รายงานวิจัยฉบับสมบูรณ์

โครงการ ตัววัดเชิงปริมาณของความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต

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สัญญาเลขที่ MRG5380277

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัย และ มหาวิทยาลัยเกษตรศาสตร์
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

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ชื่อโครงการ: ตัววัดเชิงปริมาณของความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต

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ระยะเวลาโครงการ: 2 ปี

บทคัดย่อ:

Riemann zeta function เป็นฟังก์ชันเลขคณิตซึ่งไม่สอดคล้องกับสมการผลต่างเชิงอนุพันธ์แบบพีชคณิตที่มีสัมประสิทธิ์เป็นจำนวนเชิงซ้อน หรืออาจกล่าวได้ว่า Riemann zeta function เป็น differentially transcendental ซึ่งก่อให้เกิดคำถามตามมาหลายข้อ เช่น

1. มีฟังก์ชันเลขคณิตอื่นอีกหรือไม่ที่เป็น differentially transcendental
2. สำหรับฟังก์ชันเลขคณิตที่เป็น differentially transcendental เราจะมีข้อสรุปอย่างไรเกี่ยวกับพฤติกรรมเชิงปริมาณ เมื่อแทนฟังก์ชันเลขคณิตนั้น และอนุพันธ์อันดับต่าง ๆ ของมัน ในสมการพีชคณิตซึ่งเราจะเรียกว่า ตัววัดเชิงปริมาณของ differential transcendence

โครงการวิจัยนี้มีจุดประสงค์หลักสองข้อซึ่งจะเป็นการตอบคำถามข้างต้นทั้งสองข้อ นั่นคือ

1. กำหนดเงื่อนไขที่เพียงพอที่ทำให้ฟังก์ชันเลขคณิตเป็น differentially transcendental และ
2. หาตัววัดเชิงปริมาณของ differential transcendence สำหรับฟังก์ชันเลขคณิตที่เป็น differentially transcendental

คำหลัก : จำนวน 3-5 คำ

ฟังก์ชันเลขคณิต Riemann zeta function ความเป็นอิสระต่อกัน ตัววัดเชิงปริมาณ differential transcendence

Abstract

Project Code : MRG5380277

Project Title : Quantitative measure of independence of arithmetic functions

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Project Period : 2 years

Abstract:

Riemann zeta function does not satisfy any algebraic differential difference equations with complex coefficients , that is , Riemann zeta function is a differentially transcendental function. There are some questions about this property , for example,

1. Are there other differentially transcendental functions?
2. How can we conclude about the quantitative behavier of an arithmetic function in an algebraic differential difference equations? This quantitative behavier is called measure of independence .

The two main proposes of this work are the answers of the above questions, that is,

1. find the necessary and sufficients conditions for differentially transcendental functions,
2.find the measure of independence of differentially transcendental functions.

Keywords : 3-5 words

arithmetic function , Riemann zeta function , independence, quantity measure, differential transcendence

Executive Summary

1. ชื่อโครงการ (ภาษาไทย) ตัววัดเชิงปริมาณของความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต

(ภาษาอังกฤษ) Quantitative measure of independence of arithmetic functions

2. ความสำคัญและที่มาของปัญหาที่ทำการวิจัย

Riemann zeta function เป็นฟังก์ชันเลขคณิตที่มีสมบติเป็นพังก์ชันแยกคูณบริบูรณ์ และมีความสำคัญในวิชาทฤษฎีจำนวนเป็นอย่างมาก จากการศึกษาของนักคณิตศาสตร์หลายๆ ท่าน พบว่า Riemann zeta function เป็นฟังก์ชันที่มีสมบัติ differential-transcendence นั่นคือ ไม่เป็นคำตอบของสมการเชิงอนุพันธ์ที่มีสัมประสิทธิ์เป็นจำนวนเชิงซ้อน ซึ่งในปี ค.ศ. 1956 J.Popken ได้ให้ตัววัดเชิงปริมาณของ differential-transcendence ของ Riemann zeta function และฟังก์ชันอื่นที่เกี่ยวข้องกับ Riemann zeta function ดังนั้นหากเราวิเคราะห์และสกัดวิธีการของ Popken ออกมาในรูปแบบที่เหมาะสม ก็จะทำให้ได้ตัววัดเชิงปริมาณของ differential-transcendence ของฟังก์ชันเลขคณิตอื่นที่เป็น differentially transcendental ซึ่งจะทำให้การตรวจสอบความเป็นอิสระต่อกันของฟังก์ชันเลขคณิตทำได้ง่ายขึ้น

3. วัตถุประสงค์ของโครงการ

3.1. กำหนดเงื่อนไขที่เพียงพอที่ทำให้ฟังก์ชันเลขคณิตเป็น differentially transcendental

3.2. หาตัววัดเชิงปริมาณของ differential transcendence ของฟังก์ชันเลขคณิตทั่วไป

4. ระเบียบวิธีวิจัย

การกำหนดเงื่อนไขที่เพียงพอของการเป็น differentially transcendental ของฟังก์ชันเลขคณิต จะพิจารณาจาก สมบติการเป็นฟังก์ชันแยกคูณบริบูรณ์ของ Riemann zeta function เป็นหลัก และ ในการหาตัววัดเชิงปริมาณของ differential transcendence ของฟังก์ชันเลขคณิตทั่วไป เราจะวิเคราะห์ งานวิจัยของ J.Popken และ สกัดวิธีการออกมาในรูปแบบที่เหมาะสม

5. แผนการดำเนินงานตลอดโครงการ

ช่วงที่ 1 (ปีที่ 1)

- ศึกษาและค้นคว้างานวิจัยที่เกี่ยวข้องกับความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต
- ศึกษาสมบัติของ Riemann zeta function ที่เกี่ยวข้องกับการเป็น differentially transcendental

ช่วงที่ 2 (ปีที่ 1)

- หาเงื่อนไขที่เพียงพอของการเป็นอิสระต่อกันของฟังก์ชันเลขคณิต
- หาเงื่อนไขที่เพียงพอของการเป็น differentially transcendental ของฟังก์ชันเลขคณิต
- สรุปผล และจัดเตรียมเอกสารตีพิมพ์ผลงานวิจัย

ช่วงที่ 3 (ปีที่ 2)

- วิเคราะห์และสกัดวิธีการของ J.Popken ในงานวิจัยเรื่อง A measure for the differential-transcendence of the zeta function of Riemann, Number theory and analysis, Papers in honour of Edmund Landau, Plenum Press, New York, 1969,245-255.

- หาตัววัดเชิงปริมาณของ differential transcendence ของฟังก์ชันเลขคณิตทั่วไป

ช่วงที่ 4 (ปีที่ 2)

- นำไปประยุกต์ใช้กับฟังก์ชันเลขคณิต
- สรุปผล และ จัดเตรียมเอกสารตีพิมพ์ผลงานวิจัย

6. ผลงาน/ หัวข้อเรื่องที่คาดว่าจะตีพิมพ์ในวารสารวิชาการระดับนานาชาติในแต่ละปี

ปีที่ 1 : ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : Checking linear dependence of two and three arithmetic functions

ชื่อวารสารที่คาดว่าจะตีพิมพ์ : Journal of the Korean Mathematical Society (2007 impact factor: 0.171)

ปีที่ 2 : ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : Independence Measure of Arithmetic Function

ชื่อวารสารที่คาดว่าจะตีพิมพ์ : Monatshefte fur Mathematik (2007 impact factor : 0.382)

ผลการดำเนินงานวิจัย

การดำเนินงานวิจัย

ในปีที่ 1

1. ศึกษาและค้นคว้างานวิจัยที่เกี่ยวข้องกับความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต
2. ศึกษาสมบัติของ Riemann zeta function ที่เกี่ยวข้องกับการเป็น differentially transcendental
3. หาเงื่อนไขที่เพียงพอของการเป็นอิสระต่อกันของฟังก์ชันเลขคณิต
4. หาเงื่อนไขที่เพียงพอของการเป็น differentially transcendental ของฟังก์ชันเลขคณิต

ในปีที่ 2

1. ศึกษางานวิจัยของ J.Popken ซึ่งเกี่ยวข้องกับการหาตัววัดเชิงปริมาณของการเป็น differential transcendence ของRiemann zeta function
2. หาตัววัดเชิงปริมาณของการเป็น differential transcendence ของฟังก์ชันทั่วไปโดยเงื่อนไขที่กว้างขึ้น
3. หาเงื่อนไขที่จำเป็นและเพียงพอของการเป็นอิสระต่อกันเชิงเส้นของฟังก์ชันเลขคณิตตั้งแต่สองฟังก์ชันขึ้นไป
4. นำผลที่ได้ไปประยุกต์ใช้กับฟังก์ชันเลขคณิตทั่วไป

ผลการวิจัย

ในช่วงปีที่ 1 การดำเนินงานวิจัยในช่วง 6 เดือนแรก ได้ศึกษาและค้นคว้างานวิจัยที่เกี่ยวข้องกับความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต และศึกษาสมบัติของ Riemann zeta function ที่เกี่ยวข้องกับการเป็น differentially transcendental พบว่างานวิจัยส่วนใหญ่จะเกี่ยวข้องกับการเป็นอิสระต่อกันเชิงพีชคณิตของฟังก์ชันเลขคณิต จึงได้มุ่งความสนใจไปที่การเป็นอิสระเชิงเส้นต่อกันของฟังก์ชันเลขคณิตก่อน ซึ่งผลที่ได้คือได้เงื่อนไขที่จำเป็นและเพียงพอของการเป็นอิสระต่อกันเชิงเส้นของฟังก์ชันเลขคณิตสองฟังก์ชันโดยผ่านเครื่องมือที่เรียกว่า Wronskian และได้นำผลที่ได้ไปเสนอผลงานแบบบรรยายในการประชุมวิชาการ International Conference on Mathematical Sciences ณ เมือง Bolu ประเทศ ตุรกี ระหว่างวันที่ 23 -27พฤศจิกายน 2553

การดำเนินงานวิจัย ในช่วง 6 เดือนหลัง ได้มุ่งความสนใจไปที่การเป็นอิสระต่อกันเชิงพีชคณิตของฟังก์ชันเลขคณิต และได้พบและทำงานวิจัยร่วมกับ Prof.Takao Komatsu จาก Hirosaki University ประเทศญี่ปุ่นซึ่งเดินทางมาประเทศไทยตามคำเชิญของ ศ.ดร.วิเชียร เลาหโกศล ซึ่งเป็นนักวิจัยที่ปรึกษาของโครงการวิจัยนี้ และผลที่ได้คือได้ตัววัดเชิงปริมาณของ algebraic independence สำหรับฟังก์ชันเลขคณิตภายใต้เงื่อนไขบางประการ และได้ส่งผลงานวิจัยไปตีพิมพ์ในวารสารวิชาการ คือ T.Komatsu, V.laohakosol, P.Ruengsinsub, Independence measures of arithmetic functions, J.Number Theory 131(2011),1-17

สำหรับในช่วงปีที่ 2 สามารถขยายงานที่ได้จากช่วงปีแรก โดย ได้ตัววัดเชิงปริมาณของการเป็น differential transcendence ของฟังก์ชันทั่วไป และ นำผลที่ได้ไปประยุกต์ใช้ทำให้ได้เงื่อนไขที่จำเป็นและเพียงพอของการเป็นอิสระต่อกันเชิงเส้นของฟังก์ชันเลขคณิตตั้งแต่สองฟังก์ชันขึ้นไป และ นำไปใช้เป็นเครื่องมือในการตรวจสอบการเป็น differentially transcendental ของฟังก์ชันชันเลขคณิตได้ ซึ่งผลงานได้รับการตีพิมพ์ในวารสารวิชาการ Acta arithmetica, **153.2** (2012), 199-216.

นอกจากนี้จากการศึกษาเรื่อง algebraic dependence ของฟังก์ชันเลขคณิต ทำให้ได้ความสัมพันธ์เชิงเส้นระหว่างฟังก์ชันเลขคณิตสองฟังก์ชัน ภายใต้เซตของฟังก์ชันที่เรียกว่า prime-free function และได้นำผลงานที่ได้ไปทำวิจัยเพิ่มเติมร่วมกับ Prof.Takao Komatsu ณ Hirosaki University เมือง Hirosaki ประเทศญี่ปุ่น และนำเสนอในการประชุมวิชาการ RIMS workshop “Analytic number theory related multiple aspects of arithmetic functions” ณ Kyoto University เมือง Kyoto ประเทศญี่ปุ่น ระหว่างวันที่ 27 ตุลาคม 2554- 4 พฤศจิกายน 2554

สรุปและวิจารณ์ผลการทดลอง และข้อเสนอแนะสำหรับงานวิจัยในอนาคต

1. ได้เงื่อนไขที่จำเป็นและเพียงพอของการเป็นอิสระต่อกันเชิงเส้นของฟังก์ชันเลขตั้งแต่สองฟังก์ชันขึ้นไป
2. ได้เงื่อนไขที่จำเป็นและเพียงพอที่ทำให้ฟังก์ชันเลขคณิตเป็น differentially transcendental
3. ได้ตัววัดเชิงปริมาณของ differential transcendence ของฟังก์ชันเลขคณิตทั่วไปเพื่อเป็นเครื่องมือที่สะดวกและง่ายขึ้นในการตรวจสอบความเป็นอิสระต่อกันของฟังก์ชันเลขคณิต

Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

ผลงานวิจัยที่ตีพิมพ์ในวารสารวิชาการระดับนานาชาติ

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รายงานสรุปการเงินในรอบ 24 เดือน

ชื่อหัวหน้าโครงการวิจัยผู้รับทุน ผศ.ดร.ภัททิรา เรืองสินทรัพย์

รายงานในช่วงตั้งแต่วันที่ 15 ธันวาคม 2554 **ถึงวันที่** 14 มิถุนายน 2555

รายจ่าย

หมวด (ตามสัญญา)	รายจ่ายสะสม จากรายงาน ครั้งก่อน	ค่าใช้จ่าย งวดปัจจุบัน	รวมรายจ่าย สะสมจนถึง งวดปัจจุบัน	งบประมาณ รวมทั้งโครงการ	คงเหลือ (หรือเกิน)
1. ค่าตอบแทน	180,000	0	180,000	240,000	60,000
2. ค่าวัสดุ	387	575	762	35,000	34,613
3. ค่าใช้สอย	34,672.50	32765.68	67,438.18	185,000	117,561.82
รวม	**215,059.50**	**33,340.68**	**248,200.18**	**460,000**	**211,799.82**

จำนวนเงินที่ได้รับและจำนวนเงินคงเหลือ

จำนวนเงินที่ได้รับ		
รอบ 12 เดือน (15 มิ.ย.2553-14 มิ.ย.2554)	226,385.68	บาท
รอบ 6 เดือน (15 มิ.ย.2554 -14 ธ.ค. 2554)	131,978.27	บาท
รอบ 6 เดือน (15 ธ.ค. 2554 – 14 มิ.ย. 2555)		
ดอกเบี้ย	719.21	บาท (วันที่ 25 ธ.ค.2554)
เงินอุดหนุนจาก มก.	38,333.50	บาท (วันที่ 23 พ.ค.2555)
ดอกเบี้ย	510.61	บาท (วันที่ 21 มิ.ย.2555)
รวม	**397,927.27**	**บาท**

ค่าใช้จ่าย

รอบ 12 เดือน (15 มิ.ย.2553-14 มิ.ย.2554)	155,059.50	บาท
รอบ 6 เดือน (15 มิ.ย.2554 -14 ธ.ค. 2554)	60,000	บาท
รอบ 6 เดือน (15 ธ.ค.2554 -14 มิ.ย. 2555)		
งวดที่ 1 เป็นเงิน	5,575	บาท
งวดที่ 2 เป็นเงิน	27,765.68	บาท
รวม	**248,400.18**	**บาท**
จำนวนเงินคงเหลือ	**149,527.09**	**บาท**

.... ลงนามหัวหน้าโครงการวิจัยผู้รับทุน

.... ลงนามเจ้าหน้าที่การเงินโครงการ

ภาคผนวก

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Independence measures of arithmetic functions

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ABSTRACT

The notion of algebraic dependence in the ring of arithmetic functions with addition and Dirichlet product is considered. Measures for algebraic independence are derived.

