รายงานวิจัยฉบับสมบูรณ์

โครงการ ฟังก์ชันสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดและของกึ่งกรุปเชิงเดียวบริบูรณ์

โดย นายสายัญ ปันมา

มิ.ย. 2556

สัญญาเลขที่ MRG5480245

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ผู้วิจัย	**สังกัด**
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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัย และมหาวิทยาลัยเชียงใหม่

บทคัดย่อ

รหัสโครงการ : MRG5480245

ชื่อโครงการ : ฟังก์ชันสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดและของกึ่งกรุปเชิงเดียวบริบูรณ์

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บทคัดย่อ: ให้ S เป็นกึ่งกรุป และ $A \subseteq S$ และให้ $Cay(S,A)$ เป็นไดกราฟเคย์เลย์ของ S ที่สอดคล้องกับ A จะเรียก $Cay(S,A)$ ว่า ซีไอกราฟ (CI-graph) ถ้า สำหรับทุก ๆ $T \subseteq S$ ซึ่ง $Cay(S,A) \cong Cay(S,T)$ แล้ว $\alpha(A) = T$ สำหรับบาง $\alpha \in Aut(S)$ ในงานวิจัยนี้เราได้หากึ่งกรุปคลิฟฟอร์ด และกึ่งกรุปเชิงเดียวบริบูรณ์ที่ไดกราฟเคย์เลย์เป็นซีไอกราฟ

คำหลัก: ไดกราฟเคย์เลย์ , กึ่งกรุปคลิฟฟอร์ด, กึ่งกรุปเชิงเดียวบริบูรณ์, กรุปเชิงตั้งฉาก, ซีไอกราฟ

Abstract

Project Code : MRG5480245

Project Title : On isomorphisms of Cayley digraphs of Clifford semigroups and of completely simple semigroups

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Project Period : 2 years

Abstract: Let S be a semigroup, $A \subseteq S$ and $Cay(S, A)$ the Cayley digraph of S with respect to A. The digraph $Cay(S, A)$ is called a *CI-graph* of S if, for any $T \subseteq S$, $Cay(S, A) \cong Cay(S, T)$ implies $\alpha(A) = T$ for some $\alpha \in Aut(S)$. In this research, we determine the Clifford semigroups and Completely simple semigroups which their Cayley digraphs are CI-graphs.

Keywords : Cayley digraph, Clifford semigroup, Completely simple semigroup, rectangular group, CI-graph

1. ความสำคัญและที่มาของปัญหา

การศึกษาเกี่ยวกับ*ไดกราฟเคย์เลย์ (Cayley digraph)* และ *กราฟเคย์เลย์(Cayley graph)* มีมาแล้วกว่า 130 ปี ผู้ที่ให้นิยามและเริ่มศึกษาเป็นคนแรกคือ Prof. Arthur Cayley ในปี ค.ศ. 1878 โดยเริ่มแรกได้นิยาม ไดกราฟเคย์เลย์ และกราฟเคย์เลย์ มาจากกรุปดังนี้

ให้ G เป็นกรุป และ $A \subseteq G$ ไดกราฟเคย์เลย์ของ G ที่สอดคล้องกับ A คือ *ไดกราฟ (directed graph)* ที่มี G เป็น *เซตของจุด (vertex set)* และ $E = \{(g, ga) \mid g \in G, a \in A\}$ เป็น *เซตของเส้น (edge set)* จะเขียนแทน ไดกราฟเคย์เลย์ของ G ที่สอดคล้องกับ A ด้วยสัญลักษณ์ $Cay(G, A)$ และจะเรียก $Cay(G, A)$ สั้น ๆ ว่า ไดกราฟเคย์เลย์ ของกรุป G และ ถ้า $A = A^{-1} = \{a^{-1} \mid a \in A\}$ แล้วจะเรียก $Cay(G, A)$ ว่า กราฟเคย์เลย์ของกรุป G

จากนั้นได้มีการนำไดกราฟเคย์เลย์ของกรุปและกราฟเคย์เลย์ของกรุป ไปศึกษาอย่างกว้างขวาง เช่น ศึกษาลักษณะเฉพาะของไดกราฟที่เป็นไดกราฟเคย์เลย์ของกรุป ศึกษาลักษณะเฉพาะของกราฟที่เป็นกราฟเคย์เลย์ของกรุป ศึกษาลักษณะของคลาสของ digraph endomorphism ของไดกราฟเคย์เลย์ของกรุปทั้งหมด ศึกษาลักษณะของคลาสของ graph endomorphism ของกราฟเคย์เลย์ของกรุปทั้งหมด ศึกษาลักษณะของคลาสของ digraph automorphism ของไดกราฟเคย์เลย์ของกรุปทั้งหมด และศึกษาลักษณะของคลาสของ graph automorphism ของกราฟเคย์เลย์ของกรุปทั้งหมด เป็นต้น

ต่อมาได้มีผู้ให้ความสนใจอย่างแพร่หลาย และนำกราฟเคย์เลย์ของไปประยุกต์ใช้ในหลายแขนงวิชา เช่น Biology, Chemistry, Physics, Computer science

อาทิเช่น " Simulations between cellular automata on cayley graphs"

" Quantum expanders from any classical Cayley graph expander"

" Quantum walks on Cayley graphs"

เนื่องจากไดกราฟเคย์เลย์ของกรุปมีการศึกษาอย่างแพร่หลายแล้วจึงมีผู้สนใจที่จะขยายการศึกษาไปบนไดกราฟเคย์เลย์ของกึ่งกรุป ซึ่งนิยามของไดกราฟเคย์เลย์ของกึ่งกรุป จะนิยามเช่นเดียวกันกับนิยามของไดกราฟเคย์เลย์ของกรุป เพียงแต่เปลี่ยนพีชคณิตจากกรุป G ไปเป็นกึ่งกรุป S แทน

ไดกราฟเคย์เลย์ของกึ่งกรุปได้ถูกนำไปศึกษาอย่างกว้างขวางดูได้จาก [1],[4-5],[11-12],[14-16] ซึ่งการศึกษาในเอกสารอ้างอิงเหล่านี้ส่วนใหญ่ได้ขยายผลการศึกษามาจากไดกราฟเคย์เลย์ของกรุป

ด้วยเหตุนี้ผู้วิจัยจึงสนใจที่จะขยายงานวิจัยที่มีผู้ทำไว้บนไดกราฟเคย์เลย์ของกรุป ไปสู่ไดกราฟเคย์เลย์ของกึ่งกรุป ซึ่งเรื่องที่ผู้วิจัยสนใจคือ ปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุป นั่นคือ ปัญหาที่ว่าเมื่อไหร่ที่ไดกราฟเคย์เลย์ของกึ่งกรุปจะเป็น *ซีไอกราฟ (CI-graph)* โดยที่ ซีไอกราฟมีนิยามดังนี้

จะเรียก $Cay(S, A)$ ว่า ซีไอกราฟ ถ้าทุกๆ $T \subseteq S$ ซึ่ง $Cay(S, A) \cong Cay(S, T)$ มี $\alpha \in Aut(S)$ ซึ่ง $\alpha(A) = T$

สำหรับปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกรุป ได้มีผู้ศึกษาไว้แล้วดังนี้

ในปี ค.ศ.1998 Prof. Cai Heng Li [7] ได้สนใจปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกรุปที่เป็นไดกราฟต่อเนื่อง โดยได้หากรุป G ที่ทุกๆ ไดกราฟเคย์เลย์ของ G เป็น CI-graph ของ G

จากนั้นในปี ค.ศ. 2001 Prof. Cai Heng Li และ Prof. Sanming Zhou [8] ได้ศึกษาปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกรุปที่เป็น*ไดกราฟเคย์เลย์เล็กสุดเฉพาะกลุ่ม (minimal Cayley graph)*

โดยที่ ไดกราฟเคย์เลย์เล็กสุดเฉพาะกลุ่ม มีนิยามดังนี้ จะเรียก $Cay(S,A)$ ว่า *ไดกราฟเคย์เลย์เล็กสุดเฉพาะกลุ่ม*ถ้า $Cay(S,A)$ เป็นไดกราฟเชื่อมโยงและ $Cay(S,A\setminus\{a\})$ เป็นไดกราฟไม่เชื่อมโยงทุก $a\in A$
Prof. Cai Heng Li และ Prof. Sanming Zhou ได้ให้ลักษณะเฉพาะของกรุปสลับที่ G ที่ทุกไดกราฟเคย์เลย์เล็กสุดเฉพาะกลุ่ม เป็น CI-graph ของ G

ในปี ค.ศ. 2002 Prof. Cai Heng Li [9] ได้รวบรวมทฤษฎีที่เกี่ยวกับปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกรุปทั้งหมด ซึ่งผู้วิจัยได้ข้อสังเกตว่าปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุปยังไม่ได้มีการศึกษาดังนั้นผู้วิจัยจึงสนใจที่จะศึกษาปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุป

เนื่องจากกึ่งกรุปสามารถจำแนกได้หลายชนิด และ เพื่อที่จะทำให้สามารถเชื่อมโยงปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกรุป ไปสู่ปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุปได้ ผู้วิจัยจึงได้สนใจกึ่งกรุปที่นิยามมาจากกรุป กึ่งกรุปดังกล่าวคือ *กึ่งกรุปคลิฟฟอร์ด (Clifford semigroup)* และ *กึ่งกรุปเชิงเดียวบริบูรณ์ (completely simple semigroup)* ซึ่งมีนิยามดังนี้

ให้ Y เป็น *กึ่งแลตทิซ(semilattice)* และ $\{G_\alpha \mid \alpha\in Y\}$ เป็นวงศ์ของกรุป และทุกๆ $\alpha,\beta\in Y$ และ $\alpha\le\beta$ มีฟังก์ชันสาทิสสัณฐาน $f_{\alpha,\beta}:G_\alpha\to G_\beta$ ซึ่ง

(1) สำหรับทุกๆ $\alpha\in Y$ แล้ว $f_{\alpha,\alpha}=id_{G_\alpha}$ (id_{G_α} คือฟังก์ชันเอกลักษณ์บน G_α) และ

(2) สำหรับทุกๆ $\alpha,\beta,\gamma\in Y$ ซึ่ง $\alpha\le\beta\le\gamma$ แล้ว $f_{\beta,\alpha}f_{\gamma,\beta}=f_{\gamma,\alpha}$

กำหนด $S=\bigcup_{\alpha\in Y}G_\alpha$ และการดำเนินการบน S ดังนี้

สำหรับ $x\in G_\alpha$ และ $y\in G_\beta$ แล้ว $xy=f_{\alpha,\alpha\beta}(x)f_{\beta,\alpha\beta}(y)$

สามารถพิสูจน์ได้ไม่ยากว่า S กับการดำเนินการข้างต้นเป็นกึ่งกรุป
จะเรียกกึ่งกรุปนี้ว่า กึ่งกรุปคลิฟฟอร์ด หรือเรียกอีกอย่างหนึ่งว่า *กึ่งแลตทิซอย่างเข้มของกรุป (strong semilattice of groups)* เขียนแทนด้วยสัญลักษณ์ $[Y:G_\alpha,f_{\alpha,\beta}]$

ในปี ค.ศ. 2006 ผู้วิจัย Prof. U. Knauer Prof. N. Na Chiangmai และ Prof. Sr. Arworn [14] ได้ให้ลักษณะเฉพาะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด

<u>ข้อสังเกต 1.</u> ถ้า กึ่งแลตทิซ Y มีสมาชิกเพียงตัวเดียว แล้วกึ่งกรุปคลิฟฟอร์ดจะเป็นกรุป

ให้ G เป็น กรุป, I และ Λ เป็นเซตที่ไม่เป็นเซตว่าง
และ $P=\left[p_{\lambda_i}\right]$ เป็น $\Lambda\times I$ เมทริกซ์ ซึ่ง $p_{\lambda i}\in G$ ทุก $i\in I$ และ ทุก $\lambda\in\Lambda$
ให้ $S=G\times I\times\Lambda=\{(g,i,\lambda)\mid g\in G,i\in I,\lambda\in\Lambda\}$
และนิยามการดำเนินการบน S ดังนี้ $(g,i,\lambda)(h,j,\beta)=(gp_{\lambda j}h,i,\beta)$
สามารถพิสูจน์ได้ไม่อยากว่า S เป็นเซมิกรุปภายใต้การดำเนินการข้างต้น
จะเรียกกึ่งกรุปนี้ว่า กึ่งกรุปเชิงเดียวบริบูรณ์ เขียนแทนด้วยสัญลักษณ์ $\mathrm{M}(G,I,\Lambda;P)$

ในปี ค.ศ. 2006 ผู้วิจัย J. Meksawang และ Prof. U. Knauer [10] ได้ให้ลักษณะเฉพาะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์

<u>ข้อสังเกต 2.</u> ถ้า I และ Λ ต่างมีสมาชิกเพียงตัวเดียวและ P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วกึ่งกรุปเชิงเดียวบริบูรณ์จะเป็นกรุป

ให้ S เป็นกึ่งกรุป

1. จะเรียก S ว่ากึ่งกรุปศูนย์ขวา (right zero semigroup) ถ้า $xy = y$ ทุก ๆ $x, y \in S$ สำหรับกึ่งกรุปศูนย์ขวาที่มีสมาชิก n ตัว เราจะเขียนแทนด้วย R_n
2. จะเรียก S ว่ากึ่งกรุปศูนย์ซ้าย (left zero semigroup) ถ้า $xy = x$ ทุก ๆ $x, y \in S$ สำหรับกึ่งกรุปศูนย์ซ้ายที่มีสมาชิก n ตัว เราจะเขียนแทนด้วย L_n
3. จะเรียก S ว่ากรุปขวา (right group) ถ้า S เป็นผลคูณคาร์ทีเซียน (Cartesian product) ของกรุปและกึ่งกรุปศูนย์ขวา
4. จะเรียก S ว่ากรุปซ้าย (left group) ถ้า S เป็นผลคูณคาร์ทีเซียนของกรุปและกึ่งกรุปศูนย์ซ้าย
5. จะเรียก S ว่ากรุปเชิงตั้งฉาก (rectangular group) ถ้า S เป็นผลคูณคาร์ทีเซียนของกรุปและกึ่งกรุปศูนย์ขวาและกึ่งกรุปศูนย์ซ้าย

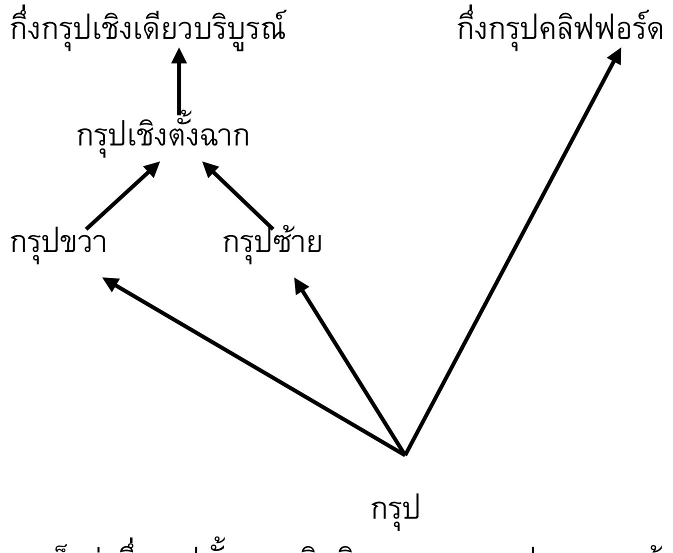
<u>ข้อสังเกต 3.</u> ให้ $S = \mathrm{M}(G, I, \Lambda; P)$ เป็นกึ่งกรุปเชิงเดียวบริบูรณ์

3.1 ถ้า P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วจะได้ว่า S เป็นกรุปเชิงตั้งฉาก

3.2 ถ้า I มีสมาชิกเพียงตัวเดียวและ P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วจะได้ว่า S เป็นกรุปขวา

3.2 ถ้า Λ มีสมาชิกเพียงตัวเดียวและ P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วจะได้ว่า S เป็นกรุปซ้าย

โดยนิยามของกึ่งกรุปคลิฟฟอร์ด และกึ่งกรุปเชิงเดียวบริบูรณ์ จะได้ว่ากึ่งกรุปทั้งสองไม่สามารถเปรียบเทียบกันได้ และยังได้อีกว่ากรุปเป็นทั้งกึ่งกรุปคลิฟฟอร์ดและกึ่งกรุปเชิงเดียวบริบูรณ์ ซึ่งเราสามารถวาดแผนภาพได้ดังนี้



จะเห็นว่ากึ่งกรุปทั้งสองชนิดนิยามมาจากกรุป และจากข้อสังเกต 1 และ 2 เรารู้ว่าเมื่อไหร่ที่ทั้งสองกึ่งกรุปดังกล่าวจะเป็นกรุป ดังนั้นเราจึงสามารถที่จะขยายทฤษฎีที่เกี่ยวกับปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกรุป ไปสู่ปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุปได้

ผู้วิจัยจึงสนใจที่จะศึกษา

1. ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็น ซีไอกราฟ

2. ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ที่เป็น ซีไอกราฟ

องค์ความรู้ใหม่ที่ได้จะทำให้ทราบคำตอบของปัญหาการสมสัณฐานของไดกราฟเคย์เลย์ของกึ่งกรุป ซึ่งยังไม่มีผู้ไม่มีผู้นำไปศึกษา

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2. วัตถุประสงค์งานวิจัย

2.1. หาลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็น ซีไอกราฟ

2.2. หาลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ที่เป็น ซีไอกราฟ

3. ระเบียบวิธีวิจัย

ปีที่ 1

1. รวบรวมความรู้พื้นฐานและงานวิจัยที่เกี่ยวข้องเกี่ยวกับไดกราฟเคล์เลย์ และกึ่งกรุปคลิฟฟอร์ด และกึ่งกรุปเชิงเดียวบริบูรณ์
2. หาลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็น ซีไอกราฟ
3. ส่งผลงานให้นักวิจัยที่ปรึกษาตรวจสอบและขอคำแนะนำเพื่อนำมาปรับปรุงงานวิจัย
4. จัดพิมพ์ และส่งงานวิจัยให้วารสารทางคณิตศาสตร์พิจารณาเพื่อตีพิมพ์

ปีที่ 2

1. หาลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ที่เป็น ซีไอกราฟ
2. ส่งผลงานให้นักวิจัยที่ปรึกษาตรวจสอบและขอคำแนะนำเพื่อนำมาปรับปรุงงานวิจัย
3. จัดพิมพ์ และส่งงานวิจัยให้วารสารทางคณิตศาสตร์พิจารณาเพื่อตีพิมพ์

4. ผลการวิจัย

4.1. ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็นซีไอกราฟ

ให้ Y เป็นกึ่งแลตทิซ จะเรียก Y ว่า *โซ่ (chain)* ถ้า สำหรับทุก ๆ $\alpha, \beta \in Y$ ได้ว่า $\alpha \le \beta$ หรือ $\beta \le \alpha$

ให้ G และ H เป็นกรุป และ f เป็นฟังก์ชันสาทิสสัณฐานจาก G ไป H จะเรียก f ว่า *การส่งศูนย์ (zero-mapping)* ถ้า $f(G) = \{e_H\}$ โดยที่ e_H เป็นสมาชิกเอกลักษณ์ใน H

