รายงานวิจัยฉบับสมบูรณ์

โครงการ ทฤษฎีบทจุดตรึงและการลู่เข้าสำหรับบางการส่งแบบไม่ขยายทั่วไป

Fixed point and convergence theorems for some generalized nonexpansive mappings

โดย

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15 มิถุนายน 2555

สัญญาเลขที่ MRG5380249

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
สำนักงานกองทุนสนับสนุนการวิจัย
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(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยนี้ได้รับทุนสนับสนุนการวิจัยจากทุนพัฒนาศักยภาพในการทำงานของอาจารย์รุ่นใหม่ ปี 2553 โดยสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และมหาวิทยาลัยนเรศวร ผู้วิจัยขอขอบพระคุณเจ้าของทุนเป็นอย่างสูงมา ณ โอกาสนี้

ขอขอบพระคุณ ศาสตราจารย์ ดร.สมยศ พลับเที่ยง นักวิจัยที่ปรึกษา ที่ได้ให้คำแนะนำ ชี้แนะผู้วิจัยทั้งในด้านการวิชาการ และการวิจัย ตลอดมา

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รหัสโครงการ: MRG5380249

ชื่อโครงการ: ทฤษฎีบทจุดตรึงและการลู่เข้าสำหรับบางการส่งแบบไม่ขยายทั่วไป

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ผู้วิจัยได้พิสูจน์ทฤษฎีบทโดยใช้ระเบียบวิธีแบบลูกผสม เพื่อใช้ในการค้นหาจุดตรึงร่วมของการส่งแบบหดเทียมที่เป็นแบบลิพชิทซ์ และการส่งแบบหดเทียมโดยแท้ ในปริภูมิฮิลเบิร์ต ผลลัพธ์ที่ได้ครอบคลุมและขยายงานของ Yao และคณะ [Y.H. Yao, Y.C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997-5002.] และงานวิจัยที่เกี่ยข้องอื่นๆ

นอกจากนั้นผู้วิจัยยังได้ค้นพบ อสมการที่มีความสำคัญเกี่ยวข้องกับการส่งแบบหดเทียมโดยแท้และการส่งที่เกิดมาจากปัญหาดุลภาพทั่วไปแบบผสมในปริภูมิบานาค อสมการที่ได้ค้นพบนั้นเป็นโครงสร้างสำคัญของการสร้างระเบียบวิธีทำซ้ำแบบหดตัวสำหรับการค้นหาผลเฉลยร่วมของปัญหาดุลภาพทั่วไปแบบผสมและปัญหาจุดตรึงของกึ่งการหดเทียมโดยแท้แบบปิด ผลลัพธ์ที่ได้เป็นจริงในปริภูมิบานาคราบเรียบที่เป็นแบบนูนอย่างเข้มและสะท้อน ที่มีสมบัติ (K) ผลลัพธ์ที่ได้จากงานวิจัยปรับปรุงและขยายผลลัพธ์ของ Zhou และ Gao [H. Zhou, E. Gao, An iterative method of fixed points for closed and quasi-strict pseudocontractions in Banach spaces, J. Appl. Math. Comput. 33 (2010) 227-237.] และงานวิจัยอื่นๆ ที่เกี่ยวข้อง

คำหลัก : การหดตัวแบบเทียม, ระเบียบวิธีลูกผสม, จุดตรึงร่วม, ปัญหาดุลภาพ, ตัวดำเนินการทางเดียว

Abstract

Project Code: MRG5380249

Project Title: Fixed point and convergence theorems for some generalized nonexpansive mappings

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Project Period : June 15, 2010 - June 15, 2012

We prove a strong convergence theorem by using a hybrid algorithm in order to find a common fixed point of Lipschitz pseudo-contraction and κ-strict pseudo-contraction in Hilbert spaces. Our results extend the recent ones announced by Yao et al. [Y.H. Yao, Y.C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997-5002.] and many others.

Moreover, we found the significant inequality related to quasi-strict pseudo-contractions and the mappings defined from generalized mixed equilibrium problems on Banach spaces. It was taken to create an iterative shrinking projection method for finding a common solution of generalized mixed equilibrium problems and fixed point problems of closed and quasi-strict pseudo-contractions. Its results hold in reflexive, strictly convex and smooth Banach spaces with the property (*K*). The results of this paper improve and extend the corresponding results of Zhou and Gao [H. Zhou, E. Gao, An iterative method of fixed points for closed and quasi-strict pseudo-contractions in Banach spaces, J. Appl. Math. Comput. 33 (2010) 227-237.] and many others.

Keywords: Pseudo-contraction, Hybrid algorithm, Common fixed point, Equilibrium problem, Monotone operator

Executive summary

ทฤษฎีบทจุดตรึง (Fixed point theorem) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน ที่มีบทประยุกต์และนำไปใช้อย่างกว้างขวาง ซึ่งปัญหาสำคัญประการหนึ่งคือวิธีการค้นหาจุดตรึง ของการส่งแบบไม่ขยาย (Nonexpansive mapping) และการส่งอื่นๆ ที่เกี่ยวข้อง พบว่าระเบียบวิธีทำซ้ำแบบปิคาร์ด (Picard iteration) หรือ ลำดับปิคาร์ด (Picard sequence) $\left\{T^n x_0\right\}_{n=1}^{\infty}$ นั้นมีข้อจำกัดและมักจะเกิดความล้มเหลวในการใช้ค้นหาจุดตรึง ตัวอย่างเช่นการส่งแบบไม่ขยาย (Nonexpansive mapping) $T:[0,1]\to[0,1]$ นิยามโดย $T(x)=1-x$ พบว่าเซตจุดตรึง คือ $F(T)=\left\{x\in[0,1]:Tx=x\right\}=\left\{\frac{1}{2}\right\}$ ถ้าเลือก $x_0=\frac{1}{3}$ แล้วลำดับปิคาร์ด คือ $\left\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\ldots\right\}$ ซึ่งไม่ลู่เข้าไปยังจุดตรึง เป็นตัวอย่างง่ายๆ ตัวอย่างหนึ่ง ที่ไม่สามารถใช้ลำดับปิคาร์ด เป็นเครื่องมือในการค้นหาจุดตรึงได้ ด้วยเหตุนี้ทำให้นักคณิตศาสตร์พยายามพัฒนาวิธีการค้นหาจุดตรึงของการส่งแบบต่างๆ ในปริภูมิต่างๆ ในปี ค.ศ.1953 W.R. Mann ได้เสนอวิธีทำซ้ำแบบเฉลี่ยค่า $x_{n+1}=\alpha_n x_n+(1-\alpha_n)Tx_n,\ n=0,1,2,\ldots$ โดยที่ $\{\alpha_n\}\subseteq[0,1]$ ซึ่งสามารถแก้ไขปัญหาการไม่ลู่เข้าของตัวอย่างข้างต้นได้ โดยที่สามารถเลือก x_0 เป็นสมาชิกตัวใดก็ได้ที่อยู่ในโดเมนของการส่ง T อย่างไรก็ตามลำดับของ Mann ยังมีข้อด้อย กล่าวคือ A. Genel และ J. Lindenstrass มีตัวอย่างในปริภูมิฮิลเบิร์ตที่ยืนยันว่าลำดับของ Mann ไม่สามารถลู่เข้าแบบเข้ม (Converge strongly) ไปยังจุดตรึงได้ แต่จะลู่เข้าอย่างอ่อน (Convergence weakly) ไปยังจุดตรึงได้เพียงเท่านั้น ต่อมาในปี ค.ศ. 2003 K. Nakajo and W. Takahashi ได้เสนอวิธีการเพื่อดัดแปลงลำดับของ Mann ซึ่งสามารถทำให้ลำดับที่ถูกสร้างขึ้นลู่เข้าแบบเข้มไปยังจุดตรึงได้ ซึ่งปัจจุบันเป็นที่รู้จักกันดีในชื่อว่าวิธีการลูกผสม (Hybrid method) จากนั้นมาได้มีนักคณิตศาสตร์หลายท่านได้นำแนวทางวิธีการแบบลูกผสม ไปใช้ในการค้นหาจุดตรึงและจุดตรึงร่วมของการส่งที่ไม่เป็นเชิงเส้นอย่างกว้างขวาง ทั้งในปริภูมิฮิลเบิร์ต และปริภูมิบานาค ซึ่งการส่งแบบหดเทียม (Pseudo contractive mapping) ถือได้ว่าเป็นการส่งเป็นการส่งที่ไม่เป็นเชิงเส้นชนิดหนึ่งที่ครอบคลุมการส่งแบบไม่ขยาย และมีความสัมพันธ์อย่างใกล้ชิดกับการส่งแบบทางเดียว (Monotone mapping) ในทางปฏิบัติแล้วการส่งแบบหดเทียมจะมีศักยภาพในเชิงการประยุกต์ได้มากกว่าการส่งแบบไม่ขยายในการแก้ไขปัญหาเชิงผกผัน (Inverse problems) และปัญหาอื่นๆ

ในงานวิจัยนี้ผู้วิจัยมีความสนใจและได้พัฒนาระเบียบวิธีทำซ้ำโดยอาศัยวิธีการลูกผสม เพื่อใช้ในการค้นหาจุดตรึงร่วมของการส่งลิพชิทซ์แบบหดเทียม และการส่งแบบหดเทียมโดยแท้ในปริภูมิฮิลเบิร์ต โดยสร้างระเบียบวิธีการทำซ้ำดังนี้

$$\begin{cases} x_0 \in H \\ C_1 = C,\ x_1 = P_{C_1}(x_0) \\ y_n = (1-\alpha_n)x_n + \alpha_n T z_n \\ z_n = (1-\beta_n)x_n + \beta_n S x_n \\ C_{n+1} = \left\{ v\in C_n \ : \ \begin{array}{l} \|\alpha_n(I-T)y_n\|^2 + (1-\kappa)\|(I-S)x_n\|^2 \\ \le 2\alpha_n\langle x_n - v, (I-T)y_n\rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n\rangle \\ + 2\alpha_n\beta_n L_T \|x_n - Sx_n\| \|y_n - x_n + \alpha_n(I-T)y_n\| \\ + \beta_n\left(\left(\dfrac{2\beta_n}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_n\|^2 \end{array} \right\} \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{cases} \tag{1.1}$$

ซึ่ง (1.1) จะครอบคลุมงานวิจัยของ Y.H. Yao และคณะ [Y.H. Yao, Y.C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997-5002.]

