



รายงานวิจัยฉบับสมบูรณ์

โครงการวิจัยเรื่องการจำแนกประเภทเชิงกลุ่มของสมการของ
ไฮลหนึ่งมิติซึ่งมีพลังงานภายในอยู่ในรูป

$$\zeta(\rho, s)\sqrt{\alpha} + \varphi(\rho, |\nabla\rho|) + \phi(\rho, s)$$

โดยนางประครอง วรกา และคณะ

พฤษภาคม 2556

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โครงการวิจัยเรื่องการจำแนกประเภทเชิงกลุ่มของสมการของไหลหนึ่งมิติซึ่งมี
พลังงานภายในอยู่ในรูป

$$\zeta(\rho, s)\sqrt{\alpha} + \varphi(\rho, |\nabla\rho|) + \phi(\rho, s)$$

ผู้จัดทำ

นางประครอง วรรณ มหาวิทยลัยเกษตรศาสตร์

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษาและสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

การวิเคราะห์เชิงกลุ่มเกี่ยวข้องกับการหาผลเฉลยแท้จริงของแบบจำลองทางคณิตศาสตร์ที่มีฟังก์ชันไม่ทราบค่าใดๆ อยู่ในแบบจำลอง โดยการวิเคราะห์เชิงกลุ่มจะจำแนกแบบจำลองออกเป็นหมวดหมู่ตามกลุ่มยอมรับของดี การวิจัยครั้งนี้จะนำการวิเคราะห์เชิงกลุ่มมาจำแนกแบบจำลองที่เป็นสมการของไหลหนึ่งมิติที่มีพลังงานภายในขึ้นอยู่กับความหนาแน่น เกรเดียนต์ของความหนาแน่น และเอนโทรปี :

$$\begin{aligned}\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) &= 0 \\ \frac{\partial}{\partial t} (\rho s) + \frac{\partial}{\partial x} (\rho s u) &= 0 \\ \frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + \Pi) &= 0\end{aligned}$$

เมื่อ,

$$\Pi = P + \rho \lambda \rho_x^2, P = \rho^2 \frac{\partial}{\partial \rho} \varepsilon - \rho \frac{\partial}{\partial x} (\rho \lambda \rho_x), \lambda = 2 \frac{\partial}{\partial \alpha} \varepsilon, \alpha = \rho_x^2,$$

t คือเวลา, ρ_x เกรเดียนต์ของ ρ ที่ขึ้นกับ x , s คือ เอนโทรปี, $\varepsilon = \varepsilon(\rho, \alpha, s)$ พลังงานภายใน, P คือความดัน การวิจัยครั้งนี้ได้ กลุ่มสมมูลของดี กลุ่มยอมรับของดี และได้จำแนกแบบจำลองได้ 82 กลุ่มที่สอดคล้องกับกลุ่มยอมรับของดี

คำสำคัญ : สมการของไหลหนึ่งมิติ กลุ่มยอมรับของดี กลุ่มสมมูลของดี การจำแนกประเภทเชิงกลุ่ม

Abstract

Group analysis provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This research presents the group classification of one-dimensional equations of fluids with internal inertia where the specific energy is a function of density, density gradient and entropy :

$$\begin{aligned}\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) &= 0 \\ \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}(\rho su) &= 0 \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + \Pi) &= 0\end{aligned}$$

where,

$$\Pi = P + \rho\lambda\rho_x^2, P = \rho^2 \frac{\partial}{\partial \rho}\varepsilon - \rho \frac{\partial}{\partial x}(\rho\lambda\rho_x), \lambda = 2 \frac{\partial}{\partial \alpha}\varepsilon, \alpha = \rho_x^2,$$

t is the time , ρ_x is the gradient of ρ with respect to x , s is the entropy, $\varepsilon = \varepsilon(\rho, \alpha, s)$ is the specific energy, P is the pressure. The equivalence Lie group and the admitted Lie group are provided. The group classification will separate all models into 82 classes according to the admitted Lie group.

Keywords: One-dimensional equations of fluids; admitted lie group; equivalence Lie group; group classification

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Let us consider the governing equations of a medium whose specific internal energy ε is a function of density ρ , density gradient $\nabla\rho$, and entropy s . (Ref: Gavriluk, S.L. Shugrin, S.M. Media with equations of state that depend on derivatives. Journal of Applied Mechanics and Technical Physics, Val.37, No.2, Page 177-189 (1996))

$$\left. \begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho u^k) &= 0 \\ \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s u^k) &= 0 \\ \frac{\partial}{\partial t}(\rho u^j) + \frac{\partial}{\partial x^k}(\rho u^j u^k + \Pi^{jk}) &= 0 \end{aligned} \right\} \quad (1.1)$$

where,

$$\Pi^{jk} = P\delta^{jk} + \rho\lambda \frac{\partial \rho}{\partial x^j} \frac{\partial \rho}{\partial x^k}, \quad P = \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x^j}(\rho\lambda \frac{\partial \rho}{\partial x^j}), \quad \lambda = 2 \frac{\partial \varepsilon}{\partial |\nabla \rho|^2}, \quad \varepsilon = \varepsilon(\rho, |\nabla \rho|^2, s)$$

This research deals with one-dimensional equations (1.1), which are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad (1.2)$$

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}(\rho s u) = 0 \quad (1.3)$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + \Pi) = 0 \quad (1.4)$$

where,

$$\Pi = P + \rho\lambda \left(\frac{\partial \rho}{\partial x} \right)^2, \quad P = \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x}(\rho\lambda \frac{\partial \rho}{\partial x}), \quad \lambda = 2 \frac{\partial \varepsilon}{\partial \alpha}, \quad \alpha = \left(\frac{\partial \rho}{\partial x} \right)^2, \quad \varepsilon = \varepsilon(\rho, \alpha, s)$$

Consider (1.4),

$$\begin{aligned} \rho u_t + u \rho_t + (\rho u u)_x + \Pi_x &= 0 \\ \rho u_t + u \rho_t + \rho u u_x + u(\rho u)_x + \Pi_x &= 0. \end{aligned}$$

Using (1.2) one obtains

$$\rho(u_t + uu_x) + \Pi_x = 0$$

Consider (1.3),

$$\rho s_t + s\rho_t + \rho us_x + s(\rho u)_x = 0$$

By (1.2) one obtains

$$s_t + us_x = 0$$

Consider

$$\Pi = P + \rho\lambda\left(\frac{\partial\rho}{\partial x}\right)^2, \quad P = \rho^2 \frac{\partial\varepsilon}{\partial\rho} - \rho \frac{\partial}{\partial x}(\rho\lambda \frac{\partial\rho}{\partial x}), \quad \lambda = 2\frac{\partial\varepsilon}{\partial\alpha}, \quad \alpha = \left(\frac{\partial\rho}{\partial x}\right)^2,$$

One obtains

$$\begin{aligned} \Pi &= \rho^2 \varepsilon_\rho - \rho(\rho\lambda\rho_x)_x + \rho\lambda(\rho_x)^2 \\ &= \rho^2 \varepsilon_\rho - \rho(\rho 2\varepsilon_\alpha \rho_x)_x + \rho 2\varepsilon_\alpha (\rho_x)^2 \\ &= \rho^2 \varepsilon_\rho + 2\rho\varepsilon_\alpha (\rho_x)^2 - 2\rho(\rho\varepsilon_\alpha \rho_x)_x \end{aligned}$$

Then we have new equations like this

$$\rho_t + (\rho u)_x = 0 \tag{1.5}$$

$$s_t + us_x = 0 \tag{1.6}$$

$$\rho(u_t + uu_x) + \Pi_x = 0 \tag{1.7}$$

where,

$$\Pi = \rho^2 \varepsilon_\rho + 2\rho\varepsilon_\alpha (\rho_x)^2 - 2\rho(\rho\varepsilon_\alpha \rho_x)_x.$$

Equivalence Lie group

New we obtained an equivalence Lie group for equations (1.2)-(1.4), which is the following:

An infinitesimal operator X^e of the equivalence Lie group is sought in the form

$$X^e = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^\alpha \partial_\alpha + \zeta^s \partial_s + \zeta^\varepsilon \partial_\varepsilon,$$

Where all coefficients $\zeta^x, \zeta^t, \zeta^\rho, \zeta^u, \zeta^\alpha, \zeta^s, \zeta^\varepsilon$ are functions of $x, t, \rho, u, \alpha, s, \varepsilon$.

The prolonged generator is

$$\begin{aligned}\bar{X}^e = X^e &+ \zeta^{\rho_x} \partial_{\rho_x} + \zeta^{\rho_t} \partial_{\rho_t} + \zeta^{u_x} \partial_{u_x} + \zeta^{u_t} \partial_{u_t} + \zeta^{\alpha_x} \partial_{\alpha_x} + \zeta^{\alpha_t} \partial_{\alpha_t} + \zeta^{s_x} \partial_{s_x} + \zeta^{s_t} \partial_{s_t} + \\ &\zeta^{\varepsilon_x} \partial_{\varepsilon_x} + \zeta^{\varepsilon_t} \partial_{\varepsilon_t} + \zeta^{\varepsilon_\rho} \partial_{\varepsilon_\rho} + \zeta^{\varepsilon_u} \partial_{\varepsilon_u} + \zeta^{\varepsilon_\alpha} \partial_{\varepsilon_\alpha} + \zeta^{\varepsilon_s} \partial_{\varepsilon_s} + \dots\end{aligned}$$

$x, t, \rho, u, \alpha, s, \varepsilon, \rho_x, \rho_t, u_x, u_t, \alpha_x, \alpha_t, s_x, s_t, \varepsilon_x, \varepsilon_t, \varepsilon_\rho, \varepsilon_u, \varepsilon_\alpha, \varepsilon_s$ are independent

where

$$\begin{aligned}\zeta^{\rho_x} &= D_x^e \zeta^\rho - \rho_x D_x^e \zeta^x - \rho_t D_x^e \zeta^t \\ \zeta^{\rho_t} &= D_t^e \zeta^\rho - \rho_x D_t^e \zeta^x - \rho_t D_t^e \zeta^t \\ \zeta^{u_x} &= D_x^e \zeta^u - u_x D_x^e \zeta^x - u_t D_x^e \zeta^t \\ \zeta^{u_t} &= D_t^e \zeta^u - u_x D_t^e \zeta^x - u_t D_t^e \zeta^t \\ \zeta^{\alpha_x} &= D_x^e \zeta^\alpha - \alpha_x D_x^e \zeta^x - \alpha_t D_x^e \zeta^t \\ \zeta^{\alpha_t} &= D_t^e \zeta^\alpha - \alpha_x D_t^e \zeta^x - \alpha_t D_t^e \zeta^t \\ \zeta^{s_x} &= D_x^e \zeta^s - s_x D_x^e \zeta^x - s_t D_x^e \zeta^t \\ \zeta^{s_t} &= D_t^e \zeta^s - s_x D_t^e \zeta^x - s_t D_t^e \zeta^t \\ \zeta^{\varepsilon_x} &= \tilde{D}_x^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_x^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_x^e \zeta^\alpha - \varepsilon_s \tilde{D}_x^e \zeta^s \\ \zeta^{\varepsilon_t} &= \tilde{D}_t^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_t^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_t^e \zeta^\alpha - \varepsilon_s \tilde{D}_t^e \zeta^s \\ \zeta^{\varepsilon_\rho} &= \tilde{D}_\rho^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_\rho^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_\rho^e \zeta^\alpha - \varepsilon_s \tilde{D}_\rho^e \zeta^s \\ \zeta^{\varepsilon_u} &= \tilde{D}_u^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_u^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_u^e \zeta^\alpha - \varepsilon_s \tilde{D}_u^e \zeta^s \\ \zeta^{\varepsilon_\alpha} &= \tilde{D}_\alpha^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_\alpha^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_\alpha^e \zeta^\alpha - \varepsilon_s \tilde{D}_\alpha^e \zeta^s \\ \zeta^{\varepsilon_s} &= \tilde{D}_s^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_s^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_s^e \zeta^\alpha - \varepsilon_s \tilde{D}_s^e \zeta^s\end{aligned}$$

The operators D_x^e, D_t^e are operators of the total derivatives with respect to x and t , respectively in the space of the independent variables x and t :

$$\begin{aligned}D_x^e &= \partial_x + \rho_x \partial_\rho + u_x \partial_u + \alpha_x \partial_\alpha + s_x \partial_s + (\varepsilon_x + \varepsilon_\rho \rho_x + \varepsilon_\alpha \alpha_x + \varepsilon_s s_x) \partial_\varepsilon \\ D_t^e &= \partial_t + \rho_t \partial_\rho + u_t \partial_u + \alpha_t \partial_\alpha + s_t \partial_s + (\varepsilon_t + \varepsilon_\rho \rho_t + \varepsilon_\alpha \alpha_t + \varepsilon_s s_t) \partial_\varepsilon\end{aligned}$$

The operator $\tilde{D}_x^e, \tilde{D}_t^e, \tilde{D}_\rho^e, \tilde{D}_u^e, \tilde{D}_\alpha^e, \tilde{D}_s^e$ are operators of the total derivative with respect to x, t, ρ, u, α, s in the space of the independent variables x, t, ρ, u, α, s :

$$\tilde{D}_x^e = \partial_x$$

$$\tilde{D}_t^e = \partial_t$$

$$\tilde{D}_\rho^e = \partial_\rho + \varepsilon_\rho \partial_\varepsilon$$

$$\tilde{D}_u^e = \partial_u$$

$$\tilde{D}_\alpha^e = \partial_\alpha + \varepsilon_\alpha \partial_\varepsilon$$

$$\tilde{D}_s^e = \partial_s + \varepsilon_s \partial_\varepsilon$$

Since $\varepsilon_x = 0$, then

$$\bar{X}^e \varepsilon_x = 0$$

$$\zeta^{\varepsilon_x} = 0$$

$$\tilde{D}_x^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_x^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_x^e \zeta^\alpha - \varepsilon_s \tilde{D}_x^e \zeta^s = 0$$

$$\zeta_x^\varepsilon - \varepsilon_\rho \zeta_x^\rho - \varepsilon_\alpha \zeta_x^\alpha - \varepsilon_s \zeta_x^s = 0$$

So

$$\zeta_x^\rho = 0$$

$$\zeta_x^\alpha = 0$$

$$\zeta_x^s = 0$$

$$\zeta_x^\varepsilon = 0$$

Since $\varepsilon_t = 0$, then

$$\bar{X}^e \varepsilon_t = 0$$

$$\zeta^{\varepsilon_t} = 0$$

$$\tilde{D}_t^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}_t^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_t^e \zeta^\alpha - \varepsilon_s \tilde{D}_t^e \zeta^s = 0$$

$$\zeta_t^\varepsilon - \varepsilon_\rho \zeta_t^\rho - \varepsilon_\alpha \zeta_t^\alpha - \varepsilon_s \zeta_t^s = 0$$

So,

$$\zeta_t^\rho = 0$$

$$\zeta_t^\alpha = 0$$

$$\zeta_t^s = 0$$

$$\zeta_t^\varepsilon = 0$$

Since $\varepsilon_u = 0$, then

$$\bar{X}^e \varepsilon_u = 0$$

$$\zeta^{\varepsilon_u} = 0$$

$$\tilde{D}_u^e \zeta^\varepsilon - \varepsilon_\rho \tilde{D}^e \zeta^\rho - \varepsilon_\alpha \tilde{D}_u^e \zeta^\alpha - \varepsilon_s \tilde{D}_u^e \zeta^s = 0$$

$$\zeta_u^\varepsilon - \varepsilon_\rho \zeta_u^\rho - \varepsilon_\alpha \zeta_u^\alpha - \varepsilon_s \zeta_u^s = 0$$

So,

$$\zeta_u^\rho = 0$$

$$\zeta_u^\alpha = 0$$

$$\zeta_u^s = 0$$

$$\zeta_u^\varepsilon = 0$$

Now we obtain

$$\zeta^\rho = \zeta^\rho(\rho, \alpha, s, \varepsilon)$$

$$\zeta^\alpha = \zeta^\alpha(\rho, \alpha, s, \varepsilon)$$

$$\zeta^s = \zeta^s(\rho, \alpha, s, \varepsilon)$$

$$\zeta^\varepsilon = \zeta^\varepsilon(\rho, \alpha, s, \varepsilon)$$

Next all necessary calculations were carried on a computer using the symbolic manipulation program REDUCE

(**Ref** : 1.Hearn, A.C. : 1987 , REDUCE Users Mannual, ver. 3.3. Santa Moica: The Rand Corporation CP 78.)

We obtain

$$X^e = C_1 X_1^e + C_2 X_2^e + C_3 X_3^e + C_4 X_4^e + C_5 X_5^e + C_6 X_6^e + X_7^e + X_8^e + X_9^e + X_{10}^e$$

where,

$$\begin{aligned} X_1^e &= \rho \partial_\rho + 2\alpha \partial_\alpha \\ X_2^e &= t \partial_t - u \partial_u - 2\varepsilon \partial_\varepsilon \\ X_3^e &= \partial_t \\ X_4^e &= t \partial_x + \partial_u \\ X_5^e &= \partial_x \\ X_6^e &= x \partial_x + u \partial_u - 2\alpha \partial_\alpha + 2\varepsilon \partial_\varepsilon \\ X_7^e &= f(\rho) \sqrt{\alpha} \partial_\varepsilon \\ X_8^e &= g(s) \partial_\varepsilon \\ X_9^e &= \frac{1}{\rho} \partial_\varepsilon \\ X_{10}^e &= h(s) \partial_s \end{aligned}$$

Next Consider Lie Equations

$$1. \quad X_1^e = \rho \partial_\rho + 2\alpha \partial_\alpha$$

Lie equations:

$$\frac{d\tilde{x}}{da} = 0 \tag{1.8}$$

$$\frac{d\tilde{t}}{da} = 0 \tag{1.9}$$

$$\frac{d\tilde{\rho}}{da} = \tilde{\rho} \tag{1.10}$$

$$\frac{d\tilde{u}}{da} = 0 \tag{1.11}$$

$$\frac{d\tilde{\alpha}}{da} = 2\tilde{\alpha} \tag{1.12}$$

$$\frac{d\tilde{s}}{da} = 0 \tag{1.13}$$

$$\frac{d\tilde{\varepsilon}}{da} = 0 \quad (1.14)$$

$$a = 0; \tilde{x} = x, \tilde{t} = t, \tilde{\rho} = \rho, \tilde{u} = u, \tilde{\alpha} = \alpha, \tilde{s} = s, \tilde{\varepsilon} = \varepsilon$$

Consider (1.8)

$$\int d\tilde{x} = \int 0 da$$

$$\tilde{x} = 0a + c$$

Because of $a = 0; \tilde{x} = x$ one obtain

$$x = c$$

Then

$$\tilde{x} = x$$

Consider (1.9),(1.11),(1.13) and (1.14) similar to (1.8) one obtain

$$\tilde{t} = t, \tilde{u} = u, \tilde{s} = s, \tilde{\varepsilon} = \varepsilon$$

consider (1.10)

$$\int \frac{1}{\tilde{\rho}} d\tilde{\rho} = \int da$$

$$\ln \tilde{\rho} = a + \ln c$$

$$\ln \tilde{\rho} - \ln c = a$$

$$\ln \frac{\tilde{\rho}}{c} = a$$

$$\frac{\tilde{\rho}}{c} = e^a$$

$$\tilde{\rho} = ce^a$$

Since $a = 0; \tilde{\rho} = \rho$ one obtain

$$\rho = ce^0$$

So,

$$c = \rho$$

Then,

$$\tilde{\rho} = \rho e^a$$

From (1.12)

$$\begin{aligned}\int \frac{1}{\tilde{\alpha}} d\tilde{\alpha} &= \int 2da \\ \ln \tilde{\alpha} &= 2a + \ln c \\ \ln \tilde{\alpha} - \ln c &= 2a \\ \ln \frac{\tilde{\alpha}}{c} &= 2a \\ \frac{\tilde{\alpha}}{c} &= e^{2a} \\ \tilde{\alpha} &= ce^{2a}\end{aligned}$$

Since $a=0; \tilde{\alpha}=\alpha$ one obtain

$$\alpha = ce^0$$

Then,

$$\tilde{\alpha} = \alpha e^{2a}$$

So, one obtain the transformation

$$X_1^e : \begin{cases} \tilde{\rho} = \rho e^a \\ \tilde{\alpha} = \alpha e^{2a} \end{cases} \quad (1.15)$$

New we check it is an equivalence transformation

Let $b = e^a$ so,

$$X_1^e : \begin{cases} \tilde{\rho} = \rho b \\ \tilde{\alpha} = \alpha b^2 \end{cases}$$

Since $\tilde{x} = x, \tilde{t} = t, \tilde{u} = u, \tilde{s} = s, \tilde{\varepsilon} = \varepsilon$ let $\tilde{f} = \tilde{f}(\tilde{x}, \tilde{t}, \tilde{\rho}, \tilde{u}, \tilde{s}, \tilde{\varepsilon})$ then

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial x} &= \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial x}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \\ \frac{\partial \tilde{f}}{\partial t} &= \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial t}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}} \\ \frac{\partial \tilde{f}}{\partial \rho} &= \frac{\partial \tilde{f}}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} = \frac{\partial \tilde{f}}{\partial \tilde{\rho}} \frac{\partial \rho b}{\partial \rho} = \frac{b \partial \tilde{f}}{\partial \tilde{\rho}} \\ \frac{\partial \tilde{f}}{\partial \alpha} &= \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} \frac{\partial \tilde{\alpha}}{\partial \alpha} = \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} \frac{\partial \alpha b^2}{\partial \alpha} = \frac{b^2 \partial \tilde{f}}{\partial \tilde{\alpha}}\end{aligned}$$

So,

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x}, \frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t}, \frac{\partial}{\partial \tilde{\rho}} = \frac{1}{b} \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \tilde{\alpha}} = \frac{1}{b^2} \frac{\partial}{\partial \alpha}.$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = (\rho b)_t + (\rho b u)_x = b\rho_t + b(\rho u)_x = b[\rho_t + (\rho u)_x]$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = (\rho b s)_t + (\rho b s u)_x = b(\rho s)_t + b(\rho s u)_x = b[(\rho s)_t + (\rho s u)_x]$$

Since

$$\begin{aligned} \tilde{\lambda} &= 2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\alpha}} = 2 \frac{\partial \varepsilon}{b^2 \partial \alpha} = \frac{1}{b^2} \lambda \\ \tilde{P} &= \tilde{\rho}^2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\rho}} - \tilde{\rho} \frac{\partial}{\partial \tilde{x}} (\tilde{\rho} \tilde{\lambda} \frac{\partial \tilde{\rho}}{\partial \tilde{x}}) \\ &= b^2 \rho^2 \frac{\partial \varepsilon}{b \partial \rho} - \rho b \frac{\partial}{\partial x} (\rho b \frac{1}{b^2} \lambda \frac{b \partial \rho}{\partial x}) \\ &= b \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho b \frac{\partial}{\partial x} (\rho \lambda \frac{\partial \rho}{\partial x}) \\ &= b \left[\rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x} (\rho \lambda \frac{\partial \rho}{\partial x}) \right] = bP \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho} \tilde{\lambda} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right)^2 = bP + \rho b \frac{1}{b^2} \lambda \left(\frac{\partial \rho b}{\partial x} \right)^2 = bP + \rho b \frac{1}{b^2} \lambda \left(\frac{b \partial \rho}{\partial x} \right)^2 \\ &= bP + \rho b \frac{1}{b^2} \lambda b^2 \left(\frac{\partial \rho}{\partial x} \right)^2 = b \left[P + \rho \lambda \left(\frac{\partial \rho}{\partial x} \right)^2 \right] = b\Pi \end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho b u)_t + (\rho b u^2 + b\Pi)_x = b[(\rho u)_t + (\rho u^2 + \Pi)_x]$$

Hench (1.15) is an equivalence transformation.

$$2. X_2^e = t\partial_t - u\partial_u - 2\varepsilon\partial_\varepsilon$$

Consider Lie equations:

$$\frac{d\tilde{t}}{da} = \tilde{t} \tag{1.16}$$

$$\frac{d\tilde{u}}{da} = -\tilde{u} \quad (1.17)$$

$$\frac{d\tilde{\varepsilon}}{da} = -2\tilde{\varepsilon} \quad (1.18)$$

$$a = 0; \tilde{t} = t, \tilde{u} = u, \tilde{\varepsilon} = \varepsilon$$

Consider (1.16)

$$\int \frac{1}{\tilde{t}} d\tilde{t} = \int da$$

$$\ln \tilde{t} = a + \ln c$$

$$\ln \tilde{t} - \ln c = a$$

$$\ln \frac{\tilde{t}}{c} = a$$

$$\frac{\tilde{t}}{c} = e^a$$

$$\tilde{t} = ce^a$$

Since $a = 0; \tilde{t} = t$ one obtain

$$t = ce^0$$

Then,

$$\tilde{t} = te^a$$

Consider (1.17)

$$\int \frac{1}{\tilde{u}} d\tilde{u} = \int -da$$

$$\ln \tilde{u} = -a + \ln c$$

$$\ln \tilde{u} - \ln c = -a$$

$$\ln \frac{\tilde{u}}{c} = -a$$

$$\frac{\tilde{u}}{c} = e^{-a}$$

$$\tilde{u} = ce^{-a}$$

Since $a = 0; \tilde{u} = u$ one obtain

$$u = ce^0$$

Then,

$$\tilde{u} = ue^{-a}$$

Consider (1.18)

$$\begin{aligned}\int \frac{1}{\tilde{\varepsilon}} d\tilde{\varepsilon} &= \int -2da \\ \ln \tilde{\varepsilon} &= -2a + \ln c \\ \ln \tilde{\varepsilon} - \ln c &= -2a \\ \ln \frac{\tilde{\varepsilon}}{c} &= -2a \\ \frac{\tilde{\varepsilon}}{c} &= e^{-2a} \\ \tilde{\varepsilon} &= ce^{-2a}\end{aligned}$$

Since $a=0; \tilde{\varepsilon} = \varepsilon$ one obtain

$$\varepsilon = ce^0$$

Then,

$$\tilde{\varepsilon} = \varepsilon e^{-2a}$$

So, one obtain the transformation

$$X_2^e : \begin{cases} \tilde{t} = te^a \\ \tilde{u} = ue^{-a} \\ \tilde{\varepsilon} = \varepsilon e^{-2a} \end{cases} \quad (1.19)$$

New we check it is an equivalence transformation

Let $b = e^a$ so,

$$X_2^e : \begin{cases} \tilde{t} = tb \\ \tilde{u} = u \frac{1}{b} \\ \tilde{\varepsilon} = \varepsilon \frac{1}{b^2} \end{cases}$$

Since $\tilde{x} = x, \tilde{\rho} = \rho, \tilde{\alpha} = \alpha, \tilde{s} = s$ then

$$\frac{\partial \tilde{f}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial bt}{\partial t} = b \frac{\partial \tilde{f}}{\partial \tilde{t}}$$

So,

$$\frac{\partial}{\partial \tilde{t}} = \frac{1}{b} \frac{\partial}{\partial t}$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \frac{1}{b} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} \frac{1}{b} = \frac{\partial \rho}{b \partial t} + \frac{1}{b} \frac{\partial \rho u}{\partial x} = \frac{1}{b} [\rho_t + (\rho u)_x]$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = \frac{1}{b} \frac{\partial \rho s}{\partial t} + \frac{\partial \rho s u}{\partial x} \frac{1}{b} = \frac{\partial \rho s}{b \partial t} + \frac{1}{b} \frac{\partial \rho s u}{\partial x} = \frac{1}{b} [(\rho s)_t + (\rho s u)_x]$$

Since

$$\begin{aligned} \tilde{\lambda} &= 2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\alpha}} = 2 \frac{\partial \varepsilon}{\partial \alpha} \frac{1}{b^2} = 2 \frac{\partial \varepsilon}{b^2 \partial \alpha} = \frac{1}{b^2} \lambda \\ \tilde{P} &= \tilde{\rho}^2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\rho}} - \tilde{\rho} \frac{\partial}{\partial \tilde{x}} \left(\tilde{\rho} \tilde{\lambda} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right) = \rho^2 \frac{\partial \varepsilon}{\partial \rho} \frac{1}{b^2} - \rho \frac{\partial}{\partial x} \left(\rho \frac{1}{b^2} \lambda \frac{\partial \rho}{\partial x} \right) \\ &= \frac{1}{b^2} \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \frac{1}{b^2} \rho \frac{\partial}{\partial x} \left(\rho \lambda \frac{\partial \rho}{\partial x} \right) = \frac{1}{b^2} \left[\rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x} \left(\rho \lambda \frac{\partial \rho}{\partial x} \right) \right] = \frac{1}{b^2} P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho} \tilde{\lambda} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right)^2 = \frac{1}{b^2} P + \rho \frac{1}{b^2} \lambda \left(\frac{\partial \rho}{\partial x} \right)^2 = \frac{1}{b^2} \left[P + \rho \lambda \left(\frac{\partial \rho}{\partial x} \right)^2 \right] = \frac{1}{b^2} \Pi \end{aligned}$$

Consider (1.4)

$$\begin{aligned} (\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} &= \frac{1}{b} \frac{\partial \rho u}{\partial t} + \frac{\partial \left(\rho \left(u \frac{1}{b} \right)^2 + \frac{1}{b^2} \Pi \right)}{\partial x} = \frac{1}{b^2} \frac{\partial \rho u}{\partial t} + \frac{1}{b^2} \frac{\partial (\rho u^2 + \Pi)}{\partial x} \\ &= \frac{1}{b^2} [(\rho u)_t + (\rho u^2 + \Pi)_x] \end{aligned}$$

Hench (1.19) is an equivalence transformation

3. $X_3^e = \partial_t$

Consider Lie equations:

$$\frac{d\tilde{t}}{da} = 1 \quad (1.20)$$

$$a = 0; \tilde{t} = t$$

Consider (1.20)

$$\int d\tilde{t} = \int da$$

$$\tilde{t} = a + c$$

Since $a = 0; \tilde{t} = t$ one obtain

$$t = 0 + c$$

Then

$$\tilde{t} = a + t$$

So, one obtain the transformation

$$X_3^e : \tilde{t} = a + t$$

New we check it is an equivalence transformation

Since $\tilde{x} = x, \tilde{\rho} = \rho, \tilde{u} = u, \tilde{\alpha} = \alpha, \tilde{s} = s, \tilde{\varepsilon} = \varepsilon$ then

$$\frac{\partial \tilde{f}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial (a+t)}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{t}}$$

So,

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t}$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned}\tilde{\lambda} &= 2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\alpha}} = 2 \frac{\partial \varepsilon}{\partial \alpha} = \lambda \\ \tilde{P} &= \tilde{\rho}^2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\rho}} - \tilde{\rho} \frac{\partial}{\partial \tilde{x}} \left(\tilde{\rho} \tilde{\lambda} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right) = \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x} \left(\rho \lambda \frac{\partial \rho}{\partial x} \right) = P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho} \tilde{\lambda} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right)^2 = P + \rho \lambda \left(\frac{\partial \rho}{\partial x} \right)^2 = \Pi\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho u)_t + (\rho u^2 + \Pi)_x$$

Hench X_3^e is an equivalence transformation

$$4. X_4^e = t\partial_x + \partial_u$$

Consider Lie equations:

$$\frac{d\tilde{x}}{da} = \tilde{t} \tag{1.21}$$

$$\frac{d\tilde{u}}{da} = 1 \tag{1.22}$$

$$a = 0; \tilde{x} = x, \tilde{u} = u$$

Since

$$\tilde{t} = t$$

Consider (1.21)

$$\begin{aligned}\int d\tilde{x} &= \int \tilde{t} da = \int t da \\ \tilde{x} &= ta + c\end{aligned}$$

Because of $a = 0; \tilde{x} = x$ one obtain

$$x = t0 + c$$

Then

$$\tilde{x} = ta + x$$

Consider (1.22)

$$\int d\tilde{u} = \int 1 da$$

$$\tilde{u} = a + c$$

Because of $a = 0; \tilde{u} = u$ one obtain

$$u = 0 + c$$

Then

$$\tilde{u} = a + u$$

So, one obtain the transformation

$$X_4^e : \begin{cases} \tilde{x} = ta + x \\ \tilde{u} = u + a \end{cases}$$

New we check it is an equivalence transformation

Since $\tilde{t} = t, \tilde{\rho} = \rho, \tilde{\alpha} = \alpha, \tilde{s} = s, \tilde{\varepsilon} = \varepsilon$ then

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial ta + x}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}}$$

$$\frac{\partial \tilde{f}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial t} + \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial ta + x}{\partial t} + \frac{\partial \tilde{f}}{\partial \tilde{t}} \frac{\partial t}{\partial t} = a \frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial \tilde{f}}{\partial \tilde{t}} = a \frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial \tilde{t}}$$

So,

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} - a \frac{\partial}{\partial x}$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \frac{\partial \rho}{\partial t} - a \frac{\partial \rho}{\partial x} + \frac{\partial \rho(u+a)}{\partial x} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = \frac{\partial \rho s}{\partial t} - a \frac{\partial \rho s}{\partial x} + \frac{\partial \rho s(u+a)}{\partial x} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned}\tilde{\lambda} &= 2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\alpha}} = 2 \frac{\partial \varepsilon}{\partial \alpha} = \lambda \\ \tilde{P} &= \tilde{\rho}^2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\rho}} - \tilde{\rho} \frac{\partial}{\partial \tilde{x}} \left(\tilde{\rho} \tilde{\lambda} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right) = \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x} \left(\rho \lambda \frac{\partial \rho}{\partial x} \right) = P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho} \tilde{\lambda} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right)^2 = P + \rho \lambda \left(\frac{\partial \rho}{\partial x} \right)^2 = \Pi\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = \frac{\partial \rho(u+a)}{\partial t} - a \frac{\partial \rho(u+a)}{\partial x} + \frac{\partial (\rho(u+a)^2 + \Pi)}{\partial x} = (\rho u)_t + (\rho u^2 + \Pi)_x$$

because of $\rho_t + (\rho u)_x = 0$.

Hench X_4^e is an equivalence transformation.

5. $X_5^e = \partial_x$

Consider Lie equations:

$$\frac{\partial \tilde{x}}{\partial a} = 1 \tag{1.23}$$

$$a = 0; \tilde{x} = x$$

Consider (1.23)

$$\begin{aligned}\int d\tilde{x} &= \int da \\ \tilde{x} &= a + c\end{aligned}$$

Because of $a = 0; \tilde{x} = x$ one obtain

$$x = 0 + c$$

Then

$$X_5^e : \tilde{x} = a + x$$

New we check it is an equivalence transformation

Since $\tilde{t} = t, \tilde{\rho} = \rho, \tilde{u} = u, \tilde{\alpha} = \alpha, \tilde{s} = s, \tilde{\varepsilon} = \varepsilon$ then

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial x + a}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}}$$

So,

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x}$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned}\tilde{\lambda} &= 2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\alpha}} = 2 \frac{\partial \varepsilon}{\partial \alpha} = \lambda \\ \tilde{P} &= \tilde{\rho}^2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\rho}} - \tilde{\rho} \frac{\partial}{\partial \tilde{x}} \left(\tilde{\rho} \tilde{\lambda} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right) = \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x} \left(\rho \lambda \frac{\partial \rho}{\partial x} \right) = P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho} \tilde{\lambda} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right)^2 = P + \rho \lambda \left(\frac{\partial \rho}{\partial x} \right)^2 = \Pi\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho u)_t + (\rho u^2 + \Pi)_x$$

Hench X_5^e is an equivalence transformation.

$$6. X_6^e = x\partial_x + u\partial_u - 2\alpha\partial_\alpha + 2\varepsilon\partial_\varepsilon$$

Consider Lie equations:

$$\frac{d\tilde{x}}{da} = \tilde{x} \tag{1.24}$$

$$\frac{d\tilde{u}}{da} = \tilde{u} \quad (1.25)$$

$$\frac{d\tilde{\alpha}}{da} = -2\tilde{\alpha} \quad (1.26)$$

$$\frac{d\tilde{\varepsilon}}{da} = 2\tilde{\varepsilon} \quad (1.27)$$

$$a = 0; \tilde{x} = x, \tilde{u} = u, \tilde{\alpha} = \alpha, \tilde{\varepsilon} = \varepsilon$$

Consider (1.24)

$$\begin{aligned} \int \frac{1}{\tilde{x}} d\tilde{x} &= \int da \\ \ln \tilde{x} &= a + \ln c \\ \ln \tilde{x} - \ln c &= a \\ \ln \frac{\tilde{x}}{c} &= a \\ \frac{\tilde{x}}{c} &= e^a \\ \tilde{x} &= ce^a \end{aligned}$$

Because of $a = 0; \tilde{x} = x$ one obtain

$$x = ce^0$$

Then

$$\tilde{x} = xe^a$$

Consider (1.25)

$$\begin{aligned} \int \frac{1}{\tilde{u}} d\tilde{u} &= \int da \\ \ln \tilde{u} &= a + \ln c \\ \ln \tilde{u} - \ln c &= a \\ \ln \frac{\tilde{u}}{c} &= a \\ \frac{\tilde{u}}{c} &= e^a \\ \tilde{u} &= ce^a \end{aligned}$$

Since $a=0; \tilde{u}=u$ one obtain

$$u = ce^0$$

Then,

$$\tilde{u} = ue^a$$

Consider (1.26)

$$\begin{aligned}\int \frac{1}{\tilde{\alpha}} d\tilde{\alpha} &= \int -2da \\ \ln \tilde{\alpha} &= -2a + \ln c \\ \ln \tilde{\alpha} - \ln c &= -2a \\ \ln \frac{\tilde{\alpha}}{c} &= -2a \\ \frac{\tilde{\alpha}}{c} &= e^{-2a} \\ \tilde{\alpha} &= ce^{-2a}\end{aligned}$$

Since $a=0; \tilde{\alpha}=\alpha$ one obtain

$$\alpha = ce^0$$

Then,

$$\tilde{\alpha} = \alpha e^{-2a}$$

Consider (1.27)

$$\begin{aligned}\int \frac{1}{\tilde{\varepsilon}} d\tilde{\varepsilon} &= \int 2da \\ \ln \tilde{\varepsilon} &= 2a + \ln c \\ \ln \tilde{\varepsilon} - \ln c &= 2a \\ \ln \frac{\tilde{\varepsilon}}{c} &= 2a \\ \frac{\tilde{\varepsilon}}{c} &= e^{2a} \\ \tilde{\varepsilon} &= ce^{2a}\end{aligned}$$

Since $a=0; \tilde{\varepsilon}=\varepsilon$ one obtain

$$\varepsilon = ce^0$$

Then,

$$\tilde{\varepsilon} = \varepsilon e^{2a}$$

So, one obtain the transformation

$$X_6^e : \begin{cases} \tilde{x} = xe^a \\ \tilde{u} = ue^a \\ \tilde{\alpha} = \alpha e^{-2a} \\ \tilde{\varepsilon} = \varepsilon e^{2a} \end{cases}$$

Let $b = e^a$ then

$$X_6^e : \begin{cases} \tilde{x} = xb \\ \tilde{u} = ub \\ \tilde{\alpha} = \alpha \frac{1}{b^2} \\ \tilde{\varepsilon} = \varepsilon b^2 \end{cases}$$

New we check it is an equivalence transformation

Since $\tilde{t} = t, \tilde{\rho} = \rho, \tilde{s} = s$ then

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial x} &= \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} = \frac{\partial \tilde{f}}{\partial \tilde{x}} \frac{\partial xb}{\partial x} = \frac{b \partial \tilde{f}}{\partial \tilde{x}} \\ \frac{\partial \tilde{f}}{\partial \alpha} &= \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} \frac{\partial \tilde{\alpha}}{\partial \alpha} = \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} \frac{\partial \frac{1}{b^2} \alpha}{\partial \alpha} = \frac{1}{b^2} \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} \end{aligned}$$

So,

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}} &= \frac{1}{b} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial \tilde{\alpha}} &= b^2 \frac{\partial}{\partial \alpha} \end{aligned}$$

Consider (1.2)

$$\tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \frac{\partial \rho}{\partial t} + \frac{1}{b} \frac{\partial \rho u b}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{b \partial \rho u}{b \partial x} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_t + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = \frac{\partial \rho s}{\partial t} + \frac{1}{b} \frac{\partial \rho s u b}{\partial x} = \frac{\partial \rho s}{\partial t} + \frac{b \partial \rho s u}{b \partial x} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned} \tilde{\lambda} &= 2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\alpha}} = 2b^2 \frac{\partial \varepsilon b^2}{\partial \alpha} = 2 \frac{b^4 \partial \varepsilon}{\partial \alpha} = b^4 \lambda \\ \tilde{P} &= \tilde{\rho}^2 \frac{\partial \tilde{\varepsilon}}{\partial \tilde{\rho}} - \tilde{\rho} \frac{\partial}{\partial \tilde{x}} \left(\tilde{\rho} \tilde{\lambda} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right) = \rho^2 \frac{\partial \varepsilon b^2}{\partial \rho} - \rho \frac{1}{b} \frac{\partial}{\partial x} \left(\rho b^4 \lambda \frac{1}{b} \frac{\partial \rho}{\partial x} \right) \\ &= \rho^2 \frac{b^2 \partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{b \partial x} \left(\rho b^4 \lambda \frac{\partial \rho}{b \partial x} \right) = \rho^2 \frac{b^2 \partial \varepsilon}{\partial \rho} - \rho b^2 \frac{\partial}{\partial x} \left(\rho \lambda \frac{\partial \rho}{\partial x} \right) = b^2 P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho} \tilde{\lambda} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \right)^2 = b^2 P + \rho b^4 \lambda \left(\frac{\partial \rho}{\partial x b} \right)^2 = b^2 P + \rho b^4 \lambda \frac{1}{b^2} \left(\frac{\partial \rho}{\partial x} \right)^2 = b^2 \Pi \end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = \frac{\partial \rho u b}{\partial t} + \frac{\partial (\rho (u b)^2 + b^2 \Pi)}{\partial (x b)} = \frac{b \partial \rho u}{\partial t} + \frac{b^2 \partial (\rho u^2 + \Pi)}{b \partial x} = b(\rho u)_t + (b \rho u^2 + \Pi)_x$$

Hench x_6^e is an equivalence transformation.

$$7. X_7^e = f(\rho) \sqrt{\alpha} \partial_{\varepsilon}$$

Consider Lie equations:

$$\frac{d\tilde{\varepsilon}}{da} = f(\tilde{\rho}) \sqrt{\tilde{\alpha}} \quad (1.28)$$

$$a = 0; \tilde{\varepsilon} = \varepsilon$$

Because of $\tilde{\rho} = \rho, \tilde{\alpha} = \alpha$ then (1.28) becomes

$$\begin{aligned} \int d\tilde{\varepsilon} &= \int f(\rho) \sqrt{\alpha} da \\ \tilde{\varepsilon} &= f(\rho) \sqrt{\alpha} a + c \end{aligned}$$

Since $a = 0; \tilde{\varepsilon} = \varepsilon$ one obtain

$$\varepsilon = f(\rho)\sqrt{\alpha}0 + c$$

Then,

$$\tilde{\varepsilon} = f(\rho)\sqrt{\alpha}a + \varepsilon$$

New we check it is an equivalence transformation.

Since $\tilde{x} = x, \tilde{t} = t, \tilde{\rho} = \rho, \tilde{u} = u, \tilde{\alpha} = \alpha, \tilde{s} = s$ then

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial \rho} &= \frac{\partial \tilde{f}}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} + \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial \tilde{\varepsilon}}{\partial \rho} = \frac{\partial \tilde{f}}{\partial \tilde{\rho}} + \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial (f(\rho)\sqrt{\alpha}a + \varepsilon)}{\partial \rho} = \frac{\partial \tilde{f}}{\partial \tilde{\rho}} + f'(\rho)\sqrt{\alpha}a \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \\ \frac{\partial \tilde{f}}{\partial \alpha} &= \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} \frac{\partial \tilde{\alpha}}{\partial \alpha} + \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial \tilde{\varepsilon}}{\partial \alpha} = \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} + \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial (f(\rho)\sqrt{\alpha}a + \varepsilon)}{\partial \alpha} = \frac{\partial \tilde{f}}{\partial \tilde{\alpha}} + \frac{1}{2\sqrt{\alpha}} f(\rho)a \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \\ \frac{\partial \tilde{f}}{\partial \varepsilon} &= \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial \tilde{\varepsilon}}{\partial \varepsilon} = \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial (f(\rho)\sqrt{\alpha}a + \varepsilon)}{\partial \varepsilon} = \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}}\end{aligned}$$

So,

$$\begin{aligned}\frac{\partial}{\partial \tilde{\varepsilon}} &= \frac{\partial}{\partial \varepsilon} \\ \frac{\partial}{\partial \tilde{\rho}} &= \frac{\partial}{\partial \rho} - f'(\rho)\sqrt{\alpha}a \frac{\partial}{\partial \varepsilon} \\ \frac{\partial}{\partial \tilde{\alpha}} &= \frac{\partial}{\partial \alpha} - \frac{1}{2\sqrt{\alpha}} f(\rho)a \frac{\partial}{\partial \varepsilon}\end{aligned}$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned}
\tilde{\lambda} &= 2\tilde{\varepsilon}_{\tilde{a}} = 2\frac{\partial \tilde{\varepsilon}}{\partial \tilde{a}} = 2\left[\frac{\partial}{\partial \alpha}(f(\rho)\sqrt{\alpha}a + \varepsilon) - \frac{1}{2\sqrt{\alpha}}f(\rho)a\frac{\partial}{\partial \varepsilon}(f(\rho)\sqrt{\alpha}a + \varepsilon)\right] = 2\varepsilon_{\alpha} = \lambda \\
\tilde{P} &= \tilde{\rho}^2\tilde{\varepsilon}_{\tilde{\rho}} - \tilde{\rho}(\tilde{\rho}\tilde{\lambda}\tilde{\rho}_{\tilde{x}})_{\tilde{x}} = \rho^2\left[\frac{\partial}{\partial \rho}(f(\rho)\sqrt{\alpha}a + \varepsilon) - f'(\rho)\sqrt{\alpha}a\frac{\partial}{\partial \varepsilon}(f(\rho)\sqrt{\alpha}a + \varepsilon)\right] - \rho[\rho\lambda\rho_x]_x \\
&= \rho^2\varepsilon_{\rho} - \rho(\rho\lambda\rho_x)_x \\
&= P \\
\tilde{\Pi} &= \tilde{P} + \tilde{\rho}\tilde{\lambda}(\tilde{\rho}_x)^2 = P + \rho\lambda(\rho_x)^2 = \Pi
\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho u)_t + (\rho u^2 + \Pi)_x$$

Hench X_7^e is an equivalence transformation.

$$8. X_8^e = g(s)\partial_{\varepsilon}$$

Consider Lie equations:

$$\begin{aligned}
\frac{d\tilde{\varepsilon}}{da} &= g(s) \\
a &= 0; \tilde{\varepsilon} = \varepsilon
\end{aligned} \tag{1.29}$$

Because of $\tilde{s} = s$ then (1.29) becomes

$$\begin{aligned}
\int d\tilde{\varepsilon} &= \int g(s)da \\
\tilde{\varepsilon} &= g(s)a + c
\end{aligned}$$

Since $a = 0; \tilde{\varepsilon} = \varepsilon$ one obtain

$$\varepsilon = g(s)0 + c$$

Then,

$$\tilde{\varepsilon} = g(s)a + \varepsilon$$

New we check it is an equivalence transformation.

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned}\tilde{\lambda} &= 2\tilde{\varepsilon}_{\tilde{a}} = 2(g(s)a + \varepsilon)_a = 2\varepsilon_a = \lambda \\ \tilde{P} &= \tilde{\rho}^2 \tilde{\varepsilon}_{\tilde{\rho}} - \tilde{\rho}(\tilde{\rho}\tilde{\lambda}\tilde{\rho}_{\tilde{x}})_{\tilde{x}} = \rho^2(g(s)a + \varepsilon)_\rho - \rho(\rho\lambda\rho_x)_x \\ &= \rho^2\varepsilon_\rho - \rho(\rho\lambda\rho_x)_x = P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho}\tilde{\lambda}(\tilde{\rho}_x)^2 = P + \rho\lambda(\rho_x)^2 = \Pi\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho u)_t + (\rho u^2 + \Pi)_x$$

Hench x_8^e is an equivalence transformation.

$$9. x_9^e = \frac{1}{\rho} \partial_\varepsilon$$

Consider Lie equations:

$$\frac{d\tilde{\varepsilon}}{da} = \frac{1}{\tilde{\rho}} \quad (1.30)$$

$$a = 0; \tilde{\varepsilon} = \varepsilon$$

Because of $\tilde{\rho} = \rho$ then (1.30) becomes

$$\begin{aligned}\int d\tilde{\varepsilon} &= \int \frac{1}{\rho} da \\ \tilde{\varepsilon} &= \frac{1}{\rho} a + c\end{aligned}$$

Since $a = 0; \tilde{\varepsilon} = \varepsilon$ one obtain

$$\varepsilon = \frac{1}{\rho} 0 + c$$

Then,

$$\tilde{\varepsilon} = \frac{1}{\rho} a + \varepsilon$$

New we check it is an equivalence transformation.

Since $\tilde{x} = x, \tilde{t} = t, \tilde{\rho} = \rho, \tilde{u} = u, \tilde{\alpha} = \alpha, \tilde{s} = s$ then

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial \rho} &= \frac{\partial \tilde{f}}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} + \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial \tilde{\varepsilon}}{\partial \rho} = \frac{\partial \tilde{f}}{\partial \tilde{\rho}} + \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial(\frac{1}{\rho}a + \varepsilon)}{\partial \rho} = \frac{\partial \tilde{f}}{\partial \tilde{\rho}} - \frac{1}{\rho^2} \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \\ \frac{\partial \tilde{f}}{\partial \varepsilon} &= \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial \tilde{\varepsilon}}{\partial \varepsilon} = \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}} \frac{\partial(\frac{1}{\rho}a + \varepsilon)}{\partial \varepsilon} = \frac{\partial \tilde{f}}{\partial \tilde{\varepsilon}}\end{aligned}$$

So,

$$\begin{aligned}\frac{\partial}{\partial \tilde{\varepsilon}} &= \frac{\partial}{\partial \varepsilon} \\ \frac{\partial}{\partial \tilde{\rho}} &= \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial}{\partial \varepsilon}\end{aligned}$$

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} = (\rho s)_t + (\rho s u)_x$$

Since

$$\begin{aligned}\tilde{\lambda} &= 2\tilde{\varepsilon}_{\tilde{\alpha}} = 2\left(\frac{1}{\rho}a + \varepsilon\right)_{\alpha} = 2\varepsilon_{\alpha} = \lambda \\ \tilde{P} &= \tilde{\rho}^2 \tilde{\varepsilon}_{\tilde{\rho}} - \tilde{\rho}(\tilde{\rho}\tilde{\lambda}\tilde{\rho}_{\tilde{x}})_{\tilde{x}} = \rho^2 \left[\frac{\partial}{\partial \rho} \left(\frac{1}{\rho}a + \varepsilon \right) + \frac{1}{\rho^2} \frac{\partial}{\partial \varepsilon} \left(\frac{1}{\rho}a + \varepsilon \right) \right] - \rho(\rho\lambda\rho_x)_x \\ &= \rho^2 \varepsilon_{\rho} - \rho(\rho\lambda\rho_x)_x = P \\ \tilde{\Pi} &= \tilde{P} + \tilde{\rho}\tilde{\lambda}(\tilde{\rho}_x)^2 = P + \rho\lambda(\rho_x)^2 = \Pi\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho u)_t + (\rho u^2 + \Pi - a)_x = (\rho u)_t + (\rho u^2 + \Pi)_x$$

Hench x_9^e is an equivalence transformation.

$$10. X_{10}^e = h(s)\partial_s$$

Consider Lie equations:

$$\begin{aligned}\frac{d\tilde{s}}{da} &= h(\tilde{s}) \\ a=0; \tilde{s} &= s\end{aligned}\tag{1.31}$$

Consider (1.31)

$$\begin{aligned}\int \frac{1}{h(\tilde{s})} d\tilde{s} &= \int da \\ \int H(\tilde{s}) d\tilde{s} &= \int da \\ \tilde{H}(\tilde{s}) &= a + c\end{aligned}$$

Since $a=0; \tilde{s}=s$ one obtain

$$\tilde{H}(s) = 0 + c$$

Then,

$$\tilde{H}(\tilde{s}) = a + \tilde{H}(s)$$

If

$$\tilde{H}(\tilde{s}) = \tilde{s} \text{ then } \tilde{H}(s) = s$$

So,

$$\tilde{s} = a + s$$

New we check it is an equivalence transformation.

Consider (1.2)

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{u})_{\tilde{x}} = \rho_t + (\rho u)_x$$

Consider (1.3)

$$\begin{aligned}(\tilde{\rho}\tilde{s})_{\tilde{t}} + (\tilde{\rho}\tilde{s}\tilde{u})_{\tilde{x}} &= (\rho(a+s))_t + (\rho(a+s)u)_x = a\rho_t + (\rho s)_t + a(\rho u)_x + (\rho su)_x \\ &= (\rho s)_t + (\rho su)_x, \text{ since } a\rho_t + a(\rho u)_x = 0\end{aligned}$$

Since

$$\begin{aligned}
\tilde{\lambda} &= 2\tilde{\varepsilon}_{\tilde{\alpha}} = 2\varepsilon_{\alpha} = \lambda \\
\tilde{P} &= \tilde{\rho}^2 \tilde{\varepsilon}_{\tilde{\rho}} - \tilde{\rho}(\tilde{\rho}\tilde{\lambda}\tilde{\rho}_{\tilde{x}})_{\tilde{x}} = \rho^2 \varepsilon_{\rho} - \rho(\rho\lambda\rho_x)_x = P \\
\tilde{\Pi} &= P + \rho\lambda\rho_x^2 = \Pi
\end{aligned}$$

Consider (1.4)

$$(\tilde{\rho}\tilde{u})_{\tilde{t}} + (\tilde{\rho}\tilde{u}^2 + \tilde{\Pi})_{\tilde{x}} = (\rho u)_t + (\rho u^2 + \Pi)_x$$

Hench X_{10}^e is an equivalence transformation.