ทฤษฎีบทต่อไปนี้เราได้หากึ่งกรุปคลิฟฟอร์ดที่ทุก ๆ ไดกราฟเคย์เลย์เป็น ซีไอกราฟ เราเรียกกึ่งกรุปดังกล่าวว่า *ซีไอกึ่งกรุป (CI-semigroup)*

ทฤษฎีบท 1 (Theorem 14 เอกสารภาคผนวก 6.1) ให้ $S = [Y : G_\alpha, f_{\alpha,\beta}]$ เป็นกึ่งกรุปคลิฟฟอร์ด จะได้ว่า ถ้า Y เป็นโซ่ และ G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $f_{\alpha,\beta}$ เป็นการส่งศูนย์ ทุก ๆ $\alpha, \beta \in Y$ แล้ว S เป็น ซีไอกึ่งกรุป

ทฤษฎีบท 2 (Corollary 15 เอกสารภาคผนวก 6.1) ให้ $S = [Y : G_\alpha, f_{\alpha,\beta}]$ เป็นกึ่งกรุปคลิฟฟอร์ด จะได้ว่า ถ้า Y เป็นโซ่ และ G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $p_\alpha \ne p_\beta$ ทุก ๆ $\alpha, \beta \in Y$ แล้ว S เป็น ซีไอกึ่งกรุป

เนื่องจากกึ่งกรุปคลิฟฟอร์ดไม่จำเป็นต้องเป็นซีไอกึ่งกรุป ดังนั้นเราจึงหาลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็นซีไอกราฟ ทฤษฎีบทต่อไปได้ให้ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็นซีไอกราฟ

ทฤษฎีบท 3 (Theorem 16 เอกสารภาคผนวก 6.1) ให้ $S=[Y:G_\alpha, f_{\alpha,\beta}]$ เป็นกึ่งกรุปคลิฟฟอร์ด จะได้ว่า ถ้า Y เป็นโซ่ และ G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $A \subseteq G_\rho$ บาง $\rho \in Y$ แล้ว $Cay(S,A)$ เป็น ซีไอกราฟ

ให้ $m \in Y$ จะเรียก m ว่า *สมาชิกมากที่สุด(maximum element)* ใน Y ถ้า $\alpha \le m$ ทุก ๆ $\alpha \in Y$ ทฤษฎีบทต่อไปได้ให้ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็นซีไอกราฟ โดยใช้สมบัติของสมาชิกมากที่สุดใน Y

ทฤษฎีบท 4 (Theorem 17 เอกสารภาคผนวก 6.1) ให้ $S=[Y:G_\alpha, f_{\alpha,\beta}]$เป็นกึ่งกรุปคลิฟฟอร์ด และ m เป็นสมาชิกมากที่สุดใน Y จะได้ว่า ถ้า G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $A \subseteq G_m$ แล้ว S เป็น ซีไอกราฟ

4.2. ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ที่เป็นซีไอกราฟ

ในหัวข้อนี้เราสนใจที่จะหาลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ที่เป็นซีไอกราฟ แต่เนื่องจากกึ่งกรุปเชิงเดียวบริบูรณ์มีอยู่หลายชนิด และเป็นการยากที่จะลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ใด ๆ ที่เป็นซีไอกราฟ ดังนั้นผู้วิจัยจึงได้เลือกกรุปซ้ายและกรุปขวาและกรุปเชิงตั้งฉาก ซึ่งทั้งสามต่างเป็นกึ่งกรุปเชิงเดียวบริบูรณ์

ทฤษฎีบทต่อไปได้ให้ลักษณะของไดกราฟเคย์เลย์ของกรุปซ้ายที่เป็นซีไอกราฟ โดยขึ้นอยู่กับความเป็นซีไอกราฟของไดกราฟเคย์เลย์ของกรุป และจำนวนสมาชิกของกึ่งกรุปศูนย์ซ้าย

ทฤษฎีบท 5 (Theorem 3.2 เอกสารภาคผนวก 6.2) ให้ $S=G\times L_n$ เป็นกรุปซ้าย และ $A \subseteq S$ จะได้ว่า $Cay(S,A)$เป็นซีไอกราฟ ก็ต่อเมื่อ $n=1$ และ $Cay(G,p_1(A))$ เป็นซีไอกราฟ โดยที่ $p_1(A)=\{g \mid (g,l)\in A\}$

ทฤษฎีบทต่อไปได้ให้ลักษณะของไดกราฟเคย์เลย์ของกรุปขวาที่เป็นซีไอกราฟ โดยได้แสดงว่า ทุกไดกราฟเคย์เลย์$Cay(S,A)$ ของกรุปขวาซึ่งเซต A ที่มีสมาชิกเพียงตัวเดียว เป็น ซีไอกราฟ

ทฤษฎีบท 6 (Theorem 3.5 เอกสารภาคผนวก 6.2) ให้ $S=G\times R_n$ เป็นกรุปขวา โดยที่ G เป็นกรุปวัฏจักร และ $R_n=\{r_1,r_2,...,r_n\}$ จะได้ว่า$Cay(S,\{(a,r_i)\})$ เป็น ซีไอกราฟ ทุก ๆ $(a,r_i)\in S$

ทฤษฎีบทต่อไปได้ให้ลักษณะของไดกราฟเคย์เลย์ของกรุปขวาที่เป็นซีไอกราฟ โดยได้แสดงว่า ไดกราฟเคย์เลย์$Cay(S,A)$ ของกรุปขวาซึ่งเซต $A\subseteq G\times\{r_i\}$ บาง $r_i \in R_n$ เป็น ซีไอกราฟ เมื่อ ไดกราฟเคย์เลย์ของกรุป G เป็น ซีไอกราฟ

ทฤษฎีบท 7 (Theorem 3.7 เอกสารภาคผนวก 6.2) ให้ $S = G \times R_n$ เป็นกรุปขวา โดยที่ $R_n = \{r_1, r_2, ..., r_n\}$ และให้ $A \subseteq G \times \{r_i\}$ บาง $r_i \in R_n$ จะได้ว่า $Cay(S, A)$ เป็นซีไอกราฟ ก็ต่อเมื่อ $Cay(G, p_1(A))$ เป็นซีไอกราฟ โดยที่ $p_1(A) = \{g \mid (g, r) \in A\}$

จากที่กล่าวไว้ข้างต้นว่ากึ่งกรุปเชิงเดียวบริบูรณ์มีอยู่หลายชนิด ผู้วิจัยจึงได้เลือกกรุปเชิงตั้งฉากเพื่อนำมาศึกษา เนื่องจากเรารู้ว่ากรุปเชิงตั้งฉากเป็นกึ่งกรุปเชิงเดียวบริบูรณ์ชนิดหนึ่ง และเรายังรู้อีกว่ากรุปซ้ายและกรุปขวาที่เราศึกษาไปข้างต้นต่างก็เป็นกรุปเชิงตั้งฉากด้วย แต่ก่อนที่เราจะหาลักษณะของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉากที่เป็นซีไอกราฟได้ เราจะต้องทราบเงื่อนไขของการสมสัณฐานกันของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉากก่อน ดังนั้น ผู้วิจัยจึงได้หาเงื่อนไขของการสมสัณฐานกันของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉาก ได้ทฤษฎีบทดังนี้

ให้ S เป็นกึ่งกรุป และ $A \subseteq S$ กำหนดสัญลักษณ์ $\langle A \rangle$ คือ กึ่งกรุปย่อยของ S ที่ก่อกำเนิดโดย A

ทฤษฎีบท 8 (Theorem 4.5 เอกสารภาคผนวก 6.3) ให้ $S = G \times L_n \times R_n$ เป็นกรุปเชิงตั้งฉาก และ $A, B \subseteq S$ จะได้ว่า $Cay(S, A) \cong Cay(S, B)$ ก็ต่อเมื่อ $Cay(\langle A' \rangle, A') \cong Cay(\langle B' \rangle, B')$ โดยที่ $A' = \{(g, r) \mid (g, l, r) \in A\}$ และ $B' = \{(h, t) \mid (h, l, t) \in B\}$

5. สรุปผลและอภิปรายผล

5.1. ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็นซีไอกราฟ

โดยทฤษฎีบท 1-4 จะได้ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดที่เป็นซีไอกราฟ ดังนี้
ให้ $S = [Y : G_\alpha, f_{\alpha,\beta}]$ เป็นกึ่งกรุปคลิฟฟอร์ด และ $A \subseteq S$
จะได้ว่า $Cay(S, A)$ เป็นซีไอกราฟ ถ้า

(i) Y เป็นโซ่ และ G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $f_{\alpha,\beta}$ เป็นการส่งศูนย์ ทุก ๆ $\alpha, \beta \in Y$ หรือ

(ii) Y เป็นโซ่ และ G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $p_\alpha \neq p_\beta$ ทุก ๆ $\alpha, \beta \in Y$ หรือ

(iii) Y เป็นโซ่ และ G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $A \subseteq G_\rho$ บาง $\rho \in Y$ หรือ

(iv) G_α เป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ p_α ทุก ๆ $\alpha \in Y$ และ $A \subseteq G_m$ โดยที่ m เป็นสมาชิกมากที่สุดใน Y

แต่ลักษณะเหล่านี้ไม่ได้เป็นเงื่อนไขที่เพียงพอของการเป็นซีไอกราฟของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด เนื่องจากกรุป G_α ไม่จำเป็นต้องเป็นกรุปวัฏจักรที่มีขนาดเป็นจำนวนเฉพาะ ทุก ๆ $\alpha \in Y$ และไม่จำเป็นต้องเป็นกรุปวัฏจักร ทุก ๆ $\alpha \in Y$ ด้วย ดังนั้นจึงมีปัญหาเปิดสำหรับเรื่องนี้อีกเป็นจำนวนมาก

ซึ่งผู้วิจัยจะได้ทำการศึกษาต่อไป

5.2. ลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ที่เป็นซีไอกราฟ

โดยทฤษฎีบท 5 เราได้ลักษณะของไดกราฟเคย์เลย์ของกรุปซ้ายที่เป็นซีไอกราฟ ดังนี้
ให้ $S = G \times L_n$ เป็นกรุปซ้าย และ $A \subseteq S$ จะได้ว่า
$Cay(S, A)$ เป็นซีไอกราฟ ก็ต่อเมื่อ $n = 1$ และ $Cay(G, p_1(A))$ เป็นซีไอกราฟ

จะเห็นว่าทฤษฎีบทนี้เราสามารถบอกลักษณะเฉพาะของของไดกราฟเคย์เลย์ของกรุปซ้ายที่เป็นซีไอกราฟ แต่ยังคงต้องอาศัยความเป็นซีไอกราฟของไดกราฟเคย์เลย์ของกรุปอยู่

โดยทฤษฎีบท 6 และ 7 เราได้ลักษณะของไดกราฟเคย์เลย์ของกรุปขวาที่เป็นซีไอกราฟ ดังนี้
ให้ $S = G \times R_n$ เป็นกรุปขวา โดยที่ $R_n = \{r_1, r_2, ..., r_n\}$ และให้ $A \subseteq S$
จะได้ว่า $Cay(S, A)$ เป็นซีไอกราฟ ถ้า
(i) G เป็นกรุปวัฏจักร และ A มีสมาชิก 1 ตัว หรือ
(ii) $A \subseteq G \times \{r_i\}$ บาง $r_i \in R_n$ และ $Cay(G, p_1(A))$ เป็นซีไอกราฟ

แต่ลักษณะข้างต้นยังไม่เพียงพอของการเป็นซีไอกราฟของไดกราฟเคย์เลย์ของกรุปขวา เนื่องจาก กรุป G ไม่จำเป็นต้องเป็นกรุปวัฏจักร และ A ไม่จำเป็นต้องมีสมาชิก 1 ตัว และ A ไม่จำเป็นต้องเป็นเซตย่อยของ $G \times \{r_i\}$ บาง $r_i \in R_n$ ดังนั้นจึงมีปัญหาเปิดสำหรับเรื่องนี้อยู่ซึ่งผู้วิจัยจะได้ทำการศึกษาต่อไป

โดยทฤษฎีบท 8 เราได้เงื่อนไขของการสมสัณฐานกันของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉาก ดังนี้
ให้ $S = G \times L_n \times R_n$ เป็นกรุปเชิงตั้งฉาก และ $A, B \subseteq S$ จะได้ว่า
$Cay(S, A) \cong Cay(S, B)$ ก็ต่อเมื่อ $Cay(\langle A' \rangle, A') \cong Cay(\langle B' \rangle, B')$ โดยที่
$A' = \{(g, r) \mid (g, l, r) \in A\}$ และ $B' = \{(h, t) \mid (h, l, t) \in B\}$

ทฤษฎีบทนี้ได้ให้เงื่อนไขที่จำเป็นและเพียงพอของการสมสัณฐานกันของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉาก ซึ่งจะเป็นเครื่องมือในการศึกษาลักษณะของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉากที่เป็นซีไอกราฟต่อไป

6. ภาคผนวก

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6.1 ผลงานวิจัยชื่อ On Cayley Isomorphisms of Clifford Semigroups
ตีพิมพ์ใน วารสาร International Journal of Pure and Applied Mathematics
Volume 79 No. 4 2012, 667-682.

6.2 ผลงานวิจัยชื่อ On Cayley Isomorphisms of Left and Right Groups
ตีพิมพ์ใน วารสาร International Journal of Pure and Applied Mathematics
Volume 80 No. 4 2012, 561-571.

6.3 ผลงานวิจัยชื่อ Isomorphism Conditions for Cayley Graphs of Rectangular groups
รอการตอบรับการตีพิมพ์จาก วารสาร Indian Journal of Pure and Applied Mathematics

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ON CAYLEY ISOMORPHISMS OF CLIFFORD SEMIGROUPS

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Abstract: In this paper, we investigate the problem about determining for which Cayley graphs of a given Clifford semigroup are CI-graphs. We give sufficient conditions for Cayley graphs of Clifford semigroups to be CI-graphs and for Clifford semigroups to be CI-semigroups.

AMS Subject Classification: 05C60, 15A66
Key Words: Cayley graph, digraph, Clifford semigroup, strong semilattice of groups, CI-graph

1. Introduction

Let S be a semigroup and let A be a subset of S. The *Cayley graph* $\mathrm{Cay}(S, A)$ of S relative to A is defined as the graph with the vertex set S and the arc set $E(\mathrm{Cay}(S, A))$ consisting of those ordered pairs (x, y) such that $xa = y$ for some $a \in A$. Clearly, if A is an empty set, then $\mathrm{Cay}(S, A)$ is an empty graph.

Arthur Cayley (1821-1895) introduced Cayley graphs of groups in 1878. One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups which possess a generating system such that the Cayley graph is planar, see [17].

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Cayley graphs of groups have been extensively studied and many interesting results have been obtained, see for examples [1], [2], [3], [5], [7], [8], [9], [10], and [11]. The Cayley graphs of semigroups have been considered by many authors. Many new interesting results on Cayley graphs of semigroups have appeared in various journals recently, see for examples [3], [4], [5], [6], [12], and [13]. In the investigation of the Cayley graphs of semigroups, the first of all interesting is finding the analogous of natural conditions which have been used in the group case.

A Cayley graph $\mathrm{Cay}(S, A)$ is called a *CI-graph* of S, CI stands for Cayley Isomorphism, if whenever B is a subset of S which $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, there exists an automorphism σ of S such that $\sigma(A) = B$. A semigroup S is called a *CI-semigroup* if all of its Cayley graphs are CI-graphs. The family of cyclic groups $\mathbb{Z}_p$, where p is prime, is the first known infinite family of CI-groups, see [11].

Necessary and sufficient conditions have been found for Cayley graphs of groups to be CI-graphs and for groups to be CI-groups, see for examples [9], [10], and [11]. After that it is natural to investigate Cayley graphs for semigroups which are unions of groups. A Clifford semigroup is such a union of groups. Here we investigate the conditions for Cayley graphs of Clifford semigroups enjoy the property of being CI-graphs and the conditions for Clifford semigroups enjoy the property of being CI-semigroups.

2. Basic Definitions and Results

All sets in this paper are assumed to be finite. Let S be a semigroup. The set $C(S) = \{c \in S \mid cs = sc \text{ for all } s \in S\}$ is called the *center* of S. The set of all idempotents of S is denoted by $E(S)$. An element $s \in S$ is called a *regular element* if $sxs = s$ for some $x \in S$. One calls S a *regular semigroup* if all of its elements are regular. A regular semigroup S is called a *Clifford semigroup* if $E(S) \subseteq C(S)$, i.e. idempotents of S commute with all elements of S.

If $(Y, \leq)$ is a nonempty partially ordered set such that the meet $a \wedge b$ of a and b exists for every a, b in Y, then we say that $(Y, \leq)$ is a *(lower) semilattice*. A semilattice Y is called a *chain* if, for all $x, y \in Y$, $x \leq y$ or $y \leq x$. Suppose that we have a semilattice Y and a set of groups G_α indexed by Y, and for all $\beta \leq \alpha$ in Y, there exists a group homomorphism $f_{\alpha,\beta} : G_\alpha \to G_\beta$ such that $f_{\alpha,\alpha} = id_{G_\alpha}$ is the identity mapping and for all α, β, γ with $\gamma \leq \beta \leq \alpha$, we have $f_{\beta,\gamma} f_{\alpha,\beta} = f_{\alpha,\gamma}$ where the multiplication on $S = \bigcup_{\alpha \in Y} G_\alpha$ is defined, for $x \in G_\alpha, y \in G_\beta$, by $xy = f_{\alpha,\alpha\wedge\beta}(x) f_{\beta,\alpha\wedge\beta}(y)$. It is easy to check that S is a semigroup, and called a *strong semilattice of groups.* We write $S = [Y; G_\alpha, f_{\alpha,\beta}]$. In 1941, A. H. Clifford proved that a semigroup is a Cilfford semigroup if and only if it is a strong semilattice of groups, see [16]. In the sequel, we will mainly use the term Cilfford semigroup instead of strong semilattice of groups.

The following proposition describes all automorphisms on Clifford semigroups $[Y; G_\alpha, f_{\alpha,\beta}]$.

Proposition 1. [15] *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. Let $\eta : Y \to Y$ be an automorphism, for each $\alpha \in Y$, let $\chi_\alpha : G_\alpha \to G_{\eta(\alpha)}$ be an isomorphism, and assume that for any $\beta \leq \alpha$, the diagram*

$$\begin{tikzcd} G_\alpha \arrow[r, "\chi_\alpha"] \arrow[d, "f_{\alpha,\beta}"'] & G_{\eta(\alpha)} \arrow[d, "f_{\eta(\alpha),\eta(\beta)}"] \\ G_\beta \arrow[r, "\chi_\beta"] & G_{\eta(\beta)} \end{tikzcd}$$

commutes. Define a mapping χ on S by $\chi(a) = \chi_\alpha(a)$ if $a \in G_\alpha$. Then χ is an automorphism on S. Conversely, every automorphism on S can be so constructed.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *(digraph) homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v))$ $\in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A (digraph) homomorphism $\varphi : (V, E) \to (V, E)$ is called an *(digraph) endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective (digraph) homomorphism and φ^{-1} is also a (digraph) homomorphism, then φ is called an *(digraph) isomorphism*, we write $(V_1, E_1) \cong (V_2, E_2)$ and say that (V_1, E_1) and (V_2, E_2) are isomorphic. An (digraph) isomorphism $\varphi : (V, E) \to (V, E)$ is called an *(digraph) automorphism.*

The following lemmas describe the structure of Cayley graphs of a given Clifford semigroup.