ในอีกทางหนึ่ง กำหนดให้ E เป็นปริภูมิบานาค C คือเซตย่อยที่ไม่เป็นเซตว่างของ E และ $\Theta: C\times C\to \mathbb{R}$ เป็นฟังก์ชันเชิงคู่ (bifunction) ปัญหาดุลภาพ (Equilibrium problem) คือการหา $x\in C$ ที่ทำให้

$$\Theta(x,y)\geq 0 \quad \text{สำหรับทุกๆ } y\in C \tag{1.2}$$

และจะเขียน $EP(\Theta)$ แทนเซตคำตอบของ (1.2) ซึ่งการค้นหาคำตอบร่วมระหว่างปัญหาดุลภาพและปัญหาจุดตรึงได้มีนักคณิตศาสตร์ให้ความสนใจอยู่ไม่น้อย ในปี ค.ศ.2007 S. Takashshi และ W. Takahashi ได้ใช้วิธีการประมาณแบบหน่วง (Viscosity approximation method) ในการค้นหาจุดตรึงในปริภูมิฮิลเบิร์ต จากนั้นปี ค.ศ. 2008 W. Takahashi และ K. Zembayashi ได้ใช้ระเบียบวิธีลูกผสมในการค้นหาคำตอบร่วมสำหรับปัญหาจุดตรึงของการส่งไม่ขยายแบบสัมพันธ์ (Relatively nonexpansive mapping) และปัญหาดุลภาพ ในปริภูมิบานาค ต่อมาปี ค.ศ. 2009 S. Zhang ได้แนะนำปัญหาดุลภาพทั่วไปแบบผสม (Generalized mixed equilibrium problem) คือการหา $x\in C$ ที่ทำให้

$$\Theta(x,y)+\langle Ax,y-x\rangle+\varphi(y)-\varphi(x)\geq 0 \qquad \text{สำหรับทุกๆ } y\in C \tag{1.3}$$

เมื่อ $A: C\to E^*$ เป็นการส่งแบบไม่เชิงเส้น และ $\varphi: C\to \mathbb{R}$ เป็นฟังก์ชันค่าจริง ซึ่งถ้า $A=0$ และ $\varphi=0$ แล้ว (1.3) จะลดรูปกลายเป็น (1.2) ในอีกด้านหนึ่งสำหรับการส่ง $T: C\to C$ จะถูกเรียกว่ากึ่งการส่งหดเทียมโดยแท้ (quasi-strict pseudo-contraction) ซึ่งนิยามโดย H. Zhou และ E. Gao ก็ต่อเมื่อ มี $k\in[0,1)$ ที่ทำให้ $\phi(p,Tx)\leq\phi(p,x)+k\phi(x,Tx)$ ทุกๆ $x\in C$ เขาได้พิสูจน์ว่าเซตของจุดตรึงของการส่งชนิดนี้เป็นเป็นเซตนูนปิด และได้สร้างระเบียบวิธีทำซ้ำแบบหดตัวสำหรับการค้นหาจุดตรึง ด้วยเหตุนี้ผู้วิจัยจึงเกิดแนวความคิดและได้สร้างระเบียบวิธีทำซ้ำแบบหดตัวสำหรับการค้นหาผลเฉลยร่วมของปัญหาดุลภาพทั่วไปแบบผสมและปัญหาจุดตรึงของกึ่งการหดเทียมโดยแท้แบบปิด ผลลัพธ์ที่ได้เป็นจริงในปริภูมิบานาคราบเรียบที่เป็นแบบนูนอย่างเข้มและสะท้อน ที่ซึ่งปริภูมิและปริภูมิภาวะคู่กัน มีสมบัติ (K) ดังนี้

$$\begin{cases} x_0\in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1=\Pi_{C_1}(x_0), \\ u_n\in C \text{ such that} \\ \quad \Theta(u_n,y)+\langle Au_n,y-u_n\rangle+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, Ju_n-JTx_n\rangle\geq 0 \ \forall y\in C, \\ C_{n+1}=\{z\in C_n : \phi(x_n,u_n)+\phi(u_n,Tx_n) \\ \qquad\qquad \leq \frac{2}{1-k}\langle x_n-z, Jx_n-JTx_n\rangle+2\langle x_n-z, JTx_n-Ju_n\rangle\}, \\ x_{n+1}=\Pi_{C_{n+1}}(x_0), \end{cases} \tag{1.4}$$

ซึ่ง (1.4) จะครอบคลุมและพัฒนางานวิจัยของ H. Zhou และ E. Gao [H. Zhou, E. Gao, An iterative method of fixed points for closed and quasi-strict pseudocontractions in Banach spaces, J. Appl. Math. Comput. 33 (2010) 227-237.]

วัตถุประสงค์ของการวิจัย

1.1 สร้างระเบียบวิธีทำซ้ำชนิด และองค์ความรู้ใหม่ๆ เกี่ยวกับการประมาณค่าจุดตรึงของบางการส่งแบบไม่ขยายทั่วไป รวมถึงการหาเงื่อนไขที่เหมาะสมเพื่อให้ระเบียบวิธีที่ได้สร้างขึ้นนั้น ลู่เข้าไปยังจุดตรึง หรือจุดตรึงร่วม ในปริภูมิฮิลเบิร์ตหรือปริภูมิบานาค

1.2 ประยุกต์ เชื่อมโยงกระบวนการทำซ้ำที่ได้สร้างขึ้นไปสู่การหาคำตอบของปัญหา อสมการแปรผัน ปัญหาดุลภาพ หรือปัญหาดุลภาพทั่วไป สมการ หรืออสมการอื่นๆ ในทางคณิตศาสตร์และเศรษฐศาสตร์ ฯลฯ

วิธีการวิจัย

2.1 รวบรวมความรู้พื้นฐานที่เกี่ยวข้องกับทฤษฎีจุดตรึง (fixed point theory) ของการส่งไม่ขยาย การส่งแบบหดเทียม และการส่งที่ไม่เป็นเชิงเส้นอื่นๆ ที่เกี่ยวข้อง ทั้งในปริภูมิฮิลเบิร์ต และปริภูมิบาค

2.2 ค้นคว้าหาเอกสารงานวิจัยต่างๆ ตำรา วารสาร และเอกสารสิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยจากแหล่งข้อมูลต่างๆ

2.3 ศึกษากระบวนการทำซ้ำและเงื่อนไขต่างๆ ที่เกี่ยวกับการประมาณค่าจุดตรึงที่เกี่ยวข้องกับการส่งแบบไม่ขยาย เช่น การส่งแบบไม่ขยายแบบหดเทียม และการส่งที่ไม่เป็นเชิงเส้นอื่นๆ ที่เกี่ยวข้อง รวมถึงการประยุกต์และนำไปใช้ เพื่อแก้ไขปัญหาอสมการแปรผัน ปัญหาดุลภาพ หรือปัญหาดุลภาพทั่วไปแบบผสม จากตำรา วารสาร และเอกสารงานวิจัยที่ได้ค้นหามา

2.4 โดยอาศัยความรู้ที่ได้จากการศึกษาตามระเบียบวิธี 2.1-2.3 และประสบการณ์ที่ได้จากการแลกเปลี่ยนความคิดเห็นและปรึกษากับนักวิจัยชาวต่างประเทศที่มีการเชื่อมโยงการทำวิจัยกันอยู่ และความรู้ที่ได้คิดค้นโดยสร้างทฤษฎีบทใหม่ๆ เกี่ยวกับการทำซ้ำเพื่อนำไปตอบปัญหา ตามวัตถุประสงค์ที่ได้กำหนดไว้ในข้อ 1.1 และ 1.2 เขียน paper โดยได้รับคำแนะนำจากนักวิจัยที่ปรึกษา และส่งตีพิมพ์ International Journal ทางคณิตศาสตร์ต่อไป

Contents of Research

1. Strong convergence by a hybrid algorithm for finding a common fixed point of Lipschitz pseudo-contraction and strict pseudo-contraction in Hilbert spaces (Published in Abstract and Applied Analysis, Volume 2011 (2011), Article ID 530683, 14 pages, impact factor 2010=1.442)

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $T : C \to C$ be L_T-Lipschitz pseudocontraction, and let $S : C \to C$ be κ-strict pseudocontraction with $\tilde{F} := F(S) \cap F(T) \neq \emptyset$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$
\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n T z_n,\\
z_n &= (1-\beta_n)x_n + \beta_n S x_n,\\
C_{n+1} &= \Big\{ v \in C_n : \|\alpha_n (I-T)y_n\|^2 + (1-\kappa)\|(I-S)x_n\|^2 \\
&\leqslant 2\alpha_n \langle x_n - v, (I-T)y_n\rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n\rangle \\
&+2\alpha_n\beta_n L_T \|x_n - Sx_n\| \, \|y_n - x_n + \alpha_n (I-T)y_n\| \\
&+\beta_n\left(\left(\frac{2\beta_n}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_n\|^2 \Big\},\\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned}
\tag{3.1}
$$

Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/(L_T + 1) < 1$ and $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \beta_n = 0$. Then $\{x_n\}$ converges strongly to $P_{\tilde{F}}(x_0)$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S, T : C \to C$ be nonexpansive mappings. Suppose that $\widetilde{F} := F(S) \cap F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/2$ and $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \beta_n = 0$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$
\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\
z_n &= (1-\beta_n)x_n + \beta_n S x_n, \\
C_{n+1} &= \Big\{ v \in C_n : \|\alpha_n (I-T)y_n\|^2 + \|(I-S)x_n\|^2 \\
&\leqslant 2\alpha_n \langle x_n - v, (I-T)y_n \rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n \rangle \\
&+ 2\alpha_n\beta_n \|x_n - Sx_n\| \|y_n - x_n + \alpha_n (I-T)y_n\| \\
&+ \beta_n (4\beta_n^2 - 1)\|(I-S)x_n\|^2 \Big\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned}
\tag{3.22}
$$

Then $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$.