So the equivalence Lie group are

$$\begin{aligned}
X_1^e &= \rho\partial_{\rho} + 2\alpha\partial_{\alpha} \\
X_2^e &= t\partial_t - u\partial_u - 2\varepsilon\partial_{\varepsilon} \\
X_3^e &= \partial_t \\
X_4^e &= t\partial_x + \partial_u \\
X_5^e &= \partial_x \\
X_6^e &= x\partial_x + u\partial_u - 2\alpha\partial_{\alpha} + 2\varepsilon\partial_{\varepsilon} \\
X_7^e &= f(\rho)\sqrt{\alpha}\partial_{\varepsilon} \\
X_8^e &= g(s)\partial_{\varepsilon} \\
X_9^e &= \frac{1}{\rho}\partial_{\varepsilon} \\
X_{10}^e &= h(s)\partial_s
\end{aligned}$$

Admitted Lie group

Consider the original equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \quad (\rho u)_t + (\rho u^2 + \Pi)_x = 0, \quad (\rho s)_t + (\rho s u)_x = 0, \\ \Pi &= \rho \frac{\delta(\rho \varepsilon)}{\delta \rho} - \rho \varepsilon = \rho^2 \varepsilon_\rho - 2\rho(\rho \rho_x \varepsilon_\alpha)_x + 2\rho \rho_x^2 \varepsilon_\alpha, \quad \alpha = |\nabla \rho|^2, \end{aligned} \quad (1)$$

Since an equivalence Lie group allows changing arbitrary elements conserving the structure of the studied equations. An infinitesimal operator X^e of the equivalence Lie group is sought in the form

$$X^e = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^\alpha \partial_\alpha + \zeta^s \partial_s + \zeta^\varepsilon \partial_\varepsilon,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^\alpha, \zeta^s$ and ζ^ε are all functions of $(x, t, \rho, u, \alpha, s, \varepsilon)$.

Calculations give the following basis of generators of the equivalence Lie group,

$$\begin{aligned} X_1^e &= \rho \partial_\rho + 2\alpha \partial_\alpha, \quad X_2^e = t \partial_t + x \partial_x - 2\alpha \partial_\alpha, \\ X_3^e &= t \partial_x - u \partial_u - 2\varepsilon \partial_\varepsilon, \quad X_4^e = f(s) \partial_s, \\ Y_1^e &= \partial_t, \quad Y_2^e = \partial_x, \quad Y_3^e = t \partial_x + \partial_u, \quad Z_1^e = \rho^{-1} \partial_\varepsilon, \\ Z_2^e &= f(s) \partial_\varepsilon, \quad Z_3^e = g(\rho) \sqrt{\alpha} \partial_\varepsilon, \end{aligned}$$

where the functions $f(s)$ and $g(\rho)$ are arbitrary.

Since the equivalence transformations corresponding to the operators $X_3^e, X_4^e, Z_1^e, Z_2^e$ and Z_3^e are applied for simplifying the function ε in the classification process, let us present these transformations. Because the function ε depends on ρ, α and s , only the transformations of these variables are presented:

$$\begin{aligned} X_3^e : \quad & \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \tilde{\varepsilon} &= e^{-2a} \varepsilon, \\ X_4^e : \quad & \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = h(s, a) & \tilde{\varepsilon} &= \varepsilon, \\ Z_1^e : \quad & \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \tilde{\varepsilon} &= \varepsilon + \rho^{-1} a, \\ Z_2^e : \quad & \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \tilde{\varepsilon} &= \varepsilon + f(s) a, \\ Z_3^e : \quad & \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \tilde{\varepsilon} &= \varepsilon + \sqrt{\alpha} g(\rho) a \end{aligned}$$

where a is the group parameter. Using the equivalence transformations corresponding to the generators Z_1^e and Z_2^e , the term $C_1 \rho^{-1} + C_2$, which appears in the function $\varepsilon(\rho, \alpha, s)$ can be omitted. Here C_1 and C_2 are constants. By virtue of the equivalence transformations corresponding to the generator Z_3^e , the function $\varepsilon(\rho, \alpha, s)$ is considered up to the term $\sqrt{\alpha} g(\rho)$.

An admitted generator X of equations (1) is sought in the form

$$X = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^\alpha \partial_\alpha + \zeta^s \partial_s,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^\alpha$ and ζ^s are functions of the variables $(x, t, \rho, u, \alpha, s)$. Calculations show that

$$\begin{aligned} \xi^x &= k_4 t x - \tilde{k}_3 x - \tilde{k}_1 x + k_5 t^3 + k_6 t^2 + k_8 t + k_9, \quad \xi^t = k_4 t^2 - 2\tilde{k}_3 t - 2\tilde{k}_1 t + k_2 t + k_7, \\ \zeta^\rho &= 2\tilde{k}_1 \rho - k_4 t \rho + \tilde{k}_3 \rho, \quad \zeta^u = k_4(x - tu) + \tilde{k}_3 u + \tilde{k}_1 u - k_2 u + 3k_5 t^2 + 2k_6 t + k_8, \\ \zeta^\alpha &= 6\tilde{k}_1 \alpha - 4k_4 t \alpha + 4\tilde{k}_3 \alpha, \quad \zeta^s = \zeta(s), \end{aligned}$$

where $k_1 = \tilde{k}_1 - \tilde{k}_3$, $k_3 = -\tilde{k}_3$, k_i , $(i = 1, 2, \dots, 8)$ are constant. The constants and the function $\zeta(s)$ satisfy the equations

$$\begin{aligned} & \zeta(2\varepsilon_{s\alpha\rho\rho}\alpha\rho + 4\varepsilon_{s\alpha\rho}\alpha - \varepsilon_{s\rho\rho}\rho - 2\varepsilon_{s\rho}) + 2\tilde{k}_1(2\varepsilon_{\alpha\rho\rho\rho}\alpha\rho^2 + 9\varepsilon_{\alpha\rho\rho}\alpha\rho \\ & + 6\varepsilon_{\alpha\rho}\alpha + 6\varepsilon_{\alpha\alpha\rho\rho}\alpha^2\rho + 12\varepsilon_{\alpha\alpha\rho}\alpha^2 - \varepsilon_{\rho\rho\rho}\rho^2 - 3\varepsilon_{\rho\rho}\rho) + 2k_2(2\varepsilon_{\alpha\rho\rho}\alpha\rho + 4\varepsilon_{\alpha\rho}\alpha \\ & - \varepsilon_{\rho\rho\rho}\rho - 2\varepsilon_{\rho}) + \tilde{k}_3(2\varepsilon_{\alpha\rho\rho\rho}\alpha\rho^2 + 8\varepsilon_{\alpha\rho\rho}\alpha\rho + 4\varepsilon_{\alpha\rho}\alpha + 8\varepsilon_{\alpha\alpha\rho\rho}\alpha^2\rho \\ & + 16\varepsilon_{\alpha\alpha\rho}\alpha^2 - \varepsilon_{\rho\rho\rho}\rho^2 - 2\varepsilon_{\rho\rho}\rho + 2\varepsilon_{\rho}) - 2k_6 q(\alpha) = 0, \end{aligned} \quad (2)$$

$$\begin{aligned} & \zeta'(4\varepsilon_{s\alpha\rho}\alpha\rho + 4\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho}\rho) + \zeta(4\varepsilon_{ss\alpha\rho}\alpha\rho + 4\varepsilon_{ss\alpha}\alpha - \varepsilon_{ss\rho}\rho) + 2\tilde{k}_1(4\varepsilon_{s\alpha\rho\rho}\alpha\rho^2 \\ & + 13\varepsilon_{s\alpha\rho}\alpha\rho + 12\varepsilon_{s\alpha\alpha\rho}\alpha^2\rho + 12\varepsilon_{s\alpha\alpha}\alpha^2 + 8\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}\rho^2) + 2k_2(4\varepsilon_{s\alpha\rho}\alpha\rho \\ & + 4\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho}\rho) + \tilde{k}_3(4\varepsilon_{s\alpha\rho\rho}\alpha\rho^2 + 12\varepsilon_{s\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha\rho}\alpha^2\rho + 16\varepsilon_{s\alpha\alpha}\alpha^2 \\ & + 8\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}\rho^2 + \varepsilon_{s\rho}\rho) = 0, \end{aligned} \quad (3)$$

$$\begin{aligned} & \zeta_{ss}^s\varepsilon_{s\alpha} + 2\zeta'\varepsilon_{ss\alpha} + \zeta\varepsilon_{sss\alpha} + 2\tilde{k}_1(\varepsilon_{ss\alpha\rho}\rho + 3\varepsilon_{ss\alpha\alpha}\alpha + 2\varepsilon_{ss\alpha}) + 2k_2\varepsilon_{ss\alpha} \\ & + \tilde{k}_3(\varepsilon_{ss\alpha\rho}\rho + 4\varepsilon_{ss\alpha\alpha}\alpha + 2\varepsilon_{ss\alpha}) = 0, \end{aligned} \quad (4)$$

$$\begin{aligned} & \zeta(\varepsilon_{s\alpha\rho}\rho + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + 2\tilde{k}_1(\varepsilon_{\alpha\rho\rho}\rho^2 + 4\varepsilon_{\alpha\rho}\rho + 6\varepsilon_{\alpha\alpha\alpha\rho}\alpha^2\rho \\ & + 6\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho\rho}\alpha\rho^2 + 17\varepsilon_{\alpha\alpha\rho}\alpha\rho + 13\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_{\alpha}) \\ & + 2k_2(\varepsilon_{\alpha\rho}\rho + 2\varepsilon_{\alpha\alpha\rho}\alpha\rho + 2\varepsilon_{\alpha\alpha}\alpha + \varepsilon_{\alpha}) \\ & + \tilde{k}_3(\varepsilon_{\alpha\rho\rho}\rho^2 + 4\varepsilon_{\alpha\rho}\rho + 8\varepsilon_{\alpha\alpha\alpha\rho}\alpha^2\rho + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho\rho}\alpha\rho^2 + 20\varepsilon_{\alpha\alpha\rho}\alpha\rho \\ & + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_{\alpha}) = 0, \end{aligned} \quad (5)$$

$$\begin{aligned} & \zeta(2\varepsilon_{s\alpha\alpha\alpha}\alpha + 3\varepsilon_{s\alpha\alpha}) + 2\tilde{k}_1(6\varepsilon_{\alpha\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\alpha\rho}\alpha\rho + 25\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha\rho}\rho \\ & + 15\varepsilon_{\alpha\alpha}) + 2k_2(2\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha}) + \tilde{k}_3(8\varepsilon_{\alpha\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\alpha\rho}\alpha\rho + 32\varepsilon_{\alpha\alpha\alpha}\alpha \\ & + 3\varepsilon_{\alpha\alpha\rho}\rho + 18\varepsilon_{\alpha\alpha}) = 0, \end{aligned} \quad (6)$$

$$\begin{aligned} & \zeta'(2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + \zeta(2\varepsilon_{ss\alpha\alpha}\alpha + \varepsilon_{ss\alpha}) + 2\tilde{k}_1(\varepsilon_{s\alpha\rho}\rho + 6\varepsilon_{s\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho \\ & + 13\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) + 2k_2(2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) \\ & + \tilde{k}_3(\varepsilon_{s\alpha\rho}\rho + 8\varepsilon_{s\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} & \zeta'\varepsilon_{s\alpha} + \zeta\varepsilon_{ss\alpha} + 2\tilde{k}_1(\varepsilon_{s\alpha\rho}\rho + 3\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) + 2k_2\varepsilon_{s\alpha} + \tilde{k}_3(\varepsilon_{s\alpha\rho}\rho \\ & + 4\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} & \zeta(2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + 2\tilde{k}_1(\varepsilon_{\alpha\rho}\rho + 6\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho}\alpha\rho + 13\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_{\alpha}) \\ & + 2k_2(2\varepsilon_{\alpha\alpha}\alpha + \varepsilon_{\alpha}) + \tilde{k}_3(\varepsilon_{\alpha\rho}\rho + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho}\alpha\rho + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_{\alpha}) = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} & k_4(2\varepsilon_{\alpha\rho\rho\rho}\alpha\rho^2 + 8\varepsilon_{\alpha\rho\rho}\alpha\rho + 4\varepsilon_{\alpha\rho}\rho + 8\varepsilon_{\alpha\alpha\rho\rho}\alpha^2\rho \\ & + 16\varepsilon_{\alpha\alpha\rho}\alpha^2 - \varepsilon_{\rho\rho\rho}\rho^2 - 2\varepsilon_{\rho\rho}\rho + 2\varepsilon_{\rho}) + 6k_5q(\alpha) = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} & k_4(4\varepsilon_{s\alpha\rho\rho}\alpha\rho^2 + 12\varepsilon_{s\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha\rho}\alpha^2\rho \\ & + 16\varepsilon_{s\alpha\alpha}\alpha^2 + 8\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}\rho^2 + \varepsilon_{s\rho}\rho) = 0, \end{aligned} \quad (11)$$

$$k_4(\varepsilon_{ss\alpha\rho}\rho + 4\varepsilon_{ss\alpha\alpha}\alpha + 2\varepsilon_{ss\alpha}) = 0, \quad (12)$$

$$\begin{aligned} & k_4(\varepsilon_{\alpha\rho\rho}\rho^2 + 4\varepsilon_{\alpha\rho}\rho + 8\varepsilon_{\alpha\alpha\alpha\rho}\alpha^2\rho + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho\rho}\alpha\rho^2 + 20\varepsilon_{\alpha\alpha\rho}\alpha\rho \\ & + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_{\alpha}) = 0, \end{aligned} \quad (13)$$

$$k_4(8\varepsilon_{\alpha\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\alpha\rho}\alpha\rho + 32\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha\rho}\rho + 18\varepsilon_{\alpha\alpha}) = 0, \quad (14)$$

$$k_4(\varepsilon_{s\alpha\rho}\rho + 8\varepsilon_{s\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0, \quad (15)$$

$$k_4(\varepsilon_{s\alpha\rho}\rho + 4\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0, \quad (16)$$

$$k_4(\varepsilon_{\alpha\rho}\rho + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\alpha\rho\varepsilon_{\alpha\alpha\rho} + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_{\alpha}) = 0. \quad (17)$$

where $q = a/\sqrt{\alpha}$ and $a^2 = 1$. The determining equations (2)-(17) define the kernel of admitted Lie algebras and its extensions. The kernel of admitted Lie algebras consists of the generators which are admitted by equations (1) for any function $\varepsilon(\rho, \alpha, s)$ and it is defined by the generators

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = t\partial_x + \partial_u.$$

The transformations corresponding to Y_1 and Y_2 are shifts with respect to time and space variable, and the transformations corresponding to Y_3 are the Galilean

transformations. Extensions of the kernel of admitted Lie algebras depend on the value of the function $\varepsilon(\rho, \alpha, s)$. They can only be operators of the form

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 + k_5 X_5 + k_6 X_6 + \zeta \partial_s,$$

where

$$\begin{aligned} X_1 &= \rho \partial_\rho + \alpha \partial_\alpha, \quad X_2 = t \partial_t - u \partial_u, \quad X_3 = u \partial_u + 2\alpha \partial_\alpha - x \partial_x - 2t \partial_t, \\ X_4 &= \rho t \partial_\rho + (ut - x) \partial_u + 4\alpha t \partial_\alpha - tx \partial_x - t^2 \partial_t, \quad X_5 = t^3 \partial_x + 3t^2 \partial_u, \quad X_6 = t^2 \partial_x + 2t \partial_u. \end{aligned}$$

Case $k_4 \neq 0$ The functions $\varepsilon(\rho, \alpha, s)$ for which there exists an admitted generator with $k_4 \neq 0$ are studied first. This generator can be rewritten in the form

$$X_4 + k_1 X_1 + k_2 X_2 + k_3 X_3 + k_5 X_5 + k_6 X_6 + \zeta \partial_s,$$

where $\zeta = \zeta(s)$. Using the equivalence transformation corresponding to the generator $X_1^e = \partial_t$, one can assume that for this generator $k_1 = 0$. Notice also that if $\zeta \neq 0$, then using the equivalence transformation corresponding to X_4^e , one can assume that $\zeta = 1$. From equation (16) one finds that Assume that there exists an admitted generator with $k_4 \neq 0$. Using the equivalence transformation corresponding to the generator $X_1^e = \partial_t$, one can assume that for this generator $k_1 = 0$. Notice also that if $\zeta' \neq 0$, then using $X_4^e = h(S) \partial_s$, one can assume that $\zeta = -S$. From (16)

$$\begin{aligned} \varepsilon_{\alpha S} &= \rho^{-2} \psi(\alpha \rho^{-4}, S). \\ \varepsilon_S &= \rho^2 \psi(\alpha \rho^{-4}, S) + \phi(\rho, S). \\ \varepsilon &= \rho^2 \psi(\alpha \rho^{-4}, S) + \phi(\rho, S) + \varphi(\rho, \alpha). \end{aligned}$$

Substituting in (17) one has

$$\rho \varphi_{\alpha \rho} + 8\alpha^2 \varphi_{\alpha \alpha \alpha} + 2\alpha \rho \varphi_{\alpha \alpha \rho} + 16\alpha \varphi_{\alpha \alpha} + 2\varphi_{\alpha} = 0$$

or introducing the function

$$g = 2\alpha \varphi_{\alpha \alpha} + \varphi_{\alpha} = 2\alpha^{1/2} (\alpha^{1/2} \varphi_{\alpha})_{\alpha}$$

this equation becomes

$$\rho g_{\rho} + 4\alpha g_{\alpha} + 2g = 0.$$

Solution of this equation is

$$g(\rho, \alpha) = \rho^{-2} q(\alpha \rho^{-4}),$$

hence,

$$\begin{aligned} \rho^{-2} \alpha^{-1/2} q(\alpha \rho^{-4}) &= 2(\alpha^{1/2} \varphi_{\alpha})_{\alpha} \\ \int \rho^{-2} \alpha^{-1/2} q(\alpha \rho^{-4}) d\alpha &= \rho^2 \int \alpha^{-1/2} q(\alpha \rho^{-4}) dz = \int z^{-1/2} q(z) dz = 2\tilde{q}(z) \\ \alpha^{1/2} \varphi_{\alpha} &= \tilde{q}(z) + h_1(\rho) \\ \varphi_{\alpha} &= \alpha^{-1/2} \tilde{q}(z) + \alpha^{-1/2} h_1(\rho) \\ \int \alpha^{-1/2} \tilde{q}(z) d\alpha &= \rho^4 \int \alpha^{-1/2} \tilde{q}(z) dz = \rho^2 \int z^{-1/2} \tilde{q}(z) dz = \rho^2 \tilde{q}(z) \end{aligned}$$

$$\begin{aligned}\varphi &= \rho^2 \tilde{q}(z) + \alpha^{1/2} h_1(\rho) + h_2(\rho) \\ \varepsilon &= \rho^2 \psi(\alpha \rho^{-4}, S) + \phi(\rho, S) + \rho^2 q(z) + \alpha^{1/2} h_1(\rho) + h_2(\rho).\end{aligned}$$

One can assume that

$$q = 0, \quad h_1 = 0, \quad h_2 = 0.$$

Thus,

$$\varepsilon = \rho^2 \psi(\alpha \rho^{-4}, S) + \phi(\rho, S).$$

Differentiating equation ss10 with respect to α , one finds that $k_5 = 0$. Equations ss10 and ss20 become

$$\begin{aligned}\rho^2 \phi_{\rho\rho\rho} + 2\rho \phi_{\rho\rho} - 2\phi_\rho &= 0. \\ \rho \phi_{\rho\rho S} - \phi_{\rho S} &= 0.\end{aligned}$$

Solution of the first equation is

$$\phi = \rho^2 \phi_1(S) + \rho^{-1} \phi_2(S) + \phi_3(S).$$

Hence,

$$\phi_{\rho S} = 2\rho \phi_1' - \rho^{-2} \phi_2',$$

and then the second equation becomes

$$\phi_2' = 0 \Rightarrow \phi_2 = C = \text{const}$$

Thus,

$$\varepsilon = \rho^2 \left(\psi(\alpha \rho^{-4}, S) + \phi_1(S) \right) + \rho^{-1} C + \phi_3(S).$$

By virtue of arbitrariness of the function ψ , one assigns $\phi_1 = 0$. Because of the equivalence transformations corresponding to Z_1^e and Z_2^e one can assume that $\phi_3 = 0$ and $C = 0$. Thus, if there exists a generator with $k_4 \neq 0$, then the function $\varepsilon(\rho, \alpha, S)$ has to be of the form

$$\varepsilon = \rho^2 \psi(z, S), \quad z = \alpha \rho^{-4}. \quad (18)$$

Let us study group properties of equations (1) with the function $\varepsilon(\rho, \alpha, S)$ of the form (18). Substituting the function $\varepsilon(\rho, \alpha, S)$ (18) into the determining equations (..)-(...), and making some manipulations (differentiations and linear combinations) one obtains that these equations are reduced to the equations $k_5 = 0$, $k_6 = 0$ and

$$2(k_3 - k_1)(2z^2 \psi_{zz} - z \psi_z + \psi) + 2k_2(2\psi_z z - \psi) + \zeta(2\psi_{zS} z - \psi_S) = 0, \quad (19)$$

$$(2(k_3 - k_1)(z \psi_z - \psi) + 2k_2 \psi + \psi_S \zeta)_S = 0. \quad (20)$$

Integrating equation (20), one has

$$2(k_3 - k_1)(z \psi_z - \psi) + 2k_2 \psi + \psi_S \zeta = \lambda, \quad (21)$$

where the function $\lambda(z)$ is an arbitrary function. Excluding k_2 from (19) using (21), one finds that

$$2z \lambda_z - \lambda = 0,$$

which means that the function $\lambda = qz^{1/2}$ is an arbitrary function or

$$\lambda = kz^{1/2},$$

where k is constant. Thus, for the group classification of equations (1) with the function $\varepsilon(\rho, \alpha, S)$ of the form (18) one needs to analyze the only equation

$$2(k_3 - k_1)(z\psi_z - \psi) + 2k_2\psi + \psi_S\zeta = kz^{1/2}. \quad (22)$$

Since $\psi_S \neq 0$, one finds

$$\zeta = \psi_S^{-1} \left(kz^{1/2} - 2((k_3 - k_1)(z\psi_z - \psi) + k_2\psi) \right) \quad (23)$$

Because of that $\zeta = \zeta(S)$ and differentiating (23) with respect to z , one has to require that

$$ak + b(k_3 - k_1) + ck_2 = 0, \quad (24)$$

where

$$a = \left(z^{1/2}\psi_S^{-1} \right)_z, \quad b = -2 \left(\psi_S^{-1}(z\psi_z - \psi) \right)_z, \quad c = -2 \left(\psi_S^{-1}\psi \right)_z.$$

Equation (24) is a classifying equation. For arbitrary function $\psi(z, S)$ one has

$$k_3 = k_1, \quad k_2 = 0, \quad k = 0 \quad (25)$$

or the kernel of admitted generators consists of the generators

$$Y_1, \quad Y_2, \quad Y_3, \quad X_1 + X_3, \quad X_4. \quad (26)$$

An extension of this Lie algebra occurs if one of the constants

$$k_3 - k_1, \quad k_2, \quad k$$

does not vanish. Operators of the extension have the form

$$\gamma X_2 + \beta X_3 + X_h,$$

where $h = h(S)$ and β, γ are constant such that

$$h^2 + \beta^2 + \gamma^2 \neq 0.$$

The table of commutators

$$\begin{aligned} Y_1 &= \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = t\partial_x + \partial_u, \quad X_1 = \rho\partial_\rho + \alpha\partial_\alpha, \quad X_2 = t\partial_t - u\partial_u, \\ X_3 &= u\partial_u + 2\alpha\partial_\alpha - x\partial_x - 2t\partial_t, \quad X_4 = t(\rho\partial_\rho + u\partial_u + 4\alpha\partial_\alpha - t\partial_t - x\partial_x) - x\partial_u \\ X_5 &= t^3\partial_x + 3t^2\partial_u, \quad X_6 = t^2\partial_x + 2t\partial_u, \end{aligned}$$

	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5	X_6
Y_1	0	0	Y_2	0	Y_1	$-2Y_1$	$X_1 + X_3$	$3X_6$	$2Y_3$
Y_2	0	0	0	0	0	$-Y_2$	$-Y_3$	0	0
Y_3	$-Y_2$	0	0	0	$-Y_3$	Y_3	0	0	0
X_1	0	0	0	0	0	0	0	0	0
X_2	$-Y_1$	0	Y_3	0	0	0	X_4	$3X_5$	$2X_6$
X_3	$2Y_1$	Y_2	$-Y_3$	0	0	0	$-2X_4$	$-5X_5$	$-3X_6$
X_4	$-X_1 - X_3$	Y_3	0	0	$-X_4$	$2X_4$	0	$-F$	$-X_5$
X_5	$-3X_6$	0	0	0	$-3X_5$	$5X_5$	F	0	0
X_6	$-2Y_3$	0	0	0	$-2X_6$	$3X_6$	X_5	0	0

where $F = 2t^3(t\partial_x + 4\partial_u)$.

	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4
Y_1	0	0	Y_2	0	Y_1	$-2Y_1$	$X_1 + X_3$
Y_2	0	0	0	0	0	$-Y_2$	$-Y_3$
Y_3	$-Y_2$	0	0	0	$-Y_3$	Y_3	0
X_1	0	0	0	0	0	0	0
X_2	$-Y_1$	0	Y_3	0	0	0	X_4
X_3	$2Y_1$	Y_2	$-Y_3$	0	0	0	$-2X_4$
X_4	$-X_1 - X_3$	Y_3	0	0	$-X_4$	$2X_4$	0

	Y_1	Y_2	Y_3	$X_1 + X_3$	X_2	X_3	X_4
Y_1	0	0	Y_2	$-2Y_1$	Y_1	$-2Y_1$	$X_1 + X_3$
Y_2	0	0	0	$-Y_2$	0	$-Y_2$	$-Y_3$
Y_3	$-Y_2$	0	0	Y_3	$-Y_3$	Y_3	0
$X_1 + X_3$	$2Y_1$	Y_2	$-Y_3$	0	0	0	$-2X_4$
X_2	$-Y_1$	0	Y_3	0	0	0	X_4
X_3	$2Y_1$	Y_2	$-Y_3$	0	0	0	$-2X_4$
X_4	$-X_1 - X_3$	Y_3	0	$2X_4$	$-X_4$	$2X_4$	0

Notice that

$$Y_1, Y_2, Y_3, X_1 + X_3, X_4,$$

compose an ideal and

$$X_2, X_3$$

compose a subalgebra of

$$Y_1, Y_2, Y_3, X_1 + X_3, X_4, X_2, X_3$$

For classifying all possibilities, it is convenient to consider the functions $a(z, S)$, $b(z, S)$ and $c(z, S)$ as coordinates of the three-dimensional vector $v = (a, b, c)$. For analyzing relations between the constants $k_3 - k_1$, k_2 and k one can study the vector space $\text{Span}(V)$, where the set V consists of the vectors v with z and S are changed. Let $\dim(\text{Span}(V)) = 3$, then

$$k_3 = k_1, \quad k_2 = 0, \quad k = 0. \quad (27)$$

There is no extensions in this case. Let $\dim(\text{Span}(V)) = 2$, then there exists a constant vector $(\gamma, \beta, q) \neq 0$ such that

$$\gamma a + \beta b + qc = 0.$$

Assume that $\gamma \neq 0$

$$a = -\beta b - qc \Rightarrow b[(k_3 - k_1) - \beta k] + c(k_2 - qk) = 0$$

If one of the constants $(k_3 - k_1) - \beta k$ and $k_2 - qk$ does not vanish, then $\dim(V) < 2$. Hence,

$$(k_3 - k_1) = \beta k, \quad k_2 = qk$$

and

$$\zeta = kh(S),$$

where

$$h(S) = \psi_S^{-1} \left(z^{1/2} - 2(\beta z \psi_z + (q - \beta)\psi) \right). \quad (28)$$

The function $h(S)$ can be also simplified: either $h = 0$ or $h = 1$. The extension is defined by the generator

$$\beta X_3 + qX_2 + h\partial_S.$$

Case $h = 0$, $\beta \neq 0$, $q = \beta\tilde{q}$, $\tilde{q} \neq 1/2$:

$$\psi = z^{1-\tilde{q}}H(S) + z^{1/2} \frac{1}{\beta(2\tilde{q}-1)}$$

It gives $\dim(V) = 1$. Case $h = 0$, $\beta \neq 0$, $q = \beta\tilde{q}$, $\tilde{q} = 1/2$:

$$\psi = z^{1/2} \left(H(S) + \frac{1}{2\beta} \ln(z) \right).$$

It gives $\dim(V) = 1$. Case $h = 1$, $\beta = 0$:

$$\psi = e^{-qS}Q(z) + \frac{1}{2q}z^{1/2}$$

$$qX_2 + 2\partial_S.$$

Using equivalence transformation one can reduce the function

$$\psi = e^{-qS}Q(z)$$

Case $h = 1$, $\beta \neq 0$, $q = \beta\tilde{q}$, $\tilde{q} \neq 1/2$:

$$\psi = z^{1-\tilde{q}}Q(ze^{-\beta S}) + z^{1/2} \frac{1}{\beta(2\tilde{q}-1)}$$

$$\beta(X_3 + \tilde{q}X_2) + 2\partial_S.$$

Using equivalence transformation one can reduce the function

$$\psi = z^{1-\tilde{q}}Q(ze^{-\beta S})$$

Case $h = 1$, $\beta \neq 0$, $q = \beta\tilde{q}$, $\tilde{q} = 1/2$:

$$\psi = z^{1-\tilde{q}} \left(Q(ze^{-\beta S}) + \frac{1}{2\beta} \ln(z) \right)$$

$$\beta(X_3 + (1/2)X_2) + 2\partial_S.$$

It can be reduced to?

$$\psi = z^{1-\tilde{q}} \left(Q(ze^{-S}) + \ln(z) \right)$$

$$X_3 + (1/2)X_2 + 2\partial_S.$$

Assume that $\gamma = 0$, $\beta \neq 0$,

$$b = -qc \Rightarrow ak + c(k_2 - q(k_3 - k_1)) = 0$$

Similar to the previous case

$$k = 0, \quad k_2 = q(k_3 - k_1),$$

and

$$\zeta = -2(k_3 - k_1)h(S),$$

where

$$h(S) = \psi_S^{-1} (z\psi_z + (q-1)\psi). \quad (29)$$

The function $h(S)$ can be also simplified: either $h = 0$ or $h = 1$. The extension is defined by the generator

$$X_3 + qX_2 - 2h\partial_S.$$

Case $h = 0$:

$$\psi = z^{1-q}H(S),$$

$$X_3 + qX_2.$$

It gives $\dim(V) = 1$. Case $h = 1$:

$$\psi = z^{1-q}H(ze^{-S}),$$

$$X_3 + qX_2 + 2\partial_S.$$

Assume that $\gamma = 0, \beta = 0$,

$$c = 0 \Rightarrow ak + b(k_3 - k_1) = 0$$

Similar to the previous case

$$k = 0, \quad k_3 - k_1 = 0,$$

and

$$\zeta = -2k_2h(S),$$

where

$$h(S) = \psi_S^{-1}\psi. \quad (30)$$

The function $h(S)$ can be also simplified $h = 1$. In this case

$$\psi = e^{-S}H(z),$$

and the extension is defined by the generator

$$X_2 - 2\partial_S.$$

Let $\dim(\text{Span}(V)) = 1$, then there exists a constant vector $(\gamma, \beta, q) \neq 0$ such that

$$a = \gamma g, \quad b = \beta g, \quad c = qg,$$

with some function $g(z, S) \neq 0$ which is not constant. Assume that $\gamma \neq 0$

$$ak + b(k_3 - k_1) + ck_2 = 0,$$

$$g = a, \quad b = \beta a, \quad c = qa \Rightarrow k = -\beta(k_3 - k_1) - qk_2,$$

and

$$\zeta = -(k_3 - k_1)h_1(S) - k_2h_2(S),$$

where

$$h_1(S) = \psi_S^{-1} \left(\beta z^{1/2} + 2(z\psi_z - \psi) \right). \quad (31)$$

$$h_2(S) = \psi_S^{-1} (qz^{1/2} + 2\psi). \quad (32)$$

The extension is defined by the generators

$$X_3 - h_1 \partial_S, \quad X_2 - h_2 \partial_S.$$

Since, $\psi_S \neq 0$, one can assume that $h_2 = 1$. Notice that the commutator

$$[X_3 - h_1 \partial_S, X_2 - h_2 \partial_S] = h_1' \partial_S = 0.$$

Hence, $h_1 = \lambda$, where λ is constant.

$$z\psi_z - (\lambda + 1)\psi = \frac{(q\lambda - \beta)}{2} z^{1/2}. \quad (33)$$

$$\psi_S = 2\psi + qz^{1/2}. \quad (34)$$

$$\psi = \beta z^{1/2} + K z^{1+\lambda} e^{-2S}$$

$$X_3 - \lambda \partial_S, \quad X_2 + \partial_S.$$

Using equivalence transformation

$$\psi = K z^{1+\lambda} e^{-2S}$$

Assume that $\gamma = 0$ and $\beta \neq 0$

$$ak + b(k_3 - k_1) + ck_2 = 0,$$

$$g = b, \quad a = 0, \quad c = qb \Rightarrow (k_3 - k_1) = -qk_2,$$

and

$$\zeta = \psi_S^{-1} \left(kz^{1/2} - 2((k_3 - k_1)z\psi_z + (k_2 - k_3 + k_1)\psi) \right)$$

$$\zeta = kh_1(S) + 2k_2h_2(S),$$

where

$$h_1(S) = \psi_S^{-1} z^{1/2},$$

$$h_2(S) = \psi_S^{-1} (q(z\psi_z - \psi) - \psi).$$

The extension is defined by the generators

$$h_1 \partial_S, \quad X_2 - qX_3 + 2h_2 \partial_S.$$

Here one has to assume that $h_1 \neq 0$ or $h_1 = 1$. The commutator is

$$[h_1 \partial_S, X_2 - qX_3 + 2h_2 \partial_S] = 2h_2' \partial_S = 2\lambda \partial_S.$$

Hence, $h_2 = \lambda S$ and

$$\begin{aligned} \psi_S &= z^{1/2}, \\ \lambda S \psi_S &= qz\psi_z - (q+1)\psi. \end{aligned} \quad (35)$$

Case $q = 0$

$$\psi = z^{1/2} S$$

$$X_2 - 2S \partial_S, \quad \partial_S.$$

It gives $\dim(V) = 0$. Case $q \neq 0$

$$\psi = Kz^{1+1/q} + z^{1/2}S$$

$$X_2 - qX_3 - 2S\partial_S, \partial_S.$$

If $K = 0$, then $\dim(V) = 0$, hence $K \neq 0$. Assume that $\gamma = 0$, $\beta = 0$ and $q \neq 0$

$$ak + b(k_3 - k_1) + ck_2 = 0,$$

$$g = c, \quad a = 0, \quad b = 0, \quad \Rightarrow k_2 = 0$$

$$\zeta = kh_1(S) - 2(k_3 - k_1)h_2(S),$$

where

$$h_1(S) = \psi_S^{-1}z^{1/2},$$

$$h_2(S) = \psi_S^{-1}(z\psi_z - \psi).$$

The extension is defined by the generators

$$h_1\partial_S, X_3 - 2h_2\partial_S.$$

As in the previous case one can assume that $h_1 = 1$. The commutator is

$$[\partial_S, X_3 - 2h_2\partial_S] = 2h_2'\partial_S = 2\lambda\partial_S.$$

Hence, $h_2 = \lambda S$ and

$$\begin{aligned} \psi_S &= z^{1/2}, \\ \lambda Sz^{1/2} &= z\psi_z - \psi. \end{aligned} \tag{36}$$

$$\psi = Kz + z^{1/2}S$$

$$X_3 + S\partial_S, \partial_S.$$

If $K = 0$, then $\dim(V) = 0$, hence $K \neq 0$. Let $\dim(\text{Span}(V)) = 0$, then there exists a constant vector (γ, β, q) such that

$$a = \gamma, \quad b = \beta, \quad c = q.$$

$$\gamma k + \beta(k_3 - k_1) + qk_2 = 0,$$

$$\zeta = \psi_S^{-1} \left(kz^{1/2} - 2((k_3 - k_1)z\psi_z + (k_2 - k_3 + k_1)\psi) \right)$$

$$\zeta = kh_1(S) + (k_3 - k_1)h_2(S) + k_2h_3(S),$$

$$a = \left(z^{1/2}\psi_S^{-1} \right)_z, \quad b = -2 \left(\psi_S^{-1}(z\psi_z - \psi) \right)_z, \quad c = -2 \left(\psi_S^{-1}\psi \right)_z.$$

where

$$h_1(S) + \gamma z = z^{1/2}\psi_S^{-1}, \quad h_2(S) + \beta z = -2\psi_S^{-1}(z\psi_z - \psi), \quad h_3(S) + qz = -2\psi_S^{-1}\psi.$$

The extension is defined by the generators

$$X_3 + h_2\partial_S, X_2 + h_3\partial_S, h_1(S)\partial_S.$$

If $h_1 \neq 0$, then one can assume that $h_1 = 1$, and

$$[\partial_S, X_3 + h_2\partial_S] = h_2'\partial_S = \lambda_2\partial_S,$$

$$[\partial_S, X_2 + h_3 \partial_S] = h_3' \partial_S = \lambda_3 \partial_S,$$

where λ_2 and λ_3 are constant.

$$\psi_S = \frac{z^{1/2}}{1 + \gamma z}, \quad \lambda_2 S \frac{z^{1/2}}{1 + \gamma z} + \beta z = -2(z\psi_z - \psi), \quad \lambda_3 S \frac{z^{1/2}}{1 + \gamma z} + qz = -2\psi. \quad (37)$$

Case $\beta = 0$

$$\psi = z^{1/2} S, \\ X_3 + S \partial_S, \quad X_2 - 2S \partial_S, \quad \partial_S.$$

Case $\beta \neq 0$

$$\psi = z^{1/2} (S - \beta z), \\ X_2 - 2X_3 - 4S \partial_S, \quad \partial_S.$$

If $h_1 = 0$, then $\gamma \neq 0$

$$\psi = \gamma^{-1} z^{-1/2} S, \\ X_3 + 3S \partial_S, \quad X_2 - 2S \partial_S.$$

Using equivalence transformation

$$\psi = z^{-1/2} S,$$

Case $k_4 = 0$

Case where all operators such that $k_4 = 0$:

$$2(3k_1 - 5k_3)\alpha\varepsilon_\alpha + (2k_1 - 3k_3)\rho\varepsilon_\rho + \varepsilon_S \zeta = 2(k_1 - k_2 - 2k_3)\varepsilon - \alpha^{1/2}g + \varphi - k\rho^{-1}, \quad (38)$$

where $g = g(\rho)$ and $\varphi = \varphi(S)$.

One-dimensional case

Here we use the algebraic approach. This approach supposes that using algebraic properties of admitted Lie algebras, during the first step of solving the determining equations one defines unknown constants and functions of an admitted generator. In particular, according to the last comment of the previous section, one-dimensional Lie algebras can be reduced to one of two cases, either $\zeta = 0$ or $\zeta = 1$. The set of possible basis generators containing the generators X_1 , X_2 and X_3 is exhausted by the following generators

$\zeta = 0$		$\zeta = 1$	
1.	$X_1 + \beta X_2 + \gamma X_3$	4.	$X_1 + \beta X_2 + \gamma X_3 + \partial_S$
2.	$\beta X_2 + X_3$	5.	$\beta X_2 + X_3 + \partial_s$
3.	X_2	6.	$X_2 + \partial_S$

In the next step, one has to substitute the coefficients of each generator into the determining equation (38) and solve it with respect to the function $\varepsilon(\rho, \alpha, s)$. Here we present the calculations of the first case. Substituting

$$k_1 = 1, \quad k_1 = \beta, \quad k_3 = \gamma, \quad \zeta = 0$$

into equation (38), one obtains

$$2(3 - 5\gamma)\alpha\varepsilon_\alpha + (2 - 3\gamma)\rho\varepsilon_\rho = 2(1 - \beta - 2\gamma)\varepsilon - \alpha^{1/2}g(\rho) + \varphi(s) - k\rho^{-1}. \quad (39)$$

The characteristic system of this equation is

$$\frac{d\alpha}{2(3-5\gamma)\alpha} = \frac{d\rho}{(2-3\gamma)\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(1-\beta-2\gamma)\varepsilon - \alpha^{1/2}g(\rho) + \varphi(s) - k\rho^{-1}}.$$

Invariants of the characteristic system depend on the vanishing of the expression

$$\kappa = (2-3\gamma)(1-\beta-2\gamma)(4-2\beta-7\gamma).$$

If $\kappa \neq 0$, then the solution of (39) is

$$\varepsilon(\rho, \alpha, s) = \alpha^{1/2}\tilde{g}(s) + \tilde{\varphi}(s) + \tilde{k}\rho^{-1} + h(s, \alpha\rho^{-2(3-5\gamma)/(2-3\gamma)})\rho^{2(1-\beta-2\gamma)/(2-3\gamma)}$$

where $\tilde{g}, \tilde{\varphi}, h$ and constant $\tilde{k}$ are arbitrary elements. Using the equivalence transformations corresponding to Z_1^e, Z_2^e and Z_3^e , one gets that

$$\varepsilon(\rho, \alpha, s) = h(s, \alpha\rho^{-2(3-5\gamma)/(2-3\gamma)})\rho^{2(1-\beta-2\gamma)/(2-3\gamma)}.$$

In this case the system of equations (1) admits the generator

$$X_1 + \beta X_2 + \gamma X_3, \quad (2-3\gamma)(1-\beta-2\gamma)(4-2\beta-7\gamma) \neq 0.$$

Similar, one finds that If $\beta = 1-2\gamma, 2-3\gamma \neq 0$, then

$$\varepsilon(\rho, \alpha, s) = \phi(s) \ln \rho + h(s, \alpha\rho^{-2(3-5\gamma)/(2-3\gamma)}),$$

$$X_1 + (1-2\gamma)X_2 + \gamma X_3.$$

If $\beta = \frac{7\gamma-4}{2}, 2-3\gamma \neq 0$, then

$$\varepsilon(\rho, \alpha, s) = \rho^{-1} \left(C \ln \rho + h(s, \alpha\rho^{-2(3-5\gamma)/(2-3\gamma)}) \right),$$

$$X_1 + \left(\frac{7\gamma-4}{2}\right)X_2 + \gamma X_3.$$

If $\gamma = \frac{2}{3}, (1+3\beta)(1+6\beta) \neq 0$, then

$$\varepsilon(\rho, \alpha, s) = h(s, \rho)\alpha^{1+3\beta},$$

$$X_1 + \beta X_2 + \frac{2}{3}X_3, \quad (1+3\beta)(1+6\beta) \neq 0.$$

If $\gamma = \frac{2}{3}, \beta = -\frac{1}{3}$, then

$$\varepsilon(\rho, \alpha, s) = (\phi(s) - C\rho^{-1}) \ln \alpha + h(\rho, s),$$

$$X_1 - \frac{1}{3}X_2 + \frac{2}{3}X_3.$$

If $\gamma = \frac{2}{3}, \beta = -\frac{1}{6}$, then

$$\varepsilon(\rho, \alpha, s) = (\psi(\rho) \ln \alpha + h(\rho, s))\alpha^{1/2},$$

$$X_1 - \frac{1}{6}X_2 + \frac{2}{3}X_3.$$

Here C, ψ, ϕ and h are arbitrary.

Two-dimensional case

Considering the basis generators

$$X = \beta_1 X_1 + q_1 X_2 + \gamma_1 X_3 + X_{\zeta_1}, \quad Y = \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3 + X_{\zeta_2},$$

and their commutator satisfies the equation

$$[X, Y] = p_1 X + p_2 Y.$$

Hence,

$$[X_{\zeta_1}, X_{\zeta_2}] = p_1 X_{\zeta_1} + p_2 X_{\zeta_2},$$

and

$$(p_1 \beta_1 + p_2 \beta_2) X_1 + (p_1 q_1 + p_2 q_2) X_2 + (p_1 \gamma_1 + p_2 \gamma_2) X_3 = 0.$$

From these conditions one finds that

$$\zeta_2' \zeta_1 - \zeta_1' \zeta_2 = p_1 \zeta_1 + p_2 \zeta_2, \quad (40)$$

and

$$p_1 \beta_1 + p_2 \beta_2 = 0, \quad p_1 q_1 + p_2 q_2 = 0, \quad p_1 \gamma_1 + p_2 \gamma_2 = 0. \quad (41)$$

For two-dimensional algebras there are only two possibilities $p_1 \neq 0, p_2 = 0$ and $p_1 = 0, p_2 \neq 0$. Let us consider the case $p_1 \neq 0, p_2 = 0$. For this case one finds that the basis of the algebra consists of the generators

$$X = X_{\zeta_1}, \quad Y = \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3 + X_{\zeta_2}.$$

Since the algebra is two-dimensional, one obtains that $\zeta_1 \neq 0$. Using the equivalence transformation, one can assume that $\zeta_1 = 1$. The general solution of equation (40) is

$$\zeta_2 = p_1 S + c_0,$$

where the constant c_0 can be assumed to be zero. Thus, in the case $p_1 \neq 0, p_2 = 0$ one the admitted algebra has the form (up to equivalence transformations):

$$\{\partial_S, \beta X_1 + q X_2 + \gamma X_3 + S \partial_S\}.$$

The list of such algebras is exhausted by the following algebras

$$\{\partial_S, X_1 + \beta X_2 + \gamma X_3 + S \partial_S\}. \quad (42)$$

$$\{\partial_S, X_2 + \gamma X_3 + S \partial_S\}. \quad (43)$$

$$\{\partial_S, X_3 + S \partial_S\}. \quad (44)$$

$$\{\partial_S, S \partial_S\}. \quad (45)$$

Let us consider the case $p_1 = 0, p_2 \neq 0$. For this case equation (40) becomes

$$\zeta_2' \zeta_1 - \zeta_1' \zeta_2 = 0. \quad (46)$$

Notice that if $\zeta_1^2 + \zeta_2^2 \neq 0$, then one can assume that $\zeta_1 = 1$. In this case equation (46) gives that $\zeta_2 = k \zeta_1$. Hence, one also can assume that $\zeta_2 = 0$. Thus, admitted algebras in the case $p_1 = 0, p_2 \neq 0$ have the following forms

$$\{\beta_1 X_1 + q_1 X_2 + \gamma_1 X_3 + \partial_S, \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3\}, \quad (47)$$

$$\{\beta_1 X_1 + q_1 X_2 + \gamma_1 X_3, \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3\}, \quad (48)$$

The set of algebras of form (47) is exhausted by the list

$$\{q_1 X_2 + \gamma_1 X_3 + \partial_S, X_1 + q_2 X_2 + \gamma_2 X_3\}, \quad (49)$$

$$\{\beta_1 X_1 + \gamma_1 X_3 + \partial_S, X_2 + \gamma_2 X_3\}, \quad (50)$$

$$\{\beta_1 X_1 + q_1 X_2 + \partial_S, X_3\}. \quad (51)$$

The set of algebras of form (48) is exhausted by the list

$$\{q_1 X_2 + \gamma_1 X_3, X_1 + q_2 X_2 + \gamma_2 X_3\}, \quad (52)$$

$$\{\beta_1 X_1 + \gamma_1 X_3, X_2 + \gamma_2 X_3\}, \quad (53)$$

$$\{\beta_1 X_1 + q_1 X_2, X_3\}. \quad (54)$$

Consider $\{\partial_s, X_1 + \beta X_2 + \gamma X_3 + s\partial_s\}$

The characteristic system of equation (38) is

$$\frac{d\alpha}{2(5\gamma_2 - 3)\alpha} = \frac{d\rho}{(3\gamma_2 - 2)\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(2\gamma_2 + q_2 - 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}}.$$

Case 1 $\gamma_2 \neq 2/3$, then

$$\begin{aligned} s &= z_1, \\ \alpha &= z_2 \rho^{2(5\gamma_2 - 3)/(3\gamma_2 - 2)}, \\ z_2 &= \alpha \rho^{-2(5\gamma_2 - 3)/(3\gamma_2 - 2)}, \\ \varepsilon &= C \rho^\mu. \end{aligned}$$

Where

$$\begin{aligned} \mu &= 2(2\gamma_2 + q_2 - 1)/(3\gamma_2 - 2), \\ C' \rho^\mu &= \frac{1}{(3\gamma_2 - 2)\rho} (\alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}), \\ C &= \frac{1}{(3\gamma_2 - 2)} \int \rho^{-1-\mu} (z_2^{1/2} \rho^{(5\gamma_2 - 3)/(3\gamma_2 - 2)} g_2(\rho) - \varphi_2(z_1) + k_2\rho^{-1}) d\rho + h(z_1, z_2). \end{aligned}$$

Case 1.1 $(\mu + 1)\mu \neq 0$ or $(7\gamma_2 + 2q_2 - 4)(2\gamma_2 + q_2 - 1) \neq 0$ One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0,$$

then

$$\varepsilon = h(s, z_2) \rho^\mu.$$

Differentiate equation (38) with respect to ρ one obtains

$$\frac{d}{d\rho} (\rho^{(-7\gamma_2 - 2q_2 + 3)/(3\gamma_2 - 2)} (z_2^{1/2} \rho^{(8\gamma_2 - 5)/(3\gamma_2 - 2)} g_1(\rho) - \varphi_1(s)\rho + k_1)) = 0. \quad (55)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{(\gamma_2 - 2q_2 - 1)}{(3\gamma_2 - 2)} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-(\gamma_2 - 2q_2 - 1)/(3\gamma_2 - 2)}.$$

Equation (38) becomes

$$2(2\gamma_2 + q_2 - 1)\varphi_1\rho - (7\gamma_2 + 2q_2 - 4)k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$(2\gamma_2 + q_2 - 1)\varphi_1 = 0.$$

and

$$(7\gamma_2 + 2q_2 - 4)k_1 = 0.$$

Since $(\mu + 1)\mu \neq 0$, then $\varphi_1 = 0$ and $k_1 = 0$. Then equation (38) becomes

$$h_s + \frac{2\gamma_1}{3\gamma_2 - 2}z_2h_{z_2} = 2\left(\frac{3q_2 + 1}{3\gamma_2 - 2}\gamma_1 - q_1\right)h - c_3z_2^{1/2}.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{(3\gamma_2 - 2)dz_2}{2\gamma_1z_2} = \frac{dh}{2\left(\frac{3q_2 + 1}{3\gamma_2 - 2}\gamma_1 - q_1\right)h - c_3z_2^{1/2}}.$$

Then

$$z_3 = z_2e^{\frac{-2\gamma_1}{3\gamma_2 - 2}s},$$

$$\frac{dh}{ds} = 2\left(\frac{3q_2 + 1}{3\gamma_2 - 2}\gamma_1 - q_1\right)h - c_3z_3^{1/2}e^{\frac{\gamma_1}{3\gamma_2 - 2}s}$$

$$h = Ce^{\lambda s}, \quad \lambda = 2\left(\frac{3q_2 + 1}{3\gamma_2 - 2}\gamma_1 - q_1\right)$$

$$C' = -c_3e^{-\lambda s}z_3^{1/2}e^{-\frac{\gamma_1}{3\gamma_2 - 2}s} = -c_3z_3^{1/2}e^{-(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})s}$$

Case 1.1.1 $\lambda - \frac{\gamma_1}{3\gamma_2 - 2} = 0$, then $\lambda = \frac{\gamma_1}{3\gamma_2 - 2}$, or $q_1 = \frac{6q_2 + 1}{2(3\gamma_2 - 2)}$.

$$C = -c_3z_3^{1/2} \int ds = -c_3z_3^{1/2}s + h(z_3)$$

Then

$$h = (-c_3z_3^{1/2}s + h(z_3))e^{\lambda s} = (-c_3z_2^{1/2}e^{\frac{-\gamma_1}{3\gamma_2 - 2}s}s + h(z_3))e^{\frac{\gamma_1}{3\gamma_2 - 2}s}$$

$$h = -c_3z_2^{1/2}s + h(z_3)e^{\frac{\gamma_1}{3\gamma_2 - 2}s}.$$

Then

$$\varepsilon = (-c_3z_2^{1/2}s + h(z_3)e^{\frac{\gamma_1}{3\gamma_2 - 2}s})\rho^\mu.$$

Case 1.1.2 $\lambda - \frac{\gamma_1}{3\gamma_2 - 2} \neq 0$, or $q_1 = \frac{6q_2 + 1}{2(3\gamma_2 - 2)}$, then

$$C = -c_3z_3^{1/2} \int e^{-(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})s} ds = \frac{c_3z_3^{1/2}e^{-(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})s}}{(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})} + h(z_3)$$

Then

$$\begin{aligned} h &= \left(\frac{c_3z_3^{1/2}e^{-(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})s}}{(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})} + h(z_3) \right) e^{\lambda s} \\ &= \frac{c_3(z_2^{1/2}e^{\frac{-\gamma_1}{3\gamma_2 - 2}s})e^{\frac{\gamma_1}{3\gamma_2 - 2}s}}{(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})} + h(z_3)e^{\lambda s} \end{aligned}$$

$$\begin{aligned}
&= \frac{c_3 z_2^{1/2}}{(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})} + h(z_3)e^{\lambda s} \\
&= \frac{c_3 \alpha^{1/2} \rho^{-(5\gamma_2 - 3)/(3\gamma_2 - 2)}}{(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})} + h(z_3)e^{\lambda s}
\end{aligned}$$

Then one can assume $c_3 = 0$, then

$$h = h(z_3)e^{\lambda s}.$$

Then

$$\varepsilon = h(z_3)e^{\lambda s}\rho^\mu.$$

Consider $\{q_1 X_2 + \gamma_1 X_3 + \partial_s, X_1 + q_2 X_2 + \gamma_2 X_3\}$.

