x_{n+1} = a_n u + b_n x_n + c_n z_n, \quad \forall n \geq 1, \end{cases} \tag{2.18}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$. If the following conditions are satisfied:

(i) *$(I - \lambda A)$ and $(I - \rho B)$ are nonexpansive mappings, where ρ and λ are positive constants;*
(ii) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*
(iii) *$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*

then the sequence $\{x_n\}$ defined by the iterative algorithm (2.18) converges strongly to a point $\widetilde{x} = P_{F(D)} u$. Moreover, if $\widetilde{y} = P_C(\widetilde{x} - \rho B \widetilde{x})$, then $(\widetilde{x}, \widetilde{y})$ is a solution to the problem (1.3).

Corollary 2.7. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Assume that the Condition (Δ) is satisfied. Let $u \in C$ be fixed and $\{u_n\}, \{x_n\}$ be two sequences in C generated by*

$$\begin{cases} \Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \dfrac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = a_n u + b_n x_n + c_n u_n, \quad \forall n \geq 1, \end{cases} \tag{2.19}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$. If the following conditions are satisfied:

(i) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*
(ii) *$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,*

then the sequence $\{x_n\}$ defined by the iterative algorithm (2.19) converges strongly to a point $\widetilde{x} = P_{GEP(C,Q,\Phi,\varphi)} u$.

Corollary 2.8. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $S : C \to C$ be a nonexpansive mappings with $F(S) \neq \emptyset$. Let $u \in C$ be fixed and $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = a_n u + b_n x_n + c_n S x_n, \quad \forall n \geq 1, \tag{2.20}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$,

then the sequence $\{x_n\}$ *defined by the iterative algorithm* (2.20) *converges strongly to a point* $\widetilde{x} = P_{F(S)}u$.

Remark 2.9. If $f : C \to C$ is a contractive mapping and we replace u by $f(x_n)$ in the Algorithm (I), then we can obtain the so-called viscosity iteration method (see [22] for more details).

3. Applications

Let $\{S_n\}$ be a family of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\mu_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \mu_n = 1$. From Lemma 1.4, we know that the mapping $S : C \to C$ defined by

$$Sx = \sum_{n=1}^{\infty} \mu_n S_n x, \quad \forall x \in C,$$

is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$.

Using this fact, as an application of Theorem 2.1, we have the following result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : C \to \mathcal{H}$ be two nonlinear mappings and $\{S_n\}$ be a family of nonexpansive mappings from C into itself. Assume that the Condition (Δ) is satisfied and*

$$\Omega = \bigcap_{n=1}^{\infty} (GEP(C, Q, \Phi, \varphi) \cap F(S_n) \cap F(D)) \neq \emptyset,$$

where the mapping D is defined by Lemma 1.5. Let $u \in C$ be fixed and $\{u_n\}, \{y_n\}, \{z_n\}, \{x_n\}$ be four sequences generated by Algorithm (I) with $S = \sum_{n=1}^{\infty} \mu_n S_n$, where $\{\mu_n\}$ is a sequence of positive numbers with $\sum_{n=1}^{\infty} \mu_n = 1$. If the following conditions are satisfied:

(i) $(I - \lambda A)$ *and* $(I - \rho B)$ *are nonexpansive mappings, where* λ *and* ρ *are positive constants;*
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$,

then the sequence $\{x_n\}$ defined by the Algorithm (I) converges strongly to a point $\widetilde{x} = P_\Omega u$. Moreover, if $\widetilde{y} = P_C(\widetilde{x} - \rho B\widetilde{x})$, then $(\widetilde{x}, \widetilde{y})$ is a solution to the problem (1.3).

Recall that a mapping $W : C \to C$ is called a *τ-strict pseudo-contraction* with the coefficient $\tau \in [0, 1)$ if

$$\|Wx - Wy\|^2 \le \|x - y\|^2 + \tau\|(I - W)x - (I - W)y\|^2, \quad \forall x, y \in C.$$

It is obvious that every nonexpansive self-mapping is a 0-strict pseudo-contraction and, furthermore, the following result is well known:

Lemma 3.2 ([23]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $W : C \to C$ a τ-strict pseudo-contraction. Define a mapping $W^{(\zeta)} : C \to C$ by $W^{(\zeta)}x = \zeta x + (1 - \zeta)Wx$ for all $x \in C$, where $\zeta \in [\tau, 1)$ is a fixed constant. Then $W^{(\zeta)}$ is a nonexpansive mapping such that $F(W^{(\zeta)}) = F(W)$.*

Now, let $\{W_n\}$ be a family of τ_n-strict pseudo-contractions for each $n \ge 1$. Observe that, from Lemma 3.2, it follows that $\{W_n^{(\tau_n)}\}$ is a family of nonexpansive mappings from C into itself, where $W_n^{(\tau_n)}$ is defined as in Lemma 3.2 for each $n \ge 1$.

Using this observation, as an application of the Theorem 2.1, we have the following result.

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $A, B : C \to \mathcal{H}$ be two nonlinear mappings. Let $\{W_n\}$ be a family of τ_n-strict pseudo-contractions from C into itself with coefficient τ_n for each $n \ge 1$. Assume that the Condition (Δ) is satisfied and*

$$\Omega = \bigcap_{n=1}^{\infty} (GEP(C, Q, \Phi, \varphi) \cap F(W_n) \cap F(D)) \neq \emptyset,$$

where the mapping D is defined by Lemma 1.5. Let $u \in C$ be fixed and $\{u_n\}, \{y_n\}, \{z_n\}, \{x_n\}$ be four sequences generated by Algorithm (I) with $S = \sum_{n=1}^{\infty} \mu_n W_n^{(\tau_n)}$, where $\{\mu_n\}$ is a sequence of positive numbers with $\sum_{n=1}^{\infty} \mu_n = 1$ and $W_n^{(\tau_n)}$ is defined as in Lemma 3.2 for each $n \ge 1$. If the following conditions are satisfied:

(i) $(I - \lambda A)$ *and* $(I - \rho B)$ *are nonexpansive mappings, where* λ *and* ρ *are positive constants;*
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$,

then the sequence $\{x_n\}$ *defined by the Algorithm* (I) *converges strongly to a point* $\widetilde{x} = P_{\Omega}u$. *Moreover, if* $\widetilde{y} = P_C(\widetilde{x} - \rho B\widetilde{x})$, *then* $(\widetilde{x}, \widetilde{y})$ *is a solution to the problem* (1.3).

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* $\mathcal{H}$ *and* $A, B : C \to \mathcal{H}$ *be two nonlinear mappings. Let* $W : C \to C$ *be a* τ*-strict pseudo-contraction. Assume that the Condition* (Δ) *is satisfied and*

$$\Omega = GEP(C, Q, \Phi, \varphi) \cap F(W) \cap F(D) \neq \emptyset,$$

where the mapping D *is defined by Lemma* 1.5. *Let* $u \in C$ *be fixed and* $\{u_n\}, \{y_n\}, \{z_n\}, \{x_n\}$ *be four sequences generated by Algorithm* (I) *with* $S = W^{(\tau)}$. *If the following conditions are satisfied:*

(i) $(I - \lambda A)$ *and* $(I - \rho B)$ *are nonexpansive mappings, where* λ *and* ρ *are positive constants;*
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,

then the sequence $\{x_n\}$ *defined by the Algorithm* (I) *converges strongly to a point* $\widetilde{x} = P_{\Omega}u$. *Moreover, if* $\widetilde{y} = P_C(\widetilde{x} - \rho B\widetilde{x})$, *then* $(\widetilde{x}, \widetilde{y})$ *is a solution to the problem* (1.3).

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ภาคผนวก 6

Some existence theorems for nonconvex variational inequalities problems

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Research Article

Some Existence Theorems for Nonconvex Variational Inequalities Problems

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By using nonsmooth analysis knowledge, we provide the conditions for existence solutions of the variational inequalities problems in nonconvex setting. We also show that the strongly monotonic assumption of the mapping may not need for the existence of solutions. Consequently, the results presented in this paper can be viewed as an improvement and refinement of some known results from the literature.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], provides us with a simple, natural, general, and unified framework to study a wide class of problems arising in pure and applied sciences. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

It should be pointed out that almost all the results regarding the existence and iterative schemes for solving variational inequalities and related optimizations problems are being considered in the convexity setting; see [2–5] for examples. Moreover, all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. Notice that the convexity assumption, made by researchers, has been used for guaranteeing the well definedness of the proposed iterative algorithm which depends on the projection mapping. In fact, the convexity assumption may not require for the well definedness of the projection mapping because it may be well defined, even in the nonconvex case (e.g., when the considered set is a closed subset of a finite dimensional space or a compact subset of a Hilbert space, etc.).

The main aim of this paper is intending to consider the conditions for the existence solutions of some variational inequalities problems in nonconvex setting. We will make use of some recent nonsmooth analysis techniques to overcome the difficulties that arise from the nonconvexity. Also, it is worth mentioning that we have considered when the mapping may not satisfy the strongly monotonic assumption. In this sense, our result represents an improvement and refinement of the known results.

2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed subset of $\mathscr{H}$. We denote by $d_C(\cdot)$ the usual distance function to the subset C; that is, $d_C(u)=\inf_{v\in C}\|u-v\|$. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u\in\mathscr{H}$ be a point not lying in C. A point $v\in C$ is called a closest point or a projection of u onto C if $d_C(u)=\|u-v\|$. The set of all such closest points is denoted by $\operatorname{proj}_C(u)$; that is,

$$\operatorname{proj}_C(u)=\{v\in C: d_C(u)=\|u-v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of $\mathscr{H}$. The proximal normal cone to C at x is given by

$$N_C^P(x)=\{z\in\mathscr{H}:\exists\rho>0; x\in\operatorname{proj}_C(x+\rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [6].

Lemma 2.3. *Let C be a closed subset of a Hilbert space $\mathscr{H}$. Then,*

$$z\in N_C^P(x)\Longleftrightarrow\exists\sigma>0,\qquad\langle z,y-x\rangle\le\sigma\|y-x\|^2,\quad\forall y\in C. \tag{2.3}$$

Clarke et al. [7] and Poliquin et al. [8] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r\in(0,+\infty]$, a subset C of $\mathscr{H}$ is said to be uniformly prox-regular with respect to r if, for all $\overline{x}\in C$ and for all $0\neq z\in N_C^P(\overline{x})$, one has

$$\left\langle\frac{z}{\|z\|},x-\overline{x}\right\rangle\le\frac{1}{2r}\|x-\overline{x}\|^2,\quad\forall x\in C. \tag{2.4}$$

We make the convention $1/r=0$ for $r=+\infty$.

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r>0$. Thus, in view of Definition 2.4, for the case of $r=\infty$, the uniform r-prox-regularity C is equivalent to the convexity of C. Moreover, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $\mathscr{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [6, 8].

Now, let us state the following facts, which summarize some important consequences of the uniform prox-regularity. The proof of this result can be found in [7, 8].

Lemma 2.5. *Let C be a nonempty closed subset of $\mathscr{H}$, $r \in (0, +\infty]$ and set $C_r := \{x \in \mathscr{H}; d(x, C) < r\}$. If C is uniformly r-uniformly prox-regular, then the following hold:*

(1) *for all $x \in C_r$, $\text{proj}_C(x) \neq \emptyset$,*

(2) *for all $s \in (0, r)$, proj_C is Lipschitz continuous with constant $r/(r - s)$ on C_s,*

(3) *the proximal normal cone is closed as a set-valued mapping.*

In this paper, we are interested in the following classes of nonlinear mappings.

Definition 2.6. A mapping $T : C \to \mathscr{H}$ is said to be

(a) *γ-strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|x - y\|^2, \quad \forall x, y \in C, \tag{2.5}$$

(b) *μ-Lipschitz* if there exist a constants $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \quad \forall x, y \in C. \tag{2.6}$$

3. System of Nonconvex Variational Inequalities Involving Nonmonotone Mapping

Let $\mathscr{H}$ be a real Hilbert space, and let C be a nonempty closed subset of $\mathscr{H}$. In this section, we will consider the following problem: find $x^*, y^* \in C$ such that

$$\begin{aligned} y^* - x^* - \rho T y^* &\in N_C^P(x^*), \\ x^* - y^* - \eta T x^* &\in N_C^P(y^*), \end{aligned} \tag{3.1}$$

where ρ and η are fixed positive real numbers, C is a closed subset of $\mathscr{H}$, and $T : C \to \mathscr{H}$ is a mapping.

The iterative algorithm for finding a solution of the problem (3.1) was considered by Moudafi [9], when C is r-uniformly prox-regular and T is a strongly monotone mapping. He also remarked that two-step models (3.1) for nonlinear variational inequalities are relatively more challenging than the usual variational inequalities since it can be applied to problems arising, especially from complementarity problems, convex quadratic programming, and other variational problems. In this section, we will generalize such result by considering the conditions for existence solution of problem (3.1) when T is not necessary strongly monotone. To do so, we will use the following algorithm as an important tool.

Algorithm 3.1. Let C be an r-uniformly prox-regular subset of $\mathscr{H}$. Assume that $T : C \to \mathscr{H}$ is a nonlinear mapping. Letting x_0 be an arbitrary point in C, we consider the following two-step projection method:

$$y_n = \operatorname{proj}_C[x_n - \eta(Tx_n)],$$
$$x_{n+1} = \operatorname{proj}_C[y_n - \rho(Ty_n)], \tag{3.2}$$

where ρ, η are positive reals number, which were appeared in problem (3.1).

Remark 3.2. The projection algorithm above has been introduced in the convex case, and its convergence was proved see [10]. Observe that (3.2) is well defined provided the projection on C is not empty. Our adaptation of the projection algorithm will be based on Lemma 2.5.

Now we will prove the existence theorems of problem (3.1), when C is a closed uniformly r-prox-regular. Moreover, from now on, the number r will be understood as a finite positive real number (if not specified otherwise). This is because, as we know, if $r = \infty$, then such a set C is nothing but the closed convex set.

We start with an important remark.

Remark 3.3. Let C be a uniformly r-prox-regular closed subset of $\mathscr{H}$. Let $T_1, T_2 : C \to \mathscr{H}$ be such that T_1 is a μ_1-Lipschitz continuous, γ-strongly monotone mapping and T_2 is a μ_2-Lipschitz continuous mapping. If $\xi = r[\mu_1^2 - \gamma\mu_2 - \sqrt{(\mu_1^2 - \gamma\mu_2)^2 - \mu_1^2(\gamma - \mu_2)^2}]/\mu_1^2$, then for each $s \in (0, \xi)$ we have

$$\gamma t_s - \mu_2 > \sqrt{(\mu_1^2 - \mu_2^2)(t_s^2 - 1)}, \tag{3.3}$$

where $t_s = r/(r - s)$.

It is worth to point out that, in Remark 3.3, we have to assume that $\mu_2 < \mu_1$. Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.4. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathscr{H}$, and let $T : C \to \mathscr{H}$ be a nonlinear mapping. Let $T_1, T_2 : C \to \mathscr{H}$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and the following conditions are satisfied:*

(a) *$M^{\rho,\eta}\delta_{T(C)} < \xi$, where $\delta_{T(C)} = \sup\{\|u - v\|; u, v \in T(C)\}$;*

(b) *there exists $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \rho, \eta < \min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s\mu_2}\right\}, \tag{3.4}$$

where $M^{\rho,\eta} = \max\{\rho, \eta\}$, $t_s = r/(r - s)$, and $\zeta = \sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}/t_s(\mu_1^2 - \mu_2^2)$.