Lemma 2. [14] *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup and $A \subseteq S$. Let $x'_\alpha \in G_\alpha,\ y'_\beta \in G_\beta$. If (x'_α, y'_β) is an arc in $\mathrm{Cay}(S, A)$, then $\beta \leq \alpha$ and for each $x_\alpha \in G_\alpha$, there exists $y_\beta \in G_\beta$ such that (x_α, y_β) is an arc in $\mathrm{Cay}(S, A)$.*

Lemma 3. [14] *Let Y be a chain, $S = [Y; G_\alpha, f_{\alpha,\beta}]$ a Clifford semigroup and $A \subseteq S$. Then*

1. *the Cayley graph $\mathrm{Cay}(S, A)$ contains $|Y|$ disjoint induced subdigraphs $(G_\alpha, E_\alpha),\ \alpha \in Y$ where $(G_\alpha, E_\alpha) \cong \mathrm{Cay}(G_\alpha, A_\alpha)$ and $A_\alpha = \{f_{\gamma,\alpha}(a) \mid a \in A \cap G_\gamma,\ \alpha \leq \gamma\},\ \alpha \in Y$.*

2. *for* $\alpha \neq \beta$, $x_\alpha \in G_\alpha$, $y_\beta \in G_\beta$, (x_α, y_β) *is an arc in the Cayley graph* $\mathrm{Cay}(S, A)$ *if and only if* $\beta < \alpha$ *and* $y_\beta = f_{\alpha,\beta}(x_\alpha)a$ *for some* $a \in A \cap G_\beta$.

By Lemma 3(1) and the definition of an induced subdigraph, for each $x \in G_\alpha$, $|A_\alpha|$ is the number of s in G_α such that (s, x) is an arc in $\mathrm{Cay}(S, A)$. Let (V, E) be a digraph. Recall that a directed cycle of order n in (V, E) is a sequence of vertices $(x_1, x_2, ..., x_n, x_1)$ in V such that $(x_1, x_2), ..., (x_{n-1}, x_n)$, $(x_n, x_1) \in E$. Denote the identity element of a group G_α by e_α.

Now we prove several preparatory lemmas about the Cayley graphs of Clifford semigroups.

Lemma 4. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup and* $A \subseteq S$. *If* $(x_1, x_2, ..., x_n, x_1)$ *is a directed cycle of order* n *in* $\mathrm{Cay}(S, A)$, *then* $x_1, x_2, ..., x_n \in G_\alpha$ *for some* $\alpha \in Y$ *such that* $|G_\alpha| \geq n$. *Moreover,* $A_\alpha \setminus \{e_\alpha\} \neq \emptyset$ *where* $A_\alpha = \{f_{\gamma,\alpha}(a) \mid a \in A \cap G_\gamma,\ \gamma \geq \alpha\}$.

Proof. Suppose that $x_i \in G_{\alpha_i}$ for all $i = 1, ..., n$. By Lemma 2, $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n \geq \alpha_1$, that is, $\alpha_1 = \alpha_2 = ... = \alpha_n$. Thus $x_1, x_2, ..., x_n \in G_{\alpha_1}$. It follows immediately that $|G_{\alpha_1}| \geq n$. By Lemma 3(1), $(x_1, x_2, ..., x_n, x_1)$ is a directed cycle of order n in $\mathrm{Cay}(G_{\alpha_1}, A_{\alpha_1})$. Then there exists $a \in A_{\alpha_1}$ such that $x_1 a = x_2$ by the definition. Since $x_1 \neq x_2$, $a \neq e_{\alpha_1}$. Hence $A_{\alpha_1} \setminus \{e_{\alpha_1}\} \neq \emptyset$. $\square$

Given two groups G_α and G_β, the group homomorphism $f : G_\alpha \to G_\beta$ such that $f(g) = e_\beta$ for all $g \in G_\alpha$ is called a *zero-mapping*.

Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$ and $\rho \in Y$. Then we put

$$\begin{aligned} A_\rho &= \{f_{\gamma,\rho}(a) \mid a \in A \cap G_\gamma,\ \gamma \geq \rho\} \\ Y_\rho &= \{\gamma \in Y \mid \gamma > \rho\} \\ Y_\rho^0 &= \{\gamma \in Y_\rho \mid f_{\gamma,\rho} : G_\gamma \to G_\rho \text{ is a zero-mapping}\} \\ Y_\rho^1 &= \{\gamma \in Y_\rho \mid f_{\gamma,\rho} : G_\gamma \to G_\rho \text{ is an isomorphism}\}. \end{aligned}$$

The *indegree* $\overrightarrow{d}(x)$ of a vertex x of a digraph D is the number of vertices of D that end in x.

Lemma 5. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup and* $A \subseteq S$. *If* Y *is a chain and all groups* G_α *are cyclic groups of order prime* p_α, *then, for all* $x \in G_\alpha$,

$$\overrightarrow{d}(x) = \begin{cases} |A_\alpha| + |Y_\alpha^1||A \cap G_\alpha| + \sum_{\gamma \in Y_\alpha^0} |G_\gamma|, & \text{if } x \in A \\ |A_\alpha| + |Y_\alpha^1||A \cap G_\alpha|, & \text{if } x \notin A. \end{cases}$$

Proof. Let $x \in G_\alpha$. By the definition, $\overrightarrow{d}(x)$ is the number of s in S such that (s,x) is an arc in $\mathrm{Cay}(S,A)$. By Lemma 2, we get that all those s must belong to G_γ for some $\gamma \geq \alpha$. We denote by $\overrightarrow{d_*}(x)$, $\overrightarrow{d_1}(x)$, $\overrightarrow{d_0}(x)$ the number of s in G_α, $\bigcup_{\gamma \in Y_\alpha^1} G_\gamma$, $\bigcup_{\gamma \in Y_\alpha^0} G_\gamma$, respectively such that (s,x) is an arc in $\mathrm{Cay}(S,A)$. Since $\{\alpha\} \cup Y_\alpha^1 \cup Y_\alpha^0 = \{\gamma \mid \gamma \geq \alpha\}$ and $\{\alpha\}$, Y_α^1, Y_α^0 are pairwise disjoint, $\overrightarrow{d}(x) = \overrightarrow{d_*}(x) + \overrightarrow{d_1}(x) + \overrightarrow{d_0}(x)$. Let $\gamma \in Y$. Consider 3 cases:

Case1. $\gamma = \alpha$. By Lemma 3(1), $\overrightarrow{d_*}(x) = |A_\alpha|$.

Case2. $\gamma \in Y_\alpha^1$. Then $\gamma > \alpha$ and $f_{\gamma,\alpha}$ is an isomorphism. Let $s \in G_\gamma$. By Lemma 3(2), we get that (s,x) is an arc in $\mathrm{Cay}(S,A)$ if and only if $x = f_{\gamma,\alpha}(s)a$ for some $a \in A \cap G_\alpha$. Hence (s,x) is an arc in $\mathrm{Cay}(S,A)$ if and only if $s = f_{\gamma,\alpha}^{-1}(x) f_{\gamma,\alpha}^{-1}(a^{-1}) = f_{\gamma,\alpha}^{-1}(xa^{-1})$. Then the number of those s in G_γ is $|A \cap G_\alpha|$. Therefore, $\overrightarrow{d_1}(x) = |Y_\alpha^1||A \cap G_\alpha|$.

Case3. $\gamma \in Y_\alpha^0$. Then $\gamma > \alpha$ and $f_{\gamma,\alpha}$ is a zero-mapping. Let $s \in G_\gamma$. By Lemma 3(2), we get that (s,x) is an arc in $\mathrm{Cay}(S,A)$ if and only if $x = f_{\gamma,\alpha}(s)a = e_\alpha a = a$ for some $a \in A \cap G_\alpha$. Therefore, for $x \in G_\alpha$, (s,x) is an arc in $\mathrm{Cay}(S,A)$ if and only if $x \in A$. Then

$$\overrightarrow{d_0}(x) = \begin{cases} \sum_{\gamma \in Y_\alpha^0} |G_\gamma|, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

□

For the Clifford semigroup $S = [Y; G_\alpha, f_{\alpha,\beta}]$ which Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings, we have $Y_\alpha^0 = Y_\alpha$ and $Y_\alpha^1 = \emptyset$ for all $\alpha \in Y$.

Lemma 6. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. Let $A, B \subseteq S$ and $g : \mathrm{Cay}(S,A) \to \mathrm{Cay}(S,B)$ be a graph isomorphism. Then $g(G_\alpha) = G_\alpha$ for all $\alpha \in Y$ such that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$.*

Proof. Suppose that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$. Let $x \in G_\alpha$ and $a \in (A \cap G_\alpha) \setminus \{e_\alpha\}$, then $(x, xa, xa^2, ..., xa^{p_\alpha} = x)$ is a directed cycle of order $p_\alpha = |G_\alpha|$ in $\mathrm{Cay}(S,A)$ and $(g(x), g(xa), g(xa^2), ..., g(x))$ is also a directed cycle of order p_α in $\mathrm{Cay}(S,B)$. By Lemma 4, we have $g(x), g(xa), g(xa^2), ..., g(xa^{(p_\alpha - 1)}) \in G_\gamma$ for some $\gamma \in Y$ such that $|G_\gamma| = p_\gamma \geq p_\alpha$ and there exists $b \in B_\gamma \setminus \{e_\gamma\} = (B \cap G_\gamma) \setminus \{e_\gamma\}$. Thus $(g(x), g(x)b, g(x)b^2, ..., g(x)b^{p_\gamma} = g(x))$ is a directed cycle of order $p_\gamma = |G_\gamma|$ in $\mathrm{Cay}(S,B)$ and $(g^{-1}(g(x)) = x, g^{-1}(g(x)b), g^{-1}(g(x)b^2), ..., x)$ is also a directed cycle of order p_γ in $\mathrm{Cay}(S,A)$. Since $x \in G_\alpha$, by *Lemma 4*,

$x, g^{-1}(g(x)b), g^{-1}(g(x)b^2), ...$
$, g^{-1}(g(x)b^{(p_\gamma-1)}) \in G_\alpha$ and $p_\gamma \leq |G_\alpha| = p_\alpha \leq p_\gamma$, that is, $p_\alpha = p_\gamma$. Now we have $g(G_\alpha) = G_\gamma$. Hence the induced subdigraph with vertex set G_α in $\mathrm{Cay}(S, A)$ is isomorphic to the induced subdigraph with vertex set G_γ in $\mathrm{Cay}(S, B)$. By Lemma 3(1), we get that $\mathrm{Cay}(G_\alpha, A_\alpha) \cong \mathrm{Cay}(G_\gamma, B_\gamma)$ and thus $|A_\alpha| = |B_\gamma|$. Next we will show that $\alpha = \gamma$. If $\alpha \neq \gamma$, let we assume that $\alpha < \gamma$. By Lemma 5, in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}(a) = |A_\alpha| + \sum_{\rho\in Y_\alpha} |G_\rho| = |A_\alpha| + \sum_{\alpha<\rho<\gamma} |G_\rho| + |G_\gamma| + \sum_{\rho\in Y_\gamma} |G_\rho| \geq |A_\alpha| + |G_\gamma| + \sum_{\rho\in Y_\gamma} |G_\rho| > |A_\alpha| + \sum_{\rho\in Y_\gamma} |G_\rho| = |B_\gamma| + \sum_{\rho\in Y_\gamma} |G_\rho|$. By Lemma 5 again, in $\mathrm{Cay}(S, B)$, we have $|B_\gamma| + \sum_{\rho\in Y_\gamma} |G_\rho| \geq \overrightarrow{d}(w)$ for all $w \in G_\gamma$. Then $\overrightarrow{d}(a) > \overrightarrow{d}(w)$ for all $w \in G_\gamma$, this contradicts our assumption that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Therefore, $\alpha = \gamma$. □

Lemma 7. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. Let $A, B \subseteq S$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $\mathrm{Cay}(G_\alpha, A\cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B\cap G_\alpha)$ for all $\alpha \in Y$ such that $(A\cap G_\alpha)\setminus\{e_\alpha\} \neq \emptyset$.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Suppose that $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$. By Lemma 6, $g(G_\alpha) = G_\alpha$. We obtain that $\mathrm{Cay}(G_\alpha, A_\alpha) \cong \mathrm{Cay}(G_\alpha, B_\alpha)$ and $|A_\alpha| = |B_\alpha|$. It follows easily that $\mathrm{Cay}(G_\alpha, A_\alpha \setminus \{e_\alpha\}) \cong \mathrm{Cay}(G_\alpha, B_\alpha \setminus \{e_\alpha\})$. Since all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings, $A_\alpha \setminus \{e_\alpha\} = (A \cap G_\alpha) \setminus \{e_\alpha\}$ and $B_\alpha \setminus \{e_\alpha\} = (B \cap G_\alpha) \setminus \{e_\alpha\}$. Thus $\mathrm{Cay}(G_\alpha, (A \cap G_\alpha) \setminus \{e_\alpha\}) \cong \mathrm{Cay}(G_\alpha, (B \cap G_\alpha) \setminus \{e_\alpha\})$ and hence $|(A\cap G_\alpha)\setminus\{e_\alpha\}| = |(B\cap G_\alpha)\setminus\{e_\alpha\}|$. If α is not the maximum of Y, then $\sum_{\rho\in Y_\alpha} |G_\rho| \neq 0$. Let $m = |A_\alpha| = |B_\alpha|$ and $n = |A_\alpha| + \sum_{\rho\in Y_\alpha} |G_\rho|$, then $m \neq n$. By Lemma 5, for all $x \in G_\alpha \setminus A$, $\overrightarrow{d}(x) = m$ and for all $x \in A \cap G_\alpha$, $\overrightarrow{d}(x) = n$. The increasing sequence of indegree of all elements in G_α in $\mathrm{Cay}(S, A)$ is

$$(\overbrace{m, m, ..., m}^{|G_\alpha|-|(A\cap G_\alpha)\setminus\{e_\alpha\}|-1 \quad terms}, \overrightarrow{d}(e_\alpha), \overbrace{n, n, ..., n}^{|(A\cap G_\alpha)\setminus\{e_\alpha\}| \quad terms})$$

and the increasing sequence of indegree of all elements in G_α in $\mathrm{Cay}(S, B)$ is

$$(\overbrace{m, m, ..., m}^{|G_\alpha|-|(B\cap G_\alpha)\setminus\{e_\alpha\}|-1 \quad terms}, \overrightarrow{d}(e_\alpha), \overbrace{n, n, ..., n}^{|(B\cap G_\alpha)\setminus\{e_\alpha\}| \quad terms}).$$

Since $|(A\cap G_\alpha)\setminus\{e_\alpha\}| = |(B\cap G_\alpha)\setminus\{e_\alpha\}|$ and $m \neq n$, indegree of e_α in $\mathrm{Cay}(S, A)$ and in $\mathrm{Cay}(S, B)$ must be equal. Thus $e_\alpha \in A \cap G_\alpha$ if and only if $e_\alpha \in B \cap G_\alpha$. Hence $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. If α is the maximum of Y, then $A_\alpha = A\cap G_\alpha$ and $B_\alpha = B\cap G_\alpha$. Hence $\mathrm{Cay}(G_\alpha, A\cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B\cap G_\alpha)$. □

Lemma 8. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. Let $A, B \subseteq S$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$ for all $\alpha \in Y$ such that $A \cap G_\alpha \neq \emptyset$.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Suppose that $A \cap G_\alpha \neq \emptyset$, we need to prove only 2 cases:

Case1. $(A \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$. By Lemma 7, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$.

Case2. $A \cap G_\alpha = \{e_\alpha\}$. Clearly, $A_\alpha = \{e_\alpha\}$. If α is the maximum element of Y, then $(s, se_\alpha = s)$ is a loop in $\mathrm{Cay}(S, A)$ for all $s \in S$. It follows immediately that every vertex in $\mathrm{Cay}(S, B)$ has a loop. By Lemma 3(1), $e_\delta \in B_\delta$ for all $\delta \in Y$. Suppose that $B_\alpha = B \cap G_\alpha \neq \{e_\alpha\}$, then $(B \cap G_\alpha) \setminus \{e_\alpha\} \neq \emptyset$ and $|B \cap G_\alpha| > 1$. By Lemma 7, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. Thus $|A \cap G_\alpha| = |B \cap G_\alpha| > 1 = |\{e_\alpha\}|$, a contradiction. Hence $B \cap G_\alpha = \{e_\alpha\}$, that is, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. If α is not the maximum element of Y, then $Y_\alpha \neq \emptyset$ and thus $\sum_{\rho \in Y_\alpha} |G_\rho| \neq 0$. By Lemma 5, in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}(e_\alpha) = 1 + \sum_{\rho \in Y_\alpha} |G_\rho|$ and $\overrightarrow{d}(x) = 1$ for all $x \in G_\alpha \setminus \{e_\alpha\}$. Thus $g(e_\alpha) \in B \cap G_\delta \subseteq B_\delta$ for some $\delta \in Y$ such that $\sum_{\rho \in Y_\alpha} |G_\rho| = \sum_{\rho \in Y_\delta} |G_\rho|$ and $|B_\delta| = 1$. Hence $\alpha = \delta$ and $\{g(e_\alpha)\} = B \cap G_\alpha = B_\alpha$. Since $A \cap G_\alpha = \{e_\alpha\}$, (e_α, e_α) is a loop in $\mathrm{Cay}(S, A)$. Obviously, $(g(e_\alpha), g(e_\alpha))$ is a loop in $\mathrm{Cay}(S, B)$. Thus $e_\alpha \in B_\alpha$, that is, $\{e_\alpha\} = \{g(e_\alpha)\} = B \cap G_\alpha = B_\alpha$. Therefore, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$. □

From now on, N_0^H denotes the number of vertices u in a digraph H such that $\overrightarrow{d}(u) = 0$. Clearly, for two given digraphs H and T, if $H \cong T$, then $N_0^H = N_0^T$.

The next lemma is proved on the Clifford semigroup $S = [Y; G_\alpha, f_{\alpha,\beta}]$ which Y is a semilattice.

