รายงานวิจัยฉบับสมบูรณ์

โครงการระเบียบวิธีทำซ้ำสำหรับการประมาณค่าเหมาะสมของลำดับทางเดียวของเบิร์กแมนในปริภูมิบานาค

Bregman monotone optimization algorithms in Banach spaces

โดยผู้ช่วยศาสตราจารย์ ดร.วีรยุทธ นิลสระคู

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

TABLE OF CONTENTS

บทคัดย่อ

ในรายงานนี้ ได้นำเสนอผลลัพธ์ที่หลากหลายในปัญหาการหาจุดตรึงของระเบียบวิธีทำซ้ำสำหรับการประมาณค่าเหมาะสมของลำดับทางเดียวของเบิร์กแมนในปริภูมิบานาค เราได้แนะนำและศึกษาระเบียบวิธีทำซ้ำต่าง ๆ สำหรับการส่งที่ประกอบไปด้วยการส่งแบบไม่ขยาย การส่งแบบกึ่งไม่ขยาย การส่งแบบไม่ขยายสัมพัทธ์ การส่งแบบไม่ขยายสัมพัทธ์แบบเข้ม และการส่งแบบไม่ขยายสัมพัทธ์ของเบิร์กแมน ในปริภูมิบานาคและปริภูมิฮิลเบิร์ต เราได้พิสูจน์ทฤษฎีบทการลู่เข้าทั้งแบบอ่อนและแบบเข้มสำหรับแต่ละระเบียบวิธีทำซ้ำอีกด้วย ผลลัพธ์ที่ได้เป็นการพัฒนาและครอบคลุมผลงานของนักคณิตศาสตร์จำนวนมาก นอกจากนี้เรายังได้แก้ไขผลงานที่คลุมเครือขอลผลงานที่มีมาก่อนหน้านี้ สุดท้ายยังได้ประยุกต์ใช้งานระเบียบวิธีทำซ้ำดังกล่าวในการหาคำตอบร่วมของปัญหาการหาจุดตรึงของการส่งแบบไม่ขยายสัมพัทธ์กับปัญหาดุลยภาพและปัญหาการหาจุดศูนย์ของตัวดำเนินการทางเดียวแบบแมกซิมัลอีกด้วย

คำสำคัญ: การส่งแบบไม่ขยายสัมพัทธ์ของเบิร์กแมน การส่งแบบไม่ขยายสัมพัทธ์แบบเข้ม การส่งแบบไม่ขยายสัมพัทธ์ การส่งแบบไม่ขยาย ปัญหาดุลยภาพ ตัวดำเนินการทางเดียวแบบแมกซิมัล ทฤษฎีบทการลู่เข้าแบบอ่อนและแบบเข้ม

ABSTRACT

In this report, we present various results related to the problem of finding a fixed point of Bregman monotone optimization algorithms in Banach spaces. We introduce and study many iteration schemes for various kinds of mappings, including nonexpansive mappings, quasi-nonexpansive mappings, relatively nonexpansive mappings, strongly relatively nonexpansive mappings, and Bregman strongly nonexpansive mappings in Banach spaces and Hilbert spaces. We also prove several weak and strong convergence theorems of each iteration. Our results improve and unify the corresponding known results studied by many authors. Moreover, we correct some misleading results appeared in the literature. Finally, we apply them to the solution of equilibrium problems and the problem of finding a zero of a maximal monotone operator.

Keywords: Bregman strongly nonexpansive mapping, strongly relatively nonexpansive mapping, relatively nonexpansive mapping, nonexpansive mapping, equilibrium problem, maximal monotone operator, weak and strong convergence theroems

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยอุบลราชธานีที่ได้ให้โอกาส ผู้วิจัยได้รับทุน ในการทำวิจัยครั้งนี้

รองศาสตราจารย์ ดร.สาธิต แซ่จึง นักวิจัยที่ปรึกษาที่ได้ให้คำแนะนำในการทำวิจัยที่มีคุณภาพ การเป็นนักวิจัยที่ดี และหลักการทำงานร่วมกับผู้อื่น

รองศาสตราจารย์ ดร.อุทิศ อินทร์ประสิทธิ์ ที่คอยอบรมสั่งสอน ถ่ายทอดความรู้ด้านต่าง ๆ ทั้งยังเป็นผู้ร่วมวิจัยในครั้งนี้อีกด้วย

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คะแนะนำตลอดทั้ง ปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ สถิติและคอมพิวเตอร์ คณะวิทยาศาสตร์ มหาวิทยาลัยอุบลราชธานี ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยครั้งนี้

CHAPTER I
EXECUTIVE SUMMARY

Let C be a nonempty set and $T : C \to C$ be a mapping. A *fixed point* of a mapping T is a point which is invariant under T, that is, if x is a fixed point of T, then $x = Tx$. The set of all fixed points of T is called the *fixed-point set* of T and denoted by $F(T)$. The presence or absence of a fixed point is an intrinsic property of T. One of the most celebrated result was proved by Banach in 1922. The Banach fixed point theorem appeared in explicit form in Banach's thesis where it was used to establish the existence of a solution for an integral equation. Since then it has become an important tool in many branches of mathematics and other applications.

The following are among the most basic questions asked in the study of a given mapping T.

(1) What additional assumptions must be added regarding the structure of the space and/or restrictions on T to assure the existence of at least one fixed point?

(2) What is the structure of the fixed-point set of T?

(3) If $F(T) \neq \varnothing$, then how can we approximate an element of $F(T)$?

The study of this project is concerned about question (3).

Let C be a subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We know that every contraction is nonexpansive. Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a (quasi-)nonexpansive mapping in a Hilbert space. To extend this theory to a Banach space, we encounter some difficulties because many of useful examples of nonexpansive mappings in Hilbert spaces are no longer nonexpansive in Banach spaces. There are several ways to overcome these difficulties.

One of them is to use the Lyapunov functional instead of the norm. In 2004, Matsushita and Takahashi studied and investigated the the weak and strong convergence theorems for relatively nonexpansive mappings (coincides with the one in the usual sense in a Hilbert space) in Banach spaces which were first introduced by Butnariu et al. Several articles have appeared providing methods for approximating fixed points of relatively nonexpansive mappings.

In 1967, Brègman discovered an elegant and effective technique for the using of the so-called Bregman distance function in the process of designing and analyzing feasibility and optimization algorithm and so on. The method of cyclic Bregman projections produces a sequence converging to a solution of the convex feasibility problem. In 1997, Albert and Butnariu investigated the method of cyclic Bregman projections in a reflexive Banach space. They proved that the method of cyclic Bregman projections produces a sequence weakly converging to a solution of the convex feasibility proble and norm convergence under additional conditions on the convex sets. In 2004, Lee and Park studied quasi-Bregman firmly nonexpansive mappings and the weak convergence theorem of a Bergman projection method for finding an asymptotic fixed point of a quasi-Bregman firmly nonexpansive mapping in a reflexive Banach space. In 2010, Reich and Sabach studied the existence and approximation of fixed points of a Bregman firmly (strongly) nonexpansive mapping in reflexive Banach spaces.

A purpose of this research is to introduce and study a new iterative algorithm to find a fixed point of a Bregman (strongly) nonexpansive mapping in a reflexive Banach space. Moreover, we prove several weak and strong convergence theorems of such schemes under some suitable assumptions. Finally, we modify these methods in order to solve the equilibrium problem.

CHAPTER II
MAIN RESULTS

2.1 Convergence theorems for relatively nonexpansive mappings

Let E be a smooth Banach space and let E^* be the dual of E. In 1996, Alber considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$. A mapping $T : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(T) \neq \varnothing$, where $F(T)$ denotes the fixed points set of T;

(R2) $\varphi(p, Tx) \leq \varphi(p, x)$ for all $p \in F(T)$ and $x \in C$;

(R3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

Let C be a nonempty closed convex subset of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{2.1.1}$$

The set of solutions of (2.5.1) is denoted by $EP(F)$. For solving the equilibrium problem, we usually assume

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \le F(x, y)$;

(A4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semicontinuous.

The equilibrium problems include fixed point problems, optimization problems, variational inequality problems and Nash equilibrium problems as special cases. Some methods have been proposed to solve the equilibrium problems. In 2005, Combettes and Hirstoaga introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved a strong convergence theorem. In 2011, the author and Saejung [Appl. Math. Comput. 217 (2011) 6577–6586.] modified Halpern and Mann's iterations for finding a fixed point of a relatively nonexpansive mapping in a Banach space as follows: $x \in E, x_1 \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx + \beta_n Jx_n + \gamma_n JTx_n), \quad n = 1, 2, \ldots, \tag{2.1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n \equiv 1$ and they proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$.

The purpose of this work is to present a strong convergence theorem of a new modified Halpern-Mann iterative scheme to find a common element of the set of fixed points of a relatively nonexpansive mapping and the set of common solutions to a system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth as follows:

Theorem 2.1.1 *Let E be a uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E, $\{F_i\}_{i=1}^m$ be a finite family of bifunctions of $C \times C$ into $\mathbb{R}$ satisfying conditions* (A1)-(A4) *and $S : C \to E$ be a relatively nonexpansive mapping such that $\Omega := F(S) \cap (\cap_{i=1}^m EP(F_i)) \neq \varnothing$. Let $\{T_{r_{i,n}}^{F_i}\}_{i=1}^m$ be a finite family of the resolvents of F_i with positive real sequences $\{r_{i,n}\}$ such that $\liminf_{n\to\infty} r_{i,n} > 0$ for all $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n Jx + \beta_n Jx_n + \gamma_n JST_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \cdots T_{r_{1,n}}^{F_1} x_n) \quad (n \ge 1) \tag{2.1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iv) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x$.

We also prove a strong convergence theorem for finding an element of the set of solutions to a system of equilibrium problems in a uniformly convex and uniformly smooth Banach space. In Hilbert spaces, if S is quasi-nonexpansive such that $I - S$ is demiclosed at zero, then S is relatively nonexpansive. So, we also present strong convergence theorem for quasi-nonexpansive mappings in a Hilbert space.

2.2 Convergence theorems for strongly relatively nonexpansive mappings

We say that a relatively nonexpansive mapping $T : C \to E$ is *strongly relatively nonexpansive* if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p, x_n) - \varphi(p, Tx_n) \to 0$ for some $p \in F(T)$ it follows that $\varphi(Tx_n, x_n) \to 0$. Another well-known family of mappings is the class of firmly nonexpansive mappings, where a mapping $T : C \to E$ is called *firmly nonexpansive type* if

$$\varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(Tx, x) + \varphi(Ty, y) \leq \varphi(Tx, y) + \varphi(Ty, x)$$

for all $x, y \in C$. It is easy to see that if T is firmly nonexpansive type with $I - T$ is demi-closed at zero, then it is strongly relatively nonexpansive. Furthermore, there is a mapping which is strongly relatively nonexpansive but is not firmly nonexpansive type as the following example shows.

Example 2.2.1 Let E be a smooth, strictly convex, and reflexive Banach space and let $T : E \to E$ be a mapping defined by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \left(\frac{2}{3}\sin\frac{1}{\|x\|}\right)x & \text{if } x \neq 0. \end{cases}$$

The purpose of this work is to prove for a class of strongly relatively nonexpansive mappings that only Conditions (C1) and (C2) are sufficient for the strong convergence theorem of Halpern's iterations to a fixed point of T without the assumption of the nonempty interior of the fixed point set of T as follows:

Theorem 2.2.2 *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E and let $T : C \to E$ be a strongly relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $u \in E, x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)JTx_n), n = 1, 2, 3, \ldots, \tag{2.2.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

Consequently, a strong convergence theorem for a relatively nonexpansive mapping is deduced and a correction for Theorem 4.1 of Zhang et al. [Comput. Math. Appl. 61 (2011) 262-276] is presented.

Theorem 2.2.3 *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to E$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by (2.2.1), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to z, where $z = \lim_{n\to\infty} \Pi_{F(T)}x_n$.*

Using a concept of duality theorems, we obtain an analogue result for a strongly generalized nonexpansive mapping. Moreover, two corresponding strong convergence theorems for a firmly nonexpansive type mapping and a firmly generalized nonexpansive type mapping are deduced. Finally, we discuss two strong convergence theorems concerning two types of resolvents of a maximal monotone operator in a Banach space.

2.3 Convergence theorems for Bregman strongly nonexpansive mappings

We first recall some definitions.

Definition 2.3.1 The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$ is defined by

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle,$$

is called the *Bregman distance* with respect to f.

Remark 2.3.2 The Bregman distance has the three point identity: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle; \tag{2.3.1}$$

Definition 2.3.3 The Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty closed and convex set $C \subseteq \operatorname{dom} f$ is the necessarily unique vector $\operatorname{Proj}_C^f(x) \in C$ satisfying

$$D_f(\operatorname{Proj}_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 2.3.4 From the definition of Bregman distance above, we have the following.

(i) If X is a Hilbert space and $f(x) = \|x\|^2$, then the Bregman projection $\operatorname{Proj}_C^f(x)$ is reduced to the metric projection of x onto C.

(ii) If X is a smooth Banach space and $f(x) = \|x\|^2$, then the Bregman projection $\operatorname{Proj}_{\mathrm{C}}^{\mathrm{f}}(x)$ is reduced to the generalized projection $\Pi_C x$ which defined by

$$\varphi(\Pi_C x, x) = \min_{y \in C} \varphi(y, x).$$

where $\varphi(y, x) = \|y\|^2 - 2\langle Jx, y \rangle + \|x\|^2$.

We next list significant types of nonexpansivity with respect to the Bregman distance.

Definition 2.3.5 Let C be a nonempty, closed and convex subset of $\operatorname{int}(\operatorname{dom} f)$. We say that a mapping $T : C \to \operatorname{int}(\operatorname{dom} f)$ with $F(T) \neq \varnothing$ is

(i) *Bregman nonexpansive* if

$$D_f(Tx, Ty) \leq D_f(x, y) \tag{2.3.2}$$

for all $x, y \in C$;

(ii) *quasi-Bregman nonexpansive* if

$$D_f(p, Tx) \leq D_f(p, x) \tag{2.3.3}$$

for all $x \in C$ and $p \in F(T)$;

(iii) *Bregman firmly nonexpansive* if

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ \leq D_f(Tx, y) + D_f(Ty, x), \end{aligned} \tag{2.3.4}$$

for all $x, y \in C$, or equivalently,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle; \tag{2.3.5}$$

(iv) *quasi-Bregman firmly nonexpansive* if

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x) \tag{2.3.6}$$

for all $x \in C$ and $p \in F(T)$,

(v) *Bregman strongly nonexpansive* if

$$D_f(p, Tx) \leq D_f(p, x) \tag{2.3.7}$$

for all $x \in C$ and $p \in F(T)$, and if whenever $\{x_n\} \subset C$ is bounded, $p \in F(T)$, and

$$D_f(p, x_n) - D_f(p, Tx_n) \to 0,$$

it follows that

$$D_f(Tx_n, x_n) \to 0.$$

From the definitions of mappings above, we have the following properties.

Proposition 2.3.6 *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T : C \to C$ be a mapping. Then:*

(i) *If T is a Bregman nonexpansive mapping, then T is quasi-Bregman nonexpansive.*

(ii) *If T is a Bregman firmly nonexpansive mapping, then T is quasi-Bregman firmly nonexpansive.*

(iii) *If T is a quasi-Bregman firmly nonexpansive mapping, then T is Bregman strongly nonexpansive.*

(iv) *If T is a quasi-Bregman nonexpansive mapping, then $F(T)$ is closed and convex.*

(v) *If f is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of X and T is Bregman firmly nonexpansive, then $F(T) = \widehat{F}(T)$.*

We now present the weak and strong convergence theorems for new ieteration.

Proposition 2.3.7 *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let T be a quasi-Bregman nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \operatorname{Proj}_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)), \quad n \geq 1, \tag{2.3.8}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Then $\{x_n\}$ is bounded and $\{\operatorname{Proj}_{F(T)}^f(x_n)\}$ converges strongly to a fixed point of T.

Theorem 2.3.8 *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let T be a Bregman strongly nonexpansive mapping of C into itself such that $F(T) = \widehat{F}(T) \neq \varnothing$. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. If ∇f is weakly sequentially continuous, then the sequence $\{x_n\}$ defined by (2.3.8) converges weakly to $u \in F(T)$, where $u = \lim_{n\to\infty} \mathrm{Proj}^f_{F(T)}(x_n)$.*

Theorem 2.3.9 *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let T be a Bregman strongly nonexpansive mapping of C into itself such that $F(T) = \widehat{F}(T) \neq \varnothing$. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. If $\mathrm{int}(F(T)) \neq \varnothing$, then the sequence $\{x_n\}$ defined by (2.3.8) converges strongly to $u \in F(T)$, where $u = \lim_{n\to\infty} \mathrm{Proj}^f_{F(T)}(x_n)$.*

Consequently, three convergence theorems for a relatively (quasi-) nonexpansive mapping are deduced and then apply them to the solution of equilibrium problem. Various special cases are discussed.

2.4 Deduced theorems in Hilbert spaces

In Hilbert spaces, if T is nonexpansive or quasi-nonexpansive such that $I-T$ isdemiclosed at zero, then T is relatively nonexpansive and then is Bregman nonexpansive. We obtain the following results.

Theorem 2.4.1 *Let H be a Hilbert space, C be a nonempty closed convex subset of H, $\{F_i\}_{i=1}^m$ be a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)-(A4) and $S : C \to E$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $\Omega := F(S) \cap (\cap_{i=1}^m EP(F_i)) \neq \varnothing$. Let $\{T^{F_i}_{r_{i,n}}\}_{i=1}^m$ be a finite family of the resolvents of F_i with real sequences $\{r_{i,n}\}$ such that $\liminf_{n\to\infty} r_{i,n} > 0$ for all $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \cdots T^{F_1}_{r_{1,n}} x_n \quad (n \geq 1) \tag{2.4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $P_\Omega x$.

Recently, In 2014, Sakurai and Iiduka introduced the Halpern algorithm based on steepest descent method for solving an example of a fixed point problem and they can formulate novel fixed point algorithm by using conjugate gradient method, which can accelerate steepest descent method and the number of iterations is less than steepest descent method. They present strong convergence theorem of their algorithm for finding a fixed point of a nonexpansive mapping in Hilbert space as follows: let $\{x_n\}$ be a sequence in H defined by $x_0 \in H$, $\mu \in (0,1]$, $\lambda > 0$, $d_0 = \frac{1}{\lambda}(T(x_0) - x_0)$ and

$$\begin{cases} d_{n+1} = \frac{1}{\lambda}(T(x_n) - x_n) + \beta_n d_n, \\ \quad y_n \ = x_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\alpha_n x_0 + (1-\mu\alpha_n) y_n. \end{cases} \tag{2.4.2}$$

for each $n \geq 0$, where $\{\alpha_n\}^{\infty} \subset (0,1)$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0,\infty)$. Then $(x_n)_{n=0}^{\infty}$ generated by (2.4.2) converges strongly to $P_{F(T)}(x_0)$. The purpose of this work is to introduce the implicit midpoint rule based on conjugate gradient method for finding a fixed point of a nonexpansive mapping.

Step 0. Choose $\lambda > 0$, and $x_0 \in H$ arbitrarily,
and set $(\alpha_n)_{n=0}^{\infty} \subset (0,1)$, $(\beta_n)_{n=0}^{\infty} \subset [0,\infty)$.
Compute $d_0 = \frac{1}{\lambda}(T(x_0) - x_0)$.

Step 1. Given $x_n, d_n \in H$, compute x_{n+1}, $d_{n+1} \in H$ as follows:

$$\begin{cases} x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\left(T\left(\frac{x_n+x_{n+1}}{2}\right) + \lambda\beta_n d_n\right), \\ d_{n+1} = \frac{1}{\lambda}\left(T\left(\frac{x_n+x_{n+1}}{2}\right) - \left(\frac{x_n+x_{n+1}}{2}\right)\right) + \beta_n d_n. \end{cases}$$

Put $n = n+1$, and go to Step 1.

We present the strong convergence theorems in a Hilbert space of this method. Setting certain parameters, as a consequence, strong convergence theorems for finding a fixed point of a nonexpansive mapping which studied in are deduced. Finally, we give some examples to support our main results.

In the other hand, let C be a non-empty subset of a Hilbert space H and $T : C \to C$ be a mapping. An attractive point of T is a point x in H such that

$$\|Ty - x\| \leq \|y - x\| \text{ for all } y \in C. \tag{2.4.3}$$

The set of all attractive points of T is denoted by $A(T)$. The purpose of this work to introduce an algorithm to accelerate the Halpern algorithm by using the conjugate gradient method.

Theorem 2.4.2 *Let C be a non-empty subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $A(T) \neq \varnothing$. Let $(x_n)_{n=0}^{\infty}$ be the acceleration of the Halpern algorithm generated by $x_0 \in C, d_0 \in C$ and*

$$\begin{cases} d_{n+1} &= \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n, \\ y_n &= x_n + \alpha d_{n+1}, \\ x_{n+1} &= \mu\alpha_n x_0 + (1-\mu\alpha_n)y_n, \end{cases} \tag{2.4.4}$$

where $\mu \in (0,1], \alpha > 0, (\alpha_n)_{n=0}^{\infty} \subset (0,1)$ and $(\beta_n)_{n=0}^{\infty} \subset [0,\infty)$. Suppose that

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C4) $\beta_n \leq \alpha_n^2$ *for all* $n \geq 0$, (C5) $(Tx_n - x_n)_{n=0}^{\infty}$ *is bounded.*

Then $(x_n)_{n=0}^{\infty}$ *defined by* (2.4.4) *converges strongly to* $x^* = P_{A(T)}x_0$.

Setting some certain parameters, the such algorithm is deduced to the Halpern algorithm. Consequence, the strong convergence theorem of acceleration of the Halpern algorithm for finding an attractive point of a nonexpansive mapping in Hilbert space is presented. When domain of a nonexpansive mapping is whole space, the attractive point set and the fixed point set are coincides and then strong convergence theorem for finding to a fixed point of a nonexpansive mapping is shown. Finally, we give some example to support our main results.

2.5 Applications

2.5.1 Equilibrium problems

Let C be a nonempty, closed and convex subset of X. Let $g : C \times C \to \mathbb{R}$ be a bifunction that satisfies the following conditions:

(A1) $g(x, x) = 0$ for all $x \in C$;

(A2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \leq 0,$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \to 0} g(tz + (1 - t)x, y) \leq g(x, y);$$

(A4) for all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

The *equilibrium problem* of g is to find $x \in C$ such that

$$g(x, y) \geq 0 \quad \text{for all } y \in C. \tag{2.5.1}$$

The set of solutions of (2.5.1) is denoted by $EP(g)$. Set

$$\operatorname{Res}_g^f(x) := \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \forall y \in C\}.$$

The following two lemmas give several properties of these resolvents.

Lemma 2.5.1 *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of X. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4). Then, the followings hold:*

(i) Res_g^f *is single-valued;*

(ii) Res_g^f *is a Bregman firmly nonexpansive mapping (is also Bregman strongly nonexpansive);*

(iii) $F(\mathrm{Res}_g^f) = EP(g)$.

Theorem 2.5.2 *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of X. Let C be a nonempty, closed and convex subset of X and let $g : C \times C \to \mathbb{R}$ be a bifunction satisfies conditions (A1)-(A4) and $EP(g) \neq \varnothing$. Suppose that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \mathrm{Proj}_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(\mathrm{Res}_g^f x_n)), \quad n \geq 1.$$

Then:

(i) *If ∇f is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $u \in EP(g)$, where $u = \lim_{n\to\infty} \mathrm{Proj}_{EP(g)}^f(x_n)$.*

(ii) *If $\mathrm{int}(EP(g)) \neq \varnothing$, then $\{x_n\}$ converges strongly to $u \in EP(g)$, where $u = \lim_{n\to\infty} \mathrm{Proj}_{EP(g)}^f(x_n)$.*

2.5.2 Maximal Monotone Operators

Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a set-valued mapping with range $R(A) = \{x^* \in E^* : x^* \in Ax\}$ and domain $D(A) = \{x \in E : Ax \neq \varnothing\}$. Then the mapping A is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* if A is monotone and there is no monotone operator from E into E^* whose graph properly contains the graph of A. It is known that if $A \subset E \times E^*$ is maximal monotone, then $A^{-1}0$ is closed and convex.

Theorem 2.5.3 *Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Using We obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

The single valued mapping $J_r : E \to D(A)$ by $J_r x = x_r$, that is, $J_r = (J + rA)^{-1}J$ is called the *resolvent* of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$. We prove a strong convergence theorem for resolvents of maximal monotone operators in a Banach space.

Theorem 2.5.4 *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E^*$ be a maximal monotone operator. Let J_r be the resolvent of A, where $r > 0$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)JJ_r x_n),$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}u$.

