รายงานวิจัยฉบับสมบูรณ์

วิธีเอ็กตร้าเกรเดียนต์ไฮบริดสตีปเปสท์เดสเซนท์สำหรับระบบปัญหาเชิงดุลยภาพผสมและระบบปัญหาอสมการเชิงแปรผันสำหรับการส่งแบบไม่เชิงเส้น

The extragradient hybrid steepest descent methods for system of mixed equilibrium problems and system of variational inequality problems involving nonlinear mappinqs

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มิถุนายน 2556

สัญญาเลขที่ MRG5480206

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานฉบับนี้เป็นของผู้วิจัย สกว. และสกอ. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยเรื่อง วิธีเอ็กตร้าเกรเดียนต์ไฮบริดสตีปเปสท์เดสเซนท์สำหรับระบบปัญหาเชิงดุลยภาพผสมและระบบปัญหาอสมการเชิงแปรผันสำหรับการส่งแบบไม่เชิงเส้น (MRG5480206) นี้ สำเร็จลุล่วงด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และสำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) ประจำปี 2554-2556 และขอขอบคุณ รศ.ดร.ภูมิ คำเอม ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าธนบุรี นักวิจัยที่ปรึกษาที่ได้ให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา และสุดท้ายขอขอบคุณมหาวิทยาลัยเทคโนโลยีราชมงคลรัตนโกสินทร์ สถาบันต้นสังกัด

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ABSTRACT

The propose of this project is to consider and study the fixed points of an infinite family of nonexpansive mappings, the variational inequality problem for relaxed cocoercive and Lipschitz continuous, the system of variational inclusions problem and the system of equilibrium problems by using a hybrid steepest descent methods and viscosity approximation method under certain hypotheses. We prove strong convergence theorem for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the variational inequality problem for relaxed cocoercive and Lipschitz continuous, the set of solutions of system of variational inclusions problem and the system of equilibrium problems in Hilbert spaces.

Keywords: Nonexpansive mappings/ Variational inequality problem/ system of equilibrium problems/system of variational inclusions problem/ Nonlinear mappings/hybrid steepest descent methods/ viscosity approximation method

บทคัดย่อ

จุดประสงค์ของโครงการวิจัยนี้เพื่อพิจารณาและศึกษาจุดตรึงของวงศ์อนันต์สำหรับการส่งแบบไม่ขยาย ปัญหาอสมการเชิงแปรผัน ปัญหาระบบเชิงแปรผันรวมและปัญหาระบบเชิงดุลยภาพ โดยวิธีไฮบริดสตีปเปสท์เดสเซนท์ และวิธีการประมาณค่าแบบหน่วงภายใต้บางเงื่อนไข ซึ่งเราพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มเพื่อหาสมาชิกร่วมของ เซตคำตอบของจุดตรึงวงศ์อนันต์สำหรับการส่งแบบไม่ขยาย เซตคำตอบของปัญหาอสมการเชิงแปรผันสำหรับริแลคซโคโดเอ๊อซิฟและความต่อเนื่องลิปสซิทซ์ เซตคำตอบของปัญหาระบบเชิงแปรผันรวมและเซตคำตอบของปัญหาระบบเชิงดุลยภาพในริภูมิฮิลเบิร์ต

คำสำคัญ: การส่งแบบไม่ขยาย/ ปัญหาอสมการเชิงแปรผัน/ ปัญหาระบบเชิงดุลยภาพ/ ปัญหาระบบเชิงแปรผันรวม / การส่งแบบไม่เชิงเส้น/ วิธีไฮบริดสตีปเปสท์เดสเซนท์/ วิธีการประมาณค่าแบบหน่วง

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CHAPTER I

INTRODUCTION

1.1 Background

Fixed-point iterations process for nonlinear mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to approximate fixed point of various classes of operators and to solve variational inequalities in both Hilbert spaces and Banach spaces; see also, for example [9], [20], [26], [51] and the references therein.

The variational inequality problem was first introduced by Hartman and Stampacchia [25] in 1966, has had a great impact and influence in the development of several branches of pure and applied sciences. The ideas and techniques of this theory are being used in a variety of diverse fields and proved to be productive and innovative, see [1-27] and the references therein. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. As a result of the interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various algorithms for solving variational inequalities and related optimization problems. Using the projection technique, one can establish the equivalence between the variational inequalities and fixed point problems. This alternative equivalent formulation has played an important role in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. It is now well-known that the variational inequalities are equivalent to the fixed-point problems, the origin of which can be traced back to Lions and Stampacchia [35]. This alternative formulation has been used to suggest and analyze projection iterative methods for solving the variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous. These conditions are very strict and rule out its application in several important problems. To overcome this drawback, Korpelevich [34] suggested and analyzed the extragradient method by using the technique of updating the solution. It has been shown that if the underlying operator is only monotone and Lipschitz continuous, then the approximate solution converges to the exact solution. Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the current interest in functional analysis. It is natural to consider a unified approach to these different problems, see, for example, [29, 39, 45, 48].

Equilibrium problem which were introduced by Blum and Oettli [4]. The Equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization which has been extended and generalized in many directions using novel and innovative techniques, see [4]. Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems, the set solutions of the variational inequality problems for a nonlinear mapping and a set of the fixed points of an infinite (a finite) family of nonexpansive mappings. For the detail, see [16], [53] and the references therein.

In 1952, the original Mann iteration was defined in a matrix formulation by Mann [37]. In 1974, Ishikawa [27] introduced the iterative scheme which later, it is said to be Ishikawa iteration and studied its strong convergence theorem for lipschitzian pseudo-contractive mapping in Hilbert spaces.

In 1989, Nadezhkina and Takahashi [39] introduced the following iterative scheme for finding an element of fixed point problem and variational inequalities and studied the weak convergence theorem for monotone and Lipschitz continuous mapping nonexpansive mappings in a real Hilbert space.

In 1997 Combettes and Hirstoaga [16] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

1.2 Some Existing Algorithms of Fixed Points

Let X be a nonempty set and $T : X \longrightarrow X$ a self map. We say that $p \in X$ is a *fixed point* of T if $p = Tp$ and denote by $F(T)$ the set of all fixed points of T. Having in view that many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator, on the other hand, the metrical fixed point theory has developed significantly in the second part of the 20th century.

As the constructive methods used in metrical fixed point theory are prevailingly iterative procedures, that is, approximate methods, it is also of crucial importance to have a priori or/and a posteriori error estimates or rate of convergence for such method. For example, the Banach fixed point theorem concerns certain contractions mappings of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point and it also given a constructive procedure for obtaining better and better approximations to the fixed point. By definition, this is a method such that we choose an arbitrary x_0 in a given set and calculate recursively a sequence $x_0, x_1, x_2, ...$ from a relation of the form

$$x_n = Tx_{n-1} = T^n x_0 \;\; n = 1, 2, 3, .. \tag{1.1}$$

That is, we choose an arbitrary x_0 and determine successively $x_1 = Tx_0, x_2 =$

$Tx_1, x_3 = Tx_2, \ldots$. It is also known as the Picard iteration starting at x_0.

Iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach fixed point theorem (or more difficult fixed point theorems). Many researchers are interested in obtaining (additional) condition on T and E as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed point of T. Moreover, if the Picard iteration converges to a fixed point of T, they will be interested in evaluating the error estimate (or alternatively, the rate of convergence) of the method, that is, in obtaining a stopping criterion for the sequence of successive approximations. However, the Picard iteration may not converge even in the weak topology.

Construction of fixed point iteration processes of nonlinear mappings is an important subject in the theory of nonlinear mappings, and finds application in a number of applied areas. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings, relatively nonexpansive mappings, hemirelatively nonexpansive mappings, generalized nonexpansive mappings and maximal monotone operators in various space have been studied by many mathematicians.

Let $(X, \|\cdot\|)$ be a real normed space and $C \subset X$ be a closed and convex. Three classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $S : C \longrightarrow C$.

If an equation can be put into the form $Sx = x$, and a solution x is an attractive fixed point of the function S, then one may begin with a point x_1 in the basin of attraction of x, and let $x_{n+1} = Sx_n$ for $n \geq 1$, and the sequence x_n will converge to the solution x. If the function S is continuously differentiable, a sufficient condition for convergence is that the spectral radius of the derivative is strictly bounded by one in a neighborhood of the fixed point. If this condition holds at the fixed point, then a sufficient small neighborhood must exist.

Mann iteration

In 1953, Mann [37] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [43]. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence (see [19] and [3]). Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made.

Halpern iteration

In 1967, Halpern [24] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S x_n \tag{1.3}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$ and prove strong convergence theorem under some certain control condition.

Ishikawa iteration

In 1974, Ishikawa [27] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S[\beta_n x_n + (1 - \beta_n) S x_n], \ \forall n \geq 0, \tag{1.4}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ and $\{\alpha_n\}$ are real sequence in $[0, 1]$ and and prove weak convergence theorem under some certain control condition.

Marino and Xu [38] studied an explicit algorithm, which generated a sequence $\{x_n\}$ recursively by the formula: For the initial guess $x_0 \in C$ is arbitrary

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)[\alpha_n V x_n + (1 - \alpha_n) T x_n], \ \forall n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$ satisfy some conditions. Let $T,\ V : C \to C$ are two nonexpansive self mappings and f is a contraction on C. Then $\{x_n\}$ converges strongly to a solution, which solves another variational inequality.

In general not much has been known regarding the convergence of the iteration processes (1.3)-(1.4) unless the underlying space has elegant properties which be briefly mention here.

Process (1.4) is indeed more general than process (1.2). But research has been concentrated on the latter due probably to the reasons that the formulation of process (1.2) is simpler than that of (1.4) and that a convergence theorem for process (1.2) may possibly lead to a convergence theorem for process (1.4) provided the sequence $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of process (1.4) has its own right. As a matter of fact, process (1.2) may fail to converge while process (1.4) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space. Both processes (1.2) and (1.4) have only weak convergence, in general. For example, Reich [43] proved that if X is a uniformly convex Banach space with a Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the Mann's iteration converges weakly to a fixed point of T. However, we note that Mann's iteration have only weak convergence even in a Hilbert space.

Viscosity Approximation Method

In 2007, Yao et al. [53] introduced the following so-called viscosity approximation method:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[\beta_n x_n + (1 - \beta_n) S x_n], \ \ \forall n \geq 0, \tag{1.6}$$

where S is a nonexpansive mapping of C into itself and f is a contraction on C.

They obtained a strong convergence theorem under some mild restrictions on the parameters.

Hybrid Steepest Descent Method

Yamada [52] introduced the following iterative scheme called the hybrid steepest descent method:

$$x_{n+1} = Sx_n - \alpha_n \mu BSx_n, \;\; \forall n \geq 1, \tag{1.7}$$

where $x_1 = x \in H$, $\{\alpha_n\} \subset (0, 1)$, let $B : H \longrightarrow H$ be a strongly monotone and Lipschitz continuous mapping and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly under a controlled condition on the sequence $\{\alpha_n\}$.

Extragradient Method

In 1976, Korpelevich [34] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n) \end{cases} \tag{1.8}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C in to $\mathbb{R}^n$. He proved that if $VI(A, C)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.8), converge to the same point $z \in VI(A, C)$.

Recently, motivated by the idea of Korpelevichs extragradient method [34], Nadezhkina and Takahashi[39] introduced the following iterative scheme for finding an element of $F(S) \cap VI(A, C)$ and proved the following weak convergence theorem.

1.3 The system of equilibrium problem and the variational inclusion problem

1.3.1 The system of equilibrium problem

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself and let B be a β-inverse-strongly monotone of C into H. The equilibrium problem for $F : C \times C \longrightarrow \mathbb{R}$ is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \ \ \forall y \in C. \tag{1.9}$$

The set of solutions of (1.9) is denoted by $EP(F)$.

Let $\{F_i, \ \ i = 1, 2, \ldots, N\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\{F_1, F_2, \ldots, F_N\}$ is to find a common element $x^* \in C$ such that

$$\begin{cases} F_1(x^*, y) \geq 0, \ \ \forall y \in C, \\ F_2(x^*, y) \geq 0, \ \ \forall y \in C, \\ \quad \vdots \\ F_N(x^*, y) \geq 0, \ \ \forall y \in C. \end{cases} \tag{1.10}$$

We denote the set of solutions of (1.10) by $\cap_{i=1}^{N} SEP(F_i)$, where $SEP(F_i)$ is the set of solutions to the equilibrium problems, that is,

$$F_i(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.11}$$

If $N = 1$, then the problem (1.10) is reduced to the equilibrium problems.
If $N = 1$ and $F(x^*, y) = \langle Bx^*, y - x^* \rangle$, then the problem (1.10) is reduced to the variational inequality problems of finding $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.12}$$

The set of solutions of (1.12) is denoted by $VI(C, B)$.

1.3.2 The variational inclusion problem

Let $B : H \longrightarrow H$ be a single-valued nonlinear mapping and $M : H \longrightarrow 2^H$ be a muiti-valued mapping. The variational inclusion problem is to find $\tilde{x} \in H$ such that

$$\theta \in B(\tilde{x}) + M(\tilde{x}), \tag{1.13}$$

where θ is the zero vecter in H. The set of solutions of problem (1.13) is denoted by $I(B, M)$. If $M = \partial\psi_C$, where C is a nonempty closed convex subset of H and $\partial\psi_C : H \longrightarrow [0, +\infty]$ is the indicator function of C, that is,

$$\psi_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then, the variational inclusion problem (1.13) is equivalent to the variational inequality problems (1.12)

Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [4], Combettes and Hirstoaga [17]. Recently, Takahashi and Zembayashi [50] consider the following equilibrium problem with a bifunction defined on the dual space of a Banach space. Moreover, they proved a strong convergence theorem for finding a solution of the equilibrium problem which generalized the result of Combettes and Hirstoaga [17].

The purpose of this project is to consider the extragradient hybrid steepest descent methods for finding a common element of the set of solutions for system of mixed equilibrium problems, the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problems in a real Hilbert space. Then, we prove a strong convergence theorem of the iterative sequence generated by the extragradient hybrid steepest descent methods under some suitable conditions in a real Hilbert space. As applications, at the end of the paper we utilize our results to study the optimization problem. All in all, we plan to construct the algorithms by using the extragradient hybrid steepest descent method and discuss the convergence criterion for the perturbed iterative algorithms to approximate the solutions of the above three sets are obtained. Furthermore, we also plan to study the relationships between the above problem and an interesting topic, as fixed point theory. We plan to organize this project as following: In the first year, we will give some new theorems about system of the variational inequality problems with inverse strongly monotone mapping and relaxed cocoercive mapping and system of mixed equilibrium problem with inverse strongly monotone mapping in the Hilbert space. Also, some fixed point problems will be discussed and studied. In the second year, the main results and some applications of this project will be presented, that is, we plan to study a form system of variational inequality problems with nonlinear mapping and system of mixed equilibrium problems for inverse strongly monotone mappings. In conclusion, we point out that the results of this project unify, extend, and improve some well-known results in literature, and moreover, the study of this area is a fruitful and growing field of intellectual endeavor. Much work is needed to develop this interesting subject.

This research is divided into 4 chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with some preliminaries and give some useful results that will be used in later chapters. Chapter 3 we prove strong convergence theorems for finding a common element of the fixed point set. The conclusion output of research is in Chapter 4.

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

2.1 Linear Spaces and Metric Spaces

Definition 2.1. Let X be a nonempty set, and assume that each pair of elements x and y in X can be combined by a process called *addition* to yield an element z in X denoted by $x + y$. Assume also that this operation of addition satisfies the following condition (1)–(4):

(1) $(x + y) + z = x + (y + z)$;

(2) $x + y = y + x$;

(3) there exists a unique element in X, denoted by 0 and called the zero element, or the origin, such that $x + 0 = x$ for all $x \in X$;

(4) each $x \in X$ there corresponds a unique element in X, denoted by $-x$ and called the negative of x, such that $x + (-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ and each element x in X can be combined by a process called *scalar multiplication* to yield an element y in X denoted by $y = \alpha x$ satisfying (5)–(8):

(5) $\alpha(\beta x) = (\alpha\beta)x$;

(6) $1 \cdot x = x$;

(7) $(\alpha + \beta)x = \alpha x + \beta x$;

(8) $\alpha(x + y) = \alpha x + \alpha y$.

The system $(X, \cdot, +)$ is called a *linear space* over $\mathbb{R}$ if it satisfies the conditions (1)–(8). A linear space is often called a *vector space*, and its elements are spoken as vectors.

Definition 2.2. Let X be a nonempty set. A mapping $d : X \times X \longrightarrow \mathbb{R}$, satisfying the following conditions for all x, y and z in X:

(A1) $d(x, y) = 0 \Longleftrightarrow x = y$;

(A2) $d(x, y) = d(y, x)$;

(A2) $d(x, y) \leq d(x, z) + d(z, y)$. The conditions (A1)-(A3) are usually called the *metric axioms*.

The function d assigns to each pair (x, y) of element of X a nonnegative real number $d(x, y)$, which does not on the order of the elements; $d(x, y)$ is called the *distance* between x and y. The set X together with a metric, denoted by (X, d), is called a *metric space*.

2.2 Normed Spaces and Banach Spaces

Definition 2.3. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\| \cdot \| : X \longrightarrow \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in X$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(4) $\|\alpha x\| = |\alpha| \|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

From this norm we can define a metric, induced by the norm $\| \cdot \|$, by

$$d(x, y) = \|x - y\|, \quad (x, y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space.*

Definition 2.4. A normed space $(X, \|\cdot\|)$ is called strictly convex if for all $x, y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0,1)$.

Definition 2.5. Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.

Definition 2.6. A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n \longrightarrow \infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converges weakly to x.

Definition 2.7. Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\ m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Theorem 2.8. [47] *Let $\{x_n\}$ be a sequence of a normed space $(X, \|\cdot\|)$, $x \in X$ and let $x_n \rightarrow x$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x.*

Definition 2.9. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.10. A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$.

Lemma 2.11. [45] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n \longrightarrow \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \longrightarrow \infty} \|y_n - x_n\| = 0$.*

Lemma 2.12. [51] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + c_n, \ n \geq 0,$$

where $\{b_n\}$ *is a sequence in* $(0, 1)$ *and* $\{c_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} b_n = \infty$,

(2) $\limsup_{n \longrightarrow \infty} \frac{c_n}{b_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |c_n| < \infty$,

Then $\lim_{n \longrightarrow \infty} a_n = 0$.

Definition 2.13. Let F and X be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow X$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on* F if T is a linear operator.

Definition 2.14. Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \longrightarrow F$ a linear operator. T is said to be *bounded* on X if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.15. Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.16. A subset C of a normed linear space X is said to be *convex subset in* X if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

2.3 Inner Product Spaces and Hilbert Spaces

Definition 2.17. The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ is called *inner product* on a real vector space X if for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$ the following conditions are satisfied:

(1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(2) $\langle x, y\rangle = \langle y, x\rangle$;

(3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.18. A *Hilbert spaces* is an inner product space which is complete under the norm induced by its inner product.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.19. [47]*(The Schwarz inequality) If x and y are any two vector in an inner product space X, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Remark 2.20. In a Hilbert space H, weak convergence is defined by $\lim_{n\longrightarrow\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Remark 2.21. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Definition 2.22. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.

Lemma 2.23. [47] *Let X be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:*

$$\|x\| \leq \lim_{n\longrightarrow\infty} \inf \|x_n\|.$$

2.4 Basic Concepts in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. We have the following are hold:

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.14}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.15}$$

$$\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle, \tag{2.16}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.17}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Lemma 2.24. [40] *Let $(E, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.25. [41] *A Hilbert space H satisfies the* ***Opial condition*** *that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.*

2.5 Some Nonlinear Mappings in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $T : C \to C$ a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Definition 2.26. A mapping $S : C \to C$ is called *L-Lipschitz-continuous* if there exists a positive real number L such that

$$\|Su - Sv\| \leq L\|u - v\|, \quad \forall u, v \in C. \tag{2.18}$$

Definition 2.27. A mapping $f : C \to C$ is called a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|. \tag{2.19}$$

Definition 2.28. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall x, y \in C. \tag{2.20}$$

Theorem 2.29. [47] *(Banach's Contraction Mapping Principle) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, i.e. there exists a unique $x^* \in X$ such that $Tx^* = x^*$.*

Definition 2.30. The *metric (nearest point) projection* P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Lemma 2.31. [47] *Let H be a real Hilbert spaces, there hold the following identities:*

(i) for each $x \in H$ and $x^* \in C$, $x^* = P_C x \iff \langle x - x^*, y - x^* \rangle \leq 0$, $\forall y \in C$;

(ii) $P_C : H \longrightarrow C$ is nonexpansive, that is, $\|P_C x - P_C y\| \leq \|x - y\|$, $\forall x, y \in H$;

(iii) P_C is firmly nonexpansive, that is,
$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$, $\forall x, y \in H$;

Definition 2.32. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u, v \in C. \tag{2.21}$$

Definition 2.33. A is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in C. \tag{2.22}$$

Lemma 2.34. *Let $A : H \to H$ be a α-inverse-strongly monotone mapping. If $\lambda \leq 2\alpha$, for any $\lambda > 0$ and $\alpha > 0$ then $I - \lambda A$ is a nonexpansive mapping from H into itself.*

Proof. *Let $u, v \in H$ and $\lambda > 0$,*

$$\begin{aligned}
\|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\
&= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\
&\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.
\end{aligned}$$

Lemma 2.35. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, let $S : C \longrightarrow C$ be a nonexpansive mapping and let $B : C \longrightarrow H$ be a ξ-inverse strongly monotone. If $0 < \alpha_n \leq 2\xi$, then $S - \alpha_n BS$ is a nonexpansive mapping in H.*

Proof. *For any $x, y \in C$ and $0 < \alpha_n \leq 2\xi$, we have*

$$\begin{aligned}
&\|(S - \alpha_n BS)x - (S - \alpha_n BS)y \\
\|^2 &= \|(Sx - Sy) - \alpha_n(BSx - BSy)\|^2 \\
&= \|Sx - Sy\|^2 - 2\alpha_n\langle Sx - Sy, BSx - BSy\rangle + \alpha_n^2\|BSx - BSy\|^2 \\
&\leq \|x - y\|^2 - 2\alpha_n\xi\|BSx - BSy\| + \alpha_n^2\|BSx - BSy\|^2 \\
&= \|x - y\|^2 + \alpha_n(\alpha_n - 2\xi)\|BSx - BSy\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

So, $S - \alpha_n BS$ is a nonexpansive mapping of C into H.

Remark 2.36. It is easy to see that if A is an α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$–Lipschitz continuous.

Definition 2.37. The mapping $S : C \longrightarrow C$ is called a *κ-strict pseudo-contraction mapping* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.23}$$

Definition 2.38. A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping defined on a real Hilbert space H:

$$\min_{x \in F} \left[\frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle \right],$$

where F is the fixed point set of a nonexpansive mapping S defined on H and b is a given point in H.

Definition 2.39. A linear bounded operator A is *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Lemma 2.40. [38] *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.41. [38] *Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\eta \in (0, 1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\eta}$,*

$$\Big\langle x - y, (A - \gamma f)x - (A - \gamma f)y \Big\rangle \geq (\bar{\gamma} - \eta\gamma)\|x - y\|^2, \quad \forall x, y \in H.$$

That is, $A - \gamma f$ is a strongly monotone with coefficient $\bar{\gamma} - \gamma\eta$.

2.6 Basic Concept of Convex Analysis

Definition 2.42. [13] Let H be a Hilbert space and let C be nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] =$

$\mathbb{R}\cup\{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\}$$

is closed. f is also called *convex* on if for any $x, y \in C$ and $t \in [0,1]$, then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Theorem 2.43. [13](Minimization theorem)
Let C be a nonempty bounded closed convex subset of a Hilbert space H and let f be a proper lower semicontinuous convex function of C into $(-\infty, \infty]$. Then there exists $x_0 \in D(f)$ such that

$$f(x_0) = \min_{x\in C} f(x).$$

Definition 2.44. [13] Let H be a Hilbert space and let $f : H \longrightarrow (-\infty, \infty]$ be a proper convex function. Then, we define the subdifferential ∂f of f by

$$\partial f(x) = \{x \in H : f(y) \geq \langle y - x, z\rangle + f(x),\ \forall y \in H\}$$

for all $x \in H$. If $f(x) = \infty$, then $\partial f(x) = \emptyset$.