Corollary 3.4. *Let $A : H \to H$ be L_A-Lipschitz monotone mapping and let $B : H \to H$ be an $\widehat{\gamma}$-inverse strongly monotone which $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/(L_A + 2)$, $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty}\beta_n = 0$ and $\gamma \in (0, 1/2]$ such that $\widehat{\gamma} \geqslant \gamma$. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = P_{C_1}(x_0) = x_0$, define a sequence $\{x_n\}$ as follows:*

$$
\begin{aligned}
y_n &= x_n - \alpha_n (x_n - z_n) - \alpha_n A z_n, \\
z_n &= x_n - \beta_n B x_n, \\
C_{n+1} &= \Bigg\{ v \in C_n : \|\alpha_n A y_n\|^2 + 2\gamma \|Bx_n\|^2 \\
&\leqslant 2\alpha_n \langle x_n - v, A y_n \rangle + 2\langle x_n - v, B z_n + B x_n \rangle \\
&+ 2\alpha_n \beta_n (L_A + 1)\|Bx_n\| \|y_n - x_n + \alpha_n A y_n\| \\
&+ \beta_n \left(\left(\frac{\beta_n}{\gamma} \right)^2 - 1 \right) \|Bx_n\|^2 \Bigg\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned}
\tag{3.25}
$$

Then $\{x_n\}$ converges strongly to $P_{A^{-1}(0)\cap B^{-1}(0)}(x_0)$.

2. An iterative method for finding generalized mixed equilibrium problems and fixed points problems of closed and quasi-strict pseudo-contraction in Banach spaces (Under review in Abstract and Applied

Analysis)

3. Main result

Theorem 3.1. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $A : C \to E^*$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that} \\ \quad \Theta(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - JTx_n\rangle \geq 0 \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, u_n) + \phi(u_n, Tx_n) \\ \qquad \leq \frac{2}{1-k}\langle x_n - z, Jx_n - JTx_n\rangle + 2\langle x_n - z, JTx_n - Ju_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

where $k \in [0,1)$ and $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Corollary 3.2. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-nonexpansive mapping, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $A : C \to E^*$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that} \\ \quad \Theta(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - JTx_n\rangle \geq 0 \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, u_n) + \phi(u_n, Tx_n) \leq 2\langle x_n - z, Jx_n - Ju_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

where $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Corollary 3.3. *Let H be a Hilbert space. Assume that C is a nonempty closed convex subset of H. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $A : C \to H$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that} \\ \quad \Theta(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - Tx_n\rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|x_n - u_n\|^2 + \|u_n - Tx_n\|^2 \\ \qquad\qquad \leq \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle + 2\langle x_n - z, Tx_n - u_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

where $k \in [0, 1)$ and $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Corollary 3.4. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ such that $\Lambda := F(T) \cap EP(\Theta) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - JTx_n\rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, u_n) + \phi(u_n, Tx_n) \\ \qquad\qquad \leq \frac{2}{1-k}\langle x_n - z, Jx_n - JTx_n\rangle + 2\langle x_n - z, JTx_n - Ju_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

where $k \in [0, 1)$ and $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Out Put **จากโครงการวิจัยได้รับทุนจาก สกอ. และ สกว.**

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 Kasamsuk Ungchittrakool, Strong convergence by a hybrid algorithm for finding a common fixed point of Lipschitz pseudo-contraction and strict pseudo-contraction in Hilbert spaces, Abstract and Applied Analysis, Volume 2011 (2011), Article ID 530683, 14 pages doi:10.1155/2011/530683. Impact factor : 2010 =1.442.

2 ผลงานวิจัยที่ส่งเพื่อพิจารณาตีพิมพ์ในวารสารวิชาการนานาชาติ

2.1 Kasamsuk Ungchittrakool, An iterative method for finding generalized mixed equilibrium problems and fixed points problems of closed and quasi-strict pseudo-contraction in Banach spaces, (revising manuscript).

3. การนำผลงานวิจัยไปใช้ประโยชน์

3.1 เชิงวิชาการ (มีการพัฒนาการเรียนการสอน/สร้างนักวิจัยใหม่)

ผลงานวิจัยที่ได้สร้างขึ้นนี้ ได้ปรากฏสู่สายตาต่อนักคณิตศาสตร์ ทั้งในและต่างประเทศ และรวมไปถึงนิสิตในระดับปริญญาโท และเอก ของผู้วิจัย ได้มีการถ่ายทอด ต่อยอด เพื่อให้เกิดองค์ความรู้ใหม่ ซึ่งงานวิจัยที่ผู้วิจัยได้คิดค้นนี้ เป็นองค์ความรู้ หรือเป็นแนวทางที่สำคัญ ที่จะทำให้นักวิจัยรุ่นใหม่ได้คิดค้นหรือเกิด ที่จะเป็นองค์ความรู้ทางด้านคณิตศาสตร์ และศาสตร์สาขาอื่นๆ ต่อไปในอนาคต

4. อื่นๆ: การเสนอผลงานในที่ประชุมวิชาการ

4.1 นำเสนอผลงานในการประชุมวิชาการระดับนานาชาติ The 16th Annual Meeting in Mathematics (AMM2011), March 10-11, 2011, Kosa Hotel, Khonkaen, Thailand. เรื่อง Strong convergence by a hybrid algorithm for finding a common fixed point of Lipschitz pseudo-contraction and strict pseudo-contraction in Hilbert spaces.

4.2 มีโครงการนำเสนอผลงานในการประชุมวิชาการระดับนานาชาติ The International Conference on MATHEMATICAL INEQUALITIES and NONLINEAR FUNCTIONAL ANALYSIS with APPLICATIONS, Gyeongsang National University, Chinju, Korea, July 25-29, 2012. เรื่อง An iterative method for finding generalized mixed equilibrium problems and fixed points problems of closed and quasi-strict pseudo-contraction in Banach spaces.

ภาคผนวก 1

Strong convergence by a hybrid algorithm for finding a common fixed point of Lipschitz pseudo-contraction and strict pseudo-contraction in Hilbert spaces

Kasamsuk Ungchittrakool

Abstract and Applied Analysis

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 530683, 14 pages
doi:10.1155/2011/530683

Research Article

Strong Convergence by a Hybrid Algorithm for Finding a Common Fixed Point of Lipschitz Pseudocontraction and Strict Pseudocontraction in Hilbert Spaces

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Received 26 January 2011; Revised 21 April 2011; Accepted 9 June 2011

Academic Editor: P. J. Y. Wong

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We prove a strong convergence theorem by using a hybrid algorithm in order to find a common fixed point of Lipschitz pseudocontraction and κ-strict pseudocontraction in Hilbert spaces. Our results extend the recent ones announced by Yao et al. (2009) and many others.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$. Recall that T is said to be a pseudocontraction if

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

is equivalent to

$$\langle x - y, (I - T)x - (I - T)y \rangle \geqslant 0, \tag{1.2}$$

for all $x, y \in C$, and T is said to be a strict pseudocontraction if there exists a constant $0 \leqslant \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \tag{1.3}$$

for all $x, y \in C$. For the second case, we say that T is a κ-strict pseudocontraction. We use $F(T)$ to denote the set of fixed points of T.

The class of strict pseudocontractions extend the class of nonexpansive mapping. (A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$, for all $x, y \in C$) that is, T is nonexpansive if and only if T is a 0-strict pseudocontraction. The pseudocontractive mapping includes the strict pseudocontractive mapping.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem [1–4], the split feasibility problem [5–7] and image recovery and signal processing [3, 8, 9], and so forth. However, the Picard sequence $\{T^n x\}_{n=0}^{\infty}$ often fails to converge even in the weak topology. Thus, averaged iterations prevail. The Mann iteration [10] is one of the types and is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geqslant 0, \tag{1.4}$$

where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\} \subset [0, 1]$. Reich [11] proved that if E is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a fixed point of T. However, we note that Mann iterations have only weak convergence even in a Hilbert space (see e.g., [12]). From a practical point of view, strict pseudocontractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [13]). Therefore, it is important to develop theory of iterative methods for strict pseudocontractions. Indeed, Browder and Petryshyn [14] prove that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha x_n + (1 - \alpha) T x_n, \quad n \geqslant 0, \tag{1.5}$$

for any starting point $x_0 \in C$, α is a constant such that $\kappa < \alpha < 1$, $\{x_n\}$ converges weakly to a fixed point of strict pseudocontraction. Marino and Xu [15] extended the result of Browder and Petryshyn [14] to Mann iteration (1.4); they proved $\{x_n\}$ converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty} (\alpha_n - k)(1 - \alpha_n) = \infty$.

The well-known strong convergence theorem for pseudocontractive mapping was proved by Ishikawa [16] in 1974. More precisely, he got the following theorem.

Theorem 1.1 (see [16]). *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be a Lipschitzian pseudocontractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geqslant 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying

(i) $\alpha_n \leqslant \beta_n$, $n \geqslant 1$,

(ii) $\lim_{n \to \infty} \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.2. (i) Since $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$, $n \geqslant 1$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, the iterative sequence (1.6) could not be reduced to a Mann iterative sequence (1.4). Therefore, the iterative sequence (1.6) has some particular cases.

(ii) The iterative sequence (1.6) is usually called the Ishikawa iterative sequence.

(iii) Chidume and Mutangadura [17] gave an example to show that the Mann iterative sequence failed to be convergent to a fixed point of Lipschitzian pseudocontractive mapping.

In an infinite-dimensional Hilbert spaces, Mann and Ishikawa's iteration algorithms have only weak convergence, in general, even for nonexpansive mapping. In order to obtain a strong convergence theorem for the Mann iteration method (1.4) to nonexpansive mapping, Nakajo and Takahashi [18] modified (1.4) by employing two closed convex sets that are created in order to form the sequence via metric projection so that strong convergence is guaranteed. Later, it is often referred as the hybrid algorithm or the CQ algorithm. After that the hybrid algorithm have been studied extensively by many authors (see e.g., [19–23]). Particularly, Martinez-Yanes and Xu [24] and Plubtieng and Ungchittrakool [20] extended the same results of Nakajo and Takahashi [18] to the Ishikawa iteration process. In 2007, Marino and Xu [15] further generalized the hybrid algorithm from nonexpansive mappings to strict pseudocontractive mappings. In 2008, Zhou [25] established the hybrid algorithm for pseudocontractive mapping in the case of the Ishikawa iteration process.