If E is reflexive, strictly convex and smooth and $B \subset E^* \times E (= E^* \times E^{**})$ is a maximal monotone operator, then $R(J^{-1} + rB) = E$ for all $r > 0$. Thus, if $r > 0$ and $x \in E$, then there exists $z \in E$ such that

$$x = J^{-1}(Jx) \in J^{-1}(Jz) + rB(Jz) = z + rBJz.$$

It follows from the strict convexities of E and E^* that such a point z is unique. Thus we can define the *generalized resolvent* Q_r of B by

$$Q_r x = z = (I + rBJ)^{-1}x.$$

We also prove a strong convergence theorem for generalized resolvents of maximal monotone operators in a Banach space.

Theorem 2.5.5 *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. Let Q_r be the generalized resolvent of B, where $r > 0$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in E$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q_r x_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) *and* (C2)*. If $B^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}u$, where $R_{(BJ)^{-1}0}$ is the unique sunny generalized nonexpansive retraction from E onto $(BJ)^{-1}0$.*

Remark 2.5.6 In Theorem 2.5.5, we present a strong convergence theorem for the generalized resolvent with a new control condition. This is complementary to Ibaraki and Takahashi's result.

APPENDIX

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Research Article

A Halpern-Mann Type Iteration for Fixed Point Problems of a Relatively Nonexpansive Mapping and a System of Equilibrium Problems

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A new modified Halpern-Mann type iterative method is constructed. Strong convergence of the scheme to a common element of the set of fixed points of a relatively nonexpansive mapping and the set of common solutions to a system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth is proved. The results presented in this work improve on the corresponding ones announced by many others.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space, E^* the dual space of E, and C a nonempty closed convex subset of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. The equilibrium problems include fixed point problems, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases. Some methods have been proposed to solve the equilibrium problems (see, e.g., [1, 2]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty, and they also proved a strong convergence theorem.

Let E be a smooth Banach space and J the normalized duality mapping from E to E^*. Alber [4] considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E). \tag{1.2}$$

Using this functional, Matsushita and Takahashi [5, 6] studied and investigated the following mappings in Banach spaces. A mapping $S : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(S) \neq \emptyset$,

(R2) $\varphi(p, Sx) \leq \varphi(p, x)$ for all $p \in F(S)$ and $x \in C$,

(R3) $F(S) = \widehat{F}(S)$,

where $F(S)$ and $\widehat{F}(S)$ denote the set of fixed points of S and the set of asymptotic fixed points of S, respectively. It is known that S satisfies condition (R3) if and only if $I - S$ is demiclosed at zero, where I is the identity mapping; that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Sx_n\}$ converges strongly to 0, it follows that $p \in F(S)$. In a Hilbert space H, the duality mapping J is an identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $S : C \to H$ is nonexpansive (i.e., $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$), then it is relatively nonexpansive. Several articles have appeared providing methods for approximating fixed points of relatively nonexpansive mappings (see, e.g., [5–19] and the references therein). Matsushita and Takahashi [5] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n) \quad n = 1, 2, \ldots, \tag{1.3}$$

where $x_1 \in C$ is arbitrary, $\{\alpha_n\}$ is an appropriate sequence in $[0, 1]$, S is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges *weakly* to a fixed point of T. Moreover, Matsushita and Takahashi [6] proposed the following modification of iteration (1.3):

$$\begin{aligned}
&x_1 \in C \quad \text{is arbitrary},\\
&y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n),\\
&C_n = \{z \in C : \varphi(z, y_n) \leq \varphi(z, x_n)\},\\
&Q_n = \{z \in C : \langle x_n - z, Jx_1 - Jx_n\rangle \geq 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \quad n = 1, 2, \ldots,
\end{aligned} \tag{1.4}$$

and proved that the sequence $\{x_n\}$ converges *strongly* to $\Pi_{F(S)} x_1$. The iteration (1.4) is called *the hybrid method.* To generate the iterative sequence, we use the generalized metric projection onto $C_n \cap Q_n$ for $n \in \mathbb{N}$. It always exists, because each $C_n \cap Q_n$ is nonempty, closed, and convex. However, in a practical case, it is not easy to be calculated. In particular, as n becomes larger, the shape of $C_n \cap Q_n$ becomes more complicate, and the projection will take much more time to be calculated.

In order to overcome this difficulty, Nilsrakoo and Saejung [15] modified Halpern and Mann's iterations for finding a fixed point of a relatively nonexpansive mapping in a Banach space as follows: $x \in E,\ x_1 \in C$ and

$$x_{n+1} = \Pi_C J^{-1}\left(\alpha_n J x + \beta_n J x_n + \gamma_n J S x_n\right), \quad n = 1, 2, \ldots, \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are appropriate sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n \equiv 1$, and they proved that $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$.

Many authors studied the problems of finding a common element of the set of fixed points for a mapping and the set of common solutions to a system of equilibrium problems in the setting of Hilbert space and uniformly smooth and uniformly convex Banach space, respectively (see, e.g., [20–33] and the references therein). In a Hilbert space H, S. Takahashi and W. Takahashi [34] introduced the iteration as follows: sequence $\{x_n\}$ generated by $x, x_1 \in C$,

$$\begin{gathered} u_n \in C \quad \text{such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) S u_n, \quad n = 1, 2, \ldots, \end{gathered} \tag{1.6}$$

where $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$, S is nonexpansive, and $\{r_n\}$ is an appropriate positive real sequence. They proved that $\{x_n\}$ converges strongly to an element in $F(S) \cap \mathrm{EP}(F)$. In 2009, Takahashi and Zembayashi [30] proposed the iteration in a uniformly smooth and uniformly convex Banach space as follows: a sequence $\{x_n\}$ generated by $u_1 \in E$,

$$\begin{gathered} x_n \in C \quad \text{such that } F(x_n, y) + \frac{1}{r_n}\langle y - x_n, J x_n - J u_n\rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n), \quad n = 1, 2, \ldots, \end{gathered} \tag{1.7}$$

where S is relatively nonexpansive, $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$, and $\{r_n\}$ is an appropriate positive real sequence. They proved that if J is weakly sequentially continuous, then $\{x_n\}$ converges *weakly* to an element in $F(S) \cap \mathrm{EP}(F)$. Consequently, there are many results presented strong convergence theorems for finding a common element of the set of fixed points for a mapping and the set of common solutions to a system of equilibrium problems by using the hybrid method. However, Nilsrakoo [35] introduced the Halpern-Mann iteration guaranteeing the strong convergence as follows: $x \in C,\ u_1 \in E$ and

$$\begin{gathered} x_n \in C \quad \text{such that } F(x_n, y) + \frac{1}{r_n}\langle y - x_n, J x_n - J u_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \Pi_C J^{-1}(\alpha_n J x + (1-\alpha_n) J x_n), \\ u_{n+1} = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J S y_n\right), \quad n = 1, 2, \ldots, \end{gathered} \tag{1.8}$$

and proved that $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{F(S)\cap \mathrm{EP}(F)}x$.

Motivated by Nilsrakoo and Saejung [15] and Nilsrakoo [35], we present a strong convergence theorem of a new modified Halpern-Mann iterative scheme to find a common element of the set of fixed points of a relatively nonexpansive mapping and the set of common solutions to a system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth. The results in this work improve on the corresponding ones announced by many others.

2. Preliminaries

We collect together some definitions and preliminaries which are needed in this paper. We say that a Banach space E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \quad \text{imply} \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x-y\| \geq \varepsilon \quad \text{imply} \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{2.2}$$

It is known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. We say that E is *uniformly smooth* if the dual space E^* of E is uniformly convex. A Banach space E is *smooth* if the limit $\lim_{t\to 0}((\|x+ty\|-\|x\|)/t)$ exists for all norm one elements x and y in E. It is not hard to show that if E is reflexive, then E is smooth if and only if E^* is strictly convex.

Let E be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ (see [4]) is defined by

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E), \tag{2.3}$$

where the *duality mapping* $J : E \to E^*$ is given by

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2 \quad (x \in E). \tag{2.4}$$

It is obvious from the definition of the function φ that

$$(\|x\| - \|y\|)^2 \leq \varphi(x,y) \leq (\|x\| + \|y\|)^2, \tag{2.5}$$

$$\varphi(x,y) = \varphi(x,z) + \varphi(z,y) + 2\langle x-z, Jz - Jy\rangle, \tag{2.6}$$

for all $x, y, z \in E$. Moreover,

$$\varphi\left(x, J^{-1}\left(\sum_{i=1}^{n} \lambda_i J y_i\right)\right) \leq \sum_{i=1}^{n} \lambda_i \varphi(x, y_i), \tag{2.7}$$

for all $\lambda_i \in [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $x, y_i \in E$.

The following lemma is an analogue of Xu's inequality [36, Theorem 2] with respect to φ.

Lemma 2.1 (see [15, Lemma 2.2]). *Let E be a uniformly smooth Banach space and $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\varphi\Big(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\Big) \leq \lambda\varphi(x, y) + (1-\lambda)\varphi(x, z) - \lambda(1-\lambda)g(\|Jy - Jz\|), \tag{2.8}$$

for all $\lambda \in [0, 1]$, $x \in E$ and $y, z \in B_r := \{z \in E : \|z\| \leq r\}$.

It is also easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0$ implies that $\varphi(x_n, y_n) \to 0$.

Lemma 2.2 (see [37, Proposition 2]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

Remark 2.3. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\varphi(x_n, y_n) \longrightarrow 0 \iff x_n - y_n \longrightarrow 0 \iff Jx_n - Jy_n \longrightarrow 0. \tag{2.9}$$

Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. It is known that [4, 37] for any $x \in E$, there exists a unique point $\hat{x} \in C$ such that

$$\varphi(\hat{x}, x) = \min_{y \in C} \varphi(y, x). \tag{2.10}$$

Following Alber [4], we denote such an element $\hat{x}$ by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that in a Hilbert space, the mapping Π_C coincides with the metric projection P_C. Concerning the generalized projection, the followings are well known.

Lemma 2.4 (see [37, Propositions 4 and 5]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E, $x \in E$ and $\hat{x} \in C$. Then,*

(a) $\hat{x} = \Pi_C x$ *if and only if* $\langle y - \hat{x}, Jx - J\hat{x}\rangle \leq 0$ *for all* $y \in C$,

(b) $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$ *for all* $y \in C$.

Remark 2.5. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Let E be a reflexive, strictly convex, and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*; that is, $J^* = J^{-1}$. We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in Alber [4]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{2.11}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma (see [4] and [38, Lemma 3.2]).

Lemma 2.6. *Let E be a reflexive, strictly convex, and smooth Banach space, and let V be as in* (2.11). *Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*), \tag{2.12}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.7 (see [39, Lemma 2.1]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n \tag{2.13}$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0,1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy conditions: $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, *and* $\limsup_{n\to\infty}\delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 (see [40, Lemma 3.1]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \qquad a_k \leq a_{m_k+1}. \tag{2.14}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

For solving the equilibrium problem, we usually assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions (see, e.g., [1, 3, 30]):

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$,

(A3) for all $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$,

(A4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semicontinuous.

The following lemma is a result which appeared in Blum and Oettli [1, Corollary 1].

Lemma 2.9 (see [1, Corollary 1]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0 \quad \forall y \in C. \tag{2.15}$$

The following lemma gives a characterization of a solution of an equilibrium problem.

Lemma 2.10 (see [30, Lemma 2.8]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and uniformly smooth Banach space E. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* *conditions (A1)–(A4). For $r > 0$, define a mapping $T_r^F : E \to C$ so-called the resolvent of F as follows:*

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \ \forall y \in C \right\}, \tag{2.16}$$

for all $x \in E$. Then, the followings hold:

(i) *T_r is single-valued,*

(ii) *T_r is a firmly nonexpansive-type mapping [11], that is, for all $x, y \in E$*

$$\left\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y \right\rangle \leq \left\langle T_r^F x - T_r^F y, Jx - Jy \right\rangle, \tag{2.17}$$

(iii) *for all $x \in E$ and $p \in \mathrm{EP}(F)$,*

$$\varphi\left(p, T_r^F x\right) \leq \varphi\left(z, T_r^F x\right) + \varphi\left(T_r^F x, x\right) \leq \varphi(p, x), \tag{2.18}$$

(iv) $F(T_r^F) = \mathrm{EP}(F)$,

(v) $\mathrm{EP}(F)$ *is closed and convex.*

Remark 2.11. Some well-known examples of resolvents of bifunctions satisfying conditions (A1)–(A4) are presented in [3, Lemma 2.15].

Lemma 2.12 (see [8, Lemma 2.3]). *Let C be a nonempty closed convex subset of a Banach space E, F a bifunction from $C \times C \to \mathbb{R}$ satisfying conditions (A1)–(A4), and $z \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $F(y,z) \leq 0$ for all $y \in C$.*

Lemma 2.13 (see [6], Proposition 2.4). *Let C be a nonempty closed convex subset of a strictly convex and smooth Banach space E and $S : C \to E$ a relatively nonexpansive mapping. Then $F(S)$ is closed and convex.*

3. Main Results

In this section, we introduce a modified Halpern-Mann type iteration without *using the generalized metric projection* and prove a strong convergence theorem for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions to a system of equilibrium problems in a uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let E a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, $\{F_i\}_{i=1}^m$ be a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a relatively nonexpansive mapping such that $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) \neq \emptyset$. Let $\{T_{r_{i,n}}^{F_i}\}_{i=1}^m$ be a finite family of the resolvents of F_i with positive real sequences $\{r_{i,n}\}$ such that*

$\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JST^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \cdots T^{F_1}_{r_{1,n}} x_n\right) \quad (n \geq 1), \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_\Omega x$.

Proof. For each $n \geq 1$, setting

$$\begin{aligned} z_n^k &= T^{F_k}_{r_{k,n}} T^{F_{k-1}}_{r_{k1,n}} \cdots T^{F_1}_{r_{1,n}} x_n, \quad (k = 1, 2, \ldots, m), \\ y_n &= J^{-1}\left(\frac{\beta_n}{1-\alpha_n} Jx_n + \frac{\gamma_n}{1-\alpha_n} JSz_n^m\right). \end{aligned} \tag{3.2}$$

We can see that $z_n^k = T^{F_k}_{k,n} z_n^{k-1}$. Since Ω is nonempty, closed, and convex, we put $\hat{x} = \Pi_\Omega x$. By Lemma 2.10(iii), we get

$$\begin{aligned} \varphi(\hat{x}, z_n^m) &\leq \varphi\left(\hat{x}, z_n^{m-1}\right) - \varphi\left(z_n^m, z_n^{m-1}\right) \\ &\leq \varphi\left(\hat{x}, z_n^{m-2}\right) - \varphi\left(z_n^{m-1}, z_n^{m-2}\right) - \varphi\left(z_n^m, z_n^{m-1}\right) \\ &\vdots \\ &\leq \varphi(\hat{x}, x_n) - \sum_{k=1}^{m} \varphi\left(z_n^k, z_n^{k-1}\right), \end{aligned} \tag{3.3}$$

where $z_n^0 = x_n$. This together with (2.7) gives

$$\begin{aligned} \varphi(\hat{x}, y_n) &\leq \frac{\beta_n}{1-\alpha_n} \varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n} \varphi(\hat{x}, Sz_n^m) \\ &\leq \frac{\beta_n}{1-\alpha_n} \varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n} \varphi(\hat{x}, z_n^m) \\ &\leq \varphi(\hat{x}, x_n). \end{aligned} \tag{3.4}$$

By Lemma 2.6, we obtain

$$\begin{aligned}
\varphi(\hat{x}, x_{n+1}) &= V(\hat{x}, Jx_{n+1}) \\
&\le V(\hat{x}, Jx_{n+1} - \alpha_n(Jx - J\hat{x})) - 2\langle x_{n+1} - \hat{x}, -\alpha_n(Jx - J\hat{x})\rangle \\
&= \varphi\Big(\hat{x}, J^{-1}(\alpha_n J\hat{x} + (1-\alpha_n)Jy_n)\Big) + 2\alpha_n\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle \\
&\le \alpha_n\varphi(\hat{x}, \hat{x}) + (1-\alpha_n)\varphi(\hat{x}, y_n) + 2\alpha_n\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle \\
&\le (1-\alpha_n)\varphi(\hat{x}, x_n) + 2\alpha_n\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle.
\end{aligned} \tag{3.5}$$

Next, we show that $\{x_n\}$ is bounded. We consider

$$\begin{aligned}
\varphi(\hat{x}, x_{n+1}) &\le \varphi\Big(\hat{x}, J^{-1}(\alpha_n Jx + \beta_n Jx_n + \gamma_n JSz_n^m)\Big) \\
&= \varphi\Big(\hat{x}, J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n\Big) \\
&\le \alpha_n\varphi(\hat{x}, x) + (1-\alpha_n)\varphi(\hat{x}, y_n) \\
&\le \alpha_n\varphi(\hat{x}, x) + (1-\alpha_n)\varphi(\hat{x}, x_n) \\
&\le \max\{\varphi(\hat{x}, x), \varphi(\hat{x}, x_n)\}.
\end{aligned} \tag{3.6}$$

By induction, we have

$$\varphi(\hat{x}, x_{n+1}) \le \max\{\varphi(\hat{x}, x), \varphi(\hat{x}, x_1)\}, \tag{3.7}$$

for all $n \ge 1$. This implies that $\{x_n\}$ is bounded, and so are $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n^m\}$, and $\{Sz_n^m\}$. Let $g : [0, 2r] \to [0, \infty)$ be a function satisfying the properties of Lemma 2.1, where $r = \sup\{\|x_n\|, \|Sz_n^m\| : n \ge 1\}$. It follows from (3.3) that

$$\begin{aligned}
\varphi(\hat{x}, y_n) &\le \frac{\beta_n}{1-\alpha_n}\varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(\hat{x}, Sz_n^m) - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}g(\|Jx_n - JSz_n^m\|) \\
&\le \frac{\beta_n}{1-\alpha_n}\varphi(\hat{x}, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(\hat{x}, z_n^m) - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}g(\|Jx_n - JSz_n^m\|) \\
&\le \varphi(\hat{x}, x_n) - \frac{\gamma_n}{1-\alpha_n}\sum_{k=1}^{m}\varphi\Big(z_n^k, z_n^{k-1}\Big) - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}g(\|Jx_n - JSz_n^m\|).
\end{aligned} \tag{3.8}$$

The rest of the proof will be divided into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\hat{x}, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(\hat{x}, x_n)\}$ is then convergent. Then,

$$\varphi(\hat{x}, x_n) - \varphi(\hat{x}, x_{n+1}) \longrightarrow 0. \tag{3.9}$$

Notice that

$$\varphi(\widehat{x}, x_{n+1}) \leq \alpha_n \varphi(\widehat{x}, x) + (1-\alpha_n)\varphi(\widehat{x}, y_n). \tag{3.10}$$

From condition (ii),

$$\begin{aligned}\varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, y_n) &= \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) + \varphi(\widehat{x}, x_{n+1}) - \varphi(\widehat{x}, y_n)\\ &\leq \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) + \alpha_n(\varphi(\widehat{x}, x) - \varphi(\widehat{x}, y_n)) \longrightarrow 0.\end{aligned} \tag{3.11}$$

It follows from (3.8) that

$$\frac{\gamma_n}{1-\alpha_n}\sum_{k=1}^{m}\varphi\left(z_n^k, z_n^{k-1}\right) + \frac{\beta_n\gamma_n}{(1-\alpha_n)^2} g(\|Jx_n - JSz_n^m\|) \longrightarrow 0. \tag{3.12}$$

By the assumptions (i), (ii), and (iv),

$$\varphi\left(z_n^k, z_n^{k-1}\right) \longrightarrow 0 \quad (k = 1, 2, \ldots, m), \qquad g(\|Jx_n - JSz_n^m\|) \longrightarrow 0. \tag{3.13}$$

By Remark 2.3, we get

$$z_n^k - z_n^{k-1} \longrightarrow 0 \quad (k = 1, 2, \ldots, m). \tag{3.14}$$

From g is continuous strictly increasing with $g(0) = 0$, we have

$$z_n^m - Sz_n^m \longrightarrow 0, \qquad \varphi(x_n, Sz_n^m) \longrightarrow 0. \tag{3.15}$$

Consequently,

$$\begin{aligned}\varphi(x_n, y_n) &\leq \frac{\beta_n}{1-\alpha_n}\varphi(x_n, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(x_n, Sz_n^m) = \frac{\gamma_n}{1-\alpha_n}\varphi(x_n, Sz_n^m) \longrightarrow 0,\\ \varphi(y_n, x_{n+1}) &\leq \alpha_n\varphi(y_n, x) + (1-\alpha_n)\varphi(y_n, y_n) = \alpha_n\varphi(y_n, x) \longrightarrow 0.\end{aligned} \tag{3.16}$$

This implies that

$$x_{n+1} - x_n \longrightarrow 0. \tag{3.17}$$

Since $\{x_n\}$ is bounded and E is reflexive, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle x_n - \widehat{x}, Jx - J\widehat{x}\rangle = \lim_{j\to\infty}\left\langle x_{n_j} - \widehat{x}, Jx - J\widehat{x}\right\rangle = \langle w - \widehat{x}, Jx - J\widehat{x}\rangle. \tag{3.18}$$

Let $k = 1, 2, \ldots, m$ be fixed. Then, $z_{n_j}^k \rightharpoonup w$ as $j \to \infty$. From $\liminf_{n\to\infty} r_{k,n} > 0$ and (3.14), we have

$$\lim_{n\to\infty} \frac{1}{r_{k,n}} \left\| Jz_n^k - Jz_n^{k-1} \right\| = 0. \tag{3.19}$$

Then,

$$F_k\left(z_n^k, y\right) + \frac{1}{r_{k,n}} \left\langle y - z_n^k, Jz_n^k - Jz_n^{k-1} \right\rangle \geq 0, \quad \forall y \in C. \tag{3.20}$$

Replacing n by n_j, we have from (A2) that

$$\frac{1}{r_{k,n_j}} \left\langle y - z_{n_j}^k, Jz_{n_j}^k - Jz_{n_j}^{k-1} \right\rangle \geq -F_k\left(z_{n_j}^k, y\right) \geq F_k\left(y, z_{n_j}^k\right), \quad \forall y \in C. \tag{3.21}$$

Letting $j \to \infty$, we have from (3.19) and (A4) that

$$F_k(y, w) \leq 0, \quad \forall y \in C. \tag{3.22}$$

From Lemma 2.12, we have $w \in \mathrm{EP}(F_k)$. Since S satisfies condition (R3) and $z_n^m - Sz_n^m \to 0$, we have $w \in F(S)$. It follows that $w \in \Omega$. By Lemma 2.4(a), we immediately obtain that

$$\limsup_{n\to\infty} \langle x_{n+1} - \hat{x}, Jx - J\hat{x} \rangle = \limsup_{n\to\infty} \langle x_n - \hat{x}, Jx - J\hat{x} \rangle = \langle w - \hat{x}, Jx - J\hat{x} \rangle \leq 0. \tag{3.23}$$

It follows from Lemma 2.7 and (3.5) that $\varphi(\hat{x}, x_n) \to 0$. Then, $x_n \to \hat{x}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\hat{x}, x_{n_i}) < \varphi(\hat{x}, x_{n_i+1}), \tag{3.24}$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exists a nondecreasing sequence of positive integer numbers $\{\ell_j\}$ such that $\ell_j \to \infty$,

$$\varphi\left(\hat{x}, x_{\ell_j}\right) \leq \varphi\left(\hat{x}, x_{\ell_j+1}\right), \qquad \varphi(\hat{x}, x_j) \leq \varphi\left(\hat{x}, x_{\ell_j+1}\right), \tag{3.25}$$

for all sufficiently large numbers j. We may assume without loss of generality that $\alpha_{\ell_j} > 0$ for all sufficiently large numbers j. Since

$$\varphi\left(\hat{x}, x_{\ell_j+1}\right) \leq \alpha_{\ell_j} \varphi(\hat{x}, x) + \left(1 - \alpha_{\ell_j}\right) \varphi\left(\hat{x}, y_{\ell_j}\right), \tag{3.26}$$

we obtain

$$\begin{aligned}\varphi\left(\widehat{x}, x_{\ell_j}\right) - \varphi\left(\widehat{x}, y_{\ell_j}\right) &= \varphi\left(\widehat{x}, x_{\ell_j}\right) - \varphi\left(\widehat{x}, x_{\ell_j+1}\right) + \varphi\left(\widehat{x}, x_{\ell_j+1}\right) - \varphi\left(\widehat{x}, y_{\ell_j}\right) \\ &\leq \alpha_{\ell_j}\left(\varphi(\widehat{x}, x) - \varphi\left(\widehat{x}, y_{\ell_j}\right)\right) \longrightarrow 0.\end{aligned} \tag{3.27}$$

It follows from (3.8) that

$$\frac{\gamma_{\ell_j}}{1-\alpha_{\ell_j}} \sum_{k=1}^{m} \varphi\left(z_{\ell_j}^{k}, z_{\ell_j}^{k-1}\right) + \frac{\beta_{\ell_j}\gamma_{\ell_j}}{\left(1-\alpha_{\ell_j}\right)^2} g\left(\left\| Jx_{\ell_j} - JSz_{\ell_j}^{m}\right\|\right) \longrightarrow 0. \tag{3.28}$$

Using the same proof of *Case 1*, we also obtain

$$\limsup_{j\to\infty}\left\langle x_{\ell_j+1} - \widehat{x}, Jx - J\widehat{x}\right\rangle \leq 0. \tag{3.29}$$

From (3.5), we have

$$\varphi\left(\widehat{x}, x_{\ell_j+1}\right) \leq \left(1-\alpha_{\ell_j}\right)\varphi\left(\widehat{x}, x_{\ell_j}\right) + 2\alpha_{\ell_j}\left\langle x_{\ell_j+1} - \widehat{x}, Jx - J\widehat{x}\right\rangle. \tag{3.30}$$

Since $\varphi(\widehat{x}, x_{\ell_j}) \leq \varphi(\widehat{x}, x_{\ell_j+1})$, we have

$$\begin{aligned}\alpha_{\ell_j}\varphi(\widehat{x}, x_{\alpha}) &\leq \varphi\left(\widehat{x}, x_{\ell_j}\right) - \varphi\left(\widehat{x}, x_{\ell_j+1}\right) + 2\alpha_{\ell_j}\left\langle x_{\ell_j+1} - \widehat{x}, Jx - J\widehat{x}\right\rangle \\ &\leq 2\alpha_{\ell_j}\left\langle x_{\ell_j+1} - \widehat{x}, Jx - J\widehat{x}\right\rangle.\end{aligned} \tag{3.31}$$

In particular, since $\alpha_{\ell_j} > 0$, we get

$$\varphi(\widehat{x}, x_{m_k}) \leq 2\left\langle x_{\ell_j+1} - \widehat{x}, Jx - J\widehat{x}\right\rangle. \tag{3.32}$$

It follows from (3.29) that $\varphi(\widehat{x}, x_{\ell_j}) \to 0$. This together with (3.30) gives

$$\varphi\left(\widehat{x}, x_{\ell_j+1}\right) \longrightarrow 0. \tag{3.33}$$

But $\varphi(\widehat{x}, x_j) \leq \varphi(\widehat{x}, x_{\ell_j+1})$ for all sufficiently large numbers j, we conclude that $x_j \to \widehat{x}$.