Lemma 2.45. [13] Let H be a Hilbert space and let $f : H \longrightarrow (-\infty, \infty]$ be a proper convex function. Let $z \in H$. Then

$$0 \in \partial f(z) \Leftrightarrow f(z) = \min_{x\in H} f(x).$$

Lemma 2.46. [13] Let E be a Banach space and let $f : E \longrightarrow (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E : f(y) \geq \langle y - x, x^*\rangle + f(x),\ \forall y \in E\}$$

for each $x \in E$. Then, ∂f is a maximal monotone operator.

Lemma 2.47. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Define the *indicator function* i_C of C by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & otherwise. \end{cases}$$

Then, i_C is proper, convex and semicontinuous and ∂i_C is a maximal monotone operator.

Definition 2.48. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H and $x \in C$. Then we define the set $N_C(x)$ of H by

$$N_C(x) = \{z \in H : \langle u - x, z \rangle \leq 0, \forall u \in C\}.$$

Such a set $N_C(x)$ is called the *normal cone* of C.

Remark The set $N_C(x)$ is a closed convex cone of H.

Definition 2.49. [13] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let B be an operator of C into H. Then B is called *hemicotinuous* if for any $u, v \in C$ and $w \in H$, the function

$$t \mapsto \langle w, B(tu + (1-t)v) \rangle$$

of [0,1] into $\mathbb{R}$ is contonuous.

Theorem 2.50. [47] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $B : C \longrightarrow H$ be monotone and hemicontinuous and let $N_C(x)$ denote the normal cone of C at $x \in C$. Define

$$Tx = \begin{cases} Bx + N_C x, & \text{if} \quad v \in C, \\ \emptyset, & \text{if} \quad v \notin C. \end{cases}$$

Then $T : H \longrightarrow 2^H$ is a maximal monotone and $0 \in Tx$ iff $x \in VI(C, B)$.

Lemma 2.51. [47] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let B be an operator of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u \in VI(C, B) \Leftrightarrow u = P_C(I - \lambda B)u.$$

where P_C is the metric projection of H onto C.

Theorem 2.52. [47] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\beta > 0$ and let $B : C \longrightarrow H$ be β-inverse strongly monotone. Then $VI(C, B) \neq \emptyset$.

Definition 2.53. [13] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. A mapping $T : C \longrightarrow C$ is called *strictly pseudocontractive* if there exists k with $0 \leq k < 1$ such that:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad for\ all\ x, y \in C.$$

Remark. If $k = 0$, then T is nonexpansive. Put $B = I - T$, where $T : C \longrightarrow C$ is a strictly pseudocontractive mapping with k. Then B is $\frac{1-k}{2}$-inverse-strongly monotone.

we assume that a bifunction $F : C \times C \longrightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Then, we have the following lemmas.

Lemma 2.54. [4] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.55. [16] *Assume that* $F : C \times C \longrightarrow \mathbb{R}$ *satisfies (A1)-(A4). For* $r > 0$ *and* $x \in H$*, define a mapping* $J_r^F : H \longrightarrow C$ *as follows:*

$$J_r^F(x) = \left\{ \ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \ \right\}$$

for all $z \in H$*. Then, the following hold:*

(1) J_r^F *is single-valued;*

(2) J_r^F *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\left\| J_r^F x - J_r^F y \right\|^2 \leq \left\langle J_r^F x - J_r^F y, x - y \right\rangle;$$

(3) $F(J_r^F) = EP(F)$*;*

(4) $EP(F)$ *is closed and convex.*

CHAPTER III

MAIN CONVERGENCE RESULTS

3.1 Hybrid Steepest Descent Methods

3.1.1 An infinite family of nonexpansive mappings

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a family of infinitely of nonexpansive mappings of C into itself and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in [0,1]. For any $n \geq 1$, define a mapping $W_n : C \longrightarrow C$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\
&\vdots \\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\
&\vdots \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I,
\end{aligned} \tag{3.24}$$

such a mappings W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$ (see [46]).

Lemma 3.56. [46, 54] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then,* *(1) for every* $x \in C$ *and* $k \in \mathbb{N}$*, the limit* $\lim_{n\longrightarrow\infty} U_{n,k}x$ *exists;*

(2) the mapping W *of* C *into itself as follows:*

$$Wx = \lim_{n\longrightarrow\infty} W_n x = \lim_{n\longrightarrow\infty} U_{n,1}x, \quad x \in C, \tag{3.25}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$*, which it is called the* W*-mapping generated by* $T_1, T_2, \ldots$ *and* $\mu_1, \mu_2, \ldots$*;*

(3) $F(W_n) = \cap_{n=1}^{\infty} F(T_n)$*, for each* $n \geq 1$*;*

(4) If E *is any bounded subset of* C*, then* $\lim_{n\to\infty} \sup_{x\in E} \|Wx - W_n x\| = 0$.

Theorem 3.57. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*, let* $F_k, k \in \{1, 2, 3, \ldots, M\}$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4), let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of* C *into itself and let* B *be* ξ*-inverse strongly monotone such that*

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \left(\cap_{k=1}^{M} SEP(F_k)\right) \cap VI(C, B) \neq \emptyset.$$

Let f *be a contraction of* H *into itself with* $\eta \in (0,1)$ *and let* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{\bar{\gamma}}{\eta}$*. Let* $\{x_n\}$*,* $\{y_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = (1-\delta_n)x_n + \delta_n P_C(x_n - \alpha_n B x_n), \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} y_n, \\ x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big) P_C(W_n u_n - \lambda_n B W_n u_n), \quad \forall n \geq 1, \end{cases} \tag{3.26}$$

where $\{W_n\}$ *is the sequence generated by (3.24) and* $\{\epsilon_n\}$*,* $\{\beta_n\}$ *are two sequences in* $(0,1)$ *and* $\{r_{k,n}\}, k \in \{1, 2, 3, \ldots, M\}$ *are a real sequence in* $(0, \infty)$ *satisfy the following conditions:*

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$*,*

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\{\alpha_n\}, \{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \lambda_n = 0$,

(C4) $\{\delta_n\} \subset [0, b]$, *for some* $b \in (0, 1)$ *and* $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,

(C5) $\liminf_{n\longrightarrow\infty} r_{k,n} > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, 3, \ldots, M\}$,

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$*, which is the unique solution of the variational inequality*

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.27}$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$*.*

Proof**.** From the restrictions on control sequence, without loss of generality, that $\epsilon_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \geq 1$. From Lemma 2.40, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. We will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\Big\{ \big|\langle Ax, x\rangle\big| : x \in H, \|x\| = 1 \Big\}.$$

Observe that

$$\Big\langle \big((1 - \beta_n)I - \epsilon_n A\big)x, x \Big\rangle = 1 - \beta_n - \epsilon_n\langle Ax, x\rangle \geq 1 - \beta_n - \epsilon_n\|A\| \geq 0,$$

this show that $(1 - \beta_n)I - \epsilon_n A$ is positive. It follows that

$$\begin{aligned} \|(1 - \beta_n)I - \epsilon_n A\| &= \sup\Big\{ \Big|\Big\langle \big((1 - \beta_n)I - \epsilon_n A\big)x, x \Big\rangle\Big| : x \in H, \|x\| = 1 \Big\} \\ &= \sup\Big\{ 1 - \beta_n - \epsilon_n\langle Ax, x\rangle : x \in H, \|x\| = 1 \Big\} \\ &\leq 1 - \beta_n - \epsilon_n\bar{\gamma}. \end{aligned}$$

We divide the proof of Theorem 3.57 into seven steps.

Step 1. We show that the mapping $P_\Theta(\gamma f + (I - A))$ has a unique fixed point.

Since f be a contraction of C into itself with coefficient $\eta \in (0, 1)$. Then, we have

$$\begin{aligned}
& \|P_\Theta(\gamma f + (I - A))(x) - P_\Theta(\gamma f + (I - A))(y)\| \\
\leq\ & \|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\| \\
\leq\ & \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
\leq\ & \gamma\eta\|x - y\| + (1 - \bar{\gamma})\|x - y\| \\
=\ & (1 - (\bar{\gamma} - \eta\gamma))\|x - y\|, \quad \forall x, y \in C.
\end{aligned}$$

Since $0 < 1 - (\bar{\gamma} - \eta\gamma) < 1$, it follows that $P_\Theta(\gamma f + (I - A))$ is a contraction of C into itself. Therefore by the Banach Contraction Mapping Principle, has a unique fixed point, say $z \in C$, that is,

$$z = P_\Theta(\gamma f + (I - A))(z).$$

Step 2. We show that $W_n - \lambda_n BW_n$ is nonexpansive.

For all $x, y \in C$, let W_n is the sequence defined by (3.24) and $\lambda_n \in (0, 2\xi)$, we obtain $W_n - \lambda_n BW_n$ is a nonexpansive. Indeed,

$$\begin{aligned}
& \|(W_n - \lambda_n BW_n)x - (W_n - \lambda_n BW_n)y\|^2 \\
=\ & \|(W_n x - W_n y) - \lambda_n(BW_n x - BW_n y)\|^2 \\
=\ & \|W_n x - W_n y\|^2 - 2\lambda_n\langle W_n x - W_n y, BW_n x - BW_n y\rangle + \lambda_n^2\|BW_n x - BW_n y\|^2 \\
\leq\ & \|x - y\|^2 - 2\lambda_n\xi\|BW_n x - BW_n y\| + \lambda_n^2\|BW_n x - BW_n y\|^2 \\
=\ & \|x - y\|^2 + \lambda_n(\lambda_n - 2\xi)\|BW_n x - BW_n y\|^2 \\
\leq\ & \|x - y\|^2, \qquad (3.28)
\end{aligned}$$

which implies that $W_n - \lambda_n BW_n$ is a nonexpansive.

Step 3. We show that the sequence $\{x_n\}$ is bounded.
In fact, let $\tilde{x} \in \Theta$, then

$$\tilde{x} = P_C(\tilde{x} - \alpha_n B\tilde{x}).$$

Setting $v_n = P_C(x_n - \alpha_n Bx_n)$ and $I - \alpha_n B$ is a nonexpansive mapping , we obtain

$$\begin{aligned} \|v_n - \tilde{x}\| &= \|P_C(x_n - \alpha_n Bx_n) - P_C(\tilde{x} - \alpha_n B\tilde{x})\| \\ &\leq \|(x_n - \alpha_n Bx_n) - (\tilde{x} - \alpha_n B\tilde{x})\| \\ &= \|(I - \alpha_n B)x_n - (I - \alpha_n B)\tilde{x}\| \\ &\leq \|x_n - \tilde{x}\| \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \|y_n - \tilde{x}\| &\leq (1 - \delta_n)\|x_n - \tilde{x}\| + \delta_n\|v_n - \tilde{x}\| \\ &\leq (1 - \delta_n)\|x_n - \tilde{x}\| + \delta_n\|x_n - \tilde{x}\| \\ &= \|x_n - \tilde{x}\|. \end{aligned} \tag{3.30}$$

Let $\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1, 2, 3, \dots, M\}$ and $\Im_n^0 = I$ for all n. Because $J_{r_{k,n}}^{F_k}$ is nonexpansive for each $k = 1, 2, 3, \dots, M$, $\tilde{x} = \Im_n^k \tilde{x}$ and (3.125), we note that $u_n = \Im_n^M y_n$. It follows that

$$\|u_n - \tilde{x}\| = \|\Im_n^M y_n - \Im_n^M \tilde{x}\| \leq \|y_n - \tilde{x}\| \leq \|x_n - \tilde{x}\|. \tag{3.31}$$

Let $e_n = P_C(W_n u_n - \lambda_n BW_n u_n)$, we can prove that

$$\begin{aligned} \|e_n - \tilde{x}\| &= \|P_C(W_n u_n - \lambda_n BW_n u_n) - P_C(W_n\tilde{x} - \lambda_n BW_n\tilde{x})\| \\ &\leq \|(W_n u_n - \lambda_n BW_n u_n) - (W_n\tilde{x} - \lambda_n BW_n\tilde{x})\| \\ &= \|(W_n - \lambda_n BW_n)u_n - (W_n - \lambda_n BW_n)\tilde{x}\| \\ &\leq \|u_n - \tilde{x}\| \leq \|x_n - \tilde{x}\|, \end{aligned} \tag{3.32}$$

which yields that

$$\begin{aligned}
& \|x_{n+1} - \tilde{x}\| \qquad (3.33) \\
= \ & \|\epsilon_n(\gamma f(u_n) - A\tilde{x}) + \beta_n(x_n - \tilde{x}) + \big((1-\beta_n)I - \epsilon_n A\big)(e_n - \tilde{x})\| \\
\leq \ & \epsilon_n\|\gamma f(u_n) - A\tilde{x}\| + \beta_n\|x_n - \tilde{x}\| + \|(1-\beta_n)I - \epsilon_n A\|\|e_n - \tilde{x}\| \\
\leq \ & \epsilon_n\gamma\|f(u_n) - f(\tilde{x})\| + \epsilon_n\|\gamma f(\tilde{x}) - A\tilde{x}\| + \beta_n\|x_n - \tilde{x}\| + (1-\beta_n-\epsilon_n\bar{\gamma})\|e_n - \tilde{x}\| \\
\leq \ & \epsilon_n\gamma\eta\|u_n - \tilde{x}\| + \epsilon_n\|\gamma f(\tilde{x}) - A\tilde{x}\| + \beta_n\|x_n - \tilde{x}\| + (1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - \tilde{x}\| \\
\leq \ & \epsilon_n\gamma\eta\|x_n - \tilde{x}\| + \epsilon_n\|\gamma f(\tilde{x}) - A\tilde{x}\| + \beta_n\|x_n - \tilde{x}\| + (1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - \tilde{x}\| \\
= \ & (1 - (\overline{\gamma} - \gamma\eta)\epsilon_n)\|x_n - \tilde{x}\| + \frac{(\overline{\gamma} - \gamma\eta)\epsilon_n}{(\overline{\gamma} - \gamma\eta)}\|\gamma f(\tilde{x}) - A\tilde{x}\|.
\end{aligned}$$

By induction, we have

$$\|x_n - \tilde{x}\| \leq \max\left\{\|x_1 - \tilde{x}\|, \frac{\|\gamma f(\tilde{x}) - A\tilde{x}\|}{\bar{\gamma} - \gamma\eta}\right\}, \quad \forall n \in \mathbb{N}. \qquad (3.34)$$

This implies that $\{x_n\}$ is bounded, and hence so are $\{u_n\}$, $\{e_n\}$, $\{y_n\}$, $\{BW_nu_n\}$, $\{Bx_n\}$, $\{Ae_n\}$, $\{v_n - x_n\}$, and $\{f(u_n)\}$.

Step 4. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
We claim that, if ω_n be a bounded sequence in C. Then

$$\lim_{n\longrightarrow\infty}\|\Im_n^k\omega_n - \Im_{n+1}^k\omega_n\| = 0, \qquad (3.35)$$

for every $k \in \{1,2,3,\ldots,M\}$. From Step 2 of the proof Theorem 3.1 in [18], we have that for $k \in \{1,2,3,\ldots,M\}$,

$$\lim_{n\longrightarrow\infty}\|J_{r_{k,n+1}}^{F_k}\omega_n - J_{r_{k,n}}^{F_k}\omega_n\| = 0. \qquad (3.36)$$

Note that for every $k \in \{1,2,3,\ldots,M\}$, we obtain

$$\Im_n^k = J_{r_{k,n}}^{F_k}J_{r_{k-1,n}}^{F_{k-1}}J_{r_{k-2,n}}^{F_{k-2}}\ldots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1} = J_{r_{k,n}}^{F_k}\Im_n^{k-1}.$$

Thus

$$\begin{aligned}
&\|\Im_n^k\omega_n - \Im_{n+1}^k\omega_n\| \qquad (3.37)\\
= \;& \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k}\Im_{n+1}^{k-1}\omega_n\|\\
\leq \;& \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}\omega_n\| + \|J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k}\Im_{n+1}^{k-1}\omega_n\|\\
\leq \;& \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}\omega_n\| + \|\Im_n^{k-1}\omega_n - \Im_{n+1}^{k-1}\omega_n\|\\
\leq \;& \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}\omega_n\| + \|J_{r_{k-1,n}}^{F_{k-1}}\Im_n^{k-2}\omega_n - J_{r_{k-1,n+1}}^{F_{k-1}}\Im_n^{k-2}\omega_n\|\\
&+\|\Im_n^{k-2}\omega_n - \Im_{n+1}^{k-2}\omega_n\|\\
\leq \;& \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}\omega_n\| + \|J_{r_{k-1,n}}^{F_{k-1}}\Im_n^{k-2}\omega_n - J_{r_{k-1,n+1}}^{F_{k-1}}\Im_n^{k-2}\omega_n\|\\
&+\ldots + \|J_{r_{2,n}}^{F_2}\Im_n^1\omega_n - J_{r_{2,n+1}}^{F_2}\Im_n^1\omega_n\| + \|J_{r_{1,n}}^{F_1}\omega_n - J_{r_{1,n+1}}^{F_1}\omega_n\|.
\end{aligned}$$

Now, apply (3.36) to conclude (3.35).

Since T_n and $U_{n,n}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &= \|\mu_1T_1U_{n+1,2}x_n - \mu_1T_1U_{n,2}x_n\|\\
&\leq \mu_1\|U_{n+1,2}x_n - U_{n,2}x_n\|\\
&= \mu_1\|\mu_2T_2U_{n+1,3}x_n - \mu_2T_2U_{n,3}x_n\|\\
&\leq \mu_1\mu_2\|U_{n+1,3}x_n - U_{n,3}x_n\|\\
&\leq \ldots\\
&\leq \mu_1\mu_2\ldots\mu_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\|\\
&\leq M_1\prod_{i=1}^{n}\mu_i, \qquad (3.38)
\end{aligned}$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ for all $n \geq 0$.

From $I - \alpha_nB$ is nonexpansive, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|P_C(x_{n+1} - \alpha_{n+1}Bx_{n+1}) - P_C(x_n - \alpha_nBx_n)\|\\
&\leq \|(x_{n+1} - \alpha_{n+1}Bx_{n+1}) - (x_n - \alpha_nBx_n)\|\\
&\leq \|(x_{n+1} - \alpha_{n+1}Bx_{n+1}) - (x_n - \alpha_{n+1}Bx_n)\| + |\alpha_{n+1} - \alpha_n|\|Bx_n\|\\
&\leq \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|Bx_n\|. \qquad (3.39)
\end{aligned}$$

From (3.26) and (3.39), we have

$$
\begin{aligned}
& \|y_{n+1} - y_n\| && (3.40)\\
= \ & \|(1-\delta_{n+1})(x_{n+1} - x_n) + \delta_{n+1}(v_{n+1} - v_n) + (\delta_{n+1} - \delta_n)(v_n - x_n)\| \\
\leq \ & (1-\delta_{n+1})\|x_{n+1} - x_n\| + \delta_{n+1}\|v_{n+1} - v_n\| + |\delta_{n+1} - \delta_n|\|v_n - x_n\| \\
\leq \ & (1-\delta_{n+1})\|x_{n+1} - x_n\| + \delta_{n+1}\Big\{\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|Bx_n\|\Big\} \\
& + |\delta_n - \delta_{n+1}|\|x_n - v_n\| \\
= \ & \|x_{n+1} - x_n\| + \delta_{n+1}|\alpha_{n+1} - \alpha_n|\|Bx_n\| + |\delta_n - \delta_{n+1}|\|x_n - v_n\|. && (3.41)
\end{aligned}
$$

Now, we compute $\|u_{n+1} - u_n\|$ and $\|e_{n+1} - e_n\|$. Consider the following computation:

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \|\Im^M_{n+1} y_{n+1} - \Im^M_n y_n\| \\
&\leq \|\Im^M_{n+1} y_{n+1} - \Im^M_{n+1} y_n\| + \|\Im^M_{n+1} y_n - \Im^M_n y_n\| \\
&\leq \|y_{n+1} - y_n\| + \|\Im^M_{n+1} y_n - \Im^M_n y_n\| \\
&\leq \|x_{n+1} - x_n\| + \delta_{n+1}|\alpha_{n+1} - \alpha_n|\|Bx_n\| + |\delta_n - \delta_{n+1}|\|x_n - v_n\| \\
&\quad + \|\Im^M_{n+1} y_n - \Im^M_n y_n\| \qquad (3.42)
\end{aligned}
$$

and

$$
\begin{aligned}
& \|e_{n+1} - e_n\| \\
= \ & \|P_C(W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - P_C(W_n u_n - \lambda_n BW_n u_n)\| \\
\leq \ & \|(W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - (W_n u_n - \lambda_n BW_n u_n)\| \\
= \ & \|(W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - (W_{n+1}u_n - \lambda_{n+1}BW_{n+1}u_n) \\
& + (W_{n+1}u_n - \lambda_{n+1}BW_{n+1}u_n) - (W_n u_n - \lambda_n BW_n u_n)\| \\
\leq \ & \|(W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - (W_{n+1}u_n - \lambda_{n+1}BW_{n+1}u_n)\| \\
& + \|W_{n+1}u_n - W_n u_n\| + \|\lambda_n BW_n u_n - \lambda_{n+1}BW_{n+1}u_n\| \\
\leq \ & \|u_{n+1} - u_n\| + M_1 \prod_{i=1}^{n} \mu_i + \lambda_n\|BW_n u_n\| + \lambda_{n+1}\|BW_{n+1}u_n\| \\
\leq \ & \|x_{n+1} - x_n\| + \delta_{n+1}|\alpha_{n+1} - \alpha_n|\|Bx_n\| + |\delta_n - \delta_{n+1}|\|x_n - v_n\| \\
& + \|\Im^M_{n+1} y_n - \Im^M_n y_n\| + M_1 \prod_{i=1}^{n} \mu_i + \lambda_n\|BW_n u_n\| + \lambda_{n+1}\|BW_{n+1}u_n\| \qquad (3.43)
\end{aligned}
$$

Setting

$$l_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\epsilon_n \gamma f(u_n) + \big((1-\beta_n)I - \epsilon_n A\big)e_n}{1 - \beta_n},$$

we have $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$, $n \geq 1$. It follows that

$$\begin{aligned}
& l_{n+1} - l_n \\
= \; & \frac{\epsilon_{n+1}\gamma f(u_{n+1}) + \big((1-\beta_{n+1})I - \epsilon_{n+1}A\big)e_{n+1}}{1-\beta_{n+1}} \\
& - \frac{\epsilon_n \gamma f(u_n) + \big((1-\beta_n)I - \epsilon_n A\big)e_n}{1-\beta_n} \\
= \; & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\gamma f(u_{n+1}) - Ae_{n+1}\Big) \\
& + \frac{\epsilon_n}{1-\beta_n}\Big(Ae_n - \gamma f(u_n)\Big) + (e_{n+1} - e_n). \qquad (3.44)
\end{aligned}$$

It follows from (3.43) and (3.44) that

$$\begin{aligned}
& \|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\
\leq \; & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big\|\gamma f(u_{n+1}) - Ae_{n+1}\big\| + \frac{\epsilon_n}{1-\beta_n}\big\|Ae_n - \gamma f(u_n)\big\| \\
& + \delta_{n+1}|\alpha_{n+1} - \alpha_n|\|Bx_n\| + |\delta_n - \delta_{n+1}|\|x_n - v_n\| \\
& + \|\Im_{n+1}^M y_n - \Im_n^M y_n\| + M_1\prod_{i=1}^n \mu_i + \lambda_n\|BW_n u_n\| \\
& + \lambda_{n+1}\|BW_{n+1}u_n\| \\
\leq \; & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\Big(\|\gamma f(u_{n+1})\| + \|Ae_{n+1}\|\Big) + \frac{\epsilon_n}{1-\beta_n}\Big(\|Ae_n\| + \|\gamma f(u_n)\|\Big) \\
& + \delta_{n+1}|\alpha_{n+1} - \alpha_n|\|Bx_n\| + |\delta_n - \delta_{n+1}|\|x_n - v_n\| \\
& + \|\Im_{n+1}^M y_n - \Im_n^M y_n\| + M_1\prod_{i=1}^n \mu_i + \lambda_n\|BW_n u_n\| \\
& + \lambda_{n+1}\|BW_{n+1}u_n\|. \qquad (3.45)
\end{aligned}$$

This together with conditions (C1)-(C4) and (3.35) imply that

$$\limsup_{n\longrightarrow\infty}\big(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

By Lemma 2.11, we obtain

$$\lim_{n\longrightarrow\infty}\|l_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|l_n - x_n\| = 0. \qquad (3.46)$$

Applying (3.35), (3.46) and conditions (C3), (C4) to (3.39) and (3.42), we obtain that

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = \lim_{n\longrightarrow\infty}\|v_{n+1}-v_n\| = 0. \tag{3.47}$$

Step 5. We show that $\lim_{n\to\infty}\|W_n e_n - e_n\| = 0$.