Lemma 9. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, ω the minimum element of Y. If $A \subseteq G_\omega$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $B \subseteq G_\omega$ and $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. If $A = \emptyset$, then $\mathrm{Cay}(S, A)$ is an empty graph. Obviously, $\mathrm{Cay}(S, B)$ is also an empty graph, that is, $B = \emptyset \subseteq G_\omega$. Hence $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$. Let $A \neq \emptyset$. Then for all $s \in S$ and $a \in A$, $sa \in G_\omega$. Thus in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}(s) = 0$ for all $s \in S \setminus G_\omega$. By Lemma 5, $\overrightarrow{d}(x) \geq |A_\omega| = |A| > 0$ for all $x \in G_\omega$. Hence $N_0^{\mathrm{Cay}(S,A)} = |S \setminus G_\omega|$. Suppose that $B \nsubseteq G_\omega$. Then there exists $b \in B \cap G_\beta$ for some $\beta > \omega$. Then

$B_\alpha \neq \emptyset$ for all $\alpha \leq \beta$. By Lemma 5 again, $\overrightarrow{d}(x) \geq |B_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \beta$. Hence $N_0^{\mathrm{Cay}(S,B)} \leq |S \setminus \bigcup_{\alpha \leq \beta} G_\alpha| \leq |S \setminus \{G_\beta, G_\omega\}| < |S \setminus G_\omega| = N_0^{\mathrm{Cay}(S,A)}$, this contradicts our assumption that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$. Now we claim that $B \subseteq G_\omega$. As in $\mathrm{Cay}(S,A)$, we can see that , in $\mathrm{Cay}(S,B)$, $\overrightarrow{d}(s) = 0$ for all $s \in S \setminus G_\omega$ and $\overrightarrow{d}(x) \geq |A_\omega| = |A| > 0$ for all $x \in G_\omega$, so $g(G_\omega) = G_\omega$. By Lemma 3(1), $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$. $\square$

Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$ and $x, y \in G_\alpha$, $z \in G_\gamma$, $\alpha \neq \gamma$. Suppose that (x,y) and (z,y) are arcs in $\mathrm{Cay}(S,A)$. By Lemma 3(1), (x,y) is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$. Then there exists $a \in A_\alpha$ such that $y = xa$. If $a = e_\alpha$, then $y = x$ and thus (x,y) is a loop. If $a \neq e_\alpha$, then $(x, xa = y, xa^2, ..., xa^{|a|} = x)$ is a directed cycle of order $|a|$ in $\mathrm{Cay}(S,A)$. Thus (x,y) is either a loop or an arc which is contained in a directed cycle. It is easily seen that $y \neq z$, so (z,y) is not a loop. By Lemma 4, (z,y) is not contained in any directed cycles. By Lemma 3(2), $\gamma > \alpha$, that is, $\gamma \in Y_\alpha$. Let us denote by $\overrightarrow{d}_{**}(y)$ the number of vertices in $\bigcup_{\rho \in Y_\alpha} G_\rho$ that end in y and $N_{\neq 0}^{**\mathrm{Cay}(S,A)}$ the number of vertices u in $\mathrm{Cay}(S,A)$ that $\overrightarrow{d}_{**}(u) \neq 0$.

Given two subsets A, B of S. Clearly, for a graph isomorphism $g : \mathrm{Cay}(S,A) \to \mathrm{Cay}(S,B)$, it is not only $\overrightarrow{d}(y) = \overrightarrow{d}(g(y))$, but also $\overrightarrow{d}_{**}(y) = \overrightarrow{d}_{**}(g(y))$. Moreover, $N_{\neq 0}^{**\mathrm{Cay}(S,A)} = N_{\neq 0}^{**\mathrm{Cay}(S,B)}$.

Analysis similar to the proof of Lemma 5 shows that the following lemma is hold.

Lemma 10. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup and $A \subseteq S$. If Y is a chain and all groups G_α are cyclic groups of order prime p_α, then, for all $x \in G_\alpha$,*

$$\overrightarrow{d}_{**}(x) = \begin{cases} |Y_\alpha^1||A \cap G_\alpha| + \sum_{\gamma \in Y_\alpha^0} |G_\gamma|, & \text{if } x \in A \\ |Y_\alpha^1||A \cap G_\alpha|, & \text{if } x \notin A. \end{cases}$$

Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup which there exists the maximum element π in a semilattice Y. Let $A \subseteq G_\pi$ and $\rho < \pi$. Suppose that $f_{\pi,\rho}$ is a zero-mapping, then $A_\rho = \{e_\rho\}$. Thus the arc set $E(\mathrm{Cay}(G_\rho, A_\rho)) = \{(x,x) | x \in G_\rho\}$. Suppose that $f_{\pi,\rho}$ is an isomorphism. Define a mapping $\Pi_\rho : \mathrm{Cay}(G_\pi, A) \to \mathrm{Cay}(G_\rho, A_\rho)$ by $\Pi_\rho(x) = f_{\pi,\rho}(x)$ for all $x \in G_\pi$. It is clear that Π_ρ is a bijective. Let (x,y) be an arc in $\mathrm{Cay}(G_\pi, A)$, then there exists $a \in A$ such that $y = xa$. Since $f_{\pi,\rho}$ is an isomorphism, $\Pi_\rho(y) = \Pi_\rho(xa) = f_{\pi,\rho}(xa) = f_{\pi,\rho}(x) f_{\pi,\rho}(a) = \Pi_\rho(x)\Pi_\rho(a)$. Becauce $\Pi_\rho(a) = f_{\pi,\rho}(a) \in A_\rho$, $(\Pi_\rho(x), \Pi_\rho(y))$ is an arc in $\mathrm{Cay}(G_\rho, A_\rho)$. Let (z,w) be an arc in $\mathrm{Cay}(G_\rho, A_\rho)$, then there exists $a_\rho \in A_\rho = f_{\pi,\rho}(A)$ such that $w = za_\rho$. Since $f_{\pi,\rho}$ is an isomorphism, there exists unique $a_\pi \in A$ such that $f_{\pi,\rho}(a_\pi) = a_\rho$. Thus $\Pi_\rho^{-1}(w) = \Pi_\rho^{-1}(za_\rho) = f_{\pi,\rho}^{-1}(za_\rho) = f_{\pi,\rho}^{-1}(z)f_{\pi,\rho}^{-1}(a_\rho) = \Pi_\rho^{-1}(z)a_\pi$. Hence $(\Pi_\rho^{-1}(z), \Pi_\rho^{-1}(w))$ is an arc in $\mathrm{Cay}(G_\pi, A)$. Therefore, Π_ρ is a graph isomorphism. We thus get $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\rho, A_\rho)$ for all $\rho < \pi$ such that $f_{\pi,\rho}$ is an isomorphism.

Lemma 11. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, π the maximum element of Y. If $A \subseteq G_\pi$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $B \subseteq G_\pi$ and $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$.*

Proof. Let $g : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ be a graph isomorphism. Since π be the maximum element of Y, $Y_\pi^1 = Y_\pi^0 = \emptyset$. If $A \subseteq G_\pi$, then $A \cap G_\alpha = \emptyset$ for all $\alpha < \pi$. By Lemma 10, in $\mathrm{Cay}(S, A)$, $\overrightarrow{d}_{**}(x) = |Y_\alpha^1||A \cap G_\alpha| = 0$ for all $x \in G_\alpha, \alpha < \pi$ and $\overrightarrow{d}_{**}(x) \leq |Y_\pi^1||A \cap G_\pi| + \sum_{\gamma \in Y_\pi^0} |G_\gamma| = 0$ for all $x \in G_\pi$. Thus $\overrightarrow{d}_{**}(x) = 0$ for all $x \in \mathrm{Cay}(S, A)$. We must have $\overrightarrow{d}_{**}(x) = 0$ for all $x \in \mathrm{Cay}(S, B)$. Suppose that $B \nsubseteq G_\pi$, so there exists $b \in B \cap G_\beta$ for some $\beta < \pi$. We have $f_{\pi,\beta}(e_\pi)b = e_\beta b = b$. By Lemma 3(2), (e_π, b) is an arc in $\mathrm{Cay}(S, B)$, that is, $\overrightarrow{d}_{**}(b) \neq 0$, this contradicts our assumption that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Thus $B \subseteq G_\pi$.

Now we want to show that $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$. If $A = \{e_\pi\}$, then $A_\alpha = \{e_\alpha\}$ for all $\alpha \in Y$. Thus the arc set $E(\mathrm{Cay}(S, A))$ $= \{(x, x)|x \in S\}$. Of course, $E(\mathrm{Cay}(S, B)) = \{(x, x)|x \in S\}$. Hence $B = \{e_\pi\}$. Therefore, $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$. If $A \setminus \{e_\pi\} \neq \emptyset$, then there exists $a \in A \setminus \{e_\pi\}$. Let $x \in G_\pi$, then $(x, xa, xa^2, ..., xa^{p_\pi} = x)$ is a directed cycle of order $p_\pi = |G_\pi|$ in $\mathrm{Cay}(S, A)$. Obviously, $g(x) \notin G_\alpha$ where $f_{\pi,\alpha}$ is a zero-mapping. We have $(g(x), g(xa), g(xa^2), ..., g(x))$ is also directed cycle of order p_π in $\mathrm{Cay}(S, B)$. By Lemma 4, $g(x), g(xa), g(xa^2), ..., g(xa^{(p_\pi - 1)}) \in G_\gamma$ for some $\gamma \in Y$ such that $|G_\gamma| = p_\gamma \geq p_\pi$. If $p_\gamma \neq p_\pi$, then $f_{\pi,\gamma}$ is a zero-mapping. Thus $g(G_\pi) \subseteq G_\rho$ for some G_ρ such that $|G_\rho| = p_\rho = p_\pi$ and $f_{\pi,\rho}$ is an isomorphism. Hence $g(G_\pi) = G_\rho$ for some G_ρ such that $f_{\pi,\rho}$ is an isomorphism, that is, $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\rho, B_\rho) \cong \mathrm{Cay}(G_\pi, B)$. □

Lemma 12. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup where Y is a chain, all groups G_α are cyclic groups of order prime p_α. If $A \subseteq G_\rho$ for some $\rho \in Y$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, then $B \subseteq G_\rho$ and $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay}G_\rho, B)$.*

Proof. Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. If $A = \emptyset$, then $\mathrm{Cay}(S, A)$ is an empty graph. Obviously, $\mathrm{Cay}(S, B)$ is also an empty graph and $B = \emptyset \subseteq G_\rho$. If ρ is the maximum element of Y, then, by Lemma 11, $B \subseteq G_\rho$ and $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay}G_\rho, B)$. Let $A \neq \emptyset$ and ρ is not the maximum element of Y. We claim that

$A \cap G_\alpha = \emptyset$ for all $\alpha \neq \rho$, $A_\alpha \neq \emptyset$ for all $\alpha \leq \rho$ and $A_\alpha = \emptyset$ for all $\alpha > \rho$. By Lemma 5, in $\mathrm{Cay}(S,A)$, $\overrightarrow{d}(x) \geq |A_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \rho$ and $\overrightarrow{d}(y) = |A_\gamma| + |Y_\gamma^1||A \cap G_\gamma| = 0$ for all $y \in G_\gamma, \gamma > \rho$. Thus $N_0^{\mathrm{Cay}(S,A)} = \sum_{\gamma \in Y_\rho} |G_\gamma|$. By Lemma 10, in $\mathrm{Cay}(S,A)$, $\overrightarrow{d}_{**}(y) = |Y_\gamma^1||A \cap G_\gamma| = 0$ for all $y \in G_\gamma, \gamma \neq \rho$, that is, $N_{\neq 0}^{**\mathrm{Cay}(S,A)} \leq |G_\rho|$. Moreover, $\overrightarrow{d}_{**}(x) \leq |Y_\rho^1||A| + \sum_{\gamma \in Y_\rho^0} |G_\gamma|$ for all $x \in G_\rho$. Thus

$$\text{in } \mathrm{Cay}(S,A), \overrightarrow{d}_{**}(s) \leq |Y_\rho^1||A| + \sum_{\gamma \in Y_\rho^0} |G_\gamma| \text{ for all } s \in S. \tag{1}$$

If $B \not\subseteq G_\rho$, then we need to consider 4 cases:

Case1. There exists $b \in B \cap G_\beta$ for some $\beta > \rho$. then $B_\alpha \neq \emptyset$ for all $\alpha \leq \beta$. By Lemma 5, in $\mathrm{Cay}(S,B)$, $\overrightarrow{d}(x) \geq |B_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \beta$. Thus $N_0^{\mathrm{Cay}(S,B)} \leq \sum_{\alpha > \beta} |G_\alpha| < \sum_{\alpha > \beta} |G_\alpha| + |G_\beta| \leq \sum_{\alpha \in Y_\rho} |G_\alpha| = N_0^{\mathrm{Cay}(S,A)}$, this contradicts our assumption that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$.

Case2. $B \cap G_\rho = \emptyset$. By case1, $B \cap G_\gamma = \emptyset$ for all $\gamma > \rho$. Let $x \in G_\gamma$ for some $\gamma \geq \rho$ and $b \in B \cap G_\alpha$, then $\alpha < \rho$ and $xb \in G_\alpha$. Thus, in $\mathrm{Cay}(S,B)$, $\overrightarrow{d}(x) = 0$ for all $x \in \bigcup_{\gamma \geq \rho} G\gamma$, that is, $N_0^{\mathrm{Cay}(S,B)} \geq \sum_{\gamma \geq \rho} |G_\gamma| = \sum_{\gamma \in Y_\rho} |G_\gamma| + |G_\rho| > \sum_{\gamma \in Y_\rho} |G_\gamma| = N_0^{\mathrm{Cay}(S,A)}$, this contradicts our assumption that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$.

Case3. There exists $b \in B \cap G_\beta$ for some $\beta < \rho$ where $f_{\rho,\beta}$ is a zero-mapping. Then $f_{\alpha,\beta}$ is s zero-mapping for all $\alpha \in Y_\rho$, that is, $(Y_\rho \cup \{\rho\}) \subseteq Y_\beta^0$. By Lemma 10, in $\mathrm{Cay}(S,B)$, $\overrightarrow{d}_{**}(b) = |Y_\beta^1||B \cap G_\beta| + \sum_{\gamma \in Y_\beta^0} |G_\gamma| \geq \sum_{\gamma \in Y_\beta^0} |G_\gamma| \geq \sum_{\gamma \in Y_\rho} |G_\gamma| + |G_\rho| > \sum_{\gamma \in Y_\rho} |G_\gamma| = \sum_{\gamma \in Y_\rho^1} |G_\gamma| + \sum_{\gamma \in Y_\rho^0} |G_\gamma| = |Y_\rho^1||G_\rho| + \sum_{\gamma \in Y_\rho^0} |G_\gamma| \geq |Y_\rho^1||A| + \sum_{\gamma \in Y_\rho^0} |G_\gamma|$. By (1), we have $\overrightarrow{d}_{**}(b) \neq \overrightarrow{d}_{**}(s)$ for all s in $\mathrm{Cay}(S,A)$, this contradicts our assumption that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$.

Case4. There exists $b \in B \cap G_\beta$ for some $\beta < \rho$ where $f_{\rho,\beta}$ is an isomorphism. Obviously, $B \cap G_\beta \neq \emptyset, Y_\beta^1 \neq \emptyset$ and $|G_\beta| = |G_\rho|$. We have $Y_\rho^0 \subseteq Y_\beta^0, Y_\rho^1 \subseteq Y_\beta^1$ and $f_{\alpha,\beta}$ is an isomorphism for all $\beta < \alpha \leq \rho$. Thus $Y_\rho^0 = Y_\beta^0$ and $(Y_\rho^1 \cup \{\rho\}) \subseteq Y_\beta^1$ By Lemma 10, in $\mathrm{Cay}(S,B)$, $\overrightarrow{d}_{**}(x) \geq |Y_\beta^1||B \cap G_\beta| > 0$ for all $x \in G_\beta$. By case2, there exists $b_1 \in B \cap G_\rho$. There exists $e_\gamma \in G_\gamma, \gamma > \rho$ such that $(e_\gamma, e_\gamma b_1 = f_{\gamma,\rho}(e_\gamma) b_1 = e_\rho b_1 = b_1)$ is an arc in $\mathrm{Cay}(S,B)$, so $\overrightarrow{d}_{**}(b_1) \neq 0$. Hence $N_{\neq 0}^{**\mathrm{Cay}(S,B)} \geq |G_\beta| + 1 > |G_\beta| = |G_\rho| \geq N_{\neq 0}^{**\mathrm{Cay}(S,A)}$, this contradicts our assumption that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$.

These 4 cases give $B \subseteq G_\rho$. As in $\mathrm{Cay}(S, A)$, Lemma 5 gives, in $\mathrm{Cay}(S, B)$, $\overrightarrow{d}(x) \geq |B_\alpha| > 0$ for all $x \in G_\alpha, \alpha \leq \rho$ and $\overrightarrow{d}(y) = |B_\gamma| + |Y_\gamma^1||B \cap G_\gamma| = 0$ for all $y \in G_\gamma, \gamma > \rho$. Thus $g(\bigcup_{\alpha\leq\rho} G_\alpha) = \bigcup_{\alpha\leq\rho} G_\alpha$, that is, $\mathrm{Cay}(\bigcup_{\alpha\leq\rho} G_\alpha, A) \cong \mathrm{Cay}(\bigcup_{\alpha\leq\rho} G_\alpha, B)$. Since ρ is a maximum element of $\{\alpha|\alpha \leq \rho\}$, analysis similar to that in the proof of Theorem 11 shows that $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay} G_\rho, B)$. $\square$

3. Main Results

We first give an example of a Cayley graph of a Clifford semigroup which is not a CI-graph and that Clifford semigroup is also not a CI-semigroup.

Example 13. Let Y be a semilattice $\{\alpha, \beta, \gamma\}$ such that $\alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma = \gamma$. Let $G_\alpha = \mathbb{Z}_2 = \{\overline{0}_\alpha, \overline{1}_\alpha\}$, $G_\beta = \mathbb{Z}_3 = \{\overline{0}_\beta, \overline{1}_\beta, \overline{2}_\beta\}$, $G_\gamma = \mathbb{Z}_5 = \{\overline{0}_\gamma, \overline{1}_\gamma, \overline{2}_\gamma, \overline{3}_\gamma, \overline{4}_\gamma\}$ and let $f_{\alpha,\gamma}$, $f_{\beta,\gamma}$ be zero-mappings, i.e. $f_{\alpha,\gamma}(G_\alpha) = f_{\beta,\gamma}(G_\beta) = \{\overline{0}_\gamma\}$. Then $S = [Y; G_\alpha, f_{\alpha,\beta}]$ is a Clifford semigroup (see Figure 1).

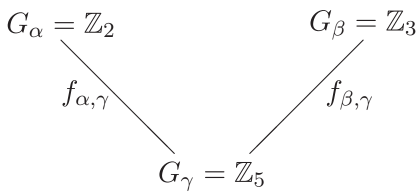

Figure 1: $S = [Y; G_\alpha, f_{\alpha,\beta}]$

Consider two subsets $A = \{\overline{1}_\alpha, \overline{1}_\beta\}$ and $B = \{\overline{1}_\alpha, \overline{1}_\beta, \overline{0}_\gamma\}$ of S. Then $\mathrm{Cay}(S, A) = \mathrm{Cay}(S, B)$ (see Figure 2).