Recently, Yao et al. [26] introduced the hybrid iterative algorithm which just involved one closed convex set for pseudocontractive mapping in Hilbert spaces as follows.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudocontraction. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows.

$$\begin{gathered} y_n = (1 - \alpha_n) x_n + \alpha_n T z_n, \\ C_{n+1} = \left\{ v \in C_n : \left\| \alpha_n (I - T) y_n \right\|^2 \leqslant 2\alpha_n \left\langle x_n - v, (I - T) y_n \right\rangle \right\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{gathered} \tag{1.7}$$

Theorem 1.3 (see [26]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudocontraction such that $F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1/(L+1))$. Then the sequence $\{x_n\}$ generated by* (1.7) *converges strongly to $P_{F(T)}(x_0)$.*

Very recently, Tang et al. [27] generalized the hybrid algorithm (1.7) in the case of the Ishikawa iterative precess as follows:

$$\begin{gathered} y_n = (1 - \alpha_n) x_n + \alpha_n T z_n, \\ z_n = \left(1 - \beta_n\right) x_n + \beta_n T x_n, \\ C_{n+1} = \Big\{ v \in C_n : \left\| \alpha_n (I - T) y_n \right\|^2 \leqslant 2\alpha_n \left\langle x_n - v, (I - T) y_n \right\rangle \\ +2\alpha_n \beta_n L \| x_n - T x_n \| \left\| y_n - x_n + \alpha_n (I - T) y_n \right\| \Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{gathered} \tag{1.8}$$

Under some appropriate conditions of $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that (1.8) converges strongly to $P_{F(T)}(x_0)$.

Motivated and inspired by the above works, in this paper, we generalize (1.7) to the Ishikawa iterative process in the case of finding the common fixed point of Lipschitz pseudocontraction and κ-strict pseudocontraction. More precisely, we provide some applications of the main theorem to find the common zero point of the Lipshitz monotone mapping and γ-inverse strongly monotone mapping in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C, \tag{2.1}$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.2}$$

holds for every $y \in H$ with $y \neq x$.

For a given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Now we collect some Lemmas which will be used in the proof of the main result in the next section. We note that Lemmas 2.1 and 2.2 are well known.

Lemma 2.1. *Let H be a real Hilbert space. There holds the following identities:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$,

(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$, *for all* $x, y \in H$ *and* $\lambda \in [0,1]$.

Lemma 2.2. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leqslant 0, \quad \forall y \in C. \tag{2.3}$$

Proposition 2.3 (see [15, Proposition 2.1]). *Assume C is a closed convex subset of a Hilbert space H; let $T : C \to C$ be a self-mapping of C. If T is a κ-strict pseudocontraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leqslant \frac{1+\kappa}{1-\kappa}\|x - y\|, \quad \forall x, y \in C. \tag{2.4}$$

Lemma 2.4 (see [28]). *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $T : C \to C$ be a continuous pseudocontractive mapping, then*

(i) $F(T)$ *is closed convex subset of* C,

(ii) $I - T$ *is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup z$ *and* $(I-T)x_n \to 0$, *then* $(I-T)z = 0$.

Lemma 2.5 (see [24]). *Let* C *be a closed convex subset of* H. *Let* $\{x_n\}$ *be a sequence in* H, *and let* $u \in H$. *Let* $q = P_C u$. *If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subset C$ *and satisfies the condition*

$$\|x_n - u\| \leqslant \|u - q\|, \quad \forall n, \tag{2.5}$$

then $x_n \to q$.

3. Main Result

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *let* $T : C \to C$ *be* L_T*-Lipschitz pseudocontraction, and let* $S : C \to C$ *be* κ*-strict pseudocontraction with* $\widetilde{F} := F(S) \cap F(T) \neq \emptyset$. *Let* $x_0 \in H$. *For* $C_1 = C$ *and* $x_1 = P_{C_1}(x_0)$, *define a sequence* $\{x_n\}$ *of* C *as follows:*

$$\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n T z_n,\\
z_n &= (1-\beta_n)x_n + \beta_n S x_n,\\
C_{n+1} &= \Big\{ v \in C_n : \|\alpha_n (I-T)y_n\|^2 + (1-\kappa)\|(I-S)x_n\|^2 \\
&\leqslant 2\alpha_n \langle x_n - v, (I-T)y_n\rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n\rangle \\
&+2\alpha_n\beta_n L_T \|x_n - Sx_n\| \|y_n - x_n + \alpha_n (I-T)y_n\| \\
&+\beta_n\left(\left(\frac{2\beta_n}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_n\|^2 \Big\},\\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned} \tag{3.1}$$

Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ *be such that* $0 < a \leqslant \alpha_n \leqslant b < 1/(L_T + 1) < 1$ *and* $0 < \beta_n \leqslant 1$ *for all* $n \in \mathbb{N}$ *with* $\lim_{n\to\infty} \beta_n = 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{\widetilde{F}}(x_0)$.

Proof. By Lemma 2.4(i), we see that $F(S)$ and $F(T)$ are closed and convex, then $\widetilde{F}$ is as well. Hence, $P_{\widetilde{F}}$ is well defined. Next, we will prove by induction that $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. Note that $\widetilde{F} \subset C = C_1$. Assume that $\widetilde{F} \subset C_k$ holds for $k \geqslant 1$. Let $p \in \widetilde{F}$, thus $p \in C_k$, and we observe that

$$\begin{aligned}
\left\|x_k - p - \alpha_k(I-T)y_k\right\|^2 &= \left\|x_k - p\right\|^2 - \left\|\alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - p - \alpha_k(I-T)y_k\right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|\alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k - (I-T)p, y_k - p\right\rangle \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&\leqslant \left\|x_k - p\right\|^2 - \left\|\alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|(x_k - y_k) + (y_k - x_k + \alpha_k(I-T)y_k)\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|x_k - y_k\right\|^2 - \left\|y_k - x_k + \alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\left\langle x_k - y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&\leqslant \left\|x_k - p\right\|^2 - \left\|x_k - y_k\right\|^2 - \left\|y_k - x_k + \alpha_k(I-T)y_k\right\|^2 \\
&\quad + 2\left|\left\langle x_k - y_k - \alpha_k(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle\right|.
\end{aligned} \tag{3.2}$$

Consider the last term of (3.2), we obtain

$$\begin{aligned}
&\left|\left\langle x_k - y_k - \alpha_k(I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad = \alpha_k\left|\left\langle x_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad = \alpha_k\left|\left\langle x_k - Tx_k + Tx_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad = \alpha_k\left|\left\langle (I-T)x_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle + \left\langle Tx_k - Tz_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad \leqslant \alpha_k(L_T + 1)\left\|x_k - y_k\right\|\left\|y_k - x_k + \alpha_k(I-T)y_k\right\| \\
&\qquad + \alpha_k L_T\left\|x_k - z_k\right\|\left\|y_k - x_k + \alpha_k(I-T)y_k\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_k (L_T+1) \| x_k - y_k \| \| y_k - x_k + \alpha_k (I-T) y_k \| \\
&\quad + \alpha_k \beta_k L_T \| x_k - S x_k \| \| y_k - x_k + \alpha_k (I-T) y_k \| \\
&\leqslant \frac{\alpha_k (L_T+1)}{2} \left(\| x_k - y_k \|^2 + \| y_k - x_k + \alpha_k (I-T) y_k \|^2 \right) \\
&\quad + \alpha_k \beta_k L_T \| x_k - S x_k \| \| y_k - x_k + \alpha_k (I-T) y_k \|.
\end{aligned}
\tag{3.3}
$$

Substituting (3.3) into (3.2), we obtain

$$
\begin{aligned}
\| x_k - p - \alpha_k (I-T) y_k \|^2 &\leqslant \| x_k - p \|^2 - \| x_k - y_k \|^2 - \| y_k - x_k + \alpha_k (I-T) y_k \|^2 \\
&\quad + \alpha_k (L_T+1) \left(\| x_k - y_k \|^2 + \| y_k - x_k + \alpha_k (I-T) y_k \|^2 \right) \\
&\quad + 2\alpha_k \beta_k L_T \| x_k - S x_k \| \| y_k - x_k + \alpha_k (I-T) y_k \| \\
&\leqslant \| x_k - p \|^2 + 2\alpha_k \beta_k L_T \| x_k - S x_k \| \| y_k - x_k + \alpha_k (I-T) y_k \|.
\end{aligned}
\tag{3.4}
$$

Notice that

$$
\| x_k - p - \alpha_k (I-T) y_k \|^2 = \| x_k - p \|^2 - 2\alpha_k \langle x_k - p, (I-T) y_k \rangle + \| \alpha_k (I-T) y_k \|^2. \tag{3.5}
$$

Therefore, from (3.4) and (3.5), we get

$$
\| \alpha_k (I-T) y_k \|^2 \leqslant 2\alpha_k \langle x_k - p, (I-T) y_k \rangle + 2\alpha_k \beta_k L_T \| x_k - S x_k \| \| y_k - x_k + \alpha_k (I-T) y_k \|. \tag{3.6}
$$

On the other hand, we found that

$$
\begin{aligned}
\| x_k - p - \beta_k (I-S) z_k \|^2 &= \| x_k - p \|^2 - \| \beta_k (I-S) z_k \|^2 - 2\beta_k \langle (I-S) z_k, x_k - p - \beta_k (I-S) z_k \rangle \\
&= \| x_k - p \|^2 - \| \beta_k (I-S) z_k \|^2 - 2\beta_k \langle (I-S) z_k - (I-S) p, z_k - p \rangle \\
&\quad - 2\beta_k \langle (I-S) z_k, x_k - z_k - \beta_k (I-S) z_k \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \|x_k - p\|^2 - \|\beta_k(I-S)z_k\|^2 - 2\langle \beta_k(I-S)z_k, x_k - z_k - \beta_k(I-S)z_k\rangle \\
&= \|x_k - p\|^2 - \|\beta_k(I-S)z_k\|^2 \\
&\quad + \left(\|\beta_k(I-S)z_k\|^2 - \|x_k - z_k\|^2 + \|x_k - z_k - \beta_k(I-S)z_k\|^2\right) \\
&= \|(x_k - z_k) + (z_k - p)\|^2 - \|x_k - z_k\|^2 + \|\beta_k(I-S)x_k - \beta_k(I-S)z_k\|^2 \\
&= \|x_k - z_k\|^2 + 2\langle x_k - z_k, z_k - p\rangle + \|z_k - p\|^2 - \|x_k - z_k\|^2 \\
&\quad + \|\beta_k(I-S)x_k - \beta_k(I-S)z_k\|^2 \\
&= 2\langle x_k - z_k, (z_k - x_k) + (x_k - p)\rangle + \|(1-\beta_k)(x_k - p) + \beta_k(Sx_k - p)\|^2 \\
&\quad + \|\beta_k(I-S)x_k - \beta_k(I-S)z_k\|^2 \\
&\leqslant 2\langle x_k - p, \beta_k(I-S)x_k\rangle + (1-\beta_k)\|x_k - p\|^2 + \beta_k\|Sx_k - p\|^2 \\
&\quad - \beta_k(1-\beta_k)\|x_k - Sx_k\|^2 - 2\beta_k^2\|(I-S)x_k\|^2 \\
&\quad + \beta_k^2\left(\frac{1+\kappa}{1-\kappa} + 1\right)^2\|x_k - z_k\|^2 \\
&\leqslant 2\langle x_k - p, \beta_k(I-S)x_k\rangle + (1-\beta_k)\|x_k - p\|^2 + \beta_k\|x_k - p\|^2 \\
&\quad + \beta_k\kappa\|(I-S)x_k\|^2 - \beta_k(1-\beta_k)\|(I-S)x_k\|^2 - 2\beta_k^2\|(I-S)x_k\|^2 \\
&\quad + \beta_k^4\left(\frac{2}{1-\kappa}\right)^2\|(I-S)x_k\|^2 \\
&= 2\langle x_k - p, \beta_k(I-S)x_k\rangle + \|x_k - p\|^2 - \beta_k(1-\kappa)\|(I-S)x_k\|^2 \\
&\quad - \beta_k^2\|(I-S)x_k\|^2 + \beta_k^4\left(\frac{2}{1-\kappa}\right)^2\|(I-S)x_k\|^2.
\end{aligned}
\tag{3.7}
$$