Then the problem (3.1) has a solution. Moreover, the sequence (x_n, y_n) which is generated by (3.2) strongly converges to a solution $(x^*, y^*) \in C \times C$ of the problem (3.1).

Proof. Firstly, by condition (b), we can easily check that $y_n - \rho T y_n$ and $x_n - \eta T x_n$ belong to the set C_s, for all $n = 1, 2, 3, \ldots$. Thus, from Lemma 2.5 (1), we know that (3.2) is well defined. Consequently, from (3.2) and Lemma 2.5 (2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\mathrm{proj}_C(y_n - \rho T y_n) - \mathrm{proj}_C(y_{n-1} - \rho T y_{n-1})\| \\ &\le t_s \|y_n - y_{n-1} - \rho(T y_n - T y_{n-1})\| \\ &\le t_s \left[\|y_n - y_{n-1} - \rho(T_1 y_n - T_1 y_{n-1})\| + \rho \|T_2 y_n - T_2 y_{n-1}\|\right]. \end{aligned} \tag{3.5}$$

Since the mapping T_1 is γ-strongly monotone and μ_1-Lipschitz continuous, we obtain

$$\begin{aligned} &\|y_n - y_{n-1} - \rho(T_1 y_n - T_1 y_{n-1})\|^2 \\ &\quad = \|y_n - y_{n-1}\|^2 - 2\rho\langle y_n - y_{n-1}, T_1 y_n - T_1 y_{n-1}\rangle + \rho^2 \|T_1 y_n - T_1 y_{n-1}\|^2 \\ &\quad \le \|y_n - y_{n-1}\|^2 - 2\rho\gamma \|y_n - y_{n-1}\| + \rho^2 \mu_1^2 \|y_n - y_{n-1}\|^2 \\ &\quad = \left(1 - 2\rho\gamma + \rho^2\mu_1^2\right)\|y_n - y_{n-1}\|^2. \end{aligned} \tag{3.6}$$

On the other hand, since T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 y_n - T_2 y_{n-1}\| \le \mu_2 \|y_n - y_{n-1}\|. \tag{3.7}$$

Thus, by (3.5), (3.6), and (3.7), we obtain

$$\|x_{n+1} - x_n\| \le t_s \left[\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}\right] \|y_n - y_{n-1}\|. \tag{3.8}$$

Similarly, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\mathrm{proj}_C(x_n - \eta T x_n) - \mathrm{proj}_C(x_{n-1} - \eta T x_{n-1})\| \\ &\le t_s \|x_n - x_{n-1} - \eta(T x_n - T x_{n-1})\| \\ &\le t_s \left[\|x_n - x_{n-1} - \eta(T_1 x_n - T_1 x_{n-1})\| + \eta \|T_2 x_n - T_2 x_{n-1}\|\right] \\ &\le t_s \left[\eta\mu_2 + \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2}\right] \|x_n - x_{n-1}\|. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\|x_{n+1} - x_n\| \le t_s^2 \theta_\rho \theta_\eta \|x_n - x_{n-1}\|, \tag{3.10}$$

where $\theta_\rho := \rho\mu_2 + \sqrt{1-2\rho\gamma+\rho^2\mu_1^2}$ and $\theta_\eta := \eta\mu_2 + \sqrt{1-2\eta\gamma+\eta^2\mu_1^2}$. Moreover, by (3.4), we know that $t_s\theta_\rho$ and $t_s\theta_\eta$ are elements of the interval $(0,1)$. Thus, from (3.10), it follows that

$$\|x_{n+1}-x_n\| \leq \kappa^n \|x_1 - x_0\| \tag{3.11}$$

for all $n = 1,2,3,\ldots$, where $\kappa := t_s^2\theta_\rho\theta_\eta$. Hence, for any $m \geq n > 1$, it follows that

$$\|x_m - x_n\| \leq \sum_{i=n}^{m-1}\|x_{i+1}-x_i\| \leq \sum_{i=n}^{m-1}\kappa^i\|x_1-x_0\| \leq \frac{\kappa^n}{1-\kappa}\|x_1-x_0\|. \tag{3.12}$$

Since $\kappa < 1$, it follows that $\kappa^n \to 0$ as $n \to \infty$, and this implies that $\{x_n\} \subset C$ is a Cauchy sequence. Consequently, from (3.9), we also have that $\{y_n\}$ is a Cauchy sequence in C. Thus, by Lemma 2.5 (3), the closedness property of C implies that there exists $(x^*, y^*) \in C \times C$ such that $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$.

We claim that $(x^*, y^*) \in C \times C$ is a solution of the problem (3.1). Indeed, by the definition of the proximal normal cone, from (3.2), we have

$$\begin{gathered}(x_n - y_n) - \eta(Tx_n) \in N_C^P(y_n),\\ (y_n - x_{n+1}) - \rho(Ty_n) \in N_C^P(x_{n+1}).\end{gathered} \tag{3.13}$$

By letting $n \to \infty$, using the closedness property of the proximal cone together with the continuity of T, we have

$$\begin{gathered}x^* - y^* - \eta(Tx^*) \in N_C^P(y^*),\\ y^* - x^* - \rho(Ty^*) \in N_C^P(x^*).\end{gathered} \tag{3.14}$$

This completes the proof. □

Immediately, by setting $T_2 = 0$, we have the following result.

Theorem 3.5. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathscr{H}$. Let $T : C \to \mathscr{H}$ be a μ-Lipschitz continuous and γ-strongly monotone mapping. If the following conditions are satisfied:*

(a) $M^{\rho,\eta}\delta_{T(C)} < \xi$, *where* $\delta_{T(C)} = \sup\{\|u-v\|; u, v \in T(C)\}$;

(b) *there exists* $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ *such that*

$$\frac{\gamma}{\mu^2} - \zeta < \rho, \eta < \frac{\gamma}{\mu^2} + \zeta, \tag{3.15}$$

where $\zeta = \sqrt{(t_s\gamma)^2 - (\mu_1^2)(t_s^2-1)}/t_s(\mu_1^2)$ *and* $t_s = r/(r-s)$.

Then the problem (3.1) *has a solution. Moreover, the sequence* (x_n, y_n) *which is generated by* (3.2) *strongly converges to a solution* $(x^*, y^*) \in C \times C$ *of the problem* (3.1).

In view of proving Theorem 3.4, we can obtain the following result, which contains a recent result presented by Moudafi [9] as a special case.

Theorem 3.6. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathscr{H}$, and let $T : C \to \mathscr{H}$ be a mapping. Let $T_1, T_2 : C \to \mathscr{H}$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists $s \in (0, \xi)$ such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \rho < \min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s\mu_2}, \frac{s}{1 + \|Ty_n\|}\right\},$$
$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \eta < \min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s\mu_2}, \frac{s}{1 + \|Tx_n\|}\right\} \tag{3.16}$$

for all $n = 1, 2, 3, \ldots$, where $t_s = r/(r - s)$, $\zeta = \sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}/t_s(\mu_1^2 - \mu_2^2)$ and the sequence (x_n, y_n) was generated by (3.2)*, then the sequence (x_n, y_n) strongly converges to a solution $(x^*, y^*) \in C \times C$ of the problem* (3.1).

Remark 3.7. (i) An inspection of Theorem 3.6 shows that the sequences $\{Tx_n\}$ and $\{Ty_n\}$ are bounded.

(ii) By setting $T_2 =: 0$, we see that Theorem 3.6 reduces to a result presented by Moudafi [9].

Remark 3.8. If C is a convex set, by the definition of the proximal normal cone, we can reformulate (3.1) as follows: find $x^*, y^* \in C \times C$ such that

$$\langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,$$
$$\langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C. \tag{3.17}$$

The problem (3.17) was introduced and studied by Verma [10], when T is a strong monotone mapping. Hence, Theorem 3.4 extends and improves the results presented by Verma [10]. For further recent results related to the problem (3.17), see also [2, 3, 5, 11–13].

4. Further Results

By using the techniques as in Theorem 3.4, we can also obtain an existence theorem of the following problem: find $x^* \in C$ such that

$$-Tx^* \in N_C^P(x^*). \tag{4.1}$$

The problem of type (4.1) was studied by Noor [14] but in a finite dimension Hilbert space setting. In this section, we intend to consider the problem (4.1) in an infinite dimension Hilbert space. To do this, the following remark is useful.

Remark 4.1. Let $T : C \to C$ be a γ-strongly monotone and μ-Lipschitz continuous mapping. Then, the function $f : (1, \mu^2/(\mu^2 - \gamma^2)) \to (0, \infty)$ which is defined by

$$f(t) = \frac{\sqrt{t^2(\gamma^2 - \mu^2) + \mu^2}}{t\mu^2}, \quad \forall t \in \left(1, \frac{\mu^2}{\mu^2 - \gamma^2}\right), \tag{4.2}$$

is a continuous decreasing function on its domain.

We now close this section by proving an existence theorem to the problem (4.1) in a nonconvex infinite dimensional setting.

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathcal{H}$, and let $T : C \to \mathcal{H}$ be a γ-strongly monotone and μ-Lipschitz continuous mapping. If $0 < \delta_{T(C)} \le \gamma r$, then the problem* (4.1) *has a solution.*

Proof. Firstly, by using an elementary calculation, we know that the function $h : [1, \mu^2/(\mu^2 - \gamma^2)) \to (0, \infty)$ which is defined by

$$h(t) = \frac{r(t-1)}{t\delta_{T(C)}} + f(t), \quad \forall t \in \left[1, \frac{\mu^2}{\mu^2 - \gamma^2}\right), \tag{4.3}$$

is a continuous increasing function on $[1, \sqrt{(\mu^2 r^2 - \delta^2_{T(C)})/r^2(\mu^2 - \gamma^2)}]$. Moreover, we see that the net $\{t_s\}_{s\in(0,r)}$ which is defined by $t_s =: r/(r-s)$ converges to 1 as $s \downarrow 0$. Using these observations, together with the fact that $h(t) \downarrow \gamma/\mu^2$ as $t \downarrow 1$, we can find $s^* \in (0, r(r^2\gamma^2 - \delta^2_{T(C)})/(\mu^2 r^2 - \delta^2_{T(C)}))$ such that $\mu^2 h(t_{s^*}) > \gamma$. It is worth to notice that, from the choice of s^*, we have $\gamma/\mu^2 - f(t_{s^*}) < s^*/\delta_{T(C)}$.

Now, we choose a fixed positive real number ρ such that

$$\frac{\gamma}{\mu^2} - f(t_{s^*}) < \rho < min\left\{\frac{\gamma}{\mu^2} + f(t_{s^*}), \frac{s^*}{\delta_{T(C)}}\right\}. \tag{4.4}$$

Next, let us start with an element $x_0 \in C$ and use an induction process to obtain a sequence $\{x_n\} \subset C$ satisfying

$$x_{n+1} = \text{proj}_C(x_n - \rho T x_n), \quad \forall n = 0, 1, 2, \ldots. \tag{4.5}$$

Note that, because of the choice of ρ, we can easily check that $x_n - \rho T x_n \in C_{s^*}$ for all $n = 1, 2, 3, \ldots$. Following the proof of Theorem 3.4, we know that $\{x_n\}$ is a Cauchy sequence in C. If $x_n \to x^*$ as $n \to \infty$, the closedness property of the proximal cone together with the continuity of T, from (4.5), we see that x^* is a solution of the problem (4.1). This completes the proof. □

Remark 4.3. Theorems 3.4, 3.5, and 4.2 not only give the conditions for the existence solution of the problems (3.1) and (4.1), respectively, but also provide the algorithm to find such solutions for any initial vector $x_0 \in C$.

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ภาคผนวก 7

Existence and iterative approximation of solutions of generalized mixed quasi-variational-like inequality problem in Banach spaces

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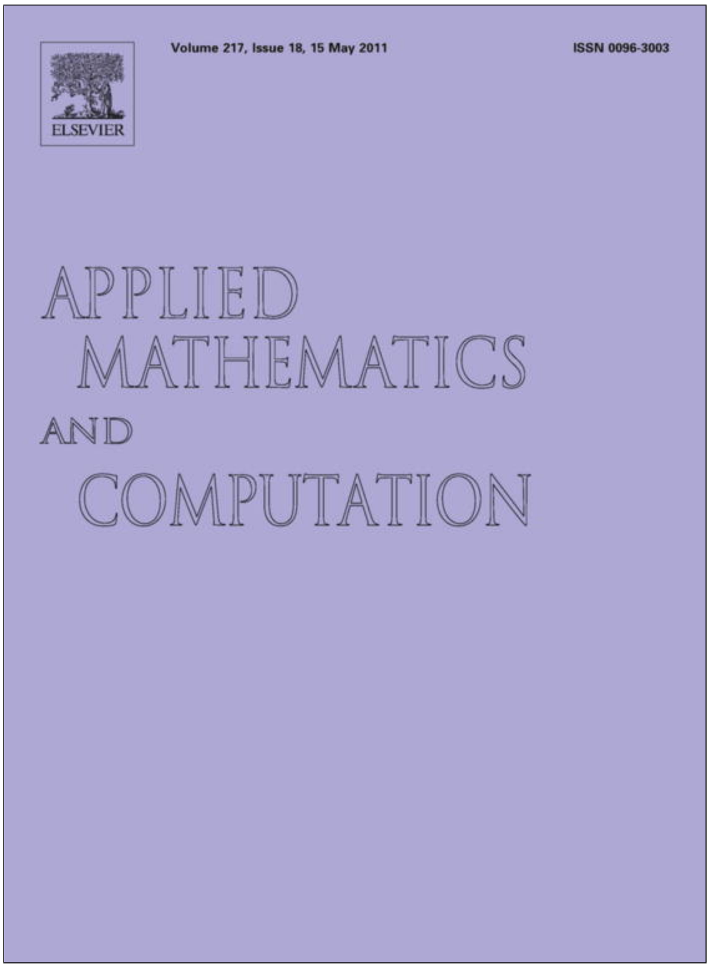
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Existence and iterative approximation of solutions of generalized mixed quasi-variational-like inequality problem in Banach spaces

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ABSTRACT

In this paper, some existence theorems for the mixed quasi-variational-like inequalities problem in a reflexive Banach space are established. The auxiliary principle technique is used to suggest a novel and innovative iterative algorithm for computing the approximate solution for the mixed quasi-variational-like inequalities problem. Consequently, not only the existence of theorems of the mixed quasi-variational-like inequalities is shown, but also the convergence of iterative sequences generated by the algorithm is also proven. The results proved in this paper represent an improvement of previously known results.

1. Introduction

The concept of variational inequality was introduced by Hartman and Stampacchia [9] in early 1960s. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, economics and transportation equilibrium, etc. The generalized mixed variational-like inequalities, which are generalized forms of variational inequalities, have potential and significant applications in optimization theory, structural analysis, and economics (see [4,18,16]).

It is well-known that due to the presence of the nonlinear bi-function, projection method and its variant forms including the Wiener–Hopf equations, descent methods cannot be extended to suggest iterative methods for solving the general mixed quasi variational inequalities, since it is not possible to find the projection of the solution. Thus, the development of an efficient and implementable technique for solving variational-like inequalities is one of the most interesting and important problems in variational inequality theory. To overcome this drawback, in recent years, a tremendous amount of work was applying the auxiliary problem principle, which does not depend on the projection, in finite- as well as in infinite-dimensional space settings, on the approximation-solvability of various classes of variational inequalities and complementarity problems.