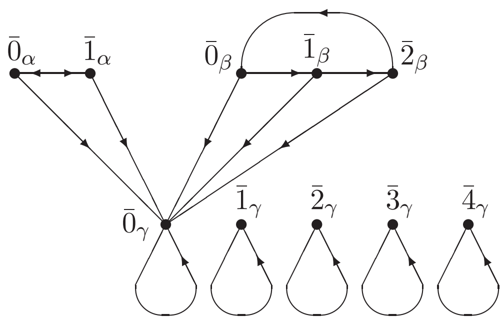

Figure 2: $\mathrm{Cay}(S, A) = \mathrm{Cay}(S, B)$

Since $|A| \neq |B|$, there is no $\sigma \in \mathrm{Aut}(S)$ such that $\sigma(A) = B$. Therefore, $\mathrm{Cay}(S, A)$ and $\mathrm{Cay}(S, B)$ are not CI-graphs, and S is not a CI-semigroup.

Here we investigate the conditions for Clifford semigroups enjoy the property of being CI-semigroups.

Theorem 14. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. If Y is a chain, all groups G_α are cyclic groups of order prime p_α and all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings, then S is a CI-semigroup.*

Proof. Let $A \subseteq S$. Suppose that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 8, $\mathrm{Cay}(G_\alpha, A \cap G_\alpha) \cong \mathrm{Cay}(G_\alpha, B \cap G_\alpha)$ for all $\alpha \in Y$ such that $A \cap G_\alpha \neq \emptyset$. Since $G_\alpha = \mathbb{Z}_{p_\alpha}$ is a CI-group for all $\alpha \in Y$, there exists $g_\alpha \in \mathrm{Aut}(G_\alpha)$ such that $g_\alpha(A \cap G_\alpha) = B \cap G_\alpha$ for all $\alpha \in Y$ such that $A \cap G_\alpha \neq \emptyset$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be an identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha = \begin{cases} g_\alpha, & \text{if } A \cap G_\alpha \neq \emptyset \\ id_{G_\alpha}, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\alpha$. Clearly, $\chi(A) = B$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. Hence $f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta}(\chi_\alpha(x)) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. $\square$

Corollary 15. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. If Y is a chain, all groups G_α are different cyclic groups of order prime p_α, then S is a CI-semigroup.*

Proof. Since all groups G_α are different cyclic groups of order prime p_α, all group homomorphisms $f_{\alpha,\beta}$ are zero-mappings. By Theorem 14, S is a CI-semigroup. $\square$

Now we investigate the conditions for Cayley graphs of Clifford semigroups enjoy the property of being CI-graphs.

Theorem 16. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup. If Y is a chain, all groups G_α are cyclic groups of order prime p_α and $A \subseteq G_\rho$ for some $\rho \in Y$, then $\mathrm{Cay}(S, A)$ is a CI-graph.*

Proof. Let $A \subseteq G_\rho \subseteq S$. Suppose that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 12, $B \subseteq G_\rho$ and $\mathrm{Cay}(G_\rho, A) \cong \mathrm{Cay}(G_\rho, B)$. Since $G_\rho = \mathbb{Z}_{p_\rho}$ is a CI-group, there exists $g_\rho \in \mathrm{Aut}(G_\rho)$ s.t. $g_\rho(A) = B$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be the identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha(x) = \begin{cases} f_{\rho,\alpha} g_\rho f_{\rho,\alpha}^{-1}(x), & \text{if } \alpha \leq \rho \text{ and } f_{\rho,\alpha} \text{ is an isomorphism} \\ f_{\alpha,\rho}^{-1} g_\rho f_{\alpha,\rho}(x), & \text{if } \rho < \alpha \text{ and } f_{\alpha,\rho} \text{ is an isomorphism} \\ x, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\alpha$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. If $\alpha = \beta$, then it is easily seen that $f_{\alpha,\beta}\chi_\alpha(x) = \chi_\beta f_{\alpha,\beta}(x)$. If $\beta < \alpha$ and $f_{\alpha,\beta}$ is a zero-mapping, then $f_{\alpha,\beta}\chi_\alpha(x) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. For $\beta < \alpha$ and $f_{\alpha,\beta}$ is an isomorphism, we need to prove 5 cases:

Case1. $\beta < \alpha \leq \rho$ and $f_{\rho,\alpha}$ is an isomorphism. Then $f_{\rho,\beta}$ is an isomorphism. Thus $f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta} f_{\rho,\alpha} g_\rho f_{\rho,\alpha}^{-1}(x) = f_{\rho,\beta} g_\rho f_{\rho,\alpha}^{-1}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\rho,\beta}^{-1} f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\rho,\alpha}^{-1}(x)$.

Case2. $\beta < \alpha \leq \rho$ and $f_{\rho,\alpha}$ is a zero-mapping. Then $f_{\rho,\beta}$ is a zero-mapping. Thus$f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\alpha,\beta}(x)$.

Case3. $\rho < \beta < \alpha$ and $f_{\beta,\rho}$ is an isomorphism. Then $f_{\alpha,\rho}$ is an isomorphism. Thus$f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta} f_{\alpha,\rho}^{-1} g_\rho f_{\alpha,\rho}(x) = f_{\beta,\rho}^{-1} g_\rho f_{\alpha,\rho}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\beta,\rho}^{-1} g_\rho f_{\beta,\rho} f_{\alpha,\beta}(x) = f_{\beta,\rho}^{-1} g_\rho f_{\alpha,\rho}(x)$.

Case4. $\rho < \beta < \alpha$ and $f_{\beta,\rho}$ is a zero-mapping. Then $f_{\alpha,\rho}$ is a zero-mapping. Thus $f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\alpha,\beta}(x)$.

Case5. $\beta \leq \rho \leq \alpha$. Then $f_{\alpha,\rho}$ and $f_{\rho,\beta}$ are isomorphisms. Thus$f_{\alpha,\beta}\chi_\alpha(x) = f_{\alpha,\beta} f_{\alpha,\rho}^{-1} g_\rho f_{\alpha,\rho}(x) = f_{\rho,\beta} g_\rho f_{\alpha,\rho}(x)$ and $\chi_\beta f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\rho,\beta}^{-1} f_{\alpha,\beta}(x) = f_{\rho,\beta} g_\rho f_{\alpha,\rho}(x)$.

□

Theorem 17. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, ω the minimum element of Y. If all groups G_α are cyclic groups of order prime p_α and $A \subseteq G_\omega$, then* $\mathrm{Cay}(S, A)$ *is a CI-graph.*

Proof. Let $A \subseteq G_\omega$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 9, $B \subseteq G_\omega$ and $\mathrm{Cay}(G_\omega, A) \cong \mathrm{Cay}(G_\omega, B)$. Since $G_\omega = \mathbb{Z}_{p_\omega}$ is a CI-group, then there exists $g_\omega \in \mathrm{Aut}(G_\omega)$ such that $g_\omega(A) = B$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be an identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha(x) = \begin{cases} f_{\alpha,\omega}^{-1} g_\omega f_{\alpha,\omega}(x), & \text{if } f_{\alpha,\omega} \text{ is an isomorphism} \\ x, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\omega$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. If $\alpha = \beta$, then it is easily seen that $f_{\alpha,\beta}\chi_\alpha(x) = \chi_\beta f_{\alpha,\beta}(x)$. If $\beta < \alpha$ and $f_{\alpha,\beta}$ is a zero-mapping, then $f_{\alpha,\beta}\chi_\alpha(x) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. For the other cases that $\beta < \alpha$ and $f_{\alpha,\beta}$ is an isomorphism see case3 and case4 in Theorem 16, with ρ replaced by ω. $\square$

Theorem 18. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, π the maximum element of Y. If all group G_α are cyclic groups of order prime p_α and $A \subseteq G_\pi$, then* $\mathrm{Cay}(S, A)$ *is a CI-graph.*

Proof. Let $A \subseteq G_\pi$ and $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 11, $B \subseteq G_\pi$ and $\mathrm{Cay}(G_\pi, A) \cong \mathrm{Cay}(G_\pi, B)$. Since $G_\pi = \mathbb{Z}_{p_\pi}$ is a CI-group, there exists $g_\pi \in \mathrm{Aut}(G_\pi)$ such that $g_\pi(A) = B$. Now we will construct an automorphism on S as Proposition 1. Let $\eta : Y \to Y$ be an identity mapping id_Y and for each $\alpha \in Y$, let

$$\chi_\alpha(x) = \begin{cases} f_{\pi,\alpha} g_\pi f_{\pi,\alpha}^{-1}(x), & \text{if } f_{\pi,\alpha} \text{ is an isomorphism} \\ x, & \text{otherwise.} \end{cases}$$

Define a mapping χ on S by $\chi(x) = \chi_\alpha(x)$ if $x \in G_\alpha$. To show that $\chi \in \mathrm{Aut}(S)$, it is sufficient to show that $f_{\alpha,\beta}\chi_\alpha = \chi_\beta f_{\alpha,\beta}$. Let $x \in G_\alpha$. If $\alpha = \beta$, then it is easily seen that $f_{\alpha,\beta}\chi_\alpha(x) = \chi_\beta f_{\alpha,\beta}(x)$. If $\beta < \alpha$ and $f_{\alpha,\beta}$ is a zero-mapping, then $f_{\alpha,\beta}\chi_\alpha(x) = e_\beta$ and $\chi_\beta f_{\alpha,\beta}(x) = \chi_\beta(e_\beta) = e_\beta$. For the other cases that $\beta < \alpha$ and $f_{\alpha,\beta}$ is an isomorphism see case1 and case2 in Theorem 16, with ρ replaced by π. $\square$

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6.2 ผลงานวิจัยชื่อ On Cayley Isomorphisms of Left and Right Groups

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ON CAYLEY ISOMORPHISMS OF LEFT AND RIGHT GROUPS

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Abstract: In this paper, we investigate the characterization of CI-graphs on Cayley digraphs of left groups. We also determine which Cayley digraphs of right groups with given connection sets are CI-graphs.

AMS Subject Classification: 05C60, 20M99
Key Words: Cayley isomorphism, CI-graph, left group, right group

1. Introduction

Let S be a semigroup and A a subset of S. The Cayley digraph $\mathrm{Cay}(S, A)$ of S relative to a connection set A is defined as the graph with the vertex set S and the arc set $E(\mathrm{Cay}(S, A))$ consisting of those ordered pairs (x, y) such that $xa = y$ for some $a \in A$. Clearly, if A is an empty set, then $\mathrm{Cay}(S, A)$ is an empty graph.

Arthur Cayley (1821-1895) introduced Cayley graphs of groups in 1878. One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups which possess a generating system such that the Cayley graph is planar.

Cayley graphs of groups have been extensively studied and many interesting results have been obtained, see for examples [1], [8], [9], [10], [11], and [17]. The Cayley graphs of semigroups have been considered by many authors. Many new interesting results on Cayley graphs of semigroups have recently appeared in various journals, see for examples [3], [4], [5], [6], [7], [8], [13], [14], [15], and

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[16]. In the investigation of the Cayley graphs of semigroups, it is first of all interesting to find the analogous of natural conditions which have been used in the group case.

A Cayley digraph $\mathrm{Cay}(S, A)$ is called a *CI-graph* of a semigroup S, CI stands for *Cayley Isomorphism*, if whenever B is a subset of S for which $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, there exists an automorphism σ of S such that $\sigma(A) = B$. A semigroup S is called a *CI-semigroup* if all of its Cayley digraphs are CI-graphs.

Necessary and sufficient conditions have been found for Cayley graphs of groups to be CI-graphs and for groups to be CI-groups, see for examples [9], [10], [11], and [12]. Such a problem is called Cayley isomorphism. Here we shall investigate this problem on left and right groups which both of them are the cartesian product between a group and a semigroup. Graphs considered in this paper are directed graphs. The terminology and notation which related to our paper will be defined in the next section.

2. Basic Definitions and Results

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $u, v \in E_1$ implies $((\varphi(u)), (\varphi(v))) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism.* A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph automorphism.*

A digraph (V, E) is called a *semigroup* (*group*) *digraph* or *digraph of a semigroup* (*group*) if there exists a semigroup (group) S and a connection set $A \subseteq S$ such that (V, E) is isomorphic to the Cayley graph $\mathrm{Cay}(S, A)$.

A semigroup S is called a *left* (*right*) *zero semigroup* if, for any $x, y \in S$, $xy = x$ $(xy = y)$.

A semigroup S is called a *left* (*right*) *group* if $S = G \times L_n$ $(S = G \times R_n)$ where G is a group and L_n (R_n) is an n-element left (right) zero semigroup. Then the operation on a left group S is defined by $(g, l)(g', l') = (gg', l)$ for $g, g' \in G$ and $l, l' \in L_n$. Similarly, the operation on a right group S is defined by $(g, r)(g', r') = (gg', r')$ for $g, g' \in G$ and $r, r' \in R_n$.

Now we recall some lemmas and theorems which are needed in the sequel.

Theorem 2.1. [11] *A cyclic group G is called a 2-DCI-group, that is, all Cayley digraphs of G of valency at most 2 are CI-graphs.*

The following lemmas give the structure of the Cayley digraphs of left groups and right groups, respectively. From now on, p_i denotes the projection map on the i^{th} coordinate of an ordered pair.

Let $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ be graphs and $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := (V_1 \cup V_2 \cup ... \cup V_n, E_1 \cup E_2 \cup ... \cup E_n)$.

Lemma 2.2. [16] *Let $S = G \times L_n$ be a left group and $A \subseteq S$. Then the following conditions hold:*

1. *for each $i \in \{1, 2, ..., n\}$,*
   $\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \mathrm{Cay}(G, p_1(A))$
2. $\mathrm{Cay}(S, A) = \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\})$.

Lemma 2.3. [16] *Let $S = G \times R_n$ be a right group and $A \subseteq S$. If $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$, then* $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G, p_1(A))$.

The next lemma shows the condition when any two Cayley digraphs of a given right group with a one-element connection set are isomorphic.

Lemma 2.4. [12] *Let $S = G \times R_n$ be a right group, and $(g, r), (g', r') \in S$ where $g, g' \in G$ and $r, r' \in R_n$. Then* $\mathrm{Cay}(S, \{(g, r)\})$
$\cong \mathrm{Cay}(S, \{(g', r')\})$ *if and only if* $|g| = |g'|$.

3. Main Results

This section is divided into two parts. We first characterize CI-graphs of left groups. We will end the section by introducing about CI-graphs of right groups which the connection set is a subset of $G \times \{r_i\}$ where $\{r_i\}$ is a singleton subset of the n-element right zero semigroup R_n.

3.1. CI-Graphs of Left Groups

We start with the lemma that will be used in Theorem 3.2. The condition for two Cayley digraphs of an arbitrary left group which can be isomorphic will be given.

Lemma 3.1. *Let $S = G \times L_n$ be a left group and $A, B \subseteq S$. Then* $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ *if and only if* $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.

Proof. ($\Longrightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ and $i \in \{1, 2, ..., n\}$. By Lemma 2.2, we have $\dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$

and $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G\times\{l_i\}, p_1(A)\times\{l_i\}) \cong \mathrm{Cay}(G\times\{l_i\}, p_1(B)\times\{l_i\}) \cong \mathrm{Cay}(G, p_1(B))$ as required.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Then $\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$ for all $i \in \{1, 2, ..., n\}$ by Lemma 2.2 (1). Therefore $\dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$. Thus we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ by Lemma 2.2 (2). □

The next result characterizes the CI-graphs of left groups.

Theorem 3.2. *Let $S = G\times L_n$ be a left group and $A \subseteq S$. Then $\mathrm{Cay}(S, A)$ is a CI-graph if and only if $n = 1$ and $\mathrm{Cay}(G, p_1(A))$ is a CI-graph.*

Proof. ($\Longrightarrow$) Let $\emptyset \neq A \subseteq G \times L_n$ and let $\mathrm{Cay}(S, A)$ be a CI-graph and $n \neq 1$. We start the proof by choosing an element $(g, l_i) \in A$ to consider. Since $n \neq 1$, so $n \geq 2$. Then there exists $k \in \{1, 2, ..., n\}$ such that $k \neq i$ and $l_k \in L_n$. We will consider the following two cases:

Case 1: if there exists $(g, l_k) \in A$, consider $B = A \setminus \{(g, l_k)\}$. We will see that $p_1(A) = p_1(B)$ and $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Thus we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ by Lemma 3.1, but $|A| \neq |B|$. So it is easy to see that there is no any functions $f \in \mathrm{Aut}(S)$ such that $f(A) = B$ which satisfy the definition of CI-graph.

Case 2: if $(g, l_k) \notin A$, consider $B = A \cup \{(g, l_k)\}$. Similarly to the case 1, $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, but we can't find any functions $f \in \mathrm{Aut}(S)$ such that $f(A) = B$ since $|A| \neq |B|$. It contradicts the assumption by these two cases. Therefore $n = 1$.

Next, we will show that $\mathrm{Cay}(G, p_1(A))$ is a CI-graph. Suppose that

$$\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, X).$$

Take $B = X \times \{l_1\}$, then $p_1(B) = X$. By Lemma 3.1, we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Since $\mathrm{Cay}(S, A)$ is a CI-graph, there exists $\alpha \in \mathrm{Aut}(S)$ such that $\alpha(A) = B$. Define $f : G \to G$ by $g \mapsto p_1(\alpha(g, l_1))$. Since $\alpha \in \mathrm{Aut}(G \times L_1)$, we have f is bijective. Therefore f is a group homomorphism since $f(g_1)f(g_2) = p_1(\alpha(g_1, l_1))p_1(\alpha(g_2, l_1)) = p_1(\alpha(g_1, l_1)\alpha(g_2, l_1)) = p_1(\alpha(g_1g_2, l_1)) = f(g_1g_2)$ for $g_1, g_2 \in G$. Moreover, $f(p_1(A)) = p_1(\alpha(A)) = p_1(B) = X$. Hence $f \in \mathrm{Aut}(G)$ and $f(p_1(A)) = p_1(B) = X$. Thus $\mathrm{Cay}(G, p_1(A))$ is a CI-graph.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A))$ be a CI-graph. Let $n = 1$. Suppose that $\mathrm{Cay}(G \times L_1, A) \cong \mathrm{Cay}(G \times L_1, B)$. So, by Lemma 3.1, we have $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Since $\mathrm{Cay}(G, p_1(A))$ is a CI-graph, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(p_1(A)) = p_1(B)$. Then we define $\beta : G \times \{l_1\} \to G \times \{l_1\}$ by $\beta(g, l_1) =$

$(\alpha(g), l_1)$. Since $\alpha \in \mathrm{Aut}(G)$, it is easy to see that β is also bijective. Therefore β is a group homomorphism since $\beta(g_1, l_1)\beta(g_2, l_1) = (\alpha(g_1), l_1)(\alpha(g_2), l_1) = (\alpha(g_1)\alpha(g_2), l_1) = (\alpha(g_1 g_2), l_1) = \beta(g_1 g_2, l_1) = \beta((g_1, l_1)(g_2, l_1))$ for $(g_1, l_1), (g_2, l_1) \in G \times \{l_1\}$. In addition, $\beta(A) = \beta(p_1(A) \times \{l_1\}) = \alpha(p_1(A)) \times \{l_1\} = p_1(B) \times \{l_1\} = B$. Hence $\mathrm{Cay}(S, A)$ is a CI-graph. $\square$

The next example shows that if $n \geq 2$, then $\mathrm{Cay}(S, A)$ is not a CI-graph.