For any $\tilde{x}\in\Theta$ and (3.28), we obtain

$$\begin{aligned}\|v_n-\tilde{x}\|^2 &= \|P_C(x_n-\alpha_n Bx_n)-P_C(\tilde{x}-\alpha_n B\tilde{x})\|^2\\ &\le \|(x_n-\alpha_n Bx_n)-(\tilde{x}-\alpha_n B\tilde{x})\|^2\\ &\le \|x_n-\tilde{x}\|^2+(\alpha_n^2-2\alpha_n\xi)\|Bx_n-B\tilde{x}\|^2. \end{aligned}\tag{3.48}$$

By Lemma 2.31(iv) and (3.48), we have

$$\begin{aligned}\|y_n-\tilde{x}\|^2 &\le (1-\delta_n)\|x_n-\tilde{x}\|^2+\delta_n\|v_n-\tilde{x}\|^2\\ &\le (1-\delta_n)\|x_n-\tilde{x}\|^2+\delta_n\Big\{\|x_n-\tilde{x}\|^2+(\alpha_n^2-2\alpha_n\xi)\|Bx_n-B\tilde{x}\|^2\Big\}\\ &= \|x_n-\tilde{x}\|^2+(\alpha_n^2-2\alpha_n\xi)\delta_n\|Bx_n-B\tilde{x}\|.^2 \end{aligned}\tag{3.49}$$

So, from (3.31) and (3.49), we derive

$$\|e_n-\tilde{x}\|^2\le\|u_n-\tilde{x}\|^2\le\|x_n-\tilde{x}\|^2+(\alpha_n^2-2\alpha_n\xi)\delta_n\|Bx_n-B\tilde{x}\|.^2 \tag{3.50}$$

From (3.26), we have

$$\begin{aligned}&\|x_{n+1}-\tilde{x}\|^2\\ =\;& \|((1-\beta_n)I-\epsilon_n A)(e_n-\tilde{x})+\beta_n(x_n-\tilde{x})+\epsilon_n(\gamma f(u_n)-A\tilde{x})\|^2\\ =\;& \|((1-\beta_n)I-\epsilon_n A)(e_n-\tilde{x})+\beta_n(x_n-\tilde{x})\|^2\\ &+\epsilon_n^2\|\gamma f(u_n)-A\tilde{x}\|^2+2\beta_n\epsilon_n\langle x_n-\tilde{x},\gamma f(u_n)-A\tilde{x}\rangle\\ &+2\epsilon_n\langle((1-\beta_n)I-\epsilon_n A)(e_n-\tilde{x}),\gamma f(u_n)-A\tilde{x}\rangle\\ \le\;& \Big((1-\beta_n-\epsilon_n\bar{\gamma})\|e_n-\tilde{x}\|+\beta_n\|x_n-\tilde{x}\|\Big)^2+\epsilon_n L_n\\ \le\;& (1-\beta_n-\epsilon_n\bar{\gamma})^2\|e_n-\tilde{x}\|^2+\beta_n^2\|x_n-\tilde{x}\|^2\\ &+2(1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\|e_n-\tilde{x}\|\|x_n-\tilde{x}\|+\epsilon_n L_n\end{aligned}$$

$$
\begin{aligned}
&\leq \Big[(1-\epsilon_n\bar{\gamma})^2 - 2(1-\epsilon_n\bar{\gamma})\beta_n + \beta_n^2\Big]\|e_n - \tilde{x}\|^2 \\
&\quad + (1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\Big\{\|e_n-\tilde{x}\|^2 + \|x_n-\tilde{x}\|^2\Big\} + \beta_n^2\|x_n-\tilde{x}\|^2 + \epsilon_n L_n \\
&= (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|e_n-\tilde{x}\|^2 + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2 + \epsilon_n L_n \qquad (3.51) \\
&\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n-\tilde{x}\|^2 + (\alpha_n^2 - 2\alpha_n\xi)\delta_n\|Bx_n - B\tilde{x}\|^2\Big\} \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2 + \epsilon_n L_n \\
&= (1-\epsilon_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(\alpha_n^2 - 2\alpha_n\xi)\delta_n\|Bx_n - B\tilde{x}\|^2 + \epsilon_n L_n \\
&\leq \|x_n-\tilde{x}\|^2 + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(\alpha_n^2 - 2\alpha_n\xi)\delta_n\|Bx_n - B\tilde{x}\|^2 + \epsilon_n L_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(2g\xi - e^2)b\|Bx_n - B\tilde{x}\|^2 \\
\leq\ &(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(2\alpha_n\xi - \alpha_n^2)\delta_n\|Bx_n - B\tilde{x}\|^2 \\
\leq\ &\|x_n-\tilde{x}\|^2 - \|x_{n+1}-\tilde{x}\|^2 + \epsilon_n L_n \\
\leq\ &\|x_n - x_{n+1}\|(\|x_n-\tilde{x}\| + \|x_{n+1}-\tilde{x}\|) + \epsilon_n L_n,
\end{aligned}
$$

where

$$
\begin{aligned}
L_n &= \epsilon_n\|\gamma f(u_n) - A\tilde{x}\|^2 + 2\beta_n\langle x_n - \tilde{x}, \gamma f(u_n) - A\tilde{x}\rangle \\
&\quad + 2\langle((1-\beta_n)I - \epsilon_n A)(e_n - \tilde{x}), \gamma f(u_n) - A\tilde{x}\rangle.
\end{aligned}
$$

By conditions (C1), (C2) and (3.46), we obtain

$$
\lim_{n\to\infty}\|Bx_n - B\tilde{x}\| = 0. \qquad (3.52)
$$

Since P_C is firmly nonexpansive mapping, we have

$$\begin{aligned}
\|v_n-\tilde{x}\|^2 &= \|P_C(x_n-\alpha_nBx_n)-P_C(\tilde{x}-\alpha_nB\tilde{x})\|^2\\
&\le \big\langle(x_n-\alpha_nBx_n)-(\tilde{x}-\alpha_nB\tilde{x}),v_n-\tilde{x}\big\rangle\\
&= \frac{1}{2}\Big\{\|(x_n-\alpha_nBx_n)-(\tilde{x}-\alpha_nB\tilde{x})\|^2+\|v_n-\tilde{x}\|^2\\
&\quad -\|(x_n-\alpha_nBx_n)-(\tilde{x}-\alpha_nB\tilde{x})-(v_n-\tilde{x})\|^2\Big\}\\
&\le \frac{1}{2}\Big\{\|x_n-\tilde{x}\|^2+\|v_n-\tilde{x}\|^2-\|(x_n-v_n)-\alpha_n(Bx_n-B\tilde{x})\|^2\Big\}\\
&\le \frac{1}{2}\Big\{\|x_n-\tilde{x}\|^2+\|v_n-\tilde{x}\|^2-\|x_n-v_n\|^2\\
&\quad -\alpha_n^2\|Bx_n-B\tilde{x}\|^2+2\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|\Big\}.
\end{aligned}$$

Hence, we have

$$\|v_n-\tilde{x}\|^2 \le \|x_n-\tilde{x}\|^2-\|x_n-v_n\|^2+2\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|$$

and so

$$\begin{aligned}
&\|y_n-\tilde{x}\|^2\\
\le\ & (1-\delta_n)\|x_n-\tilde{x}\|^2+\delta_n\|v_n-\tilde{x}\|^2\\
\le\ & (1-\delta_n)\|x_n-\tilde{x}\|^2+\delta_n\big\{\|x_n-\tilde{x}\|^2-\|x_n-v_n\|^2+2\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|\big\}\\
=\ & \|x_n-\tilde{x}\|^2-\delta_n\|x_n-v_n\|^2+2\delta_n\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|. \qquad (3.53)
\end{aligned}$$

Using (3.51) and (3.53), we also have

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|e_n-\tilde{x}\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2+\epsilon_nL_n\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|u_n-\tilde{x}\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2+\epsilon_nL_n\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|y_n-\tilde{x}\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2+\epsilon_nL_n\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n-\tilde{x}\|^2-\delta_n\|x_n-v_n\|^2+2\delta_n\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|\Big\}\\
& +(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2+\epsilon_nL_n\\
\le\ & \|x_n-\tilde{x}\|^2-(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\delta_n\|x_n-v_n\|^2\\
& +2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\delta_n\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|+\epsilon_nL_n.
\end{aligned}$$

It follow that

$$\begin{aligned} & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\delta_n\|x_n-v_n\|^2 \\ \leq\ & \|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|) \\ & +2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\delta_n\alpha_n\|x_n-v_n\|\|Bx_n-B\tilde{x}\|+\epsilon_nL_n. \end{aligned}$$

From conditions (C1), C(4), (3.46) and (3.52), we obtain

$$\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{3.54}$$

Observe also that if $e_n=P_C(W_nu_n-\lambda_nBW_nu_n)$, then

$$\begin{aligned} \|e_n-\tilde{x}\|^2 &= \|P_C(W_nu_n-\lambda_nBW_nu_n)-P_C(\tilde{x}-\lambda_nB\tilde{x})\|^2 \\ &\leq \|(W_nu_n-\lambda_nBW_nu_n)-(\tilde{x}-\lambda_nB\tilde{x})\|^2 \\ &= \|(W_nu_n-\lambda_nBW_nu_n)-(W_n\tilde{x}-\lambda_nBW_n\tilde{x})\|^2 \\ &\leq \|u_n-\tilde{x}\|^2+(\lambda_n^2-2\lambda_n\xi)\|BW_nu_n-B\tilde{x}\|^2 \\ &\leq \|x_n-\tilde{x}\|^2+(\lambda_n^2-2\lambda_n\xi)\|BW_nu_n-B\tilde{x}\|^2. \end{aligned} \tag{3.55}$$

Substituting (3.55) in (3.51), we have

$$\begin{aligned} & \|x_{n+1}-\tilde{x}\|^2 \\ \leq\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|e_n-\tilde{x}\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2+\epsilon_nL_n \\ \leq\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n-\tilde{x}\|^2+(\lambda_n^2-2\lambda_n\xi)\|BW_nu_n-B\tilde{x}\|^2\Big\} \\ & +(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-\tilde{x}\|^2+\epsilon_nL_n \\ \leq\ & \|x_n-\tilde{x}\|^2+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(\lambda_n^2-2\lambda_n\xi)\|BW_nu_n-B\tilde{x}\|^2+\epsilon_nL_n. \end{aligned}$$

It follows that

$$\begin{aligned} & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(2g\xi-e^2)\|BW_nu_n-B\tilde{x}\|^2 \\ \leq\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(2\lambda_n\xi-\lambda_n^2)\|BW_nu_n-B\tilde{x}\|^2 \\ \leq\ & \|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|)+\epsilon_nL_n \end{aligned}$$

Since $\|x_{n+1}-x_n\|\longrightarrow 0\,(n\longrightarrow\infty)$ and conditions (C1) and (C2), we obtain

$$\lim_{n\to\infty}\|BW_nu_n-B\tilde{x}\|=0. \tag{3.56}$$

Since P_C is firmly nonexpansive (Lemma 2.31 (iii)), we have

$$\begin{aligned}
\|e_n - \tilde{x}\|^2 &= \|P_C(W_n u_n - \lambda_n BW_n u_n) - P_C(\tilde{x} - \lambda_n B\tilde{x})\|^2 \\
&\leq \big\langle (W_n u_n - \lambda_n BW_n u_n) - (\tilde{x} - \lambda_n B\tilde{x}), e_n - \tilde{x}\big\rangle \\
&= \frac{1}{2}\Big\{\|(W_n u_n - \lambda_n BW_n u_n) - (\tilde{x} - \lambda_n B\tilde{x})\|^2 + \|e_n - \tilde{x}\|^2 \\
&\quad - \|(W_n u_n - \lambda_n BW_n u_n) - (\tilde{x} - \lambda_n B\tilde{x}) - (e_n - \tilde{x})\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|u_n - \tilde{x}\|^2 + \|e_n - \tilde{x}\|^2 - \|(W_n u_n - e_n) - \lambda_n(BW_n u_n - B\tilde{x})\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - \tilde{x}\|^2 + \|e_n - \tilde{x}\|^2 - \|W_n u_n - e_n\|^2 \\
&\quad - \lambda_n^2\|BW_n u_n - B\tilde{x}\|^2 + 2\lambda_n\|W_n u_n - e_n\|\|BW_n u_n - B\tilde{x}\|\Big\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|e_n - \tilde{x}\|^2 \\
\leq\ & \|x_n - \tilde{x}\|^2 - \|W_n u_n - e_n\|^2 + 2\lambda_n\|W_n u_n - e_n\|\|BW_n u_n - B\tilde{x}\|. \quad (3.57)
\end{aligned}$$

Using (3.51) and (3.57), we also have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|e_n - \tilde{x}\|^2 + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - \tilde{x}\|^2 + \epsilon_n L_n \\
&\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\Big\{\|x_n - \tilde{x}\|^2 - \|W_n u_n - e_n\|^2 \\
&\quad + 2\lambda_n\|W_n u_n - e_n\|\|BW_n u_n - B\tilde{x}\|\Big\} + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - \tilde{x}\|^2 + \epsilon_n L_n \\
&\leq \|x_n - \tilde{x}\|^2 - (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|W_n u_n - e_n\|^2 \\
&\quad + 2(1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\lambda_n\|W_n u_n - e_n\|\|BW_n u_n - B\tilde{x}\| + \epsilon_n L_n.
\end{aligned}$$

It follow that

$$\begin{aligned}
&(1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|W_n u_n - e_n\|^2 \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) \\
&+ 2(1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\lambda_n\|W_n u_n - e_n\|\|BW_n u_n - B\tilde{x}\| + \epsilon_n L_n.
\end{aligned}$$

From condition (C1), (3.46) and (3.56), we obtain

$$\lim_{n\to\infty}\|W_n u_n - e_n\| = 0. \tag{3.58}$$

For any $\tilde{x} \in \Theta$, note that $J^{F_k}_{r_{k,n}}$ is firmly nonexpansive (Lemma 2.55(2)) for $k \in \{1, 2, 3, \ldots, M\}$, then we have

$$\begin{aligned}
\|\Im^k_n y_n - \tilde{x}\|^2 &= \|J^{F_k}_{r_{k,n}} \Im^{k-1}_n y_n - J^{F_k}_{r_{k,n}} \tilde{x}\|^2 \\
&\leq \left\langle J^{F_k}_{r_{k,n}} \Im^{k-1}_n y_n - J^{F_k}_{r_{k,n}} \tilde{x}, \Im^{k-1}_n y_n - \tilde{x} \right\rangle \\
&= \left\langle \Im^k_n y_n - \tilde{x}, \Im^{k-1}_n y_n - \tilde{x} \right\rangle \\
&= \frac{1}{2}\Big(\|\Im^k_n y_n - \tilde{x}\|^2 + \|\Im^{k-1}_n y_n - \tilde{x}\|^2 - \|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2 \Big).
\end{aligned}$$

So, we obtain

$$\|\Im^k_n y_n - \tilde{x}\|^2 \leq \|\Im^{k-1}_n y_n - \tilde{x}\|^2 - \|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2, \quad k = 1, 2, 3, \ldots, M$$

which implies that for each $k \in \{1, 2, 3, \ldots, M-1\}$,

$$\begin{aligned}
\|\Im^k_n y_n - \tilde{x}\|^2 &\leq \|\Im^0_n y_n - \tilde{x}\|^2 - \|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2 \\
&\quad - \|\Im^{k-1}_n y_n - \Im^{k-2}_n y_n\|^2 - \ldots - \|\Im^2_n y_n - \Im^1_n y_n\|^2 - \|\Im^1_n y_n - \Im^0_n y_n\|^2 \\
&\leq \|y_n - \tilde{x}\|^2 - \|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2 \\
&\leq \|x_n - \tilde{x}\|^2 - \|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2.
\end{aligned}$$

Consequently, from (3.51) we derive that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|e_n - \tilde{x}\|^2 + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - \tilde{x}\|^2 + \epsilon_n L_n \\
&\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|u_n - \tilde{x}\|^2 + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - \tilde{x}\|^2 + \epsilon_n L_n \\
&= (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|\Im^k_n y_n - \tilde{x}\|^2 + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - \tilde{x}\|^2 + \epsilon_n L_n \\
&\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\Big\{ \|x_n - \tilde{x}\|^2 - \|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2 \Big\} \\
&\quad + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - \tilde{x}\|^2 + \epsilon_n L_n \\
&\leq \|x_n - \tilde{x}\|^2 - (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2 + \epsilon_n L_n.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|\Im^k_n y_n - \Im^{k-1}_n y_n\|^2 \\
&\leq \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \epsilon_n L_n.
\end{aligned}$$

By $\liminf_{n\longrightarrow\infty} \beta_n > 0$, $\epsilon_n \longrightarrow 0$ as $n \to \infty$ and (3.46), so we deduce that

$$\lim_{n\longrightarrow\infty} \|\Im_n^k y_n - \Im_n^{k-1} y_n\| = 0, \quad k = 1, 2, \ldots, M-1, \tag{3.59}$$

that is,

$$\|u_n^{(k)} - u_n^{(k-1)}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore, we have

$$\begin{aligned} \|y_n - u_n\| &= \|\Im_n^0 y_n - \Im_n^k y_n\| \\ &\leq \|\Im_n^0 y_n - \Im_n^1 y_n\| + \|\Im_n^1 y_n - \Im_n^2 y_n\| + \ldots + \|\Im_n^{M-1} y_n - \Im_n^M y_n\|. \end{aligned}$$

From (3.59), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.60}$$

Since $x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)e_n$, we have

$$\begin{aligned} \|x_n - e_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - e_n\| \\ &= \|x_n - x_{n+1}\| + \|\epsilon_n \gamma f(u_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)e_n - e_n\| \\ &= \|x_n - x_{n+1}\| + \|\epsilon_n(\gamma f(u_n) - Ae_n) + \beta_n(x_n - e_n)\| \\ &\leq \|x_n - x_{n+1}\| + \epsilon_n(\|\gamma f(u_n)\| + \|Ae_n\|) + \beta_n\|x_n - e_n\|, \end{aligned}$$

that is,

$$\|x_n - e_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\big(\|\gamma f(u_n)\| + \|Ae_n\|\big).$$

By conditions (C1), (C2) and (3.46) it follows that

$$\lim_{n\longrightarrow\infty} \|x_n - e_n\| = 0. \tag{3.61}$$

On the other hand, from (3.26), we have

$$\|y_n - x_n\| = \delta_n \|v_n - x_n\|.$$

Since $\lim_{n\longrightarrow\infty} \|x_n - v_n\| = 0$, we get

$$\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0. \tag{3.62}$$

We observe that

$$\begin{aligned}\|W_ne_n - e_n\| &\leq \|W_ne_n - W_nu_n\| + \|W_nu_n - e_n\| \\ &\leq \|e_n - x_n + x_n - y_n + y_n - u_n\| + \|W_nu_n - e_n\| \\ &\leq \|e_n - x_n\| + \|x_n - y_n\| + \|y_n - u_n\| + \|W_nu_n - e_n\|.\end{aligned}$$

Consequently, we obtain

$$\lim_{n\to\infty} \|W_ne_n - e_n\| = 0. \tag{3.63}$$

Let W be the mapping defined by (3.25). Since $\{e_n\}$ is bounded, applying Lemma 3.56(4) and (3.63), we have

$$\|We_n - e_n\| \leq \|We_n - W_ne_n\| + \|W_ne_n - e_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.64}$$

Step 6. We show that $q \in \Theta$, where $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C, B)$.

Since $\{x_n\}$ is bounded, we see that there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to q. It follows from (3.62) and (3.61) that $y_{n_i} \rightharpoonup q$ and $e_{n_i} \rightharpoonup q$. From (3.60), we obtain that $\Im_{n_i}^k y_{n_i} \rightharpoonup q$ for $k = 1, 2, ..., M$.

First, we show that $q \in \cap_{k=1}^{M} SEP(F_k)$. Since $u_n = \Im_n^k y_n$ for $k = 1, 2, 3, \ldots, M$, we also have

$$F_k(\Im_n^k y_n, y) + \frac{1}{r_n}\langle y - \Im_n^k y_n, \Im_n^k y_n - \Im_n^{k-1} y_n\rangle \geq 0, \quad \forall y \in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y - \Im_n^k y_n, \Im_n^k y_n - \Im_n^{k-1} y_n\rangle \geq -F_k(\Im_n^k y_n, y) \geq F_k(y, \Im_n^k y_n).$$

Replacing n by n_i, we have

$$\left\langle y - \Im_{n_i}^k y_{n_i}, \frac{\Im_{n_i}^k y_{n_i} - \Im_{n_i}^{k-1} y_{n_i}}{r_{n_i}}\right\rangle \geq F_k(y, \Im_{n_i}^k y_{n_i}).$$

Since $\frac{\Im_{n_i}^k y_{n_i} - \Im_{n_i}^{k-1} y_{n_i}}{r_{n_i}} \longrightarrow 0$ and $\Im_{n_i}^k y_{n_i} \rightharpoonup q$, it follows by (A4) that

$$F_k(y, q) \leq 0 \quad \forall y \in C,$$

for each $k = 1, 2, 3, \dots, M$.

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$ and hence $F_k(y_t, q) \leq 0$. So, from (A1) and (A4) we have

$$0 \;=\; F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, q) \leq tF_k(y_t, y)$$

and hence $F_k(y_t, y) \geq 0$. From (A3), we have $F_k(q, y) \geq 0$ for all $y \in C$ and hence $q \in SEP(F_k)$ for $k = 1, 2, 3, \dots, M$, that is, $q \in \cap_{k=1}^{M} SEP(F_k)$.

Next, we show that $q \in \cap_{n=1}^{\infty} F(T_n)$. By Lemma 3.56(2), we have $F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume $q \notin F(W)$. Since $e_{n_i} \rightharpoonup q$ and $q \neq Wq$, it follows by the Opial's condition (Lemma 2.25) that

$$\begin{aligned}
\liminf_{i \longrightarrow \infty} \|e_{n_i} - q\| &< \liminf_{i \longrightarrow \infty} \|e_{n_i} - Wq\| \\
&\leq \liminf_{i \longrightarrow \infty} \big\{ \|e_{n_i} - We_{n_i}\| + \|We_{n_i} - Wq\| \big\} \\
&\leq \liminf_{i \longrightarrow \infty} \|e_{n_i} - q\|
\end{aligned}$$

which derives a contradiction. Thus, we have $q \in F(W) = \cap_{n=1}^{\infty} F(T_n)$.

Finally, Now we prove that $q \in VI(C, B)$.