Then (38) becomes

$$10\gamma_1 \alpha \varepsilon_\alpha + 3\gamma_1 \rho \varepsilon_\rho - \varepsilon_s = 2(2\gamma_1 + q_1)\varepsilon + \alpha^{1/2} g_1(\rho) - \varphi_1 + k_1 \rho^{-1} \quad (56)$$

and

$$2(5\gamma_2 - 3)\alpha \varepsilon_\alpha + (3\gamma_2 - 2)\rho \varepsilon_\rho = 2(2\gamma_2 + q_2 - 1)\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}. \quad (57)$$

The characteristic system of equation (57) is

$$\frac{d\alpha}{2(5\gamma_2 - 3)\alpha} = \frac{d\rho}{(3\gamma_2 - 2)\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(2\gamma_2 + q_2 - 1)\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}}.$$

Case 1 $\gamma_2 \neq 2/3$, then

$$\begin{aligned}
z_1 &= s \\
z_2 &= \alpha \rho^{-2(5\gamma_2 - 3)/(3\gamma_2 - 2)} \\
\frac{d\varepsilon}{d\rho} &= \frac{1}{(3\gamma_2 - 2)\rho} (2(2\gamma_2 + q_2 - 1)\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(z_1) + k_2 \rho^{-1}) \\
\varepsilon &= C \rho^\mu, \mu = \frac{2(2\gamma_2 + q_2 - 1)}{(3\gamma_2 - 2)}
\end{aligned}$$

Where

$$\begin{aligned}
C' \rho^\mu &= \frac{1}{(3\gamma_2 - 2)\rho} (\alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}), \\
C' &= \frac{1}{(3\gamma_2 - 2)} \rho^{-1-\mu} (z_2^{1/2} \rho^{(5\gamma_2 - 3)/(3\gamma_2 - 2)} g_2(\rho) - \varphi_2(z_1) + k_2 \rho^{-1}).
\end{aligned}$$

Case 1.1 $(\mu + 1)\mu \neq 0$ or $(7\gamma_2 + 2q_2 - 4)(2\gamma_2 + q_2 - 1) \neq 0$

$$\begin{aligned}
C &= \frac{1}{(3\gamma_2 - 2)} (z_2^{1/2} \tilde{g}_2(\rho) + \frac{\varphi_2(z_1)}{\mu} \rho^{-\mu} - \frac{k_2}{1 + \mu} \rho^{-1-\mu}) + h(z_1, z_2) \\
C &= \frac{1}{(3\gamma_2 - 2)} (\alpha^{1/2} \rho^{-(5\gamma_2 - 3)/(3\gamma_2 - 2)} \tilde{g}_2(\rho) + \frac{\varphi_2(z_1)}{\mu} \rho^{-\mu} - \frac{k_2}{1 + \mu} \rho^{-1-\mu}) + h(z_1, z_2)
\end{aligned}$$

Then

$$\varepsilon = \frac{1}{(3\gamma_2 - 2)} (\alpha^{1/2} \tilde{g}_2(\rho) + \frac{\varphi_2(z_1)}{\mu} \rho^{-\mu} - \frac{k_2}{1 + \mu} \rho^{-1}) + h(z_1, z_2) \rho^\mu$$

One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0,$$

then

$$\varepsilon = h(s, z_2)\rho^\mu.$$

Differentiate equation (56) with respect to ρ one obtains

$$\frac{d}{d\rho}(\rho^{(-7\gamma_2-2q_2+3)/(3\gamma_2-2)}(z_2^{1/2}\rho^{(8\gamma_2-5)/(3\gamma_2-2)}g_1(\rho) - \varphi_1(s)\rho + k_1)) = 0. \quad (58)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{(\gamma_2 - 2q_2 - 1)}{(3\gamma_2 - 2)}g_1 = 0,$$

then

$$g_1 = c_3\rho^{-(\gamma_2-2q_2-1)/(3\gamma_2-2)}.$$

Equation (58) becomes

$$2(2\gamma_2 + q_2 - 1)\varphi_1\rho - (7\gamma_2 + 2q_2 - 4)k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$(2\gamma_2 + q_2 - 1)\varphi_1 = 0.$$

and

$$(7\gamma_2 + 2q_2 - 4)k_1 = 0.$$

Since $(\mu + 1)\mu \neq 0$, then $\varphi_1 = 0$ and $k_1 = 0$. Then equation (56) becomes

$$h_s + \frac{2\gamma_1}{3\gamma_2-2}z_2h_{z_2} = 2(\frac{3q_2+1}{3\gamma_2-2}\gamma_1 - q_1)h - c_3z_2^{1/2}.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{(3\gamma_2 - 2)dz_2}{2\gamma_1z_2} = \frac{dh}{2(\frac{3q_2+1}{3\gamma_2-2}\gamma_1 - q_1)h - c_3z_2^{1/2}}.$$

Then

$$z_3 = z_2e^{\frac{-2\gamma_1}{3\gamma_2-2}s}$$

$$\frac{dh}{ds} = 2(\frac{3q_2+1}{3\gamma_2-2}\gamma_1 - q_1)h - c_3z_3^{1/2}e^{\frac{\gamma_1}{3\gamma_2-2}s}$$

$$h = Ce^{\lambda s}, \lambda = 2(\frac{3q_2+1}{3\gamma_2-2}\gamma_1 - q_1)$$

$$C'e^{\lambda s} = -c_3z_3^{1/2}e^{\frac{\gamma_1}{3\gamma_2-2}s}$$

$$C' = -c_3e^{-\lambda s}z_3^{1/2}e^{\frac{\gamma_1}{3\gamma_2-2}s} = -c_3z_3^{1/2}e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}$$

Case 1.1.1 $\lambda - \frac{\gamma_1}{3\gamma_2-2} = 0$, then

$$C = -c_3z_3^{1/2}s + h(z_3)$$

Then

$$h = (-c_3z_3^{1/2}s + h(z_3))e^{\lambda s} = (-c_3z_2^{1/2}e^{\frac{-\gamma_1}{3\gamma_2-2}s}s + h(z_3))e^{\lambda s}$$

$$h = -c_3 z_2^{1/2} s + h(z_3) e^{\lambda s}.$$

Then

$$\varepsilon = (-c_3 z_2^{1/2} s + h(z_3) e^{\lambda s}) \rho^\mu.$$

Case 1.1.2 $\lambda - \frac{\gamma_1}{3\gamma_2-2} \neq 0$, then

$$\begin{aligned} C &= \frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\beta_1}{3})s}}{-(\lambda - \frac{\beta_1}{3})} + h(z_3) \\ h &= \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}}{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) \right) e^{\lambda s} \\ &= \frac{c_3 (z_2^{1/2} e^{\frac{-\gamma_1}{3\gamma_2-2}s}) e^{\frac{\gamma_1}{3\gamma_2-2}s}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) e^{\lambda s} \\ &= \frac{c_3 z_2^{1/2}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) e^{\lambda s} \\ &= \frac{c_3 \alpha^{1/2} \rho^{-(5\gamma_2-3)/(3\gamma_2-2)}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) e^{\lambda s} \end{aligned}$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(z_3) e^{\lambda s} \rho^\mu.$$

Case 1.2 $\mu = -1$, then $q_2 = 2 - \frac{7}{2}\gamma_2$

$$C = \frac{1}{(3\gamma_2 - 2)} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \rho + k_2 \ln \rho) + h(z_1, z_2).$$

Then

$$\varepsilon = \frac{1}{(3\gamma_2 - 2)} (\alpha^{1/2} \tilde{g}_2(\rho) - \varphi_2(s) + k_2 \rho^{-1} \ln \rho) + h(s, z_2) \rho^{-1}$$

One can assume that

$$g_2 = 0, \varphi_2 = 0,$$

then

$$\varepsilon = \left[\frac{k_2}{(3\gamma_2 - 2)} \ln \rho + h(s, z_2) \right] \rho^{-1}.$$

So equation (56) becomes

$$\begin{aligned} h_s + \frac{2\gamma_1}{3\gamma_2-2} z_2 h_{z_2} &= -(7\gamma_1 + 2q_1)h - z_2^{1/2} \rho^{(8\gamma_2-5)/(3\gamma_2-2)} g_1(\rho) \\ &\quad + \varphi_1(s) \rho - k_2 \frac{7\gamma_1+2q_1}{3\gamma_2-2} \ln \rho + k_2 \frac{3\gamma}{3\gamma_2-2} - k_1. \end{aligned} \quad (59)$$

Differentiate this equation with respect to ρ one obtains

$$\frac{d}{d\rho} (z_2^{1/2} \rho^{(8\gamma_2-5)/(3\gamma_2-2)} g_1(\rho) - \varphi_1(s) \rho + k_2 \frac{7\gamma_1+2q_1}{3\gamma_2-2} \ln \rho) = 0. \quad (60)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{(8\gamma_2 - 5)}{(3\gamma_2 - 2)} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-(8\gamma_2-5)/(3\gamma_2-2)}.$$

Equation (60) becomes

$$\varphi_1 \rho - \frac{(7\gamma_1 + 2q_1)}{(3\gamma_2 - 2)} k_2 = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi_1 = 0,$$

and

$$\frac{(7\gamma_1 + 2q_1)}{(3\gamma_2 - 2)} k_2 = 0.$$

Case 1.2.1 $7\gamma_1 + 2q_1 \neq 0$, or $q_1 \neq -7\gamma_1/2$ then $k_2 = 0$. Equation (59) becomes

$$h_s + \frac{2\gamma_1}{(3\gamma_2-2)} z_2 h_{z_2} = -(7\gamma_1 + 2q_1)h - c_3 z_2^{1/2} - k_1.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{(3\gamma_2 - 2)dz_2}{2\gamma_1 z_2} = \frac{dh}{-(7\gamma_1 + 2q_1)h - c_3 z_2^{1/2} - k_1}.$$

Then

$$z_3 = z_2 e^{\frac{-2\gamma_1}{3\gamma_2-2}s}$$

$$\frac{dh}{ds} = -(7\gamma_1 + 2q_1)h - c_3 z_3^{1/2} e^{\frac{\gamma_1}{3\gamma_2-2}s} - k_1$$

$$h = C e^{\lambda s}, \lambda = -(7\gamma_1 + 2q_1)$$

$$C' e^{\lambda s} = -c_3 z_3^{1/2} e^{\frac{\gamma_1}{3\gamma_2-2}s} - k_1$$

$$C' = -c_3 e^{-\lambda s} z_3^{1/2} e^{\frac{\gamma_1}{3\gamma_2-2}s} - k_1 e^{-\lambda s} = -c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s} - k_1 e^{-\lambda s}$$

Case 1.2.1.1 $\lambda - \frac{\gamma_1}{3\gamma_2-2} = 0$, or $q_1 = -\frac{1}{2}(7 + \frac{1}{3\gamma_2-2})\gamma_1$ then

$$C = -c_3 z_3^{1/2} s + k_1 \frac{e^{-\lambda s}}{\lambda} + h(z_3)$$

Then

$$h = (-c_3 z_3^{1/2} s + k_1 \frac{e^{-\lambda s}}{\lambda} + h(z_3)) e^{\lambda s} = (-c_3 z_2^{1/2} e^{\frac{-\gamma_1}{3\gamma_2-2}s} s + k_1 \frac{e^{-\lambda s}}{\lambda} + h(z_3)) e^{\lambda s}$$

$$h = -c_3 z_2^{1/2} s + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s}.$$

Then

$$\varepsilon = (-c_3 z_2^{1/2} s + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s}) \rho^{-1}.$$

One can assume that

$$k_1 = 0,$$

then

$$\varepsilon = (-c_3 z_2^{1/2} s + h(z_3) e^{\lambda s}) \rho^{-1}.$$

Case 1.2.1.2 $\lambda - \frac{\gamma_1}{3\gamma_2-2} \neq 0$, or $q_1 \neq -\frac{1}{2}(7 + \frac{1}{3\gamma_2-2})\gamma_1$ then

$$\begin{aligned}
C &= \frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}}{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + k_1 \frac{e^{-\lambda s}}{\lambda} + h(z_3) \\
h &= \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}}{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + k_1 \frac{e^{-\lambda s}}{\lambda} + h(z_3) \right) e^{\lambda s} \\
&= \frac{c_3 (z_2^{1/2} e^{\frac{-\gamma_1}{3\gamma_2-2}s}) e^{\frac{\gamma_1}{3\gamma_2-2}s}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s} \\
&= \frac{c_3 z_2^{1/2}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s} \\
&= \frac{c_3 \alpha^{1/2} \rho^{-(5\gamma_2-3)/(3\gamma_2-2)}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s}
\end{aligned}$$

Then one can assume $c_3 = 0, k_1 = 0$ so

$$\varepsilon = h(z_3) e^{\lambda s} \rho^{-1}.$$

Case 1.1.2 $\lambda - \frac{\gamma_1}{3\gamma_2-2} \neq 0$, then

$$\begin{aligned}
C &= \frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\beta_1}{3})s}}{-(\lambda - \frac{\beta_1}{3})} + h(z_3) \\
h &= \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}}{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) \right) e^{\lambda s} \\
&= \frac{c_3 (z_2^{1/2} e^{\frac{-\gamma_1}{3\gamma_2-2}s}) e^{\frac{\gamma_1}{3\gamma_2-2}s}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) e^{\lambda s} \\
&= \frac{c_3 z_2^{1/2}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) e^{\lambda s} \\
&= \frac{c_3 \alpha^{1/2} \rho^{-(5\gamma_2-3)/(3\gamma_2-2)}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + h(z_3) e^{\lambda s}
\end{aligned}$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(z_3) e^{\lambda s} \rho^\mu.$$

Case 1.2.2 $7\gamma_1 + 2q_1 = 0$ or $q_1 = -7\gamma_1/2$ Then equation (59) becomes

$$h_s + \frac{2\gamma_1}{(3\gamma_2-2)} z_2 h_{z_2} = -c_3 z_2^{1/2} - k_1 + \frac{3\gamma_1}{(3\gamma_2-2)} k_2.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{(3\gamma_2 - 2)dz_2}{2\gamma_1 z_2} = \frac{dh}{-c_3 z_2^{1/2} - k_1 + \frac{3\gamma_1}{(3\gamma_2-2)} k_2},$$

Then

$$\begin{aligned}
 z_3 &= z_2 e^{\frac{-2\gamma_1}{3\gamma_2-2}s} \\
 \frac{dh}{ds} &= -c_3 z_2^{1/2} - k_1 + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 \\
 h &= \int -c_3 z_2^{1/2} - k_1 + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 ds + h(z_3) \\
 &= \int -c_3 z_3^{1/2} e^{\frac{\gamma_1}{3\gamma_2-2}s} - k_1 + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 ds + h(z_3)
 \end{aligned}$$

Case 1.2.2.1 $\frac{\gamma_1}{3\gamma_2-2} = 0$.

$$h = -c_3 z_3^{1/2} s - k_1 s + h(z_3).$$

Then

$$\varepsilon = \left[\frac{k_2}{(3\gamma_2-2)} \ln \rho - c_3 z_3^{1/2} s - k_1 s + h(z_3) \right] \rho^{-1}.$$

Case 1.2.2.2 $\frac{\gamma_1}{3\gamma_2-2} \neq 0$.

$$h = -c_3 z_3^{1/2} \frac{e^{\frac{\gamma_1}{3\gamma_2-2}s}}{\frac{\gamma_1}{3\gamma_2-2}} - k_1 s + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 s + h(z_3)$$

Then

$$\varepsilon = \left[\frac{k_2}{(3\gamma_2-2)} \ln \rho - c_3 z_3^{1/2} \frac{e^{\frac{\gamma_1}{3\gamma_2-2}s}}{\frac{\gamma_1}{3\gamma_2-2}} - k_1 s + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 s + h(z_3) \right] \rho^{-1}.$$

Case 1.3 $\mu = 0$ or $q_2 = 1 - 2\gamma_2$

$$C = \frac{1}{(3\gamma_2-2)} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \ln \rho - k_2 \rho^{-1}) + h(z_1, z_2).$$

Then

$$\varepsilon = \frac{1}{(3\gamma_2-2)} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \ln \rho - k_2 \rho^{-1}) + h(z_1, z_2)$$

One can assume that

$$g_2 = 0, k_2 = 0,$$

then

$$\varepsilon = \frac{-1}{(3\gamma_2-2)} \varphi_2(s) \ln \rho + h(s, z_2)$$

So equation (56) becomes

$$\begin{aligned}
 h_s + \frac{2\gamma_1}{3\gamma_2-2} z_2 h_{z_2} &= -2(2\gamma_1 + q_1)h - z_2^{1/2} \rho^{(5\gamma_2-3)/(3\gamma_2-2)} g_1(\rho) \\
 &\quad - \varphi_2(s) \frac{-2(2\gamma_1+q_1)\ln \rho + 3\gamma_1}{3\gamma_2-2} + \varphi_1(s) - k_1 \rho^{-1}.
 \end{aligned} \tag{61}$$

Differentiate equation (61) with respect to ρ one obtains

$$\frac{d}{d\rho} (z_2^{1/2} \rho^{(5\gamma_2-3)/(3\gamma_2-2)} g_1(\rho) + \varphi_2(s) \frac{(-4\gamma_1-2q_1)\ln \rho + 3\gamma_1}{3\gamma_2-2} + k_1 \rho^{-1}) = 0. \tag{62}$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{(5\gamma_2 - 3)}{(3\gamma_2 - 2)} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-(5\gamma_2 - 3)/(3\gamma_2 - 2)}.$$

Equation (60) becomes

$$\frac{2(2\gamma_1 + q_1)}{(3\gamma_2 - 2)} \rho \varphi_2 + k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$k_1 = 0,$$

and

$$\frac{(2\gamma_1 + q_1)}{(3\gamma_2 - 2)} \varphi_2 = 0.$$

Case 1.3.1 $2\gamma_1 + q_1 \neq 0$, or $q_1 \neq -2\gamma_1$ then $\varphi_2 = 0$. Then (61) becomes

$$h_s + \frac{2\gamma_1}{3\gamma_2 - 2} z_2 h_{z_2} = -2(2\gamma_1 + q_1)h - z_2^{1/2} c_3 + \varphi_1(s).$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{(3\gamma_2 - 2)dz_2}{2\gamma_1 z_2} = \frac{dh}{-2(2\gamma_1 + q_1)h - c_3 z_2^{1/2} + \varphi_1(s)}.$$

Then

$$z_3 = z_2 e^{\frac{-2\gamma_1}{3\gamma_2 - 2}s}$$

$$\frac{dh}{ds} = -2(2\gamma_1 + q_1)h - z_2^{1/2} c_3 + \varphi_1(s)$$

$$h = C e^{\lambda s}, \lambda = -2(2\gamma_1 + q_1)$$

$$C' e^{\lambda s} = -c_3 z_2^{1/2} + \varphi_1(s)$$

$$C' = -c_3 e^{-\lambda s} z_3^{1/2} e^{\frac{\gamma_1}{3\gamma_2 - 2}s} + \varphi_1(s) e^{-\lambda s} = -c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2 - 2})s} + \tilde{\varphi}_1(s)$$

Case 1.3.1.1 $\lambda - \frac{\gamma_1}{3\gamma_2 - 2} = 0$, or $q_1 = -(2 + \frac{1}{2(3\gamma_2 - 2)})\gamma_1$ then

$$C = -c_3 z_3^{1/2} s + \tilde{\varphi}_1(s) + h(z_3)$$

Then

$$h = (-c_3 z_3^{1/2} s + \tilde{\varphi}_1(s) + h(z_3)) e^{\lambda s} = (-c_3 z_2^{1/2} e^{\frac{-\gamma_1}{3\gamma_2 - 2}s} s + \tilde{\varphi}_1(s) + h(z_3)) e^{\lambda s}$$

$$h = -c_3 z_2^{1/2} s + \tilde{\tilde{\varphi}}_1(s) + h(z_3) e^{\lambda s}.$$

$$\varepsilon = -c_3 z_2^{1/2} s + \tilde{\tilde{\varphi}}_1(s) + h(z_3) e^{\lambda s}.$$

One can assume that

$$\varphi_1 = 0,$$

then

$$\varepsilon = -c_3 z_2^{1/2} s + h(z_3) e^{\lambda s}.$$

Case 1.3.1.2 $\lambda - \frac{\gamma_1}{3\gamma_2-2} \neq 0$, or $q_1 \neq -(2 + \frac{1}{2(3\gamma_2-2)})\gamma_1$ then

$$\begin{aligned}
C &= \frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}}{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \tilde{\varphi}_1(s) + h(z_3) \\
h &= \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})s}}{-(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \tilde{\varphi}_1(s) + h(z_3) \right) e^{\lambda s} \\
&= \frac{c_3 (z_2^{1/2} e^{\frac{-\gamma_1}{3\gamma_2-2}s}) e^{\frac{\gamma_1}{3\gamma_2-2}s}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \tilde{\varphi}_1(s) + h(z_3) e^{\lambda s} \\
&= \frac{c_3 z_2^{1/2}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \tilde{\varphi}_1(s) + h(z_3) e^{\lambda s} \\
&= \frac{c_3 \alpha^{1/2} \rho^{-(5\gamma_2-3)/(3\gamma_2-2)}}{(\lambda - \frac{\gamma_1}{3\gamma_2-2})} + \tilde{\varphi}_1(s) + h(z_3) e^{\lambda s}
\end{aligned}$$

Then one can assume $c_3 = 0, \varphi_1 = 0$ so

$$\varepsilon = h(z_3) e^{\lambda s}.$$

Case 1.3.2 $2\gamma_1 + q_1 = 0$ or $q_1 = -2\gamma_1$ Then equation (61) becomes

$$h_s + \frac{2\gamma_1}{3\gamma_2-2} z_2 h_{z_2} = -z_2^{1/2} c_3 + \varphi_1(s) - \frac{3\gamma_1}{3\gamma_2-2} \varphi_2(s).$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{(3\gamma_2 - 2)dz_2}{2\gamma_1 z_2} = \frac{dh}{-z_2^{1/2} c_3 + \varphi_1(s) - \frac{3\gamma_1}{3\gamma_2-2} \varphi_2(s)},$$

Then

$$\begin{aligned}
z_3 &= z_2 e^{\frac{-2\gamma_1}{3\gamma_2-2}s} \\
\frac{dh}{ds} &= -z_2^{1/2} c_3 + \varphi_1(s) - \frac{3\gamma_1}{3\gamma_2-2} \varphi_2(s) \\
h &= \int -c_3 z_2^{1/2} + \varphi_1(s) - \frac{3\gamma_1}{3\gamma_2-2} \varphi_2(s) ds + h(z_3) \\
&= \int -c_3 z_3^{1/2} e^{\frac{\gamma_1}{3\gamma_2-2}s} + \varphi(s) ds + h(z_3)
\end{aligned}$$

Case 1.2.2.1 $\frac{\gamma_1}{3\gamma_2-2} = 0$.

$$h = -c_3 z_3^{1/2} s + \tilde{\varphi}(s) + h(z_3).$$

Then

$$\varepsilon = \frac{-1}{(3\gamma_2 - 2)} \varphi_2(s) \ln \rho - c_3 z_3^{1/2} s + \tilde{\varphi}(s) + h(z_3)$$

one can assume that $\varphi = 0$.

$$\varepsilon = \frac{-1}{(3\gamma_2 - 2)} \varphi_2(s) \ln \rho - c_3 z_3^{1/2} s + h(z_3)$$

Case 1.3.2.2 $\frac{\gamma_1}{3\gamma_2-2} \neq 0$.

$$h = -c_3 z_3^{1/2} \frac{e^{\frac{\gamma_1}{3\gamma_2-2}s}}{\frac{\gamma_1}{3\gamma_2-2}} - k_1 s + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 s + h(z_3)$$

Then

$$\varepsilon = \left[\frac{k_2}{(3\gamma_2-2)} \ln \rho - c_3 z_3^{1/2} \frac{e^{\frac{\gamma_1}{3\gamma_2-2}s}}{\frac{\gamma_1}{3\gamma_2-2}} - k_1 s + \frac{3\gamma_1}{(3\gamma_2-2)} k_2 s + h(z_3) \right] \rho^{-1}.$$

Case 1.3.2 $2\gamma_1 + q_1 = 0$, then equation (61) becomes

$$h_{z_1} + \frac{2\gamma_1}{3\gamma_2-2} z_2 h_{z_2} = -z_2^{1/2} c_3 + \varphi_1(s) - \frac{3\gamma_1}{3\gamma_2-2} \varphi_2(s).$$

Differentiate this equation with respect to s one obtains

$$\varphi_1'(s) = \frac{3\gamma_1}{3\gamma_2-2} \varphi_2'(s),$$

then

$$\varphi_1(s) = \frac{3\gamma_1}{3\gamma_2-2} \varphi_2(s) + c_1.$$

So equation (61) becomes

$$h_{z_1} + \frac{2\gamma_1}{3\gamma_2-2} z_2 h_{z_2} = -c_3 z_2^{1/2} + c_1.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{(3\gamma_2-2)dz_2}{2\gamma_1 z_2} = \frac{dh}{-c_3 z_2^{1/2} + c_1},$$

Case 1.3.2.1 $\gamma_1 \neq 0$ then

$$e^{z_1} = z z_2^{\frac{3\gamma_2-2}{2\gamma_1}},$$

$$h = \frac{(3\gamma_2-2)}{2\gamma_1} \int z_2^{-1} (-c_3 z_2^{1/2} - c_1) dz_2 + h(z).$$

one can assume that

$$c_3 = 0.$$

Then

$$h = \frac{(3\gamma_2-2)}{2\gamma_1} c_1 \ln z_2 + h(z).$$

Case 1.3.2.2 $\gamma_1 = 0$ then

$$z_2 = z,$$

$$h = \int (-c_3 z^{1/2} - c_1) dz_1 + h(z).$$

one can assume that

$$c_1 = 0.$$

Then

$$h = (-c_3 z^{1/2}) z_1 + h(z).$$

Case 2 $\gamma_2 = 2/3$, then the characteristic is

$$\begin{aligned}\frac{3d\alpha}{2\alpha} &= \frac{d\rho}{0} = \frac{ds}{0} = \frac{d\varepsilon}{2(1/3 + q_2)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}}. \\ s &= z_1, \\ \rho &= z_2, \\ \varepsilon &= C\alpha^{1+3q_2}.\end{aligned}$$

Where

$$C'\alpha^{1+3q_2} = \frac{3}{2\alpha}(\alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}).$$

Then

$$C = \frac{3}{2} \int \alpha^{-2-3q_2}(\alpha^{1/2}g_2(z_2) - \varphi_2(z_2) + k_2z_2^{-1})d\alpha + h(z_1, z_2).$$

Case 2.1 $(q_2 + 1/6)(q_2 + 1/3) \neq 0$. One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0.$$

Then,

$$\varepsilon = h(z_1, z_2)\alpha^{1+3q_2}.$$

Differentiate equation (61) with respect to α one obtains

$$-(3q_2 + 1/2)\alpha^{(1/2)}g_1(\rho) + (3q_2 + 1)\varphi_1(s) - z_2^{-1}k_1(3q_2 + 1) = 0. \quad (63)$$

Since $(q_2 + 1/6)(q_2 + 1/3) \neq 0$, then differentiate this equation with respect to α and z_2 one obtains

$$g_1 = 0, k_1 = 0.$$

Then equation (63) becomes

$$(3q_2 + 1)\varphi_1(s) = 0.$$

Since $(q_2 + 1/3) \neq 0$, then $\varphi_1 = 0$. So equation (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = 2((15q_2 + 3)\gamma_1 - q_1)h.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{-dz_2}{3\gamma_1 z_2} = \frac{dh}{2((15q_2 + 3)\gamma_1 - q_1)h}.$$

Case 2.1.1 $\gamma_1 \neq 0$ then

$$\begin{aligned}e^{z_1} &= z z_2^{\frac{-1}{3\gamma_1}}, \\ h &= h(z)z_2^\lambda.\end{aligned}$$

Where

$$\lambda = -2((15q_2 + 3)\gamma_1 - q_1)/3\gamma_1,$$

Case 2.1.2 $\gamma_1 = 0$ then

$$\begin{aligned}z_2 &= z, \\ h &= h(z)e^{-2q_1 z_1},\end{aligned}$$

Case 2.2 $q_2 = -1/6$. One can assume that

$$\varphi_2 = 0, k_2 = 0.$$

Then,

$$\varepsilon = (3/2)g_2(\rho) \ln \alpha \alpha^{1/2} + h(z_1, z_2)\alpha^{1/2}.$$

Differentiate equation (56) with respect to α one obtains

$$-(3\gamma_1 - 6q_1)g_2(\rho) + \alpha^{-1/2}\varphi_1(s) - \alpha^{-1/2}z_2^{-1}k_1 = 0. \quad (64)$$

Differentiate this equation with respect to z_2 and α one obtains

$$k_1 = 0, \varphi_1 = 0.$$

Then equation (64) becomes

$$(\gamma_1 - 2q_1)g_2(\rho) = 0.$$

Case 2.2.1 $\gamma_1 - 2q_1 \neq 0$, then $g_2 = 0$. So equation (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = (\gamma_1 - 2q_1)h - g_1(\rho).$$

Differentiate this equation with respect to ρ one obtains

$$g = c_3.$$

Then equation (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = (\gamma_1 - 2q_1)h - c_3.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{-dz_2}{3\gamma_1 z_2} = \frac{dh}{(\gamma_1 - 2q_1)h - c_3}.$$

Case 2.2.1.1 $\gamma_1 \neq 0$ then

$$e^{z_1} = z z_2^{\frac{-1}{3\gamma_1}},$$

$$h = C z_2^\lambda.$$

Where

$$\lambda = -(\gamma_1 - 2q_1)/3\gamma_1,$$

$$C' z_2^\lambda = \frac{c_3}{3\gamma_1 z_2},$$

$$C' = \frac{1}{3\gamma_1} \int z_2^{-1-\lambda} c_3 dz_2 + h(z).$$

Since $\gamma_1 - 2q_1 \neq 0$, then one can assume that

$$c_3 = 0.$$

Then

$$h = h(z) z_2^\lambda.$$

Case 2.2.1.2 $\gamma_1 = 0$ then

$$\begin{aligned} z_2 &= z, \\ h &= Ce^{-2qz_1}, \end{aligned}$$

where

$$C' e^{-2qz_1} = -c_3.$$

Then

$$C = \int -c_3 e^{2q_1 z_1} dz_1 + h(z),$$

Since $\gamma_1 - 2q_1 \neq 0$, then $q_1 \neq 0$, one can assume that

$$c_3 = 0.$$

Then

$$h = h(z)e^{-2q_1 z_1}.$$

Case 2.2.2 $\gamma_1 - 2q_1 = 0$, then equation (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = 15\gamma_1 g_2(\rho) - g_1(\rho).$$

Differentiate this equation with respect to ρ one obtains

$$g_1'(\rho) = 15\gamma_1 g_2'(\rho).$$

Then

$$g_1(\rho) = 15\gamma_1 g_2(\rho) + c_3.$$

Equation (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = -c_3.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{-dz_2}{3\gamma_1 z_2} = \frac{dh}{-c_3}.$$

Case 2.2.2.1 $\gamma_1 \neq 0$ then

$$\begin{aligned} e^{z_1} &= z z_2^{\frac{-1}{3\gamma_1}}, \\ h &= \frac{1}{3\gamma_1} \int z_2^{-1} c_3 dz_2 + h(z). \end{aligned}$$

Then

$$h = \left(\frac{1}{3\gamma_1} c_3 \ln z_2 + h(z) \right) z_2^{\frac{-1}{3\gamma_1}}.$$

Case 2.2.2.2 $\gamma_1 = 0$ then

$$\begin{aligned} z_2 &= z, \\ h &= \int -c_3 dz_1 + h(z). \end{aligned}$$

One can assume that

$$h(z) = 0.$$

Then

$$h = -c_3 z_1.$$

Case 2.3 $q_2 = -1/3$. One can assume that

$$g_2 = 0.$$

Then,

$$\varepsilon = -(3/2)(\varphi_2(s) - k_2\rho^{-1}) \ln \alpha + h(z_1, z_2).$$

Differentiate equation (56) with respect to α one obtains

$$\alpha^{1/2}g_1(\rho) - 2(6\gamma_1 + 3q_1)\varphi_2(s) + z_2^{-1}3k_2(7\gamma_1 + 2q_1) = 0. \quad (65)$$

Differentiate this equation with respect to α one obtains

$$g_1 = 0,$$

Differentiate equation (65) with respect to z_2 one obtains

$$k_2(21\gamma_1 + 6q_1) = 0.$$

Case 2.3.1 $7\gamma_1 + 2q_1 \neq 0$, then $k_2 = 0$. Then equation (65) becomes

$$(2\gamma_1 + q_1)\varphi_2(s) = 0.$$

Case 2.3.1.1 $2\gamma_1 + q_1 \neq 0$, then $\varphi_2 = 0$. So equation (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = -2(2\gamma_1 + q_1)h + \varphi_1 - k_1 z_2^{-1}.$$

Differentiate this equation with respect to s one obtains

$$\varphi_1 = c_1,$$

Then one obtains the equation

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = -2(2\gamma_1 + q_1)h + c_1 - k_1 z_2^{-1}.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{-dz_2}{3\gamma_1 z_2} = \frac{dh}{-2(2\gamma_1 + q_1)h + c_1 - k_1 z_2^{-1}}.$$

Case 2.3.1.1.1 $\gamma_1 \neq 0$ then

$$e^{z_1} = z z_2^{\frac{-1}{3\gamma_1}},$$

$$h = C z_2^\lambda.$$

Where

$$\lambda = 2(2\gamma_1 + q_1)/3\gamma_1,$$

$$C' z_2^\lambda = \frac{-1}{3\gamma_1 z_2} (c_1 - k_1 z_2^{-1}).$$

$$C' = \frac{1}{3\gamma_1} \int z_2^{-1-\lambda} (-c_1 + k_1 z_2^{-1}) dz_2 + h(z).$$

Remark $\lambda \neq 0$. and $\lambda \neq -1$, then One can assume that

$$c_1 = 0, k_1 = 0,$$

then

$$h = h(z)z_2^\lambda,$$

Case 2.3.1.1.2 $\gamma_1 = 0$ then

$$\begin{aligned} z_2 &= z, \\ h &= Ce^{-2qz_1}, \end{aligned}$$

where

$$C'e^{-2qz_1} = c_1 - k_1z_2^{-1}.$$

Then

$$C = \int (c_1 - k_1z^{-1})e^{2q_1z_1}dz_1 + h(z),$$

Since $2\gamma_1 + q_1 \neq 0$, and $\gamma_1 = 0$ then $q_1 \neq 0$. One can assume that

$$c_1 = 0, k_1 = 0,$$

then

$$h = h(z)e^{-2qz_1}.$$

Case 2.3.1.2 $2\gamma_1 + q_1 = 0$, then equation (56) becomes

$$h_{z_1} - 3\gamma_1z_2h_{z_2} = \varphi_1 - 15\gamma_1\varphi_2 - k_1z_2^{-1}.$$

Differentiate this equation with respect to s one obtains

$$\varphi_1' = 15\gamma_1\varphi_2',$$

then

$$\varphi_1 = 15\gamma_1\varphi_2 + c_1,$$

Then equation (56) becomes

$$h_{z_1} - 3\gamma_1z_2h_{z_2} = c_1 - k_1z_2^{-1}.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{-dz_2}{3\gamma_1z_2} = \frac{dh}{c_1 - k_1z_2^{-1}}.$$

Case 2.3.1.2.1 $\gamma_1 \neq 0$ then

$$e^{z_1} = zz_2^{\frac{-1}{3\gamma_1}},$$

$$h = \frac{-1}{3\gamma_1} \int z_2^{-1}(c_1 - k_1z_2^{-1})dz_2 + h(z).$$

one can assume that

$$k_1 = 0.$$

Then

$$h = \frac{-1}{3\gamma_1}c_1 \ln z_2 + h(z).$$

Case 2.3.1.2.2 $\gamma_1 = 0$ then

$$z_2 = z,$$

$$h = \int (c_1 - k_1z^{-1})dz_1 + h(z).$$

one can assume that

$$c_1 = 0.$$

Then

$$h = (-k_1 z^{-1})z_1 + h(z).$$

Case 2.3.2 $7\gamma_1 + 2q_1 = 0$, then (65) becomes

$$\gamma_1 \varphi_2 = 0,$$

Case 2.3.2.1 $\gamma_1 \neq 0$, then $\varphi_2 = 0$. Then (56) becomes

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = 3\gamma_1 h + \varphi_1 + (15\gamma_1 k_2 - k_1)z_2^{-1}.$$

Differentiate this equation with respect to s one obtains

$$\varphi_1 = c_1,$$

Then one obtains

$$h_{z_1} - 3\gamma_1 z_2 h_{z_2} = 3\gamma_1 h + c_1 + (15\gamma_1 k_2 - k_1)z_2^{-1}.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{-dz_2}{3\gamma_1 z_2} = \frac{dh}{3\gamma_1 h + c_1 + (15\gamma_1 k_2 - k_1)z_2^{-1}}.$$

then

$$e^{z_1} = z z_2^{\frac{-1}{3\gamma_1}},$$

$$h = C z_2^{-1},$$

where

$$C' z_2^{-1} = \frac{-1}{3\gamma_1 z_2} (c_1 + (15\gamma_1 k_2 - k_1)z_2^{-1}).$$

Then

$$C = \frac{-1}{3\gamma_1} \int (c_1 + (15\gamma_1 k_2 - k_1)z_2^{-1}) dz_2 + h(z).$$

Then

$$h = \frac{-1}{3\gamma_1} (c_1 z_2 + (15\gamma_1 k_2 - k_1) \ln z_2) + h(z).$$

Case 2.3.2.2 $\gamma_1 = 0$, then (56) becomes

$$h_{z_1} = \varphi_1 - k_1 z_2^{-1}.$$

Differentiate this equation with respect to s one obtains

$$\varphi_1 = c_1,$$

then (56) becomes

$$h_{z_1} = c_1 - k_1 z_2^{-1}.$$

The characteristic system of this equation is

$$\frac{dz_1}{1} = \frac{dz_2}{0} = \frac{dh}{c_1 - k_1 z_2^{-1}}.$$

then

$$z_2 = z,$$

$$h = \int (c_1 - k_1 z^{-1}) dz_1 + h(z).$$

one can assume that

$$c_1 = 0.$$

Then

$$h = (-k_1 z^{-1}) z_1 + h(z).$$

Consider $\{\beta_1 X_1 + \gamma_1 X_3 + \partial_s, X_2 + \gamma_2 X_3\}$.

Then (38) becomes

$$2(3\beta_1 - 5\gamma_1)\alpha\varepsilon_\alpha + (2\beta_1 - 3\gamma_1)\rho\varepsilon_\rho + \varepsilon_s = 2(\beta_1 - 2\gamma_1)\varepsilon - \alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1} \quad (66)$$

and

$$10\gamma_2\alpha\varepsilon_\alpha + 3\gamma_2\rho\varepsilon_\rho = 2(2\gamma_2 + 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}. \quad (67)$$

If $\gamma_2 = 0$, then $\varepsilon = (-1/2)(\alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1})$. So $\varepsilon(\rho, \alpha, s)$ can be transformed to zero thus we will consider in case $\gamma_2 \neq 0$. The characteristic system of equation (67) is

$$\frac{d\alpha}{10\gamma_2\alpha} = \frac{d\rho}{3\gamma_2\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(2\gamma_2 + 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}}.$$

Then

$$z_1 = s$$

$$z_2 = \alpha\rho^{-10/3}$$

$$\frac{d\varepsilon}{d\rho} = \frac{1}{3\gamma_2\rho}(2(2\gamma_2 + 1)\varepsilon + \alpha^{1/2}\rho^{5/3}g_2(\rho) - \varphi_2(z_1) + k_2\rho^{-1})$$

$$\varepsilon = C\rho^\mu, \mu = (2/3)(2 + 1/\gamma_2)$$

$$C' = \frac{\rho^{-1-\mu}}{3\gamma_2}(z_2^{1/2}\rho^{5/3}g_2(\rho) - \varphi_2(z_1) + k_2\rho^{-1})$$

Case 1 $(\mu + 1)\mu \neq 0$ or $(\gamma_2 + 1/2)(\gamma_2 + 2/7) \neq 0$. Then

$$C = \frac{1}{3\gamma_2}(z_2^{1/2}\tilde{g}_2(\rho) + \frac{\varphi_2(z_1)}{\mu}\rho^{-\mu} - \frac{k_2}{1+\mu}\rho^{-1-\mu}) + h(z_1, z_2)$$

Then

$$\varepsilon = (\frac{1}{3\gamma_2}(z_2^{1/2}\tilde{g}_2(\rho) + \frac{\varphi_2(z_1)}{\mu}\rho^{-\mu} - \frac{k_2}{1+\mu}\rho^{-1-\mu}) + h(s, z_2))\rho^\mu$$

One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0,$$

then

$$\varepsilon = h(s, z_2)\rho^\mu.$$

Differentiate equation (66) with respect to ρ one obtains

$$\frac{d}{d\rho}(\rho^{(\gamma_2-2)/(3\gamma_2)}(z_2^{1/2}g_1(\rho) - \varphi_1(s)\rho^{-5/3} + k_1\rho^{-8/3})) = 0. \quad (68)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + (3 - 6/\gamma_2)g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-3+6/\gamma_2}.$$

Equation (68) becomes

$$2(1 + 2\gamma_2)\varphi_1\rho - (2 + 7\gamma_2)k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$(1 + 2\gamma_2)\varphi_1 = 0.$$

and

$$(2 + 7\gamma_2)k_1 = 0.$$

Since $(\mu + 1)\mu \neq 0$, then $\varphi_1 = 0$ and $k_1 = 0$. Then equation (66) becomes

$$h_s - \frac{2\beta_1}{3}z_2h_{z_2} = -(4\beta_1/3\gamma_2 + 2\beta_1/3 - 2\gamma_1/\gamma_2)h - c_3z_2^{1/2}.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{-3dz_2}{2\beta_1z_2} = \frac{dh}{-(\beta_1(4/\gamma_2 + 2)/3 - 2\gamma_1/\gamma_2)h - c_3z_2^{1/2}}.$$

Then

$$z_3 = z_2 e^{\frac{2\beta_1}{3}s}$$

$$\frac{dh}{ds} = -(\beta_1(4/\gamma_2 + 2)/3 - 2\gamma_1/\gamma_2)h - c_3z_3^{1/2}e^{-\frac{\beta_1}{3}s}$$

$$h = Ce^{\lambda s}, \lambda = -2(\beta_1(2/\gamma_2 + 1)/3 - \gamma_1/\gamma_2)$$

$$C' = -c_3e^{-\lambda s}z_3^{1/2}e^{-\frac{\beta_1}{3}s} = -c_3z_3^{1/2}e^{-(\lambda + \frac{\beta_1}{3})s}$$

Case 1.1 $\lambda + \frac{\beta_1}{3} = 0$, then

$$C = -c_3z_3^{1/2}s + h(z_3)$$

Then

$$h = (-c_3z_3^{1/2}s + h(z_3))e^{\lambda s} = (-c_3z_2^{1/2}e^{\frac{\beta_1}{3}s}s + h(z_3))e^{-\frac{\beta_1}{3}s}$$

$$h = -c_3z_2^{1/2}s + h(z_3)e^{-\frac{\beta_1}{3}s}.$$

Then

$$\varepsilon = (-c_3z_2^{1/2}s + h(z_3)e^{-\frac{\beta_1}{3}s})\rho^\mu.$$

Case 1.2 $\lambda + \frac{\beta_1}{3} \neq 0$, then

$$C = \frac{-c_3z_3^{1/2}e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + h(z_3)$$

$$h = \left(\frac{-c_3z_3^{1/2}e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + h(z_3) \right) e^{\lambda s}$$

$$\begin{aligned}
&= \frac{c_3(z_2^{1/2} e^{\frac{\beta_1}{3}s}) e^{\frac{-\beta_1}{3}s}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s} \\
&= \frac{c_3 z_2^{1/2}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s} \\
&= \frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s}
\end{aligned}$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(z_3) e^{\lambda s} \rho^\mu.$$

Case 2 $\mu = -1$, or $\gamma_2 = -2/7$, then

$$C = \frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \rho + k_2 \ln \rho) + h(z_1, z_2).$$

Then

$$\varepsilon = [\frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(s) \rho + k_2 \ln \rho) + h(s, z_2)] \rho^{-1}.$$

One can assume that

$$g_2 = 0, \varphi_2 = 0,$$

then

$$\varepsilon = [\frac{k_2}{3\gamma_2} \ln \rho + h(s, z_2)] \rho^{-1}.$$

Differentiate equation (66) with respect to ρ one obtains

$$\frac{d}{d\rho} (z_2^{1/2} \rho^{8/3} g_1(\rho) - \varphi_1(s) \rho + k_2 \frac{7(4\beta_1 - 7\gamma_1)}{6} \ln \rho) = 0. \quad (69)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{8}{3} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-8/3}.$$

Equation (69) becomes

$$\varphi_1 \rho - \frac{7(4\beta_1 - 7\gamma_1)}{6} k_2 = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi_1 = 0,$$

and

$$(4\beta_1 - 7\gamma_1) k_2 = 0.$$

Case 2.1 $4\beta_1 - 7\gamma_1 \neq 0$, then $k_2 = 0$. Equation (66) becomes

$$h_s - \frac{2\beta_1}{3} z_2 h_{z_2} = (4\beta_1 - 7\gamma_1) h - c_3 z_2^{1/2} - k_1.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{-3dz_2}{2\beta_1 z_2} = \frac{dh}{(4\beta_1 - 7\gamma_1)h - c_3 z_2^{1/2} - k_1}.$$

Then

$$z_3 = z_2 e^{\frac{2\beta_1}{3}s}$$

$$\frac{dh}{ds} = (4\beta_1 - 7\gamma_1)h - c_3 z_3^{1/2} e^{-\frac{\beta_1}{3}s} - k_1$$

$$h = C e^{\lambda s}, \lambda = 4\beta_1 - 7\gamma_1$$

$$C' = (-c_3 z_3^{1/2} e^{-\frac{\beta_1}{3}s} - k_1) e^{-\lambda s} = -c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s} - k_1 e^{-\lambda s}$$

Case 2.1.1 $\lambda + \frac{\beta_1}{3} = 0$, then $\lambda = -\frac{\beta_1}{3}$.

$$C = -c_3 z_3^{1/2} s + \frac{k_1 e^{-\lambda s}}{\lambda} + h(z_3)$$

Then

$$h = (-c_3 z_3^{1/2} s + \frac{k_1 e^{-\lambda s}}{\lambda} + h(z_3)) e^{\lambda s} = (-c_3 z_2^{1/2} e^{\frac{\beta_1}{3}s} s + \frac{k_1 e^{-\lambda s}}{\lambda} + h(z_3)) e^{\lambda s}$$

$$h = -c_3 z_2^{1/2} e^{\frac{\beta_1}{3}s} s e^{\lambda s} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s} = -c_3 z_2^{1/2} s + \frac{k_1}{\lambda} + h(z_3) e^{-\frac{\beta_1}{3}s}.$$

Then

$$\varepsilon = [-c_3 z_2^{1/2} s + \frac{k_1}{\lambda} + h(z_3) e^{-\frac{\beta_1}{3}s}] \rho^{-1}.$$

One can assume that

$$k_1 = 0,$$

Then

$$\varepsilon = (-c_3 z_2^{1/2} s + h(z_3) e^{-\frac{\beta_1}{3}s}) \rho^{-1}.$$

Case 2.1.2 $\lambda + \frac{\beta_1}{3} \neq 0$, then

$$C = \frac{-c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + \frac{k_1 e^{-\lambda s}}{\lambda} + h(z_3)$$

$$h = \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + \frac{k_1 e^{-\lambda s}}{\lambda} + h(z_3) \right) e^{\lambda s}$$

$$= \frac{c_3 (z_2^{1/2} e^{\frac{\beta_1}{3}s}) e^{-\frac{\beta_1}{3}s}}{(\lambda + \frac{\beta_1}{3})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s}$$

$$= \frac{c_3 z_2^{1/2}}{(\lambda + \frac{\beta_1}{3})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s}$$

$$= \frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s}$$

Then

$$\varepsilon = \left[\frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + \frac{k_1}{\lambda} + h(z_3) e^{\lambda s} \right] \rho^{-1}.$$

Then one can assume $c_3 = 0$, $k_1 = 0$, so

$$\varepsilon = h(z_3) e^{\lambda s} \rho^{-1}.$$

Case 2.2 $4\beta_1 - 7\gamma_1 = 0$. Then equation (66) becomes

$$h_s - \frac{2\beta_1}{3} z_2 h_{z_2} = -c_3 z_2^{1/2} - k_2 \frac{7\gamma_1}{2} - k_1.$$

then

$$h_s - \frac{2\beta_1}{3} z_2 h_{z_2} = -c_3 z_2^{1/2} + k_3.$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{-3dz_2}{2\beta_1 z_2} = \frac{dh}{-c_3 z_2^{1/2} + k_3}.$$

Then

$$\begin{aligned} z_3 &= z_2 e^{\frac{2\beta_1}{3}s} \\ \frac{dh}{ds} &= -c_3 z_2^{1/2} + k_3 \\ h &= \int -c_3 z_3^{1/2} e^{-\frac{\beta_1}{3}s} + k_3 ds + h(z_3) \end{aligned}$$

Case 2.2.1 $\beta_1 \neq 0$, then

$$\begin{aligned} h &= 3c_3 z_3^{1/2} \frac{e^{-\frac{\beta_1}{3}s}}{\beta_1} + k_3 s + h(z_3) \\ &= 3c_3 (z_2^{1/2} e^{\frac{\beta_1}{3}s}) \frac{e^{-\frac{\beta_1}{3}s}}{\beta_1} + k_3 s + h(z_3) \\ &= 3c_3 (\alpha^{1/2} \rho^{-5/3}) / \beta_1 + k_3 s + h(z_3), \end{aligned}$$

Then

$$\varepsilon = \left[\frac{k_2}{3\gamma_2} \ln \rho + \frac{3c_3}{\beta_1} (\alpha^{1/2} \rho^{-5/3}) + k_3 s + h(z_3) \right] \rho^{-1}.$$

then one can assume that $c_3 = 0$. So

$$\varepsilon = \left[\frac{k_2}{3\gamma_2} \ln \rho + k_3 s + h(z_3) \right] \rho^{-1}.$$

Case 2.2.2 $\beta_1 = 0$, then

$$\begin{aligned} z_3 &= z_2. \\ \frac{dh}{ds} &= -c_3 z_2^{1/2} - k_1 \end{aligned}$$

$$h = \int -c_3 z_2^{1/2} - k_1 ds + h(z_2) = (-c_3 z_2^{1/2} - k_1)s + h(z_2) = (-c_3 z_2^{1/2} - k_1)s + h(z_2),$$

Then

$$\varepsilon = \left[\frac{k_2}{3\gamma_2} \ln \rho - (c_3 z_2^{1/2} + k_1)s + h(z_2) \right] \rho^{-1}.$$

Case 3 $\mu = 0$. then

$$\begin{aligned} C &= \frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \ln \rho - k_2 \rho^{-1}) + h(z_1, z_2). \\ \varepsilon &= \frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(s) \ln \rho - k_2 \rho^{-1}) + h(s, z_2). \end{aligned}$$

One can assume that

$$g_2 = 0, k_2 = 0,$$

then

$$\varepsilon = \frac{1}{3\gamma_2} \varphi_2(s) \ln \rho + h(s, z_2).$$

Differentiate equation (66) with respect to ρ one obtains

$$\frac{d}{d\rho} (z_2^{1/2} \rho^{5/3} g_1(\rho) + \ln \rho ((2/3) \varphi_2'(s) + (8\gamma_1 - 4\beta_1)/3) \varphi_2(s)) - ((4\beta_1 - 6\gamma_1)/3) \varphi_2(s) - \varphi_1(s) + k\rho^{-1} = 0. \quad (70)$$

Differentiate this equation with respect to z_2 one obtains

$$g_1' + \frac{5}{3} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-5/3}.$$

Equation (70) becomes

$$(2\varphi_2' + 2(4\gamma_1 - 2\beta_1)\varphi_2)\rho - 3k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$k_1 = 0,$$

and

$$\varphi_2' + (4\gamma_1 - 2\beta_1)\varphi_2 = 0.$$

Then

$$\varphi_2 = c_2 e^{(2\beta_1 - 4\gamma_1)s}.$$

Then equation (66) becomes

$$h_s - \frac{2\beta_1}{3} z_2 h_{z_2} = 2(\beta_1 - 2\gamma_1)h - z_2^{1/2} c_3 - \frac{4\beta_1 c_2}{3 - 2\gamma_1} e^{(2\beta_1 - 4\gamma_1)s} + \varphi_1(s).$$

Then

$$h_s - \frac{2\beta_1}{3} z_2 h_{z_2} = 2(\beta_1 - 2\gamma_1)h - z_2^{1/2} c_3 + \phi_1(s).$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{-3dz_2}{2\beta_1 z_2} = \frac{dh}{2(\beta_1 - 2\gamma_1)h - c_3 z_2^{1/2} + \phi_1(s)}.$$

Then

$$z_3 = z_2 e^{\frac{2\beta_1}{3}s}$$

$$\frac{dh}{ds} = 2(\beta_1 - 2\gamma_1)h - c_3 z_2^{1/2} + \phi_1(s)$$

$$h = C e^{\lambda s}, \lambda = 2(\beta_1 - 2\gamma_1)$$

$$C' = (-c_3 z_2^{1/2} + \phi_1(s)) e^{-\lambda s} = -c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s} + \phi_2(s).$$

Case 3.1 $\lambda + \frac{\beta_1}{3} = 0$ then $\lambda = -\frac{\beta_1}{3}$

$$C = -c_3 z_3^{1/2} s + \phi_3(s) + h(z_3)$$

Then

$$h = (-c_3 z_3^{1/2} s + \phi_3(s) + h(z_3))e^{\lambda s} = (-c_3 z_2^{1/2} e^{\frac{\beta_1}{3}s} s + \phi_3(s) + h(z_3))e^{\lambda s}$$

$$h = -c_3 z_2^{1/2} e^{\frac{\beta_1}{3}s} s e^{\lambda s} + \phi_4(s) + h(z_3)e^{\lambda s} = -c_3 z_2^{1/2} s + \phi_4(s) + h(z_3)e^{\lambda s}.$$

Then

$$\varepsilon = \frac{1}{3\gamma_2} \varphi_2(s) \ln \rho - c_3 z_2^{1/2} s + \phi_4(s) + h(z_3)e^{\lambda s}$$

One can assume that

$$\phi_4(s) = 0,$$

Then

$$\varepsilon = \frac{1}{3\gamma_2} \varphi_2(s) \ln \rho - c_3 z_2^{1/2} s + h(z_3)e^{\lambda s}$$

Case 3.2 $\lambda + \frac{\beta_1}{3} \neq 0$, then

$$C = \frac{-c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + \phi_3(s) + h(z_3)$$

$$\begin{aligned} h &= \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + \phi_3(s) + h(z_3) \right) e^{\lambda s} \\ &= \frac{c_3 (z_2^{1/2} e^{\frac{\beta_1}{3}s}) e^{-\frac{\beta_1}{3}s}}{(\lambda + \frac{\beta_1}{3})} + \phi_4(s) + h(z_3)e^{\lambda s} \\ &= \frac{c_3 z_2^{1/2}}{(\lambda + \frac{\beta_1}{3})} + \phi_4(s) + h(z_3)e^{\lambda s} \\ &= \frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + \phi_4(s) + h(z_3)e^{\lambda s} \end{aligned}$$

Then

$$\varepsilon = \frac{1}{3\gamma_2} \varphi_2(s) \ln \rho + \frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + \phi_4(s) + h(z_3)e^{\lambda s}$$

One can assume $c_3 = 0$, $\phi_4 = 0$, so

$$\varepsilon = \frac{1}{3\gamma_2} \varphi_2(s) \ln \rho + h(z_3)e^{\lambda s}$$

Consider $\{\beta_1 X_1 + q_1 X_2 + \partial_s, X_3\}$.