Example 1. Let $S = \mathbb{Z}_5 \times L_2$. Consider $A = \{(\overline{1}, l_1), (\overline{1}, l_2)\}$ and $B = \{(\overline{1}, l_1)\}$.

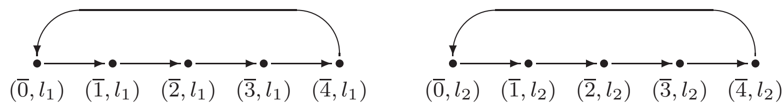

Figure 1: $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$

By the definition of a Cayley digraph, we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, see Figure 1. Since $|A| \neq |B|$, then we can't find any automorphisms f in S such that $f(A) = B$.

3.2. CI-Graphs of Right Groups

The next lemma will be useful for the proof of Lemma 3.4. We mention about the degree of vertices of right groups. Let $\overrightarrow{d}(u)$ denote the *in-degree* of an arbitrary vertex u of a given right group S.

Lemma 3.3. *Let $S = G \times R_n$ be a right group and $A \subseteq S$. Let $i \in \{1, 2, ..., n\}$. Then $A \cap (G \times \{r_i\}) = \emptyset$ if and only if $\overrightarrow{d}(u) = 0$ for all $u \in (G \times \{r_i\})$.*

Proof. Let $i \in \{1, 2, ..., n\}$.

($\Longrightarrow$) Assume that $A \cap (G \times \{r_i\}) = \emptyset$. Suppose that there exists $u \in (G \times \{r_i\})$ such that $\overrightarrow{d}(u) \neq 0$. Hence there exists an element $a \in A$ such that $xa = u$ for some $x \in S$. Since S is a right group, we have $a \in (G \times \{r_i\})$. Then $a \in A \cap (G \times \{r_i\})$, contrary to $A \cap (G \times \{r_i\}) = \emptyset$. Therefore $\overrightarrow{d}(u) = 0$ for all $u \in (G \times \{r_i\})$.

($\Longleftarrow$) Let $u, v \in (G \times \{r_i\})$ and $\overrightarrow{d}(u) = 0$, $\overrightarrow{d}(v) = 0$. Suppose that $A \cap (G \times \{r_i\}) \neq \emptyset$. So there exists an element $a \in A \cap (G \times \{r_i\})$ such that (u, v) is an arc in $\mathrm{Cay}(S, A)$, and then $\overrightarrow{d}(v) \neq 0$, a contradiction. Hence $A \cap (G \times \{r_i\}) = \emptyset$. $\square$

The following lemma gives the conditions when any two Cayley digraphs of an arbitrary right group which each of its connection set is a subset of the cartesian product of a group G and a singleton subset of the n-element right zero semigroup R_n. Throughout the proof, N_0^H denotes the number of vertices u in a graph H such that $\overrightarrow{d}(u) = 0$.

Lemma 3.4. *Let $S = G \times R_n$ be a right group. Let $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$. Then $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ if and only if the following conditions hold:*

1. *$B \subseteq G \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$,*
2. *there exists a graph isomorphism*

$$f : \mathrm{Cay}(G \times \{r_i\}, A) \to \mathrm{Cay}(G \times \{r_j\}, B)$$

such that $((g, r_k), (g', r_i)) \in E(\mathrm{Cay}(S, A))$ if and only if

$$(f(g, r_k), f(g', r_i)) \in E(\mathrm{Cay}(S, B)) \text{ for any } k \in \{1, 2, ..., n\}.$$

Proof. ($\Longrightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$.

1. Suppose that $B \nsubseteq G \times \{r_j\}$ for all $j \in \{1, 2, ..., n\}$. Then $|\{j|B \cap (G \times \{r_j\}) = \emptyset\}| \neq |\{j|A \cap (G \times \{r_j\}) = \emptyset\}|$. By Lemma 3.3, $N_0^{\mathrm{Cay}(S,B)} = |\{j|B \cap (G \times \{r_j\}) = \emptyset\}||G|$ and $N_0^{\mathrm{Cay}(S,A)} = |\{j|A \cap (G \times \{r_j\}) = \emptyset\}||G|$. Therefore $N_0^{\mathrm{Cay}(S,A)} \neq N_0^{\mathrm{Cay}(S,B)}$, which contradicts $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Then $B \subseteq G \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$.

2. Since $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, there exists a graph isomorphism $s : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$. Next, we can define $t : \mathrm{Cay}(G \times \{r_i\}, A) \to \mathrm{Cay}(G \times \{r_j\}, B)$ as the restriction of s to $G \times \{r_i\}$, i.e. $t = s_{|G \times \{r_i\}}$ by Lemma 3.3. It is obvious that t is also a graph isomorphism by the definition of s. Therefore $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G \times \{r_j\}, B)$. The statement $((g, r_k), (g', r_i)) \in E(\mathrm{Cay}(S, A))$ if and only if $(t(g, r_k), t(g', r_i)) \in E(\mathrm{Cay}(S, B))$ for any $k \in \{1, 2, ..., n\}$ is also true by the assumption.

($\Longleftarrow$) We define $\varphi : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ by

$$\varphi(g, r) = \begin{cases} (p_1 f(g, r_i), r_j), & \text{if} \quad r = r_i \\ (p_1 f(g, r_i), r_i), & \text{if} \quad r = r_j \\ (p_1 f(g, r_i), r), & \text{otherwise.} \end{cases}$$

By the assumption, it is obviously concluded that φ is a graph isomorphism from $\mathrm{Cay}(S, A)$ to $\mathrm{Cay}(S, B)$. Therefore $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. $\square$

Now we introduce the theorem about being CI-graphs of any right groups with a one-element connection set. Theorem 2.1 will be helpful in the proof.

Theorem 3.5. *Let $S = G \times R_n$ be a right group where G is a cyclic group and R_n is an n-element right zero semigroup. Let $(a, r_i) \in S$ where $i \in \{1, 2, ..., n\}$. Then* $\mathrm{Cay}(S, \{(a, r_i)\})$ *is a CI-graph.*

Proof. Suppose that $\mathrm{Cay}(S, \{(a, r_i)\}) \cong \mathrm{Cay}(S, \{(b, r_j)\})$ where $(b, r_j) \in S$ for some $j \in \{1, 2, ..., n\}$. By Theorem 2.1, we know that $\mathrm{Cay}(G, \{a\})$ is a CI-graph. So for all $b \in G$ such that $\mathrm{Cay}(G, \{b\}) \cong \mathrm{Cay}(G, \{a\})$, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = b$. Then we define $t : S \to S$ by

$$t(g, r) = \begin{cases} (\alpha(g), r_j), & \text{if} \quad r = r_i \\ (\alpha(g), r_i), & \text{if} \quad r = r_j \\ (\alpha(g), r), & \text{otherwise.} \end{cases}$$

It is obvious that t is bijective. Let $(g, r), (g', r') \in S$. Since S is a right group, there are only 3 cases to be considered depend on r'.

Case 1: $r' = r_i$. Then $t((g, r)(g', r_i)) = t(gg', r_i) = (\alpha(gg'), r_j)$ and

$$t(g, r)t(g', r_i) = (p_1(t(g, r))\alpha(g'), r_j) = (\alpha(g)\alpha(g'), r_j) = (\alpha(gg'), r_j).$$

Case 2: $r' = r_j$. Then $t((g, r)(g', r_j)) = t(gg', r_j) = (\alpha(gg'), r_i)$ and

$$t(g, r)t(g', r_j) = (p_1(t(g, r))\alpha(g'), r_i) = (\alpha(g)\alpha(g'), r_i) = (\alpha(gg'), r_i).$$

Case 3: $r' \neq r_i \neq r_j$. Then $t((g, r)(g', r')) = t(gg', r')$
$= (\alpha(gg'), r')$ and

$$t(g, r)t(g', r') = (p_1(t(g, r))\alpha(g'), r') = (\alpha(g)\alpha(g'), r') = (\alpha(gg'), r').$$

Thus we have t is a semigroup homomorphism. Since $t \in \mathrm{Aut}(S)$ and $t(a, r_i) = (\alpha(a), r_j) = (b, r_j)$, $\mathrm{Cay}(S, \{(a, r_i)\})$ is a CI-graph. □

The following lemma is similar to Lemma 3.1. We give the condition for two Cayley digraphs of a right group can be isomorphic. The connection set which will be considered is a subset of the cartesian product of a group G and a one-element subset of the right zero semigroup R_n.

Lemma 3.6. *Let $S = G \times R_n$ be a right group, $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$, and $B \subseteq S$. Then* $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ *if and only if* $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.

Proof. Let $i \in \{1, 2, ..., n\}$ and $A \subseteq G \times \{r_i\}$.

($\Longrightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 3.4, there exists $j \in \{1, 2, ..., n\}$ such that $B \subseteq G \times \{r_j\}$ and $\mathrm{Cay}(G \times \{r_i\}, A)$ $\cong \mathrm{Cay}(G \times \{r_j\}, B)$. Therefore, by Lemma 2.3, we have $\mathrm{Cay}(G, p_1(A))$ $\cong \mathrm{Cay}(G, p_1(B))$.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Then there exists $\varphi : \mathrm{Cay}(G, p_1(A)) \to \mathrm{Cay}(G, p_1(B))$ which is a digraph isomorphism. We define $f : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ by

$$f(g, r) = \begin{cases} (\varphi(g), r_j), & \text{if} \quad r = r_i, \\ (\varphi(g), r_i), & \text{if} \quad r = r_j, \\ (\varphi(g), r), & \text{otherwise.} \end{cases}$$

It is obvious that f is bijective. Let $(g, r_a), (g', r_b) \in \mathrm{Cay}(S, A)$ and $((g, r_a), (g', r_b)) \in E(\mathrm{Cay}(S, A))$. There exists $(a, r_i) \in A$ such that $(g', r_b) = (g, r_a)(a, r_i)$. Then $g' = ga$ and $r_b = r_i$. Hence $(g, g') \in E(\mathrm{Cay}(G, p_1(A)))$ and $f(g', r_b) = f(g', r_i) = (\varphi(g'), r_j)$. Thus we have $(\varphi(g), \varphi(g')) \in E(\mathrm{Cay}(G, p_1(B)))$ by the assumption. Then there exists $b \in p_1(B)$ such that $\varphi(g') = \varphi(g)b$. Since

$$\begin{aligned} f(g', r_b) = (\varphi(g'), r_j) = (\varphi(g)b, r_j) &= (\varphi(g), r_a)(b, r_j) \\ &= f(g, r_a)(b, r_j), (f(g, r_a), f(g', r_b)) \in E(\mathrm{Cay}(S, B)), \end{aligned}$$

where $(b, r_j) \in B$. Thus we have f preserves arcs, and then f^{-1} preserves arcs can prove in the same way. Therefore $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. $\square$

Here we come to our main theorem of the right group. The preceding lemma will be used in the proof.

Theorem 3.7. *Let $S = G \times R_n$ be a right group and $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$. Then* $\mathrm{Cay}(S, A)$ *is a CI-graph if and only if* $\mathrm{Cay}(G, p_1(A))$ *is a CI-graph.*

Proof. Let $i \in \{1, 2, ..., n\}$.

($\Longrightarrow$) Let $\mathrm{Cay}(S, A)$ be a CI-graph. Suppose that $\mathrm{Cay}(G, p_1(A))$ $\cong \mathrm{Cay}(G, B)$. Take $X = B \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$. By Lemma 3.6, we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, X)$. So there exists $f \in \mathrm{Aut}(S)$ such that $f(A) = X$. Define $\varphi : G \to G$ by $g \mapsto p_1(f(g, r_i))$. Clearly, φ is bijective. Then φ is also a group homomorphism since $\varphi(g_1)\varphi(g_2) = p_1(f(g_1, r_i))p_1(f(g_2, r_i)) = p_1(f(g_1, r_i)f(g_2, r_i)) = p_1 f(g_1 g_2, r_i) = \varphi(g_1 g_2)$. Let $t \in \varphi(p_1(A))$, i.e. $t = p_1(f(x, r_i))$ for some $(x, r_i) \in A$. Then $t \in p_1(f(A)) = p_1(X) = B$. Conversely, let $t \in B = p_1(X)$, i.e. $t = p_1(t, r_j)$. Since $f(A) = X$, there exists $(h, r_i) \in$

A such that $f(h,r_i) = (t,r_j)$ and thus $t = p_1(f(h,r_i)) \in \varphi(p_1(A))$. Hence $\varphi(p_1(A)) = B$. Therefore $\mathrm{Cay}(G,p_1(A))$ is a CI-graph.

($\Longleftarrow$) Let $\mathrm{Cay}(G,p_1(A))$ be a CI-graph. Suppose that $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$. By Lemma 3.6, we have $\mathrm{Cay}(G,p_1(A)) \cong \mathrm{Cay}(G,p_1(B))$ where $B \subseteq G \times \{r_j\}$ for some $j \in \{1,2,...,n\}$. Then there exists $f \in \mathrm{Aut}(G)$ such that $f(p_1(A)) = p_1(B)$. Define $\varphi : S \to S$ by

$$\varphi(g,r) = \begin{cases} (f(g),r_j), & \text{if} \quad r = r_i \\ (f(g),r_i), & \text{if} \quad r = r_j \\ (f(g),r), & \text{otherwise.} \end{cases}$$

It is easy to check that φ is bijective. About to prove that φ is a semigroup homomorphism is similar to Theorem 3.5. Next, we will prove that $\varphi(A) = B$. Let $t \in \varphi(A) = \varphi(p_1(A) \times \{r_i\})$. Then $t = \varphi(x,r_i)$ for some $x \in p_1(A)$. So $t = (f(x),r_j) \in B$. Therefore $\varphi(A) \subseteq B$. Conversely, let $t \in B$. Suppose that $t = (g,r_j)$ for some $g \in G$. Since $f(p_1(A)) = p_1(B)$, there exists $h \in p_1(A)$, i.e. $(h,r_i) \in A$ such that $f(h) = g$. Hence $t = (f(h),r_j) = \varphi(h,r_i) \in \varphi(A)$. Therefore $B \subseteq \varphi(A)$. So we can conclude that $\mathrm{Cay}(S,A)$ is a CI-graph. $\square$

We now show another example which can be concluded by Theorem 3.7.

Example 2. Let $G = \mathbb{Z}_9$ and $S = \mathbb{Z}_9 \times R_n$. Consider $A = \{\overline{1},\overline{4},\overline{6},\overline{7}\}$ and $B = \{\overline{1},\overline{3},\overline{4},\overline{7}\}$.

Define $\beta : \mathrm{Cay}(G,A) \to \mathrm{Cay}(G,B)$ by $0 \mapsto 6, 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 7, 5 \mapsto 8, 6 \mapsto 0, 7 \mapsto 4$ and $8 \mapsto 5$. We have $\mathrm{Cay}(G,A) \cong \mathrm{Cay}(G,B)$, but there is no Cayley isomorphisms mapping A to B, that is, $\mathrm{Cay}(G,A)$ is not a CI-graph. Therefore, by Theorem 3.7, we can conclude that $\mathrm{Cay}(S,A \times \{r_i\})$ is not a CI-graph for all $i \in \{1,2,...,n\}$.

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6.3 ผลงานวิจัยชื่อ Isomorphism Conditions for Cayley Graphs of Rectangular groups

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ISOMORPHISM CONDITIONS FOR CAYLEY GRAPHS OF RECTANGULAR GROUPS

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ABSTRACT. A semigroup S is called rectangular group if S is a cartesian product of a group and left zero and right zero semigroup. This paper we introduce the conditions for Cayley graphs of rectangular groups are isomorphic.

1. PRELIMINARIES

One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups G which possess a generating system A such that the Cayley graph $Cay(G, A)$ is planar, see for example [19]. In [18] Cayley graphs which represent groups are described. The result for groups originates from [18] and is meanwhile folklore, see for example [2]. After this it is natural to investigate Cayley graphs for semigroups which are unions of groups, so-called completely regular semigroups, see for example [14]. In [1],[10] and [11] Cayley graphs which represent completely regular semigroups with are right(left) groups, rectangular group and finite simple semigroups, respectively are characterized. We now introduce the conditions for Cayley graphs of rectangular groups are isomorphic.

All sets in this paper are assume to be finite. An element z of a semigroup S is a *left(right) zero* of S if $zs = z(sz = z)$ for all $s \in S$, z is a *zero* of S if it is both a left and right zero of S. A semigroup all of whose elements are left(right) zeros is a *left(right) zero semigroup.* A direct product of a group and a left(right) zero semigroup is called a *left(right) group.* A direct product of a left zero and a right zero semigroup is called a *rectangular band.* A *rectangular groups* is a direct product of a group and a rectangular band.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism.* If a digraph isomorphism $\varphi : (V_1, E_1) \to (V_2, E_2)$ exists, then the graphs are called isomorphic and we write $(V_1, E_1) \cong (V_2, E_2)$. A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph automorphism.*

Let S be a semigroup and $A \subseteq S$. We define the *Cayley graph* $Cay(S, A)$ as follows: S is the vertex set and (u, v), $u, v \in S$, is an arc in $Cay(S, A)$ if there exists an element $a \in A$ such that $v = ua$. The set A is called the *connection set* of $Cay(S, A)$.

Key words and phrases. Cayley graph, digraph, rectangular band, right group, rectangular group.

A digraph (V, E) is called a *semigroup digraph* or *digraph of a semigroup* if there exists a semigroup S and a connection set $A \subseteq S$ such that (V, E) is isomorphic to the Cayley graph $Cay(S, A)$.

A subdigraph F of a digraph G is called a strong subdigraph of G if and only if whenever u and v are vertices of F and (u, v) is an arc in G, then (u, v) is an arc in F as well.

2. Cayley Graphs of Rectangular Band

We consider a isomorphism of Cayley graphs of rectangular bands in this section. By the definition of right zero semigroup, we get the following lemma.

Lemma 2.1. *Let $v \in V(Cay(R_n, A))$ where R_n is right zero semigroup, and let $A \subseteq R_n$. Then*

(1) $\overrightarrow{d}(v) = |R_n|$ *if and only if* $v \in A$;

(2) $\overrightarrow{d}(v) = 0$ *if and only if* $v \notin A$.

From above lemma, we have the following theorem.

Theorem 2.2. *Let R_n be a right zero semigroup and $A, B \subseteq R_n$. Then $Cay(R_n, A) \cong Cay(R_n, B)$ if and only if $|A| = |B|$.*

Since a rectangular band $S = L_m \times R_n$ isomorphic to the finite simple semigroup $\mathcal{M}(G, I, \Lambda, P)$, where $G = \{e\}$ is the trivial group, $m = |I|$ and $n = |\Lambda|$. By Lemma 2 in [12] , we have the following lemma.

Lemma 2.3. *Let $S = L_m \times R_n$ be a rectangular band, $L_m = \{l_1, l_2 \ldots, l_m\}$ a left zero semigroup, $R_n = \{r_1, r_2 \ldots, r_n\}$ a right zero semigroup, and $A \subseteq S$. Then $Cay(S, A)$ is the disjoint union of m isomorphic strong subdigraphs $Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A))$ for $i \in \{1, 2, \ldots, m\}$.*

Theorem 2.4. *Let $S = L_m \times R_n$ be a rectangular band and $A, B \subseteq S$. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $|p_2(A)| = |p_2(B)|$.*

Proof. $(\Rightarrow)$ Let $Cay(S, A) \cong Cay(S, B)$. By Lemma 2.3, we get $Cay(S, A) \cong \dot\bigcup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \dot\bigcup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(B)) \cong Cay(S, B)$. Then $Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$ and thus $Cay(R_n, p_2(A)) \cong Cay(R_n, p_2(B))$. By Theorem 2.2, we get $|p_2(A)| = |p_2(B)|$.