We define the maximal monotone operator

$$Qq_1 = \begin{cases} Bq_1 + N_C q_1, & q_1 \in C, \\ \emptyset, & q_1 \notin C. \end{cases}$$

For any given $(q_1, q_2) \in G(Q)$, hence $q_2 - Bq_1 \in N_C q_1$. Since $e_n \in C$ we see from the definition of N_C that

$$\langle q_1 - e_n, q_2 - Bq_1 \rangle \geq 0.$$

On the other hand, from $e_n = P_C(W_n u_n - \alpha_n BW_n u_n)$, we have

$$\big\langle q_1 - e_n, e_n - (W_n u_n - \alpha_n BW_n u_n) \big\rangle \geq 0,$$

that is

$$\left\langle q_1 - e_n, \frac{e_n - W_n u_n}{\alpha_n} + BW_n u_n \right\rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned}
\langle q_1 - e_{n_i}, q_2\rangle &\geq \langle q_1 - e_{n_i}, Bq_1\rangle \\
&\geq \langle q_1 - e_{n_i}, Bq_1\rangle - \left\langle q_1 - e_{n_i}, \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}} + BW_n u_{n_i}\right\rangle \\
&= \left\langle q_1 - e_{n_i}, Bq_1 - BW_n u_{n_i} - \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}}\right\rangle \\
&= \langle q_1 - e_{n_i}, Bq_1 - Be_{n_i}\rangle + \langle q_1 - e_{n_i}, Be_{n_i} - BW_n u_{n_i}\rangle \\
&\quad - \left\langle q_1 - e_{n_i}, \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}}\right\rangle \\
&\geq \langle q_1 - e_{n_i}, Be_{n_i} - BW_n u_{n_i}\rangle - \left\langle q_1 - e_{n_i}, \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}}\right\rangle. \qquad (3.65)
\end{aligned}$$

Since $\|e_{n_i} - W_n u_{n_i}\| \to 0$ as $i \to \infty$ and B is Lipschitz continuous we obtain that

$$\langle q_1 - q, q_2\rangle \geq 0.$$

Notice that Q is maximal monotone, we obtain that $q \in Q^{-1}0$ and hence $q \in VI(C, B)$. This implies $q \in \Theta$. Since $z = P_\Theta(\gamma f + (I - A))(z)$, we have

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\left\langle x_n - z, \gamma f(z) - Az\right\rangle &= \lim_{i\longrightarrow\infty}\left\langle x_{n_i} - z, \gamma f(z) - Az\right\rangle \\
&= \left\langle q - z, \gamma f(z) - Az\right\rangle \leq 0. \qquad (3.66)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\left\langle e_n - z, \gamma f(z) - Az\right\rangle &= \left\langle e_n - x_n, \gamma f(z) - Az\right\rangle + \left\langle x_n - z, \gamma f(z) - Az\right\rangle \\
&\leq \|e_n - x_n\|\|\gamma f(z) - Az\| + \left\langle x_n - z, \gamma f(z) - Az\right\rangle.
\end{aligned}$$

From (3.61) and (3.66), we obtain that

$$\limsup_{n\longrightarrow\infty}\left\langle e_n - z, \gamma f(z) - Az\right\rangle \leq 0. \qquad (3.67)$$

Step 7. Finally, we show that $\{x_n\}$ converges strongly to $z = P_\Theta(I - A + \gamma f)(z)$.

Indeed, from (3.26) , we have

$$
\begin{aligned}
& \|x_{n+1}-z\|^2 \\
= \;& \|((1-\beta_n)I-\epsilon_n A)(e_n-z)+\beta_n(x_n-z)+\epsilon_n(\gamma f(u_n)-Az)\|^2 \\
= \;& \|((1-\beta_n)I-\epsilon_n A)(e_n-z)+\beta_n(x_n-z)\|^2 \\
& +\epsilon_n^2\|\gamma f(u_n)-Az\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(u_n)-Az\rangle \\
& +2\epsilon_n\langle((1-\beta_n)I-\epsilon_n A)(e_n-z),\gamma f(u_n)-Az\rangle \\
\leq \;& \Big((1-\beta_n-\epsilon_n\bar{\gamma})\|e_n-z\|+\beta_n\|x_n-z\|\Big)^2+\epsilon_n^2\|\gamma f(u_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\gamma\langle x_n-z,f(u_n)-f(z)\rangle+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\epsilon_n\gamma\langle e_n-z,f(u_n)-f(z)\rangle+2\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle \\
& -2\beta_n\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle(A(e_n-z),\gamma f(u_n)-Az\rangle \\
\leq \;& (1-\beta_n-\epsilon_n\bar{\gamma})^2\|e_n-z\|^2+\beta_n^2\|x_n-z\|^2+2(1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\|e_n-z\|\|x_n-z\| \\
& +\epsilon_n^2\|\gamma f(u_n)-Az\|^2+2\beta_n\epsilon_n\gamma\|x_n-z\|\|f(u_n)-f(z)\| \\
& +2(1-\beta_n)\epsilon_n\gamma\|e_n-z\|\|f(u_n)-f(z)\|+2\beta_n\epsilon_n\|x_n-z\|\|\gamma f(z)-Az\| \\
& -2\beta_n\epsilon_n\|e_n-z\|\|\gamma f(z)-Az\|-2\epsilon_n^2\|A(e_n-z)\|\|\gamma f(u_n)-Az\| \\
& +2\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle \\
\leq \;& \Big[(1-\epsilon_n\bar{\gamma})^2-2(1-\epsilon_n\bar{\gamma})\beta_n+\beta_n^2\Big]\|e_n-z\|^2+\beta_n^2\|x_n-z\|^2 \\
& +(1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\Big\{\|e_n-z\|^2+\|x_n-z\|^2\Big\}+\epsilon_n^2\|\gamma f(u_n)-Az\|^2 \\
& +2\beta_n\epsilon_n\gamma\eta\|x_n-z\|\|u_n-z\|+2(1-\beta_n)\epsilon_n\gamma\eta\|e_n-z\|\|u_n-z\| \\
& +2\beta_n\epsilon_n\|x_n-z\|\|\gamma f(z)-Az\|-2\beta_n\epsilon_n\|e_n-z\|\|\gamma f(z)-Az\| \\
& -2\epsilon_n^2\|A(e_n-z)\|\|\gamma f(u_n)-Az\|+2\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle \\
\leq \;& (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|e_n-z\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-z\|^2 \\
& +\epsilon_n^2\|\gamma f(u_n)-Az\|^2+2\beta_n\epsilon_n\gamma\eta\|x_n-z\|^2+2(1-\beta_n)\epsilon_n\gamma\eta\|x_n-z\|^2 \\
& +2\beta_n\epsilon_n\|x_n-z\|\|\gamma f(z)-Az\|-2\beta_n\epsilon_n\|e_n-z\|\|\gamma f(z)-Az\| \\
& +2\epsilon_n^2\|A(e_n-z)\|\|\gamma f(u_n)-Az\|+2\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n-z\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-z\|^2\\
&\quad +\epsilon_n^2\|\gamma f(u_n)-Az\|^2+2\epsilon_n\gamma\eta\|x_n-z\|^2\\
&\quad +2\beta_n\epsilon_n\|x_n-z\|\|\gamma f(z)-Az\|-2\beta_n\epsilon_n\|x_n-z\|\|\gamma f(z)-Az\|\\
&\quad +2\epsilon_n^2\|A(e_n-z)\|\|\gamma f(u_n)-Az\|+2\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle\\
&= \big(1-2\epsilon_n\bar{\gamma}+\epsilon_n^2\bar{\gamma}^2+2\epsilon_n\gamma\eta\big)\|x_n-z\|^2+\epsilon_n^2\|\gamma f(u_n)-Az\|^2\\
&\quad +2\epsilon_n^2\|A(e_n-z)\|\|\gamma f(u_n)-Az\|+2\epsilon_n\langle e_n-z,\gamma f(z)-Az\rangle\\
&= \big[1-2(\bar{\gamma}-\gamma\eta)\epsilon_n\big]\|x_n-z\|^2+\epsilon_n\Big\{2\langle e_n-z,\gamma f(z)-Az\rangle+\epsilon_n K\Big\}.
\end{aligned}
$$

where K is an appropriate constant such that

$$
K \geq \max\left\{\sup_{n\geq 1}\Big\{\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(u_n)-Az\|^2+2\|A(e_n-z)\|\|\gamma f(u_n)-Az\|\Big\}\right\},
$$

Set $b_n=2(\bar{\gamma}-\gamma\eta)\epsilon_n$ and $c_n=\epsilon_n\Big\{2\langle e_n-z,\gamma f(z)-Az\rangle+\epsilon_n K\Big\}$. Then we have

$$
\|x_{n+1}-z\|^2\leq (1-b_n)\|x_n-z\|^2+c_n,\quad \forall n\geq 0. \tag{3.68}
$$

From the condition (C1) and (3.124), we see that

$$
\lim_{n\to\infty} b_n=0,\quad \sum_{n=0}^{\infty} b_n=\infty \quad\text{and}\quad \limsup_{n\to\infty} c_n\leq 0.
$$

Therefore, applying Lemma 2.12 to (3.68), we get that $\{x_n\}$ converges strongly to $z\in\Theta$. This completes the proof. □

Corollary 3.58. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k\in\{1,2,3,\ldots,M\}$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be ξ-inverse strongly monotone such that*

$$
\Theta := \big(\cap_{k=1}^{M} SEP(F_k)\big)\cap VI(C,B)\neq\emptyset.
$$

Let f be a contraction of H into itself with $\eta\in(0,1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$
\begin{cases}
x_1=x\in C \ \textit{chosen arbitrary},\\
y_n=(1-\delta_n)x_n+\delta_n P_C(x_n-\alpha_n Bx_n),\\
u_n=J^{F_M}_{r_{M,n}}J^{F_{M-1}}_{r_{M-1,n}}J^{F_{M-2}}_{r_{M-2,n}}\ldots J^{F_2}_{r_{2,n}}J^{F_1}_{r_{1,n}}y_n,\\
x_{n+1}=\epsilon_n f(u_n)+\beta_n x_n+\big(1-\beta_n-\epsilon_n\big)P_C(u_n-\lambda_n Bu_n),\quad \forall n\geq 1,
\end{cases}
$$

where $\{\epsilon_n\}$, $\{\beta_n\}$ *are two sequences in* $(0,1)$ *and* $\{r_{k,n}\}, k \in \{1,2,3,\ldots,M\}$ *are a real sequence in* $(0,\infty)$ *satisfy the following conditions:*

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\{\alpha_n\}, \{\lambda_n\} \subset [e,g] \subset (0, 2\xi)$, $\lim_{n\rightarrow\infty} \alpha_n = 0$ *and* $\lim_{n\rightarrow\infty} \lambda_n = 0$,

(C4) $\{\delta_n\} \subset [0,b]$, *for some* $b \in (0,1)$ *and* $\lim_{n\rightarrow\infty} |\delta_{n+1} - \delta_n| = 0$,

(C5) $\liminf_{n\longrightarrow\infty} r_{k,n} > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1,2,3,\ldots,M\}$,

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\Big\langle (f(z) - z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta f(z)$.

Proof. Put $T_n \equiv I$ for all $n \geq 1$ and for all $x \in C$. Then $W_n = I$, $A = I$ and $\gamma = 1$. The conclusion follows from Theorem 3.57. This completes the proof. $\square$

If $\delta_n = 0$ and $M = 1$, in Theorem 3.57, then we can obtain the following result immediately.

Corollary 3.59. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *let* $F_k, k \in \{1,2,3,\ldots,M\}$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4), let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of* C *into itself and let* B *be* ξ*-inverse strongly monotone such that*

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)P_C(W_n u_n - \lambda_n B W_n u_n), \ \ \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (3.24) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$ and $\{r_n\}$ are a real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\{\lambda_n\} \subset [e,g] \subset (0, 2\xi)$ *and* $\lim_{n\to\infty} \lambda_n = 0$,

(C4) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

3.2 Relaxed hybrid Steepest Descent Methods

let $D : C \longrightarrow H$ be a nonlinear mapping, $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a real-valued function and let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction such that $C \cap \text{dom}\varphi \neq \emptyset$, where $\mathbb{R}$ is the set of real numbers and $\text{dom}\varphi = \{x \in C : \varphi(x) < +\infty\}$.
The *generalized mixed equilibrium problem* for finding $x \in C$ such that

$$F(x,y) + \langle Dx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{3.69}$$

The set of solutions of (3.69) is denoted by $GMEP(F, \varphi, D)$, that is,

$$GMEP(F, \varphi, D) = \{x \in C : F(x, y) + \langle Dx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}.$$

We see that if x is a solution of a problem (3.69) then $x \in \text{dom}\varphi$.

If $D = 0$, then the problem (3.69) is reduced into the *mixed equilibrium problem* is denoted by $MEP(F, \varphi)$.

If $\varphi = 0$, then the problem (3.69) is reduced into the *generalized equilibrium problem* is denoted by $GEP(F, D)$.

If $D = 0$ and $\varphi = 0$, then the problem (3.69) is reduced into the *equilibrium problem* is denoted by $EP(F)$.

If $F = 0$ and $\varphi = 0$, then the problem (3.69) is reduced into the *variational inequality problem* is denoted by $VI(C, D)$.

For solving the generalized mixed equilibrium problem and the mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function φ and the set C:

(H1) $F(x, x) = 0, \ \forall x \in C$;

(H2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0 \quad \forall x, y \in C$;

(H3) for each $y \in C, \ x \mapsto F(x, y)$ is weakly upper semicontinuous;

(H4) for each $x \in C, \ y \mapsto F(x, y)$ is convex;

(H5) for each $x \in C, \ y \mapsto F(x, y)$ is lower semicontinuous;

(B1) for each $x \in H$ and $\lambda > 0$, there exist abounded subset $G_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus G_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{\lambda}\langle y_x - z, z - x\rangle < 0; \tag{3.70}$$

(B2) C is a bounded set.

Lemma 3.60. [14] *Let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (H1)-(H5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $\lambda > 0$ and $x \in H$, define a mapping $T_\lambda^{(F,\varphi)} : H \to C$ as follows:*

$$T_\lambda^{(F,\varphi)}(x) = \Big\{z \in C : \ F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\Big\}, \ \ \forall z \in H.$$

Then, the following properties hold:

1. *For each $x \in H$, $T_\lambda^{(F,\varphi)}(x) \neq \emptyset$;*
2. *$T_\lambda^{(F,\varphi)}$ is single-valued;*
3. *$T_\lambda^{(F,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*
$$\|T_\lambda^{(F,\varphi)}x - T_\lambda^{(F,\varphi)}y\|^2 \leq \Big\langle T_\lambda^{(F,\varphi)}x - T_\lambda^{(F,\varphi)}y, x - y\Big\rangle;$$
4. *$F(T_\lambda^{(F,\varphi)}) = MEP(F,\varphi)$;*
5. *$MEP(F,\varphi)$ is closed and convex.*

Theorem 3.61. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H5) and let $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let B, D be two ξ, β-inverse strongly monotone mapping of C into H, respectively. Let $S : C \longrightarrow C$ be a nonexpansive mapping. Let $f : C \to C$ be a contraction mapping with $\eta \in (0,1)$ and let A be a strongly positive linear bounded operator with $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\eta}$. Assume that $\Theta := F(S) \cap VI(C,B) \cap GMEP(F,\varphi,D) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 = x \in C \ \text{chosen arbitrary}, \\ u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n D x_n), \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n A) P_C(S u_n - \alpha_n B S u_n), \\ x_{n+1} = (1 - \delta_n) y_n + \delta_n P_C(S y_n - \alpha_n B S y_n), \ \forall n \geq 1, \end{cases} \tag{3.71}$$

where $\{\delta_n\}$, $\{\beta_n\}$ *be two sequences in* $(0,1)$ *satisfy the following conditions:*

(C1) $\lim_{n\longrightarrow\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}\beta_n = \infty$,

(C2) $\{\delta_n\}\subset[0,b]$, *for some* $b\in(0,1)$ *and* $\lim_{n\longrightarrow\infty}|\delta_{n+1}-\delta_n| = 0$,

(C3) $\{\lambda_n\}\subset[c,d]\subset(0,2\beta)$ *and* $\lim_{n\longrightarrow\infty}|\lambda_{n+1}-\lambda_n| = 0$,

(C4) $\{\alpha_n\}\subset[e,g]\subset(0,2\xi)$ *and* $\lim_{n\longrightarrow\infty}|\alpha_{n+1}-\alpha_n| = 0$.

Then, $\{x_n\}$ *converges strongly to* $z\in\Theta$, *which is the unique solution of the variational inequality*

$$\Big\langle \gamma f(z) - Az,\ x - z\Big\rangle \le 0,\ \ \forall x\in\Theta. \tag{3.72}$$

***Proof*.** We may assume, in view $\beta_n \longrightarrow 0$ as $n\longrightarrow\infty$, that $\beta_n\in(0,\|A\|^{-1})$. By Lemma 2.40, we obtain $\|I-\beta_n A\|\le 1-\beta_n\bar{\gamma},\ \forall n\in\mathbb{N}$.

We divide the proof of Theorem 3.61 into six steps.

Step 1. We claim that the sequence $\{x_n\}$ is bounded.

Now, let $p\in\Theta$, It is clear that

$$p = Sp = P_C(p-\alpha_n Bp) = T_{\lambda_n}^{(F,\varphi)}(p-\lambda_n Dp).$$

Let $u_n = T_{\lambda_n}^{(F,\varphi)}(x_n-\lambda_n Dx_n)\in \text{dom }\varphi$, D be β-inverse strongly monotone and $0\le\lambda_n\le 2\beta$, we have

$$\|u_n - p\| \le \|x_n - p\|. \tag{3.73}$$

Let $z_n = P_C(Su_n - \alpha_n BSu_n)$ and $S-\alpha_n BS$ be a nonexpansive mapping, we have from Lemma 2.35 that

$$\|z_n - p\| \le \|u_n - p\| \le \|x_n - p\| \tag{3.74}$$

and

$$\begin{aligned}
\|y_n - p\| &\leq \beta_n\|\gamma f(x_n) - Ap\| + \|1 - \beta_n A\|\|z_n - p\| \\
&\leq \beta_n\|\gamma f(x_n) - Ap\| + (1 - \beta_n\overline{\gamma})\|z_n - p\| \\
&\leq \beta_n\gamma\|f(x_n) - f(p)\| + \beta_n\|\gamma f(p) - Ap\| + (1 - \beta_n\overline{\gamma})\|x_n - p\| \\
&\leq \beta_n\gamma\eta\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\| + (1 - \beta_n\overline{\gamma})\|x_n - p\| \\
&= (1 - (\overline{\gamma} - \eta\gamma)\beta_n)\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\|.
\end{aligned}$$

Similarly, and let $w_n = P_C(Sy_n - \alpha_n BSy_n)$ in (3.74), we can prove that

$$\|w_n - p\| \leq \|y_n - p\| \leq (1 - (\overline{\gamma} - \eta\gamma)\beta_n)\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\|, \tag{3.75}$$

which yields that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \delta_n)\|y_n - p\| + \delta_n\|w_n - p\| \\
&\leq (1 - \delta_n)\|y_n - p\| + \delta_n\|y_n - p\| \\
&= \|y_n - p\|| \\
&\leq (1 - (\overline{\gamma} - \eta\gamma)\beta_n)\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\| \\
&= (1 - (\overline{\gamma} - \eta\gamma)\beta_n)\|x_n - p\| + \frac{(\overline{\gamma} - \eta\gamma)\beta_n}{(\overline{\gamma} - \eta\gamma)}\|\gamma f(p) - Ap\| \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \eta\gamma)}\right\} \\
&\leq \dots \\
&\leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \eta\gamma)}\right\}, \quad \forall n \geq 1.
\end{aligned}$$

This show that $\{x_n\}$ is bounded. Hence $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{w_n\}$, $\{BSu_n\}$, $\{BSy_n\}$, $\{Az_n\}$ and $\{f(x_n)\}$ are also bounded.

We can choose some appripriate constant $M > 0$ such that

$$\begin{aligned}
M \geq \max\Bigg\{ &\sup_{n\geq 1}\{\|BSu_n\|\}, \sup_{n\geq 1}\{\|BSy_n\|\}, \sup_{n\geq 1}\{\|\gamma f(x_n) - Az_n\|\}, \\
&\sup_{n\geq 1}\{\|u_n - x_n\|\}, \sup_{n\geq 1}\{\|w_n - y_n\|\}\Bigg\}.
\end{aligned} \tag{3.76}$$

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

It follows from Lemma 3.60 that $u_{n-1} = T^{(F,\varphi)}_{\lambda_{n-1}}(x_{n-1} - \lambda_{n-1}Dx_{n-1})$ and $u_n = T^{(F,\varphi)}_{\lambda_n}(x_n - \lambda_n Dx_n)$ for all $n \geq 1$, we get

$$\begin{aligned} &F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \langle Dx_{n-1}, y - u_{n-1}\rangle \\ &+\frac{1}{\lambda_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0,\ \forall y \in C \end{aligned} \tag{3.77}$$

and

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C. \tag{3.78}$$

Take $y = u_{n-1}$ in (3.78) and $y = u_n$ in (3.131), we have

$$\begin{aligned} &F(u_{n-1}, u_n) + \varphi(u_n) - \varphi(u_{n-1}) \\ &+\langle Dx_{n-1}, u_n - u_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0 \end{aligned}$$

and

$$F(u_n, u_{n-1}) + \varphi(u_{n-1}) - \varphi(u_n) + \langle Dx_n, u_{n-1} - u_n\rangle + \frac{1}{\lambda_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geq 0.$$

Adding the above two inequalities, the monotonicity of F implies that

$$\langle Dx_n - Dx_{n-1}, u_{n-1} - u_n\rangle + \left\langle u_{n-1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n-1} - x_{n-1}}{\lambda_{n-1}}\right\rangle \geq 0$$

and

$$\begin{aligned} 0 &\leq \left\langle u_{n-1} - u_n, \lambda_{n-1}(Dx_n - Dx_{n-1}) + \frac{\lambda_{n-1}}{\lambda_n}(u_n - x_n) - (u_{n-1} - x_{n-1})\right\rangle \\ &= \left\langle u_n - u_{n-1}, u_{n-1} - u_n + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)u_n + (x_n - \lambda_{n-1}Dx_n) \right. \\ &\quad \left. - (x_{n-1} - \lambda_{n-1}Dx_{n-1}) - x_n + \frac{\lambda_{n-1}}{\lambda_n}x_n \right\rangle \\ &= \left\langle u_n - u_{n-1}, u_{n-1} - u_n + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)(u_n - x_n) + (x_n - \lambda_{n-1}Dx_n) \right. \\ &\quad \left. - (x_{n-1} - \lambda_{n-1}Dx_{n-1}) \right\rangle. \end{aligned}$$

Without loss of generality, let us assume that there exists $c \in \mathbb{R}$ such that $\lambda_n > c > 0, \quad \forall n \geq 1$, we have

$$\|u_n - u_{n-1}\|^2 \leq \|u_n - u_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{\lambda_{n-1}}{\lambda_n}\right| \|u_n - x_n\| \right\}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{\lambda_n}|\lambda_n - \lambda_{n-1}|\|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{1}{c}|\lambda_n - \lambda_{n-1}|M. \end{aligned} \tag{3.79}$$

Since $S - \alpha_n BS$ is nonexpansive for each $n \geq 1$, we have

$$\begin{aligned} &\|z_n - z_{n-1}\| \\ = &\|P_C(Su_n - \alpha_n BSu_n) - P_C(Su_{n-1} - \alpha_{n-1} BSu_{n-1})\| \\ \leq &\|(Su_n - \alpha_n BSu_n) - (Su_{n-1} - \alpha_{n-1} BSu_{n-1})\| \\ = &\|(Su_n - \alpha_n BSu_n) - (Su_{n-1} - \alpha_n BSu_{n-1}) + (\alpha_{n-1} - \alpha_n) BSu_{n-1}\| \\ \leq &\|(Su_n - \alpha_n BSu_n) - (Su_{n-1} - \alpha_n BSu_{n-1})\| + |\alpha_{n-1} - \alpha_n|\|BSu_{n-1}\| \\ \leq &\|u_n - u_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|BSu_{n-1}\|. \end{aligned} \tag{3.80}$$

Substitution (3.79) into (3.80), we obtain

$$\|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{c}|\lambda_n - \lambda_{n-1}|M + |\alpha_{n-1} - \alpha_n|\|BSu_{n-1}\|. \tag{3.81}$$