Notice that

$$
\|x_k - p - \beta_k(I-S)z_k\|^2 = \|x_k - p\|^2 - 2\beta_k\langle x_k - p, (I-S)z_k\rangle + \beta_k^2\|(I-S)z_k\|^2. \tag{3.8}
$$

Combining (3.7) and (3.8) and then it implies that

$$
\beta_k(1-\kappa)\|x_k - Sx_k\|^2 \leqslant 2\beta_k\langle x_k - p, (I-S)z_k + (I-S)x_k\rangle + \beta_k^2\left(\left(\frac{2\beta_k}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_k\|^2. \tag{3.9}
$$

Since $\beta_n > 0$ for all n, so we get

$$(1-\kappa)\|x_k - Sx_k\|^2 \leqslant 2\langle x_k - p, (I-S)z_k + (I-S)x_k\rangle + \beta_k\left(\left(\frac{2\beta_k}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_k\|^2. \tag{3.10}$$

It follows from (3.6) and (3.10) that we obtain

$$\begin{aligned}
&\left\|\alpha_k(I-T)y_k\right\|^2 + (1-\kappa)\|(I-S)x_k\|^2 \\
&\quad \leqslant 2\alpha_k\langle x_k - v, (I-T)y_k\rangle + 2\langle x_k - v, (I-S)z_k + (I-S)x_k\rangle \\
&\qquad + 2\alpha_k\beta_k L_T\|x_k - Sx_k\|\left\|y_k - x_k + \alpha_k(I-T)y_k\right\| + \beta_k\left(\left(\frac{2\beta_k}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_k\|^2.
\end{aligned} \tag{3.11}$$

Therefore, $p \in C_{k+1}$. By mathematical induction, we have $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. It is easy to check that C_n is closed and convex, and then $\{x_n\}$ is well defined. From $x_n = P_{C_n}(x_0)$, we have $\langle x_0 - x_n, x_n - y\rangle \geqslant 0$ for all $y \in C_n$. Using $\widetilde{F} \subset C_n$, we also have $\langle x_0 - x_n, x_n - u\rangle \geqslant 0$ for all $u \in \widetilde{F}$. So, for $u \in \widetilde{F}$, we have

$$\begin{aligned}
0 &\leqslant \langle x_0 - x_n, x_n - u\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u\rangle \\
&\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned} \tag{3.12}$$

Hence, $\|x_0 - x_n\| \leqslant \|x_0 - u\|$, for all $u \in \widetilde{F}$. In particular,

$$\|x_0 - x_n\| \leqslant \left\|x_0 - q\right\|, \quad \text{where } q = P_{\widetilde{F}}(x_0). \tag{3.13}$$

This implies that $\{x_n\}$ is bounded, and then $\{y_n\}$, $\{Ty_n\}$, $\{z_n\}$, $\{Sz_n\}$, and $\{Sx_n\}$ are as well.

From $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geqslant 0. \tag{3.14}$$

Hence

$$\begin{aligned}
0 &\leqslant \langle x_0 - x_n, x_n - x_{n+1}\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|,
\end{aligned} \tag{3.15}$$

and; therefore,

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|, \tag{3.16}$$

which implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From Lemma 2.1 and (3.14), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0.\end{aligned} \tag{3.17}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\begin{aligned}&\left\|\alpha_n(I-T)y_n\right\|^2 + (1-\kappa)\|(I-S)x_n\|^2 \\ &\quad\leqslant 2\alpha_n\left\langle x_n - x_{n+1}, (I-T)y_n\right\rangle + 2\langle x_n - x_{n+1}, (I-S)z_n + (I-S)x_n\rangle \\ &\qquad + 2\alpha_n\beta_n L_T\|x_n - Sx_n\|\left\|y_n - x_n + \alpha_n(I-T)y_n\right\| \\ &\qquad + \beta_n\left(\left(\frac{2\beta_n}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_n\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.18}$$

therefore, we obtain

$$\left\|y_n - Ty_n\right\| \longrightarrow 0, \qquad \|x_n - Sx_n\| \longrightarrow 0. \tag{3.19}$$

We note that

$$\begin{aligned}\|x_n - Tx_n\| &\leqslant \left\|x_n - y_n\right\| + \left\|y_n - Ty_n\right\| + \left\|Ty_n - Tx_n\right\| \\ &\leqslant (L_T+1)\left\|x_n - y_n\right\| + \left\|y_n - Ty_n\right\| \\ &\leqslant \alpha_n(L_T+1)\|x_n - Tz_n\| + \left\|y_n - Ty_n\right\| \\ &\leqslant \alpha_n(L_T+1)\|x_n - Tx_n\| + \alpha_n(L_T+1)\|Tx_n - Tz_n\| + \left\|y_n - Ty_n\right\| \\ &\leqslant \alpha_n(L_T+1)\|x_n - Tx_n\| + \alpha_n\beta_n L_T(L_T+1)\|x_n - Sx_n\| + \left\|y_n - Ty_n\right\|,\end{aligned} \tag{3.20}$$

that is,

$$\|x_n - Tx_n\| \leqslant \frac{\alpha_n\beta_n L_T(L_T+1)}{1-\alpha_n(L_T+1)}\|x_n - Sx_n\| + \frac{1}{1-\alpha_n(L_T+1)}\left\|y_n - Ty_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

By Lemma 2.4(ii), $I - T$ and $I - S$ are demiclosed at zero. Together with the fact that $\{x_n\}$ is bounded, which guarantees that every weak limit point of $\{x_n\}$ is a fixed point of T

and S, that is $\omega_w(x_n) \subset F(T) \cap F(S) = \widetilde{F}$, therefore, by inequality (3.13) and Lemma 2.5, we know that $\{x_n\}$ converges strongly to $q = P_{\widetilde{F}}(x_0)$. This completes the proof. □

If $S = I$, then we obtain the following corollary.

Corollary 3.2 (Yao et al. [26, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be L-Lipschitz pseudocontraction such that $F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/(L+1) < 1$ for all n. Then the sequence $\{x_n\}$ generated by* (1.7) *converges strongly to $P_{F(T)}(x_0)$.*

If T and S are nonexpansive, then we also have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S, T : C \to C$ be nonexpansive mappings. Suppose that $\widetilde{F} := F(S) \cap F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/2$ and $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \beta_n = 0$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n S x_n, \\ C_{n+1} &= \Big\{ v \in C_n : \left\|\alpha_n (I-T) y_n\right\|^2 + \|(I-S)x_n\|^2 \\ &\leqslant 2\alpha_n \langle x_n - v, (I-T)y_n \rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n \rangle \\ &\quad + 2\alpha_n\beta_n \|x_n - Sx_n\| \left\| y_n - x_n + \alpha_n (I-T) y_n \right\| \\ &\quad + \beta_n \left(4\beta_n^2 - 1\right) \|(I-S)x_n\|^2 \Big\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{aligned} \tag{3.22}$$

Then $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$.

Recall that a mapping A is said to be monotone if $\langle x - y, Ax - Ay \rangle \geqslant 0$ for all $x, y \in H$ and inverse strongly monotone if there exists a real number $\gamma > 0$ such that $\langle x - y, Ax - Ay \rangle \geqslant \gamma \|Ax - Ay\|^2$ for all $x, y \in H$. For the second case, A is said to be γ-inverse strongly monotone. It follows immediately that if A is γ-inverse strongly monotone, then A is monotone and Lipschitz continuous, that is, $\|Ax - Ay\| \leqslant (1/\gamma)\|x - y\|$. It is well known (see e.g., [29]) that if A is monotone, then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Therefore, it is important to focus on finding the zero point of monotone mappings. The pseudocontractive mapping and strictly pseudocontractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

(i) A is monotone $\Leftrightarrow T := (I - A)$ is pseudocontractive,

(ii) A is inverse strongly monotone $\Leftrightarrow T := (I - A)$ is strictly pseudocontractive.

Indeed, for (ii), we notice that the following equality always holds in a real Hilbert space:

$$\left\|(I-A)x - (I-A)y\right\|^2 = \left\|x - y\right\|^2 + \left\|Ax - Ay\right\|^2 - 2\langle x - y, Ax - Ay \rangle, \quad \forall x, y \in H. \tag{3.23}$$

Without loss of generality, we can assume that $\gamma \in (0,1/2]$, and then it yields

$$
\begin{aligned}
\langle x-y, Ax-Ay\rangle \geqslant \gamma\|Ax-Ay\|^2 & \\
&\Longleftrightarrow -2\langle x-y, Ax-Ay\rangle \leqslant -2\gamma\|Ax-Ay\|^2 \\
&\Longleftrightarrow \|(I-A)x-(I-A)y\|^2 \leqslant \|x-y\|^2 + (1-2\gamma)\|Ax-Ay\|^2 \\
&\qquad\qquad (\text{via } (3.23)) \\
&\Longleftrightarrow \|Tx-Ty\|^2 \leqslant \|x-y\|^2 + \kappa\|(I-T)x-(I-T)y\|^2 \\
&\qquad\qquad (\text{where } T := (I-A),\ \kappa := 1-2\gamma).
\end{aligned}
\tag{3.24}
$$

Due to Theorem 3.1, we have the following corollary which generalize the corresponding results of Yao et al. [26].