$(\Leftarrow)$ Let $|p_2(A)| = |p_2(B)|$. By Theorem 2.2, we get $Cay(R_n, p_2(A)) \cong Cay(R_n, p_2(B))$. Then $\dot\bigcup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \dot\bigcup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$. By Theorem 2.3, we get $Cay(S, A) \cong Cay(S, B)$. □

3. Cayley Graphs of Right Groups

In this section, we introduce the condition for Cayley graphs of a given right group are isomorphic.

By the definition of a right group we get the following lemma.

Lemma 3.1. *Let $S = G \times R_n$ be a right group where G is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup, and A a nonempty subset of S. Then, for $g, g' \in G$ and $r, r' \in R_n$, $((g, r), (g', r'))$ is an arc in $Cay(S, A)$ if and only if there exists $(a, r') \in A$ such that $g' = ga$ and $((g, r'), (g', r'))$ is an arc in $Cay(S, A)$.*

The next result gives some description for cayley graphs of right groups.

Theorem 3.2. *Let $S = G \times R_n$ be a right group where G is a group and $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup. Let A be a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle\ , g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ a set of distinct left coset of $\langle p_1(A)\rangle$ in G , and $(g_i\langle p_1(A)\rangle \times p_2(A)\ , E_i)$ a strong subdigraph of $Cay(S\ , A)$. Then $Cay(S, A) = \dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$, where $E_A = \{((s,t), (u,v)) \mid t \notin p_2(A),\ ((s,v),(u,v)) \in E_i$ for all $i\}$.*

Proof. We define $f : Cay(S, A) \to \dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_{A_i})$ by the identity mapping. Since $S = \dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A)) \cup\ S$, f is a bijection. We will prove that f and f^{-1} are homomorphisms. Let $((g, r), (g', r'))$ be an arc in $Cay(S, A)$. By Lemma 3.1, there exists $(a, r') \in A$ and $g' = ga$. Hence $g' \in g_{k_1}\langle p_1(A)\rangle, g \in g_{k_2}\langle p_1(A)\rangle$ for some $k_1, k_2 \in I$. We need only consider two cases:

(case1) If $r \in p_2(A)$, then $(g, r), (g', r') \in \dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A))$. Since $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S, A)$, $((g, r), (g', r'))$ is an arc in $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore $((g, r), (g', r')) = (f(g, r), f(g', r'))$ is an arc in $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$.

(case2) If $r \notin p_2(A)$, then $((g, r'), (g', r'))$ is also an arc in $Cay(S, A)$ by Lemma 3.1 and $((g, r), (g', r'))$ is an arc in $Cay(S, A)$. This implied that $((g, r'), (g', r')) \in E_i$. Then $((g, r), (g', r')) \in E_A$. Hence $((g, r), (g', r')) = (f(g, r), f(g', r'))$ is an arc in $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$.

Therefore f is a homomorphism.

Let $(f(g, r), f(g', r'))$ be an arc in $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$. We consider two cases.

(case1) If $(f(g, r), f(g', r'))$ is an arc in $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$, then it is an arc in $Cay(S, A)$ because $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A),\ E_i)$ is strong subdigraph of $Cay(S, A)$.

(case2) If $(f(g, r), f(g', r'))$ is an arc in (S, E_A), then $((g, r), (g', r')) \in E_A$. We get that $((g, r'), (g', r')) \in E_i$ for some $i \in I$ and this implied that $((g, r'), (g', r'))$ is an arc in $Cay(S, A)$. By Lemma 3.1, we have $((g, r), (g', r'))$ is also an arc in $Cay(S, A)$.

Then f^{-1} is a homomorphism. Hence we prove that $Cay(S, A) = \dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$. □

By Lemma 3.1 and the definition of E_A in Theorem 3.2 we have the next lemma.

Lemma 3.3. *Let $S = G \times R_n$ be a right group where G is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup. Let A be a nonempty subset of S, $(g_i\langle p_1(A)\rangle \times p_2(A)\ , E_i)$ a strong subdigraph of $Cay(S\ , A)$ and $r' \in R_n \setminus p_2(A)$. If $((u, r), (v, r))$ is an arc in $\dot\cup_{j \in I}(g_j\langle p_1(A)\rangle \times p_2(A), E_j)$, then $((u, r'), (v, r)) \in E_A$.*

Theorem 3.4. *Let $S = G \times R_n$ be a right group where G is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup. Let A be a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle\ , g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ the set of distinct left coset of $\langle p_1(A)\rangle$ in G and $(g_i\langle p_1(A)\rangle \times p_2(A)\ , E_i)$ a strong subdigraph of $Cay(S\ , A)$. Then $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong Cay(\langle A\rangle, A)$ for $i = 1, 2, ..., w$.*

Proof. We define $f : (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \to Cay(\langle A\rangle, A)$ by $(g_ia, r) \mapsto (a, r)$ for all $a \in \langle p_1(A)\rangle$ and $r \in p_2(A)$. Clearly, f is a bijection. We will prove that f and f^{-1} are homomorphisms.

For $(g_ia, r), (g_ia', r') \in g_i\langle p_1(A)\rangle \times p_2(A)$, let $((g_ia, r), (g_ia', r'))$ be an arc in $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Since $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S, A)$, $((g_ia, r'), (g_ia', r'))$ is an arc in $Cay(S, A)$. There exist $(a'', r') \in A$ such that $g_ia' = g_iaa''$ so $a' = aa''$. Since $f(g_ia', r') = (a', r') = (aa'', r') = (a, r)(a'', r') = f(g_ia, r)(a'', r')$, $(f(g_ia, r), f(g_ia', r'))$ is an arc in $Cay(\langle A\rangle, A)$. Therefore f is a homomorphism.

Let $(f(g_ia, r), f(g_ia', r'))$ is an arc in $Cay(\langle A\rangle, A)$. Then there exist $(a'', r'') \in A$ such that $f(g_ia', r') = f(g_ia, r)(a'', r'')$. Therefore $(a', r') = (a, r)(a'', r'') = (aa'', r'')$, $a' = aa''$ and $r' = r''$. Hence $(g_ia', r') = (g_iaa'', r'') = (g_ia, r)\ (a'', r'')$, so $((g_ia, r), (g_ia', r'))$ is an arc in $Cay(S, A)$. Since $(g_ia, r), (g_ia',\ r') \in g_i\langle p_1(A)\rangle \times p_2(A)$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S, A)$, then $((g_ia, r), (g_ia', r'))$ is an arc in $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore f^{-1} is a homomorphism. This mean that $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong Cay(\langle A\rangle, A)$. □

Lemma 3.5. *Let $S = G \times R_n$ be a right group where G is a group and $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup. Let A be a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle\, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ the set of distinct left coset of $\langle p_1(A)\rangle$ in G and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S, A)$. Then for all $v \in V(Cay(S, A))$, $\overrightarrow{d}(v) \neq 0$ if and only if $v \in \dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$.*

Proof. $(\Rightarrow)$ Let $v = (g_1, r_1) \in S$ and $\overrightarrow{d}(v) \neq 0$. Then there exist $u = (g_2, r_2) \in S$ such that (u, v) is an arc in $Cay(S, A)$. Hence there exist $a = (g', r') \in A$ such that $v = ua$. Therefore $(g_1, r_1) = (g_2, r_2)(g', r') = (g_2g', r')$, we have $r_1 = r' \in p_2(A)$. Since $g_1 \in G = \dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big)$, then $v = (g_1, r_1) \in \dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$.

$(\Leftarrow)$ Let $v = (g_1, r) \in \dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$, we get that $g_1 \in G$ and $r \in p_2(A)$. We need consider the two cases.

- (case1) If $v \in A$, since G is a group ,then there exist identity e of G such that $(e, r) \in S$ and $(e, r)(g_1, r) = (eg_1, r) = (g_1, r) = v$. Hence there is an edge from (e, r) to v. Therefore $\overrightarrow{d}(v) \neq 0$.
- (case2) If $v \notin A$, then there exists $(g_2, r) \in A$ for some $g_2 \in G$. Because G is a group and $g_1, g_2 \in G$, this implies that $g_2^{-1} \in G$ and $g_1g_2^{-1} \in G$. Then we have $(g_1g_2^{-1}, r) \in S$. Since $(g_1g_2^{-1}, r)(g_2, r) = (g_1g_2^{-1}g_2, r) = (g_1, r) = v$, there exist an arc from $(g_1g_2^{-1}, r)$ to v. Therefore $\overrightarrow{d}(v) \neq 0$.

□

Lemma 3.6. *Let $S = G \times R_n$ be a right group where G is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup, and let A and B be nonempty subsets of S. If $\dot\bigcup_{i\in I}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A) \cong \dot\bigcup_{j\in I}^{w}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$, then $\dot\bigcup_{i\in I}^{w}\ (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\bigcup_{j\in I}^{w}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$.*

Proof. Let $\dot\bigcup_{i\in I}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A) \cong \dot\bigcup_{j\in I}^{w}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$. Then there exists an isomorphism $f : \dot\bigcup_{i\in I}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A) \to \dot\bigcup_{j\in I}^{w}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$. By Lemma 3.5, we get that $|\dot\bigcup_{i\in I}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)| = |\dot\bigcup_{j\in I}^{w}\big(g_j\langle p_1(B)\rangle\big) \times p_2(B)|$ and we have $f(\dot\bigcup_{i\in I}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)) = \dot\bigcup_{j\in I}^{w}\big(g_j\langle p_1(B)\rangle\big) \times p_2(B)$. Since f is an isomorphism, the restrictions of f on $\dot\bigcup_{i\in I}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$ is an isomorphism from $\dot\bigcup_{i\in I}^{w}$

$(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ to $\dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $\dot\cup^w_{i\in I}\ (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. □

Lemma 3.7. *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. If $Cay(S, A) \cong Cay(S, B)$, then $\big|p_2(A)\big| = \big|p_2(B)\big|$.*

Proof. Let $Cay(S, A) \cong Cay(S, B)$. By Lemma 3.5, we get that $\big|\dot\cup_{i\in I} g_i\langle p_1(A)\rangle \times p_2(A)\big| = \big|\dot\cup_{j\in I} g_j\langle p_1(B)\rangle \times p_2(B)\big|$ for all $g_i, g_j \in G$. Since $\dot\cup_{i\in I} g_i\langle p_1(A)\rangle = G = \dot\cup_{j\in I} g_j\langle p_1(B)\rangle$, $\big|G \times p_2(A)\big| = \big|G \times p_2(B)\big|$. Therefore $\big|G\big| \times \big|p_2(A)\big| = \big|G\big| \times \big|p_2(B)\big|$. Hence $\big|p_2(A)\big| = \big|p_2(B)\big|$. □

By Theorem 4 in [12], we have the next lemma.

Lemma 3.8. *Let $S = G\times R_n$ be right group and let $(g, \lambda), (h, \beta) \in S$ where $g, h \in G$ and $\lambda, \beta \in R_n$. Then $Cay(S, \{(g, \lambda)\}) \cong Cay(S, \{(h, \beta)\}$ if and only if $|g| = |h|$.*

Theorem 3.9. *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r := \{v \in \langle p_1(A)\rangle \times \{r\}\big|r \in p_2(A)\}$, $B_r := \{v \in \langle p_1(B)\rangle \times \{r\}\big|r \in p_2(B)\}$, $\hat{A} := \{\hat{A}_r\big|\hat{A}_r = A\cap A_r\}$ and $\hat{B} := \{\hat{B}_r\big|\hat{B}_r = B\cap B_r\}$. If $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ then $\big|\hat{A}\big| = \big|\hat{B}\big|$ and $\big|\langle p_1(A)\rangle\big| = \big|\langle p_1(B)\rangle\big|$.*

Proof. Let $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$. By Lemma 3.7, we have $\big|p_2(A)\big| = \big|p_2(B)\big|$ and then $\big|\hat{A}\big| = \big|\hat{B}\big|$. Since $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$, $\big|\langle A\rangle\big| = \big|\langle B\rangle\big|$. Therefore

$$\begin{aligned}
\big|\langle p_1(A)\rangle \times \langle p_2(A)\rangle\big| &= \big|\langle p_1(B)\rangle \times \langle p_2(B)\rangle\big|\\
\big|\langle p_1(A)\rangle\big| \times \big|\langle p_2(A)\rangle\big| &= \big|\langle p_1(B)\rangle\big| \times \big|\langle p_2(B)\rangle\big|\\
\big|\langle p_1(A)\rangle\big| \times \big|p_2(A)\big| &= \big|\langle p_1(B)\rangle\big| \times \big|p_2(B)\big|\\
\big|\langle p_1(A)\rangle\big| &= \big|\langle p_1(B)\rangle\big|.
\end{aligned}$$

□

Theorem 3.10. *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r := \{v \in \langle p_1(A)\rangle \times \{r\}\big|r \in p_2(A)\}$, $B_r := \{v \in \langle p_1(B)\rangle \times \{r\}\big|r \in p_2(B)\}$, $\hat{A} := \{\hat{A}_r\big|\hat{A}_r = A \cap A_r\}$ and $\hat{B} := \{\hat{B}_r\big|\hat{B}_r = B \cap B_r\}$. Then $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ if the following conditions hold*

1. $\big|\hat{A}\big| = \big|\hat{B}\big|$ *and* $\big|\langle p_1(A)\rangle\big| = \big|\langle p_1(B)\rangle\big|$;
2. *There exists a bijection $f : \hat{A} \to \hat{B}$ such that $\big|\hat{A}_r\big| = \big|f(\hat{A}_r)\big|$ for all $\hat{A}_r \in \hat{A}$;*
3. *For each $\hat{A}_r \in \hat{A}$, there exists a bijection $h : \hat{A}_r \to f(\hat{A}_r)$ such that $\big|p_1(a)\big| = \big|p_1(h(a))\big|$ for all $a \in \hat{A}_r$.*

Proof. By (1) we get that $\big|\langle A\rangle\big| = \big|\langle B\rangle\big|$. By Lemma 3.8 and (3), we get that $Cay(\langle A\rangle, \{a\}) \cong Cay(\langle B\rangle, \{h(a)\})$ for all $a \in \hat{A}_r$. Then $Cay(\langle A\rangle, \hat{A}_r) = \oplus_{a\in\hat{A}_r} Cay(\langle A\rangle, \{a\}) \cong \oplus_{a\in\hat{A}_r} Cay(\langle B\rangle, \{h(a)\}) = Cay(\langle B\rangle, h(\hat{A}_r))$.

By (2), we get that $Cay(\langle A\rangle, \hat{A}_r) \cong Cay(\langle B\rangle, f(\hat{A}_r))$ for all $\hat{A}_r \in \hat{A}$. Then

$$\begin{aligned}
\oplus_{\hat{A}_r\in\hat{A}} Cay\big(\langle A\rangle, \hat{A}_r\big) &\cong \oplus_{\hat{A}_r\in\hat{A}} Cay\big(\langle B\rangle, f(\hat{A}_r)\big)\\
Cay\big(\langle A\rangle, \cup_{\hat{A}_r\in\hat{A}} \hat{A}_r\big) &\cong Cay\big(\langle B\rangle, \cup_{\hat{A}_r\in\hat{A}} f(\hat{A}_r)\big)\\
Cay\big(\langle A\rangle, A\big) &\cong Cay\big(\langle B\rangle, B\big).
\end{aligned}$$

□

Theorem 3.11. *Let $S = G \times R_n$ be a right group where G is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup, and let A and B be nonempty subsets of S. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.*

Proof. ($\Rightarrow$) Let $Cay(S, A) \cong Cay(S, B)$, then there exists an isomorphism $f : Cay(S, A) \to Cay(S, B)$. It then follows by Theorem 3.2 that $\dot\cup^w_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A) \cong \dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$. By Lemma 3.6, we get $\dot\cup^w_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. By Theorem 3.4, we get $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.

($\Leftarrow$) Let $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$. By Theorem 3.4, we get $\dot\cup^w_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Then there exist an isomorphism $f : \dot\cup^w_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \to \dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Then $|\dot\cup^w_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A)| = |\dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B)|$. Since $\dot\cup^w_{i\in I} g_i\langle p_1(A)\rangle = G = \dot\cup^w_{j\in I} g_j\langle p_1(B)\rangle$, $\big|G \times p_2(A)\big| = \big|G \times p_2(B)\big|$. Therefore $\big|G\big| \times \big|p_2(A)\big| = \big|G\big| \times \big|p_2(B)\big|$ and thus $\big|p_2(A)\big| = \big|p_2(B)\big|$. Suppose that $R_n \setminus p_2(A) = \{q_1, q_2, \ldots, q_m\}$ and $R_n \setminus p_2(B) = \{q'_1, q'_2, \ldots, q'_m\}$. Let $r \in p_2(A)$. Define $T : \dot\cup^w_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A) \to \dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$ by

$$T(s, r_l) = \begin{cases} f(s, r_l) & \text{if } r_l \in p_2(A) \\ (p_1(f(s, r)), q'_k) & \text{if } r_l = q_k \text{ for some } q_k \in R_n \setminus p_2(A) \end{cases}$$

Clearly, T is well defined and bijective. We will prove that T and T^{-1} are homomorphisms.

Assume that $((x, r_c), (y, r_d))$ is an arc in $\dot\cup_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$. Then $(y, r_d) = (x, r_c)(a, r_t)$ for some $(a, r_t) \in A$. Hence $(y, r_d) = (xa, r)$ and thus $r_d = r_f \in p_2(A)$ and $y = xa$. We need only consider 2 cases:

(case1) $r_c \in p_2(A)$. Then $\big(T(x, r_c), T(y, r_d)\big) = \big(f(x, r_c), f(y, r_d)\big)$ is an arc in $\dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$ since f is an isomorphism.

(case2) $r_c \in R_n \setminus p_2(A)$. Then $r_c = q_k$ for some $k \in \{1, 2, \ldots, n\}$. Hence $((x, r_c), (y, r_d)) \in E_A$. Then $((x, r_d), (y, r_d))$ is an arc in $\dot\cup_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. By Lemma 3.1, $((x, r), (y, r_d))$ is also an arc in $\dot\cup_{i\in I}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$, it follows that $(f(x, r), f(y, r_d))$ is an arc in $\dot\cup_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Let $f(x, r) = (x', r')$ and $f(y, r_d) = (y', r'_d)$. Therefore $\big((x', r'), (y', r'_d)\big)$ is an arc in $\dot\cup_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$ and thus $\big((x', r'_d), (y', r'_d)\big)$ is also an arc in $\dot\cup_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. By Lemma 3.3, $((x', q'_k), (y', r'_d)) \in E_B$. This mean $(T(x, r_c), T(y, r_d)) = \big((p_1(f(s, r)), q'_k), (y', r'_d)\big) \in E_B$. Hence $(T(x, r_c), T(y, r_d))$ is an arc in $\dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$.