From (3.71), we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\beta_n \gamma f(x_n) + (I - \beta_n A) z_n - \beta_{n-1} \gamma f(x_{n-1}) - (I - \beta_{n-1} A) z_{n-1}\| \\ &= \|\beta_n \gamma (f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1}) \gamma f(x_{n-1}) \\ &\quad + (I - \beta_n A)(z_n - z_{n-1}) - (\beta_n - \beta_{n-1}) A z_{n-1}\| \\ &= \|\beta_n \gamma (f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1})(\gamma f(x_{n-1}) - A z_{n-1}) \\ &\quad + (I - \beta_n A)(z_n - z_{n-1})\| \\ &\leq \beta_n \gamma \|f(x_n) - f(x_{n-1})\| + |\beta_n - \beta_{n-1}|\|\gamma f(x_{n-1}) - A z_{n-1}\| \\ &\quad + (I - \beta_n A)\|z_n - z_{n-1}\| \\ &\leq \beta_n \gamma \eta \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|\gamma f(x_{n-1}) - A z_{n-1}\| \\ &\quad + (1 - \beta_n \overline{\gamma})\|z_n - z_{n-1}\|. \end{aligned} \tag{3.82}$$

Substitution (3.81) into (3.82) yields that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
\leq\ & \beta_n\gamma\eta\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|\gamma f(x_{n-1}) - Az_{n-1}\| \\
& + (1-\beta_n\overline{\gamma})\Big\{\|x_n - x_{n-1}\| + \frac{1}{c}|\lambda_n - \lambda_{n-1}|M + |\alpha_{n-1} - \alpha_n|\|BSu_{n-1}\|\Big\} \\
=\ & (1-(\overline{\gamma}-\gamma\eta)\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|\gamma f(x_{n-1}) - Az_{n-1}\| \\
& + \frac{(1-\beta_n\overline{\gamma})}{c}|\lambda_n - \lambda_{n-1}|M + (1-\beta_n\overline{\gamma})|\alpha_{n-1} - \alpha_n|\|BSu_{n-1}\|. \qquad (3.83)
\end{aligned}$$

From $w_n = P_C(Sy_n - \alpha_n BSy_n)$ and $S - \alpha_n BS$ is nonexpansive mapping, we have

$$\begin{aligned}
\|w_n - w_{n-1}\| &= \|P_C(Sy_n - \alpha_n BSy_n) - P_C(Sy_{n-1} - \alpha_{n-1}BSy_{n-1})\| \\
&\leq \|(Sy_n - \alpha_n BSy_n) - (Sy_{n-1} - \alpha_{n-1}BSy_{n-1})\| \\
&= \|(Sy_n - \alpha_n BSy_n) - (Sy_{n-1} - \alpha_n BSy_{n-1}) + (\alpha_{n-1} - \alpha_n)BSy_{n-1}\| \\
&\leq \|y_n - y_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|BSy_{n-1}\|. \qquad (3.84)
\end{aligned}$$

Also, from (3.71) and (3.83), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
=\ & \|(1-\delta_n)y_n + \delta_n w_n - \{(1-\delta_{n-1})y_{n-1} + \delta_{n-1}w_{n-1}\}\| \\
=\ & \|(1-\delta_n)(y_n - y_{n-1}) + \delta_n(w_n - w_{n-1}) + (\delta_n - \delta_{n-1})(w_{n-1} - y_{n-1})\| \\
\leq\ & (1-\delta_n)\|y_n - y_{n-1}\| + \delta_n\|w_n - w_{n-1}\| + |\delta_n - \delta_{n-1}|\|w_{n-1} - y_{n-1}\| \\
\leq\ & (1-\delta_n)\|y_n - y_{n-1}\| + \delta_n\big\{\|y_n - y_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|BSy_{n-1}\|\big\} \\
& + |\delta_n - \delta_{n-1}|\|w_{n-1} - y_{n-1}\| \\
=\ & \|y_n - y_{n-1}\| + \delta_n|\alpha_{n-1} - \alpha_n|\|BSy_{n-1}\| + |\delta_n - \delta_{n-1}|\|w_{n-1} - y_{n-1}\| \\
\leq\ & (1-(\overline{\gamma}-\gamma\eta)\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|\gamma f(x_{n-1}) - Az_{n-1}\| \\
& + \frac{(1-\beta_n\overline{\gamma})}{c}|\lambda_n - \lambda_{n-1}|M + (1-\beta_n\overline{\gamma})|\alpha_{n-1} - \alpha_n|\|BSu_{n-1}\| \\
& + \delta_n|\alpha_{n-1} - \alpha_n|\|BSy_{n-1}\| + |\delta_n - \delta_{n-1}|\|w_{n-1} - y_{n-1}\| \\
\leq\ & (1-(\overline{\gamma}-\gamma\eta)\beta_n)\|x_n - x_{n-1}\| + \Big\{|\beta_n - \beta_{n-1}| + \frac{(1-\beta_n\overline{\gamma})}{c}|\lambda_n - \lambda_{n-1}| \\
& + (1-\beta_n\overline{\gamma} + \delta_n)|\alpha_{n-1} - \alpha_n| + |\delta_n - \delta_{n-1}|\Big\}M. \qquad (3.85)
\end{aligned}$$

Set $b_n = (\overline{\gamma} - \gamma\eta)\beta_n$ and
$c_n = \Big\{|\beta_n - \beta_{n-1}| + \frac{(1-\beta_n\overline{\gamma})}{c}|\lambda_n - \lambda_{n-1}| + (1-\beta_n\overline{\gamma}+\delta_n)|\alpha_{n-1} - \alpha_n| + |\delta_n - \delta_{n-1}|\Big\}M.$
Then, we have

$$\|x_{n+1} - x_n\| \leq (1-b_n)\|x_n - x_{n-1}\| + c_n, \ \ \forall n \geq 0. \tag{3.86}$$

From the conditions (C1)-(C4), we see that

$$\lim_{n\to\infty} b_n = 0, \ \sum_{n=0}^{\infty} b_n = \infty \ \text{ and } \ \limsup_{n\to\infty} c_n \leq 0.$$

Therefore, applying Lemma 2.12 to (3.86), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.87}$$

Step 3. We claim that $\lim_{n\to\infty} \|Sw_n - w_n\| = 0$.
For any $p \in \Theta$ and Lemma 2.35, we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(Su_n - \alpha_n BSu_n) - P_C(p - \alpha_n Bp)\|^2 \\
&\leq \|(Su_n - \alpha_n BSu_n) - (p - \alpha_n Bp)\|^2 \\
&= \|(Su_n - \alpha_n BSu_n) - (Sp - \alpha_n BSp)\|^2 \\
&\leq \|x_n - p\|^2 + (\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2.
\end{aligned} \tag{3.88}$$

From (3.71) and (3.88), we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
= \ & \|\beta_n(\gamma f(x_n) - Ap) + (I - \beta_n A)(z_n - p)\|^2 \\
= \ & \|(I - \beta_n A)(z_n - p)\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
& + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq \ & (1 - \beta_n\overline{\gamma})^2\|z_n - p\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
& + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq \ & (1 - \beta_n\overline{\gamma})^2\Big\{\|x_n - p\|^2 + (\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2\Big\} \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (1-\beta_n\overline{\gamma})^2\|x_n-p\|^2+(1-\beta_n\overline{\gamma})^2(\alpha_n^2-2\alpha_n\xi)\|BSu_n-Bp\|^2 \\
&\quad +\beta_n^2\|\gamma f(x_n)-Ap\|^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\gamma f(x_n)-Ap\rangle \\
&\leq \|x_n-p\|^2+(1-\beta_n\overline{\gamma})^2(\alpha_n^2-2\alpha_n\xi)\|BSu_n-Bp\|^2 \\
&\quad +\beta_n^2\|\gamma f(x_n)-Ap\|^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\gamma f(x_n)-Ap\rangle. \qquad (3.89)
\end{aligned}
$$

From (3.71), (3.75), (3.89) and Lemma 2.31(iv), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\leq (1-\delta_n)\|y_n-p\|^2+\delta_n\|w_n-p\|^2 \\
&\leq (1-\delta_n)\|y_n-p\|^2+\delta_n\|y_n-p\|^2 \\
&\leq \|y_n-p\|^2 \\
&\leq \|x_n-p\|^2+(1-\beta_n\overline{\gamma})^2(\alpha_n^2-2\alpha_n\xi)\|BSu_n-Bp\|^2 \\
&\quad +\beta_n^2\|\gamma f(x_n)-Ap\|^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\gamma f(x_n)-Ap\rangle. \qquad (3.90)
\end{aligned}
$$

It follow that

$$
\begin{aligned}
&(1-\beta_n\overline{\gamma})^2(2g\xi-e^2)\|BSu_n-Bp\|^2 \\
&\leq (1-\beta_n\overline{\gamma})^2(2\alpha_n\xi-\alpha_n^2)\|BSu_n-Bp\|^2 \\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|\gamma f(x_n)-Ap\|^2 \\
&\quad +2\beta_n\langle (I-\beta_nA)(z_n-p),\gamma f(x_n)-Ap\rangle \\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\beta_n^2\|\gamma f(x_n)-Ap\|^2 \\
&\quad +2\beta_n\langle (I-\beta_nA)(z_n-p),\gamma f(x_n)-Ap\rangle. \qquad (3.91)
\end{aligned}
$$

From condition (C1) and (3.87), we obtain

$$\lim_{n\to\infty}\|BSu_n-Bp\|=0. \qquad (3.92)$$

From $w_n = P_C(Sy_n - \alpha_n BSy_n)$, (3.89) and Lemma 2.35, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|P_C(Sy_n - \alpha_n BSy_n) - P_C(p - \alpha_n Bp)\|^2 \\
&\leq \|(Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp)\|^2 \\
&= \|(Sy_n - \alpha_n BSy_n) - (Sp - \alpha_n BSp)\|^2 \\
&\leq \|y_n - p\|^2 + (\alpha_n^2 - 2\alpha_n\xi)\|BSy_n - Bp\|^2 \\
&\leq \Big\{\|x_n - p\|^2 + (1-\beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
&\quad + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle\Big\} \\
&\quad + (\alpha_n^2 - 2\alpha_n\xi)\|BSy_n - Bp\|.^2 \qquad (3.93)
\end{aligned}$$

Using (3.71), (3.89) and (3.93), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\delta_n)\|y_n - p\|^2 + \delta_n\|w_n - p\|^2 \\
&\leq (1-\delta_n)\Big\{\|x_n - p\|^2 + (1-\beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
&\quad + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle\Big\} \\
&\quad + \delta_n\Big\{\|x_n - p\|^2 + (1-\beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
&\quad + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
&\quad + (\alpha_n^2 - 2\alpha_n\xi)\|BSy_n - Bp\|^2\Big\} \\
&= \|x_n - p\|^2 + (1-\beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
&\quad + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
&\quad + (\alpha_n^2 - 2\alpha_n\xi)\delta_n\|BSy_n - Bp\|.^2 \qquad (3.94)
\end{aligned}$$

It follows that

$$\begin{aligned}
&(2g\xi - e^2)b\|BSy_n - Bp\|^2 \\
\leq\ & \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&+ (1-\beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
&+ 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle. \qquad (3.95)
\end{aligned}$$

From condition (C1), (3.87) and (3.92), we obtain

$$\lim_{n\to\infty}\|BSy_n - Bp\| = 0. \qquad (3.96)$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|P_C(Sy_n - \alpha_n BSy_n) - P_C(p - \alpha_n Bp)\|^2 \\
&\leq \big\langle (Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp), w_n - p \big\rangle \\
&= \frac{1}{2}\Big\{ \|(Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp)\|^2 + \|w_n - p\|^2 \\
&\quad - \|(Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp) - (w_n - p)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|y_n - p\|^2 + \|w_n - p\|^2 - \|(Sy_n - w_n) - \alpha_n(BSy_n - Bp)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big(\|x_n - p\|^2 + (1 - \beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
&\quad + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap \rangle \Big) \\
&\quad + \frac{1}{2}\Big\{ \|w_n - p\|^2 - \|Sy_n - w_n\|^2 \\
&\quad - \alpha_n^2\|BSy_n - Bp\|^2 + 2\alpha_n\langle Sy_n - w_n, BSy_n - Bp \rangle \Big\}. \qquad (3.97)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|w_n - p\|^2 &\leq \|x_n - p\|^2 - \|Sy_n - w_n\|^2 + (1 - \beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
&\quad + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap \rangle \\
&\quad + 2\alpha_n\|Sy_n - w_n\|\|BSy_n - Bp\|. \qquad (3.98)
\end{aligned}$$

Using (3.94) and (3.98), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
\leq\ & (1 - \delta_n)\|y_n - p\|^2 + \delta_n\|w_n - p\|^2 \\
\leq\ & (1 - \delta_n)\Big\{ \|x_n - p\|^2 + (1 - \beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap \rangle \Big\} \\
& + \delta_n\Big\{ \|x_n - p\|^2 - \|Sy_n - w_n\|^2 \\
& + (1 - \beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 + 2\alpha_n\|Sy_n - w_n\|\|BSy_n - Bp\| \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap \rangle \Big\} \\
=\ & \|x_n - p\|^2 - \delta_n\|Sy_n - w_n\|^2 \\
& + (1 - \beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 + 2\alpha_n\delta_n\|Sy_n - w_n\|\|BSy_n - Bp\| \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap \rangle. \qquad (3.99)
\end{aligned}$$

It follow that

$$\begin{aligned}
& b\|Sy_n - w_n\|^2 \\
\leq\ & \delta_n\|Sy_n - w_n\|^2 \leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
& + (1-\beta_n\overline{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)\|BSu_n - Bp\|^2 + 2\alpha_n\delta_n\|Sy_n - w_n\|\|BSy_n - Bp\| \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle. \qquad (3.100)
\end{aligned}$$

Observing condition (C1), (3.87), (3.92) and (3.96), we obtain

$$\lim_{n\to\infty}\|Sy_n - w_n\| = 0. \qquad (3.101)$$

Note that

$$\begin{aligned}
& \|y_n - p\|^2 \\
\leq\ & (1-\beta_n\overline{\gamma})^2\|z_n - p\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq\ & (1-\beta_n\overline{\gamma})^2\|u_n - p\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq\ & (1-\beta_n\overline{\gamma})^2\Big\{\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\beta)\|Dx_n - Dp\|^2\Big\} + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
& + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq\ & \|x_n - p\|^2 + (1-\beta_n\overline{\gamma})^2\lambda_n(\lambda_n - 2\beta)\|Dx_n - Dp\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
& + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle. \qquad (3.102)
\end{aligned}$$

From (3.71) and (3.102), we can compute

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
\leq\ & (1-\delta_n)\|y_n - p\|^2 + \delta_n\|w_n - p\|^2 \\
\leq\ & (1-\delta_n)\|y_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
=\ & \|y_n - p\|^2 \\
\leq\ & \|x_n - p\|^2 + (1-\beta_n\overline{\gamma})^2\lambda_n(\lambda_n - 2\beta)\|Dx_n - Dp\|^2 \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle. \ (3.103)
\end{aligned}$$

It follow that

$$\begin{aligned}
&(1-\beta_n\overline{\gamma})^2 d(2\beta - c)\|Dx_n - Dp\|^2 \\
\leq\;& \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
&+ 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle, \qquad (3.104)
\end{aligned}$$

which imply that

$$\lim_{n\to\infty}\|Dx_n - Dp\| = 0. \qquad (3.105)$$

In addition, from the firmly nonexpansivity of $T_{\lambda_n}^{(F,\varphi)}$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n Dx_n) - T_{\lambda_n}^{(F,\varphi)}(p - \lambda_n Dp)\|^2 \\
&\leq \Big\langle (x_n - \lambda_n Dx_n) - (p - \lambda_n Dp), u_n - p\Big\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - \lambda_n Dx_n) - (p - \lambda_n Dp)\|^2 + \|u_n - p\|^2 \\
&\quad - \|(x_n - \lambda_n Dx_n) - (p - \lambda_n Dp) - (u_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - \lambda_n(Dx_n - Dp)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2\lambda_n\langle x_n - u_n, Dx_n - Dp\rangle - \lambda_n^2\|Dx_n - Dp\|^2\Big\}.
\end{aligned}$$

So, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\|x_n - u_n\|\|Dx_n - Dp\|. \qquad (3.106)$$

Substituting (3.106) into (3.102) to get

$$\begin{aligned}
&\|y_n - p\|^2 \\
\leq\;& (1-\beta_n\overline{\gamma})^2\|u_n - p\|^2 + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq\;& (1-\beta_n\overline{\gamma})^2\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\|x_n - u_n\|\|Dx_n - Dp\|\Big\} \\
&+ \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \\
\leq\;& \|x_n - p\|^2 - (1-\beta_n\overline{\gamma})^2\|x_n - u_n\|^2 + 2(1-\beta_n\overline{\gamma})^2\lambda_n\|x_n - u_n\|\|Dx_n - Dp\| \\
&+ \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle (I-\beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle \qquad (3.107)
\end{aligned}$$

and hence

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
\leq\ & \|y_n - p\|^2 \\
\leq\ & \|x_n - p\|^2 - (1 - \beta_n\overline{\gamma})^2\|x_n - u_n\|^2 \\
& + 2(1 - \beta_n\overline{\gamma})^2\lambda_n\|x_n - u_n\|\|Dx_n - Dp\| \\
& + \beta_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\langle(I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle. \quad (3.108)
\end{aligned}$$

It follows that

$$\begin{aligned}
& (1 - \beta_n\overline{\gamma})^2\|x_n - u_n\|^2 \\
\leq\ & \|x_{n+1} - x_n\|(\|x_{n+1} - p\| + \|x_n - p\|) \\
& + 2(1 - \beta_n\overline{\gamma})^2\lambda_n\|x_n - u_n\|\|Dx_n - Dp\| + \beta_n^2\|\gamma f(x_n) - Ap\|^2 \\
& + 2\beta_n\langle(I - \beta_n A)(z_n - p), \gamma f(x_n) - Ap\rangle. \qquad (3.109)
\end{aligned}$$

This together with $\|x_{n+1} - x_n\| \to 0$, $\|Dx_n - Dp\| \to 0$, $\beta_n \to 0$ as $n \to \infty$ and the condition on λ_n implies that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \text{ and } \lim_{n\to\infty} \frac{\|x_n - u_n\|}{\lambda_n} = 0. \qquad (3.110)$$

Consequently, from (3.87) and (3.158)

$$\|x_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \|x_n - u_n\| \to 0 \text{ as } n \to \infty. \qquad (3.111)$$

From (3.71) and condition (C1), we have

$$\|y_n - z_n\| = \|\beta_n\gamma f(x_n) + (1 - \beta_n A)z_n - z_n\| \leq \beta_n\|\gamma f(x_n) - Az_n\| \to 0 \text{ as } n \to \infty. \qquad (3.112)$$

From $S - \alpha_n BS$ is nonexpansive mapping(Lemma 2.35), we have

$$\begin{aligned}
\|w_n - z_n\| &= \|P_C(Sy_n - \alpha_n BSy_n) - P_C(Su_n - \alpha_n BSu_n)\| \\
&\leq \|(S - \alpha_n BS)y_n - (S - \alpha_n BS)u_n\| \\
&\leq \|y_n - u_n\|. \qquad (3.113)
\end{aligned}$$

Next, we will show that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

We consider $x_{n+1} - y_n = \delta_n(w_n - y_n) = \delta_n(w_n - z_n + z_n - y_n)$.

From (3.113), we have

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \delta_n(\|w_n - z_n\| + \|z_n - y_n\|) \\ &\leq \delta_n(\|y_n - u_n\| + \|z_n - y_n\|) \\ &\leq \delta_n(\|x_{n+1} - y_n\| + \|x_{n+1} - u_n\| + \|z_n - y_n\|). \end{aligned} \tag{3.114}$$

Observing condition (C2), (3.111) and (3.112), it follow that

$$\|x_{n+1} - y_n\| \leq \frac{\delta_n}{1-\delta_n}(\|x_{n+1} - u_n\| + \|z_n - y_n\|) \leq \frac{b}{1-b}(\|x_{n+1} - u_n\| + \|z_n - y_n\|) \to 0. \tag{3.115}$$

From (3.87) and (3.115), we obtain

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.116}$$

We observe that

$$\begin{aligned} \|Sw_n - w_n\| &\leq \|Sw_n - Sz_n\| + \|Sz_n - Sy_n\| + \|Sy_n - w_n\| \\ &\leq \|w_n - z_n\| + \|z_n - y_n\| + \|Sy_n - w_n\| \\ &\leq \|y_n - u_n\| + \|z_n - y_n\| + \|Sy_n - w_n\| \\ &\leq \|y_n - x_n\| + \|x_n - u_n\| + \|z_n - y_n\| + \|Sy_n - w_n\|. \end{aligned} \tag{3.117}$$

Consequently, we obtain

$$\lim_{n\to\infty} \|Sw_n - w_n\| = 0. \tag{3.118}$$

Step 4. We prove that the mapping $P_\Theta(\gamma f + (I - A))$ has a unique fixed point.

Since f be a contraction of C into itself with coefficient $\eta \in (0, 1)$. Then, we have

$$\begin{aligned}
&\|P_\Theta(\gamma f + (I - A))(x) - P_\Theta(\gamma f + (I - A))(y)\| \\
\leq\ & \|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\| \\
\leq\ & \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
\leq\ & \gamma\eta\|x - y\| + (1 - \bar{\gamma})\|x - y\| \\
=\ & (1 - (\bar{\gamma} - \eta\gamma))\|x - y\|, \quad \forall x, y \in C.
\end{aligned}$$

Since $0 < 1 - (\bar{\gamma} - \eta\gamma) < 1$, it follows that $P_\Theta(\gamma f + (I - A))$ is a contraction of C into itself. Therefore by the Banach Contraction Mapping Principle, has a unique fixed point, say $z \in C$, that is,

$$z = P_\Theta(\gamma f + (I - A))(z).$$

Step 5. We claim that $q \in F(S) \cap VI(C, B) \cap GMEP(F, \varphi, D)$.

First, we show that $q \in F(S)$.

Assume $q \notin F(S)$. Since $w_{n_i} \rightharpoonup q$ and $q \neq Sq$, it follows by the Opial's condition (Lemma 2.25) that

$$\begin{aligned}
\liminf_{i \longrightarrow \infty} \|w_{n_i} - q\| &< \liminf_{i \longrightarrow \infty} \|w_{n_i} - Sq\| \\
&\leq \liminf_{i \longrightarrow \infty} \big\{\|w_{n_i} - Sw_{n_i}\| + \|Sw_{n_i} - Sq\|\big\} \\
&= \liminf_{i \longrightarrow \infty} \big\|Sw_{n_i} - Sq\| \\
&\leq \liminf_{i \longrightarrow \infty} \|w_{n_i} - q\|.
\end{aligned}$$

This is a contradiction. Thus, we have $q \in F(S)$.

Next, we prove that $q \in GMEP(F, \varphi, D)$.

From Lemma 3.60 that $u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n Dx_n)$ for all $n \geq 1$ is equivalent to

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (H2), we also have

$$\varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n).$$

Replacing n by n_i, we obtain

$$\varphi(y) - \varphi(u_{n_i}) + \langle Dx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \geq F(y, u_{n_i}). \tag{3.119}$$

Let $y_t = ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. Since $y \in C$ and $q \in C$, we obtain $y_t \in C$. So, from (3.119) we have

$$\begin{aligned}
\langle y_t - u_{n_i}, Dy_t\rangle &\geq \langle y_t - u_{n_i}, Dy_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle Dx_{n_i}, y_t - u_{n_i}\rangle \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\
&\geq \langle y_t - u_{n_i}, Dy_t - Du_{n_i}\rangle + \langle y_t - u_{n_i}, Du_{n_i} - Dx_{n_i}\rangle - \varphi(y_t) \\
&\quad + \varphi(u_{n_i}) - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_t, u_{n_i}). \qquad (3.120)
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \longrightarrow 0$, $i \to \infty$, we obtain $\|Du_{n_i} - Dx_{n_i}\| \longrightarrow 0$. Furthermore, by the monotonicity of D, we have

$$\langle y_t - u_{n_i}, Dy_t - Du_{n_i}\rangle \geq 0.$$

So, from (H4), (H5) and the weak lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup q$, we have

$$\langle y_t - q, Dy_t\rangle \geq -\varphi(y_t) + \varphi(q) + F(y_t, q) \ \ as \ \ i \longrightarrow \infty. \tag{3.121}$$

From (H1), (H4) and (3.121), we also get

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, q) + t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t) \\
&= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[F(y_t, q) + \varphi(q) - \varphi(y_t)] \\
&\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - q, Dy_t\rangle \\
&= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - q, Dy_t\rangle.
\end{aligned}$$

Dividing by t, we get

$$F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - q, Dy_t\rangle \geq 0.$$

Letting $t \to 0$ in the above inequality, we arrive that, for each $y \in C$

$$F(q, y) + \varphi(y) - \varphi(q) + \langle y - q, Dq\rangle \geq 0.$$

This implies that, $q \in GMEP(F, \varphi, D)$.