1. Introduction

Denote by $(\mathcal{A}, +, *)$ the unique factorization domain of arithmetic functions equipped with addition and convolution (or Dirichlet product) defined by

$$(f+g)(n) := f(n) + g(n), \qquad (f * g)(n) = \sum_{ij=n} f(i)g(j) \quad (f, g \in \mathcal{A},\ n \in \mathbb{N}),$$

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and write $f^{*i} = f * \cdots * f$, where the right-hand expression is a convolution of $i \in \mathbb{N}$ terms. The convolution identity, I, is defined by $I(1) = 1$ and $I(n) = 0$ for all $n > 1$. It is well known [17, Chapter 4] that $(\mathcal{A}, +, *)$ is isomorphic to $(\mathcal{D}, +, \cdot)$, where

$$\mathcal{D} := \left\{ D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right\}$$

is the ring of formal Dirichlet series equipped with addition and multiplication, through the isomorphism $f \leftrightarrow D$; the addition in both domains is the customary addition while the multiplication of formal Dirichlet series corresponds to the convolution of the appropriate arithmetic functions appearing as coefficients of the two formal Dirichlet series. For $f \in \mathcal{A}$, its *valuation* ([17, Chapter 4], [16]) is defined as

$$|f| := \frac{1}{O(f)},$$

where $O(f)$ is the least integer n for which $f(n) \neq 0$. Correspondingly, for a formal Dirichlet series $D(s) := \sum_{n \geqslant 1} f(n)/n^s$, its valuation is defined as

$$|D| = |f|,$$

where the same valuation symbols are used for convenience sake. With such valuation, the isomorphism $(\mathcal{A}, +, *) \leftrightarrow (\mathcal{D}, +, \cdot)$ is indeed an isometry. Because of this isometry, we often refer to each domain interchangeably.

A set of arithmetic functions $f_1, \ldots, f_r$ is said to be algebraically dependent over $\mathbb{C}$ or $\mathbb{C}$-algebraically dependent if there exists

$$P[X_1, \ldots, X_r] := \sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r} X_1^{i_1} \cdots X_r^{i_r} \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\}$$

such that

$$\sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r} f_1^{*i_1} * \cdots * f_r^{*i_r} \equiv 0,$$

and is said to be $\mathbb{C}$-algebraically independent otherwise. If the polynomial P is homogeneous of degree one in each variable, we say that $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent and $\mathbb{C}$-linearly independent otherwise. The first investigation of dependence of arithmetic functions was due to Carlitz [3] in 1952. Popken [9] in 1962 considered the problem of algebraic dependence in a more general setting of functions defined over a unique factorization semigroup with values in a ring. His main results give necessary and sufficient conditions for algebraic dependence by analyzing the Taylor expansion of the polynomial defining the dependence. In subsequent papers [10–12], he made applications to Dirichlet series and multiplicative functions. In the direction of Dirichlet series, Popken [13] gave a measure of the so-called *differential transcendence* of certain Dirichlet series closely connected to the Riemann zeta function, ζ [8]. More recent works can be found in [18], where algebraic independence of Dirichlet series and transcendence over $\mathbb{C}[\zeta]$ are considered. The works of Popken mentioned above were simplified and sharpened in [6].

In the present work, our main objectives are first to derive some algebraic independence criteria and then to prove general quantitative results about measure of such independence of arithmetic functions which simultaneously implies corresponding results for formal Dirichlet series. We also apply our results to a number of interesting cases in particular to the formal Fibonacci and Lucas zeta series.

To do so, we require certain related concepts which we briefly recall now. A *derivation* d [16,17] over $\mathcal{A}$ is a map : $\mathcal{A} \to \mathcal{A}$ satisfying

$$d(f * g) = df * g + f * dg, \qquad d(c_1 f + c_2 g) = c_1\, df + c_2\, dg,$$

where $f, g \in \mathcal{A}$ and $c_1, c_2 \in \mathbb{C}$. Derivations of higher orders are defined in the usual manner. Two typical examples of derivation are

- the p-*basic derivation*, p prime, defined by

$$(d_p f)(n) = f(np)\nu_p(np) \quad (n \in \mathbb{N}),$$

  where $\nu_p(m)$ denotes the exponent of the highest power of p dividing m,
- the *log-derivation* defined by

$$(d_L f)(n) = f(n)\log n \quad (n \in \mathbb{N}).$$

Although, there are arithmetic sequences $f(n)$ for which the corresponding Dirichlet series $D(s) := \sum_n f(n)/n^s$ are divergent, through the isometry between $\mathcal{A}$ and $\mathcal{D}$, it is legitimate to define the formal derivation $\tilde{d}$ of (formal) Dirichlet series via the derivation d of the associated arithmetic function as

$$\tilde{d}D(s) = \sum_{n=1}^{\infty} \frac{df(n)}{n^s}.$$

Thus, the formal differentiation of the formal Dirichlet series, $D(s)$, with respect to the variable s, i.e.,

$$D'(s) = \sum_{n=1}^{\infty} \frac{-f(n)\log n}{n^s},$$

corresponds to the (negative) log-derivation $-d_L$ of the associated arithmetic function f, and the p-basic derivation d_p over $\mathcal{A}$ corresponds to the formal p-basic derivation $\tilde{d}_p$ over $\mathcal{D}$ defined by

$$\tilde{d}_p D(s) = \sum_{n=1}^{\infty} \frac{(d_p f)(n)}{n^s}.$$

For convenience, in the sequel we use the same derivation symbol d for both the domains $\mathcal{A}$ and $\mathcal{D}$. Our investigation concerning Dirichlet series will be **formal** throughout, noting that should the Dirichlet series involved converge, the results so obtained are then valid (analytically) and coincide with results proved for convergent Dirichlet series in the domain of convergence.

2. Some criteria

To state some preliminary results, we need another notion. For $f \in \mathcal{A}$, $f(1) > 0$, the Rearick logarithmic operator of f (or *logarithm* of f [14,15,7]), denoted by $\operatorname{Log} f \in \mathcal{A}$, is defined via

$$(\operatorname{Log} f)(1) = \log f(1),$$

$$(\operatorname{Log} f)(n) = \frac{1}{\log n}\sum_{k|n} f(k) f^{-1}\left(\frac{n}{k}\right)\log k = \frac{1}{\log n}\left(d_L f * f^{-1}\right)(n) \quad (n > 1),$$

where d_L denotes the log-derivation. For $h \in \mathcal{A}$, the Rearick exponential $\operatorname{Exp} h$ is defined as the unique element $f \in \mathcal{A}$, $f(1) > 0$ such that $h = \operatorname{Log} f$.

We start with some simple results.

Proposition 2.1. *Let $f \in \mathcal{A} \setminus \{0\}$.*

1. *Then f is $*$-algebraic over $\mathbb{C}$ if and only if $f = cI$ for some constant $c \in \mathbb{C}$.*
2. *Assuming $f(1) > 0$, then f and $\operatorname{Log} f$ are $\mathbb{C}$-algebraically dependent if and only if $f = cI$ for some constant $c \in \mathbb{C}$.*
3. *Assuming $f(1) \in \mathbb{R}$, then f and $\operatorname{Exp} f$ are $\mathbb{C}$-algebraically dependent if and only if $f = cI$ for some constant $c \in \mathbb{C}$.*

Proof. We give only a proof for assertion 1 as those for the other two assertions are similar.

The sufficiency part is trivial. To prove the necessity part, assume that f satisfies an algebraic equation of the form

$$a_k f^{*k} + \cdots + a_1 f + a_0 I = 0,$$

with least degree $k \geqslant 1$ and $a_k \neq 0$. Taking the log-derivation, we get

$$\left(k a_k f^{*k-1} + \cdots + a_1 I\right) * d_L f = 0.$$

By the minimality of k, we must have $d_L f = 0$ which is the result. $\square$

Shapiro–Sparer's criterion for $\mathbb{C}$-algebraic dependence of arithmetic functions in [18] states that:

Theorem 2.2. *Let $f_1, \ldots, f_t \in \mathcal{A}$ and $p_1, \ldots, p_t$ be distinct primes with corresponding p-basic derivations $d_1\ (:= d_{p_1}), \ldots, d_t\ (:= d_{p_t})$. If the Jacobian relative to $d_1, \ldots, d_t$*

$$J := J(f_1, \ldots. f_t; d_1, \ldots, d_t) = \begin{vmatrix} d_1 f_1 & \cdots & d_t f_1 \\ \vdots & & \vdots \\ d_t f_1 & \cdots & d_t f_t \end{vmatrix} \neq 0,$$

where the multiplication in the determinant expansion is interpreted as convolution $*$, then $f_1, \ldots, f_t$ are $\mathbb{C}$-algebraically independent.

Evaluating the Jacobian at $n \in \mathbb{N}$ in Theorem 2.2, we get

$$J(n) = \sum_{(i)} e_{(i)} (d_1 f_{i_1} * \cdots * d_t f_{i_t})(n),$$

where the sum is taken over all possible permutations $(i) = (i_1, \ldots, i_t)$ of $(1, \ldots, t)$ with

$$e_{(i)} = \begin{cases} 1 & \text{if } (i) \text{ is an even permutation,} \\ -1 & \text{otherwise.} \end{cases}$$

Consequently, writing $\nu_1, \ldots, \nu_t$ for $\nu_{p_1}, \ldots, \nu_{p_t}$, respectively, we have

$$\begin{aligned}
J(n) &= \sum_{(i)} e_{(i)} \sum_{k_1\cdots k_t=n} d_1 f_{i_1}(k_1)\cdots d_t f_{i_t}(k_t)\\
&= \sum_{k_1\cdots k_t=n} \sum_{(i)} e_{(i)} f_{i_1}(k_1p_1)\cdots f_{i_t}(k_tp_t)\nu_1(k_1p_1)\cdots \nu_t(k_tp_t)\\
&= \sum_{k_1\cdots k_t=n} \nu_1(k_1p_1)\cdots \nu_t(k_tp_t)\begin{vmatrix} f_1(k_1p_1) & \cdots & f_1(k_tp_t)\\ \vdots & & \vdots\\ f_t(k_1p_1) & \cdots & f_t(k_tp_t)\end{vmatrix},
\end{aligned}$$

which yields

Corollary 2.3. *Let $f_1,\dots,f_t$ be arithmetic functions and $p_1,\dots,p_t$ be distinct primes with corresponding p-basic derivations $d_1\ (:= d_{p_1}),\dots,d_t\ (:= d_{p_t})$ and corresponding p-exponent functions $\nu_1\ (:= \nu_{p_1}),\dots,\nu_t\ (:= \nu_{p_t})$. If there exists $n\in\mathbb{N}$ such that*

$$\sum_{k_1\cdots k_t=n} \nu_1(k_1p_1)\cdots \nu_t(k_tp_t)\begin{vmatrix} f_1(k_1p_1) & \cdots & f_1(k_tp_t)\\ \vdots & & \vdots\\ f_t(k_1p_1) & \cdots & f_t(k_tp_t)\end{vmatrix} \neq 0,$$

then $f_1,\dots,f_t$ are $\mathbb{C}$-algebraically independent.

Specializing the values of n, we deduce the following simple tests of algebraic independence.

Test I. The simplest test is obtained by taking $n=1$ in Corollary 2.3. If

$$\begin{vmatrix} f_1(p_1) & \cdots & f_1(p_t)\\ \vdots & & \vdots\\ f_t(p_1) & \cdots & f_t(p_t)\end{vmatrix} \neq 0,$$

then $f_1,\dots,f_t$ are $\mathbb{C}$-algebraically independent.

An immediate consequence of Test I is the following convenient test of algebraic independence.

Corollary 2.4. *Let $f_1,\dots,f_t\in\mathcal{A}$. If there are t distinct primes $p_1,\dots,p_t$ such that the set of vectors $\{(f_1(p_i),\dots,f_t(p_i))\colon i=1,\dots,t\}$ is $\mathbb{C}$-linearly independent, then $f_1,\dots,f_t$ are $\mathbb{C}$-algebraically independent.*

Test II. Taking $n=p_1$, if

$$\begin{aligned}
0 \neq 2&\begin{vmatrix} f_1(p_1^2) & f_1(p_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(p_1^2) & f_t(p_2) & \cdots & f_t(p_t)\end{vmatrix} + \begin{vmatrix} f_1(p_1) & f_1(p_1p_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(p_1) & f_t(p_1p_2) & \cdots & f_t(p_t)\end{vmatrix} + \cdots\\
&+\begin{vmatrix} f_1(p_1) & f_1(p_2) & \cdots & f_1(p_1p_t)\\ \vdots & & & \vdots\\ f_t(p_1) & f_t(p_2) & \cdots & f_t(p_1p_t)\end{vmatrix},
\end{aligned}$$

then $f_1,\dots,f_t$ are $\mathbb{C}$-algebraically independent.