Then we prove that T is a homomorphism.

Assume that $(T(x, r_c), T(y, r_d))$ is an arc in $\dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$, we have $(T(x, r_c), T(y, r_d)) \in E(S, E_B)$. Let $T(y, r_d) = f(y, r_d) = (y', r'_d)$. Then $((p_1(f(x, r)), q'_k), (y', r'_d)) \in E(S, E_B)$ and so $((p_1(f(x, r)), r'_d), (y', r'_d))$ is an arc in $\dot\cup^w_{j\in I}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Hence there exists $(b, r'_d) \in B$ such that

$(y', r'_d) = (p_1(f(x,r)), r'_d)(b, r'_d)$. Then

$$\begin{aligned} f(y, r_d) &= (x', r'_d)(b, r'_d) \\ &= (x'b, r'_d) \\ &= (x', r')(b, r'_d) \\ &= f(x,r)(b, r'_d). \end{aligned}$$

This means that $(f(x,r), f(y,r_d))$ is an arc in $\dot{\bigcup}_{j\in I}^{w}(g_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Then $((x,r),(y,r_d))$ is an arc in $\dot{\bigcup}_{i\in I}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore $((x,r_c),(y,r_d)) \in E_A$ and it is also an arc in $\dot{\bigcup}_{i\in I}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A)$. Thus T^{-1} is a homomorphism. Hence $\dot{\bigcup}_{i\in I}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (S, E_A) \cong \dot{\bigcup}_{j\in I}^{w}(g_j\langle p_1(B)\rangle \times p_2(B), E_j) \cup (S, E_B)$. By Theorem 3.2, we have $Cay(S,A) \cong Cay(S,B)$. $\square$

Example 1. Let $S = \mathbb{Z}_4 \times R_2$ be a right group, $A, B \subseteq S$ where $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, $R_2 = \{r_1, r_2\}$ and $A = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{1}, r_2)\}$, $B = \{(\bar{3}, r_1), (\bar{0}, r_2), (\bar{2}, r_2), (\bar{3}, r_2)\}$.

We have $\hat{A} = \{\hat{A}_{r_1}, \hat{A}_{r_2}\}$, $\hat{B} = \{\hat{B}_{r_1}, \hat{B}_{r_2}\}$ and $|\hat{A}| = |\hat{B}|$. Since $\langle p_1(A)\rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} = \langle p_1(B)\rangle$, $|\langle p_1(A)\rangle| = |\langle p_1(B)\rangle|$ makes condition (1) in Theorem 3.10 satisfied.

We have $\hat{A}_{r_1} = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1)\}$, $\hat{A}_{r_2} = \{(\bar{1}, r_2)\}$ and $\hat{B}_{r_1} = \{(\bar{3}, r_1)\}$, $\hat{B}_{r_2} = \{(\bar{0}, r_2), (\bar{2}, r_2), (\bar{3}, r_2)\}$. Then $|\hat{A}_{r_1}| = 3 = |\hat{B}_{r_2}|$ and $|\hat{A}_{r_2}| = 1 = |\hat{B}_{r_1}|$. There exists a bijective function f from $\hat{A}$ to $\hat{B}$ such that $f(\hat{A}_{r_1}) = \hat{B}_{r_2}$ and $f(\hat{A}_{r_2}) = \hat{B}_{r_1}$ makes condition (2) in Theorem 3.10 satisfied.

Moreover, there are bijective functions

$$\begin{aligned} h_1 : \hat{A}_{r_1} \to \hat{B}_{r_2} \text{ such that } h_1(\bar{0}, r_1) &= (\bar{0}, r_2) \\ h_1(\bar{1}, r_1) &= (\bar{3}, r_2) \\ h_1(\bar{2}, r_1) &= (\bar{2}, r_2) \\ \text{and } h_2 : \hat{A}_{r_2} \to \hat{B}_{r_1} \text{ such that } h_2(\bar{1}, r_2) &= (\bar{3}, r_1) \end{aligned}$$

makes condition (3) in Theorem 3.10 satisfied and it follows that $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$. By Theorem 3.11, we get that $Cay(S,A) \cong Cay(S,B)$. See Fig. 1 and Fig. 2.

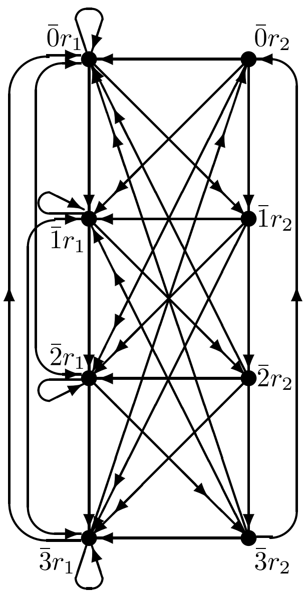

Fig. 1. $Cay(S, A)$

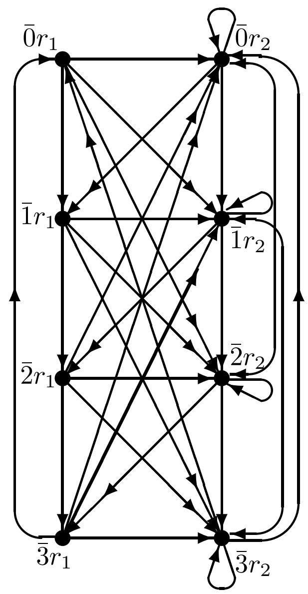

Fig. 2. $Cay(S, B)$

4. Cayley Graphs of Rectangular Groups

By [12], we have the conditions for two Cayley graphs of rectangular group $Cay(S, \{a\})$ and $Cay(S, \{b\})$ being isomorphic.

Theorem 4.1. *Let $S = G \times L_n \times R_m$ be a rectangular group, $a = (g_1, l_1, r_1)$, $b = (g_2, l_2, r_2) \in S$. Then $Cay(S, \{a\}) \cong Cay(S, \{b\})$ if and only if $|g_1| = |g_2|$.*

Lemma 4.2. *Let $S = G \times L_n \times R_m$ be a rectangular group, A nonempty subset of S, and $(g_1, l_1, r_1), (g_2, l_2, r_2) \in S$. Then $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$ if and only if there exists $(a, l, r_2) \in A$ such that $g_2 = g_1 a$ and $l_1 = l_2$.*

Proof. $(\Rightarrow)$ Let $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$. Then there is $(a, l, r) \in A$ such that $(g_2, l_2, r_2) = (g_1, l_1, r_1)(a, l, r) = (g_1 a, l_1, r)$. We have $g_2 = g_1 a, l_2 = l_1$ and $r_2 = r_1$.

$(\Leftarrow)$ Let $(a, l, r_2) \in A$, $g_2 = g_1 a$ and $l_1 = l_2$. Thus $(g_1, l_1, r_1)(a, l, r_2) = (g_1 a, l_1, r_2) = (g_2, l_2, r_2)$. Therefore $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$. □

Next, we describe the Cayley graph of rectangular group.

Lemma 4.3. *Let $S = G \times L_n \times R_m$ be a rectangular group, A nonempty subset of S. Then $Cay(S, A)$ is the disjoint union of n isomorphic strong subdigraphs $(G \times \{l_i\} \times R_m, E_i)$ for $i = 1, 2, ..., n$.*

Proof. For $i = 1, 2, ..., n$, let $V_i := G \times \{l_i\} \times R_m$ and $E_i := E\big(Cay(S, A)\big) \cap (V_i \times V_i)$. Hence (V_i, E_i) is a strong subdigraph of $Cay(S, A)$ and $S = \dot{\cup}_{i=1}^{n} V_i$. Since $E_i \subseteq E\big(Cay(S, A)\big)$, $\dot{\cup}_{i=1}^{n} E_i \subseteq E\big(Cay(S, A)\big)$. Let $\big((g, l_j, r), (g', l_k, r')\big) \in$

$E(Cay(S, A)$. By Lemma 4.2, $l_j = l_k$ and thus $\big((g, l_j, r), (g', l_k, r')\big) \in E_k$. Then $\big((g, l_j, r), (g', l_k, r')\big) \in \dot\cup_{i=1}^{n} E_i$. Hence $E\big(Cay(S, A)\big) \subseteq \dot\cup_{i=1}^{n} E_i$ and so $E\big(Cay(S, A)\big) = \dot\cup_{i=1}^{n} E_i$. Therefore $Cay(S, A) = \dot\cup_{i=1}^{n} (V_i, E_i)$.

We show that (V_i, E_i), $i = 1, 2, ..., n$, are isomorphic. Let $p, q \in \{1, 2, ..., n\}$, $p \neq q$, define $f : (V_p, E_p) \to (V_q, E_q)$ by $f\big((g, l_p, r)\big) = (g, l_q, r)$. Since $|V_p| = |V_q|$, f is a bijection. To prove that f and f^{-1} are digraph homomorphisms. Let $(g, l_p, r), (g', l_p, r') \in V_p$ and $\big((g, l_p, r), (g', p, r')\big) \in E_p$. Since $E_p \subseteq E(Cay(S,$ $A))$, $\big((g, l_p, r), (g', l_p, r')\big)$ is an arc in $Cay(S, A)$. By Lemma 4.2, there exists $(a, l, r'') \in A$ such that $g' = ga$, $r' = r''$, and thus $(g', l_q, r') = (ga, l_q, r'') = (g, l_q, r)$ (a, l, r''). Then $\big((g, l_q, r), (g', l_q, r')\big)$ is an arc in $Cay(S, A)$. It follows that $\big((g, l_q, r),$ $(g', l_q, r')\big) \in E_q$. This shows that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence f is a digraph isomorphism. □

Lemma 4.4. *Let $S = G \times L_n \times R_m$ be a rectangular group. Let A be nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle\ , g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ the set of distinct left coset of $\langle p_1(A)\rangle$ in G, and $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$ a strong subdigraph of $Cay(S\ , A)$. Then the following conditions hold:*

1. *$(G \times \{l_i\} \times R_m, E_i) = \dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$;*
2. *$(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik}) = Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, A^i)$ where $A^i = \{(g, l_i, r) | (g, l, r) \in A$ for all $l \in L_n\}$.*

Proof. (1) We define $f : (G \times \{l_i\} \times R_m, E_i) \to \dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$ by identity mapping. Since $G = \dot\cup_{k=1}^{w} g_k\langle p_1(A)\rangle$, $G \times \{l_i\} \times R_m = \dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m)$ it follows that f is a bijection. We will prove that f and f^{-1} are homomorphisms. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E_i$. By Lemma 4.2, there exists $(a, l, r') \in A$ such that $g' = ga$. Hence $g \in g_p\langle p_1(A)\rangle$, $g' \in g_q\langle p_1(A)\rangle$ for some $p, q \in \{1, 2, ..., w\}$. We get that $(g, , l_i, r), (g', l_i, r') \in \dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m)$. Because $\dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$ is the union of strong subdigraph of $Cay(S, A)$ therefore $\big((g, , l_i, r), (g', l_i, r')\big)$ is an arc in $\dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$. This show that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence f is a digraph isomorphism.

(2) We define $h : (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik}) \to Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, A^i)$ by identity mapping. Clearly, h is a bijection. We will prove that h and h^{-1} are homomorphisms. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E_{ik}$. By Lemma 4.2, there exists $(a, l, r') \in A$ such that $g' = ga$. We get that $(a, l_i, r') \in A^i$ and then $(g', l_i, r') = (ga, l_i, r') = (g, l_i, r')(a, l_i, r')$ it follows that $\big((g, l_i, r), (g', l_i, r')\big)$ is an arc in $Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, A^i)$. This show that h is a digraph homomorphism. Let $\big((g, l_i, r), (g', l_i, r')\big)$ is an arc in $Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, A^i)$. By Lemma 4.2, there exists $(a, l_i, r') \in A^i$ such that $g' = ga$. We get that $(a, j, r') \in A$ for some $j \in L_n$. Then $(g', l_i, r') = (ga, l_i, r') = (g, l_i, r')(a, j, r')$ it follows that $\big((g, l_i, r), (g', l_i, r')\big)$ is an arc in $Cay(S, A)$. Since $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$ is the strong subdigraph of $Cay(S, A)$, $\big((g, l_i, r), (g', l_i, r')\big)$ is an arc in $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$. This show that h^{-1} is a digraph homomorphism. Hence h is a digraph isomorphism. □

Since right groups are some kinds of rectangular groups, we get a condition for Cayley graphs of rectangular groups are isomorphic.

Theorem 4.5. *Let $S = G \times L_n \times R_m$ be a rectangular group, A, B nonempty subsets of S. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(\langle A' \rangle, A') \cong Cay(\langle B' \rangle, B')$ where $A' = \{(g, r) \mid (g, l, r) \in A\}$ and $B' = \{(g', r') \mid (g', l', r') \in B\}$.*

Proof. Let $(G \times \{l_i\} \times R_m, E_i^A)$, $A_i^k = (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, E_{ik})$ be a strong subdigraphs of $Cay(S, A)$ and $(G \times \{l_i\} \times R_m, E_i^B)$, $B_i^t = (g_t\langle p_1(B)\rangle \times \{l_i\} \times R_m, E_{it})$ be a strong subdigraphs of $Cay(S, B)$. By Lemma 4.3 and Lemma 4.4(1), we have
$Cay(S, A) \cong Cay(S, B)$
$\Leftrightarrow Cay(G \times L_n \times R_m, A) \cong Cay(G \times L_n \times R_m, B)$
$\Leftrightarrow \dot\bigcup_{i=1}^{n}(G \times \{l_i\} \times R_m, E_i^A) \cong \dot\bigcup_{i=1}^{n}(G \times \{l_i\} \times R_m, E_i^B)$
$\Leftrightarrow \dot\bigcup_{i=1}^{n}\dot\bigcup_{k=1}^{w} A_i^k \cong \dot\bigcup_{i=1}^{n}\dot\bigcup_{t=1}^{p} B_i^t$

Since A_i^k and B_i^t are connected subdigraphs, $w = p$. Then $A_i^k \cong B_i^t$. Let $D_k^A = (g_k\langle p_1(A)\rangle \times p_2(A'), E_k)$ and $D_t^B = (g_t\langle p_1(B)\rangle \times p_2(B'), E_t)$ be a strong subdigraphs of $Cay(g_k\langle p_1(A)\rangle \times R_m, A')$ and $Cay(g_t\langle p_1(B)\rangle \times R_m, B')$ respectively, and let $A^i = \{(g, l_i, r) \mid (g, l, r) \in A\}$, $B^i = \{(g, l_i, r) \mid (g, l, r) \in B\}$. By Lemma 4.4(2) and Theorem 3.2, we have
$Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_m, A^i) \cong Cay(g_t\langle p_1(B)\rangle \times \{l_i\} \times R_m, B^i)$
$\Leftrightarrow Cay(g_k\langle p_1(A)\rangle \times R_m, A') \cong Cay(g_t\langle p_1(B)\rangle \times R_m, B')$
$\Leftrightarrow \dot\bigcup_{k=1}^{w} D_k^A \cup (g_k\langle p_1(A)\rangle \times R_m, E_{A'}) \cong \dot\bigcup_{t=1}^{p} D_t^B \cup (g_t\langle p_1(B)\rangle \times R_m, E_{B'})$
By Lemma 3.6 and Theorem 3.4, we have $\dot\bigcup_{k=1}^{w} D_k^A \cong \dot\bigcup_{t=1}^{p} D_t^B \Leftrightarrow D_k^A \cong D_t^B \Leftrightarrow Cay(\langle A' \rangle, A') \cong Cay(\langle B' \rangle, B')$. □

Example 2. Let $S = \mathbb{Z}_4 \times L_2 \times R_2$ be a rectangular group, $A, B \subseteq S$ where $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, $L_3 = \{l_1, l_2\}$, $R_2 = \{r_1, r_2\}$ and $A = \{(\bar{0}, l_1, r_1), (\bar{1}, l_2, r_1), (\bar{2}, l_2, r_1), (\bar{1}, l_2, r_2)\}$, $B = \{(\bar{3}, l_1, r_1), (\bar{0}, l_1, r_2), (\bar{2}, l_2, r_2), (\bar{3}, l_2, r_2)\}$.

We have $A^{'} = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{1}, r_2)\}$ and $B^{'} = \{(\bar{3}, r_1), (\bar{0}, r_2), (\bar{2}, r_2), (\bar{3}, r_2)\}$. By Example 1, $Cay(\langle A^{'} \rangle, A^{'}) \cong Cay(\langle B^{'} \rangle, B^{'})$ therefore $Cay(S, A) \cong Cay(S, B)$. See Fig.3 and Fig.4.

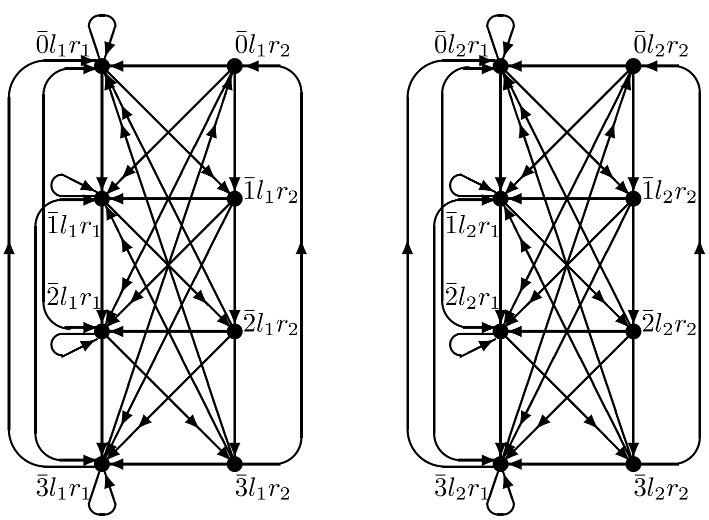

Fig. 3. $Cay(S, A)$

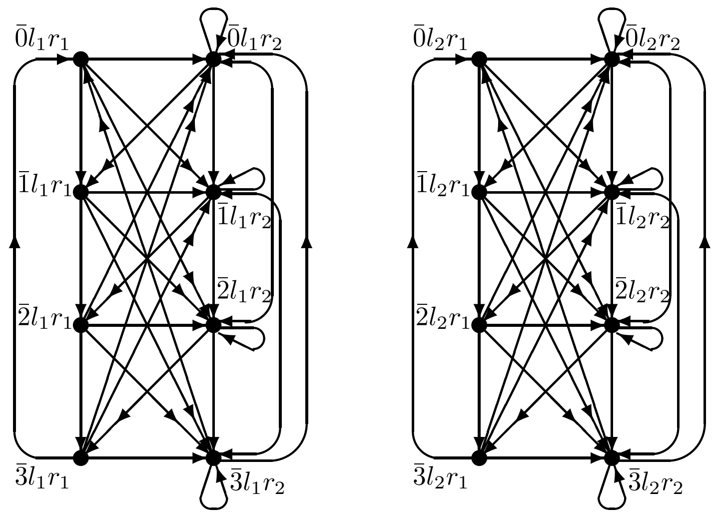

Fig. 4. $Cay(S, B)$

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