Recently, the auxiliary principle technique was extended by Huang and Deng [11] to study the existence and iterative approximation of solutions of the set-valued strongly nonlinear mixed variational-like inequality, under the assumptions that the operators are bounded closed values. On the other hand, by applying the auxiliary principle technique, Verma [19] introduced a new class of predictor–corrector iterative algorithms for solving general variational inequalities and generalized mixed variational inequalities. Furthermore, Ding [7] suggested some new predictor–corrector iterative algorithms for solving generalized mixed variational-like inequality problems and proved the convergence of the iterative sequence generated by the predictor–corrector iterative algorithm.

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Motivated and inspired by the recent research work going on in this fascinating and interesting field, in this paper, some existence theorems for the mixed quasi-variational-like inequality problem in a reflexive Banach space are provided. Also, the existence theorem for auxiliary problem of the mixed quasi-variational-like inequality problem is studied. Consequently, we construct and analyze an iterative algorithm for finding the solution of the mixed quasi-variational-like inequality problem. Finally, we discuss the convergence analysis of iterative sequence generated by the iterative algorithm.

2. Preliminaries

Let E be a real Banach space with its topological dual E^*, and $\langle\cdot,\cdot\rangle$ be the generalized duality pairing between E and E^*. Let $CB(E^*)$ be the family of all nonempty bounded and closed subsets of E^*. The Hausdorff metric, $H(\cdot,\cdot)$, on $CB(E^*)$ is defined by

$$H(C,D) = \max\left\{\sup_{x\in C} d(x,D), \sup_{y\in D} d(C,y)\right\}, \quad \forall C,D \in CB(E^*).$$

Let K be a nonempty convex subset of E, in this paper, we devote our study to a class of generalized mixed quasi-variational-like inequality problem, which is stated as follows:

Let T, $A : K \to CB(E^*)$ be two set-valued mappings. $N : E^* \times E^* \to E^*$ and $\eta : K \times K \to E$ be two single-valued mappings. Let $\varphi : E \times E \to (-\infty, +\infty]$ be a real bi-function. For a given $w^* \in E^*$, we shall study the following problem:

$$GMQVLIP(T,A,N,\eta,\varphi)\begin{cases} \text{find } u \in K,\ x,y \in E^* \text{ such that } x \in T(u),\ y \in A(u), \\ \langle N(x,y) - w^*,\ \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geqslant 0,\ \forall v \in K. \end{cases} \tag{2.1}$$

In case of (2.1), we will denote by $(u,x,y) \in GMQVLIP(T,A,N,\eta,\varphi)$.

Now, let us consider some special cases of problem (2.1).

(a) If T,A are single valued, then the problem (2.1) collapses to finding $u \in K$ such that

$$\langle N(Tu,Au) - w^*, \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geqslant 0, \quad \forall v \in K. \tag{2.2}$$

The problem (2.2) was considered and studied in Ding [6].

(b) if $E = \mathcal{H}$ is a Hilbert space, and $w^* = 0$, then the problem (2.1) is equivalent to finding $u \in K$, $x \in T(u)$, $y \in A(u)$ such that

$$\langle N(x,y), \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geqslant 0, \quad \forall v \in K. \tag{2.3}$$

This kind of problem is called the set-valued strongly nonlinear mixed variational-like inequality and was considered by Huang and Deng [11], when $K = \mathcal{H}$.

(c) If $N(Tu,Av) = Tu - Av$ for all u, $v \in K$, the problem (2.2) reduces to the general nonlinear variational-like inequality problem: for a given $w^* \in E^*$, find $u \in K$ such that

$$\langle Tu - Au - w^*, \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geqslant 0, \quad \forall v \in K. \tag{2.4}$$

Problem (2.4) with $w^* = 0$ is introduced and studied by Ding [5].

(d) If $\varphi(u,v) = f(v)$, $\forall u$, $v \in E$, where $f : E \to \mathbb{R}$, then the problem (2.2) is equivalent to that of finding $u \in K$ such that

$$\langle Tu - Au - w^*, \eta(v,u)\rangle \geqslant f(u) - f(v), \quad \forall v \in K. \tag{2.5}$$

Problem (2.5) with $w^* = 0$ is introduced and studied by Chen and Liu [3] in a reflexive Banach space.

(e) If $E = \mathcal{H}$ is a Hilbert space, $A = 0$, $w^* = 0$ then the problem (2.5) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle \geqslant f(u) - f(v), \quad \forall v \in K. \tag{2.6}$$

Problem (2.6) was considered by Verma [20].

(f) If $E = \mathcal{H}$ is a Hilbert space, $A = 0$, $\eta(v,u) = v - u$, and $f = 0$, then the problem (2.4) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geqslant \langle w^*, v\rangle - \langle w^*, u\rangle, \quad \forall v \in K. \tag{2.7}$$

Problem (2.7) was introduced and studied by Zeng [21].

In brief, for appropriate and suitable choice of the mappings T, A, N, η, the bi-function φ, and the linear continuous functional w^*, one can obtain a wide class of variational inequalities and complementarity problems. Furthermore, problem (2.1) has an important applications in various branches of pure and applied sciences (see [2–16,18–23]).

The following basic concepts will be used in the sequel.

Definition 2.1. Let K be a nonempty subset of a Banach space E. Let T, $A : K \to CB(E^*)$ be two set-valued mappings. Let $N : E^{ast} \times E^* \to E^*$, $\eta : K \times K \to K$ be mappings. Then

(i) T is said to be η-cocoercive with respect to the first argument of $N(\cdot,\cdot)$, if there exists a constant $\tau > 0$, such that

$$\langle N(x,\cdot) - N(x',\cdot), \eta(u,v)\rangle \geqslant \tau\|N(x,\cdot) - N(x',\cdot)\|^2, \quad \forall u,\ v \in K, \quad x \in T(u), \quad x' \in T(v);$$

(ii) $N(\cdot,\cdot)$ is Lipschitz continuous in the second argument with respect to the set-valued mapping A, if there exists a constant $\alpha > 0$ such that

$$\|N(\cdot,y) - N(\cdot,y')\| \leqslant \alpha\|u - v\|, \quad \forall u, v \in K, \quad y \in A(u), \quad y' \in A(v);$$

(iii) $N(\cdot,\cdot)$ is η-strongly monotone in the first argument with respect to the set-valued mapping T if there exists a constant $\xi > 0$ such that

$$\langle N(x,\cdot) - N(x',\cdot), \eta(u,v)\rangle \geqslant \xi\|u - v\|^2, \quad \forall u, v \in K, \quad x \in T(u), \quad x' \in T(v).$$

Similarly, η-strongly monotone of $N(\cdot,\cdot)$ in the second argument with respect to the set-valued mapping A can be defined;

(iv) T is said to be H-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$H((T(u), T(v)) \leqslant \gamma\|u - v\|, \quad \forall u, v \in K;$$

(v) η is Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leqslant \delta\|u - v\|,$$

for any $u,\ v \in K$.

Definition 2.2. The bifunction $\varphi : E \times E \to (-\infty, +\infty]$ is said to be skew-symmetric, if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geqslant 0,$$

for all $u,\ v \in E$.

Remark 2.3. The skew-symmetric bifunctions have properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for a convex function. As for the investigations of the skew-symmetric bifunction, we refer the reader to [1].

Definition 2.4 ([2,10]). Let K be a nonempty convex subset of a Banach space E. Let $\psi : K \to (-\infty, +\infty)$ be a Fréchet differentiable function and $\eta : K \times K \to E$. Then ψ is said to be:

(i) η-convex, if

$$\psi(v) - \psi(u) \geqslant \langle \psi'(u), \eta(v,u)\rangle,$$

for all $u,\ v \in K$;

(ii) η-strongly convex, if there exists a constant $\mu > 0$ such that

$$\psi(v) - \psi(u) - \langle \psi'(u), \eta(v,u)\rangle \geqslant \frac{\mu}{2}\|u - v\|^2,$$

for all $u,\ v \in K$.

Note that, if $\eta(u,v) = u - v$ for all $u,\ v \in K$, then ψ is said to be strongly convex.

Throughout this paper, we shall use the notations "$\rightharpoonup$" and "$\to$" for weak convergence and strong convergence, respectively.

Remark 2.5

(i) Assume that for each fixed $v \in K$ the mapping $\eta(v,\cdot) : K \to E$ is continuous from the weak topology to the weak topology. Let $g : K \to (-\infty, +\infty)$ be a functional defined by

$$g(u) = \langle f, \eta(v,u)\rangle,$$

where $v \in K$ and $f \in E^*$ are fixed. Then g is a weakly continuous functional on K.

(ii) Let $\psi : K \to (-\infty, +\infty)$ be a Fréchet differentiable function and $\eta : K \times K \to K$ be a mapping such that $\eta(u,v) + \eta(v,u) = 0$, $\forall\ u,\ v \in K$. If ψ is a Fréchet differentiable η-strongly convex functional with constant $\mu > 0$ on a convex subset K of E then ψ' is η-strongly monotone with constant $\mu > 0$ (see [23, Proposition 2.1]).

The following lemma due to Zeng et al. [23] will be needed in proving our results.

Lemma 2.6 [23, Lemma 2]. *Let K be a nonempty convex subset of a topological vector space X and let $\Phi : K \times K \to [-\infty, +\infty]$ be such that*

(i) *for each* $v \in K$, $u \mapsto \Phi(v,u)$ *is lower semicontinuous on each nonempty compact subset of K;*
(ii) *for each finite set* $\{v_1,\ldots,v_m\} \subset K$ *and for each* $u = \sum_{i=1}^m \lambda_i v_i (\lambda_i \geqslant 0, \sum_{i=1}^m \lambda_i = 1)$, $\min_{1\leqslant i\leqslant m} \Phi(v_i,u) \leqslant 0$;
(iii) *there exists a nonempty compact convex subset* K_0 *of K such that for some* $v_0 \in K_0$, *there holds:*

$$\Phi(v_0,u) > 0, \quad \forall u \in K \setminus K_0.$$

Then there exists $\hat{u} \in K$, *such that* $\Phi(v,\hat{u}) \leqslant 0$, *for all* $v \in K$.

We also need the following lemma.

Lemma 2.7 [17]. *Let* (X,d) *be a complete metric space and let* $B_1, B_2 \in CB(X)$ *and* $r > 1$ *be any real number. Then, for every* $b_1 \in B_1$ *there exists* $b_2 \in B_2$ *such that* $d(b_1,b_2) \leqslant rH(B_1,B_2)$.

In the sequel, we assume that N and η satisfy the following assumption.

Assumption 2.8. Let $N : E^* \times E^* \to E^*$, $\eta : K \times K \to E$ be two mappings satisfying the following conditions:

(a) $\eta(u,v) = \eta(u,z) + \eta(z,v)$ for each $u, v, z \in K$;
(b) for each fixed $(u,x,y) \in K \times E^* \times E^*$, $v \mapsto \langle N(x,y), \eta(u,v)\rangle$ is a concave function.
(c) for each fixed $v \in K$, the functional $(u,x,y) \mapsto \langle N(x,y), \eta(u,v)\rangle$ is weakly lower semi-continuous function from $K \times E^* \times E^*$ to $\mathbb{R}$, i. e.,

$$u_n \rightharpoonup u, \quad x_n \rightharpoonup x \quad \text{and} \quad y_n \rightharpoonup y \quad \text{imply} \quad \langle N(x,y), \eta(u,v)\rangle \leqslant \liminf_{n\to\infty} \langle N(x_n,y_n), \eta(u_n,v)\rangle.$$

Remark 2.9. It follows from Assumption 2.8(a) that $\eta(u,u) = 0$ and $\eta(u,v) = -\eta(v,u)$, $\forall u, v \in K$.

3. The existence theorems

Theorem 3.1. *Let E be a real reflexive Banach space with the dual space* E^*, *and K be a nonempty convex subset of E. Let T,* $A : K \to CB(E^*)$ *be two set-valued mappings. Let* $N : E^* \times E^* \to E^*$, *and* $\eta : K \times K \to E$. *Let* $\varphi : E \times E \to (-\infty, +\infty]$ *be skew-symmetric and weakly continuous such that* $\mathrm{int}\{u \in K : \varphi(u,u) < \infty\} \neq \emptyset$ *and* $\varphi(u,\cdot)$ *is proper convex, for each* $u \in E$. *Suppose that:*

(i) *T is* η*-cocoercive with respect to the first argument of* $N(\cdot,\cdot)$ *with constant* τ;
(ii) η *is Lipschitz continuous with constant* $\delta > 0$;
(iii) $N(\cdot,\cdot)$ *is Lipschitz continuous and* η*-strongly monotone in the second argument with respect to A with constant* $\alpha > 0$ *and* $\beta > 0$, *respectively.*

If Assumption 2.8 is satisfied, then $GMQVLIP(T,A,N,\eta,\varphi) \neq \emptyset$.

Proof. For any $u, v \in K$, we define a function $\Phi : K \times K \to \mathbb{R}$ by

$$\Phi(v,u) = \langle N(x,y) - w^*, \eta(u,v)\rangle + \varphi(u,u) - \varphi(u,v) \quad \forall u, v \in K,$$

where $x \in T(u)$, $y \in A(u)$.

Note, by $\varphi(\cdot,\cdot)$ is weakly continuous functional and since each fixed $v \in K$ the functional $(u,x,y) \mapsto \langle N(x,y), \eta(u,v)\rangle$ is weakly lower semi-continuous, we have the functional $u \mapsto \Phi(v,u)$ is weakly lower semicontinuous for each $v \in K$. This shows that condition (i) in Lemma 2.6 holds. Now we claim that $\Phi(v,u)$ satisfied condition (ii) in Lemma 2.6. If it is not true, then there exist a finite set $\{v_1, v_2, \ldots, v_m\} \subset K$ and $u = \sum_{i=1}^m \varepsilon_i v_i (\varepsilon_i \geqslant 0, \sum_{i=1}^m \varepsilon_i = 1)$, such that $\Phi(v_i,u) > 0$ for all $i = 1,2,\ldots,m$, that is,

$$\langle N(x,y) - w^*, \eta(u,v_i)\rangle + \varphi(u,u) - \varphi(u,v_i) > 0 \quad \forall i = 1,2,\ldots,m.$$

It follows that

$$\sum_{i=1}^m \varepsilon_i \langle N(x,y) - w^*, \eta(u,v_i)\rangle + \varphi(u,u) - \sum_{i=1}^m \varepsilon_i \varphi(u,v_i) > 0.$$

Noting that for each $u \in E$, $\varphi(u,\cdot)$ is a convex functional, that is $\sum_{i=1}^m \varepsilon_i \varphi(u,v_i) \geqslant \varphi(u, \sum_{i=1}^m \varepsilon_i v_i) = \varphi(u,u)$. Hence,

$$\sum_{i=1}^m \varepsilon_i \langle N(x,y) - w^*, \eta(u,v_i)\rangle > 0.$$

From Assumption 2.8 (a) and (b), we obtain

$$0 < \sum_{i=1}^m \varepsilon_i \langle N(x,y) - w^*, \eta(u,v_i)\rangle \leqslant \left\langle N(x,y) - w^*, \eta\left(u, \sum_{i=1}^m \varepsilon_i v_i\right)\right\rangle = \langle N(x,y) - w^*, \eta(u,u)\rangle = 0,$$

a contradiction. Thus condition (ii) in Lemma 2.6 holds. Since for each $u \in E$, $v \mapsto \varphi(u,v)$ is a proper convex weakly lower semicontinuous functional and $\text{int}\{u \in K : \varphi(u,u) < \infty\} \neq \emptyset$, we take $u^* \in \text{int}\{u \in K : \varphi(u,u) < \infty\}$. By Proposition I.2.6 of Pascali and Sburlan [[15, p. 27]], $\varphi(u^*,\cdot)$ is subdifferentiable at u^*. Hence we have

$$\varphi(u^*, v) - \varphi(u^*, u^*) \geqslant \langle r, v - u^*\rangle, \quad \forall r \in \partial\varphi(u^*,\cdot), \quad v \in E.$$