Test III. Taking $n=q$, prime distinct from $p_1,\dots,p_t$, if

$$0\neq\begin{vmatrix} f_1(qp_1) & f_1(p_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(qp_1) & f_t(p_2) & \cdots & f_t(p_t)\end{vmatrix}+\begin{vmatrix} f_1(p_1) & f_1(qp_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(p_1) & f_t(qp_2) & \cdots & f_t(p_t)\end{vmatrix}+\cdots$$
$$+\begin{vmatrix} f_1(p_1) & f_1(p_2) & \cdots & f_1(qp_t)\\ \vdots & & & \vdots\\ f_t(p_1) & f_t(p_2) & \cdots & f_t(qp_t)\end{vmatrix},$$

then $f_1,\dots,f_t$ are $\mathbb{C}$-algebraically independent.

Test IV. Taking $n=p_1^2$, if

$$0\neq 3\begin{vmatrix} f_1(p_1^3) & f_1(p_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(p_1^3) & f_t(p_2) & \cdots & f_t(p_t)\end{vmatrix}+\begin{vmatrix} f_1(p_1) & f_1(p_1^2p_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(p_1) & f_t(p_1^2p_2) & \cdots & f_t(p_t)\end{vmatrix}+\cdots$$
$$+\begin{vmatrix} f_1(p_1) & f_1(p_2) & \cdots & f_1(p_1^2p_t)\\ \vdots & & & \vdots\\ f_t(p_1) & f_t(p_2) & \cdots & f_t(p_1^2p_t)\end{vmatrix}$$
$$+2\begin{vmatrix} f_1(p_1^2) & f_1(p_1p_2) & \cdots & f_1(p_t)\\ \vdots & & & \vdots\\ f_t(p_1^2) & f_t(p_1p_2) & \cdots & f_t(p_t)\end{vmatrix}+\cdots+2\begin{vmatrix} f_1(p_1^2) & f_1(p_2) & \cdots & f_1(p_1p_t)\\ \vdots & & & \vdots\\ f_t(p_1^2) & f_t(p_2) & \cdots & f_t(p_1p_t)\end{vmatrix}$$
$$+\begin{vmatrix} f_1(p_1) & f_1(p_1p_2) & f_1(p_1p_3) & \cdots & f_1(p_t)\\ \vdots & & & & \vdots\\ f_t(p_1) & f_t(p_1p_2) & f_t(p_1p_3) & \cdots & f_t(p_t)\end{vmatrix}+\cdots$$
$$+\begin{vmatrix} f_1(p_1) & f_1(p_2) & \cdots & f_1(p_{t-2}) & f_1(p_1p_{t-1}) & f_1(p_1p_t)\\ \vdots & & & & \vdots & \\ f_t(p_1) & f_t(p_2) & \cdots & f_t(p_{t-2}) & f_t(p_1p_{t-1}) & f_t(p_1p_t)\end{vmatrix},$$

then $f_1,\dots,f_t$ are $\mathbb{C}$-algebraically independent.

Let us now look at some examples. Let $\{F_n\}_{n\geqslant 1}$ be the sequence of Fibonacci numbers defined by

$$F_1=F_2=1,\qquad F_{n+2}=F_{n+1}+F_n\quad(n\in\mathbb{N}).$$

The six formal Fibonacci zeta series are defined as

$$\mathcal{F}^+(s):=\sum_{n=1}^{\infty}\frac{1}{F_n^s}=\sum_{n=1}^{\infty}\frac{f^+(n)}{n^s},\qquad \mathcal{F}_e^+(s):=\sum_{n=1}^{\infty}\frac{1}{F_{2n}^s}=\sum_{n=1}^{\infty}\frac{f_e^+(n)}{n^s},$$
$$\mathcal{F}_o^+(s):=\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^s}=\sum_{n=1}^{\infty}\frac{f_o^+(n)}{n^s},\qquad \mathcal{F}^-(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_n^s}=\sum_{n=1}^{\infty}\frac{f^-(n)}{n^s},$$
$$\mathcal{F}_e^-(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{2n}^s}=\sum_{n=1}^{\infty}\frac{f_e^-(n)}{n^s},\qquad \mathcal{F}_o^-(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{2n-1}^s}=\sum_{n=1}^{\infty}\frac{f_o^-(n)}{n^s}.$$

Let $\{L_n\}_{n\geqslant 1}$ be the sequence of Lucas numbers defined by

$$L_1 = 1, \qquad L_2 = 3, \qquad L_{n+2} = L_{n+1} + L_n \quad (n \in \mathbb{N}).$$

The six formal Lucas zeta series are defined as

$$\mathcal{L}^+(s) := \sum_{n=1}^{\infty} \frac{1}{L_n^s} = \sum_{n=1}^{\infty} \frac{\ell^+(n)}{n^s}, \qquad \mathcal{L}_e^+(s) := \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s} = \sum_{n=1}^{\infty} \frac{\ell_e^+(n)}{n^s},$$
$$\mathcal{L}_o^+(s) := \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{\ell_o^+(n)}{n^s}, \qquad \mathcal{L}^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n^s} = \sum_{n=1}^{\infty} \frac{\ell^-(n)}{n^s},$$
$$\mathcal{L}_e^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n}^s} = \sum_{n=1}^{\infty} \frac{\ell_e^-(n)}{n^s}, \qquad \mathcal{L}_o^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{\ell_o^-(n)}{n^s}.$$

These twelve (formal) Fibonacci and Lucas zeta series were considered in [5] in order to prove that they are hypertranscendental. We now establish some of their dependence relations.

Proposition 2.5.

1. *Three functions in each of the following sets of arithmetic functions are $\mathbb{C}$-algebraically independent*:

$$\{f^+, f_e^+, f_e^-\}, \{f^+, f_e^+, f_o^-\}, \{f^+, f_o^+, f_e^-\}, \{f^+, f_o^+, f_o^-\},$$
$$\{f^+, f^-, f_e^-\}, \{f^+, f^-, f_o^-\}, \{f^+, f_e^-, f_o^-\};$$
$$\{f_e^+, f_o^+, f_e^-\}, \{f_e^+, f_o^+, f_o^-\}, \{f_e^+, f^-, f_e^-\}, \{f_e^+, f^-, f_o^-\}, \{f_e^+, f_e^-, f_o^-\};$$
$$\{f_o^+, f^-, f_e^-\}, \{f_o^+, f^-, f_o^-\}, \{f_o^+, f_e^-, f_o^-\};$$
$$\{f^-, f_e^-, f_o^-\}.$$

2. *We have $f^+ = f_e^+ + f_o^+$, $f^+ = 2f_o^+ - f^-$, $f^- = f_o^+ - f_e^+$, $f^+ = f^- + 2f_e^+$, i.e., three functions in each of the following sets are $\mathbb{C}$-linearly dependent*

$$\{f^+, f_e^+, f_o^+\}, \{f^+, f_o^+, f^-\}, \{f^-, f_o^+, f_e^+\}, \{f^+, f^-, f_e^+\}.$$

Proof. The results of assertion 2 are clear, so we need only check those in assertion 1. We only provide two of them using different tests (Tests I and III).

By Test III, we have

$$\begin{vmatrix} f^+(2\times 11) & f^+(3=F_4) & f^+(5=F_5) \\ f_e^+(2\times 11) & f_e^+(3=F_4) & f_e^+(5=F_5) \\ f_e^-(2\times 11) & f_e^-(3=F_4) & f_e^-(5=F_5) \end{vmatrix} + \begin{vmatrix} f^+(2=F_3) & f^+(3\times 11) & f^+(5=F_5) \\ f_e^+(2=F_3) & f_e^+(3\times 11) & f_e^+(5=F_5) \\ f_e^-(2=F_3) & f_e^-(3\times 11) & f_e^-(5=F_5) \end{vmatrix}$$
$$+ \begin{vmatrix} f^+(2=F_3) & f^+(3=F_4) & f^+(5\times 11=F_{10}) \\ f_e^+(2=F_3) & f_e^+(3=F_4) & f_e^+(5\times 11=F_{10}) \\ f_e^-(2=F_3) & f_e^-(3=F_4) & f_e^-(5\times 11=F_{10}) \end{vmatrix} = 0 + 0 + \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 2 \neq 0,$$

i.e. f^+, f_e^+, f_e^- are $\mathbb{C}$-algebraically independent.

By Test I, we have

$$\begin{vmatrix} f^+(2=F_3) & f^+(3=F_4) & f^+(5=F_5) \\ f_e^+(2=F_3) & f_e^+(3=F_4) & f_e^+(5=F_5) \\ f_o^-(2=F_3) & f_o^-(3=F_4) & f_o^-(5=F_5) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2 \neq 0,$$

i.e. f^+, f_e^+, f_o^- are $\mathbb{C}$-algebraically independent. □

The situation for formal Lucas zeta series is much the same and we merely state the result.

Proposition 2.6.

1. *Three functions in each of the following sets of arithmetic functions are $\mathbb{C}$-algebraically independent*:

$$\{\ell^+, \ell_e^+, \ell_e^-\}, \{\ell^+, \ell_e^+, \ell_o^-\}, \{\ell^+, \ell_o^+, \ell_e^-\}, \{\ell^+, \ell_o^+, \ell_o^-\},$$
$$\{\ell^+, \ell^-, \ell_e^-\}, \{\ell^+, \ell^-, \ell_o^-\}, \{\ell^+, \ell_e^-, \ell_o^-\};$$
$$\{\ell_e^+, \ell_o^+, \ell_e^-\}, \{\ell_e^+, \ell_o^+, \ell_o^-\}, \{\ell_e^+, \ell^-, \ell_e^-\}, \{\ell_e^+, \ell^-, \ell_o^-\}, \{\ell_e^+, \ell_e^-, \ell_o^-\};$$
$$\{\ell_o^+, \ell^-, \ell_e^-\}, \{\ell_o^+, \ell^-, \ell_o^-\}, \{\ell_o^+, \ell_e^-, \ell_o^-\};$$
$$\{\ell^-, \ell_e^-, \ell_o^-\}.$$

2. *We have $\ell^+ = \ell_e^+ + \ell_o^+$, $\ell^+ = 2\ell_o^+ - \ell^-$, $\ell^- = \ell_o^+ - \ell_e^+$, $\ell^+ = \ell^- + 2\ell_e^+$, i.e., three functions in each of the following sets are $\mathbb{C}$-linearly dependent*

$$\{\ell^+, \ell_e^+, \ell_o^+\}, \{\ell^+, \ell_o^+, \ell^-\}, \{\ell^-, \ell_o^+, \ell_e^+\}, \{\ell^+, \ell^-, \ell_e^+\}.$$

3. *Three functions with at least one from each of the two sets $\{f^+, f_e^+, f_e^-, f^-, f_e^-, f_o^-\}$ and $\{\ell^+, \ell_e^+, \ell_o^+, \ell^-, \ell_e^-, \ell_o^-\}$ are $\mathbb{C}$-algebraically independent.*

3. Measure of algebraic independence

We start with an auxiliary result whose proof resembles that of [6, Theorem 2].

Lemma 3.1. *Let $f_1, \dots, f_r \in \mathcal{A}$ and $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r] \setminus \{0\}$. For $t = 1, \dots, r$, define the following formal Dirichlet series*

$$D_t(s) = \sum_{n \geqslant 1} \frac{f_t(n)}{n^s}, \qquad P(D_1, \dots, D_r) = \sum_{n \geqslant 1} \frac{F(n)}{n^s}, \qquad \frac{\partial P}{\partial X_t}(D_1, \dots, D_r) = \sum_{n \geqslant 1} \frac{F_t(n)}{n^s}.$$

Then for each $n \in \mathbb{N}$ and for each prime p, we have

$$F(pn)\nu_p(pn) = \sum_{j=1}^{r} \sum_{k|n} f_j(pk) F_j\left(\frac{n}{k}\right) \nu_p(pk), \tag{3.1}$$

$$F(n)\log n = \sum_{j=1}^{r} \sum_{k|n} f_j(k) F_j\left(\frac{n}{k}\right) \log k, \tag{3.2}$$

where the Dirichlet series and their operations are considered formally.

Proof. For each prime p, let $d = d_p$ be its p-basic derivation. Through the isometry $\mathcal{A} \leftrightarrow \mathcal{D}$, the correspondence of p-basic derivation in both domains and the fact that a product of formal Dirichlet series is isomorphic to a convolution of arithmetic functions, we formally have

$$\begin{aligned}\sum_{n\geqslant 1}\frac{F(np)\nu_p(np)}{n^s} &= \sum_{n\geqslant 1}\frac{dF(n)}{n^s} = dP(D_1,\dots,D_r) = \sum_{j=1}^{r} dD_j\cdot\frac{\partial P}{\partial X_j}(D_1,\dots,D_r)\\ &= \sum_{j=1}^{r}\biggl(\sum_{n\geqslant 1}\frac{df_j(n)}{n^s}\biggr)\biggl(\sum_{n\geqslant 1}\frac{F_j(n)}{n^s}\biggr) = \sum_{j=1}^{r}\sum_{n\geqslant 1}\sum_{k\mid n}\frac{df_j(k)F_j(\frac{n}{k})}{n^s}\\ &= \sum_{n\geqslant 1}\sum_{j=1}^{r}\sum_{k\mid n}\frac{f_j(pk)F_j(\frac{n}{k})\nu_p(pk)}{n^s}. \qquad (3.3)\end{aligned}$$

Analytically, Eq. (3.3) is true only if the two Dirichlet series on the left-hand side converge absolutely, and this might not be the case for certain sequences $f_j, F_j \in \mathcal{A}$. However, the above proof is treated formally in the sense that it holds true for formal Dirichlet series and formal operations.

The relation (3.1) follows from equating the terms with $n \geqslant 2$. The relation (3.2) follows in the same manner by taking log-derivation and equating the terms with $n \geqslant 2$. $\square$

Our main result reads:

Theorem 3.2. *Let $P(X_1,\dots,X_r) \in \mathbb{C}[X_1,\dots,X_r] \setminus \{0\}$ be of total degree $\deg P = g$. For $t = 1,\dots,r$, define the following formal Dirichlet series*

$$D_t(s) = \sum_{n\geqslant 1}\frac{f_t(n)}{n^s}, \qquad P(D_1,\dots,D_r) = \sum_{n\geqslant 1}\frac{F(n)}{n^s}, \qquad \frac{\partial P}{\partial X_t}(D_1,\dots,D_r) = \sum_{n\geqslant 1}\frac{F_t(n)}{n^s}.$$

Let $\{p_1 < p_2 < p_3 < \cdots < p_r\}$ be a set of primes. If the set of vectors $\{(f_1(p_i),\dots,f_r(p_i)):\ i = 1,\dots,r\}$ is linearly independent over $\mathbb{C}$, then

$$\bigl|P(D_1,\dots,D_r)\bigr| \geqslant p_r^{-g},$$

where the Dirichlet series, their derivatives and operations are considered formally.