Corollary 3.4. *Let $A : H \to H$ be L_A-Lipschitz monotone mapping and let $B : H \to H$ be an $\widehat{\gamma}$-inverse strongly monotone which $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/(L_A+2)$, $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty}\beta_n = 0$ and $\gamma \in (0,1/2]$ such that $\widehat{\gamma} \geqslant \gamma$. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = P_{C_1}(x_0) = x_0$, define a sequence $\{x_n\}$ as follows:*

$$
\begin{aligned}
y_n &= x_n - \alpha_n(x_n - z_n) - \alpha_n A z_n, \\
z_n &= x_n - \beta_n B x_n, \\
C_{n+1} &= \Bigg\{ v \in C_n : \|\alpha_n A y_n\|^2 + 2\gamma\|Bx_n\|^2 \\
&\quad \leqslant 2\alpha_n\langle x_n - v, Ay_n\rangle + 2\langle x_n - v, Bz_n + Bx_n\rangle \\
&\quad +2\alpha_n\beta_n(L_A+1)\|Bx_n\|\|y_n - x_n + \alpha_n A y_n\| \\
&\quad +\beta_n\left(\left(\frac{\beta_n}{\gamma}\right)^2 - 1\right)\|Bx_n\|^2 \Bigg\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned}
\tag{3.25}
$$

Then $\{x_n\}$ converges strongly to $P_{A^{-1}(0)\cap B^{-1}(0)}(x_0)$.

Proof. Let $T := (I-A)$ and let $S := (I-B)$. Then T and S are pseudocontractive and $(1-2\gamma)$-pseudocontractive, respectively. Moreover, T is also (L_A+1)-Lipschitz, and if we set $\kappa := 1-2\gamma$, S is also $((1-\gamma)/\gamma)$-Lipschitz, and then $(2/(1-\kappa))^2 = 1/\gamma^2$. Hence, it follows from Theorem 3.1 that we have the desired result. □

If $B = 0$ (zero mapping), then $z_n = x_n$ and $B^{-1}(0) = H$. So, we obtain the following corollary.

Corollary 3.5 (Yao et al. [26, Corollary 3.2]). *Let* $A : H \to H$ *be a* L_A*-Lipschitz monotone mapping for which* $A^{-1}(0) \neq \emptyset$*. Assume that the sequence* $\{\alpha_n\}$ *be as in Corollary 3.4. Then the sequence* $\{x_n\}$ *generated by*

$$y_n = x_n - \alpha_n A z_n,$$
$$C_{n+1} = \left\{ v \in C_n : \left\| \alpha_n A y_n \right\|^2 \leqslant 2\alpha_n \langle x_n - v, A y_n \rangle \right\}, \tag{3.26}$$
$$x_{n+1} = P_{C_{n+1}}(x_0)$$

strongly converges to $P_{A^{-1}(0)}(x_0)$.

Acknowledgments

The author is supported by the Naresuan University and the Thailand Research Fund under Grant no. MRG5380249. Finally, the author would like to thank the referees for reading this paper carefully, providing valuable suggestions and comments, and pointing out a major error in the original version of this paper.

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ภาคผนวก 2

An iterative method for finding generalized mixed equilibrium problems and fixed points problems of closed and quasi-strict pseudo-contraction in Banach spaces

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(Under review)

AN ITERATIVE METHOD FOR FINDING GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINTS PROBLEMS OF CLOSED AND QUASI-STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES

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Abstract. In this paper, we found the significant inequality related to quasi-strict pseudo-contractions and the mappings defined from generalized mixed equilibrium problems on Banach spaces. It was taken to create an iterative shrinking projection method for finding a common solution of generalized mixed equilibrium problems and fixed point problems of closed and quasi-strict pseudo-contractions. Its results hold in reflexive, strictly convex and smooth Banach spaces with the property (K). The results of this paper improve and extend the corresponding results of Zhou and Gao [H. Zhou, E. Gao, An iterative method of fixed points for closed and quasi-strict pseudocontractions in Banach spaces, J. Appl. Math. Comput. 33 (2010) 227-237.] and many others.

Keywords: Quasi-strict pseudo-contraction; Generalized projection; Hybrid algorithm; Generalized mixed equilibrium problems; The property (K)

AMS Subject Classification: 47H09; 47H10; 47J25

1. Introduction

The equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization, and it has been extended and generalized in many directions; see [3, 18]. In particular, equilibrium problems are related to the problem of finding fixed points problems of some non linear mappings. Therefore it is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of the set of the solutions of the equilibrium problems and the set of the fixed points, (see also [8, 10, 22, 26, 29–31] and the references therein). In this paper, we suggest and analyze a hybrid algorithm for solving generalized mixed equilibrium problems and fixed point problems of closed and quasi-strict pseudo-contractions in the framework of reflexive, strictly convex and smooth Banach spaces with the property (K).

Let E be a real Banach space, and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real-valued function, and $A : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem, is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $GMEP(\Theta, A, \varphi)$, i.e.,

$$GMEP(\Theta, A, \varphi) = \{x \in C : \Theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}$$

If $A = 0$, the problem (1.1) reduces to the mixed equilibrium problem for Θ, denoted by $MEP(\Theta, \varphi)$, which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

If $\Theta = 0$, the problem (1.1) reduces to the mixed variational inequality of Browder type, denoted by $VI(C, A, \varphi)$, which is to find $x \in C$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

If $A = 0$ and $\varphi = 0$, the problem (1.1) reduces to the equilibrium problem for Θ (for short, EP), denoted by $EP(\Theta)$, which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

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Let $\Theta(x,y) = \langle Ax, y-x\rangle$ for all $x,y \in C$. Then $p \in EP(\Theta)$ if and only if for all $\langle Ap, y-p\rangle \geq 0$ for all $y \in C$, i.e., p is a solution of the variational inequality; there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.

It is well known that, in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm [13] has only weak convergence, in general, even for nonexpansive mappings. Consequently, in order to obtain strong convergence, Nakajo and Takahashi [17] modified the normal Mann's iteration algorithm, later well known as hybrid projection iteration algorithm (HIPA). Since then, (HIPA) has received rapid developments. For the details, the readers are referred to papers [12, 14, 16, 19] and the references therein. In 2005, Matsushita and Takahashi [15] proposed the hybrid iteration method with generalized projection for relatively nonexpansive mapping T in the framework of uniformly smooth and uniformly convex Banach spaces E as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JTx_n\right), \\ C_n = \left\{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\right\}, \\ Q_n = \left\{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\right\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.3}$$

where J is the duality mapping on E, and $\Pi_C(\cdot)$ is the generalized projection from E onto a nonempty closed convex subset C. Based on the guidelines of Matsushita and Takahashi [15], Plubtieng and Ungchittrakool [20, 21] studied and developed (1.3) to the case of two relatively nonexpansive mappings and finite family of relatively nonexpansive mappings, respectively. In 2007, Tada and Takahashi [24, 25] and Takahashi and Takahashi [26] proved weak and strong convergence theorems for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Takahashi et al. [27] studied a strong convergence theorem by the hybrid method for a family of nonexpansive mappings in Hilbert spaces as follows: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1}x_0$ and let

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leqslant \alpha_n \leqslant a < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = \varnothing$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)}x_0$.

Motivated by Takahashi et al. [27], Takahashi and Zembayashi [29] (see also [30]) introduced and proved a hybrid projection algorithm for solving equilibrium problems and fixed point problems of a relatively nonexpansive mapping S in the framework of uniformly smooth and uniformly convex Banach space as follows:

$$\begin{cases} x_0 = x, \ C_0 = C, \\ y_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n)JSx_n\right) \\ u_n \in C \text{ such that } \Theta(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \left\{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\right\}, \\ x_{n+1} = \Pi_{C_{n+1}}x \end{cases}$$

where $\Pi_{C_{n+1}}(\cdot)$ is the generalized projection from E onto C_{n+1}. Under some appropriate assumptions on Θ, $\{\alpha_n\}$ and $\{r_n\}$, they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP(\Theta)}(x_0)$.

In 2010, Zhou and Gao [32] introduced the definition of a quasi-strict pseudo contraction related to the function ϕ and proved a hybrid projection algorithm for finding a fixed point of a closed and quasi-strict pseudo contraction in more general framework than uniformly smooth and uniformly convex

Banach spaces as follows:

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that} \\ C_{n+1} = \left\{ z \in C_n : \phi(x_n, Tx_n) \le \frac{2}{1-k} \langle x_n - z, Jx_n - JTx_n \rangle \right\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{1.4}$$

where $\Pi_{C_{n+1}}$ is the generalized projection from E onto C_{n+1}. They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}(x_0)$.

Motivated and inspired by the above research work, in this paper, by employing the inequality that appeared in Lemma 2.10 together with (1.4) and some facts of Zhou and Gao [32], we create an iterative shrinking projection method for finding a common solution of generalized mixed equilibrium problems and fixed point problems of closed and quasi-strict pseudo-contractions in the framework of reflexive, strictly convex and smooth Banach spaces with the property (K). The results of this research improve and extend the corresponding results of Zhou and Gao [32] and many others.

2. Preliminaries

In this paper, we denote by E and E^* a Banach space and the dual space of E, respectively. Let C be a nonempty closed convex subset of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\},$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between E and E^*. It is well know that if E^* is reflexive and smooth, then $J : E \to 2^{E^*}$ is single-valued and demi-continuous.

It is also very well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach spaces E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a real smooth Banach space. Let us consider the functional defined as [15] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \text{ for all } x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert spaces H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, for all $, x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \tag{2.2}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 7, 11, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \text{ for all } x, y \in E, \tag{2.3}$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \text{ for all } x, y, z \in E. \tag{2.4}$$

Remark 2.1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of J, we have $Jx = Jy$. That is, $x = y$; one may consult [7, 28] for the details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [23] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [4–6, 15] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [4–6, 15].