Since φ is skew-symmetric, it follows that

$$\varphi(v, v) - \varphi(v, u^*) \geqslant \varphi(u^*, v) - \varphi(u^*, u^*) \geqslant \langle r, v - u^*\rangle, \quad \forall r \in \partial\varphi(u^*,\cdot), \quad v \in E.$$

Let $x^* \in T(u^*)$, $y^* \in A(u^*)$, $w \in T(u)$, $z \in A(u)$ and $r \in \partial\varphi(u^*,\cdot)$ be fixed, by using conditions (i)–(iii), and equality $\eta(u,v) = -\eta(v,u)$, we get that

$$\begin{aligned}\Phi(u^*,u) &= \langle N(w,z) - w^*, \eta(u,u^*)\rangle + \varphi(u,u) - \varphi(u,u^*) \geqslant \langle N(x^*,y^*) - N(w,z), \eta(u^*,u)\rangle - \langle N(x^*,y^*), \eta(u^*,u)\rangle \\ &\quad - \langle w^*, \eta(u,u^*)\rangle + \langle r, u - u^*\rangle = \langle N(x^*,y^*) - N(w,y^*), \eta(u^*,u)\rangle + \langle N(w,y^*) - N(w,z), \eta(u^*,u)\rangle \\ &\quad - \langle N(x^*,y^*), \eta(u^*,u)\rangle - \langle w^*, \eta(u,u^*)\rangle + \langle r, u - u^*\rangle \geqslant \tau\|N(x^*,y^*) - N(w,y^*)\|^2 + \beta\|u^* - u\|^2 \\ &\quad - \delta\|N(x^*,y^*)\|\|u^* - u\| - (\|r\| + \delta\|w^*\|)\|u - u^*\| \geqslant \beta\|u - u^*\|^2 - (\|r\| + \delta(\|w^*\| + \|N(x^*,y^*)\|))\|u - u^*\| \\ &= \|u - u^*\|[\beta\|u - u^*\| - (\|r\| + \delta(\|w^*\| + \|N(x^*,y^*)\|))].\end{aligned}$$

Let $\theta = \frac{1}{\beta}[\|r\| + \delta(\|w^*\| + \|N(x^*,y^*)\|]$ and $K_0 = \{u \in K : \|u - u^*\| \leqslant \theta\}$. Then K_0 is a weakly compact convex subset of K. Putting $v_0 = u^*$, we have that $\Phi(v_0,u) > 0$ for all $u \in K \backslash K_0$. Thus, condition (iii) of Lemma 2.6 is satisfied. By Lemma 2.6, there exists $\hat{u} \in K$ such that $\Phi(v,\hat{u}) \leqslant 0$ for all $v \in K$, that is,

$$\langle N(\hat{x},\hat{y}) - w^*, \eta(v,\hat{u})\rangle + \varphi(\hat{u}, v) - \varphi(\hat{u},\hat{u}) \geqslant 0 \quad \forall v \in K,$$

where $\hat{x} \in T(\hat{u})$, $\hat{y} \in A(\hat{u})$. Hence, $(\hat{u},\hat{x},\hat{y}) \in GMQVLIP(T,A,N,\eta,\varphi)$ This completes the proof. □

Remark 3.2. If the conditions of Theorem 3.1 are hold, and $N(\cdot,\cdot)$ is η-strongly monotone in the first argument with respect to T with constant $\xi > 0$, then the solution of the problem (2.1) is unique up to the element $u \in K$. Indeed, supposing $(\hat{u},\hat{x},\hat{y})$ and $(\tilde{u},\tilde{x},\tilde{y})$ are elements in $GMQVLIP(T,A,N,\eta,\varphi)$, we have

$$\langle N(\hat{x},\hat{y}) - w^*, \eta(v,\hat{u})\rangle \geqslant \varphi(\hat{u},\hat{u}) - \varphi(\hat{u}, v), \quad \forall v \in K, \tag{3.1}$$

$$\langle N(\tilde{x},\tilde{y}) - w^*, \eta(v,\tilde{u})\rangle \geqslant \varphi(\tilde{u},\tilde{u}) - \varphi(\tilde{u}, v), \quad \forall v \in K. \tag{3.2}$$

Taking $v = \tilde{u}$ in (3.1) and $v = \hat{u}$ in (3.2) and adding two inequalities, since φ is skew-symmetric, we obtain

$$\langle N(\hat{x},\hat{y}) - w^*, \eta(\tilde{u},\hat{u})\rangle + \langle N(\tilde{x},\tilde{y}) - w^*, \eta(\hat{u},\tilde{u})\rangle \geqslant 0.$$

Moreover, by Remark 2.9, we have

$$\langle N(\tilde{x},\tilde{y}) - N(\hat{x},\hat{y}), \eta(\hat{u},\tilde{u})\rangle \geqslant 0.$$

Since $N(\cdot,\cdot)$ is η-strongly monotone in the first argument with respect to T with the constant ξ, and η-strongly monotone in the second argument with respect to A with constant β, we get

$$(\beta + \xi)\|\hat{u} - \tilde{u}\|^2 \leqslant \langle N(\tilde{x},\tilde{y}) - N(\hat{x},\tilde{y}), \eta(\tilde{u},\hat{u})\rangle + \langle N(\hat{x},\tilde{y})\rangle - N(\hat{x},\hat{y}), \eta(\tilde{u},\hat{u})\rangle \leqslant 0.$$

Since β, $\xi > 0$, we must have $\hat{u} = \tilde{u}$.

4. Convergence analysis

4.1. Constructive Approximation

In this section, we extend the auxiliary principle technique to study the mixed quasi-variational-like inequality problem (2.1) in a reflexive Banach space E. We first establish an existence theorem for the auxiliary problem for the mixed quasi-variational-like inequality problem (2.1). By using this existence theorem, we construct the iterative algorithm for solving the problem of type (2.1).

Let $\eta : K \times K \to E$ be a mapping, $\psi : K \to (-\infty, +\infty]$ be a given Fréchet differentiable η-convex functional and $\rho > 0$ be a given positive number. Given $u \in K$, $x \in T(u)$, $y \in A(u)$, we consider the following problem $P(u,x,y)$: find $w \in K$ such that

$$\langle \rho N(x,y) - \rho w^* + \psi'(w) - \psi'(u), \eta(v,w)\rangle + \rho\varphi(w, v) - \rho\varphi(w,w) \geqslant 0, \quad \forall v \in K. \tag{4.1}$$

The problem $P(u,x,y)$ is called the auxiliary problem for fuzzy mixed variational-like inequality problem (2.1).

Theorem 4.1. *If the conditions of Theorem 3.1 are hold, and for each fixed $v \in K$, $w \mapsto \eta(v,w)$ is continuous from the weak topology to the weak topology. If the function ψ is η-strongly convex with constant μ and the functional $w \mapsto \langle \psi'(w), \eta(v,w)\rangle$ is weakly upper semicontinuous on K for each $v \in K$, then the auxiliary problem $P(u,x,y)$ has a unique solution.*

Proof. Let $\rho > 0$ and $u \in K$, $u \in K$, $x \in T(u)$, $y \in A(u)$ be fixed. Define a functional $\Omega : K \times K \to [-\infty, +\infty]$ by

$$\Omega(v,w) = \langle \psi'(u) - \psi'(w) - \rho N(x,y) + \rho w^*, \eta(v,w)\rangle + \rho\varphi(w,w) - \rho\varphi(w,v) \quad \forall v,w \in K.$$

Note that for each fixed $v \in K$, the functional $w \mapsto \langle \psi\prime(w), \eta(v,w)\rangle$ is weakly upper semi-continuous on K, and $w \mapsto \eta(v,w)$ is continuous from the weak topology to the weak topology, and $\varphi(\cdot,\cdot)$ is weakly continuous, thus, it is easy to see that, for each fixed $v \in K$ the function $w \mapsto \Omega(v,w)$ is weakly lower semi-continuous continuous on each weakly compact subset of K and so condition (i) in Lemma 2.6 is satisfied. We claim that condition (ii) in Lemma 2.6 holds. If this is false, then there exist a finite set $\{v_1, v_2, \ldots, v_m\} \subset K$ and a $w = \sum_{i=1}^m \varepsilon_i v_i$ with $\varepsilon_i \geqslant 0$ and $\sum_{i=1}^m \varepsilon_i = 1$, such that

$$\Omega(v_i,w) = \langle \psi'(u) - \psi'(w) - \rho N(x,y) + \rho w^*, \eta(v_i,w)\rangle + \rho\varphi(w,w) - \rho\varphi(w,v_i) > 0 \quad \forall i = 1,2,\ldots,m.$$

By Assumption 2.8, in light of Remark 2.9, together with the convexity of $\varphi(w,\cdot)$, we have

$$\begin{aligned} 0 &< \sum_{i=1}^m \varepsilon_i[\langle \psi'(u) - \psi'(w) - \rho(N(x,y)) + \rho w^*, \eta(v_i,w)\rangle + \rho\varphi(w,w) - \rho\varphi(w,v_i)] \\ &\leqslant \langle \psi'(u) - \psi'(w) - \rho(N(x,y)) + \rho w^*, \eta(w,w)\rangle + \rho\varphi(w,w) - \rho\sum_{i=1}^m \varepsilon_i\varphi(w,v_i) \leqslant 0, \end{aligned}$$

which is a contradiction. Thus, condition (ii) in Lemma 2.6 is satisfied. Note that the η-strong convexity of ψ implies that ψ' is η-strongly monotone with constant $\mu > 0$, see Remark 2.5(ii). By using the similar argument as in the proof of Theorem 3.1, we can readily prove that condition (iii) of Lemma 2.6 is also satisfied. By Lemma 2.6 there exists a point $w \in K$, such that $\Omega(v,w) \leqslant 0$ for all $v \in K$. This implies that w is a solution to the problem $P(u,x,y)$.

Now we prove that the solution of problem $P(u,x,y)$ is unique. Let w_1 and w_2 be two solutions of problem (4.1). Then,

$$\langle \rho N(x,y) - \rho w^* + \psi'(w_1) - \psi'(u), \eta(v,w_1)\rangle \geqslant \rho\varphi(w_1,w_1) - \rho\varphi(w_1,v), \quad \forall v \in K, \tag{4.2}$$

and

$$\langle \rho N(x,y) - \rho w^* + \psi'(w_2) - \psi'(u), \eta(v,w_2)\rangle \geqslant \rho\varphi(w_2,w_2) - \rho\varphi(w_2,v), \quad \forall v \in K. \tag{4.3}$$

Taking $v = w_2$ in (4.2) and $v = w_1$ in (4.3), and adding these two inequalities, since $\eta(w_2,w_1) + \eta(w_1,w_2) = 0$ and $\varphi(\cdot,\cdot)$ is skew-symmetric, we obtain

$$\langle \psi'(w_2) - \psi'(w_1), \eta(w_1,w_2)\rangle \geqslant 0.$$

Thus, by ψ' is η-strongly monotone, we have

$$\mu\|w_1 - w_2\|^2 \leqslant \langle \psi'(w_1) - \psi'(w_2), \quad \eta(w_1,w_2) \leqslant 0,$$

This implies that $w_1 = w_2$, and the proof is completed. □

By virtue of Theorem 4.1, we now construct an iterative algorithm for solving the problem (2.1) in a reflexive Banach space E.

Let $\rho > 0$ be fixed. For given $u_0 \in K$, $x_0 \in T(u_0)$, $y_0 \in A(u_0)$, from Theorem 4.1, there is $u_1 \in K$ such that

$$\langle \rho N(x_0,y_0) - \rho w^* + \psi'(u_1) - \psi'(u_0), \eta(v,u_1)\rangle + \rho\varphi(u_1,v) - \rho\varphi(u_1,u_1) \geqslant 0, \quad \forall v \in K.$$

Since $x_0 \in T(u_0) \in CB(E^*)$, $y_0 \in A(u_0) \in CB(E^*)$, by Lemma 2.7, there exist $x_1 \in T(u_1)$ and $y_1 \in A(u_1)$ such that

$$\begin{aligned} &\|x_0 - x_1\| \leqslant (1+1)H((T(u_0), T(u_1)), \\ &\|y_0 - y_1\| \leqslant (1+1)H(A(u_0), A(u_1)). \end{aligned}$$

Again by Theorem 4.1, there is $u_2 \in K$ such that

$$\langle \rho N(x_1,y_1) - \rho w^* + \psi'(u_2) - \psi'(u_1), \eta(v,u_2)\rangle + \rho\varphi(u_2,v) - \rho\varphi(u_2,u_2) \geqslant 0, \quad \forall v \in K.$$

Since $x_1 \in T(u_1) \in CB(E^*)$, $y_1 \in A(u_1) \in CB(E^*)$, by Lemma 2.7, there exist $x_2 \in T(u_2)$ and $y_2 \in A(u_2)$ such that

$$\begin{aligned} &\|x_1 - x_2\| \leqslant \left(1 + \frac{1}{2}\right)H(T(u_1), T(u_2)), \\ &\|y_1 - y_2\| \leqslant \left(1 + \frac{1}{2}\right)H(A(u_1), A(u_2)). \end{aligned}$$

Continuing in this way, we can obtain the iterative algorithm for solving problem (2.1) as follows:

Algorithm 1. Let $\rho > 0$ be fixed. For given $u_0 \in K$, $x_0 \in T(u_0)$, $y_0 \in A(u_0)$ there exist the sequence $\{u_n\} \subset K$ and $\{x_n\}$, $\{y_n\} \subset E^*$ such that

$$\langle \rho N(x_n,y_n) - \rho w^* + \psi'(u_{n+1}) - \psi'(u_n), \eta(v,u_{n+1})\rangle + \rho\varphi(u_{n+1},v) - \rho\varphi(u_{n+1},u_{n+1}) \geqslant 0, \quad \forall v \in K$$

and

$$x_n \in T(u_n), \quad \|x_n - x_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(T(u_n), T(u_{n+1})), \quad \forall n \in \mathbb{N},$$

$$y_n \in A(u_n), \quad \|y_n - y_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(A(u_n), A(u_{n+1})), \quad \forall n \in \mathbb{N}.$$

4.2. Convergence theorems

Now, we shall prove that the sequences $\{u_n\} \subset K$ and $\{x_n\}, \{y_n\} \subset E^*$ generated by Algorithm 1 converge strongly to a solution of problem (2.1).