Proof. If $\deg P = 0$, then clearly $|P(D_1,\dots,D_r)| = 1$. If $\deg P = 1$, then

$$P(X_1,\dots,X_r) = a_0 I + a_1 X_1 + \cdots + a_r X_r,$$

where all the coefficients a_j $(j = 1,\dots,r)$ do not vanish simultaneously. Equating coefficients, we get

$$F(p_j) = a_1 f_1(p_j) + \cdots + a_r f_r(p_j).$$

Since the set of vectors $\{(f_1(p_j),\dots,f_r(p_j)):\ j = 1,\dots,r\}$ is linearly independent over $\mathbb{C}$, then at least one of the values $F(p_1),\dots,F(p_r)$ must be non-zero, which renders

$$\bigl|P(D_1,\dots,D_r)\bigr| \geqslant p_r^{-1}.$$

Now proceed by induction on $\deg P$. Let P be of total degree $g + 1 \geqslant 2$, and assume that the assertion has already been proved for polynomials of degree $\leqslant g$. Consider the polynomials $\partial P/\partial X_t$ $(t =$

$1, \ldots, r$), which are all of degree $\leqslant g$. Unless each $\partial P/\partial X_t$ vanishes identically, then by induction we have

$$\left|\frac{\partial P}{\partial X_t}(D_1, \ldots, D_r)\right| \geqslant p_r^{-g},$$

which implies that not all of the p_r^g vectors

$$\left\{\left(F_1(1), \ldots, F_r(1)\right), \left(F_1(2), \ldots, F_r(2)\right), \ldots, \left(F_1\left(p_r^g\right), \ldots, F_r\left(p_r^g\right)\right)\right\} \tag{3.4}$$

can be a zero vector. Let $(F_1(m), \ldots, F_r(m))$ be the first non-zero vector in the sequence (3.4) so that

$$\left(F_1(d), \ldots, F_r(d)\right) = (0, \ldots, 0) \quad \text{for } d = 1, 2, \ldots, m-1.$$

By the minimality of m and the result of Lemma 3.1, we get

$$F(pm)\nu(pm) = f_1(p)F_1(m) + \cdots + f_r(p)F_r(m).$$

Since the set $\{(f_1(p_j), \ldots, f_r(p_j)): \ j = 1, \ldots, r\}$ is linearly independent over $\mathbb{C}$, among the r values of $F(p_1 m), \ldots, F(p_r m)$ at least one must be non-zero. This yields

$$\left|P(D_1, \ldots, D_r)\right| \geqslant (mp_r)^{-1} \geqslant \left(p_r^{g+1}\right)^{-1}. \qquad \square$$

As a simple example, we make quantitative one of the algebraic independence results of Proposition 2.5. Taking the first three primes $2 = F_3$, $3 = F_4$, $5 = F_5$. As seen in the proof of Proposition 2.5, part 1, the set

$$\left\{\left(f^+(2), f_e^+(2), f_o^-(2)\right), \left(f^+(3), f_e^+(3), f_o^-(3)\right), \left(f^+(5), f_e^+(5), f_o^-(5)\right)\right\}$$

is $\mathbb{C}$-linearly independent. By Theorem 3.2, we have

$$\left|P\left(\mathcal{F}^+, \mathcal{F}_e^+, \mathcal{F}_o^-\right)\right| \geqslant 5^{-g},$$

for any $P(X_1, X_2, X_3) \in \mathbb{C}[X_1, X_2, X_3] \setminus \{0\}$ of total degree g.

For a more complex example, let us note that the four Lucas zeta functions ℓ^+, ℓ^-, ℓ_e^-, ℓ_o^- are algebraically independent over $\mathbb{C}$ because by Test I, we have

$$\begin{vmatrix} \ell^+(3 = L_2) & \ell^+(7 = L_4) & \ell^+(11 = L_5) & \ell^+(29 = L_7) \\ \ell^-(3 = L_2) & \ell^-(7 = L_4) & \ell^-(11 = L_5) & \ell^-(29 = L_7) \\ \ell_e^-(3 = L_2) & \ell_e^-(7 = L_4) & \ell_e^-(11 = L_5) & \ell_e^-(29 = L_7) \\ \ell_o^-(3 = L_2) & \ell_o^-(7 = L_4) & \ell_o^-(11 = L_5) & \ell_o^-(29 = L_7) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -8 \neq 0.$$

By Theorem 3.2, we have

$$\left|P\left(\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}_e^-, \mathcal{L}_o^-\right)\right| \geqslant 29^{-g},$$

for any $P(X_1, X_2, X_3, X_4) \in \mathbb{C}[X_1, X_2, X_3, X_4] \setminus \{0\}$ of total degree g.

Theorem 3.2 enables us to derive a measure of the so-called differential transcendence of formal Dirichlet series encompassing the special case of the Riemann zeta function.

Corollary 3.3. *Let $D(s) = \sum_{n \geqslant 1} f(n)n^{-s} \in \mathcal{D}$ and $P(X_0, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r] \setminus \{0\}$ be of total degree g. If there is a set of $r+1$ primes $\{p_1 < \dots < p_{r+1}\}$ such that $f(p_i) \neq 0$ $(i = 1, \dots, r+1)$, then*

$$\left|P\left(D, D', \dots, D^{(r)}\right)\right| \geqslant \left(p_{r+1}^g\right)^{-1},$$

where the Dirichlet series, their derivatives and operations are considered formally.

Proof. Formally differentiating the Dirichlet series with respect to s for $j \in \mathbb{N}$ times, we get

$$D^{(j)}(s) = \sum_{n \geqslant 1} \frac{f(n)(-\log n)^j}{n^s}.$$

For each $i \in \{1, \dots, r+1\}$, since $f(p_i)(-\log p_i)^j \neq 0$, the determinant

$$\begin{aligned} &\begin{vmatrix} f(p_1) & f(p_1)(-\log p_1) & \cdots & f(p_1)(-\log p_1)^r \\ \vdots & & & \vdots \\ f(p_{r+1}) & f(p_{r+1})(-\log p_{r+1}) & \cdots & f(p_{r+1})(-\log p_{r+1})^r \end{vmatrix} \\ &\quad = f(p_1)\cdots f(p_{r+1}) \begin{vmatrix} 1 & (-\log p_1) & \cdots & (-\log p_1)^r \\ \vdots & & & \vdots \\ 1 & (-\log p_{r+1}) & \cdots & (-\log p_{r+1})^r \end{vmatrix} \neq 0, \end{aligned}$$

implying that the set of vectors

$$\begin{aligned} &\left\{\left(f(p_1), \dots, f(p_{r+1})\right), \left(-f(p_1)\log p_1, \dots, -f(p_{r+1})\log p_{r+1}\right), \dots, \right. \\ &\quad \left. \left(-f(p_1)(\log p_1)^r, \dots, -f(p_{r+1})(\log p_{r+1})^r\right)\right\} \end{aligned}$$

is $\mathbb{C}$-linearly independent. The assertion now follows from Theorem 3.2. $\square$

Applying the result of Corollary 3.3 to the formal Riemann zeta series, we get a nice measure

$$\left|P\left(\zeta(s), \zeta'(s), \dots, \zeta^{(r)}(s)\right)\right| \geqslant p_{r+1}^{-g},$$

for any $P(X_0, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r] \setminus \{0\}$ of total degree g.

The condition of linear independence at primes in Theorem 3.2 can be relaxed at the expense of an extra condition, as we show next.

Theorem 3.4. *Let $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r] \setminus \{0\}$ be of total degree g and let*

$$D_t(s) = \sum_{n \geqslant 1} \frac{f_t(n)}{n^s}, \qquad P(D_1, \dots, D_r) = \sum_{n \geqslant 1} \frac{F(n)}{n^s},$$
$$\frac{\partial P}{\partial X_t}(D_1, \dots, D_r) = \sum_{n \geqslant 1} \frac{F_t(n)}{n^s} \quad (t = 1, \dots, r).$$

Assume that there are a set of r primes $\{p_1 < p_2 < \dots < p_r\}$ and a set of r positive integers $\{n_1, \dots, n_r\}$ such that

$$f_t(p_i n_i) \neq 0 \quad \textit{but} \quad f_t(p_i k) = 0 \quad \textit{for } 1 \leqslant k < n_i \ (t = 1, \dots, r;\ i = 1, \dots, r).$$

If the vectors $\{(f_1(p_i n_i), \dots, f_r(p_i n_i));\ i = 1, \dots, r\}$ *are* $\mathbb{C}$*-linearly independent, then*

$$\left|P(D_1, \dots, D_r)\right| \geqslant M_1^{-g},$$

where $M_1 = \max\{p_1 n_1, \dots, p_r n_r\}$*, and the Dirichlet series, their derivatives together with operations are considered formally.*

Proof. All the Dirichlet series, their derivatives and operations are formally treated here. If $\deg P = 0$, then $|P(D_1, \dots, D_r)| = 1$. If $\deg P = 1$, then

$$P(D_1, \dots, D_r) = a_0 + \sum_{t=1}^{r} a_t D_t$$

with not all a_i's vanishing simultaneously. Now

$$\sum_{n \geqslant 1} \frac{F(n)}{n^s} = a_0 + \sum_{n \geqslant 1} \sum_{t=1}^{r} a_t \frac{f_t(n)}{n^s}.$$

Then

$$F(n) = \sum_{t=1}^{r} a_t f_t(n) \quad (n \geqslant 2).$$

Since the vectors $\{(f_1(p_i n_i), \dots, f_r(p_i n_i));\ i = 1, \dots, r\}$ are $\mathbb{C}$-linearly independent and not all a_i's are zero, at least one of the values $F(p_1 n_1), \dots, F(p_r n_r)$ must be non-zero yielding

$$\left|P(D_1, \dots, D_r)\right| \geqslant M_1^{-1}.$$

Assume that $\deg P = g + 1 \geqslant 2$ and for any polynomial Q of degree $d \leqslant g$, we have

$$\left|Q(D_1, \dots, D_r)\right| \geqslant M_1^{-d}.$$

For each $t = 1, \dots, r$, if $\partial P/\partial X_t = 0$, then $|\partial P/\partial X_t(D_1, \dots, D_r)| = 0$, while if $\partial P/\partial X_t \neq 0$, we have $|\partial P/\partial X_t(D_1, \dots, D_r)| \geqslant M_1^{-g}$. Consequently, not all of the M_1^g vectors

$$\left\{\left(F_1(1), \dots, F_r(1)\right), \left(F_1(2), \dots, F_r(2)\right), \dots, \left(F_1\left(M_1^g\right), \dots, F_r\left(M_1^g\right)\right)\right\}$$

can be zero vector. Let $(F_1(m), \dots, F_r(m))$ be the first non-zero such vector. Then for $t = 1, \dots, r$,

$$1 \leqslant m \leqslant M_1^g, \qquad F_t(d) = 0, \quad \text{for } 1 \leqslant d < m.$$

By Lemma 3.1 and the minimality of m, for each $i = 1, \dots, r$, we have

$$F(p_i n_i m)\nu_{p_i}(p_i n_i m) = \nu_{p_i}(p_i n_i) \sum_{t=1}^{r} f_t(p_i n_i) F_t(m)$$

with at least one $F_t(m) \neq 0$. Since the vectors $\{(f_1(p_i n_i), \ldots, f_r(p_i n_i));\ i = 1, \ldots, r\}$ are $\mathbb{C}$-linearly independent, at least one of the values $F(p_1 n_1 m), \ldots, F(p_r n_r m)$ must be non-zero. Thus,

$$\bigl|P(D_1, \ldots, D_r)\bigr| \geqslant \frac{1}{M_1 m} \geqslant \frac{1}{M_1^{g+1}}. \qquad \square$$

A counterpart of Corollary 3.3 is:

Corollary 3.5. *Let $D(s) = \sum_{n \geqslant 1} f(n) n^{-s} \in \mathcal{D}$ and $P(X_0, \ldots, X_r) \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\}$ be of total degree g. If there are a set of $r+1$ primes $\{p_1 < p_2 < \cdots < p_{r+1}\}$ and a set of $r+1$ positive integers $\{n_1, \ldots, n_{r+1}\}$ such that*

$$f(p_i n_i) \neq 0 \quad \text{and} \quad f(p_i k) = 0 \quad \text{for } 1 \leqslant k < n_i \ (i = 1, \ldots, r+1),$$

then

$$\bigl|P\bigl(D, D', \ldots, D^{(r)}\bigr)\bigr| \geqslant M_2^{-g},$$

where $M_2 = \max\{p_1 n_1, \ldots, p_{r+1} n_{r+1}\}$, and the Dirichlet series, its derivatives and operations are considered formally.

Proof. Differentiating formally with respect to s, we have

$$D^{(j)}(s) = \sum_{n \geqslant 1} \frac{f(n)(-\log n)^j}{n^s} \quad (j \in \mathbb{N}),$$

and since

$$f(p_i n_i)(-\log p_i n_i)^j \neq 0, \qquad f(p_i k)(-\log p_i k)^j = 0 \quad \bigl(i \in \{1, \ldots, r+1\}\bigr),$$

we see that the determinant

$$\begin{vmatrix} f(p_1 n_1) & f(p_1 n_1)(-\log p_1 n_1) & \cdots & f(p_1 n_1)(-\log p_1 n_1)^r \\ \vdots & & & \vdots \\ f(p_{r+1} n_{r+1}) & f(p_{r+1} n_{r+1})(-\log p_{r+1} n_{r+1}) & \cdots & f(p_{r+1} n_{r+1})(-\log p_{r+1} n_{r+1})^r \end{vmatrix}$$
$$= f(p_1 n_1) \cdots f(p_{r+1} n_{r+1}) \begin{vmatrix} 1 & (-\log p_1 n_1) & \cdots & (-\log p_1 n_1)^r \\ \vdots & & & \vdots \\ 1 & (-\log p_{r+1} n_{r+1}) & \cdots & (-\log p_{r+1} n_{r+1})^r \end{vmatrix} \neq 0.$$

This implies that the vectors

$$\bigl(f(p_1 n_1), \ldots, f(p_{r+1} n_{r+1})\bigr), \bigl(-f(p_1 n_1) \log p_1 n_1, \ldots, -f(p_{r+1} n_{r+1}) \log p_{r+1} n_{r+1}\bigr), \ldots,$$
$$\bigl(-f(p_1 n_1)(\log p_1 n_1)^r, \ldots, -f(p_{r+1} n_{r+1})(\log p_{r+1} n_{r+1})^r\bigr)$$

are $\mathbb{C}$-linearly independent. By Theorem 3.4,

$$\bigl|P\bigl(D, D', \ldots, D^{(r)}\bigr)\bigr| \geqslant M_2^{-g}. \qquad \square$$

In connection with the Jacobian, the result of Shapiro–Sparer (Theorem 2.2 above) can be made quantitative as follows:

Corollary 3.6. *Let $f_1, \dots, f_r \in \mathcal{A}$, $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r] \setminus \{0\}$ and*

$$D_i(s) = \sum_{n \geqslant 1} \frac{f_i(n)}{n^s} \quad (i = 1, \dots, r).$$

Assume that there are a set of r primes $\{p_1 < p_2 < \cdots < p_r\}$ and a set of positive integers $\{n_1, \dots, n_r\}$ such that

$$f_t(p_i n_i) \neq 0 \quad \text{but} \quad f_t(p_i k) = 0 \quad \text{for } 1 \leqslant k < n_i \ (t = 1, \dots, r;\ i = 1, \dots, r).$$

If the value of the Jacobian

$$J(f_1, \dots, f_r; p_1, \dots, p_r)(n_1 \cdots n_r) := \begin{vmatrix} d_{p_1} f_1 & \cdots & d_{p_1} f_r \\ \vdots & & \vdots \\ d_{p_r} f_1 & \cdots & d_{p_r} f_r \end{vmatrix} (n_1 \cdots n_r)$$

(*where the product in the expansion of the determinant is taken as the convolution*) *is non-zero, then*

$$\left| P(D_1, \dots, D_r) \right| \geqslant M_1^{-g},$$

where $M_1 = \max\{p_1 n_1, \dots, p_r n_r\}$, and the Dirichlet series, their derivatives together with operations are considered formally.