From the two cases, we can conclude that $\{x_n\}$ converges strongly to $\widehat{x}$ and the proof is finished. □

Setting $m = 1$, $F_1 = F \equiv 0$, and $r_{1,n} \equiv r_n$ in Theorem 3.1, we have the following.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$*

be a relatively nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \neq \emptyset$. *Let* $T_{r_n}^F$ *be the resolvent of* F *with a positive real sequence* $\{r_n\}$ *such that* $\liminf_{n\to\infty} r_n > 0$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JST_{r_n}^F x_n\right) \quad (n \geq 1), \tag{3.34}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap \mathrm{EP}(F)} x$.

Setting $F_1 \equiv 0$ and $r_{1,n} \equiv 1$ in Corollary 3.2, we have the following result.

Corollary 3.3. *Let* E *be a uniformly convex and uniformly smooth Banach space,* C *a nonempty closed convex subset of* E, *and* $S : C \to E$ *a relatively nonexpansive mapping such that* $F(S) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JS\Pi_C x_n\right) \quad (n \geq 1), \tag{3.35}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{F(S)} x$.

Next, we prove a strong convergence theorem for finding an element of the set of solutions to a system of equilibrium problems in a uniformly convex and uniformly smooth Banach space.

Theorem 3.4. *Let* E *be a uniformly convex and uniformly smooth Banach space,* C *a nonempty closed convex subset of* E, $\{F_i\}_{i=1}^m$ *a finite family of a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying conditions (A1)–(A4), and* $\cap_{i=1}^m \mathrm{EP}(F_i) \neq \emptyset$. *Let* $\{T_{r_{i,n}}^{F_i}\}_{i=1}^m$ *be a finite family of the resolvents of* F_i *with positive real sequences* $\{r_{i,n}\}$ *such that* $\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in E$ *and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + \beta_n Jx_n + \gamma_n JT_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n\right) \quad (n \geq 1), \tag{3.36}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *or* $\liminf_{n\to\infty} \beta_n = 0$.

Then, $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i=1}^m \mathrm{EP}(F_i)} x$.

Proof. For each $n \geq 1$, setting

$$\begin{aligned} z_n^k &= T_{r_{k,n}}^{F_k} T_{r_{k1,n}}^{F_{k-1}} \cdots T_{r_{1,n}}^{F_1} x_n, \quad (k = 1, 2, \ldots, m), \\ y_n &= J^{-1}\left(\frac{\beta_n}{1-\alpha_n} J x_n + \frac{\gamma_n}{1-\alpha_n} J z_n^m\right). \end{aligned} \tag{3.37}$$

Since $\cap_{i=1}^m \mathrm{EP}(F_i)$ is nonempty, closed, and convex, we put $\widehat{x} = \Pi_{\cap_{i=1}^m \mathrm{EP}(F_i)} x$. Using the same proof of Theorem 3.1 when S is the identity operator, we can see that

$$\varphi(\widehat{x}, y_n) \leq \varphi(\widehat{x}, x_n) - \frac{\gamma_n}{1-\alpha_n} \sum_{k=1}^{m} \varphi\left(z_n^k, z_n^{k-1}\right), \tag{3.38}$$

$$\varphi(\widehat{x}, x_{n+1}) \leq (1-\alpha_n)\varphi(\widehat{x}, x_n) + 2\alpha_n \langle x_{n+1} - \widehat{x}, Jx - J\widehat{x}\rangle. \tag{3.39}$$

The rest of the proof will be divided into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\widehat{x}, x_n)\}_{n=n_0}^{\infty}$ is non-increasing. In this situation, $\{\varphi(\widehat{x}, x_n)\}$ is then convergent. Then,

$$\varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) \longrightarrow 0. \tag{3.40}$$

Notice that

$$\varphi(\widehat{x}, x_{n+1}) \leq \alpha_n \varphi(\widehat{x}, x) + (1-\alpha_n)\varphi(\widehat{x}, y_n). \tag{3.41}$$

From condition (ii),

$$\begin{aligned} \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, y_n) &= \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) + \varphi(\widehat{x}, x_{n+1}) - \varphi(\widehat{x}, y_n) \\ &\leq \varphi(\widehat{x}, x_n) - \varphi(\widehat{x}, x_{n+1}) + \alpha_n\left(\varphi(\widehat{x}, x) - \varphi(\widehat{x}, y_n)\right) \longrightarrow 0. \end{aligned} \tag{3.42}$$

It follows from (3.38) that

$$\frac{\gamma_n}{1-\alpha_n} \sum_{k=1}^{m} \varphi\left(z_n^k, z_n^{k-1}\right) \longrightarrow 0. \tag{3.43}$$

By the assumptions (i), (ii), and (iv),

$$\varphi\left(z_n^k, z_n^{k-1}\right) \longrightarrow 0 \quad (k = 1, 2, \ldots, m). \tag{3.44}$$

By Remark 2.3, we get

$$z_n^k - z_n^{k-1} \longrightarrow 0 \quad (k = 1, 2, \ldots, m). \tag{3.45}$$

Consequently,

$$\begin{aligned}\varphi(x_n, y_n) &\leq \frac{\beta_n}{1-\alpha_n}\varphi(x_n, x_n) + \frac{\gamma_n}{1-\alpha_n}\varphi(x_n, z_n^m) = \frac{\gamma_n}{1-\alpha_n}\varphi\left(z_n^0, z_n^m\right) \longrightarrow 0, \\ \varphi(y_n, x_{n+1}) &\leq \alpha_n\varphi(y_n, x) + (1-\alpha_n)\varphi(y_n, y_n) = \alpha_n\varphi(y_n, x) \longrightarrow 0.\end{aligned} \tag{3.46}$$

This implies that

$$x_{n+1} - x_n \longrightarrow 0. \tag{3.47}$$

Since $\{x_n\}$ is bounded and E is reflexive, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle x_n - \hat{x}, Jx - J\hat{x}\rangle = \lim_{j\to\infty}\left\langle x_{n_j} - \hat{x}, Jx - J\hat{x}\right\rangle = \langle w - \hat{x}, Jx - J\hat{x}\rangle. \tag{3.48}$$

Let $k = 1, 2, \ldots, m$ be fixed. Then, $z_{n_j}^k \rightharpoonup w$ as $j \to \infty$. From $\liminf_{n\to\infty} r_{k,n} > 0$ and (3.14), we have

$$\lim_{n\to\infty}\frac{1}{r_{k,n}}\left\|Jz_n^k - Jz_n^{k-1}\right\| = 0. \tag{3.49}$$

Then,

$$F_k\left(z_n^k, y\right) + \frac{1}{r_{k,n}}\left\langle y - z_n^k, Jz_n^k - Jz_n^{k-1}\right\rangle \geq 0, \quad \forall y \in C. \tag{3.50}$$

Replacing n by n_j, we have from (A2) that

$$\frac{1}{r_{k,n_j}}\left\langle y - z_{n_j}^k, Jz_{n_j}^k - Jz_{n_j}^{k-1}\right\rangle \geq -F_k\left(z_{n_j}^k, y\right) \geq F_k\left(y, z_{n_j}^k\right), \quad \forall y \in C. \tag{3.51}$$

Letting $j \to \infty$, we have from (3.49) and (A4) that

$$F_k(y, w) \leq 0, \quad \forall y \in C. \tag{3.52}$$

From Lemma 2.12, we have $w \in \mathrm{EP}(F_k)$. By Lemma 2.4(a), we immediately obtain that

$$\limsup_{n\to\infty}\langle x_{n+1} - \hat{x}, Jx - J\hat{x}\rangle = \limsup_{n\to\infty}\langle x_n - \hat{x}, Jx - J\hat{x}\rangle = \langle w - \hat{x}, Jx - J\hat{x}\rangle \leq 0. \tag{3.53}$$

It follows from Lemma 2.7 and (3.39) that $\varphi(\hat{x}, x_n) \to 0$. Then, $x_n \to \hat{x}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\hat{x}, x_{n_i}) < \varphi(\hat{x}, x_{n_i+1}), \tag{3.54}$$

for all $i \in \mathbb{N}$. Using the same proof of *Case 2* in Theorem 3.1, we also conclude that $x_j \to \hat{x}$.

From the two cases, we can conclude that $\{x_n\}$ converges strongly to $\hat{x}$. □

Finally, we give two explicit examples validating the assumptions in Theorem 3.1 as follows.

Example 3.5 (Optimization). *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty bounded closed convex subset of E, and $f : C \to \mathbb{R}$ a lower semicontinuous and convex functional. For instance, let $E = \mathbb{R}$, $C = [0,1]$ and $f : [0,1] \to \mathbb{R}$ be defined dy*

$$f(x) = \begin{cases} 0, & \text{if } \ x = 0, 1; \\ x \log x + (1 - x)\log(1 - x), & \text{if } \ x \in (0, 1). \end{cases} \tag{3.55}$$

Then f is lower semicontinuous and convex. For each $i = 1, 2, \ldots, m$, let $F_i : C \times C \to \mathbb{R}$ be defined by $F_i(x, y) := f(y) - f(x)$ for all $x, y \in C$. It is known [1, 11] that F_i satisfies conditions (A1)–(A4), and $\mathrm{EP}(F_i) \neq \emptyset$. Let $S = \Pi_C$. Then, S is relatively nonexpansive of E into C (see [5, 6]) and $F(S) = C$. Then, $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) = \mathrm{EP}(F_i) \neq \emptyset$. Applying Theorem 3.1, we conclude that the sequence defined by (3.1) converges strongly to $\Pi_\Omega x$.

Example 3.6 (The convex feasibility problem). *Let E be a real Hilbert space, let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of E satisfying $C := \cap_{i=1}^m C_i \neq \emptyset$ (e.g., $C_1 = C_2 = \cdots = C_m = C \neq \emptyset$). Let $\{F_i\}_{i=1}^m$ be a finite family of bifunctions of $E \times E$ into $\mathbb{R}$ defined by*

$$F_i(x, y) = \frac{1}{2}\langle y - x, x - P_{C_i}x \rangle \quad \forall x, y \in E, \tag{3.56}$$

where P_{C_i} is a metric projection from E onto C_i. It is known [3, Lemma 2.15(iv)] that F_i satisfies conditions (A1)–(A4) and $\mathrm{EP}(F_i) = C_i$. Let $S = P_C$. Then, S is relatively nonexpansive of E into C (see [5, 6]) and then $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) = C \neq \emptyset$. Applying Theorem 3.1, we conclude that the sequence defined by (3.1) converges strongly to $\Pi_\Omega x$.

4. Deduced Theorems in Hilbert Spaces

In Hilbert spaces, if S is quasi-nonexpansive such that $I - S$ is demiclosed at zero, then S is relatively nonexpansive. We obtain the following result.

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{F_i\}_{i=1}^m$ a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) \neq \emptyset$. Let*

$\{T^{F_i}_{r_{i,n}}\}_{i=1}^m$ *be a finite family of the resolvents of* F_i *with real sequences* $\{r_{i,n}\}$ *such that* $\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in H$ *and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \ldots, T^{F_1}_{r_{1,n}} x_n \quad (n \geq 1), \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $P_\Omega x$.

Applying Theorem 4.1 and using the technique in [41], we have the following result.

Theorem 4.2. *Let* H *be a Hilbert space,* C *a nonempty closed convex subset of* H, f *a contraction of* H *into itself (i.e., there is* $a \in (0,1)$ *such that* $\|f(x) - f(y)\| \leq a\|x - y\|$ *for all* $x, y \in H$*),* $\{F_i\}_{i=1}^m$ *a finite family of a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying conditions (A1)–(A4), and* $S : C \to E$ *be a nonexpansive mapping such that* $\Omega := F(S) \cap (\cap_{i=1}^m \mathrm{EP}(F_i)) \neq \emptyset$. *Let* $\{T^{F_i}_{r_{i,n}}\}_{i=1}^m$ *be a finite family of the resolvents of* F_i *with real sequences* $\{r_{i,n}\}$ *such that* $\liminf_{n\to\infty} r_{i,n} > 0$ *for all* $i = 1, 2, \ldots, m$. *Let* $\{x_n\}$ *be a sequence generated by* $x, x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \ldots, T^{F_1}_{r_{1,n}} x_n, \quad (n \geq 1), \tag{4.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n (1 - \beta_n) > 0$.

Then, $\{x_n\}$ *converges strongly to* z *such that* $z = P_\Omega f(z)$.

Proof. We note that $P_\Omega f$ is contraction. By Banach contraction principle, let z be the fixed point of $P_\Omega f$ and $\{y_n\}$ a sequence generated by $y_1 = x_1 \in H$ and

$$y_{n+1} = \alpha_n f(z) + \beta_n y_n + \gamma_n S T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \ldots, T^{F_1}_{r_{1,n}} y_n, \quad (n \geq 1). \tag{4.3}$$

Using Theorem 4.1, we have $y_n \to z = P_\Omega f(z)$. Since S and $T_{r_{k,n}}^{F_k}(k = 1,2,\ldots,m)$ are nonexpansive,

$$\begin{aligned}\|y_{n+1} - x_{n+1}\| \le{}& \alpha_n \|f(x_n) - f(z)\| + \beta_n \|y_n - x_n\| \\ &+ \gamma_n \left\| ST_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} y_n - ST_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n \right\| \\ \le{}& \alpha_n a \|x_n - z\| + (\beta_n + \gamma_n) \|y_n - n_n\| \\ \le{}& \alpha_n a (\|x_n - y_n\| + \|y_n - z\|) + (\beta_n + \gamma_n) \|x_n - y_n\| \\ ={}& (1 - \alpha_n(1-a)) \|y_n - x_n\| + \alpha_n (1-a) \left(\frac{a}{1-a} \|y_n - z\| \right).\end{aligned} \tag{4.4}$$

Applying Lemma 2.7, $y_n - x_n \to 0$ and so $x_n \to z = P_\Omega f(z)$. □

Setting $m = 1$, $F_1 = F \equiv 0$, and $r_{1,n} \equiv r_n$ in Theorem 4.1, we have the following.

Corollary 4.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $F(S) \cap \mathrm{EP}(F) \ne \emptyset$. Let $T_{r_n}^F$ be the resolvent of F with a positive real sequence $\{r_n\}$ such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S T_{r_n}^F x_n \quad (n \ge 1), \tag{4.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{EP}(F)} x$.

Corollary 4.4. *Let H be a Hilbert space, C a nonempty closed convex subset of H, f a contraction of H into itself, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $S : C \to E$ a nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \ne \emptyset$. Let $T_{r_n}^F$ be the resolvent of F with a positive real sequence $\{r_n\}$ such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S T_{r_n}^F x_n \quad (n \ge 1), \tag{4.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, $\{x_n\}$ converges strongly to z such that $z = P_{F(S)\cap \mathrm{EP}(F)} f(z)$.

Remark 4.5. Corollary 4.4 improves and extends [42, Theorem 5]. More precisely, the conditions $\lim_{n\to\infty}(r_{n+1} - r_n) = \infty$ are removed.

Setting $F \equiv 0$ and $r_n \equiv 1$ in Corollary 4.3, we have the following.

Corollary 4.6. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $S : C \to E$ a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S P_C x_n \quad (n \geq 1), \tag{4.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\liminf_{n\to\infty}\beta_n(1 - \beta_n) > 0$.

Then, $\{x_n\}$ converges strongly to $P_{F(S)}x$.

Applying Theorem 3.4, we have the following result.

Theorem 4.7. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{F_i\}_{i=1}^{m}$ a finite family of a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $\cap_{i=1}^{m}\mathrm{EP}(F_i) \neq \emptyset$. Let $\{T_{r_{i,n}}^{F_i}\}_{i=1}^{m}$ be a finite family of the resolvents of F_i with positive real sequences $\{r_{i,n}\}$ such that $\liminf_{n\to\infty} r_{i,n} > 0$ for all $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{1,n}}^{F_1} x_n \quad (n \geq 1), \tag{4.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n \equiv 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,*s*

(iv) $\liminf_{n\to\infty}\beta_n(1 - \beta_n) > 0$ *or* $\liminf_{n\to\infty}\beta_n = 0$.

Then $\{x_n\}$ converges strongly to $P_{\cap_{i=1}^{m}\mathrm{EP}(F_i)}x$.

Setting $m = 1$, $F_1 = F \equiv 0$, $r_{1,n} \equiv r_n$, and $\beta_n \equiv 0$ in Theorem 4.4, we have the following result.

Corollary 4.8 (see [35, Corollary 4.4]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)–(A4), and $\mathrm{EP}(F) \neq \emptyset$. Let $T_{r_n}^{F}$ the resolvent of F with a positive real sequence $\{r_n\}$ such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated by $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_{r_n}^{F} x_n \quad (n \geq 1), \tag{4.9}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,

Then, $\{x_n\}$ *converges strongly to* $P_{\mathrm{EP}(F)}x$*.*

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A2 **Weerayuth Nilsrakoo**, Halpern-type iterations for strongly relatively nonexpansive mappings in Banach spaces, Comp. Math. Appl. 62 (2011) 4656-4666

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Halpern-type iterations for strongly relatively nonexpansive mappings in Banach spaces

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ABSTRACT

In this paper, we note that the main convergence theorem in Zhang et al. (2011) [21] is incorrect and we prove a correction. We also modify Halpern's iteration for finding a fixed point of a strongly relatively nonexpansive mapping in a Banach space. Consequently, two strong convergence theorems for a relatively nonexpansive mapping and for a mapping of firmly nonexpansive type are deduced. Using the concept of duality theorems, we obtain analogue results for strongly generalized nonexpansive mappings and for mappings of firmly generalized nonexpansive type. In addition, we study two strong convergence theorems concerning two types of resolvents of a maximal monotone operator in a Banach space.

1. Introduction

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping of a closed and convex subset of a Banach space E (that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). In 1953, Mann [1] introduced the following iterative method: a sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, 3, \ldots, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. It is known that under appropriate conditions the sequence $\{x_n\}$ converges only *weakly* to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly; for example, see [2].

Several attempts to construct the iteration method guaranteeing the strong convergence have been made. For example, Halpern [3] proposed the following so-called Halpern iteration: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, 3, \ldots, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ or $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Another approach was proposed by Bauschke and Combettes [4]. More precisely, their algorithm is defined by

$$\begin{cases} x_1 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \ldots, \end{cases}$$

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where $\limsup_{n\to\infty} \alpha_n < 1$ and P_K denotes the metric projection from a Hilbert space H onto a closed and convex subset K of H. It should be noted here that the iteration above works only in Hilbert space setting. To extend this iteration to a Banach space, a *relatively nonexpansive* mapping [5–7] was introduced. Before we give its definition, we recall some notations. The strong and weak convergences of a sequence $\{x_n\}$ in a Banach space E to an element $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let E be a smooth Banach space and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the pairing between E and E^*. Let J be the normalized duality mapping from E to E^*. Alber [8] considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$. Using this functional, Matsushita and Takahashi [5–7,9] studied and investigated the following mappings in Banach spaces. Suppose that C is a subset of a smooth Banach space E. A mapping $T : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(T) \neq \varnothing$, where $F(T)$ denotes the fixed points set of T;
(R2) $\varphi(p, Tx) \leq \varphi(p, x)$ for all $p \in F(T)$ and $x \in C$;
(R3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

In a Hilbert space H, the duality mapping J is the identity mapping and $\varphi(x, y) = \|x-y\|^2$ for all $x, y \in H$. Hence, if $T : C \to H$ is a nonexpansive mapping of a nonempty, closed and convex subset C of H, then it is relatively nonexpansive.

There are many methods for approximating fixed points of relatively nonexpansive mappings (see, *e.g.*, [5–7,10–21]). In 2004, Matsushita and Takahashi [5] studied the Mann-type iteration for a relatively nonexpansive mapping defined by

$$\begin{cases} x_1 \in C \quad \text{is arbitrary,} \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \quad n = 1, 2, 3, \ldots, \end{cases}$$

where $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, the interior of $F(T)$ is nonempty and Π_C denotes the generalized projection from E onto C. Moreover, they proposed the following analogue of the Bauschke and Combettes algorithm:

$$\begin{cases} x_1 \in C \quad \text{is arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \varphi(z, y_n) \leq \varphi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_1 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \ldots, \end{cases}$$

where $\limsup_{n\to\infty} \alpha_n < 1$.

Recently, Zhang et al. [21] modify Halpern's iteration for finding fixed point of relatively nonexpansive mappings in the following result.

Theorem 1.1 (*Cf.* [21, *Theorem 4.1*]). *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to C$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)JTx_n), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to a fixed point of T.

Careful reading of the proof of Theorem 1.1, leads to the fact that the inequality (4.5) is not correct. Indeed, the assumptions, for each $u \in F(T)$,

$$\varphi(u, x_{n+1}) \leq \alpha_n \varphi(u, x_1) + (1-\alpha_n)\varphi(u, Tx_n),$$

$\lim_{n\to\infty} \alpha_n = 0$ and $\varphi(u, Tx_n) \leq \varphi(u, x_n)$ are *not enough* to guarantee that

$$\varphi(u, x_{n+1}) \leq \varphi(u, x_n).$$

Consequently, the inequalities (4.10)–(4.15) are also not correct. Moreover, we know that the interior of the singleton fixed point set of T is empty and there are many relatively nonexpansive mappings whose fixed point sets are singleton.

We say that a relatively nonexpansive mapping $T : C \to E$ is *strongly relatively nonexpansive* [9,22] if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p, x_n) - \varphi(p, Tx_n) \to 0$ for some $p \in F(T)$ it follows that $\varphi(Tx_n, x_n) \to 0$. Note that the notion of a strongly nonexpansive mapping with respect to the norm was first introduced and studied in [23] (see also [24]).

Example 1.2 (*Cf.* [25,26]). Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a relatively nonexpansive mapping. Suppose that there exists $\kappa > 0$ such that

$$\varphi(p, Tx) + \kappa\varphi(Tx, x) \leq \varphi(p, x)$$

for all $p \in F(T)$ and $x \in C$. Then T is a strongly nonexpansive mapping.

Many authors studied weak and strong convergence theorems of strongly relatively nonexpansive mappings (see, for instance, [10,12,19,25–30] and the references therein).