Theorem 4.2. *Suppose that conditions of Theorem 4.1 are hold, and the mapping T, A are Lipschitzian continuous mappings with Lipschitzian constant γ and ζ, respectively. If $\rho \in \left(0, \frac{2\tau\mu\beta}{\delta^2(\tau\alpha^2+\beta)}\right)$ then the iterative sequence $\{u_n\}, \{x_n\}, \{y_n\}$ obtained from Algorithm 1 converge strongly to a solution of problem (2.1).*

Proof. Let $(\hat{u}, \hat{x}, \hat{y}) \in FMVLIP(T, A, N, \eta, \varphi)$. Define a function $\Delta : K \to (-\infty, +\infty]$ by

$$\Delta(u) = \psi(\hat{u}) - \psi(u) - \langle \psi'(u), \eta(\hat{u}, u) \rangle.$$

By the η-strong convexity of ψ, we have

$$\Delta(u) = \psi(\hat{u}) - \psi(u) - \langle \psi'(u), \eta(\hat{u}, u) \rangle \geqslant \frac{\mu}{2} \|u - \hat{u}\|^2. \tag{4.4}$$

Note that $\eta(u, v) = -\eta(v, u)$ for all $u, v \in K$ and $\varphi(\cdot,\cdot)$ is skew-symmetric. Since $u_{n+1} \in K$ and $(\hat{u}, \hat{x}, \hat{y}) \in FMVLIP(T, A, N, \eta, \varphi)$, from the η-strong convexity of ψ, and Algorithm 1 with $v = \hat{u}$ it follows that

$$\begin{aligned}
\Delta(u_n) - \Delta(u_{n+1}) &= \psi(u_{n+1}) - \psi(u_n) - \langle \psi'(u_n), \eta(u_{n+1}, u_n) \rangle + \langle \psi'(u_{n+1}) - \psi'(u_n), \eta(\hat{u}, u_{n+1}) \rangle \\
&\geqslant \frac{\mu}{2} \|u_n - u_{n+1}\|^2 + \rho \langle N(x_n, y_n) - w^*, \eta(u_{n+1}, \hat{u}) \rangle + \rho[\varphi(u_{n+1}, u_{n+1}) - \varphi(u_{n+1}, \hat{u})] \\
&\geqslant \frac{\mu}{2} \|u_n - u_{n+1}\|^2 + \rho \langle N(x_n, y_n) - N(\hat{x}, \hat{y}), \eta(u_{n+1}, \hat{u}) \rangle + \rho[\langle N(\hat{x}, \hat{y}) - w^*, \eta(u_{n+1}, \hat{u}) \rangle + \varphi(\hat{u}, u_{n+1}) \\
&\quad - \varphi(\hat{u}, \hat{u})] \geqslant \frac{\mu}{2} \|u_n - u_{n+1}\|^2 + \rho \langle N(x_n, y_n) - N(\hat{x}, \hat{y}), \eta(u_{n+1}, \hat{u}) \rangle = \frac{\mu}{2} \|u_n - u_{n+1}\|^2 + Q,
\end{aligned} \tag{4.5}$$

where $Q = \rho \langle N(x_n, y_n) - N(\hat{x}, \hat{y}), \eta(u_{n+1}, \hat{u}) \rangle$.

Consider,

$$\begin{aligned}
Q &= \rho \langle N(x_n, y_n) - N(\hat{x}, \hat{y}), \eta(u_{n+1}, \hat{u}) \rangle = \rho \langle N(x_n, y_n) - N(\hat{x}, \hat{y}), \eta(u_{n+1}, u_n) \rangle + \rho \langle N(x_n, y_n) - N(\hat{x}, \hat{y}), \eta(u_n, \hat{u}) \rangle \\
&= \rho \langle N(x_n, y_n) - N(\hat{x}, y_n), \eta(u_n, \hat{u}) \rangle + \rho \langle N(\hat{x}, y_n) - N(\hat{x}, \hat{y}), \eta(u_n, \hat{u}) \rangle + \rho \langle N(x_n, y_n) - N(\hat{x}, y_n), \eta(u_{n+1}, u_n) \rangle + \rho \langle N(\hat{x}, y_n) \\
&\quad - N(\hat{x}, \hat{y}), \eta(u_{n+1}, u_n) \rangle \geqslant \rho\tau \|N(x_n, y_n) - N(\hat{x}, y_n)\|^2 + \rho\beta \|u_n - \hat{u}\|^2 - \rho\delta \|N(x_n, y_n) - N(\hat{x}, y_n)\| \|u_{n+1} - u_n\| - \rho\alpha\delta \|u_n \\
&\quad - \hat{u}\| \|u_{n+1} - u_n\| = \rho[\tau \|N(x_n, y_n) - N(\hat{x}, y_n)\|^2 - \delta \|N(x_n, y_n) - N(\hat{x}, y_n)\| \|u_{n+1} - u_n\|] - \rho\alpha\delta \|u_n - \hat{u}\| \|u_{n+1} - u_n\| \\
&\quad + \rho\beta \|u_n - \hat{u}\|^2 \geqslant \rho \left[-\frac{\delta^2}{4\tau} \right] \|u_{n+1} - u_n\|^2 - \rho\alpha\delta \|u_n - \hat{u}\| \|u_{n+1} - u_n\| + \rho\beta \|u_n - \hat{u}\|^2.
\end{aligned} \tag{4.6}$$

Therefore, we have

$$\begin{aligned}
\Delta(u_n) - \Delta(u_{n+1}) &\geqslant \frac{1}{2} \left(\mu - \frac{\rho\delta^2}{2\tau} \right) \|u_{n+1} - u_n\|^2 - \rho\alpha\delta \|u_n - \hat{u}\| \|u_{n+1} - u_n\| + \rho\beta \|u_n - \hat{u}\|^2 \\
&\geqslant \left[\rho\beta - \frac{\rho^2\alpha^2\delta^2}{2(\mu - \rho\delta^2/2\tau)} \right] \|u_n - \hat{u}\|^2.
\end{aligned} \tag{4.7}$$

Since $\rho \in \left(0, \frac{2\tau\mu\beta}{\delta^2(\tau\alpha^2+\beta)}\right)$, the inequality (4.7) implies that the sequence $\{\Delta(u_n)\}$ is strictly decreasing (unless $u_n = \hat{u}$) and is nonnegative by (4.4). Hence it converges to some number. Thus, the difference of two consecutive terms of the sequence $\{\Delta(u_n)\}$ goes to zero, and so the sequence $\{u_n\}$ converges strongly to $\hat{u}$. Further, from Algorithm 1, we have

$$\|x_n - x_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(T(u_n), T(u_{n+1})) \leqslant \gamma \|u_n - u_{n+1}\|,$$

$$\|y_n - y_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(A(u_n), A(u_{n+1})) \leqslant \zeta \|u_n - u_{n+1}\|.$$

These imply that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in E^*, since $\{u_n\}$ is a convergence sequence. Thus, we can assume that $x_n \to x$ and $y_n \to y$ (as $n \to \infty$). Note $x_n \in T(u_n)$ and $y_n \in A(u_n)$, so we have

$$d(x, T(\hat{u})) \leqslant \|x - x_n\| + d(x_n, T(u_n)) + H(T(u_n), T(\hat{u})) \leqslant \|x - x_n\| + 0 + \gamma\|u_n - \hat{u}\| \to 0 \quad (n \to \infty).$$

Hence we must have $x \in T(\hat{u})$. Similarly, we can obtain $y \in A(\widehat{u})$. Now we shall show that $(\hat{u}, x, y) \in GMQVLIP(T, A, N, \eta, \varphi)$. In view of Assumption 2.8(c), for each fixed $v \in K$ we have the functional $(u, x, y) \mapsto \langle N(x, y), \eta(v, u)\rangle$ is an upper semi-continuous functional. Using this one, together with the weak continuity of the function $\varphi(\cdot, \cdot)$, we obtain

$$\begin{aligned} 0 &\leqslant \limsup_{n\to\infty} [\langle \rho N(x_n, y_n) - \rho w^* + \psi'(u_{n+1}) - \psi'(u_n), \eta(v, u_{n+1})\rangle + \rho\varphi(u_{n+1}, v) - \rho\varphi(u_{n+1}, u_{n+1})] \\ &\leqslant \rho[\langle N(x, y) - w^*, \eta(v, \hat{u})\rangle + \varphi(\hat{u}, v) - \varphi(\hat{u}, \hat{u})]. \end{aligned}$$

This implies that $(\hat{u}, x, y) \in GMQVLIP(T, A, N, \eta, \varphi)$, and the proof is completed. □

Remark 4.3. Since for appropriate and suitable choice of the mappings T, A, N, η, the bi-function φ, and the linear continuous functional w^*, we can obtain a number of known class of variational inequalities and variational-like inequalities as spacial cases from the problem (2.1), hence, our results can be view as a refinement and improvement of the previously known results for variational inequalities.

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ภาคผนวก 8

A New Class of General Nonlinear Random Set-valued Variational Inclusion Problems Involving A-maximal m-relaxed η-accretive Mappings and Random Fuzzy Mappings in Banach Spaces

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A new class of general nonlinear random set-valued variational inclusion problems involving A-maximal m-relaxed η-accretive mappings and random fuzzy mappings in Banach spaces

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Abstract

At the present article, we consider a new lass of general nonlinear random A-maximal m-relaxed η-accretive equations with random relaxed cocoercive mappings and random fuzzy mappings in q-uniformly smooth Banach spaces. By using the resolvent mapping technique for A-maximal m-relaxed η-accretive mappings due to Lan et al. and Chang's lemma, we construct a new iterative algorithm with mixed errors for finding the approximate solutions of this class of nonlinear random equations. We also verify that the approximate solutions obtained by the our proposed algorithm converge to the exact solution of the general nonlinear random A-maximal m-relaxed η-accretive equations with random relaxed cocoercive mappings and random fuzzy mappings in q-uniformly smooth Banach spaces.

Keywords: variational inclusions; A-maximal m-relaxed η-accretive mapping; random relaxed cocoercive mapping; resolvent operator technique; random iterative algorithm; random fuzzy mapping; q-uniformly smooth Banach space.

Mathematical Subject Classification 2010: Primary, 47B80; Secondary, 47H40, 60H25.

1 Introduction

The theory of variational inequalities was extended and generalized in many different directions because of its applications in mechanics, physics, optimization, economics and engineering sciences. For the applications, physical formulation, numerical methods and other aspects of variational inequalities (see [1–63] and the references therein). Quasi-variational inequalities are generalized forms of variational inequalities in which the constraint set depend on the solution. These were introduced and studied by Bensoussan et al. [11]. In 1991, Chang and Huang [16, 17] introduced and studied some new classes of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. An useful and important generalization of the variational inequalities is called the variational inclusions, due to Hassouni and Moudafi [34], which have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science.

Meanwhile, it is known that accretivity of the underlying operator plays indispensable roles in the theory of variational inequality and its generalizations. In 2001, Huang and Fang [41] were the first to introduce generalized m-accretive mapping and gave the definition of the resolvent operator for generalized m-accretive mappings in Banach spaces. Subsequently, Verma [59,60] introduced and studied new notions of A-monotone and (A, η)-monotone operators and studied some properties of them in Hilbert spaces. In [52], Lan et al. first introduced the concept of (A, η)-accretive mappings, which generalizes the existing η-subdifferential operators, maximal η-monotone operators, H-monotone operators, A-monotone operators, (H, η)-monotone operators, (A, η)-monotone operators in Hilbert spaces, H-accretive mapping, generalized m-accretive mappings and (H, η)-accretive mappings in Banach spaces.

On the other hand, the fuzzy set theory which was introduced by Professor Lotfi Zadeh [62] at the university of California in 1965 has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of the fuzzy set theory can be found in many branches of regional, physical, mathematical and engineering sciences (see, for example [10, 32, 63]). In 1989, by using the concept of fuzzy set, Chang and Zu [20] first introduced and studied a class of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities with fuzzy mappings were considered by Chang and Haung [15], Ding [30], Ding and Park [31], Haung [36], Kumam and Petrot [48], Noor [55] and Park and Jeong [56, 57] in Hilbert spaces. Recently, Huang and Lan [43], considered nonlinear equations with fuzzy mapping in fuzzy normed spaces and subsequently Lan and Verma [54] considered fuzzy variational inclusion problems in Banach spaces. It is worth to mention that variational inequalities with fuzzy mapping have been useful in the study of equilibrium and optimal control problem (see, for example [14]).

The random variational inequality and random quasi-variational inequality problems, random variational inclusion problems and random quasi-complementarity problems have been introduced and studied by Chang [13], Chang and Huang [18, 19], Chang and Zhu [21], Cho et al. [22], Ganguly and Wadhawa [33], Huang and Cho [40], Khan et al. [47] and Lan [51], etc. Recently, Lan et al. [53] introduced and studied a class of general nonlinear random set-valued operator equations involving generalized m-accretive mappings in Banach spaces. They also established the existence theorems of the solution and convergence theorems of the generalized random iterative procedures with errors for these nonlinear random set-valued operator equations in q-uniformly smooth Banach spaces. Cho and Lan [23] considered and studied a class of generalized nonlinear random (A, η)-accretive equations with random relaxed cocoercive mappings in Banach spaces and by introducing some random iterative algorithms, they proved the convergence of iterative sequences generated by proposed algorithms. Further, by considering the concepts of random mappings and fuzzy mappings, Haung [39] was first introduced the concept of random fuzzy mapping. Subsequently, the random variational inclusion problem for random fuzzy mappings is studied by Ahmad and Bazan [4]. Very recently, Onjai-Uea and Kumam [58] introduced and studied a class of general nonlinear random (H, η)-accretive equations with random fuzzy mappings in Banach spaces and by using the resolvent mapping technique for the (H, η)-accretive mappings proved the existence and convergence theorems of the generalized random iterative algorithms for these nonlinear random equations with random fuzzy mappings in q-uniformly smooth Banach spaces.

At the present article, inspired and motivated by recent researches in this field, we shall introduce and study a new class of general nonlinear random A-maximal m-relaxed η-accretive (so called (A, η)-accretive [52]) equations with random relaxed cocoercive mappings and random fuzzy mappings in Banach spaces. By using the resolvent mapping technique for A-maximal m-relaxed η-accretive mappings due to Lan et al. and Chang's lemma [12], we construct a new iterative algorithm with mixed errors for finding the approximate solutions of this class of nonlinear random equations. We also prove the existence of random solutions and the convergence of random iterative sequences generated by the our proposed algorithm in q-uniformly smooth Banach spaces. The results presented in this article improve and extend the corresponding results of [13, 18, 22–24, 33, 34, 37–40, 42, 44, 46, 49, 53, 58] and many other recent works.

2 Preliminaries

Throughout this article, let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and X be a separable real Banach space endowed with dual space X^*, the norm $.$ and the dual pair $.,.$ between X and X^*. We denote by $\mathcal{B}(X)$, $CB(X)$ and $\hat{H}(.,.)$ the class of Borel σ-fileds in X, the family of all nonempty closed bounded subsets of X and the Hausdorff metric

$$\hat{H}(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\right\}$$

on $CB(X)$, respectively.

The *generalized duality mapping* $J_q : X \multimap X^*$ is defined by

$$J_q(x) = \{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \ \|f^*\| = \|x\|^{q-1} \}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is single-valued. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

The *modulus of smoothness* of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \ \|y\| \leq t \right\}.$$

A Banach space X is called *uniformly smooth* if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

Further, a Banach space X is called *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq c t^q, \quad q > 1.$$

It is well-known that Hilbert spaces, L_p(or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth.

Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces, Xu [61] proved the following result.

Lemma 2.1. *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q \|y\|^q.$$

Definition 2.2. A mapping $x : \Omega \to X$ is said to be *measurable* if, for any $B \in \mathcal{B}(X)$, $\{t \in \Omega : x(t) \in B\} \in \mathcal{A}$.

Definition 2.3. A mapping $T : \Omega \times X \to X$ is called a *random* mapping if, for any $x \in X$, $T(.,x) : \Omega \to X$ is measurable. A random mapping T is said to be *continuous* if, for any $t \in \Omega$, the mapping $T(t,.) : X \to X$ is continuous.