Then (38) becomes

$$6\beta_1 \alpha \varepsilon_\alpha + 2\beta_1 \rho \varepsilon_\rho + \varepsilon_s = 2(\beta_1 - q_1) \varepsilon - \alpha^{1/2} g_1(\rho) + \varphi_1(s) - k_1 \rho^{-1} \quad (71)$$

and

$$10\alpha \varepsilon_\alpha + 3\rho \varepsilon_\rho = 4\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}. \quad (72)$$

The characteristic system of equation (72) is

$$\frac{d\alpha}{10\alpha} = \frac{d\rho}{3\rho} = \frac{ds}{0} = \frac{d\varepsilon}{4\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}}.$$

Then

$$\begin{aligned}
z_1 &= s \\
z_2 &= \alpha \rho^{-10/3} \\
\frac{d\varepsilon}{d\rho} &= \frac{1}{3\rho}(4\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}) \\
\varepsilon &= C\rho^{4/3} \\
C' &= \frac{\rho^{-7/3}}{3}(z_2^{1/2}\rho^{5/3}g_2(\rho) - \varphi_2(z_1) + k_2\rho^{-1}) \\
C &= \frac{1}{3}(z_2^{1/2}\tilde{g}_2(\rho) - \frac{3}{4}\varphi_2(z_1)\rho^{-4/3} - k_2\rho^{-7/3}) + h(z_1, z_2).
\end{aligned}$$

Then

$$\varepsilon = \left(\frac{1}{3}(z_2^{1/2}\tilde{g}_2(\rho) - \frac{3}{4}\varphi_2(z_1)\rho^{-4/3} - k_2\rho^{-7/3}) + h(z_1, z_2)\right)\rho^{4/3}$$

One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0,$$

then

$$\varepsilon = h(s, z_2)\rho^{4/3}.$$

Equation (71) becomes

$$h_s - \frac{2\beta_1}{3}z_2h_{z_2} = -2\left(\frac{\beta_1 + 3q_1}{3}\right)h - \rho^{1/3}(z_2^{1/2}g_1(\rho) - \rho^{-5/3}\varphi_1(s) + \rho^{-8/3}k_1)$$

Differentiate equation (71) with respect to ρ one obtains

$$\frac{d}{d\rho}(\rho^{1/3}(z_2^{1/2}g_1(\rho) + \rho^{-5/3}\varphi_1(s) - \rho^{-8/3}k_1)) = 0. \quad (73)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{1}{3}g_1 = 0,$$

then

$$g_1 = c_3\rho^{-1/3}.$$

Equation (73) becomes

$$4\varphi_1\rho - 7k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi_1 = 0, k_1 = 0.$$

Then equation (71) becomes

$$h_s - \frac{2\beta_1}{3}z_2h_{z_2} = -2\left(\frac{\beta_1 + 3q_1}{3}\right)h - z_2^{1/2}c_3$$

The characteristic system of this equation is

$$\frac{ds}{1} = \frac{-3dz_2}{2\beta_1z_2} = \frac{dh}{-2\left(\frac{\beta_1 + 3q_1}{3}\right)h - z_2^{1/2}c_3}.$$

Then

$$\begin{aligned}
 z_3 &= z_2 e^{\frac{2\beta_1}{3}s} \\
 \frac{dh}{ds} &= -2\left(\frac{\beta_1 + 3q_1}{3}\right)h - z_2^{1/2}c_3 \\
 h &= Ce^{\lambda s}, \lambda = \frac{-2(\beta_1 + 3q_1)}{3} \\
 C' &= -c_3 e^{-\lambda s} z_3^{1/2} e^{-\frac{\beta_1}{3}s} = -c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}
 \end{aligned}$$

Case 1 $\lambda + \frac{\beta_1}{3} = 0$, then

$$C = -c_3 z_3^{1/2} s + h(z_3)$$

Then

$$\begin{aligned}
 h &= (-c_3 z_3^{1/2} s + h(z_3)) e^{\lambda s} = (-c_3 z_2^{1/2} e^{\frac{\beta_1}{3}s} s + h(z_3)) e^{-\frac{\beta_1}{3}s} \\
 h &= -c_3 z_2^{1/2} s + h(z_3) e^{-\frac{\beta_1}{3}s}.
 \end{aligned}$$

Then

$$\varepsilon = (-c_3 z_2^{1/2} s + h(z_3) e^{-\frac{\beta_1}{3}s}) \rho^{4/3}.$$

Case 2 $\lambda + \frac{\beta_1}{3} \neq 0$, then

$$\begin{aligned}
 C &= \frac{-c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + h(z_3) \\
 h &= \left(\frac{-c_3 z_3^{1/2} e^{-(\lambda + \frac{\beta_1}{3})s}}{-(\lambda + \frac{\beta_1}{3})} + h(z_3) \right) e^{\lambda s} \\
 &= \frac{c_3 (z_2^{1/2} e^{\frac{\beta_1}{3}s}) e^{-\frac{\beta_1}{3}s}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s} \\
 &= \frac{c_3 z_2^{1/2}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s} \\
 &= \frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s}
 \end{aligned}$$

Then

$$\varepsilon = \left(\frac{c_3 \alpha^{1/2} \rho^{-5/3}}{(\lambda + \frac{\beta_1}{3})} + h(z_3) e^{\lambda s} \right) \rho^{4/3}.$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(z_3) e^{\lambda s} \rho^{4/3}.$$

Consider $\{q_1 X_2 + \gamma_1 X_3, X_1 + q_2 X_2 + \gamma_2 X_3\}$.

Then (38) becomes

$$10\gamma_1 \alpha \varepsilon_\alpha + 3\gamma_1 \rho \varepsilon_\rho = 2(2\gamma_1 + q_1) \varepsilon + \alpha^{1/2} g_1(\rho) - \varphi_1(s) + k_1 \rho^{-1}. \quad (74)$$

and

$$\begin{aligned}
 2(3 - 5\gamma_2) \alpha \varepsilon_\alpha + (2 - 3\gamma_2) \rho \varepsilon_\rho &= 2(1 - q_2 - 2\gamma_2) \varepsilon - \alpha^{1/2} g_2(\rho) \\
 &\quad + \varphi_2(s) - k_2 \rho^{-1}
 \end{aligned} \quad (75)$$

If $\gamma_1 = 0$, then $q_1 \neq 0$. So $\varepsilon = (-1/q_1)(\alpha^{1/2}g_1(\rho) - \varphi_1(s) + k_1\rho^{-1})$. Then $\varepsilon(\rho, \alpha, s)$ can be transformed to zero thus we will consider in case $\gamma_1 \neq 0$. Then one can assume that $\gamma_1 = 1$ and $\gamma_2 = 0$ Then (74) and (75) becomes

$$10\alpha\varepsilon_\alpha + 3\rho\varepsilon_\rho = 2(2 + q_1)\varepsilon + \alpha^{1/2}g_1(\rho) - \varphi_1(s) + k_1\rho^{-1}. \quad (76)$$

$$6\alpha\varepsilon_\alpha + 2\rho\varepsilon_\rho = 2(1 - q_2)\varepsilon - \alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1} \quad (77)$$

The characteristic system of equation (74-1) is

$$\frac{d\alpha}{10\alpha} = \frac{d\rho}{3\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(2 + q_1)\varepsilon + \alpha^{1/2}g_1(\rho) - \varphi_1(s) + k_1\rho^{-1}}.$$

Then

$$\begin{aligned} z_1 &= s \\ z_2 &= \alpha\rho^{-10/3} \\ \frac{d\varepsilon}{d\rho} &= \frac{1}{3\rho}(2(2 + q_1)\varepsilon + \alpha^{1/2}g_1(\rho) - \varphi_1(s) + k_1\rho^{-1}) \\ \varepsilon &= C\rho^\mu, \mu = \frac{2(2 + q_1)}{3} \\ C' &= \frac{\rho^{-1-\mu}}{3}(z_2^{1/2}\rho^{5/3}g_1(\rho) - \varphi_1(z_1) + k_1\rho^{-1}) \end{aligned}$$

Case 1 $(\mu + 1)\mu \neq 0$, then

$$C = \frac{1}{3}(z_2^{1/2}\tilde{g}_1(\rho) + \frac{\varphi_1(z_1)}{\mu}\rho^{-\mu} - \frac{k_1}{1 + \mu}\rho^{-1-\mu}) + h(z_1, z_2)$$

Then

$$\varepsilon = \left(\frac{1}{3}(z_2^{1/2}\tilde{g}_1(\rho) + \frac{\varphi_1(z_1)}{\mu}\rho^{-\mu} - \frac{k_1}{1 + \mu}\rho^{-1-\mu}) + h(z_1, z_2)\right)\rho^\mu$$

One can assume that

$$g_1 = 0, \varphi_1 = 0, k_1 = 0,$$

then

$$\varepsilon = h(s, z_2)\rho^\mu.$$

Equation (75) becomes

$$\frac{2}{3}z_2h_{z_2} = \left(\frac{4q_1}{3} + 2q_2 + \frac{2}{3}\right)h + \rho^{(1-2q_1)/3}(z_2^{1/2}g_2(\rho) - \rho^{-5/3}\varphi_2(s) + \rho^{-8/3}k_2)$$

Differentiate equation this equation with respect to ρ one obtains

$$\frac{d}{d\rho}(\rho^{(1-2q_1)/3}(z_2^{1/2}g_2(\rho) - \rho^{-5/3}\varphi_2(s) + \rho^{-8/3}k_2)) = 0. \quad (78)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_2' + (1 - 2q_1)/(3)g_2 = 0,$$

then

$$g_2 = c_3\rho^{-(1-2q_1)/(3)}.$$

Equation (78) becomes

$$2(q_1 + 2)\varphi_2\rho - (2q_1 + 7)k_2 = 0.$$

Splitting this equation with respect to ρ one obtains

$$(q_1 + 2)\varphi_2 = 0.$$

and

$$(2q_1 + 7)k_2 = 0.$$

Since $(\mu + 1)\mu \neq 0$, then $\varphi_2 = 0$ and $k_2 = 0$. Then equation (75) becomes

$$\frac{2}{3}z_2 h_{z_2} = \left(\frac{4q_1}{3} + 2q_2 + \frac{2}{3}\right)h + c_3 z_2^{1/2}$$

The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2z_2} = \frac{dh}{\left(\frac{4q_1}{3} + 2q_2 + \frac{2}{3}\right)h + c_3 z_2^{1/2}}.$$

Then

$$\frac{dh}{dz_2} = \frac{3}{2z_2} \left(\left(\frac{4q_1}{3} + 2q_2 + \frac{2}{3} \right) h + c_3 z_2^{1/2} \right)$$

$$h = C z_2^\lambda, \lambda = \frac{3}{2} \left(\frac{4q_1}{3} + 2q_2 + \frac{2}{3} \right)$$

$$C' z_2^\lambda = \frac{3}{2z_2} (c_3 z_2^{1/2})$$

$$C' = \frac{3}{2} c_3 z_2^{-1-\lambda+1/2}$$

Case 1.1 $\lambda = 1/2$, then

$$C = \frac{3}{2} c_3 \ln z_2 + h(s)$$

Then

$$h = \left(\frac{3}{2} c_3 \ln z_2 + h(s) \right) z_2^\lambda$$

Then

$$\varepsilon = \left(\left(\frac{3}{2} c_3 \ln z_2 + h(s) \right) z_2^\lambda \right) \rho^\mu.$$

Case 1.2 $\lambda \neq 1/2$, then

$$C = \frac{3c_3 z_2^{-\lambda+1/2}}{2(-\lambda+1/2)} + h(s)$$

$$h = \left(\frac{3c_3 z_2^{-\lambda+1/2}}{2(-\lambda+1/2)} + h(s) \right) z_2^\lambda$$

$$= \frac{3c_3 z_2^{1/2}}{2(-\lambda+1/2)} + h(s) z_2^\lambda$$

$$= \frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2(-\lambda+1/2)} + h(s) z_2^\lambda$$

Then

$$\varepsilon = \left(\frac{3c_3\alpha^{1/2}\rho^{-5/3}}{2(-\lambda + 1/2)} + h(s)z_2^\lambda \right) \rho^\mu.$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(s)z_2^\lambda \rho^\mu$$

Case 2 $\mu = -1$, then

$$C = \frac{1}{3}(z_2^{1/2}\tilde{g}_1(\rho) - \varphi_1(z_1)\rho + k_1 \ln \rho) + h(z_1, z_2).$$

Then

$$\varepsilon = \left[\frac{1}{3}(z_2^{1/2}\tilde{g}_{21}(\rho) - \varphi_1(z_1)\rho + k_1 \ln \rho) + h(z_1, z_2) \right] \rho^{-1}.$$

One can assume that

$$g_1 = 0, \varphi_1 = 0,$$

then

$$\varepsilon = \left[\frac{1}{3}k_1 \ln \rho + h(s, z_2) \right] \rho^{-1}.$$

Differentiate equation (77) with respect to ρ one obtains

$$\frac{d}{d\rho}(z_2^{1/2}\rho^{8/3}g_2(\rho) - \varphi_2(s)\rho + k_1 \frac{2q_2-4}{3} \ln \rho) = 0. \quad (79)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_2' + \frac{8}{3}g_2 = 0,$$

then

$$g_2 = c_3 \rho^{-8/3}.$$

Equation (79) becomes

$$\varphi_2 \rho - \frac{2(q_2 - 2)}{3}k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi_2 = 0,$$

and

$$2(q_2 - 2)k_1 = 0.$$

Case 2.1 $q_2 - 2 \neq 0$, then $k_1 = 0$. Equation (77) becomes

$$\frac{2}{3}z_2 h_{z_2} = 2(q_2 - 2)h + c_3 z_2^{1/2} + k_2.$$

The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2z_2} = \frac{dh}{2(q_2 - 2)h + c_3 z_2^{1/2} + k_2}.$$

Then

$$\frac{dh}{dz_2} = \frac{3}{2z_2}(2(q_2 - 2)h + c_3 z_2^{1/2} + k_2)$$

$$h = Cz_2^\lambda, \lambda = 3(q_2 - 2)$$

$$C' = \frac{3}{2}z_2^{-1-\lambda}(c_3z_2^{1/2} + k_2).$$

Remark $\lambda \neq 0$,

Case 2.1.1 $\lambda = 1/2$, then

$$C = \frac{3}{2}(c_3 \ln z_2 - 2k_2z_2^{-1/2}) + h(s).$$

Then

$$h = (\frac{3}{2}(c_3 \ln z_2 - 2k_2z_2^{-1/2}) + h(s))z_2^{1/2} = \frac{3}{2}(c_3 \ln z_2z_2^{1/2} - 2k_2) + h(s)z_2^{1/2}$$

Then

$$\varepsilon = [\frac{3}{2}(c_3 \ln z_2z_2^{1/2} - 2k_2) + h(s)z_2^{1/2}]\rho^{-1}.$$

One can assume that

$$k_2 = 0,$$

Then

$$\varepsilon = [\frac{3}{2}c_3 \ln z_2 + h(s)]z_2^{1/2}\rho^{-1}.$$

Case 2.1.2 $\lambda \neq 1/2$, then

$$C = \frac{3}{2}(\frac{c_3z_2^{-\lambda+1/2}}{-\lambda+1/2} + \frac{k_2z_2^{-\lambda}}{-\lambda}) + h(s).$$

Then

$$h = \frac{3}{2}(\frac{c_3z_2^{1/2}}{-\lambda+1/2} + \frac{k_2}{-\lambda}) + h(s)z_2^\lambda = \frac{3}{2}(\frac{c_3\alpha^{1/2}\rho^{-5/3}}{-\lambda+1/2} + \frac{k_2}{-\lambda}) + h(s)z_2^\lambda.$$

Then

$$\varepsilon = [\frac{3}{2}(\frac{c_3\alpha^{1/2}\rho^{-5/3}}{-\lambda+1/2} + \frac{k_2}{-\lambda}) + h(s)z_2^\lambda]\rho^{-1}.$$

One can assume that

$$c_3 = 0, k_2 = 0.$$

Then

$$\varepsilon = [h(s)z_2^\lambda]\rho^{-1}.$$

Case 2.2 $q_2 - 2 = 0$ then Then equation (77) becomes

$$\frac{2}{3}z_2h_{z_2} = c_3z_2^{1/2} + k_2 + \frac{2}{3}k_1.$$

or

$$\frac{2}{3}z_2h_{z_2} = c_3z_2^{1/2} + k_3.$$

The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2z_2} = \frac{dh}{c_3z_2^{1/2} + k_3}.$$

Then

$$h = \frac{3}{2} \int z_2^{-1} (c_3 z_2^{1/2} + k_3) dz_2 + h(s)$$

$$h = \frac{3}{2} (2c_3 z_2^{1/2} + k_3 \ln z_2) + h(s) = \frac{3}{2} (2c_3 \alpha^{1/2} \rho^{-5/3} + k_3 \ln z_2) + h(s).$$

Then

$$\varepsilon = \left[\frac{1}{3} k_1 \ln \rho + \frac{3}{2} (2c_3 \alpha^{1/2} \rho^{-5/3} + k_3 \ln z_2) + h(s) \right] \rho^{-1}.$$

One can assume that

$$c_3 = 0.$$

Then

$$\varepsilon = \left[\frac{1}{3} k_1 \ln \rho + \frac{3}{2} k_3 \ln z_2 + h(s) \right] \rho^{-1}.$$

Case 3 $\mu = 0$. then

$$C = \frac{1}{3} (z_2^{1/2} \tilde{g}_1(\rho) - \varphi_1(z_1) \ln \rho - k_1 \rho^{-1}) + h(z_1, z_2).$$

Then

$$\varepsilon = \frac{1}{3} (z_2^{1/2} \tilde{g}_1(\rho) - \varphi_1(s) \ln \rho - k_1 \rho^{-1}) + h(s, z_2).$$

One can assume that

$$g_1 = 0, k_1 = 0,$$

then

$$\varepsilon = \frac{1}{3} \ln \rho \varphi_1(s) + h(s, z_2).$$

Differentiate equation (77) with respect to ρ one obtains

$$\frac{d}{d\rho} (z_2^{1/2} \rho^{5/3} g_1(\rho) + \varphi_1(s) (2(1 - q_2)/(3) \ln \rho - 2/3) - \varphi_2(s) + k_2 \rho^{-1}) = 0. \quad (80)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_2' + \frac{5}{3} g_2 = 0,$$

then

$$g_2 = c_3 \rho^{-5/3}.$$

Equation (80) becomes

$$\frac{2}{3} (1 - q_2) \rho \varphi_1 - k_2 = 0.$$

Splitting this equation with respect to ρ one obtains

$$k_2 = 0,$$

and

$$(1 - q_2) \varphi_1 = 0.$$

Case 3.1 $1 - q_2 \neq 0$, then $\varphi_1 = 0$. Equation (77) becomes

$$\frac{2}{3} z_2 h_{z_2} = -2(q_2 - 1)h + c_3 z_2^{1/2} - \varphi_2(s).$$

The characteristic system of this equation is

$$\begin{aligned}\frac{ds}{0} &= \frac{3dz_2}{2z_2} = \frac{dh}{-2(q_2 - 1)h + c_3 z_2^{1/2} - \varphi_2(s)}. \\ \frac{dh}{dz_2} &= \frac{3}{2z_2}(-2(q_2 - 1)h + c_3 z_2^{1/2} - \varphi_2(s)) \\ h &= C z_2^\lambda, \lambda = 3(1 - q_2) \\ C' &= \frac{3}{2} z_2^{-1-\lambda} (c_3 z_2^{1/2} - \varphi_2(s)).\end{aligned}$$

Remark $\lambda \neq 0$,

Case 3.1.1 $\lambda = 1/2$, then

$$C = \frac{3}{2}(c_3 \ln z_2 - 2\varphi_2(s)z_2^{-1/2}) + h(s).$$

Then

$$h = \left(\frac{3}{2}(c_3 \ln z_2 - 2\varphi_2(s)z_2^{-1/2}) + h(s)\right)z_2^{1/2} = \frac{3}{2}(c_3 \ln z_2 z_2^{1/2} - 2\varphi_2(s)) + h(s)z_2^{1/2},$$

then

$$\varepsilon = \frac{3}{2}(c_3 \ln z_2 z_2^{1/2} - 2\varphi_2(s)) + h(s)z_2^{1/2}.$$

One can assume that

$$\varphi_2 = 0,$$

Then

$$\varepsilon = \left[\frac{3}{2}c_3 \ln z_2 + h(s)\right]z_2^{1/2}.$$

Case 3.1.2 $\lambda \neq 1/2$, then

$$C = \frac{3}{2}\left(\frac{c_3 z_2^{-\lambda+1/2}}{-\lambda+1/2} - \frac{\varphi_2(s)z_2^{-\lambda}}{-\lambda}\right) + h(s).$$

Then

$$h = \frac{3}{2}\left(\frac{c_3 z_2^{1/2}}{-\lambda+1/2} + \frac{\varphi_2(s)}{\lambda}\right) + h(s)z_2^\lambda = \frac{3}{2}\left(\frac{c_3 \alpha^{1/2} \rho^{-5/3}}{-\lambda+1/2} + \frac{\varphi_2(s)}{\lambda}\right) + h(s)z_2^\lambda.$$

Then

$$\varepsilon = \frac{3}{2}\left(\frac{c_3 \alpha^{1/2} \rho^{-5/3}}{-\lambda+1/2} + \frac{\varphi_2(s)}{\lambda}\right) + h(s)z_2^\lambda.$$

One can assume that

$$c_3 = 0, \varphi_2 = 0.$$

Then

$$\varepsilon = h(s)z_2^\lambda.$$

Case 3.2 $1 - q_2 = 0$, then equation (77) becomes

$$\frac{2}{3}z_2 h_{z_2} = c_3 z_2^{1/2} - (2/3)\varphi_1(s) - \varphi_2(s).$$

or

$$\frac{2}{3}z_2 h_{z_2} = c_3 z_2^{1/2} + \varphi_3(s).$$

The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2z_2} = \frac{dh}{c_3 z_2^{1/2} + \varphi_3(s)}.$$

$$h = \frac{3}{2} \int z_2^{-1} (c_3 z_2^{1/2} + \varphi_3(s)) dz_2 + h(s)$$

$$h = \frac{3}{2} (2c_3 z_2^{1/2} + \varphi_3(s) \ln z_2) + h(s) = \frac{3}{2} (2c_3 \alpha^{1/2} \rho^{-5/3} + \varphi_3(s) \ln z_2) + h(s).$$

Then

$$\varepsilon = \frac{1}{3} \ln \rho \varphi_1(s) + \frac{3}{2} (2c_3 \alpha^{1/2} \rho^{-5/3} + \varphi_3(s) \ln z_2) + h(s).$$

One can assume that

$$c_3 = 0.$$

Then

$$\varepsilon = \frac{1}{3} \ln \rho \varphi_1(s) + \frac{3}{2} \varphi_3(s) \ln z_2 + h(s).$$

Consider $\{\beta_1 X_1 + \gamma_1 X_3, X_2 + \gamma_2 X_3\}$.

Then (38) becomes

$$2(3\beta_1 - 5\gamma_1)\alpha\varepsilon_\alpha + (2\beta_1 - 2\gamma_1)\rho\varepsilon_\rho = 2(\beta_1 - 2\gamma_1)\varepsilon - \alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (81)$$

and

$$10\gamma_2\alpha\varepsilon_\alpha + 3\gamma_2\rho\varepsilon_\rho = 2(2\gamma_2 + 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1} \quad (82)$$

If $\gamma_2 = 0$, then $\varepsilon = (-1/2)(\alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1})$. So $\varepsilon(\rho, \alpha, s)$ can be transformed to zero thus we will consider in case $\gamma_2 \neq 0$. The characteristic system of equation (82) is

$$\frac{d\alpha}{10\gamma_2\alpha} = \frac{d\rho}{3\gamma_2\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(2\gamma_2 + 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}}.$$

Then

$$z_1 = s$$

$$z_2 = \alpha\rho^{-10/3}$$

$$\frac{d\varepsilon}{d\rho} = \frac{1}{3\gamma_2\rho} (2(2\gamma_2 + 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1})$$

$$\varepsilon = C\rho^\mu, \mu = \frac{2}{3\gamma_2}(2\gamma_2 + 1)$$

$$C' = \frac{\rho^{-1-\mu}}{3\gamma_2} (z_2^{1/2}\rho^{5/3}g_2(\rho) - \varphi_2(z_1) + k_2\rho^{-1})$$

Case 1 $(\mu + 1)\mu \neq 0$, then

$$C = \frac{1}{3\gamma_2} (z_2^{1/2}\tilde{g}_2(\rho) + \frac{\varphi_1(z_1)}{\mu}\rho^{-\mu} - \frac{k_2}{1+\mu}\rho^{-1-\mu}) + h(z_1, z_2)$$

Then

$$\varepsilon = \left(\frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) + \frac{\varphi_1(s)}{\mu} \rho^{-\mu} - \frac{k_2}{1+\mu} \rho^{-1-\mu}) + h(s, z_2) \rho^\mu \right)$$

One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0,$$

then

$$\varepsilon = h(s, z_2) \rho^\mu.$$

Equation (81) becomes

$$\frac{2}{3} \beta_1 z_2 h_{z_2} = \left(\frac{4\beta_1}{3\gamma_2} + \frac{2\beta_1}{3} - \frac{2\gamma_1}{\gamma_2} \right) h + \rho^{(\gamma_2-2)/(3\gamma_2)} (z_2^{1/2} g_1(\rho) - \rho^{-5/3} \varphi_1(s) + \rho^{-8/3} k_1)$$

Differentiate this equation with respect to ρ one obtains

$$\frac{d}{d\rho} (\rho^{(\gamma_2-2)/(3\gamma_2)} (z_2^{1/2} g_1(\rho) - \rho^{-5/3} \varphi_1(s) + \rho^{-8/3} k_1)) = 0. \quad (83)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + (\gamma_2 - 2)/(3\gamma_2) g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-(\gamma_2-2)/(3\gamma_2)}.$$

Equation (83) becomes

$$2(1 + 2\gamma_2) \varphi_1 \rho - (2 + 7\gamma_2) k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$(1 + 2\gamma_2) \varphi_1 = 0.$$

and

$$(2 + 7\gamma_2) k_1 = 0.$$

Since $(\mu + 1)\mu \neq 0$, then $\varphi_1 = 0$ and $k_1 = 0$. Then equation (81) becomes

$$\frac{2}{3} \beta_1 z_2 h_{z_2} = \left(\frac{4\beta_1}{3\gamma_2} + \frac{2\beta_1}{3} - \frac{2\gamma_1}{\gamma_2} \right) h + c_3 z_2^{1/2}.$$

For nonisentropic, $\beta_1 \neq 0$. The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2\beta_1 z_2} = \frac{dh}{\left(\frac{4\beta_1}{3\gamma_2} + \frac{2\beta_1}{3} - \frac{2\gamma_1}{\gamma_2} \right) h + c_3 z_2^{1/2}}.$$

Then

$$\frac{dh}{dz_2} = \frac{3}{2\beta_1 z_2} \left(\left(\frac{4\beta_1}{3\gamma_2} + \frac{2\beta_1}{3} - \frac{2\gamma_1}{\gamma_2} \right) h + c_3 z_2^{1/2} \right)$$

$$h = C z_2^\lambda, \lambda = \frac{3}{2\beta_1} \left(\frac{4\beta_1}{3\gamma_2} + \frac{2\beta_1}{3} - \frac{2\gamma_1}{\gamma_2} \right)$$

$$C' z_2^\lambda = \frac{3}{2\beta_1 z_2} c_3 z_2^{1/2}$$

$$C' = \frac{3c_3}{2\beta_1} z_2^{-1-\lambda+1/2}$$

Case 1.1 $\lambda = 1/2$, then

$$C = \frac{3c_3}{2\beta_1} \ln z_2 + h(s)$$

Then

$$h = \left(\frac{3c_3}{2\beta_1} \ln z_2 + h(s)\right) z_2^\lambda$$

Then

$$\varepsilon = \left(\left(\frac{3c_3}{2\beta_1} \ln z_2 + h(s)\right) z_2^\lambda\right) \rho^\mu.$$

Case 1.2 $\lambda \neq 1/2$, then

$$\begin{aligned} C &= \frac{3c_3 z_2^{-\lambda+1/2}}{2\beta_1(-\lambda+1/2)} + h(s) \\ h &= \left(\frac{3c_3 z_2^{-\lambda+1/2}}{2\beta_1(-\lambda+1/2)} + h(s)\right) z_2^\lambda \\ &= \frac{3c_3 z_2^{1/2}}{2\beta_1(-\lambda+1/2)} + h(s) z_2^\lambda \\ &= \frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2\beta_1(-\lambda+1/2)} + h(s) z_2^\lambda \end{aligned}$$

Then

$$\varepsilon = \left(\frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2\beta_1(-\lambda+1/2)} + h(s) z_2^\lambda\right) \rho^\mu.$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(s) z_2^\lambda \rho^\mu$$

Case 2 $\mu = -1$, then

$$\begin{aligned} C &= \frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \rho + k_2 \ln \rho) + h(z_1, z_2). \\ &= \frac{1}{3\gamma_2} (\alpha^{1/2} \rho^{-5/3} \tilde{g}_2(\rho) - \varphi_2(z_1) \rho + k_2 \ln \rho) + h(z_1, z_2). \end{aligned}$$

Then

$$\varepsilon = \left[\frac{1}{3\gamma_2} (\alpha^{1/2} \rho^{-5/3} \tilde{g}_2(\rho) - \varphi_2(s) \rho + k_2 \ln \rho) + h(s, z_2)\right] \rho^{-1}.$$

One can assume that

$$g_2 = 0, \varphi_2 = 0,$$

then

$$\varepsilon = \left[\left(\frac{1}{3\gamma_2} k_2 \ln \rho + h(s, z_2)\right)\right] \rho^{-1}.$$

Differentiate equation (81) with respect to ρ one obtains

$$\frac{d}{d\rho} (z_2^{1/2} \rho^{8/3} g_1(\rho) - \varphi_1(s) \rho + k_2 \frac{7(4\beta_1 - 7\gamma_1)}{6} \ln \rho) = 0. \quad (84)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{8}{3} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-8/3}.$$

Equation (84) becomes

$$\varphi_1 \rho - \frac{7(4\beta_1 - 7\gamma_1)}{6} k_2 = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi_1 = 0,$$

and

$$(4\beta_1 - 7\gamma_1)k_2 = 0.$$

Case 2.1 $4\beta_1 - 7\gamma_1 \neq 0$, then $k_2 = 0$. Equation (81) becomes

$$\frac{2}{3}\beta_1 z_2 h_{z_2} = -(4\beta_1 - 7\gamma_1)h + c_3 z_2^{1/2} + k_1.$$

For nonisentropic, $\beta_1 \neq 0$. The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2\beta_1 z_2} = \frac{dh}{-(4\beta_1 - 7\gamma_1)h + c_3 z_2^{1/2} + k_1}.$$

Then

$$\frac{dh}{dz_2} = \frac{3}{2\beta_1 z_2} (-(4\beta_1 - 7\gamma_1)h + c_3 z_2^{1/2} + k_1)$$

$$h = C z_2^\lambda, \lambda = \frac{-3}{2\beta_1} (4\beta_1 - 7\gamma_1)$$

$$C' z_2^\lambda = \frac{3}{2\beta_1 z_2} (c_3 z_2^{1/2} + k_1)$$

$$C' = \frac{3c_3}{2\beta_1} (z_2^{-1-\lambda+1/2} + k_1 z_2^{-1-\lambda})$$

remark $\lambda \neq 0$,

Case 2.1.1 $\lambda = 1/2$, then

$$C = \frac{3c_3}{2\beta_1} (\ln z_2 - \frac{k_1 z_2^{-\lambda}}{\lambda}) + h(s)$$

Then

$$h = (\frac{3c_3}{2\beta_1} (\ln z_2 + \frac{k_1 z_2^{-\lambda}}{-\lambda}) + h(s)) z_2^\lambda = \frac{3c_3}{2\beta_1} (\ln z_2 z_2^\lambda - \frac{k_1}{\lambda}) + h(s) z_2^\lambda$$

Then

$$\varepsilon = [\frac{3c_3}{2\beta_1} (\ln z_2 z_2^\lambda - \frac{k_1}{\lambda}) + h(s) z_2^\lambda] \rho^{-1}.$$

One can assume that $k_1 = 0$, then

$$\varepsilon = [\frac{3c_3}{2\beta_1} \ln z_2 + h(s)] z_2^\lambda \rho^{-1}.$$

Case 2.1.2 $\lambda \neq 1/2$, then

$$C = \frac{3c_3 z_2^{-\lambda+1/2}}{2\beta_1 (-\lambda + 1/2)} - \frac{k_1 z_2^{-\lambda}}{\lambda} + h(s)$$

$$\begin{aligned}
h &= \left(\frac{3c_3 z_2^{-\lambda+1/2}}{2\beta_1(-\lambda+1/2)} - \frac{k_1 z_2^{-\lambda}}{\lambda} + h(s) \right) z_2^\lambda \\
&= \frac{3c_3 z_2^{1/2}}{2\beta_1(-\lambda+1/2)} - \frac{k_1}{\lambda} + h(s) z_2^\lambda \\
&= \frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2\beta_1(-\lambda+1/2)} - \frac{k_1}{\lambda} + h(s) z_2^\lambda
\end{aligned}$$

Then

$$\varepsilon = \left[\frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2\beta_1(-\lambda+1/2)} - \frac{k_1}{\lambda} + h(s) z_2^\lambda \right] \rho^{-1}.$$

Then one can assume $c_3 = 0, k_1 = 0$, so

$$\varepsilon = h(s) z_2^\lambda \rho^{-1}$$

Case 2.2 $4\beta_1 - 7\gamma_1 = 0$, then Then equation (81) becomes

$$\frac{2}{3}\beta_1 z_2 h_{z_2} = c_3 z_2^{1/2} - 7\left(\frac{\gamma_1}{2} - \frac{\beta_1}{3}\right)k_2 + k_1.$$

or

$$\frac{2}{3}\beta_1 z_2 h_{z_2} = c_3 z_2^{1/2} + k_3.$$

For 2 dimensional, $\beta_1 \neq 0$. The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2\beta_1 z_2} = \frac{dh}{c_3 z_2^{1/2} + k_3}.$$

Then

$$\begin{aligned}
h &= \frac{3}{2\beta_1} \int z_2^{-1} (c_3 z_2^{1/2} + k_3) dz_2 + h(s) \\
h &= \frac{3}{2\beta_1} (2c_3 z_2^{1/2} + k_3 \ln z_2) + h(s) = \frac{3}{2\beta_1} (2c_3 \alpha^{1/2} \rho^{-5/3} + k_3 \ln z_2) + h(s).
\end{aligned}$$

Then

$$\varepsilon = \left[\frac{1}{3\gamma_2} k_2 \ln \rho + \frac{3}{2\beta_1} (2c_3 \alpha^{1/2} \rho^{-5/3} + k_3 \ln z_2) + h(s) \right] \rho^{-1}.$$

One can assume that $c_3 = 0$,

$$c_3 = 0.$$

Then

$$\varepsilon = \left[\frac{1}{3\gamma_2} k_2 \ln \rho + \frac{3}{2\beta_1} k_3 \ln z_2 + h(s) \right] \rho^{-1}.$$

Case 3 $\mu = 0$ then

$$C = \frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(z_1) \ln \rho - k_2 \rho^{-1}) + h(z_1, z_2).$$

Then

$$\varepsilon = \frac{1}{3\gamma_2} (z_2^{1/2} \tilde{g}_2(\rho) - \varphi_2(s) \ln \rho - k_2 \rho^{-1}) + h(s, z_2).$$

One can assume that

$$g_2 = 0, k_2 = 0,$$

then

$$\varepsilon = \frac{-1}{3\gamma_2} \ln \rho \varphi_2(s) + h(s, z_2).$$

Differentiate equation (81) with respect to ρ one obtains

$$\frac{d}{d\rho}(z_2^{1/2} \rho^{5/3} g_1(\rho) + \frac{4}{3}(2\gamma_1 - \beta_1) \ln \rho \varphi_2(s) + k_1 \rho^{-1}) = 0. \quad (85)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{5}{3} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-5/3}.$$

Equation (85) becomes

$$\frac{4}{3}(2\gamma_1 - \beta_1) \rho \varphi_2 - k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$k_1 = 0,$$

and

$$(2\gamma_1 - \beta_1) \varphi_2 = 0.$$

Case 3.1 $2\gamma_1 - \beta_1 \neq 0$, then $\varphi_2 = 0$. Equation (81) becomes

$$\frac{2}{3} \beta_1 z_2 h_{z_2} = 2(2\gamma_1 - \beta_1) h + c_3 z_2^{1/2} - \varphi_1(s).$$

For nonisentropic, $\beta_1 \neq 0$. The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2\beta_1 z_2} = \frac{dh}{2(2\gamma_1 - \beta_1)h + c_3 z_2^{1/2} - \varphi_1(s)}.$$

$$\frac{dh}{dz_2} = \frac{3}{2\beta_1 z_2} (2(2\gamma_1 - \beta_1)h + c_3 z_2^{1/2} - \varphi_1(s))$$

$$h = C z_2^\lambda, \lambda = \frac{3}{\beta_1} (2\gamma_1 - \beta_1)$$

$$C' = \frac{3}{2\beta_1} z_2^{-1-\lambda} (c_3 z_2^{1/2} - \varphi_1(s)).$$

Remark $\lambda \neq 0$, **Case 3.1.1** $\lambda = 1/2$, then

$$C = \frac{3}{2\beta_1} (c_3 \ln z_2 + 2\varphi_1(s) z_2^{-1/2}) + h(s).$$

Then

$$h = \left(\frac{3}{2\beta_1} (c_3 \ln z_2 + 2\varphi_1(s) z_2^{-1/2}) + h(s) \right) z_2^{1/2} = \frac{3}{2\beta_1} (c_3 \ln z_2 z_2^{1/2} + 2\varphi_1(s)) + h(s) z_2^{1/2},$$

then

$$\varepsilon = \frac{3}{2\beta_1} (c_3 \ln z_2 z_2^{1/2} + 2\varphi_1(s)) + h(s) z_2^{1/2}.$$

One can assume that

$$\varphi_1 = 0,$$

Then

$$\varepsilon = \left[\frac{3}{2\beta_1} c_3 \ln z_2 + h(s) \right] z_2^{1/2}.$$

Case 3.1.2 $\lambda \neq 1/2$, then

$$C = \frac{3}{2\beta_1} \left(\frac{c_3 z_2^{-\lambda+1/2}}{-\lambda+1/2} - \frac{\varphi_1(s) z_2^{-\lambda}}{-\lambda} \right) + h(s).$$

Then

$$h = \frac{3}{2\beta_1} \left(\frac{c_3 z_2^{1/2}}{-\lambda+1/2} + \frac{\varphi_1(s)}{\lambda} \right) + h(s) z_2^\lambda = \frac{3}{2\beta_1} \left(\frac{c_3 \alpha^{1/2} \rho^{-5/3}}{-\lambda+1/2} + \frac{\varphi_1(s)}{\lambda} \right) + h(s) z_2^\lambda.$$

Then

$$\varepsilon = \frac{3}{2\beta_1} \left(\frac{c_3 \alpha^{1/2} \rho^{-5/3}}{-\lambda+1/2} + \frac{\varphi_1(s)}{\lambda} \right) + h(s) z_2^\lambda.$$

One can assume that

$$c_3 = 0, \varphi_1 = 0.$$

Then

$$\varepsilon = h(s) z_2^\lambda.$$

Case 3.2 $2\gamma_1 - \beta_1 \neq 0$, then equation (81) becomes

$$\frac{2}{3} \beta_1 z_2 h_{z_2} = c_3 z_2^{1/2} - 2 \left(\frac{2}{3} \beta_1 - \gamma_1 \right) \varphi_2(s) - \varphi_1(s).$$

or

$$\frac{2}{3} z_2 \beta_1 h_{z_2} = c_3 z_2^{1/2} + \varphi_3(s).$$

For 2 dimensional, $\beta_1 \neq 0$. The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2\beta_1 z_2} = \frac{dh}{c_3 z_2^{1/2} + \varphi_3(s)}.$$

$$h = \frac{3}{2\beta_1} \int z_2^{-1} (c_3 z_2^{1/2} + \varphi_3(s)) dz_2 + h(s)$$

$$h = \frac{3}{2\beta_1} (2c_3 z_2^{1/2} + \varphi_3(s) \ln z_2) + h(s) = \frac{3}{2\beta_1} (2c_3 \alpha^{1/2} \rho^{-5/3} + \varphi_3(s) \ln z_2) + h(s).$$

Then

$$\varepsilon = \frac{-1}{3\gamma_2} \ln \rho \varphi_2(s) + h(s, z_2).$$

$$\varepsilon = \frac{-1}{3\gamma_2} \ln \rho \varphi_2(s) + \frac{3}{2\beta_1} (2c_3 \alpha^{1/2} \rho^{-5/3} + \varphi_3(s) \ln z_2) + h(s).$$

One can assume that

$$c_3 = 0.$$

Then

$$\varepsilon = \frac{-1}{3\gamma_2} \ln \rho \varphi_2(s) + \frac{3}{2\beta_1} \varphi_3(s) \ln z_2 + h(s).$$

Consider $\{\beta_1 X_1 + q_1 X_2, X_3\}$.

Then (38) becomes

$$6\beta_1 \alpha \varepsilon_\alpha + 2\beta_1 \rho \varepsilon_\rho = 2(\beta_1 - q_1) \varepsilon - \alpha^{1/2} g_1(\rho) + \varphi_1(s) - k_1 \rho^{-1}. \quad (86)$$

and

$$10\alpha \varepsilon_\alpha + 3\rho \varepsilon_\rho = 4\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1} \quad (87)$$

The characteristic system of equation (87) is

$$\frac{d\alpha}{10\alpha} = \frac{d\rho}{3\rho} = \frac{ds}{0} = \frac{d\varepsilon}{4\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}}.$$

Then

$$\begin{aligned} z_1 &= s \\ z_2 &= \alpha \rho^{-10/3} \\ \frac{d\varepsilon}{d\rho} &= \frac{1}{3\rho} (4\varepsilon + \alpha^{1/2} g_2(\rho) - \varphi_2(s) + k_2 \rho^{-1}) \\ \varepsilon &= C \rho^{4/3} \\ C' &= \frac{\rho^{-7/3}}{3} (z_2^{1/2} \rho^{5/3} g_2(\rho) - \varphi_2(z_1) + k_2 \rho^{-1}) \\ C &= \frac{1}{3} (z_2^{1/2} \tilde{g}_2(\rho) + \frac{3\varphi_2(z_1)}{4} \rho^{-4/3} - \frac{3k_2}{7} \rho^{-7/3}) + h(z_1, z_2) \\ &= \frac{1}{3} (\alpha^{1/2} \rho^{-5/3} \tilde{g}_2(\rho) + \frac{3\varphi_2(z_1)}{4} \rho^{-4/3} - \frac{3k_2}{7} \rho^{-7/3}) + h(z_1, z_2) \end{aligned}$$

Then

$$\varepsilon = \frac{1}{3} (\alpha^{1/2} \tilde{g}_2(\rho) + \frac{3\varphi_2(z_1)}{4} - \frac{3k_2}{7} \rho^{-1}) + h(z_1, z_2) \rho^{4/3}$$

One can assume that

$$g_2 = 0, \varphi_2 = 0, k_2 = 0,$$

then

$$\varepsilon = h(s, z_2) \rho^{4/3}.$$

Equation (86) becomes

$$\frac{2}{3} \beta_1 z_2 h_{z_2} = 2(\frac{\beta_1}{3} + q_1) h + \rho^{1/3} (z_2^{1/2} g_1(\rho) - \rho^{-5/3} \varphi_1(s) + \rho^{-8/3} k_1)$$

Differentiate this equation with respect to ρ one obtains

$$\frac{d}{d\rho} (\rho^{1/3} (z_2^{1/2} g_1(\rho) - \rho^{-5/3} \varphi_1(s) + \rho^{-8/3} k_1)) = 0. \quad (88)$$

Differentiate this equation with respect to z_2 one obtains

$$\rho g_1' + \frac{1}{3} g_1 = 0,$$

then

$$g_1 = c_3 \rho^{-1/3}.$$

Equation (88) becomes

$$4\varphi_1 \rho - 7k_1 = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi_1 = 0, k_1 = 0.$$

Then equation (86) becomes

$$\frac{2}{3}\beta_1 z_2 h_{z_2} = 2\left(\frac{\beta_1}{3} + q_1\right)h + c_3 z_2^{1/2}.$$

For nonisentropic, $\beta_1 \neq 0$. The characteristic system of this equation is

$$\frac{ds}{0} = \frac{3dz_2}{2\beta_1 z_2} = \frac{dh}{2\left(\frac{\beta_1}{3} + q_1\right)h + c_3 z_2^{1/2}}.$$

$$\frac{dh}{dz_2} = \frac{3}{2\beta_1 z_2} \left(2\left(\frac{\beta_1}{3} + q_1\right)h + c_3 z_2^{1/2}\right)$$

$$h = C z_2^\lambda, \lambda = \frac{3}{\beta_1} \left(\frac{\beta_1}{3} + q_1\right)$$

$$C' = \frac{3c_3}{2\beta_1} z_2^{-1-\lambda+1/2}.$$

Remark $\lambda \neq 0$,

Case 1 $\lambda = 1/2$, then

$$C = \frac{3}{2\beta_1} c_3 \ln z_2 + h(s)$$

Then

$$h = \left(\frac{3}{2\beta_1} c_3 \ln z_2 + h(s)\right) z_2^\lambda$$

Then

$$\varepsilon = \left(\left(\frac{3}{2\beta_1} c_3 \ln z_2 + h(s)\right) z_2^\lambda\right) \rho^{4/3}.$$

Case 2 $\lambda \neq 1/2$, then

$$C = \frac{3c_3 z_2^{-\lambda+1/2}}{2\beta_1(-\lambda+1/2)} + h(s)$$

$$h = \left(\frac{3c_3 z_2^{-\lambda+1/2}}{2\beta_1(-\lambda+1/2)} + h(s)\right) z_2^\lambda$$

$$= \frac{3c_3 z_2^{1/2}}{2\beta_1(-\lambda+1/2)} + h(s) z_2^\lambda$$

$$= \frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2\beta_1(-\lambda+1/2)} + h(s) z_2^\lambda$$

Then

$$\varepsilon = \left(\frac{3c_3 \alpha^{1/2} \rho^{-5/3}}{2\beta_1(-\lambda+1/2)} + h(s) z_2^\lambda\right) \rho^{4/3}.$$

Then one can assume $c_3 = 0$, so

$$\varepsilon = h(s) z_2^\lambda \rho^{4/3}$$

Three-dimensional algebras.

Let the basis generators are

$$X = \widetilde{X} + X_{\zeta_1}, \quad Y = \widetilde{Y} + X_{\zeta_2}, \quad Z = \widetilde{Z} + X_{\zeta_3},$$

where

$$\widetilde{X} = \beta_1 X_1 + q_1 X_2 + \gamma_1 X_3, \quad \widetilde{Y} = \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3, \quad \widetilde{Z} = \beta_3 X_1 + q_3 X_2 + \gamma_3 X_3.$$

Notice that

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = 0, \quad [\widetilde{Y}, \widetilde{Z}] = 0. \quad (89)$$

Let us study the Abelian Lie algebra. In this case one has

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

which means that

$$\zeta_1 \zeta_2' - \zeta_1' \zeta_2 = 0, \quad \zeta_1 \zeta_3' - \zeta_1' \zeta_3 = 0, \quad \zeta_2 \zeta_3' - \zeta_2' \zeta_3 = 0, \quad (90)$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = 0, \quad [\widetilde{Y}, \widetilde{Z}] = 0.$$

If $\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 0$, then the basis of this Lie algebra is

$$X, \quad Y, \quad Z.$$

If, for example, $\zeta_1 \neq 0$, then one can assume that $\zeta_1 = 1$ and, hence,

$$\zeta_2 = c_1, \quad \zeta_3 = c_2,$$

where c_1 and c_2 are constant. This gives the following

$$\gamma X_3 + \partial_S, \quad X_1 + q_1 X_3, \quad X_2 + q_2 X_3, \quad (91)$$

$$\beta X_2 + \partial_S, \quad X_1 + q_1 X_2, \quad X_3, \quad (92)$$

$$X_1 + \partial_S, \quad X_2, \quad X_3, \quad (93)$$

$$\partial_S, \quad X_2, \quad X_3, \quad (94)$$

where γ, β, q_1 and q_2 are arbitrary constants.

$$X_1 + \beta X_2 + \gamma X_3 + \partial_S, \quad X_1 + q_1 X_3, \quad X_2 + q_2 X_3,$$

$$X_1 + \beta X_2 + \gamma X_3 + \partial_S, \quad X_1 + q_1 X_2, \quad X_3,$$

$$X_1 + \beta X_2 + \gamma X_3 + \partial_S, \quad X_2, \quad X_3,$$

$$X_2 + \gamma X_3 + \partial_S, \quad , \quad X_1 + q_1 X_3, \quad X_2 + q_2 X_3,$$

$$X_2 + \gamma X_3 + \partial_S, \quad X_1 + q_1 X_2, \quad X_3,$$

$$X_2 + \gamma X_3 + \partial_S, \quad X_2, \quad X_3,$$

$$X_3 + \partial_S, \quad X_1 + q_1 X_3, \quad X_2 + q_2 X_3,$$

$$X_3 + \partial_S, \quad X_1 + q_1 X_2, \quad X_3,$$

$$X_3 + \partial_S, \quad X_2, \quad X_3,$$

$$\partial_S, \quad X_1 + q_1 X_3, \quad X_2 + q_2 X_3,$$

$$\partial_S, \quad X_1 + q_1 X_2, \quad X_3,$$

$$\partial_S, \quad X_2, \quad X_3,$$

All three-dimensional non-abelian algebras are classified:

$$\begin{aligned}
L(3,1) &= \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & 0 \\ e_2 & & 0 & e_1 \\ e_3 & & & 0 \end{array}, \quad L(3,2,p) = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & & 0 & pe_2 \\ e_3 & & & 0 \end{array}, \quad 0 < |p| \leq 1, \\
L(3,3) &= \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & & 0 & e_1 + e_2 \\ e_3 & & & 0 \end{array}, \quad L(3,4,p) = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & pe_1 - e_2 \\ e_2 & & 0 & e_1 + pe_2 \\ e_3 & & & 0 \end{array}, \quad p \geq 0, \\
L(3,5) &= \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & 2e_2 \\ e_2 & & 0 & e_3 \\ e_3 & & & 0 \end{array}, \quad L(3,6) = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_3 & -e_2 \\ e_2 & & 0 & e_1 \\ e_3 & & & 0 \end{array}, \quad L(3,-1) = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & 0 \\ e_2 & & 0 & 0 \\ e_3 & & & 0 \end{array}.
\end{aligned}$$

Here e_i , ($i = 1, 2, 3$) are basis vectors of a Lie algebra and equality is conditional in order to define the commutator table. Let us study $L(3,1)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = X,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = 0, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = \zeta_1. \quad (95)$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = 0, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{X}.$$

Because of (89), one has that $\widetilde{X} = 0$. Hence, $\zeta_1 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (95) become contradictive. Let us study $L(3,2,p)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = X, \quad [Y, Z] = pY,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = \zeta_1, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = p\zeta_2. \quad (96)$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = \widetilde{X}, \quad [\widetilde{Y}, \widetilde{Z}] = p\widetilde{Y}.$$

Because of (89), one has that $\widetilde{X} = 0$ and $\widetilde{Y} = 0$. Hence, $\zeta_1 \zeta_2 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (96) give that ζ_2 is constant, which contradicts to the property that X , Y and Z compose a basis of the Lie algebra. Let us study $L(3,3)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = X, \quad [Y, Z] = X + Y,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = \zeta_1, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = \zeta_1 + \zeta_2. \quad (97)$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = \widetilde{X}, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{X} + \widetilde{Y}.$$

Because of (89), one has that $\widetilde{X} = 0$ and $\widetilde{Y} = 0$. Hence, $\zeta_1\zeta_2 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (97) give that ζ_2 is constant, which contradicts to the property that X, Y and Z compose a basis of the Lie algebra. Let us study $L(3, 4, p)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = pX - Y, \quad [Y, Z] = X + pY, \quad (p \geq 0),$$

which means that

$$\zeta_1\zeta'_2 - \zeta'_1\zeta_2 = 0, \quad \zeta_1\zeta'_3 - \zeta'_1\zeta_3 = p\zeta_1 - \zeta_2, \quad \zeta_2\zeta'_3 - \zeta'_2\zeta_3 = \zeta_1 + p\zeta_2. \quad (98)$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = p\widetilde{X} - \widetilde{Y}, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{X} + p\widetilde{Y}.$$

Because of (89), one has that $\widetilde{X} = 0$ and $\widetilde{Y} = 0$. Hence, $\zeta_1\zeta_2 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (98) give that ζ_2 is constant, which contradicts to the property that X, Y and Z compose a basis of the Lie algebra. Let us study $L(3, 5)$. In this case one has

$$[X, Y] = X, \quad [X, Z] = 2Y, \quad [Y, Z] = Z,$$

which means that

$$\zeta_1\zeta'_2 - \zeta'_1\zeta_2 = \zeta_1, \quad \zeta_1\zeta'_3 - \zeta'_1\zeta_3 = 2\zeta_2, \quad \zeta_2\zeta'_3 - \zeta'_2\zeta_3 = \zeta_3. \quad (99)$$

and

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{X}, \quad [\widetilde{X}, \widetilde{Z}] = 2\widetilde{Y}, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{Z}.$$

Because of (89), one has that $\widetilde{X} = 0$, $\widetilde{Y} = 0$ and $\widetilde{Z} = 0$. Hence, $\zeta_1\zeta_2\zeta_3 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (99) become

$$\zeta'_2 = 1, \quad \zeta'_3 = 2\zeta_2, \quad \zeta_3 = \zeta_2^2. \quad (100)$$

$$\zeta_2 = S + c_1, \quad \zeta_3 = S^2 + 2c_1S + c_1^2. \quad (101)$$

Thus, the basis generators are

$$X = \partial_S, \quad Y = S\partial_S, \quad Z = S^2\partial_S.$$

Let us study $L(3, 6)$. In this case one has

$$[X, Y] = Z, \quad [X, Z] = -Y, \quad [Y, Z] = X,$$

which means that

$$\zeta_1\zeta'_2 - \zeta'_1\zeta_2 = \zeta_3, \quad \zeta_1\zeta'_3 - \zeta'_1\zeta_3 = -\zeta_2, \quad \zeta_2\zeta'_3 - \zeta'_2\zeta_3 = \zeta_1. \quad (102)$$

and

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{Z}, \quad [\widetilde{X}, \widetilde{Z}] = -\widetilde{Y}, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{X}.$$

Because of (89), one has that $\widetilde{X} = 0$, $\widetilde{Y} = 0$ and $\widetilde{Z} = 0$. Hence, $\zeta_1\zeta_2\zeta_3 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (102) become

$$\zeta'_2 = \zeta_3, \quad \zeta'_3 = -\zeta_2, \quad -\zeta_2^2 - \zeta_3^2 = \zeta_1. \quad (103)$$

$$\zeta_2 = c_1 \sin(S) + c_2 \cos(S), \quad \zeta_3 = c_1 \cos(S) - c_2 \sin(S), \quad -c_1^2 - c_2^2 = 1. \quad (104)$$

Which is a contradiction. Let us study $L(3, -1)$. In this case one has

$$[X, Y] = X, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = \zeta_1, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = 0, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = 0. \quad (105)$$

and

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{X}, \quad [\widetilde{X}, \widetilde{Z}] = 0, \quad [\widetilde{Y}, \widetilde{Z}] = 0.$$

Because of (89), one has that $\widetilde{X} = 0$. Hence, $\zeta_1 \neq 0$, and where one can assume that $\zeta_1 = 1$. Equations (105) become

$$\zeta'_2 = 1, \quad \zeta'_3 = 0, \quad \zeta_3 = 0.$$

Thus, the basis generators have the form

$$X = \partial_S, \quad Y = S\partial_S + \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3, \quad Z = \beta_3 X_1 + q_3 X_2 + \gamma_3 X_3.$$

Thus, one needs only to study the following three-dimensional algebras

$$X = \partial_S, \quad Y = S\partial_S, \quad Z = S^2\partial_S; \quad (106)$$

and

$$X = \partial_S, \quad Y = S\partial_S + \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3, \quad Z = \beta_3 X_1 + q_3 X_2 + \gamma_3 X_3.$$

The latter is

$$X = \partial_S, \quad Y = S\partial_S + q_2 X_2 + \gamma_2 X_3, \quad Z = X_1 + q_3 X_2 + \gamma_3 X_3. \quad (107)$$

$$X = \partial_S, \quad Y = S\partial_S + \beta_2 X_1 + \gamma_2 X_3, \quad Z = X_2 + \gamma_3 X_3. \quad (108)$$

$$X = \partial_S, \quad Y = S\partial_S + \beta_2 X_1 + q_2 X_2, \quad Z = X_3. \quad (109)$$

Consider $\{\gamma X_3 + \partial_s, X_1 + q_1 X_3, X_2 + q_2 X_3\}$.