Another well-known family of mappings is the class of firmly nonexpansive mappings, where a mapping $T : C \to E$ is called *firmly nonexpansive type* [27] if

$$\varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(Tx, x) + \varphi(Ty, y) \le \varphi(Tx, y) + \varphi(Ty, x)$$

for all $x, y \in C$. See [19,27–31] for more information on firmly nonexpansive type mappings. It is easy to see that if T is firmly nonexpansive type with $I - T$ is demi-closed at zero, then it is strongly relatively nonexpansive. Furthermore, there is a mapping which is strongly relatively nonexpansive but is not firmly nonexpansive type as the following example shows.

Example 1.3. Let E be a smooth, strictly convex, and reflexive Banach space and let $T : E \to E$ be a mapping defined by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \left(\frac{2}{3}\sin\frac{1}{\|x\|}\right)x & \text{if } x \neq 0. \end{cases}$$

Then $F(T) = \{0\}$. We observe that

$$\begin{aligned} \varphi(Tx, x) &= \left(\frac{4}{9}\sin^2\frac{1}{\|x\|}\right)\|x\|^2 - 2\left(\frac{2}{3}\sin^2\frac{1}{\|x\|}\right)\langle x, Jx\rangle + \|x\|^2 \\ &= \left(1 - \frac{2}{3}\sin\frac{1}{\|x\|}\right)^2\|x\|^2 \\ &\le \left(1 + \frac{2}{3}\right)^2\|x\|^2 = \frac{25}{9}\|x\|^2. \end{aligned}$$

Then

$$\begin{aligned} \varphi(0, Tx) &= \left(\frac{4}{9}\sin^2\frac{1}{\|x\|}\right)\|x\|^2 \le \frac{4}{9}\|x\|^2 \\ &= \|x\|^2 - \frac{5}{9}\|x\|^2 \\ &\le \varphi(0, x) - \frac{1}{5}\varphi(Tx, x) \end{aligned}$$

for all $x \in E$. This implies that T is relatively nonexpansive and

$$\frac{1}{5}\varphi(Tx, x) \le \varphi(0, x) - \varphi(0, Tx)$$

for all $x \in E$. It follows that $\varphi(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence such that $\varphi(0, x_n) - \varphi(0, Tx_n) \to 0$. That is, T is strongly relatively nonexpansive. Let $x_0 \in S_E$ be fixed. Put $x = \frac{2}{\pi}x_0$ and $y = \frac{2}{3\pi}x_0$. Then $Tx = \frac{4}{3\pi}x_0$ and $Ty = -\frac{4}{9\pi}x_0$. It follows that

$$\varphi(Tx, Ty) = \frac{16}{9\pi^2} - 2\left\langle \frac{4}{3\pi}x_0, J\left(-\frac{4}{9\pi}x_0\right)\right\rangle + \frac{16}{81\pi^2} = \frac{256}{81\pi^2} = \varphi(Ty, Tx).$$

Consequently,

$$\begin{aligned} \varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(Tx, x) + \varphi(Ty, y) &= 2\frac{256}{81\pi^2} + \frac{4}{9\pi^2} + \frac{100}{81\pi^2} \\ &= \frac{4}{9\pi^2} + \frac{612}{81\pi^2} \\ &> \frac{4}{9\pi^2} + \frac{484}{81\pi^2} = \varphi(Tx, y) + \varphi(Ty, x). \end{aligned}$$

Hence, T is not firmly nonexpansive type.

The purpose of this paper is to prove for a class of strongly relatively nonexpansive mappings that only Conditions (C1) and (C2) are sufficient for the strong convergence theorem of Halpern's iterations to a fixed point of T without the assumption of the nonempty interior of the fixed point set of T. Consequently, a strong convergence theorem for a relatively nonexpansive mapping is deduced and a correction for [21, Theorem 4.1] is presented. Using a concept of duality theorems (see, for instance, [32,33]), we obtain an analogue result for a strongly generalized nonexpansive mapping. Moreover, two corresponding strong convergence theorems for a firmly nonexpansive type mapping [27] and a firmly generalized nonexpansive type mapping [34] are deduced. Finally, we discuss two strong convergence theorems concerning two types of resolvents of a maximal monotone operator in a Banach space.

2. Preliminaries

We present several definitions and preliminaries which are needed in this paper. We say that a Banach space E is *strictly convex* if the following implication holds for any $x, y \in E$:

$$\|x\| = \|y\| = 1 \quad \text{and} \quad x \neq y \text{ imply } \left\| \frac{x+y}{2} \right\| < 1.$$

We say that E is *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{and} \quad \|x - y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex.

Let S_E denote the unit sphere of E, that is, $S_E := \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S_E$. In this case, E is said to be *smooth*. The norm of E is said to be *uniformly Gâteaux differentiable* (resp. *Fréchet differentiable*) if for each $y \in S_E$ (resp. for each $x \in S_E$) the limit (2.1) is attained uniformly for any $x \in S_E$ (resp. uniformly for any $y \in S_E$). The norm of E is said to be *uniformly Fréchet differentiable* (and E is called *uniformly smooth*) if the limit (2.1) is attained uniformly for any $x, y \in S_E$. This is well-known that

(1) if E is reflexive, then E is smooth if and only if E^* is strictly convex;
(2) E is uniformly smooth if and only if E^* is uniformly convex.

The value of $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^* \rangle$. The *duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

We also know the following properties (see, *e.g.*, [35] for details):

(a) $J(x) \neq \varnothing$ for each $x \in E$.
(b) If E is smooth, then J is single valued.
(c) If E is strictly convex, then $J(x) \cap J(y) = \varnothing$ for all $x \neq y$.
(d) If E has a uniformly Gâteaux differentiable norm, then J is uniformly norm-to-weak* continuous on each bounded subset of E.
(e) If E has a Fréchet differentiable norm, then J is norm-to-norm continuous.
(f) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(g) If E is a Hilbert space, then J is the identity operator.

Let E be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ (see [8]) is defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2.$$

It is obvious from the definition of the function φ that

$$(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2$$

and

$$\varphi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \leq \lambda\varphi(x, y) + (1-\lambda)\varphi(x, z) \tag{2.2}$$

for all $\lambda \in [0, 1]$ and $x, y, z \in E$. It is also easy to check that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0$ implies that $\varphi(x_n, y_n) \to 0$.

Lemma 2.1 (*Cf.* [36, *Proposition 2*]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

Remark 2.2. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\varphi(x_n, y_n) \to 0 \Longleftrightarrow x_n - y_n \to 0 \Longleftrightarrow Jx_n - Jy_n \to 0.$$

Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. It is known that [8,36] for any $x \in E$ there exists a unique point $\widehat{x} \in C$ such that

$$\varphi(\widehat{x}, x) = \min_{y \in C} \varphi(y, x).$$

Following Alber [8], we denote such an element $\widehat{x}$ by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that, in a Hilbert space, the mapping Π_C coincides with the metric projection P_C.

Lemma 2.3 (*Cf. [36, Propositions 4 and 5]*). *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E, $x \in E$ and $\widehat{x} \in C$. Then*

(a) $\widehat{x} = \Pi_C x$ *if and only if* $\langle y - \widehat{x}, Jx - J\widehat{x}\rangle \le 0$ *for all* $y \in C$;
(b) $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \le \varphi(y, x)$ *for all* $y \in C$.

Remark 2.4. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Let E be a reflexive, strictly convex and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. We will use the following mapping $V : E \times E^* \to \mathbb{R}$ studied in [8]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.5 (*Cf. [8] and [37, Lemma 3.2]*). *Let E be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \le V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 (*Cf. [38, Lemma 2.1]*). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0, 1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy the following conditions: $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n \le 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 (*Cf. [39, Lemma 3.1]*). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \quad \text{and} \quad a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.8 (*Cf. [40, Lemma 1]*). *Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \le a_n + b_n, \quad n = 1, 2, 3, \ldots.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

3. Strongly relatively nonexpansive mappings

In this section, we use Halpern's idea [3] for finding fixed point of strongly relatively nonexpansive mappings in a uniformly convex and smooth Banach space.

A mapping $T : C \to E$ is said to be *relatively quasi-nonexpansive* [15] if it satisfies only (R1) and (R2). In a Hilbert space H, the duality mapping J is the identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $T : C \to H$ is relatively quasi-nonexpansive, then it is quasi-nonexpansive, that is, $\|p - Tx\| \le \|p - x\|$ for all $p \in F(T)$ and $x \in C$. In the sequel, we shall need the following lemmas.

Lemma 3.1 (*Cf. [15, Lemma 2.5]*). *Let C be a nonempty, closed and convex subset of a strictly convex and smooth Banach space E and let $T : C \to E$ be a relatively quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 3.2. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E, $T : C \to E$ be a relatively nonexpansive mapping, $x \in E$ and $\widehat{x} = \Pi_{F(T)}x$. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences such that $\varphi(Tx_n, x_n) \to 0$ and $\varphi(Tx_n, y_n) \to 0$. Then*

$$\limsup_{n\to\infty} \langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle \le 0.$$

Proof. From the uniform convexity of E and Lemma 2.1,

$$Tx_n - x_n \to 0 \quad \text{and} \quad y_n - x_n \to 0.$$

From property (R3) of the mapping T, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y \in F(T)$ and

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle = \limsup_{n\to\infty}\langle x_n - \widehat{x}, Jx - J\widehat{x}\rangle = \lim_{i\to\infty}\langle x_{n_i} - \widehat{x}, Jx - J\widehat{x}\rangle.$$

From Lemma 2.3(a), we immediately obtain that

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle = \langle y - \widehat{x}, Jx - J\widehat{x}\rangle \le 0. \quad \square$$

Using the technique in [16,39], we obtain the following theorem.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E and let $T : C \to E$ be a strongly relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $u \in E$, $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)JTx_n), \quad n = 1, 2, 3, \ldots, \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

Proof. Let

$$y_n \equiv J^{-1}(\alpha_n Ju + (1-\alpha_n)JTx_n).$$

Then $x_{n+1} \equiv \Pi_C y_n$. Since $F(T)$ is nonempty, closed and convex, we put

$$\widehat{u} = \Pi_{F(T)}u.$$

We first show that $\{x_n\}$ is bounded. From Remark 2.4 and (2.2), we have

$$\begin{aligned}\varphi(\widehat{u}, x_{n+1}) &\le \varphi(\widehat{u}, y_n)\\ &\le \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, Tx_n)\\ &\le \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, x_n)\\ &\le \max\{\varphi(\widehat{u}, u), \varphi(\widehat{u}, x_n)\}.\end{aligned}$$

By induction, we have

$$\varphi(\widehat{u}, x_{n+1}) \le \max\{\varphi(\widehat{u}, u), \varphi(\widehat{u}, x_1)\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is the sequence $\{Tx_n\}$. From Condition (C1) and (2.2), we obtain

$$\varphi(Tx_n, y_n) \le \alpha_n\varphi(Tx_n, u) + (1-\alpha_n)\varphi(Tx_n, Tx_n) = \alpha_n\varphi(Tx_n, u) \to 0. \tag{3.2}$$

From Remark 2.4, Lemma 2.5 and (2.2), we have

$$\begin{aligned}\varphi(\widehat{u}, x_{n+1}) &\le \varphi(\widehat{u}, y_n) = V(\widehat{u}, Jy_n)\\ &\le V(\widehat{u}, Jy_n - \alpha_n(Ju - J\widehat{u})) - 2\langle y_n - \widehat{u}, -\alpha_n(Ju - J\widehat{u})\rangle\\ &= V(\widehat{u}, \alpha_n J\widehat{u} + (1-\alpha_n)JTx_n) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle\\ &= \varphi(\widehat{u}, J^{-1}(\alpha_n J\widehat{u} + (1-\alpha_n)JTx_n)) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle\\ &\le \alpha_n\varphi(\widehat{u}, \widehat{u}) + (1-\alpha_n)\varphi(\widehat{u}, Tx_n) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle\\ &\le (1-\alpha_n)\varphi(\widehat{u}, x_n) + 2\alpha_n\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle,\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$.

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\widehat{u}, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(\widehat{u}, x_n)\}$ is then convergent. Then

$$\varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, x_{n+1}) \to 0. \tag{3.4}$$

Notice that

$$\varphi(\widehat{u}, x_{n+1}) \le \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, Tx_n).$$

It follows from (3.4) and Condition (C1) that

$$\begin{aligned}\varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, Tx_n) &= \varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, x_{n+1}) + \varphi(\widehat{u}, x_{n+1}) - \varphi(\widehat{u}, Tx_n)\\ &\le \varphi(\widehat{u}, x_n) - \varphi(\widehat{u}, x_{n+1}) + \alpha_n(\varphi(\widehat{u}, u) - \varphi(\widehat{u}, Tx_n)) \to 0.\end{aligned}$$

Since T is strongly relatively nonexpansive,

$$\varphi(Tx_n, x_n) \to 0.$$

It follows from (3.2) and Lemma 3.2 that

$$\limsup_{n\to\infty}\langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle \le 0.$$

Hence the conclusion follows from Lemmas 2.6 and 2.1, and (3.3).

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\widehat{u}, x_{n_i}) < \varphi(\widehat{u}, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\varphi(\widehat{u}, x_{m_k}) \le \varphi(\widehat{u}, x_{m_k+1}) \quad \text{and} \quad \varphi(\widehat{u}, x_k) \le \varphi(\widehat{u}, x_{m_k+1})$$

for all $k \in \mathbb{N}$. This together with Condition (C1) gives

$$\begin{aligned}\varphi(\widehat{u}, x_{m_k}) - \varphi(\widehat{u}, Tx_{m_k}) &= \varphi(\widehat{u}, x_{m_k}) - \varphi(\widehat{u}, x_{m_k+1}) + \varphi(\widehat{u}, x_{m_k+1}) - \varphi(\widehat{u}, Tx_{m_k})\\ &\le \alpha_{m_k}\left(\varphi(\widehat{u}, u) - \varphi(\widehat{u}, Tx_{m_k})\right) \to 0.\end{aligned}$$

This implies that

$$\varphi(Tx_{m_k}, x_{m_k}) \to 0.$$

It now follows from (3.2) and Lemma 3.2 that

$$\limsup_{k\to\infty}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \le 0. \tag{3.5}$$

From (3.3), we have

$$\varphi(\widehat{u}, x_{m_k+1}) \le (1-\alpha_{m_k})\varphi(\widehat{u}, x_{m_k}) + 2\alpha_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle. \tag{3.6}$$

Since $\varphi(\widehat{u}, x_{m_k}) \le \varphi(\widehat{u}, x_{m_k+1})$, we have

$$\begin{aligned}\alpha_{m_k}\varphi(\widehat{u}, x_{m_k}) &\le \varphi(\widehat{u}, x_{m_k}) - \varphi(\widehat{u}, x_{m_k+1}) + 2\alpha_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle\\ &\le 2\alpha_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle.\end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\varphi(\widehat{u}, x_{m_k}) \le 2\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle.$$

It follows from (3.5) that $\varphi(\widehat{u}, x_{m_k}) \to 0$. This together with (3.6) gives

$$\varphi(\widehat{u}, x_{m_k+1}) \to 0.$$

But $\varphi(\widehat{u}, x_k) \le \varphi(\widehat{u}, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to \widehat{u}$.

This implies that $x_n \to \widehat{u}$ and the proof is complete. □

Remark 3.4. The result [41, Corollary 8] is a special case of our result.

Lemma 3.5 (*Cf.* [12, *Lemmas 3.1 and 3.2*]). *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be a relatively nonexpansive mapping. Let U be the mapping defined by*

$$U = J^{-1}(\lambda J + (1-\lambda)JT),$$

where $\lambda \in (0, 1)$, then $U : C \to E$ is strongly relatively nonexpansive and $F(U) = F(T)$.

Applying Theorem 3.3 and Lemma 3.5, we have the following result.

Theorem 3.6 (*Cf.* [16, *Corollary 5*]). *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to E$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $u \in E$, $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)(\lambda Jx_n + (1-\lambda)JTx_n))$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2), and $\lambda \in (0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

Remark 3.7. In Theorems 3.3 and 3.6, the condition of the nonempty interior of fixed point set of T is not needed.

We next prove a strong convergence theorem of our iteration in the presence of another condition and give the revised version of Theorem 1.1.

Theorem 3.8. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to E$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to z, where $z = \lim_{n\to\infty} \Pi_{F(T)}x_n$.*

Proof. We show only that $\{Jx_n\}$ is a Cauchy sequence. To this end, we revise inequalities (4.5) and (4.15) in the proof of [21, Theorem 4.1] as follows: for each $w \in F(T)$,

$$\varphi(w, x_{n+1}) \leq \varphi(w, x_n) + \alpha_n\varphi(w, x_1),$$

and there exist $p \in F(T)$, $h \in E$ with $\|h\| \leq 1$ and $r > 0$ such that $p + rh \in F(T)$ and

$$\|Jx_m - Jx_n\| \leq \frac{1}{2r}(\varphi(p, x_m) - \varphi(p, x_n)) + \frac{\varphi(p + rh, x_1)}{r}\sum_{i=m}^{n-1} \alpha_i,$$

for each $m < n$, respectively. Since $\sum_{n=1}^{\infty} \alpha_n < \infty$ from Lemma 2.8, we have $\lim_{n\to\infty} \varphi(w, x_n)$ exists and so $\{Jx_n\}$ is a Cauchy sequence. The rest of the proof is similar to the proof of [21, Theorem 4.1], so it is left for the reader to verify. □

From Theorem 3.3, we apply the result for finding fixed point of a firmly nonexpansive type mapping. Since firmly nonexpansive type mappings in a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable are strongly relatively nonexpansive [27, Theorem 5.2], we have the following results.

Theorem 3.9. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $T : C \to E$ be a firmly nonexpansive type mapping such that $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $\Pi_{F(T)}u$.*

Theorem 3.10. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be a firmly nonexpansive type mapping such that $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. If the interior of $F(T)$ is nonempty, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to z, where $z = \lim_{n\to\infty} \Pi_{F(T)}x_n$.*

4. Strongly generalized nonexpansive mappings

Let C be a subset of a smooth Banach space E. In 2007, Ibaraki and Takahashi [42] introduced the following mapping. A mapping $T : C \to E$ is *generalized nonexpansive* if the following properties are satisfied:

(G1) $F(T) \neq \varnothing$;
(G2) $\varphi(Tx, p) \leq \varphi(x, p)$ for all $x \in C$ and $p \in F(T)$.

A generalized nonexpansive mapping $T : C \to E$ is *strongly generalized nonexpansive* [16] if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(x_n, p) - \varphi(Tx_n, p) \to 0$ for some $p \in F(T)$ it follows that $\varphi(x_n, Tx_n) \to 0$. A mapping $T : C \to E$ satisfies *property* (G3) if whenever $\{x_n\}$ is a sequence in C such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $Jx_n - JTx_n \to 0$ it follows that $p \in F(T)$. Here $\overset{*}{\rightharpoonup}$ denotes the weak* convergence in the dual space. A mapping $R : E \to C$ is said to be a *sunny generalized nonexpansive retraction* if the following properties are satisfied:

(1) R is generalized nonexpansive;
(2) $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$;
(3) $Rx = x$ for all $x \in C$.

A nonempty subset C of E is said to be a *sunny generalized nonexpansive retract* (resp. *generalized nonexpansive retract*) of E if there exists a sunny generalized nonexpansive retraction (resp. generalized nonexpansive retraction) of E onto C (see [42] for more details). We know the following result.

Lemma 4.1 (*Cf. [43, Theorem 3.3]*). *Let C be a nonempty and closed subset of a reflexive, strictly convex and smooth Banach space E. Then the following are equivalent:*

(i) *C is a sunny generalized nonexpansive retract of E;*
(ii) *C is a generalized nonexpansive retract of E;*
(iii) *JC is closed and convex.*

In this case, the sunny generalized nonexpansive retraction from E onto C is given by $J^{-1}\Pi_{JC}J$, where Π_{JC} is the generalized projection from E^* onto JC.

In 2007, Ibaraki and Takahashi [44] proved that the sequence $\{x_n\}$ defined by (1.1) converges *weakly* to a fixed point of a generalized nonexpansive mapping T.

Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let $T : C \to E$ be a mapping. We define the *duality* $T^* : JC \to E^*$ of T by (see [32])

$$T^*x^* = JTJ^{-1}x^* \quad \text{for all } x^* \in JC.$$

The following duality theorem is proved in [33, Theorem 20].

Lemma 4.2. *Let C be a nonempty subset of a reflexive, strictly convex and smooth Banach space E. Let $T : C \to E$ be a strongly generalized nonexpansive mapping with property* (G3) *and let $T^* : JC \to E^*$ be the duality of T. Then T^* is strongly relatively nonexpansive and $F(T^*) = JF(T)$.*

Theorem 4.3. *Let C be a nonempty, closed and sunny generalized nonexpansive retract of a uniformly smooth Banach space E whose dual space has a Fréchet differentiable norm. Let $T : C \to E$ be a strongly generalized nonexpansive mapping with property* (G3). *Let $\{x_n\}$ be a sequence in C defined by $u \in E$, $x_1 \in C$ and*

$$x_{n+1} = R_C(\alpha_n u + (1 - \alpha_n)Tx_n), \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ *satisfying Conditions* (C1) *and* (C2). *Then $\{x_n\}$ converges strongly to $R_{F(T)}u$, where $R_{F(T)}$ is the unique sunny generalized nonexpansive retraction from E onto $F(T)$.*

Proof. Suppose that $T^* : JC \to E^*$ is the duality of T. From Lemma 4.2, T^* is strongly relatively nonexpansive and $F(T^*) = JF(T)$. Let $x_n^* \equiv Jx_n$ and $u^* = Ju$. Using (4.1) and $R_C = J^{-1}\Pi_{JC}J$, we obtain

$$\begin{aligned} x_{n+1}^* &= \Pi_{JC}J(\alpha_n J^{-1}u^* + (1 - \alpha_n)J^{-1}T^*x_n^*) \\ &= \Pi_{JC}J^{*-1}\left(\alpha_n J^*u^* + (1 - \alpha_n)J^*T^*x_n^*\right) \end{aligned}$$

for all $n \in \mathbb{N}$. Applying Theorem 3.3 gives $x_n^* \to \Pi_{F(T^*)}u^*$. Since J^{-1} is norm-to-norm continuous,

$$x_n = J^{-1}x_n^* \to J^{-1}\Pi_{F(T^*)}u^* = J^{-1}\Pi_{JF(T)}(Ju) = R_{F(T)}u. \quad \square$$

If the mapping T in Theorem 4.3 is a self mapping, then we have the following corollary.

Corollary 4.4. *Let C be a nonempty, closed, convex and sunny generalized nonexpansive retract of a uniformly smooth Banach space E whose dual space has a Fréchet differentiable norm. Let $T : C \to C$ be a strongly generalized nonexpansive mapping with property* (G3). *Let $\{x_n\}$ be a sequence in C defined by $u, x_1 \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ *satisfying Conditions* (C1) *and* (C2). *Then $\{x_n\}$ converges strongly to $R_{F(T)}u$.*

Let C be a nonempty subset of a smooth Banach space E. Recall that a mapping $T : C \to E$ is *firmly generalized nonexpansive type* [34] if

$$\varphi(Tx, Ty) + \varphi(Ty, Tx) + \varphi(x, Tx) + \varphi(y, Ty) \leq \varphi(x, Ty) + \varphi(y, Tx)$$

for all $x, y \in C$. It is not hard to show that the duality of a firmly generalized nonexpansive type mapping is firmly nonexpansive type. As a consequence of Theorem 3.9, we have the following result.

Theorem 4.5. *Let C be a nonempty, closed and sunny generalized nonexpansive retract of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be a firmly generalized nonexpansive type mapping with $F(T)$ is nonempty. Suppose that $\{\alpha_n\}$ is a sequence in* $(0, 1)$ *satisfying Conditions* (C1) *and* (C2). *Then the sequence $\{x_n\}$ defined by* (4.1) *converges strongly to $R_{F(T)}u$.*

5. Maximal monotone operators

Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a set-valued mapping with range $R(A) = \{x^* \in E^* : x^* \in Ax\}$ and domain $D(A) = \{x \in E : Ax \neq \emptyset\}$. Then the mapping A is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* if A is monotone and there is no monotone operator from E into E^* whose graph properly contains the graph of A. It is known that if $A \subset E \times E^*$ is maximal monotone, then $A^{-1}0$ is closed and convex.

Theorem 5.1 (*Cf. [45]*)**.** *Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Using Theorem 5.1, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

The single valued mapping $J_r : E \to D(A)$ by $J_r x = x_r$, that is, $J_r = (J + rA)^{-1}J$ is called the *resolvent* of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$ (see [27,28,37] for more details).