Similarly, we can define a random mapping $a : \Omega \times X \times X \to X$. We shall write $T_t(x) = T(t, x(t))$ and $a_t(x, y) = a(t, x(t), y(t))$ for all $t \in \Omega$ and $x(t), y(t) \in X$.

It is well-known that a measurable mapping is necessarily a random mapping.

Definition 2.4. A set-valued mapping $V : \Omega \multimap X$ is said to be *measurable* if, for any $B \in \mathcal{B}(X)$, $V^{-1}(B) = \{t \in \Omega : V(t) \cap B \neq \emptyset\} \in \mathcal{A}$.

Definition 2.5. A mapping $u : \Omega \to X$ is called a *measurable selection* of a set-valued measurable mapping $V : \Omega \multimap X$ if, u is measurable and for any $t \in \Omega$, $u(t) \in V(t)$.

Definition 2.6. A set-valued mapping $V : \Omega \times X \multimap X$ is called a *random* set-valued mapping if, for any $x \in X$, $V(.,x)$ is measurable. A random set-valued mapping $V : \Omega \times X \multimap X$ is said to be $\hat{H}$-*continuous* if, for any $t \in \Omega$, $V(t,.)$ is continuous in the Hausdorff metric on $CB(X)$.

Definition 2.7. Let X be a q-uniformly smooth Banach space, $T, A : \Omega \times X \to X$ and $\eta : \Omega \times X \times X \to X$ be random single-valued mappings. Then

(a) T is said to be *accretive* if

$$\langle T_t(x) - T_t(y), J_q(x(t) - y(t)) \rangle \geq 0, \quad \forall x(t), y(t) \in X, \quad t \in \Omega;$$

(b) T is called *strictly accretive* if T is accretive and

$$T_t(x) - T_t(y), J_q(x(t) - y(t)) = 0,$$

if and only if $x(t) = y(t)$ for all $t \in \Omega$;

(c) T is said to be r-*strongly accretive* if there exists a measurable function $r : \Omega \to (0, \infty)$ such that

$$\langle T_t(x) - T_t(y), J_q(x(t) - y(t)) \rangle \geq r(t) \|x(t) - y(t)\|^q, \quad \forall x(t), y(t) \in X, \quad t \in \Omega;$$

(d) T is said to be (θ, κ)-*relaxed cocoercive* if there exist measurable functions $\theta, \kappa : \Omega \to (0, \infty)$ such that

$$\langle T_t(x) - T_t(y), J_q(x(t) - y(t)) \rangle \geq -\theta(t) \|T_t(x) - T_t(y)\|^q + \kappa(t) \|x(t) - y(t)\|^q, \quad \forall x(t), y(t) \in X, \quad t \in \Omega;$$

(e) T is called ϱ-*Lipschitz continuous* if there exists a measurable function $\varrho : \Omega \to (0, \infty)$ such that

$$\|T_t(x) - T_t(y)\| \leq \varrho(t) \|x(t) - y(t)\|, \quad \forall x(t), y(t) \in X, \quad t \in \Omega;$$

(f) η is said to be τ-*Lipschitz continuous* if there exists a measurable function $\tau : \Omega \to (0, \infty)$ such that

$$\|\eta_t(x, y)\| \leq \tau(t) \|x(t) - y(t)\|, \quad \forall x(t), y(t) \in X, \quad t \in \Omega;$$

(g) η is said to be μ-*Lipschitz continuous in the second argument* if there exists a measurable function $\mu : \Omega \to (0, \infty)$ such that

$$\|\eta_t(x, u) - \eta_t(y, u)\| \leq \mu(t) \|x(t) - y(t)\|, \quad \forall x(t), y(t), u(t) \in X, \quad t \in \Omega.$$

In a similar way to part (g), we can define the Lipschitz continuity of the mapping η in the third argument.

Definition 2.8. Let X be a q-uniformly smooth Banach space, $\eta : \Omega \times X \times X \quad X$ and $H, A : \Omega \times X \quad X$ be three random single-valued mappings. Then set-valued mapping $M : \Omega \times X \multimap X$ is said to be:

(a) *accretive* if

$$u(t) - v(t), J_q(x(t) - y(t)) \geq 0, \quad x(t), y(t) \quad X, \quad u(t) \quad M_t(x), \quad v(t) \quad M_t(y), \quad t \quad \Omega;$$

(b) η*-accretive* if

$$u(t) - v(t), J_q(\eta_t(x, y)) \geq 0, \quad x(t), y(t) \quad X, \quad u(t) \quad M_t(x), \quad v(t) \quad M_t(y), \quad t \quad \Omega;$$

(c) *strictly* η*-accretive* if M is η-accretive and the equality holds if and only if $x(t) = y(t)$, $t \quad \Omega$;

(d) r*-strongly* η*-accretive* if there exists a measurable function $r : \Omega \quad (0, \quad)$ such that

$$u(t) - v(t), J_q(\eta_t(x, y)) \geq r(t) \; x(t) - y(t) \; ^q, \quad x(t), y(t) \quad X, \quad u(t) \quad M_t(x), \quad v(t) \quad M_t(y), \quad t \quad \Omega;$$

(e) α*-relaxed* η*-accretive* if there exists a measurable function $\alpha : \Omega \quad (0, \quad)$ such that

$$u(t) - v(t), J_q(\eta_t(x, y)) \geq -\alpha(t) \; x(t) - y(t) \; ^q, \quad x(t), y(t) \quad X, \; u(t) \quad M_t(x), \quad v(t) \quad M_t(y), \quad t \quad \Omega;$$

(f) m*-accretive* if M is accretive and $(I_t + \rho(t)M_t)(X) = X$ for all $t \quad \Omega$ and for any measurable function $\rho : \Omega \quad (0, \quad)$, where I denotes the identity mapping on X, $I_t(x) = x(t)$, for all $x(t) \quad X$, $t \quad \Omega$;

(g) *generalized* m*-accretive* if M is η-accretive and $(I_t + \rho(t)M_t)(X) = X$ for all $t \quad \Omega$ and any measurable function $\rho : \Omega \quad (0, \quad)$;

(h) H*-accretive* if M is accretive and $(H_t + \rho(t)M_t)(X) = X$ for all $t \quad \Omega$ and any measurable function $\rho : \Omega \quad (0, \quad)$, where $H_t(.) = H(t, .)$ for all $t \quad \Omega$;

(i) (H,η)-*accretive* if M is η-accretive and $(H_t+\rho(t)M_t)(X)=X$ for all $t\in\Omega$ and any measurable function $\rho:\Omega\to(0,\infty)$;

(j) A-*maximal* m-*relaxed* η-*accretive* if M is m-relaxed η-accretive and $(A_t+\rho(t)M_t)(X)=X$ for all $t\in\Omega$ and any measurable function $\rho:\Omega\to(0,\infty)$, where $A_t(.)=A(t,.)$ for all $t\in\Omega$;

(k) β-$\hat{H}$-*Lipschitz continuous* if there exists a measurable function $\beta:\Omega\to(0,+\infty)$ such that

$$\hat{H}(M_t(x),M_t(y))\leq\beta(t)\|x(t)-y(t)\|,\quad\forall x(t),y(t)\in X,\quad t\in\Omega.$$

Remark 2.9. (1) If $X=\mathcal{H}$ is a Hilbert space, then parts (a)–(i) of Definition 2.8 reduce to the definitions of monotone operators, η-monotone operators, strictly η-monotone operators, strongly η-monotone operators, relaxed η-monotone operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators and (H,η)-monotone operators, respectively.

(2) For appropriate and suitable choices of m, A, η and X, it is easy to see that part (j) of Definition 2.8 includes a number of definitions of monotone operators and accretive mappings (see [52]).

Proposition 2.10. [52] *Let $A:\Omega\times X\to X$ be an r-strongly η-accretive mapping and $M:\Omega\times X\multimap X$ be an A-maximal m-relaxed η-accretive mapping. Then the operator $(A_t+\rho(t)M_t)^{-1}$ is single-valued for any measurable function $\rho:\Omega\to(0,+\infty)$ and $t\in\Omega$.*

Definition 2.11. Let $A:\Omega\times X\to X$ be a strictly η-accretive mapping and $M:\Omega\times X\multimap X$ be an A-maximal m-relaxed η-accretive mapping. Then, for any measurable function

$\rho : \Omega \quad (0, + \quad)$, the resolvent operator $J^{\eta_t, M_t}_{\rho(t), A_t} : X \quad X$ is defined by:

$$J^{\eta_t, M_t}_{\rho(t), A_t}(u(t)) = (A_t + \rho(t) M_t)^{-1}(u(t)), \quad t \quad \Omega, \quad u(t) \quad X.$$

Proposition 2.12. [52] *Let X be a q-uniformly smooth Banach space and $\eta : \Omega \times X \times X$ X be τ-Lipschitz continuous, $A : \Omega \times X \quad X$ be an r-strongly η-accretive mapping and $M : \Omega \times X \multimap X$ be an A-maximal m-relaxed η-accretive mapping. Then the resolvent operator $J^{\eta_t, M_t}_{\rho(t), A_t} : X \quad X$ is $\frac{\tau^{q-1}(t)}{r(t) - \rho(t) m(t)}$-Lipschitz continuous, i.e.,*

$$\left\| J^{\eta_t, M_t}_{\rho(t), A_t}(x(t)) - J^{\eta_t, M_t}_{\rho(t), A_t}(y(t)) \right\| \leq \frac{\tau^{q-1}(t)}{r(t) - \rho(t) m(t)} \; x(t) - y(t) \;, \quad x(t), y(t) \quad X, \quad t \quad \Omega,$$

where $\rho(t) \quad \left(0, \frac{r(t)}{m(t)}\right)$ is a real-valued random variable for all $t \quad \Omega$.

3 A new random variational inclusion problem and random iterative algorithm

In what follows, we denote the collection of all fuzzy sets on X by $\mathfrak{F}(X) = \quad A \; A : X \quad [0,1] \;$. For any set K, a mapping $\mathcal{S}$ from K into $\mathfrak{F}(X)$ is called a *fuzzy mapping*. If $\mathcal{S} : K \quad \mathfrak{F}(X)$ is a fuzzy mapping, then $\mathcal{S}(x)$, for any $x \quad K$, is a fuzzy set on $\mathfrak{F}(X)$ (in the sequel, we denote $\mathcal{S}(x)$ by $\mathcal{S}_x$) and $\mathcal{S}_x(y)$, for any $y \quad X$, is the degree of membership of y in $\mathcal{S}_x$. For any $A \quad \mathfrak{F}(X)$ and $\alpha \quad [0,1]$, the set

$$(A)_\alpha = \quad x \quad X : A(x) \geq \alpha$$

is called a *α-cut* set of A.

Definition 3.1. A fuzzy mapping $\mathcal{S} : \Omega \quad \mathfrak{F}(X)$ is called *measurable* if, for any $\alpha \quad (0,1]$, $(\mathcal{S}(.))_\alpha : \Omega \multimap X$ is a measurable set-valued mapping.

Definition 3.2. A fuzzy mapping $\mathcal{S} : \Omega \times X \to \mathfrak{F}(X)$ is called a *random fuzzy mapping* if, for any $x \in X$, $\mathcal{S}(., x) : \Omega \to \mathfrak{F}(X)$ is a measurable fuzzy mapping.

Now, let us introduce our main considered problem.

Suppose that $\mathcal{S}, \mathcal{T}, \mathcal{P}, \mathcal{Q}, \mathcal{G} : \Omega \times X \to \mathfrak{F}(X)$ are random fuzzy mappings, $A, p : \Omega \times X \to X$ and $\eta : \Omega \times X \times X \to X$, $N : \Omega \times X \times X \times X \to X$ are random single-valued mappings. Further, let $a, b, c, d, e : X \to [0, 1]$ be any mappings and $M : \Omega \times X \times X \multimap X$ be a random set-valued mapping such that, for each fixed $t \in \Omega$ and $z(t) \in X$, $M(t, ., z(t)) : X \multimap X$ be an A-maximal m-relaxed η-accretive mapping with $\mathrm{Im}(p) \cap \mathrm{dom}\, M(t, ., z(t)) \neq \emptyset$. Now, we consider the following problem:

For any element $h : \Omega \to X$ and any measurable function $\lambda : \Omega \to (0, +\infty)$, find measurable mappings $x, \nu, u, v, \vartheta, w : \Omega \to X$ such that for each $t \in \Omega$, $x(t) \in X$, $\mathcal{S}_{t,x(t)}(\nu(t)) \geq a(x(t))$, $\mathcal{T}_{t,x(t)}(u(t)) \geq b(x(t))$, $\mathcal{P}_{t,x(t)}(v(t)) \geq c(x(t))$, $\mathcal{Q}_{t,x(t)}(\vartheta(t)) \geq d(x(t))$, $\mathcal{G}_{t,x(t)}(w(t)) \geq e(x(t))$ and

$$h(t) \in N_t(\nu, u, v) + \lambda(t) M_t(p_t(x) - \vartheta, w), \quad \forall t \in \Omega. \tag{3.1}$$

The problem (3.1) is called *the general nonlinear random A-maximal m-relaxed η-accretive equation with random relaxed cocoercive mappings and random fuzzy mappings in Banach spaces.*

Remark 3.3. Obviously, the random fuzzy mapping includes set-valued mapping, random set-valued mapping and fuzzy mapping as the special cases. These mean that for appropriate and suitable choices of X, A, η, λ, p, M, N, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$ and h, one can obtain many known classes of random variational inequalities, random quasi-variational inequalities, random complementarity and random quasi-complementarity problems as special cases of the problem (3.1), (see, for example [1–3, 22, 23, 34, 37, 45, 49, 50, 53, 58] and the references therein).

In the sequel, we will develop and analyze a new class of iterative methods and construct a new random iterative algorithm with mixed errors for solving the problem (3.1). For this end, we need the following lemmas.

Lemma 3.4. [12] *Let* $M : \Omega \times X \quad CB(X)$ *be a* $\hat{H}$*-continuous random set-valued mapping. Then, for any measurable mapping* $x : \Omega \quad X$*, the set-valued mapping* $M(., x(.)) : \Omega \quad CB(X)$ *is measurable.*

Lemma 3.5. [12] *Let* $M, V : \Omega \quad CB(X)$ *be two measurable set-valued mappings,* $\epsilon > 0$ *be a constant and* $x : \Omega \quad X$ *be a measurable selection of* M*. Then there exists a measurable selection* $y : \Omega \quad X$ *of* V *such that, for any* $t \quad \Omega$*,*

$$x(t) - y(t) \leq (1 + \epsilon)\hat{H}(M(t), V(t)).$$

The following lemma offers a good approach for solving the problem (3.1).

Lemma 3.6. *The set of measurable mappings* $x, \nu, u, v, \vartheta, w : \Omega \quad X$ *is a random solution of the problem* (3.1) *if and only if, for each* $t \quad \Omega$, $\nu(t) \quad S_t(x)$, $u(t) \quad T_t(x)$, $v(t) \quad P_t(x)$, $\vartheta(t) \quad Q_t(x)$, $w(t) \quad G_t(x)$ *and*

$$p_t(x) = \vartheta(t) + J^{\eta_t, M_t(.,w)}_{\rho(t)\lambda(t), A_t}[A_t(p_t(x) - \vartheta) - \rho(t)(N_t(\nu, u, v) - h(t))],$$

where $J^{\eta_t, M_t(.,w)}_{\rho(t)\lambda(t), A_t} = (A_t + \rho(t)\lambda(t)M_t(., w))^{-1}$ *and* $\rho : \Omega \quad (0, \quad)$ *is a measurable function.*

Proof. The fact follows directly from the definition of $J^{\eta_t, M_t(.,w)}_{\rho(t)\lambda(t), A_t}$.