Finally, Now we prove that $q \in VI(C, B)$.

We define the maximal monotone operator

$$Qq_1 = \begin{cases} Bq_1 + N_C q_1, & q_1 \in C, \\ \emptyset, & q_1 \notin C. \end{cases}$$

Since B is ξ-inverse strongly monotone and condition (C4), we have

$$\langle Bx - By, x - y\rangle \geq \xi\|Bx - By\|^2 \geq 0.$$

Then Q is maximal monotone. Let $(q_1, q_2) \in G(Q)$. Since $q_2 - Bq_1 \in N_C q_1$ and $w_n \in C$, we have $\langle q_1 - w_n, q_2 - Bq_1\rangle \geq 0$. On the other hand, from $w_n = P_C(Sy_n - \alpha_n BSy_n)$, we have

$$\big\langle q_1 - w_n, w_n - (Sy_n - \alpha_n BSy_n)\big\rangle \geq 0,$$

that is

$$\left\langle q_1 - w_n, \frac{w_n - Sy_n}{\alpha_n} + BSy_n \right\rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned}
& \langle q_1 - w_{n_i}, q_2 \rangle \\
\geq \ & \langle q_1 - w_{n_i}, Bq_1 \rangle \\
\geq \ & \langle q_1 - w_{n_i}, Bq_1 \rangle - \Big\langle q_1 - w_{n_i}, \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}} + BSy_{n_i} \Big\rangle \\
= \ & \Big\langle q_1 - w_{n_i}, Bq_1 - BSy_{n_i} - \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}} \Big\rangle \\
= \ & \langle q_1 - w_{n_i}, Bq_1 - Bw_{n_i} \rangle + \langle q_1 - w_{n_i}, Bw_{n_i} - BSy_{n_i} \rangle \\
& - \Big\langle q_1 - w_{n_i}, \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}} \Big\rangle \\
\geq \ & \langle q_1 - w_{n_i}, Bw_{n_i} - BSy_{n_i} \rangle - \Big\langle q_1 - w_{n_i}, \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}} \Big\rangle. \qquad (3.122)
\end{aligned}$$

Noting that $\|w_{n_i} - Sy_{n_i}\| \to 0$ as $i \to \infty$, we obtain

$$\langle q_1 - q, q_2 \rangle \geq 0.$$

Since Q is maximal monotone, we obtain that $q \in Q^{-1}0$ and hence $q \in VI(C, B)$. This implies $q \in \Theta$. Since $z = P_\Theta(\gamma f + (I - A))(z)$, we have

$$\begin{aligned}
\limsup_{n \longrightarrow \infty} \Big\langle \gamma f(z) - Az, x_n - z \Big\rangle &= \lim_{i \longrightarrow \infty} \Big\langle \gamma f(z) - Az, x_{n_i} - z \Big\rangle \\
&= \Big\langle \gamma f(z) - Az, q - z \Big\rangle \leq 0. \qquad (3.123)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Big\langle \gamma f(z) - Az, y_n - z \Big\rangle &= \Big\langle \gamma f(z) - Az, y_n - x_n \Big\rangle + \Big\langle \gamma f(z) - Az, x_n - z \Big\rangle \\
&\leq \|\gamma f(z) - Az\| \|y_n - x_n\| + \Big\langle \gamma f(z) - Az, x_n - z \Big\rangle.
\end{aligned}$$

From (3.116) and (3.123), we obtain that

$$\limsup_{n \longrightarrow \infty} \Big\langle \gamma f(z) - Az, y_n - z \Big\rangle \leq 0. \qquad (3.124)$$

Step 6. Finally, we claim that $x_n \to z$, where $z = P_\Theta(\gamma f + (I - A))(z)$.

We note that

$$\begin{aligned}
& \|y_n - z\|^2 \\
= \;& \|(I-\beta_n A)(z_n - z) + \beta_n(\gamma f(x_n) - Az)\|^2 \\
\leq \;& \|(I-\beta_n A)(z_n - z)\|^2 + 2\beta_n\langle(\gamma f(x_n) - Az), (I-\beta_n A)(z_n - z) \\
& + \beta_n(\gamma f(x_n) - Az)\rangle \\
= \;& \|(I-\beta_n A)(z_n - z)\|^2 + 2\beta_n\langle(\gamma f(x_n) - Az), y_n - z\rangle \qquad (3.125) \\
\leq \;& \|I-\beta_n A\|^2\|z_n - z\|^2 + 2\beta_n\gamma\langle f(x_n) - f(z), y_n - z\rangle + 2\beta_n\langle \gamma f(z) - Az, y_n - z\rangle \\
\leq \;& (1-\beta_n\overline{\gamma})^2\|z_n - z\|^2 + 2\beta_n\gamma\eta\|x_n - z\|\|y_n - z\| + 2\beta_n\langle \gamma f(z) - Az, y_n - z\rangle \\
\leq \;& (1-\beta_n\overline{\gamma})^2\|x_n - z\|^2 + \beta_n\gamma\eta(\|x_n - z\|^2 + \|y_n - z\|^2) + 2\beta_n\langle \gamma f(z) - Az, y_n - z\rangle \\
= \;& (1-2\beta_n\overline{\gamma} + \beta_n^2\overline{\gamma}^2 + \beta_n\gamma\eta)\|x_n - z\|^2 + \beta_n\gamma\eta\|y_n - z\|^2 + 2\beta_n\langle \gamma f(z) - Az, y_n - z\rangle
\end{aligned}$$

which implies that

$$\begin{aligned}
& \|y_n - z\|^2 \\
\leq \;& \left(1 - \frac{\big(2\overline{\gamma} - \gamma\eta\big)\beta_n}{1-\gamma\eta\beta_n}\right)\|x_n - z\|^2 \\
& + \frac{\beta_n}{1-\gamma\eta\beta_n}\left[\beta_n\overline{\gamma}^2\|x_n - z\|^2 + 2\langle\gamma f(z) - Az, y_n - z\rangle\right]. \qquad (3.126)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \;\leq\; & \|y_n - z\|^2 \\
\leq\; & \left(1 - \frac{\big(2\overline{\gamma} - \gamma\eta\big)\beta_n}{1-\gamma\eta\beta_n}\right)\|x_n - z\|^2 \\
& + \frac{\beta_n}{1-\gamma\eta\beta_n}\left[\beta_n\overline{\gamma}^2\|x_n - z\|^2 + 2\langle\gamma f(z) - Az, y_n - z\rangle\right] \\
\leq\; & \left(1 - \frac{\big(2\overline{\gamma} - \gamma\eta\big)\beta_n}{1-\gamma\eta\beta_n}\right)\|x_n - z\|^2 \\
& + \frac{\beta_n}{1-\gamma\eta\beta_n}\left[2\langle\gamma f(z) - Az, y_n - z\rangle + \beta_n\overline{\gamma}^2 K\right]. \qquad (3.127)
\end{aligned}$$

where K is an appropriate constant such that $K \geq \sup_{n\geq 1}\{\|x_n - z\|^2\}$.

Set $l_n = \frac{\big(2\overline{\gamma}-\gamma\eta\big)\beta_n}{1-\gamma\eta\beta_n}$ and $e_n = \frac{\beta_n}{1-\gamma\eta\beta_n}\left[2\langle\gamma f(z) - Az, y_n - z\rangle + \beta_n\overline{\gamma}^2 K\right]$. Then we have

$$\|x_{n+1} - z\|^2 \leq (1-b_n)\|x_n - z\|^2 + c_n, \;\; \forall n \geq 0. \qquad (3.128)$$

From the condition (C1) and (3.124), we see that

$$\lim_{n\to\infty} l_n = 0, \quad \sum_{n=0}^{\infty} l_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} e_n \le 0.$$

Therefore, applying Lemma 2.12 to (3.128), we get that $\{x_n\}$ converges strongly to $z \in \Theta$. This completes the proof. □

Corollary 3.62. *Let C be a nonempty closed convex subset of a real Hilbert space H, let B be ξ-inverse-strongly monotone mapping of C into H and $S : C \longrightarrow C$ be a nonexpansive mapping. Let $f : C \to C$ be a contraction mapping with $\eta \in (0,1)$ and let A be a strongly positive linear bounded operator with $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\eta}$. Assume that $\Theta := F(S) \cap VI(C,B) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 = x \in C \ \textit{chosen arbitrary}, \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n A) P_C(Sx_n - \alpha_n BSx_n), \\ x_{n+1} = (1-\delta_n) y_n + \delta_n P_C(Sy_n - \alpha_n BSy_n), \quad \forall n \ge 1, \end{cases}$$

where $\{\delta_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2) $\{\delta_n\} \subset [0,b]$,*for some* $b \in (0,1)$ *and* $\lim_{n\longrightarrow\infty} |\delta_{n+1} - \delta_n| = 0$,

(C3) $\{\alpha_n\} \subset [e,g] \subset (0,2\xi)$ *and* $\lim_{n\longrightarrow\infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$, which is the unique solution of the variational inequality

$$\left\langle \gamma f(z) - Az, \ x - z \right\rangle \le 0, \quad \forall x \in \Theta.$$

Proof. Put $F(x,y) = \varphi = D = 0$ for all $x, y \in C$ and $\lambda_n = 1$ for all $n \ge 1$ in Theorem 3.61, we get $u_n = x_n$. So $\{x_n\}$ converges strongly to $z \in \Theta$, which is the unique solution of the variational inequality. □

Corollary 3.63. *Let C be a nonempty closed convex subset of a real Hilbert space H and let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H5). Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \to C$ be a contraction mapping with $\eta \in (0,1)$. Assume that $\Theta := F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = \beta_n f(x_n) + (1-\beta_n) S T^F_{\lambda_n} x_n, \\ x_{n+1} = (1-\delta_n) y_n + \delta_n S y_n, \ \forall n \geq 1, \end{cases}$$

where $\{\delta_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2) $\{\delta_n\} \subset [0,b]$, *for some* $b \in (0,1)$ *and* $\lim_{n \longrightarrow \infty} |\delta_{n+1} - \delta_n| = 0$,

(C3) $\lim_{n \longrightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$.

***Proof*.** Put $\varphi = D = 0$, $\gamma = 1$, $A = I$ and $\alpha_n = 0$ in Theorem 3.61. Then we have $P_C(Su_n) = Su_n$ and $P_C(Sy_n) = Sy_n$. So $\{x_n\}$ converges strongly to $z \in \Theta$. □

3.3 Viscosity Approximation Methods

3.3.1 A countable family of nonexpansive mappings

In this section, we will use the viscosity approximation method to prove a strong convergence theorem for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of the variational inequality problem for relaxed cocoercive and Lipschitz continuous mappings, the set of solutions of system of variational inclusions and the set of solutions of equilibrium problem in a real Hilbert space.

Definition 3.64. Let $M : H \longrightarrow 2^H$ be a multi-valued maximal monotone mapping, then the set-valued mapping $J_{M,\lambda} : H \longrightarrow H$ defined by

$$J_{M,\lambda}(\tilde{x}) = (I + \lambda M)^{-1}(\tilde{x}), \quad \forall \tilde{x} \in H \tag{3.129}$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Lemma 3.65. [5] *Let $M : H \longrightarrow 2^H$ be a maximal monotone mapping and let $B : H \longrightarrow H$ be a Lipshitz continuous mapping. Then the mapping $M + B : H \longrightarrow 2^H$ is a maximal monotone mapping.*

Lemma 3.66. [36, 5]

(1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H \quad \textit{and} \quad \forall \lambda > 0.$$

(2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \quad \forall x, y \in H.$$

Lemma 3.67. [36]

(1) Let $\tilde{x} \in H$ is a solution of problem (1.13) if and only if $\tilde{x} = J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$, that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0.$$

(2) If $\lambda \in [0, 2\beta]$, then $I(B, M)$ is a closed convex subset in H.

Lemma 3.68. [49] *Let H be a Hilbert space and M a maximal monotone on H. Then, the following holds:*

$$\|J_{M,r}x - J_{M,s}x\|^2 \leq \frac{r-s}{r}\Big\langle J_{M,r}x - J_{M,s}x, J_{M,r}x - x\Big\rangle, \ \forall s,\, r > 0,\ x \in H,$$

where $J_{M,r} = (I + rM)^{-1}$ and $J_{M,s} = (I + sM)^{-1}$.

Lemma 3.69. [2] *Let C be a nonempty closed subset of a Banach space and let $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by*

$$Sy = \lim_{n \longrightarrow \infty} S_n y \ \text{ for all } y \in C.$$

Then $\lim_{n\longrightarrow\infty} \sup\{\|Sz - S_n z\| : z \in C\} = 0$.

Theorem 3.70. *Let C be a nonempty closed convex subset of a real Hilbert space H and $B : C \longrightarrow H$ be relaxed (ϕ, ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi, \omega, \mu > 0$. Let $\mathcal{G} = \{G_k : k = 1, 2, 3, \ldots, N\}$ be a finite family of β-inverse strongly monotone mappings from C into H and let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $f : C \longrightarrow C$ be a contraction with coefficient ψ $(0 \leq \psi < 1)$ and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that*

$$\Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \Big(\bigcap_{k=1}^{N} I(G_k, M_k)\Big) \cap VI(C, B) \cap EP(F) \neq \emptyset.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily}, \\ y_n = J_{M_N,\lambda_{N,n}}(I - \lambda_{N,n}G_n) \ldots J_{M_2,\lambda_{2,n}}(I - \lambda_{2,n}G_2)J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}G_1)T_{r_n}x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(y_n - \xi_n By_n), \ \forall n \geq 1, \end{cases} \tag{3.130}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $\{\xi_n\}, \{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C4) $\{\xi_n\} \subset [a,b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(\omega-\phi\mu^2)}{\mu^2}$ *and* $\lim_{n\longrightarrow\infty} |\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_{k,n}\}_{k=1}^{N} \subset [c,d] \subset (0, 2\beta)$ *and* $\lim_{n\to\infty} |\lambda_{k,n+1} - \lambda_{k,n}| = 0$, *for each* $k \in \{1, 2, \ldots, N\}$,

(C6) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ *for any bounded subset* K *of* C. *Let* S *be a mapping of* C *into itself defined by* $Sy = \lim_{n\longrightarrow\infty} S_n y$ *for all* $y \in C$ *and suppose that* $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$.

***Proof*.** First, we prove that the mapping $P_\Omega f : H \longrightarrow C$ has a unique fixed point.

In fact, since $f : C \longrightarrow C$ is a contraction with $\psi \in [0, 1)$ and $P_\Omega f : H \longrightarrow \Omega$ is also a contraction, we obtain

$$\|P_\Omega f(x) - P_\Omega f(y)\| \leq \|f(x) - f(y)\| \leq \psi\|x - y\|, \;\; \forall x, y \in C.$$

Therefore, there exists a unique element $x^* \in C$ such that $x^* = P_\Omega f(x^*)$, where

$$\Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \Big(\bigcap_{k=1}^{N} I(G_k, M_k)\Big) \cap VI(C, B) \cap EP(F).$$

Now, we prove that $(I - \xi_n B)$ is nonexpansive.

Indeed, for any $x, y \in C$, since $B : C \longrightarrow H$ be a μ-Lipschitz continuous and relaxed (ϕ, ω)-cocoercive mappings with $\omega > \phi\mu^2$ and $\xi_n \leq \frac{2(\omega-\phi\mu^2)}{\mu^2}$, we obtain

$$\begin{aligned}
&\|(I-\xi_n B)x-(I-\xi_n B)y\|^2\\
=\;& \|(x-y)-\xi_n(Bx-By)\|^2\\
=\;& \|x-y\|^2-2\xi_n\langle x-y, Bx-By\rangle+\xi_n^2\|Bx-By\|^2\\
\leq\;& \|x-y\|^2-2\xi_n\Big\{-\phi\|Bx-By\|^2+\omega\|x-y\|^2\Big\}+\xi_n^2\|Bx-By\|^2\\
\leq\;& \|x-y\|^2+2\xi_n\phi\mu^2\|x-y\|^2-2\xi_n\omega\|x-y\|^2+\xi_n^2\mu^2\|x-y\|^2\\
=\;& (1+2\xi_n\phi\mu^2-2\xi_n\omega+\xi_n^2\mu^2)\|x-y\|^2\\
=\;& \left(1-\xi_n\mu^2\left[\frac{2(\omega-\phi\mu^2)}{\mu^2}-\xi_n\right]\right)\|x-y\|^2\\
\leq\;& \left(1-\xi_n\mu^2\left[\frac{2(\omega-\phi\mu^2)}{\mu^2}-b\right]\right)\|x-y\|^2.
\end{aligned}$$

Setting

$$\zeta=\frac{\mu^2}{2}\left[\frac{2(\omega-\phi\mu^2)}{\mu^2}-b\right]>0,$$

thus,

$$\|(I-\xi_n B)x-(I-\xi_n B)y\|^2\leq(1-2\xi_n\zeta)\|x-y\|^2\leq(1-\xi_n\zeta)^2\|x-y\|^2,$$

which implies that

$$\|(I-\xi_n B)x-(I-\xi_n B)y\|\leq(1-\xi_n\zeta)\|x-y\|\leq\|x-y\|. \tag{3.131}$$

Hence $(I-\xi_n B)$ is nonexpansive.

We divide the proof of Theorem 3.70 into five steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded.

Now, let $\tilde{x}\in\Omega$ and if $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.55. Then $\tilde{x}=P_C(\tilde{x}-\lambda_n B\tilde{x})=T_{r_n}\tilde{x}$ and let $u_n=T_{r_n}x_n$. So, we have

$$\|u_n-\tilde{x}\|=\|T_{r_n}x_n-T_{r_n}\tilde{x}\|\leq\|x_n-\tilde{x}\|. \tag{3.132}$$

For $k\in\{1,2,\ldots,N\}$, and for any positive integer number n, we define the operator $\Upsilon_n^k : C\longrightarrow H$ as follows:

$$\Upsilon_n^0 x=x,$$

$$\Upsilon_n^k x = J_{M_k,\lambda_{k,n}}(I-\lambda_{k,n}G_k)\dots J_{M_2,\lambda_{2,n}}(I-\lambda_{2,n}G_2)J_{M_1,\lambda_{1,n}}(I-\lambda_{1,n}G_1)x$$

for all n, we get $y_n = \Upsilon_n^N u_n$. On the other hand, since $G_k : C \longrightarrow H$ is β-inverse strongly monotone and $\lambda_{k,n} \subset [c,d] \subset (0,2\beta)$, then $J_{M_k,\lambda_{k,n}}(I-\lambda_{k,n}G_k)$ is nonexpansive. Thus Υ_n^k is nonexpansive. From Lemma 3.67(1), we have $\tilde{x} = \Upsilon_n^N \tilde{x}$. It follows that

$$\|y_n - \tilde{x}\| = \|\Upsilon_n^N u_n - \Upsilon_n^N \tilde{x}\| \le \|u_n - \tilde{x}\| \le \|x_n - \tilde{x}\|. \tag{3.133}$$

Setting $v_n = P_C(y_n - \xi_n B y_n)$ and $I - \xi_n B$ is a nonexpansive mapping, we obtain

$$\begin{aligned}
\|v_n - \tilde{x}\| &= \|P_C(y_n - \xi_n B y_n) - P_C(\tilde{x} - \xi_n B\tilde{x})\| \\
&\le \|(y_n - \xi_n B y_n) - (\tilde{x} - \xi_n B \tilde{x})\| \\
&= \|(I - \xi_n B) y_n - (I - \xi_n B)\tilde{x}\| \\
&\le \|y_n - \tilde{x}\| \le \|x_n - \tilde{x}\|.
\end{aligned} \tag{3.134}$$

From (3.130) and (3.134), we deduce that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - \tilde{x}\| \\
&\le \alpha_n \|f(x_n) - \tilde{x}\| + \beta_n \|x_n - \tilde{x}\| + \gamma_n \|v_n - \tilde{x}\| \qquad (3.135) \\
&\le \alpha_n \|f(x_n) - f(\tilde{x})\| + \alpha_n \|f(\tilde{x}) - \tilde{x}\| + \beta_n \|x_n - \tilde{x}\| + \gamma_n \|x_n - \tilde{x}\| \\
&\le \alpha_n \psi \|x_n - \tilde{x}\| + \alpha_n \|f(\tilde{x}) - \tilde{x}\| + (1-\alpha_n)\|x_n - \tilde{x}\| \\
&\le \big(1 - \alpha_n(1-\psi)\big)\|x_n - \tilde{x}\| + \alpha_n \|f(\tilde{x}) - \tilde{x}\| \\
&= \big(1 - \alpha_n(1-\psi)\big)\|x_n - \tilde{x}\| + \alpha_n(1-\psi)\frac{\|f(\tilde{x}) - \tilde{x}\|}{(1-\psi)} \\
&\le \max\left\{\|x_n - \tilde{x}\|, \frac{\|f(\tilde{x}) - \tilde{x}\|}{1-\psi}\right\}.
\end{aligned}$$

It follows from induction that

$$\|x_n - \tilde{x}\| \le \max\left\{\|x_1 - \tilde{x}\|, \frac{\|f(\tilde{x}) - \tilde{x}\|}{1-\psi}\right\}, \quad \forall n \ge 1.$$

Therefore, $\{x_n\}$ is bounded and hence so are $\{v_n\}$, $\{y_n\}$, $\{u_n\}$, $\{By_n\}$ and $\{S_n v_n\}$.

Step 2. We claim that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

By the definition of T_r, $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we get

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all} \quad y \in H \tag{3.136}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \text{for all} \quad y \in H. \tag{3.137}$$

Take $y = u_{n+1}$ in (3.136) and $y = u_n$ in (3.137), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and hence

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\
&\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|,
\end{aligned} \tag{3.138}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$.