Proof. By the minimality of $n_1, \dots, n_r$, we get

$$0 \neq J(f_1, \dots, f_r; p_1, \dots, p_r)(n_1 \cdots n_r) = \begin{vmatrix} d_{p_1} f_1 & \cdots & d_{p_1} f_r \\ \vdots & & \vdots \\ d_{p_r} f_1 & \cdots & d_{p_r} f_r \end{vmatrix} (n_1 \cdots n_r)$$

$$= \sum_{c_1 \cdots c_r = n_1 \cdots n_r} \prod_{i=1}^r \nu_{p_i}(p_i c_i) \prod_{i=2}^r \nu_{p_i}\left(p_i^2 c_i\right) \cdots \nu_{p_r}\left(p_r^r c_r\right) \begin{vmatrix} f_1(p_1 c_1) & \cdots & f_r(p_1 c_1) \\ \vdots & & \vdots \\ f_1(p_1 c_r) & \cdots & f_r(p_r c_r) \end{vmatrix}$$

$$= \prod_{i=1}^r \nu_{p_i}(p_i n_i) \prod_{i=2}^r \nu_{p_i}\left(p_i^2 n_i\right) \cdots \nu_{p_r}\left(p_r^r n_r\right) \begin{vmatrix} f_1(p_1 n_1) & \cdots & f_r(p_1 n_1) \\ \vdots & & \vdots \\ f_1(p_1 n_r) & \cdots & f_r(p_r n_r) \end{vmatrix},$$

and so $\det(f_t(p_i n_i))_{i,t=1}^r \neq 0$, implying that the vectors

$$\left(f_1(p_1 n_1), \dots, f_r(p_1 n_1)\right), \dots, \left(f_1(p_r n_r), \dots, f_r(p_r n_r)\right)$$

are $\mathbb{C}$-linearly independent. The desired result follows at once from Theorem 3.4. $\square$

Regarding linear dependence, using the notion of Wronskian, we have:

Corollary 3.7. *Let $f_1, \dots, f_r \in \mathcal{A}$, $D_i(s) = \sum_{n \geqslant 1} f_i(n) n^{-s}$ $(i = 1, \dots, r)$ and $P(X_1, \dots, X_r) = c_0 + \sum_{i=1}^r c_i X_i \in \mathbb{C}[X_1, \dots, X_r] \setminus \{0\}$. Assume that there is a prime p and a set of positive integers $\{n_1, \dots, n_r\}$ such that*

$$f_t\bigl(p^i n_i\bigr) \neq 0 \quad \text{but} \quad f_t\bigl(p^i k\bigr) = 0 \quad \text{for } 1 \leqslant k < n_i \ (t = 1, \dots, r;\ i \in \mathbb{N}).$$

If the value of the Wronskian

$$W(d_p f_1, \dots, d_p f_r)(n_1 \cdots n_r) := \begin{vmatrix} d_p f_1 & \cdots & d_p f_r \\ d_p^2 f_1 & \cdots & d_p^2 f_r \\ \vdots & & \vdots \\ d_p^r f_1 & \cdots & d_p^r f_r \end{vmatrix} (n_1 \cdots n_r)$$

(*where the product in the expansion of the determinant is taken as the convolution*) *is non-zero, then*

$$\bigl|P(D_1, \dots, D_r)\bigr| \geqslant M_3^{-1},$$

where $M_3 = \max\{pn_1, p^2 n_2, \dots, p^r n_r\}$, and the Dirichlet series, their derivatives together with operations are considered formally.

Proof. By the minimality of $n_1, \dots, n_r$, we get

$$\begin{aligned}
0 \neq W(d_p f_1, \dots, d_p f_r)(n_1 \cdots n_r) &= \begin{vmatrix} d_p f_1 & \cdots & d_p f_r \\ d_p^2 f_1 & \cdots & d_p^2 f_r \\ \vdots & & \vdots \\ d_p^r f_1 & \cdots & d_p^r f_r \end{vmatrix} (n_1 \cdots n_r) \\
&= \sum_{c_1 \cdots c_r = n_1 \cdots n_r} \prod_{i=1}^r \nu_p(pc_i) \prod_{i=2}^r \nu_p\bigl(p^2 c_i\bigr) \cdots \nu_p\bigl(p^r c_r\bigr) \begin{vmatrix} f_1(pc_1) & \cdots & f_r(pc_1) \\ f_1(p^2 c_2) & \cdots & f_r(p^2 c_2) \\ \vdots & & \vdots \\ f_1(p^r c_r) & \cdots & f_r(p^r c_r) \end{vmatrix} \\
&= \prod_{i=1}^r \nu_p(pc_i) \prod_{i=2}^r \nu_p\bigl(p^2 c_i\bigr) \cdots \nu_p\bigl(p^r c_r\bigr) \begin{vmatrix} f_1(pn_1) & \cdots & f_r(pn_1) \\ f_1(p^2 n_2) & \cdots & f_r(p^2 n_2) \\ \vdots & & \vdots \\ f_1(p^r n_r) & \cdots & f_r(p^r n_r) \end{vmatrix} \\
&:= \prod_{i=1}^r \nu_p(pc_i) \prod_{i=2}^r \nu_p\bigl(p^2 c_i\bigr) \cdots \nu_p\bigl(p^r c_r\bigr) \det\bigl(f_t\bigl(p^i n_i\bigr)\bigr)
\end{aligned}$$

showing that $\det(f_t(p^i n_i)) \neq 0$. Putting

$$\sum_{n \geqslant 1} \frac{F(n)}{n^s} := P(D_1, \dots, D_r) = c_0 + \sum_{i=1}^r c_i D_i = c_0 + \sum_{n \geqslant 1} \sum_{i=1}^r \frac{c_i f_i(n)}{n^s},$$

we get

$$F(n) = \sum_{i=1}^r c_i f_i(n) \quad (n \geqslant 2).$$

Since $\det(f_t(p^i n_i)) \neq 0$, the vectors $(f_1(pn_1), \ldots, f_1(p^r n_r)), \ldots, (f_r(pn_1), \ldots, f_r(p^r n_r))$ are $\mathbb{C}$-linearly independent and since the c_i's do not all vanish simultaneously, at least one of the values $F(pn_1), \ldots, F(p^r n_r)$ must be non-zero and the result follows. $\square$

4. Other cases

It is to be observed that one of the main hypotheses in Theorems 3.2 and 3.4 is the linear independence of the set of vectors of functional values at different primes. This restricts their applicability to many interesting cases, such as the independence of the formal Riemann zeta series and the formal log zeta series. However, using direct approach, in this particular case, we have the following independence measure:

Theorem 4.1. *Let $D_1 = \sum_{n\geqslant 1} \frac{f(n)}{n^s}$, $D_2 = \sum_{n\geqslant 1} \frac{g(n)}{n^s}$ be formal Dirichlet series. Assume that*

$$f(1) = f(p_1 \cdots p_r) = c_f \in \mathbb{C} \setminus \{0\} \quad (r \geqslant 1), \tag{4.1}$$

$$g(p) = c_g \in \mathbb{C} \setminus \{0\}, \qquad g(1) = g(p_1 \cdots p_s) = 0 \quad (s \geqslant 2), \tag{4.2}$$

where p and the p_i's are distinct primes. Let $P(X, Y) = \sum_{i,j} a_{ij} X^i Y^j \in \mathbb{C}[X, Y] \setminus \{0\}$ with total degree g and formally put $P(D_1, D_2) := \sum_{n\geqslant 1} F(n)/n^s \in \mathcal{D}$. Then there is a positive, absolute and computable constant c such that

$$\left|P(D_1, D_2)\right| \geqslant \left\{c^{g(g+1)} \prod_{j=2}^{g(g+1)} j \log j\right\}^{-1},$$

where the Dirichlet series and their operations are considered formally.

Proof. Formally setting the product of formal Dirichlet series

$$D_1(s)^i D_2(s)^j := \sum_{n\geqslant 1} \frac{f_{ij}(n)}{n^s}$$

and noting that this corresponds to the convolution of associated arithmetic functions, we have

$$f_{ij}(n) = \sum_{a_1 \cdots a_i b_1 \cdots b_j = n} f(a_1) \cdots f(a_i) g(b_1) \cdots g(b_j).$$

Taking $k \geqslant i + j$, $n = p_1 p_2 \cdots p_k$, where $p_1 < p_2 < \cdots < p_k$ are primes and using the assumptions (4.1) and (4.2) we get

$$f_{ij}(p_1 p_2 \cdots p_k) = i^{k-j} c_f^i j! \binom{k}{j} c_g^j.$$

Thus,

$$F(p_1 p_2 \cdots p_k) = \sum_{i=0}^{g} \sum_{j=0}^{g} a_{ij} i^{k-j} j! \binom{k}{j}.$$

The right-hand side is an exponential polynomial in k with the maximum degree in the polynomial part and the number of frequencies both being at most g. By a well-known result about the number of zeros of exponential polynomials (see e.g. the lemma in [4, Chapter 12]) the number of zeros of this exponential polynomial is at most $(g+1)g-1$ and so

$$\left|P(D_1, D_2)\right| \geqslant \{p_1 p_2 \cdots p_{(g+1)g}\}^{-1}.$$

The result now follows from Chebychev's inequality (see e.g. [1, Theorem 4.7, p. 84]) that $p_r \leqslant c_1 r \log r$ $(r \geqslant 2)$ for some computable constant c_1. □

Applying Theorem 4.1 to the case of zeta and log zeta series, we have:

Corollary 4.2. *Let $P(X, Y) = \sum_{i,j} a_{ij} X^i Y^j \in \mathbb{C}[X, Y] \setminus \{0\}$ with total degree g and put $P(\zeta, \log\zeta) := \sum_{n\geqslant 1} F(n)/n^s$. Then there is a positive, absolute and computable constant c such that*

$$\left|P(\zeta, \log\zeta)\right| \geqslant \left\{c^{g(g+1)} \prod_{j=2}^{g(g+1)} j \log j\right\}^{-1},$$

where the zeta, log zeta series and their operations are considered formally.

Proof. The result follows immediately from Theorem 4.1 through the observation that [2,14]

$$\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}, \qquad \log\zeta(s) = \sum_{n\geqslant 1} \frac{\operatorname{Log} \Lambda(n)}{n^s},$$

where Λ is the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^j \text{ is a prime positive power,} \\ 0 & \text{otherwise.} \end{cases} \qquad \Box$$

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Independence measures of arithmetic functions II

by

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1. Introduction. In our earlier work, the notion of independence measure of arithmetic functions was introduced and two main results ([3, Theorems 3.2 and 3.4]) about such measure were proved. These results are proved under the hypothesis that there is a set of distinct primes for which the set of vectors of function values at points depending on these primes is linearly independent over $\mathbb{C}$, and the proofs make use of the first assertion of [3, Lemma 3.3] where the p-basic derivation is the main tool. Our first objective here is to improve upon these results by replacing the set of primes by any set of distinct natural numbers enjoying similar properties. This is accomplished by making use of the second assertion of [3, Lemma 3.3] where the log-derivation is employed instead.

To systematize our presentation, we first recall all relevant terminology. Denote by $(\mathcal{A}, +, *)$ the unique factorization domain of arithmetic functions equipped with addition and convolution (or Dirichlet product) defined by

$$(f+g)(n) := f(n) + g(n),\ (f*g)(n) = \sum_{ij=n} f(i)g(j) \quad (f, g \in \mathcal{A},\ n \in \mathbb{N}),$$

and write $f^{*i} = f * \cdots * f$ (i terms). The convolution identity, I, is defined by $I(1) = 1$ and $I(n) = 0$ for all $n > 1$. An arithmetic function f is called a *unit* (in $\mathcal{A}$) if its convolution inverse f^{-1} exists, and this is the case if and only if $f(1) \neq 0$. It is well-known, [8, Chapter 4], that $(\mathcal{A}, +, *)$ is isomorphic to $(\mathcal{D}, +, \cdot)$, where

$$\mathcal{D} := \left\{ D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right\}$$

is the ring of formal Dirichlet series equipped with addition and multipli-

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cation, through the isomorphism $f \leftrightarrow D$; addition in both domains is the customary addition while the multiplication of formal Dirichlet series corresponds to the convolution of the appropriate arithmetic functions appearing as coefficients of formal Dirichlet series. For $f \in \mathcal{A}$, its *valuation*, [8, Chapter 4], is defined as

$$|f| := \frac{1}{O(f)},$$

where $O(f)$ is the least integer n for which $f(n) \neq 0$. Correspondingly, for a formal Dirichlet series $D(s) := \sum_{n\geq 1} f(n)/n^s$, its valuation is defined as

$$|D| = |f|,$$

where the same symbols are used for convenience. With this valuation, the isomorphism $(\mathcal{A}, +, *) \leftrightarrow (\mathcal{D}, +, \cdot)$ is indeed an isometry. Therefore, we often refer to these domains interchangeably.