Then (38) become

$$10\gamma\alpha\varepsilon_\alpha + 3\gamma\rho\varepsilon_\rho - \varepsilon_s = 4\gamma\varepsilon + \alpha^{1/2}g_1(\rho) - \varphi_1(s) + k_1\rho^{-1}. \quad (110)$$

$$2(5q_1 - 3)\alpha\varepsilon_\alpha + (3q_1 - 2)\rho\varepsilon_\rho = 2(2q_1 - 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}. \quad (111)$$

and

$$10q_2\alpha\varepsilon_\alpha + 3q_2\rho\varepsilon_\rho = 2(2q_2 + 1)\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (112)$$

Consider equation (112) if $q_2 = 0$, then $\varepsilon = (-1/2)(\alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1})$. So $\varepsilon(\rho, \alpha, s)$ can be transformed to zero thus we will consider in case $q_2 \neq 0$.

Case 1. $\gamma \neq 0$ The characteristic system of equation (112) is

$$\frac{d\alpha}{10q_2\alpha} = \frac{d\rho}{3q_2\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(2q_2 + 1)\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}}.$$

Then

$$z_1 = s$$

$$z_2 = \alpha\rho^{-10/3}$$

$$\frac{d\varepsilon}{d\rho} = \frac{1}{3q_2\rho}(2(2q_2+1)\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1})$$

$$\varepsilon = C\rho^\mu, \mu = \frac{2(2q_2+1)}{3q_2}$$

$$C' = \frac{\rho^{-1-\mu}}{3q_2}(z_2^{1/2}\rho^{5/3}g_3(\rho) - \varphi_3(z_1) + k_3\rho^{-1})$$

Case 1.1 $(\mu+1)\mu \neq 0$, or $(q_2+1/2)(q_2+2/7) \neq 0$,

$$\begin{aligned} C &= \frac{1}{3q_2}(z_2^{1/2}\tilde{g}_3(\rho) + \frac{\varphi_3(z_1)}{\mu}\rho^{-\mu} - \frac{k_3}{1+\mu}\rho^{-1-\mu}) + h(z_1, z_2) \\ &= \frac{1}{3q_2}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) + \frac{3\varphi_3(z_1)}{4}\rho^{-4/3} - \frac{3k_3}{7}\rho^{-7/3}) + h(z_1, z_2) \end{aligned}$$

Then

$$\varepsilon = (\frac{1}{3q_2}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) + \frac{3\varphi_3(z_1)}{4}\rho^{-4/3} - \frac{3k_3}{7}\rho^{-7/3}) + h(z_1, z_2))\rho^\mu$$

One can assume that

$$g_3 = 0, \varphi_3 = 0, k_3 = 0,$$

then

$$\varepsilon = h(s, z_2)\rho^\mu.$$

Equation (110) and equation (111) becomes

$$h_s = \frac{2\gamma}{q_2}h - \rho^{(q_2-2)/(3q_2)}(z_2^{1/2}g_1(\rho) - \rho^{-5/3}\varphi_1(s) + \rho^{-8/3}k_1) \quad (113)$$

(SS2) becomes

$$\frac{2}{3}z_2h_{z_2} = 2(\frac{-2q_1}{q_2} + \frac{4}{3q_2} + \frac{2}{3})h + \rho^{(q_2-2)/(3q_2)}(z_2^{1/2}g_2(\rho) - \rho^{-5/3}\varphi_2(s) + \rho^{-8/3}k_2). \quad (114)$$

Differentiate equation (113) and equation (114) with respect to ρ one obtains

$$\frac{d}{d\rho}(\rho^{(q_2-2)/(3q_2)}(z_2^{1/2}g(\rho) - \rho^{-5/3}\varphi(s) + \rho^{-8/3}k)) = 0. \quad (115)$$

Where $g_1 = g = g_2, \varphi_1 = \varphi = \varphi_2, k_1 = k = k_2$, Differentiate this equation with respect to z_2 one obtains

$$\rho g' - (\frac{2-q_2}{3q_2})g = 0,$$

then

$$g = c\rho^{(2-q_2)/(3q_2)}.$$

Then

$$g_1 = c_1\rho^{(2-q_2)/(3q_2)}, g_2 = c_2\rho^{(2-q_2)/(3q_2)}.$$

Equation (115) becomes

$$2(1+2q_2)\varphi\rho - (2+7q_2)k = 0.$$

Splitting this equation with respect to ρ one obtains

$$(1+2q_2)\varphi = 0.$$

Then

$$(1 + 2q_2)\varphi_1 = 0, (1 + 2q_2)\varphi_2 = 0.$$

and

$$(2 + 7q_2)k = 0.$$

Then

$$(2 + 7q_2)k_1 = 0, (2 + 7q_2)k_2 = 0.$$

Since $(\mu + 1)\mu \neq 0$, then $\varphi_1 = 0, \varphi_2 = 0$ and $k_1 = 0, k_2 = 0$. Then equation (113) and equation (114) becomes

$$h_s = \frac{2\gamma}{q_2}h - c_1 z_2^{1/2} \quad (116)$$

$$\frac{2}{3}z_2 h_{z_2} = \left(\frac{-2q_1}{q_2} + \frac{4}{3q_2} + \frac{2}{3}\right)h + c_2 z_2^{1/2} \quad (117)$$

The characteristics of equation (144) is

$$\frac{ds}{1} = \frac{dz_2}{0} = \frac{dh}{\frac{2\gamma}{q_2}h - c_1 z_2^{1/2}}$$

$$\frac{dh}{ds} = \frac{2\gamma}{q_2}h - c_1 z_2^{1/2}$$

$$h = Ce^{\lambda s}, \lambda = \frac{2\gamma}{q_2}$$

$$C' = -c_1 e^{-\lambda s} z_2^{1/2}$$

$$C = \frac{c_1 z_2^{1/2} e^{-\lambda s}}{\lambda} + \tilde{h}(z_2)$$

Then

$$\begin{aligned} h &= \left(\frac{c_1 z_2^{1/2} e^{-\lambda s}}{\lambda} + \tilde{h}(z_2)\right)e^{\lambda s} \\ &= \frac{c_1 z_2^{1/2}}{\lambda} + \tilde{h}(z_2)e^{\lambda s} \\ &= \frac{c_1 \alpha^{1/2} \rho^{-5/3}}{\lambda} + \tilde{h}(z_2)e^{\lambda s} \end{aligned}$$

Then

$$\varepsilon = \left(\frac{c_1 \alpha^{1/2} \rho^{-5/3}}{\lambda} + \tilde{h}(z_2)e^{\lambda s}\right)\rho^\mu.$$

Then one can assume $c_1 = 0$, so

$$h = \tilde{h}(z_2)e^{\lambda s}$$

Equation (145) become

$$\frac{2}{3}z_2 \tilde{h}_{z_2} = \left(\frac{-2q_1}{q_2} + \frac{4}{3q_2} + \frac{2}{3}\right)\tilde{h} + c_2 z_2^{1/2} e^{(\lambda s)} \quad (118)$$

Differentiate this equation with respect to s one obtains

$$c_2 \lambda = 0.$$

Since $\gamma \neq 0$, then $\lambda \neq 0$, so $c_2 = 0$. Then

$$\frac{2}{3}z_2\tilde{h}_{z_2} = \left(\frac{-2q_1}{q_2} + \frac{4}{3q_2} + \frac{2}{3}\right)\tilde{h}$$

Thus

$$\begin{aligned}\tilde{h} &= c_7 z_2^{(-3q_1+2+q_2)/q_2} \\ h &= c_7 z_2^{(-3q_1+2+q_2)/q_2} e^{\lambda s}\end{aligned}$$

Then

$$\varepsilon = (c_7 z_2^{(-3q_1+2+q_2)/q_2} e^{\lambda s}) \rho^\mu.$$

Case 1.2 $\mu = 0$, or $q_2 = -1/2$,

$$\begin{aligned}C &= -\frac{2}{3}(z_2^{1/2}\tilde{g}_3(\rho) - \varphi_3(z_1)\ln\rho - k_3\rho^{-1}) + h(z_1, z_2) \\ &= -\frac{2}{3}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) - \varphi_3(z_1)\ln\rho - k_3\rho^{-1}) + h(z_1, z_2)\end{aligned}$$

Then

$$\varepsilon = -\frac{2}{3}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) - \varphi_3(z_1)\ln\rho - k_3\rho^{-1}) + h(z_1, z_2)$$

One can assume that

$$g_3 = 0, k_3 = 0,$$

then

$$\varepsilon = \frac{2}{3}\varphi_3(s)\ln\rho + h(s, z_2).$$

Equation (110) and equation (111) becomes

$$\begin{aligned}h_s &= -4\gamma h - z_2^{1/2}\rho^{5/3}g_1(\rho) - \frac{2}{3}(\varphi_3'(s) - 4\gamma\varphi_3(s))\ln\rho + 2\gamma\varphi_3(s) \\ &\quad + \varphi_1(s) - k_1\rho^{-1}\end{aligned} \tag{119}$$

(SS2) becomes

$$\begin{aligned}\frac{2}{3}z_2\tilde{h}_{z_2} &= 2(2q_1 - 1)h + z_2^{1/2}\rho^{5/3}g_2(\rho) \\ &\quad + \frac{4}{3}(2q_1 - 1)\varphi_3(s)\ln\rho - 2(q_1 - \frac{2}{3})\varphi_3(s) - \varphi_2(s) + k_2\rho^{-1}\end{aligned} \tag{120}$$

Differentiate equation (119) and equation (120) with respect to ρ one obtains

$$\frac{d}{d\rho}(z_2^{1/2}\rho^{5/3}g_1(\rho) + \frac{2}{3}(\varphi_3'(s) - 4\gamma\varphi_3(s))\ln\rho + k_1\rho^{-1}) = 0. \tag{121}$$

$$\frac{d}{d\rho}(z_2^{1/2}\rho^{5/3}g_2(\rho) + \frac{4}{3}(2q_1 - 1)\varphi_3(s)\ln\rho + k_2\rho^{-1}) = 0. \tag{122}$$

Differentiate these equation with respect to z_2 one obtains

$$\rho g' + \frac{5}{3}g = 0,$$

Where $g_1 = g = g_2$, then

$$g = c\rho^{-5/3}.$$

Then

$$g_1 = c_1\rho^{-5/3}, g_2 = c_2\rho^{-5/3}.$$

Equation (121) and (122) becomes

$$\frac{2}{3}(\varphi_3'(s) + 4\gamma\varphi_3(s))\rho - k_1 = 0.$$

$$\frac{4}{3}(2q_1 - 1)\varphi_3(s)\rho - k_2 = 0.$$

Splitting these equation with respect to ρ one obtains $k_1 = 0, k_2 = 0$ and

$$(\varphi_3'(s) + 4\gamma\varphi_3(s)) = 0.$$

$$(2q_1 - 1)\varphi_3(s) = 0.$$

Case 1.2.1 $2q_1 - 1 = 0$ then

$$\varphi_3(s) = c_3 e^{-4\gamma s}.$$

Equation (120) and equation (120) becomes

$$h_s = -4\gamma h - z_2^{1/2} c_1 + 2\gamma c_3 e^{-4\gamma s} + \varphi_1(s)$$

$$\frac{2}{3} z_2 h_{z_2} = z_2^{1/2} c_2 + \frac{1}{3} c_3 e^{-4\gamma s} - \varphi_2(s)$$

The characteristics of first equation is

$$\frac{ds}{1} = \frac{dz_2}{0} = \frac{dh}{-4\gamma h - z_2^{1/2} c_1 + 2\gamma c_3 e^{-4\gamma s} + \varphi_1(s)}$$

$$\frac{dh}{ds} = -4\gamma h - z_2^{1/2} c_1 + 2\gamma c_3 e^{-4\gamma s} + \varphi_1(s)$$

$$h = C e^{-4\gamma s}$$

$$C' = (-z_2^{1/2} c_1 + 2\gamma c_3 e^{-4\gamma s} + \varphi_1(s)) e^{4\gamma s} = -z_2^{1/2} c_1 e^{4\gamma s} + 2\gamma c_3 + \varphi_1(s) e^{4\gamma s}$$

$$C = \frac{-c_1 z_2^{1/2} e^{4\gamma s}}{4\gamma} + \varphi(s) + \tilde{h}(z_2)$$

Then

$$\begin{aligned} h &= \left(\frac{-c_1 z_2^{1/2} e^{4\gamma s}}{4\gamma} + \varphi(s) + \tilde{h}(z_2) \right) e^{-4\gamma s} \\ &= \frac{-c_1 z_2^{1/2}}{4\gamma} + \varphi(s) e^{-4\gamma s} + \tilde{h}(z_2) e^{-4\gamma s} \\ &= \frac{-c_1 \alpha^{1/2} \rho^{-5/3}}{4\gamma} + \varphi(s) e^{-4\gamma s} + \tilde{h}(z_2) e^{-4\gamma s} \end{aligned}$$

Since

$$\varepsilon = \frac{2}{3} c_3 e^{-4\gamma s} \ln \rho + h(s, z_2)$$

one can assume that $c_1 = 0, \varphi = 0, \varphi_1 = 0$, then

$$h = \tilde{h}(z_2) e^{-4\gamma s}$$

Equation (145) become

$$\frac{2}{3} z_2 \tilde{h}_{z_2} = (z_2^{1/2} c_2 - \varphi_2(s)) e^{4\gamma s}$$

Differentiate this equation with respect to s one obtains

$$4\gamma z_2^{1/2} c_2 - \varphi_2'(s) - 4\gamma \varphi_2(s) = 0.$$

Differentiate this equation with respect to z_2 one obtains

$$\gamma c_2 = 0.$$

Since $\gamma \neq 0$ then $c_2 = 0$, then

$$\varphi_2 = c_5 e^{-4\gamma s}$$

and

$$\frac{2}{3} z_2 \tilde{h}_{z_2} = c_5$$

Then

$$\tilde{h} = -\frac{3}{2} c_5 \ln z_2 + c_7$$

Then

$$h = \left(-\frac{3}{2} c_5 \ln z_2 + c_7\right) e^{-4\gamma s}$$

$$\varepsilon = \frac{2}{3} c_3 e^{-4\gamma s} \ln \rho + \left(-\frac{3}{2} c_5 \ln z_2 + c_7\right) e^{-4\gamma s}$$

One can assume that $c_7 = 0$ then

$$\varepsilon = \frac{2}{3} c_3 e^{-4\gamma s} \ln \rho - \frac{3}{2} c_5 \ln z_2 e^{-4\gamma s}$$

Case 1.2.2 $2q_1 - 1 \neq 0$ then $\varphi_3 = 0, k_1 = 0, k_2 = 0$, Equation (120) and equation (120) becomes

$$h_s = -4\gamma h - z_2^{1/2} c_1 + \varphi_1(s)$$

$$\frac{2}{3} z_2 h_{z_2} = 2(2q_1 - 1)h + z_2^{1/2} c_2 - \varphi_2(s)$$

The characteristics of first equation is

$$\frac{ds}{1} = \frac{dz_2}{0} = \frac{dh}{-4\gamma h - z_2^{1/2} c_1 + \varphi_1(s)}$$

$$\frac{dh}{ds} = -4\gamma h - z_2^{1/2} c_1 + \varphi_1(s)$$

$$h = C e^{-4\gamma s}$$

$$C' = (-z_2^{1/2} c_1 + \varphi_1(s)) e^{4\gamma s} = -z_2^{1/2} c_1 e^{4\gamma s} + \varphi_1(s) e^{4\gamma s}$$

$$C = \frac{-c_1 z_2^{1/2} e^{4\gamma s}}{4\gamma} + \varphi(s) + \tilde{h}(z_2)$$

Then

$$h = \left(\frac{-c_1 z_2^{1/2} e^{4\gamma s}}{4\gamma} + \varphi(s) + \tilde{h}(z_2)\right) e^{-4\gamma s}$$

$$= \frac{-c_1 z_2^{1/2}}{4\gamma} + \varphi(s) e^{-4\gamma s} + \tilde{h}(z_2) e^{-4\gamma s}$$

$$= \frac{-c_1 \alpha^{1/2} \rho^{-5/3}}{4\gamma} + \varphi(s) e^{-4\gamma s} + \tilde{h}(z_2) e^{-4\gamma s}$$

Since

$$\varepsilon = h(s, z_2)$$

one can assume that $c_1 = 0, \varphi = 0$, then

$$h = \tilde{h}(z_2)e^{-4\gamma s}$$

Equation (145) become

$$\frac{2}{3}z_2\tilde{h}_{z_2} = (z_2^{1/2}c_2 - \varphi_2(s))e^{4\gamma s}$$

Differentiate this equation with respect to s one obtains

$$4\gamma z_2^{1/2}c_2 - \varphi_2'(s) - 4\gamma\varphi_2(s) = 0.$$

Differentiate this equation with respect to z_2 one obtains

$$\gamma c_2 = 0.$$

Since $\gamma \neq 0$ then $c_2 = 0$, then

$$\varphi_2 = c_5 e^{-4\gamma s}$$

and

$$\begin{aligned} \frac{2}{3}z_2\tilde{h}_{z_2} &= 2(2q_1 - 1)h - c_5 \\ \frac{d\tilde{h}}{dz_2} &= \frac{3}{2z_2}(2(2q_1 - 1)h - c_5) \end{aligned}$$

Then

$$\begin{aligned} \tilde{h} &= C z_2^\lambda, \lambda = 3(2q_1 - 1) \\ C' &= \frac{-3c_5}{2} z_2^{-1-\lambda} \end{aligned}$$

Since $\lambda \neq 0$ then

$$\begin{aligned} C &= \frac{-3c_5}{2} \frac{z_2^{-\lambda}}{-\lambda} + c_7 \\ \tilde{h} &= \left(\frac{-3c_5}{2} \frac{z_2^{-\lambda}}{-\lambda} + c_7 \right) z_2^\lambda = \frac{3c_5}{2\lambda} + c_7 z_2^\lambda \\ h &= \left(\frac{3c_5}{2\lambda} + c_7 z_2^\lambda \right) e^{-4\gamma s} \end{aligned}$$

Since

$$\varepsilon = h(s, z_2).$$

One can assume that $c_5 \neq 0$, then

$$\varepsilon = c_7 z_2^\lambda e^{-4\gamma s}$$

Case 1.3 $\mu = -1$, or $q_2 = -2/7$,

$$\begin{aligned} C &= -\frac{7}{6}(z_2^{1/2}\tilde{g}_3(\rho) - \varphi_3(z_1)\rho + k \ln \rho) + h(z_1, z_2) \\ &= -\frac{7}{6}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) - \varphi_3(z_1)\rho + k \ln \rho) + h(z_1, z_2) \end{aligned}$$

Then

$$\varepsilon = (-\frac{7}{6}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) - \varphi_3(s)\rho + k_3 \ln \rho) + h(s, z_2))\rho^{-1}$$

One can assume that

$$g_3 = 0, \varphi_3 = 0,$$

then

$$\varepsilon = (-\frac{7}{6}k_3 \ln \rho + h(s, z_2))\rho^{-1}$$

Equation (110) and equation (111) becomes

$$h_s = -7\gamma h - z_2^{1/2}\rho^{5/3}g_1(\rho) + \frac{49}{6}\gamma \ln \rho k_3 - \frac{7}{2}\gamma k_3 + \varphi_1(s)\rho - k_1 \quad (123)$$

(SS2) becomes

$$\begin{aligned} \frac{2}{3}z_2 h_{z_2} = (7q_1 - 4)h + z_2^{1/2}\rho^{5/3}g_2(\rho) - (\frac{49}{6}q_1 - \frac{14}{3})\ln \rho k_3 + (\frac{7}{2}q_1 - \frac{7}{3})k_3 \\ - \varphi_2(s)\rho + k_2 \end{aligned} \quad (124)$$

Differentiate equation (123) and equation (124) with respect to ρ one obtains

$$\frac{d}{d\rho}(z_2^{1/2}\rho^{5/3}g_1(\rho) + \frac{49}{6}\gamma \ln \rho k_3 + \varphi_1(s)\rho) = 0. \quad (125)$$

$$\frac{d}{d\rho}(z_2^{1/2}\rho^{5/3}g_2(\rho) - (\frac{49}{6}q_1 - \frac{14}{3})\ln \rho k_3 - \varphi_2(s)\rho) = 0. \quad (126)$$

Differentiate these equation with respect to z_2 one obtains

$$\rho g' + \frac{8}{3}g = 0,$$

Where $g_1 = g = g_2$, then

$$g = c\rho^{-8/3}.$$

Then

$$g_1 = c_4\rho^{-8/3}, g_2 = c_5\rho^{-8/3}.$$

Equation (125) and (126) becomes

$$\varphi_1(s)\rho + \frac{49}{6}\gamma k_3 = 0.$$

$$\varphi_2(s)\rho - (\frac{49}{6}q_1 - \frac{14}{3})k_3 = 0.$$

Splitting these equation with respect to ρ one obtains $\varphi_1 = 0, \varphi_2 = 0$ and

$$\gamma k_3 = 0.$$

$$(\frac{49}{6}q_1 - \frac{14}{3})k_3 = 0.$$

Since $\gamma \neq 0$ then $k_3 = 0$ Equation (123) and equation (124) becomes

$$h_s = -7\gamma h - z_2^{1/2}c_4 - k_1$$

$$\frac{2}{3}z_2 h_{z_2} = (7q_1 - 4)h + z_2^{1/2}c_5 + k_2$$

The characteristics of first equation is

$$\frac{ds}{1} = \frac{dz_2}{0} = \frac{dh}{-7\gamma h - z_2^{1/2}c_4 - k_1}$$

$$\frac{dh}{ds} = -7\gamma h - z_2^{1/2}c_4 - k_1$$

$$h = Ce^{-7\gamma s}$$

$$C' = (-z_2^{1/2}c_4 - k_1)e^{7\gamma s}$$

$$C = \frac{(-z_2^{1/2}c_4 - k_1)e^{7\gamma s}}{7\gamma} + \tilde{h}(z_2)$$

Then

$$\begin{aligned} h &= \left(\frac{(-z_2^{1/2}c_4 - k_1)e^{7\gamma s}}{7\gamma} + \tilde{h}(z_2) \right) e^{-7\gamma s} \\ &= \frac{(-z_2^{1/2}c_4 - k_1)}{7\gamma} + \tilde{h}(z_2)e^{-7\gamma s} \\ &= \frac{(-\alpha^{1/2}\rho^{-5/3}c_4 - k_1)}{7\gamma} + \tilde{h}(z_2)e^{-7\gamma s} \end{aligned}$$

Since

$$\varepsilon = h(s, z_2)\rho^{-1}$$

one can assume that $c_4 = 0, k_1 = 0$, then

$$h = \tilde{h}(z_2)e^{-7\gamma s}$$

Equation (124) become

$$\frac{2}{3}z_2\tilde{h}_{z_2} = (7q_1 - 4)\tilde{h} + z_2^{1/2}c_5e^{7\gamma s} + k_2e^{7\gamma s}$$

Differentiate this equation with respect to s one obtains

$$7\gamma z_2^{1/2}c_5 + 7\gamma k_2 = 0.$$

Differentiate this equation with respect to z_2 one obtains

$$\gamma c_5 = 0.$$

Since $\gamma \neq 0$ then $c_5 = 0, k_2 = 0$, then

$$\frac{2}{3}z_2\tilde{h}_{z_2} = (7q_1 - 4)\tilde{h}$$

Then

$$\tilde{h} = c_7 z_2^\lambda, \lambda = \frac{3}{2}(7q_1 - 4)$$

Then

$$h = c_7 z_2^\lambda e^{-7\gamma s}$$

So,

$$\varepsilon = c_7 z_2^\lambda e^{-7\gamma s} \rho^{-1}$$

Case 2. $\gamma = 0$ Then (38) becomes

$$\varepsilon_s = -\alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (127)$$

$$2(5q_1 - 3)\alpha\varepsilon_\alpha + (3q_1 - 2)\rho\varepsilon_\rho = 2(2q_1 - 1)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}. \quad (128)$$

and

$$10q_2\alpha\varepsilon_\alpha + 3q_2\rho\varepsilon_\rho = 2(2q_2 + 1)\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (129)$$

Consider equation (129) if $q_2 = 0$, then $\varepsilon = (-1/2)(\alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1})$. So $\varepsilon(\rho, \alpha, s)$ can be transformed to zero thus we will consider in case $q_2 \neq 0$. Equation (127) gives

$$\varepsilon = (\alpha^{1/2}g_1(\rho) + k_1\rho^{-1})s + h(\alpha, \rho)$$

Differentiate equation (128) and equation (129) with respect to s one obtains

$$g_1(\rho) = c_4\rho^{(2-q_2)/(3q_2)}$$

$$c_4\left(\frac{-q_1}{q_2} + \frac{2}{3q_2} + \frac{1}{6}\right) = 0$$

Case 2.1 $\frac{-q_1}{q_2} + \frac{2}{3q_2} + \frac{1}{6} = 0$ one obtains $\varphi_2 = c_2, \varphi_3 = c_3$ and

$$k_1\left(\frac{7}{6}q_2 + \frac{2}{3}\right) = 0$$

$$k_1(7q_2 + 2) = 0$$

Case 2.1.1 $\frac{7}{6}q_2 + \frac{2}{3} = 0$ Equation (128) and equation (129) becomes

$$\frac{2}{7}\alpha h_\alpha + \frac{2}{7}\rho h_\rho = -\frac{2}{7}h - \alpha^{1/2}g_2 - k_2\rho^{-1} + c_2 \quad (130)$$

The characteristics of first equation is

$$\frac{7d\alpha}{2\alpha} = \frac{7d\rho}{2\rho} = \frac{dh}{-\frac{2}{7}h - \alpha^{1/2}g_2 - k_2\rho^{-1} + c_2}$$

$$z = \alpha\rho^{-1}$$

$$\frac{dh}{d\rho} = \frac{7}{2\rho}\left(-\frac{2}{7}h - \alpha^{1/2}g_2 - k_2\rho^{-1} + c_2\right)$$

$$h = C\rho^{-1}$$

$$C' = \frac{7}{2}\left(-\alpha^{1/2}g_2 - k_2\rho^{-1} + c_2\right)$$

$$C' = \frac{7}{2}\left(-z^{1/2}\rho^{1/2}g_2 - k_2\rho^{-1} + c_2\right)$$

$$C = \frac{7}{2}\left(-z^{1/2}\tilde{g}_2 - k_2\ln\rho + c_2\rho\right) + \tilde{h}(z)$$

Then

$$h = \left(\frac{7}{2}\left(-z^{1/2}\tilde{g}_2 - k_2\ln\rho + c_2\rho\right) + \tilde{h}(z)\right)\rho^{-1}$$

One can assume that $g_2 = 0, c_2 = 0$, then

$$h = \left(-\frac{7}{2}k_2\ln\rho + \tilde{h}(z)\right)\rho^{-1}$$

Equation (129) gives

$$g_3 = c_6 \rho^{-3/2}$$

and

$$7k_2 - c_3 \rho = 0$$

Splitting this equation with respect to ρ one obtains $c_3 = 0, k_2 = 0$. Equation (129) becomes

$$4z\tilde{h}_z = 2\tilde{h} - z^{1/2}c_6 - k_3 \quad (131)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{1}{4z}(2\tilde{h} - z^{1/2}c_6 - k_3)$$

$$\tilde{h} = Cz^{1/2}$$

$$C' = \frac{1}{4}z^{-1-1/2}(-z^{1/2}c_6 - k_3)$$

$$C = \frac{1}{4}(-c_6 \ln z + 2k_3 z^{-1/2}) + c_7$$

Then

$$\tilde{h} = \left(\frac{1}{4}(-c_6 \ln z + 2k_3 z^{-1/2}) + c_7\right)z^{1/2} = \frac{1}{4}(-c_6 \ln z z^{1/2} + 2k_3) + c_7 z^{1/2}$$

Then

$$h = \left(\left(\frac{1}{4}(-c_6 \ln z z^{1/2} + 2k_3) + c_7 z^{1/2}\right)\rho^{-1}\right)$$

$$\varepsilon = (\alpha^{1/2}c_4\rho^{-3/2} + k_1\rho^{-1})s + \left(\left(\frac{1}{4}(-c_6 \ln z z^{1/2} + 2k_3) + c_7 z^{1/2}\right)\rho^{-1}\right)$$

One can assume that $k_3 = 0, c_7 = 0$ then

$$\varepsilon = (\alpha^{1/2}c_4\rho^{-3/2} + k_1\rho^{-1})s - \frac{1}{4}c_6(\ln z)z^{1/2}\rho^{-1}$$

Case 2.1.2 $\frac{7}{6}q_2 + \frac{2}{3} \neq 0$, then $k_1 = 0$, Equation (128) and equation (129) becomes

$$10q_2\alpha h_\alpha + 3q_2\rho h_\rho = 2(2q_2 + 1)h + \alpha^{1/2}g_3 + k_3\rho^{-1} - c_3 \quad (132)$$

The characteristics of first equation is

$$\frac{d\alpha}{10q_2\alpha} = \frac{d\rho}{3q_2\rho} = \frac{dh}{2(2q_2 + 1)h + \alpha^{1/2}g_3(\rho) - c_3 + k_3\rho^{-1}}.$$

Then

$$z = \alpha\rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{1}{3q_2\rho}(2(2q_2 + 1)h + \alpha^{1/2}g_3(\rho) - c_3 + k_3\rho^{-1})$$

$$h = C\rho^\mu, \mu = \frac{2(2q_2 + 1)}{3q_2}$$

$$C' = \frac{\rho^{-1-\mu}}{3q_2}(\alpha^{1/2}g_3(\rho) - c_3 + k_3\rho^{-1})$$

Case 2.1.2.1 $(\mu + 1)\mu \neq 0$, or $(q_2 + 1/2)(q_2 + 2/7) \neq 0$,

$$C = \frac{1}{3q_2}(\alpha^{1/2}\tilde{g}_3(\rho) + \frac{c_3}{\mu}\rho^{-\mu} - \frac{k_3}{1+\mu}\rho^{-1-\mu}) + \tilde{h}(z)$$

$$= \frac{1}{3q_2}(\alpha^{1/2}\tilde{g}_3(\rho) + \frac{c_3}{\mu}\rho^{-\mu} - \frac{k_3}{1+\mu}\rho^{-1-\mu}) + \tilde{h}(z)$$

Then

$$h = (\frac{1}{3q_2}(\alpha^{1/2}\tilde{g}_3(\rho) + \frac{c_3}{\mu}\rho^{-\mu} - \frac{k_3}{1+\mu}\rho^{-1-\mu}) + \tilde{h}(z))\rho^\mu$$

One can assume that

$$g_3 = 0, c_3 = 0, k_3 = 0,$$

then

$$h = h(z)\rho^\mu.$$

One obtains

$$g_2 = c_5\rho^{(-q_2+2)/(3q_2)}$$

$$k_2(2+7q_2) = 0$$

and

$$c_2(1+2q_2) = 0$$

Since $(\mu+1)\mu \neq 0$, then $k_2 = 0, c_2 = 0$, Equation (128) becomes

$$\frac{2}{3}z\tilde{h}_z = \frac{1}{3}\tilde{h} + z^{1/2}c_5 \quad (133)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2z}(\frac{1}{3}\tilde{h} + z^{1/2}c_5)$$

$$\tilde{h} = Cz^{1/2}$$

$$C' = \frac{3}{2}z^{-1-1/2}(z^{1/2}c_5)$$

$$C = \frac{3}{2}c_5 \ln z + c_7$$

Then

$$\tilde{h} = (\frac{3}{2}c_5 \ln z + c_7)z^{1/2}$$

Then

$$h = (\frac{3}{2}c_5 \ln z + c_7)z^{1/2}\rho^\mu$$

$$\varepsilon = (\alpha^{1/2}c_4\rho^{(2-q_2)/(3q_2)})s + (\frac{3}{2}c_5 \ln z + c_7)z^{1/2}\rho^\mu$$

One can assume that $c_7 = 0$ then

$$\varepsilon = (\alpha^{1/2}c_4\rho^{(2-q_2)/(3q_2)})s + \frac{3}{2}c_5(\ln z)z^{1/2}\rho^\mu$$

Case 2.1.2.2 $\mu = 0$, or $q_2 = -1/2$

$$C = -\frac{2}{3}(\alpha^{1/2}\tilde{g}_3(\rho) - c_3 \ln \rho - k_3\rho^{-1}) + h(z)$$

Then

$$h = -\frac{2}{3}(\alpha^{1/2}\tilde{g}_3(\rho) - c_3 \ln \rho - k_3\rho^{-1}) + h(z)$$

One can assume that

$$g_3 = 0, k_3 = 0,$$

then

$$h = \frac{2}{3}c_3 \ln \rho + \tilde{h}(z).$$

One obtains

$$g_2 = c_5 \rho^{-5/3}$$

$$k_2 = 0$$

and

$$c_3 = 0$$

Equation (128) becomes

$$\frac{2}{3}z\tilde{h}_z = \frac{1}{3}\tilde{h} + z^{1/2}c_5 - c_2 \quad (134)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2z}\left(\frac{1}{3}\tilde{h} + z^{1/2}c_5 - c_2\right)$$

$$\tilde{h} = Cz^{1/2}$$

$$C' = \frac{3}{2}z^{-1-1/2}(z^{1/2}c_5 - c_2)$$

$$C = \frac{3}{2}c_5 \ln z + 2c_2 z^{-1/2} + c_7$$

Then

$$\tilde{h} = \left(\frac{3}{2}c_5 \ln z + 2c_2 z^{-1/2} + c_7\right)z^{1/2}$$

Then

$$h = \left(\frac{3}{2}c_5 \ln z + 2c_2 z^{-1/2} + c_7\right)z^{1/2}$$

One can assume that $c_2 = 0, c_7 = 0$ then

$$\varepsilon = (\alpha^{1/2}c_4\rho^{(2-q_2)/(3q_2)})s + \frac{3}{2}c_5(\ln z)z^{1/2}$$

Case 2.1.2.3 $\mu = -1$, or $q_2 = -2/7$,

$$C = -\frac{7}{6}(\alpha^{1/2}\tilde{g}_3(\rho) - c_3\rho + k_3 \ln \rho) + \tilde{h}(z)$$

Then

$$h = \left(-\frac{7}{6}(\alpha^{1/2}\tilde{g}_3(\rho) - c_3\rho + k_3 \ln \rho) + \tilde{h}(z)\right)\rho^{-1}$$

One can assume that

$$g_3 = 0, c_3 = 0,$$

Then

$$h = \left(-\frac{7}{6}k_3 \ln \rho + \tilde{h}(z)\right)\rho^{-1}$$

One obtains

$$g_2 = c_5 \rho^{-5/3}$$

$$k_3 = 0$$

and

$$c_2 = 0$$

Equation (128) becomes

$$\frac{2}{3}z\tilde{h}_z = \frac{1}{3}\tilde{h} + z^{1/2}c_5 + k_2 \quad (135)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2z}\left(\frac{1}{3}\tilde{h} + z^{1/2}c_5 + k_2\right)$$

$$\tilde{h} = Cz^{1/2}$$

$$C' = \frac{3}{2}z^{-1-1/2}(z^{1/2}c_5 + k_2)$$

$$C = \frac{3}{2}c_5 \ln z - 2k_2z^{-1/2} + c_7$$

Then

$$\tilde{h} = \left(\frac{3}{2}c_5 \ln z - 2k_2z^{-1/2} + c_7\right)z^{1/2}$$

Then

$$h = \left(\left(\frac{3}{2}c_5 \ln z - 2k_2z^{-1/2} + c_7\right)z^{1/2}\right)\rho^{-1}$$

One can assume that $k_2 = 0, c_7 = 0$ then

$$\varepsilon = (\alpha^{1/2}c_4\rho^{(2-q_2)/(3q_2)})_s + \frac{3}{2}c_5(\ln z)z^{1/2}\rho^{-1}$$

Case 2.2 $\frac{-q_1}{q_2} + \frac{2}{3q_2} + \frac{1}{6} \neq 0$ then $c_4 = 0$ then

$$\varepsilon = (k_1\rho^{-1})_s + h(\alpha, \rho)$$

$$k_1(7q_1 - 4) = 0$$

$$k_1(7q_2 + 2) = 0$$

For nonisentropic $k_1 \neq 0$ then

$$7q_1 - 4 = 0, \quad 7q_2 + 2 = 0$$

Equation (128) and equation (129) becomes

$$\frac{2}{7}\alpha h_\alpha + \frac{2}{7}\rho h_\rho = -\frac{2}{7}h - \alpha^{1/2}g_2 - k_2\rho^{-1} + c_2 \quad (136)$$

The characteristics of first equation is

$$\frac{7d\alpha}{2\alpha} = \frac{7d\rho}{2\rho} = \frac{dh}{-\frac{2}{7}h - \alpha^{1/2}g_2 - k_2\rho^{-1} + c_2}$$

$$z = \alpha\rho^{-1}$$

$$\frac{dh}{d\rho} = \frac{7}{2\rho}\left(-\frac{2}{7}h - \alpha^{1/2}g_2 - k_2\rho^{-1} + c_2\right)$$

$$h = C\rho^{-1}$$

$$C' = \frac{7}{2}(-\alpha^{1/2}g_2 - k_2\rho^{-1} + c_2)$$

$$C' = \frac{7}{2}(-z^{1/2}\rho^{1/2}g_2 - k_2\rho^{-1} + c_2)$$

$$C = \frac{7}{2}(-z^{1/2}\tilde{g}_2 - k_2 \ln \rho + c_2 \rho) + \tilde{h}(z)$$

Then

$$h = \left(\frac{7}{2}(-z^{1/2}\tilde{g}_2 - k_2 \ln \rho + c_2 \rho) + \tilde{h}(z)\right)\rho^{-1}$$

One can assume that $g_2 = 0, c_2 = 0$, then

$$h = \left(-\frac{7}{2}k_2 \ln \rho + \tilde{h}(z)\right)\rho^{-1}$$

One obtains

$$g_3 = c_6 \rho^{-3/2}$$

and

$$c_3 = 0$$

Equation (129) becomes

$$2z\tilde{h}_z = -z^{1/2}c_6 + 3k_2 - k_3 \quad (137)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{1}{2z}(-z^{1/2}c_6 + 3k_2 - k_3) \\ \tilde{h} &= \frac{1}{2} \int z^{-1}(-z^{1/2}c_6 + k_4)dz + c_7 = \frac{1}{2}(2c_6 z^{1/2} + k_4 \ln z) + c_7 \end{aligned}$$

Then

$$h = \left(-\frac{7}{2}k_2 \ln \rho + \frac{1}{2}(2c_6 z^{1/2} + k_4 \ln z) + c_7\right)\rho^{-1}$$

One can assume that $c_6 = 0, c_7 = 0$ then

$$\varepsilon = (k_1 \rho^{-1})s - \left(\frac{7}{2}k_2 \ln \rho - \frac{1}{2}k_4 \ln z\right)\rho^{-1}$$

Consider $\{\beta X_2 + \partial_s, X_1 + q_1 X_2, X_3\}$.

Then (SS1) becomes

$$\varepsilon_s = -2\beta\varepsilon - \alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (138)$$

(SS2) becomes

$$6\alpha\varepsilon_\alpha + 2\rho\varepsilon_\rho = -2(q_1 - 1)\varepsilon - \alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1}. \quad (139)$$

(SS3) becomes

$$10\alpha\varepsilon_\alpha + 3\rho\varepsilon_\rho = 4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (140)$$

Case 1. $\beta \neq 0$ The characteristic system of equation (112) is

$$\frac{d\alpha}{10\alpha} = \frac{d\rho}{3\rho} = \frac{ds}{0} = \frac{d\varepsilon}{4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}}.$$

Then

$$\begin{aligned} z_1 &= s \\ z_2 &= \alpha\rho^{-10/3} \end{aligned}$$

$$\frac{d\varepsilon}{d\rho} = \frac{1}{3\rho}(4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1})$$

$$\varepsilon = C\rho^{4/3}$$

$$\begin{aligned} C' &= \frac{\rho^{-1-4/3}}{3}(z_2^{1/2}\rho^{5/3}g_3(\rho) - \varphi_3(z_1) + k_3\rho^{-1}) \\ &= \frac{1}{3}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) + \frac{3\varphi_3(z_1)}{4}\rho^{-4/3} - \frac{3k_3}{7}\rho^{-7/3}) + h(z_1, z_2) \end{aligned}$$

Then

$$\varepsilon = (\frac{1}{3}(\alpha^{1/2}\rho^{-5/3}\tilde{g}_3(\rho) + \frac{3\varphi_3(s)}{4}\rho^{-4/3} - \frac{3k_3}{7}\rho^{-7/3}) + h(z_1, z_2))\rho^{4/3}$$

One can assume that

$$g_3 = 0, \varphi_3 = 0, k_3 = 0,$$

then

$$\varepsilon = h(s, z_2)\rho^{4/3}.$$

Equation (110) and equation (111) becomes

$$h_s = -2\beta h - z_2^{1/2}\rho^{1/3}g_1(\rho) + \rho^{-4/3}\varphi_1(s) - \rho^{-7/3}k_1 \quad (141)$$

$$\frac{2}{3}z_2h_{z_2} = (2q_1 + \frac{2}{3})h + z_2^{1/2}\rho^{1/3}g_2(\rho) - \rho^{-4/3}\varphi_2(s) + \rho^{-8/3}k_2 \quad (142)$$

Differentiate equation (141) and equation (142) with respect to ρ one obtains

$$\frac{d}{d\rho}(\rho^{(q_2-2)/(3q_2)}(z_2^{1/2}g(\rho) - \rho^{-5/3}\varphi(s) + \rho^{-8/3}k)) = 0. \quad (143)$$

Where $g_1 = g = g_2, \varphi_1 = \varphi = \varphi_2, k_1 = k = k_2$, Differentiate this equation with respect to z_2 one obtains

$$\rho g' + \frac{1}{3}g = 0,$$

then

$$g = c\rho^{-1/3}.$$

Then

$$g_1 = c_4\rho^{-1/3}, g_2 = c_5\rho^{-1/3}.$$

Equation (115) becomes

$$\frac{4}{3}\varphi\rho - \frac{7}{3}k = 0.$$

Splitting this equation with respect to ρ one obtains

$$\varphi = 0, k = 0$$

Then

$$\varphi_1 = 0, \varphi_2 = 0, k_1 = 0, k_2 = 0.$$

Then equation (141) and equation (142) becomes

$$h_s = 2\beta h - c_1z_2^{1/2} \quad (144)$$

$$\frac{2}{3}z_2h_{z_2} = (2q_1 + \frac{2}{3})h + c_2z_2^{1/2} \quad (145)$$

The characteristics of equation (144) is

$$\frac{ds}{1} = \frac{dz_2}{0} = \frac{dh}{2\beta h - c_1 z_2^{1/2}}$$

$$\frac{dh}{ds} = 2\beta h - c_1 z_2^{1/2}$$

$$h = C e^{2\beta s}$$

$$C' = -c_1 e^{-2\beta s} z_2^{1/2}$$

$$C = \frac{c_1 z_2^{1/2} e^{-2\beta s}}{-2\beta} + \tilde{h}(z_2)$$

Then

$$\begin{aligned} h &= \left(\frac{c_1 z_2^{1/2} e^{-2\beta s}}{-2\beta} + \tilde{h}(z_2) \right) e^{2\beta s} \\ &= \frac{c_1 z_2^{1/2}}{-2\beta} + \tilde{h}(z_2) e^{2\beta s} \\ &= \frac{c_1 \alpha^{1/2} \rho^{-5/3}}{-2\beta} + \tilde{h}(z_2) e^{2\beta s} \end{aligned}$$

Then

$$\varepsilon = \left(\frac{c_1 \alpha^{1/2} \rho^{-5/3}}{-2\beta} + \tilde{h}(z_2) e^{2\beta s} \right) \rho^{4/3}.$$

Then one can assume $c_1 = 0$, so

$$h = \tilde{h}(z_2) e^{2\beta s}$$

Equation (145) become

$$\frac{2}{3} z_2 \tilde{h}_{z_2} = 2q_1 \tilde{h} + c_2 z_2^{1/2} e^{(\lambda s)} \quad (146)$$

Differentiate this equation with respect to s one obtains

$$c_2 \beta = 0.$$

Since $\beta \neq 0$, then $c_2 = 0$. Then

$$\frac{2}{3} z_2 \tilde{h}_{z_2} = (2q_1 + \frac{2}{3}) \tilde{h}$$

Thus

$$\begin{aligned} \tilde{h} &= c_7 z_2^{3q_1+1} \\ h &= c_7 z_2^{3q_1+1} e^{2\beta s} \end{aligned}$$

Then

$$\varepsilon = c_7 z_2^{3q_1+1} e^{2\beta s} \rho^{4/3}.$$

Case 2. $\beta = 0$

Then (38) become

$$\varepsilon_s = -\alpha^{1/2} g_1(\rho) + \varphi_1(s) - k_1 \rho^{-1}. \quad (147)$$

$$6\alpha \varepsilon_\alpha + 2\rho \varepsilon_\rho = -2(q_1 - 1)\varepsilon - \alpha^{1/2} g_2(\rho) + \varphi_2(s) - k_2 \rho^{-1}. \quad (148)$$

and

$$10\alpha\varepsilon_\alpha + 3\rho\varepsilon_\rho = 4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (149)$$

Equation (147) gives

$$\varepsilon = (\alpha^{1/2}g_1(\rho) + k_1\rho^{-1})s + h(\alpha, \rho)$$

Differentiate equation (148) and equation (149) with respect to s one obtains

$$g_1(\rho) = c_4\rho^{-1/3}$$

Equation (148) becomes

$$c_4(q_1 + \frac{1}{6}) = 0$$

If $q_1 + \frac{1}{6} \neq 0$ then $c_4 = 0$. Splitting equation (148) and equation (149) with respect to ρ one obtains

$$\varphi_2 = 2(q_1 - 1) \int \varphi_1(s)ds + c_2$$

$$\varphi_3 = 4 \int \varphi_1(s)ds + c_3$$

$$k_1 = 0$$

Then is not nonisentropics, So $q_1 + \frac{1}{6} = 0$ Splitting equation (148) and equation (149) with respect to ρ one obtains

$$\varphi_2 = -\frac{1}{7} \int \varphi_1(s)ds + c_2$$

$$\varphi_3 = 4 \int \varphi_1(s)ds + c_3$$

$$k_1 = 0$$

Equation (148) and equation (149) becomes

$$(150)$$

$$10\alpha h_\alpha + 3\rho h_\rho = 4h + \alpha^{1/2}g_3 + k_3\rho^{-1} - c_3 \quad (151)$$

The characteristics of first equation is

$$\frac{d\alpha}{10\alpha} = \frac{d\rho}{3\rho} = \frac{dh}{4h + \alpha^{1/2}g_3 + k_3\rho^{-1} - c_3}$$

$$z = \alpha\rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{1}{3\rho}(4h + \alpha^{1/2}g_3 + k_3\rho^{-1} - c_3)$$

$$h = C\rho^{4/3}$$

$$C'\rho^{4/3} = \frac{1}{3\rho}(\alpha^{1/2}g_3 + k_3\rho^{-1} - c_3)$$

$$C' = \frac{1}{3}\rho^{-7/3}(z^{1/2}\rho^{5/3}g_3 + k_3\rho^{-1} - c_3)$$

$$C = \frac{1}{3}(z^{1/2}\tilde{g}_3 - \frac{3}{7}k_3\rho^{-7/3} + c_3\frac{3}{4}\rho^{-4/3}) + \tilde{h}(z)$$

Then

$$h = (\frac{1}{3}(z^{1/2}\tilde{g}_3 - \frac{3}{7}k_3\rho^{-7/3} + c_3\frac{3}{4}\rho^{-4/3}) + \tilde{h}(z))\rho^{4/3}$$

$$\varepsilon = (\alpha^{1/2}c_4\rho^{-1/3})s + h(\rho, \alpha)$$

One can assume that $g_3 = 0, k_3 = 0, c_3 = 0$, then

$$h = \tilde{h}(z)\rho^{4/3}$$

Differentiate (148) with respect to z one obtains

$$g_2 = c_5\rho^{-1/3}$$

and

$$7k_2 - 4c_2\rho = 0$$

Splitting this equation with respect to ρ one obtains $c_2 = 0, k_2 = 0$. Equation (149) becomes

$$\frac{2}{3}z\tilde{h}_z = \frac{1}{3}\tilde{h} + z^{1/2}c_5 \quad (152)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}(\frac{1}{3}\tilde{h} + z^{1/2}c_5) \\ \tilde{h} &= Cz^{1/2} \\ C' &= \frac{3}{2}z^{-1-1/2}(z^{1/2}c_5) \\ C &= \frac{3}{2}(c_5 \ln z) + c_7 \end{aligned}$$

Then

$$\tilde{h} = (\frac{3}{2}(c_5 \ln z) + c_7)z^{1/2} = \frac{3}{2}c_5 \ln z z^{1/2} + c_7 z^{1/2}$$

Then

$$\begin{aligned} h &= (\frac{3}{2}c_5 \ln z z^{1/2} + c_7 z^{1/2})\rho^{4/3} \\ \varepsilon &= (\alpha^{1/2}c_4\rho^{-1/3})s + (\frac{3}{2}c_5 \ln z z^{1/2} + c_7 z^{1/2})\rho^{4/3} \end{aligned}$$

One can assume that $c_7 = 0$ then

$$\varepsilon = (\alpha^{1/2}c_4\rho^{-1/3})s + \frac{3}{2}c_5(\ln z)z^{1/2}\rho^{4/3}$$

Consider $\{X_1 + \partial_s, X_2, X_3\}$.

Then (38) become

$$6\alpha\varepsilon_\alpha + 2\rho\varepsilon_\rho + \varepsilon_s = 2\varepsilon - \alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (153)$$

$$2\varepsilon = -\alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1}. \quad (154)$$

and

$$10\alpha\varepsilon_\alpha + 3\rho\varepsilon_\rho = 4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (155)$$

Consider equation (154) one obtains $\varepsilon = (1/2)(-\alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1})$. So $\varepsilon(\rho, \alpha, s)$ can be transformed to zero.