T is said to be a quasi-strict pseudo-contraction [32, p. 230] if there exists a constant $k \in [0, 1)$ and $F(T) \neq \emptyset$ such that $\phi(p, Tx) \leq \phi(p, x) + k\phi(x, Tx)$ for all $x \in C$ and $p \in F(T)$. In particular, T is said to be quasi-nonexpansive if $k = 0$ and T is said to be quasi-pseudo-contractive if $k = 1$.

Remark 2.2. A relatively nonexpansive mapping is a quasi-strict pseudo-contraction, but the converse may be not true.

Example 2.3. Let $E = \mathbb{R}$ and define $T : E \to E$ by $Tx = -3x$. Then, T is a quasi-strict pseudo-contraction but not a relatively nonexpansive mapping.

Example 2.4. Let E be a reflexive, strictly convex and smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone mapping such that $A^{-1}(0)$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed and quasi-strict pseudo-contraction mapping from E onto $D(A)$ and $F(J_r) = A^{-1}(0)$.

Example 2.5. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed and quasi-strict pseudo-contraction from E onto C with $F(\Pi_C) = C$.

Recall that a Banach space E has the property (K) if for any sequence and $\{x_n\} \subset E$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. For more information concerning property (K) the reader is referred to [9] and references cited there.

For solving the equilibrium problem for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following condition:

$(A1)$ $\Theta(x, x) = 0$ for all $x \in C$;
$(A2)$ Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$,
$\lim_{t\downarrow 0}\Theta(tz + (1-t)x, y) \leq \Theta(x, y)$;
$(A4)$ for each $x \in C$, $y \longmapsto \Theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.6 (Blum and Oettli [3])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)$ - $(A4)$. Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq \text{ for all } y \in C.$$

Lemma 2.7 (Takahashi and Zembayashi [30])**.** *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$ - $(A4)$. For $r > 0$and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \text{ for all } y \in C\right\}$$

for all $x \in C$. Then, the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive-type mapping, i.e., for any $x, y \in H$,*
$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;
(iii) *$F(T_r) = EP(\Theta)$;*
(iv) *$EP(\Theta)$ is closed and convex.*

Lemma 2.8 (Takahashi and Zembayashi [30])**.** *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$;*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

Lemma 2.9 (Zhang [33]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \ \forall y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \left\{u \in C : \Theta(u, y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \ \forall y \in C\right\}$$

for all $x \in C$. Then, the following conclusions hold:

(1) *K_r is single-valued;*
(2) *K_r is firmly nonexpansive type, i.e., for all $x, y \in E$,*
$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle;$$
(3) *$F(K_r) = GMEP(\Theta, A, \varphi)$*
(4) *$GMEP(\Theta, A, \varphi)$ is closed and convex,*
(5) *$\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x)$ $\forall p \in F(K_r)$, $x \in E$.*

Lemma 2.10. *Let E be a reflexive, strictly convex and smooth Banach space. Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a quasi-strict pseudo-contraction and $K_r : C \to C$ be as in Lemma 2.9 such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Then*

$$\phi(x, K_r Tx) + \phi(K_r Tx, Tx) \leq \frac{2}{1-k}\langle x - p, Jx - JTx\rangle + 2\langle x - p, JTx - JK_r Tx\rangle$$

for all $x \in C$ and $p \in \Omega$.

Proof. Let $x \in C$ and $p \in \Omega$. By the quasi-strict pseudo-contractility of T and equation (2.4) we have

$$\begin{aligned} &\phi(p, Tx) \leq \phi(p, x) + k\phi(x, Tx) \\ &\qquad \Leftrightarrow \phi(p, x) + \phi(x, Tx) + 2\langle p - x, Jx - JTx\rangle \leq \phi(p, x) + k\phi(x, Tx) \\ &\qquad \Leftrightarrow \phi(x, Tx) \leq \frac{2}{1-k}\langle x - p, Jx - JTx\rangle. \end{aligned} \tag{2.5}$$

It follows from (2.4), Lemma 2.9 (5) and (2.5) we obtain

$$\begin{aligned} &\phi(p, x) + \phi(x, K_r Tx) + 2\langle p - x, Jx - JK_r Tx\rangle \\ &\quad = \phi(p, K_r Tx) \\ &\quad \leq \phi(p, Tx) - \phi(K_r Tx, Tx) \\ &\quad \leq \phi(p, x) + k\phi(x, Tx) - \phi(K_r Tx, Tx) \\ &\quad \leq \phi(p, x) + 2\frac{k}{1-k}\langle x - p, Jx - JTx\rangle - \phi(K_r Tx, Tx), \end{aligned}$$

and then

$$\begin{aligned} &\phi(x, K_r Tx) + \phi(K_r Tx, Tx) \\ &\qquad \leq 2\frac{k}{1-k}\langle x - p, Jx - JTx\rangle + 2\langle x - p, Jx - JK_r Tx\rangle \\ &\qquad = 2\frac{k}{1-k}\langle x - p, Jx - JTx\rangle + 2\langle x - p, Jx - JTx\rangle + 2\langle x - p, JTx - JK_r Tx\rangle \\ &\qquad = \left(2\frac{k}{1-k} + 2\right)\langle x - p, Jx - JTx\rangle + 2\langle x - p, JTx - JK_r Tx\rangle \\ &\qquad = \frac{2}{1-k}\langle x - p, Jx - JTx\rangle + 2\langle x - p, JTx - JK_r Tx\rangle \end{aligned}$$

■

The following Lemmas are crucial for the proofs of the main results in this paper.

Lemma 2.11 (Alber [1]). *Let C be a nonempty closed convex subset of a smooth Banach space E, $x_0 \in C$ and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.12 (Alber [2]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \text{ for all } y \in C.$$

3. Main result

Theorem 3.1. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $A : C \to E^*$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that} \\ \quad \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - JTx_n \rangle \geq 0 \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, u_n) + \phi(u_n, Tx_n) \\ \qquad\qquad \leq \frac{2}{1-k} \langle x_n - z, Jx_n - JTx_n \rangle + 2 \langle x_n - z, JTx_n - Ju_n \rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

where $k \in [0, 1)$ and $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Proof. The proof is divided into seven steps.

Step 1. Show that Ω is closed and convex.

From step 1. of Zhou and Gao [32], $F(T)$ is closed and convex and by Lemma 2.9 (4) $GMEP(\Theta, A, \varphi)$ is closed and convex. So, $\Omega := F(T) \cap GMEP(\Theta, A, \varphi)$ is closed and convex.

Step 2. Show that C_n is closed and convex for all $n \geq 1$.

For $n = 1$, $C_1 = C$ is closed and convex. Assume that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_{k+1}$, one obtains that

$$\phi(x_k, u_k) + \phi(u_k, Tx_k) \leq \frac{2}{1-k} \langle x_k - z, Jx_k - JTx_k \rangle + 2 \langle x_k - z, JTx_k - Ju_k \rangle$$

It is not hard to see that the continuity and linearity of $\langle \cdot, Jx_k - JTx_k \rangle$ and $\langle \cdot, JTx_k - Ju_k \rangle$ allow C_{k+1} to be closed and convex. Then, for all $n \geq 1$, C_n is closed and convex.

Step 3. Show that $\Omega \subset C_n$ for all $n \geq 1$.

It is obvious that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \subset C = C_1$. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. For any $p \in \Omega$, we have $p \in C_k$. Notice that $x_k = \Pi_{C_k}(x_0)$ and then by Lemma 2.9 we obtain $u_k = K_{r_k} Tx_k$. So, it follows from Lemma 2.10 that

$$\phi(x_k, u_k) + \phi(u_k, Tx_k) \leq \frac{2}{1-k} \langle x_k - p, Jx_k - JTx_k \rangle + 2 \langle x_k - p, JTx_k - Ju_k \rangle.$$

This means that $p \in C_{k+1}$. By mathematical induction, $\Omega \subset C_n$ for all $n \geq 1$. Therefore $\Omega \subset \bigcap_{n=1}^{\infty} C_n =: D \neq \varnothing$

Step 4. Show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

By $x_n = \Pi_{C_n}x_0$ and Lemma 2.12, we have

$$\phi\left(x_n, x_0\right) = \phi\left(\Pi_{C_n}x_0, x_0\right) \leq \phi\left(w, x_0\right) - \phi\left(w, x_n\right) \leq \phi\left(w, x_0\right),$$

for each $w \in \Omega \subset C_n$ and for all $n \geq 1$. Therefore the sequence $\{\phi\left(x_n, x_0\right)\}$ is bounded.

On the other hand, noticing that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, so $\phi(x_n, x_0) = \min\limits_{z\in C_n} \phi(z, x_0) \leq \phi(x_{n+1}, x_0)$ for all $n \geq 1$. Therefore, $\phi(x_n, x_0)$ is nondecreasing. It follows that the limit of $\phi(x_n, x_0)$ exists.

Step 5. Show that $x_n \to q$ as $n \to \infty$, where $q = \Pi_D x_0$.

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$ for any positive integer n and by Lemma 2.12, we have

$$\phi\left(x_{n+1}, x_n\right) = \phi\left(x_{n+1}, \Pi_{C_n}x_0\right) \leq \phi\left(x_{n+1}, x_0\right) - \phi\left(\Pi_{C_n}x_0, x_0\right) = \phi\left(x_{n+1}, x_0\right) - \phi\left(x_n, x_0\right). \tag{3.2}$$

Letting $n \to \infty$ in 3.2, one has $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Without loss of generality, we can assume that $x_n \to q$ weakly as $n \to \infty$ (passing to a subsequence if necessary). It is easy to show that $q \in C_n$ for all $n \geq 1$. Hence $q \in \bigcap\limits_{n=1}^{\infty} C_n = D$. Noticing that $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(q, x_0)$, we have

$$(q, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(q, x_0)$$

which implies that $\phi(x_n, x_0) \to \phi(q, x_0)$ as $n \to \infty$. Hence $\|x_n\| \to \|q\|$. By the property (K) of E, we have $x_n \to q$. From Lemma 2.11, we have

$$\langle x_n - y, Jx_0 - Jx_n\rangle \geq 0 \text{ for all } y \in D.$$

Hence

$$\langle q - y, Jx_0 - Jq\rangle \geq 0 \text{ for all } y \in D.$$

which implies that $q = \Pi_D x_0$.

Step 6. Show that $q \in \Omega$.