In order to prove our main result, the following concepts are also needed. Let $\mathcal{S}, \mathcal{T}, \mathcal{P}, \mathcal{Q}, \mathcal{G}$: $\Omega \times X \quad \mathfrak{F}(X)$ be five random fuzzy mappings satisfying the following condition ( ): There exist five mappings $a, b, c, d, e : X \quad [0,1]$ such that

$$(\mathcal{S}_{t,x(t)})_{a(x(t))} \quad CB(X), \ (\mathcal{T}_{t,x(t)})_{b(x(t))} \quad CB(X), \ (\mathcal{P}_{t,x(t)})_{c(x(t))} \quad CB(X),$$

$$(\mathcal{Q}_{t,x(t)})_{d(x(t))} \quad CB(X), \ (\mathcal{G}_{t,x(t)})_{e(x(t))} \quad CB(X), \qquad (t, x(t)) \quad \Omega \times X.$$

By using the random fuzzy mapping $\mathcal{S}$ satisfying ( ) with the corresponding function a : $X \quad [0,1]$, we can define a random set-valued mapping S as follows:

$$S : \Omega \times X \quad CB(X), \quad (t, x(t)) \quad (\mathcal{S}_{t,x(t)})_{a(x(t))}, \qquad (t, x(t)) \quad \Omega \times X,$$

where $\mathcal{S}_{t,x(t)} = \mathcal{S}(t, x(t))$. From now on, the random fuzzy mappings $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{G}$, are assumed to satisfying the condition ( ) and we will let S, T, P, Q and G are the random set-valued mappings induced by those five random fuzzy mappings, respectively.

Now, by using Chang's lemma [12] and based on Lemma 3.6, we can construct the new following iterative algorithm for solving the problem (3.1).

Algorithm 3.7. *Let A, p, η, M, N, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, h, λ be the same as in the problem* (3.1) *and let S, T, P, Q, G be $\hat{H}$-continuous random set-valued mappings induced by $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{G}$, respectively. Assume that $\alpha : \Omega \quad (0,1]$ is a measurable step size function. For any measurable mapping $x_0 : \Omega \quad X$, the set-valued mappings $S(., x_0(.)), T(., x_0(.)), P(., x_0(.)), Q(., x_0(.))$, $G(., x_0(.)) : \Omega \quad CB(X)$ are measurable by Lemma* 3.4. Hence there exist measurable selections $\nu_0 : \Omega \quad X$ of $S(., x_0(.))$, $u_0 : \Omega \quad X$ of $T(., x_0(.))$, $v_0 : \Omega \quad X$ of $P(., x_0(.))$, $\vartheta_0 : \Omega \quad X$ of $Q(., x_0(.))$ and $w_0 : \Omega \quad X$ of $G(., x_0(.))$ by Himmelberg [35]. For each $t \quad \Omega$, set

$$x_1(t) = (1 - \alpha(t))x_0(t) + \alpha(t) \; x_0(t) - p_t(x_0) + \vartheta_0(t) + J^{\eta_t, M_t(.,w_0)}_{\rho(t)\lambda(t), A_t}[A_t(p_t(x_0) - \vartheta_0)$$

$$- \rho(t)(N_t(\nu_0, u_0, v_0) - h(t))] \; + \alpha(t)e_0(t) + r_0(t),$$

where $\rho(t)$ is the same as in Lemma 3.6 and $e_0, r_0 : \Omega \quad X$ are measurable functions. It is easy to know that $x_1 : \Omega \quad X$ is measurable. Since $\nu_0(t) \quad S_t(x_0) \quad CB(X)$, $u_0(t) \quad T_t(x_0) \quad CB(X)$, $v_0(t) \quad P_t(x_0) \quad CB(X)$, $\vartheta_0(t) \quad Q_t(x_0) \quad CB(X)$ and $w_0(t) \quad G_t(x_0) \quad CB(X)$, by Lemma 3.5, there exist measurable selections $\nu_1, u_1, v_1, w_1, \vartheta_1 : \Omega \quad X$ of the set-valued measurable mappings $S(., x_1(.))$, $T(., x_1(.))$, $P(., x_1(.))$, $Q(., x_1(.))$ and $G(., x_1(.))$, respectively, such that, for all $t \quad \Omega$,

$$\begin{aligned}
\nu_0(t) - \nu_1(t) &\leq \left(1 + \frac{1}{1}\right) \hat{H}(S_t(x_0), S_t(x_1)),\\
u_0(t) - u_1(t) &\leq \left(1 + \frac{1}{1}\right) \hat{H}(T_t(x_0), T_t(x_1)),\\
v_0(t) - v_1(t) &\leq \left(1 + \frac{1}{1}\right) \hat{H}(P_t(x_0), P_t(x_1)),\\
\vartheta_0(t) - \vartheta_1(t) &\leq \left(1 + \frac{1}{1}\right) \hat{H}(Q_t(x_0), Q_t(x_1)),\\
w_0(t) - w_1(t) &\leq \left(1 + \frac{1}{1}\right) \hat{H}(G_t(x_0), G_t(x_1)).
\end{aligned}$$

Letting

$$x_2(t) = (1 - \alpha(t))x_1(t) + \alpha(t) \; x_1(t) - p_t(x_1) + \vartheta_1(t) + J^{\eta_t, M_t(.,w_1)}_{\rho(t)\lambda(t), A_t}[A_t(p_t(x_1) - \vartheta_1)$$

$$- \rho(t)(N_t(\nu_1, u_1, v_1) - h(t))] \; + \alpha(t)e_1(t) + r_1(t), \qquad t \quad \Omega,$$

then $x_2 : \Omega \quad X$ is measurable. By induction, we can define the sequences $\; x_n(t) \;$, $\; \nu_n(t) \;$,

$u_n(t)$, $v_n(t)$, $\vartheta_n(t)$ and $w_n(t)$ for solving the problem (3.1) inductively satisfying

$$\begin{cases} x_{n+1}(t) = (1 - \alpha(t))x_n(t) + \alpha(t)\Big\{x_n(t) - p_t(x_n) + \vartheta_n(t) + J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t), A_t}[A_t(p_t(x_n) - \vartheta_n) \\ \qquad\qquad - \rho(t)(N_t(\nu_n, u_n, v_n) - h(t))]\Big\} + \alpha(t)e_n(t) + r_n(t), \quad t \quad \Omega, \\ \nu_n(t) \quad S_t(x_n), \quad \nu_n(t) - \nu_{n+1}(t) \ \leq \ \left(1 + \frac{1}{1+n}\right) \hat{H}(S_t(x_n), S_t(x_{n+1})), \\ u_n(t) \quad T_t(x_n), \quad u_n(t) - u_{n+1}(t) \ \leq \ \left(1 + \frac{1}{1+n}\right) \hat{H}(T_t(x_n), T_t(x_{n+1})), \\ v_n(t) \quad P_t(x_n), \quad v_n(t) - v_{n+1}(t) \ \leq \ \left(1 + \frac{1}{1+n}\right) \hat{H}(P_t(x_n), P_t(x_{n+1})), \\ \vartheta_n(t) \quad Q_t(x_n), \quad \vartheta_n(t) - \vartheta_{n+1}(t) \ \leq \ \left(1 + \frac{1}{1+n}\right) \hat{H}(Q_t(x_n), Q_t(x_{n+1})), \\ w_n(t) \quad G_t(x_n), \quad w_n(t) - w_{n+1}(t) \ \leq \ \left(1 + \frac{1}{1+n}\right) \hat{H}(G_t(x_n), G_t(x_{n+1})), \end{cases} \tag{3.2}$$

where for all $n \geq 0$ and $t \quad \Omega$, $e_n(t), r_n(t) \quad X$ are real-valued random errors to take into account a possible inexact computation of the random resolvent operator point satisfying the following conditions:

$$\begin{cases} \lim\limits_{n\to\infty} \ e_n(t) \ = \lim\limits_{n\to\infty} \ r_n(t) \ = 0, \quad t \quad \Omega; \\ \sum\limits_{n=0}^{\infty} \ e_n(t) - e_{n-1}(t) \ < \quad , \quad t \quad \Omega; \\ \sum\limits_{n=0}^{\infty} \ r_n(t) - r_{n-1}(t) \ < \quad , \quad t \quad \Omega. \end{cases} \tag{3.3}$$

Remark 3.8. For a suitable and appropriate choice of the mappings A, p, η, M, N, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, S, T, P, Q, G, α, h, λ, the sequences e_n , r_n and the space X, Algorithm 3.7 includes many known algorithms which due to classes of variational inequalities and variational inclusions (see, for example [13, 18, 22–24, 33, 34, 37–40, 42, 44, 46, 53, 58]).

4 Main result

In this section, we prove the existence of solutions for the problem (3.1) and the convergence of iterative sequences generated by Algorithm 3.7 in Banach spaces.

Theorem 4.1. *Let X be a q-uniformly smooth Banach space, A, p, η, M, N, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, h, λ be the same as in the problem* (3.1) and $S, T, P, Q, G : \Omega \times X \to CB(X)$ be five random set-valued mappings induced by $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, respectively. Further, suppose that

(a) p is (γ, ϖ)-relaxed cocoercive and π-Lipschitz continuous;

(b) A is r-strongly η-accretive and σ-Lipschitz continuous;

(c) η is τ-Lipschitz continuous;

(d) S, T, P, Q and G are ξ-$\hat{H}$-Lipschitz continuous, ζ-$\hat{H}$-Lipschitz continuous, ς-$\hat{H}$-Lipschitz continuous, ϱ-$\hat{H}$-Lipschitz continuous and ι-$\hat{H}$-Lipschitz continuous, respectively;

(e) N is ϵ-Lipschitz continuous in the second argument, δ-Lipschitz continuous in the third argument and κ-Lipschitz continuous in the fourth argument;

(f) There exist the measurable functions $\mu : \Omega \to (0, +\infty)$ and $\rho : \Omega \to (0, +\infty)$ with $\rho(t) \in \left(0, \frac{r(t)}{\lambda(t)m(t)}\right)$, for all $t \in \Omega$, such that

$$\left\| J^{\eta_t, M_t(.,x)}_{\rho(t)\lambda(t), A_t}(z(t)) - J^{\eta_t, M_t(.,y)}_{\rho(t)\lambda(t), A_t}(z(t)) \right\| \leq \mu(t) \|x(t) - y(t)\|, \quad \forall t \in \Omega, x(t), y(t), z(t) \in X \tag{4.1}$$

and

$$\begin{cases} \varphi(t) = \varrho(t) + \mu(t)\iota(t) + \sqrt[q]{1 - q\varpi(t) + (q\gamma(t) + c_q)\pi^q(t)} < 1, \\ \sigma(t)(\pi(t) + \varrho(t)) + \rho(t)(\epsilon(t)\xi(t) + \delta(t)\zeta(t) + \kappa(t)\varsigma(t)) \\ < \tau^{1-q}(t)(1 - \varphi(t))(r(t) - \rho(t)\lambda(t)m(t)), \end{cases} \tag{4.2}$$

where c_q is the same as in Lemma 2.1. Then there exists a set of measurable mappings $x^*, \nu^*, u^*, v^*, \vartheta^*, w^* : \Omega \to X$ which is a random solution of the problem (3.1) and for each $t \in \Omega$, $x_n(t) \to x^*(t)$, $\nu_n(t) \to \nu^*(t)$, $u_n(t) \to u^*(t)$, $v_n(t) \to v^*(t)$, $\vartheta_n(t) \to \vartheta^*(t)$, $w_n(t) \to w^*(t)$ as $n \to \infty$, where $\{x_n(t)\}$, $\{\nu_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{\vartheta_n(t)\}$ and $\{w_n(t)\}$ are the iterative sequences generated by Algorithm 3.7.

Proof. Firstly, for each $n \geq 0$, by considering (3.2) and (4.1), in view of Proposition 2.12, we see that

$$
\begin{aligned}
& x_{n+1}(t) - x_n(t) \\
\leq\ & (1-\alpha(t))x_n(t) + \alpha(t)\Big\{x_n(t) - p_t(x_n) + \vartheta_n(t) + J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_n) - \vartheta_n) \\
& - \rho(t)(N_t(\nu_n, u_n, v_n) - h(t))]\Big\} + \alpha(t)e_n(t) + r_n(t) - (1-\alpha(t))x_{n-1}(t) \\
& - \alpha(t)\Big\{x_{n-1}(t) - p_t(x_{n-1}) + \vartheta_{n-1}(t) + J^{\eta_t, M_t(.,w_{n-1})}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_{n-1}) - \vartheta_{n-1}) \\
& - \rho(t)(N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) - h(t))]\Big\} - \alpha(t)e_{n-1}(t) - r_{n-1}(t) \\
\leq\ & (1-\alpha(t))\ \ x_n(t) - x_{n-1}(t)\ \ + \alpha(t)\Big(\ \ x_n(t) - x_{n-1}(t) - (p_t(x_n) - p_t(x_{n-1})) \\
& +\ \ \vartheta_n(t) - \vartheta_{n-1}(t)\ \ +\ \ J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_n) - \vartheta_n) - \rho(t)(N_t(\nu_n, u_n, v_n) - h(t))] \\
& - J^{\eta_t, M_t(.,w_{n-1})}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_{n-1}) - \vartheta_{n-1}) - \rho(t)(N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) - h(t))]\ \ \Big) \\
& + \alpha(t)\ \ e_n(t) - e_{n-1}(t)\ \ +\ \ r_n(t) - r_{n-1}(t) \\
\leq\ & (1-\alpha(t))\ \ x_n(t) - x_{n-1}(t)\ \ + \alpha(t)\Big(\ \ x_n(t) - x_{n-1}(t) - (p_t(x_n) - p_t(x_{n-1})) \\
& +\ \ \vartheta_n(t) - \vartheta_{n-1}(t)\ \ +\ \ J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_n) - \vartheta_n) - \rho(t)(N_t(\nu_n, u_n, v_n) - h(t))] \\
& - J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_{n-1}) - \vartheta_{n-1}) - \rho(t)(N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) - h(t))] \\
& +\ \ J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_{n-1}) - \vartheta_{n-1}) - \rho(t)(N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) - h(t))] \\
& - J^{\eta_t, M_t(.,w_{n-1})}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x_{n-1}) - \vartheta_{n-1}) - \rho(t)(N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) - h(t))]\ \ \Big) \\
& + \alpha(t)\ \ e_n(t) - e_{n-1}(t)\ \ +\ \ r_n(t) - r_{n-1}(t) \\
\leq\ & (1-\alpha(t))\ \ x_n(t) - x_{n-1}(t)\ \ + \alpha(t)\Big(\ \ x_n(t) - x_{n-1}(t) - (p_t(x_n) - p_t(x_{n-1}))\ \ +\ \ \vartheta_n(t) - \vartheta_{n-1}(t) \\
& + \mu(t)\ \ w_n(t) - w_{n-1}(t)\ \ + \frac{\tau^{q-1}(t)}{r(t) - \rho(t)\lambda(t)m(t)}(\ \ A_t(p_t(x_n) - \vartheta_n) - A_t(p_t(x_{n-1}) - \vartheta_{n-1}) \\
& + \rho(t)\ \ N_t(\nu_n, u_n, v_n) - N_t(\nu_{n-1}, u_{n-1}, v_{n-1})\ \ )\Big) + \alpha(t)\ \ e_n(t) - e_{n-1}(t)\ \ +\ \ r_n(t) - r_{n-1}(t)\ \ .
\end{aligned}
\tag{4.3}
$$