Consider $\{\partial_s, X_2, X_3\}$.

Then (38) become

$$\varepsilon_s = -\alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (156)$$

$$2\varepsilon = -\alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1}. \quad (157)$$

and

$$10\alpha\varepsilon_\alpha + 3\rho\varepsilon_\rho = 4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (158)$$

Consider equation (157) one obtains $\varepsilon = (1/2)(-\alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1})$. So $\varepsilon(\rho, \alpha, s)$ can be transformed to zero.

Consider $\{\partial_s, s\partial_s, s^2\partial_s\}$. Then (38) become

$$\varepsilon_s = -\alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (159)$$

$$s\varepsilon_s = -\alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1}. \quad (160)$$

and

$$s^2\varepsilon_s = -\alpha^{1/2}g_3(\rho) + \varphi_3(s) - k_3\rho^{-1}. \quad (161)$$

Equation (159) gives

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + \int \varphi_1(s)ds - k_1\rho^{-1}s + h(\rho, \alpha).$$

One can assume that $\varphi_1 = 0$. Equation (160) and equation (161) become

$$\alpha^{1/2}(-g_1(\rho)s + g_2(\rho)) - \varphi_2(s) - (k_1s - k_2)\rho^{-1} = 0. \quad (162)$$

$$\alpha^{1/2}(-g_1(\rho)s^2 + g_3(\rho)) - \varphi_3(s) - (k_1s^2 - k_3)\rho^{-1} = 0. \quad (163)$$

Differentiate equation (160) and equation (161) with respect to α one obtains $g_1 = 0$, $g_2 = 0$. and $g_3 = 0$. and differentiate equation (160) and equation (161) with respect to ρ one obtains $k_1 = 0$, $k_2 = 0$. and $k_3 = 0$. then $\varphi_2 = 0$, and $\varphi_3 = 0$, Then $\varepsilon = h(\rho, \alpha)$, which is gas dynamics.

Consider $\{\partial_s, s\partial_s + q_2X_2 + \gamma_2X_3, X_1 + q_3X_2 + \gamma_3X_3\}$. Then (38) becomes

$$\varepsilon_s = -\alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (164)$$

$$10\gamma_2\alpha\varepsilon_\alpha + 3\gamma_2\rho\varepsilon_\rho - s\varepsilon_s = 2(2\gamma_2 + q_2)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1}. \quad (165)$$

and

$$2(5\gamma_3 - 3)\alpha\varepsilon_\alpha + (3\gamma_3 - 2)\rho\varepsilon_\rho = 2(2\gamma_3 + q_3 - 1)\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (166)$$

Equation (164) gives

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + \int \varphi_1(s)ds - k_1\rho^{-1}s + h(\rho, \alpha).$$

One can assume that $\varphi_1 = 0$. then

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s - k_1\rho^{-1}s + h(\rho, \alpha).$$

Case 1..1 $\gamma_2 \neq 0, \gamma_3 \neq 2/3$ then one obtains

$$\gamma_2\rho g_1' + \frac{1}{3}(\gamma_2 - 2q_2 - 1)g_1 = 0 \quad (167)$$

$$(\gamma_3 - \frac{2}{3})\rho g_1' + \frac{1}{3}(\gamma_3 - 2q_3 - 1)g_1 = 0 \quad (168)$$

Then

$$g_1 = c_4 \rho^{-\frac{1}{3\gamma_2}(\gamma_2 - 2q_2 - 1)}$$

and

$$c_4 \left(\frac{\gamma_3 q_2}{\gamma_2} + \frac{\gamma_3}{2\gamma_2} - \frac{2q_2}{3\gamma_2} - \frac{1}{3\gamma_2} - q_3 - \frac{1}{6} \right) = 0$$

Case 1.1.1 $\frac{\gamma_3 q_2}{\gamma_2} + \frac{\gamma_3}{2\gamma_2} - \frac{2q_2}{3\gamma_2} - \frac{1}{3\gamma_2} - q_3 - \frac{1}{6} \neq 0$, then $c_4 = 0$ Then

$$\varepsilon = -k_1 \rho^{-1} s + h(\rho, \alpha).$$

equation (165) and equation (166) give $\varphi_2 = c_2$, $\varphi_3 = c_3$ and for nonisentropics $k_1 \neq 0$ then

$$q_2 = -\frac{7}{2}\gamma_2 - \frac{1}{2}, \quad q_3 = -\frac{7}{2}\gamma_3 + 2$$

The characteristic system of equation (165) is

$$\frac{d\alpha}{10\gamma_2\alpha} = \frac{d\rho}{3\gamma_2\rho} = \frac{dh}{(-3\gamma_2 - 1)h + \alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2}$$

Then

$$z = \alpha\rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{1}{3\gamma_2\rho} ((-3\gamma_2 - 1)h + \alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2)$$

$$h = C\rho^\mu, \mu = \frac{-3\gamma_2 - 1}{3\gamma_2}$$

$$C' = \frac{\rho^{-1-\mu}}{3\gamma_2} (\alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2)$$

$$= \frac{1}{3\gamma_2} (z^{1/2}\tilde{g}_2(\rho) + k_2\rho^{-2-\mu} - c_2\rho^{-1-\mu})$$

Remark $\mu \neq -1$

Case 1.1.1.1 $\mu \neq 0$ or $\gamma_2 \neq -1/3$ then

$$C = \frac{1}{3\gamma_2} (z^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z)$$

$$= \frac{1}{3\gamma_2} (\alpha^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z)$$

Then

$$h = \left(\frac{1}{3\gamma_2} (\alpha^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z) \right) \rho^\mu$$

One can assume that

$$g_2 = 0, k_2 = 0, c_2 = 0,$$

then

$$h = \tilde{h}(z)\rho^\mu$$

$$g_3 = c_6 \rho^{-(1+8\gamma_2)/(3\gamma_2)}$$

$$k_3 = 0, c_3 = 0$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = \frac{1}{\gamma_2}(\gamma_3 - \frac{2}{3})\tilde{h} + z^{1/2}c_6 \quad (169)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2z}(\frac{1}{\gamma_2}(\gamma_3 - \frac{2}{3})\tilde{h} + z^{1/2}c_6)$$

$$\tilde{h} = Cz^\lambda, \lambda = \frac{3}{2\gamma_2}(\gamma_3 - \frac{2}{3})$$

$$C' = \frac{3}{2}z^{-1-\lambda+1/2}c_6$$

Case 1.1.1.1.1 $\lambda = 1/2$ or $\gamma_3 = \frac{1}{3}(\gamma_2 + 2)$

$$C = \frac{3}{2}c_6 \ln z + c_7$$

Then

$$\tilde{h} = (\frac{3}{2}c_6 \ln z + c_7)z^{1/2}$$

Then

$$h = ((\frac{3}{2}c_6 \ln z + c_7)z^{1/2})\rho^\mu$$

One can assume that $c_7 = 0$ then

$$\varepsilon = -k_1\rho^{-1}s + \frac{3}{2}c_6(\ln z)z^{1/2}\rho^\mu.$$

Case 1.1.1.1.2 $\lambda \neq 1/2$ or $\gamma_3 \neq \frac{1}{3}(\gamma_2 + 2)$

$$C' = \frac{3}{2}z^{-1-\lambda+1/2}c_6$$

$$C = \frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda + 1/2} + c_7$$

Then

$$\tilde{h} = (\frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda + 1/2} + c_7)z^\lambda$$

$$h = ((\frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda + 1/2} + c_7)z^\lambda)\rho^\mu$$

One can assume that $c_6 = 0$ then

$$\varepsilon = -k_1\rho^{-1}s + c_7z^\lambda\rho^\mu.$$

Case 1.1.1.2 $\mu = 0$ then $\gamma_2 = -1/3$

$$C' = -(z^{1/2}\tilde{g}_2(\rho) + k_2\rho^{-2} - c_2\rho^{-1})$$

Then

$$C = -(z^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2 \ln \rho) + \tilde{h}(z)$$

$$h = -(z^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2 \ln \rho) + \tilde{h}(z)$$

One can assume that

$$g_2 = 0, k_2 = 0,$$

then

$$h = c_2 \ln \rho + \tilde{h}(z).$$

$$g_3 = c_6 \rho^{-5/3}$$

$$k_3 = 0$$

$$c_2(3\gamma_3 - 2) = 0$$

Since $3\gamma_3 - 2 \neq 0$ then

$$k_3 = 0, c_2 = 0$$

Then

$$h = \tilde{h}(z)$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = (-3\gamma_3 + 2)\tilde{h} + z^{1/2}c_6 - c_3 \quad (170)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2z}((-3\gamma_3 + 2)\tilde{h} + z^{1/2}c_6 - c_3)$$

$$\tilde{h} = Cz^\lambda, \lambda = \frac{3}{2}(-3\gamma_3 + 2)$$

$$C' = \frac{3}{2}z^{-1-\lambda}(z^{1/2}c_6 - c_3)$$

$$= \frac{3}{2}(c_6z^{-1-\lambda+1/2} - c_3z^{-1-\lambda})$$

Remark $\lambda \neq 0$

Case 1.1.1.2.1 $\lambda = 1/2$ or $\gamma_3 = 5/9$

$$C = \frac{3}{2}(c_6 \ln z + 2c_3z^{-1/2}) + c_7$$

Then

$$\tilde{h} = (\frac{3}{2}(c_6 \ln z + 2c_3z^{-1/2}) + c_7)z^{1/2}$$

Then

$$h = (\frac{3}{2}(c_6 \ln z + 2c_3z^{-1/2}) + c_7)z^{1/2}$$

One can assume that $c_3 = 0, c_7 = 0$ then

$$\tilde{h} = \frac{3}{2}c_6(\ln z)z^{1/2}$$

$$\varepsilon = -k_1\rho^{-1}s + \frac{3}{2}c_6(\ln z)z^{1/2}.$$

Case 1.1.1.2.2 $\lambda \neq 1/2$ or $\gamma_3 \neq 5/9$

$$C = \frac{3}{2}(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + \frac{c_3}{-\lambda}z^{-\lambda}) + c_7$$

Then

$$\tilde{h} = \left(\frac{3}{2} \left(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + \frac{c_3}{-\lambda} z^{-\lambda} \right) + c_7 \right) z^\lambda$$

One can assume that $c_6 = 0, c_3 = 0$ then

$$\tilde{h} = c_7 z^\lambda$$

Then

$$\varepsilon = -k_1 \rho^{-1} s + c_7 z^\lambda.$$

Case 1.1.2 $\frac{\gamma_3 q_2}{\gamma_2} + \frac{\gamma_3}{2\gamma_2} - \frac{2q_2}{3\gamma_2} - \frac{1}{3\gamma_2} - q_3 - \frac{1}{6} = 0$ Then

$$\varepsilon = -\alpha^{1/2} g_1(\rho) s - k_1 \rho^{-1} s + h(\rho, \alpha)$$

equation (165) and equation (166) give $\varphi_2 = c_2, \varphi_3 = c_3$ and

$$k_1(7\gamma_2 + 2q_2 + 1) = 0$$

$$k_1(\dots) = 0$$

Case 1.1.2.1 $7\gamma_2 + 2q_2 + 1 \neq 0$, then $k_1 = 0$ then

$$\varepsilon = -\alpha^{1/2} g_1(\rho) s + h(\rho, \alpha)$$

The characteristic system of equation (165) is

$$\frac{d\alpha}{10\gamma_2\alpha} = \frac{d\rho}{3\gamma_2\rho} = \frac{dh}{2(2\gamma_2 + q_2)h + \alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2}$$

Then

$$z = \alpha \rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{1}{3\gamma_2\rho} (2(2\gamma_2 + q_2)h + \alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2)$$

$$h = C\rho^\mu, \mu = \frac{2(2\gamma_2 + q_2)}{3\gamma_2}$$

$$\begin{aligned} C' &= \frac{\rho^{-1-\mu}}{3\gamma_2} (\alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2) \\ &= \frac{1}{3\gamma_2} (z^{1/2}\tilde{g}_2(\rho) + k_2\rho^{-2-\mu} - c_2\rho^{-1-\mu}) \end{aligned}$$

Case 1.1.2.1.1 $\mu(\mu + 1) \neq 0$ or $(2\gamma_2 + q_2)(7\gamma_2 + 2q_2) \neq 0$ then

$$\begin{aligned} C &= \frac{1}{3\gamma_2} (z^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z) \\ &= \frac{1}{3\gamma_2} (\alpha^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z) \end{aligned}$$

Then

$$h = \left(\frac{1}{3\gamma_2} (\alpha^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z) \right) \rho^\mu$$

One can assume that

$$g_2 = 0, k_2 = 0, c_2 = 0,$$

then

$$\begin{aligned} h &= \tilde{h}(z)\rho^\mu \\ g_3 &= c_6\rho^{(-\gamma_2+2q_2)/(3\gamma_2)} \\ k_3 &= 0, c_3 = 0 \end{aligned}$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = \left(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3}\right)\tilde{h} + z^{1/2}c_6 \quad (171)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}\left(\left(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3}\right)\tilde{h} + z^{1/2}c_6\right) \\ \tilde{h} &= Cz^\lambda, \lambda = \frac{3}{2}\left(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3}\right) \\ C' &= \frac{3}{2}z^{-1-\lambda+1/2}c_6 \end{aligned}$$

Since $\gamma_3 \neq 2/3$ then $\lambda \neq 1/2$

$$\begin{aligned} C' &= \frac{3}{2}z^{-1-\lambda+1/2}c_6 \\ C &= \frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + c_7 \end{aligned}$$

Then

$$\begin{aligned} \tilde{h} &= \left(\frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + c_7\right)z^\lambda \\ h &= \left(\left(\frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + c_7\right)z^\lambda\right)\rho^\mu \end{aligned}$$

One can assume that $c_6 = 0$ then

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + c_7z^\lambda\rho^\mu.$$

Case 1.1.2.1.2 $\mu = 0$ or $q_2 = -2\gamma_2$ then

$$\begin{aligned} C &= \frac{1}{3\gamma_2}(z^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2\ln\rho) + \tilde{h}(z) \\ &= \frac{1}{3\gamma_2}(\alpha^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2\ln\rho) + \tilde{h}(z) \end{aligned}$$

Then

$$h = \frac{1}{3\gamma_2}(\alpha^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2\ln\rho) + \tilde{h}(z)$$

One can assume that

$$g_2 = 0, k_2 = 0,$$

then

$$\begin{aligned} h &= \frac{-1}{3\gamma_2}c_2\ln\rho + \tilde{h}(z) \\ g_3 &= c_6\rho^{-5/3} \end{aligned}$$

$$k_3 = 0$$

$$c_2(\gamma_2 + 3\gamma_3 - 2) = 0$$

Case 1.1.2.1.2.1 $\gamma_2 + 3\gamma_3 - 2 = 0$ then Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = z^{1/2}c_6 + c_3 + \frac{1}{3}c_2 \quad (172)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}(z^{1/2}c_6 + c_3 + \frac{1}{3}c_2) \\ \tilde{h} &= \frac{3}{2} \int (z^{-1/2}c_6 + (c_3 + \frac{1}{3}c_2)z^{-1})dz + c_7 = \frac{3}{2}(2z^{1/2}c_6 + (c_3 + \frac{1}{3}c_2)\ln z) + c_7 \end{aligned}$$

One can assume that

$$c_6 = 0, c_7 = 0,$$

then

$$\tilde{h} = -\frac{3}{2}((c_3 + \frac{1}{3}c_2)\ln z)$$

Then

$$\begin{aligned} h &= \frac{-1}{3\gamma_2}c_2 \ln \rho - \frac{3}{2}((c_3 + \frac{1}{3}c_2)\ln z) \\ \varepsilon &= -\alpha^{1/2}g_1(\rho)s + \frac{-1}{3\gamma_2}c_2 \ln \rho - \frac{3}{2}((c_3 + \frac{1}{3}c_2)\ln z) \end{aligned}$$

Case 1.1.2.1.2.2 $\gamma_2 + 3\gamma_3 - 2 \neq 0$ then $c_2 = 0$ Then

$$h = \tilde{h}(z)$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = (\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3})\tilde{h} + z^{1/2}c_6 - c_3 \quad (173)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}((\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3})\tilde{h} + z^{1/2}c_6 - c_3) \\ \tilde{h} &= Cz^\lambda, \lambda = \frac{3}{2}(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3}) \\ C' &= \frac{3}{2}z^{-1-\lambda}(z^{1/2}c_6 - c_3) \end{aligned}$$

Since $\gamma_3 \neq 2/3$ then $\lambda \neq 1/2$ and since $\gamma_2 + 3\gamma_3 - 2 \neq 0$ then $\lambda \neq 0$

$$C' = \frac{3}{2}z^{-1-\lambda+1/2}c_6$$

$$C = \frac{3}{2}(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} - c_3 \frac{z^{-\lambda}}{-\lambda}) + c_7$$

Then

$$\tilde{h} = (\frac{3}{2}(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} - c_3 \frac{z^{-\lambda}}{-\lambda}) + c_7)z^\lambda$$

One can assume that

$$c_6 = 0, c_3 = 0,$$

Then

$$\tilde{h} = c_7 z^\lambda$$

$$h = c_7 z^\lambda$$

$$\varepsilon = -\alpha^{1/2} g_1(\rho) s + c_7 z^\lambda$$

Case 1.1.2.1.3 $\mu = -1$ or $q_2 = -\frac{7}{2}\gamma_2$ then

$$C = \frac{1}{3\gamma_2} (z^{1/2} \tilde{g}_2(\rho) + k_2 \ln \rho - c_2 \rho) + \tilde{h}(z)$$

$$= \frac{1}{3\gamma_2} (\alpha^{1/2} \tilde{g}_2(\rho) + k_2 \ln \rho - c_2 \rho) + \tilde{h}(z)$$

Then

$$h = \left(\frac{1}{3\gamma_2} (\alpha^{1/2} \tilde{g}_2(\rho) + k_2 \ln \rho - c_2 \rho) + \tilde{h}(z) \right) \rho^{-1}$$

One can assume that

$$g_2 = 0, c_2 = 0,$$

then

$$h = \left(\frac{1}{3\gamma_2} k_2 \ln \rho + \tilde{h}(z) \right) \rho^{-1}$$

$$g_3 = c_6 \rho^{-8/3}$$

+++++

$$c_3 = 0$$

$$k_2(\gamma_2 + 3\gamma_3 - 2) = 0$$

Case 1.1.2.1.3.1 $\gamma_2 + 3\gamma_3 - 2 = 0$ then Equation (166) becomes

$$\frac{2}{3} z \tilde{h}_z = z^{1/2} c_6 + k_3 + \frac{1}{3} k_2 \quad (174)$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2z} (z^{1/2} c_6 + k_3 + \frac{1}{3} k_2)$$

$$\tilde{h} = \frac{3}{2} \int (z^{-1/2} c_6 + (k_3 + \frac{1}{3} k_2) z^{-1}) dz + c_7 = \frac{3}{2} (2z^{1/2} c_6 + (k_3 + \frac{1}{3} k_2) \ln z) + c_7$$

One can assume that

$$c_6 = 0, c_7 = 0,$$

then

$$\tilde{h} = \frac{3}{2} ((k_3 + \frac{1}{3} k_2) \ln z)$$

Then

$$h = \left(\frac{1}{3\gamma_2} k_2 \ln \rho + \frac{3}{2} ((k_3 + \frac{1}{3} k_2) \ln z) \right) \rho^{-1}$$

$$\varepsilon = -\alpha^{1/2} g_1(\rho) s + \left(\frac{1}{3\gamma_2} k_2 \ln \rho + \frac{3}{2} ((k_3 + \frac{1}{3} k_2) \ln z) \right) \rho^{-1}$$

Case 1.1.2.1.3.2 $\gamma_2 + 3\gamma_3 - 2 \neq 0$ then $k_2 = 0$ Then

$$h = \tilde{h}(z)\rho^{-1}$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = \left(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3}\right)\tilde{h} + z^{1/2}c_6 + k_3 \quad (175)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z} \left(\left(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3} \right) \tilde{h} + z^{1/2}c_6 + k_3 \right) \\ \tilde{h} &= Cz^\lambda, \lambda = \frac{3}{2} \left(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3} \right) \\ C' &= \frac{3}{2}z^{-1-\lambda}(z^{1/2}c_6 + k_3) \end{aligned}$$

Since $\gamma_3 \neq 2/3$ then $\lambda \neq 1/2$ and since $\gamma_2 + 3\gamma_3 - 2 \neq 0$ then $\lambda \neq 0$

$$C = \frac{3}{2} \left(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + k_3 \frac{z^{-\lambda}}{-\lambda} \right) + c_7$$

Then

$$\tilde{h} = \left(\frac{3}{2} \left(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + k_3 \frac{z^{-\lambda}}{-\lambda} \right) + c_7 \right) z^\lambda$$

One can assume that

$$c_6 = 0, k_3 = 0,$$

Then

$$\begin{aligned} \tilde{h} &= c_7 z^\lambda \\ h &= c_7 z^\lambda \rho^{-1} \\ \varepsilon &= -\alpha^{1/2} g_1(\rho) s + c_7 z^\lambda \rho^{-1} \end{aligned}$$

Case 1.1.2.2 $7\gamma_2 + 2q_2 + 1 = 0$, then $q_2 = (-1 - 7\gamma_2)/2$ then $k_1 = 0$. So

$$\varepsilon = -\alpha^{1/2} g_1(\rho) s + h(\rho, \alpha)$$

The characteristic system of equation (165) is

$$\frac{d\alpha}{10\gamma_2\alpha} = \frac{d\rho}{3\gamma_2\rho} = \frac{dh}{(-3\gamma_2 - 1)h + \alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2}$$

Then

$$\begin{aligned} z &= \alpha\rho^{-10/3} \\ \frac{dh}{d\rho} &= \frac{1}{3\gamma_2\rho} ((-3\gamma_2 - 1)h + \alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2) \\ h &= C\rho^\mu, \mu = \frac{-3\gamma_2 - 1}{3\gamma_2} \\ C' &= \frac{\rho^{-1-\mu}}{3\gamma_2} (\alpha^{1/2}g_2(\rho) + k_2\rho^{-1} - c_2) \end{aligned}$$

$$= \frac{1}{3\gamma_2}(z^{1/2}\tilde{g}_2(\rho) + k_2\rho^{-2-\mu} - c_2\rho^{-1-\mu})$$

Remark $\mu \neq -1$

Case 1.1.2.2.1 $\mu \neq 0$ or $\gamma_2 \neq -1/3$ then

$$\begin{aligned} C &= \frac{1}{3\gamma_2}(z^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z) \\ &= \frac{1}{3\gamma_2}(\alpha^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z) \end{aligned}$$

Then

$$h = (\frac{1}{3\gamma_2}(\alpha^{1/2}\tilde{g}_2(\rho) - \frac{k_2}{1+\mu}\rho^{-1-\mu} + \frac{c_2}{\mu}\rho^{-\mu}) + \tilde{h}(z))\rho^\mu$$

One can assume that

$$g_2 = 0, k_2 = 0, c_2 = 0,$$

then

$$\begin{aligned} h &= \tilde{h}(z)\rho^\mu \\ g_3 &= c_6\rho^{(-1-8\gamma_2)/(3\gamma_2)} \\ k_3 &= 0, c_3 = 0 \end{aligned}$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = (\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3})\tilde{h} + z^{1/2}c_6 \quad (176)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}((\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3})\tilde{h} + z^{1/2}c_6) \\ \tilde{h} &= Cz^\lambda, \lambda = \frac{3}{2}(\frac{\gamma_3}{\gamma_2} - \frac{2}{3\gamma_2} + \frac{1}{3}) \\ C' &= \frac{3}{2}z^{-1-\lambda+1/2}c_6 \end{aligned}$$

Since $\gamma_3 \neq 2/3$ then $\lambda \neq 1/2$

$$\begin{aligned} C' &= \frac{3}{2}z^{-1-\lambda+1/2}c_6 \\ C &= \frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + c_7 \end{aligned}$$

Then

$$\begin{aligned} \tilde{h} &= (\frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + c_7)z^\lambda \\ h &= ((\frac{3}{2}c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} + c_7)z^\lambda)\rho^\mu \end{aligned}$$

One can assume that $c_6 = 0$ then

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + c_7z^\lambda\rho^\mu.$$

Case 1.1.2.2.2 $\mu = 0$ or $\gamma_2 = -1/3$ then

$$\begin{aligned} C &= -(z^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2 \ln \rho) + \tilde{h}(z) \\ &= -(\alpha^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2 \ln \rho) + \tilde{h}(z) \end{aligned}$$

Then

$$h = -(\alpha^{1/2}\tilde{g}_2(\rho) - k_2\rho^{-1} - c_2 \ln \rho) + \tilde{h}(z)$$

One can assume that

$$g_2 = 0, k_2 = 0,$$

then

$$h = c_2 \ln \rho + \tilde{h}(z)$$

$$g_3 = c_6\rho^{-5/3}$$

$$k_3 = 0$$

$$c_2(9\gamma_3 - 7) = 0$$

Case 1.1.2.2.2.1 $\gamma_3 = 7/9$ then Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = z^{1/2}c_6 - c_3 - \frac{1}{3}k_2 \quad (177)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}(z^{1/2}c_6 - c_3 - \frac{1}{3}k_2) \\ \tilde{h} &= \int \frac{3}{2}(z^{-1/2}c_6 - (c_3 + \frac{1}{3}k_2)z^{-1})dz + c_7 = \frac{3}{2}(2z^{1/2}c_6 - (c_3 + \frac{1}{3}k_2) \ln z) + c_7 \end{aligned}$$

One can assume that

$$c_6 = 0, c_7 = 0$$

then

$$\tilde{h} = \frac{3}{2}(-(c_3 + \frac{1}{3}k_2) \ln z)$$

Then

$$\begin{aligned} h &= c_2 \ln \rho + \frac{3}{2}(-(c_3 + \frac{1}{3}k_2) \ln z) \\ \varepsilon &= -\alpha^{1/2}g_1(\rho)s + c_2 \ln \rho + \frac{3}{2}(-(c_3 + \frac{1}{3}k_2) \ln z). \end{aligned}$$

Case 1.1.2.2.2.2 $\gamma_3 \neq 7/9$ then $c_2 = 0$ then

$$h = \tilde{h}(z)$$

Equation (166) becomes

$$\frac{2}{3}z\tilde{h}_z = (-3\gamma_3 + \frac{7}{3})\tilde{h} + z^{1/2}c_6 - c_3 \quad (178)$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2z}((-3\gamma_3 + \frac{7}{3})\tilde{h} + z^{1/2}c_6 - c_3) \\ \tilde{h} &= Cz^\lambda, \lambda = \frac{3}{2}(-3\gamma_3 + \frac{7}{3}) \end{aligned}$$

$$C' = \frac{3}{2}z^{-1-\lambda}(z_3^{1/2}c_6 - c_3)$$

Remark $\lambda \neq 0$

Since $\gamma_3 \neq 2/3$ then $\lambda \neq 1/2$

$$C = \frac{3}{2}(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} - c_3 \frac{z^{-\lambda}}{-\lambda}) + c_7$$

Then

$$\tilde{h} = (\frac{3}{2}(c_6 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} - c_3 \frac{z^{-\lambda}}{-\lambda}) + c_7)z^\lambda$$

One can assume that $c_6 = 0, c_3 = 0$ then

$$\tilde{h} = c_7 z^\lambda$$

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + c_7 z^\lambda.$$

Consider $\{\partial_s, s\partial_s + \beta_2 X_1 + \gamma_2 X_3, X_2 + \gamma_3 X_3\}$. Then (38) becomes

$$\varepsilon_s = -\alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (179)$$

$$2(5\gamma_2 - 3\beta_2)\alpha\varepsilon_\alpha + (3\gamma_2 - 2\beta_2)\rho\varepsilon_\rho - s\varepsilon_s = 2(2\gamma_2 - \beta_2)\varepsilon + \alpha^{1/2}g_2(\rho) - \varphi_2(s) + k_2\rho^{-1} \quad (180)$$

and

$$10\gamma_3\alpha\varepsilon_\alpha + 3\gamma_3\rho\varepsilon_\rho = 2(2\gamma_3 + 1)\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (181)$$

If $\gamma_3 = 0$, then $\varepsilon = (-1/2)(\alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1})$. Then $\varepsilon(\rho, \alpha, s)$ can be transformed to zero thus we will consider in case $\gamma_3 \neq 0$. Equation (179) gives

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + \int \varphi_1(s)ds - k_1\rho^{-1}s + h(\rho, \alpha).$$

One can assume that $\varphi_1 = 0$. then

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s - k_1\rho^{-1}s + h(\rho, \alpha).$$

Differentiate equation (180) and equation (181) with respect to s one obtains

$$\alpha^{1/2}((3\gamma_2 - 2\beta_2)\rho g_1'(\rho) + (\gamma_2 - \beta_2 - 1)g_1(\rho)) - \varphi_2'(s) + (4\beta_2 - 7\gamma_2 - 1)k_1\rho^{-1} = 0. \quad (182)$$

(181) becomes

$$\alpha^{1/2}(3\gamma_3\rho g_1'(\rho) + (\gamma_3 - 2)g_1(\rho)) - \varphi_3'(s) - (2 + 7\gamma_3)k_1\rho^{-1} = 0. \quad (183)$$

Differentiate equation (182) and equation (183) with respect to α one obtains

$$(\frac{3\gamma_2}{2} - \beta_2)\rho g_1'(\rho) + (\frac{\gamma_2}{2} - \frac{\beta_2}{2} - \frac{1}{2})g_1(\rho) = 0. \quad (184)$$

(181) becomes

$$\frac{3\gamma_3}{2}\rho g_1'(\rho) + (\frac{\gamma_3}{2} - 1)g_1(\rho) = 0. \quad (185)$$

Since $\gamma_3 \neq 0$, then

$$g_1(\rho) = c_4 \rho^{-(\gamma_3-2)/(3\gamma_3)}.$$

Then

$$\varepsilon = -\alpha^{1/2} c_4 \rho^{-(\gamma_3-2)/(3\gamma_3)} s - k_1 \rho^{-1} s + h(\rho, \alpha).$$

Equation (184) becomes

$$c_4 \left(\frac{\gamma_2}{\gamma_3} - \frac{2\beta_2}{3\gamma_3} - \frac{\beta_2}{6} - \frac{1}{2} \right) = 0 = 0.$$

Case 1. $\frac{\gamma_2}{\gamma_3} - \frac{2\beta_2}{3\gamma_3} - \frac{\beta_2}{6} - \frac{1}{2} \neq 0$, so $c_4 = 0$ then

$$\varepsilon = -k_1 \rho^{-1} s + h(\rho, \alpha).$$

Equation (182) and equation (183) becomes

$$\rho \varphi_2'(s) - (4\beta_2 - 7\gamma_2 - 1)k_1 = 0.$$

$$\rho \varphi_3'(s) + (7\gamma_3 + 2)k_1 = 0.$$

Splitting above equation with respect to ρ one obtains

$$\varphi_2 = c_2, \varphi_3 = c_3.$$

$$(4\beta_2 - 7\gamma_2 - 1)k_1 = 0.$$

$$(7\gamma_3 + 2)k_1 = 0.$$

For nonisentropics

$$\gamma_3 = -\frac{2}{7}$$

$$4\beta_2 - 7\gamma_2 - 1 = 0$$

Equation (180) and equation (181) becomes

$$\frac{1}{7}(\beta_2 + 10)\alpha h_\alpha + \frac{1}{7}(2\beta_2 + 3)\rho h_\rho = -\frac{1}{7}(\beta_2 - 4)h - \alpha^{1/2}g_2(\rho) - k_2\rho^{-1} + c_2. \quad (186)$$

(181) becomes

$$\left(\frac{20}{7}\right)\alpha h_\alpha + \frac{6}{7}\rho h_\rho = -\frac{6}{7}h - \alpha^{1/2}g_3(\rho) - k_3\rho^{-1} + c_3. \quad (187)$$

The characteristic system of equation (187) is

$$\frac{7d\alpha}{20\alpha} = \frac{7d\rho}{6\rho} = \frac{dh}{-\frac{6}{7}h - \alpha^{1/2}g_3(\rho) - k_3\rho^{-1} + c_3}.$$

Then

$$z = \alpha \rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{7}{6\rho} \left(-\frac{6}{7}h - \alpha^{1/2}g_3(\rho) - k_3\rho^{-1} + c_3 \right)$$

$$h = C\rho^{-1}$$

$$C' = \frac{7\rho^{-2}}{6} (-\alpha^{1/2}g_3(\rho) - k_3\rho^{-1} + c_3)$$

$$C = \frac{7}{6}(z^{1/2}\tilde{g}_3(\rho) + \frac{k_3}{2}\rho^{-2} - \frac{c_3}{2}\rho^{-1}) + h(z)$$

Then

$$h = \frac{7}{6}(z^{1/2}\tilde{g}_3(\rho) + \frac{k_3}{2}\rho^{-1} - \frac{c_3}{2}) + h(z)\rho^{-1}$$

One can assume that

$$g_3 = 0, k_3 = 0, c_3 = 0,$$

then

$$h = \tilde{h}(z)\rho^{-1}.$$

Equation (186) becomes

$$\frac{2}{3}\beta_2 z \tilde{h}_z = -\tilde{h} + k_2 + \rho(z^{1/2}\rho^{5/3}g_2(\rho) - c_2). \quad (188)$$

Differentiate this equation with respect to ρ one obtains

$$\frac{d}{d\rho}\rho(z^{1/2}\rho^{5/3}g_2(\rho) - c_2) = 0. \quad (189)$$

Differentiate this equation with respect to z one obtains

$$\rho g_2'(\rho) + \frac{8}{3}g_2 = 0.$$

Then

$$g_2 = c_5\rho^{-8/3}.$$

Equation (189) becomes

$$c_2 = 0.$$

Equation (188) becomes

$$\frac{2}{3}\beta_2 z \tilde{h}_z = -\tilde{h} + k_2 + z^{1/2}c_5.$$

Case 1.1 $\beta_2 \neq 0$, The characteristic system of this equation is

$$\frac{3dz}{2\beta_2 z} = \frac{d\tilde{h}}{-\tilde{h} + k_2 + z^{1/2}c_5}.$$

Then

$$\begin{aligned} \frac{d\tilde{h}}{dz} &= \frac{3}{2\beta_2 z}(-\tilde{h} + k_2 + z^{1/2}c_5) \\ \tilde{h} &= Cz^\lambda, \lambda = -3/2\beta_2 \\ C' &= \frac{3z^{-1-\lambda}}{2\beta_2}(k_2 + z^{1/2}c_5) \end{aligned}$$

Remark $\lambda \neq 0$,

Case 1.1.1 $\lambda = 1/2$, then

$$C = \frac{3}{2\beta_2}(-2k_2 z^{-1/2} + c_5 \ln z) + C_7.$$

Then

$$\tilde{h} = \frac{3}{2\beta_2}(-2k_2 + c_5 \ln z z^{1/2}) + C_7 z^{1/2} = \frac{3}{2\beta_2}(-2k_2 + c_5 \ln z z^{1/2}) + C_7 \alpha^{1/2} \rho^{-5/2}.$$

One can assume that

$$k_2 = 0, C_7 = 0.$$

Then

$$h = \frac{3}{2\beta_2}(c_5 \ln z z^{1/2})\rho^{-1}.$$

Then

$$\varepsilon = -k_1\rho^{-1}s + \frac{3}{2\beta_2}(c_5 \ln z z^{1/2})\rho^{-1}.$$

Case 1.1.2 $\lambda \neq 1/2$, then

$$C' = \frac{3z^{-1-\lambda}}{2\beta_2}(k_2 + z^{1/2}c_5)$$

$$C = \frac{3}{2\beta_2}(k_2 \frac{z^{-\lambda}}{-\lambda} + c_5 \frac{z^{-\lambda+1/2}}{(-\lambda+1/2)}) + C_7.$$

Then

$$\tilde{h} = \frac{3}{2\beta_2}(\frac{k_2}{-\lambda} + c_5 \frac{z^{1/2}}{(-\lambda+1/2)}) + C_7 z^\lambda = \frac{3}{2\beta_2}(\frac{k_2}{-\lambda} + c_5 \frac{\alpha^{1/2}\rho^{-5/2}}{(-\lambda+1/2)}) + C_7 \alpha^{1/2}\rho^{-5/2}.$$

One can assume that

$$k_2 = 0, c_5 = 0, C_7 = 0$$

Then

$$h = 0.$$

So

$$\varepsilon = -k_1\rho^{-1}s.$$

Which is not depend on $\alpha \neq 0$,

Equation (188) becomes

$$\frac{2}{3}\beta_2 z \tilde{h}_z = -\tilde{h} + k_2 + z^{1/2}c_5.$$

Case 1. $2\beta_2 = 0$, then

$$\tilde{h} = k_2 + z^{1/2}c_5.$$

$$h = (k_2 + z^{1/2}c_5)\rho^{-1}.$$

One can assume that

$$k_2 = 0, c_5 = 0$$

Then

$$h = 0.$$

So

$$\varepsilon = -k_1\rho^{-1}s.$$

Which is not depend on $\alpha \neq 0$,

Case 2. $\frac{\gamma_2}{\gamma_3} - \frac{2\beta_2}{3\gamma_3} - \frac{\beta_2}{6} - \frac{1}{2} = 0$, then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-(\gamma_3-2)/(3\gamma_3)}s - k_1\rho^{-1}s + h(\rho, \alpha).$$

Equation (182) and equation (183) gives

$$\varphi_2 = c_2, \varphi_3 = c_3.$$

$$(7\beta_2\gamma_3 + 4\beta_2 + 21\gamma_3 + 6)k_1 = 0.$$

$$(7\gamma_3 + 2)k_1 = 0.$$

Case 2.1 $7\gamma_3 + 2 = 0$, then fist equation gives $\beta_2 = 0$,
The characteristic system of equation is

$$\frac{7d\alpha}{20\alpha} = \frac{7d\rho}{6\rho} = \frac{dh}{-\frac{6}{7}h - \alpha^{1/2}g_3(\rho) - k_3\rho^{-1}}.$$

Then

$$z = \alpha\rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{7}{6\rho}(-\frac{6}{7}h - \alpha^{1/2}g_3(\rho) - k_3\rho^{-1})$$

$$h = C\rho^{-1}$$

$$C' = \frac{7\rho^{-2}}{6}(-\alpha^{1/2}g_3(\rho) - k_3\rho^{-1})$$

$$C = \frac{7}{6}(z^{1/2}\tilde{g}_3(\rho) + \frac{k_3}{2}\rho^{-2}) + h(z)$$

Then

$$h = \frac{7}{6}(z^{1/2}\tilde{g}_3(\rho) + \frac{k_3}{2}\rho^{-1}) + \tilde{h}(z)\rho^{-1}$$

One can assume that

$$g_3 = 0, k_3 = 0,$$

then

$$h = \tilde{h}(z)\rho^{-1}.$$

$$\rho g_2'(\rho) + \frac{8}{3}g_2 = 0.$$

Then

$$g_2 = c_5\rho^{-8/3}.$$

Equation (189) becomes

$$c_2 = 0.$$

Equation (188) becomes

$$\tilde{h} = k_2 + z^{1/2}c_5.$$

Then

$$h = (k_2 + z^{1/2}c_5)\rho^{-1}.$$

One can assume that

$$k_2 = 0, c_5 = 0$$

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-(\gamma_3-2)/(3\gamma_3)}s - k_1\rho^{-1}s$$

Case 2.2 $7\gamma_3 + 2 \neq 0$, then $k_1 = 0$, then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-(\gamma_3-2)/(3\gamma_3)}s + h(\rho, \alpha).$$

The characteristic system of the equation is

$$\frac{d\alpha}{10\gamma_3\alpha} = \frac{d\rho}{3\gamma_3\rho} = \frac{dh}{2(2\gamma_3 + 1)h + \alpha^{1/2}g_3(\rho) + k_3\rho^{-1} - c_3}.$$

Then

$$\begin{aligned}
 z &= \alpha \rho^{-10/3} \\
 \frac{dh}{d\rho} &= \frac{1}{3\gamma_3 \rho} (2(2\gamma_3 + 1)h + \alpha^{1/2} g_3(\rho) + k_3 \rho^{-1} - c_3) \\
 h &= C \rho^\mu, \mu = \frac{1}{3\gamma_3} (2(2\gamma_3 + 1)) \\
 C' &= \frac{\rho^{-1-\mu}}{3\gamma_3} (\alpha^{1/2} g_3(\rho) + k_3 \rho^{-1} - c_3) \\
 &= \frac{\rho^{-1-\mu}}{3\gamma_3} (z^{1/2} \tilde{g}_3(\rho) + k_3 \rho^{-1} - c_3)
 \end{aligned}$$

Remark $\mu = -1$,

Case 2.2.1 $\mu = 0$ or $\gamma_3 = -1/2$ then

$$C = \frac{1}{3\gamma_3} (z^{1/2} \tilde{g}_3(\rho) - k_3 \rho^{-1} - c_3 \ln \rho) + \tilde{h}(z)$$

Then

$$h = \frac{1}{3\gamma_3} (z^{1/2} \tilde{g}_3(\rho) - k_3 \rho^{-1} - c_3 \ln \rho) + \tilde{h}(z)$$

One can assume that

$$g_3 = 0, k_3 = 0,$$

then

$$\begin{aligned}
 h &= \frac{1}{3\gamma_3} (-c_3 \ln \rho) + \tilde{h}(z) \\
 g_2 &= c_5 \rho^{-5/3}.
 \end{aligned}$$

Equation (189) becomes

$$k_2 = 0.$$

$$c_3 \left(\frac{\beta_2}{3} - 1 \right) = 0.$$

Case 2.2.1.1 $\beta_2 = 3$ then Equation (188) becomes

$$2z \tilde{h}_z = z^{1/2} c_5 - c_2 + c_3 = z^{1/2} c_5 + c_7.$$

$$\frac{d\tilde{h}}{dz} = \frac{1}{2z} (z^{1/2} c_5 + c_7).$$

Then

$$\tilde{h} = \frac{1}{2} (2z^{1/2} c_5 + c_7 \ln z) + c_8$$

Then

$$h = \frac{1}{3\gamma_3} (-c_3 \ln \rho) + \frac{1}{2} (2z^{1/2} c_5 + c_7 \ln z) + c_8$$

One can assume that

$$c_5 = 0, c_8 = 0.$$

Then

$$h = \frac{-c_3}{3\gamma_3} \ln \rho + \frac{1}{2} c_7 \ln z$$

$$\varepsilon = -\alpha^{1/2} c_4 \rho^{-(\gamma_3-2)/(3\gamma_3)} s + \frac{-c_3}{3\gamma_3} \ln \rho + \frac{1}{2} c_7 \ln z.$$

Case 2.2.1.2 $\beta_2 \neq 3$ then $c_3 = 0$ Then

$$h = \tilde{h}(z)$$

Equation (188) becomes

$$\frac{2}{3} \beta_2 z \tilde{h}_z = (\frac{1}{3} \beta_2 - 1) \tilde{h} + z^{1/2} c_5 - c_2.$$

Case 2.2.1.2.1 $\beta_2 \neq 0$

$$\frac{d\tilde{h}}{dz} = \frac{3}{2\beta_2 z} ((\frac{1}{3} \beta_2 - 1) \tilde{h} + z^{1/2} c_5 - c_2).$$

$$\tilde{h} = C z^\lambda, \lambda = \frac{3}{2\beta_2} (\frac{1}{3} \beta_2 - 1)$$

$$C' = \frac{3}{2\beta_2} z^{-1-\lambda} (z^{1/2} c_5 - c_2)$$

Remark $\lambda \neq 0, \lambda \neq 1/2$ then

$$C = \frac{3}{2\beta_2} (c_5 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} - c_2 \frac{z^{-\lambda}}{-\lambda}) + C c_7$$

Then

$$\tilde{h} = (\frac{3}{2\beta_2} (c_5 \frac{z^{-\lambda+1/2}}{-\lambda+1/2} - c_2 \frac{z^{-\lambda}}{-\lambda}) + c_7) z^\lambda$$

One can assume that

$$c_5 = 0, c_2 = 0.$$

Then

$$\tilde{h} = c_7 z^\lambda$$

Then

$$h = c_7 z^\lambda$$

$$\varepsilon = -\alpha^{1/2} c_4 \rho^{-(\gamma_3-2)/(3\gamma_3)} s + c_7 z^\lambda.$$

Case 2.2.1.2.2 $\beta_2 = 0$

$$\tilde{h} = z^{1/2} c_5 - c_2.$$

One can assume that $\tilde{h} = 0$ then

$$\varepsilon = -\alpha^{1/2} c_4 \rho^{-(\gamma_3-2)/(3\gamma_3)} s$$

Case 2.2.2 $\mu \neq 0$ or $\gamma_3 \neq -1/2$ then

$$C = \frac{1}{3\gamma_3} (\alpha^{1/2} g_3(\rho) + k_3 \frac{\rho^{-1-\mu}}{-1-\mu} - c_3 \frac{\rho^{-\mu}}{-\mu}) + \tilde{h}(z)$$

Then

$$h = (\frac{1}{3\gamma_3} (\alpha^{1/2} g_3(\rho) + k_3 \frac{\rho^{-1-\mu}}{-1-\mu} - c_3 \frac{\rho^{-\mu}}{-\mu}) + \tilde{h}(z)) \rho^\mu$$

One can assume that

$$g_3 = 0, k_3 = 0, c_3 = 0$$

then

$$h = \tilde{h}(z)\rho^\mu$$

$$g_2 = c_5\rho^{(2-\gamma_3)/(3\gamma_3)}.$$

Equation (189) becomes

$$c_2(1 + 2\gamma_3) = 0.$$

$$k_2(2 + 7\gamma_3) = 0.$$

Since $1 + 2\gamma_3 \neq 0, 2 + 7\gamma_3 \neq 0$ then $c_2 = 0, k_2 = 0$ Equation (188) becomes

$$\frac{2}{3}\beta_2 z \tilde{h}_z = (\frac{1}{3}\beta_2 - 1)\tilde{h} + z^{1/2}c_5.$$

Case 2.2.2.1 $\beta_2 \neq 0$

$$\frac{d\tilde{h}}{dz} = \frac{3}{2\beta_2 z}((\frac{1}{3}\beta_2 - 1)\tilde{h} + z^{1/2}c_5).$$

$$\tilde{h} = Cz^\lambda, \lambda = \frac{3}{2\beta_2}(\frac{1}{3}\beta_2 - 1)$$

$$C' = \frac{3}{2\beta_2}z^{-1-\lambda}(z^{1/2}c_5)$$

Remark $\lambda \neq 1/2$ then

$$C = \frac{3}{2\beta_2}(c_5 \frac{z^{-\lambda+1/2}}{-\lambda+1/2}) + Cc_7$$

Then

$$\tilde{h} = (\frac{3}{2\beta_2}(c_5 \frac{z^{-\lambda+1/2}}{-\lambda+1/2}) + c_7)z^\lambda$$

One can assume that

$$c_5 = 0.$$

Then

$$\tilde{h} = c_7 z^\lambda$$

Then

$$h = c_7 z^\lambda \rho^\mu$$

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-(\gamma_3-2)/(3\gamma_3)}s + c_7 z^\lambda \rho^\mu.$$

Case 2.2.2.2 $\beta_2 = 0$

$$\tilde{h} = z^{1/2}c_5.$$

One can assume that $\tilde{h} = 0$ then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-(\gamma_3-2)/(3\gamma_3)}s$$

Consider $\{\partial_s, s\partial_s + \beta_2 X_1 + q_2 X_2, X_3\}$. Then (38) becomes

$$\varepsilon_s = -\alpha^{1/2}g_1(\rho) + \varphi_1(s) - k_1\rho^{-1}. \quad (190)$$

$$6\beta_2\alpha\varepsilon_\alpha + 2\beta_2\rho\varepsilon_\rho + s\varepsilon_s = 2(\beta_2 - q_2)\varepsilon - \alpha^{1/2}g_2(\rho) + \varphi_2(s) - k_2\rho^{-1} \quad (191)$$

and

$$10\alpha\varepsilon_\alpha + 3\rho\varepsilon_\rho = 4\varepsilon + \alpha^{1/2}g_3(\rho) - \varphi_3(s) + k_3\rho^{-1}. \quad (192)$$

Equation (179) gives

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s + \int \varphi_1(s)ds - k_1\rho^{-1}s + h(\rho, \alpha).$$

One can assume that $\varphi_1 = 0$. then

$$\varepsilon = -\alpha^{1/2}g_1(\rho)s - k_1\rho^{-1}s + h(\rho, \alpha).$$

$$g_1(\rho) = c_4\rho^{-1/3}$$

Then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-1/3}s - k_1\rho^{-1}s + h(\rho, \alpha)$$

Equation (191) becomes

$$c_4(\beta_2 + 6q_2 + 3) = 0$$

If $\beta_2 + 6q_2 + 3 \neq 0$, then $c_4 = 0$ so

$$\varepsilon = -k_1\rho^{-1}s + h(\rho, \alpha).$$

$$\varphi_2 = c_2, \varphi_3 = c_3.$$

$$k_1 = 0.$$

Then

$$\varepsilon = h(\rho, \alpha).$$

Then for nonisentropics $\beta_2 + 6q_2 + 3 = 0$

$$\varphi_2 = c_2, \varphi_3 = c_3.$$

$$k_1 = 0.$$

Then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-1/3}s + h(\rho, \alpha)$$

The characteristic system of equation (192) is

$$\frac{d\alpha}{10\alpha} = \frac{d\rho}{3\rho} = \frac{dh}{4h + \alpha^{1/2}g_3(\rho) + k_3\rho^{-1} - c_3}.$$

Then

$$z = \alpha\rho^{-10/3}$$

$$\frac{dh}{d\rho} = \frac{1}{3\rho}(4h + \alpha^{1/2}g_3(\rho) + k_3\rho^{-1} - c_3)$$

$$h = C\rho^{4/3}$$

$$C' = \frac{\rho^{-7/3}}{3}(\alpha^{1/2}g_3(\rho) + k_3\rho^{-1} - c_3)$$

$$C = \frac{1}{3}(z^{1/2}\tilde{g}_3(\rho) - \frac{3k_3}{7}\rho^{-7/3} + \frac{3c_3}{4}\rho^{-4/3}) + \tilde{h}(z)$$

Then

$$h = \left(\frac{1}{3}(z^{1/2}\tilde{g}_3(\rho) - \frac{3k_3}{7}\rho^{-7/3} + \frac{3c_3}{4}\rho^{-4/3}) + \tilde{h}(z)\right)\rho^{4/3}$$

One can assume that

$$g_3 = 0, k_3 = 0, c_3 = 0$$

then

$$h = \tilde{h}(z)\rho^{4/3}$$

$$g_2 = c_5\rho^{-1/3}$$

$$c_2 = 0, k_2 = 0$$

$$\frac{2}{3}\beta_2 z \tilde{h}_z = \left(\frac{1}{3}\beta_2 - 1\right)\tilde{h} + z^{1/2}c_5.$$

Case 1 $\beta_2 \neq 0$ The characteristic system of the equation is

$$\frac{3dz}{2\beta_2 z} = \frac{d\tilde{h}}{\left(\frac{1}{3}\beta_2 - 1\right)\tilde{h} + z^{1/2}c_5}.$$

Then

$$\frac{d\tilde{h}}{dz} = \frac{3}{2\beta_2 z} \left(\left(\frac{1}{3}\beta_2 - 1\right)\tilde{h} + z^{1/2}c_5\right)$$

$$\tilde{h} = Cz^\lambda, \lambda = \frac{3}{2\beta_2} \left(\frac{1}{3}\beta_2 - 1\right)$$

$$C' = \frac{3z^{-1-\lambda}}{2\beta_2} (z^{1/2}c_5)$$

Since $\lambda \neq 1/2$, then

$$C = \frac{3}{2\beta_2} \left(c_5 \frac{z^{-\lambda+1/2}}{-\lambda+1/2}\right) + C_7.$$

Then

$$\tilde{h} = \left(\frac{3}{2\beta_2} \left(c_5 \frac{z^{-\lambda+1/2}}{-\lambda+1/2}\right) + C_7\right)z^\lambda$$

One can assume that

$$c_5 = 0.$$

Then

$$h = C_7 z^\lambda \rho^{4/3}$$

Then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-1/3}s + C_7 z^\lambda \rho^{4/3}$$

Equation (191) becomes

$$\frac{2}{3}\beta_2 z \tilde{h}_z = \left(\frac{1}{3}\beta_2 - 1\right)\tilde{h} + z^{1/2}c_5.$$

Case 2 $\beta_2 = 0$ then

$$\tilde{h} = z^{1/2}c_5$$

One can assume that $\tilde{h} = 0$ then

$$\varepsilon = -\alpha^{1/2}c_4\rho^{-1/3}s$$

Executive Summary

จากแบบเสนอโครงการวิจัยที่เสนอขอทุนวิจัย ขอบเขตของการวิจัยจะเป็นการจำแนกประเภทเชิงกลุ่มของสมการของไหลหนึ่งมิติที่พลังงานภายใน $\varepsilon = \zeta(\rho, s)\sqrt{\alpha} + \varphi(\rho, \nabla\rho) + \phi(\rho, s)$ เมื่อ ρ คือ ความหนาแน่น, $\nabla\rho$ คือเกรเดียนต์ของความหนาแน่น และ s คือเอนโทรปี $\zeta(\rho, s)$ เป็นฟังก์ชันของ ρ, s , $\varphi(\rho, \nabla\rho)$ เป็นฟังก์ชันของ $\rho, \nabla\rho$ และ $\phi(\rho, s)$ เป็นฟังก์ชันของ ρ, s

การวิจัยครั้งนี้ได้คือการจำแนกประเภทเชิงกลุ่มของสมการของไหลหนึ่งมิติที่พลังงานภายในคือ $\varepsilon = \varepsilon(\rho, s, \nabla\rho)$ ซึ่งอยู่ในรูปทั่วไป (general) และรูปแบบของพลังงานภายในที่เสนอไว้ในแบบเสนอโครงการวิจัย เป็นส่วนหนึ่งของการวิจัยครั้งนี้ ผลการวิจัยมีรายละเอียดดังต่อไปนี้