Theorem 5.2 (*Cf. [27, Lemma 2.3]*). *Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Let J_r be the resolvent of A, where $r > 0$. If $A^{-1}0$ is nonempty, then J_r is firmly nonexpansive type.*

Using this result and Theorem 3.9, we prove a strong convergence theorem for resolvents of maximal monotone operators in a Banach space.

Theorem 5.3. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E^*$ be a maximal monotone operator. Let J_r be the resolvent of A, where $r > 0$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)JJ_r x_n),$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}u$.

From Theorem 5.1, if E is reflexive, strictly convex and smooth and $B \subset E^* \times E (= E^* \times E^{**})$ is a maximal monotone operator, then $R(J^{-1} + rB) = E$ for all $r > 0$. Thus, if $r > 0$ and $x \in E$, then there exists $z \in E$ such that

$$x = J^{-1}(Jx) \in J^{-1}(Jz) + rB(Jz) = z + rBJz.$$

It follows from the strict convexities of E and E^* that such a point z is unique. Thus we can define the *generalized resolvent* Q_r of B by

$$Q_r x = z = (I + rBJ)^{-1}x.$$

For more details, see [42,46].

Lemma 5.4 (*Cf. [34, Lemma 3.5]*). *Let E be a reflexive, strictly convex Banach space whose dual space has a uniformly Gâteaux differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator. Let Q_r be the generalized resolvent of B, where $r > 0$. If $B^{-1}0$ is nonempty, then Q_r is firmly generalized nonexpansive.*

Using this result and Theorem 4.5, we prove a strong convergence theorem for generalized resolvents of maximal monotone operators in a Banach space.

Theorem 5.5. *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. Let Q_r be the generalized resolvent of B, where $r > 0$. Let $\{x_n\}$ be a sequence defined by $u, x_1 \in E$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_r x_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying Conditions (C1) and (C2). If $B^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}u$, where $R_{(BJ)^{-1}0}$ is the unique sunny generalized nonexpansive retraction from E onto $(BJ)^{-1}0$.

Remark 5.6. In Theorem 5.5, we present a strong convergence theorem for the generalized resolvent with a new control condition. This is complementary to Ibaraki and Takahashi's result [46, Theorem 4.2].

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A3 **Weerayuth Nilsrakoo**, Convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces, Proceedings of the 7th International Conference on Nonlinear Analysis and Convex Analysis II-(Busan, Korea, 2011), 29–42

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CONVERGENCE THEOREMS FOR BREGMAN STRONGLY NONEXPANSIVE OPERATORS IN REFLEXIVE BANACH SPACES

WEERAYUTH NILSRAKOO

ABSTRACT. In this paper, we deal with weak and strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces and then apply them to the solution of equilibrium problem. Various special cases are discussed.

1. INTRODUCTION

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a (quasi-)nonexpansive mapping in a Hilbert space. To extend this theory to a Banach space, we encounter some difficulties because many of useful examples of nonexpansive mappings in Hilbert spaces are no longer nonexpansive in Banach spaces. There are several ways to overcome these difficulties. One of them is to use the Lyapunov functional (see [1]) instead of the norm. Matsushita and Takahashi [24,25] studied and investigated the the weak and strong convergence theorems for relatively nonexpansive mappings (coincides with the one in the usual sense in a Hilbert space) in Banach spaces which were first introduced by Butnariu et al. [15]. Several articles have appeared providing methods for approximating fixed points of relatively nonexpansive mappings (see, *e.g.*, [3,4,11,18,21–32,34,40,41]).

In 1967, Brègman [12] discovered an elegant and effective technique for the using of the so-called Bregman distance function (see, Section 2, Definition 2.1) in the process of designing and analyzing feasibility and optimization algorithm and so on (see, *e.g.*, [2,5–8,12,14,16,17,20,33,35–39] and the references therein). The method of cyclic Bregman projections produces a sequence converging to a solution of the convex feasibility problem (see, *e.g.*, [2,5,12,16,17]).

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Albert and Butnariu [2] investigated the method of cyclic Bregman projections in a reflexive Banach space. They proved that the method of cyclic Bregman projections produces a sequence weakly converging to a solution of the convex feasibility proble and norm convergence under additional conditions on the convex sets. In 2004, Lee and Park [33] studied quasi-Bregman firmly nonexpansive mappings (see, Section 2, Definition 2.10) and the weak convergence theorem of a Bergman projection method for finding an asymptotic fixed point of a quasi-Bregman firmly nonexpansive mapping in a reflexive Banach space. Recently, Reich and Sabach [37–39] studied the existence and approximation of fixed points of a Bregman firmly (strongly) nonexpansive mapping (see, Section 2, Definition 2.10) in reflexive Banach spaces.

Inspired and motivated by the works, we introduce the Mann-type iterative algorithm to find a fixed point of a Bregman strongly nonexpansive mapping in a reflexive Banach space and then the weak and strong convergence theorems are proved under some suitable assumptions. Moreover, we modify these methods in order to solve the equilibrium problem. The main results in this paper extend, and improve the corresponding results in the literature.

2. Preliminaries

Throughout this paper, without other specifications,we denote by $\mathbb{R}$ the set of real numbers. Let X be a real reflexive Banach space with the dual space X^*. The norm and the dual pair between X^* and X are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous and $f^* : X^* \to (-\infty, +\infty]$ be a Fenchel conjugate of f. We denote by $\operatorname{dom} f$ and $\operatorname{int}(\operatorname{dom} f)$ the set $\{x \in X : f(x) < \infty\}$ and the interior of the domain of f, respectively. For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in X$, the *right-hand derivative* of f at x in the direction y defined by

$$f^{\circ}(x,y) =: \lim_{t \searrow 0} \frac{f(x+ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteaux differentiable* at x if the limit as $t \to 0$ in (2.1) exists for any y. In this case, $f^{\circ}(x,y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. The function f is called *Gâteaux differentiable* if it is *Gâteaux differentiable* for everywhere. The function f is said to be *Fréchet differentiable* at x if this limit is attained uniformly for $\|y\| = 1$. We say f is *uniformly Fréchet differentiable on a subset C of X* if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. From now on, we assume that the function f is a proper convex, lower semicontinuous and Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$.

Legendre function f is defined in [6]. From [6], if X is a reflexive Banach space, then f is Legendre if and only if it satisfies the following conditions:

(L1) $\operatorname{int}(\operatorname{dom} f)$ is nonempty, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$, and $\operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f)$;

(L2) $\operatorname{int}(\operatorname{dom} f^*)$ is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$, and $\operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$.

Since X is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ see, *e.g.*, [10]. This, by (L1) and (L2), implies

$$\nabla f = (\nabla f^*)^{-1},$$
$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$$

and

$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f).$$

By [10, Theorem 5.4], conditions (L1) and (L2) also yield that the functions f and f^* are strictly convex on the interior of their respective domains. Several interesting examples of Legendre functions are presented in [5] and [10]. Among them are the functions $\frac{1}{s}\|\cdot\|^s$ with $s \in (1, ?\infty)$, where the Banach space X is smooth and strictly convex and, in particular, a Hilbert space.

We first recall some definitions and lemmas which are needed in our main results.

Definition 2.1 ([7,12]). The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$ is defined by

$$D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle,$$

is called the *Bregman distance* with respect to f.

Remark 2.2 ([7,37]). The Bregman distance has the three point identity: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle; \tag{2.2}$$

From the definition of Bregman distance above, we have the following lemma.

Lemma 2.3. *Let f be a Legendre function. Then, for all $x \in \operatorname{int}(\operatorname{dom} f)$, $y, z \in \operatorname{dom} f$ and $\alpha \in [0,1]$,*

$$D_f(x, \nabla f^*(\alpha \nabla f(y) + (1-\alpha)\nabla f(z))) \le \alpha D_f(x,y) + (1-\alpha) D_f(x,z).$$

Definition 2.4 ([13] (see also [14])). The function f is called:

(i) *totally convex at a point $x \in \operatorname{int}(\operatorname{dom} f)$* if its modulus of total convexity at x, that is, the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty)$, defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|x - y\| = t\},$$

is positive whenever $t > 0$;

(ii) *totally convex* when it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$;
(iii) *totally convex on bounded sets* if $\nu_f(B,t)$ is positive for any nonempty bounded subset B of X and $t > 0$, where the modulus of total convexity of the function f on the set B is the function defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \operatorname{int}(\operatorname{dom} f)\}.$$

The next proposition turns out to be very useful in the proof of the main results.

Proposition 2.5 ([35]). *If $x \in \operatorname{int}(\operatorname{dom} f)$, then the function f is totally convex at x if and only if for any bounded sequence $\{y_n\} \subseteq \operatorname{dom} f$,*

$$D_f(y_n, x) \to 0 \quad \Longrightarrow \quad \|y_n - x\| \to 0. \tag{2.3}$$

Proposition 2.6 ([2,33,35]). *Let $f : X \to (-\infty, +\infty]$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then:*

(i) *f is uniformly continuous on bounded sets;*
(ii) *f is coercive i.e., $\lim_{\|x\|\to+\infty} f(x)/\|x\| = +\infty$;*
(iii) *the modulus of convexity $\mu_f : [0,+\infty) \to [0,+\infty)$ of f, defined by*

$$\mu_f(t) := \inf\left\{\frac{\alpha f(x)+(1-\alpha)f(y) - f(\alpha x+(1-\alpha)y)}{\alpha(1-\alpha)} : \alpha \in (0,1), \|x-y\| = t\right\},$$

is strictly convex, $\mu_f(0) = 0$ and

$$\mu_f(\|x-y\|) \le D_f(x,y) \quad \forall x,y \in \operatorname{int}(\operatorname{dom} f).$$

Definition 2.7 ([12]). The Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty closed and convex set $C \subseteq \operatorname{dom} f$ is the necessarily unique vector $\operatorname{Proj}_C^f(x) \in C$ satisfying

$$D_f(\operatorname{Proj}_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

Remark 2.8 ([7,39]). (i) If X is a Hilbert space and $f(x) = \|x\|^2$, then the Bregman projection $\operatorname{Proj}_C^f(x)$ is reduced to the metric projection of x onto C.
(ii) If X is a smooth Banach space and $f(x) = \|x\|^2$, then the Bregman projection $\operatorname{Proj}_C^f(x)$ is reduced to the generalized projection $\Pi_C x$ (see, *e.g.*, [1]) which defined by

$$\varphi(\Pi_C x, x) = \min_{y \in C} \varphi(y,x).$$

where $\varphi(y,x) = \|y\|^2 - 2\langle Jx, y\rangle + \|x\|^2$ and J is the normalized duality mapping from X to X^*.

Similarly to the metric (generalized) projection, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Lemma 2.9 ([35]). *Let* $f : X \to (-\infty, +\infty]$ *be a Gâteaux differentiable and totally convex on* $\operatorname{int}(\operatorname{dom} f)$. *Let* $x \in \operatorname{int}(\operatorname{dom} f)$ *and* $C \subseteq \operatorname{int}(\operatorname{dom} f)$ *be a nonempty closed convex set. If* $\widetilde{x} \in C$, *then the following statements are equivalent*

(i) $\widetilde{x} = \operatorname{Proj}_C^f(x)$;

(ii) *the vector* $\widetilde{x}$ *is the unique solution of the variational inequality:*

$$\langle \nabla f(x) - \nabla f(\widetilde{x}), \widetilde{x} - y \rangle \geq 0, \ \forall y \in C;$$

(ii) *the vector* $\widetilde{x}$ *is the unique solution of the inequality:*

$$D_f(y, \widetilde{x}) + D_f(\widetilde{x}, x) \leq D_f(y, x), \ \forall y \in C.$$

The strong and weak convergences of a sequence $\{x_n\}$ in a Banach space X to an element $x \in X$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let C be a nonempty subset of a Banach space X, and let $T : C \to X$ be a nonlinear mapping. We denote the set of fixed points of T by $F(T)$. A point $w \in C$ is called an *asymptotic fixed point* of T (see, *e.g.*, [1]) if C contains a sequence $\{x_n\}$ which converges weakly to w such that $x_n - Tx_n \to 0$. We denote the set of asymptotic fixed points of T by $\widehat{F}(T)$. We next list significant types of nonexpansivity with respect to the Bregman distance.

Definition 2.10 (see [37]). Let C be a nonempty, closed and convex subset of $\operatorname{int}(\operatorname{dom} f)$. We say that a mapping $T : C? \to \operatorname{int}(\operatorname{dom} f)$ with $F(T) \neq \varnothing$ is

(i) *Bregman nonexpansive* if

$$D_f(Tx, Ty) \leq D_f(x, y) \tag{2.4}$$

for all $x, y \in C$;

(ii) *quasi-Bregman nonexpansive* if

$$D_f(p, Tx) \leq D_f(p, x) \tag{2.5}$$

for all $x \in C$ and $p \in F(T)$;

(iii) *Bregman firmly nonexpansive* if

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ \leq D_f(Tx, y) + D_f(Ty, x), \end{aligned} \tag{2.6}$$

for all $x, y \in C$, or equivalently,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle; \tag{2.7}$$

(iv) *quasi-Bregman firmly nonexpansive* if

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x) \tag{2.8}$$

for all $x \in C$ and $p \in F(T)$,

(v) *Bregman strongly nonexpansive* if

$$D_f(p, Tx) \leq D_f(p, x) \tag{2.9}$$

for all $x \in C$ and $p \in F(T)$, and if whenever $\{x_n\} \subset C$ is bounded, $p \in F(T)$, and

$$D_f(p, x_n) - D_f(p, Tx_n) \to 0,$$

it follows that

$$D_f(Tx_n, x_n) \to 0.$$

From the definitions of mappings above, we have the following properties.

Proposition 2.11 ([37–39]). *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T : C \to C$ be a mapping. Then:*

(i) *If T is a Bregman nonexpansive mapping, then T is quasi-Bregman nonexpansive.*

(ii) *If T is a Bregman firmly nonexpansive mapping, then T is quasi-Bregman firmly nonexpansive.*

(iii) *If T is a quasi-Bregman firmly nonexpansive mapping, then T is Bregman strongly nonexpansive.*

(iv) *If T is a quasi-Bregman nonexpansive mapping, then $F(T)$ is closed and convex.*

(v) *If f is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of X and T is Bregman firmly nonexpansive, then $F(T) = \widehat{F}(T)$.*

3. Main results

In this section, we present the weak and strong convergence theorems of the iterative scheme (3.1). To this end, we need the following proposition.

Proposition 3.1. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let T be a quasi-Bregman nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \operatorname{Proj}_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(Tx_n)), \quad n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *Then* $\{x_n\}$ *is bounded and* $\{\mathrm{Proj}^f_{F(T)}(x_n)\}$ *converges strongly to a fixed point of* T.

Proof. Let $p \in F(T)$. From Lemmas 2.3 and 2.9, we have

$$\begin{aligned} D_f(p, x_{n+1}) &\leq D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, Tx_n) \\ &\leq D_f(p, x_n). \end{aligned} \tag{3.2}$$

Hence, $\lim_{n\to\infty} D_f(p, x_n)$ exists and, in particular, $\{D_f(p, x_n)\}$ is bounded. This implies that $\{x_n\}$ is bounded. Also $\{Tx_n\}$ is bounded. Let $u_n = \mathrm{Proj}^f_{F(T)}(x_n)$. Then, we get

$$D_f(u_n, x_{n+1}) \leq D_f(u_n, x_n)$$

and

$$\begin{aligned} D_f(u_{n+1}, x_{n+1}) &\leq D_f(u_n, x_{n+1}) - D_f(u_n, u_{n+1}) \\ &\leq D_f(u_n, x_n) - D_f(u_n, u_{n+1}) \\ &\leq D_f(u_n, x_n). \end{aligned}$$

It follows that $\lim_{n\to\infty} D_f(u_n, x_n)$ exists. Moreover, we get

$$D_f(u_n, u_{n+1}) \leq D_f(u_n, x_n) - D_f(u_{n+1}, x_{n+1}).$$

By induction, we have

$$D_f(u_n, u_m) \leq D_f(u_n, x_n) - D_f(u_m, x_m)$$

for all $m > n \in \mathbb{N}$. From Proposition 2.6, we get

$$\mu_f(\|u_m - u_n\|) \leq D_f(u_n, u_m) \leq D_f(u_n, x_n) - D_f(u_m, x_m).$$

Then the properties of μ_f yield that $\{u_n\}$ is a Cauchy sequence. Since X is complete and $F(T)$ is closed, $\{u_n\}$ converges strongly to some point in $F(T)$. □

Corollary 3.2 ([24, Proposition 3.1]). *Let* X *be a uniformly convex and uniformly smooth Banach space, let* C *be a nonempty, closed and convex subset of* X *and let* T *be a relatively quasi-nonexpansive mapping of* C *into itself; that is,* $F(T) \neq \varnothing$ *and* $\varphi(p, Tx) \leq \varphi(p, x)$ *for all* $p \in F(T)$ *and* $x \in C$. *Let* $\{x_n\}$ *be a sequence defined by* $x_1 \in C$ *and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTx_n), \quad n \geq 1, \tag{3.3}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *Then* $\{\Pi_{F(T)}(x_n)\}$ *converges strongly to a fixed point of* T.

Now, we present a weak convergence theorem.

Theorem 3.3. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let T be a Bregman strongly nonexpansive mapping of C into itself such that $F(T) = \widehat{F}(T) \neq \varnothing$. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. If ∇f is weakly sequentially continuous, then the sequence $\{x_n\}$ defined by (3.1) converges weakly to $u \in F(T)$, where $u = \lim_{n\to\infty} \mathrm{Proj}^f_{F(T)}(x_n)$.*

Proof. Let $p \in F(T)$. From (3.2), we get $\lim_{n\to\infty} D_f(p, x_n)$ exists and so

$$(1-\alpha_n)(D_f(p,x_n) - D_f(p,Tx_n)) \leq D_f(p,x_n) - D_f(p,x_{n+1}) \to 0.$$

Since $\limsup_{n\to\infty} \alpha_n < 1$, we get

$$D_f(p,x_n) - D_f(p,Tx_n) \to 0.$$

Since T is Bregmam strongly nonexpansive,

$$D_f(Tx_n, x_n) \to 0$$

and hence

$$\|Tx_n - x_n\| \to 0.$$

This implies that if there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some v, then by $F(T) = \widehat{F}(T)$, $v \in F(T)$.

Let $u_n = \mathrm{Proj}^f_{F(T)}(x_n)$. Then, we get

$$\langle \nabla f(x_n) - \nabla f(u_n), u_n - z\rangle \geq 0 \tag{3.4}$$

for all $z \in F(T)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. Then $v \in F(T)$. From (3.4), we have

$$\langle \nabla f(x_{n_i}) - \nabla f(u_{n_i}), u_{n_i} - v\rangle \geq 0.$$

By Theorem 3.1, there is $u \in F(T)$ such that $u_n \to u$. Since ∇f is weakly sequentially continuous and letting $i \to \infty$, we get

$$\langle \nabla f(v) - \nabla f(u), u - v\rangle \geq 0. \tag{3.5}$$

Since ∇f is monotone, we have

$$\langle \nabla f(v) - \nabla f(u), v - u\rangle \geq 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\langle \nabla f(v) - \nabla f(u), v - u\rangle = 0.$$

From (2.2), we get $D_f(u,v) = 0$. Since f is totally convex and Proposition 2.5, we get $u = v$. Therefore, $\{x_n\}$ converges weakly to $u = \lim_{n\to\infty} \mathrm{Proj}^f_{F(T)}(x_n)$. This completes the proof. □

Corollary 3.4. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed and convex subset of X and let T be a strongly relatively nonexpansive mapping of C into itself. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$. If J is weakly sequentially continuous, then the sequence $\{x_n\}$ defined by (3.3) converges weakly to $u \in F(T)$, where $u = \lim_{n\to\infty}\Pi_{F(T)}(x_n)$.*

Next, we also consider the strong convergence of (3.1).

Theorem 3.5. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let T be a Bregman strongly nonexpansive mapping of C into itself such that $F(T) = \widehat{F}(T) \neq \varnothing$. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$. If $\operatorname{int}(F(T)) \neq \varnothing$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $u \in F(T)$, where $u = \lim_{n\to\infty}\operatorname{Proj}^f_{F(T)}(x_n)$.*

Proof. Since $\operatorname{int}(F(T)) \neq \varnothing$, there exist $p \in F(T)$ and $r > 0$ such that

$$p + rh \in F(T)$$

whenever $\|h\| \leq 1$. By (2.2), we have

$$D_f(x_{n+1}, x_n) + \langle \nabla f(x_n) - \nabla f(x_{n+1}), x_{n+1} - u\rangle = D_f(u, x_n) - D_f(u, x_{n+1}) \tag{3.7}$$

for all $u \in F(T)$. We also have

$$\begin{aligned} &\langle \nabla f(x_n) - \nabla f(x_{n+1}), x_{n+1} - p\rangle \\ &= \langle \nabla f(x_n) - \nabla f(x_{n+1}), x_{n+1} - (p + rh)\rangle + r\langle \nabla f(x_n) - \nabla f(x_{n+1}), h\rangle. \end{aligned} \tag{3.8}$$

From (3.2) and $p + rh \in F(T)$, we get

$$D_f(p + rh, x_{n+1}) \leq D_f(p + rh, x_n).$$

It follows from (3.7) that

$$D_f(x_{n+1}, x_n) + \langle \nabla f(x_n) - \nabla f(x_{n+1}), x_{n+1} - (p + rh)\rangle \geq 0.$$

$$r\langle \nabla f(x_n) - \nabla f(x_{n+1}), x_{n+1} - h\rangle \geq 0.$$

From (3.7) and (3.8), we have

$$\begin{aligned} r\langle \nabla f(x_n) - \nabla f(x_{n+1}), h\rangle &\leq D_f(x_{n+1}, x_n) + \langle \nabla f(x_n) - \nabla f(x_{n+1}), x_{n+1} - p\rangle \\ &= D_f(p, x_n) - D_f(p, x_{n+1}) \end{aligned}$$

and hence

$$\langle \nabla f(x_n) - \nabla f(x_{n+1}), h\rangle \leq \frac{1}{r}(D_f(p, x_n) - D_f(p, x_{n+1})).$$

Since h with $\|h\| \leq 1$ is arbitrary, we have

$$\|\nabla f(x_n) - \nabla f(x_{n+1})\| \leq \frac{1}{r}(D_f(p, x_n) - D_f(p, x_{n+1})).$$

So, if $m > n$, then

$$\begin{aligned}\|\nabla f(x_n) - \nabla f(x_m)\| &\leq \sum_{i=n}^{m-1} \|\nabla f(x_i) - \nabla f(x_{i+1})\| \\ &\leq \frac{1}{r}\sum_{i=n}^{m-1}(D_f(p, x_i) - D_f(p, x_{i+1})).\end{aligned}$$

We know that $\{D_f(p, x_n)\}$ converges. This implies that $\{\nabla f(x_n)\}$ is a Cauchy sequence. Since X^* is complete, we have $\{\nabla f(x_n)\}$ converges strongly to some $x^* \in X^*$. Since ∇f is bijective, there is $x \in X$ such that $\nabla f(x) = x^*$. It follows that

$$x_n = \nabla f^*(\nabla f(x_n)) \to \nabla f^*(\nabla f(x)) = x.$$

As in the proof of Theorem 3.2, we also have $\|x_n - Tx_n\| \to 0$. So, we get $u \in F(T)$ with $u = \lim_{n\to\infty} \mathrm{Proj}^f_{F(T)}(x_n)$. □

Corollary 3.6. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed and convex subset of X and let T be a strongly relatively nonexpansive mapping of C into itself. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. If $\mathrm{int}(F(T)) \neq \varnothing$, then the sequence $\{x_n\}$ defined by (3.3) converges strongly to $u \in F(T)$, where $u = \lim_{n\to\infty} \Pi_{F(T)}(x_n)$.*

4. Equilibrium problems

Let C be a nonempty, closed and convex subset of X. Let $g : C \times C \to \mathbb{R}$ be a bifunction that satisfies the following conditions (see, for instance, [9,19,37]):

(A1) $g(x, x) = 0$ for all $x \in C$;
(A2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \leq 0$, for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\to 0} g(tz + (1-t)x, y) \leq g(x, y);$$

(A4) for all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

The *equilibrium problem* of g is to find $x \in C$ such that

$$g(x, y) \geq 0 \quad \text{for all } y \in C. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $EP(g)$. Set

$$\mathrm{Res}^f_g(x) := \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \forall y \in C\}.$$

The following two lemmas give several properties of these resolvents.

Lemma 4.1 ([37, Lemma 1]). *Let $f : X \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of X. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then* $\mathrm{dom}(\mathrm{Res}_g^f) = X$.