Notice from Lemma 3.68 that

$$
\begin{aligned}
& \|y_{n+1}-y_n\| \\
= \;& \|\Upsilon_{n+1}^N u_{n+1}-\Upsilon_n^N u_n\| \\
\leq \;& \|(u_{n+1}-\lambda_{k,n+1}G_k\Upsilon_{n+1}^k u_{n+1})-(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)\| \\
& +\|J_{M_k},\lambda_{k,n+1}(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)-J_{M_k},\lambda_{k,n}(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)\| \\
\leq \;& \|u_{n+1}-u_n\|+|\lambda_{k,n+1}-\lambda_{k,n}|\|G_k\Upsilon_n^k u_n\| \\
& +\frac{|\lambda_{k,n+1}-\lambda_{k,n}|}{\lambda_{k,n+1}}\|J_{M_k},\lambda_{k,n+1}(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)-(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)\| \\
\leq \;& \|u_{n+1}-u_n\|+2M_2|\lambda_{k,n+1}-\lambda_{k,n}| \\
\leq \;& \|x_{n+1}-x_n\|+\frac{M_1}{c}|r_{n+1}-r_n|+2M_2|\lambda_{k,n+1}-\lambda_{k,n}|,
\end{aligned}
\tag{3.139}
$$

where M_2 is an appropriate constant such that

$$
\begin{aligned}
M_2 \;=\; & \max\Bigg\{\sup_{n\geq 1}\{\|G_k\Upsilon_n^k u_n\|\}, \\
& \sup_{n\geq 1}\Bigg\{\frac{\|J_{M_k},\lambda_{k,n+1}(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)-(u_n-\lambda_{k,n}G_k\Upsilon_n^k u_n)\|}{J_{M_k},\lambda_{k,n+1}}\Bigg\}\Bigg\}.
\end{aligned}
$$

Since $I-\xi_n B$ is nonexpansive mappings, we have the following estimates:

$$
\begin{aligned}
\|v_{n+1}-v_n\| &\leq \|P_C(y_{n+1}-\xi_{n+1}By_{n+1})-P_C(y_n-\xi_n By_n)\| \\
&\leq \|(y_{n+1}-\xi_{n+1}By_{n+1})-(y_n-\xi_n By_n)\| \\
&= \|(y_{n+1}-\xi_{n+1}By_{n+1})-(y_n-\xi_{n+1}By_n)+(\xi_n-\xi_{n+1})By_n\| \\
&\leq \|(y_{n+1}-\xi_{n+1}By_{n+1})-(y_n-\xi_{n+1}By_n)\|+|\xi_n-\xi_{n+1}|\|By_n\| \\
&= \|(I-\xi_{n+1}B)y_{n+1}-(I-\xi_{n+1}B)y_n\|+|\xi_n-\xi_{n+1}|\|By_n\| \\
&\leq \|y_{n+1}-y_n\|+|\xi_n-\xi_{n+1}|\|By_n\|.
\end{aligned}
\tag{3.140}
$$

Substituting (3.139) into (3.140), we obtain

$$
\begin{aligned}
\|v_{n+1}-v_n\| \leq\; & \|x_{n+1}-x_n\|+\frac{M_1}{c}|r_{n+1}-r_n|+2M_2|\lambda_{k,n+1}-\lambda_{k,n}| \\
& +|\xi_n-\xi_{n+1}|\|By_n\|.
\end{aligned}
\tag{3.141}
$$

Indeed, define $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$ for all $n\in\mathbb{N}$. It follows that

$$
z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}=\frac{\alpha_n f(x_n)+\gamma_n S_n v_n}{1-\beta_n}.
$$

Thus, we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \left\|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}S_{n+1}v_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n S_n v_n}{1-\beta_n}\right\| \\
&= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(S_{n+1}v_{n+1}-S_n v_n)\right. \\
&\quad \left.+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)S_n v_n\right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|S_{n+1}v_{n+1}-S_n v_n\| \\
&\quad +\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)-S_n v_n\| \\
&\leq \frac{\psi\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|S_{n+1}v_{n+1}-S_n v_n\| \\
&\quad +\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)-S_n v_n\|. \qquad (3.142)
\end{aligned}$$

Now, compute

$$\begin{aligned}
\|S_{n+1}v_{n+1}-S_n v_n\| &\leq \|S_{n+1}v_{n+1}-S_{n+1}v_n\|+\|S_{n+1}v_n-S_n v_n\| \\
&\leq \|v_{n+1}-v_n\|+\|S_{n+1}v_n-S_n v_n\| \\
&\leq \|x_{n+1}-x_n\|+\frac{M_1}{c}|r_{n+1}-r_n|+|\xi_n-\xi_{n+1}|\|By_n\| \\
&\quad +2M_2|\lambda_{k,n+1}-\lambda_{k,n}|+\|S_{n+1}v_n-S_n v_n\|. \qquad (3.143)
\end{aligned}$$

Combining (3.142) and (3.143), we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &\leq \frac{\psi\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left\{\|x_{n+1}-x_n\|+\frac{M_1}{c}|r_{n+1}-r_n|\right. \\
&\quad \left.+|\xi_n-\xi_{n+1}|\|By_n\|+2M_2|\lambda_{k,n+1}-\lambda_{k,n}|+\|S_{n+1}v_n-S_n v_n\|\right\} \\
&\quad +\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)-S_n v_n\| \\
&\leq \|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left\{\frac{M_1}{c}|r_{n+1}-r_n|+|\xi_n-\xi_{n+1}|\|By_n\|\right. \\
&\quad \left.+2M_2|\lambda_{k,n+1}-\lambda_{k,n}|\right\}+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|S_{n+1}v_n-S_n v_n\| \\
&\quad +\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)-S_n v_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \\
\leq \; & \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Big\{\frac{M_1}{c}|r_{n+1}-r_n| + |\xi_n-\xi_{n+1}|\|By_n\| + 2M_2|\lambda_{k,n+1}-\lambda_{k,n}|\Big\} \\
& + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|S_{n+1}v_n - S_n v_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)-S_n v_n\| \\
\leq \; & \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Big\{\frac{M_1}{c}|r_{n+1}-r_n| + |\xi_n-\xi_{n+1}|\|By_n\| + 2M_2|\lambda_{k,n+1}-\lambda_{k,n}|\Big\} \\
& + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\sup\Big\{\|S_{n+1}z - S_n z\| : z \in \{v_n\}\Big\} \\
& + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)-S_n v_n\|.
\end{aligned}$$

This together with conditions (C1)-(C6) and $\lim_{n\longrightarrow\infty}\sup\Big\{\|S_{n+1}z - S_n z\| : z \in \{v_n\}\Big\} = 0$ imply that

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Hence, by Lemma 2.12, we obtain $\|z_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. It then follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.144}$$

By (3.141), we also have

$$\lim_{n\longrightarrow\infty}\|v_{n+1}-v_n\| = 0. \tag{3.145}$$

Step 3. We claim that $\lim_{n\longrightarrow\infty}\|Sv_n - v_n\| = 0$.

Since $\{G_k : k = 1,2,3,\dots N\}$ is β-inverse strongly monotone mappings, by the choice of $\{\lambda_{k,n}\}$ for given $\tilde{x} \in \Omega$ and $k \in \{0, 1, 2, \ldots, N-1\}$, we also have

$$
\begin{aligned}
& \|\Upsilon_n^{k+1} u_n - \tilde{x}\|^2 \\
= \;& \|J_{M_{k+1},\lambda_{k+1,n}}(I - \lambda_{k+1,n} G_{k+1})\Upsilon_n^k u_n - J_{M_{k+1},\lambda_{k+1,n}}(I - \lambda_{k+1,n} G_{k+1})\tilde{x}\|^2 \\
\leq \;& \|(I - \lambda_{k+1,n} G_{k+1})\Upsilon_n^k u_n - (I - \lambda_{k+1,n} G_{k+1})\tilde{x}\|^2 \\
= \;& \|(\Upsilon_n^k u_n - \lambda_{k+1,n} G_{k+1}\Upsilon_n^k u_n) - (\tilde{x} - \lambda_{k+1,n} G_{k+1}\tilde{x})\|^2 \\
= \;& \|(\Upsilon_n^k u_n - \tilde{x}) - \lambda_{k+1,n}(G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x})\|^2 \\
= \;& \|\Upsilon_n^k u_n - \tilde{x}\|^2 - 2\lambda_{k+1,n}\Big\langle \Upsilon_n^k u_n - \tilde{x}, G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\Big\rangle \\
& + \lambda_{k+1,n}^2 \|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2 \\
\leq \;& \|\Upsilon_n^k u_n - \tilde{x}\|^2 - 2\lambda_{k+1,n}\beta\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\| + \lambda_{k+1,n}^2\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2 \\
\leq \;& \|u_n - \tilde{x}\|^2 - 2\lambda_{k+1,n}\beta\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\| + \lambda_{k+1,n}^2\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2 \\
\leq \;& \|x_n - \tilde{x}\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\beta)\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2. \qquad (3.146)
\end{aligned}
$$

Form (3.135), we have

$$
\begin{aligned}
& \|x_{n+1} - \tilde{x}\|^2 \\
\leq \;& \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|v_n - \tilde{x}\|^2 \\
\leq \;& \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|y_n - \tilde{x}\|^2 \\
= \;& \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|\Upsilon_n^N u_n - \tilde{x}\|^2 \\
\leq \;& \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|\Upsilon_n^{k+1} u_n - \tilde{x}\|^2 \qquad (3.147) \\
\leq \;& \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 \\
& + \gamma_n\Big\{\|x_n - \tilde{x}\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\beta)\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2\Big\} \\
\leq \;& \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + \gamma_n\lambda_{k+1,n}(\lambda_{k+1,n} - 2\beta)\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \gamma_n\lambda_{k+1,n}(2\beta - \lambda_{k+1,n})\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2 \\
\leq \;& \gamma_n c(2\beta - d)\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2 \\
\leq \;& \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n\|f(x_n) - \tilde{x}\|^2.
\end{aligned}
$$

By condition (C2), (3.144) and $\liminf_{n\to\infty}\gamma_n > 0$, we obtain

$$\lim_{n\to\infty}\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\| = 0. \tag{3.148}$$

From Lemma 3.66(2) and $I-\lambda_{k+1,n}G_{k+1}$ is nonexpansive, we have

$$\begin{aligned}
&\|\Upsilon_n^{k+1}u_n - \tilde{x}\|^2 \\
= \;& \|J_{M_{k+1},\lambda_{k+1,n}}(I-\lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - J_{M_{k+1},\lambda_{k+1,n}}(I-\lambda_{k+1,n}G_{k+1})\tilde{x}\|^2 \\
\le \;& \Big\langle (I-\lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I-\lambda_{k+1,n}G_{k+1})\tilde{x}, \Upsilon_n^{k+1}u_n - \tilde{x}\Big\rangle \\
= \;& \frac{1}{2}\Big\{\|(I-\lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I-\lambda_{k+1,n}G_{k+1})\tilde{x}\|^2 + \|\Upsilon_n^{k+1}u_n - \tilde{x}\|^2 \\
& - \|(I-\lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I-\lambda_{k+1,n}G_{k+1})\tilde{x} - (\Upsilon_n^{k+1}u_n - \tilde{x})\|^2\Big\} \\
\le \;& \frac{1}{2}\Big\{\|\Upsilon_n^k u_n - \tilde{x}\|^2 + \|\Upsilon_n^{k+1}u_n - \tilde{x}\|^2 - \|(\Upsilon_n^k u_n \\
& - \Upsilon_n^{k+1}u_n) - \lambda_{k+1,n}(G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x})\|^2\Big\} \\
\le \;& \frac{1}{2}\Big\{\|\Upsilon_n^k u_n - \tilde{x}\|^2 + \|\Upsilon_n^{k+1}u_n - \tilde{x}\|^2 - \|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|^2 \\
& - \lambda_{k+1,n}^2\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|^2 \\
& + 2\lambda_{k+1,n}\langle \Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n, G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\rangle\Big\},
\end{aligned}$$

which yields that

$$\begin{aligned}
&\|\Upsilon_n^{k+1}u_n - \tilde{x}\|^2 \\
\le \;& \|\Upsilon_n^k u_n - \tilde{x}\|^2 - \|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|^2 \\
& + 2\lambda_{k+1,n}\|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\| \\
\le \;& \|u_n - \tilde{x}\|^2 - \|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|^2 \\
& + 2\lambda_{k+1,n}\|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\| \\
\le \;& \|x_n - \tilde{x}\|^2 - \|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|^2 \\
& + 2\lambda_{k+1,n}\|\Upsilon_n^k u_n - \Upsilon_n^{k+1}u_n\|\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\|.
\end{aligned} \tag{3.149}$$

Substituting (3.149) into (3.147), we obtain

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2\\
\leq\ & \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\|\Upsilon_n^{k+1}u_n-\tilde{x}\|^2\\
\leq\ & \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\Big\{\|x_n-\tilde{x}\|^2-\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|^2\\
&+2\lambda_{k+1,n}\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|\|G_{k+1}\Upsilon_n^k u_n-G_{k+1}\tilde{x}\|\Big\}\\
\leq\ & \alpha_n\|f(x_n)-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2-\gamma_n\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|^2\\
&+2\lambda_{k+1,n}\gamma_n\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|\|G_{k+1}\Upsilon_n^k u_n-G_{k+1}\tilde{x}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\gamma_n\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|^2\\
\leq\ & \|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|)+\alpha_n\|f(x_n)-\tilde{x}\|^2\\
&+2\lambda_{k+1,n}\gamma_n\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|\|G_{k+1}\Upsilon_n^k u_n-G_{k+1}\tilde{x}\|.
\end{aligned}$$

By condition (C2), (3.144), (3.148) and $\liminf_{n\to\infty}\gamma_n>0$, we obtain

$$\lim_{n\to\infty}\|\Upsilon_n^k u_n-\Upsilon_n^{k+1}u_n\|=0. \tag{3.150}$$

For $\tilde{x}\in\Omega$, we obtain

$$\begin{aligned}
\|v_n-\tilde{x}\|^2 &= \|P_C(y_n-\xi_n By_n)-P_C(\tilde{x}-\xi_n B\tilde{x})\|^2\\
&\leq \|(y_n-\xi_n By_n)-(\tilde{x}-\xi_n B\tilde{x})\|^2\\
&= \|(y_n-\tilde{x})-\xi_n(By_n-B\tilde{x})\|^2\\
&\leq \|y_n-\tilde{x}\|^2-2\xi_n\langle y_n-\tilde{x},By_n-B\tilde{x}\rangle+\xi_n^2\|By_n-B\tilde{x}\|^2\\
&\leq \|y_n-\tilde{x}\|^2-2\xi_n\Big\{-\phi\|By_n-B\tilde{x}\|^2+\omega\|y_n-\tilde{x}\|^2\Big\}+\xi_n^2\|By_n-B\tilde{x}\|^2\\
&\leq \|y_n-\tilde{x}\|^2+2\xi_n\phi\|By_n-B\tilde{x}\|^2-2\xi_n\omega\|y_n-\tilde{x}\|^2+\xi_n^2\|By_n-B\tilde{x}\|^2\\
&\leq \|y_n-\tilde{x}\|^2+2\xi_n\phi\|By_n-B\tilde{x}\|^2-\frac{2\xi_n\omega}{\mu^2}\|By_n-B\tilde{x}\|^2+\xi_n^2\|By_n-B\tilde{x}\|^2\\
&\leq \|x_n-\tilde{x}\|^2+\left(2\xi_n\phi+\xi_n^2-\frac{2\xi_n\omega}{\mu^2}\right)\|By_n-B\tilde{x}\|^2.
\end{aligned} \tag{3.151}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n v_n-\tilde{x}\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\|S_n v_n-\tilde{x}\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\|v_n-\tilde{x}\|^2 \qquad (3.152)\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2\\ &\quad+\gamma_n\left\{\|x_n-\tilde{x}\|^2+\left(2\xi_n\phi+\xi_n^2-\frac{2\xi_n\omega}{\mu^2}\right)\|By_n-B\tilde{x}\|^2\right\}\\ &= \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\|x_n-\tilde{x}\|^2\\ &\quad+\gamma_n\left(2\xi_n\phi+\xi_n^2-\frac{2\xi_n\omega}{\mu^2}\right)\|By_n-B\tilde{x}\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2+\gamma_n\left(2\xi_n\phi+\xi_n^2-\frac{2\xi_n\omega}{\mu^2}\right)\|By_n-B\tilde{x}\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\frac{2a\omega}{\mu^2}-b^2-2b\phi\right)\gamma_n\|By_n-B\tilde{x}\|^2\\ \le\ &\left(\frac{2\xi_n\omega}{\mu^2}-\xi_n^2-2\xi_n\phi\right)\gamma_n\|By_n-B\tilde{x}\|^2\\ \le\ &\|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2+\alpha_n\|f(x_n)-\tilde{x}\|^2\\ \le\ &\|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|)+\alpha_n\|f(x_n)-\tilde{x}\|^2.\end{aligned}$$

It now follows from the last inequality, conditions (C2), (3.144) and $\liminf_{n\to\infty}\gamma_n > 0$ that

$$\lim_{n\to\infty}\|By_n-B\tilde{x}\|=0. \qquad (3.153)$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned}
\|v_n - \tilde{x}\|^2 &= \|P_C(y_n - \xi_n By_n) - P_C(\tilde{x} - \xi_n B\tilde{x})\|^2 \\
&= \|P_C(I - \xi_n B)y_n - P_C(I - \xi_n B)\tilde{x}\|^2 \\
&\leq \Big\langle (I - \xi_n B)y_n - (I - \xi_n B)\tilde{x}, v_n - \tilde{x} \Big\rangle \\
&= \frac{1}{2}\Big\{ \|(I - \alpha_n B)y_n - (I - \xi_n B)\tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 \\
&\quad - \|(I - \xi_n B)y_n - (I - \xi_n B)\tilde{x} - (v_n - \tilde{x})\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|y_n - \tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 - \|(y_n - v_n) - \xi_n(By_n - B\tilde{x})\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|y_n - \tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 \\
&\quad - \xi_n^2 \|By_n - B\tilde{x}\|^2 + 2\xi_n \langle y_n - v_n, By_n - B\tilde{x} \rangle \Big\},
\end{aligned}$$

which yields that

$$\begin{aligned}
&\|v_n - \tilde{x}\|^2 \\
\leq\ & \|y_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 + 2\xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\| \\
\leq\ & \|x_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 + 2\xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\|. \qquad (3.154)
\end{aligned}$$

Substituting(3.154) into (3.152), we obtain

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
\leq\ & \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|v_n - \tilde{x}\|^2 \qquad (3.155) \\
\leq\ & \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 \\
& + \gamma_n \Big\{ \|x_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 + 2\xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\| \Big\} \\
=\ & \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|x_n - \tilde{x}\|^2 - \gamma_n \|y_n - v_n\|^2 \\
& + 2\gamma_n \xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\| \\
\leq\ & \alpha_n \|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n \|y_n - v_n\|^2 + 2\gamma_n \xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\gamma_n \|y_n - v_n\|^2 &\leq \|x_n - x_{n+1}\| (\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n \|f(x_n) - \tilde{x}\|^2 \\
&\quad + 2\gamma_n \xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\|.
\end{aligned}$$

By condition (C2), (3.144), (3.153) and $\liminf_{n\to\infty}\gamma_n > 0$, we obtain

$$\lim_{n\to\infty}\|y_n - v_n\| = 0. \tag{3.156}$$

On the other hand, in the light of Lemma 2.55(ii) T_{r_n} is firmly nonexpansie, so we have

$$\begin{aligned}\|u_n-\tilde{x}\|^2 &= \|T_{r_n}x_n - T_{r_n}\tilde{x}\|^2\\ &\le \langle T_{r_n}x_n - T_{r_n}\tilde{x}, x_n-\tilde{x}\rangle = \langle u_n-\tilde{x}, x_n-\tilde{x}\rangle\\ &= \frac{1}{2}\big(\|u_n-\tilde{x}\|^2 + \|x_n-\tilde{x}\|^2 - \|x_n-u_n\|^2\big),\end{aligned}$$

which implies that

$$\|u_n-\tilde{x}\|^2 \le \|x_n-\tilde{x}\|^2 - \|x_n-u_n\|^2.$$

Form (3.152), we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &\le \alpha_n\|f(x_n)-\tilde{x}\|^2 + \beta_n\|x_n-\tilde{x}\|^2 + \gamma_n\|v_n-\tilde{x}\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2 + \beta_n\|x_n-\tilde{x}\|^2 + \gamma_n\|y_n-\tilde{x}\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2 + \beta_n\|x_n-\tilde{x}\|^2 + \gamma_n\|u_n-\tilde{x}\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2 + \beta_n\|x_n-\tilde{x}\|^2 + \gamma_n\Big\{\|x_n-\tilde{x}\|^2 - \|x_n-u_n\|^2\Big\}\\ &= \alpha_n\|f(x_n)-\tilde{x}\|^2 + \beta_n\|x_n-\tilde{x}\|^2 + \gamma_n\|x_n-\tilde{x}\|^2 - \gamma_n\|x_n-u_n\|^2\\ &\le \alpha_n\|f(x_n)-\tilde{x}\|^2 + \|x_n-\tilde{x}\|^2 - \gamma_n\|x_n-u_n\|^2.\end{aligned} \tag{3.157}$$

It follows that

$$\gamma_n\|x_n-u_n\|^2 \le \|x_n-x_{n+1}\|(\|x_n-\tilde{x}\| + \|x_{n+1}-\tilde{x}\|) + \alpha_n\|f(x_n)-\tilde{x}\|^2.$$

By condition (C2), (3.144) and $\liminf_{n\to\infty}\gamma_n > 0$, we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.158}$$

Observe that

$$x_{n+1}-x_n = \alpha_n(f(x_n)-x_n) + \gamma_n(S_nv_n - x_n).$$

By condition (C2) and (3.144), we have

$$\lim_{n\longrightarrow\infty}\gamma_n\|S_nv_n - x_n\| = \lim_{n\longrightarrow\infty}(\|x_{n+1} - x_n\| - \alpha_n\|f(x_n) - x_n\|) = 0. \quad (3.159)$$

Since

$$\|Sv_n - u_n\| \leq \|Sv_n - x_n\| + \|x_n - u_n\|.$$

From (3.158) and (3.159), we have

$$\lim_{n\longrightarrow\infty}\|S_nv_n - u_n\| = 0. \quad (3.160)$$

Form (3.157), we have

$$\begin{aligned}
& \|x_{n+1} - \tilde{x}\|^2 \\
\leq\ & \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n\|x_n - u_n\|^2 \\
\leq\ & \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n\|(x_n - y_n) + (y_n - u_n)\|^2 \\
\leq\ & \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 \\
& - \gamma_n\Big\{\|x_n - y_n\|^2 + 2\|x_n - y_n\|\|y_n - u_n\| + \|y_n - u_n\|^2\Big\} \\
=\ & \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 \\
& - \gamma_n\|x_n - y_n\|^2 - 2\gamma_n\|x_n - y_n\|\|y_n - u_n\| - \gamma_n\|y_n - u_n\|^2 \\
\leq\ & \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n\|x_n - y_n\|^2.
\end{aligned}$$

It follows that

$$\gamma_n\|x_n - y_n\|^2 \leq \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n\|f(x_n) - \tilde{x}\|^2.$$

By condition (C2), (3.144) and $\liminf_{n\to\infty}\gamma_n > 0$, we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \quad (3.161)$$

Since

$$\|u_n - y_n\| \leq \|u_n - x_n\| + \|x_n - y_n\|.$$

From (3.158) and (3.161), we have

$$\lim_{n\longrightarrow\infty}\|u_n - y_n\| = 0. \tag{3.162}$$

Furthermore, by the triangular inequality we also have

$$\|S_n v_n - v_n\| \leq \|S_n v_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\|. \tag{3.163}$$

From (3.156), (3.160) and (3.162), we have

$$\lim_{n\longrightarrow\infty}\|S_n v_n - v_n\| = 0. \tag{3.164}$$

Applying Lemma 3.69 and (3.164), we have

$$\begin{aligned}\|Sv_n - v_n\| &\leq \|Sv_n - S_n v_n\| + \|S_n v_n - v_n\| \\ &\leq \sup\Big\{\|Sz - S_n z\| : z \in \{v_n\}\Big\} + \|S_n v_n - v_n\| \longrightarrow 0.\end{aligned}$$

Step 4. We claim that $\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \leq 0$. Indeed, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, Sv_n - x^*\rangle = \lim_{i\longrightarrow\infty}\langle f(x^*) - x^*, Sv_{n_i} - x^*\rangle. \tag{3.165}$$

Without loss of generality, let $\{v_{n_i}\} \rightharpoonup z \in C$. From $\|Sv_n - v_n\| \longrightarrow 0$, we obtain $Sv_{n_i} \rightharpoonup z$. Then, (3.165) reduces to

$$\limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, Sv_n - x^*\rangle = \langle f(x^*) - x^*, z - x^*\rangle.$$

In order to show $\langle f(x^*) - x^*, z - x^*\rangle \leq 0$, it suffices to show that

$$z \in \Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \Big(\bigcap_{k=1}^{N} I(G_k, M_k)\Big) \cap VI(C, B) \cap EP(F)$$

Firstly, we will show $z \in F(S) = \bigcap_{n=1}^{\infty} F(S_n)$.

Assume $z \notin F(S)$. By Opial's theorem(Lemma 2.25) and $\|Sv_n - v_n\| \longrightarrow 0$, we have

$$\begin{aligned}\liminf_{i\longrightarrow\infty}\|v_{n_i}-z\| &< \liminf_{i\longrightarrow\infty}\|v_{n_i}-Sz\|\\ &= \liminf_{i\longrightarrow\infty}\|v_{n_i}-Sv_{n_i}+Sv_{n_i}-Sz\|\\ &= \liminf_{i\longrightarrow\infty}\|Sv_{n_i}-Sz\|\\ &\le \liminf_{i\longrightarrow\infty}\|v_{n_i}-z\|.\end{aligned}$$

This is a contradiction. Thus, we obtain $z \in F(S)$.

Next, we will show that $z \in VI(C,B)$.