A set of arithmetic functions $f_1, \ldots, f_r$ is said to be algebraically dependent over $\mathbb{C}$ or $\mathbb{C}$-*algebraically dependent* if there exists

$$P(X_1, \ldots, X_r) := \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r} X_1^{i_1} \cdots X_r^{i_r} \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\}$$

such that

$$\sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r} f_1^{*i_1} * \cdots * f_r^{*i_r} \equiv 0,$$

and $\mathbb{C}$-*algebraically independent* otherwise. If P is homogeneous of degree one in each variable, we say that $f_1, \ldots, f_r$ are $\mathbb{C}$-*linearly dependent*, and $\mathbb{C}$-*linearly independent* otherwise.

A *derivation*, [8], over $\mathcal{A}$ is a map $d : \mathcal{A} \to \mathcal{A}$ satisfying

$$d(f * g) = df * g + f * dg, \quad d(c_1 f + c_2 g) = c_1 df + c_2 dg,$$

where $f, g \in \mathcal{A}$ and $c_1, c_2 \in \mathbb{C}$. Derivations of higher orders are defined in the usual manner. Two typical examples of derivation are

- the *p-basic derivation*, p prime, defined by

  $$(d_p f)(n) = f(np)\nu_p(np) \quad (n \in \mathbb{N}),$$

  where $\nu_p(m)$ denotes the exponent of the highest power of p dividing m,
- the *log-derivation* defined by

  $$(d_L f)(n) = f(n) \log n \quad (n \in \mathbb{N}).$$

Although there are arithmetic sequences $f(n)$ for which the corresponding Dirichlet series $D(s) := \sum_n f(n)/n^s$ are divergent, through the isometry between $\mathcal{A}$ and $\mathcal{D}$, it is legitimate to define the formal derivation $\tilde{d}$ of (formal)

Dirichlet series via the derivation d of the associated arithmetic function as

$$\tilde{d}\,D(s)=\sum_{n=1}^{\infty}\frac{df(n)}{n^s}.$$

Thus, the formal differentiation of the formal Dirichlet series, $D(s)$, with respect to the variable s, i.e.,

$$D'(s)=\sum_{n=1}^{\infty}\frac{-f(n)\log n}{n^s}=\sum_{n=1}^{\infty}\frac{-(d_L f)(n)}{n^s},$$

corresponds to the (negative) log-derivation $-d_L$ of the associated arithmetic function f, and the p-basic derivation d_p over $\mathcal{A}$ corresponds to the formal p-basic derivation $\tilde{D}_p$ over $\mathcal{D}$ defined by

$$\tilde{d}_p D(s)=\sum_{n=1}^{\infty}\frac{(d_p f)(n)}{n^s}.$$

For convenience, we use the same derivation symbol d for both the domains $\mathcal{A}$ and $\mathcal{D}$. Our investigations concerning Dirichlet series will be **formal** throughout.

2. Algebraic independence. The following lemma, which plays a vital role in our investigation of algebraic independence, is Lemma 3.1 in [3].

LEMMA 2.1. *Let* $f_1,\dots,f_r\in\mathcal{A}$ *and* $P(X_1,\dots,X_r)\in\mathbb{C}[X_1,\dots,X_r]\setminus\{0\}$. *For* $t=1,\dots,r$, *define the following formal Dirichlet series*

$$D_t(s)=\sum_{n\geq 1}\frac{f_t(n)}{n^s},$$

$$P(D_1,\dots,D_r)=\sum_{n\geq 1}\frac{F(n)}{n^s},\qquad \frac{\partial P}{\partial X_t}(D_1,\dots,D_r)=\sum_{n\geq 1}\frac{F_t(n)}{n^s}.$$

Then for each $n\in\mathbb{N}$ *and for each prime* p, *we have*

$$F(pn)\nu_p(pn)=\sum_{j=1}^{r}\sum_{k|n}f_j(pk)F_j(\frac{n}{k})\nu_p(pk), \tag{2.1}$$

$$F(n)\log n=\sum_{j=1}^{r}\sum_{k|n}f_j(k)F_j(\frac{n}{k})\log k, \tag{2.2}$$

where the Dirichlet series and their operations are considered formally.

Our improvement of [3, Theorem 3.2] is

THEOREM 2.2. *In the notation of Lemma* 2.1, *suppose that* $P(X_1,\dots,X_r)$ *is of total degree* $\deg P=g$. *If there is a set of* r *positive integers* $\{(1<)\,n_1<$

$\cdots < n_r\}$ such that the set of vectors

$$\{(f_1(n_i),\ldots,f_r(n_i)) : i = 1,\ldots,r\}$$

is linearly independent over $\mathbb{C}$, then

$$|P(D_1,\ldots,D_r)| \geq n_r^{-g}.$$

Proof. If $\deg P = 0$, then clearly $|P(D_1,\ldots,D_r)| = 1$. If $\deg P = 1$, then

$$P(X_1,\ldots,X_r) = a_0 I + a_1 X_1 + \cdots + a_r X_r,$$

where the coefficients a_j $(j = 1,\ldots,r)$ do not vanish simultaneously. Equating coefficients, we get

$$F(n_j) = a_1 f_1(n_j) + \cdots + a_r f_r(n_j).$$

Since the set $\{(f_1(n_j),\ldots,f_r(n_j)) : j = 1,\ldots,r\}$ is linearly independent over $\mathbb{C}$, at least one of the values $F(n_1),\ldots,F(n_r)$ must be nonzero, which renders

$$|P(D_1,\ldots,D_r)| \geq n_r^{-1}.$$

Now proceed by induction on $\deg P$. Let P be of total degree $g+1 \geq 2$, and assume that the assertion has been proved for polynomials of degree $\leq g$. Consider the polynomials $\partial P/\partial X_t$ $(t = 1,\ldots,r)$, of degree $\leq g$. Unless $\partial P/\partial X_t$ vanishes identically, by induction we have

$$\left|\frac{\partial P}{\partial X_t}(D_1,\ldots,D_r)\right| \geq n_r^{-g},$$

which implies that the n_r^g vectors

$$(F_1(1),\ldots,F_r(1)),(F_1(2),\ldots,F_r(2)),\ldots,(F_1(n_r^g),\ldots,F_r(n_r^g)) \tag{2.3}$$

cannot all be zero. Let $(F_1(m),\ldots,F_r(m))$ be the first nonzero vector in (2.3) so that

$$(F_1(d),\ldots,F_r(d)) = (0,\ldots,0) \quad \text{for } d = 1,\ldots,m-1.$$

By the minimality of m and Lemma 2.1, for all $i = 1,\ldots,r$ we get

$$F(n_i m)\log(n_i m) = f_1(n_i)F_1(m) + \cdots + f_r(n_i)F_r(m).$$

Since the set $\{(f_1(n_j),\ldots,f_r(n_j)) : j = 1,\ldots,r\}$ is linearly independent over $\mathbb{C}$, at least one of $F(n_1 m),\ldots,F(n_r m)$ must be nonzero. This yields

$$|P(D_1,\ldots,D_r)| \geq (mn_r)^{-1} \geq (n_r^{g+1})^{-1}. \blacksquare$$

Recall that a formal Dirichlet series $D(s)$ is said to be *differentially algebraic* of order $r \in \mathbb{N}_0$ if D together with all its derivatives (up to order r) $D',\ldots,D^{(r)}$ $(D^{(0)} := D)$ satisfy a non-trivial algebraic equation with complex coefficients. When $r = 0$, differentially algebraic series of order 0 are simply algebraic series. The notion of differentially algebraic arithmetic functions is defined correspondingly. An immediate consequence of Theorem 2.2 is the following measure of differentially algebraic independence.

COROLLARY 2.3. *Let $D(s) = \sum_{n\geq 1} f(n)n^{-s} \in \mathcal{D}$ and $P(X_0,\dots,X_r) \in \mathbb{C}[X_1,\dots,X_r] \setminus \{0\}$ be of total degree g. For $r \in \mathbb{N}_0$, if there is a set of $r+1$ natural numbers $\{(1 <)\ n_1 < n_2 < \cdots < n_{r+1}\}$ such that $f(n_i) \neq 0$ $(i = 1,\dots,r+1)$, then*

$$|P(D, D', \dots, D^{(r)})| \geq (n_{r+1}^g)^{-1},$$

where the Dirichlet series, their derivatives and operations are considered formally.

Proof. Formally differentiating j times the Dirichlet series with respect to s, we get

$$D^{(j)}(s) = \sum_{n\geq 1} \frac{f(n)(-\log n)^j}{n^s}.$$

For each $i \in \{1,\dots,r+1\}$, since $f(n_i)(-\log n_i)^j \neq 0$, the determinant

$$\begin{vmatrix} f(n_1) & f(n_1)(-\log n_1) & \cdots & f(n_1)(-\log n_1)^r \\ \vdots & & & \vdots \\ f(n_{r+1}) & f(n_{r+1})(-\log n_{r+1}) & \cdots & f(n_{r+1})(-\log n_{r+1})^r \end{vmatrix}$$
$$= f(n_1)\cdots f(n_{r+1}) \begin{vmatrix} 1 & (-\log n_1) & \cdots & (-\log n_1)^r \\ \vdots & & & \vdots \\ 1 & (-\log n_{r+1}) & \cdots & (-\log n_{r+1})^r \end{vmatrix},$$

being Vandermonde, is nonzero, and so the set of vectors

$$\{(f(n_1), f(n_1)(-\log n_1), \dots, f(n_1)(-\log n_1)^r), \dots,$$
$$(f(n_{r+1}), f(n_{r+1})\log n_{r+1}, \dots, f(n_{r+1})(-\log n_{r+1})^r)\}$$

is $\mathbb{C}$-linearly independent. The assertion now follows from Theorem 2.2. ∎

Corollary 2.3 reveals an interesting feature of differentially algebraic arithmetic functions:

COROLLARY 2.4. *Let $r \in \mathbb{N}_0$. If $f \in \mathcal{A}$ is differentially algebraic of order r, then excluding the point 1 it can be nonzero at r distinct points at most.*

Observe that the result of Corollary 2.4 when $r = 0$ is identical with that of [3, Proposition 2.1 part 1]. An even more amazing consequence of Corollary 2.4 is the next result which substantially generalizes an old theorem of Hilbert [1] stating that the Riemann zeta function does not satisfy any algebraic differential equation over $\mathbb{C}$; Ostrowski [6] showed more generally that the Riemann zeta function does not satisfy any algebraic differential-difference equation over $\mathbb{C}$.

COROLLARY 2.5. *An arithmetic function which is nonzero at infinitely many points is not differentially algebraic, i.e., it is* hyper-transcendental, *or equivalently, every formal Dirichlet series which is not a Dirichlet polynomial is hyper-transcendental.*

The next corollary yields a measure of algebraic independence for appropriate lacunary arithmetic functions.

COROLLARY 2.6. *In the notation of Lemma 2.1, suppose that* $P(X_1, \ldots, X_r)$ *is of total degree* g. *If there is a finite sequence of positive integers* $\{m_1 < \cdots < m_r\}$ *such that for* $t \in \{1, \ldots, r\}$ *we have*

$$f_t(m_t) \neq 0 \quad \textit{but} \quad f_t(k) = 0 \textit{ for } k \in \{1, \ldots, m_r\} \setminus \{m_t\},$$

then

$$|P(D_1, \ldots, D_r)| \geq n_r^{-g}.$$

Proof. The result follows from Theorem 2.2 by noting that the set

$$\{(f_1(m_t), \ldots, f_r(m_t)) : t = 1, \ldots, r\}$$

is $\mathbb{C}$-linearly independent. ∎

Corollary 2.6 leads at once to the next result which says that lacunary arithmetic functions are roughly $\mathbb{C}$-algebraically independent.

COROLLARY 2.7. *Let* $f_1, \ldots, f_r \in \mathcal{A}$. *If there are* r *sequences of positive integers*

$$\{n_1^{(t)} < n_2^{(t)} < \cdots\} \quad (t = 1, \ldots, r)$$

such that for $t \in \{1, \ldots, r\}$ *we have*

$$f_t(n_j^{(t)}) \neq 0, \quad \textit{but}$$

$$f_t(k) = 0 \quad \textit{for } k \in \{1, \ldots, n_1^{(t)} - 1\} \cup \bigcup_{j=1}^{\infty} \{n_j^{(t)} + 1, \ldots, n_{j+1}^{(t)} - 1\},$$

then $f_1, \ldots, f_r$ *are* $\mathbb{C}$*-algebraically independent.*

We end this section by comparing two measures of independence from [3] with those obtained via Theorem 2.2. Let $\{F_n\}_{n \geq 1}$ be the sequence of *Fibonacci numbers* defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

The six formal *Fibonacci zeta series* are defined as (see [2])

$$\mathcal{F}^+(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s} = \sum_{n=1}^{\infty} \frac{f^+(n)}{n^s}, \qquad \mathcal{F}_e^+(s) := \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s} = \sum_{n=1}^{\infty} \frac{f_e^+(n)}{n^s},$$

$$\mathcal{F}_o^+(s) := \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{f_o^+(n)}{n^s}, \qquad \mathcal{F}^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n^s} = \sum_{n=1}^{\infty} \frac{f^-(n)}{n^s},$$

$$\mathcal{F}_e^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n}^s} = \sum_{n=1}^{\infty} \frac{f_e^-(n)}{n^s}, \quad \mathcal{F}_o^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{f_o^-(n)}{n^s},$$

Let $\{L_n\}_{n\geq 1}$ be the sequence of *Lucas numbers* defined by

$$L_1 = 1, \qquad L_2 = 3, \qquad L_{n+2} = L_{n+1} + L_n \quad (n \in \mathbb{N}).$$

The six formal *Lucas zeta series* are defined as

$$\mathcal{L}^+(s) := \sum_{n=1}^{\infty} \frac{1}{L_n^s} = \sum_{n=1}^{\infty} \frac{\ell^+(n)}{n^s}, \qquad \mathcal{L}_e^+(s) := \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s} = \sum_{n=1}^{\infty} \frac{\ell_e^+(n)}{n^s},$$

$$\mathcal{L}_o^+(s) := \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{\ell_o^+(n)}{n^s}, \qquad \mathcal{L}^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n^s} = \sum_{n=1}^{\infty} \frac{\ell^-(n)}{n^s},$$

$$\mathcal{L}_e^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n}^s} = \sum_{n=1}^{\infty} \frac{\ell_e^-(n)}{n^s}, \quad \mathcal{L}_o^-(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}^s} = \sum_{n=1}^{\infty} \frac{\ell_o^-(n)}{n^s},$$

In [3, p. 10], it was shown that

$$|P(\mathcal{F}^+, \mathcal{F}_e^+, \mathcal{F}_o^-)| \geq 5^{-g}, \tag{2.4}$$

for any $P(X_1, X_2, X_3) \in \mathbb{C}[X_1, X_2, X_3] \setminus \{0\}$ of total degree g, and

$$|Q(\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}_e^-, \mathcal{L}_o^-)| \geq 29^{-g}, \tag{2.5}$$

for any $Q(X_1, X_2, X_3, X_4) \in \mathbb{C}[X_1, X_2, X_3, X_4] \setminus \{0\}$ of total degree g. Since

$$\begin{vmatrix} f^+(1 = F_1) & f^+(2 = F_3) & f^+(3 = F_4) \\ f_e^+(1 = F_2) & f_e^+(2 = F_3) & f_e^+(3 = F_4) \\ f_o^-(1 = F_1) & f_o^-(2 = F_3) & f_o^-(3 = F_4) \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2 \neq 0,$$

the set of three vectors

$$\{(f^+(1), f^+(2), f^+(3)), (f_e^+(1), f_e^+(2), f_e^+(3)), (f_o^-(1), f_o^-(2), f_o^-(3))\}$$

is $\mathbb{C}$-linearly independent and Theorem 2.2 yields

$$|P(\mathcal{F}^+, \mathcal{F}_e^+, \mathcal{F}_o^-)| \geq 3^{-g}, \tag{2.6}$$

which is much better than (2.4). A simple example of linear polynomials such as

$$P(n)\ (= P(f^+, f_e^+, f_o^-)(n)) := f^+(n) - 2f_e^+(n) + f_o^-(n)$$

shows that $P(1) = P(2) = 0, P(3) = -1 \neq 0$, i.e., the bound in (2.6) is best possible.