Lemma 4.2 ([37, Lemma 2]). *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of X. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4). Then, the followings hold:*

(i) Res_g^f *is single-valued;*
(ii) Res_g^f *is a Bregman firmly nonexpansive mapping (is also Bregman strongly nonexpansive);*
(iii) $F(\mathrm{Res}_g^f) = EP(g)$.

From Theorems 3.2 and 3.5, we have the following result.

Theorem 4.3. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of X. Let C be a nonempty, closed and convex subset of X and let $g : C \times C \to \mathbb{R}$ be a bifunction satisfies conditions (A1)-(A4) and $EP(g) \neq \varnothing$. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \mathrm{Proj}_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(\mathrm{Res}_g^f x_n)), \quad n \geq 1.$$

Then:

(i) *If ∇f is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $u \in EP(g)$, where $u = \lim_{n\to\infty} \mathrm{Proj}_{EP(g)}^f(x_n)$.*
(ii) *If $\mathrm{int}(EP(g)) \neq \varnothing$, then $\{x_n\}$ converges strongly to $u \in EP(g)$, where $u = \lim_{n\to\infty} \mathrm{Proj}_{EP(g)}^f(x_n)$.*

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Convergence Theorems of the Implicit Midpoint Rule Based on Conjugate Gradient Method for a Nonexpansive Mapping in a Hilbert Space

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Abstract

The purpose of this paper is to introduce the implicit midpoint rule based on conjugate gradient method for finding a fixed point of a nonexpansive mapping. We present the strong convergence theorems in a Hilbert space of this method. Setting certain parameters, as a consequence, strong convergence theorems for finding a fixed point of a nonexpansive mapping which studied in [16] are deduced. Finally, we give some examples to support our main results.

Mathematics Subject Classification: 47H09, 47H10
Keywords: convergence theorem, fixed point, Halpern algorithm, implicit midpoint rule, nonexpansive mapping

1 Introduction

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. A mapping $T : H \to H$ is said to be *nonexpansive* if for any $x, y \in H$

$$\|T(x) - T(y)\| \leq \|x - y\|$$

We denote by $\mathrm{Fix}(T)$ the set of all fixed points of T.

The convergence theorems of nonexpansive mappings have been considered by many researchers in recent years. In 1967, Halpern [7] introduced an explicit iteration for finding a

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fixed point of a nonexpansive mapping T of a Hilbert space H, which is called the Halpern algorithm, $x_0 \in H$ and

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)T(x_n) \tag{1.1}$$

for each $n \geq 0$, where $(\alpha_n)_{n=0}^{\infty} \subset (0, 1)$ satisfying (C1) $\lim_{n\to\infty} \alpha_n = 0$, (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and (C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then $(x_n)_{n=0}^{\infty}$ generated by (1.1) converges strongly to $P_{\text{Fix}(T)}(x_0)$, where $P_{\text{Fix}(T)}$ is the metric projection from H onto Fix(T).

In 2014, Sakurai and Iiduka [14] introduced the Halpern algorithm based on steepest descent method for solving an example of a fixed point problem and they can formulate novel fixed point algorithm by using conjugate gradient method, which can accelerate steepest descent method and the number of iterations is less than steepest descent method. They present strong convergence theorem of their algorithm for finding a fixed point of a nonexpansive mapping in Hilbert space as follows: let $(x_n)_{n=0}^{\infty}$ be a sequence in H defined by $x_0 \in H$, $\mu \in (0, 1]$, $\lambda > 0$, $d_0 = \frac{1}{\lambda}(T(x_0) - x_0)$ and

$$\begin{cases} d_{n+1} = \frac{1}{\lambda}(T(x_n) - x_n) + \beta_n d_n, \\ \quad y_n \ = x_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\alpha_n x_0 + (1 - \mu\alpha_n)y_n. \end{cases} \tag{1.2}$$

for each $n \geq 0$, where $(\alpha_n)_{n=0}^{\infty} \subset (0, 1)$ and $(\beta_n)_{n=0}^{\infty} \subset [0, \infty)$. Suppose that (C1)-(C3), (C4) $\beta_n \leq \alpha_n^2$ and (C5) $(T(x_n) - x_n)_{n=0}^{\infty}$ is bounded. Then $(x_n)_{n=0}^{\infty}$ generated by (1.2) converges strongly to $P_{\text{Fix}(T)}(x_0)$, where $P_{\text{Fix}(T)}$ is the metric projection from H onto Fix(T). Many researchers have studied the conjugate gradient method such as [5, 6, 8, 9, 11].

Recently, Xu, Alghamdi and Shahzad [16] introduced the implicit algorithm for finding a fixed point of a nonexpansive mapping T, which is called the implicit midpoint rule, $x_0 \in H$ and

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right) \tag{1.3}$$

for each $n \geq 0$, where $(\alpha_n)_{n=0}^{\infty} \subset (0, 1)$ satisfying (C1), (C2) and (C3′) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$. Then $(x_n)_{n=0}^{\infty}$ generated by (1.3) converges strongly to $P_{\text{Fix}(T)}(x_0)$, where $P_{\text{Fix}(T)}$ is the metric projection from H onto Fix(T). Moreover, many researchers have studied the implicit midpoint rule such as [1, 3, 4, 12, 17].

In this paper, we have implemented the concept of implicit midpoint rule in [14]. We will modify the implicit midpoint rule with conjugate gradient method for finding a fixed point. We also present the strong convergence theorem of a nonexpansive mapping in a Hilbert space of our algorithm. Moreover, setting certain parameters in our algorithm can deduce the convergence theorems which is studied in [16]. Finally, we give some examples to support our main results.

2 Preliminaries

Throughout this paper, let H be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let $x \in H$. A sequence $(x_n)_{n=0}^{\infty}$ in H is said to

(i) converges strongly to x, denoted by $x_n \to x$, if $\lim_{n\to\infty} ||x_n - x|| = 0$;

(ii) converges weakly to x, denoted by $x_n \rightharpoonup x$, if $\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$.

It is well known that H is a Hilbert space if and only if each bounded sequence $(x_n)_{n=0}^{\infty}$ of H has a weakly convergent subsequence $(x_{n_k})_{k=0}^{\infty}$ of $(x_n)_{n=0}^{\infty}$ (see [15]).

A mapping $T : H \to H$ is said to be a *contraction* if there exists a real number r with $0 \leq r < 1$ such that

$$||T(x) - T(y)|| \leq r||x - y||$$

for all $x, y \in H$. A function $f : H \to \mathbb{R}$ is said to be *convex* if, for any $x, y \in H$ and for any $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

A mapping $g : H \to H$ is said to be *monotone* if

$$\langle x - y, g(x) - g(y)\rangle \geq 0$$

for all $x, y \in H$. A function $f : H \to \mathbb{R}$ is said to be *Fréchet-differentiable* at $x \in H$ if there exists $y \in H$ such that

$$f(x + h) = f(x) + \langle y, h\rangle + o(||h||),$$

where $\lim_{t\to 0} \frac{o(t)}{t} = 0$. The element y is called a *gradient of f at x* and is denoted by $\bigtriangledown f(x)$ [2, Definition 1.1.11]. A mapping $A : H \to H$ is said to be *Lipschitz continuous* with $L > 0$ (L-Lipschitz continuous) if

$$||A(x) - A(y)|| \leq L||x - y||$$

for all $x, y \in H$. Let C be a nonempty closed convex subset of H. Then for each point $x \in H$, there corresponds a unique point $\hat{x}$ in C such that

$$\|x - \hat{x}\| = \inf_{y\in C} \|x - y\|.$$

We call a mapping defined by $P_C(x) = \hat{x}$, the *metric projection* of H onto C (see [15]).

Lemma 2.1. [15]. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $x \in H$. Then*

(i) the projection P_C is nonexpansive with Fix$(P_C) = C$.

(ii) $\hat{x} = P_C(x)$ *if and only if* $\langle x - \hat{x}, y - \hat{x}\rangle \leq 0$ *for all* $y \in C$.

Theorem 2.2. [15, Theorem 5.4.4: Opial's Theorem]. *Let* H *be a Hilbert space and* $(x_n)_{n=0}^{\infty} \subset H$ *converges weakly to* $x \in H$*, then* $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$ *for all* $y \in H$ *with* $x \neq y$.

Lemma 2.3. [15]. *Let* H *be a Hilbert space and let* $T : H \to H$ *be a nonexpansive mapping. Then Fix*(T) *is closed and convex.*

Lemma 2.4. [14, Proposition 2.2]. *Let* $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ *and* $(\bar{\alpha}_n)_{n=0}^{\infty}$ *be sequences of nonnegative real numbers with* $a_{n+1} \leq (1 - \bar{\alpha}_n)a_n + \bar{\alpha}_n b_n + c_n$ *for each* $n \geq 0$*. Suppose that* $\sum_{n=0}^{\infty} \bar{\alpha}_n = \infty$, $\limsup_{n\to\infty} b_n = 0$*, and* $\sum_{n=0}^{\infty} c_n < \infty$*. Then* $\lim_{n\to\infty} a_n = 0$.

3 Main Results

First, we introduce the implicit midpoint rule based on conjugate gradient method for finding a fixed point of a nonexpansive mapping.

Sakurai and Iiduka [14] can formulate novel algorithm by using conjugate gradient method which is acceleration steepest descent method. This make their algorithm performs better than the Halpern algorithm. Accordingly, we will modify algorithm (1.3) by using the concept of [14] as follows: let $f : H \to \mathbb{R}$ be convex, continuously Fréchet differentiable functional. Suppose that the gradient of f, denoted by $\triangledown f$, is Lipschitz continuous with a constant $L > 0$ and define $T^f : H \to H$ by

$$T^f = I - \lambda \triangledown f, \tag{3.1}$$

where $\lambda \in (0, 2/L]$ and $I : H \to H$ stands for the identity mapping. We will show that algorithm (1.3) is based on the steepest descent method [13, Subchapter 3.3] to minimize f over H. Since T^f is nonexpansive (see, [10, Proposition 2.3]) and

$$\mathrm{Fix}(T^f) = \operatorname*{argmin}_{x\in H} f(x) = \left\{x^* \in H : f(x^*) = \min_{x\in H} f(x)\right\}.$$

Thus algorithm (1.3) with $T = T^f$ can be expressed as follows:

$$\begin{cases} d^f_{n+1} = -\triangledown f(\tilde{x}_n), \\ y_n \quad = T^f(\tilde{x}_n) = \tilde{x}_n + \lambda d^f_{n+1}, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) y_n \qquad (n \geq 0), \end{cases} \tag{3.2}$$

where $\tilde{x}_n = \dfrac{x_n + x_{n+1}}{2}$. This implies that (3.2) uses the steepest descent direction [13, Subchapter 3.3]. The steepest descent direction of f at $\tilde{x}_n$ is $d_{n+1}^{f,SDD} = -\bigtriangledown f(\tilde{x}_n)$, and so (3.2) is based on the steepest descent method. We will use conjugate gradient method [13, Chapter 5] to accelerate steepest descent method. The conjugate gradient direction of f at $\tilde{x}_n$ is $d_{n+1}^{f,CGD} = -\bigtriangledown f(\tilde{x}_n) + \beta_n d_n^{f,CGD}$ for all $n \geq 0$, where $d_0^{f,CGD} = -\bigtriangledown f(\tilde{x}_0)$ and $(\beta_n)_{n=0}^{\infty} \subset (0, \infty)$. This together with (3.1) and (3.2) gives

$$d_{n+1}^{f,CGD} = \frac{1}{\lambda}(T^f(\tilde{x}_n) - \tilde{x}_n) + \beta_n d_n^{f,CGD}. \tag{3.3}$$

We replace $d_{n+1}^{f} = -\bigtriangledown f(\tilde{x}_n)$ and $\tilde{x}_n = \dfrac{x_n + x_{n+1}}{2}$ in (3.2) with $d_{n+1}^{f,CGD}$ defined by (3.3), we get

$$\begin{cases} d_{n+1}^{f,CGD} &= \dfrac{1}{\lambda}\left(T^f\left(\dfrac{x_n + x_{n+1}}{2}\right) - \left(\dfrac{x_n + x_{n+1}}{2}\right)\right) + \beta_n d_n^{f,CGD}. \\ y_n &= T^f\left(\dfrac{x_n + x_{n+1}}{2}\right) = \left(\dfrac{x_n + x_{n+1}}{2}\right) + \lambda d_{n+1}^{f}, \\ x_{n+1} &= \alpha_n x_0 + (1-\alpha_n) y_n \qquad (n \geq 0). \end{cases} \tag{3.4}$$

We replace y_n and $d_{n+1}^{f,CGD}$ from (3.4) in x_{n+1}, we get

$$\begin{cases} x_{n+1} &= \alpha_n x_0 + (1-\alpha_n)\left(T^f\left(\dfrac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n\right), \\ d_{n+1}^{f,CGD} &= \dfrac{1}{\lambda}\left(T^f\left(\dfrac{x_n + x_{n+1}}{2}\right) - \left(\dfrac{x_n + x_{n+1}}{2}\right)\right) + \beta_n d_n \qquad (n \geq 0). \end{cases}$$

We can formulate an algorithm for finding a fixed point as follows:

Algorithm 3.1. The implicit midpoint rule based on conjugate gradient method:

Step 0. Choose $\lambda > 0$, and $x_0 \in H$ arbitrarily, and set $(\alpha_n)_{n=0}^{\infty} \subset (0, 1)$, $(\beta_n)_{n=0}^{\infty} \subset [0, \infty)$. Compute $d_0 = \dfrac{1}{\lambda}(T(x_0) - x_0)$.

Step 1. Given $x_n, d_n \in H$, compute x_{n+1}, $d_{n+1} \in H$ as follows:

$$\begin{cases} x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\left(T\left(\dfrac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n\right), \\ d_{n+1} = \dfrac{1}{\lambda}\left(T\left(\dfrac{x_n + x_{n+1}}{2}\right) - \left(\dfrac{x_n + x_{n+1}}{2}\right)\right) + \beta_n d_n. \end{cases}$$

Put $n = n + 1$, and go to Step 1.

If we set $\beta_n = 0$ for all $n \geq 0$, then Algorithm 3.1 reduces to the implicit midpoint rule defined by (1.3).

Next, we propose the following lemma showing that Algorithm 3.1 is well-defined.

Lemma 3.1. *The implicit midpoint rule based on conjugate gradient method defined by Algorithm 3.1 is well-defined.*

Proof. Fixed $\lambda > 0$ and $x_0 \in H$, set $(\alpha_n)_{n=0}^{\infty} \subset (0,1)$, $(\beta_n)_{n=0}^{\infty} \subset [0,\infty)$, for each fixed $x_n, d_n \in H$ and the mapping $S_n : H \to H$ is defined by

$$S_n(x) = \alpha_n x_0 + (1-\alpha_n)\left(T\left(\frac{x_n + x}{2}\right) + \lambda\beta_n d_n\right)$$

for all $x \in H$. For each $x, y \in H$, we obtain

$$S_n(x) = \alpha_n x_0 + (1-\alpha_n)\left(T\left(\frac{x_n + x}{2}\right) + \lambda\beta_n d_n\right)$$

and

$$S_n(y) = \alpha_n x_0 + (1-\alpha_n)\left(T\left(\frac{x_n + y}{2}\right) + \lambda\beta_n d_n\right).$$

Then, the nonexpansivity of T implies that

$$\begin{aligned}||S_n(x) - S_n(y)|| &= \left\|(1-\alpha_n)T\left(\frac{x_n + x}{2}\right) - (1-\alpha_n)T\left(\frac{x_n + y}{2}\right)\right\| \\ &= (1-\alpha_n)\left\|T\left(\frac{x_n + x}{2}\right) - T\left(\frac{x_n + y}{2}\right)\right\| \\ &\leq (1-\alpha_n)\left\|\left(\frac{x_n + x}{2}\right) - \left(\frac{x_n + y}{2}\right)\right\| \\ &= \frac{1}{2}(1-\alpha_n)\|x - y\|.\end{aligned}$$

This implies that S_n is a contraction with coefficient $\frac{1}{2}(1-\alpha_n) \in [0,1)$. So, there exists a unique fixed point of S_n, say $x_{n+1} \in H$, such that $S_n(x_{n+1}) = x_{n+1}$. That is,

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\left(T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n\right).$$

It follows that d_{n+1} is computed. This means that $(x_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$ are well-defined. This completes the proof. □

We assume the conditions to support the Algorithm 3.1 as follows:

Assumption 3.1. The sequences $(\alpha_n)_{n=0}^{\infty}$, $(\beta_n)_{n=0}^{\infty}$ and $(x_n)_{n=0}^{\infty}$ satisfy

(C1) $\lim_{n\to\infty} \alpha_n = 0$, (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3′) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, (C4) $\beta_n \le \alpha_n^2$,

(C5′) $\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - \left(\frac{x_n + x_{n+1}}{2}\right)\right)$ is bounded.

Next, we give some helpful lemmas for proving the main theorems.

Lemma 3.2. *Suppose that Algorithm 3.1 satisfies Assumption 3.1. Then* $(x_n)_{n=0}^{\infty}$ *and* $(d_n)_{n=0}^{\infty}$ *are bounded.*

Proof. By (C1) and (C4), we obtain that $\lim_{n\to\infty} \beta_n = 0$. Accordingly, there exists $n_0 \ge 0$ such that $\beta_n \le \frac{1}{2}$ for all $n \ge n_0$. By (C5′), there exists $M_0 > 0$ such that

$$\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - \left(\frac{x_n + x_{n+1}}{2}\right)\right\| \le M_0$$

for all $n \ge 0$. Define $M_1 = \max\left\{\|d_0\|, \|d_1\|, \|d_2\|, ..., \|d_{n_0}\|, \frac{2}{\lambda} M_0\right\}$. So $M_1 < \infty$. Assume that $\|d_n\| \le M_1$ for $n \ge n_0$. By the definition of $(d_n)_{n=0}^{\infty}$ and the triangle inequality, we get

$$\|d_{n+1}\| \le \frac{1}{\lambda}\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - \left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \beta_n \|d_n\| \le M_1.$$

By mathematical induction, we get $\|d_n\| \le M_1$ for all $n \ge 0$ and hence $(d_n)_{n=0}^{\infty}$ is bounded. By the definition of $(x_n)_{n=0}^{\infty}$ and the nonexpansivity of T, for all $x \in \mathrm{Fix}(T)$ and $n \ge 0$,

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&= \left\|\alpha_n(x_0 - x) + (1 - \alpha_n)\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - x\right) + (1 - \alpha_n)\lambda\beta_n d_n\right\| \\
&\le \alpha_n\|x_0 - x\| + (1 - \alpha_n)\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x\right\| + (1 - \alpha_n)\lambda\beta_n\|d_n\| \\
&\le \alpha_n\|x_0 - x\| + \frac{1}{2}(1 - \alpha_n)\|x_n - x\| + \frac{1}{2}(1 - \alpha_n)\|x_{n+1} - x\| + (1 - \alpha_n)\lambda M_1\beta_n.
\end{aligned}$$

It follows from (C4) and $\alpha_n < 1$ $(n \ge 0)$ that

$$\|x_{n+1} - x\| \le \left(1 - \frac{2\alpha_n}{1 + \alpha_n}\right)\|x_n - x\| + \left(\frac{2\alpha_n}{1 + \alpha_n}\right)(\|x_0 - x\| + \lambda M_1).$$

Induction guarantees that, for all $x \in \mathrm{Fix}(T)$ and $n \ge 0$,

$$\|x_n - x\| \le \|x_0 - x\| + \lambda M_1.$$

This means that $(x_n)_{n=0}^{\infty}$ is bounded. This completes the proof. □

Lemma 3.3. *Let $(x_n)_{n=0}^{\infty}$ be generated by Algorithm 3.1 with the Assumption 3.1. Then*

(i) $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$;

(ii) $\lim_{n\to\infty} ||x_n - T(x_n)|| = 0$;

(iii) $\limsup_{n\to\infty} \left\langle x_0 - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^* \right\rangle \leq 0$, *where* $x^* = P_{Fix(T)}(x_0)$.

Proof. (i) Since $\alpha_n \leq |\alpha_{n+1} - \alpha_n| + \alpha_{n+1}$ and $\alpha_n < 1$ for all $n \geq 0$, we get

$$\alpha_{n+1}^2 + \alpha_n^2 \leq \alpha_{n+1}^2 + \alpha_n(|\alpha_{n+1} - \alpha_n| + \alpha_{n+1}) \leq (\alpha_{n+1} + \alpha_n)\alpha_{n+1} + |\alpha_{n+1} - \alpha_n|. \quad (3.5)$$

Since $(x_n)_{n=0}^{\infty}$ is bounded and (C5′), this implies that $\left(T\left(\frac{x_n + x_{n+1}}{2}\right)\right)_{n=0}^{\infty}$ is bounded, there exists $M_2 > 0$ such that

$$\left\|T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \leq M_2. \quad (3.6)$$

For all $n > 0$, by the nonexpansivity of T, (3.6), $(1 - \alpha_n) < 1$ $(n > 0)$, $\|d_n\| \leq M_1$ and (C4), we obtain

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \left\|(\alpha_n - \alpha_{n-1})x_0 + (1 - \alpha_n)\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right)\right)\right. \\
&\quad \left. +(\alpha_{n-1} - \alpha_n)T\left(\frac{x_{n-1} + x_n}{2}\right) + (1 - \alpha_n)\lambda\beta_n d_n - (1 - \alpha_{n-1})\lambda\beta_{n-1}d_{n-1}\right\| \\
&\leq (1 - \alpha_n)\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - T\left(\frac{x_{n-1} + x_n}{2}\right)\right\| + (1 - \alpha_n)\lambda\beta_n\|d_n\| \\
&\quad + (1 - \alpha_{n-1})\lambda\beta_{n-1}\|d_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|x_0\| + \left\|T\left(\frac{x_{n-1} + x_n}{2}\right)\right\|\right) \\
&\leq (1 - \alpha_n)\left\|\left(\frac{x_n + x_{n+1}}{2}\right) - \left(\frac{x_{n-1} + x_n}{2}\right)\right\| + \lambda M_1\left(\beta_n + \beta_{n-1}\right) \\
&\quad + |\alpha_n - \alpha_{n-1}|\left(\|x_0\| + M_2\right) \\
&\leq \frac{1}{2}(1 - \alpha_n)\|x_{n+1} - x_n\| + \frac{1}{2}(1 - \alpha_n)\|x_n - x_{n-1}\| + \lambda M_1\left(\alpha_n^2 + \alpha_{n-1}^2\right) \\
&\quad + |\alpha_n - \alpha_{n-1}|\left(\|x_0\| + M_2\right).
\end{aligned}$$

By (3.5), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \frac{1}{2}(1-\alpha_n)\|x_{n+1}-x_n\| + \frac{1}{2}(1-\alpha_n)\|x_n-x_{n-1}\| \\ &\quad + |\alpha_n-\alpha_{n-1}|\left(\|x_0\|+M_2\right) + \lambda M_1\left((\alpha_n+\alpha_{n-1})\alpha_n + |\alpha_n-\alpha_{n-1}|\right) \\ &= \frac{1}{2}(1-\alpha_n)\|x_{n+1}-x_n\| + \frac{1}{2}(1-\alpha_n)\|x_n-x_{n-1}\| + M_3|\alpha_n-\alpha_{n-1}| \\ &\quad + \lambda M_1(\alpha_n+\alpha_{n-1})\alpha_n,\end{aligned}$$

where $M_3 = \|x_0\| + M_2 + \lambda M_1 < \infty$. This implies that

$$\begin{aligned}\|x_{n+1}-x_n\| \le \left(1-\frac{2\alpha_n}{1+\alpha_n}\right)\|x_n-x_{n-1}\| + \left(\frac{2\alpha_n}{1+\alpha_n}\right)\left(\lambda M_1(\alpha_n+\alpha_{n-1})\right) \\ + \frac{2M_3|\alpha_n-\alpha_{n-1}|}{1+\alpha_n}.\end{aligned} \tag{3.7}$$

From Condition (C3′), we divide it into 2 cases as follows:

Case 1: $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$. Setting

$$a_n = \|x_n-x_{n-1}\|,\ b_n = \lambda M_1(\alpha_n+\alpha_{n-1}),\ c_n = \frac{2M_3|\alpha_n-\alpha_{n-1}|}{1+\alpha_n} \text{ and } \bar{\alpha}_n = \frac{2\alpha_n}{1+\alpha_n}.$$

Then (3.7) becomes $a_{n+1} \le (1-\bar{\alpha}_n)a_n + \bar{\alpha}_n b_n + c_n$. This together with (C1), (C2), (C3′) and Lemma 2.4 gives

$$\lim_{n\to\infty}\|x_n-x_{n-1}\| = 0.$$

Case 2: $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$. Setting

$$a_n = \|x_n-x_{n-1}\|,\ b_n = \lambda M_1(\alpha_n+\alpha_{n-1}) + \frac{M_3}{\alpha_n}|\alpha_n-\alpha_{n-1}|,\ c_n = 0 \text{ and } \bar{\alpha}_n = \frac{2\alpha_n}{1+\alpha_n}.$$