We prove first that $\{Tx_n\}$ and $\{u_n\} = \{K_{r_n}Tx_n\}$ are bounded. Indeed, take $p \in \Omega = F(T) \cap GMEP\left(\Theta, A, \varphi\right) \subset C_{n+1}$, we have

$$\begin{aligned}
&\|p\|^2 - 2\|p\|\|Tx_n\| + \|Tx_n\|^2 \\
&\quad = \left(\|p\| - \|Tx_n\|\right)^2 \leq \phi(p, Tx_n) \\
&\quad \leq \phi(p, x_n) + k\phi(x_n, Tx_n) \\
&\quad \leq \phi(p, x_n) + \frac{2k}{1-k}\langle x_n - p, Jx_n - JTx_n\rangle \\
&\quad \leq \phi(p, x_n) + \frac{2k}{1-k}\|x_n - p\|\|x_n\| + \frac{2k}{1-k}\|x_n - p\|\|Tx_n\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|Tx_n\|^2 &\leq \left(\phi(p, x_n) - \|p\|^2 + \frac{2k}{1-k}\|x_n - p\|\|x_n\|\right) + \left(\frac{2k}{1-k}\|x_n - p\| + 2\|p\|\right)\|Tx_n\| \\
&\leq M + K\|Tx_n\| = M + \frac{1}{2}\left(2K\|Tx_n\|\right) \\
&\leq M + \frac{1}{2}\left(K^2 + \|Tx_n\|^2\right) = M + \frac{1}{2}K^2 + \frac{1}{2}\|Tx_n\|^2.
\end{aligned}$$

Where $M := \sup\left\{\phi(p, x_n) - \|p\|^2 + \frac{2k}{1-k}\|x_n - p\|\|x_n\| : n \in \mathbb{N}\right\}$ and $K := \sup\left\{\frac{2k}{1-k}\|x_n - p\| + 2\|p\| : n \in \mathbb{N}\right\}$. Thus

$$\|Tx_n\|^2 \leq 2M + K^2$$

for all $n \in \mathbb{N}$. Therefore $\{Tx_n\}$ is bounded. Note that $\phi(p, K_{r_n}Tx_n) \le \phi(p, Tx_n)$ for all $n \in \mathbb{N}$. Therefore $\{u_n\} = \{K_{r_n}Tx_n\}$ is also bounded. From $x_{n+1} \in C_{n+1}$, one has

$$\phi(x_n, u_n) + \phi(u_n, Tx_n) \le \frac{2}{1-k} \langle x_n - x_{n+1}, Jx_n - JTx_n \rangle + 2 \langle x_n - x_{n+1}, JTx_n - Ju_n \rangle \quad (3.3)$$

By step 5, we obtain that $x_{n+1} - x_n \to 0$. Taking limit on the both sides of (3.3), we obtain that $\phi(x_n, u_n) + \phi(u_n, Tx_n) \to 0$ as $n \to \infty$. Noting that $0 \le (\|x_n\| - \|u_n\|)^2 \le \phi(x_n, u_n)$, $0 \le (\|u_n\| - \|Tx_n\|)^2 \le \phi(u_n, Tx_n)$ and $\|x_n\| \to \|q\|$, it implies that

$$\|q\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|Tx_n\|,$$

and consequently

$$\|Jq\| = \lim_{n\to\infty} \|Jx_n\| = \lim_{n\to\infty} \|Ju_n\| = \lim_{n\to\infty} \|JTx_n\|.$$

This implies that $\{J(u_n)\}$ and $\{J(Tx_n)\}$ are bounded. Since E is reflexive, E^* is also reflexive. So we can assume that

$$J(u_n) \to f \in E^*$$

weakly. On the other hand, in view of the reflexivity of E, one has $J(E) = E^*$, which means that for $f \in E^*$, there exists $x_f \in E$, such that $Jx_f = f$. It follows that

$$\phi(x_n, u_n) = \|x_n\|^2 - 2\langle x_n, J(u_n)\rangle + \|u_n\|^2 = \|x_n\|^2 - 2\langle x_n, J(u_n)\rangle + \|J(u_n)\|^2$$

taking $\liminf_{n\to\infty}$ on the both sides of equality above, we have

$$\begin{aligned} 0 &\ge \|q\|^2 - 2\langle q, f\rangle + \|f\|^2 \\ &= \|q\|^2 - 2\langle u_0, Jx_f\rangle + \|Jx_f\|^2 \\ &= \phi(q, x_f). \end{aligned}$$

Therefore $\phi(q, x_f) = 0$ and consequently $q = x_f$, which implies that $f = Jq$. Hence

$$J(u_n) \to Jq \in E^*$$

weakly. Since $\|J(u_n)\| \to \|Jq\|$ and E^* has the property (K), we have

$$\|J(u_n) - Jq\| \to 0.$$

Noting that $J^{-1} : E^* \to E$ is demi-continuous, we have

$$u_n \to q \in E,$$

weakly. Since $\|u_n\| \to \|q\|$ and E has the property (K), we obtain that $u_n \to q$ as $n \to \infty$. Similarly, it is not difficult to show that $Tx_n \to q$ as $n \to \infty$. From $x_n \to q$ and the closeness property of T, we have $Tq = q$.

Next, we want to show that $q \in GMEP(\Theta, A, \varphi)$. Define $G : C \times C \to \mathbb{R}$ by $G(x, y) = \Theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$ for all $x, y \in C$. It is not hard to verify that G satisfies conditions $(A1) - (A4)$. It follows from $u_n = K_{r_n}Tx_n$ and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \ge G(y, u_n) \text{ for all } y \in C.$$

By using (A4) and $\liminf_{n\to\infty} r_n > 0$, we obtain $0 \ge G(y, q)$ for all $y \in C$. For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)q$. So, from $(A1)$ and $(A4)$ we have

$$0 = G(y_t, y_t) = G(y_t, ty + (1-t)q) \le tG(y_t, y) + (1-t)G(y_t, q) \le tG(y_t, y).$$

Dividing by t, we have

$$G(y_t, y) \ge 0 \text{ for all } y \in C.$$

From $(A3)$ we have $0 \le \lim_{t\to 0} G(y_t, y) = \lim_{t\to 0} G(ty + (1-t)q, y) \le G(q, y)$ for all $y \in C$, and hence $q \in GMEP(\Theta, A, \varphi)$. So, $q \in F(T) \cap GMEP(\Theta, A, \varphi) = \Omega$.

Step 7. Show that $q = \Pi_\Omega x_0$.

It follows from steps 5 and steps 6 that

$$\phi(q, x_0) \le \phi(\Pi_\Omega x_0, x_0) \le \phi(q, x_0),$$

which implies that $\phi(\Pi_\Omega x_0, x_0) = \phi(q, x_0)$. Hence, $q = \Pi_\Omega x_0$. Then $\{x_n\}$ converges strongly to $q = \Pi_\Omega x_0$. This completes the proof. ■

If T is closed quasi-nonexpansive, then Theorem 3.1 is reduced to the following corollary.

Corollary 3.2. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-nonexpansive mapping, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $A : C \to E^*$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \ \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \ \text{such that} \\ \quad \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - JTx_n \rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, u_n) + \phi(u_n, Tx_n) \leq 2 \langle x_n - z, Jx_n - Ju_n \rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

where $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

If $E = H$ is a Hilbert space, then Theorem 3.1 is reduced to the following corollary.

Corollary 3.3. *Let H be a Hilbert space. Assume that C is a nonempty closed convex subset of H. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $A : C \to H$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \ \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \ \text{such that} \\ \quad \Theta(u_n, y) + \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - Tx_n \rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|x_n - u_n\|^2 + \|u_n - Tx_n\|^2 \\ \qquad\qquad \leq \frac{2}{1-k} \langle x_n - z, x_n - Tx_n \rangle + 2 \langle x_n - z, Tx_n - u_n \rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

where $k \in [0, 1)$ and $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

If $A = 0$ and $\varphi = 0$, then we have the following corollary.

Corollary 3.4. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction, Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ such that $\Lambda := F(T) \cap EP(\Theta) \neq \varnothing$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \ \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \ \text{such that} \ \Theta(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - JTx_n \rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, u_n) + \phi(u_n, Tx_n) \\ \qquad\qquad \leq \frac{2}{1-k} \langle x_n - z, Jx_n - JTx_n \rangle + 2 \langle x_n - z, JTx_n - Ju_n \rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

where $k \in [0, 1)$ and $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Corollary 3.5 (Zhou and Gao [32, Theorem 3.1])**.** *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-strict pseudo-contraction. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ u_n \in C \text{ such that} \\ C_{n+1} = \left\{ z \in C_n : \phi(x_n, Tx_n) \leq \frac{2}{1-k} \langle x_n - z, Jx_n - JTx_n \rangle \right\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.4}$$

where $k \in [0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Proof. Put $\Theta = 0$, $A = 0$, $\varphi = 0$ and $r_n = 1$ for all $n \geq 1$ in Theorem 3.1. Then, $K_{r_n} = \Pi_C$ for all $n \geqslant 1$. So, $u_n = \Pi_C Tx_n$ for all $n \geqslant 1$(Note that $x_1 = \Pi_C x_0$). Since $x_n = \Pi_{C_n} x_0 \in C_n \subset C$ and then $Tx_n \in C$ for all $n \geqslant 1$, so we have $u_n = Tx_n$ for all $n \geqslant 1$. Thus $\phi(x_n, u_n) + \phi(u_n, Tx_n) = \phi(x_n, Tx_n)$ and $JTx_n - Ju_n = 0$ for all $n \geq 1$. For this reason, (1.4) is a special case of (3.1). Applying Theorem 3.1, we have the desired result. ∎

Remark 3.6. It is well known that every uniformly convex and uniformly smooth Banach space satisfies all assumptions of Banach space in Theorem 3.1. On the other hand, in general, Musielak-Orlicz space [9] need not to be uniformly convex or uniformly smooth, however, for any strictly convex, reflexive and smooth of this space satisfies all assumptions of Banach space in Theorem 3.1. It can be written as the following diagram:

reflexive, strictly convex, smooth Banach spaces with the property (K)
$\Uparrow \qquad \not\Downarrow$
uniformly convex and uniformly smooth Banach spaces .

For this reason, Theorem 3.1 can be viewed as a more general one and can be applied widely in both the fixed point problems and the equilibrium problems.

Acknowledgement. The author is supported by Thailand Research Fund under Grant no. MRG5380249.

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