1. ได้กลุ่มสมมูลของลี (Equivalence Lie group) คือ

$$\begin{aligned} X_1^e &= \rho\partial_\rho + 2\alpha\partial_\alpha, X_2^e = t\partial_t + x\partial_x - 2\alpha\partial_\alpha, X_3^e = t\partial_x - u\partial_u - 2\alpha\partial_\alpha, X_4^e = h(s)\partial_s, \\ Y_1^e &= \partial_t, Y_2^e = \partial_x, Y_3^e = t\partial_x + \partial_u \\ Z_1^e &= \rho^{-1}\partial_s, Z_2^e = f(s)\partial_s, Z_3^e = g(\rho)\sqrt{\alpha}\partial_s \end{aligned}$$

2. ได้กลุ่มยอมรับของลี (Admitted Lie group) คือ

$$\begin{aligned} X_1 &= \rho\partial_\rho + \alpha\partial_\alpha, X_2 = t\partial_t - u\partial_u, X_3 = u\partial_u + 2\alpha\partial_\alpha - x\partial_x - 2t\partial_t, \\ X_4 &= \rho t\partial_\rho + (ut - x)\partial_u + 4\alpha t\partial_\alpha - tx\partial_x - t^2\partial_t, \\ X_5 &= t^3\partial_x + 3t^2\partial_u, X_6 = t^2\partial_x + 2t\partial_u \\ Y_1 &= \partial_t, Y_2 = \partial_x, Y_3 = t\partial_x + \partial_u \end{aligned}$$

3. ได้การจำแนกประเภทเชิงกลุ่มตามกลุ่มยอมรับของลี ดังตารางต่อไปนี้

	$\varepsilon(\rho, \alpha, s)$	Extensions	Remarks
M_1	$\rho^2 h(\alpha\rho^{-4}e^{\beta s})e^{\gamma s}$	$X_4, \beta X_3 + (\beta + \gamma)X_2 - \partial_s$	$\beta \neq \gamma$
M_2	$\rho^2 h(\alpha\rho^{-4}e^{-\beta s})e^{2\beta s} + s\rho^{-2}\alpha^{1/2}$	$X_4, \beta(X_3 + 2X_2) + \partial_s$	
M_3	$\alpha^\lambda \rho^{2-4\lambda} h(\alpha\rho^{-4}e^{-s})$	$X_4, X_3 + (1 - \lambda)X_2 + 2\partial_s$	$\lambda \neq 0$
M_4	$\rho^2 e^{-s} h(\alpha\rho^{-4})$	$X_4, X_2 - 2\partial_s$	
M_5	$K\alpha^\lambda \rho^{2-4\lambda} e^{-2s}$	$X_4, X_3 - (\lambda + 1)\partial_s, X_2 + \partial_s$	
M_6	$K\alpha^\lambda \rho^{2-4\lambda} + s\alpha^{1/2}$	$X_4, (1 - \lambda)X_2 + X_3 + (2\lambda - 1)s\partial_s, \partial_s$	$(\lambda - 1)K \neq 0$
M_7	$K\alpha\rho^{-2} + s\alpha^{1/2}$	$X_4, X_3 + s\partial_s, \partial_s$	$K \neq 0$
M_8	$s\alpha^{1/2}$	$X_4, X_3 + s\partial_s, X_2 - 2s\partial_s, \partial_s$	
M_9	$(K\alpha\rho^{-4} + s)\alpha^{1/2}$	$X_4, X_2 - 2X_3 - 4s\partial_s, \partial_s$	$K \neq 0$
M_{10}	$\rho^4 \alpha^{-1/2} s$	$X_4, X_3 + 3s\partial_s, X_2 - 2s\partial_s$	
M_{11}	$\rho^\mu h(s, \alpha\rho^\lambda)$	$(3\lambda + 10)X_1 - (\lambda + \mu + 2)X_2 + 2(\lambda + 3)X_3$	$(\mu + 1)\mu \neq 0$ $(3\lambda + 10) \neq 0$
M_{12}	$\rho^{-1}[h(s, \alpha\rho^\lambda) + C\ln\rho]$	$(3\lambda + 10)X_1 - (\lambda + 1)X_2 + 2(\lambda + 3)X_3$	$(3\lambda + 10) \neq 0$
M_{13}	$h(s, \alpha\rho^\lambda) + \varphi(s)\ln\rho$	$(3\lambda + 10)X_1 - (\lambda + 2)X_2 + 2(\lambda + 3)X_3$	$(3\lambda + 10) \neq 0$
M_{14}	$\alpha^\lambda h(s, \rho)$	$3X_1 + (\lambda - 1)X_2 + 2X_3$	$\lambda(2\lambda - 1) \neq 0$
M_{15}	$\alpha^{1/2}[h(s, \rho) + \varphi(s)\ln\alpha]$	$6X_1 - X_2 + 4X_3$	
M_{16}	$h(s, \rho) + (\varphi(s) + C\rho^{-1})\ln\alpha$	$3X_1 - X_2 + 2X_3$	

M_{17}	$\rho^\mu h(s, \alpha \rho^{-10/3})$	$(3\mu - 4)X_2 + 2X_3$	$\mu(\mu + 1) \neq 0$
M_{18}	$\rho^{-1}[h(s, \alpha \rho^{-10/3}) + C \ln \rho]$	$-7X_2 + 2X_3$	
M_{19}	$h(s, \alpha \rho^{-10/3}) + \varphi(s) \ln \rho$	$-2X_2 + X_3$	
M_{20}	$h(\rho, \alpha s^\lambda) s^\mu$	$14X_1 - (\lambda + 7\mu + 4)X_2 + 2(\lambda + 4)X_3 + 14\partial_s$	$7\mu \neq 2(3\lambda - 2)$
M_{21}	$h(\rho, \alpha s^\lambda) s^{2(3\lambda-2)/7} + C\rho^{-1}s$	$14X_1 - 7\lambda X_2 + 2(\lambda + 4)X_3 + \partial_s$	
M_{22}	$h(\rho, \alpha s^2) s^{-1} + g(\rho) \alpha^{1/2} s^2$	$6X_1 - X_2 + 4X_3 + \partial_s$	
M_{23}	$h(\alpha e^{10s}, \rho e^{3s}) e^\lambda$	$-(\lambda + 4)X_2 + 2X_3 + 2\partial_s$	$(\lambda - 3) \neq 0$
M_{24}	$h(\alpha e^{10s}, \rho e^{3s}) e^{3s} + K\rho^{-1}s$	$-7X_2 + 2X_3 + 2\partial_s$	
M_{25}	$h(\rho, \alpha) s^{-2}$	$X_2 + \partial_s$	
M_{26}	$[\alpha^{1/2} g(\rho) + C\rho^{-1}]s + \rho^\mu h(\alpha \rho^\lambda)$	$\partial_s, (3\lambda + 10)X_1 + 2(\lambda + 3)X_2 - (\lambda + \mu + 2)X_3 + (3\lambda + 10)s\partial_s$	$(3\lambda + 10)C \neq 0$ $\mu(\mu + 1)h \neq 0$
M_{27}	$[\alpha^{1/2} g(\rho) + C\rho^{-1}]s + h(\alpha \rho^\lambda) + K \ln \rho$	$\partial_s, (3\lambda + 10)X_1 + 2(\lambda + 3)X_2 - (\lambda + 2)X_3 + (3\lambda + 10)s\partial_s$	$(3\lambda + 10) \neq 0$ $ChK \neq 0$
M_{28}	$[\alpha^{1/2} g(\rho) + C\rho^{-1}]s + \rho^{-1}h(\alpha \rho^\lambda) + K\rho^{-1} \ln \rho$	$\partial_s, (3\lambda + 10)X_1 + 2(\lambda + 3)X_2 - (\lambda + 1)X_3 + (3\lambda + 10)s\partial_s$	$(3\lambda + 10) \neq 0$ $ChK \neq 0$
M_{29}	$[\alpha^{1/2} g(\rho) + C\rho^{-1}]s + \alpha^\lambda h(\rho)$	$\partial_s, 3X_1 + 2X_2 + (\lambda - 1)X_3 + s\partial_s$	$\lambda(2\lambda - 1) \neq 0$ $Ch \neq 0$
M_{30}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s + h(\rho) + (K_1 + K_2 \rho^{-1}) \ln \alpha$	$\partial_s, 3X_1 + 2X_2 - X_3 + 3s\partial_s$	$hCK_1K_2 \neq 0$
M_{31}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s + \alpha^{1/2}[h(\rho) + f(\rho) \ln \alpha]$	$\partial_s, 6X_1 + 4X_2 - X_3 + 6s\partial_s$	$Chf \neq 0$
M_{32}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s + \rho^\mu h(\alpha \rho^{-10/3})$	$\partial_s, (3\mu - 4)X_2 + 2X_3 + (3\mu - 4)s\partial_s$	$\mu(\mu + 1) \neq 0$ $(3\mu - 4)Ch \neq 0$
M_{33}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s + h(\alpha \rho^{-10/3}) + K \ln \rho$	$\partial_s, 2X_2 - X_3 + 2s\partial_s$	$ChK \neq 0$
M_{34}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s + \rho^{-1}[h(\alpha \rho^{-10/3}) + K \ln \rho]$	$\partial_s, 7X_2 - 2X_3 + 7s\partial_s$	$ChK \neq 0$
M_{35}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s + \rho^{4/3}h(\alpha \rho^{-10/3})$	$\partial_s, X_3 + s\partial_s$	$Ch \neq 0$
M_{36}	$[\sqrt{\alpha} g(\rho) + C\rho^{-1}]s$	$\partial_s, X_2 + s\partial_s$	$Cg \neq 0$
M_{37}	$[K \ln \alpha + C \ln \rho + h(s)]\sqrt{\alpha} \rho^\mu$	$(3\mu + 1)X_2 + 2X_3, 2X_1 - (2\mu + 1)X_2$	$(3\mu + 1)C \neq 0$
M_{38}	$h(\alpha \rho^\gamma) + \varphi(s) \ln \rho + K\sqrt{\alpha} \rho^{\gamma/2} s$	$\partial_s, (3\gamma + 10)X_1 - (\gamma + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10) \neq 0$
M_{39}	$\rho^{-1}[K_2\sqrt{\alpha} \rho^{\gamma/2} + K_1 \ln \rho] + \rho^{-1}h(\alpha \rho^\gamma) + C\rho^{-1}s$	$\partial_s, (3\gamma + 10)X_1 - (\gamma + 1)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10) \neq 0$

M_{40}	$\rho^{-1}[K_1 \ln \rho + K_2 \sqrt{\alpha} \rho^{-5/3}] + \rho^{-1} h(\alpha \rho^{-10/3}) + C \rho^{-1} s$	$\partial_s, 7X_2 - 2X_3$	
M_{41}	$\rho^{-1}[K \ln \alpha + C \ln \rho] + \rho^{-1} h(s)$	$7X_1 + 4X_3, 7X_2 - 2X_3$	
M_{42}	$h(\rho e^{\gamma s}) + [K e^{\beta \gamma s} + C \ln \alpha] \rho^{-1}$	$7\gamma X_2 - 2\gamma X_3 + 2\partial_s, 3X_1 - X_2 + 2X_3$	$\gamma \neq 0$
M_{43}	$\alpha^\mu [h(\rho e^{\gamma s}) e^{\mu s} + C]$	$-qX_2 - 2\gamma X_3 + 2\partial_s,$ $3X_1 + (\mu - 1)X_2 + 2X_3$	$\mu(2\mu - 1) \neq 0$
M_{44}	$\sqrt{\alpha} h(\rho e^{\gamma s}) e^{\lambda s}$	$-(\lambda + \gamma)X_2 - 2\gamma X_3 + 2\partial_s,$ $6X_1 - X_2 + 4X_3$	$\lambda \neq 0$
M_{45}	$\sqrt{\alpha}[f(\rho) \ln \alpha + g(\rho)s] + \sqrt{\alpha}[h(\rho e^{\gamma s})]$	$-\gamma X_2 - 2\gamma X_3 + 2\partial_s, 6X_1 - X_2 + 4X_3$	
M_{46}	$C \rho^{-1} s + h(\rho e^{\gamma s}) + \varphi(s) \ln \alpha$	$2\gamma X_2 - \gamma X_3 + \partial_s, 6X_1 - X_2 + 4X_3$	
M_{47}	$h(\rho) + [Ks + C \ln \alpha] \rho^{-1} + \varphi(s) \ln \alpha$	$\partial_s, 3X_1 - X_2 + 2X_3$	
M_{48}	$[K_1 s + K_2 \ln \rho + C \sqrt{\alpha} \rho^{\gamma/2} h(\alpha \rho^{\gamma} e^{\beta s})] \rho^{-1}$	$-7\beta X_2 + 2\beta X_3 + (2(3\gamma + 10))\partial_s,$ $(3\gamma + 10)X_1 - (\gamma + 1)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10)\beta \neq 0$
M_{49}	$\varphi(s) \ln \rho + C \sqrt{\alpha} \rho^{\gamma/2} + h(\alpha \rho^{\gamma} e^{\beta s})$	$-2\beta X_2 + \beta X_3 + (3\gamma + 10)\partial_s,$ $(3\gamma + 10)X_1 - (\gamma + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10)\beta \neq 0$
M_{50}	$\rho^{-1}[K \ln \rho + Cs] + \rho^{-1}[h(\alpha \rho^{-10/3} e^{\beta s})]$	$21\beta X_1 + 12\beta X_3 + 14\partial_s, 7X_2 - 2X_3$	$\beta \neq 0$
M_{51}	$g(s) \ln \rho + f(s) \ln \alpha$	$2X_1 + X_3, 2X_2 - X_3$	$(g - K_1 e^{\lambda s}) \neq 0$ $(f - K_2 e^{\lambda s}) \neq 0$
M_{52}	$\rho^\mu e^{\lambda s} h(\alpha \rho^{\gamma} e^{\beta s})$	$(\beta(3\mu - 4) - \lambda(3\gamma + 10))X_2 + 2\beta X_3$ $+ (2(3\gamma + 10))\partial_s, (3\gamma + 10)X_1$ $-(\gamma + \mu + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10) \neq 0$ $(2\lambda + \beta) \neq 0$
M_{53}	$h(s) \alpha^\lambda \rho^{\mu - 10\lambda/3}$	$3(3\mu - 4)X_1 + 2((3\mu - 3) - \lambda)X_3,$ $(3\mu - 4)X_2 + 2X_3$	$\mu(3\mu - 4) \neq 0$ $(h - K e^{\beta s}) \neq 0$ $\lambda(2\lambda - 1) \neq 0$
M_{54}	$h(s) \alpha^\lambda \rho^{(4 - 10\lambda)/3}$	$3X_1 + (\lambda - 1)X_2, X_3$	$\lambda(2\lambda - 1) \neq 0$ $(h - K e^{\beta s}) \neq 0$
M_{55}	$h(s) \alpha^\lambda \rho^{-10\lambda/3}$	$-2X_2 + X_3, 3X_1 + (3 - \lambda)X_2$	$\lambda(2\lambda - 1) \neq 0$ $(h - K e^{\beta s}) \neq 0$
M_{56}	$\rho^\mu e^{-\beta s/2} h(\alpha \rho^{\gamma} e^{\beta s}) + C \sqrt{\alpha} \rho^{\mu + \gamma/2} s$	$\beta(3\gamma + 6\mu + 2)X_2 + 4\beta X_3$ $+ 4(3\gamma + 10)\partial_s, (3\gamma + 10)X_1 -$ $(\gamma + \mu + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10) \neq 0$
M_{57}	$\rho^\mu e^{\lambda s} h(\alpha \rho^{-10/3} e^{\beta s})$	$3\beta(3\mu - 4)X_1 + 2(3\beta(\mu - 1) + \lambda)X_3$ $+ 2(3\mu - 4)\partial_s, (3\mu - 4)X_2 + 2X_3$	$(3\mu - 4)\mu \neq 0$ $(2\lambda + \beta) \neq 0$
M_{58}	$\rho^{4/3} e^{\lambda s} h(\alpha \rho^{-10/3} e^{\beta s})$	$6\beta X_1 - (2\beta + 3\lambda)X_2 + 6\partial_s, X_3$	$\beta \neq 0$
M_{59}	$e^{\lambda s} h(\alpha \rho^{-10/3} e^{\beta s}) + \varphi(s) \ln \rho$	$6\beta X_1 + (3\beta - \lambda)X_3 + 4\partial_s, 2X_2 - X_3$	$\lambda \neq 0$
M_{60}	$\rho^\mu e^{-\beta s/2} h(\alpha \rho^{-10/3} e^{\beta s}) + C \sqrt{\alpha} \rho^{\mu - 5/3} s$	$3\beta(3\mu - 4)X_1 + \beta(6\mu - 7)X_3$ $+ 2(3\mu - 4)\partial_s, (3\mu - 4)X_2 + 2X_3$	$(3\mu - 4)\mu \neq 0$
M_{61}	$\rho^{4/3} e^{-\beta s/2} h(\alpha \rho^{-10/3} e^{\beta s}) + C \sqrt{\alpha} \rho^{-1/3} s$	$6\beta X_1 - \beta X_2 + 4\partial_s, X_3$	
M_{62}	$e^{-\beta s/2} h(\alpha \rho^{-10/3} e^{\beta s}) +$	$12\beta X_1 + 7\beta X_3 + 8\partial_s, 2X_2 - X_3$	$\beta \neq 0$
M_{63}	$e^{\lambda s} [K \ln \rho + C \ln \alpha]$	$-\lambda X_3 + 4\partial_s, 2X_1 + X_3, 2X_2 - X_3$	$\lambda \neq 0$
M_{64}	$[K \sqrt{\alpha} \rho^{-3/2} + C \rho^{-1}] s + \alpha^{1/2} \rho^{-3/2} (K \ln \alpha + C \ln \rho)$	$\partial_s, 7X_1 + 4X_3, 7X_2 - 4X_3$	
M_{65}	$K(\alpha \rho^{-10/3})^\gamma \rho^\beta + \sqrt{\alpha} g(\rho) s$	$\partial_s, 2s\partial_s + \beta(3q - 4)X_2 + 2\beta X_3,$ $6X_1 + 2(\beta(2\gamma - 1) + 2)X_3 +$ $(2\gamma + 9\beta q - 12\beta - 6q + 6)X_2$	$\beta(2\gamma - 1) \neq 0$ $q(q + 1) \neq 0$ $3\beta(q + 1) + 1 \neq 0$
M_{66}	$K(\alpha \rho^{-10/3})^\gamma + \sqrt{\alpha} g(\rho) s$	$\partial_s, s\partial_s - 2\beta X_2 + \beta X_3,$ $3X_1 + (\beta(2\gamma - 1) + 2)X_3 +$ $(2\beta(1 - 2\gamma) + \gamma - 1)X_2$	$\beta(2\gamma - 1) \neq 0$ $\gamma K \neq 0$
M_{67}	$K(\alpha \rho^{-10/3})^\gamma \rho^{-1} + \sqrt{\alpha} g(\rho) s$	$\partial_s, 2s\partial_s - 7\beta X_2 + 2\beta X_3,$ $3X_1 + (\beta(2\gamma - 1) + 2)X_3 +$ $(7\beta(1 - 2\gamma) + 2\gamma - 2)X_2$	$\beta(2\gamma - 1) \neq 0$ $K \neq 0$

M_{68}	$K\alpha^\lambda \rho^{(8-10\lambda)/3} + C\sqrt{\alpha}\rho^\mu s$	$\partial_s, (2\lambda-1)(3\mu+1)s\partial_s - 3(3\mu+1)X_2 + 2(\lambda-9\mu-4)X_3, (3\mu+1)X_2 + 2X_3$	$(3\mu+5) \neq 0$ $(3\mu+8) \neq 0$ $(3\mu+1) \neq 0$ $(2\lambda-1)K \neq 0$
M_{69}	$K\alpha^\lambda \rho^\beta + \sqrt{\alpha}g(\rho)s$	$\partial_s, 2(3\beta+10\lambda+3)s\partial_s + (4-3\beta-10\lambda)X_2 - 6X_3, X_1 + (4\lambda-3\beta-10)X_2 + 2(7-6\beta-22\lambda)X_3$	$3\beta \neq -10\lambda-3$ $(2\lambda-1)K \neq 0$
M_{70}	$\rho^{-1}[K\ln\alpha + C\ln\rho] + \sqrt{\alpha}g(\rho)s$	$\partial_s, 2s\partial_s - 7X_2 + 2X_3, 6X_1 + 5X_2 + 2X_3$	$KCg \neq 0$
M_{71}	$K\ln\alpha + C\ln\rho + \sqrt{\alpha}g(\rho)s$	$\partial_s, 3s\partial_s + 2X_2 - X_3, 9X_1 - 8X_2 + 7X_3$	$KCg \neq 0$
M_{72}	$\sqrt{\alpha}g(\rho)s$	$\partial_s, 2s\partial_s - X_2, 6X_1 - X_2 + 4X_3$	$g' \neq 0$
M_{73}	$[K\ln\alpha + C\ln\rho]\sqrt{\alpha}\rho^{-1/3} + C_1s\sqrt{\alpha}\rho^{-1/3}$	$\partial_s, 6X_1 - X_2, X_3$	$C \neq 0$
M_{74}	$[K\ln\alpha + C\ln\rho]\sqrt{\alpha}\rho^{\mu+2/3} + C_1\rho^\mu s\sqrt{\alpha}\rho^{\mu+2/3}$	$\partial_s, (6\mu-1)X_1 + 4\mu X_3, (6\mu-1)X_2 + 4X_3$	$(6\mu-1)C \neq 0$
M_{75}	$[K\ln\alpha + C\rho^\beta s]\sqrt{\alpha}\rho^{\gamma\beta-1/2}$	$\partial_s, 6\beta s\partial_s + 3\beta\gamma X_2 + 2X_3, 6X_1 - X_2 + 4X_3$	$\beta C \neq 0$
M_{76}	$(K\ln\alpha + C\ln\rho)\sqrt{\alpha}\rho^{\mu-5/3} + C_1\rho^{-1}s$	$\partial_s, 6(\mu+1)s\partial_s + (3\mu-4)X_2 - 2X_3, 18(\mu+1)X_1 + (1-6\mu)X_2 + 2(6\mu+5)X_3$	$(\mu+1) \neq 0$
M_{77}	$(K\ln\alpha + C\ln\rho)\sqrt{\alpha}\rho^{-8/3} + C\rho^{-1}s$	$\partial_s, 7s\partial_s - 21X_1 - 13X_3, 7X_2 - 2X_3$	$K \neq 0$
M_{78}	$K\alpha^\lambda \rho^{\mu-10\lambda/3} + C\rho^{-1}s$	$\partial_s, 6(\mu+1)s\partial_s + (3\mu-4)X_2 - 2X_3, 9(\mu+1)X_1 + (7\lambda-3\mu-3)X_2 + (2(3\mu+3-\lambda))X_3$	$(\mu+1) \neq 0$ $\lambda(2\lambda-1) \neq 0$
M_{79}	$\rho^{-1}[K_1\ln\alpha + K_2\ln\rho] + C\rho^{-1}s$	$\partial_s, 7X_1 + 4X_3, 7X_2 - 2X_3$	
M_{80}	$K\ln\alpha + C\ln\rho + C_1\rho^{-1}s$	$\partial_s, 3s\partial_s + 2X_2 - X_3, 3X_1 - X_2 + 2X_3$	
M_{81}	$Ce^{\beta s}\alpha^\lambda \rho^{\mu-10\lambda/3}$	$\beta X_3 + (3\mu-4)\partial_s, 3(3\mu-4)X_1 + 2(3\mu-\lambda-3)X_3, (3\mu-4)X_2 + 2X_3$	$\beta\lambda(3\mu-4) \neq 0$
M_{82}	$Ce^{\beta s}\alpha^\lambda \rho^{(4-10\lambda)/3}$	$\beta X_2 + 2\partial_s, 3X_1 + (\lambda-1)X_2, X_3$	$\beta\lambda \neq 0$

4. ผลงานวิจัยได้ส่งเพื่อตีพิมพ์เผยแพร่ ชื่อเรื่อง Group classification of one-dimensional equations of fluids with internal inertia the specific energy is a function of density, density gradient and entropy. ชื่อวารสาร International Journal of Non-Linear Mechanics โดยรายละเอียดตามภาคผนวก

Appendix

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Title: Group classification of one-dimensional equations of fluids with internal inertia where the specific energy is a function of density, density gradient and entropy

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Abstract: An application of group analysis provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This paper presents the group classification of one-dimensional equations of fluids where the internal energy is a function of the density, the gradient of the density and the entropy. The group classification separates all models into 82 different classes according to the admitted Lie group.

Group classification of one-dimensional equations of fluids with internal inertia where the specific energy is a function of density, density gradient and entropy

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Abstract

An application of group analysis provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This paper presents the group classification of one-dimensional equations of fluids where the internal energy is a function of the density, the gradient of the density and the entropy. The group classification separates all models into 82 different classes according to the admitted Lie group.

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Subject Classification (MSC 2010): 76M60, 35Q20

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1. Introduction

This manuscript is focused on the group classification of the governing equations whose specific internal energy ε is a function of the density, the density gradient, and the entropy [1]¹

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \quad (\rho u)_t + (\rho u^2 + \Pi)_x = 0, \quad (\rho s)_t + (\rho s u)_x = 0, \\ \Pi &= \rho \frac{\delta(\rho \varepsilon)}{\delta \rho} - \rho \varepsilon = \rho^2 \varepsilon_\rho - 2\rho(\rho \rho_x \varepsilon_\alpha)_x + 2\rho \rho_x^2 \varepsilon_\alpha, \quad \alpha = |\nabla \rho|^2, \end{aligned} \quad (1)$$

where t is time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field, s is the entropy, $\varepsilon(\rho, \alpha, s)$ is a given internal energy, and $\frac{\delta}{\delta \rho}$ denotes the variational derivative with respect to ρ at a fixed value of u . These models were studied in [2, 3, 4, 5, 6]. A review of these models can be found in [7, 1] and references therein. Equations (1) were obtained in [1] using the Lagrangian of the form

$$L = \frac{1}{2}|u|^2 - \varepsilon(\rho, \nabla \rho, s).$$

Another set of models where the medium behavior depends not only on thermodynamical variables but also on their derivatives with respect to space and time was constructed in [8]

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¹See also references therein.

using the Lagrangian of the form

$$L = \frac{1}{2}|u|^2 - W(\rho, \dot{\rho}, s)$$

where $\dot{} = \partial/\partial t + u\nabla$. These models are examples of a continuum where the behavior depends not only on the thermodynamical variables but also on their derivatives with respect to space and time.

One of the methods for studying properties of differential equations is group analysis [9, 10, 11]. This method is a basic method for constructing exact solutions of partial differential equations. A wide range of applications of group analysis to partial differential equations are collected in [12, 13, 14]. Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This feature of group analysis is the fundamental basis for the mathematical modeling in the present paper.

An application of group analysis employs several steps. The first step is group classification with respect to arbitrary elements. An algorithm of the group classification is applied in cases where a system of differential equations has arbitrary elements in the form of undefined parameters and functions. This step is necessary since a specialization of the arbitrary elements can lead to different admitted Lie groups. In particular, group classification selects the functions $\varepsilon(\rho, |\nabla\rho|, s)$ such that the fluid dynamics equations (1) possess additional symmetry properties extending the kernel of the admitted Lie groups. Algorithms of finding equivalence and admitted Lie groups are particular parts of the algorithm of the group classification.

A complete group classification of equations (1), where $\varepsilon = \varepsilon(\rho, |\nabla\rho|)$ is performed in [15]. Invariant solutions of some particular cases are considered there. Group classification of the class of models describing the behavior of a dispersive continuum with $W = W(\rho, \dot{\rho})$ was studied in [16] (one-dimensional case) and [17] (three-dimensional case). Invariant solutions of some particular cases which are separated out by the group classification are considered in [16, 18]. The group classification performed in these studies [16, 15, 17] followed the classical method developed by L.V.Ovsiannikov [11] for the group classification of the gas dynamics equations. Notice that an exhaustive program of studying the models appearing in the group classification of the gas dynamics equations was announced in [19]. Some results of this program were summarized in [20]. It is also worth to notice that the classical gas dynamics model corresponds to $\varepsilon(\rho, s)$ (or $W(\rho, s)$).

The classical approach [11] to nonisentropic equations of fluids with internal inertia is very complicated. Even the study of particular cases leads to cumbersome investigations [21]. In the present paper we use an algebraic approach for the group classification of nonisentropic equations of fluids with internal inertia.

The algebraic approach takes the algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of admitted Lie algebras into account, and allows for significant simplification of the group classification. In particular, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, S.Lie [22, 23], cannot be performed without using the algebraic structure of admitted Lie groups. Recently, the algebraic properties were applied in for group classification [24, 25, 26, 27, 28, 29, 30, 31]. We also note that the use of the algebraic structure of admitted Lie groups completely simplified the group classification of equations describing the behavior of fluids with internal inertia in [32, 33].

The present paper is focused on the group classification of the one-dimensional equations of fluids (1), where $\varepsilon = \varepsilon(\rho, |\nabla\rho|, s)$ with $\varepsilon_s \neq 0$.

This paper is organized as follows. The equivalence Lie group of transformations is presented in section 2. The equivalence transformations are applied for simplifying the function $\varepsilon(\rho, |\nabla\rho|, s)$ in the process of the classification. We classify all models with respect to the admitted Lie groups in section 3, where we consider 2 cases. In the first case, where $k_4 \neq 0$, the analysis is similar to the group classification of the gas dynamics equations. In the second case, where $k_4 = 0$, the analysis uses the idea of the algebraic approach which separates the study of group classification in two steps. In the first step, one makes a preliminary study of possible coefficients of the basis generators using the requirement of admitted generators to compose a Lie algebra. In the second step, one substitutes these coefficients of each generator of the Lie algebra into the determining equation. Solving the obtained system of equations, the function $\varepsilon(\rho, |\nabla\rho|, s)$ and additional restrictions for the coefficients of the basis generators are obtained. The result of the group classification and the admitted Lie algebras of equation (1) is summarized in Table 1.

2. Equivalence Lie group

An equivalence Lie group allows changing arbitrary elements conserving the structure of the studied equations. An infinitesimal operator X^e of the equivalence Lie group is sought in the form [34]

$$X^e = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^\alpha \partial_\alpha + \zeta^s \partial_s + \zeta^\varepsilon \partial_\varepsilon,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^\alpha, \zeta^s$ and ζ^ε are all functions of $(x, t, \rho, u, \alpha, s, \varepsilon)$.

Calculations give the following basis of generators of the equivalence Lie group,

$$\begin{aligned} X_1^e &= \rho \partial_\rho + 2\alpha \partial_\alpha, & X_2^e &= t \partial_t + x \partial_x - 2\alpha \partial_\alpha, \\ X_3^e &= t \partial_x - u \partial_u - 2\varepsilon \partial_\varepsilon, & X_4^e &= f(s) \partial_s, \\ Y_1^e &= \partial_t, & Y_2^e &= \partial_x, & Y_3^e &= t \partial_x + \partial_u, & Z_1^e &= \rho^{-1} \partial_\varepsilon, \\ Z_2^e &= f(s) \partial_\varepsilon, & Z_3^e &= g(\rho) \sqrt{\alpha} \partial_\varepsilon, \end{aligned}$$

where the functions $f(s)$ and $g(\rho)$ are arbitrary.

Since the equivalence transformations corresponding to the operators $X_3^e, X_4^e, Z_1^e, Z_2^e$ and Z_3^e are applied for simplifying the function ε in the classification process, let us present these transformations. Because the function ε depends on ρ, α and s , only the transformations of these variables are presented:

$$\begin{aligned} X_3^e : & \quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \quad \tilde{\varepsilon} &= e^{-2a} \varepsilon, \\ X_4^e : & \quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = h(s, a) & \quad \tilde{\varepsilon} &= \varepsilon, \\ Z_1^e : & \quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \quad \tilde{\varepsilon} &= \varepsilon + \rho^{-1} a, \\ Z_2^e : & \quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \quad \tilde{\varepsilon} &= \varepsilon + f(s) a, \\ Z_3^e : & \quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{s} = s & \quad \tilde{\varepsilon} &= \varepsilon + \sqrt{\alpha} g(\rho) a \end{aligned}$$

where a is the group parameter. Using the equivalence transformations corresponding to the generators Z_1^e and Z_2^e , the term $C_1 \rho^{-1} + C_2$, which appears in the function $\varepsilon(\rho, \alpha, s)$ can be omitted. Here C_1 and C_2 are constants. By virtue of the equivalence transformations corresponding to the generator Z_3^e , the function $\varepsilon(\rho, \alpha, s)$ is considered up to the term $\sqrt{\alpha} g(\rho)$.

3. Admitted Lie group

An admitted generator X of equations (1) is sought in the form

$$X = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^\alpha \partial_\alpha + \zeta^s \partial_s,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^\alpha$ and ζ^s are functions of the variables $(x, t, \rho, u, \alpha, s)$.

Calculations show that

$$\begin{aligned}\xi^x &= k_4 tx - \tilde{k}_3 x - \tilde{k}_1 x + k_5 t^3 + k_6 t^2 + k_8 t + k_9, \quad \xi^t = k_4 t^2 - 2\tilde{k}_3 t - 2\tilde{k}_1 t + k_2 t + k_7, \\ \zeta^\rho &= 2\tilde{k}_1 \rho - k_4 t \rho + \tilde{k}_3 \rho, \quad \zeta^u = k_4(x - tu) + \tilde{k}_3 u + \tilde{k}_1 u - k_2 u + 3k_5 t^2 + 2k_6 t + k_8, \\ \zeta^\alpha &= 6\tilde{k}_1 \alpha - 4k_4 t \alpha + 4\tilde{k}_3 \alpha, \quad \zeta^s = \zeta(s),\end{aligned}$$

where $k_1 = \tilde{k}_1 - \tilde{k}_3$, $k_3 = -\tilde{k}_3$, k_i , $(i = 1, 2, \dots, 8)$ are constant. The constants and the function $\zeta(s)$ satisfy the equations

$$\begin{aligned}\zeta(2\varepsilon_{s\alpha\rho\rho}\alpha\rho + 4\varepsilon_{s\alpha\rho}\alpha - \varepsilon_{s\rho\rho}\rho - 2\varepsilon_{s\rho}) + 2\tilde{k}_1(2\varepsilon_{\alpha\rho\rho\rho}\alpha\rho^2 + 9\varepsilon_{\alpha\rho\rho}\alpha\rho + 6\varepsilon_{\alpha\rho}\alpha + 6\varepsilon_{\alpha\alpha\rho\rho}\alpha^2\rho \\ + 12\varepsilon_{\alpha\alpha\rho}\alpha^2 - \varepsilon_{\rho\rho\rho}\rho^2 - 3\varepsilon_{\rho\rho}\rho) + 2k_2(2\varepsilon_{\alpha\rho\rho}\alpha\rho + 4\varepsilon_{\alpha\rho}\alpha - \varepsilon_{\rho\rho}\rho - 2\varepsilon_\rho) + \tilde{k}_3(2\varepsilon_{\alpha\rho\rho\rho}\alpha\rho^2 \\ + 8\varepsilon_{\alpha\rho\rho}\alpha\rho + 4\varepsilon_{\alpha\rho}\alpha + 8\varepsilon_{\alpha\alpha\rho\rho}\alpha^2\rho + 16\varepsilon_{\alpha\alpha\rho}\alpha^2 - \varepsilon_{\rho\rho\rho}\rho^2 - 2\varepsilon_{\rho\rho}\rho + 2\varepsilon_\rho) - 2k_6 q(\alpha) = 0,\end{aligned}\quad (2)$$

$$\begin{aligned}\zeta'(4\varepsilon_{s\alpha\rho}\alpha\rho + 4\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}) + \zeta(4\varepsilon_{ss\alpha\rho}\alpha\rho + 4\varepsilon_{ss\alpha}\alpha - \varepsilon_{ss\rho\rho}) + 2\tilde{k}_1(4\varepsilon_{s\alpha\rho\rho}\alpha\rho^2 \\ + 13\varepsilon_{s\alpha\rho}\alpha\rho + 12\varepsilon_{s\alpha\alpha\rho}\alpha^2\rho + 12\varepsilon_{s\alpha\alpha}\alpha^2 + 8\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}\rho^2) + 2k_2(4\varepsilon_{s\alpha\rho}\alpha\rho \\ + 4\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}) + \tilde{k}_3(4\varepsilon_{s\alpha\rho\rho}\alpha\rho^2 + 12\varepsilon_{s\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha\rho}\alpha^2\rho + 16\varepsilon_{s\alpha\alpha}\alpha^2 \\ + 8\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}\rho^2 + \varepsilon_{s\rho\rho}) = 0,\end{aligned}\quad (3)$$

$$\begin{aligned}\zeta_{ss}^s \varepsilon_{s\alpha} + 2\zeta' \varepsilon_{ss\alpha} + \zeta \varepsilon_{sss\alpha} + 2\tilde{k}_1(\varepsilon_{ss\alpha\rho\rho} + 3\varepsilon_{ss\alpha\alpha}\alpha + 2\varepsilon_{ss\alpha}) + 2k_2 \varepsilon_{ss\alpha} \\ + \tilde{k}_3(\varepsilon_{ss\alpha\rho\rho} + 4\varepsilon_{ss\alpha\alpha}\alpha + 2\varepsilon_{ss\alpha}) = 0,\end{aligned}\quad (4)$$

$$\begin{aligned}\zeta(\varepsilon_{s\alpha\rho\rho} + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + 2\tilde{k}_1(\varepsilon_{\alpha\rho\rho}\rho^2 + 4\varepsilon_{\alpha\rho\rho} + 6\varepsilon_{\alpha\alpha\alpha\rho}\alpha^2\rho + 6\varepsilon_{\alpha\alpha\alpha}\alpha^2 \\ + 2\varepsilon_{\alpha\alpha\rho\rho}\alpha\rho^2 + 17\varepsilon_{\alpha\alpha\rho}\alpha\rho + 13\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_\alpha) + 2k_2(\varepsilon_{\alpha\rho\rho} + 2\varepsilon_{\alpha\alpha\rho}\alpha\rho + 2\varepsilon_{\alpha\alpha}\alpha + \varepsilon_\alpha) \\ + \tilde{k}_3(\varepsilon_{\alpha\rho\rho}\rho^2 + 4\varepsilon_{\alpha\rho\rho} + 8\varepsilon_{\alpha\alpha\alpha\rho}\alpha^2\rho + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho\rho}\alpha\rho^2 + 20\varepsilon_{\alpha\alpha\rho}\alpha\rho \\ + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_\alpha) = 0,\end{aligned}\quad (5)$$

$$\begin{aligned}\zeta(2\varepsilon_{s\alpha\alpha\alpha}\alpha + 3\varepsilon_{s\alpha\alpha}) + 2\tilde{k}_1(6\varepsilon_{\alpha\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\alpha\rho}\alpha\rho + 25\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha\rho\rho} + 15\varepsilon_{\alpha\alpha}) \\ + 2k_2(2\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha}) + \tilde{k}_3(8\varepsilon_{\alpha\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\alpha\rho}\alpha\rho + 32\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha\rho\rho} + 18\varepsilon_{\alpha\alpha}) = 0,\end{aligned}\quad (6)$$

$$\begin{aligned}\zeta'(2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + \zeta(2\varepsilon_{ss\alpha\alpha}\alpha + \varepsilon_{ss\alpha}) + 2\tilde{k}_1(\varepsilon_{s\alpha\rho\rho} + 6\varepsilon_{s\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 13\varepsilon_{s\alpha\alpha}\alpha \\ + 2\varepsilon_{s\alpha}) + 2k_2(2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + \tilde{k}_3(\varepsilon_{s\alpha\rho\rho} + 8\varepsilon_{s\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0,\end{aligned}\quad (7)$$

$$\zeta' \varepsilon_{s\alpha} + \zeta \varepsilon_{ss\alpha} + 2\tilde{k}_1(\varepsilon_{s\alpha\rho\rho} + 3\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) + 2k_2 \varepsilon_{s\alpha} + \tilde{k}_3(\varepsilon_{s\alpha\rho\rho} + 4\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0,\quad (8)$$

$$\begin{aligned}\zeta(2\varepsilon_{s\alpha\alpha}\alpha + \varepsilon_{s\alpha}) + 2\tilde{k}_1(\varepsilon_{\alpha\rho\rho} + 6\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho}\alpha\rho + 13\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_\alpha) + 2k_2(2\varepsilon_{\alpha\alpha}\alpha + \varepsilon_\alpha) \\ + \tilde{k}_3(\varepsilon_{\alpha\rho\rho} + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho}\alpha\rho + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_\alpha) = 0,\end{aligned}\quad (9)$$

$$\begin{aligned}k_4(2\varepsilon_{\alpha\rho\rho\rho}\alpha\rho^2 + 8\varepsilon_{\alpha\rho\rho}\alpha\rho + 4\varepsilon_{\alpha\rho}\alpha + 8\varepsilon_{\alpha\alpha\rho\rho}\alpha^2\rho + 16\varepsilon_{\alpha\alpha\rho}\alpha^2 - \varepsilon_{\rho\rho\rho}\rho^2 - 2\varepsilon_{\rho\rho}\rho + 2\varepsilon_\rho) \\ + 6k_5 q(\alpha) = 0,\end{aligned}\quad (10)$$

$$k_4(4\varepsilon_{s\alpha\rho\rho}\alpha\rho^2 + 12\varepsilon_{s\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha\rho}\alpha^2\rho + 16\varepsilon_{s\alpha\alpha}\alpha^2 + 8\varepsilon_{s\alpha}\alpha - \varepsilon_{s\rho\rho}\rho^2 + \varepsilon_{s\rho\rho}) = 0,\quad (11)$$

$$k_4(\varepsilon_{ss\alpha\rho\rho} + 4\varepsilon_{ss\alpha\alpha}\alpha + 2\varepsilon_{ss\alpha}) = 0,\quad (12)$$

$$\begin{aligned}k_4(\varepsilon_{\alpha\rho\rho}\rho^2 + 4\varepsilon_{\alpha\rho\rho} + 8\varepsilon_{\alpha\alpha\alpha\rho}\alpha^2\rho + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\rho\rho}\alpha\rho^2 + 20\varepsilon_{\alpha\alpha\rho}\alpha\rho \\ + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_\alpha) = 0,\end{aligned}\quad (13)$$

$$k_4(8\varepsilon_{\alpha\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\alpha\alpha\rho}\alpha\rho + 32\varepsilon_{\alpha\alpha\alpha}\alpha + 3\varepsilon_{\alpha\alpha\rho\rho} + 18\varepsilon_{\alpha\alpha}) = 0,\quad (14)$$

$$k_4(\varepsilon_{s\alpha\rho\rho} + 8\varepsilon_{s\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{s\alpha\alpha\rho}\alpha\rho + 16\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0,\quad (15)$$

$$k_4(\varepsilon_{s\alpha\rho\rho} + 4\varepsilon_{s\alpha\alpha}\alpha + 2\varepsilon_{s\alpha}) = 0,\quad (16)$$

$$k_4(\varepsilon_{\alpha\rho\rho} + 8\varepsilon_{\alpha\alpha\alpha}\alpha^2 + 2\varepsilon_{\alpha\rho}\varepsilon_{\alpha\alpha\rho} + 16\varepsilon_{\alpha\alpha}\alpha + 2\varepsilon_\alpha) = 0.\quad (17)$$

where $q = a/\sqrt{\alpha}$ and $a^2 = 1$. The determining equations (2)-(17) define the kernel of admitted Lie algebras and its extensions. The kernel of admitted Lie algebras consists of the generators which are admitted by equations (1) for any function $\varepsilon(\rho, \alpha, s)$ and it is defined by the generators

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = t\partial_x + \partial_u.$$

The transformations corresponding to Y_1 and Y_2 are shifts with respect to time and space variable, and the transformations corresponding to Y_3 are the Galilean transformations. Extensions of the kernel of admitted Lie algebras depend on the value of the function $\varepsilon(\rho, \alpha, s)$. They can only be operators of the form

$$k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 + k_5X_5 + k_6X_6 + \zeta\partial_s,$$

where

$$X_1 = \rho\partial_\rho + \alpha\partial_\alpha, \quad X_2 = t\partial_t - u\partial_u, \quad X_3 = u\partial_u + 2\alpha\partial_\alpha - x\partial_x - 2t\partial_t, \\ X_4 = \rho t\partial_\rho + (ut - x)\partial_u + 4\alpha t\partial_\alpha - tx\partial_x - t^2\partial_t, \quad X_5 = t^3\partial_x + 3t^2\partial_u, \quad X_6 = t^2\partial_x + 2t\partial_u.$$

3.1. Case $k_4 \neq 0$

The functions $\varepsilon(\rho, \alpha, s)$ for which there exists an admitted generator with $k_4 \neq 0$ are studied first. This generator can be rewritten in the form

$$X_4 + k_1X_1 + k_2X_2 + k_3X_3 + k_5X_5 + k_6X_6 + \zeta\partial_s,$$

where $\zeta = \zeta(s)$. Using the equivalence transformation corresponding to the generator $X_1^e = \partial_t$, one can assume that for this generator $k_1 = 0$. Notice also that if $\zeta \neq 0$, then using the equivalence transformation corresponding to X_4^e , one can assume that $\zeta = 1$.

From equation (16) one finds that

$$\varepsilon(\rho, \alpha, s) = \rho^2\psi(\alpha\rho^{-4}, s) + \phi(\rho, s) + \varphi(\rho, \alpha),$$

where $\phi(\rho, s)$ and $\varphi(\rho, \alpha)$ are arbitrary functions of the integration. Substituting ε into equation (17) one has

$$\rho\varphi_{\alpha\rho} + 8\alpha^2\varphi_{\alpha\alpha\alpha} + 2\alpha\rho\varphi_{\alpha\alpha\rho} + 16\alpha\varphi_{\alpha\alpha} + 2\varphi_\alpha = 0.$$

Introducing the function

$$g = 2\alpha\varphi_{\alpha\alpha} + \varphi_\alpha = 2\alpha^{1/2}(\alpha^{1/2}\varphi_\alpha)_\alpha$$

this equation becomes

$$\rho g_\rho + 4\alpha g_\alpha + 2g = 0.$$

Solving this equation, one obtains that

$$\varepsilon(\rho, \alpha, s) = \rho^2\psi(\alpha\rho^{-4}, s) + \phi(\rho, s) + \alpha^{1/2}h_1(\rho) + h_2(\rho).$$

Because of equivalence transformations one can assume that

$$h_1 = 0, \quad h_2 = 0.$$

Differentiating equation (10) with respect to α , one finds that $k_5 = 0$. Equations (10) and (11) become

$$\rho^2\phi_{\rho\rho\rho} + 2\rho\phi_{\rho\rho} - 2\phi_\rho = 0, \quad \rho\phi_{\rho\rho s} - \phi_{\rho s} = 0.$$

The general solution of this system of equations is

$$\phi = \rho^2 \phi_1(s) + C\rho^{-1} + \phi_3(s),$$

where C is constant. Hence,

$$\varepsilon = \rho^2 (\psi(\alpha\rho^{-4}, s) + \phi_1(s)) + \rho^{-1}C + \phi_3(s).$$

Since the function ψ is arbitrary, one can set $\phi_1 = 0$. Furthermore, because of the equivalence transformations corresponding to Z_1^e and Z_2^e , one can assume that $\phi_3 = 0$ and $C = 0$. Thus, if there exists a generator with $k_4 \neq 0$, then the function $\varepsilon(\rho, \alpha, s)$ has to be of the form

$$\varepsilon = \rho^2 \psi(z, s), \quad z = \alpha\rho^{-4}. \quad (18)$$

Let us study group properties of equations (1) with the function $\varepsilon(\rho, \alpha, s)$ of the form (18). Substituting the function $\varepsilon(\rho, \alpha, s)$ into the determining equations (2)-(17), and performing some manipulations (differentiations and linear combinations) one obtains that these equations are reduced to the equations $k_5 = 0$, $k_6 = 0$, and

$$2(k_3 - k_1)(2z^2\psi_{zz} - z\psi_z + \psi) + 2k_2(2\psi_z z - \psi) + \zeta(2\psi_{zs}z - \psi_s) = 0, \quad (19)$$

$$(2(k_3 - k_1)(z\psi_z - \psi) + 2k_2\psi + \psi_s\zeta)_s = 0. \quad (20)$$

Integrating equation (20), one has

$$2(k_3 - k_1)(z\psi_z - \psi) + 2k_2\psi + \psi_s\zeta = \lambda, \quad (21)$$

where $\lambda(z)$ is an arbitrary function. Excluding k_2 from (19) using (21), one finds that

$$\lambda = kz^{1/2},$$

where k is constant. Thus, for the group classification of equations (1) with the function $\varepsilon(\rho, \alpha, s)$ of the form (18), one needs to analyze only the equation

$$2(k_3 - k_1)(z\psi_z - \psi) + 2k_2\psi + \psi_s\zeta = kz^{1/2}. \quad (22)$$

Since $\psi_s \neq 0$, one finds

$$\zeta = \psi_s^{-1} (kz^{1/2} - 2((k_3 - k_1)(z\psi_z - \psi) + k_2\psi)). \quad (23)$$

Because $\zeta = \zeta(s)$, differentiating (23) with respect to z one needs to impose the requirement that

$$ak + b(k_3 - k_1) + ck_2 = 0, \quad (24)$$

where

$$a = (z^{1/2}\psi_s^{-1})_z, \quad b = -2(\psi_s^{-1}(z\psi_z - \psi))_z, \quad c = -2(\psi_s^{-1}\psi)_z.$$

Equation (24) is a classifying equation.

For arbitrary function $\psi(z, s)$ one has

$$k_3 = k_1, \quad k_2 = 0, \quad k = 0, \quad (25)$$

that is, the generators

$$Y_1, Y_2, Y_3, X_1 + X_3, X_4 \quad (26)$$

are admitted for any function $\varepsilon(\rho, \alpha, s)$ of the form (18). An extension of this Lie algebra L_5 occurs if one of the constants

$$k_3 - k_1, k_2, k$$

does not vanish. Operators of the extension have the form

$$\gamma X_2 + \beta X_3 + X_h,$$

where β and γ are constant, $X_h = h(s)\partial_s$, and

$$h^2 + \beta^2 + \gamma^2 \neq 0.$$

For classifying all possibilities, it is convenient to consider the functions $a(z, s)$, $b(z, s)$ and $c(z, s)$ as coordinates of the three-dimensional vector $v = (a, b, c)$. For analyzing relations between the constants $k_3 - k_1$, k_2 and k one can follow to the classical method developed by L.V.Ovsiannikov [11] for the group classification of the gas dynamics equations: one studies the vector space $\text{Span}(V)$, where the set V consists of the vectors v with z and s are varying.

3.1.1. Case $\dim(\text{Span}(V)) = 3$

In this case

$$k_3 = k_1, k_2 = 0, k = 0,$$

and hence, there is no extensions of L_5 .

3.1.2. Case $\dim(\text{Span}(V)) = 2$

There exists a constant vector $(\gamma, \beta, q) \neq 0$ such that the vector field (a, b, c) is orthogonal to the vector (γ, β, q) :

$$\gamma a + \beta b + qc = 0.$$

Assuming that $\gamma \neq 0$, one can set $\gamma = 1$. Then equation (24) becomes

$$b[(k_3 - k_1) - \beta k] + c(k_2 - qk) = 0$$

If one of the constants $(k_3 - k_1) - \beta k$ or $k_2 - qk$ does not vanish, then $\dim(V) < 2$. Hence,

$$(k_3 - k_1) = \beta k, k_2 = qk$$

and

$$\zeta(s) = kh(s),$$

where

$$h = \psi_s^{-1} (z^{1/2} - 2(\beta z \psi_z + (q - \beta)\psi)). \quad (27)$$

The extension of L_5 is defined by a single generator of the form

$$\beta X_3 + qX_2 + h\partial_s.$$

The function $h(s)$ can also be simplified: either $h = 0$ or $h = 1$.

Assume that $h = 0$. Since for $\beta = 0$ one obtains that $\varepsilon_s = 0$, then $\beta \neq 0$ and

$$\begin{aligned} \psi &= z^{1-q/\beta} H + z^{1/2} \frac{1}{2q-\beta}, \quad (q \neq \beta/2), \\ \psi &= z^{1/2} \left(H + \frac{1}{2\beta} \ln(z) \right), \quad (q = \beta/2), \end{aligned}$$

where $H = H(s)$. One can check that in this case $\dim(V) = 1$. Thus one needs only to consider the case where $h = 1$. In this case the function

$$\psi(z, s) = f(ze^{-\beta s})e^{(\beta-q)s} + \frac{\sqrt{z}}{(2q-\beta)}, \quad (q \neq 2\beta), \quad (28)$$

$$\psi(z, s) = f(ze^{-\beta s})e^{2\beta s} + s\frac{\sqrt{z}}{2}, \quad (q = 2\beta).$$

Notice that using equivalence transformations, one can reduce the function (28) to

$$\psi(z, s) = f(ze^{-\beta s})e^{(\beta-q)s}, \quad (q \neq 2\beta).$$

Assuming that $\gamma = 0$ and $\beta \neq 0$, one finds that $b = -qc$ which gives

$$ak + c(k_2 - q(k_3 - k_1)) = 0.$$

Similar to the previous case, one obtains

$$k = 0, \quad k_2 = q(k_3 - k_1),$$

and

$$\zeta(s) = -2(k_3 - k_1)h(s),$$

where

$$h = \psi_s^{-1}(z\psi_z + (q-1)\psi). \quad (29)$$

The function $h(s)$ can be also simplified: either $h = 0$ or $h = 1$. The extension of L_5 is defined by the generator

$$X_3 + qX_2 - 2h\partial_s.$$

In the case $h = 0$ one finds that

$$\psi(z, s) = z^{1-q}H(s).$$

It gives that $\dim(V) = 1$. Thus one only needs to consider $h = 1$ for which one finds that

$$\psi(z, s) = z^{1-q}Q(ze^{-s}),$$

and

$$X_3 + qX_2 + 2\partial_s.$$

Assuming that $\gamma = 0$, $\beta = 0$, one obtains $c = 0$ and

$$ak + b(k_3 - k_1) = 0,$$

which leads to

$$k = 0, \quad k_3 - k_1 = 0,$$

and

$$\zeta(s) = -2k_2h(s),$$

where

$$h = \psi_s^{-1}\psi. \quad (30)$$

Using the equivalence transformation for simplifying the function $h(s)$ to $h = 1$, one finds that

$$\psi(z, s) = e^{-s}H(z),$$

and the extension is defined by the generator

$$X_2 - 2\partial_s.$$

3.1.3. Case $\dim(\text{Span}(V)) = 1$.

There exists a constant vector $(\gamma, \beta, q) \neq 0$ such that

$$a = \gamma g, \quad b = \beta g, \quad c = qg,$$

with some nonconstant function $g(z, s) \neq 0$.