Since $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1$ and C1, we get $\lim_{n\to\infty}\frac{M_3}{\alpha_n}|\alpha_n-\alpha_{n-1}| = 0$. Then (3.7) becomes $a_{n+1} \le (1-\bar{\alpha}_n)a_n + \bar{\alpha}_n b_n + c_n$. This together with (C1), (C2), (C3′) and Lemma 2.4 gives

$$\lim_{n\to\infty}\|x_n-x_{n-1}\| = 0.$$

From case 1 and case 2, this implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

(ii) By the definition of $(x_n)_{n=0}^{\infty}$, (C4) and (3.6), we obtain that

$$\begin{aligned}
&\left\|x_{n+1} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\
&\quad= \left\|\alpha_n x_0 + (1-\alpha_n)\left(T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n\right) - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\
&\quad\leq \alpha_n\left(\|x_0\| + \left\|T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|\right) + (1-\alpha_n)\lambda\beta_n\|d_n\| \\
&\quad\leq \alpha_n(\|x_0\| + M_2) + \lambda M_1\alpha_n^2.
\end{aligned}$$

This together with (C1) gives

$$\lim_{n\to\infty}\left\|x_{n+1} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| = 0. \tag{3.8}$$

Since T is nonexpansive, we get

$$\begin{aligned}
\|x_n - T(x_n)\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - T(x_n)\right\| \\
&\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \left\|\frac{x_{n+1} - x_n}{2}\right\| \\
&\leq \frac{3}{2}\|x_n - x_{n+1}\| + \left\|x_{n+1} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|.
\end{aligned}$$

This together with Lemma 3.3 (i) and (3.8) gives

$$\lim_{n\to\infty}\|x_n - T(x_n)\| = 0.$$

(iii) Let $x^* = P_{\mathrm{Fix}(T)}(x_0)$ and $z_n = T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n$ for all $n \geq 0$. Since $\left(T\left(\frac{x_n + x_{n+1}}{2}\right)\right)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$ are bounded, this implies that $(z_n)_{n=0}^{\infty}$ and $(\langle x_0 - x^*, z_n - x^*\rangle)_{n=0}^{\infty}$ are bounded. From the limit superior of $(\langle x_0 - x^*, z_n - x^*\rangle)_{n=0}^{\infty}$, there exists subsequence $(z_{n_k})_{k=0}^{\infty}$ of $(z_n)_{n=0}^{\infty}$ such that

$$\limsup_{n\to\infty}\langle x_0 - x^*, z_n - x^*\rangle = \lim_{k\to\infty}\langle x_0 - x^*, z_{n_k} - x^*\rangle.$$

Moreover, since $(z_{n_k})_{k=0}^{\infty}$ is bounded, there exists subsequence $(z_{n_{k_i}})_{i=0}^{\infty}$ of $(z_{n_k})_{k=0}^{\infty}$ converges weakly to some point $\hat{y} \in H$. We obtain

$$\limsup_{n\to\infty}\langle x_0 - x^*, z_n - x^*\rangle = \lim_{i\to\infty}\langle x_0 - x^*, z_{n_{k_i}} - x^*\rangle = \langle x_0 - x^*, \hat{y} - x^*\rangle. \tag{3.9}$$

By the triangle inequality, (C4) and $\|d_n\| \le M_1$ $(n \ge 0)$, we obtain

$$\|z_n - x_n\| \le \left\| T\left(\frac{x_n + x_{n+1}}{2}\right) - x_{n+1} \right\| + \|x_{n+1} - x_n\| + \lambda M_1 \alpha_n^2,$$

this together with Lemma 3.3 (i), (ii) and (C1), we obtain that $\lim_{n\to\infty} \|z_n - x_n\| = 0$. It follows that $(x_{n_{k_i}})_{i=0}^{\infty}$ converges weakly to $\hat{y} \in H$. Next, we will show that $\hat{y} \in \text{Fix}(T)$. Assume that $\hat{y} \notin \text{Fix}(T)$. i.e., $\hat{y} \neq T(\hat{y})$. By Theorem 2.2, Lemma 3.3 (ii) and the nonexpansivity of T, we obtain

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_{k_i}} - \hat{y}\| &< \liminf_{i\to\infty} \|x_{n_{k_i}} - T(\hat{y})\| \\
&\le \liminf_{i\to\infty} (\|x_{n_{k_i}} - T(x_{n_{k_i}})\| + \|T(x_{n_{k_i}}) - T(\hat{y})\|) \\
&= \liminf_{i\to\infty} \|T(x_{n_{k_i}}) - T(\hat{y})\| \\
&\le \liminf_{i\to\infty} \|x_{n_{k_i}} - \hat{y}\|,
\end{aligned}$$

which is contradiction. Thus $\hat{y} \in \text{Fix}(T)$. By (3.9), Lemma 2.3 (iii) and $x^* = P_{\text{Fix}(T)}(x_0)$, we get

$$\limsup_{n\to\infty} \langle x_0 - x^*, z_n - x^* \rangle = \langle x_0 - x^*, \hat{y} - x^* \rangle \le 0.$$

This completes the proof. □

Now, we are in a position to prove the main theorem.

Theorem 3.4. *Let $(x_n)_{n=0}^{\infty}$ be generated by Algorithm 3.1 with the Assumption 3.1. Then $(x_n)_{n=0}^{\infty}$ converges strongly to $P_{\text{Fix}(T)}(x_0)$.*

Proof. Let $x^* = P_{\text{Fix}(T)}(x_0)$. Since the inequality $\|x + y\|^2 \le 2\langle y, x + y\rangle$ $(x, y \in H)$, the Schwarz inequality, the triangle inequality, (C4) and $\alpha_n < 1$, we obtain that

$$\begin{aligned}
&\left\| T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^* \right\|^2 \\
&\le \left\| T\left(\frac{x_n + x_{n+1}}{2}\right) - x^* \right\|^2 + 2\lambda\beta_n \left\langle d_n, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^* \right\rangle \\
&\le \left\| \frac{x_n + x_{n+1}}{2} - x^* \right\|^2 + 2\lambda\beta_n \|d_n\| \left(\left\| T\left(\frac{x_n + x_{n+1}}{2}\right) - x^* \right\| + \lambda\beta_n \|d_n\| \right) \\
&\le \left\| \left(\frac{x_n - x^*}{2}\right) + \left(\frac{x_{n+1} - x^*}{2}\right) \right\|^2 + 2\lambda\alpha_n^2 M_1 \left(M_2 + \|x^*\| + \lambda M_1\right) \\
&\le \frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 + M_4\alpha_n^2,
\end{aligned}$$

where $M_4 = 2\lambda M_1 (M_2 + \|x^*\| + \lambda M_1) < \infty$. We obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n(x_0 - x^*) + (1-\alpha_n)\left(T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right)\right\|^2 \\ &= \alpha_n^2\|x_0 - x^*\|^2 + (1-\alpha_n)^2\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\left\langle x_0 - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\rangle,\end{aligned}$$

and so

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n^2\|x_0 - x^*\|^2 + (1-\alpha_n)\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\|^2 \\ &\quad + 2\alpha_n\left\langle x_0 - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\rangle \\ &\le \alpha_n^2\|x_0 - x^*\|^2 + \frac{1}{2}(1-\alpha_n)\|x_n - x^*\|^2 + \frac{1}{2}(1-\alpha_n)\|x_{n+1} - x^*\|^2 \\ &\quad + M_4\alpha_n^2 + 2\alpha_n\left\langle x_0 - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\rangle.\end{aligned}$$

Thus

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le \left(1 - \frac{2\alpha_n}{1+\alpha_n}\right)\|x_n - x^*\|^2 + \left(\frac{2\alpha_n}{1+\alpha_n}\right)\left(\alpha_n\|x_0 - x^*\|^2\right) \\ &\quad + \left(\frac{2\alpha_n}{1+\alpha_n}\right)\left(M_4\alpha_n + 2\left|\left\langle x_0 - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\rangle\right|\right). \qquad (3.10)\end{aligned}$$

Setting

$$a_n = \|x_n - x^*\|^2, \; b_n = \alpha_n\|x_0 - x^*\|^2 + M_4\alpha_n + 2\left|\left\langle x_0 - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) + \lambda\beta_n d_n - x^*\right\rangle\right|$$

$$c_n = 0 \text{ and } \bar{\alpha}_n = \frac{2\alpha_n}{1+\alpha_n}.$$

Thus (3.10) becomes $a_{n+1} \le (1 - \bar{\alpha}_n)a_n + \bar{\alpha}_n b_n + c_n$. Using Lemma 2.4, we get $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$. and so

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

This means that $(x_n)_{n=0}^{\infty}$ converges strongly to x^*. This completes the proof. □

Next, if we set $\beta_n = 0$ for all $n \ge 0$, then Algorithm 3.1 reduces to the implicit midpoint rule defined by (1.3) as the following corollary. This means that Theorem 3.4 is generalization of the convergence analysis of the implicit midpoint rule.

Corollary 3.5. *Let $(x_n)_{n=0}^{\infty}$ be generated by the implicit midpoint rule Suppose that (C1) $\lim_{n\to\infty} \alpha_n = 0$, (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, and (C3′) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$. Then $(x_n)_{n=0}^{\infty}$ converges strongly to $P_{Fix(T)}(x_0)$.*

Proof. Setting $\beta_n = 0$ for all $n \geq 0$, the Algorithm 3.1 reduces to the implicit midpoint rule. Next, we will show that (C4) and (C5′) hold. Since $\beta_n = 0$ for all $n \geq 0$, we obtain $\beta_n \leq \alpha_n^2$. Thus (C4) holds. For each $x \in \text{Fix}(T)$ and $n \geq 0$, by the nonexpansivity of T and the triangle inequality, we get

$$\begin{aligned}
\|x_{n+1} - x\| &= \left\| \alpha_n(x_0 - x) + (1-\alpha_n)\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - x\right)\right\| \\
&\leq \alpha_n \|x_0 - x\| + (1-\alpha_n)\left\|\frac{x_n + x_{n+1}}{2} - x\right\| \\
&\leq \alpha_n \|x_0 - x\| + \frac{1}{2}(1-\alpha_n)\|x_n - x\| + \frac{1}{2}(1-\alpha_n)\|x_{n+1} - x\|.
\end{aligned}$$

Thus

$$\|x_{n+1} - x\| \leq \left(1 - \frac{2\alpha_n}{1+\alpha_n}\right)\|x_n - x\| + \left(\frac{2\alpha_n}{1+\alpha_n}\right)\|x_0 - x\|. \tag{3.11}$$

We will show that $\|x_n - x\| \leq \|x_0 - x\|$ for all $n \geq 0$. Obviously, $\|x_0 - x\| \leq \|x_0 - x\|$. Assume that $\|x_n - x\| \leq \|x_0 - x\|$ for $n \geq 0$. By (3.11), we get

$$\|x_{n+1} - x\| \leq \left(1 - \frac{2\alpha_n}{1+\alpha_n}\right)\|x_n - x\| + \left(\frac{2\alpha_n}{1+\alpha_n}\right)\|x_0 - x\| \leq \|x_0 - x\|.$$

By mathematical induction, we get $\|x_n - x\| \leq \|x_0 - x\|$ for all $n \geq 0$. This implies that $(x_n)_{n=0}^{\infty}$ is bounded. Moreover, for all $x \in \text{Fix}(T)$ and $n \geq 0$ by the triangle inequality and the nonexpansivity of T, we obtain

$$\begin{aligned}
\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - \left(\frac{x_n + x_{n+1}}{2}\right)\right\| &\leq \left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x\right\| + \left\|x - \left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\
&\leq \left\|\left(\frac{x_n + x_{n+1}}{2}\right) - x\right\| + \left\|\left(\frac{x_n + x_{n+1}}{2}\right) - x\right\| \\
&= 2\left\|\left(\frac{x_n - x}{2}\right) + \left(\frac{x_{n+1} - x}{2}\right)\right\| \\
&\leq 2\|x_0 - x\|.
\end{aligned}$$

This implies that $\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - \left(\frac{x_n + x_{n+1}}{2}\right)\right)_{n=0}^{\infty}$ is bounded. Thus (C5′) holds.

By Theorem 3.4, we obtain that $(x_n)_{n=0}^{\infty}$ converges strongly to $P_{\mathrm{Fix}(T)}(x_0)$. □

4 Numerical Results

The purpose of this section is give two numerical examples supporting Theorem 3.4.

Example 4.1. Let $\mathbb{R}$ be the set of real numbers with the usual inner product and let $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by $T(x) = \cos(x)$ for all $x \in \mathbb{R}$. Then T is a nonexpansive mapping. Let $(x_n)_{n=0}^{\infty}$ be generated by Algorithm 3.1 with $\alpha_n = \dfrac{1}{10^5(n+2)}$ and $\beta_n = \dfrac{1}{(10^5(n+2))^2}$ $(n \geq 0)$. It is easy to show that $(\alpha_n)_{n=0}^{\infty}$ and $(\beta_n)_{n=0}^{\infty}$ satisfy the Conditions, (C1), (C2), (C3′) and (C4). We set $\lambda = 1$, $x_0 = \pi$, $d_0 = -1 - \pi$ and $\|T(x_n) - x_n\| \leq 10^{-6}$ for large enough n. We rewrite Algorithm 3.1 as follows:

$$\begin{cases} x_{n+1} = \dfrac{x_0}{10^5(n+2)} + \left(\dfrac{10^5(n+2)-1}{10^5(n+2)}\right)\left(T\left(\dfrac{x_n + x_{n+1}}{2}\right) + \dfrac{d_n}{(10^5(n+2))^2}\right), \\ d_{n+1} = \left(T\left(\dfrac{x_n + x_{n+1}}{2}\right) - \left(\dfrac{x_n + x_{n+1}}{2}\right)\right) + \dfrac{d_n}{(10^5(n+2))^2}. \end{cases} \tag{4.1}$$

We can show that (C5′) holds, since $\|T(x_n) - x_n\| \leq 10^{-6}$ for large enough n, we get $(T(x_n) - x_n)_{n=0}^{\infty}$ is bounded. Since $(T(x_n))_{n=0}^{\infty}$ is bounded. So $(x_n)_{n=0}^{\infty}$ is bounded. This implies that $\left(\dfrac{x_n + x_{n+1}}{2}\right)_{n=0}^{\infty}$ is bounded. Thus $\left(T\left(\dfrac{x_n + x_{n+1}}{2}\right) - \left(\dfrac{x_n + x_{n+1}}{2}\right)\right)_{n=0}^{\infty}$ is bounded. Therefore (C5′) holds. We used the MATLAB to solve Algorithm 4.1. We present the number of iterations in numerical results for the Halpern algorithm, HP-CGM, the implicit midpoint rule and Algorithm 3.1 in Figure 1. The results of Figure 1 show that the number of iterations for the Halpern algorithm, HP-CGM, the implicit midpoint rule and Algorithm 3.1 satisfy $\|T(x_n) - x_n\| \leq 10^{-6}$ and we found that the least number of iterations to approximate a fixed point of T is Algorithm 3.1. Moreover, we found that $(x_n)_{n=0}^{\infty}$ generated by Algorithm 3.1 approximates a fixed point of T which is 0.739085.

Example 4.2. Let $\mathbb{R}^2$ be the usual space. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by

$$T(x_1, x_2) = \left(\sin\left(\frac{x_1 + x_2}{\sqrt{2}}\right), \cos\left(\frac{x_1 + x_2}{\sqrt{2}}\right)\right)$$

for all $(x_1, x_2) \in \mathbb{R}^2$. By $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$ and trigonometric properties, we obtain that T is a nonexpansive mapping. Let $(\mathrm{x}_n)_{n=0}^{\infty}$ be generated by Algorithm 3.1 with $\alpha_n = \dfrac{1}{10^5(n+2)}$ and $\beta_n = \dfrac{1}{(10^5(n+2))^2}$ $(n \geq 0)$. It is easy to show that $(\alpha_n)_{n=0}^{\infty}$ and $(\beta_n)_{n=0}^{\infty}$

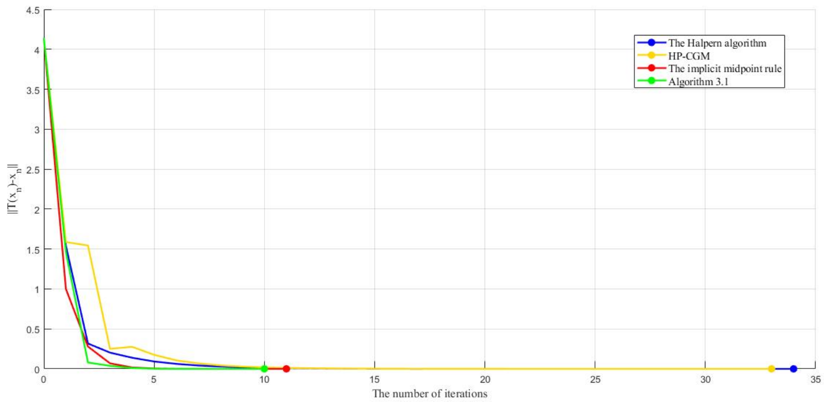

Figure 1: The number of iterations in numerical results which are satisfying $\|T(x_n) - x_n\| \leq 10^{-6}$ for the Halpern algorithm, HP-CGM, the implicit midpoint rule and Algorithm 3.1

satisfy the Conditions, (C1), (C2), (C3′) and (C4). We set $\lambda = 1$, $\mathtt{x}_0 = (x_0, y_0) = (0,0)$, $d_0 = (0,1)$ and $\|T(\mathtt{x}_n) - \mathtt{x}_n\|_2 \leq 10^{-6}$ for large enough n. We rewrite Algorithm 3.1 as follows:

$$\begin{cases} \mathtt{x}_{n+1} = \dfrac{\mathtt{x}_0}{10^5(n+2)} + \left(\dfrac{10^5(n+2)-1}{10^5(n+2)}\right)\left(T\left(\dfrac{\mathtt{x}_n + \mathtt{x}_{n+1}}{2}\right) + \dfrac{d_n}{(10^5(n+2))^2}\right), \\ d_{n+1} = \left(T\left(\dfrac{\mathtt{x}_n + \mathtt{x}_{n+1}}{2}\right) - \left(\dfrac{\mathtt{x}_n + \mathtt{x}_{n+1}}{2}\right)\right) + \dfrac{d_n}{(10^5(n+2))^2}. \end{cases} \tag{4.2}$$

We can show that (C5′) holds. Since $||T(\mathtt{x}_n) - \mathtt{x}_n\|_2 \leq 10^{-6}$ for large enough n, we get $(T(\mathtt{x}_n) - \mathtt{x}_n)_{n=0}^{\infty}$ is bounded. Since $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$, this implies that $(T(\mathtt{x}_n))_{n=0}^{\infty}$ is bounded, this implies that $(\mathtt{x}_n)_{n=0}^{\infty}$ is bounded, we also get $\left(\dfrac{\mathtt{x}_n + \mathtt{x}_{n+1}}{2}\right)_{n=0}^{\infty}$ is bounded. Thus $\left(T\left(\dfrac{\mathtt{x}_n + \mathtt{x}_{n+1}}{2}\right) - \left(\dfrac{\mathtt{x}_n + \mathtt{x}_{n+1}}{2}\right)\right)_{n=0}^{\infty}$ is bounded. Therefore (C5′) holds. We used the MATLAB to solve Algorithm 4.2. We present the number of iterations in numerical results for the Halpern algorithm, HP-CGM, the implicit midpoint rule and Algorithm 3.1 in Figure 2. The results of Figure 4.2 show that the number of iterations for the Halpern algorithm, HP-CGM, the implicit midpoint rule and Algorithm 3.1 satisfy $\|T(\mathtt{x}_n) - \mathtt{x}_n\|_2 \leq 10^{-6}$ and we found that the least number of iterations to approximate a fixed point of T is Algorithm 3.1. Moreover, we found that $(\mathtt{x}_n)_{n=0}^{\infty}$ generated by Algorithm 3.1 approximates a fixed point of T which is $(0.831021, 0.556240)$.

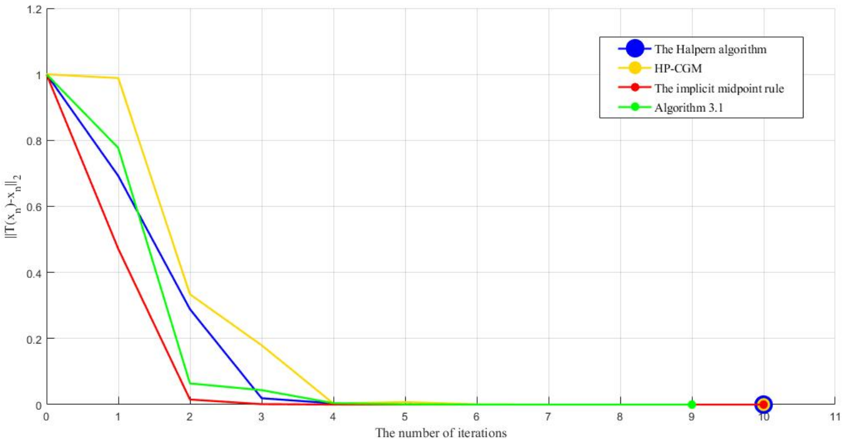

Figure 2: The number of iterations in numerical results which is satisfying $\|T(\mathtt{x}_n) - \mathtt{x}_n\|_2 \leq 10^{-6}$ for the Halpern algorithm, HP-CGM, the implicit midpoint rule and Algorithm 3.1

This paper presented the implicit midpoint rule based on conjugate gradient method for finding a fixed point of a nonexpansive mapping in a Hilbert space and its strong convergence to a fixed point of a nonexpansive mapping under certain assumptions. Moreover, the implicit midpoint rule based on conjugate gradient method can reduces to the implicit midpoint rule when $\beta_n = 0$ for all $n \geq 0$. We gave examples to support the convergence theorem. We compared the number of iterations for the Halpern algorithm, HP-CGM, the implicit midpoint rule and the implicit midpoint rule based on conjugate gradient method. The results showed that the least number of iterations to approximate a fixed point were the implicit midpoint rule based on conjugate gradient method.

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Acceleration of the Halpern Algorithm to Search for an Attractive Point of a Nonexpansive Mapping

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Abstract

The purpose of this paper is to introduce an algorithm to accelerate the Halpern algorithm by using the conjugate gradient method. Setting some certain parameters, the such algorithm is deduced to the Halpern algorithm. Consequently, the strong convergence theorem of acceleration of the Halpern algorithm for finding an attractive point of a nonexpansive mapping in Hilbert space is presented. When the domain of a nonexpansive mapping is the whole space, the attractive point set and the fixed point set coincide and then the strong convergence theorem for finding to a fixed point of a nonexpansive mapping is shown. Finally, we give an example to support our main results.

Mathematics Subject Classification: 47H10, 47H09, 47H05
Keywords: attractive point, conjugate gradient method, fixed point, nonexpansive mapping, Halpern algorithm

1 Introduction

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$ and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of all fixed points of T. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is non-empty (see [2]). We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $(x_n)_{n=0}^{\infty}$ converges strongly (weakly, resp.) to x.

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The problem of finding a fixed point of nonexpansive mappings has been widely investigated by many authors. In 1992, Wittmann [12] proved the following strong convergence theorem of Halpern's type [5] in Hilbert space as follows:

Theorem 1.1. [12] *Let C be a non-empty closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_0 \in C$, define a sequence $(x_n)_{n=0}^{\infty}$ in C by*

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n \quad \textit{for all } n \geq 0, \tag{1.1}$$

where $(\alpha_n)_{n=0}^{\infty} \subset (0,1)$ satisfies

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $x_n \to P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Let C be a non-empty subset of a Hilbert space H and $T : C \to C$ be a mapping. An attractive point of T is a point x in H such that

$$\|Ty - x\| \leq \|y - x\| \quad \text{for all } y \in C. \tag{1.2}$$

The set of all attractive points of T is denoted by $A(T)$. In 2012, Akashi and Takahashi [1] used the concept of attractive points of a nonexpansive mapping to obtain a strong convergence of Halpern's iteration in a subset of a Hilbert space as follows:

Theorem 1.2. [1, Theorem 3.5] *Let C be a non-empty subset a Hilbert space H. Let T be a nonexpansive mapping from C into itselt with $A(T) \neq \varnothing$ and let $(x_n)_{n=0}^{\infty}$ be a sequence in C defined by* (1.1) *satisfying (C1)-(C3). Then $x_n \to P_{A(T)}x_0$.*

In 2014, Sakurai and liduka [9] presented an algorithm to accelerate the Halpern algorithm in a Hilbert space of a nonexpansive mapping to search for a fixed point using the ideas of conjugate gradient methods that can accelerate the steepest descent method as follows:

Theorem 1.3. [9, Theorem 3.1] *Let H be a Hilbert space and let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Let $(x_n)_{n=0}^{\infty}$ be a sequence in H defined by $x_0 \in C, d_0 =$*

$\dfrac{1}{\alpha}(Tx_0 - x_0)$ *and*

$$\begin{cases} d_{n+1} &= \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n, \\ y_n &= x_n + \alpha d_{n+1}, \\ x_{n+1} &= \mu\alpha_n x_0 + (1-\mu\alpha_n)y_n, \end{cases} \tag{1.3}$$

where $\mu \in (0,1], \alpha > 0, (\alpha_n)_{n=0}^{\infty} \subset (0,1)$ *and* $(\beta_n)_{n=0}^{\infty} \subset [0,\infty)$. *Suppose that (C1)-(C3),*

(C4) $\beta_n \leq \alpha_n^2$ *for all* $n \geq 0$, *and*

(C5) $(T(x_n) - x_n)_{n=0}^{\infty}$ *is bounded.*

Then $x_n \to P_{F(T)}x_0$.