Meanwhile, by Lemma 2.1, there exists a constant $c_q > 0$ such that

$$\begin{aligned} & \|x_n(t) - x_{n-1}(t) - (p_t(x_n) - p_t(x_{n-1}))\|^q \\ \leq\ & \|x_n(t) - x_{n-1}(t)\|^q - q\langle p_t(x_n) - p_t(x_{n-1}), J_q(x_n(t) - x_{n-1}(t))\rangle + c_q\|p_t(x_n) - p_t(x_{n-1})\|^q. \end{aligned}$$

Consequently, since p is (γ, ϖ)-relaxed cocoercive and π-Lipschitz continuous, we obtain

$$\begin{aligned} & \|x_n(t) - x_{n-1}(t) - (p_t(x_n) - p_t(x_{n-1}))\|^q \\ \leq\ & \|x_n(t) - x_{n-1}(t)\|^q + (q\gamma(t) + c_q)\|p_t(x_n) - p_t(x_{n-1})\|^q - q\varpi(t)\|x_n(t) - x_{n-1}(t)\|^q \\ =\ & (1 - q\varpi(t) + (q\gamma(t) + c_q)\pi^q(t))\|x_n(t) - x_{n-1}(t)\|^q. \end{aligned} \tag{4.4}$$

Furthermore, by ϱ-$\hat{H}$-Lipschitz continuity of Q and ι-$\hat{H}$-Lipschitz continuity of G, from (3.2) we deduce that

$$\begin{aligned} \|\vartheta_n(t) - \vartheta_{n-1}(t)\| & \leq \left(1 + \frac{1}{n}\right)\hat{H}(Q_t(x_n), Q_t(x_{n-1})) \\ & \leq \varrho(t)\left(1 + \frac{1}{n}\right)\|x_n(t) - x_{n-1}(t)\| \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \|w_n(t) - w_{n-1}(t)\| & \leq \left(1 + \frac{1}{n}\right)\hat{H}(G_t(x_n), G_t(x_{n-1})) \\ & \leq \iota(t)\left(1 + \frac{1}{n}\right)\|x_n(t) - x_{n-1}(t)\|. \end{aligned} \tag{4.6}$$

By using (4.5) together with σ-Lipschitz continuity of A, π-Lipschitz continuity of p, we obtain

$$\begin{aligned} & \|A_t(p_t(x_n) - \vartheta_n) - A_t(p_t(x_{n-1}) - \vartheta_{n-1})\| \\ \leq\ & \sigma(t)(\|p_t(x_n) - p_t(x_{n-1})\| + \|\vartheta_n(t) - \vartheta_{n-1}(t)\|) \\ \leq\ & \sigma(t)\left(\pi(t) + \varrho(t)\left(1 + \frac{1}{n}\right)\right)\|x_n(t) - x_{n-1}(t)\|. \end{aligned} \tag{4.7}$$

Moreover, since N is ϵ-Lipschitz continuous in the second argument, δ-Lipschitz continuous in the third argument, κ-Lipschitz continuous in the fourth argument and S, T, P are

ξ-$\hat{H}$-Lipschitz continuous, ζ-Lipschitz continuous and ς-$\hat{H}$-Lipschitz continuous, respectively, by (3.2), we get

$$
\begin{aligned}
& N_t(\nu_n, u_n, v_n) - N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) \\
\leq \; & N_t(\nu_n, u_n, v_n) - N_t(\nu_{n-1}, u_n, v_n) \; + \; N_t(\nu_{n-1}, u_n, v_n) - N_t(\nu_{n-1}, u_{n-1}, v_n) \\
& + \; N_t(\nu_{n-1}, u_{n-1}, v_n) - N_t(\nu_{n-1}, u_{n-1}, v_{n-1}) \\
\leq \; & \epsilon(t) \; \nu_n(t) - \nu_{n-1}(t) \; + \delta(t) \; u_n(t) - u_{n-1}(t) \; + \kappa(t) \; v_n(t) - v_{n-1}(t) \\
\leq \; & (\epsilon(t)\xi(t) + \delta(t)\zeta(t) + \kappa(t)\varsigma(t)) \left(1 + \frac{1}{n}\right) \; x_n(t) - x_{n-1}(t) \; .
\end{aligned}
\tag{4.8}
$$

Now, substitute (4.4)–(4.8) into (4.3), we get that

$$
\begin{aligned}
x_{n+1}(t) - x_n(t) \; \leq \; & (1 - \alpha(t) + \alpha(t)\psi(t, n)) \; x_n(t) - x_{n-1}(t) \\
& + \alpha(t) \; e_n(t) - e_{n-1}(t) \; + \; r_n(t) - r_{n-1}(t) \; ,
\end{aligned}
\tag{4.9}
$$

where

$$
\psi(t, n) = (\varrho(t) + \mu(t)\iota(t)) \left(1 + \frac{1}{n}\right) + \sqrt[q]{1 - q\varpi(t) + (q\gamma(t) + c_q)\pi^q(t)} + \frac{\tau^{q-1}(t)\Gamma(t, n)}{r(t) - \rho(t)\lambda(t)m(t)},
$$
$$
\Gamma(t, n) = \sigma(t) \left(\pi(t) + \varrho(t) \left(1 + \frac{1}{n}\right)\right) + \rho(t)(\epsilon(t)\xi(t) + \delta(t)\zeta(t) + \kappa(t)\varsigma(t)) \left(1 + \frac{1}{n}\right).
$$

Let us put

$$
\theta(t, n) = 1 - \alpha(t) + \alpha(t)\psi(t, n), \quad \text{for each } \; n \geq 0, \quad t \quad \Omega.
$$

Then, for each $t \quad \Omega$, we know that

$$
\theta(t, n) \qquad \theta(t) = 1 - \alpha(t) + \alpha(t)\psi(t), \quad \text{as } \; n \qquad ,
$$

where

$$
\psi(t) = \varrho(t) + \mu(t)\iota(t) + \sqrt[q]{1 - q\varpi(t) + (q\gamma(t) + c_q)\pi^q(t)} + \frac{\tau^{q-1}(t)\Gamma(t)}{r(t) - \rho(t)\lambda(t)m(t)},
$$
$$
\Gamma(t) = \sigma(t)(\pi(t) + \varrho(t)) + \rho(t)(\epsilon(t)\xi(t) + \delta(t)\zeta(t) + \kappa(t)\varsigma(t)).
$$

It follows that, in view of the condition (4.2), we have $\psi(t) \quad (0,1)$ for all $t \quad \Omega$. This implies $0 < \theta(t) < 1$ for all $t \quad \Omega$. Hence there exist $n_0 \quad \mathbb{N}$ and a measurable function $\hat{\theta} : \Omega \quad (0, \quad)$ (Take $\hat{\theta}(t) = \frac{\theta(t)+1}{2} \quad (\theta(t), 1)$ for each $t \quad \Omega$) such that $\theta(t, n) \leq \hat{\theta}(t)$ for all $n \geq n_0$ and $t \quad \Omega$. Accordingly, for all $n > n_0$, by (4.9), deduce that, for all $t \quad \Omega$,

$$
\begin{aligned}
& x_{n+1}(t) - x_n(t) \\
& \leq \hat{\theta}(t) \;\; x_n(t) - x_{n-1}(t) \;\; + \alpha(t) \;\; e_n(t) - e_{n-1}(t) \;\; + \;\; r_n(t) - r_{n-1}(t) \\
& \leq \hat{\theta}(t)[\hat{\theta}(t) \;\; x_{n-1}(t) - x_{n-2}(t) \;\; + \alpha(t) \;\; e_{n-1}(t) - e_{n-2}(t) \;\; + \;\; r_{n-1}(t) - r_{n-2}(t) \;\;] \\
& \quad + \alpha(t) \;\; e_n(t) - e_{n-1}(t) \;\; + \;\; r_n(t) - r_{n-1}(t) \\
& = \hat{\theta}^2(t) \;\; x_{n-1}(t) - x_{n-2}(t) \;\; + \alpha(t)[\hat{\theta}(t) \;\; e_{n-1}(t) - e_{n-2}(t) \\
& \quad + \;\; e_n(t) - e_{n-1}(t) \;\;] + \hat{\theta}(t) \;\; r_{n-1}(t) - r_{n-2}(t) \;\; + \;\; r_n(t) - r_{n-1}(t) \\
& \leq \\
& \vdots \\
& \leq \hat{\theta}^{n-n_0}(t) \;\; x_{n_0+1}(t) - x_{n_0}(t) \;\; + \sum_{i=1}^{n-n_0} \alpha(t)\hat{\theta}^{i-1}(t) \;\; e_{n-(i-1)}(t) - e_{n-i}(t) \\
& \quad + \sum_{i=1}^{n-n_0} \hat{\theta}^{i-1}(t) \;\; r_{n-(i-1)}(t) - r_{n-i}(t) \;\; .
\end{aligned}
\tag{4.10}
$$

By using the inequality (4.10), it follows that, for any $m \geq n > n_0$,

$$
\begin{aligned}
x_m(t) - x_n(t) \;\; \leq \sum_{j=n}^{m-1} \;\; x_{j+1}(t) - x_j(t) \;\; & \leq \sum_{j=n}^{m-1} \hat{\theta}^{j-n_0}(t) \;\; x_{n_0+1}(t) - x_{n_0}(t) \\
& + \sum_{j=n}^{m-1} \sum_{i=1}^{j-n_0} \alpha(t)\hat{\theta}^{i-1}(t) \;\; e_{n-(i-1)}(t) - e_{n-i}(t) \\
& + \sum_{j=n}^{m-1} \sum_{i=1}^{j-n_0} \hat{\theta}^{i-1}(t) \;\; r_{n-(i-1)}(t) - r_{n-i}(t) \;\; .
\end{aligned}
\tag{4.11}
$$

Since $\hat{\theta}(t) < 1$ for all $t \quad \Omega$, it follows from (3.3) and (4.11) that $x_m(t) - x_n(t) \qquad 0$ as $n \qquad$. This means $x_n(t)$ is a Cauchy sequence in X. In view of completeness of X, there exists $x^*(t) \quad X$ such that $x_n(t) \quad x^*(t)$ for all $t \quad \Omega$.

Consequently, by using (3.2), ξ-$\hat{H}$-Lipschitz continuity of S, ζ-$\hat{H}$-Lipschitz continuity of T, ς-$\hat{H}$-Lipschitz continuity of P, ϱ-$\hat{H}$-Lipschitz continuity of Q and ι-$\hat{H}$-Lipschitz continuity of G, we know that $\{\nu_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{\vartheta_n(t)\}$ and $\{w_n(t)\}$ are also Cauchy sequences in X. Thus there are $\nu^*(t), u^*(t), v^*(t), \vartheta^*(t), w^*(t)$ in X such that, for all $t \in \Omega$, $\nu_n(t) \to \nu^*(t)$, $u_n(t) \to u^*(t)$, $v_n(t) \to v^*(t)$, $\vartheta_n(t) \to \vartheta^*(t)$ and $w_n(t) \to w^*(t)$ as $n \to \infty$. Since $\{x_n(t)\}$, $\{\nu_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{\vartheta_n(t)\}$ and $\{w_n(t)\}$ are sequences of measurable mappings, we know that $x, \nu, u, v, \vartheta, w : \Omega \to X$ are also measurable. Further, for each $t \in \Omega$, we have

$$\begin{aligned} d(\nu^*(t), S_t(x^*)) &= \inf\{\|\nu^*(t) - z\| : z \in S_t(x^*)\} \\ &\leq \|\nu^*(t) - \nu_n(t)\| + d(\nu_n(t), S_t(x^*)) \\ &\leq \|\nu^*(t) - \nu_n(t)\| + \hat{H}(S_t(x_n), S_t(x^*)) \\ &\leq \|\nu^*(t) - \nu_n(t)\| + \xi(t)\|x_n(t) - x^*(t)\|. \end{aligned}$$

Notice that, the right side of the above inequality tends to zero as $n \to \infty$, this implies that $\nu^*(t) \in S_t(x^*)$.

Similarly, we can verify that for each $t \in \Omega$, $u^*(t) \in T_t(x^*)$, $v^*(t) \in P_t(x^*)$, $\vartheta^*(t) \in Q_t(x^*)$ and $w^*(t) \in G_t(x^*)$. Moreover, the condition (4.1) and $w_n(t) \to w^*(t)$, for all $t \in \Omega$, as $n \to \infty$, imply that for each $t \in \Omega$, $J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t} \to J^{\eta_t, M_t(.,w^*)}_{\rho(t)\lambda(t),A_t}$ uniformly on X, as $n \to \infty$.

Now, since for each $t \in \Omega$, the mappings $J^{\eta_t, M_t(.,w_n)}_{\rho(t)\lambda(t),A_t}$, p_t, N_t and A_t are continuous, it follows from (3.2) and (3.3) that for each $t \in \Omega$,

$$p_t(x^*) = \vartheta^*(t) + J^{\eta_t, M_t(.,w^*)}_{\rho(t)\lambda(t),A_t}[A_t(p_t(x^*) - \vartheta^*) - \rho(t)(N_t(\nu^*, u^*, v^*) - h(t))].$$

Finally, Lemma 3.6 implies that measurable mappings $x^*, \nu^*, u^*, v^*, \vartheta^*, w^* : \Omega \to X$ are a random solution of the problem (3.1). This completes the proof.

Remark 4.2. If X is a 2-uniformly smooth Banach space and there exists a measurable function $\rho : \Omega \to (0, \infty)$ with $\rho(t) \in (0, \frac{r(t)}{\lambda(t)m(t)})$, for all $t \in \Omega$, such that

$$\varphi(t) = \varrho(t) + \mu(t)\iota(t) + \sqrt{1 - 2\varpi(t) + (2\gamma(t) + c_2)\pi^2(t)} < 1,$$

$$2\varpi(t) - (2\gamma(t) + c_2)\pi^2(t) < 1,$$

$$\rho(t) < \frac{r(t)(1 - \varphi(t)) - \tau(t)\sigma(t)(\pi(t) + \varrho(t))}{\tau(t)[(\epsilon(t)\xi(t) + \delta(t)\zeta(t) + \kappa(t)\varsigma(t)] + (1 - \varphi(t))\lambda(t)m(t)},$$

then (4.2) holds. As we know, Hilbert spaces and L_p(or l_p) spaces, $2 \leq p < \infty$, are 2-uniformly smooth.

Remark 4.3. Theorem 4.1 generalizes and improves Theorems 3.1 and 3.2 in [23], Theorems 3.1, 3.3 and 3.4 in [53] and Theorems 4.1, 4.3 and 4.4 in [58]. In brief, for an appropriate choice of the mappings A, p, η, M, N, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{G}$, S, T, P, Q, G, h, λ, the measurable step size function α, the sequences $\{e_n\}$, $\{r_n\}$ and the space X, Theorem 4.1 includes many known results of generalized variational inclusions as special cases (see [13, 18, 22–24, 33, 34, 37–40, 42, 44, 46, 49, 53, 58] and the references therein).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally in this paper. They read and approved the final manuscript.

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