If one assumes that $\gamma \neq 0$, then choosing $\gamma = 1$, one gets $b = \beta a$, $c = qa$,

$$k = -\beta(k_3 - k_1) - qk_2,$$

and

$$\zeta(s) = -(k_3 - k_1)h_1(s) - k_2h_2(s),$$

where

$$h_1 = \psi_s^{-1} (\beta z^{1/2} + 2(z\psi_z - \psi)). \quad (31)$$

$$h_2 = \psi_s^{-1} (qz^{1/2} + 2\psi). \quad (32)$$

The extension of L_5 is defined by the generators

$$X_3 - h_1\partial_s, \quad X_2 - h_2\partial_s.$$

Since $\psi_s \neq 0$, one can assume that $h_2 = 1$. Notice that the commutator of the latter generators has to vanish

$$[X_3 - h_1\partial_s, X_2 - h_2\partial_s] = h_1'\partial_s = 0.$$

Hence, $h_1 = \lambda$, where λ is constant, and the function $\psi(z, s)$ has to satisfy the equations:

$$z\psi_z - (\lambda + 1)\psi = \frac{(q\lambda - \beta)}{2}z^{1/2}, \quad (33)$$

$$\psi_s = 2\psi + qz^{1/2}. \quad (34)$$

The general solution of these equations is

$$\psi = \beta z^{1/2} + Kz^{1+\lambda}e^{-2s},$$

where K is constant, and the admitted generators are

$$X_3 - \lambda\partial_s, \quad X_2 + \partial_s.$$

Using the equivalence transformation corresponding to Z_3^e , one obtains

$$\psi = Kz^{1+\lambda}e^{-2s}.$$

Assuming that $\gamma = 0$ and $\beta \neq 0$, one sets $g = b$, and hence,

$$a = 0, \quad c = qb, \quad k_3 - k_1 = -qk_2,$$

and

$$\zeta = kh_1(s) + 2k_2h_2(s),$$

where

$$h_1 = \psi_s^{-1} z^{1/2}, \quad h_2 = \psi_s^{-1} (q(z\psi_z - \psi) - \psi).$$

The extension of L_5 is defined by the generators

$$h_1\partial_s, \quad X_2 - qX_3 + 2h_2\partial_s.$$

Since $h_1 \neq 0$, one can assume that $h_1 = 1$. The commutator of the latter generators is

$$[\partial_s, X_2 - qX_3 + 2h_2\partial_s] = 2h_2'\partial_s.$$

Hence, $h_2 = \lambda s$, where λ is constant, and

$$\psi_s = z^{1/2}, \quad qz\psi_z = (q+1)\psi + \lambda sz^{1/2}. \quad (35)$$

If $q = 0$, then $\lambda = -1$ and

$$\psi = z^{1/2}s.$$

This gives that $\dim(\text{Span}(V)) = 0$. Thus, one has to assume that $q \neq 0$. In this case

$$\psi = Kz^{1+1/q} + z^{1/2}s, \quad \lambda = -(1+q/2),$$

and the admitted generator are

$$\partial_s, \quad X_2 - qX_3 - (q+2)s\partial_s.$$

Since for $K = 0$ one has $\dim(\text{Span}(V)) = 0$, then $K \neq 0$.

Assuming that $\gamma = 0$, $\beta = 0$ and $q \neq 0$, one has

$$a = 0, \quad b = 0, \quad k_2 = 0$$

and

$$\zeta(s) = kh_1(s) - 2(k_3 - k_1)h_2(s),$$

where

$$h_1 = \psi_s^{-1}z^{1/2}, \quad h_2 = \psi_s^{-1}(z\psi_z - \psi).$$

The extension of L_5 is defined by the generators

$$h_1\partial_s, \quad X_3 - 2h_2\partial_s.$$

As in the previous case one can assume that $h_1 = 1$. Since the commutator is

$$[\partial_s, X_3 - 2h_2\partial_s] = -2h_2'\partial_s$$

one has that $h_2 = \lambda s$, where λ is constant. Hence

$$\psi_s = z^{1/2}, \quad \lambda sz^{1/2} = z\psi_z - \psi. \quad (36)$$

The general solution of these equations is $\lambda = -1/2$ and

$$\psi = Kz + z^{1/2}s.$$

The admitted generators are

$$X_3 + s\partial_s, \quad \partial_s.$$

Since for $K = 0$ one has that $\dim(\text{Span}(V)) = 0$, then $K \neq 0$.

3.1.4. Case $\dim(\text{Span}(V)) = 0$.

There exists a constant vector (γ, β, q) such that

$$a = \gamma, \quad b = \beta, \quad c = q.$$

In this case

$$\gamma k + \beta(k_3 - k_1) + qk_2 = 0, \quad (37)$$

and

$$\zeta(s) = kh_1(s) + (k_3 - k_1)h_2(s) + k_2h_3(s),$$

where

$$h_1 = -\gamma z + z^{1/2}\psi_s^{-1}, \quad h_2 = -\beta z - 2\psi_s^{-1}(z\psi_z - \psi), \quad h_3 = -qz - 2\psi_s^{-1}\psi. \quad (38)$$

The extension of L_5 is defined by the generators of the form

$$(k_3 - k_1)(X_3 + h_2\partial_s) + k_2(X_2 + h_3\partial_s) + k(h_1\partial_s).$$

The compatibility conditions for the relations (38) are reduced to the equations

$$\begin{aligned} h_1(2h_2 + h_3) &= 0, \quad h_1(2\beta - q) + \gamma(2h_2 + 3h_3) = 0, \quad \gamma(2\beta + q) = 0, \\ h_1(h'_3 + 2) - h'_1h_3 &= 0, \quad \gamma(h'_3 + 2) - qh'_1 = 0, \end{aligned} \quad (39)$$

and

$$\psi = -\frac{(h_3 + qz)z^{1/2}}{2(h_1 + \gamma z)}.$$

If $h_1 \neq 0$, then one can assume that $h_1 = 1$. Conditions (39) give

$$\gamma = 0, \quad q = 2\beta, \quad h'_2 = 1, \quad h_3 = -2h_2. \quad (40)$$

Equation (37) becomes

$$\beta((k_3 - k_1) + 2k_2) = 0.$$

Since $\gamma = 0$, one can choose

$$h_2 = s, \quad h_3 = -2s.$$

Hence

$$\psi = (s - \beta z)z^{1/2},$$

and the admitted generators are

$$\beta = 0 : \quad X_3 + s\partial_s, \quad X_2 - 2s\partial_s, \quad \partial_s,$$

$$\beta \neq 0 : \quad X_2 - 2X_3 - 4s\partial_s, \quad \partial_s.$$

If $h_1 = 0$, then $\gamma \neq 0$, and one can assume that $\gamma = 1$. Hence,

$$h_3 = -2s, \quad h_2 = 3s, \quad \psi = (s + \beta z)z^{-1/2},$$

and the extension of L_5 is defined by the admitted generators

$$X_3 + 3s\partial_s, \quad X_2 - 2s\partial_s.$$

Notice that using the equivalence transformation corresponding to Z_3^e , one has that

$$\psi = z^{-1/2}s.$$

3.2. Case $k_4 = 0$

Here we consider models for which $k_4 = 0$. Differentiating equation (10) with respect to α one also obtains that $k_5 = 0$. Notice that similar to the previous section after performing some manipulations (differentiations and linear combinations) one obtains that the determining equations (2)-(17) are reduced to $k_6 = 0$ and the equation

$$2(3k_1 - 5k_3)\alpha\varepsilon_\alpha + (2k_1 - 3k_3)\rho\varepsilon_\rho + \varepsilon_s\zeta = 2(k_1 - k_2 - 2k_3)\varepsilon - \alpha^{1/2}g + \varphi - k\rho^{-1}, \quad (41)$$

where $g = g(\rho)$, $\varphi = \varphi(s)$ and the constant k are arbitrary elements obtained during the integration.

Let us consider the Lie algebra $L_6 = \{Y_1, Y_2, Y_3, X_1, X_2, X_3\}$. Its commutator table is

	Y_1	Y_2	Y_3	X_1	X_2	X_3
Y_1	0	0	Y_2	0	Y_1	$-2Y_1$
Y_2	0	0	0	0	0	$-Y_2$
Y_3	$-Y_2$	0	0	0	$-Y_3$	Y_3
X_1	0	0	0	0	0	0
X_2	$-Y_1$	0	Y_3	0	0	0
X_3	$2Y_1$	Y_2	$-Y_3$	0	0	0

Notice that the subalgebra $\{Y_1, Y_2, Y_3\}$ is a kernel of admitted Lie algebras, and the Lie algebra $\{X_1, X_2, X_3\}$ is an Abelian subalgebra.

Since the Lie algebra $\{Y_1, Y_2, Y_3\}$ composes the kernel of admitted Lie algebras, then the basis generators of an admitted Lie algebra related with the generators X_1, X_2, X_3 , and X_ζ can be chosen in the form

$$qX_1 + \beta X_2 + \gamma X_3 + X_\zeta, \quad (42)$$

where $X_\zeta = \zeta(s)\partial_s$. The generators X_1, X_2, X_3 , and X_ζ also compose a Lie algebra, where the generator X_ζ is the center.

Notice that if $\zeta \neq 0$ for one of the basis generators, then by virtue of the equivalence transformation corresponding to the generator X_4^e , the function $\zeta(s)$ in this basis generator² can be reduced to $\zeta = 1$.

3.3. Strategy of the further study

It is well known that the set of admitted generators composes a Lie algebra [11]: the property to compose a Lie algebra is automatically satisfied for solutions of the determining equations. The idea of the algebraic approach used in the present paper is to separate the study of group classification to two steps. In the first step one makes a preliminary study of possible coefficients of the basis generators using the requirement that admitted generators compose a Lie algebra. In the second step, one substitutes the coefficients of each generator of the Lie algebra into the determining equation (41). Solving the system of equations thus obtained, the function $\varepsilon(\rho, \alpha, s)$ and additional restrictions for the coefficients of the basis generators are found.

Let us also notice that if one can choose the basis generators such that two of them have the form

$$\zeta_1(s)\partial_s, \quad \zeta_2(s)\partial_s, \quad (43)$$

then this case is reduced to $\varepsilon_s = 0$. Indeed, since the generators (43) are basis generators, then $\zeta_i \neq 0$ and $\zeta_1\zeta_2' - \zeta_1'\zeta_2 \neq 0$. By virtue of the equivalence transformation related with X_4^e , one

²Only for one basis generator: for other basis generators, $\zeta = \zeta(s)$.

can assume that $\zeta_1 = 1$, and then $\zeta_2' \neq 0$. Substituting the coefficients of the generators (43) into (41) one obtains the equations

$$\varepsilon_s = -\alpha^{1/2}g_1 + \varphi_1 - k_1\rho^{-1}, \quad (44)$$

$$\zeta_2(-\alpha^{1/2}g_1 + \varphi_1 - k_1\rho^{-1}) = -\alpha^{1/2}g_2 + \varphi_2 - k_2\rho^{-1}, \quad (45)$$

where $g_i = g_i(\rho)$ and $\varphi_i = \varphi_i(s)$, ($i = 1, 2$). Splitting equation (45), one finds that

$$k_1 = 0, \quad k_2 = 0, \quad g_1 = 0, \quad g_2 = 0, \quad \varphi_2 = \zeta_2\varphi_1,$$

which means that

$$\varepsilon(\rho, \alpha, s) = H_1(s) + H_2(\rho, \alpha),$$

Using the equivalence transformation corresponding to the generator Z_2^e , one can reduce the function $\varepsilon(\rho, \alpha, s)$ to the case where $H_1(s) = 0$.

The latter study also allows to conclude that: (a) there are no admitted Lie algebras where an extension of the kernel is more than four-dimensional; (b) there is only one possible admitted Lie algebra which has a four-dimensional extension: the basis generators of this extension have the form

$$X_1 + \zeta_1(s)\partial_s, \quad X_2 + \zeta_2(s)\partial_s, \quad X_3 + \zeta_3(s)\partial_s, \quad \partial_s. \quad (46)$$

In the preliminary study of Lie algebras of dimension greater than 1, it is sufficient for our purposes to use classifications of two- and three dimensional Lie algebras. These classifications are well-known³. For the sake of completeness they are presented here.

All two-dimensional Lie algebras have one of the following commutator tables:

$$L(2, 1) : \begin{vmatrix} & e_1 & e_2 \\ e_1 & 0 & 0 \\ e_2 & & 0 \end{vmatrix}, \quad L(2, 2) : \begin{vmatrix} & e_1 & e_2 \\ e_1 & 0 & e_1 \\ e_2 & & 0 \end{vmatrix}.$$

Here e_i are suitably chosen basis vectors of the Lie algebra.

All three-dimensional Lie algebras have in suitably chosen basis, the commutator tables

$$\begin{aligned} L(3, 1) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & 0 & 0 \\ e_2 & & 0 & e_1 \\ e_3 & & & 0 \end{vmatrix}, \quad L(3, 2, p) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & 0 & e_1 \\ e_2 & & 0 & pe_2 \\ e_3 & & & 0 \end{vmatrix}, \quad 0 < |p| \leq 1, \\ \\ L(3, 3) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & 0 & e_1 \\ e_2 & & 0 & e_1 + e_2 \\ e_3 & & & 0 \end{vmatrix}, \quad L(3, 4, p) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & 0 & pe_1 - e_2 \\ e_2 & & 0 & e_1 + pe_2 \\ e_3 & & & 0 \end{vmatrix} =, \quad p \geq 0, \\ \\ L(3, 5) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & e_1 & 2e_2 \\ e_2 & & 0 & e_3 \\ e_3 & & & 0 \end{vmatrix}, \quad L(3, 6) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & e_3 & -e_2 \\ e_2 & & 0 & e_1 \\ e_3 & & & 0 \end{vmatrix}, \end{aligned}$$

³See for example in [14].

$$L(3, 7) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & e_1 & 0 \\ e_2 & & 0 & 0 \\ e_3 & & & 0 \end{vmatrix}, \quad L(3, 0) : \begin{vmatrix} & e_1 & e_2 & e_3 \\ e_1 & 0 & 0 & 0 \\ e_2 & & 0 & 0 \\ e_3 & & & 0 \end{vmatrix}.$$

Further study depends on the dimension of a Lie algebra composed by the generators of the form (42).

3.3.1. One-dimensional extension

Here we use the algebraic approach. This approach supposes that using algebraic properties of admitted Lie algebras, during the first step of solving the determining equations one defines unknown constants and functions of an admitted generator. In particular, according to the last comment of the previous section, one-dimensional Lie algebras can be reduced to one of two cases, either $\zeta = 0$ or $\zeta = 1$. The set of possible basis generators containing the generators X_1 , X_2 and X_3 is exhausted by the following generators

$\zeta = 0$		$\zeta = 1$	
1.	$X_1 + \beta X_2 + \gamma X_3$	4.	$X_1 + \beta X_2 + \gamma X_3 + \partial_s$
2.	$\beta X_2 + X_3$	5.	$\beta X_2 + X_3 + \partial_s$
3.	X_2	6.	$X_2 + \partial_s$

In the next step, one has to substitute the coefficients of each generator into the determining equation (41) and solve it with respect to the function $\varepsilon(\rho, \alpha, s)$. Here we present the calculations of the first case, where an extension of the kernel of admitted generators Y_1 , Y_2 and Y_3 consists of the generator $X_1 + \beta X_2 + \gamma X_3$. The study of the remaining cases is similar, and the final result is presented in Table 1.

Substituting

$$k_1 = 1, \quad k_2 = \beta, \quad k_3 = \gamma, \quad \zeta = 0$$

into equation (41), one obtains

$$2(3 - 5\gamma)\alpha\varepsilon_\alpha + (2 - 3\gamma)\rho\varepsilon_\rho = 2(1 - \beta - 2\gamma)\varepsilon - \alpha^{1/2}g(\rho) + \varphi(s) - k\rho^{-1}. \quad (47)$$

The characteristic system of this equation is

$$\frac{d\alpha}{2(3 - 5\gamma)\alpha} = \frac{d\rho}{(2 - 3\gamma)\rho} = \frac{ds}{0} = \frac{d\varepsilon}{2(1 - \beta - 2\gamma)\varepsilon - \alpha^{1/2}g(\rho) + \varphi(s) - k\rho^{-1}}.$$

Invariants of the characteristic system depend on the vanishing of the expression

$$\kappa = (2 - 3\gamma)(1 - \beta - 2\gamma)(4 - 2\beta - 7\gamma).$$

If $\kappa \neq 0$, then the solution of (47) is

$$\varepsilon(\rho, \alpha, s) = \alpha^{1/2}\tilde{g}(s) + \tilde{\varphi}(s) + \tilde{k}\rho^{-1} + h(s, \alpha\rho^{-2(3-5\gamma)/(2-3\gamma)})\rho^{2(1-\beta-2\gamma)/(2-3\gamma)}$$

where $\tilde{g}, \tilde{\varphi}, h$ and constant $\tilde{k}$ are arbitrary elements. Using the equivalence transformations corresponding to Z_1^e , Z_2^e and Z_3^e , one gets that

$$\varepsilon(\rho, \alpha, s) = h(s, \alpha\rho^{-2(3-5\gamma)/(2-3\gamma)})\rho^{2(1-\beta-2\gamma)/(2-3\gamma)}.$$

In this case the system of equations (1) admits the generator

$$X_1 + \beta X_2 + \gamma X_3, \quad (2 - 3\gamma)(1 - \beta - 2\gamma)(4 - 2\beta - 7\gamma) \neq 0.$$

Similar, one finds that

If $\beta = 1 - 2\gamma$, $2 - 3\gamma \neq 0$, then

$$\varepsilon(\rho, \alpha, s) = \phi(s) \ln \rho + h(s, \alpha \rho^{-2(3-5\gamma)/(2-3\gamma)}),$$

$$X_1 + (1 - 2\gamma)X_2 + \gamma X_3.$$

If $\beta = \frac{7\gamma-4}{2}$, $2 - 3\gamma \neq 0$, then

$$\varepsilon(\rho, \alpha, s) = \rho^{-1} (C \ln \rho + h(s, \alpha \rho^{-2(3-5\gamma)/(2-3\gamma)})),$$

$$X_1 + (\frac{7\gamma-4}{2})X_2 + \gamma X_3.$$

If $\gamma = \frac{2}{3}$, $(1 + 3\beta)(1 + 6\beta) \neq 0$, then

$$\varepsilon(\rho, \alpha, s) = h(s, \rho) \alpha^{1+3\beta},$$

$$X_1 + \beta X_2 + \frac{2}{3}X_3, \quad (1 + 3\beta)(1 + 6\beta) \neq 0.$$

If $\gamma = \frac{2}{3}$, $\beta = -\frac{1}{3}$, then

$$\varepsilon(\rho, \alpha, s) = (\phi(s) - C\rho^{-1}) \ln \alpha + h(\rho, s),$$

$$X_1 - \frac{1}{3}X_2 + \frac{2}{3}X_3.$$

If $\gamma = \frac{2}{3}$, $\beta = -\frac{1}{6}$, then

$$\varepsilon(\rho, \alpha, s) = (\psi(\rho) \ln \alpha + h(\rho, s)) \alpha^{1/2},$$

$$X_1 - \frac{1}{6}X_2 + \frac{2}{3}X_3.$$

Here C , ψ , ϕ and h are arbitrary.

3.3.2. Two-dimensional extensions

Let the basis generators of an admitted Lie algebra with two-dimensional extension of the kernel be

$$X = \beta_1 X_1 + q_1 X_2 + \gamma_1 X_3 + X_{\zeta_1}, \quad Y = \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3 + X_{\zeta_2}.$$

Their commutator is $[X, Y] = [X_{\zeta_1}, X_{\zeta_2}]$.

For the Lie algebra $L(2, 2)$ one finds that

$$[X, Y] = [X_{\zeta_1}, X_{\zeta_2}] = (\zeta_1 \zeta_2' - \zeta_1' \zeta_2) \partial_s = X,$$

which means that

$$\zeta_1 \zeta_2' - \zeta_1' \zeta_2 = \zeta_1, \quad \beta_1 = 0, \quad q_1 = 0, \quad \gamma_1 = 0.$$

Hence, $\zeta_1 \neq 0$ or one can assume that $\zeta_1 = 1$. The list of such algebras is exhausted by the following Lie algebras

1.	$\partial_s, X_1 + \beta X_2 + \gamma X_3 + s \partial_s$
2.	$\partial_s, X_2 + \gamma X_3 + s \partial_s$
3.	$\partial_s, X_3 + s \partial_s$

For the Lie algebra $L(2, 1)$ one finds that

$$[X, Y] = [X_{\zeta_1}, X_{\zeta_2}] = (\zeta_1 \zeta_2' - \zeta_1' \zeta_2) \partial_s = 0,$$

which means that

$$\zeta_1 \zeta_2' - \zeta_1' \zeta_2 = 0. \quad (48)$$

Notice that if $\zeta_1^2 + \zeta_2^2 \neq 0$, then one can assume that $\zeta_1 = 1$. In this case equation (48) gives that $\zeta_2 = k\zeta_1$. Hence, one can also assume that $\zeta_2 = 0$. Thus, an extension of the kernel of admitted Lie algebras in the case of $L(2, 1)$ has one of the following forms,

$$\{\beta_1 X_1 + q_1 X_2 + \gamma_1 X_3 + \partial_s, \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3\}, \quad (49)$$

or

$$\{\beta_1 X_1 + q_1 X_2 + \gamma_1 X_3, \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3\}. \quad (50)$$

The set of algebras of form (49) and (50) is exhausted by the list

$\zeta_1^2 + \zeta_2^2 \neq 0$		$\zeta_1^2 + \zeta_2^2 = 0$	
4.	$q_1 X_2 + \gamma_1 X_3 + \partial_s, X_1 + q_2 X_2 + \gamma_2 X_3$	7.	$q_1 X_2 + \gamma_1 X_3, X_1 + q_2 X_2 + \gamma_2 X_3$
5.	$\beta_1 X_1 + \gamma_1 X_3 + \partial_s, X_2 + \gamma_2 X_3$	8.	$\beta_1 X_1 + \gamma_1 X_3, X_2 + \gamma_2 X_3$
6.	$\beta_1 X_1 + q_1 X_2 + \partial_s, X_3$	9.	$\beta_1 X_1 + q_1 X_2, X_3$

Similar to the one-dimensional case, in the next step one has to substitute the coefficients of each generator into the determining equation (41), and solve the obtained overdetermined system of equations with respect to the function $\varepsilon(\rho, \alpha, s)$. The final result of calculations is presented in Table 1 (models $M_{26} - M_{62}$).

3.3.3. Three-dimensional extensions

Let the basis generators be

$$X = \tilde{X} + X_{\zeta_1}, \quad Y = \tilde{Y} + X_{\zeta_2}, \quad Z = \tilde{Z} + X_{\zeta_3},$$

where

$$\tilde{X} = \beta_1 X_1 + q_1 X_2 + \gamma_1 X_3, \quad \tilde{Y} = \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3, \quad \tilde{Z} = \beta_3 X_1 + q_3 X_2 + \gamma_3 X_3.$$

Notice that

$$[\tilde{X}, \tilde{Y}] = 0, \quad [\tilde{X}, \tilde{Z}] = 0, \quad [\tilde{Y}, \tilde{Z}] = 0. \quad (51)$$

First let us study the Abelian Lie algebra $L(3, 0)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

which means that

$$\zeta_1 \zeta_2' - \zeta_1' \zeta_2 = 0, \quad \zeta_1 \zeta_3' - \zeta_1' \zeta_3 = 0, \quad \zeta_2 \zeta_3' - \zeta_2' \zeta_3 = 0, \quad (52)$$

and

$$[\tilde{X}, \tilde{Y}] = 0, \quad [\tilde{X}, \tilde{Z}] = 0, \quad [\tilde{Y}, \tilde{Z}] = 0.$$

If $\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 0$, then the basis of this Lie algebra is

$$X_1, \quad X_2, \quad X_3.$$

If, for example, $\zeta_1 \neq 0$, then one can assume that $\zeta_1 = 1$ and $\zeta_2 = 0, \zeta_3 = 0$. The list of Abelian Lie algebras is exhausted by the following set

1.	$\gamma X_3 + \partial_s, \quad X_1 + q_1 X_3, \quad X_2 + q_2 X_3,$
2.	$\beta X_2 + \partial_s, \quad X_1 + q_1 X_2, \quad X_3,$
3.	$\beta X_1 + \partial_s, \quad X_2, \quad X_3,$

where γ, β, q_1 and q_2 are arbitrary constants.

For $L(3, 1)$ one has

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = X,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = 0, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = \zeta_1, \quad (53)$$

and

$$[\tilde{X}, \tilde{Y}] = 0, \quad [\tilde{X}, \tilde{Z}] = 0, \quad [\tilde{Y}, \tilde{Z}] = \tilde{X}.$$

Because of (51), one has that $\tilde{X} = 0$. Hence, $\zeta_1 \neq 0$, and one can assume that $\zeta_1 = 1$. Equations (53) become contradictory.

Let us study $L(3, 2, p)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = X, \quad [Y, Z] = pY,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = \zeta_1, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = p\zeta_2, \quad (54)$$

and

$$[\tilde{X}, \tilde{Y}] = 0, \quad [\tilde{X}, \tilde{Z}] = \tilde{X}, \quad [\tilde{Y}, \tilde{Z}] = p\tilde{Y}.$$

Because of (51) and $p \neq 0$, one has that $\tilde{X} = 0$ and $\tilde{Y} = 0$. Since in this case there are two basis generators of the form (43), this case is excluded for further consideration⁴.

In the case $L(3, 3)$ one has

$$[X, Y] = 0, \quad [X, Z] = X, \quad [Y, Z] = X + Y,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = \zeta_1, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = \zeta_1 + \zeta_2, \quad (55)$$

and

$$[\tilde{X}, \tilde{Y}] = 0, \quad [\tilde{X}, \tilde{Z}] = \tilde{X}, \quad [\tilde{Y}, \tilde{Z}] = \tilde{X} + \tilde{Y}.$$

Because of (51), one has that $\tilde{X} = 0$ and $\tilde{Y} = 0$. Similar to $L(3, 3)$ one also has to exclude this case from the study.

Let us study $L(3, 4, p)$. In this case one has

$$[X, Y] = 0, \quad [X, Z] = pX - Y, \quad [Y, Z] = X + pY, \quad (p \geq 0),$$

⁴Moreover one can obtain a contradiction to the property that X, Y and Z compose a basis of the Lie algebra.

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = 0, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = p\zeta_1 - \zeta_2, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = \zeta_1 + p\zeta_2, \quad (56)$$

and

$$[\tilde{X}, \tilde{Y}] = 0, \quad [\tilde{X}, \tilde{Z}] = p\tilde{X} - \tilde{Y}, \quad [\tilde{Y}, \tilde{Z}] = \tilde{X} + p\tilde{Y}.$$

Because of (51), one has that $\tilde{X} = 0$ and $\tilde{Y} = 0$. Hence, this case is also similar to the previous case.

The same result is obtained for $L(3, 5)$ and $L(3, 6)$. Indeed, in the case $L(3, 5)$ one has

$$[X, Y] = X, \quad [X, Z] = 2Y, \quad [Y, Z] = Z,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = \zeta_1, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = 2\zeta_2, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = \zeta_3, \quad (57)$$

and

$$[\tilde{X}, \tilde{Y}] = \tilde{X}, \quad [\tilde{X}, \tilde{Z}] = 2\tilde{Y}, \quad [\tilde{Y}, \tilde{Z}] = \tilde{Z}.$$

Because of (51), one has that $\tilde{X} = 0$, $\tilde{Y} = 0$ and $\tilde{Z} = 0$.

In the case of $L(3, 6)$ one has

$$[X, Y] = Z, \quad [X, Z] = -Y, \quad [Y, Z] = X,$$

which gives that $\tilde{X} = 0$, $\tilde{Y} = 0$ and $\tilde{Z} = 0$.

Let us study $L(3, -1)$. In this case one has

$$[X, Y] = X, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

which means that

$$\zeta_1 \zeta'_2 - \zeta'_1 \zeta_2 = \zeta_1, \quad \zeta_1 \zeta'_3 - \zeta'_1 \zeta_3 = 0, \quad \zeta_2 \zeta'_3 - \zeta'_2 \zeta_3 = 0. \quad (58)$$

and

$$[\tilde{X}, \tilde{Y}] = \tilde{X}, \quad [\tilde{X}, \tilde{Z}] = 0, \quad [\tilde{Y}, \tilde{Z}] = 0.$$

Because of (51), one has that $\tilde{X} = 0$. Hence, $\zeta_1 \neq 0$, and one can assume that $\zeta_1 = 1$. Equations (58) become

$$\zeta'_2 = 1, \quad \zeta'_3 = 0, \quad \zeta_3 = 0.$$

Thus, the basis generators have the form

$$X = \partial_s, \quad Y = s\partial_s + \beta_2 X_1 + q_2 X_2 + \gamma_2 X_3, \quad Z = \beta_3 X_1 + q_3 X_2 + \gamma_3 X_3.$$

Thus, the set of non-Abelian three-dimensional Lie algebras is exhausted the following list

4.	$\partial_s, \quad s\partial_s + q_2 X_2 + \gamma_2 X_3, \quad X_1 + q_3 X_2 + \gamma_3 X_3$
5.	$\partial_s, \quad s\partial_s + \beta_2 X_1 + \gamma_2 X_3, \quad X_2 + \gamma_3 X_3$
6.	$\partial_s, \quad s\partial_s + \beta_2 X_1 + q_2 X_2, \quad X_3$

Similar to the one- and two-dimensional cases, in the next step one has to substitute the coefficients of each generator into the determining equation (41), and solve the obtained overdetermined system of equations with respect to the function $\varepsilon(\rho, \alpha, s)$. The final result of the calculations is presented in Table 1 (models $M_{63} - M_{82}$).

3.3.4. Extensions of dimension greater than 3

If the dimension of the extension is greater or equal to 4, then either one can choose a basis of generators such that two of them are of the form (43):

$$\zeta_1(s)\partial_s, \zeta_2(s)\partial_s,$$

or the admitted Lie algebra is four-dimensional and the basis generators can be chosen such as

$$X_1 + \zeta_1(s)\partial_s, \quad X_2 + \zeta_2(s)\partial_s, \quad X_3 + \zeta_3(s)\partial_s, \quad \partial_s. \quad (59)$$

Substituting the coefficients of the generators (59) into (41) one obtains reduction to the case where $\varepsilon_s = 0$.

Thus, there is no case where an extension of the kernel of admitted Lie algebras is of dimension greater than three.

4. Results of the group classification

Results of the group classification of equations (1) are summarized in Table 1, where representations of the function $\varepsilon(\rho, |\nabla\rho|, s)$ are simplified by equivalence transformations.

The first column in Table 1 presents the number of the extension, forms of the function $\varepsilon(\rho, |\nabla\rho|, s)$ are presented in the second column, extensions of the kernel of admitted Lie algebras are given in the third column, and restrictions for functions and constants are in the fourth column.

5. Conclusion

Classifying equations of fluids with internal inertia with respect to the internal energy $\varepsilon(\rho, |\nabla\rho|, s)$, group analysis provides a regular procedure for mathematical modeling. In this paper we give a group classification of equations (1), where the function $\varepsilon(\rho, |\nabla\rho|, s)$ substantially depends on the entropy s : $\varepsilon_s \neq 0$. The group classification separates all models into 82 classes, which are presented in Table 1.

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Table 1. Group classification

	$\varepsilon(\rho, \alpha, s)$	Extensions	Remarks
M_1	$\rho^2 h(\alpha \rho^{-4} e^{\beta s}) e^{\gamma s}$	$X_4, \beta X_3 + (\beta + \gamma) X_2 - \partial_s$	$\beta \neq \gamma$
M_2	$\rho^2 h(\alpha \rho^{-4} e^{-\beta s}) e^{2\beta s} + s \rho^{-2} \alpha^{1/2}$	$X_4, \beta(X_3 + 2X_2) + \partial_s$	
M_3	$\alpha^\lambda \rho^{2-4\lambda} h(\alpha \rho^{-4} e^{-s})$	$X_4, X_3 + (1 - \lambda) X_2 + 2\partial_s$	$\lambda \neq 0$
M_4	$\rho^2 e^{-s} h(\alpha \rho^{-4})$	$X_4, X_2 - 2\partial_s$	
M_5	$K \alpha^\lambda \rho^{2-4\lambda} e^{-2s}$	$X_4, X_3 - (\lambda + 1) \partial_s, X_2 + \partial_s$	
M_6	$K \alpha^\lambda \rho^{2-4\lambda} + s \alpha^{1/2}$	$X_4, (1 - \lambda) X_2 + X_3 + (2\lambda - 1) s \partial_s, \partial_s$	$(\lambda - 1) K \neq 0$
M_7	$K \alpha \rho^{-2} + s \alpha^{1/2}$	$X_4, X_3 + s \partial_s, \partial_s$	$K \neq 0$
M_8	$s \alpha^{1/2}$	$X_4, X_3 + s \partial_s, X_2 - 2s \partial_s, \partial_s$	
M_9	$(K \alpha \rho^{-4} + s) \alpha^{1/2}$	$X_4, X_2 - 2X_3 - 4s \partial_s, \partial_s$	$K \neq 0$
M_{10}	$\rho^4 \alpha^{-1/2} s$	$X_4, X_3 + 3s \partial_s, X_2 - 2s \partial_s$	
M_{11}	$\rho^\mu h(s, \alpha \rho^\lambda)$	$(3\lambda + 10) X_1 - (\lambda + \mu + 2) X_2 + 2(\lambda + 3) X_3$	$(\mu + 1) \mu \neq 0$ $(3\lambda + 10) \neq 0$
M_{12}	$\rho^{-1} [h(s, \alpha \rho^\lambda) + C \ln \rho]$	$(3\lambda + 10) X_1 - (\lambda + 1) X_2 + 2(\lambda + 3) X_3$	$(3\lambda + 10) \neq 0$
M_{13}	$h(s, \alpha \rho^\lambda) + \varphi(s) \ln \rho$	$(3\lambda + 10) X_1 - (\lambda + 2) X_2 + 2(\lambda + 3) X_3$	$(3\lambda + 10) \neq 0$
M_{14}	$\alpha^\lambda h(s, \rho)$	$3X_1 + (\lambda - 1) X_2 + 2X_3$	$\lambda(2\lambda - 1) \neq 0$
M_{15}	$\alpha^{1/2} [h(s, \rho) + \varphi(s) \ln \alpha]$	$6X_1 - X_2 + 4X_3$	
M_{16}	$h(s, \rho) + (\varphi(s) + C \rho^{-1}) \ln \alpha$	$3X_1 - X_2 + 2X_3$	
M_{17}	$\rho^\mu h(s, \alpha \rho^{-10/3})$	$(3\mu - 4) X_2 + 2X_3$	$\mu(\mu + 1) \neq 0$
M_{18}	$\rho^{-1} [h(s, \alpha \rho^{-10/3}) + C \ln \rho]$	$-7X_2 + 2X_3$	
M_{19}	$h(s, \alpha \rho^{-10/3}) + \varphi(s) \ln \rho$	$-2X_2 + X_3$	
M_{20}	$h(\rho, \alpha s^\lambda) s^\mu$	$14X_1 - (\lambda + 7\mu + 4) X_2 + 2(\lambda + 4) X_3 + 14\partial_s$	$7\mu \neq 2(3\lambda - 2)$
M_{21}	$h(\rho, \alpha s^\lambda) s^{2(3\lambda-2)/7} + C \rho^{-1} s$	$14X_1 - 7\lambda X_2 + 2(\lambda + 4) X_3 + \partial_s$	
M_{22}	$h(\rho, \alpha s^2) s^{-1} + g(\rho) \alpha^{1/2} s^2$	$6X_1 - X_2 + 4X_3 + \partial_s$	
M_{23}	$h(\alpha e^{10s}, \rho e^{3s}) e^\lambda$	$-(\lambda + 4) X_2 + 2X_3 + 2\partial_s$	$(\lambda - 3) \neq 0$
M_{24}	$h(\alpha e^{10s}, \rho e^{3s}) e^{3s} + K \rho^{-1} s$	$-7X_2 + 2X_3 + 2\partial_s$	
M_{25}	$h(\rho, \alpha) s^{-2}$	$X_2 + \partial_s$	
M_{26}	$[\alpha^{1/2} g(\rho) + C \rho^{-1}] s + \rho^\mu h(\alpha \rho^\lambda)$	$\partial_s, (3\lambda + 10) X_1 + 2(\lambda + 3) X_2 - (\lambda + \mu + 2) X_3 + (3\lambda + 10) s \partial_s$	$(3\lambda + 10) C \neq 0$ $\mu(\mu + 1) h \neq 0$
M_{27}	$[\alpha^{1/2} g(\rho) + C \rho^{-1}] s + h(\alpha \rho^\lambda) + K \ln \rho$	$\partial_s, (3\lambda + 10) X_1 + 2(\lambda + 3) X_2 - (\lambda + 2) X_3 + (3\lambda + 10) s \partial_s$	$(3\lambda + 10) \neq 0$ $ChK \neq 0$
M_{28}	$[\alpha^{1/2} g(\rho) + C \rho^{-1}] s + \rho^{-1} h(\alpha \rho^\lambda) + K \rho^{-1} \ln \rho$	$\partial_s, (3\lambda + 10) X_1 + 2(\lambda + 3) X_2 - (\lambda + 1) X_3 + (3\lambda + 10) s \partial_s$	$(3\lambda + 10) \neq 0$ $ChK \neq 0$
M_{29}	$[\alpha^{1/2} g(\rho) + C \rho^{-1}] s + \alpha^\lambda h(\rho)$	$\partial_s, 3X_1 + 2X_2 + (\lambda - 1) X_3 + s \partial_s$	$\lambda(2\lambda - 1) \neq 0$ $Ch \neq 0$
M_{30}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s + h(\rho) + (K_1 + K_2 \rho^{-1}) \ln \alpha$	$\partial_s, 3X_1 + 2X_2 - X_3 + 3s \partial_s$	$hCK_1 K_2 \neq 0$
M_{31}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s + \alpha^{1/2} [h(\rho) + f(\rho) \ln \alpha]$	$\partial_s, 6X_1 + 4X_2 - X_3 + 6s \partial_s$	$Chf \neq 0$
M_{32}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s + \rho^\mu h(\alpha \rho^{-10/3})$	$\partial_s, (3\mu - 4) X_2 + 2X_3 + (3\mu - 4) s \partial_s$	$\mu(\mu + 1) \neq 0$ $(3\mu - 4) Ch \neq 0$
M_{33}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s + h(\alpha \rho^{-10/3}) + K \ln \rho$	$\partial_s, 2X_2 - X_3 + 2s \partial_s$	$ChK \neq 0$
M_{34}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s + \rho^{-1} [h(\alpha \rho^{-10/3}) + K \ln \rho]$	$\partial_s, 7X_2 - 2X_3 + 7s \partial_s$	$ChK \neq 0$
M_{35}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s + \rho^{4/3} h(\alpha \rho^{-10/3})$	$\partial_s, X_3 + s \partial_s$	$Ch \neq 0$
M_{36}	$[\sqrt{\alpha} g(\rho) + C \rho^{-1}] s$	$\partial_s, X_2 + s \partial_s$	$Cg \neq 0$
M_{37}	$[K \ln \alpha + C \ln \rho + h(s)] \sqrt{\alpha} \rho^\mu$	$(3\mu + 1) X_2 + 2X_3, 2X_1 - (2\mu + 1) X_2$	$(3\mu + 1) C \neq 0$
M_{38}	$h(\alpha \rho^\gamma) + \varphi(s) \ln \rho + K \sqrt{\alpha} \rho^{\gamma/2} s$	$\partial_s, (3\gamma + 10) X_1 - (\gamma + 2) X_2 + 2(\gamma + 3) X_3$	$(3\gamma + 10) \neq 0$
M_{39}	$\rho^{-1} [K_2 \sqrt{\alpha} \rho^{\gamma/2} + K_1 \ln \rho] + \rho^{-1} h(\alpha \rho^\gamma) + C \rho^{-1} s$	$\partial_s, (3\gamma + 10) X_1 - (\gamma + 1) X_2 + 2(\gamma + 3) X_3$	$(3\gamma + 10) \neq 0$

Table 1. Continue

	$\varepsilon(\rho, \alpha, s)$	Extensions	Remarks
M_{40}	$\rho^{-1}[K_1 \ln \rho + K_2 \sqrt{\alpha} \rho^{-5/3}] + \rho^{-1}h(\alpha \rho^{-10/3}) + C \rho^{-1}s$	$\partial_s, 7X_2 - 2X_3$	
M_{41}	$\rho^{-1}[K \ln \alpha + C \ln \rho] + \rho^{-1}h(s)$	$7X_1 + 4X_3, 7X_2 - 2X_3$	
M_{42}	$h(\rho e^{\gamma s}) + [K e^{3\gamma s} + C \ln \alpha] \rho^{-1}$	$7\gamma X_2 - 2\gamma X_3 + 2\partial_s, 3X_1 - X_2 + 2X_3$	$\gamma \neq 0$
M_{43}	$\alpha^\mu [h(\rho e^{\gamma s}) e^{qs} + C]$	$-qX_2 - 2\gamma X_3 + 2\partial_s, 3X_1 + (\mu - 1)X_2 + 2X_3$	$\mu(2\mu - 1) \neq 0$
M_{44}	$\sqrt{\alpha} h(\rho e^{\gamma s}) e^{\lambda s}$	$-(\lambda + \gamma)X_2 - 2\gamma X_3 + 2\partial_s, 6X_1 - X_2 + 4X_3$	$\lambda \neq 0$
M_{45}	$\sqrt{\alpha}[f(\rho) \ln \alpha + g(\rho)s] + \sqrt{\alpha}[h(\rho e^{\gamma s})]$	$-\gamma X_2 - 2\gamma X_3 + 2\partial_s, 6X_1 - X_2 + 4X_3$	
M_{46}	$C \rho^{-1}s + h(\rho e^{\gamma s}) + \varphi(s) \ln \alpha$	$2\gamma X_2 - \gamma X_3 + \partial_s, 6X_1 - X_2 + 4X_3$	
M_{47}	$h(\rho) + [Ks + C \ln \alpha] \rho^{-1} + \varphi(s) \ln \alpha$	$\partial_s, 3X_1 - X_2 + 2X_3$	
M_{48}	$[K_1 s + K_2 \ln \rho + C \sqrt{\alpha} \rho^{\gamma/2} h(\alpha \rho^\gamma e^{\beta s})] \rho^{-1}$	$-7\beta X_2 + 2\beta X_3 + (2(3\gamma + 10))\partial_s, (3\gamma + 10)X_1 - (\gamma + 1)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10)\beta \neq 0$
M_{49}	$\varphi(s) \ln \rho + C \sqrt{\alpha} \rho^{\gamma/2} + h(\alpha \rho^\gamma e^{\beta s})$	$-2\beta X_2 + \beta X_3 + (3\gamma + 10)\partial_s, (3\gamma + 10)X_1 - (\gamma + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10)\beta \neq 0$
M_{50}	$\rho^{-1}[K \ln \rho + Cs] + \rho^{-1}[h(\alpha \rho^{-10/3} e^{\beta s})]$	$21\beta X_1 + 12\beta X_3 + 14\partial_s, 7X_2 - 2X_3$	$\beta \neq 0$
M_{51}	$g(s) \ln \rho + f(s) \ln \alpha$	$2X_1 + X_3, 2X_2 - X_3$	$(g - K_1 e^{\lambda s}) \neq 0$ $(f - K_2 e^{\lambda s}) \neq 0$
M_{52}	$\rho^\mu e^{\lambda s} h(\alpha \rho^\gamma e^{\beta s})$	$(\beta(3\mu - 4) - \lambda(3\gamma + 10))X_2 + 2\beta X_3 + (2(3\gamma + 10))\partial_s, (3\gamma + 10)X_1 - (\gamma + \mu + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10) \neq 0$ $(2\lambda + \beta) \neq 0$
M_{53}	$h(s) \alpha^\lambda \rho^{\mu - 10\lambda/3}$	$3(3\mu - 4)X_1 + 2((3\mu - 3) - \lambda)X_3, (3\mu - 4)X_2 + 2X_3$	$\mu(3\mu - 4) \neq 0$ $(h - K e^{\beta s}) \neq 0$ $\lambda(2\lambda - 1) \neq 0$
M_{54}	$h(s) \alpha^\lambda \rho^{(4 - 10\lambda)/3}$	$3X_1 + (\lambda - 1)X_2, X_3$	$\lambda(2\lambda - 1) \neq 0$ $(h - K e^{\beta s}) \neq 0$
M_{55}	$h(s) \alpha^\lambda \rho^{-10\lambda/3}$	$-2X_2 + X_3, 3X_1 + (3 - \lambda)X_2$	$\lambda(2\lambda - 1) \neq 0$ $(h - K e^{\beta s}) \neq 0$
M_{56}	$\rho^\mu e^{-\beta s/2} h(\alpha \rho^\gamma e^{\beta s}) + C \sqrt{\alpha} \rho^{\mu + \gamma/2} s$	$\beta(3\gamma + 6\mu + 2)X_2 + 4\beta X_3 + 4(3\gamma + 10)\partial_s, (3\gamma + 10)X_1 - (\gamma + \mu + 2)X_2 + 2(\gamma + 3)X_3$	$(3\gamma + 10) \neq 0$
M_{57}	$\rho^\mu e^{\lambda s} h(\alpha \rho^{-10/3} e^{\beta s})$	$3\beta(3\mu - 4)X_1 + 2(3\beta(\mu - 1) + \lambda)X_3 + 2(3\mu - 4)\partial_s, (3\mu - 4)X_2 + 2X_3$	$(3\mu - 4)\mu \neq 0$ $(2\lambda + \beta) \neq 0$
M_{58}	$\rho^{4/3} e^{\lambda s} h(\alpha \rho^{-10/3} e^{\beta s})$	$6\beta X_1 - (2\beta + 3\lambda)X_2 + 6\partial_s, X_3$	$\beta \neq 0$
M_{59}	$e^{\lambda s} h(\alpha \rho^{-10/3} e^{\beta s}) + \varphi(s) \ln \rho$	$6\beta X_1 + (3\beta - \lambda)X_3 + 4\partial_s, 2X_2 - X_3$	$\lambda \neq 0$
M_{60}	$\rho^\mu e^{-\beta s/2} h(\alpha \rho^{-10/3} e^{\beta s}) + C \sqrt{\alpha} \rho^{\mu - 5/3} s$	$3\beta(3\mu - 4)X_1 + \beta(6\mu - 7)X_3 + 2(3\mu - 4)\partial_s, (3\mu - 4)X_2 + 2X_3$	$(3\mu - 4)\mu \neq 0$
M_{61}	$\rho^{4/3} e^{-\beta s/2} h(\alpha \rho^{-10/3} e^{\beta s}) + C \sqrt{\alpha} \rho^{-1/3} s$	$6\beta X_1 - \beta X_2 + 4\partial_s, X_3$	
M_{62}	$e^{-\beta s/2} h(\alpha \rho^{-10/3} e^{\beta s}) +$	$12\beta X_1 + 7\beta X_3 + 8\partial_s, 2X_2 - X_3$	$\beta \neq 0$
M_{63}	$e^{\lambda s} [K \ln \rho + C \ln \alpha]$	$-\lambda X_3 + 4\partial_s, 2X_1 + X_3, 2X_2 - X_3$	$\lambda \neq 0$
M_{64}	$[K \sqrt{\alpha} \rho^{-3/2} + C \rho^{-1}]s + \alpha^{1/2} \rho^{-3/2} (K \ln \alpha + C \ln \rho)$	$\partial_s, 7X_1 + 4X_3, 7X_2 - 4X_3$	
M_{65}	$K(\alpha \rho^{-10/3})^\gamma \rho^q + \sqrt{\alpha} g(\rho)s$	$\partial_s, 2s\partial_s + \beta(3q - 4)X_2 + 2\beta X_3, 6X_1 + 2(\beta(2\gamma - 1) + 2)X_3 + (2\gamma + 9\beta q - 12\beta - 6q + 6)X_2$	$\beta(2\gamma - 1) \neq 0$ $q(q + 1) \neq 0$ $3\beta(q + 1) + 1 \neq 0$
M_{66}	$K(\alpha \rho^{-10/3})^\gamma + \sqrt{\alpha} g(\rho)s$	$\partial_s, s\partial_s - 2\beta X_2 + \beta X_3, 3X_1 + (\beta(2\gamma - 1) + 2)X_3 + (2\beta(1 - 2\gamma) + \gamma - 1)X_2$	$\beta(2\gamma - 1) \neq 0$ $\gamma K \neq 0$
M_{67}	$K(\alpha \rho^{-10/3})^\gamma \rho^{-1} + \sqrt{\alpha} g(\rho)s$	$\partial_s, 2s\partial_s - 7\beta X_2 + 2\beta X_3, 3X_1 + (\beta(2\gamma - 1) + 2)X_3 + (7\beta(1 - 2\gamma) + 2\gamma - 2)X_2$	$\beta(2\gamma - 1) \neq 0$ $K \neq 0$

Table 1. Continue

	$\varepsilon(\rho, \alpha, s)$	Extensions	Remarks
M_{68}	$K\alpha^\lambda \rho^{(8-10\lambda)/3} + C\sqrt{\alpha}\rho^\mu s$	$\partial_s, (2\lambda-1)(3\mu+1)s\partial_s -$ $3(3\mu+1)X_2 + 2(\lambda-9\mu-4)X_3,$ $(3\mu+1)X_2 + 2X_3$	$(3\mu+5) \neq 0$ $(3\mu+8) \neq 0$ $(3\mu+1) \neq 0$ $(2\lambda-1)K \neq 0$
M_{69}	$K\alpha^\lambda \rho^\beta + \sqrt{\alpha}g(\rho)s$	$\partial_s, 2(3\beta+10\lambda+3)s\partial_s +$ $(4-3\beta-10\lambda)X_2 - 6X_3,$ $X_1 + (4\lambda-3\beta-10)X_2 +$ $2(7-6\beta-22\lambda)X_3$	$3\beta \neq -10\lambda-3$ $(2\lambda-1)K \neq 0$
M_{70}	$\rho^{-1}[K\ln\alpha + C\ln\rho] + \sqrt{\alpha}g(\rho)s$	$\partial_s, 2s\partial_s - 7X_2 + 2X_3,$ $6X_1 + 5X_2 + 2X_3$	$KCg \neq 0$
M_{71}	$K\ln\alpha + C\ln\rho + \sqrt{\alpha}g(\rho)s$	$\partial_s, 3s\partial_s + 2X_2 - X_3,$ $9X_1 - 8X_2 + 7X_3$	$KCg \neq 0$
M_{72}	$\sqrt{\alpha}g(\rho)s$	$\partial_s, 2s\partial_s - X_2, 6X_1 - X_2 + 4X_3$	$g' \neq 0$
M_{73}	$[K\ln\alpha + C\ln\rho]\sqrt{\alpha}\rho^{-1/3} +$ $C_1s\sqrt{\alpha}\rho^{-1/3}$	$\partial_s, 6X_1 - X_2, X_3$	$C \neq 0$
M_{74}	$[K\ln\alpha + C\ln\rho]\sqrt{\alpha}\rho^{\mu+2/3} +$ $C_1\rho^\mu s\sqrt{\alpha}\rho^{\mu+2/3}$	$\partial_s, (6\mu-1)X_1 + 4\mu X_3,$ $(6\mu-1)X_2 + 4X_3$	$(6\mu-1)C \neq 0$
M_{75}	$[K\ln\alpha + C\rho^\beta s]\sqrt{\alpha}\rho^{\gamma\beta-1/2}$	$\partial_s, 6\beta s\partial_s + 3\beta\gamma X_2 + 2X_3,$ $6X_1 - X_2 + 4X_3$	$\beta C \neq 0$
M_{76}	$(K\ln\alpha + C\ln\rho)\sqrt{\alpha}\rho^{\mu-5/3} + C_1\rho^{-1}s$	$\partial_s, 6(\mu+1)s\partial_s + (3\mu-4)X_2 -$ $2X_3, 18(\mu+1)X_1 + (1-6\mu)X_2 +$ $2(6\mu+5)X_3$	$(\mu+1) \neq 0$
M_{77}	$(K\ln\alpha + C\ln\rho)\sqrt{\alpha}\rho^{-8/3} + C\rho^{-1}s$	$\partial_s, 7s\partial_s - 21X_1 - 13X_3, 7X_2 - 2X_3$	$K \neq 0$
M_{78}	$K\alpha^\lambda \rho^{\mu-10\lambda/3} + C\rho^{-1}s$	$\partial_s, 6(\mu+1)s\partial_s + (3\mu-4)X_2 - 2X_3,$ $9(\mu+1)X_1 + (7\lambda-3\mu-3)X_2 +$ $(2(3\mu+3-\lambda))X_3$	$(\mu+1) \neq 0$ $\lambda(2\lambda-1) \neq 0$
M_{79}	$\rho^{-1}[K_1\ln\alpha + K_2\ln\rho] + C\rho^{-1}s$	$\partial_s, 7X_1 + 4X_3, 7X_2 - 2X_3$	
M_{80}	$K\ln\alpha + C\ln\rho + C_1\rho^{-1}s$	$\partial_s, 3s\partial_s + 2X_2 - X_3,$ $3X_1 - X_2 + 2X_3$	
M_{81}	$Ce^{\beta s}\alpha^\lambda \rho^{\mu-10\lambda/3}$	$\beta X_3 + (3\mu-4)\partial_s, 3(3\mu-4)X_1 +$ $2(3\mu-\lambda-3)X_3, (3\mu-4)X_2 + 2X_3$	$\beta\lambda(3\mu-4) \neq 0$
M_{82}	$Ce^{\beta s}\alpha^\lambda \rho^{(4-10\lambda)/3}$	$\beta X_2 + 2\partial_s, 3X_1 + (\lambda-1)X_2, X_3$	$\beta\lambda \neq 0$