Many researchers have been studied the conjugate gradient method such as [3, 4, 6–8].

In this paper, we mainly consider an algorithm to accelerate the Halpern algorithm in a Hilbert space to search for an attractive point of a nonexpansive mapping by using conjugate gradient methods. We also present the strong convergence theorem for finding of a fixed point of a nonexpansive mapping. Finally, we give the example to support our main results.

2 Preliminaries

Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. Let $(x_n)_{n=0}^{\infty}$ be a sequence in H and $x \in H$.

(i) $(x_n)_{n=0}^{\infty}$ *converges strongly* to x, denoted by $x_n \to x$, if $||x_n - x|| \to 0$;

(ii) $(x_n)_{n=0}^{\infty}$ *converges weakly* to x, denoted by $x_n \rightharpoonup x$, if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$.

It's well known that if $x_n \to x$, then $x_n \rightharpoonup x$. Let C be a non-empty closed convex subset of H. Then, for any $x \in H$, there exists the nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Furthermore, we get the following theorem.

Theorem 2.1. [11, Theorem 4.1.2] *Let C be a non-empty closed convex subset of a Hilbert space H. Then $\hat{x} = P_C x$ if and only if $\langle x - \hat{x}, y - \hat{x} \rangle \leq 0$ for all $y \in C$.*

Let C be a non-empty subset of a Hilbert space H. For a mapping T of C into H, we denote by $F(T)$ the set of all *fixed points* of T and by $A(T)$ the set of all *attractive points* of T, i.e.,

(i) $F(T) = \{x \in C : x = Tx\}$;

(ii) $A(T) = \{x \in H : \|Ty - x\| \leq \|y - x\|, \ \forall y \in C\}$.

Lemma 2.2. [10, Lemma 2.3] *Let H be a Hilbert space and let C be a non-empty subset of H. Let T be a mapping from C into itself with $A(T) \neq \varnothing$. Then $A(T)$ is a closed convex subset of H.*

Lemma 2.3. [11, Theorem 6.1.3] *Let H be a Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Then $F(T)$ is a closed convex subset of H.*

To prove our main result, we need the following lemma.

Lemma 2.4. [9, Proposition 2.2] *Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ and $(\widetilde{\alpha}_n)_{n=0}^{\infty}$ be sequences of nonnegative real numbers with*

$$a_{n+1} \leq (1 - \widetilde{\alpha}_n) a_n + \widetilde{\alpha}_n b_n + c_n$$

for each $n \geq 0$. Suppose that $\sum_{n=0}^{\infty} \widetilde{\alpha}_n = \infty$, $\limsup_{n\to\infty} b_n \leq 0$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, using the method introduced by Sakurai and Iiduka [9], we obtain a strong convergence theorem for finding an attractive point of a nonexpansive mapping.

Theorem 3.1. *Let C be a non-empty subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $A(T) \neq \varnothing$. Let $(x_n)_{n=0}^{\infty}$ be the acceleration of the Halpern algorithm* *generated by* $x_0 \in C, d_0 \in C$ *and*

$$\begin{cases} d_{n+1} &= \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n, \\ y_n &= x_n + \alpha d_{n+1}, \\ x_{n+1} &= \mu\alpha_n x_0 + (1-\mu\alpha_n)y_n, \end{cases} \tag{3.1}$$

where $\mu \in (0,1], \alpha > 0, (\alpha_n)_{n=0}^{\infty} \subset (0,1)$ *and* $(\beta_n)_{n=0}^{\infty} \subset [0,\infty)$. *Suppose that*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C4) $\beta_n \leq \alpha_n^2$ *for all* $n \geq 0$,

(C5) $(Tx_n - x_n)_{n=0}^{\infty}$ *is bounded.*

Then $(x_n)_{n=0}^{\infty}$ *defined by* (3.1) *converges strongly to* $x^* = P_{A(T)}x_0$.

Proof. We first show that $(d_n)_{n=0}^{\infty}, (x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are bounded. From (C1) and (C4), we have that $\lim_{n\to\infty} \beta_n = 0$. So, there exists $n_0 \geq 0$ such that $\beta_n \leq \frac{1}{2}$ for all $n \geq n_0$. From (C5), there exists $M_0 \geq 0$ such that $\|Tx_n - x_n\| \leq M_0$ for all $n \geq 0$. Define $M_1 = \max\left\{\|d_i\|, \frac{2}{\alpha}M_0 : 0 \leq i \leq n_0\right\}$. Suppose that $\|d_n\| \leq M_1$ for $n \geq n_0$. Then

$$\begin{aligned} \|d_{n+1}\| &= \left\|\frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n\right\| \\ &\leq \frac{1}{\alpha}\|Tx_n - x_n\| + \beta_n \|d_n\| \\ &\leq \frac{M_0}{\alpha} + \frac{1}{2}M_1 \\ &\leq M_1. \end{aligned}$$

By mathematic induction, we get $\|d_n\| \leq M_1$ for all $n \geq 0$. Therefore, $(d_n)_{n=1}^{\infty}$ is bounded. Notice that, for all $x \in A(T)$ and $n \geq 0$,

$$\begin{aligned} \|y_n - x\| &= \|Tx_n + \alpha\beta_n d_n - x\| \\ &\leq \|Tx_n - x\| + \alpha\beta_n \|d_n\| \\ &\leq \|x_n - x\| + \alpha M_1 \beta_n. \end{aligned} \tag{3.2}$$

It follows from (C4) and $\alpha_n \leq 1$ that

$$\begin{aligned}\|x_{n+1} - x\| &= \|\mu\alpha_n(x_0 - x) + (1 - \mu\alpha_n)(y_n - x)\| \\ &\leq \mu\alpha_n\|x_0 - x\| + (1 - \mu\alpha_n)\|y_n - x\| \\ &\leq \mu\alpha_n\|x_0 - x\| + (1 - \mu\alpha_n)(\|x_n - x\| + \alpha M_1\beta_n) \\ &\leq (1 - \mu\alpha_n)\|x_n - x\| + \mu\alpha_n\|x_0 - x\| + \alpha M_1\alpha_n^2 \\ &\leq (1 - \mu\alpha_n)\|x_n - x\| + \mu\alpha_n\left(\|x_0 - x\| + \frac{\alpha M_1}{\mu}\right).\end{aligned}$$

Let $M_2 = \|x_0 - x\| + \frac{\alpha M_1}{\mu}$. By mathematical induction, we get $\| x_n - x\| \leq M_2$ and hence $(x_n)_{n=0}^{\infty}$ is bounded. This together with (3.2) gives $(y_n)_{n=0}^{\infty}$ is bounded.

We next will show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. From the nonexpansivity of T, $\|d_n\| \leq M_1$ and (C4), we obtain

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|(Tx_{n+1} - Tx_n) + \alpha(\beta_{n+1}d_{n+1} - \beta_n d_n)\| \\ &\leq \|Tx_{n+1} - Tx_n\| + \alpha(\beta_{n+1}\|d_{n+1}\| + \beta_n\|d_n\|) \\ &\leq \|x_{n+1} - x_n\| + \alpha M_1(\alpha_{n+1}^2 + \alpha_n^2). \qquad (3.3)\end{aligned}$$

Since $\alpha_n \leq |\alpha_{n+1} - \alpha_n| + \alpha_{n+1}$ and $\alpha_n < 1$ for all $n \geq 0$, we have that

$$\begin{aligned}\alpha_{n+1}^2 + \alpha_n^2 &\leq \alpha_{n+1}^2 + \alpha_n|\alpha_{n+1} - \alpha_n| + \alpha_n\alpha_{n+1} \\ &\leq (\alpha_{n+1} + \alpha_n)\alpha_{n+1} + |\alpha_{n+1} - \alpha_n|.\end{aligned}$$

This together with (3.3) gives

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \alpha M_1\left((\alpha_{n+1} + \alpha_n)\alpha_{n+1} + |\alpha_{n+1} - \alpha_n|\right).$$

Since

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &= \|(1 - \mu\alpha_n)(y_n - y_{n-1}) + \mu(\alpha_n - \alpha_{n-1})(x_0 - y_{n-1})\| \\ &\leq (1 - \mu\alpha_n)\|y_n - y_{n-1}\| + \mu|\alpha_n - \alpha_{n-1}|(\|x_0\| + \|y_{n-1}\|) \\ &\leq (1 - \mu\alpha_n)\|y_n - y_{n-1}\| + M_3|\alpha_n - \alpha_{n-1}|,\end{aligned}$$

we get

$$\begin{aligned}&\|x_{n+1}-x_n\|\\&\le(1-\mu\alpha_n)(\|x_n-x_{n-1}\|+\alpha M_1((\alpha_n+\alpha_{n-1})\alpha_n+|\alpha_n-\alpha_{n-1}|))+M_3|\alpha_n-\alpha_{n-1}|\\&\le(1-\mu\alpha_n)\|x_n-x_{n-1}\|+\frac{\alpha M_1}{\mu}(\alpha_n+\alpha_{n-1})\mu\alpha_n+(\alpha M_1+M_3)|\alpha_n-\alpha_{n-1}|,\end{aligned}$$

where $M_3 = \sup\{\mu(\|x_0\|+\|y_n\|) : n \ge 0\} < \infty$. Setting

$$a_n=\|x_n-x_{n-1}\|,\ b_n=\frac{\alpha M_1}{\mu}(\alpha_n+\alpha_{n-1}),\ c_n=(\alpha M_1+M_3)|\alpha_n-\alpha_{n-1}| \text{ and } \tilde{\alpha}_n=\mu\alpha_n.$$

Then

$$a_{n+1}\le(1-\tilde{\alpha}_n)a_n+\tilde{\alpha}_n b_n+c_n.$$

From (C1) and (C3), we have $\lim_{n\to\infty}\tilde{\alpha}_n=0$, $\limsup_{n\to\infty} b_n=0$ and $\sum_{n=0}^{\infty}c_n<\infty$. From Lemma 2.4, we get $\lim_{n\to\infty}\|x_n-x_{n-1}\|=0$, i.e., $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

We will show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. From (C1), we get

$$\|x_{n+1}-y_n\|=\mu\alpha_n\|x_0-y_n\|\to 0.$$

It follows that

$$\|y_n-x_n\|\le\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|\to 0.$$

and

$$\|d_{n+1}\|=\frac{1}{\alpha}\|y_n-x_n\|\to 0.$$

Therefore,

$$\|Tx_n-x_n\|=\alpha(\|d_{n+1}-\beta_n d_n\|)\le\alpha(\|d_{n+1}\|+\beta_n\|d_n\|)\to 0.$$

We now will show that

$$\limsup_{n\to\infty}\langle x_0-x^*,y_n-x^*\rangle\le 0,\quad \text{where } x^*:=P_{A(T)}x).$$

Let $x^* := P_{A(T)}x_0$. By Theorem 2.1, we get

$$\langle x_0 - x^*, y - x^*\rangle \leq 0 \text{ for all } y \in A(T). \tag{3.4}$$

Since $(y_n)_{n=0}^{\infty}$ is bounded, we obtain that $(\langle x_0 - x^*, y_n - x^*\rangle)_{n=0}^{\infty}$ is bounded. There exists a subsequence $(y_{n_k})_{k=0}^{\infty}$ of $(y_n)_{n=0}^{\infty}$ such that

$$\limsup_{n\to\infty}\langle x_0 - x^*, y_n - x^*\rangle = \lim_{k\to\infty}\langle x_0 - x^*, y_{n_k} - x^*\rangle.$$

Since $(y_{n_k})_{k=0}^{\infty}$ is bounded, there exists a subsequence $(y_{n_{k_i}})_{i=0}^{\infty}$ of $(y_{n_k})_{k=0}^{\infty}$ such that $y_{n_{k_i}} \rightharpoonup \hat{y}$ for some $\hat{y} \in H$ and then

$$\begin{aligned}\limsup_{n\to\infty}\langle x_0 - x^*, y_n - x^*\rangle &= \lim_{k\to\infty}\langle x_0 - x^*, y_{n_k} - x^*\rangle \\ &= \lim_{i\to\infty}\langle x_0 - x^*, y_{n_{k_i}} - x^*\rangle \\ &= \langle x_0 - x^*, \hat{y} - x^*\rangle.\end{aligned}$$

We have from $\lim_{n\to\infty}\| y_n - x_n\| = 0$ and $y_{n_{k_i}} \rightharpoonup \hat{y}$ that $x_{n_{k_i}} \rightharpoonup \hat{y}$. Let $y \in C$. Then

$$\begin{aligned}\limsup_{i\to\infty}\|x_{n_{k_i}} - Ty\| &\leq \limsup_{i\to\infty}\big(\|x_{n_{k_i}} - Tx_{n_{k_i}}\| + \|Tx_{n_{k_i}} - Ty\|\big) \\ &= \lim_{i\to\infty}\|x_{n_{k_i}} - Tx_{n_{k_i}}\| + \limsup_{i\to\infty}\|Tx_{n_{k_i}} - Ty\| \\ &= \limsup_{i\to\infty}\|Tx_{n_{k_i}} - Ty\| \\ &\leq \limsup_{i\to\infty}\|x_{n_{k_i}} - y\|.\end{aligned}$$

Hence

$$\limsup_{i\to\infty}\|x_{n_{k_i}} - Ty\|^2 \leq \limsup_{i\to\infty}\|x_{n_{k_i}} - y\|^2. \tag{3.5}$$

Notice that

$$\begin{aligned}\|x_{n_{k_i}} - Ty\|^2 &= \|\big(x_{n_{k_i}} - y\big) + \big(y - Ty\big)\|^2 \\ &= \|x_{n_{k_i}} - y\|^2 + \|y - Ty\|^2 + 2\langle x_{n_{k_i}} - y, y - Ty\rangle. \end{aligned} \tag{3.6}$$

Since $x_{n_{k_i}} \rightharpoonup \hat{y}$, we get

$$\lim_{i\to\infty} 2\langle x_{n_{k_i}} - y, y - Ty\rangle = 2\langle \hat{y} - y, y - Ty\rangle. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\limsup_{i\to\infty} \|x_{n_{k_i}} - Ty\|^2 = \limsup_{i\to\infty} \|x_{n_{k_i}} - y\|^2 + \|y - Ty\|^2 + 2\langle \hat{y} - y, y - Ty\rangle$$

It follows from (3.5) that

$$\|y - Ty\|^2 + 2\langle \hat{y} - y, y - Ty\rangle \leq 0.$$

This implies that

$$\|y - Ty\|^2 + \left(\|\hat{y} - Ty\|^2 - \|\hat{y} - y\|^2 - \|y - Ty\|^2\right) \leq 0$$

and hence

$$\|\hat{y} - Ty\|^2 - \|\hat{y} - y\|^2 \leq 0.$$

So, we get

$$\|\hat{y} - Ty\| \leq \|\hat{y} - y\|$$

for all $y \in C$. Therefore $\hat{y} \in A(T)$. From (3.4), this implies that

$$\limsup_{n\to\infty} \langle x_0 - x^*, y_n - x^*\rangle = \langle x_0 - x^*, \hat{y} - x^*\rangle \leq 0.$$

That is, $\limsup_{n\to\infty} \langle x_0 - x^*, y_n - x^*\rangle \leq 0$.

We finally will show that $||x_n - x^*|| \to 0$. Since $(d_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are bounded, let $M_4 = \sup\{\langle y_n - x^*, d_n\rangle : n \geq 0\}$. Then

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(Tx_n - x^*) + \alpha\beta_n d_n\|^2 \\
&\leq \|Tx_n - x^*\|^2 + 2\langle y_n - x^*, \alpha\beta_n d_n\rangle \\
&\leq \|x_n - x^*\|^2 + 2\alpha\alpha_n^2\langle y_n - x^*, d_n\rangle \\
&\leq \|x_n - x^*\|^2 + 2\alpha M_4\alpha_n^2.
\end{aligned}$$

It follows from $(1-\mu\alpha_n)<1$ that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\mu\alpha_n(x_0-x^*)+(1-\mu\alpha_n)(y_n-x^*)\|^2\\
&= \mu^2\alpha_n^2\|x_0-x^*\|^2+(1-\mu\alpha_n)^2\|y_n-x^*\|^2\\
&\quad +2\mu\alpha_n(1-\mu\alpha_n)\langle\, x_0-x^*, y_n-x^*\,\rangle\\
&\le \mu^2\alpha_n^2\|x_0-x^*\|^2+(1-\mu\alpha_n)^2(\|x_n-x^*\|^2+2\alpha M_4\alpha_n^2)\\
&\quad +2\mu\alpha_n(1-\mu\alpha_n)\langle\, x_0-x^*, y_n-x^*\,\rangle\\
&\le (1-\mu\alpha_n)\|x_n-x^*\|^2+2\alpha M_4\alpha_n^2+\mu^2\alpha_n^2\|x_0-x^*\|^2\\
&\quad +2\mu\alpha_n(1-\mu\alpha_n)\langle\, x_0-x^*, y_n-x^*\,\rangle\\
&= (1-\mu\alpha_n)\|x_n-x^*\|^2+\Big(\mu\alpha_n\|x_0-x^*\|^2+\frac{2\alpha M_4\alpha_n}{\mu}\Big)\mu\alpha_n\\
&\quad +(2(1-\mu\alpha_n)\langle x_0-x^*, y_n-x^*\rangle)\mu\alpha_n. \qquad (3.8)
\end{aligned}$$

Setting

$$\begin{aligned}
&a_n=||x_n-x^*||^2,\\
&b_n=\mu\alpha_n||x_0-x^*||^2+\frac{2\alpha M_4\alpha_n}{\mu}+(2(1-\mu\alpha_n)\langle x_0-x^*, y_n-x^*\rangle,\\
&c_n=0, \text{ and } \tilde{\alpha}_n=\mu\alpha_n.
\end{aligned}$$

From (3.8), we get

$$a_{n+1}\le(1-\tilde{\alpha}_n)a_n+\tilde{\alpha}_n b_n+c_n.$$

From (C1), we have $\lim_{n\to\infty}\tilde{\alpha}_n=0$, $\limsup_{n\to\infty}b_n=0$ and $\sum_{n=0}^{\infty}c_n<\infty$. Using Lemma 2.4, we get

$$\|x_n-x^*\|\to 0.$$

This means that $(x_n)_{n=0}^{\infty}$ converges strongly to x^*, where $x^*=P_{A(T)}x_0$. This completes the proof. □

Remark 3.2. As in Theorem 3.1, we can see that $(x_n)_{n=0}^{\infty}$ converges strongly to x^*, where $x^*\in A(T)$ and $\|x^*-x_0\|=\inf\{\|z-x_0\|: z\in F(T)\}$. Indeed, by using $\|Tx_n-x_n\|\to 0$ and T is continuous, we get $x^*=Tx^*$. That is, $x^*\in F(T)$. Moreover, since $x^*=P_{A(T)}x_0$ and $F(T)\subset A(T)$, we get

$$\|x^*-x_0\|\le\|z-x_0\| \quad \text{for all } z\in F(T)$$

and so

$$\|x^* - x_0\| = \inf\{\|z - x_0\| : z \in F(T)\}.$$

Lemma 3.3. *Let H be a Hilbert space and let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Then $A(T) = F(T)$.*

Proof. Let $p \in F(T)$. Since T is nonexpansive, we get

$$\|Ty - p\| = \|Ty - Tp\| \leq \|y - p\|, \text{ for all } y \in H.$$

Hence $F(T) \subseteq A(T)$. On the other hand, let $x \in A(T)$. Then

$$\|Ty - x\| \leq \|y - x\| \text{ for all } y \in H.$$

Since $x \in A(T) \subseteq H$. Then $\|Tx - x\| \leq \|x - x\| = 0$.Hence $Tx = x$ and so $x \in F(T)$. Therefore, $A(T) = F(T)$. This completes the proof. □

Using Theorem 3.1 and Lemma 3.3, we get the following corollary.

Corollary 3.4. [9, Theorem 3.1] *Let H be a Hilbert space H and $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Let $(x_n)_{n=0}^{\infty}$ be the acceleration of the Halpern algorithm generated by* (3.1) *satisfying (C1)-(C5). Then $(x_n)_{n=0}^{\infty}$ converges strongly to $x^* = P_{F(T)}x_0$.*

Setting $\bar{\alpha}_n = \mu\alpha_n$ and $\beta_n = 0$ for all $n \geqslant 0$, the sequence $(x_n)_{n=0}^{\infty}$ defined by (3.1) deduces to the Halpern algorithm and then the following corollary is presented. This means that Theorem 3.1 is a generalization of the convergence analysis of the Halpern algorithm.

Corollary 3.5. [1, Theorem 3.5] *Let H be a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $A(T) \neq \varnothing$. Let $(x_n)_{n=0}^{\infty}$ be the Halpern algorithm generated by* (1.1) *satisfying (C1)-(C3). Then $(x_n)_{n=0}^{\infty}$ converges strongly to $x^* = P_{A(T)}x_0$.*

Finally, we give an example to support Theorem 3.1.

Example 3.6. Let $\mathbb{R}$ be the real space. Let $C = \mathbb{Q}_+$ be the set of all nonnegative rational numbers. Let $T : C \to C$ be a mapping defined by: $Tx = \frac{x+2}{3(2x+1)}$, for all $x \in C$. Then T is a nonexpansive mapping of C into itself with $A(T) \neq \varnothing$. Let$(x_n)_{n=0}^{\infty}$ be defined by (3.1) with $\alpha = 1,\ \mu = 10^{-5},\ x_0 \in C,\ d_0 \in C,\ \alpha_n = \dfrac{1}{n+1}$ and $\beta_n = \dfrac{1}{(n+1)^2}$, for all $n \geq 0$. Then $(x_n)_{n=0}^{\infty}$ is in C defined by

$$\begin{cases} d_{n+1} &= \Big(\dfrac{x_n+2}{3(2x_n+1)} - x_n\Big) + \dfrac{1}{(n+1)^2}d_n, \\ y_n &= \dfrac{x_n+2}{3(2x_n+1)} + \dfrac{1}{(n+1)^2}d_n), \\ x_{n+1} &= \Big(\dfrac{10^{-5}}{n+1}\Big)x_0 + \Big(1 - \dfrac{10^{-5}}{n+1}\Big)(\dfrac{x_n+2}{3(2x_n+1)} + \dfrac{1}{(n+1)^2}d_n). \end{cases} \tag{3.9}$$

It can be observed that $(\alpha_n)_{n=0}^{\infty}$ and $(\beta_n)_{n=0}^{\infty}$ satisfy the conditions (C1)-(C4). Now, we set the condition (C5) by $\|Tx_n - x_n\| \leq 10^{-5}$, that is, $(Tx_n - x_n)_{n=0}^{\infty}$ is bounded.

If $x_0 = 1$ and $d_0 = \frac{1}{3}$, then the sequence $(x_n)_{n=0}^{\infty}$ defined by (3.9) can be calculated in the Table 1 as follows:

Table 1: The acceleration of the Halpern algorithm

n	x_n	$\|\|x_n - x_{n+1}\|\|$	$\|\|x_n - Tx_n\|\|$
0	1.0000000000	0.3333300000	0.2857182313
1	0.6666700000	0.3690480527	0.1824764158
2	0.2976219473	0.1414725560	0.0062139475
3	0.4390945033	0.0026293661	0.0095866034
4	0.4417238694	0.0094803491	0.0025934201
5	0.4322435203	0.0023309879	0.0004064301
6	0.4345745082	0.0003580699	0.0000541687
7	0.4342164383	0.0000492684	0.0000092118

Then $(x_n)_{n=0}^{\infty}$ approximates an attractive point of T which is 0.4342.

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