Let

$$Tw_1 = \begin{cases} Bw_1 + N_C w_1, & w_1 \in C;\\ \emptyset, & w_1 \notin C.\end{cases}$$

Since B is relaxed (ϕ,ω)-cocoercive, μ-Lipschitz continuous with $\omega > \phi\mu^2$, we obtain

$$\langle Bx-By, x-y\rangle \ge (-\phi)\|Bx-By\|^2+\omega\|x-y\|^2 \ge (\omega-\phi\mu^2)\|x-y\|^2 \ge 0, \quad (3.166)$$

which yields that B is monotone. Then T is maximal monotone (see [44]). Let $(w_1,w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_C(w_1)$ and $v_n \in C$, we have $\langle w_1 - v_n, w_2 - Bw_1\rangle \ge 0$. On the other hand, from $v_n = P_C(y_n - \xi_n By_n)$, we have

$$\big\langle w_1 - v_n, v_n - (y_n - \xi_n By_n)\big\rangle \ge 0 \quad (3.167)$$

that is,

$$\left\langle w_1 - v_n, \frac{v_n - y_n}{\xi_n} + By_n \right\rangle \ge 0. \quad (3.168)$$

Therefore, we obtain

$$\begin{aligned}
\langle w_1 - v_{n_i}, w_2 \rangle &\geq \langle w_1 - v_{n_i}, Bw_1 \rangle \\
&\geq \langle w_1 - v_{n_i}, Bw_1 \rangle - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} + By_{n_i} \right\rangle \\
&= \left\langle w_1 - v_{n_i}, Bw_1 - By_{n_i} - \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} \right\rangle \\
&= \langle w_1 - v_{n_i}, Bw_1 - Bv_{n_i} \rangle + \langle w_1 - v_{n_i}, Bv_{n_i} - By_{n_i} \rangle \\
&\quad - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} \right\rangle \\
&\geq \langle w_1 - v_{n_i}, Bv_{n_i} \rangle - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} + By_{n_i} \right\rangle \\
&= \langle w_1 - v_{n_i}, Bv_{n_i} - By_{n_i} \rangle - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} \right\rangle. \qquad (3.169)
\end{aligned}$$

Noting that $\|v_{n_i} - y_{n_i}\| \longrightarrow 0$ and B is relaxed (ϕ, ω)-cocoercive and (3.169), we obtain

$$\langle w_1 - z, w_2 \rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in VI(C, B)$.

Now, we will show that $z \in \bigcap_{k=1}^{N} I(G_k, M_k)$.

For this purpose, let $k \in \{1, 2, 3, \ldots, N\}$ and G_k is β-inverse strongly monotone, G_k is an $\frac{1}{\beta}$-Lipschitz continuous monotone mapping. It follows from Lemma 3.65, we know that $M_k + G_k$ is maximal monotone. Let $(v, g) \in G(M_k + G_k)$, that is, $g - G_k v \in M_k(v)$. On the other hand, since $\Upsilon_{n_i}^k u_{n_i} = J_{M_k, \lambda_{k,n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \lambda_{k,n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i})$, we have

$$\Upsilon_{n_i}^k u_{n_i} - \lambda_{k,n_i} G_k \Upsilon_{n_i}^k u_{n_i} \in (I + \lambda_{k,n_i} M_k)(\Upsilon_{n_i}^k u_{n_i}),$$

that is,

$$\frac{1}{\lambda_{k,n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i} - \lambda_{k,n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i}) \in M_k(\Upsilon_{n_i}^k u_{n_i}). \qquad (3.170)$$

By virtue of the maximal monotonicity of $M_k + G_k$, we have

$$\left\langle v - \Upsilon_{n_i}^k u_{n_i}, g - G_k v - \frac{1}{\lambda_{k,n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i} - \lambda_{k,n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i}) \right\rangle \geq 0, \quad (3.171)$$

and so

$$\begin{aligned}
&\left\langle v - \Upsilon_{n_i}^k u_{n_i}, g \right\rangle \\
\geq\ & \left\langle v - \Upsilon_{n_i}^k u_{n_i}, G_k v + \frac{1}{\lambda_{k,n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i} - \lambda_{k,n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i}) \right\rangle \\
=\ & \Bigg\langle v - \Upsilon_{n_i}^k u_{n_i}, G_k v - G_k \Upsilon_{n_i}^k u_{n_i} + G_k \Upsilon_{n_i}^k u_{n_i} - G_k \Upsilon_{n_i}^{k-1} u_{n_i} \\
& + \frac{1}{\lambda_{k,n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i}) \Bigg\rangle \qquad (3.172) \\
\geq\ & 0 + \langle v - \Upsilon_{n_i}^k u_{n_i}, G_k \Upsilon_{n_i}^k u_{n_i} - G_k \Upsilon_{n_i}^{k-1} u_{n_i} \rangle \\
& + \left\langle v - \Upsilon_{n_i}^k u_{n_i}, \frac{1}{\lambda_{k,n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i}) \right\rangle.
\end{aligned}$$

From $\|\Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n\| \longrightarrow 0$, we also obtain that $\Upsilon_{n_i}^k u_{n_i} \rightharpoonup z$ and $\{G_k : k = 1, 2, 3, \ldots, N\}$ are Lipschitz continuous, we have

$$\lim_{n \longrightarrow \infty} \langle v - \Upsilon_{n_i}^k u_{n_i}, g \rangle = \langle v - z, g \rangle \geq 0. \qquad (3.173)$$

Since $M_k + G_k$ is maximal monotone, we have $\theta \in (M_k + G_k)(z)$, that is, $z \in \bigcap_{k=1}^N I(G_k, M_k)$.

Finally, we will show that $z \in EP(F)$.

Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq -F(u_n, y) \geq F(y, u_n),$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup z$, it follows by (A4) that $F(y, z) \leq 0$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)z$. Since $y \in H$ and $z \in H$, we have $y_t \in H$ and hence $F(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $F(y_t, y) \geq 0$. From (A3), we have $F(z, y) \geq 0$ for all $y \in H$ and hence $z \in EP(F)$. Therefore, it follows that $z \in \Omega$.

Since $x^* = P_\Omega f(x^*)$, we have

$$\begin{aligned} \limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \limsup_{n\longrightarrow\infty}\langle f(x^*) - x^*, Sv_n - x^*\rangle \\ &= \lim_{i\longrightarrow\infty}\langle f(x^*) - x^*, Sv_{n_i} - x^*\rangle \\ &= \langle f(x^*) - x^*, z - x^*\rangle \leq 0. \end{aligned} \tag{3.174}$$

On the other hand, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle &\leq \limsup_{n\to\infty}\langle f(x^*) - x^*, x_{n+1} - x_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle. \end{aligned}$$

Since $\|x_{n+1} - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ and (3.174), we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \leq 0. \tag{3.175}$$

Step 5. We claim that $\lim_{n\longrightarrow\infty}\|x_n - x^*\| = 0$.

Indeed, from (3.130) and (3.134), we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ = \ &\langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - x^*, x_{n+1} - x^*\rangle \\ = \ &\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle + \gamma_n\langle S_n v_n - x^*, x_{n+1} - x^*\rangle \\ \leq \ &\frac{1}{2}\beta_n\Big(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\Big) + \frac{1}{2}\gamma_n\Big(\|v_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\Big) \\ &+ \alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ \leq \ &\frac{1}{2}(1 - \alpha_n)\Big(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\Big) \\ &+ \frac{1}{2}\alpha_n\Big(\|f(x_n) - f(x^*)\|^2 + \|x_{n+1} - x^*\|^2\Big) + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ \leq \ &\frac{1}{2}\Big[1 - \alpha_n(1 - \psi^2)\Big]\|x_n - x^*\|^2 + \frac{1}{2}(1 - \alpha_n)\|x_{n+1} - x^*\|^2 + \frac{1}{2}\alpha_n\|x_{n+1} - x^*\|^2 \\ &+ \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle, \end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \Big[1-\alpha_n(1-\psi^2)\Big]\|x_n-x^*\|^2+2\alpha_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle \\ &= (1-b_n)\|x_n-x^*\|^2+\delta_n, \qquad (3.176)\end{aligned}$$

where $b_n=\alpha_n(1-\psi^2)$ and $\delta_n=2\alpha_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle$. It is easy to see that $b_n\longrightarrow 0$, $\sum_{n=1}^{\infty} b_n=\infty$ and $\limsup_{n\longrightarrow\infty}\frac{\delta_n}{b_n}\leq 0$. Applying Lemma 2.12 to (3.176), we conclude that

$$x_n\longrightarrow x^*=P_\Omega f(x^*).$$

Consequently, also $\{y_n\}$ converges strongly to x^*. The proof is now complete. □

As in [[2], Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z-S_nz\|:z\in K\}<\infty$ for any bounded subset K of C by using covex combination of general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 3.71. *Let C be a nonempty closed convex subset of a real Hilbert space H and $B:C\longrightarrow H$ be relaxed (ϕ,ω)-cocoercive and μ-Lipschitz continuous with $\omega>\phi\mu^2$, for some $\phi,\omega,\mu>0$. Let $\mathcal{G}=\{G_k:k=1,2,3,\ldots,N\}$ be a finite family of β-inverse strongly monotone mappings from C into H and let F be a bifunction from $C\times C\longrightarrow\mathbb{R}$ satisfying (A1)-(A4). Let $f:C\longrightarrow C$ be a contraction with coefficient ψ $(0\leq\psi<1)$ and $\{\delta_n^k\}$ be a family of nonnegative numbers with indices $n,k\in\mathbb{N}$ with $k\leq n$ such that*

$$\Omega: F\Big(\bigcap_{k=1}^{\infty}F(T_k)\Big)\cap\Big(\bigcap_{k=1}^{N}I(G_k,M_k)\Big)\cap VI(C,B)\cap EP(F)\neq\emptyset.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1=x\in C \text{ chosen arbitrarily,} \\ y_n=J_{M_N,\lambda_{N,n}}(I-\lambda_{N,n}G_n)\ldots J_{M_2,\lambda_{2,n}}(I-\lambda_{2,n}G_2)J_{M_1,\lambda_{1,n}}(I-\lambda_{1,n}G_1)T_{r_n}x_n, \\ x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_n\sum_{k=1}^{n}\delta_n^kT_kP_C(y_n-\xi_nBy_n),\ \forall n\geq 1, \end{cases}$$

where $\{\alpha_n\},\{\beta_n\}$, $\{\gamma_n\}\subset(0,1)$ and $\{\xi_n\},\{r_n\}\subset(0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$,

(C4) $\{\xi_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(\omega - \phi\mu^2)}{\mu^2}$ *and* $\lim_{n \longrightarrow \infty} |\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_{k,n}\}_{k=1}^{N} \subset [c, d] \subset (0, 2\beta)$ *and* $\lim_{n \to \infty} |\lambda_{k,n+1} - \lambda_{k,n}| = 0$, *for each* $k \in \{1, 2, \ldots, N\}$,

(C6) $\liminf_{n \longrightarrow \infty} r_n > 0$ *and* $\lim_{n \longrightarrow \infty} |r_{n+1} - r_n| = 0$,

(C7) $\sum_{k=1}^{n} \delta_n^k, \ \forall n \in \mathbb{N}, \lim_{n \longrightarrow \infty} \delta_n^k > 0, \ \forall k \in \mathbb{N}$ *and* $\sum_{n=1}^{k} \sum_{k=1}^{n} |\delta_{n+1}^k - \delta_n^k| < \infty$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$.

In Theorem 3.70 taking $N = 1$ and $S_n = S$, then we have the following corollary.

Corollary 3.72. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $B : C \longrightarrow H$ *be relaxed* (ϕ, ω)*-cocoercive and* μ*-Lipschitz continuous with* $\omega > \phi\mu^2$, *for some* $\phi, \omega, \mu > 0$. *Let* G *be an* β*-inverse strongly monotone mappings from* C *into* H *and let* F *be a bifunction from* $C \times C \longrightarrow \mathbb{R}$ *satisfying (A1)-(A4). Let* $f : C \longrightarrow C$ *be a contraction with coefficient* ψ $(0 \leq \psi < 1)$ *and* S *be a nonexpansive mappings of* C *into itself such that*

$$\Omega : F(S) \cap I(G, M) \cap VI(C, B) \cap EP(F) \neq \emptyset.$$

Let the sequences $\{x_n\}$ *and* $\{y_n\}$ *be generated by*

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda_n}(I - \lambda_n G)u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SP_C(y_n - \xi_n By_n),\ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *and* $\{\xi_n\}, \{r_n\} \subset (0,\infty)$ *satisfy the following conditions:*

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C4) $\{\xi_n\} \subset [a,b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(\omega-\phi\mu^2)}{\mu^2}$ *and* $\lim_{n\longrightarrow\infty} |\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_n\} \subset [c,d] \subset (0,2\beta)$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(C6) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$.

CHAPTER IV

CONCLUSIONS AND OUTPUTS

4.1 Conclusions

The following results are all main theorems of this research:

(1). Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3, \ldots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive mappings of C into itself and let B be ξ-inverse strongly monotone such that

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \alpha_n B x_n), \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} y_n, \\ x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n A\big) P_C(W_n u_n - \lambda_n B W_n u_n), \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (3.24) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{r_{k,n}\}, k \in \{1, 2, 3, \ldots, M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$,

(C3) $\{\alpha_n\}, \{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \lambda_n = 0$, (C4) $\{\delta_n\} \subset [0, b]$, for some $b \in (0, 1)$ and $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,

(C5) $\liminf_{n\longrightarrow\infty} r_{k,n} > 0$ and $\lim_{n\longrightarrow\infty} |r_{k,n+1} - r_{k,n}| = 0$ for each $k \in \{1, 2, 3, \ldots, M\}$,

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$, which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

(2). Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3, \ldots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be ξ-inverse strongly monotone such that

$$\Theta := \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0, 1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \alpha_n B x_n), \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} y_n, \\ x_{n+1} = \epsilon_n f(u_n) + \beta_n x_n + \big(1 - \beta_n - \epsilon_n\big) P_C(u_n - \lambda_n B u_n), \ \ \forall n \geq 1, \end{cases}$$

where $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{r_{k,n}\}, k \in \{1, 2, 3, \ldots, M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\{\alpha_n\}, \{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \lambda_n = 0$,

(C4) $\{\delta_n\} \subset [0, b]$, for some $b \in (0, 1)$ and $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$, (C5) $\liminf_{n\longrightarrow\infty} r_{k,n} > 0$ and $\lim_{n\longrightarrow\infty} |r_{k,n+1}-r_{k,n}| = 0$ for each $k \in \{1, 2, 3, \ldots, M\}$,

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\Big\langle (f(z) - z, x - z \Big\rangle \geq 0, \quad \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta f(z)$.

(3). Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3, \ldots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive mappings of C into itself and let B be ξ-inverse strongly monotone such that

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n A\big) P_C(W_n u_n - \lambda_n B W_n u_n), \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (3.24) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{r_n\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C3) $\{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$ and $\lim_{n\rightarrow\infty} \lambda_n = 0$,

(C4) $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

(4). Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H5) and let $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let B, D be two ξ, β-inverse strongly monotone mapping of C into H, respectively. Let $S : C \longrightarrow C$ be a nonexpansive mapping. Let $f : C \to C$ be a contraction mapping with $\eta \in (0, 1)$ and let A be a strongly positive linear bounded operator with $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\eta}$. Assume that $\Theta := F(S) \cap VI(C, B) \cap GMEP(F, \varphi, D) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequence generated by the following iterative algorithm:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n D x_n), \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n A) P_C(S u_n - \alpha_n B S u_n), \\ x_{n+1} = (1 - \delta_n) y_n + \delta_n P_C(S y_n - \alpha_n B S y_n), \ \forall n \geq 1, \end{cases}$$

where $\{\delta_n\}$, $\{\beta_n\}$ be two sequences in $(0, 1)$ satisfy the following conditions:

(C1) $\lim\limits_{n \longrightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2) $\{\delta_n\} \subset [0, b]$, for some $b \in (0, 1)$ and $\lim\limits_{n \longrightarrow \infty} |\delta_{n+1} - \delta_n| = 0$,

(C3) $\{\lambda_n\} \subset [c, d] \subset (0, 2\beta)$ and $\lim\limits_{n \longrightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0$,

(C4) $\{\alpha_n\} \subset [e, g] \subset (0, 2\xi)$ and $\lim\limits_{n \longrightarrow \infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$, which is the unique solution of the variational inequality

$$\Big\langle \gamma f(z) - Az, \ x - z \Big\rangle \leq 0, \ \ \forall x \in \Theta.$$

(5). Let C be a nonempty closed convex subset of a real Hilbert space H, let B be ξ-inverse-strongly monotone mapping of C into H and $S : C \longrightarrow C$ be a nonexpansive mapping. Let $f : C \to C$ be a contraction mapping with $\eta \in (0, 1)$ and let A be a strongly positive linear bounded operator with $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\eta}$. Assume that $\Theta := F(S) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequence generated by the following iterative algorithm:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n A) P_C(Sx_n - \alpha_n BSx_n), \\ x_{n+1} = (1 - \delta_n) y_n + \delta_n P_C(Sy_n - \alpha_n BSy_n), \ \ \forall n \geq 1, \end{cases}$$

where $\{\delta_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$ satisfy the following conditions:

(C1) $\lim\limits_{n \longrightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2) $\{\delta_n\} \subset [0, b]$,for some $b \in (0, 1)$ and $\lim\limits_{n \longrightarrow \infty} |\delta_{n+1} - \delta_n| = 0$,

(C3) $\{\alpha_n\} \subset [e, g] \subset (0, 2\xi)$ and $\lim\limits_{n \longrightarrow \infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$, which is the unique solution of the variational inequality

$$\Big\langle \gamma f(z) - Az, \ x - z \Big\rangle \leq 0, \ \ \forall x \in \Theta.$$

(6). Let C be a nonempty closed convex subset of a real Hilbert space H and let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H5). Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \to C$ be a contraction mapping with $\eta \in (0, 1)$. Assume that $\Theta := F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequence generated by the following iterative algorithm:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ y_n = \beta_n f(x_n) + (1 - \beta_n) ST^F_{\lambda_n} x_n, \\ x_{n+1} = (1 - \delta_n) y_n + \delta_n Sy_n, \ \forall n \geq 1, \end{cases}$$

where $\{\delta_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\lim\limits_{n\longrightarrow\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2) $\{\delta_n\} \subset [0,b]$, for some $b \in (0,1)$ and $\lim\limits_{n\longrightarrow\infty} |\delta_{n+1} - \delta_n| = 0$,

(C3) $\lim\limits_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$.

(7). Let C be a nonempty closed convex subset of a real Hilbert space H and $B : C \longrightarrow H$ be relaxed (ϕ,ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi, \omega, \mu > 0$. Let $\mathcal{G} = \{G_k : k = 1,2,3,\dots,N\}$ be a finite family of β-inverse strongly monotone mappings from C into H and let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $f : C \longrightarrow C$ be a contraction with coefficient ψ $(0 \le \psi < 1)$ and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that

$$\Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \Big(\bigcap_{k=1}^{N} I(G_k, M_k)\Big) \cap VI(C,B) \cap EP(F) \neq \emptyset.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily}, \\ y_n = J_{M_N,\lambda_{N,n}}(I - \lambda_{N,n}G_n)\dots J_{M_2,\lambda_{2,n}}(I - \lambda_{2,n}G_2)J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}G_1)T_{r_n}x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(y_n - \xi_n B y_n),\ \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\{\xi_n\}, \{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \le \limsup_{n\longrightarrow\infty} \beta_n < 1$, (C4) $\{\xi_n\} \subset [a, b]$ for some a, b with $0 \leq a \leq b \leq \frac{2(\omega-\phi\mu^2)}{\mu^2}$ and $\lim_{n\longrightarrow\infty} |\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_{k,n}\}_{k=1}^{N} \subset [c, d] \subset (0, 2\beta)$ and $\lim_{n\to\infty} |\lambda_{k,n+1} - \lambda_{k,n}| = 0$, for each $k \in \{1, 2, \dots, N\}$,

(C6) $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of C. Let S be a mapping of C into itself defined by $Sy = \lim_{n\longrightarrow\infty} S_n y$ for all $y \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

(8).Let C be a nonempty closed convex subset of a real Hilbert space H and $B : C \longrightarrow H$ be relaxed (ϕ, ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi, \omega, \mu > 0$. Let $\mathcal{G} = \{G_k : k = 1, 2, 3, \dots, N\}$ be a finite family of β-inverse strongly monotone mappings from C into H and let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $f : C \longrightarrow C$ be a contraction with coefficient ψ $(0 \leq \psi < 1)$ and $\{\delta_n^k\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that

$$\Omega : F\Big(\bigcap_{k=1}^{\infty} F(T_k)\Big) \cap \Big(\bigcap_{k=1}^{N} I(G_k, M_k)\Big) \cap VI(C, B) \cap EP(F) \neq \emptyset.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily,} \\ y_n = J_{M_N,\lambda_{N,n}}(I - \lambda_{N,n}G_n) \dots J_{M_2,\lambda_{2,n}}(I - \lambda_{2,n}G_2)J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}G_1)T_{r_n}x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{k=1}^{n} \delta_n^k T_k P_C(y_n - \xi_n B y_n), \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $\{\xi_n\}, \{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$, (C2) $\lim_{n\longrightarrow\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\longrightarrow\infty}\beta_n \leq \limsup_{n\longrightarrow\infty}\beta_n < 1$,

(C4) $\{\xi_n\} \subset [a,b]$ for some a, b with $0 \leq a \leq b \leq \frac{2(\omega-\phi\mu^2)}{\mu^2}$ and $\lim_{n\longrightarrow\infty}|\xi_{n+1}-\xi_n| = 0$,

(C5) $\{\lambda_{k,n}\}_{k=1}^{N} \subset [c,d] \subset (0,2\beta)$ and $\lim_{n\rightarrow\infty}|\lambda_{k,n+1}-\lambda_{k,n}| = 0$, for each $k \in \{1,2,\dots,N\}$,

(C6) $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\lim_{n\longrightarrow\infty}|r_{n+1}-r_n| = 0$,

(C7) $\sum_{k=1}^{n}\delta_n^k$, $\forall n \in \mathbb{N}$, $\lim_{n\longrightarrow\infty}\delta_n^k > 0$, $\forall k \in \mathbb{N}$ and $\sum_{n=1}^{k}\sum_{k=1}^{n}|\delta_{n+1}^k - \delta_n^k| < \infty$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

(9). Let C be a nonempty closed convex subset of a real Hilbert space H and $B : C \longrightarrow H$ be relaxed (ϕ,ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi, \omega, \mu > 0$. Let G be an β-inverse strongly monotone mappings from C into H and let F be a bifunction from $C \times C \longrightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $f : C \longrightarrow C$ be a contraction with coefficient ψ $(0 \leq \psi < 1)$ and S be a nonexpansive mappings of C into itself such that

$$\Omega : F(S) \cap I(G,M) \cap VI(C,B) \cap EP(F) \neq \emptyset.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda_n}(I - \lambda_n G)u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SP_C(y_n - \xi_n By_n),\ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\{\xi_n\}, \{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(C4) $\{\xi_n\} \subset [a, b]$ for some a, b with $0 \leq a \leq b \leq \frac{2(\omega-\phi\mu^2)}{\mu^2}$ and $\lim_{n\longrightarrow\infty} |\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_n\} \subset [c, d] \subset (0, 2\beta)$ and $\lim_{n\rightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(C6) $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

4.2 Outputs

The three-published papers in international journals (MRG5480206)

1. Nawitcha Onjai-uea, **Chaichana Jaiboon** and Poom Kumam, A relaxed hybrid steepest descent methods for common solutions of generalized mixed equilibrium problems and fixed point problems, Fixed Point Theory and Applications 2011, 2011:32 **(ISI, 2010 impact factor 1.9436)**

2. N. Onjai-uea, **C. Jaiboon**, P. Kumam and U.W. Humphries, Convergence of iterative sequences for fixed points of an infinite family of nonexpansive mappings based on a hybrid steepest descent methods. Journal of Inequalities and Applications 2012, 2012:101 **(ISI, 2010 impact factor 0.88)**

3. **C. Jaiboon** and P. Kumam, Viscosity approximation method for system of variational inclusions problems and fixed point problems of a countable family of nonexpansive mappings. Volume 2012, Article ID 816529, 26 pages, 2012 **(ISI, 2010 impact factor 0. 630)**

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