Since

$$\begin{vmatrix} \ell^+(1 = L_1) & \ell^+(3 = L_2) & \ell^+(4 = L_3) & \ell^+(7 = L_4) \\ \ell^-(1 = L_1) & \ell^-(3 = L_2) & \ell^-(4 = L_3) & \ell^-(7 = L_4) \\ \ell_e^-(1 = L_1) & \ell_e^-(3 = L_2) & \ell_e^-(4 = L_3) & \ell_e^-(7 = L_4) \\ \ell_o^-(1 = L_1) & \ell_o^-(3 = L_2) & \ell_o^-(4 = L_3) & \ell_o^-(7 = L_4) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{vmatrix}$$
$$= 8 \neq 0,$$

the set of four vectors

$$\{(\ell^+(1), \ell^+(3), \ell^+(4), \ell^+(7)), (\ell^-(1), \ell^-(3), \ell^-(4), \ell^-(7)),$$
$$(\ell_e^-(1), \ell_e^-(3), \ell_e^-(4), \ell_e^-(7)), (\ell_o^-(1), \ell_o^-(3), \ell_o^-(4), \ell_o^-(7))\}$$

is $\mathbb{C}$-linearly independent and Theorem 2.2 yields

$$|Q(\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}_e^-, \mathcal{L}_o^-)| \geq 7^{-g}, \tag{2.7}$$

much better than (2.5). Again a simple example of linear polynomials such as

$$Q(n) \ (= Q(\ell^+, \ell^-, \ell_e^-, \ell_o^-)(n)) := \ell^+(n) - \ell^-(n) - 2\ell_e^-(n) + 0 \cdot \ell_o^-(n)$$

shows that $Q(1) = \cdots = Q(6) = 0$ and $Q(7) = 4 \neq 0$, i.e., the bound in (2.7) is best possible.

3. Linear dependence and Wronskian. Motivated by the case of real functions, in this section, we investigate the connection between linear dependence and the notion of Wronskian in our arithmetic setting. We start with a simple proposition, whose converse, which is much more difficult, will be examined later.

PROPOSITION 3.1. *Let $f_1, \ldots, f_r \in \mathcal{A}$ and let d be a derivation on $\mathcal{A}$. If $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent, then their Wronskian relative to d,*

$$W_d(f_1, \ldots, f_r) := \begin{vmatrix} f_1 & f_2 & \ldots & f_r \\ df_1 & df_2 & \ldots & df_r \\ \vdots & & & \\ d^{r-1}f_1 & d^{r-1}f_2 & \ldots & d^{r-1}f_r \end{vmatrix},$$

vanishes; here and throughout, the multiplication involved in the determinant expansion is the Dirichlet product.

Proof. Taking the derivations d^i for $i = 1, \ldots, r-1$ in the linear relation among $f_1, \ldots, f_r$, with coefficients $c_1, \ldots, c_r$ not all zero, we get a system of linear equations in the c_i's whose determinant is the Wronskian considered and the existence of nontrivial solutions forces the vanishing of this determinant. ■

The next result gives a sufficient condition for linear dependence.

THEOREM 3.2. *Let $f_1, \ldots, f_r \in \mathcal{A}$. If the set of positive integers $\{n_1 < \cdots < n_r\}$ is such that*

$$f_t(n_t) \neq 0 \quad \textit{but} \quad f_t(k) = 0 \textit{ for } \quad k = 1, \ldots, n_t - 1 \quad (t = 1, \ldots, r),$$

then the Wronskian (with respect to the log-derivation)

$$W_L(f_1, \ldots, f_r) := \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix}$$

(where the product in the expansion of the determinant is convolution) does not vanish, and so $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly independent.

Proof. By the minimality of $n_1, \ldots, n_r$, we get

$$W_L(f_1, \ldots, f_r)(n_1 \cdots n_r) = \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix} (n_1 \cdots n_r)$$

$$= \sum_{c_1 \cdots c_r = n_1 \cdots n_r} \begin{vmatrix} f_1(c_1) & \cdots & f_r(c_r) \\ f_1(c_1)\log(c_1) & \cdots & f_r(c_r)\log(c_r) \\ \vdots & & \vdots \\ f_1(c_1)\log^{r-1}(c_1) & \cdots & f_r(c_r)\log^{r-1}(c_r) \end{vmatrix}$$

$$= \begin{vmatrix} f_1(n_1) & \cdots & f_r(n_r) \\ f_1(n_1)\log(n_1) & \cdots & f_r(n_r)\log(n_r) \\ \vdots & & \vdots \\ f_1(n_1)\log^{r-1}(n_1) & \cdots & f_r(n_r)\log^{r-1}(n_r) \end{vmatrix}$$

$$= f_1(n_1) f_2(n_2) \cdots f_r(n_r) \begin{vmatrix} 1 & \cdots & 1 \\ \log(n_1) & \cdots & \log(n_r) \\ \vdots & & \vdots \\ \log^{r-1}(n_1) & \cdots & \log^{r-1}(n_r) \end{vmatrix} \neq 0. \ ■$$

Recall that the *norm* $N(f)$ of $f \in \mathcal{A}$ is defined as

$$N(f) = \min\{n \in \mathbb{N} : f(n) \neq 0\}.$$

Theorem 3.2 simply says that arithmetic functions whose norms are distinct are necessarily $\mathbb{C}$-linearly independent. This is worth comparing with Theorem 7 of [7] which asserts that the set of nonunit arithmetic functions whose norms are pairwise relatively prime is $\mathbb{C}$-algebraically independent.

For future use, we pause to establish an identity involving the Wronskian value evaluated at a general point.

THEOREM 3.3. *Let $f_1, \ldots, f_r \in \mathcal{A}$ and let $n \in \mathbb{N}$. Then*

$$W_L(f_1, \ldots, f_r)(n) := \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix}(n)$$

$$= \sum_{n_1 \cdots n_r = n;\, n_1 < \cdots < n_r} \left(\prod_{1 \le i < j \le r} (\log n_j - \log n_i) \right) \begin{vmatrix} f_1(n_1) & \cdots & f_1(n_r) \\ f_2(n_1) & \cdots & f_2(n_r) \\ \vdots & & \vdots \\ f_r(n_1) & \cdots & f_r(n_r) \end{vmatrix}.$$

Proof. We have

$$W_L(f_1, \ldots, f_r)(n) = \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix}(n)$$

$$= \sum_{c_1 \cdots c_r = n} \begin{vmatrix} f_1(c_1) & \cdots & f_r(c_r) \\ f_1(c_1)\log(c_1) & \cdots & f_r(c_r)\log(c_r) \\ \vdots & & \vdots \\ f_1(c_1)\log^{r-1}(c_1) & \cdots & f_r(c_r)\log^{r-1}(c_r) \end{vmatrix}$$

$$= \sum_{c_1 \cdots c_r = n} f_1(c_1) \cdots f_r(c_r) \begin{vmatrix} 1 & \cdots & 1 \\ \log(c_1) & \cdots & \log(c_r) \\ \vdots & & \vdots \\ \log^{r-1}(c_1) & \cdots & \log^{r-1}(c_r) \end{vmatrix}$$

$$= \sum_{c_1 \cdots c_r = n} f_1(c_1) \cdots f_r(c_r) \prod_{1 \le i < j \le r} (\log c_j - \log c_i)$$

$$= \sum_{n_1\cdots n_r=n;\, n_1<\cdots<n_r} \Big(\prod_{1\le i<j\le r} (\log n_j - \log n_i) \Big)$$
$$\times \sum_{i_1,\dots,i_r} \epsilon(n_{i_1},\dots,n_{i_r}) f_1(n_{i_1})\cdots f_r(n_{i_r})$$

where the inner sum on the right hand side is taken over all permutations of $(n_1,\dots,n_r)$ with $\epsilon(n_{i_1},\dots,n_{i_r}) = 1$ for an even permutation and -1 for an odd one. Thus,

$$W_L(f_1,\dots,f_r)(n_1\cdots n_r)$$
$$= \sum_{n_1\cdots n_r=n;\, n_1<\cdots<n_r} \Big(\prod_{1\le i<j\le r} (\log n_j - \log n_i) \Big) \begin{vmatrix} f_1(n_1) & \cdots & f_1(n_r) \\ f_2(n_1) & \cdots & f_2(n_r) \\ \vdots & & \vdots \\ f_r(n_1) & \cdots & f_r(n_r) \end{vmatrix}. \blacksquare$$

In the real case it is well-known (see e.g. [4]) that the converse of Proposition 3.1 is not generally true. This is also the case in the arithmetic function setting. For example, consider the two arithmetic functions

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad g(n) = \begin{cases} 1 & \text{if } n = q \ne p, \\ 0 & \text{otherwise,} \end{cases}$$

where $q \ne p$ are primes. If $c_1 I + c_2 g = 0$ $(c_1, c_2 \in \mathbb{C})$, then

$$0 = c_1 I(1) + c_2 g(1) = c_1, \qquad 0 = c_1 I(q) + c_2 g(q) = c_2,$$

showing that I and g are $\mathbb{C}$-linearly independent. However, their Wronskian relative to the p-basic derivation d_p does vanish:

$$W(I,g)(n) = \begin{vmatrix} I & g \\ d_p I & d_p g \end{vmatrix}(n)$$
$$= \sum_{ij=n} \{ I(i) g(jp) \nu_p(jp) - g(i) I(jp) \nu_p(jp) \} = 0 \qquad (n \in \mathbb{N}).$$

The converse of Theorem 3.1 does indeed hold if we stick to the log-derivation.

THEOREM 3.4. *Let $f_1,\dots,f_r \in \mathcal{A}\backslash\{0\}$. If their Wronskian $W = W_L(f_1, \dots, f_r)$ relative to the log-derivation vanishes identically, then $f_1,\dots,f_r$ are $\mathbb{C}$-linearly dependent.*

Proof. For brevity write d for d_L. First we consider the case $r = 2$. We consider two cases.

CASE 1: $f_1(1) \ne 0$. Then f_1^{-1}, the convolution inverse of f_1, exists and so

$$0 = W_L(f_1, f_2) = W_L(f_1 * f_1 * f_1^{-1}, f_2 * f_1 * f_1^{-1}) = f_1^2 * W_L(I, f_2 * f_1^{-1}),$$

yielding

$$0 = W_L(I, f_2 * f^{-1}) = d(f_2 * f_1^{-1}).$$

Thus, $f_2 * f_1^{-1} = cI$ for some $c \in \mathbb{C}$, i.e., $f_2 = cf_1$, showing that f_1 and f_2 are $\mathbb{C}$-linearly dependent.

CASE 2: $f_1(1) = f_2(1) = 0$. Since $f_1 \not\equiv 0$, let $N > 1$ be the least positive integer for which $f_1(N) \neq 0$. For $n \in \mathbb{N}$, we have

$$0 = W_L(f_1, f_2)(n) = \sum_{ab=n} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b.$$

Putting $n = 2N$, we get

$$0 = W_L(f_1, f_2)(2N) = f_1(N)f_2(2)(\log 2 - \log N),$$

i.e., $f_2(2) = 0$. By induction, for $k = 1, \dots, N-1$, we have

$$0 = W_L(f_1, f_2)(kN) = f_1(N)f_2(k)(\log k - \log N),$$

i.e., $f_2(k) = 0$. Putting $n = N^2$ and using the previously found values, we get

$$0 = W_L(f_1, f_2)(N^2) = (f_1(N)f_2(N) - f_1(N)f_2(N)) \log N,$$

yielding $f_2(N)$ arbitrary. Putting $n = N(N+1)$ and using the previously found values, we get

$$\begin{aligned} 0 &= W_L(f_1, f_2)(N(N+1)) \\ &= (f_1(N)f_2(N+1) - f_1(N+1)f_2(N)) \log(N+1), \end{aligned}$$

i.e., $f_2(N+1) = f_1(N+1)f_2(N)/f_1(N)$. In general, for $m \geq 1$, using previously found values, we have

$$\begin{aligned} 0 = W_L(f_1, f_2)(N(N+m)) &= \sum_{ab=N(N+m)} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b \\ = &\sum_{\substack{ab=N(N+m) \\ a<N}} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b \\ &+ \{f_1(N)f_2(N+m) - f_2(N)f_1(N+m)\} \log(N+m) \\ &+ \sum_{\substack{ab=N(N+m) \\ N<a\leq N+m}} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b \\ = &\{f_1(N)f_2(N+m) - f_2(N)f_1(N+m)\} \log(N+m), \end{aligned}$$

i.e., $f_2(N+m) = f_1(N+m)f_2(N)/f_1(N)$. Hence, $f_2 = cf_1$, where $c := f_2(N)/f_1(N)$.

Supposing that the assertion of the theorem holds for up to $r - 1$ (≥ 2) functions, we proceed to verify it for r functions. We again have two cases.

CASE 1: there is an $i \in \{1, \dots, r\}$ for which $f_i(1) \neq 0$. We may assume that $f_1(1) \neq 0$. Then f_1^{-1} exists and so

$$\begin{aligned} 0 = W_L(f_1, \dots, f_r) &= f_1^r * W_L(I, f_2 * f_1^{-1}, \dots, f_r * f_1^{-1}) \\ &= f_1^r * W_L(d(f_2 * f_1^{-1}), \dots, d(f_r * f_1^{-1})). \end{aligned}$$

By the induction hypothesis, $d(f_2 * f_1^{-1}), \dots, d(f_r * f_1^{-1})$ are $\mathbb{C}$-linearly dependent, which implies that so are $f_1, \dots, f_r$.

CASE 2: $f_1(1) = \cdots = f_r(1) = 0$. For brevity, write

$$A(i) = (f_1(i), \dots, f_r(i)).$$

Thus, $A(1) = (0, \dots, 0)$. Since $f_1, \dots, f_r \in \mathcal{A} \setminus \{0\}$, let N_1 be the least positive integer such that

$$A(N_1) \neq (0, \dots, 0).$$

There are two subcases.

SUBCASE 1: All the vectors $A(n)$ with $n > N_1$ are $\mathbb{C}$-multiples of $A(N_1)$, so there exist $c(n) \in \mathbb{C}$ such that $A(n) = c(n)A(N_1)$, i.e.,

$$f_1(n) = c(n) f_1(N_1), \quad \dots, \quad f_r(n) = c(n) f_r(N_1).$$

Observe that the (single) linear equation in r (≥ 4) unknowns $x_1, \dots, x_r$,

$$0 = x_1 f_1(N_1) + \cdots + x_r f_r(N_1),$$

has a nontrivial solution $(x_1, \dots, x_r) \neq (0, \dots, 0)$. This shows that

$$x_1 f_1(n) + \cdots + x_r f_r(n) = 0 \quad \text{for all } n \in \mathbb{N},$$

i.e., $f_1, \dots, f_r$ are $\mathbb{C}$-linearly dependent.

SUBCASE 2: There exists a least positive integer N_2 $(> N_1)$ such that $A(N_1), A(N_2)$ are $\mathbb{C}$-linearly independent. Again we treat two possibilities.

If all the vectors $A(n)$ with $n > N_2$ are $\mathbb{C}$-linear combinations of $A(N_1)$ and $A(N_2)$, so there exist $c_1(n), c_2(n) \in \mathbb{C}$ such that $A(n) = c_1(n)A(N_1) + c_2(n)A(N_2)$, i.e.,

$$f_1(n) = c_1(n) f_1(N_1) + c_2(n) f_1(N_2), \quad \dots, \quad f_r(n) = c_1(n) f_r(N_1) + c_2(n) f_r(N_2),$$

then the system of two equations in r (≥ 4) unknowns $x_1, \dots, x_r$,

$$\begin{aligned} 0 &= x_1 f_1(N_1) + \cdots + x_r f_r(N_1), \\ 0 &= x_1 f_1(N_2) + \cdots + x_r f_r(N_2), \end{aligned}$$

has a nontrivial solution $(x_1, \dots, x_r) \neq (0, \dots, 0)$. Then

$$x_1 f_1(n) + \cdots + x_r f_r(n) = 0 \quad \text{for all } n \in \mathbb{N},$$

showing that $f_1, \dots, f_r$ are $\mathbb{C}$-linearly dependent.

Otherwise, there exists a least positive integer N_3 $(> N_2 > N_1)$ such that $A(N_1), A(N_2), A(N_3)$ are $\mathbb{C}$-linearly independent and we continue as above. In general, assume that there is a set of $1 \leq j$ ($\leq r-1$) (lexicographically least) positive integers $N_1 < \cdots < N_j$ such that the vectors $A(N_1), \ldots, A(N_j)$ are $\mathbb{C}$-linearly independent. If all the vectors $A(n)$ with $n > N_j$ ($> N_{j-1} > \cdots > N_1$) are $\mathbb{C}$-linear combinations of $A(N_1), \ldots, A(N_j)$, so there exist $c_1(n), \ldots, c_j(n) \in \mathbb{C}$ such that $A(n) = c_1(n)A(N_1) + \cdots + c_j(n)A(N_j)$, i.e.,

$$f_1(n) = c_1(n)f_1(N_1) + \cdots + c_j(n)f_1(N_j), \ \ldots,$$
$$f_r(n) = c_1(n)f_r(N_1) + \cdots + c_j(n)f_r(N_j),$$

then the system of j ($\leq r-1$) equations in r unknowns $x_1, \ldots, x_r$,

$$\begin{aligned} 0 &= x_1 f_1(N_1) + \cdots + x_r f_r(N_1), \\ &\ \vdots \\ 0 &= x_1 f_1(N_j) + \cdots + x_r f_r(N_j), \end{aligned}$$

has a nontrivial solution $(x_1, \ldots, x_r) \neq (0, \ldots, 0)$. Then

$$x_1 f_1(n) + \cdots + x_r f_r(n) = 0 \quad \text{for all } n \in \mathbb{N},$$

showing that $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent.

There remains the case where there are (lexicographically) least positive integers $N_1 < \cdots < N_r$ such that $A(N_1), \ldots, A(N_r)$ are $\mathbb{C}$-linearly independent and so

$$\begin{vmatrix} f_1(N_1) & \cdots & f_1(N_r) \\ f_2(N_1) & \cdots & f_2(N_r) \\ \vdots & & \vdots \\ f_r(N_1) & \cdots & f_r(N_r) \end{vmatrix} \neq 0. \tag{3.1}$$

Using Theorem 3.3 together with the (lexicographically) minimal property of $N_1 < \cdots < N_r$, the hypothesis that the Wronskian vanishes shows that so does the determinant on the left hand side of (3.1). This contradiction finishes the proof. ∎

Proposition 3.1 together with Theorem 3.4 provides us with a satisfactory necessary and sufficient condition for $\mathbb{C}$-linear dependence of arithmetic functions through the use of Wronskian. This should be compared with the use of Jacobian for testing $\mathbb{C}$-algebraic independence in [9], which only works in one direction. Though Proposition 3.1 and Theorem 3.4 are not so easy to use, they do yield several independence tests; we next give an example.

THEOREM 3.5. *Let* $\alpha, \beta \in \mathbb{N}$ *and*

$$S = \{s_1, \ldots, s_\alpha\} \subseteq \mathbb{C}, \qquad K = \{0 \leq k_1 \leq \cdots \leq k_\beta\} \subseteq \mathbb{N}_0,$$
$$T = \{f_{s,k} : s \in S,\ k \in K\} \subseteq \mathcal{A},$$

with $f_{s,k}(1) \neq 0$ $(s \in S,\, k \in K)$. *Assume that for all sufficiently large primes* p,

(1) $f_{s,k}(p) \neq 0$ $(s \in S,\, k \in K)$;

(2) $\displaystyle\lim_{p\to\infty} \frac{f_{s,k_i}(p)}{f_{s,k_u}(p)} = 0$ *for* $1 \le i < u \le \beta$;

(3) $\displaystyle\lim_{p\to\infty} \frac{f_{s_j,k_a}(p)}{f_{s_v,k_b}(p)} = 0$ *for* $1 \le j < v \le \alpha$ *and* $a, b \in \{1, \ldots, \beta\}$.

Then the elements of T *are* $\mathbb{C}$*-linearly independent.*

Proof. Suppose that the elements of T are $\mathbb{C}$-linearly dependent and so their Wronskian vanishes by Proposition 3.1. Write W for

$$W_L(f_{s_1,k_1}, \ldots, f_{s_1,k_\beta}, \ldots, f_{s_\alpha,k_1}, \ldots, f_{s_\alpha,k_\beta}).$$

Let $A(i) = (f_{s_1,k_1}(i) \ \cdots\ f_{s_1,k_\beta}(i) \ \cdots\ f_{s_\alpha,k_1}(i) \ \cdots\ f_{s_\alpha,k_\beta}(i))$, and

$$\det(A(i_0), A(i_1), \cdots, A(i_{r-1}))$$
$$:= \begin{vmatrix} f_{s_1,k_1}(i_0) & \ldots & f_{s_1,k_\beta}(i_0) & \ldots & f_{s_\alpha,k_1}(i_0) & \ldots & f_{s_\alpha,k_\beta}(i_0) \\ f_{s_1,k_1}(i_1) & \ldots & f_{s_1,k_\beta}(i_1) & \ldots & f_{s_\alpha,k_1}(i_1) & \ldots & f_{s_\alpha,k_\beta}(i_1) \\ \vdots & & & & & & \\ f_{s_1,k_1}(i_{r-1}) & \ldots & f_{s_1,k_\beta}(i_{r-1}) & \ldots & f_{s_\alpha,k_1}(i_{r-1}) & \ldots & f_{s_\alpha,k_\beta}(i_{r-1}) \end{vmatrix},$$

where $r = \alpha\beta$. Then, for $\nu \in \mathbb{N}$,

$$W(\nu) = \sum_{i_0 i_1 \cdots i_{r-1} = \nu} (\log i_1)(\log i_2)^2 \ldots (\log i_{r-1})^{r-1} \det(A(i_0), A(i_1), \cdots, A(i_{r-1})).$$

Taking $\nu = p_1 \cdots p_{r-1}$, where $p_1 < \cdots < p_{r-1}$ are distinct primes, we get

$$W(p_1 \cdots p_{r-1}) = C(p_1, \ldots, p_{r-1}) \det(A(1), A(p_1), \ldots, A(p_{r-1})),$$

where $C(p_1, \ldots, p_{r-1}) \neq 0$ is the Vandermonde determinant defined by

$$C(i_1, \ldots, i_{r-1}) = \begin{vmatrix} \log i_1 & \log i_2 & \ldots & \log i_{r-1} \\ (\log i_1)^2 & (\log i_2)^2 & \ldots & (\log i_{r-1})^2 \\ \vdots & & & \\ (\log i_1)^{r-1} & (\log i_2)^{r-1} & \ldots & (\log i_{r-1})^{r-1} \end{vmatrix} \tag{3.2}$$
$$= \sum_\sigma \operatorname{sgn}(\sigma)(\log i_{\sigma(1)})^1 (\log i_{\sigma(2)})^2 (\log i_{\sigma(3)})^3 \cdots (\log i_{\sigma(r-1)})^{r-1},$$

where the summation is over all permutations σ of $\{1, \ldots, r-1\}$ with $\operatorname{sgn}(\sigma) = \pm 1$ depending on whether σ is even or odd. Since $C(i_1, \ldots, i_{r-1})$ is a Vandermonde determinant, we have $C(i_1, \ldots, i_{r-1}) \neq 0$ if and only if

$i_1, \dots, i_{r-1}$ are distinct and not equal to 1. We wish to derive a contradiction by showing that there are primes $p_1, \dots, p_{r-1}$ such that

$$D := \det(A(1), A(p_1), \dots, A(p_{r-1})) \neq 0.$$

For primes $p_1, \dots, p_{r-1}$ sufficiently large, since the function values are non-zero by condition (1), we can write

$$D = f_{s_\alpha,k_\beta}(p_{r-1}) f_{s_\alpha,k_\beta}(p_{r-2}) \cdots f_{s_\alpha,k_\beta}(p_1) f_{s_\alpha,k_\beta}(1) D^*,$$

where

$$D^* = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(1)}{f_{s_\alpha,k_\beta}(1)} & 1 \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & 1 \\ \vdots & & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & 1 \\ \frac{f_{s_1,k_1}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & 1 \end{vmatrix}.$$

It thus suffices to show that $D^* \neq 0$. Expanding D^* along the last row, keeping $p_1, \dots, p_{r-2}$ fixed for the moment and letting $p_{r-1} \to \infty$, by the asymptotic assumptions (2) and (3), we see that

$$D^* = D_1 + o(p_{r-1}),$$

where

$$D_1 = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(1)}{f_{s_\alpha,k_\beta}(1)} \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} \\ \vdots & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} \end{vmatrix}.$$

Observe that D_1 is independent of p_{r-1} and $\dim D_1 = \dim D - 1$. It is thus enough to show that $D_1 \neq 0$. Now we repeat the above steps by writing

$$D_1 = \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} \cdots \frac{f_{s_\alpha,k_{\beta-1}}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} \frac{f_{s_\alpha,k_{\beta-1}}(1)}{f_{s_\alpha,k_\beta}(1)} D_1^*,$$

where

$$D_1^* = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & 1 \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & 1 \\ \vdots & & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & 1 \end{vmatrix}.$$

It thus suffices to show that $D_1^* \neq 0$. Expanding D_1^* along the last row, keeping $p_1, \ldots, p_{r-3}$ fixed for the time being and letting $p_{r-2} \to \infty$, by the asymptotic assumptions (2) and (3), we get

$$D_1^* = D_2 + o(p_{r-2}),$$

where

$$D_2 = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} \\ \vdots & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} \end{vmatrix}.$$

Observe again that D_2 is independent of p_{r-2} and $\dim D_2 = \dim D_1 - 1$. It is again enough to show that $D_2 \neq 0$. Repeating the same reduction steps, we finally reach a nonzero determinant of dimension 1 as desired. ∎

Theorem 3.5 yields another proof of the following, slightly modified, Lemma 3 of Lucht–Schmalmack [5].

COROLLARY 3.6. *Let $\alpha \in \mathbb{N}$, $S = \{s_1, \ldots, s_\alpha\} \subseteq \mathbb{C}$ with $\Re(s_1) < \cdots < \Re(s_\alpha)$, and let $K = \{0, 1, \ldots, \beta\} \subseteq \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a fixed $a \in \mathbb{N} \setminus \{1\}$, let $T = \{a^\nu : \nu \in \mathbb{N}\}$ be a geometric progression such that $n^s \neq n^{s'}$ for all $n \in T$ and distinct $s, s' \in S$. Then the set*

$$\{I^s \log^k|_T : s \in S,\ k \in K\}$$

of arithmetic functions $(I^s \log^k)(n) := n^s (\log n)^k$, whose domain is restricted to the set T, is $\mathbb{C}$-linearly independent.

Proof. This follows immediately from Theorem 3.5 applied to the arithmetic functions

$$f_{sk}(\nu) = (I^s \log^k)(a^\nu) = a^{\nu s}(\log a^\nu)^k \quad (\nu \in \mathbb{N}). \ ∎$$

In contrast to the $\mathbb{C}$-linear independence over the domain T, it is known (see e.g. [9] or [7]) that the functions $I^s \log^k$ are indeed $\mathbb{C}$-algebraically independent over the whole $\mathbb{N}$.

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